Observability Analysis and Observer Design for Complex Dynamical Systems

Gang Zheng

Université Lille 1 - Science et Technologie

HABILITATION À DIRIGER DES RECHERCHES

Spécialité : Automatique, Génie Informatique, Traitement du Signal et Images

par

Gang Zheng

Non-A team, Inria Lille / SyNer-CO2, CRIStAL, CNRS UMR 9189

Observability Analysis and Observer Design for Complex Dynamical Systems

Soutenue le 26 Novembre 2015 devant le jury d'examen :

Président Olivier Colot Professor, Université Lille 1

Rapporteur Bernard Brogliato Directeur de Recherche, INRIA
Rapporteur Wei Kang Professor, Naval Postgraduate School, USA
Rapporteur Claude Moog Directeur de Recherche, CNRS

Examinateur Arie Levant Professor, Tel-Aviv University, Israel
Examinateur Wilfrid Perruquetti Professor, École Centrale de Lille
Examinateur Jean-Pierre Richard Professor, École Centrale de Lille

To my wife Dan, and my daughters Lisa and Linda

Acknowledgements

I would like to acknowledge firstly Bernard Brogliato, Wei Kang and Claude Moog for their kind acceptances to review my work, and thank Olivier Colot, Arie Levant and Wilfrid Perruquetti to be the members of jury. Especially, I would like to express my sincere appreciation to Jean-Pierre Richard who brought me to Lille, and accepted to supervise this work.

This document summarizes my recent theoretical works after my recruitment by Inria in 2009. Therefore, I would like to express my sincere gratitude to all my colleagues in the team Non-A (ex-Alien) of Inria, and SyNer of CRIStAL Cnrs: Alexandre, Andrey, Anne-Marie, Denis, Jean-Pierre B&R, Laurentiu, Lotfi, Nicolai, Rosane, Thierry, Wilfrid.... Also, I would especially like to thank my friends (Antoine, Arnaud, Dayan, Driss, Francisco, Jesus, Lei, Malek, Mohamed, Yury...) and all my Ph.D. students (Yingchong, Ramdane, Matteo and Zhilong). The list is too long to be finished, and I think you sincerely to kindly accept to work with me during the years.

Further thanks give to my parents and family for their incessant love and care. I would like to thank my dearest Dan who is always patient, and my lovely princesses Lisa and Linda who bring me so many happiness...

Abstract

Different types of dynamical systems have been widely used to model different plants to be controlled in many different disciplines. Although control is the final goal in the control theory, given a concrete model, the designed controller sometimes could depend not only on the output, but also its internal states. This motivates us to study whether it is possible to reconstruct system's internal states by using external measurements, named as the problem of observability analysis and observer design.

This manuscript will summarize our obtained results on observability analysis and observer design for three types of dynamical systems, respectively in three parts. Part I studies the observability analysis and observer design for ordinary systems in continuous time, where the differential geometrical approach and the technique of immersion are used to study the observation problem. Finite-time and interval observers are proposed for time-continues systems. The second Part of this manuscript concerns the observability analysis and observer design for singular system. For linear singular system, the observability will be analyzed by using elementary algebraic method, while the differential geometrical method is extended to study nonlinear singular system. Luenberger-like and interval observers are studied for different types nonlinear singular systems. The observability analysis and observer design for time-delay system is considered in Part III. By introducing delay operator, the backward (causal) and forward (non-causal) unknown input observability are defined. Sufficient conditions are given to investigate the observability of a quite general linear/nonlinear time-delay system with unknown inputs. Luenberger-like observer is also studied for linear time-delay systems.

Table of contents

Curriculum Vitae

Personal information

Name:	Gang ZHENG
Gender:	Male
Birthday:	January 3, 1979
Address:	INRIA Lille, 40 Avenue Halley, 59650, Villeneuve d'Ascq, France
Telephone:	(+33) 3 59 57 79 53
Email:	gang.zheng@inria.fr

Education

09/2004 - 11/2006	Ph.D. of Automatic control, University of Cergy-Pontoise, France
09/2001 - 07/2004	Master of Information and systems, Wuhan University, China
09/1997 - 07/2001	Bachelor of Information and systems, Wuhan University, China

Working experience

09/2011 - Present	First class researcher, INRIA
09/2009 - 09/2011	Second class researcher, INRIA
09/2008 - 08/2009	Full time ATER, ENSEA
01/2008 - 08/2008	Post-doc of LJK/CNRS
12/2006 - 12/2007	Post-doc of BiPop/INRIA

Ph.D. students

- Kokou LANGUEH : Ph.D. from October 2015 to October 2018. Co-supervised with Dr. Thierry Floquet. Subject : Observation and left invertibility of time-delay systems;
- Matteo GUERRA: Ph.D. from October 2012 to October 2015. Co-supervised with Prof. Wilfrid Perruquetti, Dr. Denis Efimov. Subject : Supervisor control of multi-robots;
- Zilong SHAO: Ph.D. from Feberury 2013 to March 2016. Co-supervised with Prof. Wilfrid Perruquetti, Dr. Denis Efimov. Subject : Robust control of manipulators;
- Ramdane TAMI: Ph.D. defended in December 2014. Co-supervised with Prof. Driss Boutat. Subject : Observability analysis of nonlinear systems.
  Now Post-doc with Prof Frédéric Kratz of INSA Centre de Loire in France.
- Yingchong MA: Ph.D. defended in December 2013. Co-supervised with Prof. Wilfrid Perruquetti. Subject : Cooperative control of mobile robots.
  Now Post-doc with Prof. Shiller Zvi at Ariel University, Isreal.

Research projects:

- ARCIR Estimation distribuée de systm̀mes dynamiques en réseaux, 2014-2016, supported by Conseil Régional Nord-Pas de Calais (participant)
- Inria ADT SLIM: Software Library for the cooperation of Multi-robots, 2012-2014, supported by Inria (coordinator)
- Interreg IV A 2 Sysiass, 2010 -2014, supported by FEDER (participant)
- CPER CIA, 'Internet of Things', 2011-2015, supported by Inria Lille and Conseil Régional Nord-Pas de Calais (participant)
- ANR ChaSliM : Chattering-free Sliding Modes, 2011-2015, supported by ANR (participant)
- Inria ADT SENSAS : SENSor network ApplicationS, 2010-2014, supported by Inria (participant)

Research interests

Observation of nonlinear system; Identification of parameters; Normal form; Control of nonlinear systems; Chaotic system and its application in communication, Robotics.

Principal contributions

My main contributions of research activities contain theoretical research on observability analysis and observer design for dynamical systems, and practical application to robotic control.

Contribution on research

During my research, I have studied different types of dynamical systems, including nonlinear, hybrid, time-delay, impulsive and singular systems. My contributions can be listed below:

- For the observation of nonlinear systems, I have proposed a new class of general normal forms, whose linear part is parameterized by its outputs. This new class of normal forms covers quite a large class of dynamical systems. The corresponding observer design problems have been studied as well. The following are my publications on this topic [28, 33, 79, 159, 178, 181, 179, 180, 194, 213, 214, 216–218, 211, 212].
- For hybrid systems, I studied the observability property, and proposed a new algebraic observer for switched system with zenon phenomenon. I also presented a result on uniform controller for hybrid systems [226, 219, 194, 202, 220].
- Concerning time-delay systems, I studied the observability of time-delay systems with unknown inputs, including causal observability and non causal one. The associated result on delay identification was reported as well during my research. My works related to this aspect ares [15, 101, 204, 210].
- For the study of impulsive systems, I investigated the observer design problem of linear and nonlinear impulsive systems with impact perturbation. By using the concept of normal form, I proposed a finite time observer, which guarantees the finite time convergence independent of the persistently acting impact. Reduce-order observer was developed as well for the proposed normal form [224, 225].

- For singular systems, the observability and detectability of a general class of singular linear systems with unknown inputs are tackled. It is shown that, under suitable assumptions, the original problem can be studied by means of a regular (non-singular) linear system with unknown inputs with algebraic constraints. The work on this topic gives [10, 14, 16, 32, 206–208].

Except the above mentioned work, my recent research covers as well Compressive Sensing (CS), which provides an alternative to Shannon/Nyquist sampling when signal under acquisition is known to be sparse or compressible. In my work, I proposed to construct the sensing matrix with chaotic sequence, introduced the cluster structured sparsity in CS, and proposed a new algorithm to solve this problem. My publications related to this topics are [195–198, 200, 201, 199]. Concerning the application of obtained results, I studied the secure data transmission based on chaos, since chaotic system is quite sensitive to the disturbance. I proposed several efficient schemes to design transmitter and receiver [82, 215, 221, 222, 203, 223].

Contribution on innovation

Except the scientific research, I paid as well attention to the technological innovations, which are mainly for the control of mobile robot, robotic manipulator and Blimp. We have developed some new platforms and validated our algorithms in ROS which contain:

- Local path planning for mobile robots based on intermediate objectives [127, 132].

- Motion planning for robots using the intelligent-PID controller and potential field [130, 129, 128, 131].

- Finite-time obstacle avoidance for mobile robot with additive input disturbances [81].

- Adaptive precise position control for a low-cost manipulator[166].

- Development of autonomous Blimp for zone surveillance.

Contribution on knowledge diffusion

In order to diffuse our scientific results and technological innovations, with my supervised engineers, we have established 2 demonstrators at Euratechnologie Inria Lille, and participated 1 scientific salon.

- The first demonstrator, called RobotCity, was developed in 2011 for the purpose of illustrating the feasibility of our path planning algorithm and robust controller.
http://researchers.lille.inria.fr/gzheng/Videos/robotcity.wmv

- In 2014, we established the second demonstrator on mobile robot, named RCG, whose objective is to surveil the area with mobile robots to detect any anomalies while avoiding obstacles.
http://researchers.lille.inria.fr/gzheng/Videos/Robcopguard.mp4

- Besides these two demonstrators, in order to diffuse our scientific results, we have participated Innorobo 2014 at Lyon.
  http://researchers.lille.inria.fr/gzheng/Videos/Innorobo.avi

Teaching

I regularly teach automatic control, mobile robots at Ecole Centrale de Lille, University of Lille 1, ISEN Lille, and ENSAM Lille.

Publications

I have published 30 journal papers, 2 book chapters, and 54 conferences. The list of my publications can be found at the end of this manuscript.

This manuscript will only summarize my results on Observability analysis and Observer design for three types of dynamical systems: continuous-time, singular and time-delay systems.

Chapter 1

General Introduction

Dynamical systems have been widely used to model different plants to be controlled in many different disciplines, ranging from biology, chemistry, to mechanics and so on. Nowadays, one of the most popular ways to model the real process is to use the state-space equation, described normally by the ordinary differential equation, which is named as continuous-time system in the literature. Of course, when the modeling by ODE is not feasible, we can try other systems, such as singular and time-delay systems, non-smooth systems [36], or even PDEs. From engineering point of view, a wide variety of information cannot directly be obtained through measurement for the modeling systems. Due to some economical or technological reasons, we cannot place as many sensors as we want to measure the internal information, since it costs expensive, or sometimes impossible. Besides, given a concrete plat, some kinds of inputs (external disturbances for example) and some parameters (constants of an electrical actuator, delay in a transmission...) are unknown or are not measured, whose estimates are sometimes needed to be used in the closed-loop controller. Similarly, more often than not, signals from sensors are distorted and tainted by measurement noises. Therefore, although control is the final goal in the control theory, given a concrete model, in order to simulate, to control or to supervise processes, and to extract information conveyed by the signals, one often has to estimate parameters, internal variables, or the unknown inputs.

Estimation techniques are, under various guises, present in many parts of control, signal processing and applied mathematics. Such an important area gave rise to a huge international literature that cannot be summarized here. Roughly speaking, in automatic control, the estimation covers at least the following topics:

- Identification of uncertain parameters in the system equations, including delays;
- Estimation of state variables, which are not measured;
- Observation fault diagnosis (unknown input) and isolation;

From control point of view, we know the problem of the parameter estimation can be converted into the problem of state observation, by regarding those parameters as additional states, and then extending the system's state with zero dynamics, provided that they are (piece-wise) constant, or slowly time-varying. Therefore, this manuscript will focus on the state estimation problem with known or unknown inputs.

When dealing with state observation problem for the studied system, normally we need to consider the following two important issues:

- **Observability:** With the available external measurement (output), is it possible to reconstruct the internal information (states)?

- **Observer design:** Which kind of observer can be designed (asymptotic or not, interval, ...), and how?

The concept of controllability and observability are two key points when Kalman established the modern control theory for linear time-invariant system around 1960 [64]. The aim to analyze the observability for the studied system is to find out the conditions (at least sufficient ones) under which the internal states of the treated systems can be estimated. Depending on the properties of the studied system (linear or nonlinear, for example), methodologically researchers adopts two quite different approaches to study this problem:

- Algebraic method;
- Differential geometric method;

As it was mentioned in [64] that the entire modern theory of linear constant dynamical systems can be viewed as a systematic development of the equivalent algebraic conditions on controllability and observability. In fact, the use of modules to study those properties can be dated back to 1920 under the influence of E. Noether. This method can be easily extended to study other types of linear systems, including linear time-delay systems [173]. However, the generalization of the similar theory to nonlinear system is not so trivial. Till the beginning of 1970, based on the works of Chow [43], Hermann [85], Sussman [176], Krener showed in his thesis that the differential geometric method is very powerful to analyze the controllability and observability for nonlinear systems. By using geometric method, in 1977 with Hermann, he gave the definitive treatment of controllability and observability for nonlinear systems [86]. This method opened a door in the control domain to study many control problems for nonlinear systems, and lots of results are published by using differential geometric method in 1980s, such as disturbance decoupling, feedback linearization, output injection [36]. However, as stated in [46] some special problems for nonlinear systems, such as system inversion, or the synthesis of dynamic feedback, could hardly be tackled with the already well-established differential geometric methods. This forces researchers turn back to algebraic method. Introduced by Fliess in [68, 71], the differential algebraic methods were used as well to solve many problems for nonlinear systems, including observability [57], invertibility [69, 70], dynamic feedback linearization [4], and so on.

Concerning the theory of observer design in control domain, it was initialized by Kalman [102] and Luenberger [124], where the observation error was used to drive the designed observer. After that, lots of different types of techniques have been developed to estimate the interval states of the studied system. From a general point of view, we can classify those techniques into two different categories:

- Estimation by using differentiator;
- Estimation by using observer.

If the studied system is fully observable, algebraically all its states can be represented as a function of the output (and input if it is known) and its derivatives [56]. This implies that an efficient differentiator, enabling to calculate the successive derivative of the signal, is enough to estimate the states. The only problem is that the not well designed differentiator will amplify the high frequency noise, if the signal is corrupted. Estimating the derivatives of noisy signals is a longstanding problem in numerical analysis, in signal processing and control. It has attracted a lot of attention due to its importance in many fields of engineering and applied mathematics. Differentiating a noisy signal at various orders is a fundamental issue. A number of different approaches have been proposed and are recalled in [120]. In the control domain, we can cite two important ones: algebraic differentiator [137] and higher-order sliding modes differentiator [118]. They both have the ability to attenuate the high frequency noise when calculating high-order derivatives for a given corrupted signal. The second category is a typical topic in the control domain. The list of different types of observers is quite long, including Kalman observer [102], Luenberger observer [125], adaptive observer [123], high-gain observer [78], sliding mode observer (finite-time or fixed-time) [59], moving horizon observer [104], dissipative observer [169], interval observer [136] ... Those observers have been studied for different types of systems.

It is worth noting that the observability and observer design are not two equivalent concepts. The system is observable implies the existence of observer, but the inverse is not always true, since the system can be detectable. In other word, the deduced observability condition is only sufficient for some types of observers, such as algebraic observer. Sometimes other sorts of observers (Luenberger-like, high-gain,...) could ask additional conditions, for example the well-known Lipschitz condition when treating nonlinear continuous-time systems.

As we have mentioned in the beginning, different plants request different types of models, such as continuous/discrete-time, delays, singular and so on. Therefore, the observability analysis and observer design are logically dependent of the properties of each kind of model. During my research activities, besides the classical continuous-time system, I encountered as well singular system and time-delay system. A concrete application is the formation of several mobile robots with delayed transmission. For each robot, their dynamics are continuous-time. In order to keep the desired formation for different robots, the algebraic constraints have to be imposed. Besides, the measurement of robot's state can be delayed in reality. This motivated me to focus on the observability analysis and observer design for these three different types of systems (continuous-time, singular, and time-delay). The following will briefly explain the differences and difficulties when treating these three types of systems. For each sort of systems, the relevant works published in the literature will be presented in the detail respectively in Chapter 2 for continuous-time system, in Chapter 7 for singular system, and in Chapter 12 for time-delay systems.

1.1 Differences and difficulties

Continuous-time systems

Considering the observability for continuous-time system, the simplest one is the linear time-invariant system. Due to the fact that the studied system is linear, we can normally use the elementary algebra (vector space, linear transformation, rank, image) to analyze its properties, such as observability and detectability for systems with known or unknown inputs. For such type of systems, several different but equivalent definitions of observability are proposed in the literature. The most common way, like for nonlinear system, is the distinguishability [86, 183], i.e. different initial conditions yield different outputs. Since zero initial conditions produce the zero outputs, the above property is equivalent to say that non-zero initial condition implies non-zero outputs. Moreover, since the output of the system is uniquely determined by its initial condition, we can also characterize the observability for LTI system thought the ability to reconstruct the initial conditions. For LTI, this expression leads to the well-known invertibility of the observability Gramian, which can be used as a criteria to test whether the studied system is observable or not. Of course, some other checkable conditions are proposed to test the observability, such as the famous Kalman rank condition [103], Popov-Belevich-Hautus test condition [151, 17, 83], unobservable space and invariant zero set [183].

For linear system with unknown input, the definition of observability was generalized in the sense that the state can be reconstructed for all unknown inputs, i.e. independent of the unknown inputs. The generalization is named as strong observability in the literature [84]. Similar expressions, as for LTI system with known inputs, were used as well to characterize it from different aspects, like distinguishability for all unknown inputs, non-zero initial condition implies non-zero outputs for all unknown inputs, and so on [183].

Unlike LTI system where we can use the elementary algebraic approach to analyze the observability, for nonlinear system, normally the differential geometric method is used to characterize the observability [86]. Sometimes, we can also mix these two methods, if necessary. But this time, since the model is nonlinear, thus we need to use the abstract algebra (meromorphic functions, field, one-form...) [46]. Generally speaking, we are seeking an injective map from the output and its derivatives to the state. Due to the nonlinear properties, the observability might be local, thus different definitions can be found in the literature. Nowadays, most of researchers adopt the definitions given by Hermann and Krener, including global, local, weak ones [86] and the algebraic definition given by Fliess [56]. For nonlinear system with known input, we have also a nice sufficient condition to judge the local observability, which is called as well the observability rank condition, by using the Lie derivative. For the system with unknown input, one can refer to the work [171].

Singular systems

Due to the characteristic that singular systems contain both differential and algebraic equations, even for the linear case, singular system might contain impulse if the initial condition is not consistent or the input is not enough differentiable. Therefore, the well-defined concept of observability for singular systems have to be reconsidered with respect to that for regular (non-singular) systems. In fact, this special characteristic (i.e, might contain impulse) leads to different definitions, including observability, R-observability and Impulse-observability [49]. Generally speaking, they characterize the state reconstruction ability from different aspects: R-observability defines the ability to estimate the reachable set of the studied system. Impulse-observability corresponds to the ability to estimate the impulse term of the studied system and the observability covers both mentioned abilities to estimate all states of the studied system.

For linear singular systems, since it might contain impulse, the state could be non-differentiable. In this situation, the solution can be represented in the framework of Schwartz distributions [45], by introducing Dirac function to allow the derivative of discontinuous signal. For the linear case, we can also use the distinguishability to characterize the different definitions of observability. R-observability requests only to distinguish two reachable states. Impulse-observability considers only the impulsive part [89]. Since the solution explicitly consists of two parts: impulsive part and non-impulsive part, due to the speciality of Dirac function, we can even more have the similar 'zero output implies zero state' statement to define the different concepts of observability. The only thing we need to pay attention is to separate the impulsive and non-impulsive parts in the output, and to precise which part in the output and in the state is zero. For example, if a zero impulsive part in the output implies the impulsive part in the state is zero, then it gives the definition of the Impulsive-observability.

For those mentioned different definitions, we can always follow similar ideas, used to analyze the linear time-invariant system, to study the observability of linear singular system. Those approaches are still based on the elementary algebra. The similar Hautus rank condition, Invariant zero set, and Monilari algorithm can be also deduced for linear singular system with known or unknown input [11]. The corresponding conditions can be then obtained as well to design the simple Luenberger-like observer for linear singular system ([90, 53]).

Considering nonlinear singular system, the distinguishability is still valid to define the observability. Therefore, like nonlinear regular system, we can use as well the differential geometric method to analyze its observability [12].

Time-delay systems

The analysis of observation for time-delay systems can be dated back to the 80's of the last century [117, 145, 160, 153]. The main difficulty on the observability analysis for time-delay system is due to the fact that its state is infinite-dimensional, which in fact is a collection of information which contains the history of the system. This characteristic can explain why the initial condition for time-delay system is normally a function. For systems without delay, the observability can be characterized by the estimation of the initial condition, which in fact is equivalent to the reconstructibility of the state. For linear systems with delay, we can also characterize the observability by this property, In [144], the so called initial observability was proposed. This definition is useful when the primary goal is to estimate the initial (past) states. If the initial condition $x(t)$ for $t < 0$ is zero or a known function, and $x(0) \in \mathbb{R}^n$ is arbitrarily unknown, the initial observability becomes as $\mathbb{R}^n$-observability, introduced by [77]. This notion is useful if the objective is to estimate the instantaneous disturbance. As it has been pointed out in the literature, for linear system with delay, the notion of observability for initial condition is not equivalent to the reconstructibility of the system's final trajectory [117]. More precisely, for systems without delay, the knowledge of the initial condition is equivalent to know the final state. However, for systems with delay, the knowledge of initial condition is only sufficient, but not necessary to estimate the final trajectory of the state. Therefore, although the natural extension of observability from the linear systems without delay to the ones with delay is to as well estimate the initial condition, it is in fact not very useful since in practice the most important purpose is indeed to estimate the final trajectory of the state at any time, not the initial condition. Due to this fact, other concepts of observability related to the reconstructibility of the final trajectory of the states were introduced. As we know that the reconstruction can be finite-time, or infinite-time (asymptotic, when the time tends to infinity), therefore two different definitions on observability are proposed in [145]: finally observable (finite-time) and infinite-time observable (asymptotic, as detectability). Besides, there exist as well two observability concepts defined by using a formal algebraic way, called as weak observability and strong observability [164]. The studied system is strongly observable (i.e. observable over the polynomial ring) if the observability matrix with delay operator is left invertible over the polynomial ring. It is said to be weakly observable (i.e. observable over the real field) if this observability matrix has a left inverse over the real field. The last four observability definitions are for the purpose of only estimating the final trajectory of the states. For the sake of simplicity and convenience (coherent to the observability definition for other types of systems), we will use the word 'state' when treating time-delay systems in this manuscript, which in fact means the final trajectory of the infinite-dimensional state of the studied system.

For linear time-delay systems, various aspects of the observability problem have been studied in the literature, using different methods such as the functional analytic approach [23] or the algebraic approach [35, 72, 173] (polynomial ring, Smith form, Hermit form...). For linear time-delay system, by using the abstract algebra with polynomial ring, the matrices are not constant any more, but polynomials of delay operator. Due to the similarity, we can still follow the same ideas (Molinari algorithm, Hautus-like condition) for LTI to deduce necessary and sufficient conditions of observability for systems with delay. For nonlinear time-delay systems, the theory of non-commutative rings was firstly proposed in [141] to disturbance decoupling problem. After that, the observability problem has been studied in [188] for systems with known inputs. The nonlinear time-delay system with unknown inputs was studied in [209] by using the same approach. The associated observer for some classes of time delay systems can be found in [47, 164, 51, 66, 76] and the references therein.

Comparisons

From the above discussions, it is clear that different types of systems have different properties, thus the observability should be analyzed differently. Generally speaking, the distinguishability is the most common way to define the observability for different types of systems, which was firstly proposed for nonlinear continuous-time systems. Compared to systems described by ODEs, the singular systems might contain impulse, therefore, the definitions of observability are adapted by considering the ability to estimate the impulsive terms. Considering the time-delay systems, in this case the observability depends on the delay, therefore the estimation could be causal (i.e. needs only the past information) or non-causal (i.e. depends as well the future information). Hence, the definitions of observability need to be adapted by taking into account this issue. Moreover, when the inputs are unknown for those types of systems (continuous-time, singular and time-delay), the observability should not depend on the unknown input, which yields the so called strong observability.

1.2 Organization

This manuscript will only summarize the obtained result on observability analysis and observer design for three types of dynamical systems: continuous-time, singular and time-delay systems.

Part I studies the observability analysis and observer design for nonlinear continuous-time system. Chapter 2 presents some existing works on these two topics for nonlinear continuous-time systems, as well as the motivations of our researches. In Chapter 3, by using the technique of immersion, we propose an extended output-depending normal form, for which a simple high-gain observer can be designed. Here, we use differential geometric method to deduce necessary and sufficient conditions which guarantee the existence of a diffeomorphism to transform the general nonlinear system into the proposed normal form. Concerning partial observable case, Chapter 4 treats this problem by employing again the differential geometric approach to study the observation problem. We investigate a special partial observer normal form, and give necessary and sufficient conditions as well to transform the studied system into this special form. For the output-depending normal form, there exists an asymptotic observer, such as high-gain observer. However, we are wondering whether it is possible to design a finite-time observer for such a form. This problem will be studied in Chapter 5. The above mentioned chapters are based on the assumption that the model of the studied system is precise, i.e. without uncertainties. If the systems suffer from unknown disturbances, normally the asymptotic estimation becomes impossible. In this situation, the last chapter of this part deals with the interval observation for such an uncertain nonlinear systems.

The second Part of this manuscript concerns the observability analysis and observer design for singular system. Like Part I, Chapter 7 gives a general introduction of these two topics for linear and nonlinear singular systems. For linear singular system, the observability will be analyzed in Chapter 8, where necessary and sufficient conditions will be deduced. Moreover, a constructive estimation method for the state is presented. The observability for nonlinear singular system is investigated in Chapter 9. In this chapter, we will use the well-known concept of zero-dynamics in nonlinear system to analyze the observability of nonlinear singular systems. Again, sufficient conditions are reported in this chapter. Like the first part of this manuscript, we are trying as well to extend the differential geometric method to study nonlinear singular system. For this, Chapter 10 studies a special class of nonlinear singular systems which can be regularized into a regular nonlinear system, and then the conventional Lie-bracket conditions are deduced. Finally, the uncertain nonlinear singular system is considered in Chapter 11. Following what we did for nonlinear continuous-time system in Part I, the same interval estimation technique is applied as well to nonlinear uncertain singular system, and an interval observer is proposed for this issue.

The observability analysis and observer design for time-delay system is considered in Part III. The motivation with respect to the existing result for linear and nonlinear time-delay systems is introduced in Chapter 12. Concerning linear time-delay system, the backward and forward unknown input observability are defined in Chapter 13. Based on these definitions, sufficient conditions are given to investigate the observability of a quite general linear time-delay system with unknown inputs. Here, we generalize the conventional algorithm for LTI system (such as Monilari, Silverman) to study linear time-delay system. Things become more complicated when studying nonlinear time-delay system. By extending the Lie-derivative in the sense of non-commutative rings, we deduce sufficient conditions on the observability for nonlinear time-delay systems with unknown inputs in Chapter 14. Moreover, the identifiability of delay in the nonlinear time-delay system is treated as well in this chapter. For the general linear time-delay system with unknown input studied in Chapter 13, Chapter 15 tries to answer the question whether it is possible to design a simple Luenberger-like observer, as what we did for LTI system. In this chapter, some sufficient conditions are deduced by using the concept of polynomial ring, and we show that they are equivalent to necessary and sufficient conditions for the existence of a Luenberger-like observer when treating LTI system without delay.

Finally, this manuscript, in the final part, ends up with some conclusions and potential research perspectives.

Part I

O&O for Continuous-Time System

Chapter 2

Introduction

Observability and observer design problem for linear systems have been exhaustively studied [183]. It becomes more complex when studying the nonlinear dynamical systems. During last four decades, many different methods have been proposed for observability analysis and observer design.

Unlike linear time-invariant system where the algebraic approach can be applied to analyze the observability, for nonlinear system, normally we use the differential geometric method to characterize the observability. Sometimes, we can also mix these two methods, if necessary. Due to the nonlinear properties, the observability might be local, thus different definitions can be found in the literature. Nowadays, most of researchers adopt the definitions given by [86], including global, local, weakly ones. For the system with unknown input, one can refer to the work of [171].

Concerning the observer design for nonlinear systems, conceptually, there exist two quite different methods. Given a general nonlinear system, the first method tries to directly design an observer, either asymptotic or finite-time. Up to now, many asymptotic observers have been widely studied, such as Luenberger observer [125], high-gain observer [39] and so on [21, 2, 37]. Compared to asymptotic observer, finite-time one was less studied in the literature, which however is well appreciated in practice. Different methods have been proposed, such as sliding mode technique [148, 74], delay measurement [161], output injection [63], algebraic methods [73, 8] and homogeneity [25, 24]. The global finite-time observer based on homogeneity was firstly introduced by [149] for the nonlinear systems which can be transformed into a linear system with output injection. After that, [168] extended this idea and proposed a semi-global finite-time observer for the special systems with triangular structure. The global finite-time observer for such a system was studied respectively in [139] by introducing the second gain, in [167] and [38] by introducing adaptive gains.

The second methods is to transform the nonlinear system into a more simple form which enables us to apply existing observers. The first idea of normal form is due to [22] for time variant dynamical systems, and to [113] for time invariant dynamical systems, where the author introduced the so-called observer canonical form with output injection with all nonlinear terms being only function of the output. Then [114] gave the associated canonical form with output injection for multi-outputs nonlinear systems without inputs, and the result for multi-outputs systems with inputs was studied in [187]. Based on the above works many algorithms are developed to generalize the existing results, including algebraic approaches ([108, 150, 158]), geometric approaches ([92, 133, 31, 126]) and the so called direct transformations stated in [121]. To enlarge the class of observer forms, the concept of output depending normal form was firstly addressed in [80, 156], then was developed in [214] and [185, 186]. The other approaches to enlarge the class of normal forms are the extended normal form introduced in ([99, 143, 6, 191, 192]), where the main idea is to add an auxiliary dynamics to the dynamical system in such a way that the extended system can satisfy the conditions proposed in the literature. The geometric characterization of the second method was addressed in [29] and [30].

Concerning the second method based on differential geometric approach, the solvability of the problem requires the restrictive commutative Lie bracket condition for the deduced vector fields. In order to relax this restriction, we can reconstruct a new family of vector fields which can satisfy the commutative Lie bracket condition. Moreover, inspired by this solution, we can also use the technique of immersion to construct a less restrictive family of vector fields. Sometimes, we can also apply a change of coordinates on the output to relax this restriction. If the system is not fully observable, the same method can be adapted to treat partially observable case [105].

For a more general normal form, such as output-depending one, due to the fact that the existing results deal only with the triangular systems with linear constant part which is of Brunovsky form, thus they cannot be applied for some nonlinear systems which cannot be transformed into this form. The design of a finite-time observer is quite challenging. Another difficult aspect is about the uncertain systems, for which an exact estimation is not possible. In this situation, it is interesting to provide an interval estimation for the studied systems.

The following summarizes our recent results on observability analysis and observer design for nonlinear continuous-time systems:

- The first result concerns the relaxation of the commutative Lie bracket conditions, by using the immersion technique. After having proposed a simple form by using this approach, sufficient geometric condition is deduced to guarantee the existence of a change of coordinates to transform a nonlinear system into the proposed normal form;

- Secondly, we investigate the estimation problem for a class of partially observable nonlinear systems. For the proposed Partial Observer Normal Form (PONF), necessary and sufficient conditions are deduced to seek a diffeomorphism which can transform the studied system into the proposed PONF;

- The third result treats the problem of global finite-time observer design for a class of nonlinear systems which can be transformed into the output depending normal form. By introducing the output-dependent gains, we extend the result in [139] to design a global finite-time observer for the studied normal form;

- Finally, we study the interval observer design for a class of nonlinear continuous systems, which can be represented as a superposition of a uniformly observable nominal subsystem with a Lipschitz nonlinear perturbation. It is shown in this case there exists an interval observer for the system that estimates the set of admissible values for the state consistent with the output measurements.

Chapter 3

Extended Output-Depending Normal Form

The observer design for nonlinear dynamical systems is an important issue in the control theory. One of the methods is to transform the nonlinear system into a more simple form which enables us to apply existing observers. Here, we will recall our recent work, published in [J11, C20], by applying the technique of immersion (i.e. adding an auxiliary dynamics into the dynamical system) in order to relax the commutative Lie bracket conditions. We propose a new observer normal form by mixing the output depending normal form and the extended normal form. Sufficient geometric condition will be deduced to guarantee the existence of a diffeomorphism to transform the studied system into the proposed normal form.

3.1 Notations and problem statement

Consider a single output nonlinear dynamical system in the following form:

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{3.1}$$

where $x \in U \subseteq \mathbb{R}^n$ is the state and $y \in \mathbb{R}$ is the output. We assume that the vector field f and the output function h are smooth. In the following, we also assume that the pair (h, f) satisfies the observability rank condition. Thus, the so-called observability differential 1-forms are independent, and given by:

$$\theta_1 = dh \tag{3.2}$$

$$\theta_i = dL_f^{i-1}h \quad \text{for } 2 \leq i \leq n \tag{3.3}$$

where $L_f^k h$ is the k^{th} Lie derivative of h along f and d is the differential operator.

Thus, according to [113], one can construct $\tau = [\tau_1, \cdots, \tau_n]$ where the first vector field τ_1 is a solution for the following algebraic equations:

$$\begin{aligned} &\theta_i(\tau_1) = 0 \quad \text{for } 1 \leq i \leq n-1 \\ &\theta_n(\tau_1) = 1 \end{aligned} \tag{3.4}$$

and the other vector fields are given by induction as follows:

$$\tau_i = [\tau_{i-1}, f] \tag{3.5}$$

for $2 \leq i \leq n$, where $[,]$ denotes the Lie bracket.

In [113], the commutativity of Lie bracket, i.e. $\left[\tau_i, \tau_j\right] = 0$ for $1 \leq i \leq n$, $1 \leq j \leq n$ is the necessary and sufficient condition to transform system (3.1) into a nonlinear observer form with output injection. If this condition is not fulfilled, then another frame $\overline{\tau} = [\overline{\tau}_1, \overline{\tau}_2 ... \overline{\tau}_n]$ can be built from τ according to [156, 214, 185, 186] as follows:

$$\begin{cases} \overline{\tau}_1 = \pi \tau_1 \\ \overline{\tau}_i = \frac{1}{\alpha_i} [\overline{\tau}_{i-1}, f] \end{cases} \tag{3.6}$$

where $\pi = \prod_{i=2}^{n} \alpha_i$, and $\alpha_i(y)$ for $2 \leq i \leq n$ being non vanishing functions of the output to be determined. If the commutativity of Lie bracket condition is fulfilled for the new frame $\overline{\tau}$, then system (3.1) can be transformed into the output depending nonlinear observer form with output injection. However, there exist as well some dynamical systems which do not fulfill the above conditions. In this situation, we are wondering whether it is possible to relax these conditions by applying the technique of immersion.

3.2 Extended output depending normal form

Consider the nonlinear system (3.1), one seeks an auxiliary dynamics $\dot{w} = \eta(y, w)$ so that the following extended dynamical system:

$$\dot{x} = f(x) \tag{3.7}$$
$$\dot{w} = \eta(y, w) \tag{3.8}$$
$$y = h(x) \tag{3.9}$$

could be transformed via a diffeomorphism $(z^T, \xi)^T = \phi(x, w)$ into the following more general extended output depending observer form:

$$\dot{z} = A(y)z + B(y, w) \tag{3.10}$$
$$\dot{\xi} = B_{n+1}(y, w) \tag{3.11}$$
$$y = Cz \tag{3.12}$$

where $\xi \in \mathbb{R}$, $w \in \mathbb{R}$, $C = [0, ..., 0, 1]$,

$$A(y) = \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \alpha_2(y) & 0 & ... & ... & 0 \\ 0 & \alpha_3(y) & ... & ... & ... \\ 0 & ... & ... & ... & ... \\ 0 & ... & ... & \alpha_n(y) & 0 \end{pmatrix} \tag{3.13}$$

For the proposed form (3.10-3.12), we can design a high-gain observer as follows [39]:

$$\begin{aligned}\dot{\hat{z}} &= A(y)\hat{z}+B(w,y)-\Gamma^{-1}(y)R_\rho^{-1}C^T(C\hat{z}-\bar{y})\\ 0 &= \rho R_\rho+G^T R_\rho+R_\rho G-C^T C\end{aligned}$$

where

$$G=\begin{pmatrix}0 & \cdots & 0 & 0\\ 1 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{pmatrix}$$

$$\Gamma(y)=diag[\prod_{i=2}^{n}\alpha_i(y),\prod_{i=3}^{n}\alpha_i(y),\cdots,\alpha_n(y),1]$$

$$R_\rho(n+1-i,n+1-j)=\frac{(-1)^{i+j}C_{i+j-2}^{j-1}}{\rho^{i+j-1}}$$

for $1\leq i\leq n$ and $1\leq j\leq n$, where $C_n^p=\frac{n!}{(n-p)!p!}$ is a binomial coefficient. The observation error will be governed by the following dynamics:

$$\dot{e}=\dot{\hat{z}}-\dot{z}=(A(y)-\Gamma^{-1}(y)R_\rho^{-1}C^T C)e$$

If y and w are bounded, then the observation error is exponentially stable by well choosing ρ.

3.3 Main result

We are going to deduce the sufficient geometric condition which guarantees the existence of an auxiliary dynamics $\dot{w}=\eta(y,w)$ and a diffeomorphism $(z^T,\xi)^T=\phi(x,w)$ for the purpose of transforming the extended system (3.7-3.9) into the proposed extended output depending observer normal form (3.10-3.12) where $\xi\in\mathbb{R}$ and $w\in\mathbb{R}$.

For this, let us consider a function $l(w)\neq 0$ to be determined later and build the following new frame σ from $\bar{\tau}$ defined in (3.6):

$$\begin{cases}\sigma_1=l(w)\bar{\tau}_1\\ \sigma_k=\frac{1}{\alpha_k}[\sigma_{k-1},F]\end{cases}\tag{3.14}$$

where α_k for $2\leq k\leq n$ is uniquely determined when constructing $\bar{\tau}$, and F is the vector field for the extended system (3.7-3.9), noted as $F=f+\eta(y,w)\frac{\partial}{\partial w}$. Then we can state the following theorem.

Theorem 3.1 *If there exists a function $l(w)\neq 0$ such that*

$$\left[\sigma_i,\sigma_j\right]=0$$

for $1 \leq i \leq n$ and $1 \leq j \leq n$ where σ_i is defined in (3.14), then there exists a diffeomorphism $(z^T, \xi)^T = \phi(x, w)$ which transforms the extended system (3.7-3.9) into the proposed normal form (3.10-3.12).

Assuming that there exists a function $l(w) \neq 0$ such that $\left[\sigma_i, \sigma_j\right] = 0$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Denote $\Delta_0 = span\{\sigma_1, \cdots, \sigma_n\}$ the distribution spanned by vector fields σ_i for $1 \leq i \leq n$. It is clear that Δ_0 is involutive. Let Δ be the global distribution of dimension $n+1$. One has $\Delta_0 \subset \Delta$ since Δ is involutive, then by Frobenius's theorem one can always find another σ_{n+1} which is independent of σ_i such that $[\sigma_i, \sigma_{n+1}] = 0$ for $1 \leq i \leq n$ and $dw(\sigma_{n+1}) = 1$. Note

$$\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_n, \sigma_{n+1}]$$

and denote the set of the observability 1-forms of the extended system as:

$$\theta_e = (dh, dL_F h, \cdots, dL_F^{n-1} h, dw)^T$$

then one can calculate the following matrix:

$$\Lambda = \theta_e \sigma = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & l & * \\ 0 & 0 & 0 & 0 & l\alpha_n & * & \vdots \\ 0 & \vdots & 0 & \ddots & * & \vdots & * \\ \vdots & 0 & \frac{l\pi}{\alpha_2\alpha_3} & * & \vdots & * & * \\ 0 & \frac{l\pi}{\alpha_2} & * & \cdots & * & * & * \\ l\pi & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that Λ is invertible, thus one can define the following multi 1-forms:

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n+1} \end{pmatrix} = \Lambda^{-1} \theta_e \tag{3.15}$$

Moreover, due to Poincaré's lemma, the condition $\left[\sigma_i, \sigma_j\right] = 0$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ is equivalent to the existence $\phi = (\phi_1, \cdots, \phi_{n+1})^T$ such that $\omega = d\phi := \phi_*$. As $\omega = d\phi = \phi_*$, then the diffeomorphism can be obtained by integration: $z_i = \phi_i(x) = \int \omega_i$.

Remark 3.2 *The following lists some special cases of the proposed normal form:*

- *If the vector field σ_{n+1} is obtained by induction as $\sigma_{n+1} = [\sigma_n, F]$, then $\frac{\partial \phi_*(F)}{\partial z_n} = \frac{\partial}{\partial \xi}$ and the normal form becomes $\dot{z} = A(y)z + B(w)$, where the second term $B(w)$ will depend only on the auxiliary variable w.*

- *As stated in the assumption of Lemma 3.1 in [30], it can be shown that*

$$[\overline{\tau}_i, \overline{\tau}_n] - \rho(y)\overline{\tau}_{i-1} \in span\{\overline{\tau}_1, \cdots, \overline{\tau}_{i-1}\}$$

is the condition to ensure the existence of $\eta(y,w)$ and $l(w)$, where $\rho(y)$ is only a function of y.

3.4 Application to SEIR model

Consider the Susceptible Exposed Infected and Recovered (SEIR) model [94]:

$$\begin{aligned}
\frac{dS}{dt} &= bN - \mu S - \beta\frac{SI}{N} - pbE - qbI \\
\frac{dE}{dt} &= \beta\frac{SI}{N} + pbE + qbI - (\mu+\varepsilon)E \\
\frac{dI}{dt} &= \varepsilon E - (r+\delta+\mu)I \\
\frac{dR}{dt} &= rI - \mu R \\
\frac{dN}{dt} &= (b-\mu)N - \delta I
\end{aligned} \tag{3.16}$$

where $S(t)$ is the susceptibility of the host population to the contagious disease, $E(t)$ is the exposed population but not yet expressing symptoms, $I(t)$ is the infected population, $R(t)$ is the recovered population, b is the rate of the natural birth, μ is the rate of fecundity, β is the transmission rate, δ is the death rate related to diseases, ε is the rate at which the exposed population becomes infected, p is the rate of the offspring from an exposed population, q is the rate of the offspring from an infected population and r is the rate at which the infected individuals are recovered.

It is supposed that one can measure the infected population $I(t)$ and the total population N which is given as follows:

$$N = S + E + I + R \tag{3.17}$$

One wants to estimate the susceptibility of the host population $S(t)$ and the exposed population $E(t)$ from the infectious population. Then $R(t)$ can be deduced from the algebraic equation (3.17).

For the sake of simplicity, let us consider the normalized model of (3.16), by setting $x_1 = \frac{S}{N}$, $x_2 = \frac{E}{N}$, $x_3 = \frac{I}{N}$, $x_4 = \frac{R}{N}$ and $y = \frac{I}{N}$. Consequently the SEIR dynamics can be rewritten as follows:

$$\dot{x}_1 = b - bx_1 + \gamma_1 x_1 x_3 - pbx_2 - qbx_3 \tag{3.18}$$
$$\dot{x}_2 = \beta x_1 x_3 + \gamma_2 x_2 + \delta x_2 x_3 + qbx_3 \tag{3.19}$$
$$\dot{x}_3 = \varepsilon x_2 + \gamma_3 x_3 + \delta x_3^2 \tag{3.20}$$
$$\dot{x}_4 = rx_3 - bx_4 + \delta x_3 x_4 \tag{3.21}$$
$$y = x_3 \tag{3.22}$$

with $\gamma_1 = -(\beta - \delta)$, $\gamma_2 = -(b + \varepsilon - pb)$, $\gamma_3 = -(r + \delta + b)$ and

$$x_1 + x_2 + x_3 + x_4 = 1 \tag{3.23}$$

Due to the above constrain, we need only study the observability of (3.18-3.20) with the measurement (3.22). If it can be transformed into the proposed normal form, then the above algebraic constrain enables us to estimate x_4. The following will show step-by-step how to seek an auxiliary dynamics and deduce such a diffeomorphism.

3.4.1 Calculation of τ

A simple calculation gives the associated observability 1-forms as follows:

$$\begin{aligned}\theta_1 &= dx_3\\ \theta_2 &= \varepsilon dx_2 + (\gamma_3 + 2\delta x_3)dx_3\\ \theta_3 &= \varepsilon\beta x_3 dx_1 + \varepsilon(\gamma_2 + \gamma_3 + 3\delta x_3)dx_2 + Q_1 dx_3\end{aligned}$$

where $Q_1 = \varepsilon\beta x_1 + 3\varepsilon\delta x_2 + \varepsilon q b + \gamma_3^2 + 6\delta\gamma_3 x_3 + 6\delta^2 x_3^2$.

Then the associated frame τ is given by:

$$\begin{aligned}&\tau_1 = \tfrac{1}{\varepsilon\beta x_3}\tfrac{\partial}{\partial x_1}\\ &\tau_2 = u\tau_1 + \tfrac{1}{\varepsilon}\tfrac{\partial}{\partial x_2} \quad \text{where } \; u = -b + \gamma_3 + (\delta + \gamma_1)x_3 + \varepsilon\tfrac{x_2}{x_3}\\ &\tau_3 = -\tfrac{pb}{\varepsilon}\tfrac{\partial}{\partial x_1} - \left(L_f u\right)\tau_1 + u\tau_2 + \tfrac{1}{\varepsilon}(\gamma_2 + \delta x_3)\tfrac{\partial}{\partial x_2} + \tfrac{\partial}{\partial x_3}\end{aligned}$$

A straightforward calculation gives $[\tau_1, \tau_2] = [\tau_1, \tau_3] = 0$ and

$$[\tau_2, \tau_3] = Q_2\tau_1 + \frac{1}{x_3}\tau_2$$

where $Q_2 = -(3\delta + 2\gamma_1 + \frac{2\gamma_2 - \gamma_3}{x_3} - 3\varepsilon\frac{x_2}{x_3^2})$.

As $[\tau_2, \tau_3] \neq 0$, then system (3.18-3.20) cannot be transformed into the observer form with output injection, but one can use them to construct a new frame $\overline{\tau}$.

3.4.2 Calculation of $\overline{\tau}$

To build $\overline{\tau}$, one needs to seek non vanishing functions $\alpha_2(y)$ and $\alpha_3(y)$ from τ. Without loss of generality, one can always assume that $\alpha_3(y) = 1$. The reason is that if $\alpha_3(y)$ is different to 1 in one normal form, one can always apply a diffeomorphism on the output $z_3 = \int_0^y \frac{1}{\alpha_3(s)}ds$ which will make $\alpha_3(y) = 1$ in the transformed normal form. Therefore, one needs only to determine $\alpha_2(y)$.

According to [214], one uses the following equation:

$$[\tau_2, \tau_3] = \lambda(y)\tau_2 \mod \tau_1$$

with $\lambda(y) = \frac{d\alpha_2(y)}{dy}\frac{1}{\alpha_2(y)}$ to determine $\alpha_2(y)$. Then one has:

$$\frac{d\alpha_2(y)}{dy}\frac{1}{\alpha_2(y)} = \frac{1}{x_3} = \frac{1}{y}$$

which yields $\alpha_2(y) = y = x_3$.

Consequently, one has $\alpha_2 = x_3$, $\alpha_3 = 1$ and $\pi = \alpha_2\alpha_3 = x_3$. Then from (3.6) one obtains:

$$\begin{aligned}
\overline{\tau}_1 &= x_3\tau_1 = \frac{1}{\varepsilon\beta}\frac{\partial}{\partial x_1}\\
\overline{\tau}_2 &= \frac{1}{x_3}[\overline{\tau}_1, f] = \frac{1}{\varepsilon\beta x_3}(-b+\gamma_1 x_3)\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial x_2}\\
\overline{\tau}_3 &= [\overline{\tau}_2, f] = (-pb\beta + \frac{(-b+\gamma_1 x_3)^2}{x_3} - b\varepsilon\frac{x_2}{x_3^2})\overline{\tau}_1\\
&+ \frac{1}{\varepsilon}(\gamma_2 - b + (\delta+\gamma_1)x_3)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}
\end{aligned}$$

A straightforward calculation gives $[\overline{\tau}_1, \overline{\tau}_2] = [\overline{\tau}_1, \overline{\tau}_3] = 0$ and $[\overline{\tau}_2, \overline{\tau}_3] = -2\frac{b}{x_3^2}\overline{\tau}_1$. One can see again that the commutativity condition for the new frame $\overline{\tau}$ is not satisfied, and according to the third point of Remark 3.2 the functions $\eta(y,w)$ and $l(w)$ exist.

3.4.3 Calculation of σ

In this step, we will seek an auxiliary dynamics $\dot{w} = \eta(y,w)$ and a non zero function $l(w)$ which fulfill the condition of Theorem 3.1. For this, set $\sigma_1 = l(w)\overline{\tau}_1 = \frac{l}{\varepsilon\beta}\frac{\partial}{\partial x_1}$, then one has:

$$\begin{aligned}
\sigma_2 &= \frac{1}{x_3}[\sigma_1, F] = \frac{1}{x_3}(lH - \eta l')\sigma_1 + \frac{l}{\varepsilon}\frac{\partial}{\partial x_2}\\
\sigma_3 &= [\sigma_2, F] = (-L_F(\frac{1}{x_3}(lH-\eta l')) - \beta pb)\sigma_1 + (lH-\eta l')\sigma_2\\
&\quad +(\frac{l}{\varepsilon}(\gamma_2+\delta x_3) - \frac{l'}{\varepsilon}\eta)\frac{\partial}{\partial x_2} + l\frac{\partial}{\partial x_3}
\end{aligned}$$

where $H = (-b+\gamma_1 x_3)$. Finally, one gets:

$$[\sigma_2, \sigma_3] = (\frac{l}{\varepsilon}L_{\frac{\partial}{\partial x_2}}(-L_F(\frac{1}{x_3}(lH-\eta l')) - lL_{\frac{\partial}{\partial x_3}}(\frac{1}{x_3}(lH-\eta l')))\sigma_1$$

Therefore, $[\sigma_2, \sigma_3] = 0$ implies that:

$$\frac{l}{\varepsilon}L_{\frac{\partial}{\partial x_2}}(-L_F(\frac{1}{x_3}(lH-\eta l')) - lL_{\frac{\partial}{\partial x_3}}(\frac{1}{x_3}(lH-\eta l')) = 0$$

which is equivalent to:

$$-lb + l'(-\eta + x_3\eta'_y) = 0 \tag{3.24}$$

where $\eta'_y = \frac{\partial \eta}{\partial y}$ and $l' = \frac{dl}{dw}$.

