



Analysis of non-stationary (seasonal/cyclical) long memory processes

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UNIVERSITÉ PARIS 1
EAST CHINA NORMAL UNIVERSITY

**ANALYSIS OF NON-STATIONARY
(SEASONAL/CYCLICAL) LONG
MEMORY PROCESSES**

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Department:	CES de l'Université Paris 1 and Economics Department
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Résumé

La mémoire longue, aussi appelée la dépendance à long terme (LRD), est couramment détectée dans l'analyse de séries chronologiques dans de nombreux domaines, par exemple, en finance, en économétrie, en hydrologie, etc. Donc l'étude des séries temporelles à mémoire longue est d'une grande valeur. L'introduction du processus ARFIMA (fractionally autoregressive integrated moving average) établit une relation entre l'intégration fractionnaire et la mémoire longue, et ce modèle a trouvé son pouvoir de prévision à long terme, d'où il est devenu l'un des modèles à mémoire longue plus populaires dans la littérature statistique. Précisément, un processus à longue mémoire ARFIMA(p, d, q) est défini comme suit:

$$\Phi(B)(I - B)^d(X_t - \mu) = \Theta(B)\varepsilon_t, \quad t \in \mathbb{Z},$$

où $\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ et $\Theta(z) = 1 + \dots + \theta_1 z + \dots + \theta_q z^q$ sont des polynômes d'ordre p et q , respectivement, avec des racines en dehors du cercle unité; ε_t est un bruit blanc Gaussien avec une variance constante σ_ε^2 . Lorsque $d \in (-\frac{1}{2}, \frac{1}{2})$, $\{X_t\}$ est stationnaire et inversible. Cependant, l'hypothèse a priori de la stationnarité des données réelles n'est pas raisonnable.

Par conséquent, de nombreux auteurs ont fait leurs efforts pour proposer des estimateurs applicables au cas non-stationnaire. Ensuite, quelques questions se lèvent que quel estimateur doit être choisi pour applications, et que à quoi on doit faire attention lors de l'utilisation de ces estimateurs. Donc à l'aide de la simulation de Monte Carlo à échantillon fini, nous effectuons une comparaison complète des estimateurs sémi-paramétriques, y compris les estimateurs de Fourier et les estimateurs d'ondelettes, dans le cadre des séries non-stationnaires. À la suite de cette étude comparative, nous avons que (i) sans bonnes échelles taillées, les estimateurs d'ondelettes sont fortement biaisés et ils ont généralement une performance inférieure à ceux de Fourier; (ii) tous les estimateurs étudiés sont robustes à la présence d'une tendance linéaire en temps dans le niveau de $\{X_t\}$ et des effets GARCH dans la variance de $\{X_t\}$; (iii) dans une situation où la probabilité de transition est basse, la consistance des estimateurs quand même tient aux changements de régime dans le niveau de $\{X_t\}$, mais les changements ont une contamination au résultat d'estimation; encore, l'estimateur d'ondelettes de log-régression fonctionne mal dans ce cas; et (iv) en général, l'estimateur complètement étendu de Whittle avec un polynôme locale (fully-extended local polynomial Whittle Fourier estimator) est préféré pour une utilisation pratique, et cet estimateur nécessite une bande (i.e. un nombre de fréquences utilisés dans l'estimation) plus grande que les autres estimateurs de Fourier considérés dans ce travail.

D'ailleurs, la mémoire longue saisonnier/cyclique (SCLM) est une caractéristique courante dans de nombreuses séries chronologiques et ils peuvent être également non-stationnaire. En particulier, nous nous penchons sur une classe des séries SCLM, des processus Gegenbauer à 1-facteur dont le paramètre de mémoire peut être n'importe quel nombre réel.

On développe dans le domaine de Fourier deux nouveaux estimateurs de paramètre de mémoire, à savoir l'estimateur semi-paramétrique de la régression du log-periodogram et l'estimateur Gaussien semi-paramétrique. La consistance des estimateurs est prouvée et leur robustesse est vérifiée à l'aide d'une simulation de Monte-Carlo. Les deux estimateurs proposés profitent des données effilées, ce qui atténue les fuites de fréquence autour du pic spectral de la série non-stationnaire SCLM. Pour illustrer la validité de nos estimateurs et combler le vide qu'il existe très peu de la littérature sur la performance à échantillon fini (finite sample performance: FSP) des estimateurs du paramètre de mémoire pour les séries SCLM, nous effectuons une étude de Monte Carlo où des processus Gegenbauer à 1-facteur stationnaires et non-stationnaires sont simulés. De cette étude de Monte Carlo, le FSP des estimateurs illustre que (i) quand $d > -\frac{1}{2}$, le GSE possède de meilleures propriétés à échantillon fini, que le LPSE; (ii) le LPSE donne des estimations fiables pour $d < -\frac{1}{2}$ tandis que le GSE ne peut pas; (iii) Nous constatons également que la localisation du pic spectral a aussi une influence importante sur le résultat de l'estimation; et (iiii) de plus, d'un point de vue pragmatique, on jette une lumière sur la sélection optimale du nombre de fréquences et de l'ordre du taper pour les deux estimateurs.

Enfin, nous présentons deux applications. La première concerne la propriété à mémoire longue dans les indices du prix de l'immobilier familial (house price index: HPI) des 50 états et du Washington D.C. aux États-Unis. Nous obtenons les estimations de paramètre de mémoire, i.e., d , de l'HPI par une procédure d'estimation semi-paramétrique et une analyse de partitionnement est effectuée par rapport à des valeurs de d . Selon le résultat de partitionnement, il est constaté que les estimations de paramètre de mémoire présentent une agrégation régionale et la valeur de l'estimation est liée à la densité de la population de l'état. Cette observation sert de référence utile pour les décideurs chinois lorsqu'ils lancent des règlements pour freiner les prix d'immobilier. Pour la réduction de la bulle du marché immobilier et la prévention de la brusque chute des prix des logements, nous suggérons que les règlements devraient varier de régions à régions et le contrôle de la population est toujours recommandé dans les régions métropolitaines. La deuxième compare la qualité de prédiction des processus ARFIMA et des processus GARMA sur la série mensuelle du taux de change de l'Euro v.s. le Dollar US (USD). Bien que le processus GARMA ait théoriquement plus de souplesse que le processus ARFIMA, d'un point de la prédiction, la capacité de prévision de celui-là est inférieure à celle-ci dans ce cas.

Mots clés: fréquence de Fourier, l'estimation semi-paramétrique Gaussienne, effilé, un processus Gegenbauer à k facteurs, échantillon fini, l'estimation semi-paramétrique de log-périodogramme, non-stationnaire, mémoire longue, ondelettes, l'analyse de partitionnement, prédiction, HPI.

Abstract

Long memory, also called long range dependence (LRD), is commonly detected in the analysis of real-life time series data in many areas; for example, in finance, in econometrics, in hydrology, etc. Therefore the study of long-memory time series is of great value. The introduction of ARFIMA (fractionally autoregressive integrated moving average) process established a relationship between the fractional integration and long memory, and this model has found its power in long-term forecasting, hence it has become one of the most popular long-memory models in the statistical literature. Specifically, an ARFIMA(p, d, q) process $\{X_t\}_{t \in \mathbb{Z}}$ is defined as follows.

$$\Phi(B)(I - B)^d(X_t - \mu) = \Theta(B)\varepsilon_t, \quad t \in \mathbb{Z},$$

where $\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ and $\Theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$ are polynomials of order p and q , respectively, with roots outside the unit circle; and ε_t is Gaussian white noise with a constant variance σ_ε^2 . When $d \in (-\frac{1}{2}, \frac{1}{2})$, $\{X_t\}$ is stationary and invertible. However, the a priori assumption on stationarity of real-life data is not reasonable. Therefore many statisticians have made their efforts to propose estimators applicable to the non-stationary case. Then questions arise that which estimator should be chosen for applications; and what we should pay attention to when using these estimators. Therefore we make a comprehensive finite sample comparison of semi-parametric Fourier and wavelet estimators under the non-stationary ARFIMA setting. In light of this comparison study, we have that (i) without proper scale trimming the wavelet estimators are heavily biased and they generally have an inferior performance to the Fourier ones; (ii) all the estimators under investigation are robust to the presence of a linear time trend in levels of $\{X_t\}$ and the GARCH effects in variance of $\{X_t\}$; (iii) the consistency of the estimators still holds in the presence of regime switches in levels of $\{X_t\}$, however, it tangibly contaminates the estimation results. Moreover, the log-regression wavelet estimator works badly in this situation with small and medium sample sizes; and (iv) fully-extended local polynomial Whittle Fourier (fextLPWF) estimator is preferred for a practical utilization, and the fextLPWF estimator requires a wider bandwidth than the other Fourier estimators.

Moreover, seasonal/cyclical long memory (SCLM) is a characteristic of many time series and they may be also non-stationary. In particular, we consider a class of SCLM series, 1-factor Gegenbauer process whose memory parameter can be any real number. We propose two new frequency-domain memory parameter estimators, i.e., the log-periodogram semi-parametric estimator and Gaussian semi-parametric estimator. The consistency of the estimators is proved and their robustness is verified via Monte Carlo simulations. The two proposed estimators take advantage of tapering the raw data, which

mitigates frequency leakage around the spectral peak of the non-stationary SCLM series. To illustrate the validity of our estimators and to fill the blank that there exists very little literature on the finite sample performance (FSP) of the memory parameter estimators for SCLM series, we carry out a Monte Carlo experiment where stationary and non-stationary 1-factor Gegenbauer processes are simulated. From this Monte Carlo experiment, the FSP of the estimators is illustrated that the GSE excels the LPSE for $d > -\frac{1}{2}$ and the LPSE yields reliable estimates for $d < -\frac{1}{2}$ while the GSE can not. We also find that the location of spectral peak also has a substantial impact on the estimation result, and furthermore from a pragmatic perspective, we cast a light on the optimal selection of parameter values, i.e., the bandwidth, the taper order, for both estimators.

Finally, we present two applications. The first is concerned with the long-memory property in the single-family house price indexes (HPI) of the U.S. 50 states and Washington D.C.. We obtain the memory parameter estimates, i.e., d , of the HPI through a semi-parametric estimation procedure and a cluster analysis is performed with respect to the values of d . According to the clustering result, it is found that the memory parameter estimates exhibit a regional aggregation phenomenon and the value of the estimate is related to the state's population density, which serves as a useful reference for the Chinese policymakers when they launch the regulations to rein in house prices. In order to achieve the reduction of housing market bubble and the abrupt plummet of house prices, we suggest that the regulations should vary from regions to regions and population control is still recommended in the metropolitan areas. The second compares the prediction quality of ARFIMA processes and GARMA processes for the monthly exchange rate series, Euro vs. U.S. Dollar (USD). Although GARMA process theoretically has more flexibility than ARFIMA processes, from a pragmatic point of view, the prediction capability of the former is inferior to the latter in this case.

Key Words: Fourier frequency, Gaussian semi-parametric estimation, tapering, k-factor Gegenbauer process, finite sample performance, log-periodogram semi-parametric estimation, non-stationary (seasonal/cyclical) long memory time series, prediction, wavelet, cluster analysis, HPI.

Contents

1	Introduction	1
1.1	Long-Memory Processes	1
1.2	Research Background	4
1.3	Our Contributions	5
2	(Non-)Stationary Long-Memory Processes	8
2.1	Assumptions and Preliminaries	10
2.1.1	Fourier Domain	11
2.1.2	Wavelet Domain	12
2.2	Fourier Semi-Parametric Estimators	14
2.2.1	Fourier Log-Periodogram Semi-Parametric Estimator	14
2.2.2	Local Whittle Fourier Estimators	15
2.3	Wavelet Semi-Parametric Estimators	16
2.3.1	Local Regression Wavelet Estimator	17
2.3.2	Local Whittle Wavelet Estimator	17
2.4	Finite Sample Performance Comparison Study	18
2.4.1	Finite Sample Performance for ARFIMA(p, d, q) Processes	19
2.4.2	Monte Carlo Results for ARFIMA(p, d, q) Processes with Heteroscedasticity in $\{u_t\}$	30
2.4.3	Monte Carlo Results for ARFIMA(p, d, q) Processes with a Linear Time Trend or an Occasional-Structural-Break Process in Levels	31
2.4.4	Discussion	35
3	Seasonal/Cyclical Long-Memory Processes	37
3.1	Seasonal/Cyclical Long-Memory Processes	37
3.2	Assumptions and Preliminaries	38
3.3	Fourier Semi-parametric Estimation Methods for Seasonal/Cyclical Long-Memory Processes	41
3.3.1	Log-Periodogram Semi-parametric Estimator	42
3.3.2	Gaussian Semi-parametric Estimator	43
3.4	Monte Carlo Study for Seasonal/Cyclical Long-Memory Processes	44

4	Applications	55
4.1	The Chinese Housing Market Outlook Based on the US House Price Analysis	55
4.1.1	Semi-parametric Estimation Procedure	56
4.1.2	Empirical Study	57
4.1.3	Policy Implications	62
4.1.4	Discussion	64
4.2	Exchange Rate Time Series: U.S. Dollar v.s. Euro	65
5	Conclusion	69
5.1	Overview of Contribution	69
5.2	Possible Directions for Future Research	70
6	Appendix	72
6.1	Notations	72
6.2	Tables for Monte Carlo Results	74
6.2.1	Results of Fourier Estimators for Long-Memory Processes	74
6.2.2	Results of Wavelet Estimators for Long-Memory Processes	113
6.2.3	Results of Fourier Estimators for Seasonal/Cyclical Long-Memory Processes	121
	Bibliography	133
	Acknowledgment	141

Chapter 1

Introduction

As is known, most historical data are documented chronologically in order to help people to know and understand what happened in the past, and this type of data is called time series. The very first time series, the record of sunspots, could be dated back to 28 B.C. Specifically speaking, a time series is a collection of observations of well-defined data, measured typically at successive points in time spaced at uniform time intervals.

In this dissertation we study scale time series with long memory. Simply put, long memory implies strong correlation between observations far apart in time. The rate of decay of statistical dependence is more slowly than an exponential decay. To visualise this point of view, we plot the sample autocorrelation function (ACF) of a simulated short-memory (SM) series in the upper subplot of Fig.(1.1), and the lower subplot shows the ACF of a simulated long-memory (LM) series. In fact, the identification of long memory, also called long range dependence (LRD), is common in the analysis of real-life time series data in many areas; for example, in finance [79], [49], [41]; in econometrics [105]; in hydrology [91]; in environmental studies [116], [59], [23], [125]; in biology [72]; in informatics [71], etc.. Therefore the study of long-memory time series is of great value. We are going to formally give some definitions of long memory in next section.

1.1 Long-Memory Processes

Before we give three definitions of long-memory, we first need to know some important concepts of a time series: strict stationarity and weak stationary.

Definition 1.1.1. (*Strict stationarity*) *The time series $\{X_t\}_{t \in \mathbb{Z}}$ is said to be strict stationary if the joint distribution of $X_{t_1}, X_{t_2}, \dots, X_{t_k}$ is the same as that of $X_{t_1+h}, X_{t_2+h}, \dots, X_{t_k+h}$, for $t_1, \dots, t_k, h \in \mathbb{Z}$.*

This definition implies that the joint distribution of a strict stationary time series only depends on the 'difference' h , not the time $(t_1, \dots, t_k)$. However, in reality, we do not require such a strong stationarity of investigated time series and hence we have the 'weak stationarity', 'second-order stationary', or 'covariance stationary'. The covariance is defined as follows:

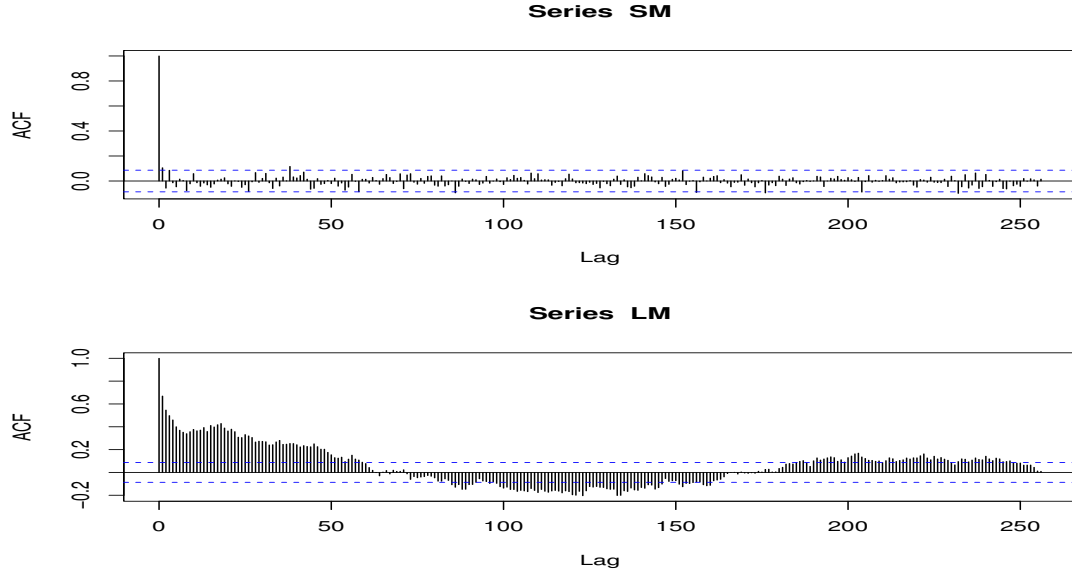


Figure 1.1: Short Memory v.s. Long Memory

Definition 1.1.2. (*Autocovariance function*) The autocovariance function of a time series $\{X_t\}_{t=-\infty}^{\infty}$ with $\text{Var}(X_t) < \infty$ is defined by

$$\gamma_X(s, t) = \text{Cov}(X_s, X_t) = E[(X_s - EX_s)(X_t - EX_t)].$$

Now we are ready to give the definition of the '(weak) stationarity'.

Definition 1.1.3. (*Stationarity or weak stationarity*) The time series $\{X_t\}_{t \in \mathbb{Z}}$ (where $\mathbb{Z}$ is the integer set) is said to be stationary if

1. $E(X_t^2) < \infty, \forall t \in \mathbb{Z};$
2. $E(X_t) = \mu, \forall t \in \mathbb{Z};$
3. $\gamma_X(s, t) = \gamma_X(s + h, t + h), \forall s, t, h \in \mathbb{Z};$

Clearly, for a stationary time series $\{X_t\}$, we have $\gamma_X(h) = \text{Cov}(X_t, X_{t+h})$, for $t, h \in \mathbb{Z}$ and $\gamma_X(h) = \gamma_X(-h)$. When $h = 0$, $\gamma_X(0) = \text{Cov}(X_t, X_t)$ is the variance of $\{X_t\}$, so the autocorrelation function for $\{X_t\}$ is defined as $\rho_X(h) = \gamma_X(h)/\gamma_X(0)$. First we give a time-domain definition for the long-memory property of a stationary time series.

