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Introduction (français)

Cette thèse est divisée en trois parties qui peuvent être lues indépendamment. Dans la première, on étudie les théorèmes d'universalité pour les *mécanismes* (aussi appelés *systèmes articulés*) dont l'espace ambiant est une surface homogène. Dans la seconde, on étudie un lien entre flots géodésiques et billards, ainsi que la dynamique de certains mécanismes. La troisième porte sur les structures de similitude transverses sur les feuilletages, ainsi que sur le théorème de décomposition de De Rham. Chacune de ces parties contient une introduction propre.

Un mécanisme est un ensemble de tiges rigides reliées par des liaisons pivots. Mathématiquement, on considère un mécanisme comme un graphe muni d'une structure supplémentaire : on associe une longueur à chaque arête (le graphe est dit *métrique*), et certains sommets sont fixés au plan tandis que d'autres évoluent librement. Une *réalisation* d'un mécanisme dans le plan est le choix d'une position dans le plan pour chaque sommet, de sorte que les longueurs associées aux arêtes correspondent aux distances dans le plan entre les sommets correspondants. En particulier, on autorise les arêtes à se croiser. Enfin, l'espace de configuration d'un système articulé est l'ensemble de ses réalisations.

Le premier chapitre constitue une introduction à cette notion de mécanisme : on y donne des éléments historiques, et des exemples fondamentaux, qui sont utiles pour les chapitres suivants.

La première partie, qui suit cette introduction, est constituée de quatre chapitres et correspond à la pré-publication [Kou14] : on y étudie des mécanismes dont l'espace ambiant n'est plus le plan, mais diverses variétés riemanniennes. Le chapitre 2 introduit la question de l'*universalité* des mécanismes : cette notion correspond à l'idée que toute courbe serait tracée par un sommet d'un mécanisme, et que toute variété différentiable serait l'espace de configuration d'un mécanisme. On y présentera des résultats déjà connus qui vont dans ce sens pour les mécanismes dans le plan : d'une part, les courbes que l'on peut tracer sont exactement les courbes semi-algébriques compactes ; et d'autre part, pour toute variété compacte connexe M, il existe[1] un espace de configuration d'un système articulé dont toutes les composantes connexes sont difféomorphes à M. Ce même chapitre contient aussi les énoncés de tous les nouveaux résultats essentiels de cette partie, qui consistent à étendre les théorèmes d'universalité au plan de Minkowski, au plan hyperbolique et enfin à la sphère. Dans chaque cas, les difficultés rencontrées diffèrent, ainsi que les techniques pour les résoudre, mais les résultats obtenus sont très similaires, sauf dans le cas du plan de Minkowski, où l'on s'affranchit de l'exigence de compacité : l'universalité est alors valable dans un sens encore plus large. Les trois derniers chapitres contiennent les démonstrations de ces énoncés, alors que les outils généraux sont donnés

[1] On ne sait toujours pas, cependant, si l'on peut exiger ou non que l'espace de configuration soit difféomorphe à la variété M elle-même.

dans le chapitre 2.

La seconde partie est constituée de trois chapitres, dont les deux derniers correspondent à la pré-publication [Kou15a]. Dans le chapitre 6, on établit un premier lien entre flots géodésiques sur des variétés à courbure négative et billards dispersifs, en mettant en parallèle les comportements de ces deux systèmes. La similitude entre ces deux systèmes est bien connue depuis les travaux de Sinaï dans les années 1960, mais elle est rarement détaillée dans la littérature. Dans le même ordre d'idée, on donne une condition suffisante pour qu'un flot géodésique sur une surface fermée soit Anosov : il suffit que toutes les solutions de l'équation de Ricatti le long des géodésiques, nulles au temps $t = 0$, soient supérieures à une même constante $m > 0$ lorsque $t = 1$. Ce théorème bien connu a été utilisé à plusieurs reprises dans la littérature, mais sans qu'aucune preuve écrite ne semble disponible : nous l'utiliserons à notre tour dans les résultats qui suivent. Dans le chapitre 7, on présente deux résultats nouveaux concernant le flot géodésique de surfaces dans $\mathbb{R}^3$ euclidien, qui ont subi une forte contraction selon l'un des axes. Toute surface dans $\mathbb{R}^3$ peut être aplatie selon l'axe des z, et la surface aplatie s'approche d'une table de billard dans $\mathbb{R}^2$. On montre que, sous certaines hypothèses, le flot géodésique de la surface converge localement uniformément vers le flot de billard. De plus, si le billard est dispersif, les propriétés chaotiques du billard remontent au flot géodésique : on montre qu'il est alors Anosov. Enfin, dans le chapitre 8, on donne des généralités sur la dynamique des systèmes articulés, puis on applique le résultat du chapitre 7 à la théorie des systèmes articulés. Ceci permet d'obtenir un nouvel exemple de mécanisme Anosov, comportant cinq tiges. C'est la première fois qu'on exhibe un système articulé Anosov dont les longueurs des arêtes sont données explicitement. Une vidéo de ce mécanisme, due à Jos Leys, est disponible sur ma page web.

La troisième partie n'a pas de lien direct avec les deux autres, si ce n'est l'étude de variétés riemanniennes : elle correspond à la pré-publication [Kou15b]. On s'intéresse d'abord aux variétés munies de connexions localement métriques, c'est-à-dire de connexions qui sont localement des connexions de Levi-Civita de métriques riemanniennes ; on donne dans ce cadre un analogue du théorème de décomposition de De Rham, qui s'applique habituellement aux variétés riemanniennes. Dans le cas où une telle connexion préserve une structure conforme, on montre que cette décomposition comporte au plus deux facteurs ; de plus, lorsqu'il y a exactement deux facteurs, l'un des deux est l'espace euclidien $\mathbb{R}^q$. On répond ainsi à une question posée dans [MN15b]. L'étude des connexions localement métriques qui préservent une structure conforme est étroitement liée à celle des "structures de similitude" sur les variétés : ce sont les structures obtenues par quotient d'une variété riemannienne M par un sous-groupe de son groupe de similitudes $\mathrm{Sim}(M)$. La démonstration des résultats de cette partie passe par l'étude des *feuilletages munis d'une structure de similitude transverse*. Sur ces feuilletages, on montre un résultat de rigidité qui peut être vu indépendamment des autres : ils sont soit transversalement plats, soit transversalement riemanniens. Remarquons que ces résultats sont valables dans le cas C^∞, alors que de tels problèmes n'avaient été étudiés précédemment que dans le cas analytique.

Introduction (English)

This thesis is divided into three parts which may be read independently. In the first one, we study universality theorems for linkages whose ambiant space is a homogeneous surface. In the second one, we study the link between geodesic flows and billiards, as well as the dynamics of some linkages. The third one is about transverse similarity structures on foliations, and De Rham's decomposition theorem. Each of these parts contains its own introduction.

A linkage is a set of rigid rods joined together by hinges. Mathematically, one considers a linkage as a graph with an additional structure: lengths are given to the edges (the graph is said to be *metric*), and some vertices are fixed to the plane while the others move freely. A *realization* of a linkage in the plane is the choice of a position in the plane for each vertex, so that the edge lengths match with the distance in the plane between the corresponding vertices. In particular, one allows the edges to cross. Finally, the configuration space of a linkage is the set of all its realizations.

The first chapter is an introduction to the notion of linkage: we will present the historical background, and fundamental examples, which are useful for the next chapters.

The first part, after this introduction, is composed of four chapters and corresponds to the preprint [Kou14]: we study linkages whose ambiant space is no longer the plane, but various Riemannian manifolds. Chapter 2 introduces the question of the *universality* of linkages: this notion corresponds to the idea that every curve would be traced out by a vertex of some linkage, and that any differentiable manifold would be the configuration space of some linkage. We shall present some results in this direction which are already known for *planar* linkages: on the one hand, the curves which may be traced out are exactly compact semi-algebraic curves; on the other hand, for any compact connected manifold M, there exists[2] a configuration space of a linkage whose connected components are all diffeomorphic to M. The same chapter also contains the statements of all the main new results of this part, which are extensions of universality theorems to the Minkowski plane, the hyperbolic plane, and finally the sphere. In each case, one encounters different difficulties, and makes use of different techniques, but the results which are obtained are very similar, except in the Minkowski case, where the compacity hypothesis is no longer necessary: universality then becomes valid in a broader sense. The last three chapters contain the proofs of these statements, while the general tools are given in Chapter 2.

The second part is composed of three chapters, where the last two correspond to the preprint [Kou15a]. In Chapter 6, we establish a first link between geodesic flows on negatively curved manifolds and dispersive billiards, by putting in parallel the behaviors of these two systems. It is well-known since Sinai's work in the 1960's that these two

[2] It is still unknown, however, whether it is possible to require that the configuration space be diffeomorphic to the manifold M itself.

systems are similar, but they are rarely studied together in the literature. In the same vein, we give a sufficient condition for a geodesic flow on a closed surface to be Anosov: it suffices that all solutions of the Ricatti equation along the geodesics, which equal zero at time $t = 0$, are greater than a single constant $m > 0$ at time $t = 1$. This well-known theorem has been used several times in the literature, but has apparently never been awarded any written proof: we will use it ourselves in the new results of this part. In Chapter 7, we present two new results concerning the geodesic flow of surfaces in the Euclidean $\mathbb{R}^3$, which undergo a strong contraction in one direction. Any surface in $\mathbb{R}^3$ can be flattened with respect to the z-axis, and the flattened surface gets close to a billiard table in $\mathbb{R}^2$. We show that, under some hypotheses, the geodesic flow of the surface converges locally uniformly to the billiard flow. Moreover, if the billiard is dispersing, the chaotic properties of the billiard also apply to the geodesic flow: we show that it is Anosov in this case. Finally, in Chapter 8, we give generalities on the dynamics of linkages, and then apply the result of Chapter 7 to the theory of linkages. This provides a new example of Anosov linkage, made of 5 rods. It is the first time that one exhibits an Anosov linkage whose edge lengths are given explicitly. A video of this linkage, by Jos Leys, is available on my website.

The third part does not have a direct link with the two others, except for the study of Riemannian manifolds: it corresponds to the preprint [Kou15b]. We first consider manifolds with locally metric connections, that is, connections which are locally Levi-Civita connections of Riemannian metrics; we give in this framework an analog of De Rham's decomposition theorem, which usually applies to Riemannian manifolds. In the case such a connection also preserves a conformal structure, we show that this decomposition has at most two factors; moreover, when there are exactly two factors, one of them is the Euclidean space $\mathbb{R}^q$. Thus, we answer a question asked in [MN15b]. The study of locally metric connections which preserve a conformal structure is closely linked to "similarity structures" on manifolds: these are the structures obtained by the quotient of a Riemannian manifold M by a subgroup of its similarity group $\mathrm{Sim}(M)$. The proofs of the results of this part use *foliations with transverse similarity structures*. On these foliations, we give a rigidity theorem of independant interest: they are either transversally flat, or transversally Riemannian. Notice that these results are valid in the C^∞ case, while such problems had only been studied in the analytic case previously.

Chapter 1

What is a linkage?

A *mechanical linkage*, or simply *linkage*, is a graph whose vertices are considered as rigid rods. Let us state precise mathematical definitions.

Definition 1.1. A *planar linkage* $\mathcal{L}$ is a graph (V, E) together with:

1. A function $l : E \to \mathbb{R}_{\geq 0}$ (which gives the length of each edge);
2. A subset $F \subseteq V$ of *fixed vertices* (represented by ■ on the figures);
3. A function $\phi_0 : F \to \mathbb{R}^2$ which indicates where the vertices of F are fixed.

Definition 1.2. Let $\mathcal{L}$ be a planar linkage, and consider the Euclidean distance δ in $\mathbb{R}^2$. A *realization* of a planar linkage $\mathcal{L}$ is a function $\phi : V \to \mathbb{R}^2$ such that:

1. For each edge $(v_1 v_2) \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1 v_2)$;
2. $\phi|_F = \phi_0$.

The configuration space $\mathrm{Conf}(\mathcal{L})$ is the set of all realizations ϕ of $\mathcal{L}$: it is naturally a subset of $(\mathbb{R}^2)^n$, where n is the number of vertices. It inherits the topology of the ambiant Euclidean space. Finally, the *workspace* of a vertex $v \in V$ is the set $\{\phi(v) \mid \phi \text{ realization of } \mathcal{L}\}$.

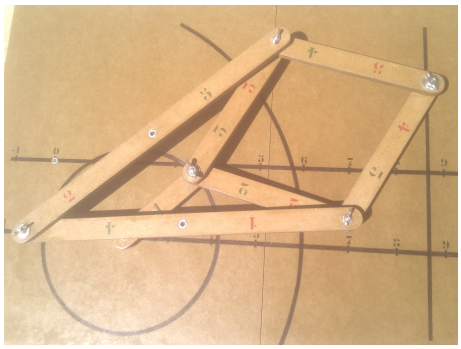


Figure 1.1 – A wooden realization of the Peaucellier straight-line motion linkage. Design: Adriane Kaïchouh, Mickaël Kourganoff, Thomas Letendre. Construction: Pierre Gallais.

Linkages are one of the simplest physical examples involving manifolds of dimension 3 or more (other than the ambiant space $\mathbb{R}^3$): they appear naturally as configuration spaces. In Section 1.1, we will give simple examples of linkages whose configuration space is diffeomorphic to $\mathbb{T}^n$ or $\mathbb{S}^n$, for any natural number n. Generically, the dimension of the configuration space of a linkage is $2(|V| - |F|) - |E|$ (twice the number of free vertices, minus the number of edges). More precisely:

Proposition 1.3. *Choose any graph (V, E), any $F \subseteq V$ and any $\phi_0 : F \to \mathbb{R}^2$. Then there is a set L of full Lebesgue measure in $\mathbb{R}^E$ such that for all choice of edge lengths $l \in L$, the configuration space of $\mathcal{L} = (V, E, l, F, \phi_0)$ is a smooth orientable manifold of dimension $2(|V| - |F|) - |E|$.*

Proof. Consider the function

$$\begin{aligned} F : (\mathbb{R}^2)^{V \setminus F} &\to \mathbb{R}^E \\ \phi &\mapsto f_\phi \end{aligned}$$

where, for any $(vw) \in E$ and any $\phi \in (\mathbb{R}^2)^{V \setminus F}$,

$$f_\phi(vw) = \delta(\phi(v), \phi(w))$$

(here, the domain of ϕ is extended to the whole V using ϕ_0).

Then for any $l \in \mathbb{R}^E$, the configuration space of $\mathcal{L} = (V, E, l, F, \phi_0)$ is $F^{-1}(l)$.

By Sard's Theorem, the regular values of F form a set of full Lebesgue measure in $\mathbb{R}^E$. For such a value l, $F^{-1}(l)$ is a smooth manifold of dimension $2(|V| - |F|) - |E|$. Moreover, since $\mathbb{R}^E$ is orientable, the normal bundle of $F^{-1}(l)$ in $(\mathbb{R}^2)^{V \setminus F}$ is orientable as well. But $(\mathbb{R}^2)^{V \setminus F}$ itself is also orientable: one obtains an orientation of the tangent bundle of M, thus M is orientable. □

Remark. Most of the linkages considered in Part I will *not* satisfy the assumptions of Proposition 1.3. In this case, the configuration space may still be a smooth manifold, but it does not need to be orientable, and it is impossible to compute its dimension from the number of edges and vertices alone.

There exist many natural problems involving linkages, which cover various fields of mathematics, such as algebraic geometry, algebraic topology, Riemannian geometry, dynamical systems, and the theory of computational complexity. Many interesting problems involving complexity may be found in [DO07]. In this thesis, we focus on two particular aspects of linkages: universality and dynamics. The examples in the rest of this chapter are chosen in view of these two problems.

1.1 Fundamental examples

1.1.1 The robotic arm

The robotic arm $\mathcal{R}_n$ (Figure 1.2) is a linkage whose underlying graph is a path (all vertices have degree 2, except the two ends), with one fixed end. It has n edges of lengths $l_1, l_2, \ldots, l_n$.

The configuration space of $\mathcal{R}_n$ is the torus $\mathbb{T}^n$: each of its configurations corresponds to n angles $\theta_1, \ldots, \theta_n$ formed by each of the n edges with the horizontal axis.

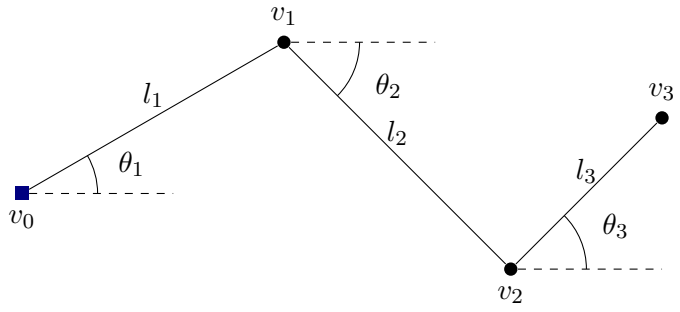


Figure 1.2 – The robotic arm $\mathcal{R}_3$. Recall that fixed vertices (here v_0) are represented by squares.

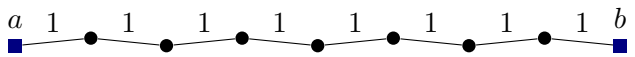
1.1.2 Polygons

A polygon is a linkage without fixed vertices, whose underlying graph is a cycle.

It is often convenient to consider a polygon with two fixed adjacent vertices. In fact, fixing those two vertices amounts to removing the factor $SO(2) \ltimes \mathbb{R}^2$ which is found in the configuration space of any linkage without fixed vertices.

After fixing these two vertices at a distance which corresponds to the edge between them, one may remove this edge which has become useless, without changing the configuration space. Thus, a polygon may be seen as a robotic arm whose two ends are fixed.

Example.



Consider the linkage $\mathcal{L}_n$ above with n edges. The vertices a and b are fixed at a distance $n - \epsilon$ with a small enough $\epsilon > 0$, and all the edges have length 1.

Proposition 1.4. *The configuration space* $\mathrm{Conf}(\mathcal{L}_n)$ *is diffeomorphic to* $\mathbb{S}^{n-2}$.

Proof. We may assume that $\phi_0(a) = (0, 0)$ and $\phi_0(b) = (n - \epsilon, 0)$.

First, consider the robotic arm $\mathcal{R}_n$. For any configuration $(\theta_1, \dots, \theta_n) \in \mathrm{Conf}(\mathcal{R}_n)$, write $s = (s_1, s_2) = (\sum \cos\theta_i, \sum \sin\theta_i) \in \mathbb{R}^2$ the position of the last vertex v_n. Then $\mathrm{Conf}(\mathcal{L}_n) = \{(\theta_1, \dots, \theta_n) \in \mathbb{T}^n \mid s = (n - \epsilon, 0)\}$: for a small $\epsilon > 0$, $\mathrm{Conf}(\mathcal{L}_n)$ is contained in an arbitrarily small neighborhood of $\mathcal{C}_0 = (0, \dots, 0)$.

Now, consider the subset of all configurations whose last vertex lies on the horizontal axis: $E = \{\mathcal{C} \in \mathbb{T}^n \mid s_2 = 0\}$. Since the gradient of s_2 is $\nabla s_2 = \begin{pmatrix} \cos\theta_1 \\ \vdots \\ \cos\theta_n \end{pmatrix}$, E is a submanifold of $\mathbb{T}^n$ in a neighborhood of $\mathcal{C}_0$.

Compute the differentials of s_1 up to order 1 and 2:

$$\nabla s_1 = -\begin{pmatrix} \sin\theta_1 \\ \vdots \\ \sin\theta_n \end{pmatrix} \text{ and } D^2 s_1 = -\begin{pmatrix} \cos\theta_1 & & 0 \\ & \ddots & \\ 0 & & \cos\theta_n \end{pmatrix}.$$

Thus, $Ds_1(\mathcal{C}_0) = 0$ and $D^2 s_1(\mathcal{C}_0)$ is non-degenerate. Moreover, s_1 reaches its global maximum n at $\mathcal{C}_0$. With Morse's Lemma, E is equipped with a global coordinate system $(x_i)_{1 \leq i \leq n-1}$ near $\mathcal{C}_0$ such that:

$$s_1 - n = -\left(\sum_{i=1}^{n-1} x_i^2\right).$$

In particular, the level of E defined by $s_1 = n - \epsilon$ is diffeomorphic to a sphere of dimension $n-2$ for a small enough $\epsilon > 0$. □

This proof gives us a glimpse of how Morse theory may be used in view of determining the topology of some configuration spaces. The first chapter of [Far08] uses this approach to describe the homology groups of the configuration spaces of polygons.

1.1.3 Spider linkages

Although less famous than polygons, spider linkages (with n legs) provide interesting examples and were studied by many authors.

A spider is made of a central vertex to which n legs are attached, each of which has one articulation (each leg is a copy of $\mathcal{R}_2$). The end of each leg is fixed somewhere in the plane. For $n = 2$, the spider is in fact a pentagonal linkage.

Since a spider has $2n$ edges and $n+1$ free vertices, its configuration space is a surface for a generic choice of the lengths of the edges: one may obtain a wide variety of surfaces in this way.

Spiders with $n = 3$. Thurston and Weeks [TW84] detailed a particular case of a spider with $n = 3$, which they called *triple linkage* (Figure 1.3).

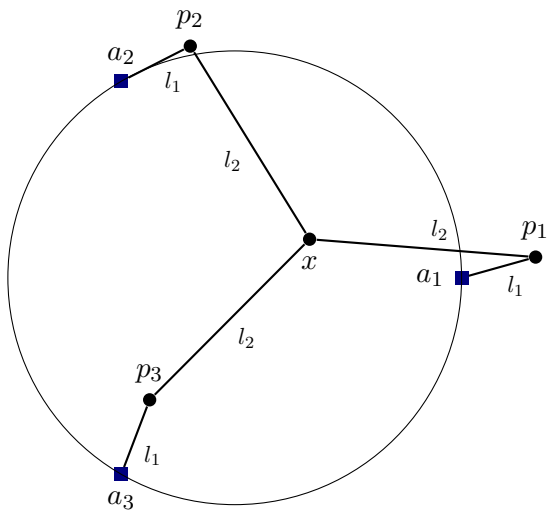


Figure 1.3 – Thurston and Weeks' triple linkage. The three fixed vertices are on the unit circle and form an equilateral triangle, and there are two length parameters l_1 and l_2.

In this case, each of the 3 legs of the spider restricts the movement of the central vertex to an annulus centered at a_i, with inner radius $|l_1 - l_2|$ and outer radius $l_1 + l_2$.

Thus, the workspace of the central vertex is the intersection of three annuli. When the lengths l_1 and l_2 vary, this intersection may take different shapes (Figure 1.4).

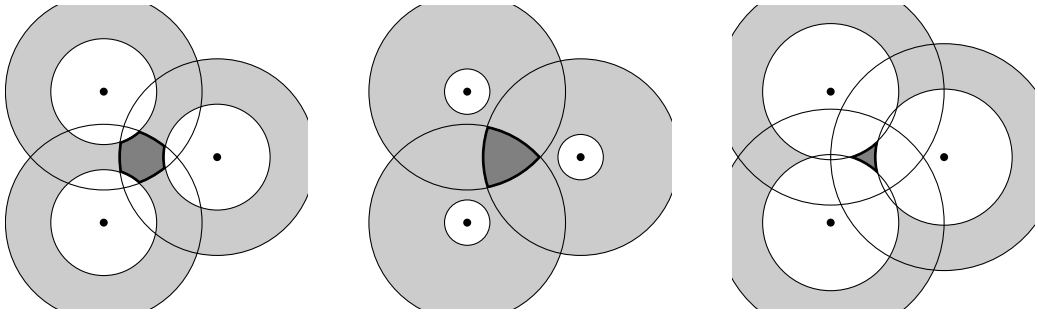


Figure 1.4 – The workspace of the central vertex in Thurston's triple linkage (in dark grey), for different choices of the lengths.

Like Thurston and Weeks in their article, let us focus on the case[1] on the left of Figure 1.4: the workspace of the central vertex x is a hexagon. For each position of x in the interior of the hexagon, there are two possible positions for p_1, which are symmetric with respect to the line through x and a_1. There are also two possible positions for each of the two other vertices, so any point in the interior of the workspace of the central vertex corresponds to 8 points in the configuration space. The boundary of the hexagon corresponds to configurations in which at least one of the arms is completely stretched or folded. Such configurations belong in fact to several hexagons. Thus, the configuration space of the linkage is made of 8 copies of the hexagon, glued together along their boundaries. Each edge belongs to two hexagons, and each vertex to four hexagons, so the polyhedron has 8 faces, 24 edges and 12 vertices. Its Euler characteristic is $8 - 24 + 12 = -4$, so[2] it is diffeomorphic to a surface of genus 3.

For different choices of the edge lengths and positions of the fixed points, the intersection of the three annuli may take many forms: in each case, it is possible to make a similar computation to determine the genus of the surface (for example, it is possible to obtain the disjoint union of 6 spheres, or a surface of genus 12). Ten different topologies for the configuration space of the triple linkage are given in [HM03].

The general case. In general, we have the following [Mou11]:

Theorem 1.5 (Mounoud, 2009)**.** *Let g be an natural number and r the biggest integer such that 2^r divides $g-1$. A compact orientable surface of genus g is diffeomorphic to a spider's configuration space if and only if $(1/2^r)(g-1) \leq 6r + 12$.*

In particular, it is impossible to realize a surface of genus 14 as a configuration space of a spider.

In 2006, O'Hara [O'H07] computed all the configuration spaces obtained by a spider whose arms all have the same length length ($l_1 = l_2 = 1$, with the notations of Figure 1.3), and whose fixed vertices are on the unit circle and form a regular polygon $\mathcal{P}$ in $\mathbb{R}^2$.

[1] This case corresponds, for example, to the condition $(3/2 - |l_1 - l_2|)^2 + 3/4 < (l_1 + l_2)^2 < (3/2 + |l_1 - l_2|)^2 + 3/4$ and $|l_1 - l_2| < \sqrt{3}/2$.

[2] In fact, we should show first that the configuration space is a smooth orientable surface, which is not so obvious, even for generic lengths l_1 and l_2. For example, one may compute directly the differential of the function F of Proposition 1.3.

Theorem 1.6 (O'Hara, 2006)**.** *Let R be the radius of the circumscribed circle to the polygon $\mathcal{P}$. There exists a critical value R_n such that the configuration space is diffeomorphic to a connected orientable closed surface Σ_g if R satisfies:*

$$0 < R < 2 \ \text{ and } \ R \neq R_n.$$

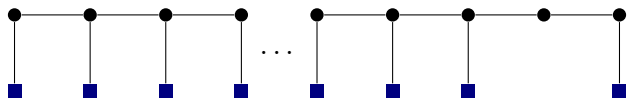
The genus g is given by

$$g = \begin{cases} 1 - 2^{n-1} + n2^{n-3} + n2^{n-1} = 1 + (5n-4)2^{n-3} & si \ \ 0 < R < R_n, \\ 1 - 2^{n-1} + n2^{n-3} \qquad\qquad\quad = 1 + (n-4)2^{n-3} & si \ \ R_n < R < 2. \end{cases}$$

In his proof, O'Hara gives two methods to compute the genus: one of them is purely topological, while the other one uses Morse Theory. In the same paper, he also describes the singularities which appear when R does not satisfy the conditions of Theorem 1.6.

1.1.4 Centipedes

A centipede[3] with n legs (Figure 1.5) is a linkage whose underlying graph has $2n+1$ vertices, where $n+1$ free vertices form a path, and n fixed vertices are attached to the 1st, 2nd, ..., $(n-1)^{\text{th}}$ and $(n+1)^{\text{th}}$ vertex of the path, as in the following figure:



with any edge lengths, and any positions for the fixed vertices.

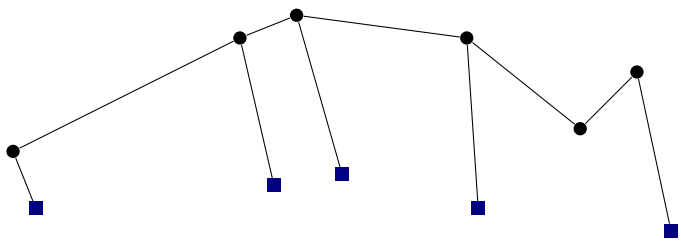


Figure 1.5 – A centipede with 5 legs.

As for spiders, the configuration spaces of centipedes are generically surfaces. It is remarkable that any connected closed oriented surface is the configuration space of some centipede, as shown in [JS01].

In general, given a connected closed manifold M, it is unknown[4] whether there exists a linkage whose configuration space is diffeomorphic to M. This kind of problem is called "universality problem": it is at the heart of Part I.

1.1.5 The pantograph

The pantograph (literally, a device which "writes everything" in Greek), invented by the astronomer Christoph Scheiner in 1603, was used to reproduce drawings at different

[3] These linkages seem to have no standard name in the literature.

[4] Even for connected closed non-orientable surfaces!

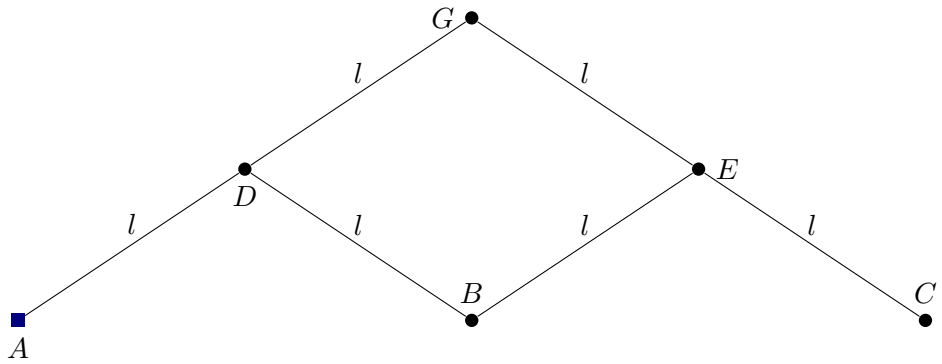


Figure 1.6 – The pantograph. There is one rigid bar joining A (resp. C) to G, with a hinge D (resp. E) at the middle. The vertex A is fixed.

scales (Figure 1.6). The point C is the image of the point B by a homothety of center A and ratio 2. It is possible to obtain any other ratio by changing the edge lengths. In practice, a pen was fixed to the vertex C and the vertex B was moved along the drawing which was to be copied. For this linkage (among others), we will be interested in the possible positions of the two vertices B and C, rather than the topology of the configuration space.

Concerning the pantograph, two remarks are in order:

1. Here, we allowed some hinges (D and E) to be at the middle of bars, while our definition of a linkage as a mathematical graph requires them to be at the end. We could change the definition to include this situation, but in our setting, it is more convenient to consider AD and DG as two different edges of length l, as well as another edge of length $2l$ between A and G. The three vertices A, D, G form a flat triangle, so they are aligned for all configurations. With this technique, it is possible to add a hinge anywhere on a bar.

2. With our definitions, there are in fact many realizations of this linkage such that C is *not* the image of B by a homothety of center A. For example, for any position of A, C, G, there is a realization such that $B = G$. This is known as the problem of *degenerated configurations*: they have to be dealt with carefully when trying to understand the topology of configuration spaces. The same problem will occur in Section 1.2.2. For more details, see [KM02] or Chapter 3.

1.2 Straight-line motion

The problem of straight-line motion appeared naturally when Watt designed his double-action steam engine in 1781. He needed a mechanism able to guide the piston of the engine along a straight line, and to transmit the energy to other elements of the system (for example, a wheel). With our definitions, the question was the following:

Question. Does there exist a linkage containing one vertex whose workspace is a line segment?

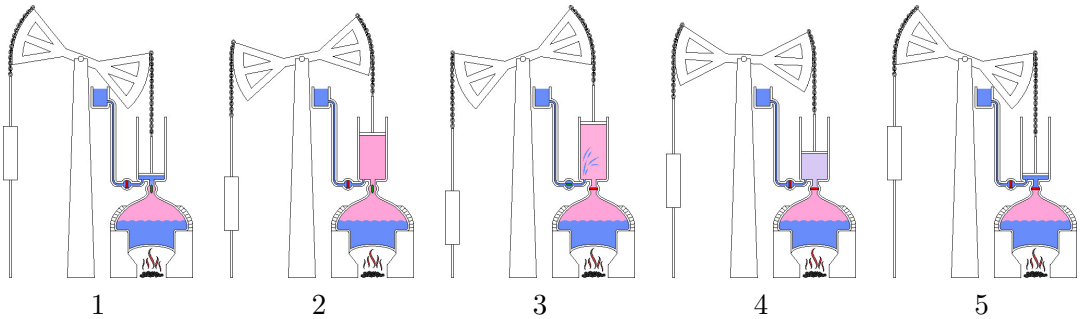


Figure 1.7 – Newcomen's steam engine[5]. The cylinder (on the right) is filled with steam while the pump (on the left) is pulled down by its own weight (2). Then cold water is injected into the cylinder (3), which condensates the steam, creates vacuum and lowers the piston (4): at the other side, the pump goes up and takes the water out from the mine.

Earlier steam engines did not require such a mechanism. Half a century before Watt, Newcomen designed another steam engine, which was widely used to pump water from the coal mines. The steam only pulled the piston to one side (contrary to Watt's double-action engine, where steam pulled it alternatively to both sides), and the mass of the pump on the other side pulled the piston back to its original position (Figure 1.7). In Newcomen's engine, it was possible to achieve straight-line motion with a simple flexible chain. In contrast, Watt needed a rigid linkage to guide the piston.

1.2.1 Watt's linkage

Watt's linkage (Figure 1.8) contains one vertex whose workspace is close to a straight line. It was used in Watt's famous *double-action steam engine*, and is still used in the suspension systems of some cars.

For engineers, the problem of straight-line motion was solved, but for mathematicians, it was only the beginning.

Elementary computation shows that the workspace of the central vertex is the curve of equation (in polar coordinates):

$$r^2 = b^2 - (a \sin\theta \pm \sqrt{c^2 - a^2 \cos^2\theta})^2.$$

It is called *Watt's curve* and, with a good choice of parameters, it has the shape of an eight (like on the figure). Near the center, it has curvature 0, so it is a straight line up to order 2, which is sufficient for most practical applications.

1.2.2 The Peaucellier inversor

In the 1860's, Peaucellier and Lipkin discovered simultaneously a linkage which achieved perfect straight-line motion. First, we introduce the Peaucellier inversor (Figure 1.9).

[5] © User:Emoscopes / Wikimedia Commons / CC-BY-SA-3.0

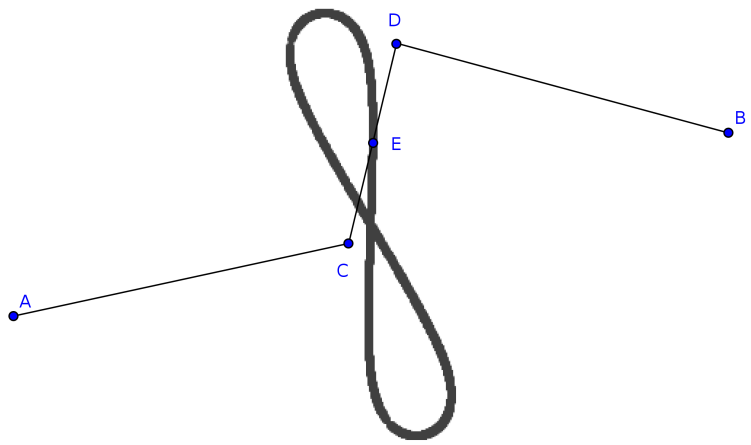


Figure 1.8 – Watt's linkage (obtained with Geogebra), with the workspace of the central vertex E. It has two fixed vertices A and B, and an additional vertex E at the middle of the edge CD. It is made of three bars AC, CD and DB: the bar CD has length $2c$, while AC and DB have the same length length b. The distance between the fixed vertices A and B is $2a$.

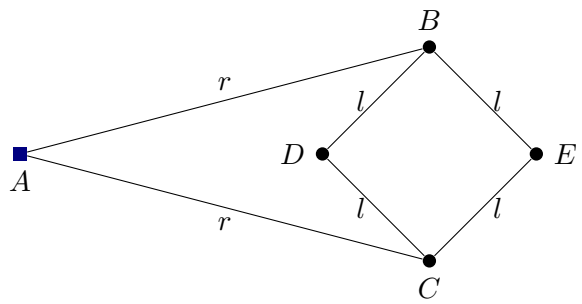


Figure 1.9 – The Peaucellier inversor. We assume that $r > l$.

Proposition 1.7. *For any position of the Peaucellier linkage, the points A, D and E are aligned, and $AD \cdot AE = r^2 - l^2$. In other words, E is the image of D by an inversion with respect to the circle with center A and radius $\sqrt{r^2 - l^2}$.*

Proof. Each of the points A, D and E is equidistant to the two points B and C, so A, D and E are aligned.

Let H be the intersection of the segments BC and DE. Then by the Pythagorean theorem, $BH^2 = l^2 - DH^2 = r^2 - AH^2$. Thus, $AH^2 - DH^2 = r^2 - l^2$, so $AD \cdot AE = r^2 - l^2$. □

In Chapters 3 and 4, this linkage will be adapted to other ambiant geometries.

1.2.3 The Peaucellier straight-line motion linkage.

It is obtained by adding one fixed vertex and one edge to the Peaucellier inversor (Figure 1.10).

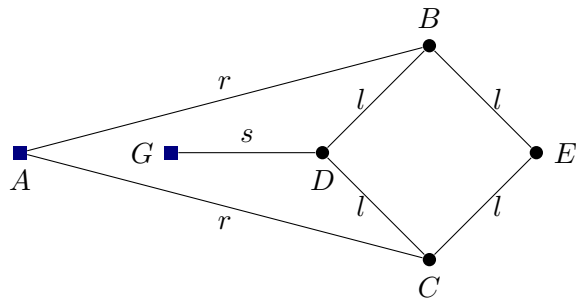


Figure 1.10 – The Peaucellier straight-line motion linkage. The distance between the two fixed points A and G is equal to s, the length of the edge GD.

The workspace of D is contained in a circle $\mathcal{C}$ centered at G, so the workspace of E is contained in the image of $\mathcal{C}$ by the inversion with respect to the circle centered at A, of radius $r^2 - l^2$. If one chooses the position of G and the length of the new edge s so that $A \in \mathcal{C}$, then the image of $\mathcal{C}$ by an inversion centered at A is a straight line. Therefore, the workspace of E is contained in a straight line (more precisely, it is a line segment).

A popular catchphrase is the following: "The Peaucellier linkage transforms linear motion into circular motion." Indeed this is true in some sense, since D's workspace is contained in a circle, while E's workspace is a line segment. However, this formulation might let think that, in a steam engine, D corresponds to a wheel and E to the piston, which cannot be the case: the vertex D does not follow a whole circle, but only goes back and forth on a circular arc! In fact, the only important fact in the Peaucellier linkage is that one vertex follows a straight line. Once this goal is achieved, is it possible to transmit the energy to a wheel using simply one bar, fixed somewhere on the wheel (Figure 1.11).

1.2.4 Hart's linkage

In 1875, Harry Hart discovered a new linkage for inversion, with only four bars (Figure 1.12). Similarly to the Peaucellier linkage, it is possible to add one fixed vertex and one edge to obtain Hart's straight-line motion linkage.

1.2.5 Other straight-line mechanisms

Many other mathematicians discovered linkages which provide approximate or exact linear motion, including Chebyshev, Kempe and Sylvester (see [Kem77] for a detailed review of such linkages).

Very recently, the Dutch artist Theo Jansen designed his own approximate straight-line motion linkage. It allows his large "creatures" made of plastic rods to "walk" smoothly on the beach.

[6] © User:Panther / Wikimedia Commons / CC-BY-SA-3.0

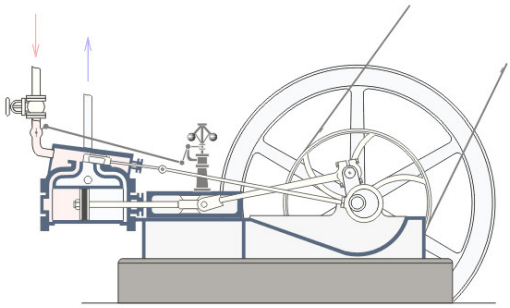


Figure 1.11 – A double-action steam engine[6]. Here, the straight-line motion linkage, used to guide the piston along a straight line, is not represented. The energy of the piston is transmitted to the wheel with only one bar.

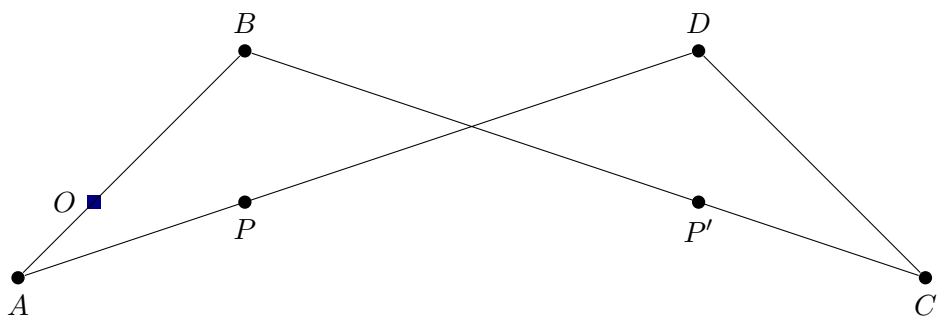


Figure 1.12 – Hart's inversor. The bars AB and DC have the same length, as well as the bars AD and BC. The point O is located on the bar AB, so that $AO/AB = \mu$. Likewise, P is on the bar AD, P' is on the bar CB, and $AP/AD = CP'/CB = \mu$. It may be shown that P' is the image of P by inversion with respect to a circle of center O (for a proof, see [DO07] for example).

Part I

Universality theorems for linkages in homogeneous surfaces

Chapter 2

Introduction and generalities on universality

Throughout Part I, we shall consider linkages which are not necessarily planar: the ambiant space may be any manifold $\mathcal{M}$ instead of $\mathbb{R}^2$.

A *realization* of a linkage $\mathcal{L}$ in a manifold $\mathcal{M}$ is a mapping which sends each vertex of the graph to a point of $\mathcal{M}$, respecting the lengths of the edges. The *configuration space* $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L})$ is the set of all realizations of $\mathcal{L}$ in $\mathcal{M}$. This supposes, classically, the ambient manifold $\mathcal{M}$ to have a Riemannian structure: thus the configuration space may be seen as the space of "isometric immersions" of the metric graph $\mathcal{L}$ in $\mathcal{M}$.

Here we will always deal with (non-trivially) marked connected graphs, that is, a non-empty set of vertices have fixed realizations (in fact, when $\mathcal{M}$ is homogeneous, considering a linkage without fixed vertices only adds a translation factor to the configuration space). Hence, our configurations spaces will be compact even if $\mathcal{M}$ is not compact, but rather complete.

2.1 Some historical background

Most existing studies deal with the special case where $\mathcal{M}$ is the Euclidean plane and some with the higher dimensional Euclidean case (see for instance [Far08] and [Kin98]). There are also studies about polygonal linkages in the standard 2-sphere (see [KM+99]), or in the hyperbolic plane (see [KM96]).

Universality theorems. When $\mathcal{M}$ is the Euclidean plane $\mathbb{E}^2$, a configuration space is an algebraic set. This set is smooth for a generic length structure on the underlying graph.

Universality theorems tend to state that, playing with mechanisms, we get any algebraic set of $\mathbb{R}^n$, and any manifold, as a configuration space! In contrast, it is a hard task to understand the topology or geometry of the configuration space of a given mechanism, even for a simple one.

Universality theorems have been announced in the ambient manifold $\mathbb{E}^2$ by Thurston in oral lectures, and then proved by Kapovich and Millson in [KM02]. They have been proved in $\mathbb{E}^n$ by King [Kin98], and in $\mathbb{R}P^2$ and in the 2-sphere by Mnëv (see [Mnë88] and [KM02]). It is our aim in Part I to prove them in the cases of: the hyperbolic plane $\mathbb{H}^2$, the sphere $\mathbb{S}^2$ and the (Lorentz-)Minkowski plane $\mathbb{M}$. These are simply connected homogeneous pseudo-Riemannian surfaces (the list of such spaces includes in addition the

Euclidean and the de Sitter planes). Then it becomes natural to ask whether universality theorems hold in a more general class of manifolds, for instance on Riemannian surfaces without a homogeneity hypothesis.

In order to be more precise, it will be useful to introduce partial configuration spaces: for W a subset of the vertices of $\mathcal{L}$, one defines $\mathrm{Conf}^W_{\mathcal{M}}(\mathcal{L})$ as the set of realizations of the subgraph induced by W that extend to realizations of $\mathcal{L}$. One has in particular a restriction map: $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) \to \mathrm{Conf}^W_{\mathcal{M}}(\mathcal{L})$.

If $W = \{a\}$ is a vertex of $\mathcal{L}$, its partial configuration space is its *workspace*, *i.e.* the set of all its positions in $\mathcal{M}$ corresponding to realizations of $\mathcal{L}$.

Euclidean planar linkages. Now regarding the algebraic side of universality, the history starts (and almost ends) in 1876 with the well-known Kempe's theorem [Kem76]:

Theorem 2.1. *Any algebraic curve of the Euclidean plane $\mathbb{E}^2$, intersected with a Euclidean ball, is the workspace of some vertex of some mechanical linkage.*

This theorem has the following natural generalization, which we will call the *algebraic universality theorem*, proved by Kapovich and Millson (see [KM02]):

Theorem 2.2. *Let A be a compact semi-algebraic subset (see Definition 2.12) of $(\mathbb{E}^2)^n$ (identified with $\mathbb{R}^{2n}$). Then, A is a partial configuration space $\mathrm{Conf}^W_{\mathbb{E}^2}(\mathcal{L})$ of some linkage $\mathcal{L}$ in $\mathbb{E}^2$. When A is algebraic, one can choose $\mathcal{L}$ such that the restriction map $\mathrm{Conf}_{\mathbb{E}^2}(\mathcal{L}) \to A = \mathrm{Conf}^W_{\mathbb{E}^2}(\mathcal{L})$ is a smooth finite trivial covering.*

When $\mathrm{Conf}_{\mathbb{E}^2}(\mathcal{L})$ is not a smooth manifold, as usual, by a smooth map on it, we mean the restriction of a smooth map defined on the ambient $\mathbb{R}^{2n}$.

From Theorem 2.2, Kapovich and Millson easily derive the *differential universality theorem* on the Euclidean plane:

Theorem 2.3. *Any compact connected smooth manifold is diffeomorphic to one connected component of the configuration space of some linkage in the Euclidean plane $\mathbb{E}^2$. More precisely, there is a configuration space whose components are all diffeomorphic to the given differentiable manifold.*

Jordan and Steiner also proved a weaker version of this theorem with more elementary techniques (see [JS99]).

How to go from the algebraic universality to the differentiable one? The differentiable universality theorems (Theorems 2.3, 2.5 and 2.7) follow immediately from the algebraic ones (Theorems 2.2, 2.4 and 2.6) once we know which smooth manifolds are diffeomorphic to algebraic sets. In 1952, Nash [Nas52] proved that for any smooth connected compact manifold M, one may find an algebraic set which has one component diffeomorphic to M. In 1973, Tognoli [Tog73] proved that there is in fact an algebraic set which is diffeomorphic to M (a proof may be found in [AK92], or in [BCR98]).

In the non-compact case (in which we will be especially interested), Akbulut and King [AK81] proved that every smooth manifold which is obtained as the interior of a compact manifold (with boundary) is diffeomorphic to an algebraic set. Note that conversely, any (non-singular) algebraic set is diffeomorphic to the interior of a compact manifold with boundary.

2.2 Results

It is very natural to ask if these algebraic and differential universality theorems can be formulated and proved for configuration spaces in a general target space $\mathcal{M}$. Our results suggest this could be true: indeed, we naturally generalize universality theorems to the cases of $\mathcal{M} = \mathbb{M}$, $\mathbb{H}^2$ and $\mathbb{S}^2$, the Minkowski and hyperbolic planes and the sphere, respectively. Notice that for a general $\mathcal{M}$, there is no notion of algebraic subset of $\mathcal{M}^n$! We will however observe that there is a natural one in the cases we are considering here. In the general case, the question around Kempe's theorem could be rather formulated as: "Characterize curves in $\mathcal{M}$ that are workspaces of some vertex of a linkage."

Minkowski planar linkages. These linkages are studied in Chapter 3. Classically, the structure of $\mathcal{M}$ needed to define realizations of a linkage is that of a Riemannian manifold. Observe however that a distance, not necessarily of Riemannian type, on $\mathcal{M}$ would also suffice for this task. But our idea here is instead to relax positiveness of the metric. Instead of a Riemannian metric, we will assume $\mathcal{M}$ has a pseudo-Riemannian one. We will actually restrict ourselves to the simple flat case where $\mathcal{M}$ is a linear space endowed with a non-degenerate quadratic form, and more specially to the 2-dimensional case, that is the Minkowski plane $\mathbb{M}$. On the graph side, weights of edges are no longer assumed to be positive numbers. This framework extension is mathematically natural, and may be related to the problem of the embedding of causal sets in physics, but the most important (as well as exciting) fact for us is that configuration spaces are (a priori) no longer compact, and we want to see what new spaces we get in this new setting.

The Lorentz-Minkowski plane $\mathbb{M}$ is $\mathbb{R}^2$ endowed with a non-degenerate indefinite quadratic form. We denote the "space coordinate" by x and the "time coordinate" by t.