As l is only a function of w then $-\eta + x_3\eta'_y$ is only a function of w. Consequently, the function $\eta(w,y)$ has the following form: $\eta = \kappa_1(w)y + \kappa_2(w)$. Then, according to (3.24) one has:

$$lb + l'\kappa_2(w) = 0 \tag{3.25}$$

which implies $l(w) = e^{-\int_0^w \frac{b}{\kappa_2(s)}ds}$. To simplify the calculations, we take $\eta = -\kappa(w)H = -\kappa(w)(\gamma_1 x_3 - b)$, i.e. $\kappa_1(w) = -\kappa(w)\gamma_1$ and $\kappa_2(w) = \kappa(w)b$. Therefore, one can add the following auxiliary dynamics:

$$\dot{w} = -\kappa(w)(\gamma_1 y - b) \tag{3.26}$$

where $w \in \mathbb{R}$ is an auxiliary variable, considered as an *extra* output and $\kappa(w)$ can be freely chosen in order to ensure the boundedness of w.

Then, the corresponding frame σ is as follows:

$$\begin{aligned}
\sigma_1 &= \frac{l}{\varepsilon\beta}\frac{\partial}{\partial x_1}, \qquad \sigma_2 = \frac{l}{\varepsilon}\frac{\partial}{\partial x_2} \\
\sigma_3 &= -\frac{pb}{\varepsilon} l \frac{\partial}{\partial x_1} + \frac{\gamma_2 + b + (\delta - \gamma_1)x_3}{\varepsilon} l \frac{\partial}{\partial x_2} + l \frac{\partial}{\partial x_3}
\end{aligned}$$

One can check that $[\sigma_1, \sigma_2] = [\sigma_1, \sigma_3] = [\sigma_2, \sigma_3] = 0$. To complete the dimension of the frame, one should find σ_4 which commutes with σ_i for $1 \le i \le 3$. For this, one can choose:

$$\sigma_4 = \frac{l'x_1}{l}\frac{\partial}{\partial x_1} + \frac{l'}{l}(x_2 + \frac{(\delta - \gamma_1)}{2\varepsilon}x_3^2)\frac{\partial}{\partial x_2} + \frac{l'x_3}{l}\frac{\partial}{\partial x3} + \frac{\partial}{\partial w}$$

which makes $[\sigma_4, \sigma_i] = 0$ for $1 \le i \le 3$.

3.4.4 Determination of diffeomorphism

After determining the auxiliary dynamics, one can calculate the observability 1-forms $\theta_e = [dh, dL_F h, dL_F^2, dw]^T$, then one obtains:

$$\Lambda = \theta_e \sigma = \begin{pmatrix} 0 & 0 & l & \frac{l'}{l}x_3 \\ 0 & l & \Lambda_{23} & \Lambda_{24} \\ x_3 l & l\,(\gamma_2 + \gamma_3 + 3\delta x_3) & \Lambda_{33} & \Lambda_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where:

$$\begin{aligned}
\Lambda_{23} &= (\gamma_2 + b + (\delta - \gamma_1)x_3)\,l + (\gamma_3 + 2\delta x_3)l \\
\Lambda_{24} &= \frac{l'}{2l}\left(2\varepsilon x_2 + 2\gamma_3 x_3 + (5\delta - \gamma_1)x_3^2\right) \\
\Lambda_{33} &= (\gamma_2 + \gamma_3 + 3\delta x_3)\,(\gamma_2 + b + (\delta - \gamma_1)x_3)\,l - pb\varepsilon\beta x_3 l + Q_1 l \\
\Lambda_{34} &= \frac{l'}{l}(x_2 + \frac{(\delta - \gamma_1)}{2\varepsilon}x_3^2)\varepsilon(\gamma_2 + \gamma_3 + 3\delta x_3) + \varepsilon\beta x_3 \frac{l'x_1}{l} + \frac{l'x_3}{l}Q_1
\end{aligned}$$

Thus one obtains $\omega = \Lambda^{-1}\theta_e = dz$ which yields the following diffeomorphism:

$$\begin{aligned}
z_1 &= \frac{\beta\varepsilon}{l(w)}x_1 + \frac{bp\beta}{l(w)}x_3 \\
z_2 &= \frac{\varepsilon}{l(w)}x_2 - \frac{(b+\gamma_2)}{l(w)}x_3 - \frac{1}{2}\frac{(\delta-\gamma_1)}{l(w)}x_3^2 \\
z_3 &= \frac{1}{l(w)}x \\
\xi &= w
\end{aligned} \tag{3.27}$$

allowing the transformation of system (3.18-3.20) and the auxiliary dynamics (3.26) into the following extended output depending form:

$$\begin{aligned}
\dot{z}_1 &= B_1(w,y) \\
\dot{z}_2 &= yz_1 + B_2(w,y) \\
\dot{z}_3 &= z_2 + B_3(w,y) \\
\dot{\xi} &= B_4(w,y) \\
\bar{y} &= z_3
\end{aligned} \tag{3.28}$$

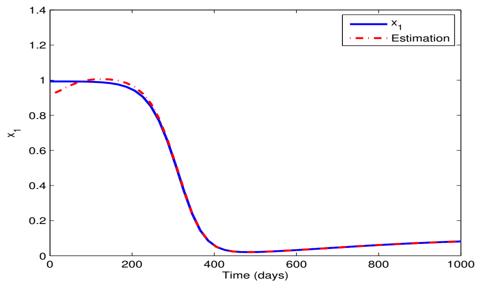
where

$$\begin{aligned}
B_1(y,w) &= b\beta(\frac{1}{l}(p(b+\gamma_3)-q\varepsilon)y + \frac{p}{l}(\delta-\gamma_1)y^2) + \frac{b\beta\varepsilon}{l} \\
B_2(y,w) &= -\beta bpy + \frac{1}{l}(\delta^2 + \frac{1}{2}\gamma_1^2 - \frac{3}{2}\delta\gamma_1)y^3 - \\
&\quad \frac{1}{l}(\delta(\gamma_2+\gamma_3+\frac{3}{2}b) - 2\gamma_1\gamma_3 - \frac{3}{2}b\gamma_1)y^2 - \\
&\quad \frac{1}{l}(b(\gamma_2+\gamma_3+b-q\varepsilon)+\gamma_2\gamma_3)y \\
B_3(y,w) &= \frac{1}{l}(3b+\gamma_2+\gamma_3)y + \frac{3}{2l}(\delta - \frac{5}{3}\gamma_1)y^2
\end{aligned}$$

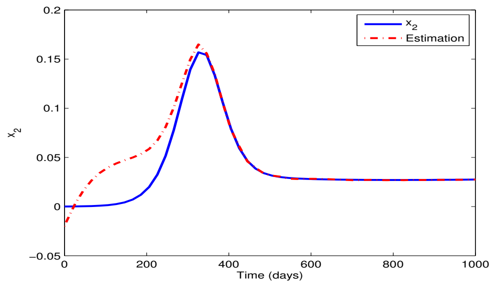
3.4.5 Simulation results

For the simulation, in order to have a bounded state for the auxiliary dynamics, we choose $\kappa(w) = \frac{\sin^2(aw)}{(aw)^2}$ where $a \in]0,1[$, and the same parameters of the SEIR model as those in

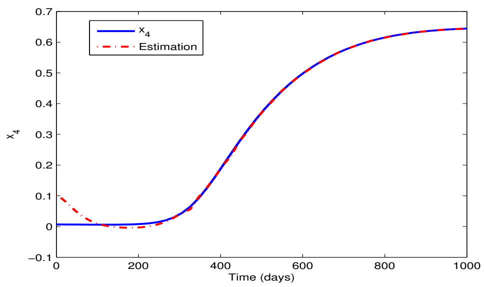
[119] are used, i.e. $N = 141$, $b = 0.221176/N$, $\delta = 0.002$, $p = 0.8$, $q = 0.95$, $\beta = 0.05$, $\varepsilon = 0.05$, $r = 0.003$. The initial conditions are $S(0) = 140$, $E(0) = 0.01$, $I(0) = 0.02$ and $N(0) = 141$. The simulation results are presented in the following figures.



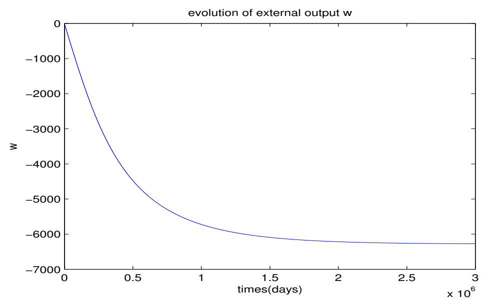
a) Estimation of x_1



b) Estimation of x_2



c) Estimation of x_4



d) Boundedness of auxiliary dynamics w

3.5 Conclusion

The first result of Part I is to introduce the auxiliary dynamics to relax the restrictive Lie bracket condition. For this, a new extended output depending normal form was proposed, which mixes both the extended normal form and the output depending normal form. This new normal form enables us to design a simple high-gain observer. Sufficient condition was given in order to guarantee the existence of a diffeomorphism which can be used to transform the extended dynamical systems into the proposed normal form.

Chapter 4

Partial Observer Normal Form

In the literature, most the existing results are devoted to designing a full-order observer, under the assumption that the whole state of the studied system is observable. Few works have been dedicated to the partial observability which however makes sense in practice when only a part of states are observable or are necessary for the controller design. Here, we will propose a more general Partial Observer Normal Form (PONF) for a class of partially observable nonlinear systems. This new form is a generalization of the normal form studied in [98]. The proposed PONF is divided into two subsystems, and we relax the form proposed in [98] by involving all states in the second subsystem. To deal with this generalization, we use the notion of commutativity of Lie bracket modulo a distribution. Our results allow as well to apply additionally a diffeomorphism on the output space. Therefore, the deduced necessary and sufficient geometric conditions are more general because of the introduction of commutativity of Lie bracket modulo a distribution. The result of this chapter has been published in [J2, C23].

4.1 Notation and problem statement

Consider the following nonlinear dynamical system with single output:

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{k=1}^{m} g_k(x)\, u_k \tag{4.1}$$

$$y = h(x) \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$, and the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g = [g_1, \cdots, g_m]$ with $g_i : \mathbb{R}^n \to \mathbb{R}^n$ for $1 \leq i \leq m$, $h : \mathbb{R}^n \to \mathbb{R}$ are supposed to be sufficiently smooth. It is assumed that $f(0) = 0$ and $h(0) = 0$.

Let $\mathscr{X} \subset \mathbb{R}^n$ be a neighborhood of 0, for system (4.1)-(4.2), if the pair $(h(x), f(x))$ locally satisfies the observability rank condition on $\mathscr{X}$, i.e. rank $\left\{dh, dL_f h, \cdots, dL_f^n h\right\}(x) = n$ for $x \in \mathscr{X}$, then the following 1-forms:

$$\theta_1 = dh \text{ and } \theta_i = dL_f^{i-1} h, \text{ for } 2 \leq i \leq n$$

are independent on $\mathscr{X}$, where $L_f^k h$ denotes the k^{th} Lie derivative of h along f. Therefore, there exists a family of vector fields $\bar{\tau} = [\bar{\tau}_1, \cdots, \bar{\tau}_n]$ proposed in [113], where the first vector field $\bar{\tau}_1$ is the solution of the following algebraic equations:

$$\begin{cases} \theta_i(\bar{\tau}_1) = 0 \text{ for } 1 \leq i \leq n-1 \\ \theta_n(\bar{\tau}_1) = 1 \end{cases} \tag{4.3}$$

and the other vector fields are obtained by induction as $\bar{\tau}_i = -ad_f \bar{\tau}_{i-1} = [\bar{\tau}_{i-1}, f]$ for $2 \leq i \leq n$, where $[,]$ denotes the Lie bracket. According to [113], if

$$\left[\bar{\tau}_i, \bar{\tau}_j\right] = 0 \text{ for } 1 \leq i, j \leq n \tag{4.4}$$
$$\left[\bar{\tau}_i, g_k\right] = 0 \text{ for } 1 \leq i \leq n-1 \text{ and } 1 \leq k \leq m \tag{4.5}$$

then system (4.1)-(4.2) can be locally transformed, by means of a local diffeomorphism $\xi = \phi(x)$, into the following nonlinear observer normal form:

$$\begin{cases} \dot{\xi} = A\xi + B(y) + \sum\limits_{k=1}^{m} \alpha_k(y) u_k \\ y = C\xi \end{cases} \tag{4.6}$$

where $A \in \mathbb{R}^{n\times n}$ is the Brunovsky matrix and $C = [0, \cdots, 0, 1] \in \mathbb{R}^{1\times n}$.

Obviously, for the nonlinear dynamical system (4.1)-(4.2), if rank $\left\{dh, \cdots, dL_f^n h\right\}(x) = r < n$ for $x \in \mathscr{X}$, which implies that only a part of states of the studied system are observable, the proposed method by [113] could not be applied.

To treat this partially observable situation, we propose the following Partial Observer Normal Form (PONF):

$$\begin{cases} \dot{\xi} = A\xi + \beta(y) + \sum\limits_{k=1}^{m} \alpha_k^1(y) u_k \\ \dot{\zeta} = \eta(\xi, \zeta) + \sum\limits_{k=1}^{m} \alpha_k^2(\xi, \zeta) u_k \\ y = C\xi \end{cases} \tag{4.7}$$

where $\xi \in \mathbb{R}^r$, $\zeta \in \mathbb{R}^{n-r}$, $y \in \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}^r$, $\eta : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$, $\alpha_k^1 : \mathbb{R} \to \mathbb{R}^r$, $\alpha_k^2 : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$, $C = (0, \cdots, 0, 1) \in \mathbb{R}^{1\times r}$ and A is the $r \times r$ Brunovsky matrix:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{r\times r}$$

For the proposed form (4.7), one can easily design a reduced-order observer to estimate the part of observable state ξ, by choosing the gain K such that $(A - KC)$ is Hurwitz. Therefore, the rest focuses only on how to deduce a diffeomorphism which transforms the nonlinear system (4.1)-(4.2) into the proposed PONF (4.7).

4.2 Geometric conditions

In the following, we will deduce necessary and sufficient conditions which guarantee the existence of a diffeomorphism to transform the studied partially observable nonlinear system into the proposed PONF. For this, considering the studied system (4.1)-(4.2), it is assumed that rank $\left\{dh, dL_f h, \cdots, dL_f^n h\right\}(x) = r < n$ for $x \in \mathscr{X} \subset \mathbb{R}^n$, where $\mathscr{X}$ is a neighborhood of 0.

Denote the observability 1-forms for $1 \leq i \leq r$ by $\theta_i = dL_f^{i-1} h$ and note

$$\Delta = \operatorname{span}\{\theta_1, \theta_2, \cdots, \theta_r\}$$

as the co-distribution spanned by the observability 1-forms. Specifically, thanks to observability rank condition, it is clear that $dL_f^i h \in \Delta$ for $i \geq r+1$. Then one can define the kernel (or the annihilator) of the co-distribution Δ as follows:

$$\Delta^{\perp} = \ker\Delta = \{X : \theta_k(X) = 0,\ \text{for } 1 \leq k \leq r\} \tag{4.8}$$

for which we have the following properties.

Lemma 4.1 *For the distribution $\Delta^{\perp}$ defined in (4.8), the following properties are satisfied:*

P1: $\Delta^{\perp}$ is involutive, i.e. for any two vector fields $H_1 \in \Delta^{\perp}$ and $H_2 \in \Delta^{\perp}$, we have $[H_1, H_2] \in \Delta^{\perp}$;

P2: there exist $(n-r)$ vector fields $\{\tau_{r+1}, \cdots, \tau_n\}$ that span $\Delta^{\perp}$ such that $[\tau_i, \tau_j] = 0$ for $r+1 \leq i, j \leq n$, i.e. $\{\tau_{r+1}, \cdots, \tau_n\}$ is the commutative basis of $\Delta^{\perp}$;

P3: $\Delta^{\perp}$ is f-invariant, i.e. for any vector field $H \in \Delta^{\perp}$, we have $[f, H] \in \Delta^{\perp}$.

Now, let $\bar{\tau}_1$ be one of the vector field solutions of the following under-determined algebraic equations:

$$\begin{cases} \theta_k(\bar{\tau}_1) = 0 \text{ for } 1 \leq k \leq r-1 \\ \theta_r(\bar{\tau}_1) = 1 \end{cases} \tag{4.9}$$

and it is obvious that this solution is not unique, since (4.9) contains only r algebraic equations. Therefore, for any $H \in \Delta^{\perp}$, $\bar{\tau}_1 + H$ is also a solution of (4.9). By following the method proposed in [113], we can construct, for any chosen $\bar{\tau}_1$ satisfying (4.9), the following vector fields:

$$\bar{\tau}_i = [\bar{\tau}_{i-1}, f] \text{ for } 2 \leq i \leq r \tag{4.10}$$

Let $\tilde{\tau}_1$ be another solution of (4.9), which enables us, by following the same method, to construct another family of $(r-1)$ vector fields:

$$\tilde{\tau}_i = [\tilde{\tau}_{i-1}, f] \text{ for } 2 \leq i \leq r \tag{4.11}$$

Then the following lemma highlights the relation between these two families of vector fields for the different chosen $\bar{\tau}_1$ and $\tilde{\tau}_1$.

Lemma 4.2 *For any two different solutions $\bar{\tau}_1$ and $\tilde{\tau}_1$ of (4.9), there exist $H_i \in \Delta^{\perp}$ for $1 \leq i \leq r$ such that $\tilde{\tau}_i = \bar{\tau}_i + H_i$ where $\bar{\tau}_i$ and $\tilde{\tau}_i$ are defined in (4.10) and (4.11), respectively.*

The above lemma reveals an important property: no matter how to choose the first vector field $\bar{\tau}_1$ satisfying (4.9), the family of vector fields $\bar{\tau}_i$ for $1 \le i \le r$ are defined modulo $\Delta^\perp$. Other important properties for this family of vector fields $\bar{\tau}_i$ are listed in the following lemma.

Lemma 4.3 *For any $\bar{\tau}_1$ satisfying (4.9) and the associated $\bar{\tau}_i$ for $2 \le i \le r$ deduced from (4.10), the following properties are satisfied:*

1) for any $H \in \Delta^\perp$, we have $[\bar{\tau}_i, H] \in \Delta^\perp$;

2) for any $H_i \in \Delta^\perp$ and $H_j \in \Delta^\perp$ with $1 \le i, j \le r$, we have

$$[\bar{\tau}_i + H_i, \bar{\tau}_j + H_j] = [\bar{\tau}_i, \bar{\tau}_j] \text{ modulo } \Delta^\perp;$$

3) for $1 \le i \le r$ and $r+1 \le j \le n$, we have $\left[\bar{\tau}_i, \tau_j\right] \in \Delta^\perp$ where τ_j was defined in P2 of Lemma 4.1, which is the commutative basis of $\Delta^\perp$.

Lemma 4.4 *For any $\bar{\tau}_1$ satisfying (4.9) and the associated $\bar{\tau}_i$ for $2 \le i \le r$ deduced from (4.10), there exists a family of vector fields $\tau_i = \bar{\tau}_i$ modulo $\Delta^\perp$ such that $\left[\tau_i, \tau_j\right] = 0$ for $1 \le i, j \le r$, if and only if $[\bar{\tau}_i, \bar{\tau}_j] \in \Delta^\perp$.*

Remark 4.5 *Any family of vector fields $\bar{\tau}_i$ for $1 \le i \le r$ given by (4.9)-(4.10) together with the family of vector fields τ_j for $r+1 \le i \le n$ defined in P2 of Lemma 4.1, are independent on $\mathscr{X}$. Thus they provide a basis of the tangent fiber bundle $T\mathscr{X}$ of $\mathscr{X}$.*

With the deduced r independent vector fields $\bar{\tau}_i$ for $1 \le i \le r$, we can then define the following matrix:

$$\Lambda_1 = \big(\theta_j(\bar{\tau}_i)\big)_{1 \le i,j \le r} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} (\bar{\tau}_1, \cdots, \bar{\tau}_r) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & \lambda_{2,r} \\ 0 & \cdots & \lambda_{r-1,r-1} & \lambda_{r-1,r} \\ 1 & \cdots & \lambda_{r,r-1} & \lambda_{r,r} \end{pmatrix} \tag{4.12}$$

with $\lambda_{i,j} = \theta_i(\bar{\tau}_j)$ for $1 \le i, j \le r$. Since it is invertible, we can define the following 1-forms:

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_r \end{pmatrix} = \Lambda_1^{-1} [\theta_1, \cdots, \theta_r]^T \tag{4.13}$$

A straightforward calculation shows that

$$\begin{aligned} \omega_r &= \theta_1 \\ \omega_{r-k} &= \left(\theta_{k+1} - \sum_{i=r-k+1}^{n} \lambda_{k+1,i}\omega_i \right) \quad \text{for } \ 1 \le k \le r-1 \end{aligned} \tag{4.14}$$

which are linear combination of the 1-forms θ_i for $1 \le i \le r$. Then it is easy to verify that the deduced 1-forms ω_i for $1 \le i \le r$ have the following properties.

Lemma 4.6 *1) the annihilator of 1-forms ω_k for $1 \leq k \leq r$ is equal to $\Delta^{\perp}$ defined in (4.8);*

2) the 1-forms ω_k for $1 \leq k \leq r$ are well defined such that they are independent of the choice of $\bar{\tau}_i$ modulo $\Delta^{\perp}$*;*

3) for $1 \leq i,k \leq r$, $\omega_k(\bar{\tau}_i) = \delta_k^i$ where δ_k^i represents the Kronecker delta, i.e. $\delta_k^i = 1$ if $k = i$, otherwise $\delta_k^i = 0$.

The above lemma shows that, for any chosen solution of (4.9), the constructed family of vector fields always yields the same 1-form ω_k for $1 \leq k \leq r$, which allows us to state the following theorem.

Theorem 4.7 *There exists a local diffeomorphism $(\xi^T, \zeta^T)^T = \phi(x)$ on $\mathscr{X}$ which transforms system (4.1)-(4.2) into the proposed normal form (4.7), if and only if the following conditions are fulfilled:*

1) there exists a family of vector fields $\tau_i = \bar{\tau}_i$ modulo $\Delta^{\perp}$ *such that $[\tau_i, \tau_l] = 0$ for $1 \leq i \leq r$ and $1 \leq l \leq r$;*

2) $\left[\tau_i, \tau_j\right] = 0$ for $1 \leq i \leq r$ and $r+1 \leq j \leq n$ where τ_j was defined in P2 of Lemma 4.1;

3) $[\tau_i, g_k] \in \Delta^{\perp}$ for $1 \leq i \leq n$, $i \neq r$ and $1 \leq k \leq m$.

If the above conditions are fulfilled, then the diffeomorphism ϕ is locally determined by its differential $\phi_* := d\phi = \omega$, i.e. $\phi(x) = \int_{\gamma} \omega$, where γ is any path on $\mathscr{X}$ with $\gamma(0) = 0$ and $\gamma(1) = x$.

For any choice of $\bar{\tau}_1$ satisfying (4.9) with the associated family of vector fields $\bar{\tau}_i$ for $2 \leq i \leq r$, Theorem 4.7 needs to seek a new family of vector fields $\tau_i = \bar{\tau}_i$ modulo $\Delta^{\perp}$ such that $[\tau_i, \tau_l] = 0$ for $1 \leq i \leq r$ and $1 \leq l \leq r$. Thanks to Lemma 4.4, we can remove at the beginning all terms in the directions of the basis of $\Delta^{\perp}$ in $\bar{\tau}_1$, noted as τ_1. It is important to emphasize that this elimination will yield a unique τ_1 for any solution of (4.9). Then, we can iteratively eliminate all terms in the directions of the basis of $\Delta^{\perp}$ in $\bar{\tau}_i$, which gives the following *simple and constructive* procedure to calculate τ_i for $1 \leq i \leq n$:

Procedure 4.8 *For the nonlinear system (4.1)-(4.2), the family of vector fields τ_i for $1 \leq i \leq n$ can be simply computed via the following steps:*

Step 1: Compute the observability 1-forms θ_i for $1 \leq i \leq r$;

Step 2: Determine $\Delta^{\perp}$ and seek a commutative basis $\{\tau_{r+1}, \cdots, \tau_n\}$ that spans $\Delta^{\perp}$;

Step 3: Choose any solution $\bar{\tau}_1$ of (4.9), then eliminate all its terms in the directions of $\{\tau_{r+1}, \cdots, \tau_n\}$ which yields τ_1;

Step 4: Iteratively, for $2 \leq i \leq r$, compute $[\tau_{i-1}, f]$ in which eliminate all terms in the directions of $\{\tau_{r+1}, \cdots, \tau_n\}$ which gives τ_i.

The following example is to highlight Theorem 4.7 by following Procedure 4.8.

Example 4.9 *Let us consider the following nonlinear system:*

$$\begin{cases} \dot{x}_1 = -x_3^2 + x_1^3 x_3 - \frac{1}{2}x_3^3 + x_2^5 \\ \dot{x}_2 = x_1 - \frac{1}{2}x_3^2 \\ \dot{x}_3 = -x_3 + x_1^3 - \frac{1}{2}x_3^2 \\ y = x_2 \end{cases} \tag{4.15}$$

Step 1:
A simple calculation gives $\operatorname{rank}\left\{dh, dL_f h, dL_f^2 h\right\} = 2$*, thus* $r = 2$*, and one has* $\theta_1 = dx_2$ *and* $\theta_2 = dx_1 - x_3 dx_3$*, since* $dL_f^2 h = 5x_2^4 \theta_1$*.*

Step 2:
Then one obtains $\Delta = \operatorname{span}\{\theta_1, \theta_2\}$*, and* $\Delta^\perp = \operatorname{span}\left\{x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right\}$*. Since* τ_3 *should be a commutative basis of* $\Delta^\perp$*, thus one has* $\tau_3 = x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$*.*

Step 3:
By solving equation (4.9), one obtains $\bar{\tau}_1 = \frac{\partial}{\partial x_1} + q_1(x)\tau_3$*. Following Procedure 4.8, by eliminating all terms in* $\bar{\tau}_1$ *in the direction of* τ_3*, we have* $\tau_1 = \frac{\partial}{\partial x_1}$*.*

Step 4:
Then we can calculate: $[\tau_1, f] = \frac{\partial}{\partial x_2} + 3x_1^2 \tau_3$*. for which the elimination of all terms in the direction of* τ_3 *yields* $\tau_2 = \frac{\partial}{\partial x_2}$ *.*

Finally Procedure 4.8 enables us to easily get the following vector fields: $\tau_1 = \frac{\partial}{\partial x_1}$*,* $\tau_2 = \frac{\partial}{\partial x_2}$ *and* $\tau_3 = x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$ *with which it is easy to check that all conditions of Theorem 4.7 are satisfied, thus there exists a diffeomorphism which can transform the studied example into the proposed form (4.7).*

For the deduction of the diffeomorphism, we use the obtained vector field $\tau = [\tau_1, \tau_2, \tau_3]$ *which gives:*

$$\Lambda_1 = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} (\tau_1, \tau_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

thus one has

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \Lambda_1^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} d\xi_1 \\ d\xi_2 \end{pmatrix} = d \begin{pmatrix} x_1 - \frac{1}{2}x_3^2 \\ x_2 \end{pmatrix}$$

Then one can solve the following equation:

$$\omega_3 \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_1} \right) = (0, 0, 1)$$

which uniquely determines $\omega_3 = dx_3$. Finally, one gets the following diffeomorphism $\phi(x) = \begin{pmatrix} x_1 - \frac{1}{2}x_3^2 \\ x_2 \\ x_3 \end{pmatrix}$ which transforms the studied system (14.22) into the following form

$$\begin{cases} \dot{\xi}_1 = 0 \\ \dot{\xi}_2 = \xi_1 \\ \dot{\zeta}_1 = \xi_1^3 - \xi_3 + \frac{\xi_3^2}{2}\left(3\xi_1^2 + \frac{3}{2}\xi_1\xi_3^2 + \frac{1}{4}\xi_3^4 - 1\right) \\ y = \xi_2 \end{cases}$$

4.3 Diffeomorphism on the output

In the following, we will deal with the case when the conditions of Theorem 4.7 are not fulfilled. Let us remark that the deduced diffeomorphism $\phi(x)$ in Section 4.2 does not modify the output. This is due to the fact that $\theta_r(\bar{\tau}_1) = 1$. One way to relax this constraint is to seek a new vector field, noted as $\bar{\sigma}_1$, such that $\theta_r(\bar{\sigma}_1)$ becomes a function of the output, and this will introduce a diffeomorphism on the output. For that purpose, we need modify the new vector field τ_1 obtained in Section 4.2 by following Procedure 4.8 and construct a new family of commutative vector fields, with which the deduced diffeomorphism $\phi(x)$ will apply as well a change of coordinates on the output (see [114, 156, 29]). Since this diffeomorphism will modify the output of the studied system (4.1)-(4.2), therefore the PONF (4.7) is adapted as follows:

$$\begin{cases} \dot{\xi} = A\xi + \beta(y) + \sum\limits_{k=1}^{m} \alpha_k^1(y)u_k \\ \dot{\zeta} = \eta(\xi,\zeta) + \sum\limits_{k=1}^{m} \alpha_k^2(\xi,\zeta)u_k \\ \bar{y} = \xi_r = \psi(y) \end{cases} \tag{4.16}$$

With the same procedure, we can calculate the observability 1-forms θ_i for $1 \leq i \leq r$ where r is the rank of observability matrix, which defines the co-distribution $\Delta = \text{span}\{\theta_1, \theta_2, \cdots, \theta_r\}$, and the kernel (or the annihilator) of the co-distribution Δ, noted as $\Delta^{\perp}$.

By following Procedure 4.8 presented in Section 4.2, it is assumed that we have already obtained a family of vector fields τ_j for $r+1 \leq j \leq n$ which are commutative basis of $\Delta^{\perp}$, and τ_i for $1 \leq i \leq r$ which do not contain any component in the direction of τ_j for $r+1 \leq j \leq n$. In this section, it is assumed that there exist $1 \leq i, l \leq r$ such that $[\tau_i, \tau_l] \neq 0$, thus Theorem 4.7 cannot be applied. In order to introduce a change of coordinate for the output, let $s(y) \neq 0$ be a function of the output which will be determined later. Then one can define a new vector field σ_1 from τ_1 as follows: $\bar{\sigma}_1 = s(y)\tau_1$. By induction, one can define the following new vector fields $\bar{\sigma}_i = [\bar{\sigma}_{i-1}, f]$ modulo $\Delta^{\perp}$, for $2 \leq i \leq r$. A straightforward calculation gives

$$\bar{\sigma}_i = \sum_{k=1}^{i} (-1)^{i-k} C_{i-1}^{k-1} L_f^{i-k} s(y)\tau_k \text{ modulo } \Delta^{\perp} \tag{4.17}$$

where C_{i-1}^{k-1} is the binomial coefficients and $L_f^0 s(y) = s(y)$.

For the same reason stated in Procedure 4.8, by eliminating all terms in the directions of τ_j for $r+1 \leq j \leq n$ (the commutative basis of $\Delta^{\perp}$) we obtain:

$$\sigma_i = \sum_{k=1}^{i} (-1)^{i-k} C_{i-1}^{k-1} L_f^{i-k} s(y) \tau_k, \text{ for } 1 \leq i \leq r \tag{4.18}$$

For the sake of completeness, we simply note

$$\sigma_j = \tau_j, \text{ for } r+1 \leq j \leq n \tag{4.19}$$

where $\{\tau_{r+1}, \cdots, \tau_n\}$ are commutative basis of $\Delta^{\perp}$, obtained by following Procedure 4.8 presented in Section 4.2.

We can then define the following matrix:

$$\tilde{\Lambda}_1 = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} (\sigma_1, \cdots, \sigma_r) = \begin{pmatrix} 0 & \cdots & 0 & s(y) \\ \vdots & \cdots & s(y) & * \\ 0 & \cdots & * & * \\ s(y) & \cdots & * & * \end{pmatrix} \tag{4.20}$$

Since $s(y) \neq 0$, thus it is invertible, and we can define the following 1-forms:

$$\begin{pmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_r \end{pmatrix} = \tilde{\Lambda}_1^{-1} [\theta_1, \cdots, \theta_r]^T \tag{4.21}$$

and using the same method in Section 4.2 we can uniquely determine $\tilde{\omega}_i$ for $r+1 \leq i \leq n$ by solving the following equations:

$$\tilde{\omega}_i(\sigma_j) = \delta_i^j \tag{4.22}$$

for $1 \leq j \leq n$. Finally we have the following theorem.

Theorem 4.10 *There exists a local diffeomorphism $(\xi^T, \zeta^T)^T = \phi(x)$ on $\mathscr{X}$ which transforms system (4.1)-(4.2) into the PONF (4.16) if and only if the following conditions are fulfilled:*

1) $[\sigma_i, \sigma_l] = 0$ for $1 \leq i \leq r$ and $1 \leq l \leq r$ where σ_i defined in (4.18);

2) $[\sigma_i, \sigma_j] = 0$ for $1 \leq i \leq r$ and $r+1 \leq j \leq n$ where σ_j defined in (4.19);

3) $[\sigma_i, g_k] \in \Delta^{\perp}$ for $1 \leq i \leq n$, $i \neq r$ and $1 \leq k \leq m$.

If the above conditions are fulfilled, then the diffeomorphism ϕ is locally determined by its differential $\phi_* := d\phi = \tilde{\omega}$, i.e. $\phi(x) = \int_{\gamma} \tilde{\omega}$, where γ is any path on $\mathscr{X}$ with $\gamma(0) = 0$ and $\gamma(1) = x$. Moreover, one has $\xi_r = \bar{y} = \psi(y)$ where $\psi(y) = \int_0^y \frac{1}{s(c)} dc$ is a change of coordinates on the output.

Example 4.11 *Let us consider the well-known SIR (Susceptible, Infected, Removed) epidemic model describing the contagious disease propagation [67] as follows:*

$$\begin{cases} \dot{S} = -\beta SI \\ \dot{I} = \beta SI - \gamma I \\ \dot{R} = \gamma I \\ y = I \end{cases}$$

where S denotes the suspected population, I denotes the infected and R denotes the removed population and the total population.

By using the same notations introduced in Section 4.2, a simple calculation gives $\theta_1 = dI$ and $\theta_2 = \beta IdS + (\beta S - \gamma)\,dI$, and we have $\Delta = \mathrm{span}\{dI, \beta IdS + (\beta S - \gamma)\,dI\}$ which gives $\Delta^{\perp} = \mathrm{span}\{\frac{\partial}{\partial R}\}$.

Calculation of $\bar{\tau}$:
Following the procedure proposed in [113], we obtain the following vector fields: $\bar{\tau}_1 = \frac{1}{\beta I}\frac{\partial}{\partial S} + q_1\frac{\partial}{\partial R}$ and $\bar{\tau}_2 = [\bar{\tau}_1, f] = \frac{\partial}{\partial I} + \frac{\beta S - \gamma - \beta I}{\beta I}\frac{\partial}{\partial S} + q_2\frac{\partial}{\partial R}$, with $q_1(S,I,R)$ is any function and $q_2 = -L_f q_1$, which gives $[\bar{\tau}_1, \bar{\tau}_2] \neq 0$. Therefore the conditions stated in [98] are not fulfilled.

Calculation of τ:
Applying Procedure 4.8 proposed in Section 4.2, we obtain the following new vector fields: $\tau_1 = \frac{1}{\beta I}\frac{\partial}{\partial S}$, $\tau_2 = \frac{\partial}{\partial I} + \frac{\beta S - \gamma - \beta I}{\beta I}\frac{\partial}{\partial S}$, and $\tau_3 = \frac{\partial}{\partial R}$. It is obvious that $[\tau_1, \tau_2] \neq 0$, which implies that the conditions of Theorem 4.7 are not fulfilled.

Calculation of σ:
In order to construct a new family of commutative vector fields by introducing a change of coordinates on the output, the method proposed in [30] is used to deduce a non-zero output function $s(y)$. For this, set $\sigma_1 = s(y)\tau_1$. According to (4.18) one has $\sigma_2 = s(y)\tau_2 - s'(y)\,(\beta SI - \gamma I)\,\tau_1$. A straightforward calculation gives $[\sigma_1, \sigma_2] = \left(\frac{2s^2(y)}{I} - 2s(y)s^{'}(y)\right)\tau_1$, thus $[\sigma_1, \sigma_2] = 0$ if and only if the function $s(y)$ fulfills the following differential equation: $\frac{s(y)}{y} - \frac{ds(y)}{dy} = 0$, which motivates us to choose $s(y) = y = I$. Then one has $\sigma_1 = \frac{1}{\beta}\frac{\partial}{\partial S}$, $\sigma_2 = -I\frac{\partial}{\partial S} + I\frac{\partial}{\partial I}$, and $\sigma_3 = \frac{\partial}{\partial R}$, with which one has $[\sigma_1, \sigma_2] = [\sigma_1, \sigma_3] = [\sigma_2, \sigma_3] = 0$, thus the first two conditions of Theorem 4.10 are satisfied. As the SIR model does not contain any input, then the third condition of Theorem 4.10 does not need to be verified.

Since

$$\tilde{\Lambda}_1 = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} (\sigma_1, \sigma_2) = \begin{pmatrix} 0 & I \\ I & -\beta I^2 + (-\gamma + S\beta)\,I \end{pmatrix}$$

which yields

$$\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix} = \tilde{\Lambda}^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} d\xi_1 \\ d\xi_2 \end{pmatrix} = d\begin{pmatrix} \beta(S+I) \\ \ln I \end{pmatrix}$$

Following (4.22), one can solve the following equation:

$$\tilde{\omega}_3\left(\sigma_1,\sigma_2,\sigma_3\right)=\tilde{\omega}_3\left(\frac{1}{\beta}\frac{\partial}{\partial S},-I\frac{\partial}{\partial S}+I\frac{\partial}{\partial I},\frac{\partial}{\partial R}\right)=(0,0,1)$$

which uniquely determines $\tilde{\omega}_3 = dR$. *Therefore, the local diffeomorphism is given as follows* $\phi(x)=[\beta(S+I),\ln I,R]^T$, *which transforms the SIR model into the following form:*

$$\begin{cases}\dot{\xi}_1=-\beta\gamma e^{\bar{y}}\\ \dot{\xi}_2=\xi_1-\beta e^{\bar{y}}-\gamma\\ \dot{\zeta}_1=\gamma e^{\bar{y}}\\ \bar{y}=C\phi=\ln I\end{cases}$$

4.4 Conclusion

The result presented in this chapter concerns the partial observability for a class of nonlinear systems. Necessary and sufficient conditions are established to guarantee the existence of a local diffeomorphism transforming the studied systems into the proposed partial observer normal form with and without a change of coordinates on the output. The introduction of a change of coordinates on the output is for the purpose of relaxing the restriction of the commutativity of Lie bracket condition. For the transformed system, a simple Luenberger observer can be designed to estimate the part of observable states.

Chapter 5

Finite-Time Observer Design

As presented in the previous chapters, besides the traditional methods to design observers directly for the studied systems, the technique of normal form seeks a transformation to convert the studied system into a simple form for which the classical observer design approaches, such as Luenberger observer, can be realized easily. Sometimes, the well-chosen desired normal form enables us to design not only asymptotic observer, but also finite-time one. Here, for the output depending normal form, we will show how to extend the global finite-time observer, proposed in [139] for the normal form of Krener and Isidori, to design a global finite-time observer for the general output depending form. Sufficient condition will be given to guarantee the existence of such a global observer. The presented result was published in [C11].

5.1 Notations and problem statement

Consider the following nonlinear system:

$$\begin{cases} \dot{x} = f(x) + b(x)u \\ y = h(x) \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ represents the smooth input, $y \in \mathbb{R}$ is the output, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, $b(x) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h(x) : \mathbb{R}^n \to \mathbb{R}$ are smooth. It is supposed that the pair (h, f) satisfies the observability rank condition, i.e. $\text{rank}\{dL_f^{i-1}h, 1 \le i \le n\} = n$. Therefore, system (5.1) with $u = 0$ is locally observable. It is assumed as well that system (5.1) can be transformed via a smooth diffeomorphism $z = \phi(x)$ into the following output depending normal form:

$$\begin{cases} \dot{z} = A(y)z + \beta(y) + g(z)u \\ y = Cz \end{cases} \tag{5.2}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$, $A(y) \in \mathbb{R}^{n\times n}$ is of the Brunovsky form:

$$A(y) = \begin{pmatrix} 0 & a_1(y) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1}(y) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{5.3}$$

with $a_i(y) \neq 0$, $C = (1, 0, \cdots, 0)$ and $g(z) \in \mathbb{R}^{n\times m}$ is of the following triangular form:

$$g(z) = \left[g_1^T(\underline{z}_1), g_2^T(\underline{z}_2), \cdots, g_i^T(\underline{z}_i), \cdots, g_n^T(\underline{z}_n)\right]^T \tag{5.4}$$

where $g_i = [g_{i,1}, \cdots, g_{i,m}]$ for $1 \leq i \leq n$ and $\underline{z}_i$ means the vector $(z_1, \cdots, z_i)$.

It is worth noting that the above output depending normal form is more general than the normal form with output injection proposed in [114, 187] and [156] where $A(y)$ is the product of a scalar function of y and the classical Brunovsky form. Asymptotic observer for the normal form (5.2) has been studied in [39] and [21], but no result on finite-time observer has been reported.

5.2 Assumptions, notations and preliminary result

Due to the physical constraint, the control and the state values of the practical systems are normally bounded. Therefore, in what follows, we make the following standard (see [78, 65] for example) assumption in the estimation theory on the boundedness of the state under a given bounded input for system (5.1).

Assumption 5.1 *For the studied system (5.1), it is assumed that the state x is bounded under a given bounded input u, i.e. there exists a positive constant u_0 with $||u||_\infty \leq u_0$, such that $x \in \mathscr{X} \subset \mathbb{R}^n$ under this u where $\mathscr{X}$ is a given nonempty compact set.*

Assumption 5.1 requires that the state x of system (5.1) is bounded under a given bounded input u, i.e. $x \in \mathscr{X}$. Since it is assumed that system (5.1) can be transformed via a smooth diffeomorphism $z = \phi(x) : \mathscr{X} \to \mathscr{Z}$ into the output depending normal form (5.2), therefore the state z and the output y of (5.2) are both bounded as well, i.e. $z \in \mathscr{Z} \subset \mathbb{R}^n$ and $y \in \mathscr{Y} \subset \mathbb{R}$ where $\mathscr{Z}$ and $\mathscr{Y}$ are two corresponding nonempty compact sets. With the above boundedness assumption, we have the following results for the transformed system (5.2).

Lemma 5.2 *Suppose that system (5.1) can be transformed via a smooth diffeomorphism $z = \phi(x)$ into the output depending normal form (5.2). If Assumption 5.1 is satisfied, for any $a_i(y) \neq 0$ of system (5.2) with $y \in \mathscr{Y}$, there always exist positive constants $\underline{a}_i > 0$, $\overline{a}_i > 0$ and $\sigma_i > 0$ for $1 \leq i \leq n-1$ such that $0 < \underline{a}_i <| a_i(y) |< \overline{a}_i < \infty$, $\forall y \in \mathscr{Y}$ and $|\frac{da_i(y(t))}{dt}| < \sigma_i < \infty$, $\forall y \in \mathscr{Y}$.*

Lemma 5.3 *Suppose that system (5.1) can be transformed via a smooth diffeomorphism $z = \phi(x)$ into the output depending normal form (5.2). Then, $g(z)$ of system (5.2) is locally Lipschitz on $\mathscr{Z}$.*

When designing a global observer for nonlinear systems, normally the *global* Lipschitz property is required, see for example [139], which however is quite restrictive and difficult to be satisfied for many systems. The above Lemma 5.3 only showed that $g(z)$ of system (5.2) is *locally* Lipschitz on $\mathscr{Z}$. However, as explained in [65], under Assumption 5.1 we can extend the local nonlinearity of $g(z)$ to the whole space $\mathbb{R}^n$. The idea is to define a smooth bounded saturation function $\sigma : \mathbb{R}^n \to \mathscr{Z}$ which coincides with z on $\mathscr{Z}$, such that $\sigma(z) = z$ for all $z \in \mathscr{Z}$, then we can define an extended function $\bar{g}(z)$ of $g(z)$ as follows: $\bar{g}(z) = g(\sigma(z))$. By the construction, it is obvious that $\bar{g}(z)$ is *globally* Lipschitz with respect to z on $\mathbb{R}^n$ but bounded for all $z \in \mathscr{Z}$. With this extension, system (5.2) can be embedded into the following dynamics:

$$\begin{cases} \dot{z} = A(y)z + \beta(y) + \bar{g}(z)u \\ y = Cz \end{cases} \tag{5.5}$$

where $\bar{g}(z) \in \mathbb{R}^{n\times m}$ is of the following triangular form: $\bar{g}(z) = \left[\bar{g}_1^T(\underline{z}_1), \cdots, \bar{g}_n^T(\underline{z}_n)\right]^T$ with $\bar{g}_i = [\bar{g}_{i,1}, \cdots, \bar{g}_{i,m}]$ for $1 \le i \le n$.

Since system (5.5) is equivalent to system (5.2) when $z \in \mathscr{Z}$ under the same input u, hence there is no difference that we consider system (5.5) instead of system (5.2) for the observer design [65]. Therefore, in what follows, we will synthesize a global finite-time observer for system (5.5) in the next section.

For the sake of simplicity and without loss of generality, let us note the Lipschitzian constant for $\bar{g}_i(\underline{z}_i)$ of system (5.5) with $1 \le i \le n$ as l on $\mathbb{R}^n$, and define $\underline{a} = \min_{1\le i\le n}\{\underline{a}_i\}$ and $\overline{a} = \max_{1\le i\le n}\{\overline{a}_i\}$.

Denote

$$A_0 = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{5.6}$$

and

$$\Gamma(y,\theta) = \mathrm{diag}\{\gamma_1, \cdots, \gamma_n\} \tag{5.7}$$

where

$$\begin{aligned} \gamma_1 &= 1 \\ \gamma_i &= \frac{\prod_{j=1}^{i-1} a_j(y)}{\theta^{i-1}}, \text{ for } 2 \le i \le n \end{aligned} \tag{5.8}$$

with positive constant θ which will be determined in the next section. Since $a_i(y) \ne 0$ and bounded for all $y \in \mathscr{Y}$, then $\Gamma(y,\theta)$ is always invertible and bounded. Define

$$\Lambda(y,\dot{y},\theta) = \dot{\Gamma}(y,\theta)\Gamma^{-1}(y,\theta) \tag{5.9}$$

where $\dot{\Gamma}(y,\theta) = \frac{d\Gamma(y,\theta)}{dt} = \frac{dy}{dt}\frac{\partial\Gamma(y,\theta)}{\partial y}$.

Lemma 5.4 *If Assumption 5.1 is satisfied, then $\Lambda(y,\dot{y},\theta)$ defined in (5.9) is bounded for all $y \in \mathscr{Y}$, and there always exists a positive constant $\bar{\lambda}$ such that each element of $\Lambda(y,\dot{y},\theta)$ satisfies the following inequality: $|\Lambda_{i,j}(y,\dot{y},\theta)| < \bar{\lambda} < \infty$, $\forall y \in \mathscr{Y}$ for $1 \le i \le n$ and $1 \le j \le n$.*

Then, let $S_\infty(\theta)$ be the solution of the following equation:

$$0 = \theta S_\infty(\theta) + A_0^T S_\infty(\theta) + S_\infty(\theta) A_0 - C^T C \tag{5.10}$$

with A_0 defined in (5.6). Because of the structure of A_0 and C, i.e. the pair (A_0, C) is observable, it has been proven that there exists a unique symmetric positive definite solution $S_\infty(\theta)$ to the above algebraic matrix equation [20]. Then it can be shown that $S_\infty(\theta) = S_\infty^T(\theta)$ with the $(i,j)^{th}$ entry as $S_\infty(\theta)_{i,j} = \frac{S_\infty(1)_{i,j}}{\theta^{i+j-1}}$ for $1 \le i \le n$ and $1 \le j \le n$, where $S_\infty(1)_{i,j} = (-1)^{i+j} \frac{(i+j-2)!}{(i-1)!(j-1)!}$.

5.3 Main result

5.3.1 Finite time observer

Let us firstly recall the following theorems.

Theorem 5.5 *[168] Suppose there is a Lyapunov function $V(x)$ defined on a neighborhood $\mathscr{U} \subset \mathbb{R}^n$ of the origin, and $\frac{dV(x)}{dt} \le -pV(x)^\beta + kV(x), \forall x \in \mathscr{U} \setminus \{0\}$ with $\beta \in (0,1)$, $p,k > 0$, then the origin is finite-time stable. The set $\Omega = \{x \mid V(x)^{1-\beta} < \frac{p}{k}\} \cap \mathscr{U}$ is contained in the domain of attraction of the origin. The settling time satisfies $T(x_0) \le \frac{\ln(1-\frac{k}{p}V(x_0)^{1-\beta})}{k(\beta-1)}, \forall x_0 \in \Omega$.*

Theorem 5.6 *[87] If a system is globally asymptotically stable and finite-time attractive on a neighborhood of the origin, then it is globally finite-time stable.*

The above theorems show a way to prove the global finite-time stability. Now let us recall that, for system (5.5) with $\beta(y) = 0$ and $A(y) = A_0$, i.e. $a_i(y) = 1$ for all $1 \le i \le n-1$, [139] has already proved that it admits the following global finite-time high-gain observer:

$$\begin{cases} \dot{\hat{z}}_1 = & \hat{z}_2 + k_1(\lceil e_1 \rfloor^{\alpha_1} + \rho e_1) + \sum_{j=1}^{m} \bar{g}_{1,j}(\hat{\underline{z}}_1) u_j \\ \dot{\hat{z}}_2 = & \hat{z}_3 + k_2(\lceil e_1 \rfloor^{\alpha_2} + \rho e_1) + \sum_{j=1}^{m} \bar{g}_{2,j}(\hat{\underline{z}}_2) u_j \\ \vdots & \\ \dot{\hat{z}}_{n-1} = & \hat{z}_n + k_{n-1}(\lceil e_1 \rfloor^{\alpha_{n-1}} + \rho e_1) + \sum_{j=1}^{m} \bar{g}_{n-1,j}(\hat{\underline{z}}_{n-1}) u_j \\ \dot{\hat{z}}_n = & k_n(\lceil e_1 \rfloor^{\alpha_n} + \rho e_1) + \sum_{j=1}^{m} \bar{g}_{n,j}(\hat{\underline{z}}_n) u_j \end{cases} \tag{5.11}$$

with $e_1 = y - C\hat{z}$ and $\lceil e_1 \rfloor^{\alpha_i} = \mid e_1 \mid^{\alpha_i} \operatorname{sign}(e_1)$ where α_i are defined as those in [149]: $\alpha_i = i\alpha - (i-1)$ for $1 \le i \le n$ and $\alpha \in (1 - \frac{1}{n}, 1)$ for α close to 1. The gains in (5.11) are determined by:

$$K = [k_1, \cdots, k_n]^T = S_\infty^{-1}(\theta) C^T \tag{5.12}$$

Inspired by the above result, for the system (5.5) with a general $A(y)$ defined in (5.3) and K defined in (5.12), one can define the following gain depending on the output:

$$L(y,\theta)=[L_1(y,\theta),\cdots,L_n(y,\theta)]^T=\Gamma^{-1}(y,\theta)K \tag{5.13}$$

based on which we can propose the following observer dynamics:

$$\begin{cases}\dot{\hat{z}}_1 = & a_1(y)\hat{z}_2+\beta_1(y)+L_1(y,\theta)\left(\lceil e_1\rfloor^{\alpha_1}+\rho e_1\right)+\sum\limits_{j=1}^{m}\bar{g}_{1,j}(\hat{\underline{z}}_1)u_j\\ \dot{\hat{z}}_2 = & a_2(y)\hat{z}_3+\beta_2(y)+L_2(y,\theta)\left(\lceil e_1\rfloor^{\alpha_2}+\rho e_1\right)+\sum\limits_{j=1}^{m}\bar{g}_{2,j}(\hat{\underline{z}}_2)u_j\\ \vdots & \\ \dot{\hat{z}}_{n-1} = & a_{n-1}(y)\hat{z}_n+\beta_{n-1}(y)+\sum\limits_{j=1}^{m}\bar{g}_{n-1,j}(\hat{\underline{z}}_{n-1})u_j+L_{n-1}(y,\theta)\left(\lceil e_1\rfloor^{\alpha_{n-1}}+\rho e_1\right)\\ \dot{\hat{z}}_n = & \beta_n(y)+L_n(y,\theta)\left(\lceil e_1\rfloor^{\alpha_n}+\rho e_1\right)+\sum\limits_{j=1}^{m}\bar{g}_{n,j}(\hat{\underline{z}}_n)u_j\end{cases} \tag{5.14}$$

where θ and ρ are both positive constants.