Definition 1.1.4. A covariance stationary process $\{X_t\}_t$ is called long-memory process if it has an autocorrelation function $\rho(k)$ which behaves like a power function decaying to zero hyperbolically as: $\rho_X(k) \sim C_\rho(k)k^{-\alpha}$, as $k \rightarrow \infty$, where $\sim$ represents the asymptotic equivalence, and $C_\rho(k)$ is a function which varies slowly to infinity.

Here we say that a function g varies slowly to infinity (or to 0) if $\forall a \in \mathbb{R}, g(ax)/g(x) \rightarrow 1$ as $x \rightarrow \infty$ (or $x \rightarrow 0$). In light of this definition, the decay rate of autocorrelations is

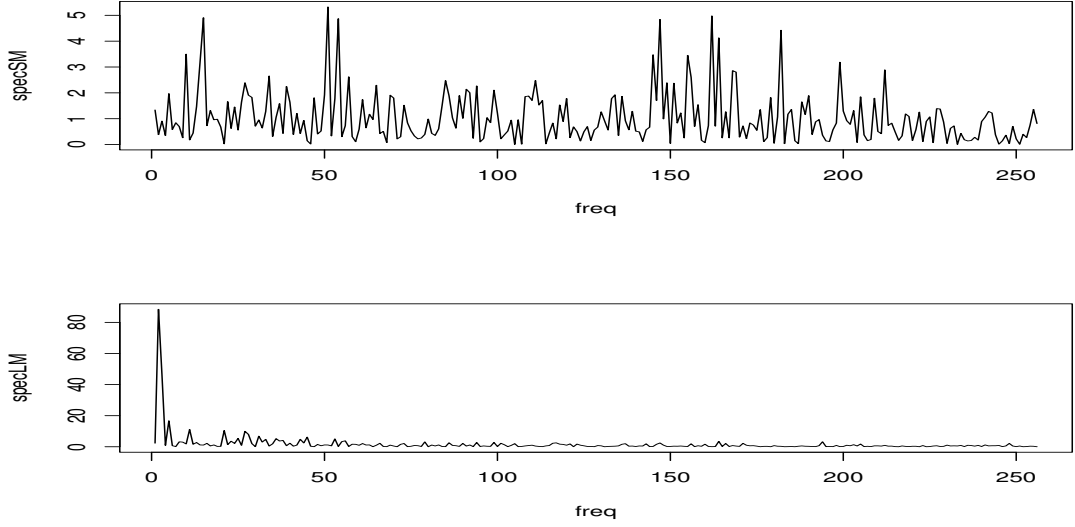


Figure 1.2: Short Memory v.s. Long Memory

very slow, and the autocorrelation series is absolutely divergent, i.e., $\sum_{k=0}^{\infty} = \infty$. Due to a powerful mathematical tool, Fourier transform, which relates a function of time into a new function of frequency, the past decades has witnessed the importance of the spectral domain analysis of long-memory property and hence the second definition is in this domain. Specifically, the long-memory property is characterized by the rate of explosions at low frequencies as follows:

Definition 1.1.5. A stationary process $\{X_t\}_t$ is called long-memory process if its spectral density function f is approximated by

$$f(\lambda) \sim C_f(\lambda)|\lambda|^{-2d}, \text{ as } \lambda \rightarrow 0^+,$$

where $C_f(\lambda)$ is a function which varies slowly to 0 at frequency zero and $d \in (0, \frac{1}{2})$.

When $d > \frac{1}{2}$, $f(\lambda)$ is not integrable so that the stationarity does not hold. Fig.(4.2) illustrates this definition by contrasting the spectrums of SM series and LM series, where the two series are the same as in Fig.(1.1). Obviously, there is a spectral peak at zero frequency for LM series while the spectrum of SM series oscillates in a much smaller range. Later, we will see that Eq.(1.1.5) is an important feature for semi-parametric methods to value the degree of dependence of the future data on the past data. It should be noted that in light of Tauber Theorem (see e.g. [39](p.421)), the equivalence of the Def.1.1.4 and Def.1.1.5 can be demonstrated.

The third definition is also in spectral domain and it defines the long-memory property in a more general way.

Definition 1.1.6. A stationary process $\{X_t\}_t$ is called long-memory process if its spectral density function f has the following property:

$$\exists \lambda_0 \in [0, \pi] \text{ such that } f(\lambda_0) \text{ unbounded.}$$

In other words, a long-memory process has at least one singularity in $[0, \pi]$. For reference, a review on definitions of long memory with respect to different applications is presented in [106].

1.2 Research Background

Due to the seminal work [53] and [60], the proposed ARFIMA model, using the difference operator $(I - B)^d$, where d can be any real number, established a relationship between the fractional integration and long memory. Furthermore, this model has found its power in long-term forecasting, and hence it has become one of the most popular parametric long-memory models in the statistical literature. For the stationary ARFIMA process, the memory parameter estimation methods can be generally divided into three categories: i) parametric estimation; ii) Fourier-domain semi-parametric estimation and iii) wavelet-domain semi-parametric estimation. Now we review these methods briefly in the following. The ARFIMA process is a special case of fractionally integrated process, which has been semi-parametrically exploited to a large extent. Now we

- Parametric estimators:
 - exact maximum likelihood (ML) estimators([112])
 - log-periodogram regression (LPR) estimator ([56]); pseudo-maximum likelihood estimator ([46]); efficient maximum likelihood estimator ([26])
- Fourier-domain semi-parametric estimators:
 - log-periodogram regression estimator (Künsch suggested the semi-parametric version of the LPR estimator of [56] in [74]; [103]); Hurvich and Chen's tapered LPR estimator ([64]); Velasco's tapered LPR estimator ([117]);
 - (feasible) exact local Whittle estimator ([109], [108]); fully-extended (polynomial) local Whittle estimator ([88]); modified local Whittle estimator ([100]); Velasco's tapered Gaussian semi-parametric estimator ([118]); Hurvich and Chen's Gaussian semi-parametric estimator ([64]).
- Wavelet-domain semi-parametric methods:
 - Local wavelet-based regression method ([3])
 - Local Whittle wavelet method ([87])

To have a good understanding of likelihood-based methods for long-memory series, see [20]. Furthermore, Nielsen and Frederiksen in [89] made a finite sample comparison of parametric, semiparametric, and wavelet estimators of memory parameter in a stationary

setting, i.e. $|d| < \frac{1}{2}$. They conclude that "(1) the frequency domain maximum likelihood procedure is superior to the time domain parametric methods; (2) all the estimators are fairly robust to conditionally heteroscedastic errors; (3) the local polynomial Whittle and bias-reduced log-periodogram regression estimators are shown to be more robust to short-run dynamics than other semiparametric (frequency domain and wavelet) estimators and in some cases even outperform the time domain parametric methods, and (4) without sufficient trimming of scales the wavelet-based estimators are heavily biased." For reference, to have an overview of long-memory time series, the book [92] is an excellent choice.

However, in the real world, we often deal with non-stationary data. Up to our knowledge, the existing non-stationarity of long memory time series can be generalised in two ways: (i) The constant memory parameter of $I(d)$ process can be bigger than $\frac{1}{2}$; (ii) Memory parameter d is time-dependent, that is, $|d(t)| < \frac{1}{2}$, and this case is studied in [67], [68], [17], [50], [108] etc. In this dissertation we only deal with the first type of non-stationarity.

We also consider an extension to ARFIMA process, the Gegenbauer ARMA (GARMA) process (introduced and studied in [59]), in this contribution. This process generated by this model exhibits seasonality/cycle. Its spectral density function is characterized by

$$f(\omega + \lambda) \sim |\lambda|^{-2d}, \lambda \rightarrow 0,$$

where $0 < d < \frac{1}{2}$. For illustration, we plot spectra of 4 example Gegenbauer processes in Fig.1.3. The top two subplots display a seasonal persistent processes at an annual period at two different degrees of persistence. The process of the bottom left subplot has a high-frequency oscillation while that of the bottom right has a very low-frequency oscillation.

More generally, GARMA process can be generalised to k -factor Gegenbauer process, which implies the process has k spectral singularities in $(0, \pi)$. We extend GARMA process to the non-stationary case and study its statistical properties and propose memory parameter estimator.

1.3 Our Contributions

This dissertation on (non-stationary) generalised long-memory series researches the memory parameter estimation problems and the application of this type of series to historical time series.

Chapter 2 provides the main results for (non-)stationary long-memory series. Since there exists a large amount of statistics literature, theoretical and empirical, on the stationary and invertible long-memory series, our effort is focussed on the non-stationary long-memory series.

First, we recall the definition of a (non-)stationary long-memory series and the difficulty we encounter to estimate the memory parameter is clarified when the long-memory series encompassing the non-stationarity. Second, we furnish a comprehensive review on the semi-parametric memory parameter estimation procedures in Fourier domain and in

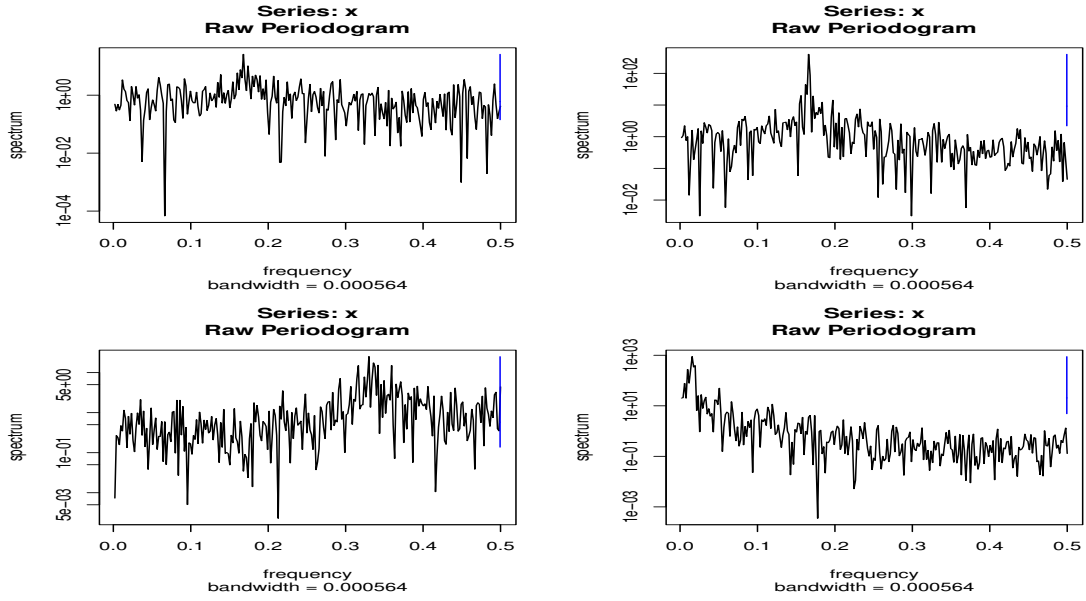


Figure 1.3: Spectra for Gegenbauer processes with in top left subplot: $d = .25$, spectral peak at $\pi/6$; in top right subplot: $d = .45$, spectral peak at $\pi/6$; in bottom left subplot: $d = .45$, spectral peak at $\pi/3$; and in bottom right subplot: $d = .45$, spectral peak at 0.001.

wavelet domain. As a result of great improvement in computer performance, computational statistics is developing rapidly. It becomes an indispensable branch of statistics, since it not only numerically verifies the theoretical statistical results, but also assesses the performance of proposed estimators so that a guideline can be furnished to applied researchers for the utilization of the estimator from a pragmatic perspective. Therefore, third, we perform an extensive simulation experiment in order to pragmatically address the following problems:

- If there exists an optimal estimator for empirical applications?
- Since the presence of heteroscedasticity in variance is common in data and there exists no theoretical results in this case, then we want to check if the estimators are still robust in this case.
- Since the amount of data under investigation sometimes is very limited, we want to know how the estimators perform with small sample size.
- How to determine user-chosen parameters, like bandwidth, pooling order, difference order, upper and lower wavelet trimming number?

Chapter 3 introduces the non-stationary seasonal/cyclical long-memory (SCLM) time series. Our emphasis is laid on the statistical results for 1-factor Gegenbauer series because the results can be readily extended to the k -factor Gegenbauer series. We propose two Fourier semi-parametric memory parameter estimators, i.e., the log-periodogram

semi-parametric estimator and Gaussian semi-parametric estimator, applicable to non-stationary SCLM series. The consistency of the estimators is proved and their robustness is verified via Monte Carlo simulations. The seasonal/cyclical property of a time series indicates that there exists at least one spectral peak, and the definition of the memory parameter d for the non-stationary SCLM series is rooted in the (successively) differentiated series. The two proposed estimators take advantage of the 'tapering' technique, which mitigates frequency leakage around the spectral peak. To illustrate the validity of our estimators and to fill the blank that there exists very little literature on the finite sample performance of the memory parameter estimators for SCLM series, we carry out a Monte Carlo experiment where stationary and non-stationary 1-factor Gegenbauer processes are simulated. As a consequence, the factors that have an impact on the estimation results are investigated and furthermore, an instruction is provided for the application of these estimators from a pragmatic perspective.

Chapter 4 supplies two applications of non-stationary long-memory series. For the first application, we discover the long-memory and non-stationarity properties in the U.S. state house price indices and obtain the memory parameter estimates for 50 states. Then we carry out a clustering analysis in order to acquire some empirical findings and moreover, we present some policy implications for the U.S. housing market. Due to the vacuum of historical Chinese house price data and the similarities the U.S. and China share, based on our findings, we offer some suggestions to the policymakers on how to stabilize house prices in China. In the second application, we compare the forecasting quality of ARFIMA processes and GARMA processes for the US Dollar v.s. Euro exchange rate series. We find that different estimation methods may result in quite different process's structure, and moreover, although GARMA process enjoys more flexibility over ARFIMA process, from a forecasting point of view, the latter process outperforms the former one in this case under investigation.

Chapter 5 contains an overview of our contributions and points out possible directions for future research.

Chapter 2

(Non-)Stationary Long-Memory Processes

In this chapter we consider a fractionally integrated $I(d)$ process $\{X_t\}$ with spectral density $f(\lambda)$ ($-\pi < \lambda < \pi$) that can be approximated by $|\lambda|^{-2d}$ up to a multiplicative for the frequency $\lambda \rightarrow 0$, where the integration order d could be any real number. Precisely speaking, the $I(d)$ process $\{X_t\}$ has the form as follows:

$$(1 - B)^d(X_t - \mu) = u_t, \quad t = 1, \dots, n, \quad (2.1)$$

where B is the lag operator and d is called memory parameter. Even more, a constraint is imposed on $\{u_t\}$ that the stationary process $\{u_t\}$ is a linear process as follows: for $t = 0, 1, \dots$,

$$u_t = C(B)\varepsilon_t \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j|c_j| < \infty, \quad C(1) \neq 0,$$

where $\varepsilon_t \sim N(0, \sigma^2)$ and σ is a constant and $E\varepsilon_t^4 < \infty$. Furthermore, when $d < \frac{1}{2}$, the process $\{X_t\}$ is said to possess long memory; and when $d > 0$, $\{X_t\}$ is called stationary. Clearly, when $0 < d < \frac{1}{2}$, this type of processes satisfies Def.1.1.5 in Chapter 1. Furthermore, the introduction of $I(d)$ in [53] has attracted the attention of many statisticians and econometricians.

As a special case of $I(d)$ process, the stationary and invertible fractionally integrated autoregressive and moving average (ARFIMA) process, has been explored to a large extent in the literature, i.e., d belongs to the interval $(-\frac{1}{2}, \frac{1}{2})$. Specifically, an ARFIMA process is defined as follows:

Definition 2.0.1. A process $\{X_t\}_{t \in \mathbb{Z}}$ is a stationary and invertible ARFIMA(p, d, q) process if

$$\Phi(B)(I - B)^d(X_t - \mu) = \Theta(B)\varepsilon_t, \quad t \in \mathbb{Z}, \quad (2.2)$$

where $d \in (-\frac{1}{2}, \frac{1}{2})$; $\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ and $\Theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$ are polynomials of order p and q , respectively, with roots outside the unit circle; and ε_t is Gaussian white noise with a constant variance σ_ε^2 .

In light of this definition, with some simple computation, we have the spectral density function of a stationary and invertible ARFIMA process as:

$$\begin{aligned} f(\lambda) &= |1 - e^{-i\lambda}|^{-2d} \frac{\sigma_\varepsilon^2}{2\pi} \frac{\Theta(e^{i\lambda})}{\Phi(e^{i\lambda})} \\ &= |2 \sin(\lambda/2)|^{-2d} \frac{\sigma_\varepsilon^2}{2\pi} \frac{\Theta(e^{i\lambda})}{\Phi(e^{i\lambda})}, \end{aligned}$$

and as $\lambda \rightarrow 0$,

$$f(\lambda) \sim G|\lambda|^{-2d}, \quad (2.3)$$

where G is a finite real number. It leaves two points to be noted:

- The ARFIMA(p, d, q) process $\{X_t\}$ is a long-memory process if $0 < d < \frac{1}{2}$.
- Although when $d > \frac{1}{2}$, the pseudo-spectral density function is not integrable, the representation in Eq. (2.3) still holds in this case.

Furthermore, we can consider a more general stationary $\{X_t\}_{t \in \mathbb{Z}}$ with respect to the ARFIMA process, whose spectral density function has the form of

$$f(\lambda) = |\lambda|^{-2d} f^*(\lambda), \quad \lambda \rightarrow 0, \quad (2.4)$$

where $d \in (-\frac{1}{2}, \frac{1}{2})$ and f^* is a continuous function bounded in $[0, 2\pi)$.

However, in the real world, macroeconomic and financial data usually exhibit long memory but non-stationarity, therefore, in this paper our emphasis is laid on the estimation of $d > \frac{1}{2}$. When the stationary process $\{X_t\}$ is extended to the non-stationary case, especially when $d > 1$, the main obstacle encountered in Fourier domain is that the most widely used periodogram will no longer properly approximate the true spectral density of the process. Three important manipulations to solve this problem could be found in the existing literature: (i) Differencing before estimation, but it incurs problems if the series is trend stationary ([109], [110]); (ii) Tapering ([118])/Tapering after differencing ([64]); and (iii) Utilization of the exact representation (or the representation derived from it) of the true spectral density proposed in [99]. We will implement the latter two manipulations of the data in our Monte Carlo experiment. In the wavelet domain scale spectrum is utilized instead of the periodogram in the estimation procedure ([87]), which will be detailed later. In this paper nine estimation procedures, for the non-stationary $I(d)$ process, are recapped and they encompass parametric and semi-parametric Fourier/wavelet methods.