The configuration space $\mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ is an algebraic subset (defined by polynomials of degree 2) of $\mathbb{M}^n = \mathbb{R}^{2n}$ (n is the number of vertices of $\mathcal{L}$), and similarly a partial configuration space $\mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L})$ is semi-algebraic (see Definiton 2.12). In contrast to the Euclidean case, these sets may be non-compact (even if $\mathcal{L}$ has some fixed vertices in $\mathbb{M}$). We will prove:

Theorem 2.4. *Let A be a semi-algebraic subset of $\mathbb{M}^n$ (identified with $\mathbb{R}^{2n}$). Then, A is a partial configuration space $\mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L})$ of some linkage $\mathcal{L}$ in $\mathbb{M}$. When A is algebraic, one can choose $\mathcal{L}$ such that the restriction map $\mathrm{Conf}_{\mathbb{M}}(\mathcal{L}) \to A$ is a smooth finite trivial covering.*

Somehow, considering Minkowskian linkages is the exact way of realizing non-compact algebraic sets! In particular, Kempe's theorem extends (globally, i.e. without taking the intersection with balls) to the Minkowski plane: any algebraic curve is the workspace of one vertex of some linkage.

Remark. If the restriction map is injective, then it is a bijective algebraic morphism from $\mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ to A, but not necessarily an algebraic isomorphism. In fact, it is true for non-singular complex algebraic sets that bijective morphisms are isomorphisms, but this is no longer true in the real algebraic case (see for instance [Mum95], Chapter 3).

We also have a differential version of the universality theorem in the Minkowski plane (which follows directly from Theorem 2.4, as explained at the end of Section 2.1):

Theorem 2.5. *For any differentiable manifold M with finite topology, i.e. diffeomorphic to the interior of a compact manifold with boundary, there is a linkage in the Minkowski*

plane with a configuration space whose components are all diffeomorphic to M. More precisely, there is a partial configuration space $\mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L})$ *which is diffeomorphic to M and such that the restriction map* $\mathrm{Conf}_{\mathbb{M}}(\mathcal{L}) \to \mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L})$ *is a smooth finite trivial covering.*

Hyperbolic planar linkages. In Chapter 4, we prove that both algebraic and differential universality theorems hold in the hyperbolic plane. The problem is that the notion of algebraic set has no intrinsic definition in the hyperbolic plane. However, it is possible to define an algebraic set in the Poincaré half-plane model $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y > 0 \right\}$ (and hence in $\mathbb{H}^2$) as an algebraic set of $\mathbb{R}^2$ which is contained in the half-plane. In fact, it turns out that the analogous definitions in the other usual models (the Poincaré disc model, the hyperboloid model, or the Beltrami-Klein model) are all equivalent. With this definition, we obtain the same results as in the Euclidean case:

Theorem 2.6. *Let A be a compact semi-algebraic subset of $(\mathbb{H}^2)^n$ (identified with a subset of $\mathbb{R}^{2n}$ using the Poincaré half-plane model). Then, A is a partial configuration space of some linkage $\mathcal{L}$ in $\mathbb{H}^2$. When A is algebraic, one can choose $\mathcal{L}$ such that the restriction map* $\mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L}) \to A$ *is a smooth finite trivial covering.*

Conversely, any partial configuration space of any linkage with at least one fixed vertex is a compact semi-algebraic subset of $(\mathbb{H}^2)^n$, so this theorem characterizes the sets which are partial configuration spaces (see Definiton 2.12 for the notion of "semi-algebraic").

In particular, Kempe's theorem holds in the hyperbolic plane.

And here follows the differential version:

Theorem 2.7. *For any compact differentiable manifold M, there is a linkage in the hyperbolic plane with a configuration space whose components are all diffeomorphic to M. More precisely, there is a partial configuration space* $\mathrm{Conf}^W_{\mathbb{H}^2}(\mathcal{L})$ *which is diffeomorphic to M and such that the restriction map* $\mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L}) \to \mathrm{Conf}^W_{\mathbb{H}^2}(\mathcal{L})$ *is a smooth finite trivial covering.*

Spherical linkages. These linkages are the subject of Chapter 5. In 1988, Mnëv [Mnë88] proved that the algebraic and differential universality theorems hold true in the real projective plane $\mathbb{R}P^2$ endowed with its usual metric as a quotient of the standard 2-sphere. Even better, he showed that the number of copies in the differential universality for $\mathbb{R}P^2$ can be reduced to 1, *i.e.* any manifold is the configuration space of some linkage. As Kapovich and Millson pointed out [KM02], a direct consequence of Mnëv's theorem is the differential universality theorem for the 2-sphere (but, this time, we get several copies of the desired manifold):

Theorem 2.8 (Mnëv-Kapovich-Millson)**.** *For any compact differentiable manifold M, there is a linkage in the sphere with a configuration space whose components are all diffeomorphic to M.*

However, it seems impossible to use Mnëv's techniques to prove the *algebraic* universality for spherical linkages: for example, all the configuration spaces of his linkages are symmetric with respect to the origin of $\mathbb{R}^3$. In order to obtain any semi-algebraic set as

a partial configuration space, we need to start again from scratch and construct linkages specifically for the sphere.

Contrary to the Minkowski and hyperbolic cases, the generalization of the theorems to higher dimensional spheres is straightforward. Thus, we are able to prove the following:

Theorem 2.9. *Let $d \geq 2$ and let A be a compact semi-algebraic subset of $(\mathbb{S}^d)^n$ (identified with a subset of $\mathbb{R}^{(d+1)n}$). Then, A is a partial configuration space of some linkage $\mathcal{L}$ in $\mathbb{S}^d$.*

In particular, Kempe's theorem holds in the sphere.

Conversely, any partial configuration space of any linkage is a compact semi-algebraic subset of $(\mathbb{S}^d)^n$ (see Section 2.4), so this theorem characterizes the sets which are partial configuration spaces.

Let us note that even when A is algebraic, our construction does not provide a linkage $\mathcal{L}$ such that the restriction map $\mathrm{Conf}_{\mathbb{S}^d}(\mathcal{L}) \to A$ is a smooth finite trivial covering. We do not know whether such a linkage exists.

Some questions. Our results suggest naturally – among many questions – the following:

1. Besides the 2-dimensional case, are the results in the Minkowski plane true for any (finite-dimensional) linear space endowed with a non-degenerate quadratic form? And what about higher-dimensional hyperbolic spaces? It is likely that the adaptation of the 2-dimensional proof hides no surprise, like in the Euclidean case, but it would probably require tedious work to prove it.

2. In our definition of linkages in the Minkowski plane, we allow some edges to have imaginary lengths (they are "timelike"). Is it possible to require the graphs of Theorems 2.4 and 2.5 to be spacelike, *i.e.* require all their edges to have real lengths?

3. In all the universality theorems that we prove, we obtain a linkage whose configuration space is diffeomorphic to the sum of a finite number of copies of the given manifold M. Is it possible to choose this sum trivial, that is, with exactly one copy of M? (This question is also open in the Euclidean plane.)

4. Is the differential universality theorem true on any Riemannian manifold?

Linkages on Riemannian manifolds. Let us give a partial answer to the last question using the following idea: just as the surface of the earth looks flat to us, any Riemannian manifold will almost behave as the Euclidean space if one considers a linkage which is small enough. However, our linkage has to be robust to small perturbations of the lengths, which is *not* the case for many of the linkages described in Part I (consider for example the *rigidified square linkage*).

Theorem 2.10. *Consider a linkage $\mathcal{L}$ in the Euclidean space $\mathbb{E}^n$, with at least one fixed vertex, such that for any small perturbation of the length vector $l \in (\mathbb{R}_{\geq 0})^E$, the configuration space $\mathrm{Conf}_{\mathbb{E}^n}(\mathcal{L})$ remains the same up to diffeomorphism. Then for any Riemannian manifold $\mathcal{M}$, there exists a linkage $\mathcal{L}_{\mathcal{M}}$ in $\mathcal{M}$ whose configuration space is diffeomorphic to $\mathrm{Conf}_{\mathbb{E}^n}(\mathcal{L})$.*

In particular, Theorem 2.10 combined with the work of Jordan and Steiner [JS01] yields directly

Corollary 2.11. *In any Riemannian surface $\mathcal{M}$, the differentiable universality theorem is true for compact orientable surfaces. In other words, any compact orientable surface is diffeomorphic to the configuration space* $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L})$ *of some linkage $\mathcal{L}$.*

This leads to the following

Question. Which manifolds can be obtained as the configuration space of some linkage in $\mathbb{R}^n$ which is robust to small perturbations (in the sense of Theorem 2.10) ?

This question is probably very difficult, but it is clear that there are restrictions on such manifolds: for example, they have to be orientable (because of Proposition 1.3).

2.3 Ingredients of the proofs

There are essentially three technical as well as conceptual tools: functional linkages, combination of elementary linkages, and regular inputs. The main idea is always the same as in all the known proofs of Universality theorems (see the proofs of Thurston, Mnëv [Mnë88], King [Kin98] or Kapovich and Millson [KM02]): one combines elementary linkages to construct a *"polynomial linkage"*.

Functional linkages. One major ingredient in the proofs is the notion of functional linkages. Here we enrich the graph structure by marking two new vertex subsets P and Q playing the role of inputs and outputs, respectively. If the partial realization of Q is determined by the partial realization of P, by means of a function $f : \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L}) \to \mathcal{M}^Q$ (called the *input-output function*), then we say that we have a functional linkage for f (for us, $\mathcal{M}$ will be the Minkowski plane $\mathbb{M}$, the hyperbolic plane $\mathbb{H}^2$ or the sphere $\mathbb{S}^d$). The Peaucellier linkage is a famous historical example: it is functional for an inversion with respect to a circle. With the notations of Figure 1.9, the input is D and the output is E.

Combination. Another major step in the proofs consists in proving the existence of functional linkages associated to any given polynomial f. This will be done by "combining" elementary functional linkages. We define combination so that combining two functional linkages for the functions f_1 and f_2 provides a functional linkage for $f_1 \circ f_2$.

Elementary linkages. All the work then concentrates in proving the existence of linkages for suitable elementary functions (observe that even for elementary linkages one uses a combination of more elementary ones). As an example, we give the list of the elementary linkages needed to prove Theorem 2.4 (in the Minkowski case):

1. The linkages for geometric operations:
   (a) The robotic arm linkage (Section 3.2.1): one of the most basic linkages, used everywhere in our proofs and in robotics in general.
   (b) The rigidified square (Section 3.2.2): a way of getting rid of degenerate configurations of the square using a well-known construction.

(c) The Peaucellier inversor (Section 3.2.3): this famous linkage of the 1860's has a slightly different behavior in the Minkowski plane but achieves basically the same goal.

(d) The partial t_0-line linkage (Section 3.2.4): it is obtained using a Peaucellier linkage, but does not trace out the whole line.

(e) The t_0-integer linkage (Section 3.2.5): it is a linkage with a discrete configuration space.

(f) The t_0-line linkage (Section 3.2.6): it draws the whole line, and is obtained by combining the two previous linkages.

(g) The horizontal parallelizer (Section 3.2.7): it forces two vertices to have the same ordinate, and it is obtained by combining several line linkages.

(h) The diagonal parallelizer (Section 3.2.8): its role is similar to the horizontal parallelizer but its construction is totally different.

2. The linkages for algebraic operations, which realize computations on the $t = 0$ line:

(a) The average function linkage (Section 3.3.1): it computes the average of two numbers, and is obtained by combining several of the previous linkages.

(b) The adder (Section 3.3.2): it is functional for addition on the $t = 0$ line, and is obtained from several average function linkages.

(c) The square function linkage (Section 3.3.3): it is functional for the square function and is obtained by combining the Peaucellier linkage (which is functional for inversion) with adders. This linkage is somewhat difficult to obtain because we want the inputs to be able to move everywhere in the line, while the inversion is of course not defined at $x = 0$.

(d) The multiplier (Section 3.3.4): it is functional for multiplication and is obtained from square function linkages.

(e) The polynomial linkage (Section 3.3.5): obtained by combining adders and multipliers, it is functional for a given polynomial function f. This linkage is used to prove the universality theorems: if the outputs are fixed to 0, the inputs are allowed to move exactly in $f^{-1}(0)$.

Regular inputs. In our theorems, we need the restriction map $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) \to \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$ to be a smooth finite trivial covering. In the differential universality Theorem, it ensures in particular that the whole configuration space consists in several copies of the given manifold M. The set of regular inputs $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L})$ is the set of all realizations of the inputs which admit a neighborhood onto which the restriction map is a smooth finite covering. We have to be very careful, because even for quite simple linkages such as the robotic arm, the restriction map is not a smooth covering everywhere! There are mainly two possible reasons for the restriction map not to be a smooth covering:

1. One realization of the inputs may correspond to infinitely many realizations of the whole linkage (for example, when the robotic arm in Section 3.2.1 has two inputs fixed at the same location, the workspace of the third vertex is a whole circle).

2. Even if it corresponds only to a finite number of realizations, these realizations may not depend smoothly on the inputs (for example, when the robotic arm in Section 3.2.1 is stretched).

New difficulties in each case. While the idea is always the same in all known proofs of universality theorems for linkages, *i.e.* combine elementary linkages to form a functional linkage for polynomials, each case has its own new difficulties due to different geometric properties, and the elementary linkages always require major changes to work correctly. Here follow examples of such differences with the Euclidean case:

The Minkowski case

1. The Minkowski plane $\mathbb{M}$ is not isotropic: its directions are not all equivalent. Indeed, these directions have a causal character in the sense that they may be spacelike, lightlike or timelike. For example, one needs different linkages in order to draw spacelike, timelike and lightlike lines.

2. In the Euclidean plane, two circles $\mathcal{C}(x,r)$ and $\mathcal{C}(x',r')$ intersect if and only if $|r-r'| \leq \|x-x'\| \leq r+r'$, but in the Minkowski plane, the condition of intersection is much more complicated to state (see Section 3.1.2).

3. In the Euclidean plane, one only has to consider compact algebraic sets. Applying a homothety, one may assume such a set to be inside a small neighborhood of zero, which makes the proof easier. Here, the algebraic sets are no longer compact, so we have to work with mechanisms which are able to deal with the whole plane.

The hyperbolic case

1. The *rigidified square linkage*, used extensively in all known proofs in the flat case, does not work anymore in its usual form, and does not have a simple analogue.

2. There is no natural notion of homothety: in particular, the *pantograph* does not compute the middle of a hyperbolic segment, contrary to the flat cases.

3. The notion of *algebraic set* is less natural than in the flat case.

4. In every standard model of the hyperbolic plane (such as the Poincaré half-plane), the expression of the distance between two points is much more complicated than in the flat case.

The spherical case

1. Just as in the hyperbolic case, the curvature prevents the *rigidified square linkage* from working correctly.

2. There is no natural notion of homothety.

3. In the Euclidean or hyperbolic planes, we only need to prove algebraic universality for bounded algebraic sets, which means that our functional linkages do not need to work on the whole surface. In the sphere, all the distances are uniformly bounded (even the lengths of the edges of our linkages), so we need to take into account the whole sphere when constructing linkages.

4. The compactness of the sphere also makes it difficult to construct linkages which deal with algebraic operations (addition, multiplication, division) since there is no proper embedding of $\mathbb{R}$ in the sphere.

2.4 Algebraic and semi-algebraic sets

In this section, we recall the standard definitions of algebraic and semi-algebraic sets. We adapt them to the Minkowski plane, the hyperbolic plane and the sphere in a natural way and state some of their properties.

Definition 2.12. An *algebraic subset of* $\mathbb{R}^n$ is a set $A \subseteq \mathbb{R}^n$ such that there exist $m \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a polynomial such that $A = f^{-1}(0)$.

We define a *semi-algebraic subset of* $\mathbb{R}^n$ (see [BCR98]) as the projection of an algebraic set[1]. More precisely, it is a set B such that there exists $N \geq n$ and an algebraic set A of $\mathbb{R}^N$ such that $B = \pi(A)$, where π is the projection onto the first coordinates

$$\begin{aligned} \pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^{N-n} &\to \mathbb{R}^n \\ (x, y) &\longmapsto x. \end{aligned}$$

We define the *(semi-)algebraic subsets of* $\mathbb{M}^n$ by identifying $\mathbb{M}^n$ with $(\mathbb{R}^2)^n = \mathbb{R}^{2n}$.

We also define the *(semi-)algebraic subsets of* $(\mathbb{H}^2)^n$, using the Poincaré half-plane model (see Definition 4.1), as the (semi-)algebraic subsets of $\mathbb{R}^{2n}$ which are contained in $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}^n$.

Finally, a *(semi-)algebraic subset of* $(\mathbb{S}^d)^n$ (for $d \geq 2$) is a semi-algebraic subset of $\mathbb{R}^{d+1}$ which is contained in the unit sphere of $\mathbb{R}^{d+1}$.

Proposition 2.13. *For any* compact *semi-algebraic subset B of $\mathbb{R}^n$, there exists $N \geq n$ and a* compact *algebraic subset A of $\mathbb{R}^N$ such that $B = \pi(A)$, where π is the projection onto the first coordinates: $\mathbb{R}^N \to \mathbb{R}^n$.*

Proof. **First case.** Assume for the moment that there exist polynomials $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ such that

$$B = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \ldots, m\} \quad f_i(x) \geq 0\}.$$

Let

$$\begin{aligned} h : \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{R}^m \\ \left(x, \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \right) &\longmapsto \begin{pmatrix} f_1(x) - y_1^2 \\ \vdots \\ f_m(x) - y_m^2 \end{pmatrix} \end{aligned}$$

[1] Our definition of semi-algebraic sets is not the standard one, but we know from the Tarski–Seidenberg theorem that the two definitions are equivalent (see [BCR98]).

and $A = h^{-1}(0)$. Then the projection of A onto the first n coordinates is obviously B. Moreover, A is compact since it is the image of B by the continuous function

$$\begin{aligned} g : B &\to \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \\ x &\longmapsto \left(x, \begin{pmatrix} \sqrt{f_1(x)} \\ \vdots \\ \sqrt{f_m(x)} \end{pmatrix} \right). \end{aligned}$$

General case. The *finiteness theorem for semi-algebraic sets* (see [BCR98], 2.7.2) states that any closed algebraic set can be described as the union of a finite number of sets $B_1, \ldots, B_k$ which satisfy the assumption of the first case: apply the first case to each of the B_i's to end the proof. □

We end this section with two analogous propositions for the hyperbolic plane and the sphere.

Proposition 2.14. *For any compact semi-algebraic subset B of $(\mathbb{H}^2)^n$, there exists $N \geq n$ and a compact algebraic subset A of $(\mathbb{H}^2)^N$ (with some $N \geq n$) such that $B = \pi(A)$, where π is the projection onto the first coordinates: $(\mathbb{H}^2)^N \to (\mathbb{H}^2)^n$.*

Proof. Let A' be a compact algebraic set of $\mathbb{R}^{N'}$ (with some $N' \geq 2n$) such that $B = \pi(A')$, where π is the projection onto the first coordinates: $\mathbb{R}^{N'} \to \mathbb{R}^{2n}$. Then the projection of the compact algebraic set

$$A := \left\{ \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_{n+1} \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_{N'-n} \\ 1 \end{pmatrix} \right) \;\middle|\; (x_1, y_1, \ldots, x_n, y_n, x_{n+1}, \ldots, x_{N'-n}) \in A' \right\}$$

(where $A \subseteq (\mathbb{H}^2)^{N'-n}$) is exactly B. □

Proposition 2.15. *For any compact semi-algebraic subset B of $(\mathbb{S}^2)^n$, there exists $N \geq n$ and a (compact) algebraic subset A of $(\mathbb{S}^2)^N$ (with some $N \geq n$) such that $B = \pi(A)$, where π is the projection onto the first coordinates: $(\mathbb{S}^2)^N \to (\mathbb{S}^2)^n$.*

Proof. Let A' be a compact algebraic set of $\mathbb{R}^{N'}$ (with some $N' \geq 3n$) such that $B = \pi(A')$, where π is the projection onto the first coordinates: $\mathbb{R}^{N'} \to \mathbb{R}^{3n}$. Since A' is compact, there is a λ such that $\pi_2(A') \in [-\lambda, \lambda]^{N'-3n}$, where π_2 is the projection onto the last coordinates: $\mathbb{R}^{N'} \to \mathbb{R}^{N'-3n}$. Then the projection of the compact algebraic set

$$A := \left\{ \left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_{N'-2n} \\ y_{N'-2n} \\ 0 \end{pmatrix} \right) \;\middle|\; \begin{array}{r} (x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \lambda x_{n+1}, \ldots, \lambda x_{N'-2n}) \in A' \\ x_{n+1}^2 + y_{n+1}^2 = 1, \ldots, x_{N'-2n}^2 + y_{N'-2n}^2 = 1 \end{array} \right\}$$

(where $A \subseteq (\mathbb{S}^2)^{N'-2n}$) is exactly B. □

Of course, Proposition 2.15 extends to $\mathbb{S}^d$ with any $d \geq 2$.

2.5 Generalities on linkages

In the present section, we develop generalities on linkages which apply to the Minkowski plane, the hyperbolic plane and the sphere. Thus, we consider a smooth manifold $\mathcal{M}$ endowed with a distance function

$$\delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}.$$

In the case of a Riemannian manifold (in particular, for the hyperbolic plane and the sphere), the metric determines a real-valued distance on $\mathcal{M}$.

In the case of the Minkowski plane, $\mathcal{M}$ is the plane $\mathbb{R}^2$. Here, we argue by a naive algebraic analogy and define a distance as

$$\delta\left(\begin{pmatrix} x_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ t_2 \end{pmatrix}\right) = \sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2} \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}.$$

Accordingly, the length structure of the linkage will be generalized by taking values in $\mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$ (instead of $\mathbb{R}_{\geq 0}$) as follows:

Definition 2.16. A *linkage* $\mathcal{L}$ in $\mathcal{M}$ is a graph (V, E) together with:

1. A function $l : E \to \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$ (which gives the length of each edge[2]);
2. A subset $F \subseteq V$ of *fixed vertices* (represented by ■ on the figures);
3. A function $\phi_0 : F \to \mathcal{M}$ which indicates where the vertices of F are fixed;

When the linkage is named $\mathcal{L}_1$, we usually write $\mathcal{L}_1 = (V_1, E_1, l_1, \dots)$ and name its vertices $a_1, b_1, c_1, \dots$. If the linkage $\mathcal{L}_1$ is a copy of the linkage $\mathcal{L}$, the vertex $a_1 \in V_1$ corresponds to the vertex $a \in V$, and so on.

Definition 2.17. Let $\mathcal{L}$ be a linkage in $\mathcal{M}$. A *realization* of a linkage $\mathcal{L}$ in $\mathcal{M}$ is a function $\phi : V \to \mathcal{M}$ such that:

1. For each edge $v_1v_2 \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1v_2)$;
2. $\phi|_F = \phi_0$.

Remark. On the figures of Part I, linkages are represented by abstract graphs. The edges are not necessarily represented by straight segments, and the positions of the vertices on the figures do not necessarily correspond to a realization (unless otherwise stated).

Definition 2.18. Let $\mathcal{L}$ be a linkage in $\mathcal{M}$. Let $W \subseteq V$. The *partial configuration space of $\mathcal{L}$ in $\mathcal{M}$ with respect to W* is

$$\mathrm{Conf}^W_{\mathcal{M}}(\mathcal{L}) = \{\phi|_W \mid \phi \text{ realization of } \mathcal{L}\}.$$

In other words, $\mathrm{Conf}^W_{\mathcal{M}}(\mathcal{L})$ is the set of all the maps $\phi : W \to \mathcal{M}$ which extend to realizations of $\mathcal{L}$. In particular, the *configuration space* $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) = \mathrm{Conf}^V_{\mathcal{M}}(\mathcal{L})$ is the set of all realizations of $\mathcal{L}$.

Definition 2.19. A *marked linkage* is a tuple $(\mathcal{L}, P, Q)$, where P and Q are subsets of V: P is called the "input set" and its elements, called the "inputs", are represented by ▼ on the figures, whereas Q is called the "output set" and its elements, called the "outputs", are represented by ▲ on the figures.

The *input map* $p : \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) \to \mathcal{M}^P$ is the map induced by the projection $\mathcal{M}^V \to \mathcal{M}^P$ (the restriction map). In other words, for all $\phi \in \mathrm{Conf}_{\mathcal{M}}(\mathcal{L})$, we have $p(\phi) = \phi|_P$.

Likewise, we define the *output map* $q : \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) \to \mathcal{M}^Q$ by $q(\phi) = \phi|_Q$.

[2] Of course, when $\mathcal{M}$ is a Riemannian manifold, we may choose all the lengths in $\mathbb{R}_{\geq 0}$!

The notion of marked linkage is not necessary to study configuration spaces. However, in the linkages we use in our proofs, some vertices play an important role (the inputs and the outputs) while others do not: this is why we always consider marked linkages. The following notion[3] accounts for the names "inputs" and "outputs":

Definition 2.20. We say that $\mathcal{L}$ is a *functional linkage* for the input-output function $f : \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L}) \to \mathcal{M}^Q$ if

$$\forall \phi \in \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}) \qquad f(p(\phi)) = q(\phi).$$

2.6 Regularity

Definition 2.21. Let $\mathcal{L}$ be a linkage. Let $W \subseteq V$ and $\psi \in \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$. Let π_W be the restriction map

$$\pi_W : \mathrm{Conf}^{W\cup P}_{\mathcal{M}}(\mathcal{L}) \to \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L}).$$

We say that ψ *is a regular input for* W if there exists an open neighborhood $U \subseteq \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$ of ψ such that $\pi_W|_{\pi_W^{-1}(U)}$ is a finite smooth covering[4].

We write $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}, W) \subseteq \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$ the set of regular inputs for W. When W is the set V of all vertices, we simply write $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L})$.

Roughly speaking, ψ is a regular input for W if it determines a finite number of realizations ϕ of W, and if these configurations are determined smoothly with respect to ψ (in other words, π_W^{-1} is a smooth multivalued function in a neighborhood of ψ).

The following fact is simple but essential:

Fact 2.22. *For any $W_1, W_2 \subseteq V$, we have*

$$\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}, W_1) \cap \mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}, W_2) \subseteq \mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}, W_1 \cup W_2).$$

Therefore, in practice, in order to prove that $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$, we only have to prove that $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L}, \{v\}) = \mathrm{Conf}^P_{\mathcal{M}}(\mathcal{L})$ for all $v \in V$.

2.7 Changing the input set

In this proposition, we take a linkage, then consider the same linkage with a different set of inputs P and analyse the impact on $\mathrm{Reg}^P_{\mathcal{M}}(\mathcal{L})$.

Fact 2.23. *Let $\mathcal{L}_1 = (V_1, E_1, l_1, F_1, \phi_{01}, P_1, Q_1)$, $P_2 \subseteq V_1$ and define*

$$\mathcal{L}_2 = (V_1, E_1, l_1, F_1, \phi_{01}, P_2, Q_1).$$

Recall that $p_1 : \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}_1) \to \mathcal{M}^{P_1}$ and $p_2 : \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}_2) \to \mathcal{M}^{P_2}$ are the respective input maps of $\mathcal{L}_1$ and $\mathcal{L}_2$. Then $\mathrm{Reg}^{P_2}_{\mathcal{M}}(\mathcal{L}_2)$ contains

$$\left\{\psi \in \mathrm{Conf}^{P_2}_{\mathcal{M}}(\mathcal{L}_2) \;\middle|\; \forall \phi \in p_2^{-1}(\psi) \quad p_1(\phi) \in \mathrm{Reg}^{P_1}_{\mathcal{M}}(\mathcal{L}_1)\right\} \cap \mathrm{Reg}^{P_2}_{\mathcal{M}}(\mathcal{L}_2, P_1).$$

Proof. This is a simple consequence of the fact that the composition of two smooth functions is a smooth function. □

[3] already defined informally at the beginning of Section 2.3

[4] We do not require U or $\pi_W^{-1}(U)$ to be smooth manifolds: recall that a smooth map on $\pi_W^{-1}(U)$ is, by definition, the restriction of a smooth map defined on the ambient $\mathcal{M}^{W\cup P}$.

2.8 Combining linkages

This notion is essential to construct complex linkages from elementary ones. The proofs in this section are straightforward and left to the reader.

Let $\mathcal{L}_1 = (V_1, E_1, l_1, F_1, \phi_{01}, P_1, Q_1)$ and $\mathcal{L}_2 = (V_2, E_2, l_2, F_2, \phi_{02}, P_2, Q_2)$ be two linkages, $W_1 \subseteq V_1$, and $\beta : W_1 \to V_2$.

The idea is to construct a new linkage $\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$ as follows:

Step 1 Consider $\mathcal{L}_1 \cup \mathcal{L}_2$, the disjoint union of the two graphs (V_1, E_1) and (V_2, E_2).

Step 2 Identify some vertices of V_1 with some vertices of V_2 via β, without removing any edge.

Since linkages are graphs which come with an additional structure, we need to clarify what happens to the other elements (l, F, ϕ_0, P, Q). In particular, note that the inputs of $\mathcal{L}_2$ which are in $\beta(W_1)$ are *not* considered as inputs in the new linkage $\mathcal{L}_3$.

Definition 2.24 (Combining two linkages)**.** We define

$$\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2 = (V_3, E_3, l_3, F_3, \phi_{03}, P_3, Q_3)$$

in the following way:

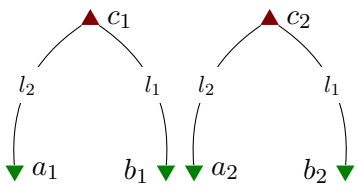
1. $V_3 = (V_1 \setminus W_1) \cup V_2$;
2. $E_3 = (E_1 \cap (V_1 \setminus W_1)^2) \cup (E_2 \cap V_2^2) \cup \{v\beta(v') \mid v \in V_1 \setminus W_1, v' \in W_1, vv' \in E_1\}$ $\cup \{\beta(v)\beta(v') \mid v, v' \in W_1, vv' \in E_1\}$;
3. For all $v_1, v_1' \in V_1 \setminus W_1$, $w_1, w_1' \in W_1$, $v_2, v_2' \in V_2$, define
$$\begin{aligned} l_3(v_1v_1') &= l_1(v_1v_1'), \quad l_3(v_1\beta(w_1)) = l_1(v_1w_1), \\ l_3(v_2v_2') &= l_2(v_2v_2'), \quad l_3(\beta(w_1)\beta(w_1')) = l_1(w_1w_1'); \end{aligned}$$
4. $F_3 = (F_1 \setminus W_1) \cup \beta(F_1 \cap W_1) \cup F_2$;
5. $\phi_{03}|_{F_1 \setminus W_1} = \phi_{01}|_{F_1 \setminus W_1}, \quad \phi_{03} \circ \beta = \phi_{01}|_{W_1}, \quad \phi_{03}|_{F_2 \setminus \beta(W_1)} = \phi_{02}|_{F_2 \setminus \beta(W_1)}$;
6. $P_3 = (P_1 \setminus W_1) \cup \beta(P_1 \cap W_1) \cup (P_2 \setminus \beta(W_1))$;
7. $Q_3 = (Q_1 \setminus W_1) \cup Q_2$.

The combination of two linkages is prohibited in the following cases:

1. There exist $a_1, b_1 \in F_1 \cap W_1$ such that $\beta(a_1) = \beta(b_1)$ and $\phi_{01}(a_1) \neq \phi_{01}(b_1)$ (two vertices are fixed at different places but should be attached to the same other vertex).
2. There exist $a_1, b_1 \in W_1$ such that $a_1b_1 \in E_1$, $\beta(a_1)\beta(b_1) \in E_2$, and $l_1(a_1b_1) \neq l_2(\beta(a_1)\beta(b_1))$ (two edges of different lengths should join one couple of vertices).

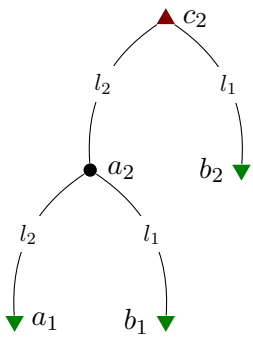
3. There exist $a_1 \in V_1$ and $b_1, c_1 \in W_1$, such that $a_1b_1 \in E_1$, $a_1c_1 \in E_1$, $\beta(b_1) = \beta(c_1)$, and $l_1(a_1b_1) \neq l_1(a_1c_1)$ (again, two edges of different lengths should join one couple of vertices).

Example 2.25. Consider the two identical linkages $\mathcal{L}_1$ and $\mathcal{L}_2$:



The inputs of $\mathcal{L}_i$ are a_i, b_i and the output is c_i.

To combine the two linkages, let $W_1 = \{c_1\}$ and $\beta(c_1) = a_2$. Then $\mathcal{L}_3 := \mathcal{L}_1 \cup_\beta \mathcal{L}_2$ is the following linkage:



The inputs of $\mathcal{L}_3$ are a_1, b_1, b_2 and the output is c_2.

We end this section with three facts whose proofs are straightforward. The first describes $\mathrm{Conf}_{\mathcal{M}}(\mathcal{L})$ when $\mathcal{L}$ is obtained as the combination of two linkages, the second one describes $\mathrm{Reg}_{\mathcal{M}}(\mathcal{L})$, while the third one establishes a link between the combination of functional linkages and the composition of functions.

Fact 2.26. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two linkages, $W_1 \subseteq V_1$, $\beta : W_1 \to V_2$, and $\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$ be defined as in Definition 2.24. Then*

$$\mathrm{Conf}_{\mathcal{M}}(\mathcal{L}_3) = \big\{\phi_3 \in \mathcal{M}^{V_3} \;\big|\; \exists(\phi_1, \phi_2) \in \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}_1) \times \mathrm{Conf}_{\mathcal{M}}(\mathcal{L}_2)$$

$$\phi_1|_{V_1 \setminus W_1} = \phi_3|_{V_1 \setminus W_1},\ \phi_1|_{W_1} = \phi_3|_{\beta(W_1)} \circ \beta,\ \phi_2 = \phi_3|_{V_2}\big\}.$$

Fact 2.27. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two linkages, $W_1 \subseteq V_1$, $\beta : W_1 \to P_2$, and $\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. Suppose that $\psi_3 \in \mathrm{Conf}^{P_3}_{\mathcal{M}}(\mathcal{L}_3)$ satisfies both of the following properties:*

1. $\exists \psi_1 \in \mathrm{Reg}^{P_1}_{\mathcal{M}}(\mathcal{L}_1)\ \ \psi_1|_{P_1 \setminus W_1} = \psi_3|_{P_1 \setminus W_1}, \psi_1|_{P_1 \cap W_1} = \psi_3|_{\beta(P_1 \cap W_1)} \circ \beta$;
2. $\forall \phi \in p_3^{-1}(\psi_3)\ \ \phi|_{P_2} \in \mathrm{Reg}^{P_2}_{\mathcal{M}}(\mathcal{L}_2)$.

Then $\psi_3 \in \mathrm{Reg}^{P_3}_{\mathcal{M}}(\mathcal{L}_3)$.

Fact 2.28. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two linkages with* $\mathrm{card}(Q_1) = \mathrm{card}(P_2)$.

Assume that $\mathcal{L}_1$ is a functional linkage for $f_1 : \mathrm{Conf}^{P_1}_{\mathcal{M}}(\mathcal{L}_1) \to \mathcal{M}^{Q_1}$ and that $\mathcal{L}_2$ is a functional linkage for $f_2 : \mathrm{Conf}^{P_2}_{\mathcal{M}}(\mathcal{L}_2) \to \mathcal{M}^{Q_2}$. Let $W_1 = Q_1$, $\beta : W_1 \to P_2$ a bijection, and $\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. The bijection β induces a bijection $\hat{\beta}$ between $\mathcal{M}^{Q_1}$ and $\mathcal{M}^{P_2}$.

Then $\mathcal{L}_3$ is functional for $f_2 \circ \hat{\beta} \circ f_1|_{\mathrm{Conf}^{P_3}_{\mathcal{M}}(\mathcal{L}_3)}$.

2.9 Appendix: Linkages on any Riemannian manifold

The aim of this section is to prove Theorem 2.10.

Consider a linkage $\mathcal{L} = (V, E, F, l, \phi_0, P, Q)$ in the Euclidean space $\mathbb{E}^n$ as in the statement of the theorem: we may assume without loss of generality that $\mathcal{L}$ is a connected graph, that the sum of the lengths of the edges is smaller than 1, and that one of the vertices is fixed to 0, so that the configuration space of $\mathcal{L}$ is a subset of B^V, where B is the unit ball of $\mathbb{E}^n$. We introduce C the set of all mappings $\phi : V \to B$ such that $\phi|_F = \phi_0$ (namely, those which map the fixed points to their assigned locations), and define the mapping

$$\begin{aligned} \Phi : C &\to (\mathbb{R}_{\geq 0})^E \\ \phi &\longmapsto \left((v, v') \mapsto \|\phi(v') - \phi(v')\|^2\right). \end{aligned}$$

Then the configuration space of $\mathcal{L}$ in $\mathbb{E}^n$ is $\Phi^{-1}(l^2)$. Making a small perturbation of l, we may assume by the Lemma of Sard that l^2 is a regular value of Φ. By assumption, this perturbation does not change $\mathrm{Conf}_{\mathbb{E}^n}(\mathcal{L})$, up to diffeomorphism.

Let U be an open neighborhood of 0 in $\mathbb{E}^n$, equipped with a metric g, such that (U, g) is isometric to an open subset of the Riemannian manifold $\mathcal{M}$, and denote by δ the associated distance on U. Applying a linear transformation to U, we may assume that g_0 (the metric g at 0) coincides with the canonical Euclidean scalar product on $\mathbb{R}^n$.

For a small enough $r > 0$, the mapping

$$\begin{aligned} \Psi : C \times ((-r, r) \setminus \{0\}) &\to \mathbb{R}^E \times \mathbb{R} \\ \begin{pmatrix} \phi \\ \lambda \end{pmatrix} &\longmapsto \begin{pmatrix} \psi : (v, v') \mapsto \frac{1}{\lambda^2}\delta^2(\lambda\phi(v), \lambda\phi(v')) \\ \lambda \end{pmatrix} \end{aligned}$$

is well-defined, smooth, and may be extended smoothly to $C \times (-r, r)$ (apply Taylor's formula).

Then for all small enough $\lambda \in \mathbb{R}$,

$$\Psi^{-1}\begin{pmatrix} l^2 \\ \lambda \end{pmatrix} = \frac{1}{\lambda}\mathrm{Conf}_{(U,g)}(\mathcal{L}_\lambda) \times \{\lambda\},$$

where $\mathcal{L}_\lambda = (V, E, F, \lambda l, \lambda\phi_0, P, Q)$. Notice that $\mathrm{Conf}_{(U,g)}(\mathcal{L}_\lambda)$ is diffeomorphic to the configuration space of some linkage in $\mathcal{M}$, since (U, g) is isometric to an open set of $\mathcal{M}$.

The key to the proof is the following fact:

Fact 2.29. *For all $\phi \in C$, $\Psi\begin{pmatrix} C \\ 0 \end{pmatrix} = \begin{pmatrix} \Phi(C) \\ 0 \end{pmatrix}$.*

Proof. In this proof, for any open set $W \in \mathbb{R}^n$, we will write $C^1([0, 1], W)$ the set of C^1 paths which take their values in W.

Let $\phi \in C$. For any small enough $\lambda \in \mathbb{R}$, we have:

$$\begin{aligned}\Psi_1 \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \cdot (v, v') &= \frac{1}{\lambda^2} \inf_{\substack{\gamma \in C^1([0,1],U) \\ \gamma(0)=\lambda\phi(v),\ \gamma(1)=\lambda\phi(v')}} \left(\int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt \right)^2 \\ &= \inf_{\substack{\gamma \in C^1([0,1],\frac{1}{\lambda}U) \\ \gamma(0)=\phi(v),\ \gamma(1)=\phi(v')}} \left(\int_0^1 \sqrt{g_{\lambda\gamma(t)}(\gamma'(t), \gamma'(t))} dt \right)^2 .\end{aligned}$$

Any C^1 path from $[0,1]$ to $\mathbb{R}^n$ takes it values in $\frac{1}{\lambda}U$ for some $\lambda > 0$. Thus, taking the limit as $\lambda \to 0$, we obtain:

$$\Psi_1 \begin{pmatrix} \phi \\ 0 \end{pmatrix} \cdot (v, v') = \inf_{\substack{\gamma \in C^1([0,1],\mathbb{R}^n) \\ \gamma(0)=\phi(v),\ \gamma(1)=\phi(v')}} \left(\int_0^1 \sqrt{g_0(\gamma'(t), \gamma'(t))} dt \right)^2 = \|\phi(v') - \phi(v)\|^2,$$

since g_0 is the canonical Euclidean scalar product. □

Fact 2.29 shows that $\begin{pmatrix} l^2 \\ 0 \end{pmatrix}$ is a regular value of Ψ, and that $\Psi^{-1} \begin{pmatrix} l^2 \\ 0 \end{pmatrix} = \mathrm{Conf}_{\mathbb{E}^n}(\mathcal{L})$.

Hence, for any small enough $\lambda \in \mathbb{R}$, $\mathrm{Conf}_{(U,g)}(\mathcal{L}_\lambda)$ is diffeomorphic to $\mathrm{Conf}_{\mathbb{E}^n}(\mathcal{L})$, which ends the proof.

Chapter 3

Linkages in the Minkowski plane

The aim of this chapter is to prove Theorem 2.4.

3.1 Generalities on the Minkowski plane

3.1.1 Notation

The Minkowski plane $\mathbb{M}$ is $\mathbb{R}^2$ equipped with the bilinear form $\varphi\left(\begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} x' \\ t' \end{pmatrix}\right) = xx' - tt'$. The pseudo-norm $\|\cdot\|$ is defined by $\|\alpha\|^2 = \varphi(\alpha, \alpha)$, and $\|\alpha\| \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}$. The "distance" between α_1 and α_2 is defined by $\delta(\alpha_1, \alpha_2) = \|\alpha_1 - \alpha_2\|$.

For $\alpha \in \mathbb{M}$, we write x_α and t_α the usual coordinates in $\mathbb{R}^2$, so that $\|\alpha\|^2 = x_\alpha^2 - t_\alpha^2$.

Sometimes, it will be more convenient to use *lightlike coordinates*, defined by $y_\alpha = x_\alpha + t_\alpha$ and $z_\alpha = x_\alpha - t_\alpha$, so that $\|\alpha\|^2 = y_\alpha z_\alpha$.

We write $\mathcal{I} = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \;\middle|\; t = 0 \right\}$.

3.1.2 Intersection of two hyperbolae

In the Minkowski plane, hyperbolae play a central role (instead of circles in the Euclidean plane): for any $\alpha \in \mathbb{M}$ and $r^2 \in \mathbb{R}$, the hyperbola $\mathcal{H}(\alpha, r)$ is the set of all $\gamma \in \mathbb{M}$ such that $\delta(\alpha, \gamma)^2 = r^2$.

Let $\alpha_0, \alpha_1 \in \mathbb{M}$ and $r_0^2, r_1^2 \in \mathbb{R}$, and write $d = \|\alpha_0 - \alpha_1\|$. Our aim in this section is to determine the cardinality of $I = \mathcal{H}(\alpha_0, r_0) \cap \mathcal{H}(\alpha_1, r_1)$.

Proposition 3.1. *If $\alpha_0 \neq \alpha_1$ and $r_0^2 r_1^2 \neq 0$, we have* $\mathrm{card}(I) \leq 2$.

Proof. We write $y_0 = y_{\alpha_0}$ and $z_0 = z_{\alpha_0}$. We may assume $\alpha_1 = 0$ and $y_0 \neq 0$. Then, I is the set of the solutions of the system with unknown (y, z):

$$\begin{cases} yz = r_1^2 \\ (y - y_0)(z - z_0) = r_0^2 \end{cases}$$

which is equivalent to

$$\begin{cases} yz = r_1^2 \\ y_0 z^2 - (y_0 z_0 + r_1^2 - r_0^2) z + r_1^2 z_0 = 0 \end{cases}$$

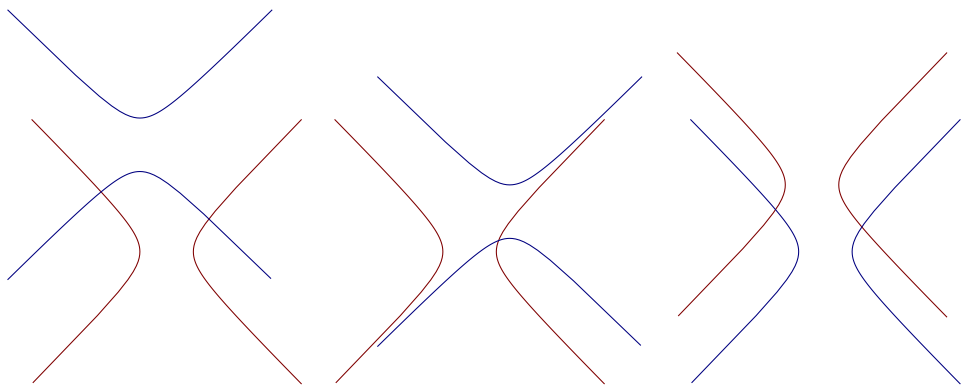
Thus, z is one of the roots of a polynomial of degree 2 and y is fully determined by z, so there are at most two solutions to the system. □

Proposition 3.2. *1. If $r_0^2 r_1^2 < 0$ and $d^2 \neq 0$, then $\operatorname{card}(I) = 2$. Moreover, if d' is the distance between the two points of I, then $d^2 d'^2 < 0$.*

2. If $r_0^2 r_1^2 < 0$ and $d^2 = 0$, then $\operatorname{card}(I) = 1$.

3. If $r_0^2 r_1^2 > 0$ and $r_0^2 d^2 < 0$, then $\operatorname{card}(I) = 2$.

Proof. Examine the following figures. □



$r_0^2 r_1^2 < 0$ and $d^2 \neq 0$. $r_0^2 r_1^2 < 0$ and $d^2 = 0$. $r_0^2 r_1^2 > 0$ and $r_0^2 d^2 < 0$.

3.1.3 The case of equality in the triangle inequality

In the Minkowski plane, the triangle inequality is not always valid, but the equality case is the same as in the Euclidean plane.

Proposition 3.3. *Let $\alpha, \beta \in \mathbb{M}$. If $\|\alpha\| + \|\beta\| = \|\alpha + \beta\|$, then α and β are colinear.*

Proof. We have

$$(\|\alpha\| + \|\beta\|)^2 = \|\alpha + \beta\|^2$$
$$\|\alpha\|^2 + \|\beta\|^2 + 2\|\alpha\|\|\beta\| = \|\alpha\|^2 + \|\beta\|^2 + 2\varphi(\alpha, \beta)$$
$$\varphi(\alpha, \beta) = \|\alpha\|\|\beta\|.$$

Therefore, the discriminant of the polynomial function

$$\lambda \longmapsto \|\beta\|^2 \lambda^2 + 2\varphi(\alpha, \beta)\lambda + \|\alpha\|^2$$

is zero. Thus, $\|\alpha + \lambda\beta\|^2$ is either nonnegative for all λ or nonpositive for all λ. This means that α and β are not linearly independent. □

3.1.4 The dual linkage

Let $\mathcal{L}_1$ be a linkage in the Minkowski plane. We define the reflection

$$s : \mathbb{C} \to \mathbb{C}$$
$$a + ib \longmapsto b + ia$$

and construct $\mathcal{L}_2$, the *dual linkage* of $\mathcal{L}_1$, by

1. $V_2 = V_1$
2. $E_2 = E_1$
3. $F_2 = F_1$
4. $P_2 = P_1$
5. $Q_2 = Q_1$
6. $l_2 = s \circ l_1$
7. $\phi_{02} = s \circ \phi_{01}$ (with $\mathbb{R}^2$ identified to $\mathbb{C}$ with the coordinates (x, t)).

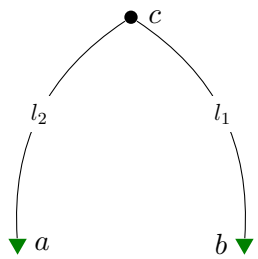
For all $W \subseteq V_1$, this linkage satisfies

$$\mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L}_2) = \left\{ s \circ \phi \;\middle|\; \phi \in \mathrm{Conf}^W_{\mathbb{M}}(\mathcal{L}_1) \right\},$$

$$\mathrm{Reg}^{P_2}_{\mathbb{M}}(\mathcal{L}_2) = \left\{ s \circ \phi \;\middle|\; \phi \in \mathrm{Reg}^W_{\mathbb{M}}(\mathcal{L}_1) \right\}.$$

3.2 Elementary linkages for geometric operations

3.2.1 The robotic arm linkage



We let $P = \{a, b\}$ and $F = \emptyset$. We assume $l_1 \neq 0$ and $l_2 \neq 0$.
We translate Proposition 3.2 in terms of linkages.

Fact 3.4. *1. If l_1^2 and l_2^2 have different signs,*

$$\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) \supseteq \mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) \supseteq \left\{ \psi \in \mathbb{M}^P \;\middle|\; \|\psi(a) - \psi(b)\| \neq 0 \right\}$$

2. If l_1^2 and l_2^2 have the same sign,

$$\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) \supseteq \mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) \supseteq \left\{ \psi \in \mathbb{M}^P \;\middle|\; \|\psi(a) - \psi(b)\|^2 \cdot l_1^2 < 0 \right\}$$

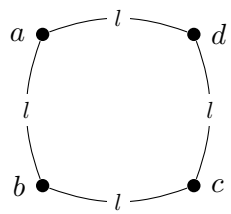
3. More generally, let $\psi \in \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$. If the intersection $\mathcal{H}(\psi(a), l_2) \cap \mathcal{H}(\psi(b), l_1)$ contains exactly two elements, then $\psi \in \mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L})$.