5.3.2 Change of coordinates

For the proposed observer described in (5.14), denote $e=z-\hat{z}$, then the observation error is governed by the following system:

$$\dot{e}= \quad (A(y)-\rho\Gamma^{-1}(y,\theta)S_\infty^{-1}(\theta)C^TC)e-\mathscr{L}(L,e)+\mathscr{G}(z,\hat{z},u) \tag{5.15}$$

with $\mathscr{G}(z,\hat{z},u)=(\bar{g}(z)-\bar{g}(\hat{z}))u$ and

$$\mathscr{L}(L,e)=[L_1(y,\theta)\lceil e_1\rfloor^{\alpha_1},\cdots,L_n(y,\theta)\lceil e_1\rfloor^{\alpha_n}]^T \tag{5.16}$$

The stability analysis directly for (5.15) is complicated since the matrix in the linear term is not constant. In order to facilitate the analysis, let us introduce the following change of coordinates: $\varepsilon=\Gamma(y,\theta)e$ where $\Gamma(y,\theta)$ is defined in (5.7) which is always invertible and bounded for all $y\in\mathscr{Y}$. From the above equation, we have $e_1=\varepsilon_1$, then we obtain: $\mathscr{L}(L,e)=\mathscr{L}(L,\varepsilon)$ and

$$\begin{aligned}\dot{\varepsilon} \;=\;& \Gamma(y,\theta)\left[(A(y)-\rho\Gamma^{-1}(y,\theta)S_\infty^{-1}(\theta)C^TC)\right]e\\ &-\Gamma(y,\theta)\mathscr{L}(L,e)+\Gamma(y,\theta)\mathscr{G}(z,\hat{z},u)+\dot{\Gamma}(y,\dot{y},\theta)e\\ =\;& \Gamma(y,\theta)\left[A(y)-\rho\Gamma^{-1}(y,\theta)S_\infty^{-1}(\theta)C^TC\right]\Gamma^{-1}(y,\theta)\varepsilon\\ &-\Gamma(y,\theta)\mathscr{L}(L,\varepsilon)+\Gamma(y,\theta)\mathscr{G}(z,\hat{z},u)+\Lambda(y,\dot{y},\theta)\varepsilon\end{aligned} \tag{5.17}$$

where is $\Lambda(y,\dot{y},\theta)$ is given by (5.9).

Since $\Gamma(y,\theta)A(y)\Gamma^{-1}(y,\theta) = \theta A_0$ and $C\Gamma^{-1}(y,\theta) = C$, one has

$$\dot{\varepsilon} \;=\; \left[\theta A_0 - \rho S_\infty^{-1}(\theta)C^T C\right]\varepsilon - \Gamma(y,\theta)\mathscr{L}(L,\varepsilon) + \Gamma(y,\theta)\mathscr{G}(z,\hat{z},u) + \Lambda(y,\dot{y},\theta)\varepsilon \tag{5.18}$$

Since $\varepsilon = \Gamma(y,\theta)e$ with $\Gamma(y,\theta)$ being always invertible and bounded, then the finite-time convergence of e to zero is equivalent to the finite-time convergence of ε to zero. In what follows, instead of proving the global finite-time convergence of e to zero, we will prove the global finite-time stability of the observation error dynamics described by (5.18).

5.3.3 Finite-time convergence

According to Theorem 5.6, we are going to prove the global finite-time convergence of the proposed observer (5.14) in two steps. More precisely, we will firstly prove that (5.18) is globally asymptotically stable on $\mathbb{R}^n \backslash \mathscr{U}_1$ and then show that it is locally finite-time stable in $\mathscr{U}_2$, where $\mathscr{U}_1$ and $\mathscr{U}_2$ are two neighborhoods of the origin. Finally, the global finite-time convergence can be achieved by proving that $\mathscr{U}_1 \subset \mathscr{U}_2$.

For this, let us define

$$\mathscr{U}_1 = \{\varepsilon : ||\varepsilon||^2_{S_\infty(\theta)} \leq 1\} \tag{5.19}$$

as the ball centered at the origin with the radius equal to 1, then we can state the following lemma.

Lemma 5.7 *If Assumption 5.1 is satisfied, then there exist positive constants* $0 < \theta_1 < \infty$ *and* $\rho_1 = \theta_1/2$*, such that for all* $\theta > \theta_1$ *and* $\rho > \rho_1$ *the observation error system* (5.18) *is globally asymptotically stable on* $\mathbb{R}^n \backslash \mathscr{U}_1$.

Lemma 5.8 *If Assumption 5.1 is satisfied, then there exist positive constants* $0 < \varepsilon < 1$, $1 < \theta_2 < \infty$ *and a neighborhood of the origin* $\mathscr{U}_2$*, such that for all* $\alpha \in (1-\varepsilon, 1)$ *and* $\theta > \theta_2$ *the observation error system* (5.18) *is locally finite-time stable on* $\mathscr{U}_2$.

Theorem 5.9 *If Assumption 5.1 is satisfied, then there exist positive constants* $0 < \varepsilon < 1$, $0 < \bar{\theta} < \infty$ *and* $0 < \bar{\rho} < \infty$ *such that for all* $\theta > \bar{\theta}$, $\rho > \bar{\rho}$ *and* $\alpha \in (1-\varepsilon, 1)$ *system (5.14) is a global finite-time observer for (5.5).*

5.4 Example

Let us consider the following example:

$$\begin{cases} \dot{x}_1 = & \exp(x_2)(x_3 + x_2^2) - \exp(-x_2)(x_1 + x_2) + \sin(x_1 + x_2)u - u \\ \dot{x}_2 = & \exp(-x_2)(x_1 + x_2) + u \\ \dot{x}_3 = & -2x_2 \exp(-x_2)(x_1 + x_2) + \sin(x_1 + 2x_2 + x_2^2 + x_3)u - 2x_2 u \\ y = & x_2 \end{cases} \tag{5.20}$$

where the input is chosen as $u = 2\cos(5t) - 5$ in order to satisfy Assumption 5.1 and the boundedness of state is $\mathscr{X} = [0,7] \times [-1,1] \times [-1,4]$ (verified via Fig. 5.2 - Fig. 5.4).

Following the procedure presented in [214], one obtains the following diffeomorphism: $z = \phi(x) = \left(x_2, x_1 + x_2, x_2^2 + x_3 \right)^T$ which transforms (5.20) into the following normal form:

$$\begin{cases} \dot{z}_1 &= \exp(-y)z_2 + u \\ \dot{z}_2 &= \exp(y)z_3 + \sin(z_2)u \\ \dot{z}_3 &= \sin(z_1 + z_2 + z_3)u \\ y &= z_1 \end{cases} \tag{5.21}$$

Since $g(z) = [1, \sin(z_2), \sin(z_1 + z_2 + z_3)]^T$ in (5.21) is already *global* Lipschitz, therefore no extension is needed, or we can simply set $\bar{g}(z) = g(z)$ in (5.5). Since Assumption 5.1 is satisfied, thus one can follow the proposed result stated in Theorem 5.9 and design the following finite-time observer:

$$\begin{cases} \dot{\hat{z}}_1 = \exp(-y)\hat{z}_2 + u + L_1(y)\left(\lceil e_1 \rfloor^{\alpha} + \rho e_1\right) \\ \dot{\hat{z}}_2 = \exp(y)\hat{z}_3 + \sin(\hat{z}_2)u + L_2(y)\left(\lceil e_1 \rfloor^{2\alpha-1} + \rho e_1\right) \\ \dot{\hat{z}}_3 = \sin(\hat{z}_1 + \hat{z}_2 + \hat{z}_3)u + L_3(y)\left(\lceil e_1 \rfloor^{3\alpha-2} + \rho e_1\right) \end{cases} \tag{5.22}$$

where $e_1 = \hat{z}_1 - z_1$. Since

$$S_\infty(\theta) = \begin{pmatrix} \frac{1}{\theta} & -\frac{1}{\theta^2} & \frac{1}{\theta^3} \\ -\frac{1}{\theta^2} & \frac{2}{\theta^3} & -\frac{3}{\theta^4} \\ \frac{1}{\theta^3} & -\frac{3}{\theta^4} & \frac{6}{\theta^5} \end{pmatrix} \tag{5.23}$$

which yields $K = [3\theta, 3\theta^2, \theta^3]^T$, thus we obtain $L(y,\theta) = [3\theta, 3\theta^3 \exp(y), \theta^5]^T$. For the simulation settings, we choose $\theta = 3$, $\alpha = 0.8$, $\rho = \theta/2 + 10\theta^{2/3}$.

After having estimated z_i of (5.21) for $1 \le i \le 3$ via the observer (5.22), we can apply the inverse of the deduced diffeomorphism: $\hat{x} = \phi^{-1}(\hat{z}) = \left(\hat{z}_2 - \hat{z}_1, \hat{z}_1, \hat{z}_3 - \hat{z}_2^2 \right)^T$ to estimate the original state x of (5.20). The corresponding simulation results are depicted in Fig. 5.1-Fig. 5.4 with a noisy measurement (random noise in $[-0.03, 0.03]$) of the output.

5.5 Conclusion

Asymptotic and finite-time observers have been widely studied for the some well-known forms, such as output injection and triangular form. Concerning the more general form like output depending one, it is not so trivial to design a finite-time observer. For this, we investigated the global finite-time observer design problem for a special class of nonlinear systems, which can be transformed into an output depending triangular normal form. By imposing some assumptions, a global finite-time observer is proposed by adapting the result in [139]. Two parameters (θ and ρ) are used to tune the convergence time for the proposed observer.

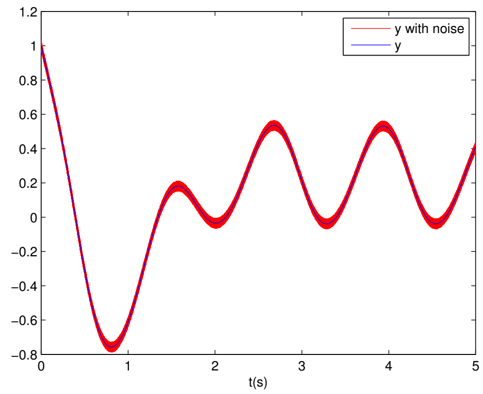


Fig. 5.1 The bounded output and the measurement with noise.

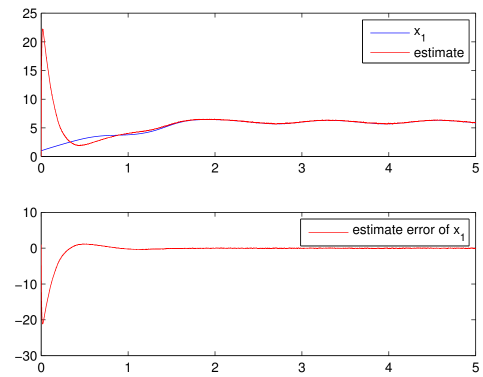


Fig. 5.2 x_1 and its estimation $\hat{x}_1$ with the initial conditions $x_1(0) = 1$ and $\hat{x}_1(0) = 0$.

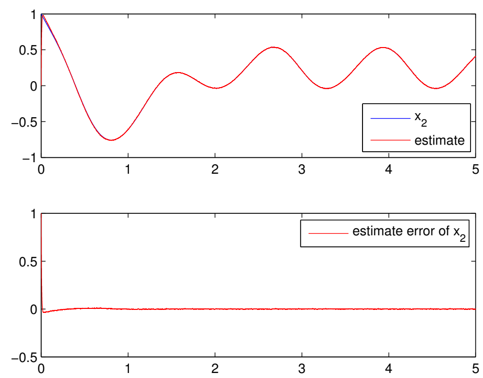


Fig. 5.3 x_2 and its estimation $\hat{x}_2$ with the initial conditions $x_2(0) = 1$ and $\hat{x}_2(0) = 0$.

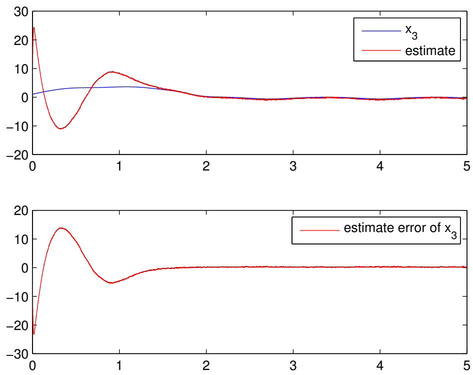


Fig. 5.4 x_3 and its estimation $\hat{x}_3$ with the initial conditions $x_3(0) = 1$ and $\hat{x}_3(0) = 0$.

Chapter 6

Interval Observer Design

Concerning nonlinear uncertain systems containing unknown parameters, despite of the existence of many solutions for observer design, a design of state estimator is rather complicated since the system is intrinsically nonlinear and it has uncertain terms in the state and in the output equations. Therefore, the whole system may be even not observable, which means that an exact estimation is not possible. Under this situation, we can relax the estimation goal to make an evaluation of the interval of admissible values for the state applying the theory of set-membership or interval estimation. The following recalls our recent results published in [J3, C18] on interval observer design for general uncertain nonlinear systems.

6.1 Notations and problem statement

Suppose that the unknown (may be time-varying) parameters θ belong to a compact set $\Theta \subset \mathbb{R}^p$, then the plant dynamics under consideration is given by

$$\begin{cases} \dot{x} = f(x) + B(x,\theta)u + \delta f(x,\theta), \\ y = h(x) + \delta h(x,\theta), \end{cases} \tag{6.1}$$

where x belongs to an open subset Ω of $\mathbb{R}^n$ (it is assumed that $0 \in \Omega$) and the initial state value belongs to a compact set $I_0(x_0) = [\underline{x}_0, \overline{x}_0]$; $y \in \mathbb{R}$ and $u \in \mathbb{R}^m$ represent respectively the output and the input. The vector fields f and h are smooth, and δf, δh and B are assumed to be locally Lipschitz continuous.

In our study, the studied system (6.1) is not assumed to be observable. Moreover, due to the unknown parameters, the exact estimation of state is impossible for such a system. Under this situation, the goal is to present a method to obtain an interval estimation.

For the state x, the basic idea of interval observer is to provide the upper and lower estimations of the state, noted as $\overline{x}$ and $\underline{x}$ respectively, and satisfying $\underline{x} \leq x \leq \overline{x}$ in the element-wise sense. In other words, we want to design observers which give us two dynamics $\dot{\overline{e}}$ and $\dot{\underline{e}}$ with $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$, such that both $\overline{e}$ and $\underline{e}$ are always positive. In this sense, the interval observer is linked to the following concept of positive system:

Corollary 6.1 *[172] Assume that A is a Metzler matrix and $b(t) \in \mathbb{R}^n_+, \forall t \geq t_0$, where t_0 represents the initial time, then the following system*

$$\frac{dx(t)}{dt} = Ax + b(t), \tag{6.2}$$

possesses, for every $x(t_0) \in \mathbb{R}_+^n$, a unique solution $x(t)$ for all $t \geq t_0$. Moreover, for any $x(t_0) \in \mathbb{R}_+^n$, the inequality $x(t) \geq 0$ holds for every $t \geq t_0$.

In other words, under conditions of Corollary 6.1, $\mathbb{R}_+^n$ is positively invariant w.r.t (6.2). The result of our work is in spirit of the above corollary.

6.2 Canonical form

For the studied system (6.1), one obtains the nominal drift-system by setting $u = 0$, $\delta f = 0$, $\delta h = 0$ in (6.1):

$$\begin{cases} \dot{x} = f(x), \\ y = h(x). \end{cases} \tag{6.3}$$

Here, it is assumed that the nominal system (6.3) satisfies the observability rank condition, *i.e.* the following map:

$$\Phi_{(6.3)} = \left(h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\right)^T \tag{6.4}$$

is a change of coordinates. With this diffeomorphism $\zeta = \Phi_{(6.3)}(x)$, it follows that, system (6.3) can be rewritten as:

$$\begin{cases} \dot{\zeta} = \tilde{A}\zeta + \tilde{b}\varphi(\zeta), \\ y = \tilde{C}\zeta, \end{cases} \tag{6.5}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \; \tilde{b} = \begin{pmatrix} 0 \\ \\ 0 \\ 1 \end{pmatrix}, \; \tilde{C} = (1, 0, \ldots, 0,), \; \varphi(\zeta) = L_f^n h(x) \tag{6.6}$$

The forthcoming analysis is based on this canonical form. We will not even assume that (6.1) is observable, but need only the observability for the nominal system (6.3).

Assumption 6.2 *The nominal system (6.3) is observable and $f(0) = 0$, $h(0) = 0$ in (6.3).*

It is obvious that, with the diffeomorphism $\Phi_{(6.3)}(x)$, system (6.1) can be transformed into the following one:

$$\begin{cases} \dot{\zeta} = \tilde{A}\zeta + \tilde{F}(\zeta, \theta) + \tilde{G}(\zeta, \theta)u, \\ y = \tilde{C}\zeta + \tilde{H}(\zeta, \theta), \end{cases} \tag{6.7}$$

where

$$\begin{aligned}
\tilde{G}(\zeta,\theta) &= \left(\tilde{G}_1(x,\theta),\ldots,\tilde{G}_n(x,\theta)\right)^T_{x=\Phi^{-1}_{(6.3)}(\zeta)},\\
\tilde{G}_i(x,\theta) &= L_{B(x,\theta)}L^{i-1}_{f(x)}h(x), i=1\ldots n,\\
\tilde{F}(\zeta,\theta) &= \left(\tilde{F}_1(x,\theta),\ldots,\tilde{F}_{n-1}(x,\theta)\right)^T_{x=\Phi^{-1}_{(6.3)}(\zeta)}+\tilde{b}\varphi(\zeta),\\
\tilde{F}_i(x,\theta) &= L_{\delta f(x,\theta)}L^{i-1}_{f(x)}h(x), i=1\ldots n,\\
\tilde{H}(\zeta,\theta) &= \delta h(x,\theta)|_{x=\Phi^{-1}_{(6.3)}(\zeta)}.
\end{aligned}$$

Then we have the following fact.

Claim 6.3 *Under Assumption 6.2, there exist a matrix $\tilde{L}$ and an invertible matrix P such that the matrix $\tilde{A}-\tilde{L}\tilde{C}$ is similar to a Metzler matrix $A-LC$, which means $A-LC = P(\tilde{A}-\tilde{L}\tilde{C})P^{-1}$.*

The conditions of the existence of such a transformation matrix P can be found in [154], they are related with solution of a Sylvester equation. By Assumption 6.2 the pair $(\tilde{A},\tilde{C})$ is observable, then there always exists a matrix $\tilde{L}$ such that the claim is satisfied [154].

Introducing the new coordinates $z = P\zeta$ we arrive at the desired representation of system (6.1):

$$\begin{cases} \dot{z} = Az + F(z,\theta) + G(z,\theta)u, \\ y = Cz + H(z,\theta), \end{cases} \tag{6.8}$$

where the matrices A, C are given in Claim 6.3, and $H(z,\theta) = \tilde{H}(P^{-1}z,\theta)$, $F(z,\theta) = P\tilde{F}(P^{-1}z,\theta)$, $G(z,\theta) = P\tilde{G}(P^{-1}z,\theta)$.

Remark 6.4 *Since the origin of (6.3) is assumed to be an equilibrium and $\Phi_{(6.3)}$ is a diffeomorphism with $\Phi_{(6.3)}(0) = 0$, thus the origin is also an equilibrium for the both transformed systems in coordinates ζ and z for $F = 0$ and $u = 0$. By construction, F, H and G are locally Lipschitz continuous.*

Let us remind that, since the initial condition x_0 for (6.1) is only known within a certain interval $I(x_0) = [\underline{x}_0, \overline{x}_0]$, then using the diffeomorphism $\Phi_{(6.3)}(x)$, the initial condition $z_0 = P\Phi_{(6.3)}(x_0)$ is also known within a certain interval $I(z_0) = [\underline{z}_0, \overline{z}_0]$. Thus our original problem turns out to a dynamical system with the input (u,y) and the outputs $\underline{z}(t)$ and $\overline{z}(t)$ such that for all $t \geq 0$ we have $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$.

6.3 Bounding functions

Since Θ is a compact set and by continuity of $F(z,\theta), H(z,\theta)$ and $G(z,\theta)$ (the functions $\delta f(x,\theta), B(x,\theta)$ and $\delta h(x,\theta)$ were assumed to be continuous and $\Phi_{(6.3)}$ given by (6.4) is a diffeomorphism), the element-wise minimum and maximum of $F(z,\theta), H(z,\theta)$ and $G(z,\theta)u$ (for a given u) in the domain $\theta \in \Theta$, $\underline{z} \leq z \leq \overline{z}$ exist. In order to built the observers, we need a more precise knowledge on these max and min functions.

For a matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$, define $\mathscr{A}^+ = \max\{0,\mathscr{A}\}$[1] and $\mathscr{A}^- = \mathscr{A}^+ - \mathscr{A}$. For a vector $x \in \mathbb{R}^n$, define $x^+ = \max\{0,x\}$ and $x^- = x^+ - x$. Then, let us firstly recall the following lemma.

Lemma 6.5 [60] *Let $\mathscr{A} \in \mathbb{R}^{n\times n}$, by the definition $\mathscr{A} = \mathscr{A}^+ - \mathscr{A}^-$ and for any $[\underline{z},\overline{z}] \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, if $\underline{z} \le z \le \overline{z}$, then $\mathscr{A}^+\underline{z} - \mathscr{A}^-\overline{z} \le \mathscr{A}z \le \mathscr{A}^+\overline{z} - \mathscr{A}^-\underline{z}$.*

Lemma 6.6 *Let $\underline{x},x,\overline{x} \in \mathbb{R}^n$ and $\underline{\mathscr{A}},\mathscr{A},\overline{\mathscr{A}} \in \mathbb{R}^{n\times m}$, then*

$$\underline{x} \le x \le \overline{x} \Longleftrightarrow \underline{x}^+ \le x^+ \le \overline{x}^+,\ \overline{x}^- \le x^- \le \underline{x}^-;$$
$$\underline{\mathscr{A}} \le \mathscr{A} \le \overline{\mathscr{A}} \Longleftrightarrow \underline{\mathscr{A}}^+ \le \mathscr{A}^+ \le \overline{\mathscr{A}}^+,\ \overline{\mathscr{A}}^- \le \mathscr{A}^- \le \underline{\mathscr{A}}^-.$$

Lemma 6.7 [60] *Let $\underline{\mathscr{A}} \le \mathscr{A} \le \overline{\mathscr{A}}$ for some $\underline{\mathscr{A}}$, $\overline{\mathscr{A}}$, $\mathscr{A} \in \mathbb{R}^{n\times n}$ and $\underline{x} \le x \le \overline{x}$ for $\underline{x}$, $\overline{x}$, $x \in \mathbb{R}^n$, then*

$$\begin{aligned}\underline{\mathscr{A}}^+\underline{x}^+ - \overline{\mathscr{A}}^+\underline{x}^- - \underline{\mathscr{A}}^-\overline{x}^+ + \overline{\mathscr{A}}^-\overline{x}^- &\le \mathscr{A}x \\ \le \overline{\mathscr{A}}^+\overline{x}^+ - \underline{\mathscr{A}}^+\overline{x}^- - \overline{\mathscr{A}}^-\underline{x}^+ + \underline{\mathscr{A}}^-\underline{x}^-&.\end{aligned} \tag{6.9}$$

To apply these lemmas, we have to introduce the following standard (see [78], for example) assumption in the estimation theory on the boundedness of the state x and the input u values for system (6.1).

Assumption 6.8 *For system (6.1), it is assumed that $x(t) \in \mathscr{X}$ and $u(t) \in \mathscr{U}$ for all $t \ge 0$, where $\mathscr{X} \subset \Omega$ and $\mathscr{U} \subset \mathbb{R}^m$ are two given compacts.*

Under this assumption, since $\zeta = \Phi_{(6.3)}(x)$ defined by (6.4) is a diffeomorphism and due to the fact that $z = P\zeta$, thus there exists a compact set $\mathscr{Z} \subset \mathbb{R}^n$ such that $z(t) \in \mathscr{Z}$ for all $t \ge 0$.

In [154] it has been assumed that uncertain terms in the system equations admit known upper and lower bounding functions. In our work we are going to prove that these functional bounds exist and satisfy some useful properties.

Lemma 6.9 *There exist two functions $\overline{F},\underline{F} : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that, for all $\theta \in \Theta$ and $\underline{z} \le z \le \overline{z}$ with $z \in \mathscr{Z}$, the following inequalities hold:*

$$\underline{F}(\underline{z},\overline{z}) \le F(z,\theta) \le \overline{F}(\underline{z},\overline{z}), \tag{6.10}$$

and for a given submultiplicative norm $\|\cdot\|$ we have:

$$\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \le \overline{l}_{\overline{F}}\|\overline{z}-z\| + \underline{l}_{\overline{F}}\|\underline{z}-z\| + l_{\overline{F}},$$
$$\|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \le \overline{l}_{\underline{F}}\|\overline{z}-z\| + \underline{l}_{\underline{F}}\|\underline{z}-z\| + l_{\underline{F}},$$

for some positive constants $\overline{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\overline{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$ and $l_{\underline{F}}$.

[1]The $\max\{\cdot\}$ operation is applied element-wise.

Remark 6.10 *Lemma 6.9 shows that the difference of functions $\overline{F}, \underline{F}$ and F has a linear growth with respect to the interval width estimates $\underline{z} - z$ and $\overline{z} - z$.*

Remark 6.11 *It is clear that the positive constants $\bar{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\bar{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$ and $l_{\underline{F}}$ depend on the choice of P. Due to the fact that $F(z,\theta) = P\tilde{F}(P^{-1}z,\theta)$, then we have $\underline{F}(\underline{z},\overline{z}) \leq F(z,\theta) \leq \overline{F}(\underline{z},\overline{z})$ where $\underline{F} = P^+\underline{\tilde{F}} - P^-\overline{\tilde{F}}$ and $\overline{F} = P^+\overline{\tilde{F}} - P^-\underline{\tilde{F}}$. The above relations imply that the result of Lemma 6.9 is equivalent to the following one:*

$$\begin{aligned}
\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &\leq \|P\|\|P^{-1}\| \left[\bar{l}_{\overline{\tilde{F}}}\|\overline{z}-z\| + \underline{l}_{\overline{\tilde{F}}}\|\underline{z}-z\| + l_{\overline{\tilde{F}}}\right], \\
\|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &\leq \|P\|\|P^{-1}\| \left[\bar{l}_{\underline{\tilde{F}}}\|\overline{z}-z\| + \underline{l}_{\underline{\tilde{F}}}\|\underline{z}-z\| + l_{\underline{\tilde{F}}}\right],
\end{aligned}$$

for some positive constants $\bar{l}_{\overline{\tilde{F}}}$, $\underline{l}_{\overline{\tilde{F}}}$, $l_{\overline{\tilde{F}}}$, $\bar{l}_{\underline{\tilde{F}}}$, $\underline{l}_{\underline{\tilde{F}}}$ and $l_{\underline{\tilde{F}}}$, which are independent of P.

Remark 6.12 *Note that the values of constants $\bar{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\bar{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$, $l_{\underline{F}}$ and functions $\underline{F}$, $\overline{F}$ are the theoretically maximal bounds. The goal of the lemma is just to show that the bounds exist and to provide some approximate outer estimates for them. For the concrete applications, more accurate values may be computed.*

Using the same arguments, a similar result can be established for H, *i.e.* there exist two functions $\overline{H}, \underline{H} : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that, for all $\theta \in \Theta$ and $\underline{z} \leq z \leq \overline{z}$ with $z \in \mathscr{Z}$, the following inequality holds:

$$\underline{H}(\underline{z},\overline{z}) \leq H(z,\theta) \leq \overline{H}(\underline{z},\overline{z}), \tag{6.11}$$

and for a given submultiplicative norm $\|\cdot\|$ we have

$$\begin{aligned}
\|\overline{H}(\underline{z},\overline{z}) - H(z,\theta)\| &\leq \bar{l}_{\overline{H}}\|\overline{z}-z\| + \underline{l}_{\overline{H}}\|\underline{z}-z\| + l_{\overline{H}}, \\
\|\underline{H}(\underline{z},\overline{z}) - H(z,\theta)\| &\leq \bar{l}_{\underline{H}}\|\overline{z}-z\| + \underline{l}_{\underline{H}}\|\underline{z}-z\| + l_{\underline{H}},
\end{aligned}$$

for some positive constants $\bar{l}_{\overline{H}}$, $\underline{l}_{\overline{H}}$, $l_{\overline{H}}$, $\bar{l}_{\underline{H}}$, $\underline{l}_{\underline{H}}$ and $l_{\underline{H}}$.

Similar relations for the term G can be also derived using Lemma 6.9, *i.e.* there exist two functions $\overline{G}, \underline{G} : \mathbb{R}^{2n+m} \to \mathbb{R}^n$ such that the following inequality holds for all $u \in \mathscr{U}$, $\theta \in \Theta$ and $\underline{z} \leq z \leq \overline{z}$:

$$\underline{G}_i(\underline{z},\overline{z},u_i) \leq u_i G_i(z,\theta) \leq \overline{G}_i(\underline{z},\overline{z},u_i) \tag{6.12}$$

for all $0 \leq i \leq m$, and for a given submultiplicative norm $\|\cdot\|$ we have

$$\begin{aligned}
\|\overline{G}_i(\underline{z},\overline{z},u_i) - G_i(z,\theta)u_i\| &\leq |u_i|(\bar{l}_{\overline{G}}\|\overline{z}-z\| + \underline{l}_{\overline{G}}\|\underline{z}-z\| + l_{\overline{G}}), \\
\|\underline{G}_i(\underline{z},\overline{z},u_i) - G_i(z,\theta)u_i\| &\leq |u_i|(\bar{l}_{\underline{G}}\|\overline{z}-z\| + \underline{l}_{\underline{G}}\|\underline{z}-z\| + l_{\underline{G}}),
\end{aligned}$$

for some positive constants $\bar{l}_{\overline{G}}$, $\underline{l}_{\overline{G}}$, $l_{\overline{G}}$, $\bar{l}_{\underline{G}}$, $\underline{l}_{\underline{G}}$ and $l_{\underline{G}}$.

6.4 Main result

Let $\bar{z}$, $\underline{z}$ be the estimates of the transformed state z, whose dynamics constitute the interval observer as follows:

$$\begin{aligned}
\dot{\bar{z}} &= A\bar{z} + \overline{G}(\underline{z},\bar{z},u) + \overline{F}(\underline{z},\bar{z}) + L(y - C\bar{z}) + L^{+}\overline{H}(\underline{z},\bar{z}) - L^{-}\underline{H}(\underline{z},\bar{z}),\\
\dot{\underline{z}} &= A\underline{z} + \underline{G}(\underline{z},\bar{z},u) + \underline{F}(\underline{z},\bar{z}) + L(y - C\underline{z}) + L^{+}\underline{H}(\underline{z},\bar{z}) - L^{-}\overline{H}(\underline{z},\bar{z}),
\end{aligned}$$

where the observer gain $L = (l_1, \ldots, l_n)^T$ has to be designed. Defining the upper error $\bar{e} = \bar{z} - z$ and the lower error $\underline{e} = z - \underline{z}$, their dynamics read as:

$$\begin{aligned}
\frac{d\bar{e}}{dt} &= (A - LC)\bar{e} + \overline{\Delta}(\underline{z},\bar{z},z,\theta,u,L),\\
\frac{d\underline{e}}{dt} &= (A - LC)\underline{e} + \underline{\Delta}(\underline{z},\bar{z},z,\theta,u,L),
\end{aligned} \tag{6.13}$$

where $\overline{\Delta}(\underline{z},\bar{z},z,\theta,u,L)$ is the sum of the following terms:

$$\begin{aligned}
\overline{\Delta}_G(\underline{z},\bar{z},z,\theta,u) &= \overline{G}(\underline{z},\bar{z},u) - G(z,\theta)u,\ \overline{\Delta}_F(\underline{z},\bar{z},z,\theta) = \overline{F}(\underline{z},\bar{z}) - F(z,\theta),\\
\overline{\Delta}_{LH}(\underline{z},\bar{z},z,\theta,L) &= L^{+}\overline{H}(\underline{z},\bar{z}) - L^{-}\underline{H}(\underline{z},\bar{z}) + LH(z,\theta),
\end{aligned}$$

and $\underline{\Delta}(\underline{z},\bar{z},z,\theta,u,L)$ is the sum of

$$\begin{aligned}
\underline{\Delta}_G(\underline{z},\bar{z},z,\theta,u) &= G(z,\theta)u - \underline{G}(\underline{z},\bar{z},u),\ \underline{\Delta}_F(\underline{z},\bar{z},z,\theta) = F(z,\theta) - \underline{F}(\underline{z},\bar{z}),\\
\underline{\Delta}_{LH}(\underline{z},\bar{z},z,\theta,L) &= -LH(z,\theta) - L^{+}\underline{H}(\underline{z},\bar{z}) + L^{-}\overline{H}(\underline{z},\bar{z})).
\end{aligned}$$

Corollary 6.13 *For all $z \in \mathscr{X}$, $u \in \mathscr{U}$ and $\theta \in \Theta$ there exist positive constants $\underline{l}_{\underline{\Delta}}$, $\bar{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$, $\bar{l}_{\overline{\Delta}}$, $l_{\overline{\Delta}}$, $l_{\underline{\Delta}}$ such that for a chosen submultiplicative norm $\|\cdot\|$*

$$\begin{aligned}
\|\overline{\Delta}(\cdot,L)\| &\leq [\bar{l}_{\overline{\Delta}}\|\bar{z} - z\| + \underline{l}_{\overline{\Delta}}\|z - \underline{z}\| + l_{\overline{\Delta}}](1 + \|L\|),\\
\|\underline{\Delta}(\cdot,L)\| &\leq [\bar{l}_{\underline{\Delta}}\|\bar{z} - z\| + \underline{l}_{\underline{\Delta}}\|z - \underline{z}\| + l_{\underline{\Delta}}](1 + \|L\|).
\end{aligned}$$

Remark 6.14 *As it has been explained in Remark 6.11, the result of Corollary 6.13 can be stated as well for some positive constants independent of P,* i.e. *there exist $\underline{l}_{\underline{\tilde{\Delta}}}$, $\bar{l}_{\underline{\tilde{\Delta}}}$, $\underline{l}_{\overline{\tilde{\Delta}}}$, $\bar{l}_{\overline{\tilde{\Delta}}}$, $l_{\overline{\tilde{\Delta}}}$, $l_{\underline{\tilde{\Delta}}}$, independent of P, such that*

$$\begin{aligned}
\|\overline{\Delta}(\cdot,L)\| &\leq [\bar{l}_{\overline{\tilde{\Delta}}}\|\bar{z} - z\| + \underline{l}_{\overline{\tilde{\Delta}}}\|z - \underline{z}\| + l_{\overline{\tilde{\Delta}}}](1 + \|L\|)\|P\|\|P^{-1}\|,\\
\|\underline{\Delta}(\cdot,L)\| &\leq [\bar{l}_{\underline{\tilde{\Delta}}}\|\bar{z} - z\| + \underline{l}_{\underline{\tilde{\Delta}}}\|z - \underline{z}\| + l_{\underline{\tilde{\Delta}}}](1 + \|L\|)\|P\|\|P^{-1}\|.
\end{aligned}$$

Lemma 6.15 *Assume that Assumptions 6.2 and 6.8 are satisfied, then for any $u \in \mathscr{L}_\infty$ in $\mathscr{U}$ and any $(\bar{e}(t_0), \underline{e}(t_0)) \geq 0$ (component-wise), the inequality $(\bar{e}(t), \underline{e}(t)) \geq 0$ holds for every $t \geq t_0$.*

Theorem 6.16 *Suppose that Assumptions 6.2 and 6.8 are satisfied. For the constants $\underline{l}_{\underline{\Delta}}$, $\bar{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$, $\bar{l}_{\overline{\Delta}}$ deduced from Corollary 6.13, if there exist positive definite and symmetric matrices S, Q, O such that the following inequality is satisfied:*

$$D^T S + SD + SO^{-1}S + \alpha\|O\|I + Q \preceq 0, \tag{6.14}$$

where $D = A - LC$ *and* $\alpha = 3(1+\|L\|)^2 \max\{\overline{l}^2_{\overline{\Delta}} + \overline{l}^2_{\underline{\Delta}}, \underline{l}^2_{\overline{\Delta}} + \underline{l}^2_{\underline{\Delta}}\}$, *then the variables* $\underline{z}(t)$ *and* $\overline{z}(t)$ *are bounded. Moreover,*

$$\underline{z}(t) \le z(t) \le \overline{z}(t). \tag{6.15}$$

is satisfied for all $t > 0$ *if it is valid for* $t = 0$.

Note that if the relation (6.15) is satisfied and the variables $\underline{z}$ and $\overline{z}$ are bounded, then by standard arguments [96] we can compute $\underline{x}(t) = \underline{\Psi}(\underline{z}(t), \overline{z}(t))$ and $\overline{x}(t) = \overline{\Psi}(\underline{z}(t), \overline{z}(t))$ (where $\underline{\Psi}, \overline{\Psi}$ depend on $\Phi_{(6.3)}$ and P) such that $\underline{x}(t) \le x(t) \le \overline{x}(t)$, for all $t \ge 0$, *i.e.* we obtain the interval estimation for the original nonlinear system (6.1).

If the output y equals to $h(x)$, *i.e.* there is no uncertainty $\delta h(x,\theta)$, then clearly the above theorem has more simple conditions.

Corollary 6.17 *Let Assumptions 6.2 and 6.8 be satisfied, and* $y = h(x)$ *in (6.1). For the deduced matrices* $\tilde{L}$ *and* P *in Claim 6.3, if there exist the positive definite and symmetric matrices* S *and* Q *such that the following LMI be true*

$$\begin{bmatrix} -I & S \\ S & D^T S + SD + \alpha I + Q \end{bmatrix} \preceq 0 \tag{6.16}$$

where $D = P\tilde{A}P^{-1} - L\tilde{C}P^{-1}$, $\alpha = 3\max\{\overline{l}^2_{\overline{\Delta}} + \overline{l}^2_{\underline{\Delta}}, \underline{l}^2_{\overline{\Delta}} + \underline{l}^2_{\underline{\Delta}}\}$ *with the constants* $\underline{l}_{\underline{\Delta}}$, $\overline{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$ *and* $\overline{l}_{\overline{\Delta}}$ *deduced from Corollary 6.13, then the variables* $\underline{z}(t), \overline{z}(t)$ *are bounded and (6.15) is satisfied for all* $t \ge 0$.

Based on the result stated in Corollary 6.17, the following algorithm is presented to summarize the design procedure for the proposed interval observer:

Step 1: Since the nominal system of (6.1) is observable, compute the diffeomorphism $\Phi_{(6.3)}$ to obtain $\tilde{A}$ and $\tilde{C}$;

Step 2: Due to the fact that the pair $(\tilde{A}, \tilde{C})$ is observable, seek a matrix $\tilde{L}$ and an invertible matrix P such that the matrix $A - LC$ is Hurwitz and Metzler, where $A = P\tilde{A}P^{-1}$, $C = \tilde{C}P^{-1}$ and $L = P\tilde{L}$;

Step 3: Transform system (6.1) by applying the change of coordinates $z = P\Phi_{(6.3)}(x)$ to (6.8) with F, H and G, and calculate the positive constants $\overline{l}_*$, $\underline{l}_*$, and l_* where $*$ represents $\overline{F}, \overline{H}, \overline{G}$ and $\underline{F}, \underline{H}, \underline{G}$, which enables us to compute $\underline{l}_{\underline{\Delta}}$, $\overline{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$ (see Corollary 6.13);

Step 4: Set $D = A - LC$ and $\alpha = 3\max\{\overline{l}^2_{\overline{\Delta}} + \overline{l}^2_{\underline{\Delta}}, \underline{l}^2_{\overline{\Delta}} + \underline{l}^2_{\underline{\Delta}}\}$. If the LMI (11.15) can be solved, then go to Step 5. Otherwise, go back to Step 2 by changing the choices of $\tilde{L}$ and P.

Step 5: Design the interval observer (6.13), whose observation error is bounded since (11.15) is satisfied.

As it has been shown that an interval observer for the uncertain nonlinear system (6.1) is proposed using the transformation of coordinates calculated for the nominal system (6.3). It is worth noting that the original system may be non-uniformly observable, but if it is possible to extract from (6.1) a nominal observable system (6.3), then the proposed approach establishes the interval observer and the corresponding transformation of coordinates providing the interval state estimation for (6.1). Moreover, if Assumption 6.8 is not satisfied for (6.1) for all $t \geq 0$, the presented interval method is still valid during a finite time T if $x(t) \in \mathscr{X}$ and $u(t) \in \mathscr{U}$ for $T \geq t \geq 0$. Let us demonstrate the advantages of this approach via an example of a nonlinear non-observable system.

6.5 Example

Consider the following example of the nonlinear system (6.1):

$$\begin{aligned}
\dot{x}_1 &= x_2 + a_1 \sin(\theta_1 x_1 x_2), \\
\dot{x}_2 &= -a_4 x_2 - a_2 \sin(\theta_2^2 x_1) + a_3 \cos(y) u, \\
y &= x_1 + c x_2 + \theta_3 x_1 x_2,
\end{aligned}$$

where $a_1 = 0.25$, $a_2 = 19$, $a_3 = 1$, $a_4 = 2$ and $c = 0.526$ are given known parameters, the unknown parameters admit the condition $|\theta_i| \leq \bar{\theta}$ for $i = 1,2,3$ with $\bar{\theta} = 0.1$. For simulation we will use $\theta_1 = 0.1$, $\theta_2 = -0.1$, $\theta_3 = -0.05[1 + 0.25\sin(3t) + 0.25\cos(5t)]$ (it is a time-varying signal representing additional disturbance/noise) and $u(t) = 0.1\sin(t) + 0.75\cos(0.25t)$. It is straightforward to check that the linearization of this system at the origin for all admissible values of parameters is stable. We assume that $|x_1(0)| \leq 0.1$, $|x_2(0)| \leq 0.1$ and that solutions stay bounded and $|x_1(t)| \leq \bar{x}_1 = 0.2$, $|x_2(t)| \leq \bar{x}_2 = 0.2$ Therefore, Assumption 6.8 is satisfied for $\mathscr{X} = [-0.2, 0.2]^2$ and $\mathscr{U} = [-1, 1]$. Moreover, since the observability matrix of this system depends on the unknown parameters, thus the system is not always observable on these compact sets.

For this example, the following nominal system has been chosen:

$$f_1(x) = x_2,\ f_2(x) = -a_4 x_2,\ h(x) = x_1 + c x_2$$

then $\delta f_1(x,\theta) = a_1 \sin(\theta_1 x_1 x_2)$,$\delta f_2(x,\theta) = -a_2 \sin(\theta_2^2 x_1)$, $\delta h(x,\theta) = \theta_3 x_1 x_2$. It is straightforward to check that the nominal system as a linear system in the canonical form is observable. Thus Assumption 6.2 is verified. Claim 6.3 is satisfied for the matrix $L = [1.9, 0]^T$. Let us compute the bounding functions for δf and δh. To this end, define the following two functions:

$$\text{Product}(\underline{x},\bar{x}) = \begin{bmatrix} \min\{\bar{x}_1\bar{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\underline{x}_2\} \\ \max\{\bar{x}_1\bar{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\underline{x}_2\} \end{bmatrix}, \quad \begin{bmatrix} \underline{\sin}(\underline{x},\bar{x}) \\ \overline{\sin}(\underline{x},\bar{x}) \end{bmatrix} = \begin{bmatrix} \sin(\underline{x}) \\ \sin(\bar{x}) \end{bmatrix}$$

corresponding to the interval of the product $x_1 x_2$ for $x = [x_1\ x_2]^T$ with $\underline{x} \leq x \leq \bar{x}$ and the interval of the function $\sin(x)$ for a scalar x with $\underline{x} \leq x \leq \bar{x}$ (for $|x| \leq \pi/2$). Then

$$\begin{aligned}
\underline{\delta f}_1(\underline{x},\bar{x}) &= a_1 \underline{\sin}\{\rho(\bar{\theta},\underline{x},\bar{x})\}, \underline{\delta f}_2(\underline{x},\bar{x}) = -a_2 \overline{\sin}\left\{\text{Product}\left(\begin{bmatrix} -\bar{\theta} \\ \bar{\theta} \end{bmatrix}, \begin{bmatrix} \bar{x}_1 \\ \bar{x}_1 \end{bmatrix}\right)\right\} \\
\overline{\delta f}_1(\underline{x},\bar{x}) &= a_1 \overline{\sin}\{\rho(\bar{\theta},\underline{x},\bar{x})\}, \overline{\delta f}_2(\underline{x},\bar{x}) = -a_2 \underline{\sin}\left\{\text{Product}\left(\begin{bmatrix} -\bar{\theta} \\ \bar{\theta} \end{bmatrix}, \begin{bmatrix} \bar{x}_1 \\ \bar{x}_1 \end{bmatrix}\right)\right\}
\end{aligned}$$

and $\begin{bmatrix} \underline{\delta h}_1(\underline{x},\overline{x}) \\ \overline{\delta h}_1(\underline{x},\overline{x}) \end{bmatrix} = \rho(\bar{\theta},\underline{x},\overline{x})$ where $\rho(\bar{\theta},\underline{x},\overline{x}) = \text{Product}\left(\begin{bmatrix} -\bar{\theta} \\ \bar{\theta} \end{bmatrix}, \text{Product}(\underline{x},\overline{x})\right)$. Take $\overline{l}_{\underline{\Delta}} = \overline{l}_{\overline{\Delta}} = \underline{l}_{\overline{\Delta}} = \underline{l}_{\underline{\Delta}} = [a_2 + (a_1+1)\bar{x}_2]\bar{\theta}$, then $\alpha = 3(1+\|L\|)^2 \max\{\overline{l}_{\overline{\Delta}}^2 + \overline{l}_{\underline{\Delta}}^2, \underline{l}_{\overline{\Delta}}^2 + \underline{l}_{\underline{\Delta}}^2\} = 1.279$. For the chosen parameters, the matrix inequality from Theorem 6.16 is satisfied for:

$$S = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 1.8 \end{bmatrix},\ O = I,\ Q = 0.8I,$$

thus all conditions of Theorem 6.16 have been verified. The results of the interval estimation are given in Fig. 6.1.

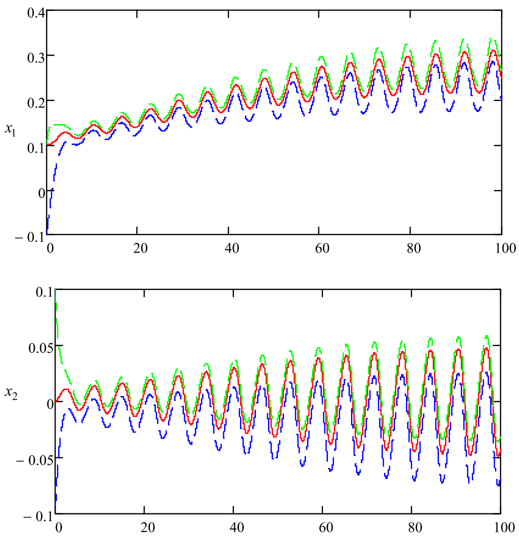


Fig. 6.1 The results of interval estimation for the coordinates x_1 and x_2

6.6 Conclusion

The problem of state estimation is studied for an uncertain nonlinear system not in a canonical form. The uncertainty is presented by a vector of unknown time-varying parameters, the system equations depend on this vector in a nonlinear fashion. It is assumed that the values of this vector of unknown parameters belong to some known compact set. The idea of the proposed approach is to extract a known nominal observable subsystem from the plant equations, next a transformation of coordinates developed for the nominal system is applied to the original one. The interval observer is designed for the transformed system. It is shown that the residual nonlinear terms dependent on the vector of unknown parameters have linear upper and lower functional bounds, that simplifies the interval observer design and stability/cooperativity analysis. As a direction of future research, the problem of estimation accuracy optimization can be posed, *i.e.* how by a selection of the observer gain L to improve the asymptotic accuracy of estimation.

Part II

O&O for Singular System

Chapter 7

Introduction

Singular system (known as well descriptor system/differential-algebraic system) was introduced to model a large class of systems in many different domains, such as physical, biological, and economic ones, for which the standard representation sometimes cannot be applied [41, 40]. The structure of this type of systems contains both the dynamic equations and the algebraic ones, and due to this characteristic, many well-defined concepts dealing with the observability problem for regular (non-singular) systems have to be reconsidered. Due to as well this special structure, singular system might contain impulse, and this leads to different definitions, including observability, R-observability and Impulse-observability [193, 49]. Generally speaking, they characterize the state reconstruction ability from different aspects: R-observability defines the ability to estimate the reachable set of the studied system; Impulse-observability corresponds to the ability to estimate the impulse term of the studied system and the observability covers both mentioned abilities to estimate all states of the studied system.

Observability and the problem of observer design have been widely studied for singular systems with perfectly known model. For linear singular system, in [193], the authors have studied the solvability, controllability and observability concepts for singular systems with regular matrix pencil. There, the observability analysis is addressed and algebraic characterizations were found. The algebraic duality between controllability and observability for singular systems with regular matrix pencil is proven by using the Schwartz distribution framework in [45]. Necessary and sufficient conditions allowing for the design of a Luenberger-like observer were found in [146]. In the three previous mentioned works, it was considered that the system has a regular matrix pencil which entails a unique state solution. Without any particular assumption over the matrix pencil of the system, the casual observability, which does not allow to use neither the derivatives of the input nor the derivatives of the output, is studied in [89]. The same authors suggest an observer design in [90]. In that work it is shown that by allowing the derivatives of the input and output to be involved in the observer (called it there as a generalized observer), detectability is enough for the convergence of the observation error. A reduced order observer is designed in [53]. In spite of the extended literature regarding the observability analysis and synthesis of singular systems, there exist few results dealing with such problems when the system contains unknown input. In [147], the observer design problem is considered for linear singular systems with unknown input and necessary and sufficient conditions are given for the design of a Luenberger-like observer. In [54], a reduced order observer is proposed. Under some regularity conditions, the observer design is studied in [44]. Meanwhile, in [110] a proportional multiple-integral observer is proposed. Using the graph-theory approach, observability conditions are found in [27].