From a practical standpoint, there are at least fifth reasons to motivate us to make a finite sample performance (FSP) comparison study of the estimators aforementioned:

1. Previously, [89] conducted a thorough investigation of the memory parameter estimators for only for stationary long-memory (LM) time series, while in this chapter we put much effort on the non-stationary $I(d)$ process, i.e., $d > \frac{1}{2}$. Furthermore,

seven semi-parametric, applicable to both stationary and non-stationary cases, are considered and examined in this chapter, thus it is natural for us to think if there exists an optimal estimator for practical applications.

2. A linear time trend or level shifts more than often could be found in the economic or financial historical data, for example, the term structure of interest rates in [61] and the U.S. real GDP in [62]. Therefore the robustness of the estimation procedures to these variations in $\{X_t\}$ is necessary to the utilization.
3. The asymptotic convergence and normality for some estimators are established under the Gaussian assumption of the process, however, for practitioners, the Gaussianity is too restrictive and sometimes is redundant. Heteroskedasticity is a common property in many financial or economic data and it has serious consequences for the OLS estimator. Although the OLS estimator remains unbiased, the confidence intervals cannot be trusted. Hence the FSP comparison of the estimators for the $I(d)$ process with heteroskedastic innovations will offer an instructive guideline for the utilization of different estimation procedures.
4. The amount of macroeconomic data available to the applied researchers or practitioners is usually very limited. For example, the historical observations of GDP is less than 200 due to the quarterly release by the statistical institutes since or after 1970, so the estimators performance for the small sample size needs to be illustrated.
5. In [24], albeit in the stationary case, there exist obvious discrepancies between the memory parameter estimators $\hat{d}$ for the exchange rates of six Asia Pacific countries, due to the different estimation procedures used; furthermore, even the same estimation procedure with different selections of frequency ordinates could result in very different memory estimators. Therefore this phenomenon requires that we assess the adequacy of the estimation procedures using various numbers of frequency ordinates in order to build up a solid background for their application to the real data.

For these reasons, we carry out a Monte Carlo experiment for the $I(d)$ process in Eq.(2.1) and its variations with $d \in [\frac{1}{2}, 2]$, the interval we encounter most in the real economic and financial world.

2.1 Assumptions and Preliminaries

Before reviewing the widely-used or recently developed estimation procedures applicable to (non-)stationary $I(d)$ processes and comparing their FSP in terms of bias and root mean square errors (RMSE), we are required to understand some assumptions on the processes and some preliminaries necessary to the estimation procedures. This section is divided into two subsections: the first is devoted to the Fourier-domain basics and the second to the wavelet-domain basics.

2.1.1 Fourier Domain

The Fourier-domain estimation procedures are based on $I(d)$ process's spectral density function and for the stationary case, the usual periodogram is a good approximation to it. However, when the $I(d)$ process is generalized to cover the non-stationary ($d > \frac{1}{2}$) or non-invertible ($d < -\frac{1}{2}$) case, the usual periodogram, which is defined in Eq.(2.6) with $h_t \equiv 1$, can not properly approximate the (pseudo-)spectral density of $\{X_t\}$. To address this problem, several modifications to the discrete Fourier transformations (DFT), and hence the periodogram were proposed in the existing literature. To begin with, we introduce four types of DFT.

- **Hurvich and Chen (HC) Tapered DFT and its (Pooled) Periodogram**

The paper [64] considered a family of complex-valued taper to deal with non-invertible time series, a result of over-differencing the process. Let

$$h_t = 1 - e^{i\lambda_j},$$

and define the HC tapered DFT of order τ as follows:

$$\omega_{x,\tau}^t(\lambda_j) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi \sum_{t=1}^n h_t^2}} \sum_{t=1}^n h_t^\tau X_t e^{it\lambda_j} \quad (2.5)$$

where $\lambda_j = \frac{2\pi j}{n}$, $j = 1, 2, \dots, n$ is the fundamental Fourier frequency. Then the tapered periodogram is

$$I_{x,\tau}^t(\lambda_j) = |\omega_{x,\tau}^t(\lambda_j)|^2. \quad (2.6)$$

When $h_t \equiv 1$, $t = 1, \dots, n$, then (2.5) and (2.6) become the usual DFT and periodogram, denoted as $\omega_x(\lambda_j)$ and $I_x(\lambda_j)$, respectively. Furthermore, the pooled

periodogram is defined as in [104], which is $I_{x,\tau,p}^{tp}(\tilde{\lambda}_k) = \sum_{j=(p+\tau)(k-1)+1}^{(p+\tau)(k-1)+p} I_{x,\tau}^t(\lambda_j)$,

where $\tilde{\lambda}_k \stackrel{\text{def}}{=} p^{-1} \sum_{j=(p+\tau)(k-1)+1}^{(p+\tau)k} \lambda_j$.

Interested readers can refer to the monograph [97] for a precise description of the manipulation 'tapering'.

- **Exact form of DFT (eDFT) and its Periodogram**

The paper [99] presented an exact form of DFT which is resulted from a pure algebraic manipulation and hence applies to all values of d . The exact form of DFT (eDFT) is defined by

$$\omega_x^e(\lambda_j; d) \stackrel{\text{def}}{=} \omega_x(\lambda_j) - D_n(e^{i\lambda_j}; d)^{-1} \frac{1}{\sqrt{2\pi n}} \tilde{X}_{\lambda_j n}(d), \quad (2.7)$$

and the exact form of periodogram then is $I_{\Delta^d X}^e(\lambda) = |D_n(e^{i\lambda_j}; d)|^2 |\omega_x^e(\lambda_j; d)|^2$, where $D_n(e^{i\lambda_j}; d) = \sum_{k=0}^n \frac{(-d)_k}{k!} e^{ik\lambda}$ and

$$\tilde{X}_{\lambda_n}(d) = \sum_{p=0}^{n-1} \tilde{d}_{\lambda p} e^{-ip\lambda} X_{n-p} \text{ with } \tilde{d}_{\lambda p} = \sum_{k=p+1}^n \frac{(-d)_k}{k!} e^{ik\lambda}.$$

- **Modified DFT (modDFT) and its Periodogram**

Inspired by (2.7), [100] conceived an approximation to (2.7) and called it the modified DFT (modDFT). This modDFT is invariant to a constant and a linear time trend, which is an advantage over the eDFT (2.7). The modDFT is defined by

$$\omega_x^{mod}(\lambda_j) \stackrel{\text{def}}{=} \omega_x(\lambda_j) + \frac{e^{i\lambda_j}}{1 - e^{i\lambda_j}} \frac{X_n - X_0}{\sqrt{2\pi n}}, \quad (2.8)$$

and the modified periodogram is then $I_x^{mod}(\lambda_j) = |\omega_x^{mod}(\lambda_j)|^2$.

- **Fully-Extended DFT (fextDFT) and its Periodogram**

Still suggested in the work of [99] for the $I(d)$ process with $d \in \mathbb{R}$, the fextDFT was proposed in [2] and the resulting fully-extended periodogram can properly approximate the spectral density function in the small neighborhood of the origin. The fully-extended DFT (fextDFT) is defined on a compact set $I := [a, b] \subset [-\frac{3}{2}, \infty)$ by

$$\omega_x^{fext}(\lambda_j; d) \stackrel{\text{def}}{=} \omega_x(\lambda_j) + k(\lambda_j; d), \quad (2.9)$$

where $k(\lambda_j; d)$ is the correction term taking constant values on the intervals $d \in I_p := [p - \frac{1}{2}, p + \frac{1}{2})$, $p = -1, 0, 1, 2, \dots$, which is defined by

$$k(\lambda_j; d) := \begin{cases} -e^{i\lambda_j} Z_0, & d \in I_{-1} = [-3/2, -1/2); \\ 0, & d \in I_0 = [-1/2, 1/2); \\ e^{i\lambda_j} \sum_{r=1}^p (1 - e^{i\lambda_j})^{-r} Z_r, & d \in I_p, p = 1, 2, \dots \end{cases} \quad (2.10)$$

where $Z_0 := \omega_x(0) = (2\pi n)^{-1/2} \sum_{t=1}^n X_t$ and $Z_r := (2\pi n)^{-1/2} (\nabla^{r-1} X_n - \nabla^{r-1} X_0)$, $r = 1, \dots, p$.

The fully-extended periodogram then is $I_x^{fext}(\lambda_j; d) = |\omega_x^{fext}(\lambda_j; d)|^2$.

2.1.2 Wavelet Domain

Nonlinearity is the nature of the world and the linearity is just a special case. As one of advanced mathematical tools for nonlinear science, the wavelet analysis has greatly developed in recent years. For introductory reference, there exist quite a lot books, see

e.g. [90], [120], [107], [21], [22], [94] and [123] from a statistical perspective; [119], [15] and [82] from an engineering perspective. Wavelet transform has been taking the place of the conventional Fourier transform in many areas, for example, molecular dynamics, image processing, multifractal analysis, to name just few. This popularity is mainly due to its time-frequency localization ability. Although the memory parameter d is assumed to be a constant, i.e., not a time-dependent element, in this dissertation, as a counterpart of Fourier estimators, we are still very interested in the wavelet estimators.

Before presenting the wavelet estimation procedures, we need to first introduce some assumptions as in [45] about the scale function $\phi \in \mathbf{L}^2(\mathbb{R})$ and the wavelet $\psi \in \mathbf{L}^2(\mathbb{R})$: Let

$$\hat{\phi}(\xi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \phi(t) e^{-i\xi t} dt \quad \text{and} \quad \hat{\psi}(\xi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi(t) e^{-i\xi t} dt$$

as the Fourier transforms of ϕ and ψ , respectively.

W-1 ϕ and ψ are compactly-supported, integrable, and $\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t) dt = 1$ and $\int_{-\infty}^{\infty} \psi^2(t) dt = 1$.

W-2 There exists $\alpha > 1$ such that $\sup_{\xi \in \mathbb{R}} |\hat{\psi}|(1 + |\xi|)^\alpha < \infty$.

W-3 The function ψ has M vanishing moments, i.e. $\int_{-\infty}^{\infty} t^m \psi(t) dt = 0$ for all $m = 0, \dots, M-1$.

W-4 The function $\sum_{k \in \mathbb{Z}} k^m \phi(\cdot - k)$ is a polynomial of degree m for all $m = 0, \dots, M-1$.

In the wavelet semi-parametric context, for the process $\{X_t\}$ with a spectral density $f(\lambda) = |1 - e^{i\lambda}|^{-2d} f^*(\lambda)$, the wavelet estimation procedure utilizes the scale spectrum $\sigma_j^2(d, f^*)$ instead of the periodogram in the Fourier domain, that is, when $j \rightarrow \infty$,

$$\sigma_j^2(d, f^*) \stackrel{\text{def}}{=} \text{var}[W_{j,0}^X] \asymp \sigma^2 2^{2dj}, \quad (2.11)$$

where $\sigma_j^2(d, f^*)$ can be estimated by the empirical variance (see [86])

$$\hat{\sigma}_j^2 \stackrel{\text{def}}{=} \frac{1}{n_j} \sum_{k=0}^{n_j-1} (W_{j,k}^X)^2 \quad \text{for } (j, k) \in l_n, \quad (2.12)$$

where $l_n \stackrel{\text{def}}{=} \{(j, k) : j \geq 0, 0 \leq k < n_j\}$ with

$$n_j = \lfloor 2^{-j}(n - T + 1) - T + 1 \rfloor \quad (2.13)$$

where $T = 2M$. Furthermore, define U the highest scale such that $U = \max\{j : \lfloor 2^{-j}(n - T + 1) - T + 1 \rfloor \geq 0\}$ as well as L the lowest scale such that $L = 0$. Clearly, for the

wavelet estimation, we are obliged to select the lower and upper scales, i.e., L_n and U_n , which satisfies $L \leq L_n < U_n \leq U$; then we have $m^{equiv} = \frac{1}{2} \sum_{j=L_n}^{U_n} n_j$, the half the number of wavelet coefficients involved in the wavelet estimations and it is a counterpart of bandwidth parameter m for the Fourier estimation procedure.

Remarque 2.1.1. [86] showed that given $M \geq \delta$, the utilization of a wavelet and a scaling function satisfying (W-4) and (W-3) implicitly performed a M^{th} -order differentiation of the time series, which is contrast to the Fourier methods, where the time series must be explicitly differentiated at least δ times and a data taper must be applied on the differenced series to avoid frequency-domain leakage; see [64].

Daubechies Wavelets

The wavelet estimators is applied with Daubechies wavelets to $I(d)$ processes in our Monte Carlo experiment. Therefore, we now make a brief explanation to the Daubechies wavelets. Daubechies proposed to construct in [28] orthogonal wavelets based on a compactly supported function characterized by a maximal number of vanishing moment. For each $M \in \mathbb{N}$, the Daubechies wavelet with M vanishing moments is supported on the interval $[0, 2M - 1]$, and in other words, the order of vanishing moments is half the filter length. The Daubechies wavelets also become smoother with the growth of M . For $M \geq 4$, the technique of spectral factorization is commonly used to compute the coefficients in the Daubechies scaling filters. Recall (W-3), the order of vanishing moments M plays an important role in the memory parameter estimation for non-stationary $I(d)$ series and hence Daubechies wavelet is an appropriate choice in our Monte Carlo study.

2.2 Fourier Semi-Parametric Estimators

In this section we will respectively discuss the Fourier and wavelet estimators.

2.2.1 Fourier Log-Periodogram Semi-Parametric Estimator

The semi-parametric log-periodogram Fourier (LPF) regression estimator in [56] is widely used in the empirical study to estimate the memory parameter d of the invertible and stationary $I(d)$ process. The first successful attempt in the semi-parametric context to deal with a non-stationary or the non-invertible $I(d)$ process was attributed to [118]. In that paper Velasco proposed to taper the data prior to the estimation and his LPF estimation procedure only involves a grid of frequency ordinates, which inflates the estimator's asymptotic variance with respect to the one in [56]. Later, Hurvich and Chen in [64] proposed a tapered and pooled LPF (tpLPF) estimator and since all the information in the small neighborhood of the origin is used in the estimation, this estimator has a smaller asymptotic variance compared to the Velasco's estimator. Therefore we do not implement the Velasco's estimation procedure in this comparison study.

The tpLPF estimator is to difference the original non-stationary process and then to deal with the resulting stationary process. We now introduce the δ^{th} -order difference of

an $I(d)$ process $\{X_t\}$. Let $Y_t \stackrel{\text{def}}{=} \Delta^\delta X_t$ with $d < \delta + \frac{1}{2}$, so $\{Y_t\}$ is stationary and its spectral density $f_y(\lambda)$ can be approximated by $C|\lambda|^{-2(d-\delta)}$. Let $d_\delta = d - \delta$ hereon. We have, as $\lambda \rightarrow 0$,

$$\log(f_y(\lambda)) \sim \log C + (-2 \log \lambda) \cdot d_\delta \stackrel{\text{def}}{=} \log C + d_\delta \cdot g_y(\lambda), \quad 0 < G < \infty. \quad (2.14)$$

The tpLPF estimator has the same spirit as the LPF estimator in that it is based on the linear regression model

$$\log(I_{y,\tau,p}^{tpLPF}) = \text{constant} - d_\delta g_y(\tilde{\lambda}_k) + \text{error}, \quad (2.15)$$

where $\tilde{\lambda}_k = p^{-1} \sum_{j=(p+\tau)(k-1)+1}^{(p+\tau)k} \lambda_j$, $k = 1, \dots, \bar{K}$ such that $(p+\tau)\bar{K} \leq m$. The tpLPF estimator is $\hat{d}^{tpLPF} = \delta + \hat{d}_\delta^{tpLPF}$, where $\hat{d}_\delta^{tpLPF}$ is the ordinary least squares (OLS) estimate for the regression (2.15). The paper [84] also showed that when $-\tau - \frac{1}{2} < d_\delta < \frac{1}{2}$ with a properly selected m , as $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}_\delta^{tpLPF} - d_\delta) \rightarrow N(0, \frac{\sigma_{p,\tau}^2}{4})$, where $\sigma_{p,\tau}^2$ is decreasing with p .

2.2.2 Local Whittle Fourier Estimators

The local Whittle Fourier (LWF) estimation method is another popular method in the Fourier domain, suggested by [74] and investigated by [103] for stationary $I(d)$ processes. The LWF estimation method is to find the minimizer $(\hat{G}, \hat{d})$ of the objective function

$$Q(G, d) = \frac{1}{m} \sum_{j=1}^m \left[\log(G\lambda_j^{-2d}) + \frac{I_x(\lambda_j)}{G\lambda_j^{-2d}} \right].$$

For the non-stationary case, the idea is the same as for the stationary case. However, since the usual periodogram is no longer a good approximation to the density function, the term $I_x(\lambda_j)$ and hence $I_x(\lambda_j)/(G\lambda_j^{-2d})$ needs to be modified with respect to the different estimators: the exact local Whittle Fourier (eLWF) estimator, modified local Whittle Fourier (modLWF) estimator and fully-extended local Whittle Fourier (fextLWF) estimator and they are denoted by $\hat{d}^{eLWF}$, $\hat{d}^{modLWF}$ and $\hat{d}^{fextLWF}$, respectively.

A special attention should be paid to the tapered local Whittle Fourier (tLWF) estimator proposed in [64]. Like the tpLPF estimator, this method is not to estimate d directly but to estimate d_δ of $\{Y_t\}$ based on the relationship (2.14) and employs the tapered periodogram. We then obtain $\hat{d}_\delta^{tLWF}$ and so $\hat{d}^{tLWF} = \hat{d}_\delta^{tLWF} + \delta$.

Moreover, with a proper selection of bandwidth m , we have the following asymptotic properties for these LWF estimators. The true value of the memory parameter is denoted by d_0 hereon.

For the eLWF estimator $\hat{d}^{eLWF}$, the paper [109] showed when $d > -\frac{1}{2}$, as $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}^{eLWF} - d_0) \rightarrow N(0, \frac{1}{4})$. Since the eLWF estimator is not invariant to an $I(d)$

process with a constant or a linear time trend, the author later proposed a feasible eLWF (feLWF) estimator in [108] to overcome this weakness and this estimator is consistent for $d > -\frac{1}{2}$ and when $d \in (-\frac{1}{2}, 2)$, as $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}^{feLWF} - d_0) \rightarrow N(0, \frac{1}{2})$.