Proof. The fact that the intersection $\mathcal{H}(\psi(a), l_2) \cap \mathcal{H}(\psi(b), l_1)$ contains exactly two elements implies that these are obtained from simple roots of a polynomial of degree 2 (see the proof of Proposition 3.1). Therefore, locally, the roots depend smoothly on the coefficients. □

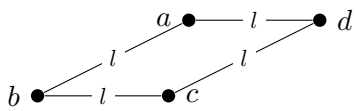
3.2.2 The rigidified square linkage

This linkage is well-known in the Euclidean plane. It is the usual solution to the problem of *degenerate configurations* of the square. It is very useful to notice that it does behave in the same way in the Minkowski plane.

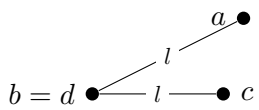
We first explain why we need to rigidify square linkages. If one considers the ordinary square linkage (see the following figure), there are many realizations ϕ in which $\phi(a)\phi(b)\phi(c)\phi(d)$ is not a parallelogram (we call these realizations *degenerate realizations of the square*).



The ordinary square linkage



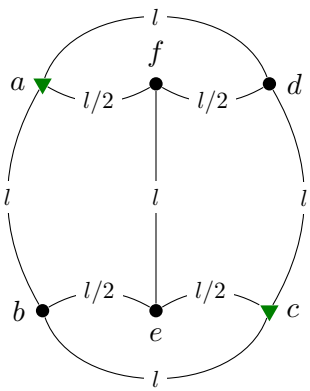
A realization of the ordinary square linkage



A degenerate realization of the ordinary square linkage

In degenerate realizations, two vertices are sent to the same point of $\mathbb{M}$.

To avoid degenerate realizations, we add two vertices and five edges to the square *abcd*. We call this operation "rigidifying the square".



This linkage is called the *rigidified square*. The input set is $P = \{a, c\}$.

We assume $l \neq 0$.

Proposition 3.5. *1. For all $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ we have*

$$\phi(b) - \phi(a) = \phi(c) - \phi(d)$$

($\phi(a)\phi(b)\phi(c)\phi(d)$ is an "affine parallelogram").

2. For all $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ such that $\phi(b) \neq \phi(d)$ and $\phi(a) \neq \phi(c)$, we have $\phi|_P \in \mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L})$. In particular, $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L})$ contains

$$\left\{\psi \in \mathbb{M}^{\{a,c\}} \,\middle|\, \|\psi(a) - \psi(c)\| \cdot l^2 < 0\right\}.$$

Proof. 1. Let $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$. From the equality case in the triangle inequality, we have $\phi(f) = \frac{\phi(a)+\phi(d)}{2}$ and $\phi(e) = \frac{\phi(b)+\phi(c)}{2}$.

Case 1: $\phi(a) = \phi(c)$**.** In this case, $\|\phi(f) - \phi(c)\| = \|\phi(f) - \phi(a)\|$. Therefore,

$$\|\phi(c) - \phi(e)\| + \|\phi(f) - \phi(c)\| = \|\phi(f) - \phi(e)\|,$$

so $\phi(e)$, $\phi(c)$ and $\phi(f)$ are aligned, so $\phi(b), \phi(c)$ and $\phi(d)$ are aligned and therefore $\phi(b) - \phi(a) = \phi(c) - \phi(d)$.

Case 2: $\phi(d) = \frac{\phi(a)+\phi(c)}{2}$**.** We have

$$\|\phi(b) - \phi(a)\| + \|\phi(b) - \phi(c)\| = \|\phi(c) - \phi(a)\| \quad (= 2l),$$

so $\phi(a)$, $\phi(c)$ and $\phi(b)$ are aligned, thus $\phi(d) = \phi(b)$. We are taken back to the first case.

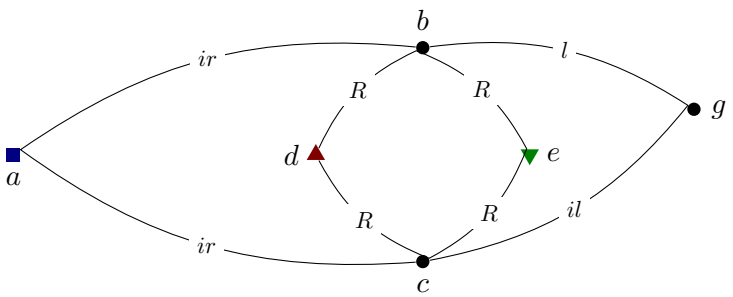
Case 3: $\phi(d) \neq \frac{\phi(a)+\phi(c)}{2}$ **and** $\phi(a) \neq \phi(c)$**.** Let $I = \mathcal{H}(\phi(a), l) \cap \mathcal{H}(\phi(c), l)$. We have $\phi(d) \in I$, $\phi(b) \in I$, and $\mathrm{card}(I) \leq 2$.

We have $\phi(a) + \phi(c) - \phi(d) \in I$. If $\phi(a) + \phi(c) - \phi(d) = \phi(d)$ then we are in the second case. If not, we have $I = \{\phi(d), \phi(a) + \phi(c) - \phi(d)\}$ and therefore, either $\phi(b) = \phi(d)$ (this is again the first case) or $\phi(b) = \phi(a) + \phi(c) - \phi(d)$, *i.e.* $\phi(b) - \phi(a) = \phi(c) - \phi(d)$.

2. This is a consequence of Fact 3.4.

□

3.2.3 The Peaucellier inversor



Choose $r, R, l > 0$ and let $F = \{a\}$, $\phi_0(a) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P = \{e\}$, $Q = \{d\}$. The square $bdce$ is rigidified (see Section 3.2.2), but for convenience, we do not draw on the figure the vertices which are necessary for the rigidification.

The vertex g and the two edges (bg) and (cg) are not part of the traditional Peaucellier inversor, but they are here to prevent $\phi(b)$ and $\phi(c)$ from being equal.

In the Euclidean plane, it is well-known that the Peaucellier linkage is a functional linkage for the inversion with respect to the circle $\mathcal{C}\left(0, \sqrt{|R^2 - r^2|}\right)$, that is, the function $\alpha \longmapsto \frac{|R^2-r^2|}{\|\alpha\|^2}\alpha$. In the Minkowski plane, we will prove that it is essentially functional for inversion with respect to the hyperbola $\mathcal{H}\left(0, \sqrt{R^2 + r^2}\right)$. More precisely, it is functional for the function $\alpha \longmapsto -\frac{R^2+r^2}{\|\alpha\|^2}\alpha$ (in the version of the Peaucellier inversor which we choose, a "-" sign appears).

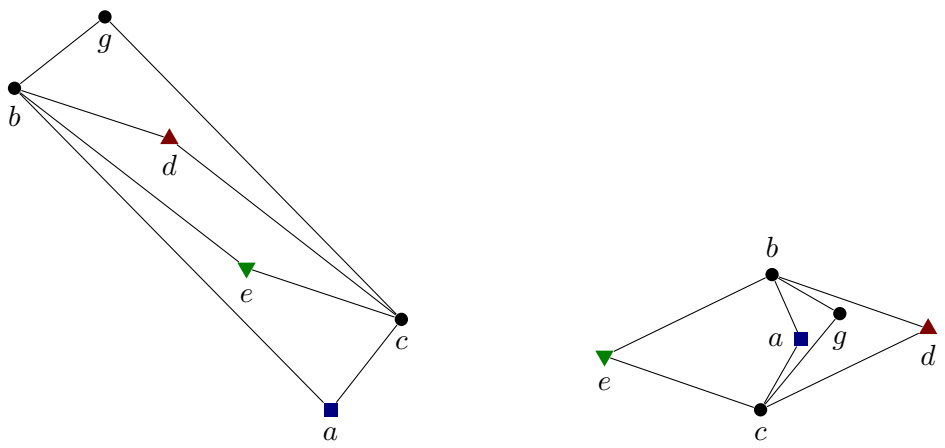


Figure 3.1 – Two realizations of the same Peaucellier inversor in the Minkowski plane.

Proposition 3.6. *For all* $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$*, we have* $\|\phi(e)\| \neq 0$ *and* $\phi(d) = -\frac{R^2+r^2}{\|\phi(e)\|^2}\phi(e)$.

Proof. Let $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$. We know that $\phi(b) \neq \phi(c)$ (because the edges (gb) and (gc) do not have the same lengths) so the intersection of the two hyperbolae $\mathcal{H}(\phi(e), R)$ and $\mathcal{H}(\phi(a), ir)$ is exactly $\{\phi(b), \phi(c)\}$. Moreover, $\|\phi(e)\| \neq 0$ (because the edges (be) and (ba) do not have the same lengths).

Then, $(y_{\phi(b)}, z_{\phi(b)})$ and $(y_{\phi(c)}, z_{\phi(c)})$ are the two solutions of the following system with unknown (y, z):

$$\begin{cases} yz = -r^2 \\ (y - y_{\phi(e)})(z - z_{\phi(e)}) = R^2. \end{cases}$$

This system is equivalent to

$$\begin{cases} yz = -r^2 \\ -y_{\phi(e)}z^2 + (y_{\phi(e)}z_{\phi(e)} - r^2 - R^2)z + r^2 z_{\phi(e)} = 0. \end{cases}$$

We deduce that

$$z_{\phi(b)} + z_{\phi(c)} = z_{\phi(e)} - \frac{r^2 + R^2}{y_{\phi(e)}}$$

and similarly

$$y_{\phi(b)} + y_{\phi(c)} = y_{\phi(e)} - \frac{r^2 + R^2}{z_{\phi(e)}}$$

which gives the desired result, since $\phi(d) = \phi(b) + \phi(c) - \phi(e)$. □

Fact 3.7. *The workspace of the vertex* e*,* $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$*, contains the spacelike cone*

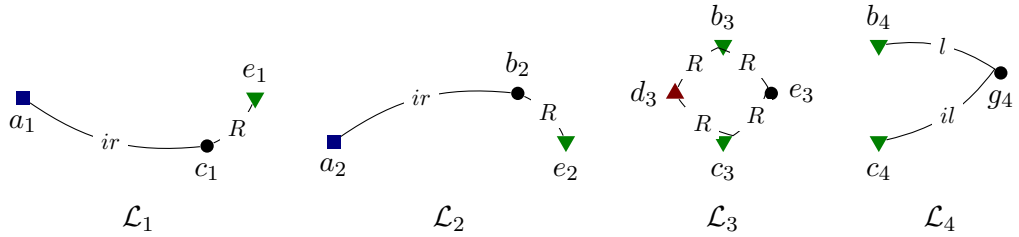
$$\left\{\alpha \in \mathbb{M} \;\middle|\; \|\alpha\|^2 > 0\right\}.$$

Proof. Here, we use Proposition 3.2 at each step. Choose any α in the spacelike cone, let $\phi(e) = \alpha$, and choose $\phi(b)$ and $\phi(c)$ such that $\{\phi(b), \phi(c)\} = \mathcal{H}(\phi_0(a), ir) \cap \mathcal{H}(\alpha, R)$. Then, $\|\phi(c) - \phi(b)\|^2 < 0$, so it is possible to choose $\phi(d)$ such that $\mathcal{H}(\phi(b), R) \cap \mathcal{H}(\phi(c), R) = \{\phi(d), \phi(e)\}$. Finally, choose $\phi(g)$ in the intersection $\mathcal{H}(\phi(b), l) \cap \mathcal{H}(\phi(c), il)$. □

Proposition 3.8. *For this linkage,* $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$.

Proof. We give a detailed proof in order to illustrate the use of Fact 2.27. This method is the key to many proofs concerning $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L})$.

The Peaucellier inversor may be seen as the combination of the following linkages:
$\mathcal{L}_1$: a robotic arm $\{a_1, c_1, e_1\}$ with one input e_1 and one fixed vertex a_1, one edge $\{a_1, c_1\}$ of length ir and one edge $\{c_1, e_1\}$ of length R;
$\mathcal{L}_2$: a robotic arm $\{a_2, b_2, e_2\}$ with one input e_2 and one fixed vertex a_2, one edge $\{a_2, b_2\}$ of length ir and one edge $\{b_2, e_2\}$ of length R;
$\mathcal{L}_3$: a rigidified square $\{b_3, d_3, c_3, e_3\}$ with inputs b_3, c_3 and four edges of length R;
$\mathcal{L}_4$: a robotic arm $\{b_4, g_4, c_4\}$ with inputs b_4, c_4, one edge $\{b_4, g_4\}$ of length l and one edge $\{g_4, c_4\}$ of length il.



We combine the linkages in the following way (observe that the name of the vertices are chosen so that each β_i preserves the letters and only changes indices):

1. Let $W_1 = \{c_1, e_1\}$. Let $\beta_1(c_1) = c_3$, $\beta_1(e_1) = e_3$. Let $\mathcal{L}_5 = \mathcal{L}_1 \cup_{\beta_1} \mathcal{L}_3$. The input set of $\mathcal{L}_5$ is $P_5 = \{e_3, b_3\}$.

2. Let $W_2 = \{b_2, e_2\}$. Let $\beta_2(b_2) = b_3$, $\beta_2(e_2) = e_3$. Let $\mathcal{L}_6 = \mathcal{L}_2 \cup_{\beta_2} \mathcal{L}_5$. The input set of $\mathcal{L}_6$ is $P_6 = \{e_3\}$.

3. Let $W_6 = \{b_3, c_3\}$. Let $\beta_6(b_3) = b_4$, $\beta_6(c_3) = c_4$. Let $\mathcal{L}_7 = \mathcal{L}_6 \cup_{\beta_6} \mathcal{L}_4$. The input set of $\mathcal{L}_7$ is $P_7 = \{e_3\}$.

The linkage $\mathcal{L}_7$ is exactly the Peaucellier linkage.

Let $\psi \in \mathrm{Conf}^{P_1}_{\mathbb{M}}(\mathcal{L}_1)$ such that the intersection $\mathcal{H}(0, ir) \cap \mathcal{H}(\psi(e_1), R)$ has cardinality 2. Facts 3.4 and 2.23 show that $\psi \in \mathrm{Reg}^{P_1}_{\mathbb{M}}(\mathcal{L}_1)$.

We may naturally identify $\mathrm{Conf}^{\{e_3, b_4\}}_{\mathbb{M}}(\mathcal{L}_7)$ with a subset C of $\mathrm{Conf}^{P_5}_{\mathbb{M}}(\mathcal{L}_5)$ (identifying b_4 with b_3). Let us show, using Fact 2.27, that C is in fact a subset of $\mathrm{Reg}^{P_5}_{\mathbb{M}}(\mathcal{L}_5)$. Let $\psi \in C$, and let $\phi \in \mathrm{Conf}^{V_7}_{\mathbb{M}}(\mathcal{L}_7)$ such that $\phi(e_3) = \psi(e_3)$ and $\phi(b_4) = \psi(b_3)$. Let $\psi_1 \in \mathbb{M}^{\{e_1\}}$ defined by $\psi_1(e_1) = \psi(e_3)$: since $\phi(b_4) \neq \phi(c_3)$, the intersection $\mathcal{H}(0, ir) \cap \mathcal{H}(\psi_1(e_1), R)$ has cardinality at least 2, but it is in fact exactly 2 from Proposition 3.1. Therefore, $\psi_1 \in \mathrm{Reg}^{P_1}_{\mathbb{M}}(\mathcal{L}_1)$, so the first hypothesis of Fact 2.27 is satisfied. For the second hypothesis, we need to show that $\phi|_{P_3} \in \mathrm{Reg}^{P_3}_{\mathbb{M}}(\mathcal{L}_3)$. We know that $\phi(b_4) \neq \phi(c_4)$, and from Proposition 3.6, we also know that $\phi(e_3) \neq \phi(d_3)$. Therefore, Proposition 3.5 tells us that $\phi|_{P_3} \in \mathrm{Reg}^{P_3}_{\mathbb{M}}(\mathcal{L}_3)$. The two hypotheses of Fact 2.27 are satisfied, so $C \subseteq \mathrm{Reg}^{P_5}_{\mathbb{M}}(\mathcal{L}_5)$.

In the same way, one may show that $\mathrm{Conf}^{\{e_3\}}_{\mathbb{M}}(\mathcal{L}_7) \subseteq \mathrm{Reg}^{P_6}_{\mathbb{M}}(\mathcal{L}_6)$, and finally, that $\mathrm{Conf}^{\{e_3\}}_{\mathbb{M}}(\mathcal{L}_7) \subseteq \mathrm{Reg}^{P_7}_{\mathbb{M}}(\mathcal{L}_7)$, so $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}_7) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}_7)$. $\square$

Proposition 3.9. *For all $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ we have the equivalence*

$$\phi(d) \in \mathcal{H}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, i\right) \Longleftrightarrow y_{\phi(e)} - z_{\phi(e)} = -(R^2 + r^2).$$

Proof. Let $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$. The following lines are equivalent:

$$\phi(d) \in \mathcal{H}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}, i\right)$$

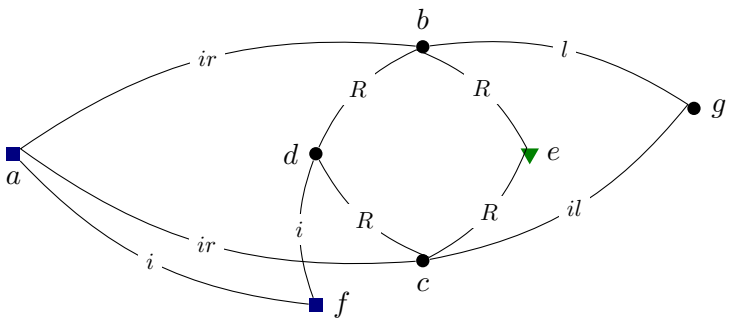
$$(y_{\phi(d)} + 1)(z_{\phi(d)} - 1) = -1$$

$$\left(-\frac{(R^2 + r^2)}{\|\phi(e)\|^2} y_{\phi(e)} + 1\right)\left(-\frac{(R^2 + r^2)}{\|\phi(e)\|^2} z_{\phi(e)} - 1\right) = -1$$

$$y_{\phi(e)} - z_{\phi(e)} = -(R^2 + r^2).$$

□

3.2.4 The partial t_0-line linkage



$$R = r = \tfrac{1}{\sqrt{2}}; l > 0; F = \{a, f\}; \phi_0(a) = \begin{pmatrix} 5 \\ t_0 + 1/2 \end{pmatrix}, \phi_0(f) = \begin{pmatrix} 5 \\ t_0 - 1/2 \end{pmatrix}; P = \{e\}.$$

Proposition 3.10. *The workspace of e, $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$, is contained in the line $t = t_0$, but does not necessarily contain the whole line. More precisely*

$$\left\{\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \;\middle|\; t = t_0, |x - 5| > 1/2\right\} \subseteq \mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) \subseteq \left\{\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{M} \;\middle|\; t = t_0\right\}.$$

Proof. We apply Fact 3.7, Propositions 3.8 and 3.9. □

For example, this linkage with the choice $t_0 = 0$ will be called the partial $(t_0 = 0)$-line linkage.

The dual of this linkage (see Section 3.1.4) is called the partial x_0-*line linkage*.

3.2.5 The t_0-integer linkage

This linkage contains four vertices a, b, c, d which are restricted to move on $\mathcal{I}$ (the x-axis) using a partial t_0-line linkages. More precisely, the t_0-integer linkage is obtained as the combination of the linkage on the figure below with is combined with four partial t_0-line linkages $\mathcal{L}_i, i = 1 \dots 4$, to form the t_0-integer linkage. The combination mappings β_i send a, b, c and d respectively to the inputs e_i of the linkages $\mathcal{L}_i$.

Take $F = \{a\}; \phi_0(a) = \begin{pmatrix} 0.5 \\ t_0 \end{pmatrix}; P = \emptyset$.

We have

$$\mathrm{Conf}_{\mathbb{M}}^{\{d\}}(\mathcal{L}) = \left\{ \begin{pmatrix} -3 \\ t_0 \end{pmatrix}, \begin{pmatrix} -2 \\ t_0 \end{pmatrix}, \dots, \begin{pmatrix} 3 \\ t_0 \end{pmatrix}, \begin{pmatrix} 4 \\ t_0 \end{pmatrix} \right\}.$$

Moreover, $\mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ is a finite set so $\mathrm{Reg}_{\mathbb{M}}^{P}(\mathcal{L}) = \mathrm{Conf}_{\mathbb{M}}^{P}(\mathcal{L})$.

We will use this linkage twice to construct more complex linkages. In Section 3.2.6, we could have used a simpler linkage with a configuration space of cardinality 2 instead of 8, but we need it to have cardinality at least 7 in Section 3.3.3.

3.2.6 The t_0-line linkage

This linkage traces out the whole horizontal line $t = t_0$: it contains a vertex e, the input, such that

$$\mathrm{Conf}_{\mathbb{M}}^{\{e\}}(\mathcal{L}) = \{\alpha \in \mathbb{M} \mid t_\alpha = t_0\}.$$

To construct it, the idea is to combine a partial t_0-line linkage with a t_0-integer linkage, as follows (to simplify the notations, we only give the construction of the $(t_0 = 0)$-line linkage). Let:

- $\mathcal{L}_1$ be a $(t_0 = \frac{1}{2})$-integer linkage;
- $\mathcal{L}_2$ a $(t_0 = -\frac{1}{2})$-integer linkage;
- $\mathcal{L}_3$ the combination (disjoint union) of the two linkages $\mathcal{L}_1$ and $\mathcal{L}_2$;
- $\mathcal{L}_4$ a linkage similar to a partial t_0-line linkage, with the only difference that $F_4 = \emptyset$ instead of $F_4 = \{a_4, f_4\}$;
- $W_3 = \{d_1, d_2\}$ and $\beta(d_1) = a_4$, $\beta(d_2) = f_4$;
- $\mathcal{L}_5 = \mathcal{L}_3 \cup_\beta \mathcal{L}_4$. Since for any $x \in \mathbb{R}$, we have either $|x - 5| > 1/2$ or $|x - 7| > 1/2$, we obtain as desired

$$\mathrm{Conf}_{\mathbb{M}}^{\{e_5\}}(\mathcal{L}_5) = \{\alpha \in \mathbb{M} \mid t_\alpha = t_0\}.$$

Using Fact 2.27, we also obtain $\mathrm{Reg}_{\mathbb{M}}^{P}(\mathcal{L}) = \mathrm{Conf}_{\mathbb{M}}^{P}(\mathcal{L})$.

For future reference, we let $a := a_1$, $f := f_1$, $e := e_4$.

The dual of this linkage is called the x_0-*line linkage*.

3.2.7 The horizontal parallelizer

This linkage has the input set $P = \{e_3, e_4\}$. It satisfies

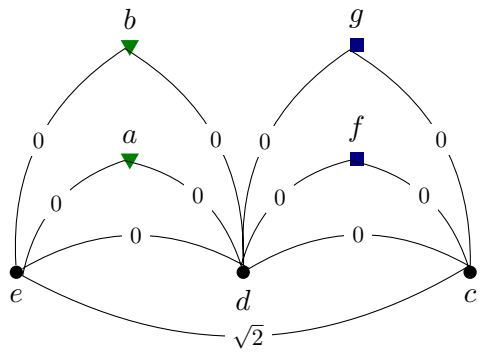
$$\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \left\{\psi \in \mathbb{M}^{\{e_3,e_4\}} \;\middle|\; t_{\psi(e_3)} = t_{\psi(e_4)}\right\}.$$

Let:

- $\mathcal{L}_1$ and $\mathcal{L}_2$ be two $(x_0 = 0)$-line linkages;
- $\mathcal{L}_3$, $\mathcal{L}_4$ two linkages similar to $(t_0 = 0)$-line linkages, but with $F_3, F_4 = \emptyset$;
- $\mathcal{L}_5$ the combination of $\mathcal{L}_1$ and $\mathcal{L}_2$;
- $W_3 = \{a_3, f_3\}, \beta(a_3) = e_1, \beta(f_3) = e_2$, and $\mathcal{L}_6 = \mathcal{L}_3 \cup_\beta \mathcal{L}_5$;
- $W_4 = \{a_4, f_4\}, \beta(a_4) = e_1, \beta(f_4) = e_2$, and $\mathcal{L}_7 = \mathcal{L}_4 \cup_\beta \mathcal{L}_6$.

$\mathcal{L}_7$ is the desired linkage.
For future reference, we let $a := e_3$ and $b := e_4$.
The dual of this linkage is called the *vertical parallelizer*.

3.2.8 The diagonal parallelizer



$P = \{a, b\}, F = \{g, f\}, \phi_0(f) = \begin{pmatrix}1\\1\end{pmatrix}, \phi_0(g) = \begin{pmatrix}0\\0\end{pmatrix}$.
In this section, we use the lightlike coordinates y and z (see Section 3.1.1).

Proposition 3.11. *We have*

$$\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \left\{\psi \in \mathbb{M}^P \;\middle|\; y_{\psi(a)} = y_{\psi(b)}\right\}.$$

Proof. The point is that for $\alpha_1, \alpha_2 \in \mathbb{M}$ such that $y_{\alpha_1} = y_{\alpha_2}$ and $\alpha_1 \neq \alpha_2$, the intersection $\mathcal{H}(\alpha_1, 0) \cap \mathcal{H}(\alpha_2, 0)$ is a straight line, more precisely:

$$\mathcal{H}(\alpha_1, 0) \cap \mathcal{H}(\alpha_2, 0) = \{\gamma \mid y_\gamma = y_{\alpha_1}\}.$$

First, let us prove the inclusion $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) \subseteq \left\{\psi \in \mathbb{M}^P \;\middle|\; y_{\psi(a)} = y_{\psi(b)}\right\}$.
For all $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$, $\phi(c) \in \mathcal{H}(\phi(g), 0) \cap \mathcal{H}(\phi(f), 0)$ and $z_{\phi(f)} = z_{\phi(g)}$, so $z_{\phi(c)} = 0$. Likewise, $z_{\phi(d)} = 0$.
Since $\phi(e) \in \mathcal{H}(\phi(d), 0)$ and $\phi(e) \notin \mathcal{H}(\phi(c), 0)$, we have $y_{\phi(d)} = y_{\phi(e)}$ and $\phi(d) \neq \phi(e)$.

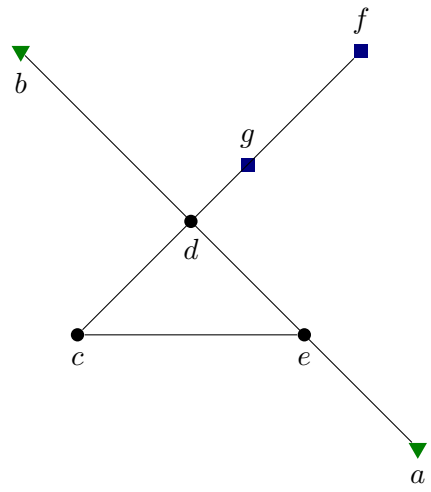


Figure 3.2 – One realization of the diagonal parallelizer.

Therefore, since $\phi(a) \in \mathcal{H}(\phi(d), 0) \cap \mathcal{H}(\phi(e), 0)$, we have $y_{\phi(a)} = y_{\phi(d)}$. Likewise, $y_{\phi(b)} = y_{\phi(d)}$ and finally, $y_{\phi(a)} = y_{\phi(b)}$.

Now, let us prove the inclusion $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) \supseteq \left\{\psi \in \mathbb{M}^P \;\middle|\; y_{\psi(a)} = y_{\psi(b)}\right\}$. Let $\psi \in \mathbb{M}^{\{a,b\}}$ such that $y_{\psi(a)} = y_{\psi(b)}$. Construct $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$ such that $\phi|_{\{a,b\}} = \psi$. Let $\phi(d) \in \mathbb{M}$ such that $z_{\phi(d)} = 0$ and $y_{\phi(d)} = y_{\psi(a)}$. Let $\phi(e) = \phi(d) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\phi(c) = \phi(d) + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (in (x, t) coordinates). Then $\phi \in \mathrm{Conf}_{\mathbb{M}}(\mathcal{L})$.

Finally, the coordinates of all vertices vary smoothly with respect to the coordinates of a and b, so $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$. □

3.3 Elementary linkages for algebraic operations

3.3.1 The average function linkage

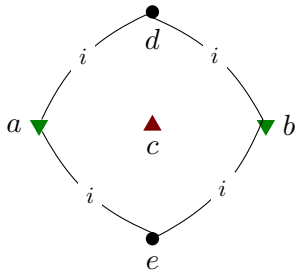
The average function linkage is a linkage with the input set $P = \{a, b\}$ and the output set $Q = \{c\}$ which is a functional linkage for the function

$$\begin{aligned} f : \mathcal{I}^2 &\to \mathcal{I} \\ (x_1, x_2) &\longmapsto \frac{x_1 + x_2}{2}, \end{aligned}$$

and such that $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P$.

Recall that by $\mathcal{I}$, we mean the x-axis, and by "$\mathcal{L}$ is a functional linkage for f", we mean that for all $\psi \in \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$

$$x_{\psi(c)} = \frac{x_{\psi(a)} + x_{\psi(b)}}{2}.$$



The vertices a, b, c are restricted to move on the line $\mathcal{I}$ using $(t_0 = 0)$-line linkages: this means that the linkage in the figure above is combined with three $(t_0 = 0)$-line linkages. Likewise, the points e, d and c are restricted to have the same x coordinate using a vertical parallelizer. The square *adbe* is rigidified (thus, the actual average function linkage has much more than these 5 vertices, but many of them are not represented on the figure).

To see that this linkage is the desired functional linkage, first notice that $\phi(c)$ is the middle of the segment $[\phi(a), \phi(b)]$ for all realization ϕ, because $\phi(a)\phi(d)\phi(b)\phi(e)$ is a parallelogram. Moreover, the expression

$$t_{\phi(d)} = \pm\sqrt{1 + \frac{\left|x_{\phi(b)} - x_{\phi(a)}\right|^2}{2}}$$

shows that the coordinates of $\phi(d)$ (and similarly, $\phi(e)$) depend on $\phi(a)$ and $\phi(b)$ in a differentiable way, so $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P$.

3.3.2 The adder

The adder is a linkage with the input set $P = \{a_1, b_1\}$ and the output set $Q = \{b_2\}$ which is a functional linkage for the function

$$\begin{aligned} f : \mathcal{I}^2 &\to \mathcal{I} \\ (x_1, x_2) &\longmapsto x_1 + x_2, \end{aligned}$$

with $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P$.

It is constructed as $\mathcal{L}_1 \cup_\beta \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are average function linkages, with $W_1 = \{c_1\}$, $\beta(c_1) = c_2$, $F_2 = \{a_2\}$, and $\phi_{02}(a_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Note that we may obtain a functional linkage for substraction by letting $P = \{b_2, b_1\}$ and $Q = \{a_1\}$. One may also construct (by induction on n) a functional linkage for $x \mapsto nx$, where n is any integer, and (by switching the input and the output) a functional linkage for $x \mapsto \frac{1}{n}x$.

3.3.3 The square function linkage

The square function linkage is a linkage with the input set $P = \{a\}$ and the output set $Q = \{b\}$: it is functional for the function

$$\begin{aligned} \mathcal{I} &\to \mathcal{I} \\ x &\longmapsto x^2, \end{aligned}$$

with $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P$.

To construct it, recall the algebraic trick described by Kapovich and Millson in [KM02]:

$$\forall x \in \mathbb{R} \setminus \{-0.5, 0.5\} \quad x^2 = 0.25 + \frac{1}{\frac{1}{x-0.5} - \frac{1}{x+0.5}}.$$

We have to find another trick to obtain a formula which works for every $x \in \mathbb{R}$.

To do this, notice that for all x and x' in $\mathbb{R}$ we have the identity

$$x^2 = 2(x+x')^2 + 2(x')^2 - (x+2x')^2.$$

Thus the expression x^2 can be rewritten

$$2\left(0.25 + \frac{1}{\frac{1}{x+x'-0.5} - \frac{1}{x+x'+0.5}}\right) + 2\left(0.25 + \frac{1}{\frac{1}{x'-0.5} - \frac{1}{x'+0.5}}\right) - \left(0.25 + \frac{1}{\frac{1}{x+2x'-0.5} - \frac{1}{x+2x'+0.5}}\right). \tag{3.1}$$

Moreover, for all $x \in \mathbb{R}$ there exists an $x' \in \{-3, -2, \ldots, 3, 4\}$ such that

$$\{x+x', x+2x', x'\} \cap \{-0.5, 0.5\} = \emptyset.$$

Start with a $(t_0 = 0)$-integer linkage $\mathcal{L}_1$: think of the vertex d_1 as the number x'. Let $\mathcal{L}_2$ be the linkage $\mathcal{L}_1$ to which one adds new fixed vertices at $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0.25 \\ 0 \end{pmatrix}$, and a new mobile vertex which will represent x and will be the input of the linkage (one does not add any new edge for now). Since Expression 3.1 is the composition of additions, subtractions and inversions, one may combine $\mathcal{L}_2$ with linkages for addition, subtraction and inversion (for the inversion, use the Peaucellier inversor), in the spirit of Fact 2.28, so that the output of the new linkage $\mathcal{L}$ corresponds to Expression 3.1. This is the desired linkage.

3.3.4 The multiplier

The multiplier is a linkage with the input set $P = \{a, b\}$ and the output set $Q = \{c\}$ which is a functional linkage for the function

$$\begin{aligned} f : \mathcal{I}^2 &\to \mathcal{I} \\ (x_1, x_2) &\longmapsto x_1 x_2, \end{aligned}$$

such that $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P$.

Simply construct the multiplier by combining square function linkages and adders, using the identity

$$\forall x_1, x_2 \in \mathbb{R} \quad x_1 x_2 = \frac{1}{4}\left((x_1+x_2)^2 - (x_1-x_2)^2\right).$$

3.3.5 The polynomial linkage

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial. We identify $\mathbb{R}$ with $\mathcal{I}$.

The polynomial linkage is a functional linkage for the function f with $\mathrm{card}(P) = n$ and

$$\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = \mathcal{I}^P.$$

The polynomial linkage is obtained by combining adders and multipliers (use Fact 2.28). The coefficients are represented by fixed vertices.

Example To illustrate the general case, we give the following example: $n = 2$, $m = 1$, $f(x, y) = 2x^3y + \pi$.

To construct a functional linkage for f, start with a linkage $\mathcal{L}$ consisting of two fixed vertices a, b with $\phi_0(a) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\phi_0(b) = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$, but also two vertices c, d which are the inputs and correspond respectively to the variables x and y.

Combine this linkage with a multiplier: the combination mapping β sends c to one of the inputs of the multiplier and d to the other one. The linkage (still called $\mathcal{L}$) is now functional for $(x, y) \longmapsto xy$.

Combine the new linkage with another multiplier: the combination mapping β sends c to one of the inputs and the output of $\mathcal{L}$ to the other one. The new linkage $\mathcal{L}$ is functional for $(x, y) \longmapsto x^2y$.

Repeating this process once, we obtain a functional linkage for x^3y, and then for $2x^3y$ (using the vertex a).

Finally, combine the linkage $\mathcal{L}$ with an adder: the combination mapping β sends the output of $\mathcal{L}$ to one of the inputs, and b to the other one.

3.4 End of the proof of Theorem 2.4

Let $n \in \mathbb{N}$. We are given A a semi-algebraic subset of $(\mathbb{R}^2)^n$, but we first assume that A is in fact an *algebraic* subset of $(\mathbb{R}^2)^n$, defined by a polynomial $f : \mathbb{R}^{2n} \to \mathbb{R}^m$ (so that $A = f^{-1}(0)$).

Take a polynomial linkage $\mathcal{L}$ for f. Name the elements of the input set: $P = \{a_1, \ldots, a_{2n}\}$. The output set Q has $2m$ elements.

The linkage $\mathcal{L}$ does not yet look like the desired linkage: since $\mathcal{L}$ has $2n$ inputs, the partial configuration space $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L})$ is a subset of $(\mathbb{R}^2)^{2n}$ (in fact, it is a subset of $\mathcal{I}^{2n}$), while A is a subset of $(\mathbb{R}^2)^n$ (in particular, we are looking for a linkage with n inputs). To obtain $\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = A \subseteq (\mathbb{R}^2)^n$, we have to modify $\mathcal{L}$ in the following way.

1. With several $(x_0 = 0)$-line linkages and diagonal parallelizers, extend the linkage $\mathcal{L}$ to a new one with new vertices $c_2, c_4, c_6, \ldots, c_{2n}$ such that for all realization ϕ and for all $k \in \{1, \ldots, n\}$
$$x_{\phi(c_{2k})} = 0;$$
$$y_{\phi(c_{2k})} = y_{\phi(a_{2k})} \quad \left(i.e.\ x_{\phi(c_{2k})} + t_{\phi(c_{2k})} = x_{\phi(a_{2k})} + t_{\phi(a_{2k})}\right).$$

2. With several vertical and horizontal parallelizers, extend this linkage to a new one with vertices $d_2, d_4, d_6, \ldots, d_{3n}$ such that for all realization ϕ and for all $k \in \{1, \ldots, n\}$
$$x_{\phi(d_{2k})} = x_{\phi(a_{2k-1})};$$
$$t_{\phi(d_{2k})} = t_{\phi(c_{2k})}.$$

Thus, for all realization ϕ and all $k \in \{1, \ldots, n\}$, we have $x_{\phi(d_{2k})} = x_{\phi(a_{2k-1})}$ and $t_{\phi(d_{2k})} = x_{\phi(a_{2k})}$.

Let $P = \{d_2, d_4, \ldots, d_{2n}\}$. Note that the input map p is a finite covering onto the simply connected set $(\mathbb{R}^2)^n$. Therefore, p is a trivial covering. The output set Q is unchanged.

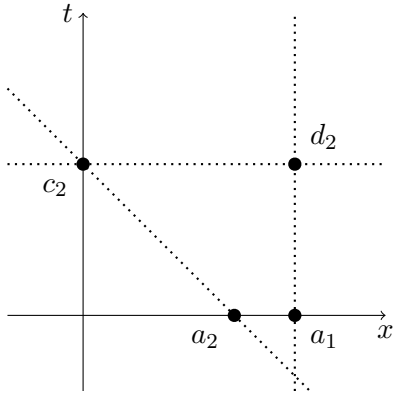


Figure 3.3 – A partial realization of the four vertices a_1, a_2, c_2, d_2. We have $x_{\phi(d_2)} = x_{\phi(a_1)}$ and $t_{\phi(d_2)} = x_{\phi(a_2)}$.

Fix the outputs to the origin: precisely, replace F by $F \cup Q$ and let

$$\forall a \in Q \quad \phi_0(a) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We obtain as desired $\mathrm{Reg}^P_{\mathbb{M}}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = A$.

Finally, if A is any *semi-algebraic* set of $(\mathbb{R}^2)^n$, then A is the projection of an algebraic set B of $(\mathbb{R}^2)^N$ for some $N \geq n$. Construct the linkage $\mathcal{L}_1$ such that $\mathrm{Conf}^{P_1}_{\mathbb{M}}(\mathcal{L}_1) = B$ and remove the unnecessary inputs $d_{2n+2}, d_{2n+4} \ldots, d_{2N}$. Then

$$\mathrm{Conf}^P_{\mathbb{M}}(\mathcal{L}) = A,$$

which ends the proof Theorem 2.4.

Chapter 4

Linkages in the hyperbolic plane

The aim of this chapter is to prove Theorem 2.6.

4.1 Generalities on the hyperbolic plane

Definition 4.1. The *Poincaré half-plane model* is the half-plane:

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}$$

endowed with the metric:

$$\frac{(dx)^2 + (dy)^2}{y^2}.$$

This model is the one we will always use in this chapter.

The distance δ on $\mathbb{H}^2$ is given by the formula:

$$\delta\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = \operatorname{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} \right).$$

4.1.1 Circles

In the Poincaré half-plane model, a hyperbolic circle with hyperbolic center α and hyperbolic radius R is in fact a Euclidean circle with center β and radius r, where $y_\beta = y_\alpha \cosh R$, $x_\beta = x_\alpha$ and $r = y_\alpha \sinh R$. Also notice that $y_\beta = \sqrt{y_\alpha^2 + r^2}$.

4.1.2 Some compact subsets of $\mathbb{H}^2$

Since we work with linkages with compact configuration spaces, whereas $\mathbb{H}^2$ is not compact, we need to introduce some compact subsets on which we will use our linkages. Fix a real parameter $\eta > 1$, and think of it as a very large number (the precise meaning of "large" will be given later).

Let

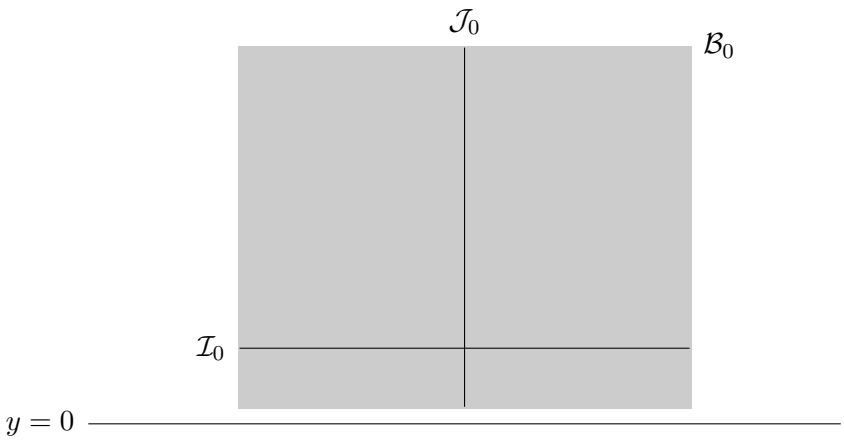
$$\mathcal{I}_0 := \left\{ \alpha \in \mathbb{H}^2 \;\middle|\; y_\alpha = 2, |x_\alpha| \leq 10\eta \right\}.$$

$$\mathcal{J}_0 := \left\{ \alpha \in \mathbb{H}^2 \;\middle|\; x_\alpha = 0,\ 2e^{-10\eta} \leq y_\alpha \leq 2e^{10\eta} \right\}.$$

$$\mathcal{B}_0 := \left\{ \alpha \in \mathbb{H}^2 \;\middle|\; |x_\alpha| \leq 10\eta,\ 2e^{-10\eta} \leq y_\alpha \leq 2e^{10\eta} \right\}.$$

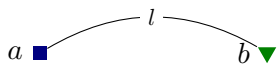
For any segment of positive length I, we write $\hat{I}$ the line containing I. For example:

$$\hat{\mathcal{I}_0} := \left\{\alpha \in \mathbb{H}^2 \;\middle|\; y_\alpha = 2\right\}.$$



4.2 Elementary linkages for geometric operations

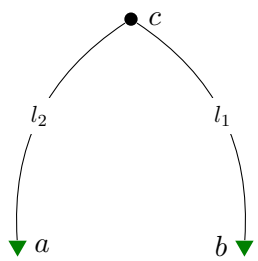
4.2.1 The circle linkage



We let $F = \{a\}$ and $P = \{b\}$ (see Section 3.1.1 for the notations).

In this linkage $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$ is a hyperbolic circle, which is also a Euclidean circle. Conversely, if $\mathcal{C}$ is a Euclidean circle contained in the half-plane, it is also a hyperbolic circle, so there is a circle linkage such that $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = \mathcal{C}$. Moreover, $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$.

4.2.2 The robotic arm linkage



We let $P = \{a, b\}, l_1 > 0, l_2 > 0$. We have:

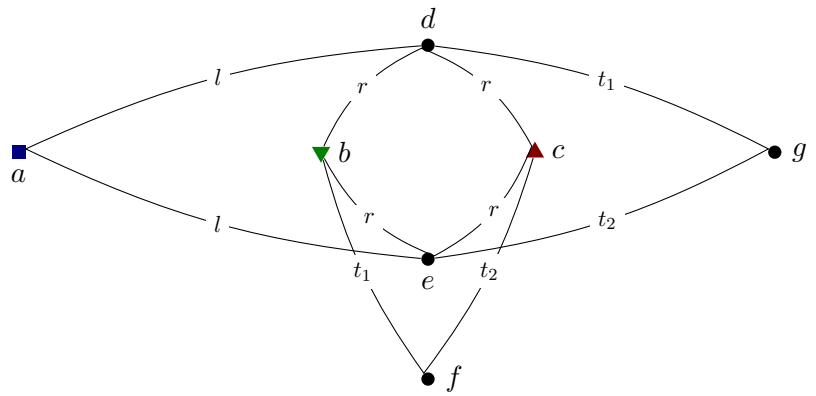
$$\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; |l_1 - l_2| \leq \delta(\psi(a), \psi(b)) \leq l_1 + l_2\right\}$$

and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains:

$$\left\{\psi \in \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \;\middle|\; |l_1 - l_2| < \delta(\psi(a), \psi(b)) < l_1 + l_2\right\}$$

(recall that δ is the hyperbolic distance on $\mathbb{H}^2$).

4.2.3 The Peaucellier inversor



We let $F = \{a\}, P = \{b\}, Q = \{c\}$. We require $l \neq r, t_1 \neq t_2, t_1 > r, t_2 > r$.

Proposition 4.2. *This linkage is functional for the (Euclidean) inversion with respect to the circle $\mathcal{C}$ with hyperbolic center $\phi_0(a)$ and hyperbolic radius* $\operatorname{arcosh} \frac{\cosh l}{\cosh r}$.

Proof. Let $\phi \in \mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L})$. Let μ be the middle of the hyperbolic segment $[\phi(d)\phi(e)]$.

First case. In this case, we assume $\phi_0(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_\mu = 0$ and $y_\mu \leq 1$.

$\phi(d)$ and $\phi(e)$ have two possible values each, and $\phi(d) \neq \phi(e)$ because $t_1 \neq t_2$. By symmetry, μ is also the middle of the hyperbolic segment $[\phi(b), \phi(c)]$ and $x_{\phi(b)} = x_{\phi(c)} = 0$. If necessary, we exchange b and c so that $y_{\phi(c)} \geq y_{\phi(b)}$.

Let α be the Euclidean center of $\mathcal{C}$. We have

$$y_\alpha = y_{\phi(a)} \cosh \operatorname{arcosh} \frac{\cosh l}{\cosh r} = \frac{\cosh l}{\cosh r}.$$

From the hyperbolic Pythagorean Theorem applied to the hyperbolic triangles $(\mu\phi(a)\phi(d))$, $(\mu\phi(b)\phi(d))$ and $(\mu\phi(c)\phi(d))$, letting $D = \delta(\phi(d), \mu)$, we get:

$$\delta(\phi(a), \mu) = \operatorname{arcosh} \frac{\cosh l}{\cosh D}$$

$$\delta(\phi(c), \mu) = \delta(\phi(b), \mu) = \operatorname{arcosh} \frac{\cosh r}{\cosh D}$$

We may now compute the coordinates of $\phi(b)$ and $\phi(c)$:

$$y_{\phi(b)} = \frac{\exp \operatorname{arcosh} \frac{\cosh r}{\cosh D}}{\exp \operatorname{arcosh} \frac{\cosh l}{\cosh D}}$$

$$y_{\phi(b)} = \frac{\cosh r + \sqrt{(\cosh r)^2 - (\cosh D)^2}}{\cosh l + \sqrt{(\cosh l)^2 - (\cosh D)^2}}$$

$$y_{\phi(c)} = \frac{1}{\left(\exp \operatorname{arcosh} \frac{\cosh r}{\cosh D}\right)\left(\exp \operatorname{arcosh} \frac{\cosh l}{\cosh D}\right)}$$

$$y_{\phi(c)} = \frac{(\cosh D)^2}{\left(\cosh r + \sqrt{(\cosh r)^2 - (\cosh D)^2}\right)\left(\cosh l + \sqrt{(\cosh l)^2 - (\cosh D)^2}\right)}.$$

Finally, we obtain as desired:

$$(y_\alpha - y_{\phi(b)})(y_\alpha - y_{\phi(c)}) = \frac{(\cosh l)^2}{(\cosh r)^2} - 1.$$

General case. Let $\Phi : \mathbb{H}^2 \mapsto \mathbb{H}^2$ be an isometry such that $\Phi(\phi(a)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $x_{\Phi(\mu)} = 0$ and $y_{\Phi(\mu)} \leq 1$. Let i be the inversion with respect to the circle with hyperbolic center $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and hyperbolic radius $\operatorname{arcosh} \frac{\cosh l}{\cosh r}$. Then $\phi(c) = \Phi^{-1} \circ i \circ \Phi(\phi(b))$, and $\Phi^{-1} \circ i \circ \Phi$ is the inversion with respect to the circle $\mathcal{C}$. □

We now study the workspace of the input b. Obviously, the input cannot be in the image of the lower half-plane by the inversion, because the output has to remain in the upper half-plane. Moreover, since the two edges (bf) and (bc) have different lengths, the input cannot be a fixed point of the inversion.