Concerning nonlinear singular systems, for the case when all the inputs are known, the algebraic observability of DAE time varying systems has been studied in [182]. [107] studied an observer for a class of nonlinear singular systems in which the system was linearized around the equilibrium point. The same technique was used in [34] to study a reduced order observer for a class of nonlinear singular systems. Other techniques, such as LMI ([122, 52]) and convex optimization [111], are proposed as well to design an observer for nonlinear singular systems with known (or unknown) inputs. Using also LMIs, a reduced order observer for a class of Lipschitz nonlinear singular systems is presented in [122]. Asymptotic observers for systems having index one were proposed in [61] and [5].

Although the singular systems are different to the conventional regular systems, but they do exist some similarities, since they both contain differential equations. The difficulty is how to treat the algebraic equations existing in singular systems. If we can overcome this problem, then we can reuse the well developed techniques for regular systems to analyze the observability and to design observer for both linear and nonlinear singular system with known or unknown inputs. This consideration motivates us to seek the conditions under which the singular systems might be regularized, if possible.

This part summarizes our recent works on observability analysis and observer design for linear and nonlinear singular systems with known or unknown inputs.

- Firstly, the strong observability and strong detectability of a general class of linear singular systems with unknown inputs are tackled by converting the singular system into a regular one with unknown inputs and algebraic constraints. Moreover, the assumption of regular matrix pencil for singular systems is removed;

- Secondly, we study a class of nonlinear singular systems with unknown inputs. Under suitable conditions, we achieve to replace the DAE of the system by ODE on a manifold of reduced dimension. This is done by searching for an invariant submanifold (called zeroing submanifold, conceived from the zero dynamics concept in [95]) where the DAE are satisfied during an interval of time. Then observability conditions are found in terms of the original system parameters;

- Thirdly, we generalize the well-known differential geometric method to study the nonlinear singular systems. By applying the technique of regularization, we are seeking a diffeomorphism to transform the regularized system into a simpler singular normal form, for which a Luenberger observer can be designed to estimate its state. Necessary and sufficient conditions are deduced to guarantee the existence of such a diffeomorphism;

- Most of the existing results are for the asymptotic estimation of the state for singular systems without uncertainties. The observer design becomes complicated when considering the systems with uncertain terms in the state and in the measurement. In this situation, the exact estimation may be not possible, and one solution is to provide the upper and lower bound estimation of the admissible values for the state by applying the theory of set-membership or interval estimation. Therefore, in the last result of this part, we propose an interval observer for such uncertain nonlinear singular systems.

Chapter 8

Linear Singular System

In this chapter, we will present our recent results on the observability analysis for a general class of singular linear systems with unknown input, published in [J14, C31]. In our study, the system is not required to have a regular matrix pencil, which is always assumed in most of the existing works when the observability is studied. Our study focuses on the estimation of the non-impulsive states (slow part) of the singular system. Based on this limitation, it is possible to reuse the existing method for linear regular system to analyze the observability for linear singular system. Necessary and sufficient conditions for the strong observability and strong detectability will be studied.

8.1 Notation and problem statement

Consider the linear singular system with unknown inputs governed by the following equations

$$\Sigma : \left\{ \begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx + Du(t) \end{array} \right. , \tag{8.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the system output, and $u(t) \in \mathbb{R}^m$ is the unknown input vector. Matrices $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are all constant. The matrix E is assumed to be singular. Given a state $x_0 \in \mathbb{R}^n$ and a function $u(t)$, we denote by $x_u(x_0, t)$ the state of Σ at time t which results from taking the initial condition equal to x_0 and the input vector is equal to u. Therefrom, in a straightforward manner we define the output $y_u(x_0, t) = Cx_u(x_0, t) + Du(t)$.

We are interested in the reconstruction of the (non-impulsive) trajectory of state vector $x(t)$ given the output information $y(\tau)_{\tau \in [0,t]}$. System Σ *is not assumed* to have a regular pencil ([100]), i.e., it is allowed that $\det(sE - A) = 0$ for all $s \in \mathbb{C}$ (then $x_u(x_0, t)$ may have more than a solution). Nevertheless, $u(t)$ must be so that $x(t)$ bepiecewise continuous for all $t > 0$; however, an impulse may occur at $t = 0$. In order to give algebraic conditions allowing the reconstruction of $x(t)$, we consider the following definitions, which are based on classical definitions for linear time invariant systems (see, e.g. [183]).

Definition 8.1 (Strong observability) *System Σ is strongly observable (SO) if for all $x_0 \in \mathbb{R}^n$ and for every input function u, the following implication is satisfied*

$$y_u(x_0, t) = 0 \ \forall t > 0 \textit{ implies } x\left(0^+\right) = 0. \tag{8.2}$$

Definition 8.2 (Strong detectability) *System Σ is strongly detectable (SD) if for all $x_0 \in \mathbb{R}^n$ and for every input function u, the following implication holds*

$$y_u(x_0,t) = 0 \; \forall t > 0 \text{ implies } \lim_{t\to\infty} x_u(x_0,t) = 0. \tag{8.3}$$

It is clear that strong observability is a necessary condition to reconstruct the entire trajectory of the state $x(t)$. Indeed, let us suppose that Σ is not strongly observable, then it means that there exist $\bar{u}$ and $\bar{x}_0$ such that $y_{\bar{u}}(\bar{x}_0,t) = 0 \; \forall t > 0$, but $x(0^+) \neq 0$. Then, since we assume that $x(t)$ is piecewise continuous, then $x_{u=0}(0,t) \equiv 0$ and $x_{\bar{u}}(\bar{x}_0,t) \neq 0$ in an open interval, however, both yield a system output identically equal to zero. Thereby, it would be impossible to reconstruct the entire state trajectory. Below it would be shown that SO is a structural necessary and sufficient condition for the reconstruction in finite time of $x(t)$. Analogously, it will be shown as well that SD is a structural necessary and sufficient condition for the asymptotic reconstruction of $x(t)$.

8.2 Observability analysis

Since E is singular, there exist non-singular matrices $T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ such that E can be transformed as follows:

$$TES = \begin{bmatrix} I_{\rho_E} & 0 \\ 0 & 0 \end{bmatrix} \tag{8.4}$$

where $\rho_E = \text{rank}E$. Thus, let us define the vector $z := \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T = S^{-1}x$, where $z_1 \in \mathbb{R}^{\rho_E}$ and $z_2 \in \mathbb{R}^{n-\rho_E}$. In these new coordinates, Σ can be rewritten as follows

$$\Psi : \begin{cases} TES\dot{z}(t) &= TASz(t) + TBu(t) \\ y(t) &= CSz(t) + Du(t) \end{cases}. \tag{8.5}$$

In view of (8.4), Ψ takes the following form

$$\dot{z}_1(t) = T_1AS_1z_1(t) + T_1AS_2z_2(t) + T_1Bu(t), \tag{8.6a}$$
$$0 = T_2AS_1z_1(t) + T_2AS_2z_2(t) + T_2Bu(t), \tag{8.6b}$$
$$y = CS_1z_1(t) + CS_2z_2(t) + Du(t). \tag{8.6c}$$

where S_1 and S_2 matrices arise from the following partition of S, $S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$ with $S_1 \in \mathbb{R}^{\rho_E\times\rho_E}$ and $S_2 \in \mathbb{R}^{\rho_E\times n-\rho_E}$. Analogously, T_1 and T_2 matrices come from the partition $T^T = \begin{bmatrix} T_1^T & T_2^T \end{bmatrix}$ with $T_1 \in \mathbb{R}^{\rho_E\times n}$ and $T_2 \in \mathbb{R}^{n-\rho_E\times n}$. It is clear that Σ is SO (resp. SD) if, and only if, Ψ is SO (resp. SD). Below we will see that a simple manner to study the observability of Ψ, and by extension of Σ, is by considering (8.6b) as part of the system output of a new pseudo system and considering z_2 as part of the vector of UI. Indeed, let us define the system Φ by means of the following equation,

$$\Phi : \begin{cases} \dot{z}_1(t) &= \bar{A}z_1(t) + \bar{B}v(t) \\ \bar{y}(t) &= \bar{C}z_1(t) + \bar{D}v(t) \end{cases}, \tag{8.7}$$

where $v(t) \in \mathbb{R}^{n-\rho_E+m}$, $\bar{y}(t) \in \mathbb{R}^{n-\rho_E+p}$ and the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are defined as follows

$$\bar{A} = T_1AS_1, \quad \bar{B} = \begin{bmatrix} T_1AS_2 & T_1B \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} T_2AS_1 \\ CS_1 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} T_2AS_2 & T_2B \\ CS_2 & D \end{bmatrix}. \tag{8.8}$$

It is clear by (8.6) that Φ looks like system Ψ. In general, they do not represent identical systems. However, both systems are identical if these two identities hold: $\bar{y}^T = \begin{bmatrix} 0^T & y^T \end{bmatrix}$ and $v^T(t) = \begin{bmatrix} z_2^T(t) & u^T(t) \end{bmatrix}$. In the next theorem it is claimed that the fulfillment of the SO (resp. SD) of Σ is equivalent to the fulfillment of the SO (resp. SD) of Φ (condition needed for the reconstruction of z_1) plus a rank condition (required for the reconstruction of z_2).

Theorem 8.3 *System Σ is SO (resp. SD) if, and only if, Φ is SO (resp. SD) and the following rank condition holds*

$$\operatorname{rank}\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} = n - \operatorname{rank}E + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix}. \tag{8.9}$$

Furthermore, this equivalence is independent of the choice of T and S.

As for Σ, we could expect that SO and SD can be completely characterized by the five-tuple (E,A,B,C,D). Indeed, let $R(s)$ be the so-called system matrix of Σ, i.e.,

$$R(s) = \begin{bmatrix} sE-A & -B \\ C & D \end{bmatrix}, s \in \mathbb{C}.$$

We say that $s_0 \in \mathbb{C}$ is a zero of Σ if $\operatorname{rank}R(s_0) < n + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix}$. Let $\sigma_z(\Sigma)$ be defined as the set of zeros of Σ. Let us characterize SO and SD in terms of the zeros of Σ.

Corollary 8.4 *System Σ is SO (resp. SD) if, and only if, $\sigma_z(\Sigma) = \varnothing$ (resp. $\sigma_z(\Sigma) \subset \mathbb{C}^-$).*

8.3 State reconstruction

Let M_k $(k \geq 1)$ be the matrices obtained by the following Molinari recursive algorithm (see, [140]):

$$M_{k+1} = N_{k+1}^{\perp\perp} N_{k+1}, \quad M_1 = \left(\bar{D}^{\perp}\bar{C}\right)^{\perp\perp} \bar{D}^{\perp}\bar{C},$$
$$N_{k+1} = T_k \begin{pmatrix} M_k\bar{A} \\ \bar{C} \end{pmatrix}, T_k = \begin{pmatrix} M_k\bar{B} \\ \bar{D} \end{pmatrix}^{\perp}. \tag{8.10}$$

where $F^{\perp}$ is the annihilator of the matrix F (i.e. $X^{\perp}X = 0$), $F^{\perp\perp}$ a full row rank matrix such that $\operatorname{rank}F^{\perp\perp}F = \operatorname{rank}F$ (then the matrix $\begin{bmatrix} \left(F^{\perp}\right)^T & \left(F^{\perp\perp}\right)^T \end{bmatrix}^T$ is nonsingular). Let us denote by l, the smallest integer such that $\text{rank}M_l = \text{rank}M_{l+1}$. For our purposes, we point out that Φ is SO if, and only if, $\text{rank}M_l = \text{rank}E$. For the case of SD we have to work a bit more with system Φ. Indeed, let us assume that $\text{rank}M_l < \text{rank}E$. Let V be a full column rank matrix so that $M_l V = 0$. There exists a pair of matrices Q and K^* such that

$$\bar{A}V + \bar{B}K^* = VQ \quad \text{and} \quad \bar{C}V + \bar{D}K^* = 0. \tag{8.11}$$

From (8.11), it is clear that the $im\left(\left(\bar{A} + \bar{B}K^*V^+\right)V\right) \subset imV$ and $\left(\bar{C} + \bar{D}K^*V^+\right)V = 0$. We can define a non-singular matrix P of dimension $\text{rank}E$ as $P = \begin{bmatrix} M_l \\ V^+ \end{bmatrix}$, $P^{-1} = \begin{bmatrix} M_l^+ & V \end{bmatrix}$, where V^+ and M_l^+ are the Moore-Penrose pseudo-inverse of V and M, respectively: $V^+ = \left(V^TV\right)^{-1}V^T$ and $M_l^+ = M_l^T\left(M_lM_l^T\right)^{-1}$. By defining the vectors $w_1 = M_lz_1$ and $w_2 = V^+z_1$, we have that $z_1 = M_l^+w_1 + Vw_2$. System Φ in these new coordinates can be rewritten as follows:

$$\begin{aligned}
\dot{w}_1 &= \bar{A}_1w_1 + \bar{B}_1\left(v - K^*w_2\right), && (8.12a)\\
\dot{w}_2 &= \bar{A}_2w_1 + \bar{A}_3w_2 + \bar{B}_2\left(v - K^*w_2\right), && (8.12b)\\
\bar{y} &= \bar{C}_1w_1 + \bar{D}\left(v - K^*w_2\right), && (8.12c)
\end{aligned}$$

where

$$\begin{array}{l}
\bar{A}_1 = M_l\left(\bar{A} + \bar{B}K^*V^+\right)M_l^+, \quad \bar{B}_1 = M_l\bar{B},\\
\bar{A}_2 = V^+\left(\bar{A} + \bar{B}K^*V^+\right)M_l^+, \quad \bar{B}_2 = V^+\bar{B},\\
\bar{A}_3 = V^+\left(\bar{A} + \bar{B}K^*V^+\right)V, \quad \bar{C}_1 = \bar{C}M_l^+.
\end{array} \tag{8.13}$$

Thus, as for SD, it is known that system Φ is SD if, and only if, $rank\begin{bmatrix} \bar{B}_1 \\ \bar{D} \end{bmatrix} = rank\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}$ and $\bar{A}_3$ is a Hurwitz matrix (see, e.g. [13]).

Coming back to system Φ. Define $\xi_1 := \left(\bar{D}^\perp\bar{C}\right)^{\perp\perp}\bar{D}^\perp\bar{y} = M_1z_1$, with M_1 defined as in (8.10). Let us derive the vector ξ_1:

$$\dot{\xi}_1(t) = M_1\bar{A}z_1(t) + M_1\bar{B}v(t). \tag{8.14}$$

Let us define a new vector ξ_2 as follows

$$\xi_2 := N_2^{\perp\perp}T_1\begin{bmatrix} \dot{\xi}_1 \\ \bar{y}(t) \end{bmatrix}, \tag{8.15}$$

with $N_2^{\perp\perp}$ and T_1 defined by (8.10). Thus, taking into account (8.7), (8.14), and (8.10), we have that

$$\frac{d}{dt}J_2\begin{bmatrix} \bar{y} \\ \int_{t_0}^{t}\bar{y}(\tau)\,d\tau \end{bmatrix} = \xi_2 = M_2z_1(t), \; t > t_0 \geq 0, \tag{8.16}$$

where

$$J_2 = N_2^{\perp\perp} T_1 \begin{bmatrix} J_1 & 0 \\ 0 & I_{\bar{p}} \end{bmatrix}, \quad J_1 = \left(\bar{D}^{\perp}\bar{C}\right)^{\perp\perp} \bar{D}^{\perp}.$$

In the first identity of (8.16), we take outside the differential operator from (8.15) and use the definition of ξ_1. Thus, we can follow an iterative procedure to obtain the following set of equations, for $k \geq 1$,

$$\frac{d^k}{dt^k} J_{k+1} \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \int_{t_0}^{t} \cdots \int_{t_0}^{\tau_2} \bar{y}(\tau_1)\, d\tau_1 \cdots d\tau_k \end{bmatrix} = M_{k+1} z_1, \tag{8.17}$$

where M_{k+1} is defined by (8.10), and J_{k+1} is defined by the following recursive algorithm, for $k \geq 1$,

$$J_1 = \left(\bar{D}^{\perp}\bar{C}\right)^{\perp\perp} \bar{D}^{\perp}, \quad J_{k+1} = N_{k+1}^{\perp\perp} T_k \begin{bmatrix} J_k & 0 \\ 0 & I_{\bar{p}} \end{bmatrix}. \tag{8.18}$$

Thus $M_k z_1$ is expressed by a high order derivative of a function of $y(t)$. In such a way a real-time differentiator could be used, two of them frequently used due to their finite time convergence can be found in [118] and [138]. For instance if $\operatorname{rank} M_l = \operatorname{rank} E$, then z_1 is algebraically observable, i.e. it could be reconstructed by using a real-time differentiator.

In order to match system Σ with system Φ, from now on, we define $\bar{y} = \begin{bmatrix} 0_{1\times n-\operatorname{rank}E} & y^T \end{bmatrix} \in \mathbb{R}^{\bar{p}}$ ($\bar{p} := n - \operatorname{rank}E + p$), and $v(t) = \begin{bmatrix} z_2^T(t) & u^T(t) \end{bmatrix}^T \in \mathbb{R}^q$ ($q = n - \operatorname{rank}E + m$), then in view of (8.6), equations (8.5) and (8.7) are identical. Below, we consider two cases: when Σ is SO and when it is SD, but not SO. Of course, since Φ is a standard linear system, there might be other methods, besides the one proposed below.

Case 1: Σ is SO.
Since Φ is SO, $\operatorname{rank} M_l = \operatorname{rank} E$. Then in this case, from (8.17), we obtain the equation

$$\frac{d^{l-1}}{dt^{l-1}} M_l^{-1} J_l \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \int_{t_0}^{t} \cdots \int_{t_0}^{\tau_2} \bar{y}(\tau_1)\, d\tau_1 \cdots d\tau_{l-1} \end{bmatrix} = z_1, \tag{8.19}$$

where $M_l \in \mathbb{R}^{\rho_E \times \rho_E}$ and $J_l \in \mathbb{R}^{\rho_E \times \bar{p}l}$. Let U be a matrix so that

$$rank \begin{bmatrix} D \\ F \end{bmatrix} U = rank \begin{bmatrix} D \\ F \end{bmatrix} =: \bar{m}, \quad U \in \mathbb{R}^{q\times\bar{m}} \tag{8.20}$$

Since (8.9) must be satisfied according to Theorem 8.3, we have that

$$z_2(t) = \begin{bmatrix} I_{n-\rho_E} & 0_{\bar{q}} \end{bmatrix} \left(\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} \begin{matrix} I & 0 \\ 0 & U \end{matrix} \right)^{+} \left(\begin{bmatrix} \dot{z}_1(t) \\ \bar{y} \end{bmatrix} - \begin{bmatrix} \bar{A} \\ \bar{C} \end{bmatrix} z_1(t) \right) \tag{8.21}$$

where $\bar{q} := n - \rho_E + \bar{m}$. Now, we are ready to give a formula to reconstruct x in finite time.

Theorem 8.5 *If system Σ is SO, then the state x can be expressed algebraically by the following formula:*

$$x(t) = \frac{d^l}{dt^l} \begin{bmatrix} S & 0_{n\times\bar{m}} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \int_{t_0}^{t} \cdots \int_{t_0}^{\tau_1} \bar{y}(\tau_1)\, d\tau_1 \cdots d\tau_l \end{bmatrix}, \tag{8.22}$$

where $H_1 \in \mathbb{R}^{\rho_E \times \bar{p}(l+1)}$ and $H_2 \in \mathbb{R}^{\bar{q}\times\bar{p}(l+1)}$ are defined as

$$H_1 := \begin{bmatrix} 0_{\rho_E\times\bar{p}} & M_l^{-1}J_l \end{bmatrix},\ H_2 := \begin{bmatrix} \bar{B}U \\ \bar{D}U \end{bmatrix}^{+} (G_1 - G_2),$$
$$G_1 := \begin{bmatrix} M_l^{-1}J_l & 0_{\rho_E\times\bar{p}} \\ 0_{\bar{p}\times\bar{p}l} & I_{\bar{p}\times\bar{p}} \end{bmatrix},\ G_2 := \begin{bmatrix} 0_{\rho_E\times\bar{p}} & \bar{A}M_l^{-1}J_l \\ 0_{\bar{p}\times\bar{p}} & \bar{C}M_l^{-1}J_l \end{bmatrix},$$

$G_1, G_2 \in \mathbb{R}^{\rho_E+\bar{p}\times\bar{p}(l+1)}$, and M_l and J defined recursively in (8.10) and (8.18), respectively, and U defined by (8.20).

Remark 8.6 *One might obtain a little more from the previous analysis, that is, one can express by an algebraic formula the part of u that can be reconstructed (assuming Σ is SO). Indeed, let $\bar{u}$ be implicitly defined by the equation $\begin{bmatrix} D^T & F^T \end{bmatrix}^T u = \begin{bmatrix} D^T & F^T \end{bmatrix}^T U\bar{u}$. With the same procedure followed to obtain (8.22), $\bar{u}$ can be expressed by the formula*

$$\bar{u}(t) = \frac{d^l}{dt^l} \begin{bmatrix} 0_{\bar{m}\times n} & I_{\bar{m}} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} \bar{y}(t) \\ \vdots \\ \int_{t_0}^{t} \cdots \int_{t_0}^{\tau_1} \bar{y}(\tau_1)\, d\tau_1 \cdots d\tau_l \end{bmatrix}. \tag{8.23}$$

Case 2: Σ is SD.

Let us assume that Σ is SD, but not SO. Next we show how to carry out the estimation of $x(t)$.

Theorem 8.7 *Assuming that Σ is SD, but not SO, we obtain that*

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0,$$

provided that $\hat{x}(t)$ is designed by following (8.24)-(8.25).

$$\hat{x}(t) = \frac{d^l}{dt^l}\left(\begin{bmatrix} S & 0_{n\times\bar{m}} \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \end{bmatrix} Y_l(t) \right) \begin{bmatrix} S & 0_{n\times m} \end{bmatrix} \begin{bmatrix} V \\ K^* \end{bmatrix} \hat{w}_2 \tag{8.24}$$

$$\dot{\hat{w}}_2 = \bar{A}_3\hat{w}_2 + \bar{A}_2 \frac{d^l}{dt^l}\left(\begin{bmatrix} 0_{\rho_E-\rho_M\times\bar{p}} & J_l \end{bmatrix} + \bar{B}_2 U \bar{H}_2 \right) Y_l \tag{8.25}$$

where $\bar{H}_1 \in \mathbb{R}^{\rho_E \times \bar{p}(l+1)}$, $\bar{H}_2 \in \mathbb{R}^{\bar{q}+\bar{p} \times \bar{p}(l+1)}$, *and* $\bar{G}_1, \bar{G}_2 \in \mathbb{R}^{\rho_M+\bar{p} \times \bar{p}(l+1)}$ ($\rho_M = \rho_{M_l}$) *satisfy the following identities,*

$$\bar{H}_1 = \left[\begin{array}{cc} 0_{\rho_E \times \bar{p}} & M_l^+ J_l \end{array}\right], \ \bar{H}_2 := \left[\begin{array}{c} \bar{B}_1 U \\ \bar{D} U \end{array}\right]^+ \left(\bar{G}_1 - \bar{G}_2\right),$$
$$\bar{G}_1 := \left[\begin{array}{cc} J_l & 0_{\rho_M \times \bar{p}} \\ 0_{\bar{p} \times \bar{p}l} & I_{\bar{p}} \end{array}\right], \ G_2 := \left[\begin{array}{cc} 0_{\rho_M \times \bar{p}} & \bar{A}_1 J_l \\ 0_{\bar{p} \times \bar{p}} & \bar{C}_1 J_l \end{array}\right].$$

Remark 8.8 *If* $\bar{u}$ *needs to be reconstructed also, then it can be done by means of* $\widehat{\bar{u}}(t)$, *defined as follows,*

$$\widehat{\bar{u}}(t) = \frac{d^l}{dt^l}\left(\left[\begin{array}{cc} 0_{\bar{m} \times n} & I_{\bar{m}} \end{array}\right] \left[\begin{array}{c} \bar{H}_1 \\ \bar{H}_2 \end{array}\right] Y_l(t)\right) \left[\begin{array}{cc} 0_{\bar{m} \times n} & I_{\bar{m}} \end{array}\right] \left[\begin{array}{c} V \\ \bar{K}^* \end{array}\right] \hat{w}(t) \tag{8.26}$$

where $\bar{K}^*$ *is implicitly defined by the equation*

$$\left[\begin{array}{c} \bar{B} \\ \bar{D} \end{array}\right] K^* = \left[\begin{array}{c} \bar{B} \\ \bar{D} \end{array}\right] \left[\begin{array}{cc} I_{n-\rho_E} & 0 \\ 0 & U \end{array}\right] \bar{K}^*$$

So, we obtain straightforwardly that $\|\hat{u}(t) - u(t)\|$ *goes to zero.*

8.4 Example

Let us consider that Σ has the following matrices values

$$E = \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], A = \left[\begin{array}{cccc} -1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right], D = \left[\begin{array}{c} 1 \\ -1 \\ 0 \\ 0 \end{array}\right]$$
$$C = \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 \end{array}\right], \quad F = \left[\begin{array}{c} 0 \\ 1 \end{array}\right].$$

It is easy to see that, in this example, $\det(sE - A) = 0$ for every $s \in \mathbb{C}$. Hence, many solutions for $x(t)$ are expected to satisfy the differential equation in (8.1). However, to each output $y(t)$ corresponds only one trajectory of $x(t)$ (a.e.). Indeed, we will see that, according to Theorem 8.3, Σ is SO.

For this case matrices S and T are chosen as follows,

$$S = \left[\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right], \quad T = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

Matrices $\bar{A}$, $\bar{C}$, $\bar{B}$, and $\bar{D}$ take the following values:

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 2 & 3 & 1 \\ -2 & -6 & -1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

Matrices M_2 and J_2 take the values

$$M_2 = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

Here, $rankM_2 = 2$, $rank\bar{B} = 3$, and $n - \rho_E = 2$. Therefore, both conditions of Theorem 8.3 are satisfied.

The reconstruction of $x(t)$ can be done by means of the formula (8.22), once we define $\bar{y}^T = \begin{bmatrix} 0 & 0 & y^T \end{bmatrix}$. For this example, the reconstruction of u is also possible following formula (8.23). Thus, we have

$$x_1 = -\frac{1}{3}y_1 + \frac{7}{12}y_2 - \frac{1}{2}\dot{y}_1 + \frac{1}{6}\dot{y}_2, \quad x_2 = \frac{2}{3}y_1 - \frac{1}{6}y_2 + \frac{1}{6}\dot{y}_2,$$
$$x_3 = -\frac{1}{4}y_2 + \frac{1}{2}\dot{y}_1, \quad x_4 = \frac{1}{3}y_1 - \frac{1}{6}y_2 + \frac{1}{6}\dot{y}_2, \quad u = \frac{1}{2}y_2 + \frac{1}{2}\dot{y}_2.$$

8.5 Conclusion

We have given necessary and sufficient conditions to estimate the slow (non-impulsive) trajectories, for singular systems in which more than one solution of the differential equation is allowed, i.e. the pencil of the system is not required to be regular as it is assumed in most of the previous works where the observability is studied. Moreover, the explicit formulas have been deduced to reconstruct the states, respectively for the strong observability and strong detectability cases.

Chapter 9

Nonlinear Singular System

In this chapter, we consider that the system contains unknown inputs and the DAE are given in an explicit form. Under suitable conditions, we try to replace the DAE of the system by ODE on a manifold of reduced dimension. This is done by searching for an invariant submanifold (called zeroing submanifold) where the DAE are satisfied during an interval of time. Then observability conditions can be found in terms of the original system parameters. The results of this chapter have been published in [J6, C24].

9.1 Notations and problem statement

Consider systems described by the following equations:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{9.1a}$$

$$0 = F(x(t)) + G(x(t))u(t) \tag{9.1b}$$

$$y(t) = h(x(t)) \tag{9.1c}$$

where the state $x(t)$ belongs to an open set $\mathbb{U} \subset \mathbb{R}^n$. The maps $f : \mathbb{U} \to \mathbb{R}^n$, $g : \mathbb{U} \to \mathbb{R}^{n\times m}$, $F : \mathbb{U} \to \mathbb{R}^q$, $G : \mathbb{U} \to \mathbb{R}^{q\times m}$, and $h : \mathbb{U} \to \mathbb{R}^p$ are all smooth maps. The input vector $u(t) \in \mathbb{R}^m$ is unknown a priori; however it should be noted that $u(t)$ has to be so that a solution for (9.1a)-(9.1b) exists. The aim is the estimation of $x(t)$ by means of the system output $y(t)$. Let N be a set defined as

$$N = \{x \in \mathbb{U} :_x \in \mathbb{R}^m \text{ s.t. } F(x) + G(x)u_x = 0\} \tag{9.2}$$

In what follows we will do our study around an $x_0 \in N$ for which $x(t;x_0)$ satisfies (9.1a)-(9.1b) in a neighborhood of $t = 0$.

As we have shown in the last chapter on the observability analysis for linear singular system, we can combine (9.1b)-(9.1c) as a whole new output, and adapt the conventional method for regular system to study the observability of singular system. However, it is not equivalent for nonlinear case, as it can be shown in the following simple example:

$$\begin{aligned} &\dot{x}_1 = f_1(x,u), \ \ \dot{x}_2 = f_2(x,u) \\ &0 = x_1^2, \ \ y = x_2 \end{aligned}$$

In the above system, if we take $\bar{y} = (0, y)^T$ as the whole new output (as what we have done for linear singular system in the last chapter), it is easy to see that the system is not observable around 0. However, from the constraint $0 = x_1^2$ we can see that the above system is always observable with $x_1 = 0$. The reason that we cannot make such a combination is that the constraint is nonlinear. In order to treat this case, the following will borrow the well-known concept of zero dynamics in nonlinear system to search a maximal zeroing submanifold from this nonlinear constraint.

9.2 Searching for a maximal zeroing submanifold

The procedure pursued here to estimate $x(t)$ lies into two main steps. Firstly, we look for a maximal zeroing submanifold (w.r.t. $F(x)$ and $G(x)$), which is a submanifold such that if $x(0)$ belongs to it then there exists an input function $u(t)$ such that $x(t; x(0))$ satisfies (9.1a)-(9.1b) for all t in a neighborhood of $t = 0$. The first part yields a coordinates transformation so that some terms of the state in the new coordinates are equal to zero (the same number as the dimension of the zeroing submanifold). This also allows for expressing the input vector as a function of the state vector. The second part consists in observability analysis on the reduced system without unknown inputs. Now, we proceed to give a formal definition of a zeroing submanifold. For it, we will need to define invariant and locally invariant submanifolds (see [1]). Let M be a smooth submanifold of $\mathbb{R}^n$.

Definition 9.1 *Let $V \subset M$ be a smooth submanifold, and f a vector field on M. Then, V is an Invariant submanifold (ISM) w.r.t. f if, for all $v \in V$, $f(v) \in T_vV \subset T_vM$, where T_vM means the tangent space to M at v.*

Definition 9.2 *M is a **locally ISM** at x_0 w.r.t. f if there exists a neighborhood U_0 of x_0 such that $M \cap U_0$ is an ISM w.r.t. f.*

Next definition is in its essence a definition found in Chapter 6 of [95]; however, we have adapted it using the previous definitions of ISM and also clause i) has been slightly modified to consider the effect of $G(x)$ of (9.1b).

Definition 9.3 *A zeroing submanifold (ZSM) at x_0 is a smooth submanifold $M \subset \mathbb{U}$ containing x_0 that satisfies i) $M \subset N$ and ii) there exists a smooth mapping $u : M \to \mathbb{R}^m$ so that M is a locally ISM at x_0 w.r.t. the vector field $\hat{f}(x) := f(x) + g(x)u(x)$.*

Remark 9.4 *Clause ii) means that there exists a neighborhood U_0 of x_0 such that if $x(0) \in M \cap U_0$ then $x(t) \in M \cap U_0$ for all t in a neighborhood of $t = 0$ (see, e.g., [1]).*

Proposition 9.5 *If a ZSM M is such that the ISM $M \cap U_0$ (w.r.t. $\hat{f}(x)$) is a closed set of $\mathbb{U}$, then $x(t)$ stays within $M \cap U_0$ for all t, provided $x(0)$ belongs to $M \cap U_0$.*

Definition 9.6 *A ZSM M is locally maximal if, for any other ZSM $\bar{M}$, there exists a neighborhood U of x_0 such that the inclusion $M \cap U \supset \bar{M} \cap U$ is satisfied.*

We will seek for a locally maximal ZSM. The proposed method is similar to the one given in [95], pp. 299-301. However, we do not assume that $q = m$ and we include the input explicitly in the algebraic equation. The proposed algorithm is a nonlinear version of the algorithm used to find the weakly unobservable subspace in linear systems with inputs appearing explicitly in the differential equations and in the system output (see, e.g., [13]). The following is our step-by-step algorithm to find a locally maximal ZSM.

Step 1. It is assumed that there exists a neighborhood U_0 containing x_0 such that the $rankG(x) = r_0$ for all $x \in U_0$, for some r_0. Let us define $M_0 = U_0$. For if, there exists a full row rank matrix $R_0(x)$ with terms being smooth functions of x in a neighborhood $U_0^{'}$ of x_0 such that

$$rankR_0(x) = q - r_0 \text{ and } R_0(x)G(x) = 0 \text{ for all } x \in U_0^{'} \tag{9.3}$$

Thus, the maps $\Phi_0(x)$ and $H_1(x)$ are defined as $H_1(x) = \Phi_0(x) := R_0(x)F(x)$. Let us assume that the rank of $dH_1(x)$ is constant in a neighborhood $U_1 \subset U_0^{'}$ of x_0. Then, the set $M_1 := \{x \in U_1 : H_1(x) = 0\}$ is a smooth submanifold.

Proposition 9.7 *M_1 satisfies the identity $M_1 = N \cap U_1$.*

Step 2. Let us assume that rank of $col(G(x), L_gH_1(x))$ is equal to a constant r_1 for all x in M_1. Then there exists a matrix $R_1(x)$ with terms being smooth functions of x in a neighborhood $U_1^{'}$ of x_0 such that, for all $x \in M_1 \cap U_1^{'}$, $R_1(x)\begin{pmatrix} G(x) \\ L_gH_1(x) \end{pmatrix} = 0$ and $rankR_1(x) = q + \dim H_1(x) - r_1$. Thus, we define $H_2(x) = col(H_1(x), \Phi_1(x))$ where

$$\Phi_1(x) = R_1(x)\, col\left(F(x), L_fH_1(x)\right)$$

Again, let us assume that $dH_2(x)$ has constant rank in a neighborhood $U_2 \subset U_1^{'}$. Thus, the set $M_2 := \{x \in U_2 : H_2(x) = 0\}$ is a smooth submanifold also.

Step k. Assuming that $rankcol(G(x), L_gH_{k-1}(x)) = r_{k-1}$ for all $x \in M_{k-1}$, then there exists a neighborhood $U_{k-1}^{'}$ of x_0 and a matrix $R_{k-1}(x)$ of smooth functions on $U_{k-1}^{'}$ such that

$$\begin{gathered} rankR_{k-1}(x) = q + \dim H_{k-1}(x) - r_{k-1} \\ R_{k-1}(x)\begin{pmatrix} G(x) \\ L_gH_{k-1}(x) \end{pmatrix} = 0 \end{gathered} \tag{9.4}$$

for all $x \in M_{k-1} \cap U_{k-1}^{'}$. By defining $H_k(x)$ as

$$H_k(x) = \begin{pmatrix} H_{k-1}(x) \\ \Phi_{k-1}(x) \end{pmatrix}, \ \Phi_{k-1}(x) = R_{k-1}(x)\begin{pmatrix} F(x) \\ L_fH_{k-1}(x) \end{pmatrix} \tag{9.5}$$

and if $dH_k(x)$ has constant rank on $U_k \subset U_{k-1}^{'}$ around x_0, we obtain the smooth manifold M_k:

$$M_k = \{x \in U_k : H_k(x) = 0\}$$

Proposition 9.8 *Under assumptions of Lemma 9.9, we obtain that $M_{k+1} \subset M_k$, for $k \geq 1$.*

Lemma 9.9 *Assume that there exist nested neighborhoods $U_{k+1} \subset U_k$ and $U'_{k+1} \subset U'_k$ ($k \in \overline{1,n}$) of x_0 such that, for every k, $dH_k(x)$ has constant rank in U_k and col $(G(x), L_g H_k(x))$ has constant rank for all x on the smooth manifold*

$$M_k := \{x \in U_k : H_k(x) = 0\} \tag{9.6}$$

for all $k \in \overline{1,n}$, and $H_k(x)$ and $R_k(x)$ satisfy (9.5) and (9.4), respectively, on U'_k. Then, there exists a $k^* \leq n$ and a neighborhood $\bar{U}_{k^*}$ so that $M_{k^*} \cap \bar{U}_{k^*} = M_{k^*+j} \cap \bar{U}_{k^*}$ for all $j \geq 1$.

Remark 9.10 *Lemma 9.9 implies that the algorithm will stop at k^* step. Moreover, k^* will be the first integer k satisfying $\operatorname{rank} dH_k(x) = \operatorname{rank} dH_{k+1}(x)$. This is true because of the dimension of M_k is $n - \operatorname{rank} dH_k(x)$ (for $k \geq 1$) and M_{k^*} and M_{k^*+1} have the same dimension.*

Proposition 9.11 *If conditions of Lemma 9.9 are satisfied with a set of matrices $R_0(x)$, $R_1(x)$, $\ldots, R_{k^*-1}(x)$, then those conditions remain valid for other choices of such a set of matrices.*

Theorem 9.12 *$Z^* := M_{k^*}$ is a locally maximal ZSM.*

Proposition 9.13 *Assuming that rank of col $(G(x), L_g H_{k^*}(x))$ is equal to m for $x \in Z^*$, there exists a unique (locally) smooth mapping $u^* : Z^* \to \mathbb{R}^m$, such that $F(x) + G(x)u^*(x) = 0$ and $\hat{f}(x) \in T_x Z^*$ ($\hat{f}(x) := f(x) + g(x)u^*(x)$). That is, the equation*

$$\begin{pmatrix} F(x) \\ L_f H_{k^*}(x) \end{pmatrix} + \begin{pmatrix} G(x) \\ L_g H_{k^*}(x) \end{pmatrix} u^*(x) = 0 \tag{9.7}$$

has a unique solution around x_0.

Remark 9.14 *Under the assumption of the previous proposition, the differential index of the DAE will be equal to k^*. This is due to the fact that the algorithm followed to calculate the zeroing submanifold introduces intrinsically a procedure with which, after k^* time derivatives of the algebraic equation, we may obtain an ODE around x_0.*

9.3 State reconstruction

Let f^* be the restriction of $\hat{f}(x) = f(x) + g(x)u^*(x)$ to Z^* (assuming that *rank* $\begin{pmatrix} G(x) \\ L_g H_{k^*}(x) \end{pmatrix} = m$ for any $x \in Z^*$). Thus on Z^*, the dynamics of system (9.1) is governed by

$$\dot{x} = f^*(x) \text{ and } y = h(x) \tag{9.8}$$

Lemma 9.15 *Under the assumptions of Lemma 9.9, system (9.1) is LWO at x_0 if, and only if, (9.8) is LWO at x_0.*

Let n^* be the dimension of Z^*. Since $rankdH_{k^*}(x) = n - n^*$ for all $x \in Z^*$, we can arrange a vector function $\bar{H}^*(x) \in \mathbb{R}^{n-n^* \times n}$ whose terms are taken from $H_{k^*}(x)$ so that $rankd\bar{H}^*(x_0) = n - n^*$. Thus, there exists a diffeomorphism $\Psi(x) = \begin{pmatrix} \bar{H}^*(x) \\ \phi(x) \end{pmatrix}$ with which, defining $z = \Psi(x)$, we obtain

$$z_1(t) = \bar{H}^*(x(t)) = 0,\ \dot{z}_2(t) = \tilde{f}_2(z_2(t)),\ \text{and}\ y(t) = \tilde{h}_2(z_2)$$

where $z_1(t) \in \mathbb{R}^{n-n^*}$ and $z_2(t) \in \mathbb{R}^{n^*}$. There, $\tilde{f}_2(z_2(t))$ and $\tilde{h}_2(z_2)$ are given by the formulas

$$\begin{aligned} \tilde{f}_2(z_2) &= \left[\tfrac{\partial \phi(x)}{\partial x} f^*(x)\right]_{x=\Psi^{-1}(z)} \\ \tilde{h}_2(z_2) &= \tilde{h}(z)\,|_{z_1=0}\ \left(\tilde{h}(z) = h\left(\Psi^{-1}(z)\right)\right) \end{aligned} \tag{9.9}$$

Thereby, the original problem is reduced to the estimation of z_2 from the knowledge of $y(t)$. However, since LWO is not enough for the design of an observer, below we will assume that (9.8) is uniformly observable (see, e.g., [78]), i.e., we assume that on

$$Z_0^* = \left\{z_2 \in \mathbb{R}^{n^*} : z_2 = \phi(x)\ \text{for}\ x \in Z^*\ \text{s.t.}\ \bar{H}^*(x) = 0\right\}$$

the rank condition (9.10) is satisfied

$$rankcol\left(d\tilde{h}_2(z_2), dL_{\tilde{f}_2}\tilde{h}_2(z_2), \ldots, dL_{\tilde{f}_2}^{n^*-1}\tilde{h}_2(z_2)\right) = n^* \tag{9.10}$$

Below, in Theorem 9.16, we show that condition (9.10) can be checked in the original coordinates.

Theorem 9.16 *Under the assumptions of Lemma 9.9, (9.1) is uniformly observable on Z^* if (9.11) is satisfied for all $x \in Z^*$.*

$$rankcol\left(dH_{k^*}(x), dh(x), dL_{f^*}h(x), \ldots, dL_{f^*}^{n^*-1}h(x)\right) = n \tag{9.11}$$

Condition (9.10) implies that z_2 can be locally expressed as a function of $\left(y, \dot{y}, \ldots, y^{(n^*-1)}\right)$ which is known as algebraic observability [56]. In fact, it was shown in [175] that for analytic systems the fulfillment of (9.10) is equivalent to the algebraic observability of z_2. Thus, the following corollary is an immediate consequence of Theorem 9.16.

Corollary 9.17 *Under the assumptions of Lemma 9.9, $rankcol(G(x_0), L_gH_{k^*}(x_0)) = m$ and (9.11), there exists a function Γ such that $x(t) = \Gamma\left(y, \dot{y}, \ldots, y^{(n^*-1)}\right)$.*

Remark 9.18 *For the case when the map u^* is not unique, i.e. that $rankcol(G(x_0), L_gH_{k^*}(x_0)) = r < m$, the state estimation may still be done. As if rank of $col(G(x), L_gH_{k^*}(x))$ is constant in a neighborhood of x_0, locally there exist matrices $D_1(x)$ and $D_2(x)$ of rank r and $m-r$, respectively, whose entries are smooth functions of x, such that $rank(col(G(x), L_gH_{k^*}(x))D_1(x)) = r$ and $col(G(x), L_gH_{k^*}(x))D_2(x) = 0$ for all x in a neighborhood of x_0. Thus, with $D(x) :=$*

$(D_1(x), D_2(x))$, and a partition of its inverse as $(D(x))^{-1} = col(\bar{D}_1(x), \bar{D}_2(x))$, we obtain that

$$col\left(F(x), L_f H_{k^*}(x)\right) + col\left(G(x), L_g H_{k^*}(x)\right) D_1(x) \bar{D}_1(x) u = 0 \tag{9.12}$$

Let us define $\alpha_1 = \bar{D}_1(x) u$ and $\alpha_2 = \bar{D}_2(x) u$. Then, (9.12) has a unique solution for $\alpha_1 = \alpha_1(x)$. Thus, defining $f^*(x) = f(x) + g(x) D_1(x) \alpha_1(x)$, we can rewrite, locally on the manifold Z^*, the dynamic equations of the system as follows

$$\dot{x} = f^*(x) + g(x) D_2(x) \alpha_2 \text{ and } y = h(x)$$

Thereby, the state estimation may be carried out by using an unknown input observer (α_2 is the UI). In particular a reduced order observer may be designed. Indeed, using the diffeomorphism defined at the beginning of this section and the change of coordinates given by $z = \Psi(x)$, we obtain the sub-vector $z_1 = \bar{H}^*(x) = 0$ and z_2 being the state of the system $\dot{z}_2 = \tilde{f}_2(z_2) + \tilde{g}_2(z_2) \alpha_2$ and $y = \tilde{h}_2(z_2)$ where

$$\tilde{f}_2(z_2) = \left[\frac{\partial \phi(x)}{\partial x} f^*(x)\right]_{x=\Psi^{-1}(z)}, \quad \tilde{g}_2(z_2) = \left[\frac{\partial \phi(x)}{\partial x} g(x) D_2(x)\right]_{x=\Psi^{-1}(z)}$$
$$\tilde{h}_2(z_2) = \tilde{h}(z)\,|_{z_1=0} \text{ with } \tilde{h}(z) = h\left(\Psi^{-1}(z)\right)$$

Hence, an UI observer for $z_2(t) \in \mathbb{R}^{n-n^*}$ could be designed.

9.4 Example

Let us consider an example with the following functions:

$$f(x) = \begin{pmatrix} x_2 & x_4 + x_6 x_1 & x_1 x_4 & x_5 & x_3 & x_7 & -x_5 x_2 \end{pmatrix}^T$$
$$g(x) = \begin{pmatrix} 0 & x_2 & 1 & x_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_6 x_7 & 1 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 \end{pmatrix}^T, \ h(x) = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$
$$F(x) = \begin{pmatrix} 0 \\ \frac{-\cos\left(\frac{x_5}{2}\right)}{10} \\ x_6 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 & 0 & 2x_5 - \sin x_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By (9.2), $N = \left\{x \in \mathbb{R}^7 : x_6 = 0, x_5 = \sin(x_3) \text{ or } x_5 = \pi\right\}$. Thus, the observability is around $x_0 = 0$.

Step 1. Since $rank G(x) = 1$ for all $x \in \mathbb{R}^7$,

$$R_0(x) = \begin{pmatrix} -1 & 2x_5 - \sin x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_1 = R_0(x) F(x) = \begin{pmatrix} -\frac{1}{10}(2x_5 - \sin x_3)\cos\left(\frac{x_5}{2}\right) \\ x_6 \end{pmatrix}$$

We see that rank of the matrix dH_1 is equal to 2 for all $x \in U_1 = \left\{x \in \mathbb{R}^7 : |x_5| < \pi \text{ and } |x_3| < \frac{\pi}{2}\right\}$. Hence, $M_1 = \{x \in U_1 : x_6 = 0, 2x_5 = \sin x_3\}$ is a 5-dimension manifold.

Step 2. For $col\left(G(x), L_g H_1(x)\right)$, we obtain

$$\begin{pmatrix} G(x) \\ L_g H_1(x) \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & x_6 x_7 \\ 2x_5 - \sin x_3 & 1 & 0 & \gamma & 0 \end{pmatrix} \tag{9.13}$$

where $\gamma = \frac{1}{10} x_1 \cos\left(\frac{x_5}{2}\right) \cos(x_3)$ and

$$\delta = \frac{1}{20} \sin\left(\frac{x_5}{2}\right)(2x_5 - \sin x_3) - \frac{1}{5} \cos\left(\frac{x_5}{2}\right) + \frac{1}{10} \cos\left(\frac{x_5}{2}\right) \cos(x_3)$$

Therefore, for all $x \in M_1$, the rank of the matrix in (9.13) is equal to 2. Thus, $R_1(x)$, $\Phi_1(x)$, and $H_2(x)$ are taken as

$$R_1(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \Phi_1(x) = \begin{pmatrix} 0 \\ x_6 \\ x_7 \end{pmatrix}, H_2(x) = \begin{pmatrix} -\frac{1}{10}(2x_5 - \sin x_3)\cos\left(\frac{x_5}{2}\right) \\ x_6 \\ x_7 \end{pmatrix}$$

Since $rankdH_2(x) = 3$ on U_1, $M_2 = \{x \in U_1 : x_6 = x_7 = 0 \text{ and } 2x_5 = \sin x_3\}$.

Step 3. Now, we have that $col\left(G(x), L_g H_2(x)\right) = col\left(G(x), L_g H_2(x), \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\right)$, which has rank equal to 3 on M_2. Thus, matrix $R_2(x) \in \mathbb{R}^{3\times 6}$ has zeros everywhere except in the entries (1,1), (2,3), and (3,5), which have a number one. As for $\Phi_2(x)$, it takes the form $\Phi_2(x) = \begin{pmatrix} 0 & x_6 & x_7 \end{pmatrix}^T$. Finally, we obtain that $H_3(x) = H_2(x)$, which implies that $Z^* = M_2$ and $H^* = H_2$. Moreover, $rankcol\left(G(x), L_g H^*(x)\right) = 3$ for all $x \in Z^*$. Therefore, $u = u^*(x)$ satisfies the equation (9.7) with $k^* = 2$, for $x \in Z^*$. Thus, we obtain $u_1^*(x) = \frac{2x_3 - \left(\frac{1}{10}x_1 \cos\left(\frac{x_5}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2}$, $u_2^*(x) = x_2 x_5$, and $u_3^*(x) = \frac{1}{10}\cos\left(\frac{x_5}{2}\right)$. Thus, $f^*(x)$ $(x \in Z^*)$ has the form:

$$\begin{aligned}
f_1^* &= x_2, \quad f_6^* = f_7^* = 0 \\
f_2^* &= x_4 + x_2 \frac{2x_3 - \left(\frac{1}{10}x_1 \cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} \\
f_3^* &= x_1 x_4 + \frac{2x_3 - \left(\frac{1}{10}x_1 \cos\left(\frac{x_5}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} + \frac{1}{10} x_1 \cos\left(\frac{2\sin(x_3)}{2}\right) \\
f_4^* &= 2\sin(x_3) + x_2 \frac{2x_3 - \left(\frac{1}{10}x_1 \cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2} \\
f_5^* &= x_3 + \frac{2x_3 - \left(\frac{1}{10}x_1 \cos\left(\frac{2\sin(x_3)}{2}\right) + x_1 x_4\right)\cos x_3}{\cos x_3 - 2}
\end{aligned}$$

In this case, matrix in (9.11) is equal to $col\left(dH^*, dh, dL_{f^*}h, dL_{f^*}^2 h\right)$, which has rank 7 in a vicinity of $x = 0$. Then, the system is (locally) uniformly observable according to Theorem 9.16. Furthermore, as the dimension of Z^* is equal to 4, at most 3 derivatives of y

are required for the estimation of the entire state x. In fact, we have that x can be expressed in terms of $(y, \dot{y}, \ddot{y})$ as follows:

$$\begin{gathered} x_1 = y_1,\, x_2 = \dot{y}_1,\, x_3 = y_2,\, x_4 = \ddot{y}_1 + \dot{y}_1 \left(y_2 - \tfrac{1}{2}\dot{y}_2 \cos y_2 \right) \\ x_5 = \tfrac{1}{2}\sin y_2,\, x_6 = x_7 = 0 \end{gathered} \tag{9.14}$$

9.5 Conclusion

A new method to carry out the state estimation for a class of nonlinear singular systems has been proposed. By means of a *zeroing manifold algorithm*, provided that suitable conditions are satisfied, the state space whose dynamics is governed by a sole system of differential equations is found. This has allowed to apply standard techniques for the state and unknown input reconstruction. Nevertheless, the observability conditions allowing the state estimation can be checked also in terms of the original system with DAE. For a future work, one could look for considering a class of system with states having no explicit differential equations governing their dynamics.