For the modLWF estimator $\hat{d}^{modLWF}$, the paper [100] showed that $\hat{d}^{modLWF}$ is consistent for $d \in (0, 2)$ and when $d \in (0, 2)$, as $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}^{modLWF} - d_0) \rightarrow N(0, \frac{1}{4})$.

For the fextLWF estimator $\hat{d}^{fextLWF}$, the paper [2] showed that when $d > -\frac{1}{2}$, $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}^{fextLWF} - d_0) \rightarrow N(0, \frac{1}{4})$. A bias-reduced estimator, the fully-extended local polynomial Whittle Fourier (fextLPWF) estimator, is proposed in [88]. It tweaks the fextLWF estimator in that the short memory component of the spectrum is modeled by a finite, positive, even polynomial $\Upsilon(\lambda) = \sum_{l=1}^r \nu_l \lambda^{2l}$, $r \in \mathbb{N}$ instead of the constant 'G' in (2.3), i.e., $f(\lambda) \sim \Upsilon(\lambda) \lambda^{-2d}$, $\lambda \rightarrow 0$.

For the tLWF estimator $\hat{d}^{tLWF}$, the paper [64] showed that as $n \rightarrow \infty$, we have $\sqrt{m}(\hat{d}_\delta^{tLWF} - d_0) \rightarrow N(0, \frac{\Psi(\tau)}{4})$, where $\Psi(\tau) = \frac{\Gamma(4\tau+1)\Gamma^4(\tau+1)}{\Gamma^4(2\tau+1)}$.

Until now, we are still left with a question of the selection of bandwidth m in the estimation. We will illustrate in Sec. 2.4 the Fourier estimators' finite sample behavior with respect to the different selections of m from a pragmatic perspective.

2.3 Wavelet Semi-Parametric Estimators

In the wavelet context, let $\{X_t\}$ be a process with a spectral density

$$f(\lambda) = |1 - e^{i\lambda}|^{-2d} f^*(\lambda), \quad (2.16)$$

and the wavelet estimation procedure uses the scale variance $\sigma_j^2(d, f^*) \stackrel{\text{def}}{=} \text{var}[W_{j,0}^X]$ and as $j \rightarrow \infty$,

$$\sigma_j^2(d, f^*) \asymp \sigma^2 2^{2dj}, \quad (2.17)$$

where ' $\asymp$ ' means the left and right hand sides are asymptotically equal and $\sigma_j^2(d, f^*)$ can be estimated by the empirical variance (see [86])

$$\hat{\sigma}_j^2 \stackrel{\text{def}}{=} \frac{1}{n_j} \sum_{k=0}^{n_j-1} (W_{j,k}^X)^2, \quad \text{for } (j, k) \in l_n. \quad (2.18)$$

Moreover, define U the highest scale such that $U = \max\{j : \lfloor 2^{-j}(n-T+1) - T+1 \rfloor \geq 0\}$ as well as L the lowest scale such that $L = 0$. Clearly, we are required to select the lower and upper scales, i.e., L_n and U_n , which satisfies $L \leq L_n < U_n \leq U$.

Remarque 2.3.1. *The paper [86] showed that given $M \geq \delta$, the use of a wavelet and a scaling function satisfying (W-3) and (W-4) implicitly performs a M^{th} -order differentiation of the time series, which is contrast to the Fourier methods, where the time series*

must be explicitly differenced at least δ times and a data taper must be applied on the differenced series to avoid frequency-domain leakage; see [64].

2.3.1 Local Regression Wavelet Estimator

Up to our knowledge, the existing literature only documents the asymptotic convergence for the first and second order moments of the local regression wavelet (LRW) estimator in a Gaussian process context. Due to Eq.(2.17) and Eq.(2.18), we have the approximate regression relationship

$$\log(\hat{\sigma}_j^2) = \text{constant} + dj(2 \log 2) + \text{error}. \quad (2.19)$$

The LRW estimator is defined as the OLS estimate for Eq.(2.19) including the scales from L_n to U_n and we will investigate the impact of selection L_n and U_n on the estimation performance through a MC experiment. Under suitable conditions, for the Gaussian process $\{X_t\}$ with a properly selected lower scale L_n , we have

$$\sqrt{n2^{-L_n}}(\hat{d}^{LRW} - d_0) \rightarrow N(0, \sum_{i,j=0}^l \mathbf{w}_j V_{i,j}(d) \mathbf{w}_j), \quad (2.20)$$

see [85] for more details.

2.3.2 Local Whittle Wavelet Estimator

The local Whittle wavelet (LWW) likelihood function is given by

$$L^{LWW}(\sigma^2, d) = \frac{1}{2\sigma^2} \sum_{(j,k) \in l} 2^{-2dj} (W_{j,k}^X)^2 + \frac{|l|}{2} \log(\sigma^2 2^{2\langle l \rangle} d), \quad (2.21)$$

where l is l_n with $L_n \leq j \leq U_n$ and $|l|$ denotes the cardinal of l and $\langle l \rangle \stackrel{\text{def}}{=} |l|^{-1} \sum_{(j,k) \in l} j$.

The estimator $\hat{d}^{LWW}$ is obtained by minimizing Eq.(2.21). We need to have some concepts clarified before presenting the asymptotic property of LWW estimator. First, the settings for the upper scale sequence (U_n) :

(S-1) $U_n - L_n$ is fixed, equal to $l > 0$;

(S-2) $U_n \leq J_n$ for all n and $U_n - L_n \rightarrow \infty$ as $n \rightarrow \infty$, where J_n is the largest available scale.

Second, we define a set of functions as follows.

Definition 2.3.2. For any $0 < \beta \leq 2$, $\gamma > 0$ and $\epsilon \in (0, \pi]$, $H(\beta, \gamma, \epsilon)$ is the set of all non-negative and even function g that satisfies $g(0) > 0$ and for all $\lambda \in (-\epsilon, \epsilon)$,

$$|g(\lambda) - g(0)| \leq \gamma g(0) |\lambda|^\beta. \quad (2.22)$$

The following theorem is proved in [87].

Theorem 2.3.3. *Let $\{X_t\}$ be an $I(d)$ process with pseudo-spectral density given by Eq.2.16 and $f^* \in H(\beta, \gamma, \epsilon)$ for some $\gamma > 0$, $\beta \in (0, 2]$ and $\epsilon \in (0, \pi]$. Assume (W-1)-(W-4) with $(1 + \beta)/2 - \alpha < d \leq M$. Let (L_n) be a sequence satisfying*

$$\lim_{n \rightarrow \infty} \{L_n^2(n2^{-L_n})^{-1/4} + L_n^{-1}\} = 0,$$

and (U_n) be a sequence such that either (S-1) or (S-2) holds. Then, as $n \rightarrow \infty$,

$$\hat{d}^{LWW} = d_0 + \mathcal{O}_{\mathbb{P}}\{(n2^{-L_n})^{-1/2} + 2^{-\beta L_n}\}.^1 \quad (2.23)$$

Hence, for $2^{L_n} \asymp n^{1/(1+2\beta)}$, the optimal rate $n^{\beta/(1+2\beta)}$ is obtained and Eq.(2.23) theoretically suggests trimming the lowest wavelet scales when the LWW estimator is implemented.

2.4 Finite Sample Performance Comparison Study

In this section we exercise a comprehensive Monte Carlo (MC) experiment to assess the accuracy of the semi-parametric estimation procedures presented above from a pragmatic perspective. In this experiment not only is considered the $I(d)$ process, but also its variations, for some of which there is no established theoretical results. The vast amount of MC result tables are not displayed in this paper, which is available upon request.

The ARFIMA(p, d, q) processes, employed in this MC experiment, for $d \in [\frac{1}{2}, 2]$ are simulated using Matlab with the ToolboxLRD toolbox; see [45]. The bias and RMSE were computed with 5,000 replications; the sample size n is selected to be $n = 150, 300, 500$, and 1000; the number of frequency ordinates involved in the estimation m (bandwidth parameter) has the form of $m = \lfloor n^\alpha \rfloor$ and $\alpha = 0.5, 0.65$ as well as 0.8. The numeric results in the Tables 6.1-6.47 were obtained by rounding the Matlab outputs to four decimal places.

In this experiment we first examine the estimators' FSP under the ARFIMA(p, d, q) setting:

$$(1 - B)^d(X_t - \mu) = u_t, \quad (2.24)$$

where

$$(a) \ u_t = \varepsilon_t; \quad (2.25)$$

$$(b) \ (1 - \phi B)u_t = \varepsilon_t; \quad (2.26)$$

$$(c) \ u_t = (1 + \theta B)\varepsilon_t, \quad (2.27)$$

where $\varepsilon_t \sim N(0, 1)$, and ϕ, θ are selected to be $-.4$ and $.4$. Then we numerically explore their FSP for the ARFIMA process with a linear time trend, occasional structural breaks (OSB) in levels and heteroscedastic innovations, i.e., GARCH effects. The tables of this Monte Carlo experiment are mainly contained in Appendix.

¹ $\mathcal{O}_{\mathbb{P}}(\cdot)$ means that if $f(x) = \mathcal{O}_{\mathbb{P}}(g(x))$, there is a sufficiently large constant $\bar{C}$ such that for all sufficiently large values of x , $f(x)$ is at most $\bar{C}$ multiplied by $g(x)$ in absolute value in probability.

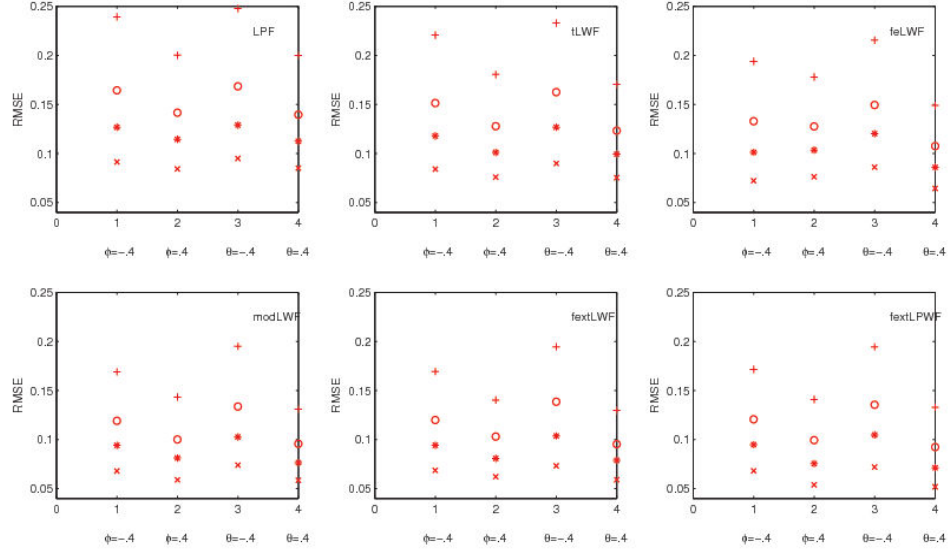


Figure 2.1: The Fourier Estimators' Performance for the ARFIMA Model with $d = .8$. The marks $+$, o , $*$ and $\times$ represent the MC results for sample size of 150, 300, 500 and 1000, respectively. The RMSE are computed with 5000 replications for the cases (2.27) and (2.27). For the tpLPF and tLWF estimators, we choose the difference order $\delta = 1$. For the fextLPWF estimator, $m = \lfloor n^{.8} \rfloor$ and for the others, $m = \lfloor n^{.65} \rfloor$.

2.4.1 Finite Sample Performance for ARFIMA(p, d, q) Processes

In this subsection we provide a description of the MC results for the semi-parametric Fourier and wavelet estimators under the ARFIMA setting.

Fourier Estimators

For each Fourier estimator investigated in this chapter, the consistency is numerically revealed in the Tab.6.1-Tab.6.39, because the bias and RMSE becomes smaller as the sample size grows. To visualize this point of view, we display in the Fig.2.1 the RMSE with respect to the cases from Eq.(2.27) to Eq.(2.27), and each subplot stands for one Fourier estimator. Moreover, to take a quick look at these Tables, we find a negative MA coefficient has the most adverse impact on the estimation results.

As is already mentioned in Sec. 2.2, the bandwidth m , closely related to the asymptotic variance of a Fourier estimator, is a user-chosen parameter, and it has a substantial impact on the semi-parametric estimation outcomes. In fact, several authors researched on this problem and we present a brief summary of their contributions as follows:

- For the untapered and unpooled LPE, i.e., $\hat{d}^{tpLPF}$ with $\delta = 0$ and $\tau = p = 1$, the expression, a function of the sample size n and f^* , for an optimal ' m ' is proposed in [63]. Unfortunately, the strict constraints imposed on f^* and $f^{**}(0) \neq 0$ even exclude an ARFIMA(0, d , 0) process.

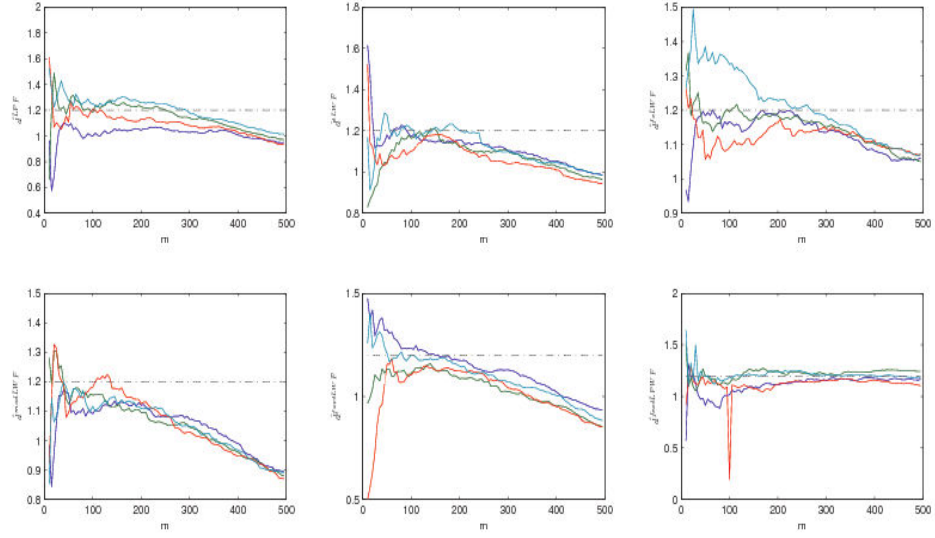


Figure 2.2: The Fourier Estimators' Performance for the ARFIMA(0,d,1) Model with $d = 1.2$ and $\theta = -.4$ Four realizations are simulated for the ARFIMA(0,1.2,1) model with $\theta = -.4$ in each subplot. The sample size is 1000.

- A theoretical selection of m is proposed in [57] and [58] for the LPE, however, it is not suitable for practical applications.
- An optimal selection of m proposed in [63] is also not satisfactory, and due to it is computationally burdened, it's not used in practice.

This reality motivates us to explore the relationship between the bandwidth selection and the estimation performance from a pragmatic perspective. Actually, although there exists no general theoretical optimal selection of m , we should bear it in mind that as $n \rightarrow \infty$, we must have $m \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \frac{m}{n} = 0$. Hence in this experiment we focus on $m = \lfloor n^{.5} \rfloor$, $\lfloor n^{.65} \rfloor$ and $\lfloor n^{.8} \rfloor$. We provide a visual illustration in the Fig.2.2 for the sensitivity of the estimates over the bandwidths, and each subplot contains the estimates $\hat{d}$ of four realizations for one Fourier estimator.

In the ARFIMA(0, d , 0) case, when m goes from $\lfloor n^{.5} \rfloor$ up to $\lfloor n^{.65} \rfloor$, the RMSE is greatly reduced sometimes at the expense of some slightly increased bias. When $m = \lfloor n^{.8} \rfloor$, for the tpLWF, tLWF and fextLPWF ($r = 1$) estimators, they fairly have the best performance in terms of both bias and RMSE, as is revealed in Tab.6.19, for example. However, for the other Fourier estimators considered in this chapter, although they have its best performance in terms of RMSE, but this reduction in RMSE incurs a big inflation in bias compared to the performance when $m = \lfloor n^{.65} \rfloor$.

Table 2.1: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 0) Model

d	n	τ	$d - \delta \in [-.5, .5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	1	-0.0271	0.2918	-0.0180	0.1715	-0.0110	0.1088
	300	1	-0.0207	0.2203	-0.0122	0.1237	-0.0061	0.0750
	500	1	-0.0161	0.1849	-0.0088	0.0985	-0.0023	0.0586
	1000	1	-0.0170	0.1447	-0.0079	0.0768	-0.0017	0.0421
.7	150	1	-0.0318	0.2876	-0.0146	0.1688	-0.0100	0.1076
	300	1	-0.0211	0.2211	-0.0096	0.1219	-0.0070	0.0762
	500	1	-0.0178	0.1856	-0.0079	0.1016	-0.0032	0.0593
	1000	1	-0.0142	0.1469	-0.0056	0.0760	-0.0034	0.0428
.9	150	1	-0.0335	0.2824	-0.0145	0.1718	-0.0102	0.1073
	300	1	-0.0215	0.2180	-0.0095	0.1214	-0.0076	0.0762
	500	1	-0.0193	0.1833	-0.0094	0.0983	-0.0050	0.0573
	1000	1	-0.0108	0.1429	-0.0057	0.0753	-0.0018	0.0427
1.1	150	1	-0.0224	0.2932	-0.0148	0.1694	-0.0089	0.1073
	300	1	-0.0148	0.2212	-0.0127	0.1219	-0.0057	0.0766
	500	1	-0.0138	0.1848	-0.0068	0.0987	-0.0060	0.0583
	1000	1	-0.0129	0.1456	-0.0057	0.0749	-0.0037	0.0423
1.3	150	1	-0.0135	0.2852	-0.0080	0.1691	-0.0078	0.1067
	300	1	-0.0106	0.2191	-0.0082	0.1237	-0.0033	0.0750
	500	1	-0.0114	0.1816	-0.0053	0.0997	-0.0022	0.0573
	1000	1	-0.0065	0.1435	-0.0050	0.0742	-0.0014	0.0425
1.5	150	2	-0.0338	0.3710	-0.0210	0.2084	-0.0127	0.1268
	300	2	-0.0231	0.2714	-0.0096	0.1464	-0.0065	0.0883
	500	2	-0.0176	0.2256	-0.0089	0.1190	-0.0053	0.0681
	1000	2	-0.0159	0.1724	-0.0054	0.0878	-0.0036	0.0491
1.7	150	2	-0.0280	0.3630	-0.0254	0.2116	-0.0124	0.1292
	300	2	-0.0297	0.2759	-0.0135	0.1476	-0.0084	0.0890
	500	2	-0.0270	0.2254	-0.0103	0.1183	-0.0057	0.0682
	1000	2	-0.0178	0.1740	-0.0085	0.0878	-0.0029	0.0495
1.9	150	2	-0.0265	0.3616	-0.0166	0.2034	-0.0118	0.1267
	300	2	-0.0279	0.2666	-0.0142	0.1466	-0.0092	0.0888
	500	2	-0.0233	0.2274	-0.0141	0.1171	-0.0034	0.0671
	1000	2	-0.0124	0.1751	-0.0067	0.0887	-0.0037	0.0494

^a The τ is the tapering order. The δ is the difference order.