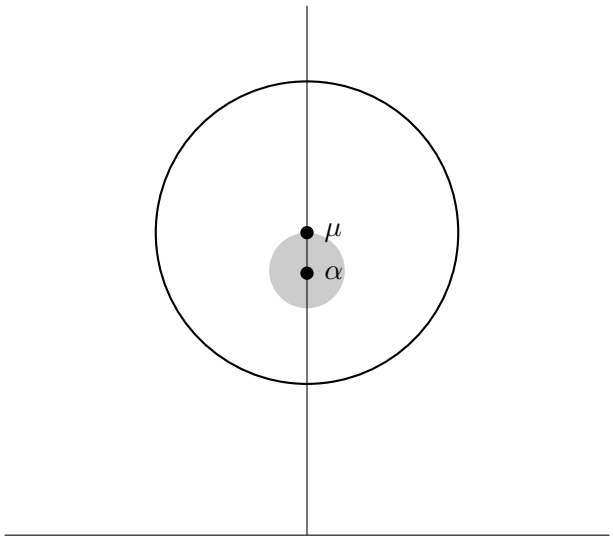


Figure 4.1 – The set K must not intersect the gray disk $\mathcal{D}$ or the black circle $\mathcal{C}$.

The following proposition tells us that these two obstructions are essentially the only ones.

Proposition 4.3. *Let $\mathcal{C}$ be a circle of hyperbolic center $\alpha \in \mathbb{H}^2$, hyperbolic radius $R > 0$, Euclidean center μ and Euclidean radius r. Let $\mathcal{D}$ be the closed disk of hyperbolic center α and hyperbolic radius $Q = \delta(\alpha, \mu)$.*

Let K be a compact set in $\mathbb{H}^2 \setminus (\mathcal{C} \cup \mathcal{D})$. Then there exists a choice of $\phi_0(a), l, r, t_1, t_2$ such that the Peaucellier linkage with these lengths is functional for the inversion with respect to $\mathcal{C}$, and such that $K \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$.

Proof. Let K' be a compact set in $\mathbb{H}^2 \setminus (\mathcal{C} \cup \mathcal{D})$ such that $K \subseteq \overset{\circ}{K'}$ (the interior of K'). Let Q', such that $Q < Q' < R$ and $K' \subseteq \mathbb{H}^2 \setminus (\mathcal{C} \cup \mathcal{D}')$, where $\mathcal{D}'$ is the closed disk of hyperbolic center α and hyperbolic radius Q'.

Let $Q'' > 0$ such that $K' \subseteq \mathcal{D}''$, where $\mathcal{D}''$ is the open disk of hyperbolic center α and hyperbolic radius Q''.

Define

$$\begin{aligned}\Phi : \mathbb{R}_{\geq 0} &\to \mathbb{R}\\ u &\to \operatorname{arcosh}((\cosh u)(\cosh R)) - u.\end{aligned}$$

Since $\lim_{u\to+\infty} \Phi(u) = \log(\cosh R) = Q$, there exists $u_0 \geq Q''$ such that $\Phi(u_0) \leq Q'$.

Let $\phi_0(a) = \alpha$, $r = u_0$ and $l = \Phi(u_0) + u_0$. Then, $R = \operatorname{arcosh} \frac{\cosh l}{\cosh r}$, $l - r \leq Q'$ and $l + r \geq Q''$.

Finally, choose t_1 and t_2 close enough to each other to have $K' \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$. Then $K \subseteq \overset{\circ}{K'} \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$. □

4.2.4 The Euclidean line linkage

The aim of this linkage is to trace out any given Euclidean segment. More precisely, let Δ be a straight line and $I \subseteq \Delta$ a Euclidean segment: we construct a linkage $\mathcal{L}$ with one input such that $I \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \Delta$.

Let $\alpha \in \mathbb{R}^2$, $k_1, k_2 > 0$ such that:

1. Δ is outside the closed disk with hyperbolic center α and hyperbolic radius k_1;
2. I is contained in the open disk with hyperbolic center α and hyperbolic radius k_2.

Let l and r such that $l + r = k_2$. Choosing l and r sufficiently close to $\frac{k_2}{2}$, we may also require $\operatorname{arcosh} \frac{\cosh l}{\cosh r} \leq k_1$. From Proposition 4.3, we deduce that there is a Peaucellier linkage $\mathcal{L}_1$ such that $I \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$.

Let i be the inversion for which $\mathcal{L}$ is functional. Then $i(\Delta)$ is a circle contained in the half-plane (in which one point has been removed). Let $\mathcal{L}_2$ be a circle linkage for this circle. Let $W_1 = \{c_1\}$ and $\beta(c_1) = b_2$. Construct the combination $\mathcal{L}_3 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. This linkage has the desired properties.

Rename the input: $b := b_1$, and the fixed vertices: $a := a_1$, $c := a_2$.

We also add an edge between the two fixed vertices a and c, of length $\delta(\phi_{03}(a), \phi_{03}(c))$. This new edge will be useful for Sections 4.2.5 and 4.2.6.

4.2.5 The vertical parallelizer

The aim of this linkage is to force two vertices to have the same x coordinate. More precisely, it has two inputs a and b with:

$$\left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; x_{\psi(a)} = x_{\psi(b)}\right\} \cap (\mathcal{B}_0)^P \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; x_{\psi(a)} = x_{\psi(b)}\right\}$$

(see the notations of Section 4.1.2).

To construct it, the idea is to allow two vertical Euclidean line linkages to move together horizontally.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two identical line linkages for the segment $\mathcal{J}_0$. Construct their disjoint union $\mathcal{L}_3 = \mathcal{L}_1 \cup \mathcal{L}_2$.

Let I be a horizontal segment of Euclidean length 20η centered at $\phi_{01}(a_1)$, and I' a horizontal segment of Euclidean length 20η centered at $\phi_{01}(c_1)$. Let $\mathcal{L}_4$ and $\mathcal{L}_5$ be line linkages for I and I' respectively, and construct their disjoint union $\mathcal{L}_6 = \mathcal{L}_4 \cup \mathcal{L}_5$.

Change F_3 to $\emptyset$, let $W_3 = \{a_1, c_1, a_2, c_2\}$, $\beta(a_1) = \beta(a_2) = b_4$, $\beta(c_1) = \beta(c_2) = b_5$ and $\mathcal{L}_7 = \mathcal{L}_3 \cup_\beta \mathcal{L}_6$.

Rename the inputs: $a := b_1, b := b_2$.

This linkage has the desired properties.

4.2.6 The hyperbolic alignment linkage

The *hyperbolic alignment linkage* and the *equidistance linkage* are not needed to prove the differential universality (Theorem 2.7), but we will use them to prove the algebraic universality (Theorem 2.6).

The *hyperbolic alignment linkage* forces its three inputs a, b, c to be on the same hyperbolic line. Fix a real constant $l > 0$, then define

$$A = \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; \psi(a), \psi(b), \psi(c) \text{ are on the same hyperbolic line}\right\}$$

and

$$B = \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; 0 < \max(\delta(\psi(a), \psi(b)), \delta(\psi(b), \psi(c)), \delta(\psi(a), \psi(c))) \leq l\right\}.$$

We want to construct a linkage such that

$$A \cap B \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq A.$$

Take a vertical hyperbolic segment J of hyperbolic length l. Note that J is also a Euclidean segment, and that any hyperbolic segment of length l is the image of J by a global isometry of $\mathbb{H}^2$.

Take three identical Euclidean line linkages $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ for J, with no fixed vertices ($F_1 = F_2 = F_3 = \emptyset$), and glue these three linkages together: let $W_1 = \{a_1, c_1\}$, $\beta(a_1) = a_2, \beta(c_1) = c_2$, and $\mathcal{L}_4 = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. Next, let $W_4 = \{a_2, c_2\}$, $\beta(a_2) = a_3, \beta(c_2) = c_3$, and $\mathcal{L}_5 = \mathcal{L}_4 \cup_\beta \mathcal{L}_3$. Rename the inputs: $a := b_1$, $b := b_2$, $c := b_3$. Since the isometries of $\mathbb{H}^2$ send the vertical line to other hyperbolic lines, the vertices a, b and c are always on the same hyperbolic line and $\mathcal{L}_5$ is the desired linkage.

4.2.7 The equidistance linkage

The equidistance linkage forces an input a to be equidistant from the two other inputs d and e.

Fix two real constants $k_1 > 0$, $k_2 > 0$ and define

$$A = \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; \delta(\psi(a), \psi(d)) = \delta(\psi(a), \psi(e))\right\}$$

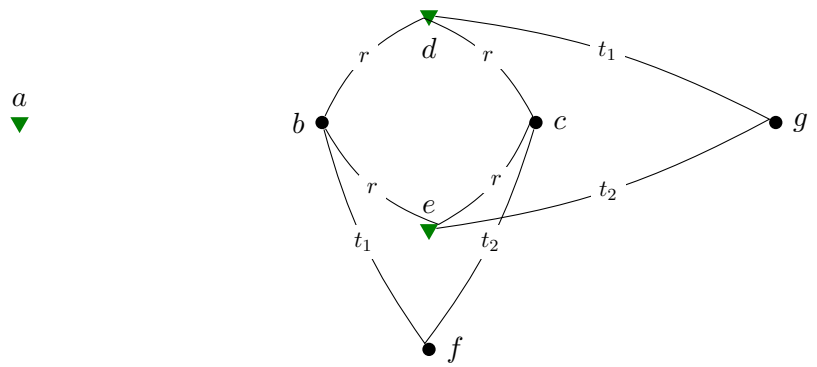
and

$$B = \left\{\psi \in (\mathbb{H}^2)^P \;\middle|\; \delta(\psi(d), \psi(e)) \geq k_1, \delta(\psi(a), \psi(d)) \leq k_2\right\}.$$

We want to construct a linkage $\mathcal{L}$ such that

$$A \cap B \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq A.$$

Start with the following linkage:



Let $P = \{a, d, e\}$, $F = \emptyset$ and $r > k_2$.

Using a hyperbolic alignement linkage with parameter $l = k_2 + 2r$, force the three vertices a, b and c to move on the same hyperbolic line. In other words, combine the linkage on the figure above with a hyperbolic alignment linkage (the combination mapping sends a, b, c to the inputs of the hyperbolic alignment linkage).

Finally, choose $t_1, t_2 > r$ with $t_1 \neq t_2$ and $|t_1 - t_2|$ sufficiently small.

4.3 Elementary linkages for algebraic operations

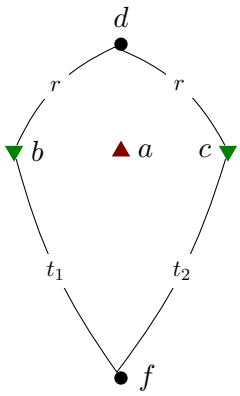
In this section, we describe linkages which are functional for algebraic operations such as addition or multiplication on real numbers. The real line is identified with $\hat{\mathcal{I}}_0 = \left\{\alpha \in \mathbb{H}^2 \;\middle|\; y_\alpha = 2\right\}$, which means that we will write simply x instead of $(x, 2)$.

4.3.1 The symmetrizer

First version

The symmetrizer is a functional linkage for:

$$\begin{aligned} f : \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})(\subseteq \hat{\mathcal{I}}_0{}^P) &\to \hat{\mathcal{I}}_0 \\ (x_1, x_2) &\longmapsto \frac{x_1 + x_2}{2}. \end{aligned}$$



Let $P = \{b, c\}, Q = \{a\}$, $t_1 > r$, $t_2 > r$, $|t_1 - t_2| = 1$, $r = 8\eta$. The vertices b and c are restricted to move on $\mathcal{I}_0$ using two line linkages: this means that the linkage on the

figure above is combined with two line linkages, which are not represented on the figure, with a combination map β such that $\beta(b)$ and $\beta(c)$ are the two inputs of the line linkages. The vertices a and d are restricted to move on the same vertical line using a vertical parallelizer.

For this linkage,

$$\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = \left\{ \psi \in \hat{\mathcal{I}_0}^P \;\middle|\; 1 \leq |\psi(b) - \psi(c)| \leq 16\eta \right\}$$

and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains

$$\left\{ \psi \in \hat{\mathcal{I}_0}^P \;\middle|\; 1 < |\psi(b) - \psi(c)| < 16\eta \right\}.$$

Modifying $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$

Since we want the symmetrizer to handle input vertices b and c which are close to each other, or even equal, the first version is not suitable for our purpose. Notice that

$$\frac{x_1 + x_2}{2} = \frac{\frac{x_1+8\eta}{2} + \frac{x_2+(-8\eta)}{2}}{2}.$$

Following this formula and the idea of Fact 2.28, take one symmetrizer $\mathcal{L}_1$, but add one input to the set of fixed vertices and set it to the coordinate $\begin{pmatrix} 8\eta \\ 2 \end{pmatrix}$. Next, take a second symmetrizer $\mathcal{L}_2$, add one input to the set of fixed vertices and set it to $\begin{pmatrix} -8\eta \\ 2 \end{pmatrix}$. Finally, take a third symmetrizer $\mathcal{L}_3$ and combine it with $\mathcal{L}_1$ and $\mathcal{L}_2$, using a combination mapping β which sends the outputs of $\mathcal{L}_1$, $\mathcal{L}_2$ to the inputs of $\mathcal{L}_3$.

Thus, by combining three symmetrizers we get a new version of the symmetrizer $\mathcal{L}$, which is functional for the same function, but such that $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ ($\subseteq \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$) contains:

$$\left\{ \psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(b)| < 7\eta, |\psi(c)| < 7\eta \right\}.$$

4.3.2 The adder

Take a symmetrizer $\mathcal{L}$. Fix the vertex b to 0, let $P = \{a\}$ and $Q = \{c\}$. We obtain a functional linkage for $x \longmapsto 2x$, for which $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains:

$$\left\{ \psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(a)| < 3\eta \right\}.$$

Combining this linkage with the symmetrizer, we get a functional linkage for

$$(x_1, x_2) \longmapsto 2\frac{x_1 + x_2}{2} = x_1 + x_2.$$

For this linkage, $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains:

$$\left\{ \psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(a)| < 3\eta, |\psi(b)| < 3\eta \right\}.$$

This linkage is called the *adder*.

4.3.3 The opposite value linkage

Take a symmetrizer $\mathcal{L}$. Fix the vertex a to 0, let $P = \{c\}$ and $Q = \{b\}$. We obtain a functional linkage for $x \longmapsto -x$, for which $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains:

$$\left\{\psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(c)| < 7\eta\right\}.$$

4.3.4 The rational homothety linkage

Let n be an integer. Using $n-1$ adders, we get a functional linkage for $x \longmapsto nx$, for which $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains

$$\left\{\psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(c)| < \frac{3}{n}\eta\right\}.$$

Switching the input and the output, we get a functional linkage for $x \longmapsto \frac{x}{n}$, for which $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains

$$\left\{\psi \in \hat{\mathcal{I}_0}^P \;\middle|\; |\psi(c)| < 3\eta\right\}.$$

4.3.5 The square function linkage

With the input set $P = \{a\}$ and the output set $Q = \{b\}$, it is a functional linkage for the function:

$$\begin{aligned} f : \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})(\subseteq \hat{\mathcal{I}_0}) &\to \hat{\mathcal{I}_0} \\ x &\longmapsto x^2, \end{aligned}$$

so that $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \hat{\mathcal{I}_0}^P$ and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains a neighborhood of $\begin{pmatrix}0\\2\end{pmatrix}$ in $\hat{\mathcal{I}_0}$.

To construct it, let $\mathcal{C}$ be the circle of Euclidean center $\begin{pmatrix}0\\2\end{pmatrix}$ and radius 1, K a compact set like in Proposition 4.3 such that $\begin{pmatrix}\pm 0.5\\2\end{pmatrix} \in \overset{\circ}{K}$ and $\begin{pmatrix}-4\\2\end{pmatrix} \in \overset{\circ}{K}$, and $\mathcal{L}$ a Peaucellier linkage such that $K \subseteq \mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$. Then $\mathcal{L}$ is functional for

$$\begin{aligned} f : \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})(\subseteq \hat{\mathcal{I}_0}) &\to \hat{\mathcal{I}_0} \\ x &\longmapsto \frac{1}{x}. \end{aligned}$$

Now, as in the Minkowski case, we use the algebraic trick first described by Kapovich and Millson [KM02]:

$$\forall x \in \mathbb{R} \setminus \{-0.5, 0.5\} \quad x^2 = 0.25 + \frac{1}{\frac{1}{x-0.5} - \frac{1}{x+0.5}},$$

Thus, the desired linkage is obtained by composition of the previous linkages.

4.3.6 The multiplier

A first version

The multiplier is a linkage with $P = \{a, b\}$ and $Q = \{c\}$, which is a functional linkage for the function:

$$\begin{aligned} f : \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})(\subseteq \hat{\mathcal{I}_0}^P) &\to \hat{\mathcal{I}_0} \\ (x_1, x_2) &\longmapsto x_1 x_2 \end{aligned}$$

such that $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \hat{\mathcal{I}_0}^P$ and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains a neighborhood $\mathcal{U}$ of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ in $\hat{\mathcal{I}_0}^P$.

We simply construct it by combining square function linkages and adders, using the identity:

$$\forall x_1, x_2 \in \mathbb{R} \quad x_1 x_2 = \frac{1}{4}((x_1 + x_2)^2 - (x_1 - x_2)^2).$$

Modifying $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$

We are now going to construct a multiplier such that $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \hat{\mathcal{I}_0}^P$ and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ contains $\left\{ \begin{pmatrix} x \\ 2 \end{pmatrix} \;\middle|\; x \in [-\eta, \eta] \right\}^P$.

Let n be an integer such that $\left\{ \begin{pmatrix} x \\ 2 \end{pmatrix} \;\middle|\; x \in [-\frac{\eta}{n}, \frac{\eta}{n}] \right\}^P \subseteq \mathcal{U}$ (where $\mathcal{U}$ is defined in Section 4.3.6). Using two rational homothety linkages and one multiplier (first version), and the formula:

$$\forall x_1, x_2 \in \mathbb{R} \quad x_1 x_2 = n^2 \left(\frac{x_1}{n}\right)\left(\frac{x_2}{n}\right)$$

we obtain the desired linkage.

4.3.7 The polynomial linkage

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial of degree $d \geq 1$ and coefficients in $[-K, K]$ with $1 \leq K \leq \eta$. We still identify $\mathbb{R}$ with $\hat{\mathcal{I}_0}$. Our aim is to construct a functional linkage for $f|_{\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})}$, with $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) \subseteq \hat{\mathcal{I}_0}^P$ and $\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L})$ containing $\mathcal{U}_{K,d,n} := [-M_{K,d,n}, M_{K,d,n}]^P$, where $M_{K,d,n} = \frac{1}{K(d+1)^n}\eta^{1/d}$.

It is obtained by combining adders and multipliers. The coefficients are represented by fixed vertices.

Remark. At this stage, it would be possible to fix the outputs of the polynomial linkage to $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ to prove directly Theorem 2.7. However, the proof of Theorem 2.6 is more complicated, since we need the input vertices to move outside the line $\hat{\mathcal{I}_0}$, as it will be explained in the next section.

4.4 End of the proof of Theorem 2.6

Let A be a compact semi-algebraic subset of $(\mathbb{H}^2)^n$. First, we assume that A is a compact *algebraic* subset of $(\mathbb{H}^2)^n$.

We want to construct a linkage with $P = \{d_2, d_4, \dots, d_{2n}\}$ such that $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = A$. The idea is to identify each point α of the Poincaré half-plane with three coordinates $X^1_\alpha, X^2_\alpha, X^3_\alpha$, defined by:

$$\forall i \in \{1,2,3\} \quad X^i_\alpha = \delta\left(\begin{pmatrix} i \\ 2 \end{pmatrix}, \alpha\right).$$

Since $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ are not aligned, these three coordinates characterize the point α.

Let $f : (\mathbb{R}^2)^n = \mathbb{R}^{2n} \to \mathbb{R}^m$ be a polynomial function (of degree d) with coefficients in $[-1, 1]$ such that $A = f^{-1}(0)$.

We may assume that A is contained in the set

$$\mathcal{V} := ((-M_{1,d,2n}, M_{1,d,2n}) \times (3, M_{1,d,2n}))^n$$

(see Section 4.3.7 for the definition of $M_{1,d,2n}$). If it is not, choose an isometry Φ of $\mathbb{H}^2$ such that $\Phi(A)$ is contained in this set (for a large enough η), construct the desired linkage, and then replace ϕ_0 by $\Phi^{-1} \circ \phi_0$.

If necessary, increase η (but do not change the definition of $\mathcal{V}$ by doing so) so that

$$M_{100,2,2} \geq \max\left\{ |x| \;\middle|\; x \in \mathbb{R}, i \in \{1,2,3\}, \delta\left(\begin{pmatrix} x \\ 2 \end{pmatrix}, \begin{pmatrix} i \\ 2 \end{pmatrix}\right) \leq \max_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathcal{V} \\ k\in\{1,\dots,n\}}} \delta\left(\alpha_k, \begin{pmatrix} i \\ 2 \end{pmatrix}\right) \right\}.$$

We are now ready to construct our linkage. Start with a linkage $\mathcal{L}$ with the input set $P = \{d_2, d_4, \dots, d_{2n}\}$ and no edge. Add other vertices

$$a_1, a_2, \dots, a_{2n}, b^1_2, b^2_2, b^3_2, b^1_4, b^2_4, b^3_4, \dots, b^1_{2n}, b^2_{2n}, b^3_{2n}, c^1_2, c^2_2, c^3_2, c^1_4, c^2_4, c^3_4, \dots, c^1_{2n}, c^2_{2n}, c^3_{2n}$$

which are restricted to move on $\hat{\mathcal{I}}_0$ using line linkages.

Combine the linkage with equidistance linkages (with parameters $k_1 = 1$ and $k_2 = M_{1,d,2n} + 4$) so that for all $\phi \in \mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L})$, all $k \in \{1, \dots, n\}$ and all $i \in \{1,2,3\}$:

$$X^i_{\phi(b^i_{2k})} = X^i_{\phi(d_{2k})}.$$

Then, use polynomial linkages so that for all $\phi \in \mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L})$, all $k \in \{1, \dots, n\}$ and all $i \in \{1,2,3\}$:

$$x_{\phi(c^i_{2k})} = x_{\phi(a_{2k})} \cdot (x_{\phi(b^i_{2k})} - i)^2$$

and

$$x_{\phi(c^i_{2k})} = 2 \cdot ((x_{\phi(a_{2k-1})} - i)^2 + (x_{\phi(a_{2k})} - 2)^2).$$

Thus we have for all i and k:

$$x_{\phi(a_{2k})} \cdot (x_{\phi(b^i_{2k})} - i)^2 = 2 \cdot ((x_{\phi(a_{2k-1})} - i)^2 + (x_{\phi(a_{2k})} - 2)^2)$$

$$\operatorname{arcosh}\left(1 + \frac{(x_{\phi(b^i_{2k})} - i)^2}{2 \cdot 2}\right) = \operatorname{arcosh}\left(1 + \frac{(x_{\phi(a_{2k-1})} - i)^2 + (x_{\phi(a_{2k})} - 2)^2}{2 \cdot x_{\phi(a_{2k})}}\right)$$

$$\delta\left(\phi(b^i_{2k}), \begin{pmatrix} i \\ 2 \end{pmatrix}\right) = \delta\left(\begin{pmatrix} x_{\phi(a_{2k-1})} \\ x_{\phi(a_{2k})} \end{pmatrix}, \begin{pmatrix} i \\ 2 \end{pmatrix}\right)$$

$$X^i_{\phi(b^i_{2k})} = X^i_{\begin{pmatrix} x_{\phi(a_{2k-1})} \\ x_{\phi(a_{2k})} \end{pmatrix}}$$

$$X^i_{\phi(d_{2k})} = X^i_{\begin{pmatrix} x_{\phi(a_{2k-1})} \\ x_{\phi(a_{2k})} \end{pmatrix}}.$$

Thus:

$$\phi(d_{2k}) = \begin{pmatrix} x_{\phi(a_{2k-1})} \\ x_{\phi(a_{2k})} \end{pmatrix}.$$

Add vertices $e_1, \ldots, e_m$ and use a polynomial linkage so that for all $\phi \in \mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L})$:

$$f(x_{\phi(a_1)}, \ldots, x_{\phi(a_{2n})}) = (x_{\phi(e_1)}, \ldots, x_{\phi(e_m)}).$$

Now, notice that $p|_{\pi^{-1}(\mathcal{V})}$ is a smooth finite covering onto $\mathcal{V}$, which is necessarily trivial since $\mathcal{V}$ is simply connected.

To finish the construction, fix the vertices $e_1, \ldots, e_m$ to the point $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Thus,

$$\mathrm{Reg}^{\{a_1,\ldots,a_{2n}\}}_{\mathbb{H}^2}(\mathcal{L}) = \mathrm{Conf}^{\{a_1,\ldots,a_{2n}\}}_{\mathbb{H}^2}(\mathcal{L}) = A \subseteq \mathbb{R}^{2n} = \hat{\mathcal{I}}_0,$$

and finally:

$$\mathrm{Reg}^P_{\mathbb{H}^2}(\mathcal{L}) = \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = A \subseteq (\mathbb{H}^2)^n \subseteq (\mathbb{R}^2)^n.$$

Moreover, the restriction map $\mathrm{Conf}_{\mathbb{H}^2}(\mathcal{L}) \to \mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L})$ is a smooth finite covering, which is trivial as the restriction of a trivial covering.

If A is only a compact semi-algebraic subset of $(\mathbb{H}^2)^n$, we know from Proposition 2.14 that A is the projection onto the first coordinates of a compact algebraic set B. Apply the above construction to B and remove some vertices from the input set to obtain $\mathrm{Conf}^P_{\mathbb{H}^2}(\mathcal{L}) = A$, which ends the proof of Theorem 2.6.

Chapter 5

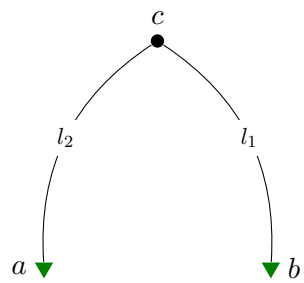
Linkages in the sphere

The aim of this chapter is to prove Theorem 2.9. In the first three sections, we focus on the two-dimensional sphere, while higher dimensions are studied in the last section.

The sphere $\mathbb{S}^2$ will be considered as the unit sphere of $\mathbb{R}^3$. Thus, a point $\alpha \in \mathbb{S}^2$ is denoted by three coordinates $x_\alpha, y_\alpha, z_\alpha$.

5.1 Elementary linkages for geometric operations

5.1.1 The articulated arm linkage

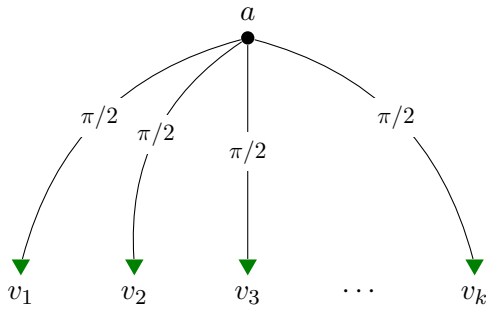


$P = \{a, b\}, Q = \emptyset, F = \emptyset$. The lengths of the edges l_1 and l_2 satisfy $0 < l_1, l_2 \leq \pi$. A standard computation gives:

$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2)^P \;\middle|\; |l_1 - l_2| \leq \delta(\psi(a), \psi(b)) \leq \min(l_1 + l_2, 2\pi - (l_1 + l_2))\right\}.$$

5.1.2 The great circle linkage

One vertex a is linked to k other vertices $v_1, \ldots, v_k$, by edges of length $\pi/2$.



$P = \{v_1, \ldots, v_k\}, Q = \emptyset, F = \emptyset$.
This linkage forces $v_1, \ldots, v_k$ to be on the same great circle of the sphere:

$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2)^P \;\middle|\; \exists f \in (\mathbb{R}^3)^* \;\; \forall i \in \{1, \ldots, k\} \;\; f(\psi(v_k)) = 0\right\}.$$

The *fixed great circle linkage* is a variant in which $a \in F$. Then

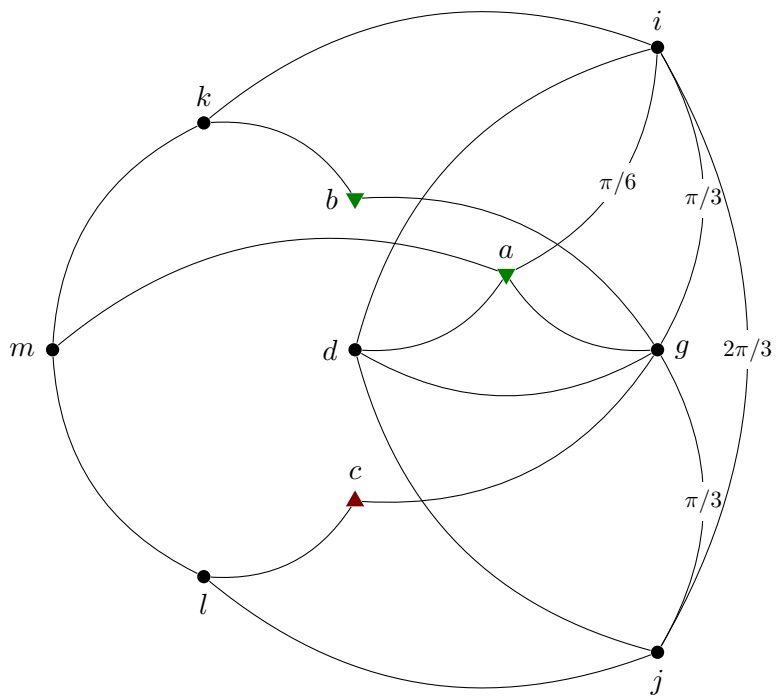
$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = (\mathbb{S}^2 \cap a^\perp)^P.$$

5.1.3 The symmetrizer

The symmetrizer is a functional linkage for symmetry with respect to a great circle (*i.e.* orthogonal symmetry with respect to a plane $\mathcal{P}$ in $\mathbb{R}^3$). It is the key to the construction of several other linkages, but it is also the most difficult to obtain.

First version

Here is a first attempt, which we shall call $\mathcal{L}_1$.



All the edges have length $\pi/2$, except when another length is indicated.

Let $P = \{a, b\}, Q = \{c\}$, and $F = \emptyset$. We want a to be the unit normal vector to the (linear) plane $\mathcal{P}$ of symmetry, and b to be the point to which we want to apply the symmetry. The result of the symmetry is c.

Proposition 5.1. *Fix some $\psi \in (\mathbb{S}^2)^P$. Let $\alpha \in \mathbb{S}^2$ be symmetric to $\psi(b)$ with respect to $\psi(a)^\perp$. Then:*

$$\left\{\phi(c) \;\middle|\; \phi \in p^{-1}(\psi)\right\} = \{\alpha, -\alpha\}.$$

Proof. Assume that $\phi \in \mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}_1)$ is such that $\phi|_P = \psi$. Applying a rotation to the sphere if necessary, we may assume that $\psi(a) = (0, 0, 1)$ and $y_{\psi(b)} = 0$. Let $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_1)$. If $x_{\psi(b)} = 0$, we may also assume up to rotation that $y_{\phi(d)} = 0$. If $x_{\psi(b)} \neq 0$, then $\phi(g)^\perp$ contains the two distinct points $\psi(a)$ and $\psi(b)$, so $\phi(g) \in \{\pm(0, 1, 0)\}$. Applying a symmetry with respect to $\phi(g)^\perp$ if necessary, we may assume that $\phi(g) = (0, 1, 0)$. But $\phi(d) \in \phi(g)^\perp \cap \phi(a)^\perp$, so that $\phi(d) \in \{\pm(1, 0, 0)\}$. Therefore, whether or not $x_{\psi(b)} = 0$, we may assume $\phi(d) \in \{\pm(1, 0, 0)\}$ and $\phi(g) = (0, 1, 0)$. Hence, $\phi(i) = (0, 1/2, \sqrt{3}/2)$ and $\phi(j) = (0, 1/2, -\sqrt{3}/2)$.

Since $\phi(k)$ is on the line $\phi(i)^\perp \cap \phi(b)^\perp$, it has two possible (opposite) values.

$\phi(k) \notin \mathbb{R}\phi(a)$ because $\phi(k) \in \phi(i)^\perp$ and $\phi(a) \notin \phi(i)^\perp$.

Since m is on the line $\phi(k)^\perp \cap \phi(a)^\perp$, it has two possible opposite values.

$\phi(j) \notin \mathbb{R}\phi(m)$ because $\phi(m) \in \phi(a)^\perp$ and $\phi(j) \notin \phi(a)^\perp$.

Since l is on the line $\phi(j)^\perp \cap \phi(m)^\perp$, $\phi(l)$ has two possible opposite values.

$\phi(l) \notin \mathbb{R}\phi(g)$ because $\phi(l) \in \phi(j)^\perp$ and $\phi(g) \notin \phi(j)^\perp$.

Since c is on the line $\phi(l)^\perp \cap \phi(g)^\perp$, $\phi(c)$ has two possible opposite values.

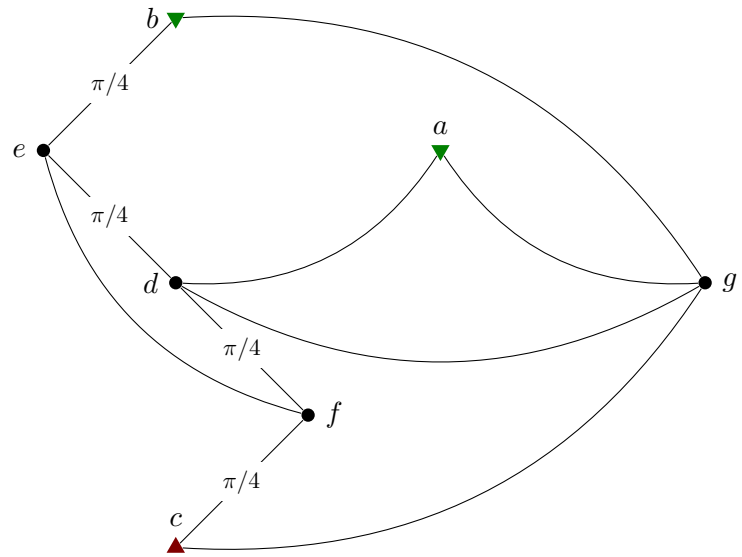
Note that the construction of ϕ described above really provides a realization of the linkage, which proves that $\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}_1)$ is the whole $(\mathbb{S}^2)^P$.

To see that one of the possible values of $\phi(c)$ is symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$, use the symmetries of the abstract linkage: take $\phi \in \mathrm{Conf}^{\{a,b,d,g,i,k,m\}}_{\mathbb{S}^2}(\mathcal{L}_1)$, and extend ϕ to V (the set of all vertices) by letting $\phi(c)$ be symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$, $\phi(l)$ symmetric to $\phi(k)$ with respect to $\phi(a)^\perp$, and $\phi(j)$ symmetric to $\phi(i)$ with respect to $\phi(a)^\perp$. Then it is clear that $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_1)$. □

In order to avoid the configurations in which $\phi(c)$ is not symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$, we introduce a second version of the symmetrizer.

Second version

Here is a different version of the symmetrizer, $\mathcal{L}_2$:



All the edges have length $\pi/2$, except when another length is indicated; $P = \{a, b\}, Q = \{c\}$, and $F = \emptyset$. As before, we want c to be symmetric to b with respect to $a^\perp$.

Proposition 5.2. *1. For all $\psi \in (\mathbb{S}^2)^P$, there exists $\phi \in p^{-1}(\psi)$ such that $\phi(c)$ is symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$.*

2. There does not exist $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_2)$ such that $-\phi(c)$ is symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$.

Proof. Let us prove the first assertion. Let $\psi \in (\mathbb{S}^2)^P$, choose $\phi(g)$ anywhere in $\psi(a)^\perp \cap \psi(b)^\perp$, and a point $\alpha \in \phi(g)^\perp \cap \phi(a)^\perp$. If $\delta(\alpha, \phi(b)) \leq \pi/2$, let $\phi(d) = \alpha$, else let $\phi(d) = -\alpha$. In any case we have $\delta(\phi(d), \phi(b)) \leq \pi/2$ so we can choose $\phi(e)$ on the intersection of the circles $\mathcal{C}(\phi(b), \pi/4)$ and $\mathcal{C}(\phi(d), \pi/4)$. Let $\phi(c)$ be symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$, and let $\phi(f)$ be the image of $\phi(e)$ by a half turn of axis $\mathbb{R}\phi(d)$. Then, ϕ is a realization of the linkage.

We now prove the second assertion. Let $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_2)$. If $\delta(\phi(c), \phi(d)) < \pi/2$ then $\delta(-\phi(c), \phi(d)) > \pi/2$ whereas $\delta(\phi(b), \phi(d)) \leq \pi/2$, so $-\phi(c)$ is not symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$. If $\delta(\phi(c), \phi(d)) = \pi/2$ then $\phi(c) = \pm\phi(a)$, which means that $\phi(c)$, $\phi(f)$, $\phi(d)$, $\phi(e)$ are on the same geodesic and $\delta(\phi(c), \phi(e)) = 3\pi/4$. Therefore $\phi(c) \neq \phi(b)$, so $-\phi(c)$ is not symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$. □

$\mathcal{L}_2$ is not a functional linkage for symmetry. There is a possible degenerate configuration which seems difficult to avoid: for any position of the inputs $\psi \in \mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}_2)$, there is a $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_2)$ such that $\phi(b) = \phi(d)$. This problem is very related to the problem of the degenerate configurations of the parallelogram, which Kempe did not see when he wrote his original proof. The solution to this problem in the plane is the rigidification of the parallelogram, but the usual rigidification does not work in the sphere.

Gluing the two versions

We now have two linkages, $\mathcal{L}_1$ and $\mathcal{L}_2$, which are almost functional linkages for symmetry, and have different degenerate configurations.

We glue them together: let $W_1 = \{a_1, b_1, c_1\}$ and $\beta(a_1) = a_2$, $\beta(b_1) = b_2$, $\beta(c_1) = c_2$, and $\mathcal{L} = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$.

We rename some vertices for future reference: $a := a_2$, $b := b_2$, $c := c_2$, $d := d_2$, $g := g_2$.

Proposition 5.3. *1.* $\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = (\mathbb{S}^2)^P$.

2. $\mathcal{L}$ is a functional linkage for symmetry: for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$, $\phi(c)$ is symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$.

Proof. This is an immediate consequence of propositions 5.1 and 5.2. □

5.1.4 The parallelizer

The parallelizer has three inputs b, c, h, such that

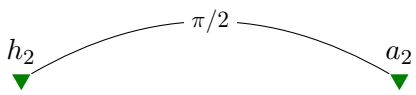
$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2)^P \;\middle|\; \delta(\psi(h), \psi(b)) = \delta(\psi(h), \psi(c))\right\}.$$

Notice that the equality $\delta(\psi(h), \psi(b)) = \delta(\psi(h), \psi(c))$ is equivalent to $(\psi(a)|\phi(h)) = (\psi(b)|\psi(h))$, where $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^3$. Therefore, for any linear form $f : \mathbb{R}^3 \to \mathbb{R}$, there exists $\alpha \in \mathbb{S}^2$ such that any realization ϕ of the parallelizer with the vertex h fixed at α satisfies:

$$f(\phi(a)) = f(\phi(b)).$$

To construct the parallelizer, we use the following characterization: $\delta(\psi(h), \psi(b)) = \delta(\psi(h), \psi(c))$ if and only if there exists a linear plane $\mathcal{P}$ containing $\psi(h)$ such that $\psi(b)$ is the reflection of $\psi(c)$ with respect to $\mathcal{P}$.

Start with a symmetrizer $\mathcal{L}_1$ and consider the following linkage $\mathcal{L}_2$:



$P_2 = \{a_2, h_2\}, Q_2 = \emptyset, F_2 = \emptyset$.

Then let $W_1 = \{a_1\}$, $\beta(a_1) = a_2$, and $\mathcal{L} = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. Change the input set of $\mathcal{L}$ so that $P = \{b_1, c_1, h_2\}$, and rename the inputs: $b = b_1$, $c = c_1$, $h = h_2$, which ends the construction.

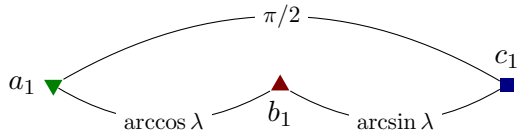
5.2 Elementary linkages for algebraic operations

5.2.1 The homothety linkage

Let $\lambda \in (0, 1)$. Our aim is to construct a linkage which takes one point $\phi(a) = (x_{\phi(a)}, y_{\phi(a)}, 0)$ and, when possible, forces another point $\phi(b) = (x_{\phi(b)}, y_{\phi(b)}, 0)$ to satisfy

$$y_{\phi(b)} = \lambda y_{\phi(a)}.$$

Start with the following linkage $\mathcal{L}_1$:



$P_1 = \{a_1\}$, $Q_1 = \{b_1\}$, $F_1 = \{c_1\}$, $\phi_{01}(c_1) = (0, 0, 1)$.

This linkage is functional for a homothety from the equator to the (smaller) circle of latitude $\arccos \lambda$: thus for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_1)$, $y_{\phi(b_1)} = \lambda y_{\phi(a_1)}$, and

$$\mathrm{Conf}^{P_1}_{\mathbb{S}^2}(\mathcal{L}_1) = \mathbb{S}^2 \cap (Oxy)$$

However, $z_{\phi(b_1)} \neq 0$ so we need to improve the construction. Let $\mathcal{L}_2$ be a parallelizer for the linear form $f(x, y, z) = y$ and $\mathcal{L}_3$ a parallelizer for $g(x, y, z) = z$. Let $W_1 = \{b_1\}$, $\beta_1(b_1) = b_2$, and $\mathcal{L}_4 = \mathcal{L}_1 \cup_{\beta_1} \mathcal{L}_2$. Let $W_4 = \{a_1, c_2\}$, $\beta_4(a_1) = b_3$, $\beta_4(c_2) = c_3$, and $\mathcal{L}_5 = \mathcal{L}_4 \cup_{\beta_4} \mathcal{L}_3$.

We get:

$$\mathrm{Conf}^{P_1}_{\mathbb{S}^2}(\mathcal{L}_1) = \mathbb{S}^2 \cap (Oxy),$$

and for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$, $y_{\phi(c_3)} = \lambda y_{\phi(a_1)}$, $z_{\phi(c_3)} = 0$.

Finally, rename the two vertices: $a = a_1$ and $b = c_3$.

5.2.2 The adder

Our aim is to construct a linkage which takes two points $\phi(a) = (x_{\phi(a)}, y_{\phi(a)}, 0)$, $\phi(b) = (x_{\phi(b)}, y_{\phi(b)}, 0)$ and, when possible, forces a third point $\phi(c) = (x_{\phi(c)}, y_{\phi(c)}, 0)$ to satisfy

$$y_{\phi(c)} = y_{\phi(a)} + y_{\phi(b)}.$$

There are several steps to construct such a linkage $\mathcal{L}$.

1. Restrict the two points a, b to move in the Oxy plane by using the fixed great circle linkage for $k = 2$.

2. Using a symmetrizer, extend this linkage to a new one having a vertex d such that d is symmetric to b with respect to the plane $\{y - z = 0\}$. Then for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$ we have $y_{\phi(b)} = z_{\phi(d)}$.

3. With two parallelizers, extend this linkage to a new one having a vertex e such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$:

   (a) $y_{\phi(e)} = y_{\phi(a)}$;
   (b) $z_{\phi(e)} = z_{\phi(d)}$.

4. With two parallelizers, extend this linkage to a new one with a vertex c such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$:

   (a) $y_{\phi(c)} + z_{\phi(c)} = y_{\phi(e)} + z_{\phi(e)}$;
   (b) $z_{\phi(c)} = 0$.

Then for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$ we have $y_{\phi(c)} = y_{\phi(a)} + y_{\phi(b)}$, as desired. Let $P = \{a, b\}$ and $Q = \{c\}$.

We have

$$\mathrm{Conf}^{P}_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2 \cap (Oxy))^P \;\middle|\; y_{\phi(a)} + y_{\phi(b)} \in [-1, 1]\right\}.$$

5.2.3 The multiplier

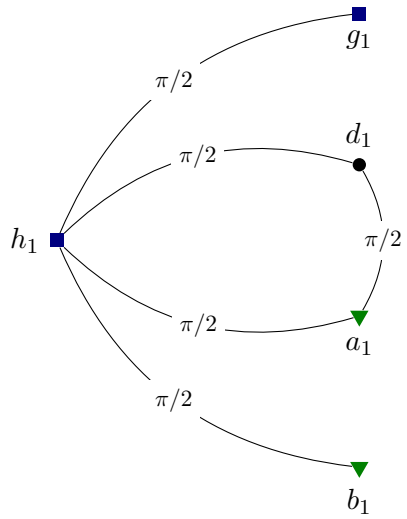
Identify the plane (Oxy) with the complex plane: to a point $(x, y, 0) \in \mathbb{R}^3$, associate the complex number $\zeta_{(x,y,0)} = x + iy$. We want to construct a functional linkage which takes two complex numbers and returns their product.

Since we work in the sphere, we only need to multiply complex numbers α_1 and α_2 in the unit circle. This corresponds to adding the arguments. We split this operation into two steps:

1. Compute $\frac{\arg(\alpha_1)+\arg(\alpha_2)}{2} \bmod \pi$;

2. Double the argument.

The following linkage $\mathcal{L}_1$ will be the basis of the construction:



$P_1 = \{a_1, b_1\}, Q = \emptyset$, $F_1 = \{g_1, h_1\}$, $\phi_{01}(g_1) = (1, 0, 0)$, $\phi_{01}(h_1) = (0, 0, 1)$.

We have $\mathrm{Conf}_{\mathbb{S}^2}^{P_1}(\mathcal{L}_1) = (\mathbb{S}^2)^{P_1}$.

Take two copies $\mathcal{L}_2$ and $\mathcal{L}_3$ of the symmetrizer. Let $W_1 = \{a_1, b_1\}$, $\beta_1(a_1) = a_2$, $\beta_1(b_1) = b_2$, and $\mathcal{L}_5 = \mathcal{L}_1 \cup_{\beta_1} \mathcal{L}_2$. Then let $W_5 = \{a_2, g_1\}$, $\beta_5(a_2) = a_3$, $\beta_5(g_1) = b_3$, $\mathcal{L}_4 = \mathcal{L}_5 \cup_{\beta_5} \mathcal{L}_3$. We write $a_4 := a_3$, $b_4 := b_2$, $c_4 := c_2$, $d_4 := d_1$, $f_4 := c_3$, $g_4 := b_3$.

Now for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L}_4)$, $\phi(c_4)$ is symmetric to $\phi(b_4)$ with respect to $\phi(a_4)^\perp$, and $\phi(f_4)$ is symmetric to $\phi(g_4)$ with respect to $\phi(a_4)^\perp$. In other words,

$$\arg \zeta_{\phi(d_4)} = \frac{\arg(\zeta_{\phi(b_4)}) + \arg(\zeta_{\phi(c_4)})}{2} \bmod \pi$$

and

$$\arg \zeta_{\phi(f_4)} = 2 \arg \zeta_{\phi(d_4)} \bmod 2\pi.$$

Let $\mathcal{L}_6$ be the linkage $\mathcal{L}_4$ with the input set $P_6 = \{b_4, c_4\}$. We have $\mathrm{Conf}_{\mathbb{S}^2}^{P_6}(\mathcal{L}_6) = (\mathbb{S}^2)^{P_6}$.

Taking $Q_6 = \{f_4\}$, $\mathcal{L}_6$ becomes a functional linkage for multiplication.

5.2.4 The polynomial linkage

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial. Our aim is now to construct a linkage with n inputs $a_1, \ldots, a_n$, such that:

$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2 \cap (Oxy))^P \;\middle|\; f(y_{\psi(a_1)}, \ldots, y_{\psi(a_n)}) = 0\right\}.$$

Let us assume first that $m = 1$.

Recall that we write $\zeta_{\psi(a_k)} = x_{\psi(a_k)} + iy_{\psi(a_k)}$. We can also write:

$$y_{\psi(a_k)} = \frac{\zeta_{\psi(a_k)} - \overline{\zeta_{\psi(a_k)}}}{2i}$$

.

Thus, there exists a polynomial $g : \mathbb{C}^{2n} \to \mathbb{C}$ such that for all $\psi \in (\mathbb{S}^2 \cap (Oxy))^P$:

$$g(\zeta_{\psi(a_1)}, \overline{\zeta_{\psi(a_1)}}, \ldots, \zeta_{\psi(a_n)}, \overline{\zeta_{\psi(a_n)}}) = f(y_{\psi(a_1)}, \ldots, y_{\psi(a_n)}).$$

We write

$$g = \sum_{j=1}^{r} g_j$$

where each g_j is a monomial:

$$g_j(\zeta_{\psi(a_1)}, \overline{\zeta_{\psi(a_1)}}, \ldots, \zeta_{\psi(a_n)}, \overline{\zeta_{\psi(a_n)}}) = \lambda_j \epsilon_j (\zeta_{\psi(a_1)})^{\gamma_{j,1}} (\overline{\zeta_{\psi(a_1)}})^{\gamma_{j,2}} \cdots (\zeta_{\psi(a_n)})^{\gamma_{j,2n-1}} (\overline{\zeta_{\psi(a_n)}})^{\gamma_{j,2n}}$$

with $\epsilon_j \in \{1, i, -1, -i\}$ and λ_j a positive real number.