Chapter 10

Luenberger-like Observer Design

This chapter proposes a new method for observer design of nonlinear singular systems from normal form point of view, by applying the differential geometric method, without any Lipschitz assumption on the nonlinearity. Here we focus only on a class of nonlinear singular systems with a single output and one algebraic equation. By introducing an intermediate artificial regular dynamical system, necessary and sufficient conditions will be deduced to transform nonlinear singular systems into a singular observable normal form, for which a simple Luenberger-like observer is proposed. The results of this chapter have been published in [C25].

10.1 Notations and problem statement

Consider the following class of nonlinear singular systems:

$$\begin{cases} E\dot{x} = f(x) + g(x)u \\ y = h(x) = Cx \end{cases} \tag{10.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g = [g_1, \cdots, g_m]$ where $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with E being canonical Nilpotent matrix with degree $n-1$ defined as follows:

$$E = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{10.2}$$

and $C = (1, 0, \cdots 0)$. It is assumed as well that (10.1) is observable.

When applying the technique of normal form, we need firstly propose a simple form which enables us to reuse the existing observers proposed in the literature. The goal is then to deduce necessary and sufficient conditions which guarantee the existence of a diffeomorphism to transform (10.1) into the proposed normal form. For this, let us firstly introduce the following singular normal form:

$$\begin{cases} E\dot{z} = z + \beta(y) + \alpha(y)u \\ y = Cz \end{cases} \tag{10.3}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}$, $\alpha(y) \in \mathbb{R}^{n\times m}$ and $\beta(y) \in \mathbb{R}^n$ are vector fields depending only on the output. The matrix E is a *canonical* Nilpotent matrix with degree $n-1$ defined in (10.2) and $C = (1, 0, \cdots 0)$. Therefore, the pair (E, C) is observable, i.e. there exists K such that $(E - KC)$ is Hurwitz.

The reason to propose such a simple form is that, by defining $\Gamma = E - KC$, we can easily design the following simple Luenberger-like dynamics:

$$\begin{cases} \dot{\eta} = \Gamma^{-1}\eta + \Gamma^{-1}\alpha(y)u + \Gamma^{-1}(\beta(y) - \Gamma^{-1}Ky) \\ \hat{z} = \eta - \Gamma^{-1}Ky \end{cases} \tag{10.4}$$

which in fact is an exponential observer for the singular dynamical system (10.3). Thus, the following focuses on seeking necessary and sufficient conditions to transform nonlinear singular system (10.1) to the proposed normal form (10.3).

10.2 Main result

The basic idea is to reuse the well-known differential geometric tools for regular systems to treat the singular ones. For this, let us regularize system (10.1) by adding an artificial dynamics on x_1. For $1 \le i \le n$, denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ the ith canonical basis of $\mathbb{R}^n$, then (10.1) can be rewritten as follows:

$$\begin{cases} E\dot{x} + \dot{x}_1 e_n = f(x) + g(x)u + \dot{y}e_n \\ y = x_1 \end{cases} \tag{10.5}$$

Let

$$P = (e_n, e_1, \cdots, e_{n-1}) \tag{10.6}$$

which gives $Pe_1 = e_n$ and $Pe_i = e_{i-1}$ for $2 \le i \le n$. Then system (10.5) is equivalent to the following one:

$$\begin{cases} P\dot{x} = f(x) + g(x)u + \dot{y}e_n \\ y = x_1 \end{cases} \tag{10.7}$$

where P is defined in (10.6) .

The above idea is to make system (10.7) to be a regular one. Since P is invertible, then by multiplying both sides of (10.7) with $Q = P^{-1}$, one obtains the following *artificial* regular system:

$$\begin{cases} \dot{x} = Qf(x) + Qg(x)u + \dot{y}e_1 \\ y = x_1 \end{cases} \tag{10.8}$$

with the fact that $Qe_n = P^{-1}e_n = e_1$. By using the same technique for (10.1), system (10.3) is equivalent to the following one as well:

$$\begin{cases} \dot{\xi} = A\xi + \tilde{\alpha}(y)u + \tilde{\beta}(y) + \dot{y}e_n \\ y = \xi_n \end{cases} \tag{10.9}$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{10.10}$$

where $\xi_n = z_1, \xi_k = z_{k+1}$, for $1 \leq k \leq n-1$. The dynamics (10.9) is in the Brunovsky form and $\tilde{\beta}(y) = \beta(y) + ye_1$.

Assumption 10.1 *For system (10.1), it is assumed that its associated regular system (10.8) fulfills the rank condition rank* $\left(dh, dL_{Qf}h, \cdots, dL_{Qf}^{n-1}h\right)^T = n$ *where* $h(x) = y = Cx$ *with* $C = (1, 0, \cdots, 0)$.

In the following, we will deduce geometric condition under which there exists a change of coordinates transforming the singular system (10.1) (or (10.8) since they are equivalent) into the observable normal form (10.3) which enables us to design the proposed observer (10.4).

Consider the following 1-forms:

$$\theta_1 = dh = dx_1 \tag{10.11}$$

$$\theta_i = dL_{Qf}^{i-1}h \text{ for } 2 \leq i \leq n \tag{10.12}$$

Since the rank condition of Assumption 10.1 is satisfied, then θ_i for $1 \leq i \leq n$ are linearly independent.

Let τ_1 be the vector field uniquely determined by the following equation:

$$\begin{cases} \theta_i(\tau_1) = 0 \text{ for } 1 \leq i \leq n-1 \\ \theta_n(\tau_1) = 1 \end{cases} \tag{10.13}$$

and by induction one constructs the family of vector fields as follows:

$$\tau_{i+1} = [\tau_i, Qf] \text{ for } 1 \leq i \leq n-1 \tag{10.14}$$

where $[,]$ denote the conventional Lie bracket.

Note $\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$ and $\tau = (\tau_1, \cdots, \tau_n)$, one can define the following matrix for the transformation: $\Lambda = \begin{pmatrix} \Lambda_{1,1} & \cdots & \Lambda_{1,n} \\ \vdots & \ddots & \vdots \\ \Lambda_{n,1} & \cdots & \Lambda_{n,n} \end{pmatrix}$ where for $1 \le i, j \le n$ we have $\Lambda_{i,j} = \theta_i(\tau_j)$. From (10.11-10.14), it is easy to see that: $\Lambda_{i,n-(i-1)} = 1$ and $\Lambda_{i,j} = 0$ *for* $j < n-(i-1)$, thus one has

$$\Lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & \Lambda_{2,n} \\ 0 & 0 & \cdots & \Lambda_{3,n-1} & \Lambda_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \Lambda_{n,2} & \cdots & \Lambda_{n,n-1} & \Lambda_{n,n} \end{pmatrix}$$

It is clear that Λ is invertible, thus one can define the following multi 1-forms:

$$\omega = \Lambda^{-1}\theta \tag{10.15}$$

from the structure of Λ one obtains $\omega_n = \theta_1$ and

$$\omega_{n-k} = \theta_{k+1} - \sum_{j=n-k-1}^{n} \Lambda_{k,j}\omega_j.$$

The following result gives necessary and sufficient conditions to guarantee the existence of the seeking diffeomorphism.

Theorem 10.2 *Suppose Assumption 10.1 is satisfied, then there exists a diffeomorphism which transforms (10.1) (or (10.8)) into (10.3) (or (10.9)) if and only if one of the following conditions is fulfilled:*

- $[\tau_i, \tau_j] = 0$ *for* $1 \le i \le n$, $1 \le j \le n$, $[g_l, \tau_i] = 0$ *for* $1 \le i \le n-1$, $1 \le l \le m$ *and* $[\tau_i, e_1] = 0$ *for* $1 \le i \le n-1$;
- $d\omega = 0$, $[g_l, \tau_i] = 0$ *for* $1 \le i \le n-1$, $1 \le l \le m$ *and* $[\tau_i, e_1] = 0$ *for* $1 \le i \le n-1$.

We have shown as well that if the transformation $\xi = \phi(x)$ exists, then its differential is such that $d\phi := \phi_* = \omega$. In other words, the seeking diffeomorphism ϕ is just the integral of the closed one-form ω.

10.3 Example

The following gives an academic example in order to highlight the proposed results. For this, consider the following nonlinear non-Lipschitz singular system:

$$\begin{cases} \dot{x}_2 = x_1 - \frac{3}{2}x_3^2 + \frac{3}{2}x_3^{\frac{1}{2}}x_2 + \frac{3}{2}x_3^{\frac{1}{2}}e^{x_1}u \\ \dot{x}_3 = x_2 - x_3^{\frac{3}{2}} + e^{x_1}u \\ 0 = x_3 + x_1 + u \\ y = x_1 \end{cases} \tag{10.16}$$

Following the technique introduced in the last section, this system can be re-formulated into an *artificial* regular nonlinear system as follows:

$$\begin{cases} \dot{x} = Qf + Qgu + e_1\dot{y} = \begin{pmatrix} x_3 + x_1 \\ x_2 - x_3^{\frac{3}{2}} \\ x_1 - \frac{3}{2}x_3^2 + \frac{3}{2}x_3^{\frac{1}{2}}x_2 \end{pmatrix} + \begin{pmatrix} u + \dot{y} \\ \frac{3}{2}x_3^{\frac{1}{2}}e^{x_1}u \\ e^{x_1}u \end{pmatrix} \\ y = x_1 \end{cases} \tag{10.17}$$

Then, one can calculate the corresponding 1-forms:

$$\begin{aligned} \theta_1 &= dh = dx_1, \;\; \theta_2 = dL_{Qf}h = dx_1 + dx_3 \\ \theta_3 &= dL_{Qf}^2 h = dx_2 - \frac{3}{2}x_3^{\frac{1}{2}}dx_3 + dx_3 + dx_1, \end{aligned}$$

then (10.16) is observable. Moreover, it can be used to uniquely determine the following vector fields:

$$\tau_1 = \frac{\partial}{\partial x_2}, \;\; \tau_2 = \frac{\partial}{\partial x_3} + \frac{3}{2}x_3^{\frac{1}{2}}\frac{\partial}{\partial x_2}, \;\; \tau_3 = \frac{\partial}{\partial x_1}.$$

It is clear that for all $1 \le i \le 3$ and $1 \le j \le 3$ one has $\left[\tau_i, \tau_j\right] = 0$.

Moreover, one can easily check that $[\tau_i, g] = 0$ and $[\tau_i, e_1] = 0$ for $1 \le i \le 2$. Denote $\theta = (\theta_1, \theta_2, \theta_3)^T$ and $\tau = (\tau_1, \tau_2, \tau_3)$, one gets

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

which is invertible. Thus one has

$$\omega = \Lambda^{-1}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} d(x_2 - x_3^{\frac{3}{2}}) \\ dx_3 \\ dx_1 \end{pmatrix}$$

which gives the following change of coordinates:

$$\begin{cases} \xi_1 = x_2 - x_3^{\frac{3}{2}} \\ \xi_2 = x_3 \\ \xi_3 = x_1 \end{cases}$$

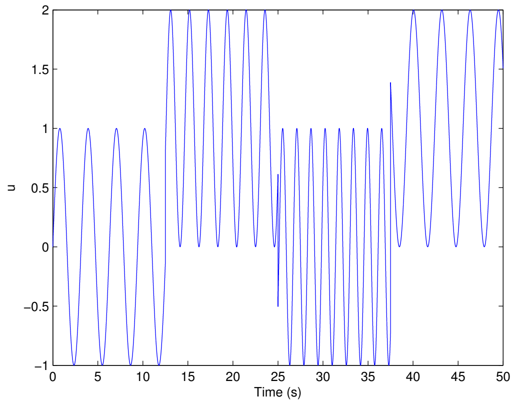


Fig. 10.1 Discontinuous input u.

This brings (10.16) into the following one:

$$\begin{cases} \dot{\xi}_1 = y \\ \dot{\xi}_2 = \xi_1 + e^y u \\ \dot{\xi}_3 = \xi_2 + y + u + \dot{y} \\ y = \xi_3 \end{cases} \tag{10.18}$$

which is of the form (10.9). Then one can apply the proposed method to design an observer for the studied system.

In the simulation, $K = (5,5,1)^T$, and the input u is chosen as follows:

$$u = \begin{cases} sin(2t), & t \in [0,12.5] \\ 1 + sin(3t), & t \in (12.5,25] \\ sin(4t), & t \in (25,37.5] \\ 1 - sin(2t), & t \in (37.5,50] \end{cases}$$

The simulation results for the observer are given in Fig. 10.1-Fig. 10.4.

10.4 Conclusion

Observer design for a class of nonlinear singular systems is studied in this chapter. We firstly proposed a normal form which represents a class of nonlinear singular systems with a linear dynamic errors, then presented a new observer for such a form. Employing differential geometric method which is widely used in nonlinear regular systems, we gave necessary and sufficient conditions which is able to deduce a diffeomorphism to transform a class of nonlinear singular systems into the proposed singular normal form.

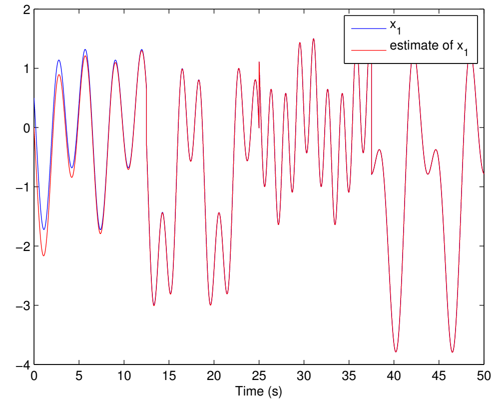


Fig. 10.2 x_1 and its estimation with initial conditions $x_1(0) = 0.5$ and $\hat{x}_1(0) = 0$.

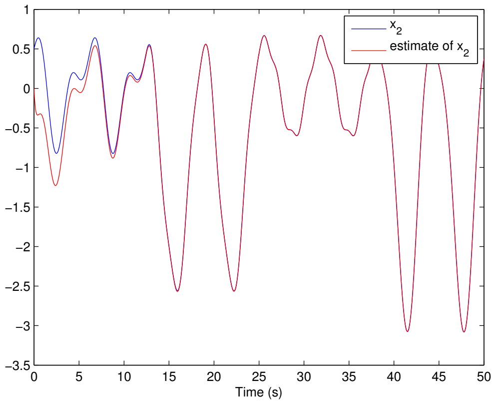


Fig. 10.3 x_2 and its estimation with initial conditions $x_2(0) = 0.5$ and $\hat{x}_2(0) = 0$.

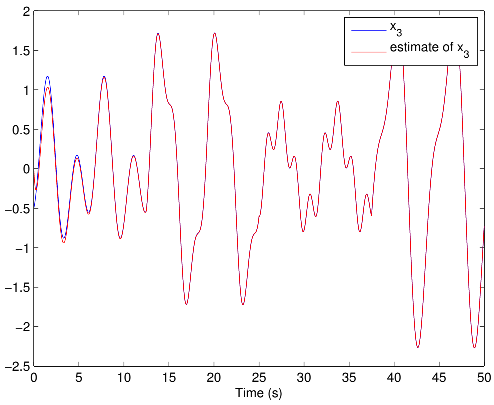


Fig. 10.4 x_3 and its estimation with initial conditions $x_3(0) = -0.5$ and $\hat{x}_3(0) = 0$.

Chapter 11

Interval Observer Design

Most of the existing results are for the asymptotic estimation of the state for singular systems without uncertainties. The observer design becomes complicated when considering the systems with uncertain terms in the state and in the measurement. In this situation, the exact estimation may be not possible, and one solution (as we did for regular systems in the previous chapter) is to provide the upper and lower bound estimation of the admissible values for the state by applying the theory of set-membership or interval estimation. This chapter recall the results, published in [J3, C1], to design an interval observer for a class of nonlinear singular systems with uncertainties.

11.1 Notations and problem statement

Consider the following uncertain nonlinear singular system:

$$\Sigma_\xi : \begin{cases} \bar{E}\dot{\xi} = \bar{A}\xi + \bar{f}(\xi,u) + v(t) \\ y = \bar{C}\xi + w(t) \end{cases} \tag{11.1}$$

where $\xi \in \mathbb{R}^n$ is the state whose initial value belongs to a compact set $I_0(\xi(t_0)) = [\underline{\xi}(t_0), \overline{\xi}(t_0)]$; $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are respectively the output and the input. $\bar{E} \in \mathbb{R}^{n\times n}$, $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{C} \in \mathbb{R}^{p\times n}$, and the vector field $\bar{f}$ represents the nonlinear term with the appropriate dimension. $w(t)$ and $v(t)$ are the disturbance in the output and in the model, respectively.

When the matrix $\bar{E}$ is nonsingular, then (11.1) can be written as:

$$\begin{cases} \dot{\xi} = \bar{E}^{-1}\bar{A}\xi + \bar{E}^{-1}\bar{f}(\xi,u) + \bar{E}^{-1}v(t) \\ y = \bar{C}\xi + w(t) \end{cases} \tag{11.2}$$

which becomes a classical regular system. When the matrix $\bar{E}$ is singular, (11.1) represents a large class of nonlinear singular systems with uncertainties in the state and in the output, covering those studied in [111] and [52]. This chapter is devoted to designing an interval observer for this larger class of uncertain nonlinear singular systems.

11.2 Assumptions and preliminary results

In this chapter, for the uncertain nonlinear singular system Σ_ξ, we are interested in the interval estimation of the (non-impulsive) trajectory of the state $\xi(t)$ with the known information of $y(t)$ and $u(t)$ for $t > 0$. Therefore, this paper considers the following definition of observability adopted from Definition 1 in [11].

Definition 11.1 *System Σ_ξ is observable if $y(\xi_1,t) = y(\xi_2,t)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$ and $t > 0$ implies $\xi_1(0^+) = \xi_2(0^+)$.*

In what follows, we make the following assumption for the original system Σ_ξ.

Assumption 11.2 *For the triple $(\bar{E},\bar{A},\bar{C})$ of system Σ_ξ defined in (11.1), it is assumed that the following rank conditions:*

$$rank \begin{bmatrix} \bar{E} \\ \bar{C} \end{bmatrix} = n \tag{11.3}$$

and

$$rank \begin{bmatrix} s\bar{E} - \bar{A} \\ \bar{C} \end{bmatrix} = n, \forall s \in \mathbb{C} \tag{11.4}$$

are satisfied.

It is obvious that, even if the rank conditions (11.3) and (11.4) are assumed to be satisfied, the studied uncertain nonlinear singular system Σ_ξ might not be observable, due to the nonlinear term $\bar{f}(\xi,u)$ and the uncertainties (v,w). Without loss of generalities, for the singular matrix $\bar{E} \in \mathbb{R}^{n\times n}$, it is assumed that rank$\bar{E} = q < n$. Due to the rank condition (11.3), there exists a non-singular matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \in \mathbb{R}^{(p+n)\times(p+n)}$ such that $P \begin{bmatrix} \bar{E} \\ \bar{C} \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, which is equivalent to:

$$\begin{cases} P_1\bar{E} + P_2\bar{C} = I_n \\ P_3\bar{E} + P_4\bar{C} = 0 \end{cases} \tag{11.5}$$

with $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times p}$, $P_3 \in \mathbb{R}^{p\times n}$ and $P_4 \in \mathbb{R}^{p\times p}$. Based on the above result, we have the following lemma.

Lemma 11.3 *Suppose Assumption 11.2 is satisfied for the triple $(\bar{E},\bar{A},\bar{C})$ of Σ_ξ. Then there always exist matrices $K \in \mathbb{R}^{n\times p}$, $L \in \mathbb{R}^{n\times p}$ and an invertible matrix $Q \in \mathbb{R}^{n\times n}$ such that the following matrix:*

$$R = QNQ^{-1} \tag{11.6}$$

is Hurwitz and Metzler, where

$$N = (P_1 + KP_3)\bar{A} + L\bar{C}, \tag{11.7}$$

with P_1 and P_3 being defined in (11.5).

With the above deduced matrix Q which transforms the Hurwitz matrix N to the Hurwitz and Metzler matrix R, since it is non-singular, then we can choose $x = Q\xi$ as a diffeomorphism, with which system Σ_ξ in (11.1) can be rewritten as follows:

$$\Sigma_x : \begin{cases} E\dot{x} = Ax + f(x,u) + v(t) \\ y = Cx + w(t) \end{cases} \tag{11.8}$$

where $E = \bar{E}Q^{-1}$, $A = \bar{A}Q^{-1}$, $C = \bar{C}Q^{-1}$ and $f(x,u) = \bar{f}(Q^{-1}x,u)$. For the transformed system (11.8), we can state the following lemma.

Lemma 11.4 *Suppose Assumption 11.2 is satisfied for the triple $(\bar{E},\bar{A},\bar{C})$ of Σ_ξ. Then for the transformed system $\Sigma_x(E,A,C)$ defined in (11.8), there always exist a matrix $K \in \mathbb{R}^{n\times p}$, two invertible matrices $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \in \mathbb{R}^{(p+n\times(p+n)}$ and $Q \in \mathbb{R}^{n\times n}$, such that the following equality is satisfied:*

$$QP_{24}C + QP_{13}E = I_n$$

where the matrices P_{13} and P_{24} are defined as follows:

$$\begin{aligned} P_{13} &= P_1 + KP_3 \\ P_{24} &= P_2 + KP_4 \end{aligned} \tag{11.9}$$

Since the initial condition ξ_0 for (11.1) is supposed to be located into a certain interval $I(\xi_0) = [\underline{\xi_0}, \overline{\xi_0}]$, then using the diffeomorphism $x = Q\xi$, the initial condition x_0 is also known within a certain interval $I(x_0) = [\underline{x}_0, \bar{x}_0]$. Thus the interval estimation of ξ in (11.1) is equivalent to estimate the interval of x in (11.8) by using the knowledge of (u,y).

11.3 Interval estimation

Prior to introduce the interval observer for (11.1), let us firstly make the following assumption which is necessary in the sequel.

Assumption 11.5 *For the studied system (11.1), it is assumed that:*

1) the state $\xi(t)$ is bounded under the bounded input $u(t) \in C^\infty$, i.e. $\xi(t) \in \Omega \subset \mathbb{R}^n$ for a given $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ for all $t \geq 0$ where Ω and $\mathscr{U}$ are two given compact sets;

2) the function $\bar{f}(\xi,u)$ for all $\xi \in \Omega$ and $u \in \mathscr{U}$ is locally Lipschtiz w.r.t (ξ,u);

3) the disturbances $w(t)$ and $v(t)$ are bounded, and the derivative of $w(t)$ is bounded for all $t \geq 0$, i.e. there exist constants $\bar{v}, \underline{v}, \overline{w}, \underline{w}, \overline{w_d}, \underline{w_d}$ such that $\underline{v} \leq v(t) \leq \bar{v}$, $\underline{w} \leq w(t) \leq \overline{w}$, and $\underline{w_d} \leq \dot{w}(t) \leq \overline{w_d}$.

Remark 11.6 *Since system (11.8) was transformed from the system (11.1) by applying the diffeomorphism $x = Q\xi : \Omega \to \mathscr{X}$, therefore if Assumption 11.5 is fulfilled for (11.1), then the state $x(t)$ of (11.8) is bounded as well for the bounded input $u(t)$, i.e. $x(t) \in \mathscr{X} \subset \mathbb{R}^n$*

for all $t \geq 0$ where $\mathscr{X}$ represents the compact set defined via the diffeomorphism. Moreover, since $f(x,u) = \bar{f}(Q^{-1}x,u)$ in (11.8), thus it is locally Lipschtiz w.r.t (x,u) for all $x \in \mathscr{X}$ and $u \in \mathscr{U}$.

With Assumption 11.5, let us recall the following boundedness results stated in [205] and [60] which will be used to design the interval observer.

Lemma 11.7 *[205] For a locally Lipschitz continuous function $f(x,u)$ w.r.t (x,u) with $x \in \mathscr{X}$ and $u \in \mathscr{U}$, there exist two functions $\overline{f}, \underline{f} : \mathbb{R}^{2n+m} \to \mathbb{R}^n$ such that, for $\underline{x} \leq x \leq \overline{x}$ with $x \in \mathscr{X}$, $\underline{f}(\underline{x},\overline{x},u) \leq f(x,u) \leq \overline{f}(\underline{x},\overline{x},u)$, and for a given submultiplicative norm $\|\cdot\|$ we have*

$$\begin{aligned}
\|\overline{f}(\underline{x},\overline{x},u) - f(x,u)\| &\leq \bar{l}_{\overline{f}}\|\overline{x}-x\| + \underline{l}_{\overline{f}}\|\underline{x}-x\| + l_{\overline{f}} \\
\|\underline{f}(\underline{x},\overline{x},u) - f(x,u)\| &\leq \bar{l}_{\underline{f}}\|\overline{x}-x\| + \underline{l}_{\underline{f}}\|\underline{x}-x\| + l_{\underline{f}}
\end{aligned}$$

for some positive constants $\bar{l}_{\overline{f}}$, $\underline{l}_{\overline{f}}$, $l_{\overline{f}}$, $\bar{l}_{\underline{f}}$, $\underline{l}_{\underline{f}}$ and $l_{\underline{f}}$.

As we have stated in Section 11.2, if Assumption 11.2 and Assumption 11.5 are satisfied, then there exist the matrices K, L, and the non-singular matrices P and Q such that the matrix N defined in (11.7) is Hurwitz, and the matrix R defined in (11.6) is Hurwitz and Metzler. For the sake of simplicity, note $Q_{P_{13}} = QP_{13}$, $Q_{P_{24}} = QP_{24}$, $Q_L = QL$ where P_{13} and P_{24} are defined in (11.9). Then one can design the following two dynamics:

$$\begin{cases} \dot{\overline{z}} = & R\overline{z} + (RQ_{P_{24}} - Q_L)y + \overline{\Delta}(\underline{z},\overline{z},u) \\ \overline{x} = & \overline{z} + Q_{P_{24}}y \end{cases} \tag{11.10}$$

and

$$\begin{cases} \dot{\underline{z}} = & R\underline{z} + (RQ_{P_{24}} - Q_L)y + \underline{\Delta}(\underline{z},\overline{z},u) \\ \underline{x} = & \underline{z} + Q_{P_{24}}y \end{cases} \tag{11.11}$$

with

$$\begin{aligned}
&\overline{\Delta}(\underline{z},\overline{z},u) = \overline{\Delta_f}(\underline{z},\overline{z},u) + \overline{\Delta_v} + \overline{\Delta_w} + \overline{\Delta_{w_d}},\ \overline{\Delta_f}(\underline{z},\overline{z},u) = Q^+_{P_{13}}\overline{f}(\underline{z},\overline{z},u) - Q^-_{P_{13}}\underline{f}(\underline{z},\overline{z},u) \\
&\overline{\Delta_v} = Q^+_{P_{13}}\overline{v} - Q^-_{P_{13}}\underline{v},\ \overline{\Delta_w} = Q^+_L\overline{w} - Q^-_L\underline{w},\ \overline{\Delta_{w_d}} = Q^-_{P_{24}}\overline{w_d} - Q^+_{P_{24}}\underline{w_d}
\end{aligned}$$

and

$$\begin{aligned}
&\underline{\Delta}(\underline{z},\overline{z},u) = \underline{\Delta_f}(\underline{z},\overline{z},u) + \underline{\Delta_v} + \underline{\Delta_w} + \underline{\Delta_{w_d}},\ \underline{\Delta_f}(\underline{z},\overline{z},u) = Q^+_{P_{13}}\underline{f}(\underline{z},\overline{z},u) - Q^-_{P_{13}}\overline{f}(\underline{z},\overline{z},u) \\
&\underline{\Delta_v} = Q^+_L\underline{v} - Q^-_L\overline{v},\ \underline{\Delta_w} = Q^+_L\underline{w} - Q^-_L\overline{w},\ \underline{\Delta_{w_d}} = Q^-_{P_{24}}\underline{w_d} - Q^+_{P_{24}}\overline{w_d}
\end{aligned}$$

By noting

$$\overline{\Gamma}(\underline{x},\overline{x},x,u) = \quad \overline{\Delta}(\underline{z},\overline{z},u) - Q_{P_{13}}f(x,u) - Q_{P_{13}}v(t) - Q_L w(t) + Q_{P_{24}}\dot{w}(t) \tag{11.12}$$

and

$$\underline{\Gamma}(\underline{x},\overline{x},x,u) = \quad -\underline{\Delta}(\underline{z},\overline{z},u) + Q_{P_{13}}f(x,u) + Q_{P_{13}}v(t) + Q_L w(x,u) - Q_{P_{24}}\dot{w}(t) \tag{11.13}$$

we have the following corollary.

Corollary 11.8 *For $\overline{\Gamma}(\underline{x},\overline{x},x,u)$ and $\underline{\Gamma}(\underline{x},\overline{x},x,u)$ defined in (11.12) and (11.13) with $x \in \mathscr{X}$ and $u \in \mathscr{U}$, we have $\overline{\Gamma}(\underline{x},\overline{x},x,u) \geq 0$ and $\underline{\Gamma}(\underline{x},\overline{x},x,u) \geq 0$. Moreover, there exist positive constants $\underline{l}_{\underline{\Gamma}}$, $\bar{l}_{\underline{\Gamma}}$, $\underline{l}_{\overline{\Gamma}}$, $\bar{l}_{\overline{\Gamma}}$ and $l_{\overline{\Gamma}}$, $l_{\underline{\Gamma}}$ such that for a chosen submultiplicative norm $\|\cdot\|$ the following inequalities:*

$$\begin{array}{rcl} \|\overline{\Gamma}(\underline{x},\overline{x},x,u)\| & \leq & \bar{l}_{\overline{\Gamma}}\|\overline{x}-x\| + \underline{l}_{\overline{\Gamma}}\|x-\underline{x}\| + l_{\overline{\Gamma}} \\ \|\underline{\Gamma}(\underline{x},\overline{x},x,u)\| & \leq & \bar{l}_{\underline{\Gamma}}\|\overline{x}-x\| + \underline{l}_{\underline{\Gamma}}\|x-\underline{x}\| + l_{\underline{\Gamma}} \end{array}$$

are satisfied for all $t \geq 0$.

Theorem 11.9 *Suppose Assumption 11.2 and Assumption 11.5 are satisfied. Then for any initial state $x(t_0)$ of (11.8) belongs to a certain interval $I(x(t_0)) = [\underline{x}(t_0), \overline{x}(t_0)]$, systems (11.10) and (11.11) form an interval observer for (11.8) such that the following inequality:*

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t), \tag{11.14}$$

holds for all $t \geq t_0$. Mover, if there exist positive definite symmetric matrices S, M and a positive scalar μ such that the following LMI is satisfied:

$$\begin{bmatrix} \mathscr{R}^T S + S\mathscr{R} + \frac{\alpha}{\mu} I + M & S \\ S & -\frac{1}{\mu} I \end{bmatrix} \preceq 0 \tag{11.15}$$

where $\mathscr{R} = diag\{R, R\}$, $\alpha = 2\max\{\underline{l}_{\underline{\Gamma}}^2, \bar{l}_{\underline{\Gamma}}^2, \underline{l}_{\overline{\Gamma}}^2, \bar{l}_{\overline{\Gamma}}^2, l_{\overline{\Gamma}}^2, l_{\underline{\Gamma}}^2\}$, then the variables $\underline{x}(t)$ and $\overline{x}(t)$ are bounded for all $t \geq 0$.

Corollary 11.10 *For system (11.8) with the initial state $\xi(t_0) \in [\underline{\xi}(t_0), \overline{\xi}(t_0)]$, if Assumption 11.2 and Assumption 11.5 are satisfied, then there exists a non-singular matrix Q such that the interval estimation of ξ in (11.1) is given as follows:*

$$\left[Q^{-1}\right]^{+}\underline{x} - \left[Q^{-1}\right]^{-}\overline{x} \leq \xi \leq \left[Q^{-1}\right]^{+}\overline{x} - \left[Q^{-1}\right]^{-}\underline{x} \tag{11.16}$$

where $\underline{x}$ and $\overline{x}$ are the states of the proposed interval observer defined in (11.10) and (11.11).

Let us remark that Assumption 11.5 imposed the state boundedness $\xi(t) \in \Omega \subset \mathbb{R}^n$ under a given bounded input $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ for all $t \geq 0$, and this property is sometimes difficult to be checked for general uncertain nonlinear singular systems, therefore the proposed result could not be applied. However, if the boundedness property of the state in Assumption 11.5 is satisfied only for $t \in [0, T]$ with T being a finite time, i.e. $\xi(t) \in \Omega \subset \mathbb{R}^n$ under a given bounded input $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ for all $t \in [0, T]$, then Theorem 11.9 is still valid during this finite time T. Therefore the proposed result can be relaxed by the following corollary.

Corollary 11.11 *Suppose Assumption 11.2 and Assumption 11.5 are valid only for $t \in [0, T]$ with T being a finite time. Then for any initial state $x(t_0)$ of (11.8) belongs to a certain interval $I(x(t_0)) = [\underline{x}(t_0), \overline{x}(t_0)]$, systems (11.10) and (11.11) give an interval estimation of $x(t)$ of (11.8) for $t \in [t_0, T]$, i.e. $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \in [t_0, T]$. And the interval estimation of $\xi(t)$ of (11.1) for all $t \in [t_0, T]$ can be still obtained by (11.16).*

11.4 Design procedure

For system $\Sigma_\xi\left(\bar{E},\bar{A},\bar{C}\right)$ defined in (11.1), supposed that Assumption 11.2 and Assumption 11.5 are both satisfied for all $t \geq 0$ (or for all $t \in [0,T]$), then the interval estimation of the state ξ for (11.1) can be obtained via the following procedure:

Step1: Due to the satisfaction of the rank condition (11.3) in Assumption 11.2, determine an invertible matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that $P\begin{bmatrix} \bar{E} \\ \bar{C} \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$;

Step2: Due to the satisfaction of the rank condition (11.4) in Assumption 11.2, choose the matrices K, L such that $N = L\bar{C} + (P_1 + KP_3)\bar{A}$ is Hurwitz;

Step3: Choose an invertible matrix Q such that $R = QNQ^{-1}$ is Hurwitz and Metzler;

Step4: Transform system $\bar{\Sigma}_\xi\left(\bar{E},\bar{A},\bar{C}\right)$ into $\Sigma_x(E,A,C)$ by applying the diffeomorphism $x = Q\xi$;

Step5: If Assumption 11.5 is satisfied for all $t \geq 0$ (or for all $t \in [0,T]$), calculate the bound of the function f in (11.8);

Step6: Design interval observer (11.10) and (11.11) for $\Sigma_x(E,A,C)$, which yields $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ (or for all $t \in [0,T]$);

Step7: Finally we obtain the interval estimation of $\xi(t)$ for all $t \geq 0$ (or for all $t \in [0,T]$):

$$\left[Q^{-1}\right]^{+}\underline{x} - \left[Q^{-1}\right]^{-}\overline{x} \leq \xi \leq \left[Q^{-1}\right]^{+}\overline{x} - \left[Q^{-1}\right]^{-}\underline{x}.$$

11.5 Example

Consider the following nonlinear RLC circuit in Fig. 11.1, modified from [189], where

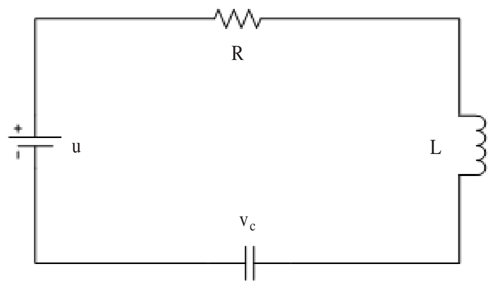


Fig. 11.1 Nonlinear RLC ciruit.

the capacitor is not linear, but satisfying nonlinear $q - v_c$ characteristic $v_c = q + 0.5\sin q^2$.

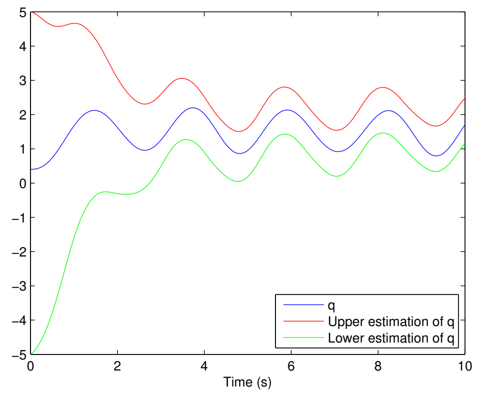


Fig. 11.2 The interval estimation for q.

According to the analysis in [189], we have

$$\begin{aligned}
\dot{q} &= \phi/L \\
\dot{\phi} &= -\phi R/L - v_c + u \\
0 &= v_c - q - 0.5\sin q^2 \\
y &= v_c
\end{aligned}$$

where u is source voltage, q and v_c represent respectively the charge and the voltage of the capacitor, and ϕ is the flux through the inductor. Denote $\xi = (q, \phi, v_c)^T$, and taking into account the uncertainties, the above dynamics can be written into the form (11.1) with

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} 0 & 1/L & 0 \\ 0 & -R/L & -1 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } \bar{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T$$

$\bar{f}(\xi, u) = \begin{bmatrix} 0 \\ 0 \\ -0.5\sin(\xi_1^2) \end{bmatrix}$ and $v(t) = \begin{bmatrix} 0.2\sin 2t \\ 0.2\sin 4t \\ 0.1\cos 3t \end{bmatrix}$, $w(t) = 0.1\sin(t)$, where $v(t)$ and $w(t)$ are the assumed disturbances.

In the simulation setting, we choose $L = 5$, $R = 10$ and $u = 1.5 + 30\sin 2.8t$. For such a simple nonlinear singular system, we can easily follow the proposed procedure to design an interval observer, thus the intermediate steps are omitted. The simulation results are depicted in Fig. 11.2-11.4.

11.6 Conclusion

When treating the uncertain nonlinear singular systems while the asymptotic estimation is not possible, this chapter is devoted to designing an interval observer for the studied uncertain system. By imposing the rank conditions and assuming the boundedness of the uncertainties

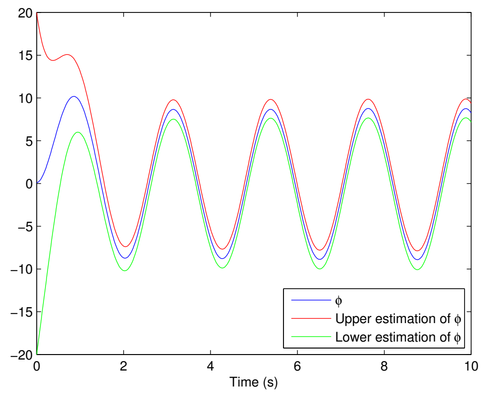


Fig. 11.3 The interval estimation for ϕ.

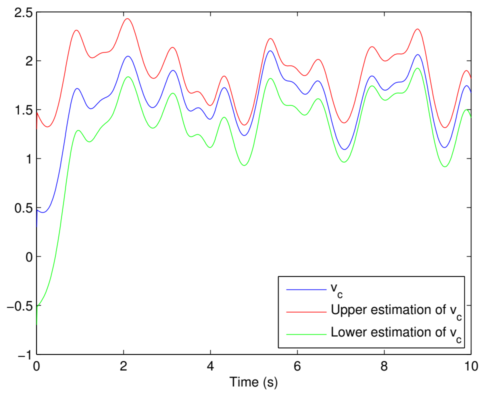


Fig. 11.4 The interval estimation for v_c.

(and the derivative of the uncertainty) in the state and in the output, we show that an interval observer can always be synthesized to provide the upper and the lower estimations of the real state.

Part III

O&O for Time-Delay System

Chapter 12

Introduction

Time-delay systems are widely used to model concrete systems in engineering sciences, such as biology, chemistry, mechanics and so on [112, 142, 157, 164]. Many results have been published to treat this kind of systems for different aspects [157, 164, 75]. Like other types of systems (such as continuous-time, singular and so on), stability and observability are as well two most important issues. Beside, since some results have been reported for the purpose of stability and observability analysis, by assuming that the delay of the studied systems is known, therefore it makes the delay identification be one of the most important topics in the field of time-delay systems. Hence, Part III will summarize our results on the observability and identifiability analysis, and observer design for time-delay system with unknown input.

As we have mentioned before, the observability property has been exhaustively studied for linear and nonlinear systems without delays. It has been characterized in [86, 114, 174, 177] from a differential point of view, and in [56] from an algebraic point of view. However, when the system is subject to time delay, such analysis is more complicated (see the surveys [157] and [164]). The analysis of observation for time-delay systems can be dated back to the 80's of the last century [117, 145, 160, 153]. For this issue, different definitions of observability have been proposed, such as strong observability, spectral observability and weak observability [116].

For linear time-delay systems, various aspects of the observability problem have been studied in the literature, using different methods such as the functional analytic approach [23] or the algebraic approach [35, 72, 173]. For nonlinear time-delay systems, by using the theory of non-commutative rings [141], the observability problem has been studied in [188] for systems with known inputs. The associated observer for some classes of time-delay systems can be found in [47, 164, 51, 66, 76] and the references therein.

The majority of the existing works on observability analysis are focused on time-delay systems whose outputs are not affected by unknown inputs. Although the inputs are commonly supposed to be known and are usually used to control the studied system, however there exists as well other cases, such as observer design for time-delay systems, in which the inputs can be unknown ([162, 55, 110, 190]). Moreover, some proposed unknown inputs observer design methods do depend on the known delay, which should be identified in advance. Concerning the delay identification problem, up to now, various techniques have been proposed, such as identification by using variable structure observers [58], modified least squares techniques [155], convolution approach [18], algebraic fast identification technique [19] as initiated in [73], and so on (see [58] for additional references). Note that most of the papers on identification in presence of delays concern linear models. Another source of complexity comes from the presence of feedback loops involving the delays. Indeed, when the delay appears only on the inputs or outputs, the system has the finite dimension. When the delays are involved in a closed-loop manner, the resulting model has delayed states and become a functional differential equation, which has the infinite dimension [157]. Motivated by this requirement, the delay identification problem for nonlinear time-delay systems with unknown inputs will be investigated as well in this part.

After the analysis of observability, we will focus on the unknown input observer design problem for time-delay system. In fact, this problem for linear systems without delays has been already solved in [26, 55, 190, 91, 115, 184, 88]. It becomes more complicated when the studied system involves delays, which might appear in the state, in the input and in the output. For this issue, different techniques have been proposed in the literature, such as infinite dimensional approach [160], polynomial approach based on the ring theory [163, 62], Lyapunov function based on LMI [50, 165] and so on.

More precisely, [66] proposed an unknown input observer with dynamic gain for linear systems with commensurate delays in state, input and output variables, while the output was not affected by the unknown inputs. Inspired by the technique of output injection [114], [93] solved this problem by transforming the studied system into a higher dimensional observer canonical form with delayed output injection. In [50, 51], the unknown input observer was designed for the systems involving only one delay in the state, and no delay appears in the input and output. The other observers for some classes of time-delay systems can be found in [47, 164, 76] and the references therein.

In summary, Part III presents our recent results on observability analysis and observer design for time-delay systems, which are listed as follows:

- Firstly, for the linear time-delay system whose model and output could be both affected by the unknown inputs, we introduced the Unknown Input Observability (UIO), Backward UIO and Forward UIO concepts. For each definition of observability, we obtain sufficient conditions that can be verified by using some matrices depending on the original system parameters. The established condition for the unknown input observability turns out to be a generalization of the already known condition for systems with unknown inputs, but without delays (in that case such condition is also a necessary one), and also it is a generalization of the known strongly observable condition for linear systems with commensurable delays, but without unknown inputs.

- Secondly, concerning nonlinear time-delay systems, by using the framework of non-commutative rings, we deduce necessary and sufficient conditions for identifying the delay in two different cases: dependent outputs over the non-commutative rings, and then independent ones. Also necessary and sufficient conditions of causal and non-causal observability for nonlinear time-delay systems with unknown inputs are studied;

- The last result of this part is to investigate an unknown input observer design for a large class of linear systems with unknown inputs and commensurate delays. We propose a Luenberger-like observer by involving only the past and actual values of the system output. The required conditions for the proposed observer are considerably relaxed in the sense that they coincide with the necessary and sufficient conditions for the unknown input observer design of linear systems without delays.

Chapter 13

Linear Time-Delay System

As we have pointed out that, the majority of the existing works on observability analysis is focused on time delay systems whose outputs are not affected by unknown inputs. However, this situation might exist in some practical applications and this motivates this work, published in [J5]. Here, we deal with linear time-delay systems whose delays are commensurable. We consider that delays may appear in the state, input, and output. The aim is searching for some conditions allowing for the reconstruction of the entire state vector using backward, actual, and/or forward output information.

13.1 Notations

The method we used to analyze the observability for linear time-delay systems with unknown inputs is based on the polynomial ring. The following recalls some basic notations which are widely used by using this method.

Denote $\mathbb{R}$ as the field of real numbers, then introduce the delay operator $\delta : x(t) \to x(t-h)$ with $\delta^k x(t) = x(t-kh)$, $k \in \mathbb{N}_0$. Let $\mathbb{R}[\delta]$ be the polynomial ring of δ over the field $\mathbb{R}$, i.e. each element $a(\delta) \in \mathbb{R}[\delta]$ can be written as follows:

$$a(\delta) = \sum_{i=0}^{d_a} a_i \delta^i \text{ with } a_i \in \mathbb{R}$$

where d_a is the maximum degree of $a(\delta)$, noted as $d_a = \deg_\delta a(\delta)$. For any $a(\delta)$ and $b(\delta) \in \mathbb{R}[\delta]$, the additional and multiplicative operations are defined as usual:

$$\begin{aligned} a(\delta) + b(\delta) &= \textstyle\sum_{i=0}^{\max\{d_a, d_b\}} (a_i + b_i)\delta^i \\ a(\delta)b(\delta) &= \textstyle\sum_{i=0}^{d_a} \sum_{j=0}^{d_b} a_i b_j \delta^{i+j} \end{aligned}$$

from which it is obvious that $\mathbb{R}[\delta]$ is a commutative ring.

$\mathbb{R}^n[\delta]$ is the $\mathbb{R}[\delta]$-module whose elements are the vectors of dimension n and whose entries are polynomials. By $\mathbb{R}^{q \times s}[\delta]$ we denote the set of matrices of dimension $q \times s$, whose entries are in $\mathbb{R}[\delta]$. For $f(\delta)$, a polynomial of $\mathbb{R}[\delta]$, $\deg f(\delta)$ is the degree of $f(\delta)$. For a matrix $M(\delta)$, $\deg M(\delta)$ (the degree of $M(\delta)$) is defined as the maximum degree of all the entries $m_{ij}(\delta)$ of $M(\delta)$. $\det M(\delta)$ is the determinant of this matrix, and $\operatorname{rank}_{\mathbb{R}[\delta]} M(\delta)$ means the rank of the matrix $M(\delta)$ over $\mathbb{R}[\delta]$. For a matrix $M(\delta)$, $\operatorname{rank}_{\mathbb{R}[\delta]} M(\delta)$ means the rank of the matrix $M(\delta)$ over $\mathbb{R}[\delta]$. $M(\delta) \sim N(\delta)$ means the similarity between two polynomial matrices $M(\delta)$ and $N(\delta)$ over $\mathbb{R}[\delta]$, i.e. there exist two unimodular matrices $U_1(\delta)$ and $U_2(\delta)$ over $\mathbb{R}[\delta]$ such that $M(\delta) = U_1(\delta)N(\delta)U_2(\delta)$. The acronym for greatest common divisor is *gcd*.

Using the same methodology, we can introduce the forward time-shift operator as δ^{-1} : $x(t) \to x(t+h)$. Then the Laurent polynomial ring is denoted as $\mathbb{R}\left[\delta, \delta^{-1}\right]$. It is obvious that $\mathbb{R}\left[\delta, \delta^{-1}\right]$ contains $\mathbb{R}[\delta]$.

13.2 Problem statement and definitions

We will deal with the following class of linear systems with commensurate delays

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_b} B_i u(t-ih) \\ y(t) &= \sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_d} D_i u(t-ih) \end{aligned} \tag{13.1}$$

where the state vector $x(t) \in \mathbb{R}^n$, the system output vector $y(t) \in \mathbb{R}^p$, and the unknown input vector $u(t) \in \mathbb{R}^m$, the initial condition $\varphi(t)$ is a piecewise continuous function $\varphi(t)$: $[-kh, 0] \to \mathbb{R}^n$ ($k = \max\{k_a, k_b, k_c, k_d\}$); thereby $x(t) = \varphi(t)$ on $[-kh, 0]$. A_i, B_i, C_i, and D_i are matrices of appropriate dimension with entries in $\mathbb{R}$.

By using the delay operator (backward time-shift operator) δ, system (13.1) may be represented in the following compact form:

$$\begin{aligned} \dot{x}(t) &= A(\delta)x(t) + B(\delta)u(t) \\ y(t) &= C(\delta)x(t) + D(\delta)u(t) \end{aligned} \tag{13.2}$$

where $A(\delta)$, $B(\delta)$, $C(\delta)$, and $D(\delta)$ are matrices over the polynomial ring $\mathbb{R}[\delta]$, defined as $A(\delta) := \sum_{i=0}^{k_a} A_i\delta^i$, $B(\delta) := \sum_{i=0}^{k_b} B_i\delta^i$, $C(\delta) := \sum_{i=0}^{k_c} C_i\delta^i$, and $D(\delta) := \sum_{i=0}^{k_d} D_i\delta^i$. As for $x(t;\varphi,u)$, we mean the solution of the delay differential equation of system (13.1) with the initial condition equal to φ, and the input vector equal to w. Analogously, we define $y(t;\varphi,u) := C(\delta)x(t;\varphi,u) + D(\delta)u(t)$, that is, to be the system output of (13.1) when $x(t) = x(t;\varphi,u)$.

Practically, what we search for is to find out conditions allowing for the estimation of $x(t)$. To tackle the problem in a more formal way, we use the following observability definitions.