In the ARFIMA(1, d , 0) and ARFIMA(0, d , 1) cases, except for the fextLPWF ($r = 1$) estimator, there is always a trade-off between the performance of bias and RMSE, that is, the estimators are less biased with $m = \lfloor n^{.5} \rfloor$ while they have smaller RMSE with $m = \lfloor n^{.65} \rfloor$. Moreover, these estimators generally suffer a lousy performance in bias and RMSE when $m = \lfloor n^{.8} \rfloor$. This finding is reflected in Tab.2.2-Tab.2.4, and for more tables, refer to Appendix. Therefore a big bandwidth m is not favored in this case.

Table 2.2: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $m = \lfloor n^{.5} \rfloor$

ϕ	d	n	fextLWF		fextLPWF		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	0.0282	0.2270	-0.1588	0.4181	0.0243	0.2285	-0.0067	0.2164	-0.1046	0.2659
		300	0.0296	0.1860	-0.0946	0.2955	0.0360	0.1830	-0.0020	0.1701	-0.0676	0.2019
		500	0.0248	0.1587	-0.0680	0.2372	0.0411	0.1578	0.0065	0.1428	-0.0526	0.1695
		1000	0.0196	0.1304	-0.0475	0.1722	0.0353	0.1302	0.0092	0.1166	-0.0343	0.1295
	.7	150	-0.0114	0.2077	-0.1484	0.4251	-0.0186	0.2222	-0.0066	0.1975	-0.0954	0.2694
		300	-0.0041	0.1677	-0.1045	0.3142	-0.0105	0.1724	-0.0067	0.1508	-0.0656	0.2003
		500	-0.0025	0.1414	-0.0711	0.2570	-0.0003	0.1387	-0.0067	0.1262	-0.0484	0.1611
		1000	-0.0045	0.1141	-0.0519	0.1969	0.0011	0.1125	-0.0054	0.1032	-0.0360	0.1249
	.9	150	-0.0326	0.2124	-0.1543	0.4442	-0.0368	0.2212	-0.0319	0.2039	-0.0812	0.2689
		300	-0.0237	0.1682	-0.1005	0.3320	-0.0260	0.1714	-0.0246	0.1632	-0.0508	0.1958
		500	-0.0175	0.1407	-0.0731	0.2685	-0.0185	0.1433	-0.0191	0.1378	-0.0394	0.1601
		1000	-0.0106	0.1138	-0.0414	0.2016	-0.0138	0.1136	-0.0147	0.1135	-0.0219	0.1252
	1.1	150	-0.0344	0.2178	-0.1442	0.4444	-0.0386	0.2229	-0.0356	0.2127	-0.0503	0.2514
		300	-0.0224	0.1699	-0.0818	0.3316	-0.0243	0.1672	-0.0285	0.1719	-0.0310	0.1824
		500	-0.0183	0.1424	-0.0628	0.2737	-0.0201	0.1438	-0.0197	0.1450	-0.0239	0.1477
		1000	-0.0117	0.1132	-0.0362	0.2012	-0.0127	0.1121	-0.0120	0.1125	-0.0142	0.1141
	1.3	150	-0.0291	0.2324	-0.1015	0.4255	-0.0434	0.2251	-0.0425	0.2367	-0.0265	0.2265
		300	-0.0110	0.1765	-0.0724	0.3216	-0.0183	0.1701	-0.0254	0.1780	-0.0210	0.1739
		500	-0.0138	0.1486	-0.0464	0.2547	-0.0118	0.1412	-0.0202	0.1508	-0.0105	0.1411
		1000	-0.0085	0.1201	-0.0391	0.1946	-0.0111	0.1132	-0.0098	0.1163	-0.0118	0.1131
	1.5	150	-0.0140	0.2230	-0.1023	0.4172	-0.0310	0.2197	-0.0715	0.2698	-0.0220	0.2203
		300	-0.0022	0.1800	-0.0555	0.3058	-0.0163	0.1697	-0.0493	0.2189	-0.0065	0.1676
		500	0.0051	0.1521	-0.0402	0.2481	-0.0099	0.1465	-0.0334	0.1838	-0.0068	0.1396
		1000	0.0129	0.1270	-0.0509	0.1808	-0.0007	0.1155	-0.0168	0.1406	-0.0024	0.1098
	1.7	150	-0.0260	0.2120	-0.0874	0.4078	-0.0227	0.2063	-0.0509	0.2457	-0.0153	0.2137
		300	-0.0101	0.1694	-0.0572	0.2980	-0.0028	0.1635	-0.0296	0.1843	-0.0027	0.1639
		500	-0.0052	0.1388	-0.0441	0.2403	0.0042	0.1399	-0.0178	0.1482	-0.0004	0.1415
		1000	-0.0020	0.1119	-0.0824	0.2026	0.0087	0.1150	-0.0143	0.1172	0.0049	0.1114
	1.9	150	-0.0356	0.2177	-0.0900	0.4178	-0.0661	0.1812	-0.0423	0.2334	-0.0022	0.2113
		300	-0.0271	0.1665	-0.0635	0.2977	-0.0364	0.1388	-0.0279	0.1749	0.0016	0.1613
		500	-0.0172	0.1403	-0.0491	0.2549	-0.0196	0.1161	-0.0185	0.1448	0.0023	0.1376
		1000	-0.0133	0.1130	-0.0511	0.2149	-0.0064	0.0937	-0.0136	0.1166	0.0038	0.1104

^a The m is the number of the frequency ordinates involved in the estimation.

Table 2.3: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $m = \lfloor n^{.65} \rfloor$

ϕ	d	n	fextLWF		fextLPWF(r=1)		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	-0.0342	0.1434	-0.0570	0.2190	-0.0062	0.1415	-0.0145	0.1396	-0.0906	0.1740
		300	-0.0184	0.1020	-0.0331	0.1490	0.0065	0.1051	-0.0016	0.1021	-0.0495	0.1203
		500	-0.0068	0.0846	-0.0245	0.1143	0.0115	0.0891	-0.0016	0.0852	-0.0341	0.0932
		1000	-0.0027	0.0634	-0.0155	0.0827	0.0156	0.0696	0.0016	0.0665	-0.0187	0.0687
	.7	150	-0.0469	0.1383	-0.0613	0.2311	-0.0444	0.1382	-0.0278	0.1220	-0.0910	0.1717
		300	-0.0276	0.1044	-0.0427	0.1710	-0.0254	0.1000	-0.0164	0.0906	-0.0606	0.1176
		500	-0.0186	0.0828	-0.0436	0.1460	-0.0172	0.0793	-0.0131	0.0730	-0.0449	0.0937
		1000	-0.0078	0.0621	-0.0142	0.1078	-0.0064	0.0600	-0.0086	0.0562	-0.0309	0.0715
	.9	150	-0.0600	0.1421	-0.0627	0.2539	-0.0622	0.1439	-0.0401	0.1388	-0.0837	0.1776
		300	-0.0350	0.1026	-0.0351	0.1738	-0.0367	0.1013	-0.0275	0.1030	-0.0485	0.1248
		500	-0.0248	0.0817	-0.0224	0.1336	-0.0265	0.0824	-0.0180	0.0828	-0.0284	0.0916
		1000	-0.0153	0.0609	-0.0137	0.0971	-0.0157	0.0610	-0.0111	0.0625	-0.0161	0.0663
	1.1	150	-0.0658	0.1472	-0.0476	0.2519	-0.0670	0.1436	-0.0432	0.1407	-0.0591	0.1520
		300	-0.0407	0.1036	-0.0290	0.1760	-0.0368	0.1023	-0.0237	0.0990	-0.0363	0.1035
		500	-0.0278	0.0831	-0.0214	0.1349	-0.0272	0.0821	-0.0170	0.0806	-0.0254	0.0822
		1000	-0.0157	0.0613	-0.0125	0.0958	-0.0167	0.0609	-0.0105	0.0597	-0.0148	0.0612
	1.3	150	-0.0611	0.1476	-0.0345	0.2425	-0.0625	0.1464	-0.0374	0.1435	-0.0455	0.1377
		300	-0.0353	0.1042	-0.0159	0.1726	-0.0357	0.1028	-0.0211	0.0996	-0.0276	0.0988
		500	-0.0237	0.0818	-0.0095	0.1352	-0.0237	0.0804	-0.0137	0.0795	-0.0213	0.0810
		1000	-0.0135	0.0607	-0.0061	0.0987	-0.0142	0.0616	-0.0088	0.0618	-0.0131	0.0604
	1.5	150	-0.0568	0.1465	-0.0359	0.2233	-0.0646	0.1481	-0.0516	0.1686	-0.0475	0.1373
		300	-0.0218	0.1054	-0.0169	0.1590	-0.0316	0.1017	-0.0255	0.1146	-0.0233	0.0977
		500	-0.0072	0.0859	-0.0191	0.1206	-0.0205	0.0829	-0.0129	0.0885	-0.0157	0.0780
		1000	0.0043	0.0673	-0.0145	0.0902	-0.0081	0.0629	-0.0037	0.0639	-0.0083	0.0593
	1.7	150	-0.0733	0.1441	-0.0441	0.2237	-0.0690	0.1473	-0.0368	0.1418	-0.0401	0.1342
		300	-0.0398	0.1034	-0.0289	0.1629	-0.0290	0.1008	-0.0200	0.1017	-0.0196	0.0989
		500	-0.0259	0.0829	-0.0469	0.1473	-0.0130	0.0811	-0.0133	0.0816	-0.0090	0.0790
		1000	-0.0127	0.0605	-0.0522	0.1313	-0.0012	0.0638	-0.0077	0.0597	-0.0016	0.0598
	1.9	150	-0.0893	0.1556	-0.0490	0.2350	-0.1011	0.1544	-0.0413	0.1398	-0.0303	0.1305
		300	-0.0513	0.1091	-0.0498	0.1883	-0.0538	0.1030	-0.0218	0.0999	-0.0131	0.0964
		500	-0.0346	0.0849	-0.0502	0.1723	-0.0305	0.0794	-0.0167	0.0802	-0.0055	0.0809
		1000	-0.0201	0.0633	-0.0140	0.1014	-0.0112	0.0581	-0.0109	0.0594	0.0000	0.0614

^a The m is the number of the frequency ordinates involved in the estimation.

Table 2.4: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $m = \lfloor n^8 \rfloor$

ϕ	d	n	fextLWF		fextLPWF(r=1)		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	-0.1800	0.1988	-0.0106	0.1228	-0.1562	0.1798	-0.0189	0.1365	-0.1312	0.1601
		300	-0.1218	0.1363	-0.0073	0.0860	-0.1055	0.1246	-0.0096	0.1025	-0.0920	0.1129
		500	-0.0956	0.1069	-0.0021	0.0675	-0.0823	0.0983	-0.0004	0.0853	-0.0694	0.0853
		1000	-0.0692	0.0779	-0.0002	0.0485	-0.0581	0.0705	0.0025	0.0682	-0.0501	0.0619
	.7	150	-0.1892	0.2035	-0.0188	0.1398	-0.1909	0.2087	-0.0278	0.1227	-0.1152	0.1469
		300	-0.1369	0.1495	-0.0116	0.1036	-0.1332	0.1461	-0.0187	0.0886	-0.0802	0.1001
		500	-0.1066	0.1174	-0.0043	0.0797	-0.1047	0.1147	-0.0123	0.0741	-0.0637	0.0782
		1000	-0.0763	0.0838	0.0006	0.0575	-0.0766	0.0838	-0.0084	0.0566	-0.0454	0.0563
	.9	150	-0.2125	0.2270	-0.0161	0.1339	-0.2135	0.2280	-0.0399	0.1354	-0.1220	0.1511
		300	-0.1469	0.1582	-0.0102	0.0955	-0.1478	0.1591	-0.0252	0.1029	-0.0827	0.1044
		500	-0.1171	0.1255	-0.0079	0.0741	-0.1183	0.1269	-0.0169	0.0835	-0.0637	0.0799
		1000	-0.0853	0.0915	-0.0020	0.0523	-0.0840	0.0906	-0.0085	0.0610	-0.0452	0.0570
	1.1	150	-0.2257	0.2404	-0.0174	0.1352	-0.2293	0.2435	-0.0426	0.1410	-0.1143	0.1432
		300	-0.1561	0.1662	-0.0096	0.0914	-0.1584	0.1688	-0.0221	0.0987	-0.0794	0.0995
		500	-0.1254	0.1334	-0.0073	0.0732	-0.1259	0.1343	-0.0166	0.0800	-0.0626	0.0789
		1000	-0.0895	0.0958	-0.0034	0.0510	-0.0909	0.0970	-0.0102	0.0594	-0.0450	0.0567
	1.3	150	-0.2346	0.2486	-0.0099	0.1322	-0.2356	0.2493	-0.0393	0.1446	-0.1099	0.1382
		300	-0.1619	0.1725	-0.0019	0.0971	-0.1632	0.1736	-0.0210	0.1013	-0.0758	0.0970
		500	-0.1281	0.1366	-0.0014	0.0742	-0.1287	0.1368	-0.0130	0.0821	-0.0601	0.0761
		1000	-0.0939	0.0999	0.0002	0.0532	-0.0935	0.0995	-0.0082	0.0599	-0.0439	0.0555
	1.5	150	-0.2404	0.2552	-0.0164	0.1237	-0.2461	0.2605	-0.0505	0.1663	-0.1038	0.1334
		300	-0.1644	0.1767	-0.0063	0.0885	-0.1689	0.1797	-0.0227	0.1152	-0.0741	0.0955
		500	-0.1292	0.1396	-0.0032	0.0699	-0.1334	0.1418	-0.0133	0.0908	-0.0577	0.0737
		1000	-0.0900	0.0993	0.0014	0.0495	-0.0949	0.1012	-0.0016	0.0639	-0.0414	0.0536
	1.7	150	-0.2443	0.2563	-0.0402	0.1466	-0.2595	0.2735	-0.0381	0.1455	-0.0970	0.1285
		300	-0.1769	0.1845	-0.0164	0.1046	-0.1758	0.1864	-0.0192	0.1001	-0.0676	0.0903
		500	-0.1443	0.1513	-0.0038	0.0775	-0.1352	0.1438	-0.0138	0.0796	-0.0521	0.0705
		1000	-0.1042	0.1099	0.0001	0.0537	-0.0964	0.1031	-0.0076	0.0610	-0.0369	0.0510
	1.9	150	-0.2872	0.2980	-0.0350	0.1478	-0.3047	0.3167	-0.0368	0.1356	-0.0871	0.1207
		300	-0.2006	0.2095	-0.0161	0.0962	-0.2164	0.2254	-0.0263	0.1009	-0.0583	0.0842
		500	-0.1562	0.1631	-0.0091	0.0725	-0.1743	0.1815	-0.0163	0.0802	-0.0461	0.0677
		1000	-0.1127	0.1179	-0.0031	0.0509	-0.1270	0.1323	-0.0092	0.0593	-0.0308	0.0486

^a The m is the number of the frequency ordinates involved in the estimation.

We still need to list some important suggestions for the application of these Fourier estimators in the following:

- For the estimation methods involving tapering, i.e. the LPF and tLWF methods, it requires $\tau \geq \delta$ (see [64]) and for simplicity, we set $\delta = \tau$ throughout this MC experiment since the RMSE expectedly climbs along with the growth of the taper order, which is in line with the theoretical result.
- In this experiment we set the pooling order $p = 1$ (non-pooling) and $p = 3$ to investigate the effect of the pooling technique on the tpLPF estimator. As is illustrated in Tab.2.5 and Tab.2.6, for the moderate or large sample size, we find the pooling expectedly reduces the RMSE at the expense of a little increase in bias. However, the pooled LPF estimator doesn't yield a better performance than the non-pooled estimator for a small sample size ($n = 150$). Therefore it is not advised to pool the periodogram for a small sample size while pooling is preferred for the moderate or large sample size.