Observe that without changing the locus

$$\left\{\psi \in (\mathbb{S}^2 \cap (Oxy))^P \;\middle|\; f(y_{\psi(a_1)}, \ldots, y_{\psi(a_n)}) = 0\right\},$$

one may assume $\lambda_j < \lambda_0$ for all j, where λ_0 is arbitrary in $(0, 1)$ (if necessary, multiply f by a small constant).

We are now ready to construct the linkage. Start with a fixed great circle linkage $\mathcal{L}$ which forces all the a_k to move in the plane (Oxy). Use symmetrizers to extend $\mathcal{L}$ to a new linkage with vertices a'_k such that for all $k \in \{1, \ldots, n\}$,

$$\zeta_{\phi(a'_k)} = \overline{\zeta_{\phi(a_k)}}.$$

For each $j \in \{1, \ldots, r\}$:

1. Use multipliers to extend $\mathcal{L}$ to a new linkage with a vertex c_j such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$,

$$\zeta_{\phi(c_j)} = (\zeta_{\psi(a_1)})^{\gamma_{j,1}} (\overline{\zeta_{\psi(a_1)}})^{\gamma_{j,2}} \cdots (\zeta_{\psi(a_n)})^{\gamma_{j,2n-1}} (\overline{\zeta_{\psi(a_n)}})^{\gamma_{j,2n}}.$$

2. Use a multiplier to extend the linkage to a new one with a vertex d_j such that:

$$\zeta_{\phi(d_j)} = i\epsilon_j \zeta_{\phi(c_j)}.$$

3. Use a homothety linkage to extend the linkage to a new one with a vertex b_j such that:

$$y_{\phi(b_j)} = \lambda_j y_{\phi(d_j)}.$$

Thus we have for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$:

$$y_{\phi(b_j)} = \Im(ig_j(\zeta_{\psi(a_1)}, \overline{\zeta_{\psi(a_1)}}, \ldots, \zeta_{\psi(a_n)}, \overline{\zeta_{\psi(a_n)}})).$$

Then, use several adders to extend the linkage to a new one, still called $\mathcal{L}$, with a vertex c such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$:

$$y_{\phi(c)} = \sum_{j=1}^{r} y_{\phi(b_j)}.$$

Thus

$$y_{\phi(c)} = \mathrm{Im}(ig(\zeta_{\psi(a_1)}, \overline{\zeta_{\psi(a_1)}}, \ldots, \zeta_{\psi(a_n)}, \overline{\zeta_{\psi(a_n)}})),$$

which means that

$$y_{\phi(c)} = f(y_{\psi(a_1)}, \ldots, y_{\psi(a_n)}).$$

Choose λ_0 so small that all the steps of the computation remain in $[-1, 1]$. Then:

$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = (\mathbb{S}^2 \cap (Oxy))^P.$$

Finally, if $m \geq 2$, just write $f = (f_1, \ldots, f_m)$ and use m linkages like above.

5.3 End of the proof of Theorem 2.9 for $d = 2$

In this section, we prove the algebraic universality in $\mathbb{S}^2$.

First, we assume that A is an *algebraic* subset of $(\mathbb{S}^2)^n$. Let $f : (\mathbb{R}^3)^n = \mathbb{R}^{3n} \to \mathbb{R}^m$ be a polynomial function such that $A = f^{-1}(0)$.

1. Take a polynomial linkage $\mathcal{L}$ with inputs $a_1, \ldots, a_{3n}$ such that:
$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = \left\{\psi \in (\mathbb{S}^2 \cap (Oxy))^P \;\middle|\; f(y_{\psi(a_1)}, \ldots, y_{\psi(a_{3n})}) = 0\right\}.$$
2. With several symmetrizers, extend this linkage to a new one with vertices b_1, b_4, b_7, …, b_{3n-2} such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$ and for all $k \in \{1, \ldots, n\}$:
$$x_{\phi(b_{3k-2})} = y_{\phi(a_{3k-2})}.$$
3. With several symmetrizers, extend this linkage to a new one with vertices c_3, c_6, c_9, …, c_{3n} such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$ and for all $k \in \{1, \ldots, n\}$:
$$z_{\phi(c_{3k})} = y_{\phi(a_{3k})}.$$
4. With several parallelizers, extend this linkage to a new one with vertices d_3, d_6, d_9, …, d_{3n} such that for all $\phi \in \mathrm{Conf}_{\mathbb{S}^2}(\mathcal{L})$ and for all $k \in \{1, \ldots, n\}$:
$$x_{\phi(d_{3k})} = x_{\phi(b_{3k-2})};$$
$$y_{\phi(d_{3k})} = y_{\phi(a_{3k-1})};$$
$$z_{\phi(d_{3k})} = z_{\phi(c_{3k})}.$$

Now, let $P = \{d_3, d_6, \ldots, d_{3n}\}$. We have:

$$\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = f^{-1}(0) = A.$$

If A is only a compact semi-algebraic subset of $(\mathbb{S}^2)^n$, we know from Proposition 2.15 that A is the projection onto the first coordinates of an algebraic subset B of the sphere: apply the above construction to B and remove some vertices from the input set to obtain $\mathrm{Conf}^P_{\mathbb{S}^2}(\mathcal{L}) = A$; thus, Theorem 2.9 is proved.

5.4 Higher dimensions

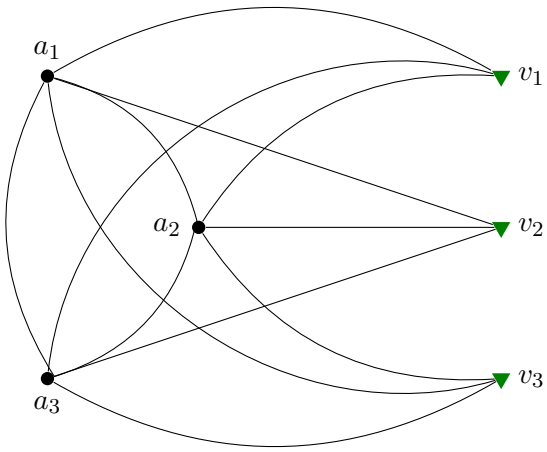
In this section, we fix a number $d \geq 2$ and consider realizations in the sphere $\mathbb{S}^d$.

5.4.1 The 3-plane linkage

This linkage forces several points to move in the same (linear) 3-plane. It is to be compared with the "great circle linkage" described in section 5.1.2, which forces several points to move in the same (linear) 2-plane.

There are k inputs $v_1, \ldots, v_k$, and $d-2$ other vertices $a_1, \ldots, a_{d-2}$. For all $i, j \in \{1, \ldots, d-2\}$, there is an edge $a_i a_j$ of length $\pi/2$. For all $i \in \{1, \ldots, d-2\}$ and $l \in \{1, \ldots, k\}$, there is an edge $a_i v_l$ of length $\pi/2$.

Here is an example with $d = 5$ and $k = 3$.



Proposition 5.4. *We have* $\mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L}) = E$, *where*

$$E = \left\{ \psi \in (\mathbb{S}^d)^P \;\middle|\; \exists \mathcal{F} \text{ subspace of } \mathbb{R}^{d+1}, \dim \mathcal{F} = 3, \forall i \in \{1, \ldots, k\} \;\; \psi(v_k) \in \mathcal{F} \right\}.$$

Proof. First, we prove that $\mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L}) \subseteq E$. Let $\psi \in \mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L})$ and $\phi \in p^{-1}(\psi)$. Let

$$\mathcal{F} = \bigcap_{1 \leq i \leq d-2} \phi(a_i)^\perp.$$

We know that $\{\phi(a_1), \ldots, \phi(a_{d-2})\}$ is an orthonormal set, so $\dim \mathcal{F} = 3$. Moreover, for all $l \in \{1, \ldots, k\}$, $\psi(v_l) \in \mathcal{F}$.

Now, we prove that $E \subseteq \mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L})$. Let $\psi \in E$. Let $\mathcal{F}$ be a subspace of $\mathbb{R}^{d+1}$ with $\dim \mathcal{F} = 3$ containing $\psi(v_l)$ for $l \in \{1, \ldots, k\}$. Construct $\phi \in (\mathbb{S}^d)^P$ by letting $\{\phi(a_1), \ldots, \phi(a_{d-2})\} \subseteq \mathcal{F}^\perp$ be an orthonormal set and let $\phi|_P = \psi$. Then $\phi \in \mathrm{Conf}_{\mathbb{S}^d}(\mathcal{L})$ so $\psi \in \mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L})$. $\square$

The *fixed 3-plane linkage* is a variant in which $a_1, \ldots, a_{d-2} \in F$ (namely, they are fixed vertices). Then there exists $\mathcal{F}$ a subspace of $\mathbb{R}^{d+1}$ with $\dim \mathcal{F} = 3$ and

$$\mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L}) = \left\{ \psi \in (\mathbb{S}^2)^P \;\middle|\; \forall i \in \{1, \ldots, k\} \;\; \psi(v_k) \in \mathcal{F} \right\}.$$

5.4.2 The d-dimensional symmetrizer

Like in the 2-dimensional case, the d-dimensional symmetrizer has two inputs a and b, and one output c. It is a functional linkage for symmetry: for all $\phi \in \mathrm{Conf}_{\mathbb{S}^d}(\mathcal{L})$, $\phi(c)$ is symmetric to $\phi(b)$ with respect to $\phi(a)^\perp$. The idea is that the symmetry takes place in a 3-plane containing $\phi(a)$, $\phi(b)$ and $\phi(c)$.

Let $\mathcal{L}_1$ be a classical symmetrizer and $\mathcal{L}_2$ be a 3-plane linkage, with $k = \mathrm{card}(V_1)$. Let $W_1 = V_1$, β a bijection between V_1 and $\{v_1, \ldots, v_k\}$ ($\subseteq V_2$), and $\mathcal{L} = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. Letting $a := \beta(a_1)$, $b := \beta(b_1)$, $c := \beta(c_1)$, we obtain as desired:

$$\mathrm{Conf}_{\mathbb{S}^d}(\mathcal{L}) = \left\{ \phi \in (\mathbb{S}^d)^V \;\middle|\; \phi(c) \text{ is symmetric to } \phi(b) \text{ with respect to } \phi(a)^\perp \right\}.$$

5.4.3 The d-dimensional parallelizer

Like in the 2-dimensional case, the d-dimensional parallelizer forces two points to have the same scalar product with a third one. We restrict the vertices of a classical parallelizer to move on a 3-plane containing its three inputs.

Let $\mathcal{L}_1$ be a classical parallelizer and $\mathcal{L}_2$ be a 3-plane linkage, with $k = \mathrm{card}(V_1)$. Let $W_1 = V_1$, β a bijection between V_1 and $\{v_1, \ldots, v_k\}$ ($\subseteq V_2$), and $\mathcal{L} = \mathcal{L}_1 \cup_\beta \mathcal{L}_2$. Let $h := \beta(h_1)$, $b := \beta(b_1)$, $c := \beta(c_1)$. Then we obtain as desired:

$$\mathrm{Conf}^P_{\mathbb{S}^d}(\mathcal{L}) = \left\{ \psi \in (\mathbb{S}^d)^P \;\middle|\; \delta(\psi(h), \psi(b)) = \delta(\psi(h), \psi(c)) \right\}.$$

5.4.4 End of the proof of Theorem 2.9 for $d \geq 2$

Here, we prove the algebraic universality in $\mathbb{S}^d$. The proof is similar to the case $d = 2$. There are only two differences.

1. The polynomial linkage $\mathcal{L}$ is attached to a fixed 3-plane linkage.

2. We use d-dimensional symmetrizers and d-dimensional parallelizers.

Part II

Anosov geodesic flows, billiards and linkages

Chapter 6

Anosov geodesic flows and dispersing billiards

6.1 Introduction

The aim of this chapter is to highlight the similarities between two uniformly hyperbolic dynamical systems: geodesic flows on negatively curved manifolds and billiard flows on negatively curved billiards.

A significant difference between these two dynamical systems is that geodesic flows are smooth, while billiard flows are not. Although the behaviors of the two systems are close, the theorems often require different proofs in each case: for example, there is no known way of deducing the ergodicity of negatively curved billiards from the ergodicity of geodesic flows on negatively curved surfaces.

A smooth billiard table D in $B = \mathbb{T}^2$ or $B = \mathbb{R}^2$ is the closure of an open set in B such that ∂D is a smooth manifold of dimension 1 without boundary: in other words, each component of ∂D is the image of a smooth embedding $\Gamma : \mathbb{T}^1 \to B$. Each curve Γ is called a *wall* of D: it has a unit tangent vector T and a unit normal vector N pointing toward Int D. The curvature of Γ is $\left\langle \frac{dT}{ds} \;\middle|\; N \right\rangle$, where s is the arc length parameter. For example, the walls of a disc are positively curved, while the walls of its complementary set are negatively curved. A billiard whose walls have negative curvature is said to be *dispersing*.

One defines the phase space $\Omega = T^1(\text{Int } D)$, and the billiard flow $\phi_t : \Omega \to \Omega$, in the following way:

1. As long as it does not hit a wall, the particle follows a straight line;
2. When it arrives to the boundary of the billiard, the particle bounces, following the billiard reflection law: the angle between the particle's speed vector and the boundary's tangent line is preserved (Figure 6.1).

The flow ϕ_t is not defined at all times :

1. It is not defined at times when the particle is on the boundary of the billiard. Of course, one could extend the definition to such t, but the flow obtained in this way would not be continuous[1].

[1] Many authors change the topology of Ω in order to make the flow continuous, but it cannot be made differentiable.

2. When the particle makes a grazing collision with a wall at a time $t_0 > 0$, *i.e.* collides with the boundary with an angle $\theta = 0$, the flow stops being defined for all times $t \geq t_0$. Although one could extend continuously the definition of the trajectory after such a collision, the differentiability of the flow would be lost.

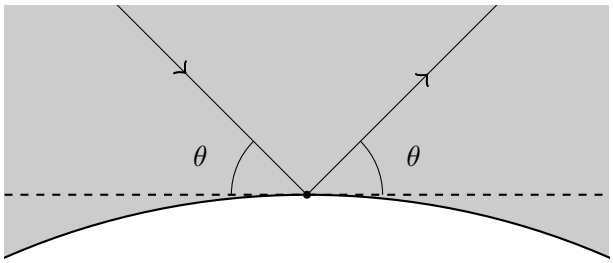


Figure 6.1 – The billiard reflection law.

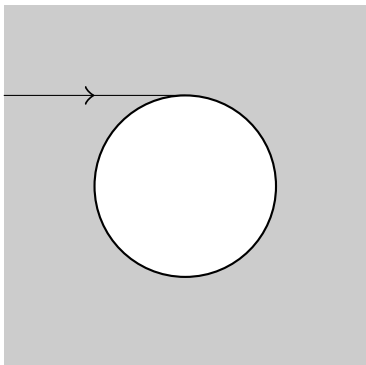


Figure 6.2 – A grazing collision on a dispersing billiard in $\mathbb{T}^2$. The flow stops being defined after this time.

We define $\tilde{\Omega}$ as the set of all $(x, v) \in \Omega$ such that the trajectory starting from (x, v) does not contain any grazing collision, in the past or the future. Notice that $\tilde{\Omega}$ is a residual set of full measure, stable under the flow ϕ_t, and that ϕ_t is C^∞ on $\tilde{\Omega}$.

We will say that a billiard has *finite horizon* if every trajectory hits the boundary at least once.

Uniform hyperbolicity. We are now ready to define the two notions of uniform hyperbolicity which we will use.

Definition 6.1. A flow $\phi : \mathbb{R} \times M \to M$ on a *closed* manifold is *Anosov* (or *uniformly hyperbolic*) if there exists a decomposition of TM, stable under the flow,

$$T_x M = E_x^0 \oplus E_x^u \oplus E_x^s$$

where $E_x^0 = \mathbb{R} \left.\frac{d}{dt}\right|_{t=0} \phi_t(x)$, such that

$$\|D\phi^t|_{E_x^s}\| \leq a\lambda^t, \ \|D\phi_x^{-t}|_{E_x^u}\| \leq a\lambda^t$$

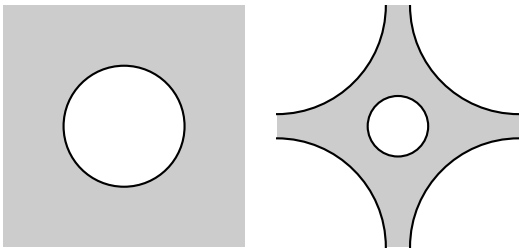


Figure 6.3 – On the left, a dispersing billiard in $\mathbb{T}^2$ with infinite horizon. On the right, a dispersing billiard in $\mathbb{T}^2$ with finite horizon.

(for some $a > 0$ and $\lambda \in (0,1)$, which do not depend on x).

This definition does not depend on the choice of the Riemannian metric on M.

Since the billiard flow is only defined on a non-compact set $\tilde{\Omega}$ (dense in Ω), we need another definition for uniform hyperbolicity in the case of billiards. This definition is given in a more abstract framework in [CM06], but here we adapt it directly to billiard flows.

Definition 6.2. The billiard flow ϕ_t is *uniformly hyperbolic* if at each point $x \in \tilde{\Omega}$, there exists a decomposition of $T_x\Omega$, stable under the flow,

$$T_x\Omega = E_x^0 \oplus E_x^u \oplus E_x^s$$

where $E_x^0 = \mathbb{R}\, \frac{d}{dt}\big|_{t=0} \phi_t(x)$, such that

$$\|D\phi_x^t|_{E_x^s}\| \leq a\lambda^t, \ \|D\phi_x^{-t}|_{E_x^u}\| \leq a\lambda^t$$

(for some $a > 0$ and $\lambda \in (0,1)$, which do not depend on x).

Structure of the chapter. In this chapter, we explain how to show the uniform hyperbolicity for surfaces of negative curvature and billiards with negatively curved walls, using the *cone criterion* introduced by Alekseev [Ale69], and its refinement by Wojtkowski [Woj85]. The two proofs use exactly the same ideas. The fundamental tool is the study of *Jacobi fields*.

All surfaces with negative curvature have an Anosov geodesic flow: according to Arnold and Avez [AA67], the first proof of this fact goes back to 1898 [Had98]. Later, it was extended to all manifolds with negative sectional curvature (a modern proof is available in [KH95]). But the negative curvature assumption is not necessary for a geodesic flow to be Anosov. To see if a geodesic flow is Anosov, we will need to examine the solutions of the Ricatti equation

$$u'(t) = -K(t) - u^2(t)$$

where K is the Gaussian curvature of the surface, and to use the following criterion:

Theorem 6.3. *Let M be a closed surface. Assume that there exists $m > 0$ such that for all geodesic $\gamma : [0,1] \to M$, and all u solution of the Ricatti equation along this geodesic such that $u(0) = 0$, u is well-defined on $[0,1]$ and $u(1) \geq m$. Then the geodesic flow $\phi_t : T^1M \to T^1M$ is Anosov.*

Theorem 6.3 was mentioned in [DP03] and [MP13], but as far as we know, no detailed proof was available.

In this chapter, we prove Theorem 6.3, and then explain how this result applies in the particular case of closed Riemannian surfaces with negative curvature. We will even show the following refinement:

Theorem 6.4. *Let M be a closed Riemannian surface with nonpositive curvature. Assume that every geodesic in M contains a point where the curvature is negative. Then, the geodesic flow on M is Anosov.*

Theorem 6.4 may also be obtained directly, without using Theorem 6.3 or the cone criterion, from Proposition 3.10 of [Ebe73]. Hunt and MacKay [HM03] used this result to exhibit the first Anosov linkage (more details will be given in Chapter 8).

For billiards, we will prove the following counterpart of Theorem 6.4:

Theorem 6.5. *If D is a smooth dispersing billiard with finite horizon, then the billiard flow is uniformly hyperbolic in $\tilde{\Omega}$.*

In Chapter 7, we will apply Theorem 6.3 to give new examples of surfaces whose geodesic flow is Anosov while their curvature is not negative everywhere.

Consequences of uniform hyperbolicity. It is shown in [PS72] that (smooth) volume-preserving Anosov flows are ergodic: every invariant subset has either zero or full measure. It was shown later (see [Dol98] and [Kli74]) that Anosov geodesic flows are even exponentially mixing.

As for billiard flows, Sinaï proved ergodicity for smooth dispersing billiards with finite horizon in [Sin70]. It was shown in [BDL15] that such flows are exponentially mixing.

Corners and cusps. In the definition of billiards, it is common to allow the walls to be only piecewise smooth. If the tangent lines at the singularity make a nonzero angle, it is called a corner, otherwise it is called a cusp (see Figure 6.4). Corners and cusps give the possibility for compact billiards in $\mathbb{R}^2$ to be dispersing.

Dispersing billiards with corners (and with finite horizon) have uniformly hyperbolic, mixing flows.

When cusps are introduced, the study of the flow becomes more complicated: the first proof of ergodicity was published no sooner than 1995 (see [Řeh95]). Such billiards still have mixing flows, and the mixing rate is greater than polynomial (see [BM08]), but it is unknown whether it is exponential[2].

6.2 The cone criterion

Definition 6.6. Consider a Euclidean space E.

A *cone*[3] in E is a set C such that there exist a decomposition $E = F \oplus G$ and a real number $\alpha \geq 0$ such that

[2] It is not to be confused with the mixing rate of the billiard *map*, which we did not define here, and which is polynomial (see [CM07]) in this case.

[3] The word "cone" has several different meanings in mathematics: here we take the same definition as [KH95].

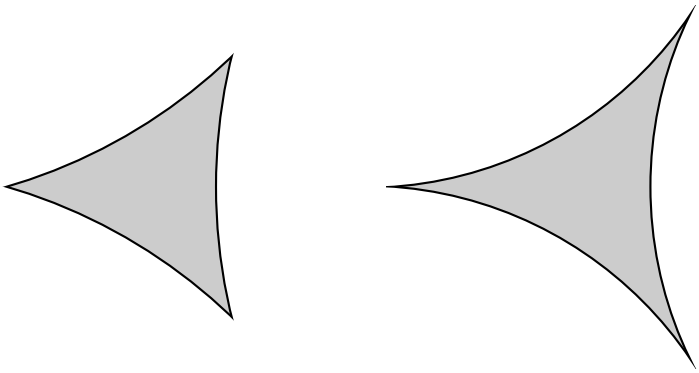


Figure 6.4 – On the left, a dispersing billiard with corners in $\mathbb{R}^2$. On the right, a dispersing billiard with cusps.

$$C = \{(x, y) \in F \oplus G \mid \|x\| \leq \alpha \|y\|\}.$$

The number $\arctan \alpha$ is called the *angle* of the cone.

Two cones C_1, C_2 are said to be *supplementary* if they correspond to decompositions $E = F_1 \oplus G_1$ and $E = F_2 \oplus G_2$ such that $F_1 = G_2$ and $F_2 = G_1$.

Proposition 6.7. *Consider a sequence of linear mappings $A_k : \mathbb{R}^n \to \mathbb{R}^n$, and a sequence of supplementary cones C_k and D_k, corresponding to the decomposition $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. Assume that there exist $a > 0$, $\lambda > 1$ such that for all $k \in \mathbb{Z}$:*

1. *$A_k(C_k) \subseteq C_{k+1}$ (invariance in the future),*
2. *$\|A_{k-1} \circ \ldots \circ A_{k-i}(v)\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in C_{k-i}$ (expansion in the future),*
3. *$A_k^{-1}(D_{k+1}) \subseteq D_k$ (invariance in the past),*
4. *$\left\|A_k^{-1} \circ \ldots \circ A_{k+i-1}^{-1}(v)\right\| \geq a\lambda^i \|v\|$ for all $i \geq 0$ and $v \in D_{k+i}$ (expansion in the past).*

Then

$$E_k^u = \bigcap_{i=0}^{+\infty} A_{k-1} \circ \ldots \circ A_{k-i}(C_{k-i})$$

is an m-dimensional subspace contained in C_k, and

$$E_k^s = \bigcap_{i=0}^{+\infty} A_k^{-1} \circ \ldots \circ A_{k+i-1}^{-1}(D_{k+i})$$

is an $(n-m)$-dimensional subspace contained in D_k.

Proof. For all $i \geq 0$, $A_{k-1} \circ \ldots \circ A_{k-i}(C_{k-i})$ is a cone, which contains a vector space V_i of dimension m. Thus, the intersection E_k^u contains a vector space V of dimension m (for example, consider a converging subsequence of orthonormal bases of V_i). Assume that

there exists $w \in E_k^u \backslash V$. Then there exists $v \in V$ and $t \in \mathbb{R}$ such that $v + tw \in \{0\} \times \mathbb{R}^{n-m}$ (notice also that $tw \in E_k^u$). Since $A_{k-i}^{-1} \circ \ldots \circ A_{k-1}^{-1}(tw)$ and $A_{k-i}^{-1} \circ \ldots \circ A_{k-1}^{-1}(v)$ lie in E_{k-i}^u, Assumption 2 gives us:

$$\left\| A_{k-i}^{-1} \circ \ldots \circ A_{k-1}^{-1}(tw) \right\| \leq \frac{1}{a\lambda^i} \|tw\| \underset{k \to +\infty}{\to} 0,$$

$$\left\| A_{k-i}^{-1} \circ \ldots \circ A_{k-1}^{-1}(v) \right\| \leq \frac{1}{a\lambda^i} \|v\| \underset{k \to +\infty}{\to} 0,$$

but at the same time, since $v + tw \in D_k$, Assumption 4 gives:

$$\left\| A_{k-i}^{-1} \circ \ldots \circ A_{k-1}^{-1}(v + tw) \right\| \geq a\lambda^i \|v + tw\| \underset{k \to +\infty}{\to} +\infty,$$

which contradicts the triangle inequality.

One obtains the result for E_k^s in the same way. □

Theorem 6.8. *Let* $A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ *(with* $k \in \mathbb{Z}$*) be a sequence of* 2×2 *matrices, with determinant* ± 1*. Fix* $\epsilon > 0$*, and consider the cone* C_ϵ *of all vectors* $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ *such that* $\epsilon y \leq x \leq \frac{1}{\epsilon} y$*. Assume that for all* k*, and all* $v = \begin{pmatrix} x \\ y \end{pmatrix}$ *with* $xy > 0$*,*

$$A_k v \in C_\epsilon.$$

Then, there exist $a > 0$ *and* $\lambda > 1$ *such that for all* $k \in \mathbb{Z}$*, for all* $i \geq 0$ *and* $v \in C_\epsilon$*,*

$$\|A_{k-1} \circ \ldots \circ A_{k-i}(v)\| \geq a\lambda^i \|v\|.$$

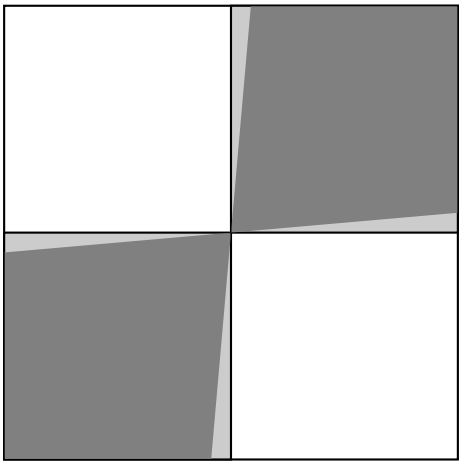


Figure 6.5 – Each A_k maps the cone $xy > 0$ (in grey) into the smaller cone C_ϵ (in dark grey).

Proof. On the basis of Wojtkowski's idea [Woj85], instead of proving expansion directly for the Euclidean norm, we consider the function

$$\begin{aligned} N : C_\epsilon &\to \mathbb{R}_{\geq 0} \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \sqrt{xy}. \end{aligned}$$

Notice that N is equivalent to the Euclidean norm on C_ϵ, *i.e.* there exists $M > 0$ such that for all $v \in C_\epsilon$,

$$\frac{1}{M} \|v\| \leq N(v) \leq M \|v\|,$$

because $\frac{\epsilon}{2}(x^2 + y^2) \leq xy \leq \frac{2}{\epsilon}(x^2 + y^2)$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon$.

We are going to show that for all $k \in \mathbb{Z}$ and $v \in C_\epsilon$, $N(A_k v) \geq \frac{1}{1-\epsilon^2} N(v)$. With the equivalence of norms, this will complete the proof.

Let $k \in \mathbb{Z}$. We may assume that $\det(A_k) = 1$, by multiplying A_k by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the left. Moreover, we may assume that all the coefficients of A_k are positive, by multiplying A_k by $-\mathrm{Id}$.

Notice that the two vectors $A_k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_k \\ c_k \end{pmatrix}$ and $A_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b_k \\ d_k \end{pmatrix}$ are in the cone C_ϵ, by continuity of A_k.

Then for $v = \begin{pmatrix} x \\ y \end{pmatrix} \in C_\epsilon$:

$$\begin{aligned} N(A_k v) &= (a_k x + b_k y)(c_k x + d_k y) \\ &\geq (a_k d_k - b_k c_k) xy + 2 b_k c_k xy \\ &\geq (1 + 2 b_k c_k) N(v) \end{aligned}$$

But $a_k d_k - b_k c_k = 1$ and $a_k \leq \frac{1}{\epsilon} b_k, d_k \leq \frac{1}{\epsilon} c_k$, so that $b_k c_k \geq \frac{1}{1-\epsilon^2} - 1$.

Finally, $N(A_k v) \geq \frac{1}{1-\epsilon^2} N(v)$. □

6.3 Anosov geodesic flows

6.3.1 Jacobi fields

To show that a geodesic flow is hyperbolic, one has to study how the geodesics move away from (or closer to) each other. Thus, one considers small variations of a given geodesic.

Definition 6.9. Let (M, g) be a Riemannian manifold and $\gamma : (a, b) \to M$ a geodesic. Consider a geodesic variation of γ, *i.e.* a smooth function

$$f(t, s) : (a, b) \times (c, d) \to M$$

such that $f(., 0)$ is the geodesic γ, and for all $s \in (c, d)$, $f(., s)$ is a geodesic.

The vector field $Y = \frac{\partial f}{\partial s}$ along the curve $\gamma(t)$ is called an *infinitesimal variation of* γ.

Proposition 6.10. *Any infinitesimal variation of γ is a solution of the Jacobi equation:*

$$\ddot{Y} = -R(\dot{\gamma}, Y)\dot{\gamma},$$

where R is the Riemann tensor. The solutions of the Jacobi equation are called Jacobi fields.

Proof. Let ∇ be the Levi-Civita connection of (M, g). Since $\left[\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right] = 0$, one obtains

$$\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} = \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}$$

so that (with $s = 0$):

$$\ddot{Y} = \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s} = \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t}.$$

On the other hand (still for $s = 0$),

$$\begin{aligned} R(\dot{\gamma}, Y)\dot{\gamma} &= R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial f}{\partial t} \\ &= \nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} - \nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} - \nabla_{[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}]} \frac{\partial f}{\partial t}. \\ &= -\nabla_{\frac{\partial f}{\partial t}} \nabla_{\frac{\partial f}{\partial s}} \frac{\partial f}{\partial t} \end{aligned}$$

Thus, $\ddot{Y} = -R(\dot{\gamma}, Y)\dot{\gamma}$. □

Proposition 6.11. *Every Jacobi field along a geodesic γ is an infinitesimal variation of γ.*

Proof. Here, we copy the proof of [KN63].

Let $t_1 \in (a, b)$. For any t_2 close enough to t_1, any solution of the Jacobi equation is determined by its values at t_1 and t_2 (since it is a second-order linear equation).

Let Y be a solution of the Jacobi equation along γ. For $i = 1, 2$, let $h_i(s)$ ($s \in (-\epsilon, \epsilon)$) be a curve such that $(h_i(0), h_i'(0)) = (\gamma(t_i), X(t_i))$. If necessary, take a smaller ϵ, and choose t_2 even closer to t_1, so that there exists for each s a unique geodesic $f(., s)$ ($t \in (a, b)$), through $h_1(s)$ and $h_2(s)$, of minimal length between $h_1(s)$ and $h_2(s)$. Let X be the Jacobi field $\frac{\partial f}{\partial s}$ along γ. Since X and Y are two solutions of the Jacobi equation which coincide at t_1 and t_2, they are equal. Thus, Y is an infinitesimal variation of γ. □

From now on, assume that M has dimension 2, that γ is a unit speed geodesic, and that Y is a Jacobi field which is orthogonal to $\dot{\gamma}$. Choose an orientation of the normal bundle of γ in M (which has dimension 1), *i.e.* a vector $e(t) \in T^1_{\gamma(t)}M$ orthogonal to $\gamma'(t)$, so that $Y(t)$ is identified by one real coordinate, noted $y(t) = g(Y(t), e(t))$.

The quantity $\dot{y}$ satisfies

$$\dot{y} = \frac{\partial f}{\partial t} \cdot g(Y, e) = g(\nabla_{\frac{\partial f}{\partial t}} Y, e) + g(Y, \nabla_{\frac{\partial f}{\partial t}} e) = g(\nabla_{\frac{\partial f}{\partial t}} Y, e).$$

Thus:

$$\dot{y} = g(\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial s}, e) = g(\nabla_{\frac{\partial f}{\partial s}} \dot{\gamma}, e).$$

In other words, $\dot{y}$ measures the infinitesimal variation of the vector $\dot{\gamma}$ with respect to s. Thus, when y and $\dot{y}$ have the same sign, the Jacobi field is diverging: the geodesics go away from each other. When y and $\dot{y}$ have opposite signs, the Jacobi field is converging. We will consider the ratio $u = \frac{\dot{y}}{y}$, when it is well-defined (*i.e.* $y \neq 0$), to measure the convergence rate.

$u > 0$ $\qquad$ $u < 0$

Proposition 6.12. *When it is well-defined, u is a solution of the Ricatti equation:*

$$\dot{u}(t) = -K(\gamma(t)) - u^2(t).$$

where K is the Gaussian curvature.

Proof. In dimension 2, the Riemann tensor may be written

$$\langle R(a,b)c \mid d\rangle = K \cdot (g(a,c)g(d,b) - g(a,d)g(c,b)).$$

Thus, in the case of a unit speed geodesic and of an orthogonal Jacobi field, the vector $R(\dot{\gamma}, Y)\dot{\gamma}$ is orthogonal to $\dot{\gamma}$, and its coordinate is Ky. The Jacobi equation then becomes:

$$\ddot{y} = -Ky.$$

Thus,

$$\dot{u} = \frac{d}{dt}\left(\frac{\dot{y}}{y}\right) = \frac{\ddot{y}y}{y^2} - \frac{(\dot{y})^2}{y^2} = K - u^2.$$

□

The solutions of this equation are not always defined for all times: it may happen that $u(t)$ explodes to $-\infty$ in positive time (or to $+\infty$ in negative time). This corresponds to the phenomenon of convergence of the wavefront: up to order 1, all the geodesics of the infinitesimal variation "gather at one point". In most cases, the Jacobi field becomes divergent just after the convergence point (Figure 6.6).

6.3.2 Proof of Theorem 6.3

Let $(x, v) \in T^1M$. The tangent plane $T_{(x,v)}(T^1M)$ is the direct sum of a vertical and a horizontal subspace $H_{(x,v)} \oplus V_{(x,v)}$, given by the metric g on M. Each of these two spaces is naturally endowed with a norm, respectively g_H and g_V: one equips T^1M with the norm $g_T = g_H + g_V$ (in particular, one decides that H is orthogonal to V).

Denote by $W_{(x,v)} \subseteq T_{(x,v)}(T^1M)$ the plane orthogonal to the direction of the flow ϕ_t, and let $(w, w') \in W_{(x,v)}$. There exists $Y(t)$ a Jacobi field such that $(Y(0), \dot{Y}(0)) = (w, w')$:

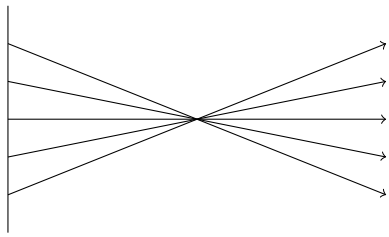


Figure 6.6 – u is not well-defined at the convergence point.

then the vectors $\dot{Y}(0)$ and $\dot{\gamma}(0)$ are orthogonal, and $(Y(t), \dot{Y}(t)) = D\phi_t(w, w')$. Moreover, notice that

$$\begin{aligned}\frac{\partial}{\partial t} g(\dot{Y}, \dot{\gamma}) &= g(\nabla_{\frac{\partial f}{\partial t}} \dot{Y}, \dot{\gamma}) + g(\dot{Y}, \nabla_{\frac{\partial f}{\partial t}} \dot{\gamma}) \\ &= g(\nabla_{\frac{\partial f}{\partial s}} \nabla_{\frac{\partial f}{\partial t}} \dot{\gamma}, \dot{\gamma}) + 0 \\ &= 0.\end{aligned}$$

Thus, $Y(t)$ and $\dot{Y}(t)$ remain orthogonal to $\dot{\gamma}$ for all t. In particular, the family of planes $(W_{(x,v)})$ (where (x, v) varies in T^1M) is stable under $D\phi_t$.

Let $(x, v) \in T^1M$, and γ the geodesic such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$. Choose an orientation of $H_{(\gamma(t),\dot{\gamma}(t))} \cap W_{(\gamma(t),\dot{\gamma}(t))}$, *i.e.* a continuous unit vector $e_1(t)$ in $H_{(\gamma(t),\dot{\gamma}(t))} \cap W_{(\gamma(t),\dot{\gamma}(t))}$. It induces naturally an orientation of $V_{(\gamma(t),\dot{\gamma}(t))}$, given by a continuous unit vector $e_2(t)$ in $V_{(\gamma(t),\dot{\gamma}(t))}$. This orthogonal basis of $W_{(\gamma(t),\dot{\gamma}(t))}$ allows us to identify it to the Euclidean $\mathbb{R}^2$.

For $k \in \mathbb{Z}$, set

$$A_k = D_{(\gamma(k),\dot{\gamma}(k))}\phi_1 : W_{(\gamma(k),\dot{\gamma}(k))} \to W_{(\gamma(k+1),\dot{\gamma}(k+1))}.$$

The A_k are linear mappings with determinant ± 1, because the flow ϕ_t preserves the Liouville measure.

We are going to show that the sequence $A_k : \mathbb{R}^2 \to \mathbb{R}^2$ (with the identification above) satisfies the assumptions of Theorem 6.8: with the notations of this theorem, one wants to show that there exists $\epsilon > 0$ such that for all vector v with positive coordinates in $\mathbb{R}^2$, we have $A_k v \in C_\epsilon$, for some ϵ. In other words, we want to show that there exists $\epsilon > 0$ such that for all geodesics $\gamma : [0, 1] \to M$ and all solutions u of the Ricatti equation $\dot{u}(t) = -K(\gamma(t)) - u^2(t)$ along this geodesic which satisfy $u(0) > 0$, u is well-defined on $[0, 1]$ and $\epsilon \leq u(1) \leq \epsilon^{-1}$.

Let $K_{\max}$ be the maximum absolute value of the curvature of M, and let $\epsilon = \min(1/4, 1/K_{\max}, m)$. We claim that such an ϵ has the desired property.

The inequality $\epsilon \leq u(1)$ results directly from the main assumption of the theorem, and from the fact that $\epsilon \leq m$. To show that $u(1) \leq \epsilon^{-1}$, we assume that $u(1) > \epsilon^{-1}$ and look for a contradiction. First, notice that for all $t \in [0, 1]$ such that $u(t) \geq \epsilon^{-1}$, $u'(t) \leq \epsilon^{-1} - u(t)^2 < 0$. Therefore, $u(t) \geq \epsilon^{-1}$ for $t \in [0, 1]$. From $u' \leq \epsilon^{-1} - u^2$, we obtain

$$\frac{u'}{u^2} \leq \frac{1}{\epsilon u^2} - 1 \leq -\frac{1}{2}$$

$$\frac{1}{u(0)} - \frac{1}{u(1)} \leq -\frac{1}{2}$$

whence $u(1) \leq 2$, which contradicts $\epsilon^{-1} > 2$ and $u(1) \geq \epsilon^{-1}$. Thus, the assumptions of Theorem 6.8 are satisfied.

Theorem 6.8 provides us with a family of cones which satisfy invariance and expansion in the future. Moreover, $\phi_t(x, v) = \phi_t^{-1}(x, -v)$, so there also exists a family of cones satisfying invariance and expansion in the past. By Proposition 6.7, the flow ϕ_t has a stable and an unstable direction: thus, it is Anosov, and Theorem 6.3 is proved.

6.3.3 Proof of Theorem 6.4

In this proof, we will use the lemma:

Lemma 6.13. *Under the assumptions of Theorem 6.4, there exist $m > 0$ and $t_0 > 0$ such that every unit speed geodesic $\gamma : [0, t_0] \to M$ satisfies:*

$$\int_0^{t_0} K(\gamma(t)) \leq -m.$$

Proof. If the conclusion is false, consider a sequence (γ_n) of unit speed geodesics defined on $[-n, n]$, such that for all n,

$$\int_{-n}^{n} K(\gamma(t)) \geq -\frac{1}{n}.$$

By the Arzelà-Ascoli theorem and a diagonal argument, one may extract a subsequence of γ_n which converges uniformly on each $[-n, n]$ to a geodesic defined on $\mathbb{R}$. By dominated convergence, it satisfies $\int_{\mathbb{R}} K(\gamma(t)) = 0$, which contradicts the assumption. □

Now, consider the values of m and t_0 given by lemma 6.13, and choose a geodesic γ. We may assume that $m < 1$ and, by dividing the metric of M by a constant if necessary, that $t_0 < 1$. Denote by u the solution of the Ricatti equation $u'(t) = -K(t) - u^2(t)$ with $u(0) = 0$: if this solution is defined on $[0, 1]$, let $t_1 = 1$; if not, write $[0, t_1)$ the maximal interval on which the solution is defined. In particular, $t_1 \leq 1$.

Set $t_2 = \sup\{t \in [0, t_1] \mid u(t) \geq m\}$ (with $t_2 = 0$ if this set is empty). Thus, for all $t \geq t_2$,

$$u'(t) = -K(t) - u^2(t) \geq -m^2.$$

If $t_2 = 0$, then for all $t \in [0, t_1)$, using the estimate given by Lemma 6.13,

$$u(t) = u(0) + \int_0^t u'(x)dx = \int_0^t -K(x) - u^2(x)dx = -\int_0^t K(x) - \int_0^t u^2(x)dx \geq m - m^2.$$

If $t_2 \neq 0$, then for all $t \in [t_2, t_1[$, using the fact that $K(t) \geq 0$,

$$u(t) = u(t_2) + \int_{t_2}^t u'(x)dx \geq u(t_2) + \int_0^t -u^2(x)dx \geq m - m^2.$$

In both cases, one gets $u(t) \geq m - m^2$ for all $t \in [t_2, t_1[$. Thus, the solution do not blow up to $-\infty$, so $t_1 = 1$, and $u(1) \geq m - m^2$. One may apply Theorem 6.3: the geodesic flow on M is Anosov and Theorem 6.4 is proved.

6.4 Smooth dispersing billiards

6.4.1 Jacobi fields for billiards

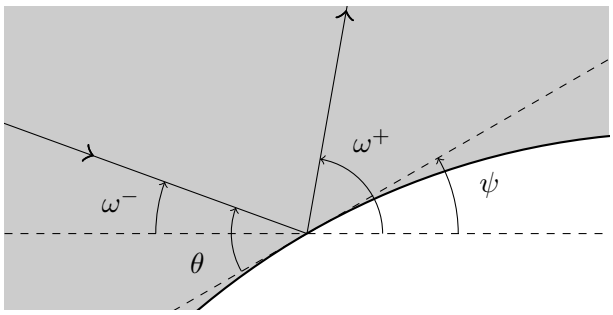
Let D be a billiard and $\gamma : (a, b) \to M$ a billiard trajectory (defined for all times $t \in (a, b)$, except for the collision times). Consider a smooth function

$$f(t, s) : (a, b) \times (c, d) \to M$$

such that $f(., 0)$ is the trajectory γ, and for all $s \in (c, d)$, $f(., s)$ is a billiard trajectory. The function $f(., s)$ is not defined at times $t_1(s), t_2(s), \ldots, t_{p_s}(s)$, which correspond to the collision times.

By analogy with the case of geodesic flows, we shall call[4] "Jacobi field" the vector field $Y = \frac{\partial f}{\partial s}$ along the curve γ. Inside the billiard, Y satisfies the equation $\ddot{Y} = 0$, since the curvature is zero. At a collision time, Y undergoes a discontinuity, which we are now going to study.

Consider a unit speed billiard trajectory variation (*i.e.* $f(., s)$ is a unit speed billiard trajectory for all s). Denote by $\tau(s)$ the first collision time of $f(., s)$. This collision occurs on some piece of the boundary Γ: assume that Γ is parametrized by arc length and define $r(s)$ so that $\Gamma(r(s))$ is the point where the collision occurs. The angle between the horizontal axis and $\frac{\partial}{\partial t}\big|_{t=t_0^\pm} f(t, s)$ is written $\omega^\pm(s)$. Define $\theta = \frac{1}{2}(\omega^+ - \omega^-)$ and $\psi = \frac{1}{2}(\omega^+ + \omega^-)$. The angle θ is the angle of incidence, while ψ is the angle between the horizontal axis and the tangent vector of the boundary $\Gamma'(r(s))$.



To state and prove the next result, it is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex notation.

Proposition 6.14. *Denoting by Y^- the corresponding Jacobi field just before the collision time, and Y^+ the Jacobi field just after it,*

$$Y^+ = -e^{2i\theta} Y^-.$$

Proof. Fix $t_- < \tau$ and $t_+ > \tau$. For all small enough s, and for t_- and t_+ sufficiently close to τ:

$$f(t_\pm, s) = \Gamma(r(s)) + (t_\pm - \tau(s)) e^{i\omega^\pm(s)}.$$

After derivation with respect to s, evaluating at $s = 0$:

$$Y(t_\pm) = \frac{\partial r}{\partial s} e^{i\psi(s)} - \frac{\partial \tau}{\partial s} e^{i\omega^\pm(s)} + i(t_\pm - \tau(s)) \frac{\partial \omega^\pm}{\partial s} e^{i\omega^\pm(s)}.$$

[4] This terminology is common in the literature: see [Don91] or [Woj94].

Let t_+ and t_- tend to τ:

$$Y^{\pm} = \frac{\partial r}{\partial s} e^{i\psi} - \frac{\partial \tau}{\partial s} e^{i\omega^{\pm}}.$$

Finally, compute:

$$Y^+ + e^{2i\theta} Y^- = \frac{\partial r}{\partial s} e^{i\psi}(1 + e^{2i\theta}) - 2\frac{\partial \tau}{\partial s} e^{i\omega^+} = 2\cos\theta \frac{\partial r}{\partial s} e^{i\omega^+} - 2\frac{\partial \tau}{\partial r}\frac{\partial r}{\partial s} e^{i\omega^+} = 0.$$

□

In the case of an orthogonal field, Y may be defined by one scalar y (using the usual orientation of $\mathbb{R}^2$). Proposition 6.14 shows in particular that an orthogonal Jacobi field remains orthogonal after a collision. We will write again $u = \frac{\dot{y}}{y}$.

In the interior of the billiard, the trajectories follows the geodesics of $B = \mathbb{R}^2$ or $\mathbb{T}^2$, whose curvature is zero: thus, the Ricatti equation is simply $u'(t) = -u^2(t)$. However, when a collision occurs, u undergoes a discontinuity.

Proposition 6.15. *Consider f a unit speed geodesic variation corresponding to an orthogonal Jacobi field.*

At a collision,

$$\begin{aligned} y^+ &= -y^- \\ \dot{y}^+ &= -\dot{y}^- + \frac{2\kappa}{\sin\theta} \\ u^+ &= u^- - \frac{2\kappa}{\sin\theta} \end{aligned}$$

where κ is the curvature of the boundary and θ is the angle of incidence.

Proof. The equality $y^+ = -y^-$ is a reformulation of Proposition 6.14 in the case of orthogonal Jacobi fields.

On the other hand,

$$\dot{y}^+ + \dot{y}^- = \frac{\partial(\omega^+ + \omega^-)}{\partial s} = 2\frac{\partial \psi}{\partial s} = 2\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial s} = 2\kappa\frac{y^-}{\sin\theta}$$

and

$$u^+ - u^- = \frac{\dot{y}^+}{y^+} - \frac{\dot{y}^-}{y^-} = -\frac{\dot{y}^+ + \dot{y}^-}{y^-} = -\frac{2\kappa}{\sin\theta}.$$

□

In particular, positively curved walls decrease the value of u (and tend to make the Jacobi field converge), just as the positive curvature of a Riemannian manifold. Likewise, negatively curved walls have the same effect on u as the negative curvature of a manifold. In particular, notice the following:

Proposition 6.16. *If D is a dispersing billiard, any orthogonal Jacobi field with $u(0) \geq 0$ satisfies $u(t) \geq 0$ for all $t \geq 0$.*

Proof. Assume that there is no collision between $t = 0$ and some time $t = t_1$. Then u is a solution of the equation $u'(t) = -u^2(t)$. If $u(0) = 0$, then $u(t) = 0$ for all $t \in [0, t_1]$. If $u(0) > 0$, then $\frac{u'}{u^2} = -1$ for all $t \in [0, t_1]$ such that $u(t) \neq 0$, so

$$\frac{1}{u(0)} - \frac{1}{u(t)} = -t,$$

which shows that the solution is well-defined and positive for $t \in [0, t_1]$.