Definition 13.1 (Unknown Input Observability) *System (13.1) is called unknown input observable (**UIO**) on the interval $[t_1, t_2]$ if there exist t_1' and t_2' ($t_1' < t_2'$) such that, for all input u and every initial condition φ,*

$$y(t;\varphi,u) = 0 \text{ for all } t \in \left[t_1', t_2'\right] \text{ implies } x(t;\varphi,u) = 0 \text{ for } t \in [t_1, t_2]$$

Definition 13.2 (Backward UIO) *System (13.1) is said to be Backward UIO (**BUIO**) on $[t_1, t_2]$ if it is **UIO** with $t_1 > t_2'$.*

Definition 13.3 (Forward UIO) *System (13.1) is said to be Forward **UIO** (**FUIO**) on $[t_1, t_2]$ if it is **UIO** with $t_1' > t_2$.*

Remark 13.4 *These definitions are essentially formulated following the observability definitions given in [103] for linear systems. Basically, UIO considers the case when the state vector can be reconstructed using past, actual, and future values of the system output. As for BUIO, it is related with the case when only actual and past values of the system output are needed for the actual state reconstruction. Finally, FUIO defines a property which theoretically allows for the reconstruction of the actual state vector using only actual and future values of the system output.*

Obviously, either BUIO or FUIO implies UIO. It should be noted that BUIO and FUIO do not exclude each other. For instance, the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + \delta x_2; \quad y_1 = \delta x_1, \quad y_2 = x_2$$

is BUIO on $[t_1, t_1 + h]$ $(t_1 \geq h)$ since, for each $\tau \in [t_1, t_1 + h]$, $y(t) = 0$ on $[t_1 - h, \tau]$ implies $x(\tau) = 0$. Moreover, it is FUIO on $[t_1, t_1 + h]$, since, $y(t) = 0$ on $[\tau, t_1 + 2h]$ implies $x(\tau) = 0$.

13.3 Basic results

The study of the observability for linear systems (without delays) has been successfully tackled by using geometric methods, in particular invariant subspaces. For the time delay case such methods cannot be followed straightforwardly, but still many of those ideas can be borrowed (see [47], [48]).

Let $P(\delta)$ be a matrix of $q \times s$ dimension with rank equal to r (clearly $r \leq \min\{q, s\}$). We know there exists an invertible matrix $T(\delta)$ over $\mathbb{R}[\delta]$ (representing elementary row operations) such that $P(\delta)$ is put into (column) Hermite form. Thus, we have that

$$T(\delta) P(\delta) = \begin{bmatrix} P_1(\delta) \\ 0 \end{bmatrix}$$

where $P_1(\delta)$ is of $r \times s$ dimension, and $\mathrm{rank}_{\mathbb{R}[\delta]} P_1 = r$. Also, there exist two invertible matrices $U(\delta)$ and $V(\delta)$ over $\mathbb{R}[\delta]$ (representing elementary row and column operations, respectively) such that $P(\delta)$ is reduced to its Smith form, i.e.,

$$U(\delta) P(\delta) V(\delta) = \begin{bmatrix} \mathrm{diag}(\psi_1(\delta) \cdots \psi_r(\delta)) & 0 \\ 0 & 0 \end{bmatrix}$$

where the $\{\psi_i(\delta)\}$'s are monic nonzero polynomials satisfying

$$\psi_i(\delta) \,|\, \psi_{i+1}(\delta) \text{ and } d_i(\delta) = d_{i-1}(\delta)\, \psi_i(\delta)$$

where $d_i(\delta)$ is the gcd of all $i \times i$ minors of $P(\delta)$ $(d_0 = 1)$. The $\{\psi_i(\delta)\}$'s are called invariant factors, and the $\{d_i(\delta)\}$'s determinant divisors.

Following the ideas of [170] and [140], let us define $\{\Delta_k(\delta)\}$ matrices generated by the following algorithm,

$$\begin{gathered}\Delta_0 \triangleq 0,\ G_0(\delta) \triangleq C(\delta),\ F_0(\delta) \triangleq D(\delta)\\ S_k(\delta) \triangleq \begin{bmatrix} \Delta_k(\delta)B(\delta)\\ F_k(\delta)\end{bmatrix},\quad k \geq 0\\ \begin{bmatrix} F_{k+1}(\delta) & G_{k+1}(\delta)\\ 0 & \Delta_{k+1}(\delta)\end{bmatrix} \triangleq T_k(\delta)\begin{bmatrix} \Delta_k(\delta)B(\delta) & \Delta_k(\delta)A(\delta)\\ F_k(\delta) & G_k(\delta)\end{bmatrix}\end{gathered} \tag{13.3}$$

where $T_k(\delta)$ is an invertible matrix over $\mathbb{R}[\delta]$ that transforms S_k into its Hermite form, and Δ_0 is of dimension 1 by n. Then, $\{M_k(\delta)\}$ matrices are defined as follows,

$$\begin{gathered}M_0(\delta) \triangleq N_0(\delta) \triangleq \Delta_0,\ N_{k+1}(\delta) \triangleq \begin{bmatrix} N_k(\delta)\\ \Delta_{k+1}(\delta)\end{bmatrix},\ \text{for } k \geq 0\\ \begin{bmatrix} M_{k+1}(\delta)\\ 0\end{bmatrix} \triangleq \begin{bmatrix} \operatorname{diag}\left(\psi_1^{k+1}(\delta),\ldots,\psi_{r_{k+1}}^{k+1}(\delta)\right) & 0\\ 0 & 0\end{bmatrix}\\ = U_{k+1}(\delta)N_{k+1}(\delta)V_{k+1}(\delta)\end{gathered} \tag{13.4}$$

with $U_k(\delta)$ and $V_k(\delta)$ being invertible matrices over $\mathbb{R}[\delta]$ that transform N_k to its Smith form. It is worth noting that, by construction, $F_k(\delta)$ and $M_k(\delta)$ matrices have both full row rank, and $M_k(\delta)$ has always n columns.

It is intuitively clear that since $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$ are invertible over $\mathbb{R}[\delta]$, then the invariant factors of $\{N_k\}$, and so $\{M_k\}$ matrices, should not depend on the particular selection of those former invertible matrices, which was guaranteed by the following lemma.

Lemma 13.5 *$\{M_k\}$ matrices given by (13.3)-(13.4) are independent of the choice of $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$.*

Lemma 13.6 *By using the notation $d_j^k(\delta)$ as the j-th determinant divisor of $M_k(\delta)$ (generated by (13.3)-(13.4)), we obtain $d_j^{k+1}(\delta) \mid d_j^k(\delta)$, for every $j \leq \operatorname{rank}_{\mathbb{R}[\delta]} M_k(\delta)$.*

Lemma 13.7 *$M_{k+1}(\delta) = M_k(\delta)$ if, and only if, $\Delta_{k+1}(\delta) = P(\delta)N_k(\delta)$ for some matrix $P(\delta)$.*

Theorem 13.8 *If $M_{k+1}(\delta) = M_k(\delta)$, then $M_{k+i}(\delta) = M_k(\delta)$ for all $i \geq 0$.*

Theorem 13.9 *After a finite number of steps, let's say k^*, the algorithm (13.3)-(13.4) converges, i.e., there exists a least integer k^* such that $M_{k^*+1}(\delta) = M_{k^*}(\delta)$. Furthermore, k^* is independent of the choice of $\{T_k\}$, $\{U_k\}$, and $\{V_k\}$ matrices used in (13.3)-(13.4).*

Next corollary is a direct consequence of the previous results.

Corollary 13.10 *Let k^* be the least integer k such that $M_{k+1} = M_k$, then for all $i \geq 0$, $M_{k^*+i} = M_{k^*}$.*

13.4 State reconstruction

From now it is possible to give sufficient observability conditions. In the way to arrive to such conditions we will draw the connection of the recursive algorithm used to obtain $\Delta_k(\delta)$ and a way that may be used for the reconstruction of the state vector. Let us define $\hat{y}^1(t) = T_0(\delta)\begin{bmatrix} 0 \\ y(t) \end{bmatrix}$. Thus, from (13.2) and (13.3) we obtain

$$\hat{y}^1(t) \triangleq \begin{bmatrix} \hat{y}_1^1(t) \\ \hat{y}_2^1(t) \end{bmatrix} = \begin{bmatrix} G_1(\delta)x(t) + F_1(\delta)u(t) \\ \Delta_1(\delta)x(t) \end{bmatrix} \tag{13.5}$$

From here, we will define a chain of vectors $\left\{\hat{y}^i(t)\right\}$. Thus, $\hat{y}_1^i(t)$ and $\hat{y}_2^i(t)$ will be the upper and lower subvectors of $\hat{y}^i(t)$, respectively, whose dimension will be implicitly defined.

We have already obtained a *virtual* output $\hat{y}_2^1(t)$ without the influence of the unknown input $u(t)$. Let us define $\alpha_1 = \deg \Delta_1(\delta)$, then, for $t \geq \alpha_1 h$, $\hat{y}_2^1(t)$ is differentiable. Hence, according to (13.5) and (13.2), we obtain the following equation for $t \geq \alpha_1$,

$$\frac{d}{dt}\hat{y}_2^1(t) = \frac{d}{dt}\Delta_1(\delta)x(t) = \Delta_1(\delta)A(\delta)x(t) + \Delta_1(\delta)B(\delta)u(t) \tag{13.6}$$

Now, let us generate an extended vector $\xi_1(t) \triangleq \begin{bmatrix} \frac{d}{dt}\hat{y}_2^1(t) \\ \hat{y}_1^1(t) \end{bmatrix}$. Thus, again, we define $\hat{y}^2(t) = T_1(\delta)\xi_1(t)$, $t \geq \alpha_1 h$. In view of (13.3) and (13.6), we obtain the following identity,

$$\hat{y}^2(t) \triangleq \begin{bmatrix} \hat{y}_1^2(t) \\ \hat{y}_2^2(t) \end{bmatrix} = \begin{bmatrix} G_2(\delta)x(t) + F_2(\delta)u(t) \\ \Delta_2(\delta)x(t) \end{bmatrix}$$

Differentiation of $\hat{y}_2^2(t)$ gives, for $t \geq h\max(\alpha_1, \alpha_2)$ (where $\alpha_2 = \deg \Delta_2(\delta)$),

$$\frac{d}{dt}\hat{y}_2^2(t) = \Delta_2(\delta)A(\delta)x(t) + \Delta_2(\delta)B(\delta)u(t)$$

Likewise, we may define a second extended vector $\xi_2(t) \triangleq \begin{bmatrix} \frac{d}{dt}\hat{y}_2^2(t) \\ \hat{y}_1^2(t) \end{bmatrix}$. After defining $\hat{y}_3(t) \triangleq T_2(\delta)\xi_2(t)$, we obtain the identities $\hat{y}_1^3(t) = G_3(\delta)x(t) + F_3(\delta)u(t)$ and $\hat{y}_2^3(t) = \Delta_3(\delta)x(t)$.

The previous procedure allows us for writing the following expressions, for $k \geq 1$ and $t \geq h \max_{1 \leq i \leq k-1} \deg \Delta_i(\delta)$,

$$\begin{aligned} \xi_0(t) &\triangleq \begin{bmatrix} 0 \\ y(t) \end{bmatrix}, \xi_k(t) \triangleq \begin{bmatrix} \frac{d}{dt}\hat{y}_2^k(t) \\ \hat{y}_1^k(t) \end{bmatrix} \\ \hat{y}^k(t) &\triangleq \begin{bmatrix} \hat{y}_1^k(t) \\ \hat{y}_2^k(t) \end{bmatrix} \triangleq T_{k-1}(\delta)\,\xi_{k-1}(t) \end{aligned} \tag{13.7}$$

Thus, we obtain the identities

$$\Delta_k(\delta)\,x(t) = \hat{y}_2^k(t),\ k \geq 1 \tag{13.8}$$

Hence, we define $Y(t)$ as

$$Y(t) \triangleq \begin{bmatrix} \hat{y}_2^1(t) \\ \hat{y}_2^2(t) \\ \vdots \\ \hat{y}_2^{k^*}(t) \end{bmatrix}$$

With the definition $t^* = h \max_{1 \leq i \leq k^*-1} \deg \Delta_i(\delta)$, by (13.8) we have that

$$N_{k^*}(\delta)\,x(t) = Y(t) \text{ for all } t \geq t^* \tag{13.9}$$

If we enlarge the case by allowing forward time-shift operator, then the following proposition is an obvious consequence of properties of $\mathbb{R}\left[\delta, \delta^{-1}\right]$.

Proposition 13.11 *$M_{k^*}(\delta)$ matrix has an inverse on $\mathbb{R}\left[\delta, \delta^{-1}\right]$ if, and only if, $M_{k^*}(\delta)$ has n invariant factors, all of them of the form $a\delta^j$, with $a \in \mathbb{R}_{\neq 0}$ and $j \in \mathbb{N}_0$.*

Let us define t_1^* as follows

$$t_1^* = h \times \deg\left(\begin{bmatrix} V_{k^*}M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*}\right) + t^* \tag{13.10}$$

Theorem 13.12 *The vector $x(t)$ can be reconstructed in finite time, for any $t > t_1^*$, if $M_{k^*}(\delta)$ has n invariant factors of the form $a\delta^j$, where $a \in \mathbb{R}_{\neq 0}$, $j \in \mathbb{N}_0$. The formula to reconstruct $x(t)$ is*

$$x(t) = \begin{bmatrix} V_{k^*}M_{k^*}^{-1} & 0 \end{bmatrix} U_{k^*}Y(t),\ t > t_1^* \tag{13.11}$$

Furthermore, the i-th entry of $x(t)$ is given by an expression of the form:

$$x_i(t) = \sum_{k,j} q_{k,j}\, y_k^{(j)}(t) \tag{13.12}$$

where $y_k^{(i)}(t)$ is the i-th derivative of the k-th entry of $y(t)$ and $0 \neq q_{j,k}^i \in \mathbb{R}\left[\delta, \delta^{-1}\right]$.

Remark 13.13 *For the trivial case $h = 0$, the condition that M_{k^*} is invertible over $\mathbb{R}$ is also a necessary condition for the system to be UIO (known it as strong observability, see, e.g. [140] and [183]).*

Remark 13.14 *For the case when $B(\delta) = 0$ and $D(\delta) = 0$, the condition that M_{k^*} is invertible over $\mathbb{R}[\delta]$ is equivalent to $\left((C(\delta))^T, (C(\delta)A(\delta))^T, \cdots, \left(C(\delta)A^{n-1}(\delta)\right)^T\right)^T$ is left invertible over $\mathbb{R}[\delta]$. This condition is known as strong observability also (see, [117]).*

Thereby, we suggest a definition of strong observability (as a generalization) for the systems considered in this work. We consider $\mathbb{R}\left[\delta, \delta^{-1}\right]$ as the ring over which the matrix (given below) may be invertible, this allows to have a less restrictive characterization of the observability.

Definition 13.15 *System (13.1) is said to be strongly observable (SO) if, and only if, M_{k^*} is invertible over $\mathbb{R}\left[\delta, \delta^{-1}\right]$, i.e. iff M_{k^*} has n invariant factors all of them of the form $a\delta^j$ ($a \in \mathbb{R}_{\neq 0}$, $j \in \mathbb{N}_0$).*

Now, we can deduce easily sufficient conditions for the UIO, BUIO and FUIO.

Corollary 13.16 *If system (13.1) is SO then it is UIO on $[t_1^*, t_2]$, for all $t_2 > t_1^*$.*

Corollary 13.17 *If system (13.1) is SO and, for all $i \in \overline{1, n}$, every polynomial $q_{j,i}$ of (13.12) belongs to $\mathbb{R}[\delta]$, then (13.1) is BUIO on $[t_1^*, t_2]$, for every $t_2 > t_1^*$.*

Proposition 13.18 *Let us assume that system (13.1) is SO. Then every polynomial $q_{j,i}$ in (13.12) belongs to $\mathbb{R}[\delta]$ if, and only if, $\det M_{k^*} = 1$.*

Corollary 13.19 *Assume that system (13.1) is SO. Then it is BUIO on $[t_1^*, t_2]$, for all $t_2 > t_1^*$, if, and only if, $\det M_{k^*} = 1$.*

As for FUIO, we have the following corollary which characterizes it, provided (13.1) is SO.

Corollary 13.20 *If system (13.1) is SO and, for all $i \in \overline{1, n}$, every polynomial $q_{j,i}$ of (13.12) belongs to $\mathbb{R}\left[\delta^{-1}\right]$, then (13.1) is FUIO on $[t_1^*, t_2]$, for all $t_2 > t_1^*$.*

13.5 Examples

Example 13.21 *Let us consider the following example:*

$$A(\delta) = \begin{pmatrix} 1 & \delta & \delta & 0 & 0 \\ -\delta^2 & 0 & \delta & 0 & -\delta \\ \delta & 1 & -\delta^2 & -1+\delta & 1-\delta+\delta^3 \\ 0 & 0 & -1 & 0 & 0 \\ \delta-\delta^2 & 0 & -1+\delta & 2 & 1-\delta \end{pmatrix}, \quad B(\delta) = \begin{pmatrix} -\delta & 0 \\ 0 & -\delta \\ 1+\delta & 1 \\ 1 & 0 \\ 0 & 1-\delta \end{pmatrix}$$

$$C(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \quad D(\delta) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1+\delta & 0 \end{pmatrix}$$

According to (13.3) and (13.4), we have that

$$T_0=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&1-\delta&1\end{bmatrix},\ \Delta_1=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&1-\delta&0&0&\delta\end{bmatrix},M_1=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\end{bmatrix}$$

Following with the described procedure we obtain the matrices

$$T_1=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&\delta\end{bmatrix},\Delta_2=\begin{bmatrix}0&0&0&0&0\\0&0&0&2\delta&0\\1&2\delta&\delta&0&0\end{bmatrix},M_2=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&\delta&0&0\\0&0&0&\delta&0\end{bmatrix}$$

and

$$T_2=\begin{bmatrix}0&0&0&1\\0&0&-\frac{1}{2}&\frac{1}{2}\delta^2\\0&\frac{1}{2}&0&-\delta\\1&0&0&0\end{bmatrix},\Delta_3=\begin{bmatrix}0&-\delta&-\delta&0&0\\0&0&0&0&0\end{bmatrix},M_3=M_4=\operatorname{diag}\left(1,1,\delta,\delta,\delta^2\right)$$

Thus, in this case $k^*=3$, and the system is UIO since all the invariant factors of M_3 belong to $\mathbb{R}\left[\delta,\delta^{-1}\right]$. Explicitly, we have that the state vector can be expressed as

$$\begin{aligned}
x_1&=y_1,\ x_2=-\delta^{-1}y_1+\delta^{-1}\dot{y}_1+\frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2-\frac{1}{2}\delta^{-1}\dot{y}_3\\
x_3&=\delta^{-1}y_1+y_2-\delta^{-1}\dot{y}_1-\frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2+\frac{1}{2}\delta^{-1}\dot{y}_3-\frac{1}{2}\left(1-\delta^{-1}\right)\ddot{y}_2+\frac{1}{2}\delta^{-1}\ddot{y}_3\\
x_4&=\frac{1}{2}\left(1-\delta^{-1}\right)\dot{y}_2-\frac{1}{2}\delta^{-1}\dot{y}_3\\
x_5&=\left(\delta^{-2}-\delta^{-1}\right)y_1+\left(\delta^{-1}-1\right)y_2+\delta^{-1}y_3+\left(\delta^{-1}-\delta^{-2}\right)\dot{y}_1+\frac{1}{2}\left(1-\delta^{-1}\right)^2\dot{y}_2\\
&\quad+\frac{1}{2}\left(\delta^{-2}-\delta^{-1}\right)\dot{y}_3
\end{aligned}$$

Therefore, according to Corollary 13.20, the system is FUIO.

Example 13.22 *Now, let us consider that the matrices of the system (13.2) are the following,*

$$A=\begin{pmatrix}0&-1&1\\-1&\delta&0\\1&0&1\end{pmatrix},\ B=\begin{pmatrix}0&0\\0&0\\0&\delta\end{pmatrix},C=\begin{pmatrix}\delta&0&0\\0&1&0\end{pmatrix},\ D=\begin{pmatrix}-2\delta&0\\0&0\end{pmatrix}$$

In this example, we have that M_1 has an invariant factor equal to 1, M_2 has the invariant factors $\{1,1\}$, and the invariant factors of M_3 are $\{1,1,1\}$. Hence the system is SO and furthermore, it is BUIO also, that is the state vector can be reconstructed using actual and past values of the system output. Indeed, it is easy to verify that the state variables can be *expressed as*

$$\begin{aligned} x_1(t) &= -\dot{y}_2(t) + \delta y_2(t), \;\; x_2(t) = y_2(t) \\ x_3(t) &= -\ddot{y}_2(t) + \delta \dot{y}_2(t) + y_2(t) \end{aligned}$$

Example 13.23 *Finally let us consider the following example:*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \delta & 1 \\ \delta & 0 & 0 \end{bmatrix}, B = 0, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ \delta \end{bmatrix}$$

In this case $M_3 = \mathrm{diag}(1, \delta, \delta)$*, so the system is UIO, however, it is neither BUIO nor FUIO. As we can see, x depends on past, actual, and future values of the system output:*

$$x_1 = -y_1 + \delta^{-1}\dot{y}_2, \;\; x_2 = -\dot{y}_1 + \delta^{-1}\ddot{y}_2, \;\; \textit{and} \;\; x_3 = -\delta y_1 + \dot{y}_2$$

13.6 Conclusion

We have tackled the observability of linear commensurable time-delay systems with unknown input using three different definitions. Essentially, the first definition (UIO) deals with the possibility of reconstructing the state vector using output information (using past, actual, and/or future values), the second definition (BUIO) is related with the state reconstruction using just actual and past output information. And the third definition (FUIO) is about the state reconstruction using future values of the system output. We have given sufficient conditions allowing for the system to be UIO, BUIO, or FUIO, respectively. As for the conditions given for the UIO we have seen that the condition obtained includes the already known conditions for systems with delays without unknown inputs and for the case of linear systems with unknown inputs without delays. However, when treating the observability problem for nonlinear time-delay system with unknown input, the generalization of the current result is not trivial, since, as we can see in the next chapter, the polynomial ring is no longer commutative.

Chapter 14

Nonlinear Time-Delay System

This chapter will analyze the observability for nonlinear time-delay system with unknown input. Moreover, concerning nonlinear time-delay system, since many results have been reported for the purpose of stability and observability analysis, by assuming that the delay of the studied systems is known, it makes the delay identification be one of the most important topics in the field of time-delay systems, and this issue will be studied as well in this chapter.

Our analysis will be still based on the theory of ring, which however becomes non-commutative when treating nonlinear time-delay system. Therefore, we will adopt the methodology proposed in [141] to study those systems. The presented results in this chapter have been published in [J13, J18, C30, C35, C39]. As usual, the following will recall some basic notations of the theory of non-commutative rings.

14.1 Notations

Consider the following nonlinear time-delay system with commensurate delay:

$$
\begin{cases}
\dot{x} = f(x(t-i\tau)) + \sum_{j=0}^{s} g^j(x(t-i\tau))u(t-j\tau), \\
y = h(x(t-i\tau)) = [h_1(x(t-i\tau)), \ldots, h_p(x(t-i\tau))]^T, \\
x(t) = \psi(t),\ u(t) = \varphi(t),\ t \in [-s\tau, 0],
\end{cases}
\tag{14.1}
$$

where the constant delays $i\tau$ are associated to the finite set of integers $i \in S_- = \{0, 1, \ldots, s\}$; $x \in W \subset \mathbb{R}^n$ refers to the state variables; $u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$ is the unknown input; $y \in \mathbb{R}^p$ is the measurable output; f, g^j and h are meromorphic functions[1]; $f(x(t-i\tau)) = f(x, x(t-\tau), \ldots, x(t-s\tau))$; $\psi : [-s\tau, 0] \to \mathbb{R}^n$ and $\varphi : [-s\tau, 0] \to \mathbb{R}^m$ denote unknown continuous functions of initial conditions. It is assumed that, for initial conditions ψ and φ, system (14.1) admits a unique solution.

Denote $\mathscr{K}$ as the field of meromorphic functions of a finite number of the variables from $\{x_j(t-i\tau), j \in [1, n], i \in S_-\}$. Like the linear time-delay system, we can introduce the delay operator δ, which means, for $i \in \mathbb{Z}^+$:

$$
\delta^i \xi(t) = \xi(t-i\tau),\ \xi(t) \in \mathscr{K},
\tag{14.2}
$$

[1] means quotients of convergent power series with real coefficients [46, 188].

$$\delta^i(a(t)\xi(t)) = \delta^i a(t)\delta^i\xi(t) = a(t-i\tau)\xi(t-i\tau). \tag{14.3}$$

Then, denote $\mathscr{K}(\delta]$ as the set of polynomials in δ over $\mathscr{K}$, of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{14.4}$$

where $a_i(t) \in \mathscr{K}$ and $r_a \in \mathbb{Z}^+$. The addition in $\mathscr{K}(\delta]$ is defined as usual, but the multiplication is given as:

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i\le r_a, j\le r_b} a_i(t)b_j(t-i\tau)\delta^k. \tag{14.5}$$

Considering (14.1) without input, differentiation of an output component $h_j(x(t-i\tau))$ with respect to t is defined as follows:

$$\dot{h}_j(x(t-i\tau)) = \sum_{i=0}^{s} \frac{\partial h_j}{\partial x(t-i\tau)}\delta^i f.$$

Thanks to the definition of $\mathscr{K}(\delta]$, (14.1) can be rewritten in a more compact form:

$$\begin{cases} \dot{x} = f(x,\delta) + G(x,\delta)u = f(x,\delta) + \sum_{i=1}^{m} G_i(x,\delta)u_i(t) \\ y = h(x,\delta) \\ x(t) = \psi(t),\ u(t) = \varphi(t),\ t \in [-s\tau, 0], \end{cases} \tag{14.6}$$

where $f(x,\delta) = f(x(t-i\tau))$ and $h(x,\delta) = h(x(t-i\tau))$, with entries belonging to $\mathscr{K}$, $u = u(t)$, and $G(x,\delta) = [G_1,\cdots,G_m]$ with $G_i(x,\delta) = \sum_{l=0}^{s} g_i^l \delta^l$.

With the standard differential operator d, denote by $\mathscr{M}$ the left module over $\mathscr{K}(\delta]$:

$$\mathscr{M} = span_{\mathscr{K}(\delta]}\{d\xi,\ \xi \in \mathscr{K}\} \tag{14.7}$$

where $\mathscr{K}(\delta]$ acts on $d\xi$ according to (14.2) and (14.3).

Unlike the polynomial ring defined in the last chapter for linear time-delay system, the polynomial ring $\mathscr{K}(\delta]$ is non-commutative, however it is proved that it is a left Ore ring [97, 188], which enables us to define the rank of a left module over $\mathscr{K}(\delta]$.

Define the vector space $\mathscr{E}$ over $\mathscr{K}$:

$$\mathscr{E} = span_{\mathscr{K}}\{d\xi : \xi \in \mathscr{K}\}$$

$\mathscr{E}$ is the set of linear combinations of a finite number of elements from $dx_j(t-i\tau)$ with row vector coefficients in $\mathscr{K}$. Since the delay operator δ and the standard differential operator are commutative, the one-form of $\omega \in \mathscr{M}$ can be written as: $\omega = \sum_{j=1}^{n} a_j(\delta]dx_j$, where $a(\delta] \in \mathscr{K}(\delta]$. For a given vector field $\beta = \sum_{j=1}^{n} b_j(\delta]\frac{\partial}{\partial x_j}$ with $b_j(\delta] \in \mathscr{K}(\delta]$, the inner product of ω and β is defined as follows:

$$\omega\beta = \sum_{j=1}^{n} a_j(\delta]b_j(\delta] \in \mathscr{K}(\delta].$$

For $0 \leq j \leq s$, let $f(x(t-j\tau))$ and $h(x(t-j\tau))$ respectively be an n and p dimensional vector with entries $f_r \in \mathscr{K}$ for $1 \leq r \leq n$ and $h_i \in \mathscr{K}$ for $1 \leq i \leq p$. Let

$$\frac{\partial h_i}{\partial x} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathscr{K}^{1\times n}(\delta], \tag{14.8}$$

where, for $1 \leq r \leq n$:

$$\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j \in \mathscr{K}(\delta].$$

Then, the Lie derivative for nonlinear systems without delays can be extended to nonlinear time-delay systems in the framework of [188] as follows

$$L_f h_i = \frac{\partial h_i}{\partial x}(f) = \sum_{r=1}^{n} \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j (f_r) \tag{14.9}$$

and in the same way one can define $L_{G_i} h_i$.

14.2 Definitions and preliminary result

Based on the above notations, the following definitions are given:

Definition 14.1 *(Change of coordinates) [135] $z = \phi(\delta, x) \in \mathscr{K}^{n\times 1}$ is a causal change of coordinates over $\mathscr{K}$ for the system (14.1) if there locally exist a function $\phi^{-1} \in \mathscr{K}^{n\times 1}$ and some constants $c_1, \cdots, c_n \in N$ such that $diag\{\delta^{c_i}\}x = \phi^{-1}(\delta, z)$. The change of coordinates is bicausal over $\mathscr{K}$ if $max\{c_i\} = 0$, that is $x = \phi^{-1}(\delta, z)$.*

Definition 14.2 *(Relative degree) System (14.6) has the relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq \mathbb{R}^n$ if the following conditions are satisfied for $1 \leq i \leq p$:*

1. *for all $x \in W$, $L_{G_j} L_f^r h_i(x) = 0$ for all $1 \leq j \leq m$ and $0 \leq r < \nu_i - 1$;*
2. *there exists $x \in W$ such that $\exists j \in \{1, \cdots, m\}$, $L_{G_j} L_f^{\nu_i - 1} h_i(x) \neq 0$.*

If the first condition is satisfied for all $r \geq 0$ and some $i \in \{1, \cdots, p\}$, we set $\nu_i = \infty$.

Moreover, for system (14.6), one can also define observability indices introduced in [114] over non-commutative rings. For $1 \leq k \leq n$, let $\mathscr{F}_k$ be the following left module over $\mathscr{K}(\delta]$:

$$\mathscr{F}_k := span_{\mathscr{K}(\delta]} \left\{dh, dL_f h, \cdots, dL_f^{k-1} h\right\}.$$

It was shown that the filtration of $\mathscr{K}(\delta]$-module satisfies $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n$, then define $d_1 = rank_{\mathscr{K}(\delta]} \mathscr{F}_1$, and $d_k = rank_{\mathscr{K}(\delta]} \mathscr{F}_k - rank_{\mathscr{K}(\delta]} \mathscr{F}_{k-1}$ for $2 \leq k \leq n$. Let $k_i = card\{d_k \geq i, 1 \leq k \leq n\}$, then $(k_1, \cdots, k_p)$ are the observability indices. Reorder, if necessary, the output components of (14.6) so that

$$rank_{\mathscr{K}(\delta]} \{\frac{\partial h_1}{\partial x}, \cdots, \frac{\partial L_f^{k_1 - 1} h_1}{\partial x}, \cdots, \frac{\partial h_p}{\partial x}, \cdots, \frac{\partial L_f^{k_p - 1} h_p}{\partial x}\} = k_1 + \cdots + k_p.$$

Based on the above definitions, let us define the following notations, which will be used in the sequel. For $1 \leq i \leq p$, denote by k_i the observability indices, ν_i the relative degree for y_i of (14.6), and

$$\rho_i = \min\{\nu_i, k_i\}.$$

Without loss of generality, suppose $\sum_{i=1}^{p} \rho_i = j$, thus $\{dh_1, \cdots, dL_f^{\rho_1 - 1}h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p - 1}h_p\}$ are j linearly independent vectors over $\mathscr{K}(\delta]$. Then note:

$$\Phi = \{dh_1, \cdots, dL_f^{\rho_1 - 1}h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p - 1}h_p\} \tag{14.10}$$

and

$$\pounds = span_{\mathbb{R}[\delta]}\left\{h_1, \cdots, L_f^{\rho_1 - 1}h_1, \cdots, h_p, \cdots, L_f^{\rho_p - 1}h_p\right\}, \tag{14.11}$$

Let $\pounds(\delta]$ be the set of polynomials in δ with coefficients over $\pounds$. The module spanned by element of Φ over $\pounds(\delta]$ is defined as follows:

$$\Omega = span_{\pounds(\delta]}\{\xi, \xi \in \Phi\}. \tag{14.12}$$

Define

$$\mathscr{G} = span_{\mathbb{R}[\delta]}\{G_1, \ldots, G_m\},$$

where G_i is given in (14.6), and its *left* annihilator:

$$\mathscr{G}^{\perp} = span_{\pounds(\delta]}\{\omega \in \mathscr{M} \mid \omega\beta = 0, \forall \beta \in \mathscr{G}\}, \tag{14.13}$$

where $\mathscr{M}$ is defined in (14.7).

After having defined the relative degree and observability indices via the extended Lie derivative for nonlinear time-delay systems in the framework of non-commutative rings, now an observable canonical form can be derived.

Theorem 14.3 *Consider the system (14.6) with outputs $(y_1, \cdots, y_p)$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. There exists a change of coordinates $\phi(x, \delta) \in \mathscr{K}^{n \times 1}$, such that (14.6) is transformed into the following form:*

$$\dot{z}_{i,j} = z_{i,j+1} \tag{14.14}$$

$$\dot{z}_{i,\rho_i} = V_i(x, \delta) = L_f^{\rho_i}h_i(x, \delta) + \sum_{j=1}^{m} L_{G_j}L_f^{\rho_i - 1}h_i(x, \delta)u_j \tag{14.15}$$

$$y_i = C_i z_i = z_{i,1} \tag{14.16}$$

$$\dot{\xi} = \alpha(z, \xi, \delta) + \beta(z, \xi, \delta)u \tag{14.17}$$

where $z_i = \left(z_{i,1}, \cdots, z_{i,\rho_i}\right)^T = \left(h_i, \cdots, L_f^{\rho_i - 1}h_i\right)^T \in \mathscr{K}^{\rho_i \times 1}$, $\alpha \in \mathscr{K}^{\mu \times 1}, \beta \in \mathscr{K}^{\mu \times 1}(\delta]$ with $\mu = n - \sum_{j=1}^{p} \rho_j$ and $C_i = (1, 0, \cdots, 0) \in \mathbb{R}^{1 \times \rho_i}$. Moreover, if $k_i < \nu_i$, one has $V_i(x, \delta) = L_f^{\rho_i}h_i = L_f^{k_i}h_i$.

Based on Theorem 14.3, noting $\rho_i = \min\{\nu_i, k_i\}$ for $1 \le i \le p$ where the k_i represent the observability indices and ν_i stands for the relative degree of y_i for (14.6), the following equality can be derived:

$$\mathscr{H}(x,\delta) = \Psi(x,\delta) + \Gamma(x,\delta)u, \tag{14.18}$$

with

$$\mathscr{H}(x,\delta) = \left(h_1^{(\rho_1)}, \cdots, h_p^{(\rho_p)}\right)^T, \ \Psi(x,\delta) = \left(L_f^{\rho_1}h_1, \cdots, L_f^{\rho_p}h_p\right)^T,$$

and

$$\Gamma(x,\delta) = \begin{pmatrix} L_{G_1}L_f^{\rho_1-1}h_1 & \cdots & L_{G_m}L_f^{\rho_1-1}h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1}L_f^{\rho_p-1}h_p & \cdots & L_{G_m}L_f^{\rho_p-1}h_p \end{pmatrix}, \tag{14.19}$$

where $\mathscr{H}(x,\delta) \in \mathscr{K}^{p\times 1}$, $\Psi(x,\delta) \in \mathscr{K}^{p\times 1}$ and $\Gamma(x,\delta) \in \mathscr{K}^{p\times m}(\delta]$. Assume that $rank_{\mathscr{K}(\delta]}\Gamma = m$. Since $\Gamma \in \mathscr{K}^{p\times m}(\delta]$ with $m \le p$, according to Lemma 4 in [134], there exists a matrix $\Xi \in \mathscr{K}^{p\times p}(\delta]$ such that:

$$\Xi\Gamma = \left[\ \bar{\Gamma}^T, \mathbf{0}\ \right]^T, \tag{14.20}$$

where $\bar{\Gamma} \in \mathscr{K}^{m\times m}(\delta]$ has full rank m. With the compact equation (14.18), identifiability and observability will be analyzed separately in Sections 14.3 and 14.4.

14.3 Identifiability

In order to study the delay identifiability of (14.6), let us firstly introduce the following definition of the identifiability of time delay, which is an adaptation of Definition 2 in [3].

Definition 14.4 *For system (14.6), an equation with delays, containing only the output and a finite number of its derivatives:*

$$\alpha(h, \dot{h} \ldots, h^{(k)}, \delta) = 0, k \in \mathbb{Z}^+$$

is said to be an output delay equation *(of order k). Moreover, this equation is said to be an* output delay-identifiable equation *for (14.6) if it cannot be written as $\alpha(h, \dot{h} \ldots, h^{(k)}, \delta) = a(\delta]\tilde{\alpha}(h, \dot{h} \ldots, h^{(k)})$ with $a(\delta] \in \mathscr{K}(\delta]$.*

As stated in [3], if there exists an output delay-identifiable equation for (14.6) (i.e. involving the delay in an essential way), then the delay can be identified for almost all y by numerically finding zeros of such an equation. For this issue, the interested reader can refer to [9] and the references therein. Thus, delay identification for (14.6) boils down to the research of such an output delay equation.

14.3.1 Dependent outputs over $\mathscr{K}(\delta]$

Let us firstly consider the simplest case for identifying the delay for (14.6), i.e., from only the outputs of (14.6), which is stated in the following result.

Theorem 14.5 *There exists an output delay-identifiable equation (of order 0) $\alpha(h,\delta)$ for (14.6) if and only if*

$$rank_{\mathscr{K}(\delta]}\frac{\partial h}{\partial x} < rank_{\mathscr{K}}\frac{\partial h}{\partial x}. \tag{14.21}$$

Example 14.6 *Consider the following dynamical system:*

$$\begin{cases} \dot{x} = f(x,u,\delta), \\ y_1 = x_1, \\ y_2 = x_1\delta x_1 + x_1^2. \end{cases} \tag{14.22}$$

It can be seen that

$$\frac{\partial h}{\partial x} = \begin{pmatrix} 1, & 0 \\ \delta x_1 + 2x_1 + x_1\delta, & 0 \end{pmatrix}$$

which yields $rank_{\mathscr{K}(\delta]}\frac{\partial h}{\partial x} = 1$ and $rank_{\mathscr{K}}\frac{\partial h}{\partial x} = 2$. Thus Theorem 14.5 is satisfied, and the delay of system (14.22) can be identified.

In fact, a straightforward calculation gives $y_2 = y_1\delta y_1 + y_1^2$, which permits to identify the delay δ by applying an algorithm to detect zero-crossing when varying δ.

Inequality (14.21) implies that the outputs of (14.6) are dependent over $\mathscr{K}(\delta]$. Theorem 14.5 can be seen as a special case of Theorem 2 in [3]. However, as it will be shown in the next section, this condition is not necessary for the case where the output of (14.6) is independent over $\mathscr{K}(\delta]$.

14.3.2 Independent outputs over $\mathscr{K}(\delta]$

Theorem 14.5 has analyzed the case where the outputs of (14.6) are dependent over $\mathscr{K}(\delta]$. In the contrary case (independence over $\mathscr{K}(\delta]$), the dynamics of system (14.6) have to be involved in order to deduce some output delay equations, which might be used to identify the delay. In the following, it will be firstly given the sufficient condition for the existence of a delay output equation for system (14.6) when the output is independent over $\mathscr{K}(\delta]$. Then a necessary and sufficient condition will be provided.

For this, denote $Q = [q_1, \cdots, q_p]$ as $1\times p$ vector with $q_j \in \mathscr{K}(\delta]$ for $1 \le j \le p$. According to (14.18), one has:

$$Q\mathscr{H} = Q(\Psi + \Gamma u) \tag{14.23}$$

where $\mathscr{H} = \left[y_1^{(\rho_1)}, \cdots, y_p^{(\rho_p)}\right]^T$.

In what follows, we will give sufficient conditions yielding $Q\Gamma = 0$, which implies that

$$Q(\mathscr{H} - \Psi) = 0 \tag{14.24}$$

is exactly the output delay equation, since it contains only the output, its derivatives and delays.

Theorem 14.7 *There exists an output delay equation for (14.6), if there exists a non zero $\omega = \sum_{c=1}^{n}\sum_{j=1}^{p} q_j \frac{\partial L_f^{\rho_j - 1} h_j}{\partial x_c} dx_c$, with $q_j \in \mathscr{K}(\delta]$ for $1 \le j \le p$, such that $\omega \in \mathscr{G}^{\perp} \cap \Omega$ and $\omega f \in \pounds$, where $\mathscr{G}^{\perp}$ is defined in (14.13), Ω in (14.12), and $\pounds$ in (14.11).*

If, in addition, the deduced output delay equation (14.24) is an output delay-identifiable equation, i.e. containing the delay δ in an essential way, then the delay of (14.6) can be identified (at least locally) by detecting zero-crossing of (14.24). The following will give necessary and sufficient conditions guaranteeing the essential involvement of δ in (14.24). But before this, let us define:

$$\mathscr{Y} = \left(h_1, \ldots, L_f^{\rho_1 - 1} h_1, \ldots, h_p, \ldots, L_f^{\rho_p - 1} h_p\right)^T,$$

and denote by $\mathscr{K}_0 \subset \mathscr{K}$ the field of meromorphic functions of x, which will be used in the following theorem (also involving Ψ defined in (14.18)).

Theorem 14.8 *The output delay equation (14.24) is an output delay-identifiable equation if and only if*

$$rank_{\mathscr{K}(\delta]} \frac{\partial \mathscr{Y}}{\partial x} < rank_{\mathscr{K}} \frac{\partial \{\mathscr{Y}, \Psi\}}{\partial x} \tag{14.25}$$

or for any element q_j of $Q \in \mathscr{K}^{1\times p}(\delta]$, $\nexists a(\delta] \in \mathscr{K}(\delta]$ such that

$$q_j = a(\delta]\bar{q}_j, \text{ with } \bar{q}_j \in \mathscr{K}_0, 1 \le j \le p \tag{14.26}$$

and

$$rank_{\mathscr{K}(\delta]} \frac{\partial \mathscr{Y}}{\partial x} = rank_{\mathscr{K}} \frac{\partial \{\mathscr{Y}, \Psi\}}{\partial x}. \tag{14.27}$$

Remark 14.9 *It is clear that Theorem 14.5 is a special case of Theorem 14.8, since the output delay-identifiable equation stated in Theorem 14.5 does not contain any derivative of the output.*

In [3], a condition similar to (15.5) of Theorem 14.8 is stated as a necessary and sufficient condition for delay identification for nonlinear systems with known *inputs. However, as we proved above, in the case of* unknown *inputs, this condition is sufficient, but not necessary.*

14.4 Observability

Similarly to the observability definitions given in [86] and [56] for nonlinear delay-free systems, it has been given in [135] a definition of observability for nonlinear time-delay systems. The following gives a more generic definition of observability in the case of systems with unknown inputs.

Definition 14.10 *System (14.6) is locally unknown input observable if the state $x(t)$ can be expressed as a function of the output and a finite number of its time derivatives with their backward and forward shifts. A locally observable system is locally backward unknown input observable if its state can be written as a function of the output and its derivatives with their backward shifts only. If, however, its state can be written as a function of the output and its derivatives with their forward shifts only, then it said to be locally forward unknown input observable.*

In the same way, the following definition is given of systems with unknown inputs.

Definition 14.11 *The unknown input $u(t)$ can be locally estimated if it can be written as a function of the output and a finite number of its time derivatives with backward and forward shifts. The input can be locally backward (or forward) estimated if $u(t)$ can be expressed as a function of the output and its time derivatives with backward (or forward) shifts only.*

Theorem 14.12 *Consider the system (14.6) with outputs $(y_1, \cdots, y_p)$ and their corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. Consider Φ and $\bar{\Gamma}$ defined in (14.10) and (14.20), respectively.*

If $rank_{\mathscr{K}(\delta]}\Phi = n$,, then there exists a change of coordinates $\phi(x, \delta)$ such that (14.6) can be transformed into (14.14-14.17) with dim $\xi = 0$.

Moreover, if the change of coordinates is locally bicausal over $\mathscr{K}$, then the state $x(t)$ of (14.6) is locally causally observable; if, in addition, $\bar{\Gamma} \in \mathscr{K}^{m\times m}(\delta]$ is unimodular over $\mathscr{K}(\delta]$, then the unknown input $u(t)$ of (14.6) can be locally backward estimated.

For the case where the condition $rank_{\mathscr{K}(\delta]}\Phi = n$ in Theorem 14.12 is not satisfied, a constructive algorithm was proposed in [7] to solve this problem for nonlinear systems without delays. In the following we are going to extend this idea to treat the observation problem for time-delay systems with unknown inputs. The objective is to generate additional variables from the available measurement and unaffected by the unknown input such that an extended canonical form similar to (14.14)-(14.15) can be obtained for the estimation of the remaining state ξ.

Theorem 14.13 *Consider the system (14.6) with outputs $y = (y_1, \cdots, y_p)^T$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. Suppose $rank_{\mathscr{K}(\delta]}\Phi < n$ where Φ is defined in (14.10). There exist l new independent outputs over $\mathscr{K}$ suitable to the backward estimation problem if and only if $rank_{\mathscr{K}}\mathscr{L} = l$ where*

$$\mathscr{L} = span_{\mathbb{R}[\delta]}\{\omega \in \mathscr{G}^{\perp} \cap \Omega \mid \omega f \notin \pounds\} \tag{14.28}$$

with f defined in (14.6), $\pounds$ defined in (14.11), Ω defined in (14.12) and $\mathscr{G}^{\perp}$ defined in (14.13). Moreover, the l additional outputs, denoted as $\bar{y}_i$, $1 \le i \le l$, are given by:

$$\bar{y}_i = \omega_i f \ mod \ \pounds$$

where $\omega_i \in \mathscr{L}$.

Remark 14.14 *Theorem 14.13 gives a constructive way to treat the case where $rank_{\mathscr{K}(\delta]}\Phi < n$. Once additional new outputs are deduced according to Theorem 14.13, it enables to define a new Φ. If $rank_{\mathscr{K}(\delta]}\Phi = n$, Theorem 14.12 can then be applied. Otherwise, if $rank_{\mathscr{K}(\delta]}\Phi < n$ and if Theorem 14.13 is still valid, then one can still deduce new outputs for the studied system. Thus a "Check-Extend" procedure is iterated until $rank_{\mathscr{K}(\delta]}\Phi = n$ is obtained.*

Like what we did for linear time-delay system, the above result can be also extended to the case of non-causal observations (i.e. contain the forward operator) of the state and the unknown inputs. For this, let us introduce the forward time-shift operator δ^{-1}, similarly to the backward time-shift operator δ defined in Section 14.1:

$$\delta^{-1} f(t) = f(t+\tau)$$

and, for $i, j \in \mathbb{Z}^+$:

$$(\delta^{-1})^i \delta^j f(t) = \delta^j (\delta^{-1})^i f(t) = f(t-(j-i)\tau).$$

Now, denote by $\bar{\mathscr{K}}$ the field of meromorphic functions of a finite number of variables from $\{x_j(t-i\tau), j \in [1,n], i \in S\}$ where $S = \{-s, \cdots, 0, \cdots, s\}$ is a finite set of relative integers. One has $\mathscr{K} \subseteq \bar{\mathscr{K}}$. Denote by $\bar{\mathscr{K}}(\delta, \delta^{-1}]$ the set of polynomials of the form:

$$a(\delta, \delta^{-1}] = \bar{a}_{r_{\bar{a}}} (\delta^{-1})^{r_{\bar{a}}} + \cdots + \bar{a}_1 \delta^{-1} + a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a}, \qquad (14.29)$$

with $a_i(t)$ and $\bar{a}_i(t)$ belonging to $\bar{\mathscr{K}}$. Keep the usual definition of addition for $\bar{\mathscr{K}}(\delta, \delta^{-1}]$ and define the multiplication as follows:

$$\begin{aligned} a(\delta, \delta^{-1}]b(\delta, \delta^{-1}] &= \sum_{i=0}^{r_a} \sum_{j=0}^{r_b} a_i \delta^i b_j \delta^{i+j} + \sum_{i=0}^{r_a} \sum_{j=1}^{r_{\bar{b}}} a_i \delta^i \bar{b}_j \delta^i (\delta^{-1})^j \\ &+ \sum_{i=1}^{r_{\bar{a}}} \sum_{j=0}^{r_b} \bar{a}_i (\delta^{-1})^i b_j \delta^{-1^i} \delta^j + \sum_{i=1}^{r_{\bar{a}}} \sum_{j=1}^{r_{\bar{b}}} \bar{a}_i (\delta^{-1})^i \bar{b}_j (\delta^{-1})^{i+j}. \end{aligned} \qquad (14.30)$$

It is clear that $\mathscr{K}(\delta] \subseteq \bar{\mathscr{K}}(\delta, \delta^{-1}]$ and that the ring $\bar{\mathscr{K}}(\delta, \delta^{-1}]$ possesses the same properties as $\mathscr{K}(\delta]$. Thus, a module $\bar{\mathscr{M}}$ can be also defined over $\bar{\mathscr{K}}(\delta, \delta^{-1}]$, as follows: $\bar{\mathscr{M}} = span_{\bar{\mathscr{K}}(\delta, \delta^{-1}]}\{d\xi, \xi \in \bar{\mathscr{K}}\}$. Then Theorem 14.12 can be extended to deal with non-causal observability as follows:

Theorem 14.15 *Consider the system (14.6) with outputs $(y_1, \cdots, y_p)$ and the corresponding $(\rho_1, \cdots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indices, respectively. If $rank_{\mathscr{K}(\delta]} \Phi = n$, where Φ is defined in (14.10), then there exists a change of coordinates $\phi(x, \delta)$ such that (14.6) can be transformed into (14.14-14.17) with $\dim \xi = 0$.*

Moreover, if the change of coordinates is locally bicausal over $\bar{\mathscr{K}}$, then the state $x(t)$ of (14.6) is at least locally non-causally observable; if, in addition, $\bar{\Gamma} \in \mathscr{K}^{m \times m}(\delta]$ is unimodular over $\bar{\mathscr{K}}(\delta, \delta^{-1}]$, then the unknown input $u(t)$ of (14.6) can be at least locally non-causally estimated.

14.5 Example

Consider:

$$\begin{cases} \dot{x}_1 = -\delta x_1^2 + \delta x_4 u_1, \quad \dot{x}_2 = -x_1^2 \delta x_3 + x_2 + x_1 \delta x_4 u_1, \\ \dot{x}_3 = x_4 - x_1^2 \delta x_4 u_1, \quad \dot{x}_4 = x_5 + \delta x_1, \quad \dot{x}_5 = \delta x_1 \delta x_3 + u_2, \\ y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_1 \delta x_1 + x_3. \end{cases} \tag{14.31}$$

One can check that $\nu_1 = k_1 = \nu_2 = k_2 = 1$, $\nu_3 = 1$, $k_3 = 3$, yielding $\rho_1 = \rho_2 = \rho_3 = 1$ and $\Phi = \{dx_1, dx_2, (\delta x_1 + x_1\delta)dx_1 + dx_3\}$. One has $rank_{\mathscr{K}(\delta]}\Phi = 3 < n$.

Set $\mathscr{G} = span_{\mathbb{R}[\delta]}\{G_1, \cdots, G_m\}$, then one has:

$$\mathscr{G}^{\perp} = span_{\mathbb{R}[\delta]} \left\{ x_1 dx_1 - dx_2, x_1^2 dx_1 + dx_3, dx_4 \right\}.$$

Since $rank_{\mathscr{K}(\delta]}\Phi = 3$, thus $\pounds = span_{\mathbb{R}[\delta]}\{x_1, x_2, x_1\delta x_1 + x_3\}$ and

$$\Omega = span_{\pounds(\delta]}\{dx_1, dx_2, dx_3\},$$

which yields:

$$\Omega \cap \mathscr{G}^{\perp} = span_{\pounds(\delta]} \left\{ x_1 dx_1 - dx_2, x_1^2 dx_1 + dx_3 \right\}.$$

In the following, identifiability and observability will be successively checked for (14.31).