Table 2.5: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $p = 1$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.0880	0.5023	0.0767	0.3068	0.1808	0.2587
		300	2	0.0636	0.4365	0.0537	0.2219	0.1261	0.1821
		500	2	0.0393	0.3274	0.0358	0.1743	0.1004	0.1449
		1000	2	0.0359	0.2522	0.0242	0.1368	0.0682	0.1014
	.7	150	2	0.0545	0.4978	0.0642	0.3121	0.1765	0.2563
		300	2	0.0442	0.4092	0.0335	0.2220	0.1175	0.1784
		500	2	0.0325	0.3275	0.0313	0.1784	0.0993	0.1427
		1000	2	0.0285	0.2544	0.0178	0.1348	0.0672	0.1016
	.9	150	2	0.0506	0.5101	0.0494	0.3045	0.1704	0.2541
		300	2	0.0254	0.4168	0.0359	0.2189	0.1150	0.1760
		500	2	0.0298	0.3206	0.0218	0.1784	0.0941	0.1393
		1000	2	0.0178	0.2552	0.0147	0.1343	0.0670	0.1000
	1.1	150	2	0.0335	0.4961	0.0506	0.3007	0.1643	0.2487
		300	2	0.0269	0.4200	0.0291	0.2169	0.1128	0.1727
		500	2	0.0206	0.3192	0.0252	0.1745	0.0941	0.1383
		1000	2	0.0100	0.2546	0.0100	0.1350	0.0622	0.0994
	1.3	150	2	0.0277	0.4994	0.0371	0.3055	0.1624	0.2495
		300	2	0.0153	0.4163	0.0275	0.2174	0.1129	0.1737
		500	2	0.0095	0.3317	0.0138	0.1766	0.0891	0.1374
		1000	2	0.0073	0.2556	0.0133	0.1332	0.0640	0.0980
	1.5	150	3	0.0862	0.6375	0.0775	0.3635	0.1792	0.2889
		300	3	0.0619	0.4856	0.0518	0.2604	0.1207	0.1957
		500	3	0.0411	0.4034	0.0394	0.2042	0.1028	0.1567
		1000	3	0.0423	0.3194	0.0255	0.1540	0.0704	0.1127
	1.7	150	3	0.0499	0.6280	0.0642	0.3643	0.1643	0.2788
		300	3	0.0564	0.4911	0.0472	0.2539	0.1200	0.1955
		500	3	0.0340	0.4086	0.0333	0.2028	0.0985	0.1547
		1000	3	0.0349	0.3242	0.0186	0.1584	0.0672	0.1113
	1.9	150	3	0.0528	0.6286	0.0583	0.3639	0.1652	0.2774
		300	3	0.0373	0.4835	0.0358	0.2552	0.1133	0.1935
		500	3	0.0262	0.4066	0.0231	0.2036	0.0927	0.1518
		1000	3	0.0175	0.3212	0.0148	0.1554	0.0671	0.1115

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 2.6: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $p = 3$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	-0.0058	0.3525	0.0320	0.2005	0.1500	0.1957
		300	1	0.0033	0.2745	0.0189	0.1424	0.1055	0.1366
		500	1	-0.0047	0.2311	0.0112	0.1139	0.0876	0.1103
		1000	1	0.0009	0.1787	0.0055	0.0859	0.0622	0.0791
	.7	150	1	0.0002	0.3422	0.0343	0.1974	0.1495	0.1946
		300	1	-0.0098	0.2677	0.0199	0.1400	0.1046	0.1362
		500	1	-0.0022	0.2268	0.0116	0.1131	0.0882	0.1109
		1000	1	-0.0098	0.1791	0.0067	0.0856	0.0612	0.0782
	.9	150	1	0.0005	0.3432	0.0314	0.2002	0.1530	0.1956
		300	1	-0.0031	0.2687	0.0211	0.1399	0.1052	0.1358
		500	1	-0.0027	0.2288	0.0130	0.1117	0.0866	0.1097
		1000	1	-0.0076	0.1726	0.0066	0.0861	0.0626	0.0790
	1.1	150	1	0.0263	0.3423	0.0437	0.2055	0.1565	0.1986
		300	1	0.0087	0.2626	0.0240	0.1402	0.1082	0.1378
		500	1	0.0042	0.2200	0.0166	0.1118	0.0907	0.1114
		1000	1	0.0057	0.1711	0.0104	0.0850	0.0637	0.0799
	1.3	150	1	0.0514	0.3432	0.0548	0.2053	0.1655	0.2055
		300	1	0.0286	0.2670	0.0359	0.1412	0.1157	0.1438
		500	1	0.0200	0.2226	0.0253	0.1127	0.0946	0.1152
		1000	1	0.0201	0.1779	0.0151	0.0860	0.0659	0.0817
	1.5	150	2	0.0105	0.5628	0.0387	0.2475	0.1698	0.2285
		300	2	0.0120	0.3692	0.0252	0.1762	0.1155	0.1554
		500	2	-0.0030	0.2953	0.0131	0.1409	0.0843	0.1191
		1000	2	0.0012	0.2156	0.0054	0.1102	0.0659	0.0883
	1.7	150	2	0.0155	0.5603	0.0359	0.2480	0.1684	0.2264
		300	2	-0.0026	0.3670	0.0171	0.1791	0.1130	0.1537
		500	2	-0.0058	0.2891	0.0124	0.1443	0.0837	0.1175
		1000	2	0.0007	0.2074	0.0069	0.1079	0.0634	0.0867
	1.9	150	2	0.0060	0.5558	0.0396	0.2490	0.1755	0.2306
		300	2	0.0051	0.3664	0.0190	0.1733	0.1158	0.1550
		500	2	-0.0031	0.2870	0.0129	0.1397	0.0858	0.1191
		1000	2	0.0000	0.2080	0.0033	0.1066	0.0649	0.0880

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

- The modLWF estimator is not an ideal choice for d in the small neighborhoods of $\frac{1}{2}$ and 2, since at these two points a jump in bias is observed.
- In this experiment the fextLPWF estimator is implemented with $r = 1$ and it yields a really bad performance when $m = \lfloor n^{.5} \rfloor$ and favors a bigger m compared to the other Fourier estimators, which is illustrated in the Fig. 2.2. This finding is theoretically supported by the fact that m should satisfy $\frac{m^{2r+1/2}}{n^{2r}} \rightarrow \infty$ for the fextLPWF ($r > 0$) estimator in order to achieve the consistency and asymptotic

normality of $\hat{d}^{fextLPWF}$. For example, if m has the form of $\lfloor n^\alpha \rfloor$, $\alpha > 0$ and $r = 1$, we get $\alpha > .8$. However, according to our MC results, α is not necessarily bigger than .8 from a pragmatic perspective.

To sum up, these Fourier estimators all have their own merits, hence the choice among them for an application is based on the user's preference, that is, the pursuit of better performance in bias or RMSE. Moreover, generally speaking, we find that for all the ARFIMA(p, d, q) ($p + q \neq 0$) series, the fextLPWF ($r = 1$) estimator with bandwidth $m = \lfloor n^{.8} \rfloor$ enjoys the best performance in terms of RMSE. However, the fextLPWF estimation procedure is more computationally burdened than the other estimation procedures. Furthermore, the selection of bandwidth m still remains as an open problem.

Wavelet Estimators

We now turn to the MC results for the wavelet estimators. The Daubechies wavelet filter with vanishing moment $M = 2$ is used in this experiment. Furthermore, the paper [87] indicates that for wavelet estimators, it makes no big differences for the estimation performance among different wavelet filters. The selection of $M = 2$ is because the bigger M is selected the smaller number of wavelet coefficients are involved in the estimation, and hence a relatively big M will cause an inflation in bias and RMSE.

In light of the MC results, we observe that (i) the consistency of the wavelet estimators holds since the RMSE decreases as the sample size increases; (ii) the wavelet estimators' performance is sensitive to the wavelet scales trimming, and the choice of the trimming number is critical in the estimation because without proper trimming the RMSE of estimates is very big; (iii) the RMSE is generally smaller with more scales involved in the wavelet estimation. These findings are numerically reflected via this experiment, and see e.g. Tab.2.7 and Tab.2.11). To have a visual illustration, we display the boxplots of the estimators in the Fig.2.3. Moreover, like the Fourier estimators, the negative MA coefficient considerably contaminates the estimation accuracy, that is, the resulting estimates are usually heavily biased and have a really large RMSE, and for more tables, see Appendix.

Table 2.7: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(0, d , 0) Model

d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.1155	0.2350	-0.1712	0.4358	-0.0677	0.2132	-0.0765	0.5059
	300	-0.0726	0.1333	-0.0949	0.2241	-0.0508	0.1195	-0.0493	0.2102
	500	-0.0807	0.1252	-0.1084	0.2013	-0.0431	0.0837	-0.0408	0.1327
	1000	-0.0513	0.0753	-0.0656	0.1169	-0.0327	0.0556	-0.0270	0.0776
.7	150	-0.1194	0.2365	-0.1640	0.4263	-0.0811	0.2183	-0.0731	0.4958
	300	-0.0794	0.1373	-0.0965	0.2293	-0.0571	0.1243	-0.0450	0.2097
	500	-0.0817	0.1265	-0.1079	0.2003	-0.0494	0.0909	-0.0430	0.1373
	1000	-0.0546	0.0784	-0.0641	0.1171	-0.0357	0.0582	-0.0288	0.0803
.9	150	-0.1252	0.2394	-0.1699	0.4314	-0.0890	0.2253	-0.0879	0.5013
	300	-0.0803	0.1404	-0.1015	0.2323	-0.0584	0.1298	-0.0548	0.2151
	500	-0.0820	0.1271	-0.1100	0.2020	-0.0502	0.0934	-0.0462	0.1389
	1000	-0.0569	0.0815	-0.0645	0.1166	-0.0381	0.0624	-0.0312	0.0842
1.1	150	-0.1248	0.2423	-0.1802	0.4545	-0.0878	0.2264	-0.0848	0.4865
	300	-0.0860	0.1495	-0.1002	0.2382	-0.0598	0.1297	-0.0561	0.2181
	500	-0.0878	0.1333	-0.1094	0.2038	-0.0532	0.0962	-0.0505	0.1425
	1000	-0.0594	0.0864	-0.0665	0.1191	-0.0394	0.0650	-0.0307	0.0854
1.3	150	-0.1310	0.2477	-0.1753	0.4393	-0.0878	0.2235	-0.0815	0.4819
	300	-0.0837	0.1500	-0.1031	0.2355	-0.0600	0.1311	-0.0551	0.2136
	500	-0.0864	0.1358	-0.1097	0.2041	-0.0521	0.0971	-0.0461	0.1414
	1000	-0.0569	0.0845	-0.0666	0.1224	-0.0402	0.0663	-0.0300	0.0871
1.5	150	-0.1245	0.2506	-0.1740	0.4384	-0.0783	0.2182	-0.0822	0.4621
	300	-0.0821	0.1484	-0.1036	0.2366	-0.0591	0.1311	-0.0541	0.2123
	500	-0.0799	0.1293	-0.1105	0.2063	-0.0482	0.0947	-0.0457	0.1423
	1000	-0.0571	0.0853	-0.0684	0.1246	-0.0381	0.0647	-0.0301	0.0884
1.7	150	-0.1139	0.2359	-0.1757	0.4233	-0.0754	0.2089	-0.0851	0.4292
	300	-0.0752	0.1469	-0.1002	0.2323	-0.0553	0.1281	-0.0519	0.2004
	500	-0.0805	0.1311	-0.1055	0.2040	-0.0450	0.0954	-0.0441	0.1377
	1000	-0.0537	0.0844	-0.0665	0.1262	-0.0349	0.0648	-0.0287	0.0880
1.9	150	-0.1093	0.2317	-0.1656	0.4204	-0.0652	0.1923	-0.0754	0.3794
	300	-0.0669	0.1405	-0.0902	0.2246	-0.0482	0.1222	-0.0485	0.1874
	500	-0.0783	0.1328	-0.1108	0.2131	-0.0408	0.0905	-0.0355	0.1312
	1000	-0.0487	0.0838	-0.0652	0.1264	-0.0310	0.0635	-0.0267	0.0863

^a K : trimming number of the highest wavelet scales; J : trimming number of the lowest wavelet scales.

Table 2.8: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(0, d , 0) Model

d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.0431	0.1748	-0.0666	0.3538	-0.0419	0.2003	-0.0764	0.4907
	300	-0.0305	0.0996	-0.0239	0.1691	-0.0333	0.1109	-0.0245	0.2046
	500	-0.0260	0.0718	-0.0157	0.1123	-0.0269	0.0741	-0.0182	0.1234
	1000	-0.0220	0.0488	-0.0112	0.0675	-0.0211	0.0483	-0.0095	0.0704
.7	150	-0.0529	0.1837	-0.0581	0.3509	-0.0612	0.2149	-0.0768	0.5040
	300	-0.0368	0.1053	-0.0314	0.1757	-0.0407	0.1155	-0.0307	0.2085
	500	-0.0308	0.0765	-0.0183	0.1175	-0.0330	0.0781	-0.0194	0.1224
	1000	-0.0250	0.0509	-0.0112	0.0704	-0.0253	0.0518	-0.0118	0.0735
.9	150	-0.0608	0.1870	-0.0668	0.3568	-0.0662	0.2141	-0.0912	0.5023
	300	-0.0413	0.1114	-0.0308	0.1801	-0.0427	0.1188	-0.0350	0.2080
	500	-0.0319	0.0791	-0.0218	0.1202	-0.0342	0.0827	-0.0214	0.1279
	1000	-0.0276	0.0552	-0.0124	0.0735	-0.0288	0.0553	-0.0128	0.0760
1.1	150	-0.0599	0.1871	-0.0743	0.3663	-0.0595	0.2093	-0.0934	0.4843
	300	-0.0445	0.1137	-0.0345	0.1827	-0.0457	0.1216	-0.0369	0.2130
	500	-0.0357	0.0827	-0.0234	0.1252	-0.0356	0.0847	-0.0207	0.1297
	1000	-0.0292	0.0571	-0.0123	0.0749	-0.0307	0.0574	-0.0134	0.0790
1.3	150	-0.0658	0.1894	-0.0743	0.3531	-0.0714	0.2176	-0.0777	0.4755
	300	-0.0422	0.1144	-0.0369	0.1836	-0.0451	0.1238	-0.0369	0.2042
	500	-0.0373	0.0850	-0.0246	0.1278	-0.0379	0.0865	-0.0231	0.1312
	1000	-0.0297	0.0580	-0.0159	0.0781	-0.0299	0.0590	-0.0148	0.0792
1.5	150	-0.0643	0.1896	-0.0788	0.3457	-0.0670	0.2096	-0.0844	0.4545
	300	-0.0431	0.1154	-0.0391	0.1822	-0.0441	0.1202	-0.0355	0.2031
	500	-0.0359	0.0854	-0.0228	0.1259	-0.0362	0.0867	-0.0234	0.1279
	1000	-0.0271	0.0581	-0.0140	0.0779	-0.0295	0.0590	-0.0134	0.0803
1.7	150	-0.0645	0.1857	-0.0859	0.3384	-0.0664	0.2032	-0.0819	0.4247
	300	-0.0425	0.1144	-0.0399	0.1787	-0.0407	0.1188	-0.0373	0.1913
	500	-0.0329	0.0841	-0.0243	0.1241	-0.0339	0.0849	-0.0245	0.1255
	1000	-0.0254	0.0568	-0.0146	0.0788	-0.0280	0.0588	-0.0162	0.0791
1.9	150	-0.0606	0.1775	-0.0930	0.3219	-0.0550	0.1838	-0.0555	0.3736
	300	-0.0390	0.1106	-0.0440	0.1737	-0.0390	0.1136	-0.0388	0.1784
	500	-0.0310	0.0840	-0.0292	0.1208	-0.0322	0.0835	-0.0259	0.1192
	1000	-0.0235	0.0576	-0.0175	0.0790	-0.0241	0.0577	-0.0150	0.0782

^a K : trimming number of the highest wavelet scales; J : trimming number of the lowest wavelet scales.

Moreover, we find that trimming the smallest lower-scales usually gains a big bias reduction, because lower-scale wavelet coefficients reflect a short or intermediate memory feature rather than a long memory feature in data, which is theoretically supported by (2.23). We also find for $d \in [\frac{1}{2}, 2]$, the RMSE of the wavelet estimator decreases as the value of d increases, which is probably because the implicit M^{th} -order differencing renders the $I(d)$ process with small d over-differenced during the computation of the wavelet coefficients.

We now give a wavelet scales trimming strategy for the LRW and LWW estimators,

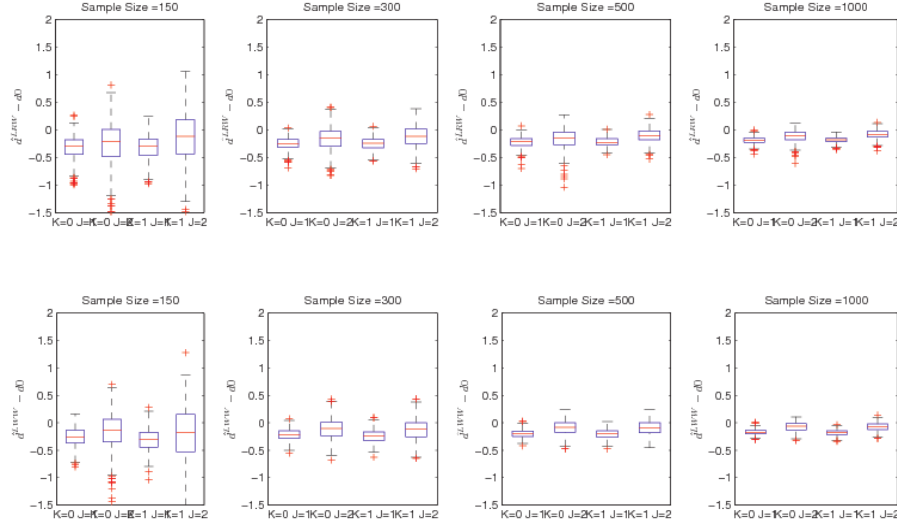


Figure 2.3: The Wavelet Estimators' Performance for the ARFIMA(0,d,1) Model with $d = 1.2$ and $\theta = -0.4$. The boxplots in the first row are for the LPW estimator and the boxplots in the second row are for the LWW estimator according to four wavelet trimming strategies. The boxplots are based on the MC results of 5000 replications.

respectively.

- For the LRW estimator, trimming the largest upper-scale may be conducive to the RMSE reduction, probably due to the very limited wavelet coefficients at the largest scale so that the empirical variance Eq.(2.18) could not be well approximated.
- Unlike the LRW estimator, for the LWW estimator, trimming the largest upper scale is not necessary at all.

Furthermore, for the innovations in Eq.(2.24) with a positive AR coefficient or negative MA coefficient, the LWW estimator generally has a better estimation performance over the LRW estimator in terms of the bias and RMSE; and for the other cases, these two estimators have their own merits.

To conclude this subsection, there leaves a remark to be highlighted.

Remarque 2.4.1. *If we are sure that the data considered is generated by Eq.(2.1), the parametric time-domain maximum likelihood estimator investigated in [113] applies to the non-stationary case, in fact, for all $d \in \mathbb{R}$. However, it is a computation intensive method.*

2.4.2 Monte Carlo Results for ARFIMA(p, d, q) Processes with Heteroscedasticity in $\{u_t\}$

Highly significant heteroscedasticity has been observed in economic and financial time series. For example, the log-returns of stock indices (see e.g. [70], [75], [77], [83]),

exchange rates (see e.g. [101], [115], [93]), futures contracts (see e.g. [5], [80]) and etc.. Actually, since the influential paper [121], work on heteroscedasticity has spawned. In [12], the rate of growth in the implicit GNP (gross national product) deflator was detected to have heteroscedastic effect and this heteroscedasticity was modeled by a GARCH (generalized ARCH) model, which is a generalization to the ARCH (autoregressive conditional heteroscedasticity) model introduced in [37]. In [9], the authors integrated both long-memory property and heteroscedasticity into one model, an ARFIMA-GARCH model, and this model depicted the evolution of CPI (consumer price index) inflations quite well for ten economies. For more reference on modeling heteroscedasticity, see e.g. [13], [29], [47], [48], [32], [33], [34], [35], [36] and etc.. Furthermore, the behavior of many financial series exhibits simultaneously long memory and heteroscedasticity. Therefore it is important to check if the estimation procedures in this chapter are robust to the non-stationary $I(d)$ process with non-Gaussian innovations in Eq.(2.24).