Moreover, Proposition 6.15 shows that u increases at each collision. Therefore, u remains positive at all times. □

6.4.2 Proof of Theorem 6.5

The ideas of the proof are the same as for Theorem 6.3. We start with the lemma:

Lemma 6.17. *Let D be a billiard with finite horizon. Then, there exists t_0 such that every trajectory in D (with unit speed) experiences at least one collision between $t = 0$ and $t = t_0$.*

Proof. Assume that the conclusion is false. Then for all $n > 0$, there exists a billiard trajectory without collision γ_n, defined on $[-n, n]$. By the Arzelà-Ascoli theorem, one may extract a subsequence which converges uniformly on each interval $[-n, n]$. The limit is still a trajectory without collision, defined on $\mathbb{R}$, which contradicts the assumption. □

Notice that for all $(x, v) \in \tilde{\Omega}$, $T_{(x,v)}\Omega$ is naturally identified with $\mathbb{R}^2 \times \mathbb{R}$, which is endowed with the standard Euclidean norm.

Now, fix $(x, v) \in \tilde{\Omega}$ and denote by $W_{(x,v)} \subseteq T_{(x,v)}(\Omega)$ the orthogonal plane to the direction of the flow ϕ_t. Let $(w, w') \in W_{(x,v)}$. There exists an orthogonal Jacobi field $Y(t)$ such that $(Y(0), \dot{Y}(0)) = (w, w')$, and the family of planes W is stable under $D\phi_t$. Let γ be the geodesic such that $(\gamma(0), \dot{\gamma}(0)) = (x, v)$. All planes $W_{(\gamma(t),\dot{\gamma}(t))}$, are identified to the same Euclidean $\mathbb{R}^2$, in the same way as in the proof of Theorem 6.3.

Denote by $(t_k)_{k\in\mathbb{Z}}$ the sequence of all collision times of the geodesic γ, and $\tilde{t}_k = (t_k + t_{k+1})/2$. For $k \in \mathbb{Z}$, set:

$$A_k = D_{(\gamma(\tilde{t}_k),\dot{\gamma}(\tilde{t}_k))}\phi_{\tilde{t}_{k+1}-\tilde{t}_k} : W_{(\gamma(\tilde{t}_k),\dot{\gamma}(\tilde{t}_k))} \to W_{(\gamma(\tilde{t}_{k+1}),\dot{\gamma}(\tilde{t}_{k+1}))}.$$

The A_k are linear mappings with determinant ± 1, because the flow ϕ_t preserves the Liouville measure on Ω (it is possible to check this from Proposition 6.15).

On the interval $]t_k, \tilde{t}_k[$, u is a solution of the equation $u'(t) = -u^2(t)$, so:

$$u(\tilde{t}_{k+1}) = \frac{1}{\frac{1}{u(t_{k+1}^+)} + (\tilde{t}_{k+1} - t_{k+1})}.$$

With Propositions 6.15 and 6.16, $u(t_{k+1}^+) \geq -\kappa_{\max}$, where $\kappa_{\max}(< 0)$ is the minimum of the curvature of the walls. Moreover, on a smooth billiard, there exist constants $\delta_{\min}$ and $\delta_{\max}$, independant of k, such that $\delta_{\min} \leq t_{k+1} - t_k \leq \delta_{\max}$ for all $k \in \mathbb{Z}$. In particular, $\delta_{\min}/2 \leq \tilde{t}_{k+1} - t_{k+1} \leq \delta_{\max}/2$, whence:

$$0 < \frac{1}{-1/\kappa_{\max} + \delta_{\max}/2} \leq u(\tilde{t}_{k+1}) \leq \frac{1}{\delta_{\min}/2} < +\infty.$$

Thus, Theorem 6.8 applies to the sequence (A_k), and Proposition 6.7 gives the desired result: Theorem 6.5 is proved.

Chapter 7

Geodesic flows of flattened surfaces

7.1 Introduction

In 1927, Birkhoff [Bir27] noticed the following fact: if one of the principal axes of an ellipsoid tends to zero, then the geodesic flow of this ellipsoid tends, at least heuristically, to the billiard flow of the limiting ellipse. In 1963, Arnold [Arn63] stated that the billiard flow in a torus with strictly convex obstacles could be approximated by the geodesic flow of a flattened surface of negative curvature, which would consist of two copies of the billiard glued together, and suggested that this might imply that such a billiard would be chaotic. Later, Sinaï [Sin70] proved the hyperbolicity of the billiard flow in this case, without using the approximation by geodesic flows. In the general case, the correspondance between billiards and geodesic flows of shrinked surfaces is well-known, but it is difficult to use in practice. One of the difficulties is the following: near tangential trajectories, some geodesics converge to "fake" billiard trajectories, which follow the boundary of the obstacle for some time and then leave (see Figure 7.1).

More precisely, for a given billiard[1] $D \subseteq \mathbb{T}^2$ or $D \subseteq \mathbb{R}^2$, Birkhoff and Arnold's idea is to consider a surface Σ in a space $E = \mathbb{T}^2 \times \mathbb{R}$ or $E = \mathbb{R}^3$, such that $D = \pi(\Sigma)$, where π

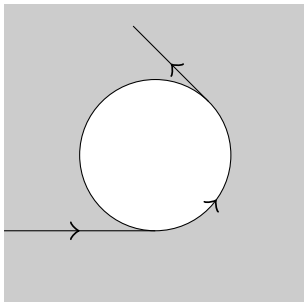


Figure 7.1 – A "fake" billiard trajectory, which is the limit of a sequence of geodesics in the flattening surface. The billiard (in grey) is in $\mathbb{T}^2$ and there is one circular obstacle (in white).

[1] See the beginning of Chapter 6 for the definition of billiards.

is the projection onto the first two coordinates; and then, to consider its image Σ_ϵ by a flattening map for $\epsilon > 0$:

$$f_\epsilon : E \to E$$
$$(x, y, z) \mapsto (x, y, \epsilon z).$$

The Euclidean metric of $\mathbb{R}^3$ induces a metric h_ϵ on Σ_ϵ. It is convenient to consider the metric $g_\epsilon = (f_\epsilon)^*(h_\epsilon)$ on Σ, which tends to a degenerate 2-form g_0 on Σ as ϵ decreases to 0. Thus, (Σ, g_0) is not a Riemannian metric, but in many cases (for example, for the ellipsoid), it remains a metric space[2] and every billiard trajectory in D corresponds to a geodesic in (Σ, g_0). The Arzelà-Ascoli theorem guarantees that every sequence of unit speed geodesics (Γ_n) in (Σ, g_{ϵ_n}), with $\epsilon_n \to 0$, converges to a geodesic of (Σ, g_0), up to a subsequence. In this chapter, we prove a stronger version of this result.

Thus, from any given billard, Arnold constructs a surface which he flattens, so that its geodesic flow converges to the billiard flow. In this chapter, we do the reverse: we prove that, under some natural hypotheses, the geodesic flow of *any* given compact surface in $\mathbb{R}^3$, or $\mathbb{T}^2 \times \mathbb{R}$, flattening to a smooth billiard, converges locally uniformly to the billiard flow, away from grazing trajectories (Theorem 7.1). We also prove that, if the limiting billiard has finite horizon and is dispersive, then the geodesic flow in (Σ, g_ϵ) is Anosov for any small enough $\epsilon > 0$ (Theorem 7.2). In this case, it is well-known that the limiting billiard is chaotic, but the surface near the limit does not necessarily have negative curvature everywhere: some small positive curvature may remain in the area corresponding to the interior of the billiard, while the negative curvature concentrates in the area near the boundary. Since the limiting billiard has finite horizon, any geodesic falls eventually in the area of negative curvature, which guarantees that the flow is Anosov. The precise statements of our results are given in Section 7.2.

Other analogies have been made between billiards and smooth dynamical systems. In [TRK98], Turaev and Rom-Kedar showed that the billiard flow could be approximated in the C^r topology by the behavior of a particle in $\mathbb{R}^2$ exposed to a potential field which explodes near the boundary[3]. In our situation, there is no potential and the particle has 3 coordinates instead of 2, but some of our techniques are similar to theirs.

Our setting has also much in common with the example of Donnay and Pugh [DP03], who exhibited in 2003 an embedded surface in $\mathbb{R}^3$ which has an Anosov flow. This surface consists of two big concentric spheres of very close radii, glued together by many tubes of negative curvature in a finite horizon pattern. In this surface, any geodesic eventually enters a tube and experiences negative curvature, while the positive curvature is small (because the spheres are big). However, in our situation, we may not choose the shape of the tubes and we need precise estimates on the curvature to show that the geodesic flow is Anosov.

7.2 Main results

Let $B = \mathbb{R}^2$ or $B = \mathbb{T}^2$, and consider a compact surface Σ immersed in $E = B \times \mathbb{R}$, whose canonical basis is written (e_x, e_y, e_z). With the notations of the introduction, we denote by Φ_t^ϵ the geodesic flow on (Σ, g_ϵ) and by Ψ_t the billiard flow on $D = \pi(\Sigma)$.

[2] In [BFK98], Burago, Ferleger and Kononeko used such degenerate spaces (*Alexandrov spaces*) to estimate the number of collisions in some billiards.

[3] These systems are called *soft billiards* in the literature: see also [BT03] for more details.

In this chapter, we always assume that D is a smooth billiard: we do not allow corners, to avoid the problems discussed in the introduction. For the definition of billiards, see the beginning of Chapter 6.

Recall that $\pi : E \to B$ is the projection onto the first two coordinates, while f_ϵ is a contraction along the z-axis. If Σ is transverse to the fibers of π above Int D, then π and f_ϵ induce mappings on the unit tangent bundles:

$$\begin{cases} \pi_* : T^1\left(\Sigma_\epsilon \cap \pi^{-1}\left(\text{Int } D\right)\right) \to T^1 D \\ \qquad\qquad (q,p) \mapsto \left(\pi(q), \dfrac{D_q\pi(p)}{\|D_q\pi(p)\|}\right) \end{cases}$$

and

$$\begin{cases} (f_\epsilon)_* : T^1\Sigma \to T^1\Sigma_\epsilon \\ \qquad (q,p) \mapsto \left(f_\epsilon(q), \dfrac{D_q f_\epsilon(p)}{\|D_q f_\epsilon(p)\|}\right). \end{cases}$$

Consider also the set A of all $(t,q,p) \in \mathbb{R} \times T^1\Sigma$ such that $\pi_*(q,p)$ and $\Psi_t \circ \pi_*(q,p)$ belong to Int D, and that the billiard trajectory between $\pi_*(q,p)$ and $\Psi_t \circ \pi_*(q,p)$ does not have a tangential collision with a wall of the billiard. Notice that A is an open dense subset of $\mathbb{R} \times T^1\Sigma$.

Theorem 7.1. *Assume that*

1. *the surface $\pi^{-1}(\text{Int } D) \cap \Sigma$ is transverse to the fibers of the projection π;*
2. *for all $q \in \pi^{-1}(\partial D) \cap \Sigma$, the curvature of $\Sigma \cap V$ is nonzero at q, where V is a neighborhood of q in the affine plane $q + \text{Vect}(e_z, (T_q\Sigma)^\perp)$.*

Then:

$$\begin{aligned} A &\to T^1(\text{Int } D) \\ (t,q,p) &\mapsto \pi_* \circ \Phi_t^\epsilon(q,p) \end{aligned}$$

converges uniformly on every compact subset of A to

$$\begin{aligned} A &\to T^1(\text{Int } D) \\ (t,q,p) &\mapsto \Psi_t \circ \pi_*(q,p) \end{aligned}$$

as $\epsilon \to 0$.

Remark. If Σ is a connected compact surface embedded in $\mathbb{R}^3$, with positive curvature everywhere, then the two assumptions of Theorem 7.1 are automatically satisfied, and the description of A is simpler.

On the other side, concerning dispersing billiards, we prove:

Theorem 7.2. *In addition to the two hypotheses of Theorem 7.1, assume that $B = \mathbb{T}^2$ and:*

3. *the walls of the billiard D have negative curvature;*
4. *the billiard D has finite horizon: it contains no geodesic of $\mathbb{T}^2$ with infinite lifetime in the past and the future.*

Then for any small enough $\epsilon > 0$, the geodesic flow on $(\Sigma_\epsilon, h_\epsilon)$ is Anosov.

In the proofs of Theorems 7.1 and 7.2, we will assume that Σ is *embedded* in E, to simplify the notations, but the same proof works for the immersed case. In the case $B = \mathbb{T}^2$, we will see D as a periodic billiard in the universal cover $\mathbb{R}^2$, and Σ as a periodic surface in $\mathbb{R}^3$.

Structure of the proofs. The main tool to study the geodesic flow is the geodesic equation which involves the position q, the speed p, and the normal vector N to Σ:

$$\dot{p} = -N(q) \langle DN(q) \cdot p \mid p \rangle . \tag{7.1}$$

It is simply obtained by taking the derivative of the equation

$$\langle p \mid N \rangle = 0.$$

Equation 7.1 involves the second fundamental form, which is closely linked to the curvature of Σ: in Section 7.3, we make precise estimates on the second fundamental form, study nongrazing collisions with the walls of the billiard (Lemma 7.11) and prove the uniform convergence of the flow (Theorem 7.1). In Section 7.4, we prove that the geodesic flow is Anosov (Theorem 7.2): for this, we also need to study grazing trajectories (Lemma 7.13), and examine the solutions of the Ricatti equation (see Theorem 6.3).

Some questions. For technical reasons, we had to introduce Assumption 2 in the statements of Theorems 7.1 and 7.2, but is it necessary? One may also wonder whether these theorems generalize to surfaces immersed in higher-dimensional spaces, or if the convergence in Theorem 7.1 holds for the C^k topology.

7.3 Proof of Theorem 7.1

In $\Sigma \cap \pi^{-1}(\text{Int } D)$, g_ϵ converges smoothly to a flat metric, so the geodesic flow converges smoothly to the billiard flow. Hence, the difficulty of the proof concentrates at the boundary of the billiard table: there, we have to show that the geodesic flow satisfies a billiard reflection law at the limit (Proposition 7.11). For this, we will need some estimates on the second fundamental form of the surface near the boundary.

First, let us fix some notations.

Definition 7.3. Given $\epsilon > 0$, we may choose a normal vector N^ϵ on any simply connected subset of Σ_ϵ. We will always assume implicitly that such a choice of orientation has been made: since we work locally on the surface, it is not necessary to have a global orientation.

Consider $N^\epsilon_x, N^\epsilon_y, N^\epsilon_z$ the three components of N^ϵ in $\mathbb{R}^3$. Thus for $q \in \Sigma$:

$$N^\epsilon_x(f_\epsilon(q)) = \frac{N^1_x}{\sqrt{(N^1_x)^2 + (N^1_y)^2 + \frac{1}{\epsilon^2}(N^1_z)^2}}, \quad N^\epsilon_y(f_\epsilon(q)) = \frac{N^1_y}{\sqrt{(N^1_x)^2 + (N^1_y)^2 + \frac{1}{\epsilon^2}(N^1_z)^2}}$$

and

$$N^\epsilon_z(f_\epsilon(q)) = \frac{\frac{1}{\epsilon}N^1_z}{\sqrt{(N^1_x)^2 + (N^1_y)^2 + \frac{1}{\epsilon^2}(N^1_z)^2}}.$$

We shall often simply write N instead of N^ϵ, when there is no possible confusion.

Finally, define

$$H(f_\epsilon(q)) = N_z^1(q).$$

The quantity $H(f_\epsilon(q))$ has the advantage of being independent of ϵ, contrary to $N^\epsilon(f_\epsilon(q))$.

For all $q \in \Sigma_\epsilon$, we know that $\pi(q) \in \partial D$ if and only if $N_z(q) = 0$, or equivalently, $H(q) = 0$. This gives us two notions of "being close to the boundary": for all $\epsilon, \delta, \nu \in (0,1)$, we define

$$V_\nu^\epsilon = \{q \in \Sigma_\epsilon \mid |N_z^\epsilon(f_\epsilon(q))| < 1 - \nu\}$$

and

$$Z_\delta^\epsilon := \{q \in \Sigma_\epsilon \mid |H(q)| \leq \delta\}.$$

To simplify the notations, we will often omit the ϵ and simply write V_ν and Z_δ. Notice that for any δ and ν, when ϵ is sufficiently small, we have $V_\nu \subseteq Z_\delta$, because the metric tends to a flat one outside Z_δ.

We now have to deal with three very small values: δ, ν and ϵ. In the whole chapter, we will assume $\epsilon \ll \delta$: in other words, we will first choose a small δ, and then choose a small ϵ which depends on the choice of δ. When the value ν will come into play, its choice will always be made between the choice of δ and the choice of ϵ, that is: $\epsilon \ll \nu \ll \delta$.

Definition 7.4 (Darboux frame)**.** For any unit speed curve $\Gamma : [0,1] \to \Sigma$, we define the tangent vector $T = \Gamma'(s)$. The normal vector N is the unit normal to $(T_{\Gamma(s)}\Sigma)$. Finally, the *normal geodesic vector* G is defined by $G = N \wedge T$.

In this frame, there exist three quantities γ_N^ϵ (normal curvature), γ_G^ϵ (geodesic curvature) and τ_G^ϵ (geodesic torsion), also written simply γ_N, γ_G and τ_G, such that

$$\begin{aligned}\frac{dT}{ds} &= \gamma_G G + \gamma_N N\\ \frac{dG}{ds} &= \gamma_G T + \tau_G N\\ \frac{dN}{ds} &= \gamma_N T + \tau_G G\end{aligned}$$

The (traditional) curvature of Γ considered as a curve in $\mathbb{R}^3$ is $k = \left\|\frac{dT}{ds}\right\|$. Thus, if $k \neq 0$, writing $n = \frac{dT/ds}{\|dT/ds\|}$, we obtain:

$$\gamma_N = k \langle N \mid n \rangle, \tag{7.2}$$

and in particular:

$$|\gamma_N| = k\sqrt{1 - \langle N \mid T \wedge n\rangle^2}. \tag{7.3}$$

For example, if Γ is the intersection of Σ with a plane P, $T \wedge n$ has the same direction as the normal vector of P, so it is convenient to use Equation 7.3.

Notice that the normal curvature at s only depends on $\Gamma(s)$ and $\Gamma'(s)$: thus we may write $\gamma_N(q,p)$ for $(q,p) \in T^1\Sigma$. Moreover, we have the relation:

$$\gamma_N(q,p) = \langle DN(q) \cdot p \mid p \rangle.$$

For any $q \in \Sigma_\epsilon$, $\gamma_+^\epsilon(q)$ and $\gamma_-^\epsilon(q)$ (sometimes written simply γ_+ and γ_-) are the principal curvatures of Σ_ϵ at q. They correspond respectively to the maximum and minimum normal curvatures at $f_\epsilon(q)$.

$K^\epsilon(q) = \gamma_+^\epsilon(q)\gamma_-^\epsilon(q)$ is the Gaussian curvature of Σ_ϵ at q.

We can now make a first remark:

Fact 7.5. *For any small enough $\delta > 0$, $H|_{Z_\delta}$ is a submersion from Z_δ to $\mathbb{R}$.*

Proof. Let $q \in \pi^{-1}(\partial D) \cap \Sigma$ and consider a curve γ which parametrizes the section of Σ by the plane containing the directions (Oy) and (Oz), with $\gamma(0) = q$. Assumption 2 of the theorem implies that γ has nonzero curvature at q, and thus $\left\langle D_{\gamma(t)}H(\gamma'(t)) \;\middle|\; \gamma'(t)\right\rangle$ is nonzero for any small enough t. Therefore, H is a submersion from Z_δ to $\mathbb{R}$. □

Lemma 7.6. *Let $(q,p) \in T^1\Sigma$ and $(q^\epsilon, p^\epsilon) = (f_\epsilon)_*(q,p)$. If $\gamma_N^\epsilon(q^\epsilon,p^\epsilon) \neq 0$ for some $\epsilon > 0$, then the sign of $\gamma_N^\epsilon(q^\epsilon,p^\epsilon)$ is the same for all $\epsilon > 0$.*

Proof. Let $\Gamma : [-1,1] \to \Sigma$ be any curve such that $(\Gamma(0), \Gamma'(0)) = (q,p)$, and consider $\Gamma^\epsilon = f_\epsilon \circ \Gamma$ for $\epsilon > 0$. Writing T^ϵ its tangent vector, the assumption implies that $\left\langle \frac{dT^\epsilon}{dt} \;\middle|\; N^\epsilon\right\rangle$ is nonzero at $t = 0$ for some ϵ, which means that $\frac{dT^\epsilon}{dt} \notin T_{\Gamma^\epsilon(0)}\Sigma_\epsilon$. Obviously, this property does not depend on ϵ, so $\gamma_N^\epsilon(q^\epsilon,p^\epsilon)$ is nonzero for all ϵ. By continuity, $\gamma_N^\epsilon(q^\epsilon,p^\epsilon)$ does not change sign. □

Lemma 7.7. *Let $\alpha \in (0,1)$, and $q_0 \in \Sigma$ such that $\pi(q_0) \in \partial D$. We assume that $N(q_0)$ is directed towards the exterior of the billiard table D, and (up to a rotation of axis e_z) that $N(q_0) = -e_y$. Then there exists $r > 0$ and ϵ_0 such that for all $\epsilon \leq \epsilon_0$, all $q \in B(f_\epsilon(q_0), r)$, and all $p \in T_q^1\Sigma_\epsilon$: $\gamma_N(q,p) > 0$ whenever $|p_x| \leq \alpha$ (where p_x is the x-coordinate of p).*

Proof. By Assumption 2 of Theorem 7.1, we know that $\gamma_N(q_0, e_z) > 0$. We define $W_\eta = \left\{p \in T_{q_0}^1\Sigma \;\middle|\; |p_x| \leq \eta\right\}$. By continuity of γ_N, there exists $\eta > 0$ such that, for all $p \in W_\eta$, $\gamma_N(q_0,p) > 0$.

Write $W_\eta^\epsilon = (f_\epsilon)_*(W_\eta)$. Notice that for all $p \in T_{q_0}^1\Sigma$, writing $(f_\epsilon)_*(q_0,p) = (f_\epsilon(q_0), p^\epsilon)$:

$$p_x^\epsilon = \frac{p_x}{\sqrt{p_x^2 + \epsilon^2 p_z^2}} \quad \text{and} \quad p_z^\epsilon = \frac{\epsilon p_z}{\sqrt{p_x^2 + \epsilon^2 p_z^2}}.$$

Thus for a small enough ϵ, W_η^ϵ contains all $p \in T_{q_0}^1\Sigma_\epsilon$ for which $|p_x| \leq \alpha$. We denote such an ϵ by ϵ_0. By Lemma 7.6, $\gamma_N > 0$ on $W_\eta^{\epsilon_0}$. Again by continuity, the property $\gamma_N > 0$ extends to a small neighborhood of the form $\left\{(q,p) \in T^1\Sigma_{\epsilon_0} \;\middle|\; q \in B(q_0, r'),\ |p_x| \leq \alpha\right\}$ for some $r' > 0$.

Finally, we use Lemma 7.6 once again, which proves that there exists $r > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $\gamma_N > 0$ on $\left\{(q,p) \in T^1\Sigma_\epsilon \;\middle|\; q \in B(q_0, r),\ |p_x| \leq \alpha\right\}$. □

Proposition 7.8. *Choose $q_0 \in \Sigma$ such that $\pi(q_0) \in \partial D$, and assume that $N(q_0)$ is directed towards the exterior of the billiard table D, and (up to a rotation of axis e_z) that $N(q_0) = -e_y$. Write $q_0^\epsilon = f_\epsilon(q_0)$.*

Then for all $\alpha \in (0,1)$, there exists $r_0 > 0$ such that for all $r \leq r_0$ and for all $\nu \in (0,1)$:

$$\inf_{\substack{q \in V_\nu \cap B(q_0^\epsilon, r) \\ p \in T_q^1\Sigma_\epsilon,\ |p_x| \leq \alpha}} \gamma_N^\epsilon(q,p) \underset{\epsilon \to 0}{\to} +\infty. \tag{7.4}$$

Moreover, under the additional assumption that the curvature of ∂D is negative at $\pi(q_0)$, there exists $r_0 > 0$ such that for all $r \leq r_0$ and for all $\nu \in (0,1)$:

$$\limsup_{\epsilon \to 0} \sup_{q \in V_\nu \cap B(q_0^\epsilon, r)} \gamma_-^\epsilon(q) < 0. \tag{7.5}$$

To prove this proposition, we first prove a 2-dimensional version in a particular case:

Lemma 7.9. *For all $\epsilon > 0$ consider the ellipse*

$$\mathcal{E}_\epsilon = \left\{ (y,z) \in \mathbb{R}^2 \;\middle|\; y^2 + \frac{z^2}{\epsilon^2} = 1 \right\}.$$

Define $N^\epsilon(q)$ as the unit normal vector of the ellipse at $q \in \mathcal{E}_\epsilon$, pointing towards the interior, and let

$$W_\nu = \{ z \in \mathcal{E}_\epsilon \mid |N_z^\epsilon| < 1 - \nu \}.$$

Then for all $\nu \in (0,1)$, if $K(q)$ denotes the curvature of $\mathcal{E}_\epsilon$ at q:

$$\inf_{q \in W_\nu} K(q) \underset{\epsilon \to 0}{\to} +\infty.$$

Proof. We parametrize $\mathcal{E}_\epsilon$ by:

$$t \mapsto \begin{pmatrix} \cos t \\ \epsilon \sin t \end{pmatrix}.$$

Then the curvature is

$$K(t) = \frac{\epsilon}{(\epsilon^2 \cos^2 t + \sin^2 t)^{3/2}}$$

while the unit normal vector is

$$\begin{pmatrix} -\epsilon \cos t \\ -\sin t \end{pmatrix} \frac{1}{\sqrt{\epsilon^2 \cos^2 t + \sin^2 t}}.$$

If $|N_z| < 1 - \nu$, then $\sin^2 t \geq (1-\nu)^2(\epsilon^2 \cos^2 t + \sin^2 t)$, whence $\tan^2 t \leq \frac{\epsilon^2(1-\nu)^2}{\nu(2-\nu)}$. Therefore,

$$K = \frac{\epsilon / \left| cos^3 t \right|}{(\epsilon^2 + \tan^2 t)^{3/2}} \geq \frac{\epsilon}{\left(\epsilon^2 + \frac{\epsilon^2(1-\nu)^2}{\nu(2-\nu)}\right)^{3/2}} = \frac{(\nu(2-\nu))^{3/2}}{\epsilon^2}$$

which tends to $+\infty$ as $\epsilon \to 0$. □

Proof of Proposition 7.8. For each $q \in B(q_0, r)$, consider the curve Γ_q resulting from the intersection of Σ with the affine plane (q, e_y, e_z) and the normal vector n of Γ_q.

With the notations of Definition 7.4, let us show that we may choose r small enough for $|\langle N \mid T \wedge n \rangle|$ to remain bounded away from 1 for all small ϵ and all $q \in B(q_0, r)$. Since $N_x^1(q_0) = 0$, we may choose r such that N_x remains close to 0 for $\epsilon = 1$. We know that N_x decreases as ϵ decreases to 0, so N_x remains close to 0 when $\epsilon \to 0$. Since $T \wedge n$ is colinear to e_x, this implies that $|\langle N \mid T \wedge n \rangle|$ remains close to 0.

Now, let $\mathcal{C}$ be a circle tangent up to order 2 to Σ_1 at q, parallel to $e_x^\perp$: the existence of such a circle is guaranteed, for a small enough r, by Assumption 2 of the theorem. This circle gives birth to a family $\mathcal{E}_\epsilon = f_\epsilon(\mathcal{C})$ of ellipses which are tangent to Σ_ϵ at $f_\epsilon(q)$

up to order 2. Lemma 7.9 tells us that as ϵ decreases to 0, the curvature of $\mathcal{E}_\epsilon$ at $f_\epsilon(q)$ (which is the same as the curvature of $f_\epsilon(\Gamma_q)$ at $f_\epsilon(q)$) tends to infinity as long as $q \in V_\nu$, uniformly with respect to q. Together with Equation 7.3, this proves that

$$\inf_{\substack{q\in V_\nu\cap B(q_0^\epsilon,r)\\ p\in T_q^1\Sigma_\epsilon,\ p_x=0}} \gamma_N^\epsilon(q,p) \underset{\epsilon\to 0}{\to} +\infty.$$

To prove (7.4), let $\alpha \in (0,1)$. Lemma 7.7 applied to q_0 and $\frac{\alpha+1}{2}$ gives us some r_0 and ϵ_0 such that for all $q \in B(q_0, r_0)$ and all $p \in T_q^1\Sigma_\epsilon$ such that $|p_x| \leq \frac{\alpha+1}{2}$, $\gamma_N(q,p) > 0$. Since $\gamma_N(q,\cdot)$ is a quadratic form on the tangent space $T_q\Sigma_\epsilon$, which takes uniformly large values for $p \in T_q^1\Sigma_\epsilon \cap (e_x)^\perp$, we deduce that it also takes uniformly large values for $|p_x| \leq \alpha$.

Finally, we prove (7.5): consider $q \in B(q_0^\epsilon, r) \cap V_\nu$, and Γ a parametrization by arc length of $\{q' \in B(q_0, r) \cap V_\nu \mid H(q') = H(q)\}$. Since $H|_{Z_\delta}$ is a submersion (for any small enough δ), the curvature of the curve $\pi \circ \Gamma$ is close to the curvature of ∂D near $\pi(q)$, which is bounded away from zero. Moreover, the unit tangent vector of Γ is bounded away from e_z because of Assumption 2, so the speed of $\pi \circ \Gamma$ is bounded away from zero, which implies that the curvature of Γ itself is bounded away from 0, uniformly with respect to ϵ and q. Moreover, $\langle e_z \mid T \wedge n\rangle$ tends uniformly to 1 as ϵ tends to 0, so $\langle N \mid T \wedge n\rangle$ is bounded away from 1 in V_ν^ϵ. With Equation 7.3, this completes the proof of (7.5). □

As a direct consequence of Lemma 7.7 and Proposition 7.8, we obtain:

Fact 7.10. *If the walls of D are negatively curved, then for any small enough δ, there exists $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$, the Gaussian curvature of Σ_ϵ in Z_δ is negative.*

In the following proposition, which is crucial for both Theorems 7.1 and 7.2, we examine nongrazing collisions with the walls of the billiards. We will consider geodesics $(q^\epsilon(t), p^\epsilon(t))_{t\in\mathbb{R}}$ in Σ^ϵ for some $\epsilon > 0$. We will say that such a geodesic is t_0-*admissible* (for some time $t_0 > 0$) if the billiard trajectory starting from $\pi_*(q^\epsilon(0), p^\epsilon(0))$ bounces against a wall of the billiard before the time t_0: in this case, we will denote by $t_c \in [0, t_0]$ the time of the first bounce. For any time $t_0 > 0$, and any t_0-admissible geodesic, we will denote by $(q^0(t), p^0(t))$ the (unique) pullback of this billiard trajectory by π in Σ_ϵ for $t \in [0, t_c]$, and write $q_c = q^0(t_c)$; moreover, we will choose the normal $N(q_c)$ so that it is directed towards the exterior of the billiard table D, and make a rotation of axis e_z so that $N(q_c) = -e_y$ (see Figure 7.2).

Notice that for any $\delta > 0$, the projection of the geodesic flow outside Z_δ is uniformly close to the billiard flow for small values of ϵ, so there exists $\epsilon_0 > 0$, which depends on δ, but not on the choice of the geodesic, such that for all $\epsilon \leq \epsilon_0$, any geodesic as above enters Z_δ at a time t_{in}^ϵ close to t_c.

Proposition 7.11. *For all $t_0 > 0$, $m > 0$ and $\alpha \in (0,1)$, there exist $l < 0$ and $\delta_0 > 0$, such that for all $\delta \leq \delta_0$, there exists $\epsilon_0 > 0$, such that for all $\epsilon \leq \epsilon_0$ and each t_0-admissible geodesic (q^ϵ, p^ϵ) in Σ_ϵ such that $|p_x^\epsilon(0)| < \alpha$:*

1. $\sup_{t\in[0,t_{\text{in}}^\epsilon+\sqrt{\delta}]} |p_x^\epsilon(t) - p_x^\epsilon(0)| \leq m$;

2. *the geodesic $(q^\epsilon(t), p^\epsilon(t))$ exits Z_δ at a time $t_{\text{out}} \leq t_{\text{in}} + \sqrt{\delta}$;*

3. if the curvature of ∂D is negative everywhere, and if $q^\epsilon(0) \notin Z_\delta$, then

$$\int_{t^\epsilon_{\text{in}}}^{t^\epsilon_{\text{out}}} K(q^\epsilon(t))dt \leq l.$$

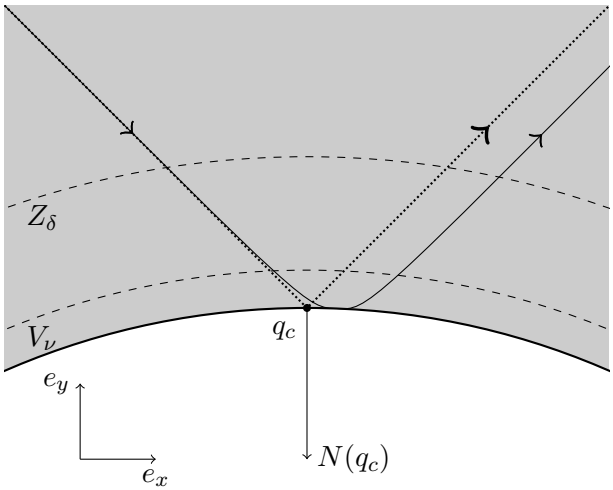


Figure 7.2 – The projection of the geodesic onto the billiard (solid line) is close to the billiard trajectory (dotted line).

Proof. Fix $t_0 > 0$, $m > 0$ and $\alpha \in (0, 1)$. In the whole proof, δ_0 and ϵ_0 are assumed to be as small as necessary. In other words, the reader may add mentally to most mathematical sentences the following prefix: "there exist $l < 0$ and $\delta_0 > 0$, such that for all $\delta \leq \delta_0$, there exists $\epsilon_0 > 0$, such that for all $\epsilon \leq \epsilon_0$ and each t_0-admissible geodesic (q^ϵ, p^ϵ) in Σ_ϵ such that $|p^\epsilon_x(0)| < \alpha$: ...".

Let us prove Statement (1). We shall often write q, p for $q^\epsilon(t), p^\epsilon(t)$ to simplify the notations.

Outside Z_δ, the geodesic flow converges uniformly to the billiard flow, so

$$\sup_{t\in[0,t_{\text{in}}]} |p^\epsilon_x(t) - p^\epsilon_x(0)| \leq m/2.$$

Thus, we only need to consider $\sup_{t\in[t_{\text{in}},t_{\text{in}}+\sqrt{\delta}]} |p^\epsilon_x(t) - p^\epsilon_x(t_{\text{in}})|$.

Let $t^1 = \inf\left\{t \in [t_{\text{in}}, t_{\text{in}} + \sqrt{\delta}] \;\middle|\; \langle DN(q) \cdot p \mid p\rangle \leq 0\right\}$ (or $t^1 = t_{\text{in}} + \sqrt{\delta}$ if this set is empty), and consider $t \in [t_{\text{in}}, t^1]$ (thus, $\langle DN(q) \cdot p \mid p\rangle \geq 0$ at time t).

The geodesics $(q(t), p(t))$ follow the geodesic equation:

$$\dot{p} = -N(q) \langle DN(q) \cdot p \mid p\rangle ,$$

which gives us the following estimates:

$$\begin{aligned}|p_x(t) - p_x(t_{\text{in}})| &\leq \int_{t_{\text{in}}}^{t} |\dot{p_x}| \\ &\leq \int_{t_{\text{in}}}^{t} \left|\frac{N_x}{N_y}\right| |\dot{p_y}| \\ &\leq \left(\sup_{B(q(t_{\text{in}}),\sqrt{\delta})} \left|\frac{N_x}{N_y}\right|\right) \int_{t_{\text{in}}}^{t} |\dot{p_y}|\end{aligned}$$

For all sufficiently small δ, the quantity N_y is negative in $B(q(t_{\text{in}}), \sqrt{\delta})$, thus $\dot{p_y} = -N_y(q) \langle DN(q) \cdot p \mid p \rangle$ is nonnegative and:

$$|p_x(t) - p_x(t_{\text{in}})| \leq \left(\sup_{B(q(t_{\text{in}}),\sqrt{\delta})} \left|\frac{N_x}{N_y}\right|\right) |p_y(t) - p_y(t_{\text{in}})|.$$

We know that $\frac{N_x}{N_y} = \frac{N_x^1}{N_y^1}$ does not depend on ϵ. Moreover, $q(t_{\text{in}})$ is close to q_c and $\frac{N_x^1}{N_y^1}(q_c) = 0$, so the quantity $\sup_{B(q(t_{\text{in}}),\sqrt{\delta})} \left|\frac{N_x}{N_y}\right|$ is close to 0. On the other hand, $|p_y(t) - p_y(t_{\text{in}})|$ remains bounded since the geodesic has unit speed, which concludes the proof of Statement (1) for $t \in [t_{\text{in}}, t^1]$.

To extend the result to $t \in [t_{\text{in}}, t_{\text{in}} + \sqrt{\delta}]$, we prove that in fact $t^1 = t_{\text{in}} + \sqrt{\delta}$: assume that $t^1 \neq t_{\text{in}} + \sqrt{\delta}$. Then $p_x(t)$ remains close to $p_x(t_{\text{in}})$ for $t \in [t_{\text{in}}, t^1]$, so it remains bounded away from 1, but $\gamma_N(q,p) = \langle DN(q) \cdot p \mid p \rangle \leq 0$ at t^1. Thus, there is a contradiction with Lemma 7.7, and Statement (1) is proved.

Now, let us prove Statement (2). We introduce the parameter ν and fix the parameters in the following way: first fix a small δ, then a small ν, and finally a small ϵ.

Let us show that the boundaries of Z_δ and V_ν near q_c are nearly parallel to the e_x axis. From Fact 7.5, the levels of $(H|_{Z_\delta})^{-1}(a)$ are smooth curves. Moreover, for a sufficiently small $a \in [-1, 1]$, near q_c, the y-coordinates of the unit tangent vectors to $H^{-1}(a)$ remain small, while the x-coordinates are bounded away from zero. In particular, this applies to the boundary of Z_δ, but also to the boundary of V_ν, which is a level of H, since N_z^ϵ depends only on H and ϵ (see Definition 7.3).

Outside V_ν, $|p_z|$ is bounded by $\sqrt{\nu(1-\nu)}$. Since p_x is bounded away from 1 and $p_x^2 + p_y^2 + p_z^2 = 1$, we deduce that p_y remains bounded away from zero, uniformly with respect to δ, ν and ϵ, for all sufficiently small ν. In particular, p_y does not change sign in $Z_\delta \setminus V_\nu$, so it is only possible to enter V_ν once with $p_y < 0$ and exit once with $p_y > 0$. Thus, the geodesic can enter V_ν at most once.

There remains to show that the time spent in each zone is small.

For any $q \in \pi^{-1}(\text{Int } D) \cap \Sigma_\epsilon$, it is natural to define $\frac{\partial H}{\partial x}$ as $DH_q(p)$, where p is the (unique) speed vector in $T_q\Sigma$ such that $D_q\pi(p) = e_x$. We also define $\frac{\partial H}{\partial y}$ in the same way.

Outside V_ν (therefore outside $\pi^{-1}(\partial D)$) we write:

$$\dot{H} = \frac{\partial H}{\partial x} p_x + \frac{\partial H}{\partial y} p_y.$$

Since the levels of the submersion H are nearly parallel to e_x, $\frac{\partial H/\partial x}{\partial H/\partial y}$ is close to 0 near q_c. Outside V_ν, with the fact that p_y is bounded away from 0, this proves that $\dot{H}$ is bounded away from 0, so the time spent in $Z_\delta \setminus V_\nu$ is $O(\delta)$.

In V_ν we have

$$\dot{p_y} = -N_y \langle DN(q) \cdot p \mid p \rangle .$$

Fix $\delta, \nu > 0$. Since N_z is bounded away from 1, and $\frac{N_x}{N_y}$ is close to zero, we deduce that N_y is bounded away from zero. Moreover, by Proposition 7.8, $\langle DN(q) \cdot p \mid p \rangle \underset{\epsilon \to 0}{\to} +\infty$ uniformly in V_ν, so $\dot{p_y} \underset{\epsilon \to 0}{\to} +\infty$. Since p_y is bounded, this implies that the time spent in V_ν tends to 0 as $\epsilon \to 0$. Thus, the total time spent in each zone is $O(\delta)$, so for any small enough δ, $t_{\text{out}} \leq t_{\text{in}} + \sqrt{\delta}$ (Statement 2).

If the curvature of ∂D is negative everywhere, and $q^0(0) \notin Z_\delta$, then the geodesic has the following behavior: it enters Z_δ with $p_y < 0$, then enters V_ν with $p_y < 0$. In V_ν, p_y changes sign, then the geodesic exits V_ν and finally, exits Z_δ. Therefore, writing t_2 and t_3 the entry and exit times in V_ν, since K is negative in Z_δ (see Fact 7.10):

$$\begin{aligned}\int_{t_2}^{t_3} K = \int_{t_2}^{t_3} \gamma_+^\epsilon \gamma_-^\epsilon \leq \left(\sup_{V_\nu} \gamma_-^\epsilon\right) \int_{t_2}^{t_3} \gamma_+^\epsilon &\leq \left(\sup_{V_\nu} \gamma_-^\epsilon\right) \int_{t_2}^{t_3} -N_y \langle DN(q) \cdot p \mid p \rangle \\ &= \left(\sup_{V_\nu} \gamma_-^\epsilon\right) (p_y(t_3) - p_y(t_2)).\end{aligned}$$

With Proposition 7.8, this proves that $\int_{t_2}^{t_3} K$ is bounded away from 0: Statement (3) is proved. □

End of the proof of Theorem 7.1. To prove the local uniform convergence, we introduce a family of elements $(t^\epsilon, q^\epsilon(0), p^\epsilon(0)) \in f_\epsilon(A)$ with parameter ϵ, and assume that $\left(t^\epsilon, (f_\epsilon)_*^{-1}(q^\epsilon(0), p^\epsilon(0))\right)$ has a limit $(t^0, q^0(0), p^0(0)) \in A$ as $\epsilon \to 0$. The geodesic of Σ_ϵ starting at $(q^\epsilon(0), p^\epsilon(0))$ is written $(q^\epsilon(t), p^\epsilon(t))_{t \in \mathbb{R}}$. We want to show that $\pi_*(q^\epsilon(t^\epsilon), p^\epsilon(t^\epsilon))$ tends to $\Psi_{t^0} \circ \pi_*(q^0(0), p^0(0))$. Since the billiard trajectory $\Psi_t \circ \pi_*(q^0(0), p^0(0))$ experiences only a finite number of bounces in any finite time interval, we may assume that the trajectory for $t \in [0, t^0]$ has only one bounce[4], at a time t_c. As in Proposition 7.11, let $(q^0(t), p^0(t))$ be the (unique) pullback of this trajectory by π in Σ_ϵ for $t \in [0, t_c]$, and let $q_c = q^0(t_c)$. Assume that $N(q_c)$ is directed towards the exterior of the billiard table D, and (up to a rotation of axis e_z) that $N(q_c) = -e_y$. The geodesic $(q^\epsilon(t), p^\epsilon(t))$ enters Z_δ at some time t_{in}^ϵ and exits at some time t_{out}^ϵ, and the only difficulty to prove the convergence is located between these two times, since g_ϵ converges uniformly to a flat metric outside Z_δ.

Since $p_x^0(0) < 1$, Proposition 7.11 shows that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} |p_x(t_{\text{out}}) - p_x(t_{\text{in}})| = 0.$$

Moreover, for all $\delta > 0$, $\lim_{\epsilon \to 0} p_z(t_{\text{in}}) = \lim_{\epsilon \to 0} p_z(t_{\text{out}}) = 0$, and since the geodesic has unit speed,

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} |p_y(t_{\text{in}})| = \lim_{\delta \to 0} \lim_{\epsilon \to 0} |p_y(t_{\text{out}})| .$$

We have already seen that the geodesic enters Z_δ with $p_y < 0$ and exits with $p_y > 0$. Thus:

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} p_y(t_{\text{in}}) = -\lim_{\delta \to 0} \lim_{\epsilon \to 0} p_y(t_{\text{out}}).$$

[4] If the billiard trajectory has no bounce at all, then the geodesic remains outside of Z_δ and the convergence is clear.

Proposition 7.11 also states that $\lim_{\delta\to 0}\lim_{\epsilon\to 0}|t_{\text{out}}-t_{\text{in}}|=0$.

Thus, the limiting trajectory satisfies the billiard reflection law and Theorem 7.1 is proved. □

7.4 Proof of Theorem 7.2

In this section, the walls of the billiard are assumed to be concave, and the billiard has *finite horizon*. The following lemma gives an important consequence of the second property.

Lemma 7.12. *Let D be a billiard in $\mathbb{T}^2$ whose walls are negatively curved. Assume that D has finite horizon (D contains no geodesic of $\mathbb{T}^2$ with infinite lifetime in the past and the future). Then, there is an $\eta>0$, a time $t^{\max}$ and an angle ϕ_0 such that every curve of length $t^{\max}$ in $\mathbb{T}^2$, which is η-close to a straight line in the C^1 metric, hits at least once the boundary with an angle $\geq\phi_0$.*

Proof. Assume that the conclusion of the lemma is false. Then there are curves $\Gamma_n:[-n,n]\to\mathbb{T}^2$ which do not hit the boundary with an angle greater than $\frac{1}{n}$, and which are $\frac{1}{n}$-close to geodesics in the C^1 metric. By a diagonal argument, one may extract a subsequence which converges to a geodesic $\Gamma:\mathbb{R}\to\mathbb{T}^2$ which does not hit the boundary with an angle greater than 0, so that Γ remains in D. □

As another consequence of the concavity of the walls, we may assume that the principal curvatures satisfy $\left|\gamma_-^\epsilon\right|\leq\left|\gamma_+^\epsilon\right|$ in Z_δ (with Proposition 7.8). We write $\kappa(\delta,\epsilon)=max_{q\notin Z_\delta}\left(\left|\gamma_+^\epsilon(q)\right|,\left|\gamma_-^\epsilon(q)\right|\right)$. Notice that for all $\delta>0$, $\kappa(\delta,\epsilon)\underset{\epsilon\to 0}{\to}0$. Later we will simply write κ for $\kappa(\delta,\epsilon)$. We also define

$$W_\kappa=\{q\in\Sigma_\epsilon\mid K(q)\leq-\kappa\}.$$

Notice that for any fixed $\delta>0$, there exists $\epsilon_0>0$ such that for $\epsilon\leq\epsilon_0$, $W_\kappa\subseteq Z_\delta$.

In the following proposition, we determine what remains of Proposition 7.11 when the geodesics are not assumed to be nongrazing, but when instead they are assumed to undergo little curvature. We will consider geodesics $(q^\epsilon(t),p^\epsilon(t))_{t\in\mathbb{R}}$ in Σ_ϵ, for some $\epsilon>0$. For such a geodesic, define t_{in}^ϵ as the first time at which the geodesic $(q^\epsilon(t),p^\epsilon(t))$ enters Z_δ: here, we will will say that a geodesic is *admissible* if this time exists with $t_{\text{in}}^\epsilon+\sqrt[3]{\delta}<1$. As before, we will choose the orientation of the normal vector N such that it points towards the exterior of the billiard table at its boundary near q_{in}, and then make a rotation of axis e_z, such that $N_x(q_{\text{in}})=0$ and $N_y(q_{\text{in}})<0$.

Proposition 7.13. *For all $m>0$, there exists $\delta_0>0$, such that for all $\delta\leq\delta_0$, there exists $\epsilon_0>0$, such that for all $\epsilon\leq\epsilon_0$ and each admissible geodesic (q^ϵ,p^ϵ) such that $\int_0^1|K(q^\epsilon(t))|\leq 3\kappa^2$:*

1. $\inf_{t\in[0,t_{\text{in}}+\sqrt[3]{\delta}]}(p_y(t)-p_y(0))\geq-2\sqrt{\kappa}$;
2. $\sup_{t\in[0,t_{\text{in}}+\sqrt[3]{\delta}]}|p_x(t)-p_x(0)|\leq m$;
3. *denoting by Z_δ^0 the connected component of Z_δ containing $q(t_{\text{in}})$, there exists $t_{\text{out}}\in[t_{\text{in}},t_{\text{in}}+\sqrt[3]{\delta}]$, at which the geodesic exits Z_δ^0, and does not come back to Z_δ^0 before visiting another component of Z_δ.*

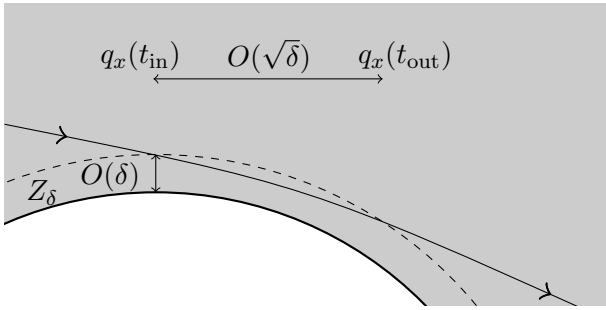


Figure 7.3 – The geodesic exits Z_δ before a time which is $O(\sqrt{\delta})$, because $p_y(t) - p_y(0)$ is bounded from below by a small negative number.

Proof. As before, we only need to consider what happens for $t \geq t_{\text{in}}$, as the metric tends to a flat metric outside Z_δ.