Identifiability analysis:

Following Theorem 14.3, one has:

$$\mathscr{H} = [\dot{y}_1, \dot{y}_2, \dot{y}_3]^T, \; \Psi = \left[-\delta x_1^2, -x_1^2 \delta x_3 + x_2, x_4\right]^T,$$

and

$$\Gamma = \begin{bmatrix} \delta x_4, & 0 \\ x_1 \delta x_4, & 0 \\ -x_1^2 \delta x_4, & 0 \end{bmatrix}.$$

Thus, by choosing $Q = [x_1, -1, 0]$, a non zero one-form can be found, such as:

$$\omega = x_1 dx_1 - dx_2 \in \Omega \cap \mathscr{G}^{\perp},$$

satisfying

$$\omega f = -x_1 \delta x_1^2 + x_1^2 \delta x_3 - x_2 \in \pounds.$$

According to Theorem 14.7, the following equation is an output delay equation:

$$Q(\mathscr{H} - \Psi) = 0, \tag{14.32}$$

since it contains only the output, its derivatives and delays.

Since $\mathscr{Y} = (x_1, x_2, x_1\delta x_1 + x_3)^T$, one has:

$$\frac{\partial \mathscr{Y}}{\partial x} = \begin{pmatrix} 1, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0 \\ \delta x_1 + x_1\delta, & 0, & 1, & 0, & 0 \end{pmatrix},$$

and

$$\frac{\partial \Psi}{\partial x} = \begin{pmatrix} -2\delta x_1\delta, & 0, & 0, & 0, & 0 \\ -2x_1\delta x_3, & 1, & -x_1^2\delta, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \end{pmatrix}.$$

Thus, one obtains:

$$rank_{\mathscr{K}(\delta]} \frac{\partial \mathscr{Y}}{\partial x} = 3 < rank_{\mathscr{K}} \frac{\partial \{\mathscr{Y}, \Psi\}}{\partial x} = 6.$$

Theorem 14.8 is satisfied and (14.32) involves δ in an essential way. A straightforward calculation gives:

$$y_1\dot{y}_1 - \dot{y}_2 = -y_1\delta y_1^2 + y_1^2\delta y_3 - y_1^2\delta y_1\delta^2 y_1 - y_2,$$

which permits to identify the delay.

Observability analysis:

From the definition of $\mathscr{L}$ in (14.28), one can check that $rank_{\mathscr{K}}\mathscr{L} = 1$, which gives the one-form $\omega = x_1^2 dx_1 + dx_3$, satisfying $\omega \in \Omega \cap \mathscr{G}^{\perp}$ and $\omega f = -x_1^2\delta x_1^2 + x_4 \notin \pounds$. Thus, according to Theorem 14.13, a new output $\bar{y}_1 = h_4$ is given by:

$$\bar{y}_1 \quad = h_4 = \omega f \bmod \pounds = x_4 = y_1^2\dot{y}_1 + \dot{y}_3 + y_1^2\delta y_1^2. \tag{14.33}$$

For the new output $\bar{y}_1$, one has $k_i = \nu_i = 1$ for $1 \leq i \leq 3$, $k_4 = \nu_4 = 2$, thus $\rho_i = 1$ for $1 \leq i \leq 3$ and $\rho_4 = 2$. Finally, one obtains the new Φ as follows:

$$\Phi = \quad \{dx_1, dx_2, (\delta x_1 + x_1\delta)dx_1 + dx_3, dx_4, \delta dx_1 + dx_5\}.$$

It can be checked that $rank_{\mathscr{K}(\delta]}\Phi = 5 = n$, and the new $\pounds$ is:

$$\pounds \quad = span_{\mathbb{R}[\delta]}\{x_1, x_2, x_1\delta x_1 + x_3, x_4, x_5 + \delta x_1\}.$$

This gives the following change of coordinates:

$$z = \phi(x, \delta) = (x_1, x_2, x_1\delta x_1 + x_3, x_4, x_5 + \delta x_1)^T.$$

It is easy to check that it is bicausal over $\mathscr{K}(\delta]$, since:

$$x = \phi^{-1} = (z_1, z_2, z_3 - z_1\delta z_1, z_4, z_5 - \delta z_1)^T.$$

When $t \geq \tau$, one gets the following estimations of states:

$$\begin{cases} x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3 - y_1 \delta y_1, \\ x_4 = \bar{y}_1,\ x_5 = -\delta y_1 + \dot{\bar{y}}_1, \end{cases}$$

with $\bar{y}_1$ defined in (14.33).

Moreover, the matrix Γ with the new output $\bar{y}_1$ can be obtained as follows:

$$\Gamma = \begin{pmatrix} \delta x_4, & 0 \\ x_1 \delta x_4, & 0 \\ x_1^2 \delta x_4, & 0 \\ 0, & 1 \end{pmatrix},$$

with $rank_{\mathscr{K}(\delta]} \Gamma = 2$. One can find matrices $\Xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ x_1 & -1 & 0 & 0 \\ x_1^2 & 0 & 1 & 0 \end{pmatrix}$, $\bar{\Gamma} = \begin{pmatrix} \delta x_4 & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{\Gamma}^{-1} = \begin{pmatrix} \frac{1}{\delta x_4} & 0 \\ 0 & 1 \end{pmatrix}$ such that $\begin{bmatrix} \bar{\Gamma}^{-1} & \mathbf{0} \end{bmatrix} \Xi \Gamma = I_{2\times 2}$. Consequently, according to Theorem 14.12, u_1 and u_2 can be causally estimated. When $t \geq 3\tau$, a straightforward computation yields the following estimates for the unknown inputs:

$$\begin{cases} u_1 = \frac{\dot{y}_1 + \delta y_1^2}{\delta \bar{y}_1}, \\ u_2 = \ddot{\bar{y}}_1 - \delta \dot{y}_1 - \delta y_1 \delta y_3 + \delta y_1^2 \delta^2 y_1. \end{cases}$$

14.6 Conclusion

This chapter has studied identifiability and observability for nonlinear time-delay systems with unknown inputs. Concerning the identification of the delay, dependent and independent outputs over the non-commutative rings have been analyzed. Concerning the observability, necessary and sufficient conditions have been deduced for both backward and non-backward cases. The backward and non-backward estimations of unknown inputs of the studied systems have been analyzed as well.

Chapter 15

Luenberger-Like Observer Design

The unknown input observer design for linear systems without delays has already been solved in the literature. This problem becomes more complicated when the studied system involves delays, which might appear in the state, in the input and in the output. Most of the existing works on unknown input observer are focused on time-delay systems whose outputs are not affected by unknown inputs. However, this situation might exist in many practical applications since most of the sensors involve computation and communication, thus introduce output delays. Compared to the existing results in the literature, this chapter deals with the unknown input observer design problem for a more general sort of linear time-delay systems where the commensurate delays are involved in the state, in the input as well as in the output. Moreover, the studied linear time-delay system admits more than one delay.

The presented result, published in [J5], adopts as well the polynomial method based on ring theory, which has already been presented at the beginning of Part III, since it enables us to reuse some useful techniques developed for systems without delays.

15.1 Problem statement

Consider the same type of systems as before:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \sum_{i=0}^{k_a} A_i x(t-ih) + \sum_{i=0}^{k_b} B_i u(t-ih) \\ y(t) & = & \sum_{i=0}^{k_c} C_i x(t-ih) + \sum_{i=0}^{k_e} D_i u(t-ih) \end{array}\right. \tag{15.1}$$

where the state vector $x(t) \in \mathbb{R}^{n_x}$, the system output vector $y(t) \in \mathbb{R}^p$, the unknown input vector $u(t) \in \mathbb{R}^m$, the initial condition $\varphi(t)$ is a piecewise continuous function $\varphi(t) : [-kh, 0] \to \mathbb{R}^n$ ($k = \max\{k_a, k_b, k_c, k_d\}$); thereby $x(t) = \varphi(t)$ on $[-kh, 0]$. A_i, B_i, C_i and D_i are the matrices of appropriate dimension with entries in $\mathbb{R}$.

After having introduced the delay operator δ, system (15.1) may be then represented in the following compact form:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A(\delta) x(t) + B(\delta) u(t) \\ y(t) & = & C(\delta) x(t) + D(\delta) u(t) \end{array}\right. \tag{15.2}$$

where $A(\delta) \in \mathbb{R}^{n_x \times n_x}[\delta]$, $B(\delta) \in \mathbb{R}^{n_x \times m}[\delta]$, $C(\delta) \in \mathbb{R}^{p \times n_x}[\delta]$, and $D(\delta) \in \mathbb{R}^{p \times m}[\delta]$ are matrices over the polynomial ring $\mathbb{R}[\delta]$, defined as $A(\delta) := \sum_{i=0}^{k_a} A_i \delta^i$, $B(\delta) := \sum_{i=0}^{k_b} B_i \delta^i$, $C(\delta) := \sum_{i=0}^{k_c} C_i \delta^i$, and $D(\delta) := \sum_{i=0}^{k_d} D_i \delta^i$.

Remark 15.1 *For the system without delay, i.e.* $A(\delta) = A$, $B(\delta) = B$, $C(\delta) = C$ *and* $D(\delta) = D$ *in (15.2), [84] proposed the following unknown input Luenberger-like observer:*

$$\begin{aligned} \dot{\xi} &= P\xi + Qy \\ \hat{x} &= \xi + Ky \end{aligned}$$

and it has been proven as well the above Lunberger-like observer exists only if the following rank condition:

$$\operatorname{rank} \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B \\ D \end{bmatrix} + \operatorname{rank} D \tag{15.3}$$

is satisfied.

When considering the general linear system (15.2) with commensurate delays which can appear in the state, in the input and in the output, the problem to design a simple unknown input Luenberger-like observer is still open. Our solution is inspired by the method proposed in [93] where only linear time-delay systems without input were studied. More precisely, we firstly try to decompose system (15.2) into a simpler form provided that some conditions are satisfied, and then transform it into a higher dimensional observer normal form with output (and the derivative of the output) injection and its delay. Finally we can design an unknown input observer for the obtained observer normal form.

15.2 Notations and definitions

Since we are going to analyze system (15.2) which is described by the polynomial matrices over $\mathbb{R}[\delta]$, therefore let us give some useful definitions of unimodular and change of coordinates over $\mathbb{R}[\delta]$.

Definition 15.2 *A given polynomial matrix* $A(\delta) \in \mathbb{R}^{n \times q}[\delta]$ *is said to be left (or right) unimodular over* $\mathbb{R}[\delta]$ *if there exists* $A_L^{-1}(\delta) \in \mathbb{R}^{q \times n}[\delta]$ *with* $n \geq q$ *(or* $A_R^{-1}(\delta) \in \mathbb{R}^{q \times n}[\delta]$ *with* $n \leq q$*), such that* $A_L^{-1}(\delta)A(\delta) = \mathrm{I}_q$ *(or* $A(\delta)A_R^{-1}(\delta) = \mathrm{I}_n$*). A square matrix* $A(\delta) \in \mathbb{R}^{n \times n}[\delta]$ *is said to be unimodular over* $\mathbb{R}[\delta]$ *if* $A_L^{-1}(\delta) = A_R^{-1}(\delta)$.

Definition 15.3 *[93] For* $x(t)$ *defined in (15.2),* $z(t) = T(\delta)x(t)$ *with* $T(\delta) \in \mathbb{R}^{n_z \times n_x}[\delta]$ *and* $n_z \geq n_x$ *is said to be a causal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $\operatorname{rank}_{\mathbb{R}[\delta]} T(\delta) = n_x$*. Moreover, it is said to be a bicausal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $T(\delta)$ *is left unimodular over* $\mathbb{R}[\delta]$.

When designing an unknown input observer for time-delay systems, it is desired to use only the actual and the past information (not the future information) of the measurements to estimate the states because of the causality. This implies that we prefer the definition of backward unknown input observability given before.

Recall that a Molinari-like algorithm has been presented before in Part III to test the backward unknown input observability. When applying such an algorithm to system (15.2), it has been proven that there always exists a least integer k^*, which is independent of the choices of $\{P_k(\delta),\Lambda_k(\delta),\Sigma_k(\delta)\}$, such that $M_{k^*+1}(\delta)=M_{k^*}(\delta)$, based on which the following assumption will be made.

Assumption 15.4 *For the quadruple $(A(\delta),B(\delta),C(\delta),D(\delta))$ of system (15.2), there exists a least integer $k^*\in\mathbb{N}_0$ such that $\operatorname{rank}_{\mathbb{R}[\delta]}M_{k^*}(\delta)=n_x$, and $M_{k^*}(\delta)$ is unimodular over $\mathbb{R}[\delta]$.*

According to the result stated before, this assumption implies in fact that system (15.2) is backward unknown input observable.

For simplicity, for any polynomial matrix $D(\delta)\in\mathbb{R}^{p\times m}[\delta]$ with $\operatorname{rank}_{\mathbb{R}[\delta]}D(\delta)=r_D\leq\min\{p,m\}$, let us denote

$$\operatorname{Inv}_S[D(\delta)]=\{\psi_i(\delta)\}_{1\leq i\leq r_D}$$

as the set of its invariant factors of the Smith form defined in (13.4). Thereby, the following statement, adapted from the result on the left unimodular stated in [93], is obvious.

Lemma 15.5 *A polynomial matrix $D(\delta)\in\mathbb{R}^{p\times m}[\delta]$ is left (or right) unimodular over $\mathbb{R}[\delta]$ if and only if $\operatorname{rank}_{\mathbb{R}[\delta]}D(\delta)=m\leq p$ (or $\operatorname{rank}_{\mathbb{R}[\delta]}D(\delta)=p\leq m$) and $\operatorname{Inv}_S[D(\delta)]\subset\mathbb{R}$.*

It is said to be unimodular over $\mathbb{R}[\delta]$ if and only if $\operatorname{rank}_{\mathbb{R}[\delta]}D(\delta)=p=m$ and $\operatorname{Inv}_S[D(\delta)]\subset\mathbb{R}$.

15.3 Main result

Before proposing an unknown input observer for the general system (15.2), we will firstly decompose system (15.2) into a simpler form under some additional conditions.

15.3.1 Preliminary result

In order to transform the general system (15.2) into a simpler form, let us make the following assumption.

Assumption 15.6 *For the polynomial matrices $B(\delta)$, $C(\delta)$ and $D(\delta)$ in system (15.2), it is assumed that*

$$\operatorname{Inv}_S\begin{bmatrix} C(\delta)B(\delta) & D(\delta)\\ D(\delta) & 0\\ B(\delta) & 0\end{bmatrix}=\operatorname{Inv}_S\begin{bmatrix} C(\delta)B(\delta) & D(\delta)\\ D(\delta) & 0\end{bmatrix}\tag{15.4}$$

Remark 15.7 *When treating linear systems without delay, the conditions imposed in Assumption 15.4 is equivalent to:*

$$\operatorname{rank}\begin{bmatrix} sI-A & -B\\ C & D\end{bmatrix}=n+\operatorname{rank}\begin{bmatrix} B\\ D\end{bmatrix}\ \textit{for all } s\in\mathbb{C}\tag{15.5}$$

which is exactly the necessary and sufficient condition such that the system is strongly observable [183].

The condition (15.4) imposed in Assumption 15.6 is equivalent to:

$$\operatorname{rank}\begin{bmatrix} CB & D \\ D & 0 \\ B & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \operatorname{rank} D$$

and it is exactly the necessary condition to ensure the existence of such a Luenberger-like observer. As we are going to propose an unknown input observer for the general time-delay system (15.2) with the same structure, thus the condition (15.4) imposed in Assumption 15.6 is not restrictive.

Lemma 15.8 *Suppose Assumption 15.6 is satisfied, then there exists a matrix $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ satisfying the following conditions:*

1. $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$;
2. *for any matrix $J(\delta)$ such that $J(\delta)\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$, then $J(\delta)W(\delta) = 0$.*

15.3.2 System decomposition

Suppose that $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ is a matrix over $\mathbb{R}[\delta]$ such that Lemma 15.8 is satisfied. Decompose $W(\delta) = [W_1(\delta), W_2(\delta)]$ with $W_1(\delta), W_2(\delta) \in \mathbb{R}^{(n_x+p)\times p}[\delta]$, then we have $W_1(\delta)D(\delta) = 0$ since $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Thus, we obtain $W_1(\delta)y = W_1(\delta)C(\delta)x$ which yields the following equation

$$W_1(\delta)\dot{y} = W_1(\delta)C(\delta)A(\delta)x + W_1(\delta)C(\delta)B(\delta)u$$

Since $W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} = \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix}$, then we have

$$\begin{aligned} W_1(\delta)\dot{y} + W_2(\delta)y &= W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} u \\ &= W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u \end{aligned}$$

Decompose again the matrix $W(\delta)$ as $W(\delta) = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$, where $K_i(\delta) \in \mathbb{R}^{n_x\times p}[\delta]$ and $\Gamma_i(\delta) \in \mathbb{R}^{p\times p}[\delta]$ for $1 \le i \le 2$. Thereby, we can substitute $\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u$

into the original system (15.2) in order to replace the unknown input u. That is, if Assumption 15.6 is satisfied, then system (15.2) can be put into the following simpler form:

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\dot{y} + K_2(\delta)y \\ y &= \tilde{C}(\delta)x + \Gamma_1(\delta)\dot{y} + \Gamma_2(\delta)y \end{aligned} \tag{15.6}$$

where $\bar{A}(\delta) = A(\delta) - K(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} \in \mathbb{R}^{n_x \times n_x}[\delta]$ and $\tilde{C}(\delta) = C(\delta) - \Gamma(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} \in \mathbb{R}^{p \times n_x}[\delta]$.

Suppose $\mathrm{rank}_{\mathbb{R}[\delta]}\tilde{C}(\delta) = r \leq p$, then there exists a unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that

$$\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} \tag{15.7}$$

with $\bar{C}(\delta) \in \mathbb{R}^{r \times n_x}[\delta]$ being full row rank over $\mathbb{R}[\delta]$. By noting $\bar{y} = \Lambda(\delta)y$, finally system (15.6) can be written into the following decomposed form:

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\Lambda^{-1}(\delta)\dot{\bar{y}} + K_2(\delta)\Lambda^{-1}(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} x + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{aligned} \tag{15.8}$$

where

$$\begin{aligned} \bar{\Gamma}_1(\delta) &= \Lambda(\delta)\Gamma_1(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{p \times p}[\delta] \\ \bar{\Gamma}_2(\delta) &= \Lambda(\delta)\Gamma_2(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{p \times p}[\delta] \end{aligned} \tag{15.9}$$

15.3.3 Observer normal form

This subsection is devoted to designing a Lunberger-like observer for the deduced simple form (15.8). Before this, define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\bar{\mathscr{O}}_l(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{l-1}(\delta) \end{bmatrix} \in \mathbb{R}^{rl \times n_x}[\delta] \tag{15.10}$$

where $l \in \mathbb{N}_0$, and let us recall a useful result stated in [91].

Theorem 15.9 *[91] There exists a bicausal generalized change of coordinates $z = T(\delta)x$ which transforms the following system:*

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x \\ \bar{y} &= \bar{C}(\delta)x \end{aligned} \tag{15.11}$$

with $\operatorname{rank}_{\mathbb{R}[\delta]}\bar{C}(\delta) = r$ *into the following observer normal form:*

$$\begin{cases} \dot{z} = A_0 z + F(\delta)\bar{y} \\ \bar{y} = C_0 z \end{cases}$$

where $F(\delta) = \left[F_1^T(\delta), \cdots, F_{l^*}^T(\delta)\right]^T$ *and*

$$A_0 = \begin{bmatrix} 0 & \mathrm{I}_r & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{I}_r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{rl^* \times rl^*}$$
$$C_0 = [\mathrm{I}_r, 0, \cdots, 0] \in \mathbb{R}^{r \times rl^*} \tag{15.12}$$

if and only if there exists a least integer $l^* \in \mathbb{N}_0$ *such that* $\bar{\mathcal{O}}_{l^*}(\delta)$ *defined in (15.10) is left unimodular over* $\mathbb{R}[\delta]$.

Moreover, the bicausal generalized change of coordinates $z = T(\delta)x$ *with*

$$T(\delta) = \operatorname{col}\{T_1(\delta), \cdots, T_{l^*}(\delta)\}$$

is defined as follows:

$$\begin{cases} T_1(\delta) = \bar{C}(\delta) \\ T_{i+1}(\delta) = T_i(\delta)\bar{A}(\delta) - F_i(\delta)\bar{C}(\delta), \text{ for } 1 \leq i \leq l^* - 1 \end{cases} \tag{15.13}$$

with $F_i(\delta)$ *being determined through the following equations:*

$$[F_{l^*}(\delta), \cdots, F_1(\delta)] = \bar{C}(\delta)\bar{A}^{l^*}(\delta)\left[\bar{\mathcal{O}}_{l^*}(\delta)\right]_L^{-1} \tag{15.14}$$

Remark 15.10 *Theorem 15.9 deduces a very simple observer normal form with constant matrices* A_0 *and* C_0 *plus a linear output delayed term* $F(\delta)\bar{y}$ *for a special form of system (15.8), which implies in fact that in the original system (15.2) the matrices* $B(\delta) = D(\delta) = 0$*, i.e. system (15.8) has no inputs. Theorem 15.9 can be also applied to design a Luenberger-like observer for system (15.2) with known input, however it cannot treat directly system (15.2) with unknown input, since the observability in this case depends as well on the matrices* $B(\delta)$ *and* $D(\delta)$.

We have pointed out that system (15.2) is backward unknown input observable if Assumption 15.4 is satisfied. In the following it will be shown that Assumptions 15.4 and 15.6 imply that the observability matrix defined in (15.10) is left unimodular. For this, we need the following result.

Lemma 15.11 *If Assumption 15.4 is satisfied for the quadruple* $(A(\delta), B(\delta), C(\delta), D(\delta))$ *in system (15.2), then there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\mathscr{O}_{l^*}(\delta) = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \tag{15.15}$$

is left unimodular over $\mathbb{R}[\delta]$.

Based on Lemma 15.11, we have the following result for the deduced system (15.6).

Lemma 15.12 *If Assumption 15.4 is satisfied for the quadruple* $(A(\delta), B(\delta), C(\delta), D(\delta))$ *defined in (15.2), then for the deduced system (15.6) there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\tilde{\mathscr{O}}_{l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\bar{A}^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \tag{15.16}$$

is left unimodular over $\mathbb{R}[\delta]$.

Lemma 15.13 *If there exists a least integer* $l^* \in \mathbb{N}_0$ *such that* $\tilde{\mathscr{O}}_{l^*}(\delta)$ *defined in (15.16) is left unimodular over* $\mathbb{R}[\delta]$*, then* $\bar{\mathscr{O}}_{l^*}(\delta)$ *defined in (15.10) is left unimodular over* $\mathbb{R}[\delta]$.

Theorem 15.14 *If Assumption 15.4 and Assumption 15.6 are both satisfied for system (15.2), then for the deduced system (15.8) there exists a least integer* $l^* \in \mathbb{N}_0$ *such that* $\bar{\mathscr{O}}_{l^*}(\delta)$ *defined in (15.10) is left unimodular over* $\mathbb{R}[\delta]$.

After having proved the left unimodularity of $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (15.10) over $\mathbb{R}[\delta]$, the following corollary is obvious due to Theorem 15.9.

Corollary 15.15 *If Assumption 15.4 and Assumption 15.6 are both satisfied for system (15.2), then there exists a bicausal generalized change of coordinates* $z = T(\delta)x$ *defined in (15.13) such that system (15.8) can be transformed into the following observer normal form:*

$$\begin{cases} \dot{z} &= A_0 z + [F(\delta), 0]\,\bar{y} + \bar{K}_1(\delta)\dot{\bar{y}} + \bar{K}_2(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} C_0 \\ 0 \end{bmatrix} z + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{cases} \tag{15.17}$$

where $\bar{\Gamma}_1(\delta)$, $\bar{\Gamma}_2(\delta)$, A_0, C_0 *and* $F(\delta)$ *are defined in (15.9), (15.12) and (15.14) respectively, with*

$$\begin{aligned} \bar{K}_1(\delta) &= T(\delta)K_1(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \\ \bar{K}_2(\delta) &= T(\delta)K_2(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{n_z \times p}[\delta] \end{aligned} \tag{15.18}$$

where $n_z = rl^*$.

15.3.4 Unknown input observer design

For the obtained observer normal form (15.17), we are ready to present our main result.

Theorem 15.16 *If Assumption 15.4 and Assumption 15.6 are both satisfied for system (15.2), then the following dynamics:*

$$\left\{\begin{array}{ll}\dot{\xi} & = L_0\xi + J(\delta)\Lambda(\delta)y\\ \hat{z} & = \xi + H(\delta)\Lambda(\delta)y\\ \hat{x} & = T_L^{-1}(\delta)\hat{z}\end{array}\right. \tag{15.19}$$

with $T_L^{-1}(\delta)$ being defined in (15.13), and

$$\begin{array}{ll}L_0 & = A_0 - G_0C_0\\ H(\delta) & = \bar{K}_1(\delta) - [G_0, 0]\,\bar{\Gamma}_1(\delta)\\ J(\delta) & = [F(\delta), 0] + \bar{K}_2(\delta) + L_0H(\delta) - [G_0, 0]\,\bar{\Gamma}_2(\delta) + [G_0, 0]\end{array} \tag{15.20}$$

where G_0 is a constant matrix which makes $(A_0 - G_0C_0)$ Hurwitz, is an exponential unknown input observer for system (15.2).

Remark 15.17 *The proposed method is based on the output injection (delayed) technique. It can be seen that the observation error dynamics $\dot{e}_z = [A_0 - G_0C_0]e_z$ is independent of the delay, which implies that this method can be applied to any commensurate and constant delay.*

15.4 Design procedure

For the given quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$, if Theorem 15.16 is valid, the following summarizes the procedure to design the proposed unknown input observer for system (15.2):

Step 1: Compute the unimodular matrix $U(\delta)$ over $\mathbb{R}[\delta]$ which transforms the following matrix into its Hermite form:

$$U(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta)\\ D(\delta) & 0\end{bmatrix} = \begin{bmatrix} V(\delta)\\ 0\end{bmatrix}$$

with $V(\delta)$ being full row rank over $\mathbb{R}[\delta]$, and calculate $\bar{V}(\delta)$ such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0\\ D(\delta) & 0\end{bmatrix}$. Then we obtain the gain matrix $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$;

Step 2: With the obtained matrix $W(\delta)$, decompose it as $W(\delta) = \begin{pmatrix} K(\delta)\\ \Gamma(\delta)\end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta)\\ \Gamma_1(\delta) & \Gamma_2(\delta)\end{bmatrix}$, then transform system (15.2) into (15.8) with $\bar{A}(\delta) = A(\delta) - K(\delta)\begin{bmatrix} C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}$,

$\tilde{C}(\delta) = C(\delta) - \Gamma \begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$, and find the unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix}$;

Step 3: After having obtained $\bar{A}(\delta)$ and $\bar{C}(\delta)$, deduce $T(\delta)$ defined in (15.13) and $F(\delta)$ defined in (15.14);

Step 4: Deduce A_0 and C_0 defined in (15.12), $\bar{\Gamma}_1(\delta)$ and $\bar{\Gamma}_2(\delta)$ defined in (15.9), $\bar{K}_1(\delta)$ and $\bar{K}_2(\delta)$ defined in (15.18);

Step 5: Design the observer of the form (15.19) by choosing the matrices L_0, $H(\delta)$ and $J(\delta)$ defined in (15.20).

15.5 Example

Consider the following example:

$$A(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \delta & 1 & 1 & -1 \end{bmatrix}, B(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix}, C(\delta) = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, D(\delta) = \begin{pmatrix} 1 & \delta \\ 0 & 0 \\ 1 & \delta \end{pmatrix}$$

For the given quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$, by applying the algorithm (13.3)-(13.4), we find that there exist $k^* = 3$ such that $M_{k^*} = M_{k^*+1} = \mathrm{I}_4$, thus Assumption 15.4 is satisfied. Moreover, by calculating the invariant factors we have

$$\mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \mathrm{Inv}_S \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \{1, 1, 1\}$$

therefore Assumption 15.6 is satisfied as well. According to Theorem 15.16, there exists a Luenberger-like observer to exponentially estimate the state of the studied system.

Step 1:

In order to transform the matrix $\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}$ into its Hermite form, we can find

$$U(\delta) = \begin{bmatrix} \delta & 0 & -\delta & 1 & 0 & \delta \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, V(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{bmatrix}$$

then we can find

$$\bar{V}(\delta) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \delta & \delta & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$, which gives us

$$W(\delta) = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta-1 & 0 & 1-\delta & 1 & 0 & \delta-1 \\ -\delta & 0 & \delta & 0 & 0 & -\delta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 2:

With the above deduced $W(\delta)$, we obtain:

$$\bar{A}(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ \delta^2-\delta & \delta^2-1 & -\delta^2+2\delta & -2\delta+2 \\ -\delta^2 & 1-\delta-\delta^2 & 1-\delta+\delta^2 & 2\delta-1 \end{bmatrix}$$

and $\tilde{C}(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$, thus we can choose the unimodular matrix over $\mathbb{R}[\delta]$ as $\Lambda(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, which gives $\bar{C}(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$.

Step 3:

With the deduced $\bar{A}(\delta)$ and $\bar{C}(\delta)$, we can check that there exists $l^* = 3$ such that

$$\bar{\mathscr{O}}_3(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & \delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -\delta & 1+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}, \left[\bar{\mathscr{O}}_3(\delta)\right]_L^{-1} = \begin{bmatrix} \delta & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \delta & 0 & -1 & 0 \\ \delta^2-1 & 1 & -2\delta & 0 & 1 & 0 \end{bmatrix}$$

which gives $[F_3(\delta), F_2(\delta), F_1(\delta)] = \bar{C}(\delta)\bar{A}^3(\delta)\left[\bar{\mathscr{O}}_3(\delta)\right]_L^{-1}$ with

$$F(\delta) = \begin{bmatrix} F_1(\delta) \\ F_2(\delta) \\ F_3(\delta) \end{bmatrix} = \begin{bmatrix} -2-\delta^2+5\delta & 0 \\ 0 & 0 \\ 1+3\delta-5\delta^2+\delta^3 & 0 \\ 0 & 0 \\ 3-4\delta-\delta^2+\delta^3 & 2\delta-2 \\ 0 & 1 \end{bmatrix} \tag{15.21}$$

Then we obtain the bicausal generalized change of coordinates $z = T(\delta)x$ where

$$T(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & 2-4\delta+\delta^2 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -2+4\delta-\delta^2 & -\delta+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$$

with

$$[T(\delta)]_L^{-1} = \begin{bmatrix} 2-4\delta+\delta^2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -19\delta^2+15\delta+8\delta^3-4-\delta^4 & 0 & -4\delta+2+\delta^2 & 0 & -1 & 0 \\ 15\delta^2-13\delta-7\delta^3+4+\delta^4 & 0 & 3\delta-2-\delta^2 & 1 & 1 & 0 \end{bmatrix}$$

Step 4:

With the deduced change of coordinates, the studied system can be transformed into the simple observer form (15.17) with $F(\delta)$ given in (15.21) and

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \bar{K}_1(\delta) & \bar{K}_2(\delta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & \delta-1 & 1-\delta & 0 & 1-\delta & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \bar{\Gamma}_1(\delta) & \bar{\Gamma}_2(\delta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the simulation setting, we can choose

$$G_0 = \begin{bmatrix} 85 & 0 & 2000 & 0 & 12500 & 0 \\ 0 & 85 & 0 & 2000 & 0 & 12500 \end{bmatrix}^T$$

such that $(A_0 - G_0C_0)$ has negative eigenvalues $(-10,-10,-25,-25,-50,-50)$. And finally we obtain the following gain matrices:

$$L_0 = \begin{bmatrix} -85 & 0 & 1 & 0 & 0 & 0 \\ 0 & -85 & 0 & 1 & 0 & 0 \\ -2000 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2000 & 0 & 0 & 0 & 1 \\ -12500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12500 & 0 & 0 & 0 & 0 \end{bmatrix}, H(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \delta-1 & 1-\delta \\ 0 & 1 & -1 \end{bmatrix}$$

$$J(\delta) = \begin{bmatrix} 83+5\delta-\delta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 2001-5\delta^2+3\delta+\delta^3 & \delta-1 & 1-\delta \\ 0 & 0 & 0 \\ 12503-4\delta-\delta^2+\delta^3 & \delta-1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5:

With the deduced L_0, $H(\delta)$ and $J(\delta)$, one can easily design the unknown input observer described in (15.19). In the simulation of the studied system, we set the unknown input as $u_1 = -10\sin 100t$ and $u_2 = 20\sin 20t$ (see Fig. 15.1). The simulation step is 0.001s, and the basic delay $h = 0.01$s. By choosing the calculated gain matrices L_0, $H(\delta)$ and $J(\delta)$, the observation errors (in log scale) are given in Fig. P3Ch3:fig:obs1, from which we can notice not only the convergence of the proposed observer, but also the delay effect in the observer which is equal to 0.04s and it is due to the term δ^4 in $[T(\delta)]_L^{-1}$. The singularity in the figure is due to the fact that the observation error passes zero and changes the sign. In order to show that the proposed method is independent of the time-delay involved in the studied system, another simulation was made with the same gains and a bigger delay $h = 0.1s$, whose results (again in log scale) were depicted in Fig. 15.3. Compared Fig. 15.2 with Fig. 15.3, we can conclude that, with two distinct delays, the resulting estimation errors converge to 0 with the same speed, depending on the eigenvalues of $T_L^{-1}(\delta)(A_0 - G_0C_0)T(\delta)$. Moreover, with a bigger delay, the estimations will have a bigger delay as well (in Fig. 15.3 the delay effect in the observer equals to 0.4s, corresponding to the term δ^4 in $[T(\delta)]_L^{-1}$).

In order to show the robustness of the proposed observer, the third simulation (the same gains with $h = 0.01s$) was made by adding a mean-zero random disturbance in the output belonging to $[-2,2]$. The estimation errors are depicted in Fig. 15.4 and it can be noticed that the estimation error is always bounded.

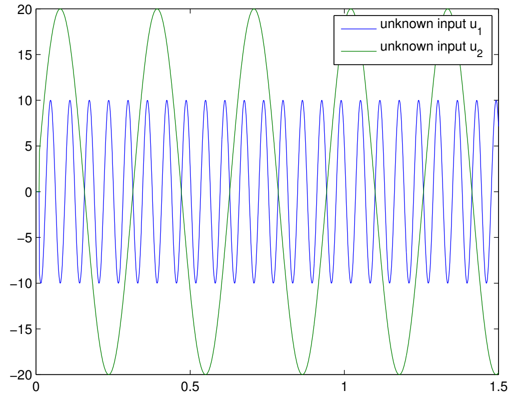


Fig. 15.1 Unknown input u of the studied system.

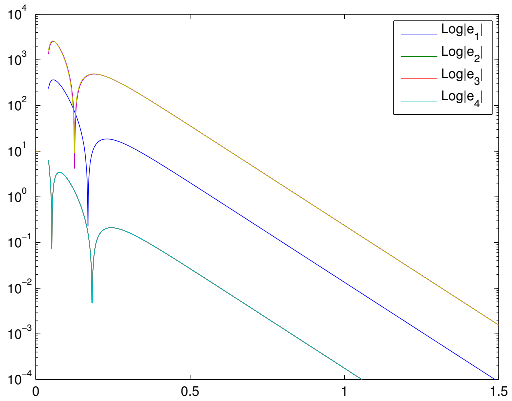


Fig. 15.2 The observation error (in log scale) for $h = 0.01s$.

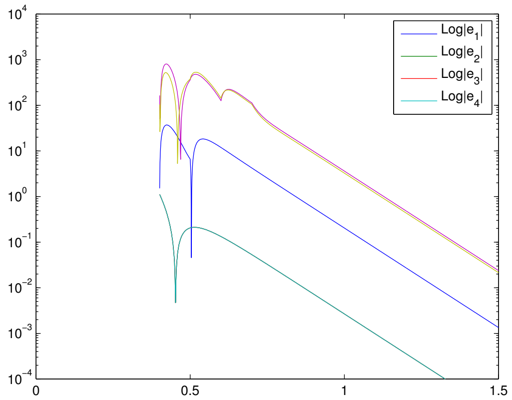


Fig. 15.3 The observation error (in log scale) for $h = 0.1s$.

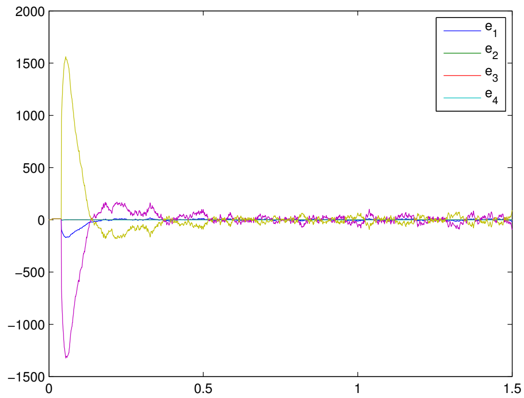


Fig. 15.4 The observation error for $h = 0.01s$ with noisy measurement.

15.6 Conclusion

The class of linear time-delay systems investigated in this chapter is quite larger than that those studied in the literature since we consider unknown inputs in both the state equation and in the system output. Moreover, commensurate delays are allowed to appear in the state, input, and in the output also. We have matched the backward unknown input observability condition recently obtained in [J10], with the observability condition required in [93] for the observer design of linear time-delay systems without inputs. The required conditions for the observer design are considerably relaxed in the sense that they coincide with the necessary and sufficient conditions for the unknown input observer design of linear systems without delays.

Part IV

Conclusions and Perspectives

Chapter 16

Conclusions and Perspectives

This manuscript summarized my main recent theoretical contributions since my recruitment at Inria in 2009 dealing with observability analysis and observer design for different types of dynamical systems, including nonlinear continuous-time systems, linear/nonlinear singular systems and linear/nonlinear time-delay systems.

For the observability analysis, this document used 2 different approaches: algebra and differential geometry, to study respectively linear and nonlinear systems:

- When treating linear systems, normally we use algebraic tools to analyze the observability. In this manuscript, we adopt this approach to analyze the observability for linear singular systems and linear time-delay systems with unknown input, respectively in Chapter 8 by using elementary algebra and in Chapter 13 by using abstract algebra.

- When studying nonlinear systems, the differential geometrical method is commonly applied. One main reason to use the differential geometrical method is to seek a simple way in order to transform the studied nonlinear systems into some simple and equivalent observable normal forms (such as triangular form, output injection, output depending and so on), for which the existing observers proposed in the literature can be reused to estimate its states. This powerful approach enables us to deduce necessary and sufficient conditions for such an equivalence. Therefore, this technique has been used in this manuscript to study nonlinear time-continuous systems (Chapter 3 and 4), nonlinear singular systems (Chapter 9) and nonlinear time-delay systems (Chapter 14).

Concerning observer design for different types of systems, this document considered 3 types of observers:

- Luenberger-like observer;

- Finite-time observer;

- Interval observer.

Luenberger-like observer has simple structure, but possesses only asymptotic estimation. This kind of observer has been designed for nonlinear singular system (Chapter 10) and linear time-delay system (Chapter 15). When finite-time estimation is needed, a non-asymptotic observer is desired, and this type of observer has been synthesized for nonlinear time-continuous system in Chapter 5. The interval observer is proposed for the systems with uncertainties, since in this case neither asymptotic nor non-asymptotic observer can provide precise estimation. This sort of observer is studied for nonlinear time-continuous systems in Chapter 6 and for nonlinear singular system in Chapter 11 in order to give the upper and lower bounded estimations.

Although the topics on observability analysis and observer design have been widely studied for a long time in the literature, but there still exist lots of interesting aspects which need to be developed from the different sizes point of view.

- **Small size**, i.e. only single system is considered. That will be the simplest and natural way to continue my current work for different types of systems. In this direction, at least two aspects can be considered.

  The first one is to enlarge the different types of systems to be studied. Except these three kinds of systems discussed in this document, we can envisage to analyze the observability for sampled-data systems with unknown inputs, since nowadays the digital device has been widely used to design the controller, and to collect the measurement with a fixed or variable frequency. Besides, since fractional order differential equations are normally used to model some biological processes, it would be nice as well to extend the techniques presented in this manuscript to analyze the observability for such kind of systems. Another interesting topic I would like to treat is linked to the recent concept of compressive sensing [42]. This new technique enables us to sample the output with a very lower frequency than that required by Shanon-Nyquist theorem, but can still successfully reconstruct the whole state. The analysis of observability from the control theory point of view has never been done, which in fact needs us to study the properties of stochastic systems.

  The second consideration is to design different types of observers for those mentioned systems, including asymptotic, non-asymptotic and interval ones. Also, the observers might not be always limited to estimate the states, as what we have only done in this document. We can also expect to propose those sorts of observers for the mentioned systems to identify parameters, to estimate the unknown inputs, to detect the faults, and so on. Most of the existing observers for the mentioned problems (parameter identification, unknown inputs estimation) are asymptotic, rare results on finite-time or interval estimation have been studied. We can adopt the technique of sliding mode to treat them. Besides, we have obtained the simultaneous delay identification and state observation for time-delay systems, however the related observers have not yet been developed in the literature. This problem can be also investigated in the future work, by synthesizing asymptotic, finite-time or interval observers. The required approaches to study those mentioned problems have already developed separately in this manuscript, therefore the future works will be focused on how to combine those techniques to have a simultaneous estimation. From my point of view, this generalization can be done in short term. In fact, those estimations for different aims are requested in many practical applications, which has plentiful economical influences. As we can see, even for this small aspect, it deserves lots of future works.

- **Medium size**, i.e. several coupled or distributed systems will be studied. With the development of technology, nowadays the process to be controlled becomes more and more complex. Frequently, the plant might contain several systems, which can belong or not belong to the same class, named as system of system. The connection among several systems will substantially increase the complexity of analysis. However, the requirement to manage system of system is in fact driven by many real applications, ranging from chemical industry to transportation. A very simple but quite illustrative example is the famous heterogeneous network. We can imagine a surveillance system with different types of robots, including mobile robots (might be different types), manipulators (with different degrees of freedom, could be even flexible), quad-rotor, blimp... Those different kinds of systems construct a heterogeneous network in order to achieve a certain goal, for example grasp a desired object. For those coupled systems, it might contain lots of restrictions, such as: the communications could be limited and delayed, the sampled frequencies for different types of robots are not the same, the measurements for a common object from different sensors equipped on different robots have different precisions

It is obvious that the control for system of system is a very difficult task. A famous example is that connecting two stable systems may not result in a stable one, even if both systems belong to the same class. This issue will not exist for observability analysis for system of system, and we can use the existing technique to analyze the global or partial observability. This argument is valid provided that the connections between systems are topologically fixed. Obviously there exists another issue for the observability analysis if the topology is time-varying, or sometimes suddenly changed. Therefore, two different cases will be studied in the future:

– Time-invariant topology connection. In this case, the observability analysis is straightforward, and we just need to reuse the well developed techniques (algebraic and differential geometric) to study the global or partial observability for system of system. However, when designing observer for this kind of coupled systems, the same issue mentioned above occurs. We will solve this problem by using two different methods. The first one is inspired by the pioneering researches on cascade systems, and we can envisage to use ISS properties for different types of systems to synthesize observer. The second solution is to design separately non-asymptotic observers (algebraic, finite-time or fixed-time) for each subsystem, and then combine all into a global one.

– Time-varying topology connection. Unlike the case of time-invariant topology connection, besides the observer design challenging, the observability analysis becomes more complicated for the second case. It is clear that, in this situation, the time-varying topology plays an important role on the observability. Different topology will lead to the lost or the reconstruction of the global or partial observability of the whole system. A similar problem has been treated for the consensus of multi-agent systems to synthesize the relative controller. But, from the best of our knowledge, such a problem has not yet been solved in the literature for observability analysis and observer design. To deal with such a problem, the first task is to seek a global model to describe all the coupled systems, including the time-varying topology. This can be done by borrowing the idea of Laplacian matrix from multi-agent systems, which however in this case is time-varying. With the deduced global model, we need then redefine the observability concept, since, unlike conventional single system, it depends now on the time-varying topology too. After the new observability definition, we can then analyze its observability in the global sense. Concerning observer design for this complicated case, a general solution seems to be difficult. We can start with some simple assumptions on the time-varying topology, such as slowly time-varying, and then design the associated observers (asymptotic or non-asymptotic) for the whole system.

Other sorts of coupled systems widely exist in many practical applications as well. Therefore, the observability analysis and observer design (data fusion) for these homogeneous or heterogeneous systems are very promising, and I do believe that efforts should be made on this issue.

- **Immense size**, i.e. thousand of systems are linked. It is obvious that, when the size of the coupled systems becomes bigger, the observability and observer design become more and more challenging. In this direction, I would like to focus on two things. The first one is related to the very popular concept, named as 'big data'. As we know, lots of smart devices (or systems) are linked together, and establish a huge Internet of things. Those devices in this internet can provide immense data (relevant or not), a hot topic on this issue nowadays is how to proceed the collected big data to get more precise decision. For example, we are used to check the weather information before leaving for the vacations, but how to make a precise and long term prediction, this is still an unsolved problem [106, 109]. In practice, the collected data are of different types, since they are from lots of different sensors, such as light, wind, humidity, temperature.... Moreover, the collected data are memorized (hours, days, months, years,...), this is due to the fact that the weather is in some sense 'periodic'. Obviously, more precise the model we want, larger dimension of the systems will be led. Not mentioned that different sensors have different reliability, thus play different impacts in the model prediction, and this will make the observation problem more complicated. We are going to use different types of observers, not only the traditional Kalman filter, in order to have a more precise estimation.

  Another interesting aspect in this direction is linked to the optimal sensor placement [152]. For a concrete application, we need to answer at least the following questions:

  - How many sensors (maximal number) we need to place?
  - Which types of sensors we need to choose?
  - Where we should place those sensors?
  - What is the sensitivity for the observation?

  If the system's dimension is not so large, we can still manage the optimal placement of sensors by hands, but when the size becomes bigger enough, we should relay on the super computer to answer the above questions. Take the soft robot as an example. When designing a soft robot, according to special requirements in the different scenarios, we should integrate different types of actuators and sensors measuring the robot position and parameters (such as applied force, shape and so on) in the soft robot. The type of the sensors and where they are mounted will determine the observability (possibility of state and posture reconstruction) of the final reduced model. Moreover, since the robot is flexible, the dimension of the precise model (by using finite element method, for example) will reach millions. Those above mentioned problems then become very difficult to be answered, even by using computer. This is still an open problem up to now. To deal with this problem, we will use the optimization framework, in which we need firstly define the objective to be optimized and the observability sensitivity index. Then a numerical algorithm will be developed to display this index according to different choices of sensors we used and the different places we putted.

References

[1] Abraham, R., Marsden, J. E., 1980. Foundations of Mechanics, 2nd Edition. Addison-Wesley Publishing Company, United States of America.

[2] Acary, V., Brogliato, B., Orlov, Y., 2012. Chattering-free digital sliding-mode control with state observer and disturbance rejection. Automatic Control, IEEE Transactions on 57 (5), 1087–1101.

[3] Anguelova, M., Wennberga, B., 2008. State elimination and identifiability of the delay parameter for nonlinear time-delay systems. Automatica 44 (5), 1373–1378.

[4] Aranda-Bricaire, E., Moog, C., Pomet, J., 1995. A linear algebraic framework for dynamic feedback linearization. IEEE Trans. Autom. Control 40, 127–132.

[5] Åslund, J., Frisk, E., 2006. An observer for non-linear differential-algebraic systems. Automatica 42 (6), 959–965.

[6] Back, J., Yu, K. T., Seo, J. H., 2006. Dynamic observer error linearization. Automatica, 42 (12), 2195–2200.

[7] Barbot, J.-P., Boutat, D., Floquet, T., 2009. An observation algorithm for nonlinear systems with unknown inputs. Automatica 45, 1970–1974.

[8] Barbot, J.-P., Fliess, M., Floquet, T., 2007. An algebraic framework for the design of nonlinear observers with unknown inputs. IEEE Conference on Decision and Control, 384–389.

[9] Barbot, J.-P., Zheng, G., Floquet, T., Boutat, D., Richard, J.-P., 2012. Delay estimation algorithm for nonlinear time-delay systems with unknown inputs. in Proc. of IFAC Workshop on Time Delay Systems.

[10] Bejarano, F. J., Floquet, T., Perruquetti, W., Zheng, G., 2011. Observability and detectability analysis of singular linear systems with unknown inputs. In: Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on. IEEE, pp. 4005–4010.

[11] Bejarano, F. J., Floquet, T., Perruquetti, W., Zheng, G., 2013. Observability and detectability of singular linear systems with unknown inputs. Automatica 49(3), 793–800.

[12] Bejarano, F. J., Floquet, T., Perruquetti, W., Zheng, G., 2015. Observation of nonlinear differential-algebraic systems with unknown inputs. IEEE Transactions on Automatic Control 60(7), 1957–1962.

[13] Bejarano, F. J., Fridman, L., Poznyak, A., 2009. Unknown input and state estimation for unobservable systems. SIAM J. Control Optim. 48 (2), 1155–1178.

[14] Bejarano, F. J., Perruquetti, W., Floquet, T., Zheng, G., 2012. State reconstruction of nonlinear differential-algebraic systems with unknown inputs. In: Decision and control (CDC), 2012 IEEE 51st annual conference on. IEEE, pp. 5882–5887.

[15] Bejarano, F. J., Zheng, G., 2014. Observability of linear systems with commensurate delays and unknown inputs. Automatica 50 (8), 2077–2083.

[16] Bejarano, F. J., Zheng, G., 2016. Unknown input functional observability of descriptor systems with delays. In: Decision and Control (CDC), 2016 IEEE 55th Conference on. IEEE, pp. 580–585.

[17] Belevitch, V., 1968. Classical Network Theory. Holden-Day, San Francisco.

[18] Belkoura, L., 2005. Identifiability of systems described by convolution equations. Automatica 41 (3), 505–512.

[19] Belkoura, L., Richard, J.-P., Fliess, M., 2009. Parameters estimation of systems with delayed and structured entries. Automatica 45 (5), 1117–1125.

[20] Bernstein, D., 2009. Matrix mathematics: Theory, facts, and formulas (second edition). Princeton University Press.

[21] Besançon, G., Ticlea, A., 2007. An immersion-based observer design for rank-observable nonlinear systems. IEEE Transactions on Automatic Control 52 (1), 83–88.

[22] Bestle, D., Zeitz, M., 1983. Canonical form observer design for nonlinear time varying systems. Internatio 38, 419–431.

[23] Bhat, K., Koivo, H., 1976. Modal characterizations of controllability and observability in time delay systems. IEEE Transactions on Automatic Control 21 (2), 292– 293.

[24] Bhat, S. P., Bernstein, D. S., 2000. Finite-time stability of continuous autonomous systems. SIAM J. Control and Optimization 38 (3), 751–766.

[25] Bhat, S. P., Bernstein, D. S., 2005. Geometric homogeneity with applications to finite-time stability. MCSS 17 (2), 101–127.