The GARCH(1,1) model for the $\{\epsilon_t\}$ in Eq.(2.24) is employed in this Monte Carlo simulation study. As this comparison study indicates that the semi-parametric estimation procedures, including Fourier and wavelet domain methods, for the series with GARCH effects produce a comparable outcome to the ones with Gaussian innovations, the Monte Carlo results for this case are not displayed in Sec.6.

2.4.3 Monte Carlo Results for ARFIMA(p, d, q) Processes with a Linear Time Trend or an Occasional-Structural-Break Process in Levels

In the Sec. 2.4.1 and Sec. 2.4.2, we analyze the estimators' FSP for processes with a constant in levels, actually $\mu = 0$. However, it's usually not the case for historical financial or economic observations, for example, the GDP data in [62], the S&P 500 absolute returns in [52] and etc. These two factors in data may yield spurious inferences of long-memory property and therefore, we carry out an experiment to check if a linear time trend or OSB in levels contaminates the memory parameter estimation results and to examine the impact of them on the estimators' FSP. We report our MC findings in the following.

- Linear time trend in $\{X_t\}$

In this part we consider the process $\{X_t\}$ endowed with a linear time trend since a linear time trend in the economic and financial data is very common, and specifically, in this Monte Carlo experiment $\{X_t\}$ has the following representation:

$$(1 - B)^d(X_t - 1 - 2t) = u_t, \quad (2.28)$$

where $\{u_t\}$ has three situations as in Eq.(2.26)-Eq.(2.27). The semi-parametric estimation procedures investigated in this chapter are theoretically applicable to the time series with a linear time trend, which is mentioned or implied in Sec.2.2 and Sec.2.3. In detail, for the GPHF and tapered LWF estimation procedures, since the interval we consider in this paper requires the difference order $\delta \geq 1$, the linear time trend term will be canceled out after differencing; for the modLWF and fextLF estimation procedures, under the data generating specification Eq.(2.28), the modified

DFT and fully-extended DFT hence their estimators are invariant to a linear time trend; for the dtr-feLWF estimation procedure, the linear time trend is detrended so that the feLWF estimator applies to the detrended data; and finally, for the LRW and LWW estimation procedures, the assumptions **(W-3)** and **(W-4)** imply that the wavelet estimators are invariant to a linear time trend. Therefore as is expected, the Monte Carlo results of data generated by Eq.(2.28) is essentially the same as those in Sec. 2.4.1 and hence the results are not contained in the Sec.6.

- Occasional-break process in $\{X_t\}$

The research on long memory and regime switching have developed mostly independently over decades. For reference on regime-switching models, see e.g. [10], [1], [11], [18], [30] and [19] etc. However, the appearance of a occasional-structural-break (OSB) property in data is usually confused with long memory. This phenomenon was documented and studied, for example, in [25], [38] and [52]. In order to shed some light on this aspect, we are motivated to evaluate the robustness of the estimation procedures for the series with exposure to occasional structural breaks in levels. In this experiment the estimators' performance is investigated under a markov switching (MS) model. The MS model has the following representation:

$$y_t = \beta z_t^{nS} + \phi_{S_t}^* z_t^S + e_t, \quad (2.29)$$

$$e_t = P(\phi_{S_t}^*), \quad t = 1, \dots, n, \quad (2.30)$$

The superscript nS (S) signifies the non-switching (switching) property of the explanatory variable z . The value of ϕ^* depends on the switching parameter S_t . The term $P(\phi^*)$ with switching effects is the assumed probability density function of the innovations $\{e_t\}$. In this experiment, z_t^S has two states (State 1 and State 2), and the parameters of the MS model is set as follows: $\beta = .2$,

$$\phi_{S_t}^* = \begin{cases} 1.2, & \text{with standard deviation .1 in } S_t = 1; \\ .1, & \text{with standard deviation .1 in } S_t = 2; \end{cases}$$

and

$$z_t^S = \begin{cases} .5, & \text{with standard deviation 1 in } S_t = 1; \\ .1, & \text{with standard deviation .5 in } S_t = 2; \end{cases}$$

and the transition matrix is

$$\begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix}$$

For more details, the interested readers can refer to [96]. The data generating specification Eq.(2.24) now becomes

$$(1 - B)^d(X_t - y_t) = u_t, \quad (2.31)$$

where $\{u_t\}$ has three situations as in Eq.(2.26)-Eq.(2.27). Given the Monte Carlo results in the Tables, although there is no theoretical result is established in this case, under the condition that only a small number of OSB happens (the transition probability from one regime to the other is set to 0.1), it is revealed that the consistency of

the semi-parametric estimators considered still practically holds. Clearly, their FSP is similar to that of the case without OSB in levels in Sec.2.4.1, but the OSB obviously contaminates the estimation performance, since the inflation in magnitude of both bias and RMSE is observed. This fact is numerically reflected e.g. in Tab.2.9 and Tab.2.10. Moreover, for the log-regression wavelet estimator, the RMSE are quite big for the median and small sample sizes ($n \leq 500$), and hence, we assert that when analyzing data with regime switching, we must use the LRW estimator with great prudence.

Table 2.9: Monte Carlo Results of the fextLPWF($r=1$) Estimation for ARFIMA($0, d, 0$) Models with OSB in levels

d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
		bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.1895	0.4347	-0.0971	0.2331	-0.0799	0.1557
		(-0.1419)	(0.4167)	(-0.0477)	(0.2229)	(-0.0064)	(0.1275)
	300	-0.1179	0.3127	-0.0602	0.1634	-0.0596	0.1110
		(-0.0887)	(0.3012)	(-0.0230)	(0.1502)	(-0.0029)	(0.0889)
	500	-0.0887	0.2490	-0.0495	0.1336	-0.0515	0.0892
		(-0.0612)	(0.2395)	(-0.0190)	(0.1249)	(0.0046)	(0.0698)
	1000	-0.0648	0.1818	-0.0314	0.0988	-0.0409	0.0671
		(-0.0481)	(0.1809)	(-0.0057)	(0.0895)	(0.0037)	(0.0507)
.9	150	-0.1730	0.4412	-0.0859	0.2531	-0.0978	0.1649
		(-0.1417)	(0.4336)	(-0.0649)	(0.2496)	(-0.0214)	(0.1322)
	300	-0.1032	0.3305	-0.0460	0.1777	-0.0677	0.1176
		(-0.0969)	(0.3329)	(-0.0303)	(0.1712)	(-0.0119)	(0.0950)
	500	-0.0764	0.2696	-0.0361	0.1487	-0.0516	0.0897
		(-0.0620)	(0.2664)	(-0.0257)	(0.1436)	(-0.0085)	(0.0735)
	1000	-0.0432	0.2050	-0.0178	0.0994	-0.0368	0.0634
		(-0.0378)	(0.2020)	(-0.0148)	(0.0973)	(-0.0031)	(0.0521)
1.3	150	-0.1256	0.4252	-0.0390	0.2630	-0.0756	0.1585
		(-0.1066)	(0.4408)	(-0.0353)	(0.2626)	(-0.0124)	(0.1352)
	300	-0.0793	0.3199	-0.0142	0.1755	-0.0397	0.1052
		(-0.0616)	(0.3230)	(-0.0115)	(0.1732)	(-0.0072)	(0.0961)
	500	-0.0585	0.2599	-0.0084	0.1400	-0.0268	0.0792
		(-0.0381)	(0.2620)	(-0.0042)	(0.1418)	(-0.0023)	(0.0737)
	1000	-0.0406	0.1921	-0.0002	0.1013	-0.0119	0.0536
		(-0.0356)	(0.1958)	(-0.0018)	(0.1038)	(0.0006)	(0.0524)
1.7	150	-0.1081	0.4156	-0.0461	0.2619	-0.0755	0.1545
		(-0.0820)	(0.4154)	(-0.0447)	(0.2575)	(-0.0425)	(0.1449)
	300	-0.0710	0.3078	-0.0159	0.1705	-0.0268	0.1094
		(-0.0496)	(0.2998)	(-0.0170)	(0.1584)	(-0.0183)	(0.1028)
	500	-0.0492	0.2535	-0.0149	0.1378	-0.0058	0.0842
		(-0.0367)	(0.2397)	(-0.0132)	(0.1331)	(-0.0084)	(0.0786)
	1000	-0.0880	0.2034	-0.0055	0.1060	0.0098	0.0564
		(-0.0781)	(0.2017)	(-0.0079)	(0.0977)	(-0.0028)	(0.0537)

The bracketed numbers are estimated for the ARFIMA($0, d, 0$) case with OSB in levels.

Table 2.10: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(0, d , 0) Model with OSB in Levels

d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.2178	0.2979	-0.2471	0.4709	-0.1936	0.2813	-0.1587	0.5153
		(-0.1155)	(0.2350)	(-0.1712)	(0.4358)	(-0.0677)	(0.2132)	(-0.0765)	(0.5059)
	300	-0.1714	0.2049	-0.1597	0.2573	-0.1545	0.1896	-0.1178	0.2374
		(-0.0726)	(0.1333)	(-0.0949)	(0.2241)	(-0.0508)	(0.1195)	(-0.0493)	(0.2102)
	500	-0.1681	0.1930	-0.1779	0.2442	-0.1398	0.1579	-0.1120	0.1689
		(-0.0807)	(0.1252)	(-0.1084)	(0.2013)	(-0.0431)	(0.0837)	(-0.0408)	(0.1327)
	1000	-0.1357	0.1474	-0.1207	0.1531	-0.1218	0.1298	-0.0891	0.1152
		(-0.0513)	(0.0753)	(-0.0656)	(0.1169)	(-0.0327)	(0.0556)	(-0.0270)	(0.0776)
	.9	-0.2623	0.3357	-0.2383	0.4741	-0.2472	0.3238	-0.1679	0.5226
		(-0.1252)	(0.2394)	(-0.1699)	(0.4314)	(-0.0890)	(0.2253)	(-0.0879)	(0.5013)
.9	300	-0.1984	0.2303	-0.1600	0.2630	-0.1973	0.2286	-0.1146	0.2370
		(-0.0803)	(0.1404)	(-0.1015)	(0.2323)	(-0.0584)	(0.1298)	(-0.0548)	(0.2151)
	500	-0.1899	0.2131	-0.1618	0.2377	-0.1717	0.1880	-0.0979	0.1621
		(-0.0820)	(0.1271)	(-0.1100)	(0.2020)	(-0.0502)	(0.0934)	(-0.0462)	(0.1389)
	1000	-0.1556	0.1665	-0.1072	0.1449	-0.1461	0.1540	-0.0764	0.1100
		(-0.0569)	(0.0815)	(-0.0645)	(0.1166)	(-0.0381)	(0.0624)	(-0.0312)	(0.0842)
	1.3	-0.2560	0.3304	-0.2021	0.4550	-0.2503	0.3272	-0.1361	0.4911
		(-0.1310)	(0.2477)	(-0.1753)	(0.4393)	(-0.0878)	(0.2235)	(-0.0815)	(0.4819)
	300	-0.1949	0.2302	-0.1304	0.2516	-0.1930	0.2263	-0.0964	0.2313
		(-0.0837)	(0.1500)	(-0.1031)	(0.2355)	(-0.0600)	(0.1311)	(-0.0551)	(0.2136)
1.3	500	-0.1829	0.2099	-0.1344	0.2210	-0.1633	0.1824	-0.0792	0.1563
		(-0.0864)	(0.1358)	(-0.1097)	(0.2041)	(-0.0521)	(0.0971)	(-0.0461)	(0.1414)
	1000	-0.1471	0.1598	-0.0848	0.1321	-0.1365	0.1461	-0.0562	0.0994
		(-0.0569)	(0.0845)	(-0.0666)	(0.1224)	(-0.0402)	(0.0663)	(-0.0300)	(0.0871)
	1.7	-0.2215	0.3069	-0.1890	0.4468	0.2074	0.2876	-0.0972	0.4303
		(-0.1139)	(0.2359)	(-0.1757)	(0.4233)	(-0.0754)	(0.2089)	(-0.0851)	(0.4292)
	300	-0.1576	0.2010	-0.1098	0.2376	-0.1541	0.1935	-0.0685	0.2081
		(-0.0752)	(0.1469)	(-0.1002)	(0.2323)	(-0.0553)	(0.1281)	(-0.0519)	(0.2004)
	500	-0.1504	0.1844	-0.1192	0.2167	-0.1279	0.1529	-0.0532	0.1430
		(-0.0805)	(0.1311)	(-0.1055)	(0.2040)	(-0.0450)	(0.0954)	(-0.0441)	(0.1377)
1.7	1000	-0.1190	0.1363	-0.0726	0.1280	-0.1078	0.1211	-0.0375	0.0904
		(-0.0537)	(0.0844)	(-0.0665)	(0.1262)	(-0.0349)	(0.0648)	(-0.0287)	(0.0880)

^a K : trimming number of the highest wavelet scales; J : trimming number of the lowest wavelet scales.

^b The bracketed numbers are are estimated for the ARFIMA(0, d , 0) case with OSB in levels.

Table 2.11: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(0, d , 0) Model with OSB in Levels

d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.1587 (-0.0431)	0.2334 (0.1748)	-0.1448 (-0.0666)	0.3771 (0.3538)	-0.1674 (-0.0419)	0.2628 (0.2003)	-0.1604 (-0.0764)	0.5138 (0.4907)
	300	-0.1286 (-0.0305)	0.1606 (0.0996)	-0.0993 (-0.0239)	0.1993 (0.1691)	-0.1389 (-0.0333)	0.1747 (0.1109)	-0.1001 (-0.0245)	0.2234 (0.2046)
	500	-0.1241 (-0.0260)	0.1414 (0.0718)	-0.0861 (-0.0157)	0.1416 (0.1123)	-0.1249 (-0.0269)	0.1421 (0.0741)	-0.0869 (-0.0182)	0.1479 (0.1234)
	1000	-0.1116 (-0.0220)	0.1195 (0.0488)	-0.0736 (-0.0112)	0.1003 (0.0675)	-0.1134 (-0.0211)	0.1217 (0.0483)	-0.0745 (-0.0095)	0.1019 (0.0704)
.9	150	-0.2169 (-0.0608)	0.2807 (0.1870)	-0.1470 (-0.0668)	0.3820 (0.3568)	-0.2339 (-0.0662)	0.3112 (0.2141)	-0.1780 (-0.0912)	0.5165 (0.5023)
	300	-0.1688 (-0.0413)	0.1987 (0.1114)	-0.0883 (-0.0308)	0.1970 (0.1801)	-0.1826 (-0.0427)	0.2143 (0.1188)	-0.1021 (-0.0350)	0.2282 (0.2080)
	500	-0.1513 (-0.0319)	0.1679 (0.0791)	-0.0732 (-0.0218)	0.1387 (0.1202)	-0.1575 (-0.0342)	0.1745 (0.0827)	-0.0752 (-0.0214)	0.1470 (0.1279)
	1000	-0.1347 (-0.0276)	0.1427 (0.0552)	-0.0597 (-0.0124)	0.0927 (0.0735)	-0.1384 (-0.0288)	0.1465 (0.0553)	-0.0605 (-0.0128)	0.0962 (0.0760)
1.3	150	-0.2078 (-0.0658)	0.2758 (0.1894)	-0.1182 (-0.0743)	0.3707 (0.3531)	-0.2435 (-0.0714)	0.3218 (0.2176)	-0.1400 (-0.0777)	0.4978 (0.4755)
	300	-0.1648 (-0.0422)	0.1972 (0.1144)	-0.0705 (-0.0369)	0.1918 (0.1836)	-0.1812 (-0.0451)	0.2152 (0.1238)	-0.0699 (-0.0369)	0.2186 (0.2042)
	500	-0.1441 (-0.0373)	0.1635 (0.0850)	-0.0539 (-0.0246)	0.1355 (0.1278)	-0.1508 (-0.0379)	0.1716 (0.0865)	-0.0548 (-0.0231)	0.1400 (0.1312)
	1000	-0.1244 (-0.0297)	0.1345 (0.0580)	-0.0385 (-0.0159)	0.0868 (0.0781)	-0.1288 (-0.0299)	0.1386 (0.0590)	-0.0408 (-0.0148)	0.0882 (0.0792)
1.7	150	-0.1734 (-0.0645)	0.2539 (0.1857)	-0.0988 (-0.0859)	0.3392 (0.3384)	-0.1979 (-0.0664)	0.2795 (0.2032)	-0.0957 (-0.0819)	0.4288 (0.4247)
	300	-0.1309 (-0.0425)	0.1712 (0.1144)	-0.0476 (-0.0399)	0.1811 (0.1787)	-0.1443 (-0.0407)	0.1847 (0.1188)	-0.0520 (-0.0373)	0.1959 (0.1913)
	500	-0.1139 (-0.0329)	0.1389 (0.0841)	-0.0338 (-0.0243)	0.1272 (0.1241)	-0.1221 (-0.0339)	0.1467 (0.0849)	-0.0366 (-0.0245)	0.1307 (0.1255)
	1000	-0.0992 (-0.0254)	0.1126 (0.0568)	-0.0264 (-0.0146)	0.0810 (0.0788)	-0.1012 (-0.0280)	0.1142 (0.0588)	-0.0250 (-0.0162)	0.0817 (0.0791)

^a K : trimming number of the highest wavelet scales; J : trimming number of the lowest wavelet scales.

^b The bracketed numbers are are estimated for the ARFIMA(0, d , 0) case with OSB in levels.

2.4.4 Discussion

In this chapter we first review popular semi-parametric memory parameter estimation procedures for non-stationary $I(d)$ processes in the existing literature. Then we carry out a Monte Carlo experiment to evaluate the FSP of these procedures via bias and RMSE. In this experiment we consider the case Eq.(2.24), and its variations: (i) the process with a linear time trend; (ii) OSB in levels; and (iii) GARCH effects in variance. According to the Monte Carlo results, we cast a light on the impact the tapering and pooling orders have on the tpLPF estimator (the tapering order has on the tLWF estimator); and propose a suitable wavelet scale trimming strategy for the wavelet estimators. In sum, this Monte Carlo comparison study indicates that (i) both the Fourier and wavelet estimators are robust to

the $I(d)$ process with a linear time trend and GARCH innovations; (ii) the consistency of the estimators still holds in the presence of regime switches in levels of $\{X_t\}$, however, it tangibly contaminates the estimation results. Moreover, the LRW estimator works badly in this situation with small and medium sample sizes; (iii) the wavelet estimators generally have an inferior performance compared to the Fourier estimators in all the cases considered in this chapter; and (iv) from a pragmatic perspective, the fextLPWF estimator is favored since they are suitable for all $d > -\frac{1}{2}$ and they generally enjoy relatively smaller RMSE compared to the other estimators.