To prove the first statement, writing $\dot{p_y} = \dot{p_y}^+ - \dot{p_y}^-$, with $\dot{p_y}^+ = \max(\dot{p_y}, 0)$ (positive part) and $\dot{p_y}^-(t) = -\min(\dot{p_y}, 0)$ (negative part), it suffices to show that $\int_{t_{\text{in}}}^t \dot{p_y}^-$ is (uniformly) close to 0. We divide this integral into two parts. In the part where $q \in W_\kappa$, the quantity $\dot{p_y} = -N_y(q) \langle DN(q) \cdot p \mid p \rangle$ is bigger than $-\sqrt{K(t)}$ (because $\left|\gamma_-^\epsilon\right| \leq \left|\gamma_+^\epsilon\right|$), so it is bigger than $-K(t) - 1$. The time spent in W_κ is smaller than $3\kappa^2/\kappa$, and the integral of $|K(t)|$ is smaller than $3\kappa^2$. In the part where $q \notin W_\kappa$, $\langle DN(q) \cdot p \mid p \rangle$ is bigger than $-\sqrt{\kappa}$. Thus, $\int_{t_{\text{in}}}^t \dot{p_y}^- \leq 3\kappa^2 + 3\kappa + \sqrt{\kappa}(t_{\text{in}} + \sqrt[3]{\delta}) \leq 2\sqrt{\kappa}$, and Statement 1 is proved.

For all $t \in [0, \sqrt[3]{\delta}]$, we may write, as in the proof of Proposition 7.11:

$$|p_x(t) - p_x(t_{\text{in}})| \leq \left(\sup_{B(q(t_{\text{in}}), \sqrt[3]{\delta})} \left| \frac{N_x}{N_y} \right| \right) \int_{t_{\text{in}}}^t |\dot{p_y}| \,.$$

Since $\frac{N_x}{N_y}$ does not depend on ϵ, $\left(\sup_{B(q(t_{\text{in}}), \sqrt[3]{\delta})} \left|\frac{N_x}{N_y}\right|\right)$ is close to 0. Moreover,

$$\int_{t_{\text{in}}}^t |\dot{p_y}| = \int_{t_{\text{in}}}^t \left(\dot{p_y} + 2\dot{p_y}^-\right) \leq |p_y(t) - p_y(t_{\text{in}})| + 2 \int_{t_{\text{in}}}^t \dot{p_y}^- .$$

The term $|p_y(t) - p_y(t_{\text{in}})|$ is bounded by 2, and $\int_{t_{\text{in}}}^t \dot{p_y}^-$ is close to 0, so $\int_{t_{\text{in}}}^t |\dot{p_y}|$ is bounded, which proves Statement 2.

Finally, to prove Statement 3, fix any $\alpha \in (0, 1)$. Lemma 7.11 implies that all trajectories such that $|p_x(0)| < \alpha$ exit Z_δ before $t = t_{\text{in}} + \sqrt{\delta}$. For the other trajectories, Statement 2 implies that p_x remains bounded away from 0. Together with Statement 1 and the uniform concavity of the walls of the billiard table, this implies that the geodesic must exit Z_δ^0 definitively before a time which is $O(\sqrt{\delta})$ (see Figure 7.3). □

Lemma 7.14. *For all $m > 0$, there exists $\epsilon_0 > 0$, such that for all $\epsilon \leq \epsilon_0$, and all geodesics $(q^\epsilon(t), p^\epsilon(t))_{t \in \mathbb{R}}$ such that $\int_0^1 |K(q^\epsilon(t))| \leq 3\kappa^2$,*

$$\sup_{t \in \left[\frac{1}{3}, \frac{2}{3}\right]} \|p(t) - p(1/3)\| \leq m.$$

In particular, the choice of ϵ_0 does not depend on the choice of the geodesic.

Proof. Outside Z_δ, $\dot{p_x}$ vanishes as $\epsilon \to 0$. Each time that the geodesic enters or exits Z_δ, Proposition 7.11 implies (with the choice of $\alpha \in (0,1)$ close to 1) that p is nearly tangent to the boundary of the billiard table (otherwise, the geodesic undergoes strong negative curvature after the entry or before the exit, which is why we consider only the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$). Moreover, Proposition 7.13 implies that the time spent in Z_δ is small. Thus, the exit point is near the entrance point and, from Statement 2 of Proposition 7.13, the speed vector p is almost preserved. Then, the geodesic goes to visit another component of Z_δ, so there is an upper bound on the number of times that it enters Z_δ. Thus, the total change in p is uniformly small. □

End of the proof of Theorem 7.2. To show that the flow has the Anosov property, we consider a small δ, a small ϵ, and a geodesic $(q^\epsilon(t), p^\epsilon(t))_{t\in\mathbb{R}}$ in Σ_ϵ, and examine the Ricatti equation:

$$\begin{cases} u(0) = 0 \\ u'(t) = -K^\epsilon(q^\epsilon(t)) - u^2(t). \end{cases}$$

It suffices to show that $u(1)$ is positive and bounded away from 0, uniformly with respect to the choice of the geodesic (see Theorem 6.3). In the following, we write $K(t) := K^\epsilon(q^\epsilon(t))$.

Applying a homothety to Σ if necessary, we may assume that $t^{\max}$ given by Lemma 7.12 is less than $\frac{1}{3}$. If $\int_0^1 |K(t)|\, dt \leq 3\kappa^2$, then Lemma 7.14 tells us that, for any small enough ϵ, $(\pi_*(q(t), p(t)))_{t\in\left[\frac{1}{3},\frac{2}{3}\right]}$ is C^1-close to a straight line in $\mathbb{T}^2$, which contradicts Lemma 7.12. Thus, there exists $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$, $\int_0^1 |K(t)|\, dt \geq 3\kappa^2$. Since $K \leq 0$ in Z_δ and $|K| \leq \kappa^2$ outside Z_δ, we deduce that $K \leq \kappa^2$ in Σ_ϵ. Therefore, considering the positive and negative parts of K,

$$\int_0^1 K = \int_0^1 (K^+ - K^-) = -\int_0^1 |K| + 2\int_0^1 K^+ \leq -3\kappa^2 + 2\kappa^2 \leq -\kappa^2.$$

Now, let us show that $u(1) \geq \kappa^2/2$, which will end the proof. To do this, we assume that $u(1) < \kappa^2/2$ and show that for all $t \in [0,1]$, $|u(t)| \leq 2\kappa^2$. Let $t^1 = \sup\left\{t \in [0,1] \;\middle|\; u(t) \geq 2\kappa^2\right\}$ (or $t^1 = 0$ if this set is empty). For $t \in [t^1, 1]$, $u'(t) = -K(t) - u^2(t) \geq -K(t) - 4\kappa^4$, so $u(1) - u(t^1) \geq -\int_{t^1}^1 K(t)dt - 4\kappa^4$, whence

$$u(t^1) \leq u(1) + \int_{t^1}^1 K(t)dt + 4\kappa^4 \leq \kappa^2/2 + \kappa^2 + 4\kappa^4 < 2\kappa^2.$$

This implies, with the definition of t^1, that $t^1 = 0$ and $u(t) \leq 2\kappa^2$ for all $t \in [0,1]$. Thus,

$$u(1) \geq u(0) - \int_0^1 K(t) - 4\kappa^4 \geq \kappa^2 - 4\kappa^4 \geq \kappa^2/2,$$

a contradiction. This ends the proof of Theorem 7.2. □

Chapter 8

Dynamics of linkages

8.1 Introduction

In this chapter, we consider the physical behavior of linkages when they are given an initial speed, without any external force and without friction. We will only consider linkages such that $\mathrm{Conf}(\mathcal{L})$ is a smooth manifold in $(\mathbb{R}^2)^n$. It is the case for a generic choice of the edge lengths (see Proposition 1.3). Of course, the dynamics depend on the distribution of the masses in the system: to simplify the problem, we will assume that the masses are all concentrated at the vertices of the graph. If one denotes the speed of each vertex by v_i, and the masses by m_i, the *principle of least action* (see [Arn78]) states that the trajectory between two times t_0 and t_1 will be a critical point of the kinetic energy

$$\mathcal{K} = \frac{1}{2}\int_{t_0}^{t_1} \sum_{i=1}^{n} m_i v_i^2(t)dt,$$

which is also a characterization of the geodesics in the manifold $\mathrm{Conf}(\mathcal{L})$ endowed with a suitable metric:

Fact 8.1. *The physical behavior of the linkage $\mathcal{L}$, when it is isolated and given an initial speed, is the geodesic flow on $\mathrm{Conf}(\mathcal{L}) \subseteq (\mathbb{R}^2)^n = \mathbb{R}^{2n}$, endowed with the metric:*

$$g = \sum_{i=1}^{n} m_i(dx_{2i-1}^2 + dx_{2i}^2),$$

provided that the metric g is nondegenerate. In particular, if all the masses are equal to 1, g is the metric induced by the Euclidean $\mathbb{R}^{2n}$.

Anosov behavior. We ask the following:

Question. Do there exist linkages with Anosov behavior?

The following theorem gives a theoretical answer to this question.

Theorem 8.2. *Let (M, g) be any connected compact Riemannian manifold and $1 \leq k < +\infty$. Then there exists a linkage $\mathcal{L}$, a choice of masses, and a Riemannian metric h on M, such that h is C^k-close to g and every connected component of $\mathrm{Conf}(\mathcal{L})$ is isometric to (M, h).*

Proof. Embed (M, g) isometrically in some $\mathbb{R}^{2n}$: this is possible by a famous theorem of Nash [Nas56]. With another theorem of Nash and Tognoli (see [Tog73], and also [Iva82], page 6, Theorem 1), this surface is C^k-approximated by a smooth algebraic set A in $\mathbb{R}^{2n}$, which is naturally equipped with the metric induced by $\mathbb{R}^{2n}$. The manifold A is diffeomorphic to M, and even isometric to (M, h) where the metric h is C^k-close to g. We may now use Theorem 2.2: any compact algebraic set $B \subseteq \mathbb{R}^{2n}$ is exactly the partial configuration space of some linkage, that is, the set of the possible positions of a subset of the vertices; moreover, if B is a smooth submanifold of $\mathbb{R}^{2n}$, each connected component of the whole configuration space may be required to be smooth and diffeomorphic to B. Thus, there is a linkage and a subset of the vertices W such that the partial configuration space of W is A: each component of the configuration space of this linkage, with masses 1 for the vertices in W and 0 for the others, is isometric to the algebraic set A endowed with the metric induced by $\mathbb{R}^{2n}$, which is itself isometric to (M, h). □

In particular, there exist configuration spaces with negative sectional curvature, and thus with Anosov behavior. This answer is somewhat frustrating, as it is difficult to construct such a linkage with this method in practice, and it would have a high number of vertices anyway, at least several hundreds.

In the 1980's, Thurston and Weeks [TW84] pointed out that the configuration spaces of quite simple linkages could have an interesting topology, by introducing the famous *triple linkage* (see Section 1.1.3 for more details): they showed that, for some choice of the lengths, its configuration space could be a compact orientable surface of genus 3. Later, Hunt and MacKay [HM03] found out that this particular linkage also had quite interesting dynamics.

Asymptotic configuration spaces. The computation of the curvature of a given configuration space is impossible in practice, most of the time. Thus, the idea of Hunt and MacKay was to give a small length ϵ to some edges[1], while the masses are fixed (0 for the vertex at the center and 1 for the others), and to consider the limit of $\mathrm{Conf}(\mathcal{L}_\epsilon)$ as $\epsilon \to 0$. At the limit, the surface is not the configuration space of a physical system anymore (it is called an *asymptotic configuration space*), but it is easier to study because the equations are simpler. In the case of the triple linkage, the miracle is that the limit surface is Schwarz's well-known "P surface" in $\mathbb{T}^3$, defined by $\sum_{i=1}^{3} \cos x_i = 0$, which has negative curvature except at a finite number of points, and thus an Anosov geodesic flow (with Theorem 6.4). The structural stability of Anosov flows allows the authors to conclude that the configuration space of $\mathcal{L}_\epsilon$ for a small enough ϵ has an Anosov geodesic flow. In particular, one does not know how small ϵ has to be for $\mathcal{L}_\epsilon$ to be an Anosov linkage.

This technique may be applied to other linkages. For example, in 2013, Pollicott and Magalhães [MP13] tried to see what happened with the asymptotic "double linkage", an equivalent of the triple linkage but with only two articulated arms (also called "pentagon"). But the asymptotic configuration space in that case has both positive and negative curvature and it is impossible to conclude that the geodesic flow is Anosov, although their computer simulation suggests that it should be the case. In fact, since Hunt and MacKay's example, no other linkage has been proved mathematically to be Anosov.

[1] with the notations of Figure 1.3, $l_1 = \epsilon$ and $l_2 = 1$

Linkages and billiards. To understand the link between linkages and billiards, consider Thurston's triple linkage, where all vertices have mass 0 except the central vertex which has mass 1. The workspace of the central vertex is a hexagon (see Figure 1.4), and its trajectories are obviously straight lines in the interior of the workspace, but what happens physically when the vertex hits the boundary of the workspace? It turns out that it reflects by a billiard law. In fact, when the masses of the non-central vertices are small, the configuration space is equipped with the metric of a flattened surface Σ_ϵ with a small $\epsilon > 0$, in the spirit of Chapter 7.

However, it may happen that the workspace of some vertex is a dispersive billiard, while the geodesic flow in the configuration space (with a small parameter ϵ) is not Anosov. For example, consider Thurston's triple linkage in the case on the right of Figure 1.4. Then the workspace of the central vertex x is a non-smooth dispersive billiard – a triangle with negatively curved walls – but the configuration space is topologically a sphere, so its geodesic flow cannot be Anosov. In fact, the corners of the billiard concentrate the positive curvature of the configuration space when it flattens.

A new Anosov linkage. In the rest of this chapter, our aim is to give a new example of an Anosov linkage (see Figures 8.1 and 8.2) by applying Theorem 7.2. In this example, the billiard is not the workspace of a single vertex: it is the partial configuration space of four vertices, that is, the set of the possible positions of these vertices. It is *a priori* a subset of $(\mathbb{R}^2)^4$, but in this particular case, it turns out that it may be seen as a subset of $\mathbb{T}^2$. The configuration space $\mathrm{Conf}(\mathcal{L})$, in turn, may be seen as an immersed surface in $\mathbb{T}^2 \times \mathbb{R}$ which flattens to the billiard table as one of the masses tends to 0 (Figure 8.3).

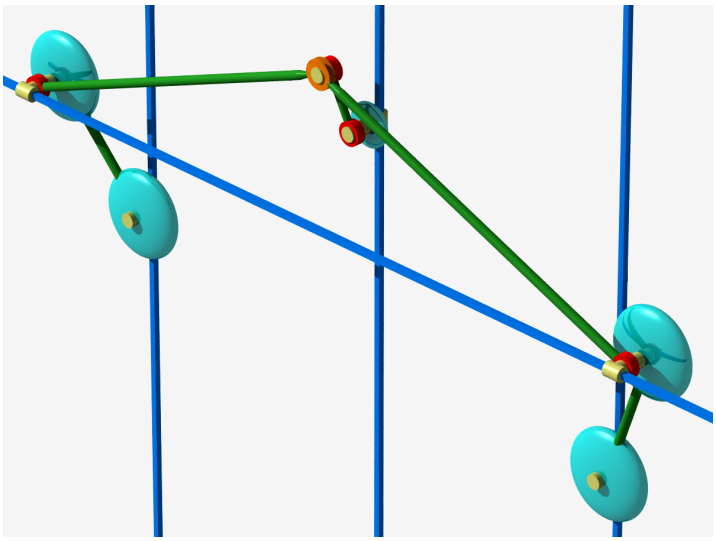


Figure 8.1 – A physical realization of our Anosov linkage.

Notice that this example is *not* an asymptotic linkage, in the following sense: there is a whole explicit range of values for the edge lengths such that the linkage has an Anosov behavior. This is the first time that a linkage with explicit lengths is proved to be Anosov.

However, one mass has to be close to 0 and our theorem does not say explicitly how close it has to be. Maybe this linkage is, in fact, Anosov even when the mass is equal

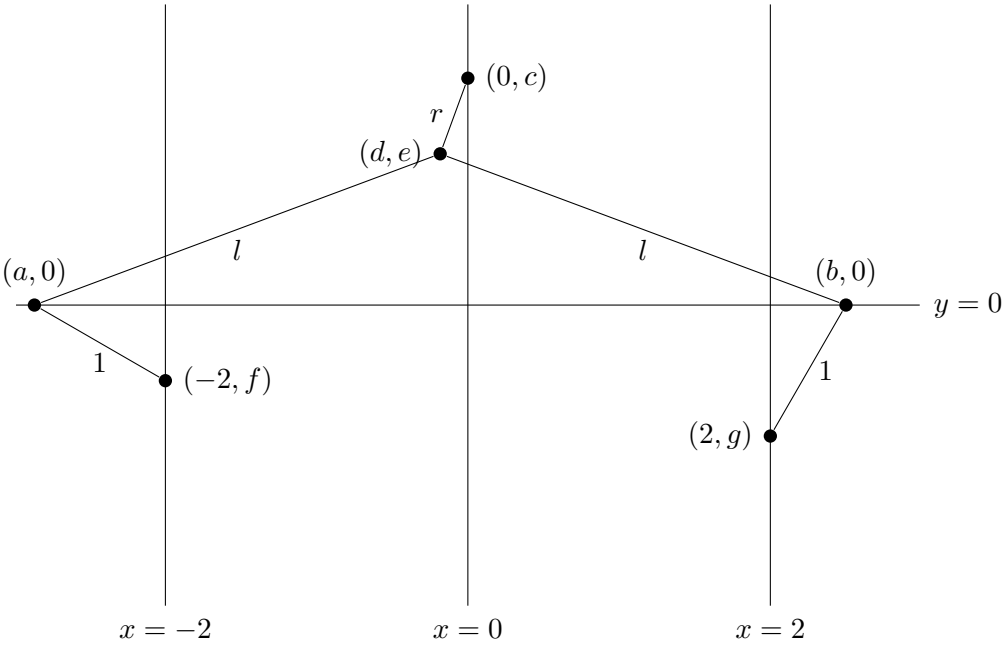


Figure 8.2 – Mathematical description of our Anosov linkage.

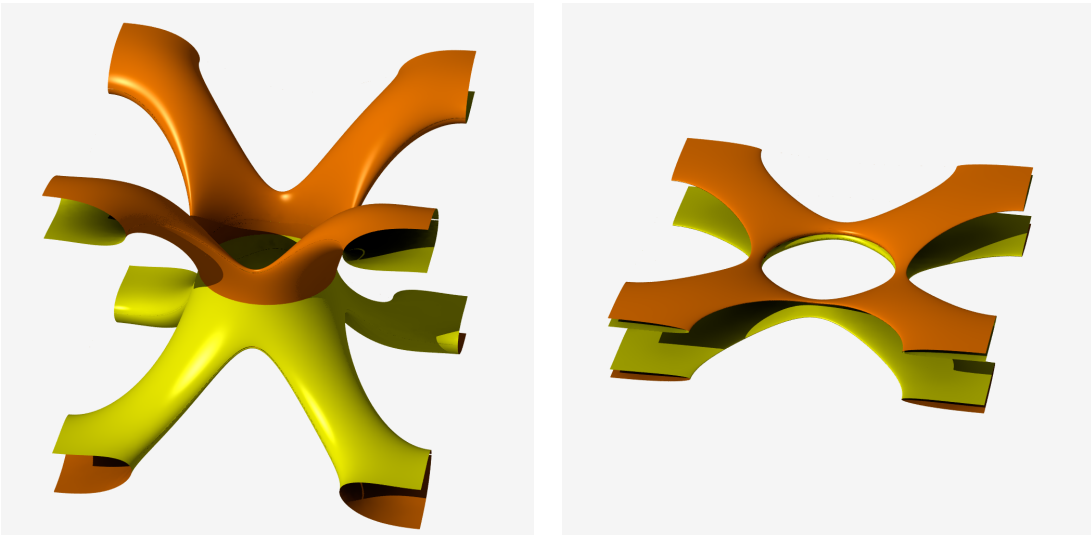


Figure 8.3 – On the left, the configuration space of our linkage in $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is horizontal and $\mathbb{R}$ vertical. It is a surface of genus 7, with a self-intersection (at the center of the picture). On the right, the flattened configuration space, which is close to a billiard table with finite horizon (see also Figure 8.5).

to 1. Notice that the masses in Hunt and MacKay's example are also unexplicit.

In our example, no vertex is fixed, but all the vertices (except one) have only one degree of freedom and move on a straight line. This may be realized physically by *prismatic joints* – or, if one wants to stick to the traditional definition of linkages, it is possible to use Peaucellier's straight line linkage, or to approximate the straight lines by

portions of arcs of large radius.

Precisely, we prove the following:

Theorem 8.3. *In the linkage of Figure 8.2, choose the lengths of the rods such that* $l + r > 3$, $l < 3$, $(l-2)^2 + r^2 < 1$ *and* $r < 1/2$. *The mass at* $(a, 0), (-2, f), (b, 0)$ *and* $(2, g)$ *is* 1, *the mass at* $(0, c)$ *is* ϵ^2, *while the mass at* (d, e) *is* 0. *Then for any sufficiently small* $\epsilon > 0$, *the geodesic flow on the configuration space of the linkage is Anosov.*

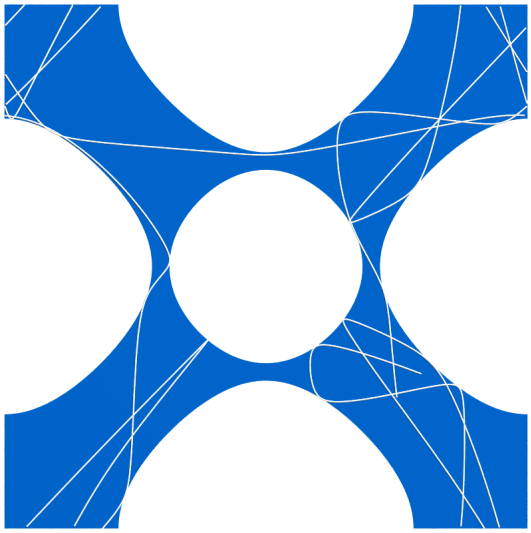


Figure 8.4 – The projection of the geodesics of the configuration space onto the billiard table. Theorem 7.1 states that the nongrazing geodesics are close to billiard trajectories. The behavior of grazing trajectories, described by Figure 7.1, is clearly visible here.

A realistic physical system. Similarly to Hunt and MacKay [HM03], we insist on the fact that our linkage is realistic from a physical point of view. For example, it is possible to add small masses to the rods and to the central vertex without losing the Anosov property (using the structural stability of Anosov flows). See Hunt and MacKay's article for more details about this aspect.

8.2 Proof of Theorem 8.3

The aim of this section is to prove that the configuration space of the linkage described in Theorem 8.3 is isometric to an immersed surface in $\mathbb{T}^2 \times \mathbb{R}$ which satisfies the 4 assumptions of Theorem 7.2.

The configuration space $\mathrm{Conf}(\mathcal{L})$ is the set of all $(a, b, c, d, e, f, g) \in \mathbb{R}^7$ such that:

$$(a+2)^2 + f^2 = (b-2)^2 + g^2 = 1;$$
$$(a-d)^2 + e^2 = (b-d)^2 + e^2 = l^2;$$
$$d^2 + (c-e)^2 = r^2.$$

Notice that $(a+2, f)$ and $(b-2, g)$ lie in the unit circle $\mathbb{T} \subseteq \mathbb{R}^2$. Thus, $\mathrm{Conf}(\mathcal{L})$ is in fact a subset of $\mathbb{T}^2 \times \mathbb{R}^3$ and any of its elements may be written (θ, ϕ, c, d, e), with the identification $a = -\cos\theta - 2$, $f = \sin\theta$, $b = \cos\phi + 2$, $g = \sin\phi$.

Fact 8.4. *For all $\mathcal{C}_0 \in \mathrm{Conf}(\mathcal{L})$ such that $e \neq 0$ and $e \neq c$, $\mathrm{Conf}(\mathcal{L})$ is locally a smooth graph above θ and ϕ near $\mathcal{C}_0$. More precisely, there exists a neighborhood U of $\mathcal{C}_0$ in $\mathrm{Conf}(\mathcal{L})$, an open set V of $\mathbb{T}^2$ and a smooth function $F : V \to \mathbb{R}^3$ such that*

$$U = \{(\mathcal{D}, F(\mathcal{D})) \mid \mathcal{D} \in V\}.$$

Proof. The function F is given by the following formulae:

$$d = \frac{-\cos\theta + \cos\phi}{2};$$
$$e = \pm\sqrt{l^2 - \left(\frac{\cos\theta + \cos\phi}{2} + 2\right)^2} = \pm\sqrt{l^2 - \left(\frac{\cos\theta + \cos\phi + 4}{2}\right)^2};$$
$$c = e \pm \sqrt{r^2 - \left(\frac{\cos\theta - \cos\phi}{2}\right)^2} = \pm\sqrt{l^2 - \left(\frac{\cos\theta + \cos\phi + 4}{2}\right)^2} \pm \sqrt{r^2 - \left(\frac{\cos\theta - \cos\phi}{2}\right)^2}$$

where the choices of the signs are made according to $\mathcal{C}_0$. □

Fact 8.5. *a) For all $\mathcal{C}_0 \in \mathrm{Conf}(\mathcal{L})$ such that $(-\cos\theta - 2, 0)$, (d, e) and $(0, c)$ are not aligned, and such that $\phi \neq 0 \mod \pi$, $\mathrm{Conf}(\mathcal{L})$ is locally a smooth graph above θ and c near $\mathcal{C}_0$.*

b) For all $\mathcal{C}_0 \in \mathrm{Conf}(\mathcal{L})$ such that $(\cos\phi + 2, 0)$, (d, e) and $(0, c)$ are not aligned, and such that $\theta \neq 0 \mod \pi$, $\mathrm{Conf}(\mathcal{L})$ is locally a smooth graph above ϕ and c near $\mathcal{C}_0$.

Proof. By symmetry, we only need to prove the first statement. The idea of the proof is the same as for Fact 8.4: on the one hand, the numbers d and e are obtained as the simple roots of a polynomial of degree 2, so they vary smoothly with respect to θ and c; on the other hand, $\phi = \pm\arccos(2d + \cos\theta)$ where the choice of the sign is made according to $\mathcal{C}_0$. □

Fact 8.6. *For all $\mathcal{C}_0 \in \mathrm{Conf}(\mathcal{L})$, $\mathrm{Conf}(\mathcal{L})$ is locally a smooth graph near $\mathcal{C}_0$:*

1. *either above θ and ϕ,*
2. *or above θ and c,*
3. *or above ϕ and c.*

Proof. Assume the opposite. Then the hypotheses of Fact 8.5 are not satisfied. If $\phi = \theta = 0 \mod \pi$, then $\phi = \theta \mod 2\pi$ because $r < 1/2$; then $\phi = \theta = \pi \mod 2\pi$ because $l < 3$, but this implies that $e \neq 0$ and $e \neq c$, so Fact 8.4 applies, which is impossible. Therefore, by symmetry, we may assume that $(-\cos\theta - 2, 0)$, (d, e) and $(0, c)$ are aligned. Now, with Fact 8.4, we have either $e = 0$ or $e = c$. In both cases, $(-\cos\theta - 2, 0)$, (d, e) and $(0, c)$ are all on the line $y = 0$, which contradicts the fact that $l + r > 3$. □

Fact 8.6 implies in particular that $\mathrm{Conf}(\mathcal{L})$ is a smooth submanifold of $\mathbb{T}^2 \times \mathbb{R}^3$.

As explained in the introduction of the chapter, $\mathrm{Conf}(\mathcal{L})$ is endowed with the metric which corresponds to its kinetic energy (recall that the masses of the vertices are ϵ^2 at $(0, c)$, 0 at (d, e), and 1 everywhere else):

$$g_\epsilon = da^2 + df^2 + db^2 + dg^2 + \epsilon^2 dc^2 = d\theta^2 + d\phi^2 + \epsilon^2 dc^2.$$

Fact 8.6 shows that the metric g_ϵ is nondegenerate (although it is induced by a degenerate metric of $\mathbb{T}^2 \times \mathbb{R}^3$!), so with Fact 8.1 the physical behavior of the linkage is the geodesic flow on $(\mathrm{Conf}(\mathcal{L}), g_\epsilon)$. Our aim is to show that it is an Anosov flow by applying Theorem 7.2.

Consider the projection onto the first coordinates:

$$\begin{aligned} p : \mathbb{T}^2 \times \mathbb{R}^3 &\to \mathbb{T}^2 \times \mathbb{R} \\ (\theta, \phi, c, d, e) &\mapsto (\theta, \phi, c). \end{aligned}$$

Again with Fact 8.6, $p|_{\mathrm{Conf}(\mathcal{L})}$ is an immersion: $\mathrm{Conf}(\mathcal{L})$ is isometric to a smooth surface Σ immersed in $\mathbb{T}^2 \times \mathbb{R}$, endowed with the metric $g_\epsilon = d\theta^2 + d\phi^2 + \epsilon^2 dc^2$. We shall now call z the third coordinate instead of c, to be consistent with the notations of Theorem 7.2.

Denote by $\pi : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$ the projection onto the first coordinates. The surface Σ projects to a smooth billiard table:

$$D = \pi(\mathrm{Conf}(\mathcal{L})) = \left\{(\theta, \phi) \in \mathbb{T}^2 \;\middle|\; |\cos\theta - \cos\phi| \leq 2r, \quad \cos\theta + \cos\phi \leq 2l - 4\right\}$$

Its boundary has three connected components in $\mathbb{T}^2$: $\{\cos\theta - \cos\phi = 2r\}$, $\{-\cos\theta + \cos\phi = 2r\}$, and $\{\cos\theta + \cos\phi = 2l - 4\}$.

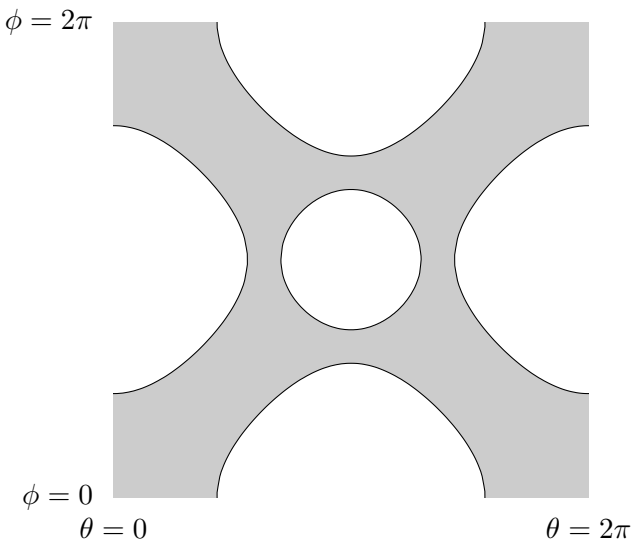


Figure 8.5 – The billiard table D (in grey) for $r = 0.4$ and $l = 2.8$. The billiard has negatively curved walls, which means that the obstacles are strictly convex.

There remains to show that the immersed surface Σ satisfies the 4 assumptions of Theorem 7.2. Assumption 1 is satisfied as a direct consequence of Fact 8.4. The following proposition proves Assumption 2.

Proposition 8.7. *For all $q \in \pi^{-1}(\partial D) \cap \Sigma$, the curvature of $\Sigma \cap V$ is nonzero at q, where V is a neighborhood of q in the affine plane $q + \mathrm{Vect}(e_z, (T_q\Sigma)^\perp)$.*

Proof. Here we will assume that $\pi(q) \in \{(\theta,\phi) \in \mathbb{T}^2 \mid \cos\phi + \cos\theta = 2l-4\}$, but the proof is identical for the other components of ∂D. Let $F(\theta,\phi) = \left(\frac{\cos\theta+\cos\phi+4}{2}\right)^2$. For any small $t \geq 0$, let $\theta(t) = q_\theta + N_\theta(q)t$, $\phi(t) = q_\phi + N_\phi(q)t$, and choose $z(t)$ of the form:

$$z(t) = \pm\sqrt{l^2 - \left(\frac{\cos\theta(t)+\cos\phi(t)+4}{2}\right)^2} \pm \sqrt{r^2 - \left(\frac{\cos\theta(t)-\cos\phi(t)}{2}\right)^2}$$

with a choice of the $\pm$ signs so that $(\theta(0),\phi(0),z(0)) = q$. Then for all small $t \geq 0$, $(\theta(t),\phi(t),z(t)) \in \Sigma$.

As t tends to 0, we may estimate:

$$z(t) = \pm\sqrt{\left(\frac{d}{dt}\bigg|_{t=0} F(\theta(t),\phi(t))\right)t + O(t^2)} \pm \sqrt{r^2 - \left(\frac{\cos\theta(0)-\cos\phi(0)}{2}\right)^2} + O(t)$$

$$(z(t)-z(0))^2 = \pm\left(\frac{d}{dt}\bigg|_{t=0} F(\theta(t),\phi(t))\right)t + o(t).$$

Notice that $\nabla F(\theta(0),\phi(0)) = -\begin{pmatrix}\sin\theta(0)\\ \sin\phi(0)\end{pmatrix}\left(\frac{\cos\theta+\cos\phi+4}{2}\right)$ is nonzero (because $2 < l < 3$). Moreover, $\begin{pmatrix}\theta'(0)\\ \phi'(0)\end{pmatrix}$ is $\begin{pmatrix}N_\theta(q)\\ N_\phi(q)\end{pmatrix}$, which is colinear to $\nabla F(\theta(0),\phi(0))$, so

$$\frac{d}{dt}\bigg|_{t=0} F(\theta(t),\phi(t)) = \begin{pmatrix}\theta'(0)\\ \phi'(0)\end{pmatrix} \cdot \nabla F(\theta(0),\phi(0)) \neq 0.$$

This gives us:

$$t = \pm\frac{1}{\frac{d}{dt}\big|_{t=0} F(\theta(t),\phi(t))}(z(t)-z(0))^2 + o((z(t)-z(0))^2).$$

Hence, $t \mapsto z(t)$ has an inverse function $z \mapsto t(z)$ which has a nonzero second derivative at $t = 0$. Since (z,t) are the coordinates in an affine (orthonormal) basis of $q + \mathrm{Vect}(e_z, (T_q\Sigma)^\perp)$, this implies that $\Sigma \cap V$ has nonzero curvature at q. □

The following proposition proves Assumption 3.

Proposition 8.8. *The walls of the billiard D have negative curvature.*

Proof. In general, the curvature of the boundary of a set defined by the inequality $F(q) \leq C$, where $C \in \mathbb{R}$ is a constant, with the normal vector pointing inwards, is the divergence of the normalized gradient of F, namely:

$$\nabla \cdot \frac{\nabla F}{\|\nabla F\|}.$$

First consider the boundary of the set $\{\cos\phi + \cos\theta \leq 2l-4\}$. Here $F(\phi,\theta) = \cos\phi + \cos\theta$. Thus:

$$\frac{\nabla F}{\|\nabla F\|} = \frac{-1}{\sqrt{\sin^2\theta + \sin^2\phi}}\begin{pmatrix}\sin\theta\\ \sin\phi\end{pmatrix}.$$

Hence, the divergence of the normalized gradient has the same sign as:

$$-\sin^2\phi\cos\theta - \sin^2\theta\cos\phi$$

which can be rewritten:

$$-(2l-4)\cos^2\theta + (2l-4)^2\cos\theta - (2l-4).$$

This is a second order polynomial in $\cos\theta$ with discriminant $(2l-4)^2((2l-4)^2-4) < 0$ (here we use the assumption $l < 3$). Since $(2l-4) > 0$ (because $l > 2$), the polynomial is everywhere negative.

Now, consider the boundary of the set $\{\cos\phi - \cos\theta \leq 2r\}$. This time, the divergence of the normalized gradient has the same sign as

$$\sin^2\phi\cos\theta - \sin^2\theta\cos\phi$$

which can be rewritten

$$-2r\cos^2\theta - 4r^2\cos\theta - 2r.$$

This time, the discriminant is $16r^2(r^2-1)$, which is negative since $r < 1$.

The third wall is the boundary of the set $\{\cos\theta - \cos\phi \leq 2r\}$. The divergence of the normalized gradient has the same sign as

$$-\sin^2\phi\cos\theta + \sin^2\theta\cos\phi$$

which can be rewritten

$$-2r\cos^2\theta + 4r^2\cos\theta - 2r.$$

Again, the discriminant is $16r^2(r^2-1)$, which is negative. □

Finally, we prove Assumption 4, which will end the proof of Theorem 8.3.

Proposition 8.9. *If $(l-2)^2 + r^2 < 1$ and $r < 1/2$, then D has finite horizon.*

Proof. Assume that there exists a geodesic $(\theta(t), \phi(t))$ with infinite lifetime in the past and in the future.

First, we prove that the slope of the geodesic is ± 1. We may assume that the slope is in $[-1, 1]$ (if not, exchange the roles θ and ϕ): thus there is a time t_0 for which $\theta(t_0) = 0 \mod 2\pi$. Then the set $G = \{\phi(t) - \phi(t_0) \mod 2\pi \mid t \in \mathbb{R},\ \theta(t) = 0 \mod 2\pi\}$ is a subgroup of $\mathbb{R}/2\pi\mathbb{Z}$. Moreover, for all $t \in G$, we have $|\cos\theta(t) - \cos\phi(t)| \leq 2r$ so $\cos\phi(t) \geq 1 - 2r > 0$, so $G \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mod 2\pi$, which means that G is reduced to a single point: the slope is either 0 or ± 1 (since we assumed it is in $[-1, 1]$). If the slope is 0, then $|\cos\theta(t) - \cos\phi(t)| \leq 2r$ applied to a t such that $\cos\theta(t) = -1$ gives us $\cos\phi(t) \leq -1 + 2r$, which is not compatible with $\cos\phi(t) \geq 1 - 2r > 0$ since $r < 1/2$, so the slope is in fact ± 1.

Changing θ into $-\theta$ if necessary, we may assume that the slope is 1. Thus, there exist t_1 and t_2 such that $\phi(t_1) + \theta(t_1) = \pi \mod 2\pi$ and $\phi(t_2) + \theta(t_2) = 0 \mod 2\pi$. We have $\theta(t_2) - \theta(t_1) = \phi(t_2) - \phi(t_1) \mod 2\pi$ (because the slope is 1), so $\phi(t_2) - \phi(t_1) = \frac{\pi}{2} \mod \pi$, so $\cos\phi(t_2)\cos\phi(t_1) = -\sin\phi(t_2)\sin\phi(t_1)$. By taking the squares of both sides of this equality we obtain:

$$\cos^2\phi(t_1) + \cos^2\phi(t_2) = 1. \tag{8.1}$$

We have $-\cos\theta(t_1)+\cos\phi(t_1) \leq 2r$ and $\cos\phi(t_2)+\cos\theta(t_2) \leq 2l-4$, which implies that $-\cos\theta(t_1) = \cos\phi(t_1) \leq r$ and $\cos\theta(t_2) = \cos\phi(t_2) \leq l-2$. Injecting this in (8.1), we obtain:

$$r^2 + (l-2)^2 \geq 1,$$

which contradicts $r^2 + (l-2)^2 < 1$. $\square$

Part III

Transverse similarity structures on foliations

Chapter 9

Transverse similarity structures on foliations

9.1 Some background and vocabulary

In this section, we consider a connected manifold M endowed with an *affine connection* ∇ and recall some basic vocabulary. See [KN63] (for example) for more details about these notions.

Holonomy group. For $x \in M$, any continuous, piecewise C^1, closed curve γ with base point x defines a linear map from the tangent space T_xM to itself, given by *parallel transport* along γ. The set of all linear maps obtained in this way has a natural group structure: it is called the *holonomy group* at x, and written $\mathrm{Hol}_x(\nabla)$.

For two base points $x, y \in M$, it is easy to see that the holonomy groups $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_y(\nabla)$ are conjugated: thus, for most purposes, we may omit the base point and write simply $\mathrm{Hol}(\nabla)$.

The *restricted holonomy group* $\mathrm{Hol}^0(\nabla)$ is the subgroup of $\mathrm{Hol}(\nabla)$ given by the *contractible* curves.

Irreducibility. We say that ∇ has *irreducible holonomy* (or simply, M is irreducible) if, for $x \in M$, the only subspaces of T_xM stable under $\mathrm{Hol}_x(\nabla)$ are T_xM and $\{0\}$. Otherwise, we say that ∇ is reducible. If the property holds for $\mathrm{Hol}^0_x(\nabla)$ instead of $\mathrm{Hol}_x(\nabla)$, we say that (M, ∇) is *locally irreducible* or *locally reducible*.

Connections preserving structures. Given a Riemannian metric g on M, we say that ∇ *preserves* g if for any curve γ joining $x \in M$ to $y \in M$, the parallel transport $P_\gamma : T_xM \to T_yM$ is an isometry for the metric g.

Likewise, for a conformal structure c on M, we say that ∇ *preserves* c if for any curve γ joining $x \in M$ to $y \in M$, the parallel transport $P_\gamma : T_xM \to T_yM$ is a conformal map.

9.2 Introduction

9.2.1 Decomposition of locally metric connections

We recall the famous decomposition theorem of De Rham [dR52]:

Theorem 9.1 (De Rham, 1952)**.** *Consider a* complete *connected Riemannian manifold* (M, g)*. Then the universal cover* $(\tilde{M}, \tilde{g})$ *of* (M, g) *admits the following decomposition:*

$$(\tilde{M}, \tilde{g}) = (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_k, g_k),$$

where $k \geq 0$*,* M_0 *is the Euclidean space* $\mathbb{R}^q$ *(for some* $q \geq 0$*), and* $M_1, \ldots, M_k$ *are non-flat, locally irreducible manifolds. Moreover,* $\pi_1(M)$ *acts on* $\tilde{M}$ *as a subgroup of*

$$\mathrm{Isom}(M_0, g_0) \times \mathrm{Isom}(M_1, g_1) \times \ldots \times \mathrm{Isom}(M_k, g_k).$$

This decomposition is unique up to the order of the factors.

One of the aims of this chapter is to extend this theorem to the case of a manifold endowed with a locally metric connection, *i.e.* a connection whose restricted holonomy group $\mathrm{Hol}^0(\nabla)$ is a relatively compact subgroup of $GL_n(\mathbb{R})$. Such a connection is locally the Levi-Civita connection of a Riemannian metric. Moreover, its pullback $\tilde{\nabla}$ to the universal cover $\tilde{M}$ of M has a relatively compact holonomy group, so $\tilde{\nabla}$ is globally the Levi-Civita connection of some Riemannian metric $\tilde{g}$ on $\tilde{M}$.

Example 9.2. Consider $N = \mathbb{R}^n \setminus \{0\}$ with the Levi-Civita connection ∇_N of the Euclidean metric, and the subgroup G of $\mathrm{Diffeo}(N)$ spanned by the homothety $\varphi : x \mapsto 2x$. Since φ preserves ∇_N, $M = N/G$ is naturally endowed with a connection ∇, which is locally metric, but not globally.

Locally metric connections behave well with respect to the product structure: if (M_1, ∇_1) and (M_2, ∇_2) are two manifolds with locally metric connections, then the product connection (∇_1, ∇_2) is again a locally metric connection on $M_1 \times M_2$. In this framework, we will prove the following theorem, which is an analogue of Theorem 9.1:

Theorem 9.3. *Consider a* compact *connected manifold* (M, ∇)*, where* ∇ *is a locally metric connection. Then its universal cover* $(\tilde{M}, \tilde{\nabla})$ *admits the following decomposition:*

$$(\tilde{M}, \tilde{\nabla}) = (M_0, \nabla_0) \times (M_1, \nabla_1) \times \ldots \times (M_k, \nabla_k)$$

where $k \geq 0$*,* M_0 *is flat, and* $M_1, \ldots, M_k$ *are non-flat, locally irreducible manifolds. Moreover,* $\pi_1(M)$ *acts on* $\tilde{M}$ *as a subgroup of*

$$\mathrm{Aff}(M_0, \nabla_0) \times \mathrm{Aff}(M_1, \nabla_1) \times \ldots \times \mathrm{Aff}(M_k, \nabla_k).$$

This decomposition is unique up to the order of the factors.

Here, $\mathrm{Aff}(M, \nabla)$ is the group of *affine transformations* of M, *i.e.* diffeomorphisms which preserve the connection ∇.

Notice the two main differences between the statements of Theorems 9.1 and 9.3:

1. In Theorem 9.3, M is assumed to be *compact* instead of *complete*. In fact, a manifold with a locally metric connection is almost never geodesically complete.

2. In Theorem 9.3, the flat manifold M_0 is not necessarily a Euclidean space. In Example 9.2, the manifold M is flat, but its universal cover is a covering of the Riemannian manifold $\mathbb{R}^n \setminus \{0\}$ which is not complete, so M_0 cannot be a Euclidean space. By contrast, in the setting of Theorem 9.1, the universal cover is always complete.

We prove Theorem 9.3 in Section 9.4. The difficulty lies in the fact that, although there always exists a global metric on $\tilde{M}$ which is preserved by $\tilde{\nabla}$, it is (most of the time) not complete, so that Theorem 9.1 does not apply. Instead, we will use the following generalization of De Rham's theorem:

Theorem 9.4 (Ponge-Reckziegel, 1993)**.** *Let M be a simply connected Riemannian manifold, whose Levi-Civita connection ∇ is reducible: thus, the tangent bundle TM admits two complementary orthogonal distributions E' and E'' invariant by parallel transport, which induce foliations $\mathcal{F}'$ and $\mathcal{F}''$. Assume that the leaves of $\mathcal{F}'$ are all complete. Then, M is globally isometric to a product of Riemannian manifolds $M' \times M''$.*

Theorem 9.4 is a consequence of the main result of [PR93], but we will give a direct proof in Appendix A, based on the ideas of the proof of the classical De Rham theorem in [KN63].

9.2.2 Conformal structures

Next, we focus on a particular case of locally metric connections: those which preserve a conformal structure.

Such a connection appears naturally on the quotient of a Riemannian manifold by a group of similarities, like in Example 9.2. It is essential to notice that these connections do not behave well with the product structure: the product of two such connections does not preserve a conformal structure in general. There is no natural notion of product for conformal structures: thus, it is much more complicated to construct examples of such connections which are reducible.

In this case, the De Rham decomposition takes a very particular form. In fact, it was conjectured recently (see [BM14]) that a locally (but not globally) metric connection which preserves a conformal structure was either flat or irreducible. Indeed, it is the case for "Riemannian cones", as proved by Gallot (see Theorem 9.11). Very recently, however, Matveev and Nikolayevsky gave a counterexample to this conjecture [MN15a], which corresponds to $k = 1$ and $M_0 = \mathbb{R}$ in the setting of Theorem 9.3. Then, they proved the following [MN15b]:

Theorem 9.5 (Matveev-Nikolayevsky, 2015)**.** *Consider a compact connected manifold (M, ∇), where ∇ is an* analytic*, locally metric connection which preserves a conformal structure, but is not globally metric.*

Then either (M, ∇) is flat, or its universal cover $(\tilde{M}, \tilde{\nabla})$ admits the following decomposition:

$$(\tilde{M}, \tilde{\nabla}) = (M_0, \nabla_0) \times (M_1, \nabla_1)$$

where $M_0 = \mathbb{R}^q$ $(q \geq 0)$, and M_1 is a non-flat, locally irreducible manifold. In this case, $\pi_1(M)$ acts on $\tilde{M}$ as a subgroup of

$$\mathrm{Aff}(M_0, \nabla_0) \times \mathrm{Aff}(M_1, \nabla_1).$$

We answer positively, by a new proof, to a question asked in [MN15b]:

Theorem 9.6. *Theorem 9.5 remains true without assuming analyticity.*

In other words, the De Rham decomposition of a locally metric connection given in Theorem 9.3 takes a particular form when the connection preserves a conformal structure: the number k of irreducible factors is always 0 or 1, and if $k = 1$, then $M_0 = \mathbb{R}^q$ for some $q \geq 0$.

9.2.3 Similarity structures

Locally metric connections which preserve a conformal structure have an important property: they preserve *similarity structures*.

Definition 9.7. 1. On a differentiable manifold M, a *similarity structure* is given by an open cover $(U_i)_{1\leq i\leq r}$ of M, together with a Riemannian metric g_i on each U_i, such that the transitions are locally similarities, *i.e.*

$$g_j = \lambda_{ij} g_i \text{ on } U_i \cap U_j, \tag{9.1}$$

where each λ_{ij} is locally constant. Of course, any Riemannian structure induces a similarity structure. Notice also that any similarity structure induces a conformal structure in a natural way.

2. If (M, g) is a Riemannian manifold, its similarity pseudogroup $\mathrm{Sim}_{\mathrm{loc}}(M)$ consists of all $\phi : U \to V$ such that $\phi^* g = \lambda g$, where U and V are open subsets of M, and $\lambda \in \mathbb{R}_{>0}$ is locally constant on U. For any $x \in M$, the number $\lambda(x)$ is called the *ratio of ϕ at x* (if M is connected, there is no need to specify the point x). The *similarity group* $\mathrm{Sim}(M)$ is the set of all $\phi \in \mathrm{Sim}_{\mathrm{loc}}(M)$ which are bijections from the whole manifold M to itself.

Here are three fundamental examples:

Example 9.8. Consider a Riemannian manifold (N, g_N), its Levi-Civita connection ∇_N, and G a subgroup of $\mathrm{Sim}(N)$ acting properly discontinuously on N. Then the quotient $M = N/G$ is naturally endowed with a similarity structure.