[26] Bhattacharyya, S., 1978. Observer design for linear systems with unknown inputs. IEEE Transactions on Automatic Control 23 (3), 483–484.

[27] Boukhobza, T., Hamelin, F., 2007. State and input observability for structured linear systems: A graph-theoretic approach. Automatica 43 (7), 1204–1210.

[28] Bououden, S., Boutat, D., Zheng, G., Barbot, J.-P., Kratz, F., 2011. A triangular canonical form for a class of 0-flat nonlinear systems. International Journal of Control 84 (2), 261–269.

[29] Boutat, D., 2007. Geometrical conditions for observer error linearization via $\int 0, 1, ..., (n-2)$. In: 7th IFAC Symposium on Nonlinear Control Systems Nolcos'07,.

[30] Boutat, D., Busawon, K., 2011. On the transformation of nonlinear dynamical systems into the extended nonlinear observable canonical form. International Journal of Control, 84 (1), 94–106.

[31] Boutat, D., Zheng, G., Barbot, J., Hammouri, H., 2006. Observer error linearization multi-output depending. In: the 45th IEEE Conference on Decision and Control,. pp. 5394–5399.

[32] Boutat, D., Zheng, G., Boutat-Baddas, L., Darouach, M., 2012. Observers design for a class of nonlinear singular systems. In: Decision and Control (CDC), 2012 IEEE 51st Annual Conference on. IEEE, pp. 7407–7412.

[33] Boutat, D., Zheng, G., Hammouri, H., 2010. A nonlinear canonical form for reduced order observer design. IFAC Proceedings Volumes 43 (14), 409–414.

[34] Boutayeb, M., Darouach, M., 1995. Observers design for non linear descriptor systems. In: in Proc. IEEE Conf. Decision Control.

[35] Brewer, J., Bunce, J., Van Vleck, F.-S., 1986. Linear systems over commutative rings. Marcel Dekker, New York.

[36] Brogliato, B., 1999. Nonsmooth Mechanics. Communications and Control Engineering. Springer Verlag, London.

[37] Brogliato, B., Heemels, W., 2009. Observer design for lur'e systems with multivalued mappings: A passivity approach. Automatic Control, IEEE Transactions on 54 (8), 1996–2001.

[38] Burlion, L., Ahmed-Ali, T., Lamnabhi-Lagarrigue, F., 2011. Updating the gain of global finite-time high gain observers. In: CDC-ECE.

[39] Busawon, K., Farza, M., Hammouri, H., 1998. A simple observer for a class of nonlinear systems. Applied Mathematics Letters 11, 27–31.

[40] Campbell, S., 1980. Singular systems of differential equations I. Pitman.

[41] Campbell, S., 1982. Singular systems of differential equations II. Pitman.

[42] Candes, E., Wakin, M., 2008. An introduction to compressive sampling. Signal Processing Magazine, IEEE 25 (2), 21–30.

[43] Chow, W., 1940/1941. ber systeme von linearen partiellen differentialgleichungen erster ordnung. Mathematische Annalen 117, 98–105.

[44] Chu, D., Mehrmann, V., 1999. Disturbance decoupled observer design for descriptor systems. Systems & Control Letters 38, 37–48.

[45] Cobb, D., 1984. Controllability, observability, and duality in singular systems. IEEE Transactions on Automatic Control 29 (12), 1076–1082.

[46] Conte, G., Moog, C., Perdon, A., 1999. Nonlinear control systems: An algebraic setting. Lecture Notes in Control and Information Sciences 242, Springer–Verlag,London.

[47] Conte, G., Perdon, A. M., Guidone-Peroli, G., 2003. Unknown input observers for linear delay systems: a geometric approach. In: Proceedings of the 42nd IEEE Conference on Decision and Control. Maui, Hawaii USA, pp. 6054–6059.

[48] Conte, G., Perdon, A. M., Moog, C. H., 2007. Applications of Time Delay Systems. Vol. 352 of Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, Ch. Inversion and tracking problems for time delay linear systems, pp. 267–284.

[49] Dai, L., 1989. Singular Control Systems. Vol. 118. Lecture Notes in Control and Information Sciences, Springer-Verlag, New York.

[50] Darouach, M., 2001. Linear functional observers for systems with delays in state variables. IEEE Transactions on Automatic Control 46 (3), 491–496.

[51] Darouach, M., 2006. Full order unknown inputs observers design for delay systems. in Proc. of IEEE Mediterranean Conference on Control and Automation.

[52] Darouach, M., Boutat-Baddas, L., 2008. Observers for a class of nonlinear singular systems. IEEE Transactions on Automatic Control 53 (11), 2627–2633.

[53] Darouach, M., Boutayeb, M., 1995. Design of observers for descriptor systems. IEEE Transactions on Automatic Control 40 (7), 1323–1327.

[54] Darouach, M., Zasadzinski, M., Hayar, M., 1996. Reduced-order observer design for descriptor systems with unknown inputs. IEEE Transactions on Automatic Control 41 (7), 1068–1072.

[55] Darouach, M., Zasadzinski, M., Xu, S., 1994. Full order observers for linear systems with unknown inputs. IEEE Transaction on Automatic Control 39 (3), 606–609.

[56] Diop, S., Fliess, M., 1991. Nonlinear observability, identifiability and persistent trajectories. in Proc. of 36th IEEE Conf. on Decision and Control.

[57] Diop, S., Fliess, M., 1991. On nonlinear observability. In: in Proc. of ECC.

[58] Drakunov, S., Perruquetti, W., Richard, J.-P., Belkoura, L., 2006. Delay identification in time-delay systems using variable structure observers. Annual Reviews in Control 30 (2), 143–158.

[59] Drakunov, S., Utkin, V., 1995. Sliding mode observers. tutorial. In: Decision and Control, 1995., Proceedings of the 34th IEEE Conference on. Vol. 4. pp. 3376–3378.

[60] Efimov, D., Fridman, L., Raïssi, T., Zolghadri, A., Seydou, R., 2012. Interval estimation for lpv systems applying high order sliding mode techniques. Automatica 48, 2365–2371.

[61] El Assoudi, A., El Yaagoubi, E. h., Hammouri, H., 2005. A high gain observer for a class of implicit systems. In: Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference. Seville, Spain, pp. 6359–6363.

[62] Emre, E., Khargonekar, P., 1982. Regulation of split linear systems over rings: Coefficient-assignment and observers. IEEE Transactions on Automatic Control 27 (1), 104–113.

[63] Engel, R., Kreisselmeier, G., 2002. A continuous-time observer which converges in finite time. IEEE Transactions on Automatic Control 47 (7), 1202C1204.

[64] Evangelisti, E., 2011. Controllability and Observability. C.I.M.E. Summer Schools. Springer Berlin Heidelberg.

[65] Farza, M., M'Saad, M., Maatoug, T., Kamoun, M., 2009. Adaptive observers for nonlinearly parameterized class of nonlinear systems. Automatica 45 (10), 2292–2299.

[66] Fattouh, A., Sename, O., Dion, J.-M., 1999. An unknown input observer design for linear time-delay systems. In: 38th IEEE Conference on Decision and Control.

[67] Feng, R., Garrido, J., 2008. Actuarial applications of epidemiological models. Concordia University. Department of Mathematics & Statistics.

[68] Fliess, M., 1985. A new approach to the noninteracting control problem in nonlinear systems theory. In: Proc. 23rd Allerton Conf.

[69] Fliess, M., 1985. Some remarks on nonlinear invertibility and dynamic state feedback. In: in Proc. MTNS'85.

[70] Fliess, M., 1986. A note on the invertibility of nonlinear input-output systems. Systems & Control Letters 8, 147–151.

[71] Fliess, M., Glad, S., 1990. An algebraic approach to linear and nonlinear control, dans Essays on Control: Perspectives in the Theory and its Applications. Birkhauser, Boston, pp. 223–267.

[72] Fliess, M., Mounier, H., 1998. Controllability and observability of linear delay systems: an algebraic approach. ESAIM: Control, Optimisation and Calculus of Variations 3, 301–314.

[73] Fliess, M., Sira-Ramirez, H., 2004. Reconstructeurs d'état. Comptes Rendus de l'Académie des Sciences - Series I 338 (1), 91–96.

[74] Floquet, T., Barbot, J.-P., Perruquetti, W., 2003. Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems. Automatica 39 (6), 1077–1083.

[75] Fridman, E., 2014. Introduction to Time-Delay Systems: Analysis and Control. Springer International Publishing.

[76] Fu, Y.-M., Duan, G.-R., Song, S.-M., 2004. Design of unknown input observer for linear time-delay systems. International Journal of Control, Automation, and Systems 2 (4), 530–535.

[77] Gabasov, R., Zhevniak, R., F. M. Killova, T. B. Kopeykina, 1972. Conditional observability of linear systems. Probl. Contr. Theory 1 (3-4), 217–238.

[78] Gauthier, J., Hammouri, H., Othman, S., 1992. A simple observer for nonlinear systems applications to bioreactors. IEEE Transactions on Automatic Control 37 (6), 875–880.

[79] Ghanes, M., Zheng, G., De Leon-Morales, J., 2008. On simultaneous parameter identification and state estimation for cascade state affine systems. In: American Control Conference, 2008. IEEE, pp. 45–50.

[80] Guay, M., 2002. Observer linearization by output-dependent time-scale transformations. IEEE Transactions on Automatic Control 47 (10), 1730 – 1735.

[81] Guerra, M., Efimov, D., Zheng, G., Perruquetti, W., 2014. Finite-time supervisory stabilization for a class of nonholonomic mobile robots under input disturbances. IFAC Proceedings Volumes 47 (3), 4867–4872.

[82] Hamiche, H., Kemih, K., Ghanes, M., Zhang, G., Djennoune, S., 2011. Passive and impulsive synchronization of a new four-dimensional chaotic system. Nonlinear Analysis: Theory, Methods & Applications 74 (4), 1146–1154.

[83] Hautus, M., 1969. Controllability and observability conditions of linear autonomous systems. In: Proc. Nederl. Akad. Wetensch., Ser. A.

[84] Hautus, M., 1983. Strong detectability and observers. Linear Algebra and its Applications 50, 353–368.

[85] Hermann, R., 1968. Differential geometry and the calculus of variations. Mathematics in science and engineering. Academic Press.

[86] Hermann, R., Krener, A., 1977. Nonlinear controllability and observability. IEEE Transactions on Automatic Control 22 (5), 728–740.

[87] Hong, Y., Xu, Y., Huang, J., 2002. Finite-time control for robot manipulators. System Control & Letters 46 (4), 243–253.

[88] Hostetter, G., Meditch, J., 1973. Observing systems with unmeasurable inputs. IEEE Transactions on Automatic Control 18, 307–308.

[89] Hou, M., Muller, P., 1999. Causal observability of descriptor systems. IEEE Transactions on Automatic Control 44 (1), 158–163.

[90] Hou, M., Muller, P., 1999. Observer design for descriptor systems. IEEE Transactions on Automatic Control 44 (1), 164–169.

[91] Hou, M., Muller, P., 1992. Design of observers for linear systems with unknown inputs. IEEE Transactions on Automatic Control 37, 871–875.

[92] Hou, M., Pugh, A., 1999. Observer with linear error dynamics for nonlinear multi-output systems. Systems & Control Letters, 37, 1–9.

[93] Hou, M., Zitek, M. H., Patton, R., 2002. An observer design for linear time-delay systems. IEEE Transactions on Automatic Control 47, 121–125.

[94] Iggidr, A., 2006. Controllability, observability and stability of mathematical models. Mathematical Models. In Encyclopedia of Life Support Systems (EOLSS). Developed under the auspices of the UNESCO, Eolss Publishers, Oxford, UK.

[95] Isidori, A., 1995. Nonlinear Control Systems, 3rd Edition. Communications and Control Engineering. Springer, London, Great Britain.

[96] Jaulin, L., Kieffer, M., Didrit, O., Walter, E., 2001. "Applied Interval Analysis". Springer, Berlin.

[97] Ježek, J., 1996. Rings of skew polynomials in algebraical approach to control theory. Kybernetika 32 (1), 63–80.

[98] Jo, N., Seo, J., 2002. Observer design for non-linear systems that are not uniformly observable. International Journal of Control, 75 (5), 369–380.

[99] Jouan, P., 2003. Immersion of nonlinear systems into linear systems modulo output injection. SIAM Journal on Control and Optimisation, 41(6), 1756–1778.

[100] Kaczorek, T., 2007. Polynomial and Rational Matrices Applications in Dynamical Systems Theory. Communications and control engineering. Springer.

[101] Kader, Z., Zheng, G., Barbot, J.-P., 2014. Left inversion of nonlinear time delay system. In: Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on. IEEE, pp. 469–474.

[102] Kalman, R., 1960. A new approach to linear filtering and prediction problems. Transactions of the ASME - Journal of Basic Engineering 82, 35–45.

[103] Kalman, R., Falb, P., Arbib, M., 1969. Topics in Mathematical Control Theory. McGraw-Hill Book Co., New York-Toronto, Ont.-London.

[104] Kang, W., 2006. Moving horizon numerical observers of nonlinear control systems. Automatic Control, IEEE Transactions on 51 (2), 344–350.

[105] Kang, W., Barbot, J., Xu, L., 2009. On the observability of nonlinear and switched systems. Emergent Problems in Nonlinear Systems and Control, 199–216.

[106] Kang, W., Xu, L., 2012. Optimal placement of mobile sensors for data assimilations. Tellus A 64.

[107] Kaprielian, S., Turi, J., 1992. An observer for a nonlinear descriptor system. in Proc. IEEE Conf. Decision Control.

[108] Keller, H., 1987. Nonlinear observer design by transformation into a generalized observer canonical form. International Journal of Control, 46(6), 1915–1930.

[109] King, S., Kang, W., Xu, L., 2014. Observability for optimal sensor locations in data assimilation. International Journal of Dynamics and Control, 1–9.

[110] Koenig, D., 2005. Unknown input proportional multiple-integral oberver design for linear descriptor systems: application to state and fault estimation. IEEE Transactions on Automatic Control 50 (2), 212–217.

[111] Koenig, D., 2006. Observers design for unknown input nonlinear descriptor systems via convex optimization. IEEE Trans. Automatic Control 51(6), 1047–1052.

[112] Kolmanovskii, V., Myshkis, A., 1999. Introduction to the theory and application of functional differential equations. Kluwer Academic Publishers, Dordrecht.

[113] Krener, A., Isidori, A., 1983. Linearization by output injection and nonlinear observers. Systems & Control Letters, 3 (1), 47–52.

[114] Krener, A., Respondek, W., 1985. Nonlinear observer with linearizable error dynamics. SIAM Journal on Control and Optimization 23 (2), 197–216.

[115] Kudva, P., Viswanadham, N., Ramakrishna, A., 1980. Observers for linear systems with unknown inputs. IEEE Transactions on Automatic Control 25, 113–115.

[116] Lee, E., Neftci, S., Olbrot, A., 1982. Canonical forms for time delay system. IEEE Transactions on Automatic Control 27 (1), 128–132.

[117] Lee, E., Olbrot, A., 1981. Observability and related structural results for linear hereditary systems. International Journal of Control 34 (6), 1061–1078.

[118] Levant, A., 2003. High-order sliding modes: differentiation and output-feedback control. International Journal of Control 76 (9-10), 924–941.

[119] Li, M. Y., Muldowney, J. S., 1996. A geometric approach to global-stability problems. SIAM Journal on Mathematical Analysis 27, 1070–1083.

[120] Liu, D., Gibaru, O., Perruquetti, W., 2009. Error analysis for a class of numerical differentiator: Application to state observation. In: IEEE Conference on Decision and Control.

[121] Lopez, M., Plestan, F., Glumineau, A., 1999. Linearization by completely generalized input ouput injection. Kybernetika, 35, 793–802.

[122] Lu, G., Ho, D. W. C., 2006. Full-order and reduced-order observers for lipschitz descriptor systems. IEEE Transactions on Circuits and Systems II: Express Briefs 53 (7), 563–567.

[123] Luders, G., Narendra, K., 1973. An adaptive observer and identifier for a linear system. Automatic Control, IEEE Transactions on 18 (5), 496–499.

[124] Luenberger, D., 1964. Observing the state of a linear system. Military Electronics, IEEE Transactions on 8 (2), 74–80.

[125] Luenberger, D., 1971. An introduction to observers. IEEE Transactions on Automatic Control 16, 596–602.

[126] Lynch, A., Bortoff, S., 2001. Nonlinear observers with approximately linear error dynamics: the multivariable case. IEEE Transactions on Automatic Control 46 (6), 927–932.

[127] Ma, Y., Zheng, G., Perruquetti, W., 2013. Cooperative path planning for mobile robots based on visibility graph. In: Control Conference (CCC), 2013 32nd Chinese. IEEE, pp. 4915–4920.

[128] Ma, Y., Zheng, G., Perruquetti, W., 2013. Real-time identification of different types of non-holonomic mobile robots. IFAC Proceedings Nonlinear Control Systems 46 (23), 791–796.

[129] Ma, Y., Zheng, G., Perruquetti, W., 2013. Real-time local path planning for mobile robots. In: Robot Motion and Control (RoMoCo), 2013 9th Workshop on. IEEE, pp. 215–220.

[130] Ma, Y., Zheng, G., Perruquetti, W., Qiu, Z., 2013. Control of nonholonomic wheeled mobile robots via i-pid controller. In: Intelligent Robots and Systems (IROS), 2013 IEEE/RSJ International Conference on. IEEE, pp. 4413–4418.

[131] Ma, Y., Zheng, G., Perruquetti, W., Qiu, Z., 2014. Motion planning for non-holonomic mobile robots using the i-pid controller and potential field. In: Intelligent Robots and Systems (IROS 2014), 2014 IEEE/RSJ International Conference on. IEEE, pp. 3618–3623.

[132] Ma, Y., Zheng, G., Perruquetti, W., Qiu, Z., 2015. Local path planning for mobile robots based on intermediate objectives. Robotica 33 (4), 1017.

[133] Marino, R., Tomei, P., 1995. Nonlinear control design: geometric, adaptive and robust. Prentice Hall, London.

[134] Marquez-Martinez, L., Moog, C., 2007. New insights on the anlysis of nonlinear time-delay systems: Application to the triangular equivalence. Systems & Control Letters 56, 133–140.

[135] Marquez-Martinez, L., Moog, C., Velasco-Villa, M., 2002. Observability and observers for nonlinear systems with time delays. Kybernetika 38 (4), 445–456.

[136] Mazenc, F., Bernard, O., 2010. Asymptotically stable interval observers for planar systems with complex poles. IEEE Trans. Autom. Control 55 (2), 523–527.

[137] Mboup, M., 2009. Parameter estimation for signals described by differential equations. Applicable Analysis 88, 29–52.

[138] Mboup, M., Join, C., Fliess, M., 2009. Numerical differentiation with annihilators in noisy environment. Numerical Algorithms 50 (4), 439–467.

[139] Menard, T., Moulay, E., Perruquetti, W., 2010. A global high-gain finite-time observer. IEEE Transactions on Automatic Control 55 (6), 1500–1506.

[140] Molinari, B., 1976. A strong controllability and observability in linear multivariable control. IEEE Transactions on Automatic Control 21 (5), 761–764.

[141] Moog, C., Castro-Linares, R., Velasco-Villa, M., Marque-Martinez, L. A., 2000. The disturbance decoupling problem for time-delay nonlinear systems. IEEE Transactions on Automatic Control 45 (2).

[142] Niculescu, S.-I., 2001. Delay effects on stability: A robust control approach. Lecture Notes in Control and Information Sciences, Springer 269.

[143] Noh, D., Jo, N., Seo, J., 2004. Nonlinear observer design by dynamic observer error linearization. IEEE Transactions on Automatic Control 49 (10), 1746–1753.

[144] Olbrot, A., 1977. A counterexample to"observability of linear systems with time-variable delays". IEEE Transactions on Automatic Control 22 (2), 281–283.

[145] Olbrot, A. W., 1981. Observability and observers for a class of linear systems with delays. IEEE Transactions on Automatic Control 26 (2), 513–517.

[146] Paraskevopoulos, P., Koumboulis, F., 1992. Observers for singular systems. IEEE Transactions on Automatic Control 37 (8), 1211 – 1215.

[147] Paraskevopoulos, P., Koumboulis, F., Tzierakis, K., Panagiotakis, G., 1992. Observer design for generalized state space systems with unknown inputs. Systems & Control Letters 18 (4), 309321.

[148] Perruquetti, W., Floquet, T., Borne, P., 1998. A note on sliding observer and controller for generalized canonical forms. IEEE Conference on Decision and Control (1920–1925).

[149] Perruquetti, W., Floquet, T., Moulay, E., 2008. Finite-time observers: Application to secure communication. IEEE Transactions on Automatic Control 53 (1), 356–360.

[150] Phelps, A., 1991. On constructing nonlinear observers. SIAM Journal on Control and Optimmisation 29, 516–534.

[151] Popov, V., 1973. Hyperstability of Control Systems. Springer-Verlag, Berlin.

[152] Qi, J., Sun, K., Kang, W., 2014. Optimal pmu placement for power system dynamic state estimation by using empirical observability gramian.

[153] Rabah, R., 1995. On observability of linear delay systems with unknown inputs. In: 3rd IEEE Mediterranean Symposium on New Directions. Limassol, Cyprus.

[154] Raïssi, T., Efimov, D., Zolghadri, A., 2012. Interval state estimation for a class of nonlinear systems. IEEE Trans. Automatic Control 57 (1), 260–265.

[155] Ren, X., Rad, A., Chan, P., Lo, W., 2005. Online identification of continuous-time systems with unknown time delay. IEEE Transaction on Automatic Control 50 (9), 1418–1422.

[156] Respondek, W., Pogromsky, A., Nijmeijer, H., 2004. Time scaling for observer design with linearizable error dynamics. Automatica, 40 (2), 277–285.

[157] Richard, J.-P., 2003. Time-delay systems:an overview of some recent advances and open problems. Automatica 39 (10), 1667–1694.

[158] Rudolph, J., Zeitz, M., 1994. A block triangular nonlinear observer normal form. Systems & Control Letters 23 (1), 1–8.

[159] Saadi, W., Boutat, D., Zheng, G., Sbita, L., 2015. Multi-output partial nonlinear observer normal form. In: Decision and Control (CDC), 2015 IEEE 54th Annual Conference on. IEEE, pp. 7654–7658.

[160] Salamon, D., 1980. Observers and duality between observation and state feedback for time delay systems. IEEE Transactions on Automatic Control 25 (6), 1187–1192.

[161] Sauvage, F., Guay, M., Dochain, D., 2007. Design of a nonlinear finite-time converging observer for a class of nonlinear systems. Journal of Control Science and Engineering.

[162] Sename, O., 1994. Unknown input robust observers for time-delay systems. in Proc. of IEEE Conference on Decision and Control.

[163] Sename, O., 1997. Unknown input robust observer for time delay system. IEEE Conference on Decision and Control.

[164] Sename, O., 2001. New trends in design of observers for time-delay systems. Kybernetika 37 (4), 427–458.

[165] Seuret, A., Floquet, T., Richard, J.-P., Spurgeon, S. K., 2007. A sliding mode observer for linear systems with unknown time varying delay. In: American Control Conference. pp. 4558–4563.

[166] Shao, Z., Zheng, G., Efimov, D., Perruquetti, W., 2015. Modelling and control for position-controlled modular robot manipulators. In: Intelligent Robots and Systems (IROS), 2015 IEEE/RSJ International Conference on. IEEE, pp. 3290–3295.

[167] Shen, Y., Huang, Y., Gu, J. J., 2011. Global finite-time observers for lipschitz nonlinear systems. IEEE Transactions on Automatic Control 56 (2), 418–424.

[168] Shen, Y., Xia, X., 2008. Semi-global finite-time observers for nonlinear systems. Automatica 44 (12), 3152–3156.

[169] Shim, H., Seo, J., Teel, A., 2003. Nonlinear observer design via passivation of error dynamics. Automatica 39 (5), 885 – 892.

[170] Silverman, L., 1969. Inversion of multivariable linear systems. IEEE Transactions on Automatic Control 14 (3), 270–276.

[171] Singh, S., 1982. Invertibility of observable multivariable nonlinear system. IEEE Transactions on Automatic and Control 27, 487–489.

[172] Smith, H., 1995. Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems. Surveys and Monographs, AMS, Providence 41.

[173] Sontag, E., 1976. Linear systems over commutative rings: A survey. Ricerche di Automatica 7, 1–34.

[174] Sontag, E., 1984. A concept of local observability. Systems & Control Letters 5, 41–47.

[175] Sontag, E. D., Wang, Y., 1991. I/O equations for nonlinear systems and observation spaces. In: Proceedings of the 30th Conference on Decision and Control. Brighton, England, pp. 720–725.

[176] Sussmann, H. J., Jurdjevic, V., 1972. Controllability of nonlinear systems. Journal of Differential Equations 12 (1), 95 – 116.

[177] Tami, R., Boutat, D., Zheng, G., 2013. Extended output depending normal form. Automatica 49 (7), 2192–2198.

[178] Tami, R., Boutat, D., Zheng, G., 2013. Nonlinear observer normal form with output injection and extended dynamic. IFAC Proceedings Volumes 46 (23), 670–675.

[179] Tami, R., Boutat, D., Zheng, G., 2014. Nonlinear observer for the pm synchronous motor. In: American Control Conference (ACC), 2014. IEEE, pp. 4133–4138.

[180] Tami, R., Boutat, D., Zheng, G., Kratz, F., 2014. Parameters and states estimation for dengue epidemic model. In: Control Conference (ECC), 2014 European. IEEE, pp. 528–533.

[181] Tami, R., Zheng, G., Boutat, D., Aubry, D., 2013. Partial observability normal form for nonlinear functional observers design. In: Control Conference (CCC), 2013 32nd Chinese. IEEE, pp. 95–100.

[182] Terrell, W. J., 1997. Observability of nonlinear differential algebraic systems. Circuits Systems Signal Processing 16 (2), 271–285.

[183] Trentelman, H., Stoorvogel, A., Hautus, M. L. J., 2001. Control Theory for Linear Systems. Communications and control engineering. Springer, New York, London.

[184] Wang, S., Davison, E., Dorato, P., 1975. Observing the states of systems with unmeasurable disturbances. IEEE Transactions on Automatic Control 20, 716–717.

[185] Wang, Y., Lynch, A., 2009. Observer design using a generalized time-scaled block triangular observer form. Systems & Control Letters, 58, 346–352.

[186] Wang, Y., Lynch, A., 2010. Multiple time scalings of a multi-output observer form. IEEE Transactions on Automatic Control 55, 966–971.

[187] Xia, X., Gao, W., 1989. Nonlinear observer with linearizable error dynamics. SIAM Journalon Control and Optimization, 27, 199–216.

[188] Xia, X., Marquez, L., Zagalak, P., Moog, C., 2002. Analysis of nonlinear time-delay systems using modules over non-commutative rings. Automatica 38, 1549–1555.

[189] Yang, C., Zhang, Q., Chai, T., 2009. Observer design for a class of nonlinear descriptor systems. In: in Proc. IEEE Conf. Decision Control.

[190] Yang, F., Wilde, R., 1988. Observers for linear systems with unknown inputs. IEEE Transactions on Automatic Control 33 (7), 677–681.

[191] Yang, J., Back, J., Seo, J., Shim, H., 2010. Reduced-order dynamic observer error linearization. In: Nonlinear Control Systems. pp. 915–920.

[192] Yang, J., Back, J., Seo, J. H., 2011. A complete solution to a simple case of dynamic observer error linearization: New approach to observer error linearization. IEICE transactions on fundamentals of electronics, communications and computer sciences 94 (1), 424–429.

[193] Yip, E., Sincovec, R., 1981. Solvability, controllability, and observability of continuous descriptor systems. IEEE Transactions on Automatic Control AC-26 (3), 702–707.

[194] Yu, L., Barbot, J.-P., Benmerzouk, D., Boutat, D., Floquet, T., Zheng, G., 2011. Discussion about sliding mode algorithms, zeno phenomena and observability. In: Sliding Modes after the First Decade of the 21st Century. Springer Berlin Heidelberg, pp. 199–219.

[195] Yu, L., Barbot, J. P., Zheng, G., Sun, H., 2010. Compressive sensing with chaotic sequence. IEEE Signal Processing Letters 17 (8), 731–734.

[196] Yu, L., Barbot, J. P., Zheng, G., Sun, H., 2010. Toeplitz-structured chaotic sensing matrix for compressive sensing. In: Communication Systems Networks and Digital Signal Processing (CSNDSP), 2010 7th International Symposium on. IEEE, pp. 229–233.

[197] Yu, L., Sun, H., Barbot, J. P., Zheng, G., 2011. Bayesian compressive sensing for clustered sparse signals. In: Acoustics, Speech and Signal Processing (ICASSP), 2011 IEEE International Conference on. IEEE, pp. 3948–3951.

[198] Yu, L., Sun, H., Barbot, J.-P., Zheng, G., 2012. Bayesian compressive sensing for cluster structured sparse signals. Signal Processing 92 (1), 259–269.

[199] Yu, L., Sun, H., Zheng, G., Barbot, J. P., 2015. Model based bayesian compressive sensing via local beta process. Signal Processing 108, 259–271.

[200] Yu, L., Wei, C., Zheng, G., 2015. Adaptive bayesian estimation with cluster structured sparsity. IEEE Signal Processing Letters 22 (12), 2309–2313.

[201] Yu, L., Wei, C., Zheng, G., 2015. Group sparse lms for multiple system identification. In: Signal Processing Conference (EUSIPCO), 2015 23rd European. IEEE, pp. 1691–1695.

[202] Yu, L., Zheng, G., Boutat, D., Barbot, J.-P., 2011. Observability and observer design for a class of switched systems. IET control theory & applications 5 (9), 1113–1119.

[203] Zheng, G., Aggoune, W., Barbot, J.-P., 2009. Analogue private communication based on hybrid chaotic systems with delays. IFAC Proceedings Volumes 42 (7), 246–251.

[204] Zheng, G., Barbot, J.-P., Boutat, D., 2011. Delay identification for nonlinear time-delay systems with unknown inputs. In: Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on. IEEE, pp. 6302–6307.

[205] Zheng, G., Barbot, J.-P., Boutat, D., 2013. Identification of the delay parameter for nonlinear time-delay systems with unknown inputs. Automatica 49 (6), 1755–1760.

[206] Zheng, G., Barbot, J.-P., Boutat, D., Floquet, T., Richard, J.-P., 2010. Causal observability of nonlinear time-delay systems with unknown inputs. In: Decision and Control (CDC), 2010 49th IEEE Conference on. IEEE, pp. 443–448.

[207] Zheng, G., Barbot, J.-P., Boutat, D., Floquet, T., Richard, J.-P., 2010. Finite time observation of nonlinear time-delay systems with unknown inputs. IFAC Proceedings Volumes 43 (14), 427–432.

[208] Zheng, G., Barbot, J.-P., Boutat, D., Floquet, T., Richard, J.-P., 2011. On observation of time-delay systems with unknown inputs. IEEE Transactions on Automatic Control 56 (8), 1973–1978.

[209] Zheng, G., Barbot, J.-P., Floquet, T., Boutat, D., Richard, J.-P., 2011. On obserability of nonlinear time-delay systems with unknown inputs. IEEE Transactions on Automatic Control 56 (8), 1973–1978.

[210] Zheng, G., Bejarano, F. J., Perruquetti, W., Richard, J.-P., 2015. Unknown input observer for linear time-delay systems. Automatica 61, 35–43.

[211] Zheng, G., Boutat, D., 2010. Reduced observer and its application to synchronization of chaotic systems. In: Communication Systems Networks and Digital Signal Processing (CSNDSP), 2010 7th International Symposium on. IEEE, pp. 201–207.

[212] Zheng, G., Boutat, D., 2011. Synchronisation of chaotic systems via reduced observers. IET Control Theory & Applications 5 (2), 308–314.

[213] Zheng, G., Boutat, D., Barbot, J., 2005. Output dependent observability linear normal form. In: Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on. IEEE, pp. 7026–7030.

[214] Zheng, G., Boutat, D., Barbot, J., 2007. Single output dependent observability normal form. SIAM Journal on Control and Optimization, 46 (6), 2242–2255.

[215] Zheng, G., Boutat, D., Barbot, J., Floquet, T., 2006. Multiple secure communication based on chaos. IFAC Proceedings Volumes 39 (8), 315–320.

[216] Zheng, G., Boutat, D., Barbot, J.-P., 2007. Single output dependent quadratic observability normal form. IFAC Proceedings Volumes 40 (12), 432–437.

[217] Zheng, G., Boutat, D., Barbot, J.-P., 2008. Geometrical conditions for output depending observability normal form. In: Decision and Control, 2008. CDC 2008. 47th IEEE Conference on. IEEE, pp. 5075–5079.

[218] Zheng, G., Boutat, D., Barbot, J.-P., 2009. Multi-output dependent observability normal form. Nonlinear Analysis: Theory, Methods & Applications 70 (1), 404–418.

[219] Zheng, G., Boutat, D., Barbot, J.-P., 2010. On uniform controller design for linear switched systems. Nonlinear Analysis: Hybrid Systems 4 (1), 189–198.

[220] Zheng, G., Boutat, D., Barbot, J.-P., 2012. Finite time observer design for linear switched system. In: Control Conference (CCC), 2012 31st Chinese. IEEE, pp. 2178–2183.

[221] Zheng, G., Boutat, D., Boutat-Baddas, L., Floquet, T., Barbot, J.-P., 2007. A multiple data secure communication based on synchronization of chaotic systems with zero dynamics. In: Control Conference (ECC), 2007 European. IEEE, pp. 5878–5883.

[222] Zheng, G., Boutat, D., Floquet, T., Barbot, J.-P., 2008. Secure data transmission based on multi-input multi-output delayed chaotic system. International Journal of Bifurcation and Chaos 18 (07), 2063–2072.

[223] Zheng, G., Boutat, D., Floquet, T., Barbot, J.-P., 2009. Secure communication based on multi-input multi-output chaotic system with large message amplitude. Chaos, Solitons & Fractals 41 (3), 1510–1517.

[224] Zheng, G., Orlov, Y., Perruquetti, W., Richard, J.-P., 2011. Finite time observer-based control of linear impulsive systems with persistently acting impact. IFAC Proceedings Volumes 44 (1), 2442–2447.

[225] Zheng, G., Orlov, Y., Perruquetti, W., Richard, J.-P., 2014. Finite-time-observer design for nonlinear impulsive systems with impact perturbation. International Journal of Control 87 (10), 2097–2105.

[226] Zheng, G., Yu, L., Boutat, D., Barbot, J.-P., 2009. Algebraic observer for a class of switched systems with zeno phenomenon. In: Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on. IEEE, pp. 3876–3881.

List of publications

Journal papers

[J1] G. Zheng, D. Efimov and W. Perruquetti, Design of interval observer for a class of uncertain unobservable nonlinear systems, Automatica, accepted, 2015

[J2] R. Tami, G. Zheng, D. Boutat, D. Aubry and H. Wang, Partial observer normal form for nonlinear system, Automatica, accepted, 2015

[J3] G. Zheng, D. Efimov, F. Bejarano, W. Perruquetti and H. Wang, Interval observer for a class of uncertain nonlinear singular systems, Automatica, accepted, 2015

[J4] L. Yu, W. Wu and G. Zheng, Adaptive Bayesian estimation with cluster structured sparsity, IEEE Signal Processing Letters, accepted, 2015

[J5] G. Zheng, F. Bejarano, W. Perruquetti and J.-P. Richard, Unknown input observer for linear time-delay systems, Automatica, vol. 61, p.35-43, 2015

[J6] F. Bejarano, T. Floquet, W. Perruquetti, and G. Zheng, Observation of nonlinear differential-algebraic systems with unknown Inputs, IEEE Transactions on Automatic Control, vol. 60(7), p. 1957-1962, 2015

[J7] L. Yu, H. Sun, G. Zheng and J.P. Barbot, Model based Bayesian compressive sensing via local beta process, Signal Processing, vol. 108, p. 259-271, 2014

[J8] G. Zheng, Y. Orlov, W. Perruquetti, J.P. Richard, Finite time observer design for nonlinear impulsive systems with impact perturbation, International Journal of Control, vol. 87(10), p. 2097-2105, 2014

[J9] Y. Ma, G. Zheng, W. Perruquetti, Z. Qiu, Local path planning for mobile robots based on intermediate objectives, Robotica, p. 1-15, 2014

[J10] F. Bejarano, G. Zheng, Observability of linear systems with commensurate delays and unknown inputs, Automatica, vol. 50(8), p. 2077-2083, 2014

[J11] R. Tami, D. Boutat, G. Zheng, Extended output depending normal form, Automatica, vol. 49(7), p. 2192-2198 , 2013

[J12] R. Tami, D. Boutat, G. Zheng, Observateur non linéaire étendu, Journal Européen des Systèmes Automatisés, Hermès, vol. 47(4-8), p.503-518, 2013

[J13] G. Zheng, J.-P. Barbot and D. Boutat, Delay identification for nonlinear time-delay systems with unknown inputs, Automatica, vol. 49(6) p. 1755-1760, 2013

[J14] F. Bejarano, T. Floquet, W. Perruquetti, and G. Zheng, Observability and detectability of singular linear systems with unknown inputs, Automatica, vol. 49(3), p. 793-800, 2013

[J15] A. Girard and G. Zheng, Verification of safety and liveness properties of metric transition systems, ACM Transactions on Embedded Computing Systems, vol. 11(S2), 2012

[J16] L. Yu, H. Sun, J.-P. Barbot, G. Zheng, Bayesian compressive sensing for cluster structured sparse signals, Signal Processing, vol. 92(1), p. 259-269, 2012

[J17] H. Hamiche, K. Kemih, M. Ghanes, G. Zheng, S. Djennoune, Passive and impulsive synchronization of a new four-dimensional chaotic system. Nonlinear Analysis: Theory, Methods and Applications, vol. 74(4), p. 1146-1154, 2011

[J18] G. Zheng, J.-P. Barbot, D. Boutat, T.Floquet, J.-P. Richard, On observation of time-delay systems with unknown input, IEEE Transactions on Automatic Control, vol. 56(8), p. 1973-1978, 2011

[J19] L. Yu, G. Zheng, D. Boutat, J.-P. Barbot, Observability and observer design for a class of switched systems, IET Control Theory & Application, vol. 5 (2), p. 308-314, 2011

[J20] G. Zheng and D. Boutat, Syncrhonization of chaotic systems via reduced observers, IET Control Theory & Applications, vol. 5(2), p. 308-314, 2011

[J21] S. Bououden, D. Boutat, G. Zheng, J.-P. Barbot, F. Kratz, A triangular canonical form for a class of 0-flat nonlinear system, International Journal of Control, vol. 84(2), p. 261-269, 2011

[J22] L. Yu, J.-P. Barbot, G. Zheng and H. Sun, Compressive sensing with chaotic sequence, IEEE Signal Processing Letters, vol. 17(8), p. 731-734, 2010

[J23] G. Zheng, D. Boutat and J.-P. Barbot, On uniform controller design for linear switched systems, Nonlinear Analysis: Hybrid Systems, vol. 40, p. 189-198, 2010.

[J24] M. Ghanes and G. Zheng, On sensorless induction motor drivers: sliding-mode observer and output feedback controller, IEEE Transaction on industrial electronics, vol. 56(9), p. 3404-3413, 2009.

[J25] G. Zheng and A. Tonnelier, Chaotic solutions in the quadratic integrate-and-fire neuron with adaptation, Cognitive Neurodynamics, vol. 3, p. 197-204, 2009.

[J26] G. Zheng, A. Tonnelier and D. Martinez, Voltage-stepping schemes for the simulation of spiking neural networks, Journal of Computational NeuroScience, vol. 26(3), p. 409-423, 2009.

[J27] G. Zheng, D. Boutat, T. Floquet and J.-P. Barbot, Secure communication based on multi-input multi-output chaotic system with large message amplitude, Chaos, Solitons and Fractals, vol. 41(3), p. 1510-1517, 2009.

[J28] G. Zheng, D. Boutat and J.-P. Barbot, Multi-output dependent observability normal form, Journal of Nonlinear Analysis Series A: Theory, Methods & Applications, vol. 70(1), p. 404-418, 2009

[J29] G. Zheng, D. Boutat, T. Floquet and J.-P. Barbot, Secure data transmission based on multi-input multi-output delayed chaotic system, International Journal of Bifurcation and Chaos, vol. 18(7), p.2063-2072, 2008.

[J30] G. Zheng, D. Boutat and J.-P. Barbot, Single output dependent observability normal form, SIAM Journal on Control and Optimization, vol. 46(6), p. 2242-2255, 2007.

Book Chapters

[B1] G. Zheng and J.-P. Richard, Identifiability and observability of nonlinear time-delay system with unknown inputs, in 'Recent Results on Nonlinear Time Delayed Systems', Edt. I. Karafyllis, M. Malisoff, F. Mazenc, P. Pepe, (Advances in Delays and Dynamics at Springer), Springer Verlag, 2015.

[B2] L. Yu, J.-P. Barbot, D. Benmerzouk, D. Boutat, T. Floquet and G. Zheng, Discussion about sliding modes algorithms, Zeno phenomena and obserervability, in 'Sliding Modes after the first Decade of the 21st Century', Lecture Notes in Control and Information Sciences, Volume 412, Spirnger, 2012.

Conference Papers

[C1] G. Zheng, D. Efimov, F. J. Bejarano, W. Perruquetti, On the interval estimation for nonlinear singular system, in Proc. of IEEE CDC, 2015

[C2] W. Saadi, D. Boutat, G. Zheng, L. Sbita, Multi-Output Partial Nonlinear Observer Normal Form, in Proc. of IEEE CDC, 2015

[C3] M. Guerra, C. Vazquez, D. Efimov, G. Zheng, L. Freidovich, W. Perruquetti, ε-Invariant Output Stabilization: Homogeneous Approach and Dead Zone Compensation, in Proc. of IEEE CDC, 2015

[C4] Z. Shao, G. Zheng, D. Efimov and W. Perruquetti, Modelling and control for position-controlled modular robot manipulators, in Proc. of IEEE/RSJ IROS, 2015

[C5] M. Guerra, G. Zheng, D. Efimov and W. Perruquetti, Robust decentralized supervisory control in a leader-follower configuration with obstacle avoidance, in Proc. of IFAC MICNON 2015

[C6] Z. Shao, G. Zheng, D. Efimov and W. Perruquetti, Modelling and control of actuators with built-in position controller, in Proc. of IFAC MICNON 2015

[C7] M. Guerra, G. Zheng, D. Efimov, W. Perruquetti, An ISS based Solution to avoid Local Minima in the Potential Field Method, in Proc. of ECC, 2015

[C8] L. Yu, G. Zheng, F. J. Bejarano, Observer design for linear systems with commensurate delays, in Proc. of IEEE CCC, 2015

[C9] L. Yu, W. Chen, G. Zheng, Group sparse LMS for multiple system identification, in Proc. of EUSIPCO; 2015

[C10] M. Guerra, D. Efimov, G. Zheng, W. Perruquetti, Finite-Time Supervisory Stabilization for a Class of Nonholonomic Mobile Robots Under Input Disturbances, in Proc. of IFAC World Congress, 2014

[C11] G. Zheng, D. Efimov, W. Perruquetti, A new finite-time observer for nonlinear systems, in Proc. of IEEE CDC 2014

[C12] Z. Kader, G. Zheng, J.P. Barbot, Left inversion of nonlinear time delay system, in Proc. of IEEE CDC 2014

[C13] R. Tami, D. Boutat , G. Zheng, F. Kratz, Parameters and states estimation for Dengue epidemic model,in Proc. of ECC, 2014

[C14] R. Tami, G. Zheng, D. Boutat, Nonlinear Observer for the PM Synchronous Motor, in Proc. of ACC, 2014

[C15] Y. Ma, G. Zheng, W. Perruquetti, Z. Qiu, Motion planning for non-holonomic mobile robots using the i-PID controller and potential field, in Proc. of IEEE IROS, 2014

[C16] G. Zheng, D. Boutat, Adaptive observer for simultaneous estimation of state and parameter for a class of nonlinear systems, in Proc. of IEEE CCC, 2014

[C17] Y. Ma, G. Zheng, W. Perruquetti, Z. Qiu, Control of nonholonomic wheeled mobile robots via i-PID controller, in Proc. of IEEE IROS 2013

[C18] G. Zheng, D. Efimov, W. Perruquetti, Interval state estimation for uncertain nonlinear systems, in Proc. of IFAC Nolcos 2013

[C19] Y. Ma, G. Zheng, W. Perruquetti, Real-time Identification of different types of non-holonomic mobile robots, in Proc. of IFAC Nolcos 2013

[C20] R. Tami, G. Zheng, D. Boutat, Nonlinear observer normal form with output injection and extended dynamic, in Proc. of IFAC Nolcos 2013

[C21] Y. Ma, G. Zheng, W. Perruquetti, Real-time local path planning for mobile robots, in Proc. of IEEE RoMoCo 2013, Poland

[C22] Y. Ma, G. Zheng, W. Perruquetti, Cooperative path planning for mobile robots based on visibility graph, in Proc. of IEEE CCC 2013, China

[C23] R. Tami, G. Zheng, D. Boutat, Partial observability normal form for nonlinear functional observers design, in Proc. of IEEE CCC 2013, China

[C24] F. J. Bejarano, W. Perruquetti, T. Floquet, G. Zheng, State Reconstruction of Nonlinear Differential-Algebraic Systems with Unknown Inputs, in Proc. of 51st IEEE CDC, 2012, USA

[C25] D. Boutat, G. Zheng, L. Boutat-Baddas, M. Darouach, Observers design for a class of nonlinear singular systems, in Proc. of 51st IEEE Conference on Decision and Control, 2012, USA

[C26] J.-P. Barbot, G. Zheng, T. Floquet and D. Boutat, Delay estimation algorithm for nonlinear time-delay systems with unknown inputs, in Proc. of IFAC TDS 2012, USA

[C27] G. Zheng, D. Boutat, J.-P. Barbot, Finite time observer design for linear switched system, in Proc. of IEEE CCC 2012, China

[C28] R. Tami, D. Boutat, G. Zheng, H. Hammouri, Une forme normale pour l'observateur d'ordre reduit, in Proc. of IEEE CIFA 2012, France

[C29] D. Boutat, G. Zheng, L. Boutat-Baddas and M. Darouach, Forme normale d'observabilité pour une classe de systèmes non linéaires singuliers, in Proc. of IEEE CIFA 2012, France

[C30] G. Zheng, J.-P. Barbot, D. Boutat, Delay identifcation for nonlinear time-delay systems, in Proc. of IEEE CDC 2011, USA

[C31] F.-J. Bejarano, T. Floquet, W. Perruquetti, G. Zheng, Observability and Detectability Analysis of Singular Linear Systems with Unknown Inputs, in Proc. of IEEE CDC 2011, USA

[C32] G. Zheng, Y. Orlov, W. Perruquetti, J.P. Richard, Finite time observer design for nonlinear impulsive systems with impact perturbation, in Proc. of IEEE CCDC 2011, China

[C33] G. Zheng, Y. Orlov, W. Perruquetti, J.P. Richard, Finite Time Observer-Based Control of Linear Impulsive Systems with Persistently Acting Impact, in Proc. of IFAC World Congress, 2011, Italy

[C34] L. Yu, H. Sun, J.P. Barbot, G. Zheng, Bayesian compressive sensing for cluster structured sparse signals, in Proc. of IEEE ICASSP, 2011, Czech

[C35] G. Zheng, J.-P. Barbot, D. Boutat, T. Floquet and J.-P. Richard, Causal obserability of nonlinear time-delay systems with unknown inputs, in Proc. of IEEE CDC, 2010

[C36] L. Yu, J.-P. Barbot, D. Benmerzouk, D. Boutat, T. Floquet and G. Zheng, Disscussion about sliding modes algorithms, Zeno phenomena and obserervability of switched systems, in Proc. of IEEE VSS, 2010

[C37] G. Zheng and D. Boutat, Reduced observers and its application to synchronization of chaotic systems, in Proc. of IEEE CSNDSP, 2010

[C38] L. Yu, J.-P. Barbot, G. Zheng and H. Sun, Toeplitz-Structured Chaotic Sensing Matrix for Compressive Sensing, in Proc. of IEEE CSNDSP, 2010

[C39] G. Zheng, J.-P. Barbot, D. Boutat, T. Floquet and J.-P. Richard, Finite time observation of nonlinear time-delay systems with unknown inputs, in Proc. of IFAC Nolcos, 2010

[C40] D. Boutat, G. Zheng and H. Hammouri, A nonlinear canonical form for reduced order observer design, in Proc. of IFAC, Nolcos, 2010

[C41] G. Zheng, L. Yu, D. Boutat and J.-P. Barbot, Algebraic observer for a class of switched systems with Zeno phenomenon, In Proc. of IEEE CDC 2009

[C42] G. Zheng, W. Aggoune and J.-P. Barbot, Private communication based on hybrid chaotic systems with delays, In Proc. of IFAC Chaos 2009

[C43] G. Zheng, D. Boutat and J.-P. Barbot, Geometrical conditions for output depending observability normal form, In Proc. of IEEE CDC, 2008

[C44] G. Zheng and A. Girard, Bounded and unbounded safety verification using bisimulation metrics, Springer Lecture Notes in Computer Science, Hybrid systems: Computation and Control, 2008.

[C45] G. Zheng, M. Ghanes and J. De-Leon-Morales, On simultaneous parameter identification and state estimation for cascade state affine systems, In Proc. of American Control Conference 2008

[C46] M. G. Kabbi, G. Zheng, A. Tonnelier and D. Martinez, Simulation de réseaux de neurons intègre-et-tire non linéaires par discrétisation du potentiel, In the second french conference of computational Neurosciences, 2008, Marseille, France

[C47] G. Zheng, D. Boutat and J.-P. Barbot, Output depending quadratic normal form, In Proc. of 7th IFAC Symposium on Nonlinear Control Systems, (2007)

[C48] G. Zheng, D. Boutat, L. Boutat-Baddas, T. Floquet and J.-P. Barbot,A multiple data secure communication based on synchronization of chaotic systems with zero dynamics, In Proc. of Europe Control Conference 2007, Kos, Greece

[C49] D. Boutat, G. Zheng, J.-P. Barbot and H. Hammouri, Observer error linearization multi-output depending, In Proc. of IEEE CDC 2006, San Diego, USA

[C50] G. Zheng, D. Boutat and J.-P. Barbot, Approche multi-sorties de la transmission de données basée sur la synchronisation de système chaotique, In Proc. of IEEE CIFA 2006, Bordeaux, France

[C51] G. Zheng, D. Boutat , T. Floquet and J.-P. Barbot, Multiple secure communication based on chaos, In Proc. of 1st IFAC Chaos 2006, Reims, France

[C52] G. Zheng, D. Boutat and J.-P. Barbot, Output dependent observability linear normal form, In Proc. of IEEE CDC-ECC 2005, Serville, Spain

[C53] G. Zheng, Discussion sur les formes normales d'observabilité et les observateurs modes glissants étape par étape,JDMACS 2005, Lyon, France

[C54] G. Zheng, L. Boutat-Baddas, D. Boutat and J.-P. Barbot, Some comments on observability normal form and step-by-step sliding mode observer, SIAM conference 2005, New Orleans, USA