Example 9.9. Let (M, g) be a Riemannian manifold, and $(\tilde{M}, \tilde{g})$ its universal cover. Any closed 1-form ω on M lifts to an exact 1-form $\tilde{\omega}$ on $\tilde{M}$. Consider a primitive f of $\tilde{\omega}$ and let $\tilde{h} = e^f \tilde{g}$. Then the fundamental group of M acts on $(\tilde{M}, \tilde{h})$ by similarities, and thus $\tilde{h}$ induces a similarity structure on M (see Example 9.8).

Example 9.10. Let N be any Riemannian manifold. The *cone over N* is the manifold $M = N \times S^1$ endowed with the similarity structure given by Example 9.9, where ω is the canonical form $d\theta$ on M.

Levi-Civita connection. Any similarity structure defines canonically a "Levi-Civita" connection on M: on each U_i, it is obtained by taking the Levi-Civita connection of g_i. Equation 9.1 ensures that this connection is well-defined. From this viewpoint, we may state Gallot's result [Gal79], which was mentioned above:

Theorem 9.11 (Gallot, 1979)**.** *If (M, ∇) is the cone over a Riemannian manifold N (see Example 9.10), then ∇ is either irreducible or flat.*

Moreover, the Levi-Civita connection of a similarity structure is locally metric and preserves the conformal structure induced by the similarity structure.

Conversely, it turns out that all locally metric connections which preserve a conformal structure are obtained as Levi-Civita connections of similarity structures. More precisely:

Fact 9.12. *If (M, ∇) is a connected manifold endowed with a locally metric connection which preserves a conformal structure, then (M, ∇) is the quotient of a Riemannian manifold (N, g) (endowed with its Levi-Civita connection) by a subgroup G of* $\mathrm{Sim}(N)$*. In other words, ∇ is the Levi-Civita connection of a similarity structure on M.*

Proof. Consider a conformal structure c on M which is preserved by ∇, a point $x \in M$, and a metric g in the conformal class c defined in a neighborhood of x. Since ∇ preserves c, all the elements of the restricted holonomy group $\mathrm{Hol}_x^0(\nabla)$ are similarities of the Euclidean space (T_xM, g_x). Moreover, since ∇ is locally metric, $\mathrm{Hol}_x^0(\nabla)$ is relatively compact, so its elements are in fact isometries.

Now, consider the universal cover $(\tilde{M}, \tilde{\nabla})$ of (M, ∇), $\tilde{x} \in \tilde{M}$ an element which projects to the point x, $\tilde{c}$ the pullback of the conformal class c, and $\tilde{g}$ the pullback of the metric g in a neighborhood of $\tilde{x}$. The elements of $\mathrm{Hol}_{\tilde{x}}(\tilde{\nabla})(= \mathrm{Hol}_{\tilde{x}}^0(\tilde{\nabla}))$ are also isometries of the Euclidean space $(T_{\tilde{x}}\tilde{M}, \tilde{g}_{\tilde{x}})$. Thus, by parallel transport of $\tilde{g}_{\tilde{x}}$, one obtains a metric $\tilde{h}$ defined on the whole manifold $\tilde{M}$, and this metric is in the conformal class $\tilde{c}$.

Let $\alpha \in \pi_1(M)$. Since α preserves the conformal class of $\tilde{h}$, there exists $\lambda : M \to \mathbb{R}_{>0}$ such that $\alpha^*\tilde{h} = \lambda\tilde{h}$. Moreover, $\tilde{h}$ is invariant under the parallel transport of $\tilde{\nabla}$, and $\tilde{\nabla}$ is preserved by $\pi_1(M)$, so λ is constant. Hence, $\pi_1(M)$ acts by similarities on $\tilde{M}$. □

Irreducible Levi-Civita connections have special properties:

Fact 9.13. *Consider a Riemannian manifold (M, g) with its Levi-Civita connection ∇. If ∇ has irreducible restricted holonomy, then:*

1. *the only metrics whose Levi-Civita connection is ∇ are the metrics $h_\lambda = \lambda g$, $\lambda > 0$;*
2. $\mathrm{Aff}(M, g) = \mathrm{Sim}(M, g)$.

Proof.
1. Let h be a metric with whose Levi-Civita connection is ∇ and let $x \in M$. Define the linear mapping $F : T_xM \to T_xM$ in the following way: for all $u \in T_xM$, $F(u)$ is the unique vector such that $g(u, \cdot) = h(F(u), \cdot)$. Since $\mathrm{Hol}_x(\nabla)$ preserves g and h, all the elements of $\mathrm{Hol}_x(\nabla)$ commute with F, so the eigenspaces of F are stable under $\mathrm{Hol}_x(\nabla)$. Since ∇ is irreducible, the only possible eigenspaces for F are $\{0\}$ and T_xM; but F is self-adjoint (for both metrics g and h), so F is a homothety, and thus g and h are proportional.
2. For all $\phi \in \mathrm{Aff}(M, g)$, the metric ϕ^*g is preserved by the Levi-Civita connection ∇ of g, so ϕ^*g is proportional to g and thus ϕ is a similarity.

□

Fact 9.13 implies that any *irreducible* locally metric connection on a manifold M preserves a conformal structure (indeed, any metric $\tilde{g}$ which is preserved by $\tilde{\nabla}$ on the universal cover $(\tilde{M}, \tilde{\nabla})$ induces a conformal structure on M which is preserved by ∇). Moreover, we now have a more precise version of Theorem 9.3:

Theorem 9.14. *Consider a compact connected manifold (M, ∇), where ∇ is a locally metric connection. Then its universal cover $(\tilde{M}, \tilde{\nabla})$ admits a metric $\tilde{g}$ which is preserved by $\tilde{\nabla}$, such that:*

$$(\tilde{M}, \tilde{g}) = (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_k, g_k)$$

where $k \geq 0$, M_0 is flat, and $M_1, \ldots, M_k$ are non-flat, locally irreducible manifolds. Moreover, $\pi_1(M)$ acts on $\tilde{M}$ as a subgroup of

$$\mathrm{Aff}(M_0, g_0) \times \mathrm{Sim}(M_1, g_1) \times \ldots \times \mathrm{Sim}(M_k, g_k).$$

This decomposition is unique up to the order of the factors.

Proof. Choose any metric $\tilde{g}$ preserved by $\tilde{\nabla}$. In the decomposition given by Theorem 9.3, the metric $\tilde{g}$ induces metrics $g_0, g_1, \ldots, g_k$ on $M_0, M_1, \ldots, M_k$. For $i \in \{1, \ldots, k\}$, since ∇_i is irreducible, Fact 9.13 implies that $\mathrm{Aff}(M_i, g_i) = \mathrm{Sim}(M_i, g_i)$. □

9.2.4 Transverse similarity structures

The main tool in the proofs of Theorems 9.3 and 9.6 is the study of transverse similarity structures on foliations. Such foliations may be seen as a particular case of (transversally) conformal foliations, or a generalization of (transversally) Riemannian foliations.

Let $(M, \mathcal{F})$ be a compact foliated manifold, and $(U_i)_{1 \leq i \leq r}$ a covering of M compatible with the foliation, such that each U_i is diffeomorphic to $V_i \times T_i$, where V_i (the plaque) is an open ball of $\mathbb{R}^p$ and T_i an open ball of $\mathbb{R}^q$. We denote by $f_i : U_i \to \mathbb{T}_i$ the projections, $T = \cup_{1 \leq i \leq r} T_i$ the (global) transversal, and define the transition maps $(\gamma_{ij})_{i,j} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ so that $f_j = \gamma_{ij} \circ f_i$ on $U_i \cap U_j$. The pseudogroup Γ spanned by the (γ_{ij}) is called the *holonomy pseudogroup* of the foliation.

A transverse similarity structure on the foliation $\mathcal{F}$ is a metric g on the transversal T such that the transition maps γ_{ij} are local similarities (*i.e.* belong to $\mathrm{Sim}_{\mathrm{loc}}(T)$). The foliation is said to be *transversally Riemannian* (or simply *Riemannian*) if it is possible to choose g such that the γ_{ij} are isometries.

Our main result on transverse similarity structures is the following:

Theorem 9.15. *Let $(M, \mathcal{F})$ be a compact foliated manifold with a transverse similarity structure. Then one of the following two facts occurs:*

1. *The transverse similarity structure on the foliation $\mathcal{F}$ is flat (*i.e. *the metric g on the transversal T is flat);*

2. *The foliation $\mathcal{F}$ is transversally Riemannian (*i.e. *there exists a metric h on the transversal T such that the transition maps are isometries).*

We prove Theorem 9.15 in Section 9.3, and give two alternative proofs in Appendix B, one of which is valid only in the analytic case.

Notice that Theorem 9.15 does not assume that M is endowed with a locally metric connection, but only that there is a transverse similarity structure on the foliation: the setting is more general than for Theorem 9.3.

About the foliated Ferrand-Obata conjecture. For transversally *conformal* foliations, there is an analogue of Theorem 9.15 (see [Tar04a]):

Theorem 9.16 (Tarquini, 2004)**.** *Any transversally analytic conformal foliation of codimension ≥ 3, on a compact connected manifold, is either transversally Möbius or Riemannian.*

It is also believed that Theorem 9.16 should be valid without the analyticity assumption: this is the *foliated Ferrand-Obata conjecture*. Our Theorem 9.15 implies the following:

Corollary 9.17. *The foliated Ferrand-Obata conjecture is true if the transverse conformal structure on the foliation is induced by a transverse similarity structure.*

9.3 Foliations with transverse similarity structures

In this section, we prove Theorem 9.15. We refer to Section 9.2.4 for the basic notations.

A foliation is said to be equicontinuous if its holonomy pseudogroup Γ is equicontinuous. If the foliation has a transverse similarity structure, equicontinuity is equivalent to the existence of a constant $m > 1$ such that the ratio of any $\gamma \in \Gamma$ at any $x \in M$ lies in the interval $[1/m, m]$.

The following proposition is crucial in the proof of Theorem 9.15:

Proposition 9.18. *Any equicontinuous foliation with a transverse similarity structure is Riemannian.*

Proof. This theorem is proved in [Tar04b]: more generally, any foliation with *compact finite type* is Riemannian. □

Now, our first step in the proof is based on a trick which was described in [FT02].

Proposition 9.19. *Let (M, g) be a Riemannian manifold whose Riemann tensor R does not vanish. Then $\mathrm{Sim}_{\mathrm{loc}}(M)$ preserves a Riemannian metric.*

Proof. If R denotes the Riemann tensor, define $\|R\|_g(x)$ as the supremum of the values of $\|R_x(u, v)w\|_g$ when u, v, w are vectors of T_xM which have unit length for g. Then the metric $\|R\|_g\, g$ is invariant by $\mathrm{Sim}_{\mathrm{loc}}(M)$. □

Thus, if $(M, \mathcal{F})$ is a foliated manifold with a transverse similarity structure, either $\mathcal{F}$ is Riemannian, or the Riemann tensor of (T, g) vanishes somewhere. Our aim is to show that, in the last case, the Riemann tensor vanishes in fact everywhere.

Another useful tool to prove flatness is the following:

Proposition 9.20. *Consider a connected Riemannian manifold (M, g), an open subset $U \subseteq M$ and a similarity $\phi \in \mathrm{Sim}_{\mathrm{loc}}(M)$, $\phi : M \to U$. Assume that ϕ has a fixed point $x \in M$, and that its ratio is $r_\phi < 1$. Then:*

1. *(M, g) is flat;*
2. *in the case that $\phi \in \mathrm{Sim}(M)$ (*i.e. *$U = M$ and ϕ is a bijection), the manifold M is isometric to the Euclidean $\mathbb{R}^q$ for some $q \geq 0$.*

Proof. Let us prove the first statement. Choose any $y \in M$ and four vectors a, b, c, d in T_yM of unit length for g. The point is that ϕ preserves R, *i.e.*

$$R(\phi_* a, \phi_* b)\phi_* c = \phi_* R(a, b)c.$$

Thus:

$$\begin{aligned}\langle R(a,b)c \mid d\rangle &= r_\phi^{-2n}\,\langle \phi_*^n R(a,b)c \mid \phi_*^n d\rangle\\ &= r_\phi^{-2n}\,\langle R(\phi_*^n a,\phi_*^n b)\phi_*^n c \mid \phi_*^n d\rangle\\ &\le r_\phi^{-2n} r_\phi^{4n}\,\|R\|_g\,(\phi^n(y)).\end{aligned}$$

Since $\phi^n(y)$ tends to the fixed point x, the quantity $\|R\|_g\,(\phi^n(y))$ is bounded. Thus, $\langle R(a,b)c \mid d\rangle = 0$ and the first statement is proved.

For the second statement, notice that since M is flat, the exponential map $\exp_x : B(0,\epsilon) \to B_g(x,\epsilon)$ is an isometry for some $\epsilon > 0$ (where $B(0,\epsilon)$ is the ball in T_xM of center 0 and radius ϵ for the Euclidean metric g_x, while $B_g(x,\epsilon)$ is the ball in M of center x and radius ϵ for the distance induced by g).

Thus, for all $n \ge 0$, $\phi^{-n} \circ \exp_x \circ D_x\phi^n$ is an isometry from $B(0, r_\phi^{-n}\epsilon)$ to $B_g(x, r_\phi^{-n}\epsilon)$. Since ϕ^n preserves the Levi-Civita connection of g, we have

$$\exp_x = \phi^{-n} \circ \exp_x \circ D_x\phi^n.$$

Hence, $\exp_x$ is an isometry from $B(0, r_\phi^{-n}\epsilon)$ to $B_g(x, r_\phi^{-n}\epsilon)$ for all $n \ge 0$. Since the balls $B_g(x, r_\phi^{-n}\epsilon)$ cover M, $\exp_x$ is an isometry from $\mathbb{R}^q$ to M. □

Until the end of this section, we consider a *compact, connected* foliated manifold $(M,\mathcal{F})$ with a transverse similarity structure. We still denote by g a metric on the transversal T such that the γ_{ij} are local similarities. This metric g induces a distance d_i on each T_i.

Lemma 9.21. *There exists $\epsilon_0 > 0$ such that for all $x \in M$, there exists $i \in \{1,\dots,r\}$ which satisfies $x \in U_i$ and $d_i(f_i(x), \partial T_i) > \epsilon_0$ (see Section 9.2.4 for the notations).*

Proof. Assume the contrary: there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that for all $i \in \{1,\dots,r\}$ with $x_n \in U_i$, we have $d_i(f_i(x_n), \partial T_i) \le 1/n$. Since M is closed, we may assume that x_n converges to some $x_\infty \in M$. Then x_∞ is in some U_{i_0}, and for any large enough n, $x_n \in U_{i_0}$. Hence, $d_{i_0}(f_{i_0}(x_n), \partial T_{i_0}) \to 0$, which contradicts the fact that $x_\infty \in U_{i_0}$. □

In the following, we fix this ϵ_0.

Lemma 9.22. *Let $\gamma \in \Gamma$, $x \in M$ and $i \in \{1,\dots,r\}$, such that γ is defined on a neighborhood of $f_i(x)$ in T_i and takes its values in T_j.*

Then there exists $\tilde\gamma = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1i_2}$ defined on a neighborhood of $f_{i_1}(x)$, which coincides with $\gamma_{ji_p} \circ \gamma \circ \gamma_{i_1i}$ near $f_{i_1}(x)$, such that for all $l \in \{1,\dots,p-1\}$, $d_{i_l}(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}(f_{i_1}(x)), \partial U_{i_l}) > \epsilon_0$, and $\gamma_{i_li_{l+1}}$ is defined on $B_g(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1i_2}(f_{i_1}(x)), \epsilon_0)$.

Proof. It results from the general theory of holonomy pseudogroups that γ is obtained by following some curve $c : [0,1] \to M$ such that $c(0) = x$, and $f_j(c(1)) = \gamma(f_i(x))$. For each $i \in \{1,\dots,r\}$, we define E_i as the set of all open intervals (a,b) such that for all $t \in (a,b)$, $c(t) \in U_i$ and $d_i(f_i(c(t)), \partial T_i) > \epsilon_0$. Lemma 9.21 implies that $\cup_{1\le i\le r} E_i$ is an open cover of $[0,1]$: it has a finite subcover $\{(a_1,b_1),\dots,(a_p,b_p)\}$, to which corresponds a sequence $i_1,\dots,i_p$ of indices. Then, γ coincides with $\gamma_{i_pj} \circ \tilde\gamma \circ \gamma_{ii_1}$, where $\tilde\gamma = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1i_2}$, near x. □

Lemma 9.23. *Let E be the set of all $x \in M$ for which there exists $m > 1$ such that for all $i \in \{1, \dots, r\}$ with $x \in U_i$, every $\gamma \in \Gamma$ defined on $f_i(x)$ has ratio $\geq 1/m$ at $f_i(x)$.*

1. *In the definition of E, it is possible to choose m independently of x.*
2. *If E is non-empty, then $E = M$ and Γ is equicontinuous.*

Proof. We start with the proof of the first statement. Assume that there is no uniform bound: then, there exist sequences (x^n), (i^n), (j^n) and (γ^n) such that $x^n \in E$, γ^n is defined on a neighborhood of $f_{i^n}(x^n)$ in T_{i^n}, takes its values in T_{j^n}, and the ratio of γ^n is $\leq 1/n$ at $f_{i^n}(x^n)$. Let $k_{\max}$ be the maximum ratio of γ_{ij} for $i, j \in \{1, \dots, r\}$.

The natural idea is that the points x^n accumulate somewhere because M is compact: if all the terms γ^n were defined on balls of the same radius ϵ_0, then an infinity of them would be well-defined on the same term x^{n_0}, which would contradict the fact that $x^{n_0} \in E$. In fact, the domain of γ^n can be arbitrarily small, so we need to consider another sequence (y^n) instead of (x^n).

For each n, Proposition 9.22 gives us a $\tilde{\gamma}^n = \gamma_{i^n_{p^n-1} i^n_{p^n}} \circ \cdots \circ \gamma_{i^n_1 i^n_2}$ defined on a neighborhood of $f_{i^n_1}(x^n)$, which coincides with $\gamma_{j^n i^n_{p^n}} \circ \gamma^n \circ \gamma_{i^n_1 i^n}$ near $f_{i^n_1}(x^n)$, such that for all $l \in \{1, \dots, p^n - 1\}$, $d_{i^n_l}(\gamma_{i^n_{l-1} i^n_l} \circ \cdots \circ \gamma_{i^n_1 i^n_2}(f_{i^n_1}(x^n)), \partial U_{i_l}) > \epsilon_0$, and $\gamma_{i_l i_{l+1}}$ is defined on $B_g(\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i^n_1}(x^n)), \epsilon_0)$. Notice that $\tilde{\gamma}^n$ has ratio $\leq k^2_{\max}/n$ at $f_{i^n_1}(x^n)$ (because $\tilde{\gamma}^n$ coincides with $\gamma_{j^n i^n_{p^n}} \circ \gamma^n \circ \gamma_{i^n_1 i^n}$ near $f_{i^n_1}(x^n)$, and the ratio of γ^n is $\leq 1/n$ at $f_{i^n}(x^n)$).

Choose $q^n \in \{1, \dots, p^n - 1\}$ which minimizes the ratio of $\gamma_{i^n_{p^n-1} i^n_{p^n}} \circ \cdots \circ \gamma_{i^n_{q^n} i^n_{q^n+1}}$ at $\gamma_{i^n_{q^n-1} i^n_{q^n}} \circ \cdots \circ \gamma_{i^n_1 i^n_2}(f_{i^n_1}(x^n))$ (in particular this ratio is $\leq k^2_{\max}/n$), and write $\tilde{\rho}^n = \gamma_{i^n_{p^n-1} i^n_{p^n}} \circ \cdots \circ \gamma_{i^n_{q^n} i^n_{q^n+1}}$. Choose y^n such that $f_{i^n_{q^n}}(y^n) = \gamma_{i^n_{q^n-1} i^n_{q^n}} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i^n_1}(x^n))$. Notice that $y^n \in E$.

Thus, $\tilde{\rho}^n$ is well-defined on $B_g(f_{i^n_{q^n}}(y^n), \epsilon_0)$ and has ratio $\leq k^2_{\max}/n$ at $f_{i^n_{q^n}}(y^n)$.

Since M is compact, we may assume up to extraction that (y^n) converges to a limit $y \in M$ (and $y \in U_i$ for some i): There exists $n_0 > 0$ such that for all $n \geq n_0$, $y^n \in U_i$ and $f_i(y^n) - f_i(y) < \epsilon_0/(3k_{\max})$. Thus, $\tilde{\rho}^n$ is well-defined on $f_{i^{n_0}_{q^{n_0}}}(y^{n_0})$ for all $n \geq n_0$, which contradicts the fact that $y^{n_0} \in E$ and ends the proof of the first statement.

To prove the second statement, first notice that for all $x \in E$, and all $i \in \{1, \dots, r\}$ such that $x \in U_i$, every $\gamma \in \Gamma$ defined on $f_i(x)$ (taking values in T_j) has ratio $\leq m$ at $f_i(x)$: otherwise, γ^{-1} would have ratio $< 1/m$ at $f_j(x)$, which contradicts the fact that $\gamma(f_i(x)) \in f_j(E)$.

Since M is connected, it suffices to show that E is open and closed in M. Thus, Γ will be equicontinuous on M.

Let us show that E is open. Let $x_0 \in E$ and i such that $d_i(f_i(x_0), \partial T_i) > \epsilon_0$. Consider V a neighborhood of x_0 such that $V \subseteq U_i$ and $f_i(V) \subseteq B_g(f_i(x_0), \epsilon_0/(2k_{\max} m))$. Let us show that $V \subseteq E$: let $y_0 \in V$, $i_0 \in \{1, \dots, r\}$, and $\gamma \in \Gamma$ defined on $f_{i_0}(y_0)$, taking its values in T_{j_0}.

With Lemma 9.22, there exists a $\tilde{\gamma} = \gamma_{i_{p-1} i_p} \circ \cdots \circ \gamma_{i_1 i_2}$ defined on a neighborhood of $f_{i_1}(y_0)$, which coincides with $\gamma_{j i_p} \circ \gamma \circ \gamma_{i_1 i}$ near $f_{i_1}(y_0)$, such that for all $l \in \{1, \dots, p-1\}$, $d_{i_l}(\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(y_0)), \partial U_{i_l}) > \epsilon_0$, and $\gamma_{i_l i_{l+1}}$ is defined on $B_g(\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(y_0)), \epsilon_0)$.

In particular, by induction on l, $\gamma_{i_l i_{l+1}}$ is well-defined on a ball centered at $\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(y_0))$ and containing $\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(x_0))$, so the ratio of $\gamma_{i_{l-1} i_l} \circ \cdots \circ \gamma_{i_1 i_2}$

on this ball is between $1/m$ and m (because $x_0 \in E$). Therefore, $\tilde{\gamma}$ is well-defined on $f_{i_1}(x_0)$, and its ratio is between $1/m$ and m at $f_{i_1}(x_0)$. The ratio of γ is $\geq 1/(k_{\max}^2 m)$ at $f_i(y_0)$, so $y_0 \in E$, and E is open.

Now, we show that $M \setminus E$ is open in M. Let $x_0 \in M \setminus E$, $i \in \{1, \ldots, r\}$, and $\gamma \in \Gamma$ defined on $f_i(x_0)$ with ratio $< 1/m$. Then γ is defined on a connected open set $W \subseteq T_i$ containing $f_i(x_0)$, and $f_i^{-1}(W)$ is an open set of M, containing x_0 and contained in $M \setminus E$, so $M \setminus E$ is open. □

End of the proof of Theorem 9.15. Assume that (T, g) is not flat, and let T' be the set of all $y \in T$ at which the Riemann tensor of g is nonzero. Notice that T' is stable under the holonomy pseudogroup Γ. Now, Proposition 9.19 gives us a Riemannian metric g' on T' which is invariant by $\mathrm{Sim}_{\mathrm{loc}}(T')$, and thus by the holonomy pseudogroup Γ. Hence, the set E defined in Lemma 9.23 is non-empty. By Lemma 9.23, Γ is equicontinuous. Finally, in view of Proposition 9.18, $\mathcal{F}$ is a Riemannian foliation, and Theorem 9.15 is proved.

9.4 End of the proofs of the main results

Let us start this section with the proof of Theorem 9.3. Consider a compact manifold (M, ∇), where ∇ is locally metric, and its universal cover $(\tilde{M}, \tilde{\nabla})$, on which there is a metric $\tilde{g}$ preserved by $\tilde{\nabla}$. Fix $x \in M$ and choose a preimage $\tilde{x} \in \tilde{M}$.

Let $E_{\tilde{x}}^0$ be the maximal linear subspace of the tangent space $T_{\tilde{x}}\tilde{M}$ on which $\mathrm{Hol}_{\tilde{x}}(\tilde{\nabla})$ acts trivially. Let $E_{\tilde{x}}^{>0}$ be the orthogonal complement of $E_{\tilde{x}}^0$. The local theorem of De Rham (see for example [KN63]) states that there is a unique decomposition of $E_{\tilde{x}}^{>0}$ (up to the order of the factors) into mutually orthogonal, invariant irreducible subspaces:

$$E_{\tilde{x}}^{>0} = E_{\tilde{x}}^1 \oplus \ldots \oplus E_{\tilde{x}}^k.$$

This induces a decomposition $T_x M = E_x^0 \oplus E_x^{>0}$, where $E_x^{>0} = E_x^1 \oplus \ldots \oplus E_x^k$.

Moreover, since $\pi_1(M)$ acts on $\tilde{M}$ by preserving the connection $\tilde{\nabla}$, this decomposition does not depend on the choice of the preimage $\tilde{x}$ of x, up to the order of the factors. Thus, the holonomy group $\mathrm{Hol}_x(\nabla)$ acts on $E_x^{>0}$ by permuting the factors: by considering a finite cover of M, one may assume that $\mathrm{Hol}_x(\nabla)$ preserves the decomposition of $T_x M$. Then, one may consider E' the distribution on $\tilde{M}$ obtained by parallel transport of E_x^k, and E'' obtained by parallel transport of $E_x^0 \oplus \ldots \oplus E_x^{k-1}$. These distributions induce transverse foliations $\mathcal{F}'$ and $\mathcal{F}''$ on M.

There exists a covering $(U_i)_{1 \leq i \leq r}$ of M compatible with the foliations $\mathcal{F}'$ and $\mathcal{F}''$, such that each U_i is diffeomorphic to $V_i \times T_i$, where V_i (the plaque of $\mathcal{F}''$) is an open ball of $\mathbb{R}^p$ and T_i (the plaque of $\mathcal{F}'$) an open ball of $\mathbb{R}^q$. The connection ∇ induces a connection ∇_T on the transversal $T = \cup_{1 \leq i \leq r} T_i$, which is preserved by the holonomy pseudogroup. Since T is simply connected, ∇_T preserves a Riemannian metric g_T. The holonomy pseudogroup of $\mathcal{F}''$ acts by affine transformations on (T, ∇_T). But since ∇_T is irreducible, these transformations are in fact local similarities of (T, g_T), which implies that $\mathcal{F}''$ has a transverse similarity structure.

By construction, the holonomy group of M does not act trivially on E', so $\mathcal{F}''$ is not transversally flat. With Theorem 9.15, $\mathcal{F}''$ is transversally Riemannian, so the leaves of $\mathcal{F}'$ are naturally endowed with a Riemannian structure. In particular, the leaves of

the pullback $\tilde{\mathcal{F}}'$ of $\mathcal{F}'$ are all complete. Therefore, by Theorem 9.4, $\tilde{M}$ is the product of two Riemannian manifolds $\tilde{M}'$ and $\tilde{M}''$. The existence of the decomposition given in Theorem 9.3 follows by induction on the dimension of M.

Uniqueness is proved in the same way as for the usual decomposition theorem of De Rham: the factors are necessarily the maximal integral manifolds of the distributions given by $E^0_{\tilde{x}}, E^1_{\tilde{x}}, \ldots, E^k_{\tilde{x}}$. Thus, Theorem 9.3 is proved.

For the proof of Theorem 9.6, we will need the following propositions:

Proposition 9.24. *Consider a complete connected Riemannian manifold (M, g). If $\mathrm{Sim}(M)$ does not act properly on M, then M is (globally) isometric to $\mathbb{R}^q$ for some $q \geq 0$.*

Proof. In this case, there exist a compact set $K \subseteq M$ and a sequence (S_n) of similarities such that $K \cap S_n(K) \neq \emptyset$ and the ratio of S_n (written r_n) tends to $+\infty$ or 0 when $n \to +\infty$. Considering S_n^{-1} instead of S_n if necessary, we may assume that $r_n \to 0$.

Let $K' = \{x \in M \mid d(x, K) \leq \epsilon\}$ for some small $\epsilon > 0$, where d is the distance induced by g in M. Then $S_n(K') = \{x \in M \mid d(x, S_n(K)) \leq r_n\epsilon\}$: in particular, for some large enough $n_0 > 0$, $S_{n_0}(K') \subseteq K'$. Thus, S_{n_0} has a fixed point and M is isometric to $\mathbb{R}^q$ by Proposition 9.20. □

Proposition 9.25. *Consider the product of two connected Riemannian manifolds, denoted by $(M, h) = (M_1, h_1) \times (M_2, h_2)$, and a subgroup G of $\mathrm{Sim}(M)$ which preserves the product structure (*i.e. *which is a subgroup of $\mathrm{Sim}(M_1) \times \mathrm{Sim}(M_2)$), and acts on M in a cocompact way. Also assume that $\mathrm{Sim}(M)$ contains elements which are not isometries. Then, either $M_1 = \mathbb{R}^q$ or $M_2 = \mathbb{R}^q$, for some $q \geq 0$.*

Proof. Assume that the conclusion is false. In view of Lemma 9.24, $\mathrm{Sim}(M_1)$ and $\mathrm{Sim}(M_2)$ act properly on M_1 and M_2 respectively.

Since G acts cocompactly on M, there is a compact set $K \subseteq M$ such that $\mathrm{Sim}(M) \cdot K = M$. We may assume that $K = K_1 \times K_2$, where $\mathrm{Sim}(M_1) \cdot K_1 = M_1$ and $\mathrm{Sim}(M_2) \cdot K_2 = M_2$.

Choose $x_1 \in K_1$. Since $\mathrm{Sim}(M_1)$ acts properly on M_1, there is a constant $R > 1$ such that for all $\gamma \in \mathrm{Sim}(M_1)$ satisfying $\gamma(x_1) \in K_1$, the ratio of γ is between R and $1/R$. Likewise, choose $x_2 \in K_2$. There is a constant, still called R, such that the ratio of any $\gamma \in \mathrm{Sim}(M_2)$ satisfying $\gamma(x_2) \in K_2$ is between R and $1/R$.

We assumed that $\mathrm{Sim}(M)$ contains elements which are not isometries, so there exists $\gamma_0 \in \mathrm{Sim}(M_1)$ whose ratio is greater than R^3. And since $G \cdot K = M$, there exists $\gamma = (\gamma_1, \gamma_2) \in G$ such that $\gamma(\gamma_0(x_1), x_2) \in K$. Then, $\gamma_1 \circ \gamma_0(x_1) \in K_1$, so the ratio of $\gamma_1 \circ \gamma_0$ is smaller than R, so the ratio of γ_1 is smaller than $1/R^2$. Meanwhile, $\gamma_2(x_2) \in K_2$, so the ratio of γ_2 is greater than $1/R$. But since $(\gamma_1, \gamma_2) \in \mathrm{Sim}(M)$, γ_1 and γ_2 should have the same ratio, which is impossible. □

Consider a connected, compact manifold (M, ∇), where ∇ is a reducible, locally metric (but not globally metric) non-flat connection which preserves a conformal structure, and consider its universal cover $\tilde{M}$. Then $\pi_1(M)$ is a subgroup of $\mathrm{Sim}(\tilde{M})$. Moreover, $\mathrm{Sim}(\tilde{M})$ contains elements which are not isometries. Thus, we may apply Proposition 9.25, which concludes the proof of Theorem 9.6.

Appendix A. A direct proof of Theorem 9.4

In this section, we adapt the ideas of the proof of De Rham's theorem available in [KN63], to give a direct proof of Theorem 9.4.

Consider a connected, simply connected Riemannian manifold M with reducible holonomy, and write $T_x(M) = E'_x \oplus E''_x$ a decomposition into subspaces invariant by holonomy (of dimension at least 1). Denoting by E' and E'' the corresponding distributions, define $M'(x)$ and $M''(x)$ the maximal integral manifolds of E' and E'' through x, for any $x \in M$.

The *local* decomposition theorem of De Rham states that such a manifold is locally a product: there is a neighborhood of x which is isometric to a product of Riemannian manifolds $V' \times V''$. But for the *global* version of the theorem (Theorem 9.1), one has to assume, classically, that M is *complete*: here, we do not make this assumption.

Definition 9.26. Let $x \in M$ and $X(t)$ a curve in T_xM. The development of $X(t)$ into M, when it exists, is the (unique) curve ϕ starting from x, such that $X(t)$ is the result of the parallel displacement of $\frac{d\phi}{dt}(t)$ along the curve $\phi|_{[0,t]}$ itself.

Proposition 9.27. *The development of $X(t)$ into M is always unique. Moreover, if M is complete, it always exists.*

Proof. It is the solution of a differential equation. For more details, see [KN63], Chapter IV, Theorem 4.1. □

Definition 9.28. For two curves $\phi, \psi : [0,1] \to M$ with $\phi(1) = \psi(0)$, we denote by $\psi \cdot \phi : [0,1] \to M$ the concatenation of the two curves, *i.e.*

$$\psi \cdot \phi(t) = \begin{cases} \phi(2t) & \text{if } t \leq 1/2 \\ \psi(2t-1) & \text{if } t \geq 1/2 \end{cases}$$

In the same spirit, we define ϕ^{-1} so that $\phi^{-1}(t) = \phi(1-t)$.

Lemma 9.29. *In this lemma, we assume that either $M'(x)$ is complete for all $x \in M$, or $M''(x)$ is complete for all $x \in M$.*

Let $\phi : [0,1] \to M$ be a curve tangent to E' and $\psi : [0,1] \to M$ a curve tangent to E'', such that $\phi(0) = \psi(0)$. Then there exists a unique smooth homotopy $f : [0,1] \times [0,1] \to M$ such that:

1. *$f(.,0) = \phi$;*
2. *$f(0,.) = \psi$;*
3. *For all $(t,s) \in [0,1] \times [0,1]$, $\frac{\partial f}{\partial t}(t,s)$ is parallel to $\frac{\partial f}{\partial t}(t,0)$ along the curve $f(t,.)|_{[0,s]}$.*
4. *For all $(t,s) \in [0,1] \times [0,1]$, $\frac{\partial f}{\partial s}(t,s)$ is parallel to $\frac{\partial f}{\partial s}(0,s)$ along the curve $f(.,s)|_{[0,t]}$.*
5. *For all $(t,s) \in [0,1] \times [0,1]$, the parallel displacement along the "parallelogram" $f(.,0)|_{[0,t]}^{-1} \cdot f(t,.)|_{[0,s]}^{-1} \cdot f(.,s)|_{[0,t]} \cdot f(0,.)|_{[0,s]}$ is trivial.*

Proof. By symmetry, we may assume that $M'(x)$ is complete for all $x \in M$.

Let $s \in [0,1]$. For all t, let $X(t)$ be the result of the parallel displacement of $\frac{d\phi}{dt}(t)$ along the curve $\phi|_{[0,t]} \cdot \psi|_{[0,s]}^{-1}$. From properties 1, 2, 3 and 5, we deduce that $f(.,s)$, if it exists, is necessarily the development of $X(t)$ into $M'(\psi(s))$, so we have unicity. For the existence, we must check that f defined in this way (for all $s \in [0,1]$) satisfies all of the desired properties.

First, consider t_0 and s_0 such that $[0,t_0] \times [0,s_0]$ lies in a open set which is isometric to a product $V' \times V''$ (obtained with the local version of De Rham's theorem). Then it is clear that Properties 3, 4 and 5 are satisfied for all $(t,s) \in [0,t_0] \times [0,s_0]$.

Cover $\phi([0,1])$ by a finite number of open sets $V_1, \dots V_r$ of the form $V_i' \times V_i''$. Let s_0 such that $f([0,1] \times [0,s_0]) \subseteq V_1 \cup \dots \cup V_r$. Applying Properties 3, 4 and 5 to each V_i one after another, one proves that these properties are satisfied for $(t,s) \in [0,1] \times [0,s_0]$.

Let s_1 be the upper bound of all s_0 such that f satisfies properties 3, 4 and 5 for all $(t,s) \in [0,1] \times [0,s_0]$. Notice that the three properties are satisfied for s_1 itself, by taking the limit when $s \to s_1$, $s < s_1$, in each property. Then, if $s_0 < 1$, we may cover $f(.,s_1)$ by a finite number of open sets of the form $V' \times V''$ and prove, as above, that the three properties are satisfied for $(t,s) \in [0,1] \times [s_1 + \epsilon]$ with $\epsilon > 0$, which contradicts the definition of s_1. Thus, $s_1 = 1$ and the lemma is proved. □

Definition 9.30. Let $\phi : [0,1] \to M$ be any curve, and write $x = \phi(0)$. Let $X(t)$ be the result of the parallel displacement of $\frac{d\phi}{dt}(t)$ along the curve $\phi|_{[0,t]}$ itself. Let $X'(t)$ and $X''(t)$ be the projections of $X(t)$ on $E_x'(M)$ and $E_x''(M)$. We define the *projection of ϕ onto $M'(x)$ (*resp. *$M''(x)$)* as the development of $X'(t)$ into $M'(x)$ (*resp. $M''(x)$*).

In view of Proposition 9.27, the projection onto $M'(x)$ is always well-defined because we assumed that $M'(x)$ is complete, but it is not the case for the projection onto $M''(x)$.

Lemma 9.31. *Let $x, y \in M$ and $\phi : [0,1] \to M$ joining x to y.*

1. *The projection of ϕ onto $M'(x)$ and the projection of ϕ^{-1} onto $M''(y)$ are well-defined. We call them τ' and τ''.*

2. *The curves τ' and τ'' have the same endpoint.*

3. *This endpoint depends only on x and y (but not on the curve ϕ). It is called the* projection of y onto $M'(x)$. *The projection of y onto $M''(x)$ is defined in the same way, by exchanging the roles of the distributions E' and E''.*

Proof. Write $\phi = \phi_r \cdot \dots \cdot \phi_1$, where each ϕ_i lies in an open set V_i of the form $V_i' \times V_i''$. Each curve ϕ_i joins two points, which we name x_i and x_{i+1}.

Consider ϕ_r' the projection of ϕ_r onto $M'(x_r)$, and let ϕ_r'' be projection of ϕ_r^{-1} onto $M''(x_{r+1})$. Since V_r is a product, it is clear that the projections are well-defined and have the same endpoint. Let $\tau_r' = \phi_r'$ and $\tau_r'' = \phi_r''$.

Continue the construction of ϕ_k', ϕ_k'', τ_k' and τ_k'' by induction for $k = r-1$ to 1: define ϕ_k' as the projection of ϕ_k onto $M'(x_k)$, and ϕ_k'' as the projection of ϕ_k^{-1} onto $M''(x_{k+1})$. Notice that the parallel displacement along ϕ_k is the same as the parallel displacement along $(\phi_k'')^{-1} \cdot \phi_k'$. Then, construct a homotopy $f(t,s)$ using the curves $\phi_k''(s)$ and $\tau_{k+1}'(t)$, and Lemma 9.29. Let $\tau_k' = f(.,1) \cdot \phi_k'$ and $\tau_k'' = f(1,.) \cdot \tau_{k+1}''$. Then τ_k' is the projection of $\phi_r \cdot \dots \cdot \phi_k$ onto $M'(x_k)$ and τ_k'' is the projection of $(\phi_r \cdot \dots \cdot \phi_k)^{-1}$ onto $M''(y)$. By

construction, τ'_k and τ''_k have the same endpoint. Thus, assertions 1 and 2 are proved by applying this result for $k = 1$.

Now, we prove that the endpoint of τ' depends only on x and y, *i.e.* does not depend on the choice of ϕ. Consider another curve ψ joining x to y: since M is simply connected, there is a homotopy between the two curves. Moreover, we may assume that this homotopy is "small", *i.e.* there exists $j \in \{0, \dots, r\}$ and $\psi_j : [0,1] \to V_j$ joining x_j to x_{j+1} such that

$$\psi = \phi_r \cdot \ldots \cdot \phi_{j+1} \cdot \psi_j \cdot \phi_{j-1} \cdot \ldots \cdot \phi_1.$$

Make the above construction for ψ. The point is that the parallel transport of E'-vectors along ψ''_j between x_{j+1} and $\psi''_j(1)$ is the same as the parallel transport along ϕ''_j (because ψ''_j and ϕ''_j are both E''-curves joining the same points). Thus, for each k, the endpoint of τ'_k does not change if we consider the curve ψ instead of ϕ. □

Lemma 9.32. *Let $x, y \in M$. Denote by y' and y'' the projections of y onto $M'(x)$ and $M''(x)$. Then y is the projection of y' onto $M'(y'')$, as well as the projection of y'' onto $M''(y')$.*

Proof. Consider a curve ϕ joining x to y, and its projections τ' and τ'', as in Lemma 9.31. Also write α'' the projection of ϕ onto $M''(x)$, and α' the projection of ϕ^{-1} onto $M'(y)$. Consider the curve $\tau = (\tau'')^{-1} \cdot \tau' \cdot (\alpha'')^{-1}$.

By Lemma 9.31, the projections of τ and $(\alpha')^{-1}$ onto $M'(y'')$ have the same endpoint. Since the projection of $(\alpha')^{-1}$ is $(\alpha')^{-1}$ itself, this endpoint is y. Thus, the endpoint of the projection of $\tau' \cdot (\alpha'')^{-1}$ onto $M'(y'')$ is also y. This means exactly that the projection of y' onto $M'(y'')$ is y. □

End of the proof of Theorem 9.4. Let $x_0 \in M$, and define $\Phi : M \to M'(x_0) \times M''(x_0)$, which to a point $y \in M$ associates its projections onto $M'(x_0)$ and $M''(x_0)$. The product $M'(x_0) \times M''(x_0)$ is naturally endowed with the product of the metrics induced by M. We claim that Φ is an isometry between M and $M'(x_0) \times M''(x_0)$.

Injectivity is a direct consequence of Lemma 9.32.

For surjectivity, consider any $(y', y'') \in M'(x_0) \times M''(x_0)$, and let y be the projection of y' onto $M'(y'')$. Since x_0 is the projection of y' onto $M''(y'')$, it results from Lemma 9.32 that the projection of y onto $M'(x_0)$ is y'. Symmetrically, the projection of y onto $M''(x)$ is y'', and thus $\Phi(y) = (y', y'')$.

The only thing left to do is to show that Φ is isometric at each point. Consider $y \in M$ and $v \in T_yM$, with the decomposition $v = v' + v''$. Let $\phi : [0,1] \to M$ be a curve joining x_0 to y, such that $\frac{d}{dt}\big|_{t=1} \phi = v$. Let τ be the projection of ϕ onto $M'(x_0)$. By definition of the projection, $\frac{d}{dt}\big|_{t=1} \tau$ is the parallel displacement of v' along $\tau \cdot \phi^{-1}$. Thus, v' has the same length as $\frac{d}{dt}\big|_{t=1} \tau$, which has itself the same length as $\Phi_*(v')$. The same result applies to v'', so Φ is an isometry, and Theorem 9.4 is proved.

Appendix B. Two other proofs of Theorem 9.15

We present two alternative proofs of Theorem 9.15. We keep the notations introduced at the beginning of Section 9.3.

B1. A simpler proof in the analytic framework

This proof uses Proposition 9.18 and Lemma 9.22.

Assume that (T, g) is not Riemannian: by Proposition 9.18, it is not equicontinuous, so for all $m > 0$, there exists $\gamma = \gamma_{i_{p-1}i_p} \circ \ldots \circ \gamma_{i_1 i_2} \in \Gamma$ with ratio $\leq m$ at some $f_{i_1}(x)$, $x \in M$. With Lemma 9.22, we may assume that for all $l \in \{1, \ldots, p-1\}$, $d_{i_l}(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(x)), \partial U_{i_l}) > \epsilon_0$, and $\gamma_{i_l i_{l+1}}$ is defined on $B_g(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(x)), \epsilon_0)$.

Choose $\epsilon \in (0, \epsilon_0)$ such that the injectivity radius of the exponential map of (T, g) is at least ϵ at every point $x \in T$ satisfying $d_i(x, \partial T_i) \geq \epsilon_0$.

If m is chosen small enough, there exist s, t with $1 \leq s \leq t \leq p$ such that:

1. $i_t = i_s$;
2. $d_{i_s}(\gamma_{i_{s-1}i_s} \circ \ldots \circ \gamma_{i_1 i_2}(f_{i_1}(x)), \gamma_{i_{t-1}i_t} \circ \ldots \circ \gamma_{i_1 i_2}(f_{i_1}(x))) < \epsilon/2$;
3. $\gamma_{i_{t-1}i_t} \circ \ldots \circ \gamma_{i_s i_{s+1}}$ has ratio $\leq 1/2$ at $\gamma_{i_{s-1}i_s} \circ \ldots \circ \gamma_{i_1 i_2}(f_{i_1}(x))$.

Let $\gamma_0 = \gamma_{i_{t-1}i_t} \circ \ldots \circ \gamma_{i_s i_{s+1}}$. In the analytic framework, there is a canonical way to extend the domain of γ_0 to the closed ball $B = B_g(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_t}(x)), \epsilon)$ so that γ_0 remains a similarity of ratio $\leq 1/2$, using the exponential map of g: the new γ_0 maps the ball B into itself, and thus it has a fixed point. Therefore, the transversal T is flat on B by Proposition 9.20, and again by analyticity, it is flat everywhere, which concludes the proof.

B2. Another proof in the C^∞ framework

This proof uses Proposition 9.18, Proposition 9.19 and Lemma 9.22.

The *equicontinuity domain* F is defined as the set of all $x \in M$ for which there exists $m > 1$ such that for all $i \in \{1, \ldots, r\}$ with $x \in U_i$, every $\gamma \in \Gamma$ defined on $f_i(x)$ has ratio between $1/m$ and m at $f_i(x)$. Assume that the foliation is not transversally flat: then the set F is non-empty by Proposition 9.19. We want to prove that $F = M$.

It is possible to show that F is open, by copying the proof that E is open in Lemma 9.23: we leave this to the reader.

Let (K_n) be an exhaustion of F by compact sets, *i.e.* an increasing sequence of compact subsets of F whose union is F. Since F is saturated (*i.e.* it is a union of leaves), it is possible to assume that every set K_n is itself saturated (take the closure of the union of all leaves which intersect K_n: this new set is saturated, since it is the closure of a saturated set).

Choose n so large that for all $i \in \{1, \ldots, r\}$ and all $x \in U_i \cap \partial F$ such that $d_i(x, \partial U_i) \geq \epsilon_0$, the ball $B_g(f_i(x), \epsilon_0)$ intersects $f_i(K_n \cap U_i)$. Then there is also an $\epsilon_1 > 0$ such that for all $i \in \{1, \ldots, r\}$ and all $x \in U_i \cap \partial F$, the ball $B_g(f_i(x), \epsilon_1)$ does not intersect $f_i(K_n \cap U_i)$.

Assuming that $F \neq M$, we choose x_0 on ∂F (the boundary of F) and look for a contradiction. Notice that the whole leaf of x_0 is contained in ∂F, because F is saturated. The idea is that, under the action of the holonomy group, the images of a transverse ball centered at x_0 remain small because their center is on ∂F, while they cannot intersect K_n: thus x_0 is in the equicontinuity domain F, which contradicts the fact that F is open. The details now follow.

Consider $i \in \{1, \ldots, r\}$ such that $x_0 \in U_i$ and $\gamma \in \Gamma$ defined on a neighborhood of $f_i(x_0)$. With Lemma 9.22, there exists a $\tilde{\gamma} = \gamma_{i_{p-1}i_p} \circ \cdots \circ \gamma_{i_1 i_2}$ defined on a neighborhood

of $f_{i_1}(x_0)$, which coincides with $\gamma_{ji_p} \circ \gamma \circ \gamma_{i_1 i}$ near $f_{i_1}(x_0)$, such that for all $l \in \{1, \ldots, p-1\}$, $d_{i_l}(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(x_0)), \partial U_{i_l}) > \epsilon_0$, and $\gamma_{i_l i_{l+1}}$ is defined on $B_g(\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}(f_{i_1}(x_0)), \epsilon_0)$.

Since K_n is saturated, the image of $B_g(f_{i_1}(x_0), \epsilon_1)$ by $\gamma_{i_{l-1}i_l} \circ \cdots \circ \gamma_{i_1 i_2}$ (when it is defined) does not intersect $f_{i_l}(U_{i_l} \cap K_n)$. Thus, this image is a ball of radius $\leq \epsilon_0$. By induction, this implies that this image is always well-defined, and thus, the ratio of γ is smaller than $k_{\max}^2 \epsilon_0 / \epsilon_1$ at $f_i(x_0)$ (where $k_{\max}$ is still the maximum ratio of the γ_{ij}). By considering γ^{-1}, we see that the ratio is also bigger than $\epsilon_1 / (k_{\max}^2 \epsilon_0)$. Thus, $x_0 \in F$, which concludes the proof.

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