Chapter 3

Seasonal/Cyclical Long-Memory Processes

The seasonality/cycle is often spotted in historical financial and economic observations, for example, the U.S. monetary aggregates M1, M2 and M3 ([98]), the IBM product revenues ([102]), the U.K. monthly inflation series ([7]) and the electricity spot prices from German energy market ([31]), each of which possesses a spectral peak not located at the origin. For more reference on applications, see e.g. [40], [42], [43] and [44]. This fact calls for a direct extension to Eq.(2.3) as follows.

3.1 Seasonal/Cyclical Long-Memory Processes

Now the spectral density function $f(\lambda)$ for a time series is assumed to have the below characteristic in the small neighborhood of frequency $\omega \in (0, \pi)$, (at least for one such ω) for a constant $0 < G < \infty$,

$$f(\omega \pm \lambda) \sim G\lambda^{-2d}, \quad \lambda \rightarrow 0^+, \quad (3.1)$$

where $|d| < \frac{1}{2}$. When $d > 0$ there is a spectral peak at frequency ω and when $d < 0$ a spectral zero. As a matter of fact, (3.1) indicates that the series is seasonal/cyclical and ω represents the seasonal/cyclical frequency. Therefore this kind of series is called a seasonal and cyclical long-memory (SCLM) process. Evidently, Eq.(3.1) depicts a local feature of $f(\lambda)$, and hence there may exist other singularities of this form in $(0, \pi)$, which implies the underlying series possesses more than one seasonal/cyclical frequency. A k-factor Gegenbauer process is one example, which is detailed in Sec.3.4.

For the methods of estimating memory parameter $|d| < \frac{1}{2}$, see e.g. [7] in which an asymmetric SCLM (ASCLM) process was additionally proposed and investigated. However, we cannot assume a priori that the memory parameter d of real data surely lies inside the interval $(-\frac{1}{2}, \frac{1}{2})$ and hence we are motivated to deal with statistical inference for SCLM series with $d \in \mathbb{R}$. If $d \geq \frac{1}{2}$, a SCLM series becomes non-stationary and now

that $f(\cdot)$ is not integrable in $(-\pi, \pi)$, we call $f(\cdot)$ in this case the pseudo-spectral density function. To obtain an adequate memory parameter estimate of d for non-stationary time series, differencing the raw data is an option, nevertheless, it may incur information loss, and hence we propose the estimators suitable for raw SCLM series with $d \in \mathbb{R}$. The estimators are both based on [117] and [118] where non-stationary LM series are considered. As is for the LM case, we take advantage of two popular semi-parametric estimation methods: the log-periodogram semi-parametric estimation (LPSE) method and the Gaussian semi-parametric estimation (GSE) method. Both methods require tapering raw observations, since when $d > 1$ the periodogram of the untapered data is no longer an acceptable approximation to the pseudo-spectral density function $f(\cdot)$ and tapering can reduce the frequency leakage near the spectral peak. We also present the consistency and normality of these estimators. In this contribution most of our effort is concentrated on the non-stationary SCLM series, i.e., $d > \frac{1}{2}$, since the $d < -\frac{1}{2}$ case is a rare case in the real world. However, these estimators are still numerically evaluated for the $d < -\frac{1}{2}$ case in the Monte Carlo experiment in Section 3.4.

Up to our knowledge, even for (A)SCLM series with $|d| < \frac{1}{2}$, there leaves a margin for a detailed analysis concerning the finite sample performance (FSP) of the semi-parametric memory parameter estimators and furthermore, there exists only a very limited amount of Monte Carlo studies in this regard in the statistical literature. For references interested readers can see [59] for the parametric estimator; see [8] for the frequency-domain semi-parametric estimators in the ASCLM case; and see [124] for the wavelet-domain semi-parametric estimators. Thus this reality stimulates us to carry out an extensive Monte Carlo experiment, where non-stationary and stationary series are accommodated, to assess the accuracy of the estimators and analyse the factors affecting the estimation results. As a consequence, a guideline for the application of these two estimators is provided from a pragmatic perspective.

3.2 Assumptions and Preliminaries

As is already known, the Eq.(3.1) provides an alternative model for stationary seasonal and cyclical processes with memory $d \in (-\frac{1}{2}, \frac{1}{2})$. Now we introduce an important example of SCLM processes as follows.

Definition 3.2.1. *Let $\{X_t\}$ be a process such that*

$$\Phi(B) \prod_{i=1}^k (I - 2\nu_i B + B^2)^{d_i} (X_t - \mu) = \Theta(B) \varepsilon_t, \quad i = 1, \dots, k, \quad (3.2)$$

where k is a positive integer, $|\nu_i| \leq 1$, d_i is a fractional number, μ is the mean of the process and ε_t is Gaussian white noise with a constant variance σ_ε^2 . The polynomials $\Phi(\cdot)$ and $\Theta(\cdot)$ are defined as in Def.2.0.1. We call this $\{X_t\}$ a k -factor GARMA process. The frequencies $\lambda_i = \arccos(\nu_i)$, $i = 1, \dots, k$ are called Gegenbauer frequencies. In

the following proposition, we summarize the statistical properties for a k -factor GARMA process. For reference see [55] and [125].

Proposition 3.2.2. *Let $d_i \neq 0$, for $i = 1, \dots, k$, such that:*

$$\begin{aligned} |d_i| &< 1/2, \text{ if } |\nu_i| < 1; \\ |d_i| &< 1/4, \text{ if } |\nu_i| = 1. \end{aligned}$$

Then the k -factor Gegenbauer process $\{X_t\}_{t \in \mathbb{Z}}$ defined by Def.?? has the following properties:

- $\{X_t\}$ is stationary and invertible.
- If $d_i > 0$, $i = 1, \dots, k$, then $\{X_t\}$ is of long memory.

When $\Phi(z) = \Theta(z) \equiv 1$, $\{X_t\}_{t \in \mathbb{Z}}$ becomes a k -factor Gegenbauer process. From now on, we dwell on the 1-factor Gegenbauer process.

Following the idea of defining a non-stationary long-memory process in [117], when $\{X_t\}$ is a non-stationary process, it is said to have memory parameter $d(\geq \frac{1}{2})$ if $\exists s \in \mathbb{Z}$,

$$Y_t = (I - 2\nu B + B^2)^s X_t, \quad |d - s| < \frac{1}{2}, \quad (3.3)$$

where s is the difference order, and $\{Y_t\}$ has the spectral density function as follows: for $\lambda \rightarrow 0^+$,

$$\begin{aligned} f_Y(\omega \pm \lambda) &= |1 - 2\nu \exp(i(\omega \pm \lambda)) + \exp(i2(\omega \pm \lambda))|^{-2(d-s)} f^*(\omega \pm \lambda) \\ &= |2(\cos(\omega \pm \lambda) - \cos \omega)|^{-2(d-s)} f^*(\omega \pm \lambda) \\ &= |4 \sin \frac{2\omega \pm \lambda}{2} \sin \frac{\lambda}{2}|^{-2(d-s)} f^*(\omega \pm \lambda), \end{aligned} \quad (3.4)$$

where $\omega = \arccos(\nu)$, $\omega \in (0, \pi)$; $f^*(\lambda)$ is bounded above, away from zero and continuous at frequency ω . Clearly $f_Y(\omega + \lambda)$ is in line with (3.1) because $\sin \lambda \sim \lambda$, $\lambda \rightarrow 0$.

Now taking a look at the situation when $s = 1$, we assume that $\{X_t\}$ is obtained by the recursive procedure as follows:

$$\begin{aligned} X_1 &= Y_1 + 2\nu X_0 + X_{-1} \\ &\vdots \\ X_t &= Y_t + 2\nu X_{t-1} + X_{t-2}, \quad t \geq 1, \end{aligned}$$

where X_{-1} and X_0 are random variables independent of time t . We call this procedure the integration of $\{Y_t\}$ and for $s > 1$, $\{X_t\}$ is the outcome of s times integration of $\{Y_t\}$.

Then the pseudo-spectral density function $f(\omega \pm \lambda)$ for $d \geq \frac{1}{2}$ is defined by

$$\begin{aligned} f(\omega \pm \lambda) &= |1 - 2\nu \exp(i(\omega \pm \lambda)) + \exp(i2(\omega \pm \lambda))|^{-2s} f_Y(\omega \pm \lambda) \\ &= |1 - 2\nu \exp(i(\omega \pm \lambda)) + \exp(i2(\omega \pm \lambda))|^{-2d} f^*(\omega \pm \lambda) \\ &= |4 \sin \frac{2\omega \pm \lambda}{2} \sin \frac{\lambda}{2}|^{-2d} f^*(\omega \pm \lambda), \quad \nu = \cos \omega. \end{aligned}$$

Thus $f(\cdot)$ conforms to Eq.(3.1) with $d \in \mathbb{R}$. When $d \geq 1$, we encounter the same problem as Velasco did in [117], i.e., the periodogram is a greatly biased substitute for $f(\cdot)$ around frequency ω . Therefore to solve this problem, we resort to the tapered periodogram, restated below.

Given a data taper series $\{h_t\}$, define the tapered discrete Fourier transform (DFT) of $\{x_t\}_{t=1}^n$, $\lambda_j = \frac{2\pi j}{n}$, $j \in \mathbb{N}$,

$$\omega^t(\lambda_j) = \frac{1}{\sqrt{2\pi}} \sum_{t=1}^n \frac{h_t}{\sqrt{\sum_{t=1}^n h_t^2}} x_t \exp(i\lambda_j t), \quad j = 1, \dots, n,$$

and then the tapered periodogram of $\{X_t\}_{t=1}^n$ is

$$I^t(\lambda_j) = |\omega^t(\lambda_j)|^2, \quad j = 1, 2, \dots, n.$$

The taper class utilized in this paper should satisfy the following properties so that tapering can help reduce the frequency leakage near the peak:

- $\sum_{t=1}^n h_t \{1 + t + t^2 + \dots + t^s\} \exp(i\lambda_j t) = 0$;
- For a function $b(n)$, $\forall n > 0$, $0 < b(n) < \infty$, $\sum_{t=1}^n = b(n)$.
- For simplicity n is assumed to be a multiple of p , the Dirichlet kernel D_p^T satisfies

$$D_p^t(\lambda) = \sum_{t=1}^n h_t \exp(i\lambda t) = \frac{a(\lambda)}{n^{p-1}} \left(\frac{\sin[n\lambda/(2p)]}{\sin[\lambda/2]} \right),$$

where $a(\lambda)$ is a complex function, whose modulus is bounded and bounded away from 0, with $p - 1$ derivatives, all bounded in modulus as n increases for $\lambda \in [-\pi, \pi]$. We say that such a sequence of data tapers $\{h_t\}_{t=1}^n$ is of order p .

Prior to the proposition of the two semi-parametric estimators, we need to give some more information on the behavior of $f(\cdot)$ around the spectral peak in the light of the assumptions in [117] and [118] so that satisfactory asymptotic properties of the semi-parametric estimators considered in this paper can be achieved.

Assumption 3.2.3. The (pseudo-)spectral density function $f_y(\lambda)$, $s = \lfloor d + \frac{1}{2} \rfloor$, satisfies for some constant $0 < G < \infty$,

$$f_y(\omega \pm \lambda) \sim G\lambda^{-2(d-s)}, \quad \lambda \rightarrow 0^+.$$

Assumption 3.2.4. When $d \geq -\frac{1}{2}$, the spectral density function $f_y(\lambda)$ satisfies, for constants $0 < \beta \leq 2$, $0 < G, E_\beta < \infty$,

$$f_y(\omega \pm \lambda) = G\lambda^{-2(d-s)} + E_\beta\lambda^{-2(d-s)+\beta} + o(\lambda^{-2(d-s)+\beta}), \quad \lambda \rightarrow 0^+.$$

Assumption 3.2.5. In a neighborhood $(-\varepsilon + \omega, \omega + \varepsilon)$ around seasonal/cyclical frequency ω ,

$$\left| \frac{d}{d\lambda} f_y(\omega \pm \lambda) \right| = \mathcal{O}(\lambda^{-1-2(d-s)}), \quad \lambda \rightarrow 0^+.$$

Then $f_y(\lambda)$ has first derivative such that $\left| \frac{d}{dt} f_y(\omega \pm \lambda) \right| = \mathcal{O}(\lambda^{-1-2d})$ as $\lambda \rightarrow 0^+$.

Assumption 3.2.6. We have $y_t = \sum_{l=0}^{\infty} \alpha_l \varepsilon_{t-l}$, $\sum_{l=0}^{\infty} \alpha_l^2 < \infty$, where $\{\varepsilon_t\}$ satisfies $E(\varepsilon_t | F_{t-1}) = 0$ and $E(\varepsilon_t^2 | F_{t-1}) = 1$ almost surely, $t = 1, 2, \dots$ in which F_t is the σ_t -field of events generated by ε , $s \leq t$ and there exists a random variable ε such that $E\varepsilon^2 < \infty$ and for all $\zeta > 0$ and some $C > 0$, $P(|\varepsilon_t| > \zeta) \leq CP(|\varepsilon| > \zeta)$.

Now we are ready to present some desirable statistical properties which are essential to obtain the proposed statistical inference for the semi-parametric estimators.

Theorem 3.2.7. Under Assumptions 3.2.4 and 3.2.5 ($d > -\frac{1}{2}$, $0 < \beta \leq 2$) for f_y , a data taper of order $p \geq 2$, $p \in \mathbb{N}$ with $p \geq s + 1$, for any sequences of positive integers $k = k(n)$ and $j = j(n)$, $1 \leq k < j$ and $\eta = j - k$, such that $\frac{j}{n} \rightarrow 0$, defining $\gamma_{j,k} = (jk)^{d-p} \log(j+1)$, $v_p^t(\lambda \pm \omega) = \omega_p^t(\lambda \pm \omega) / (G^{1/2} \lambda^{-d})$, we have

- (a) $E[v_p^t(\omega \pm \lambda_{jp}) \bar{v}_p^t(\omega \pm \lambda_{jp})] = 1 + \mathcal{O}\{\min(j^{-\beta}, j^{-1}) + (\frac{j}{n})^\beta + \gamma_{j,j}\};$
- (b) $E[v_p^t(\omega \pm \lambda_{jp}) v_p^t(\omega \pm \lambda_{jp})] = \mathcal{O}(j^{-p} + \gamma_{j,j});$
- (c) $E[v_p^t(\omega \pm \lambda_{jp}) \bar{v}_p^t(\omega \pm \lambda_{kp})] = \mathcal{O}(k^{-1} \eta^{1-p} + k^{-1} \eta^{-p} \log n + \eta^{-p} + \gamma_{k,j});$
- (d) $E[v_p^t(\omega \pm \lambda_{jp}) v_p^t(\omega \pm \lambda_{kp})] = \mathcal{O}(k^{-1} \eta^{1-p} + k^{-1} \eta^{-p} \log n + \eta^{-p} + \gamma_{k,j}).$

The deduction is similar to the proof of Theorem 6 in [117] making use of the convolution representations in [8] of the right-hand side of Eq.(a)-Eq.(d) and therefore is omitted; for details see [8] and [117].

We are presently fully prepared to proceed to next section to introduce the semi-parametric estimators applicable to non-stationary SCLM time series, i.e., $d \geq \frac{1}{2}$.

3.3 Fourier Semi-parametric Estimation Methods for Seasonal/Cyclical Long-Memory Processes

In this section we respectively propose two estimators, one based on the LPSE method and the other based on the GSE method.

3.3.1 Log-Periodogram Semi-parametric Estimator

In this section we focus on an easy-to-understand semi-parametric estimator which is build upon log-(tapered) periodogram. From Eq.(3.1) with some arithmetic manipulations, we have

$$\log I(\omega \pm \lambda) = \log G + d(-2 \log \lambda) + \log \frac{I(\omega \pm \lambda)}{f(\omega \pm \lambda)} + \text{errors}, \quad \lambda \rightarrow 0^+,$$

which implies if the term $\log \frac{I(\omega \pm \lambda)}{f(\omega \pm \lambda)} + \text{errors}$ agrees with some condition, the estimator of the memory parameter d is the coefficient of the regressor $-2 \log \lambda$ and it has the following explicit representation:

$$\begin{aligned} \hat{d}_{LPSE,1} &= \frac{\sum_k (z_k - \bar{z}) \log I(\omega + \lambda_k)}{\sum_k (z_k - \bar{z})^2}, \quad k = l + 1, \dots, m_1, \\ \hat{d}_{LPSE,2} &= \frac{\sum_k (z_k - \bar{z}) \log I(\omega - \lambda_k)}{\sum_k (z_k - \bar{z})^2}, \quad k = l + 1, \dots, m_2, \end{aligned} \quad (3.5)$$

where $z_k = -2 \log \lambda_k$, $\bar{z} = \sum_k z_k / (m_i - l)$, $i = 1, 2$; and the subscripts 1 and 2 indicates the right-and left-hand side of the seasonal/cyclical frequency ω , $\omega \in (0, \pi)$, respectively. The bandwidth m_i , $i = 1, 2$ is the number of the Fourier frequency ordinates involved in the estimation, l is the frequency trimming number and they are user-chosen parameters which can be determined with respect to user's experience or a priori information on data. This log-periodogram semi-parametric estimator (LPSE) was first proposed in the seminal paper [56] and is widely used in the empirical studies.

Now that when $d \geq \frac{1}{2}$ the non-stationarity is induced into SCLM series, we consider a tapered periodogram only utilizing the Fourier frequency λ_{jp} , $1 \leq j < \lfloor n/(2p) \rfloor$ due to Thm. 3.2.7, and hence the tapered LPSEs are thus defined as follows:

$$\hat{d}_{LPSE,1}^p = \frac{\sum_k (z_{kp} - \bar{z}) \log I_p^T(\omega + \lambda_{kp})}{\sum_k (z_{kp} - \bar{z})^2}, \quad k = l + 1, \dots, m_1, \quad (3.6)$$

$$\hat{d}_{LPSE,2}^p = \frac{\sum_k (z_{kp} - \bar{z}) \log I_p^T(\omega - \lambda_{kp})}{\sum_k (z_{kp} - \bar{z})^2}, \quad k = l + 1, \dots, m_2, \quad (3.7)$$

where z_{kp} , $\bar{z}$ are defined as in Eq.(3.5). Frequently, we set $m_1 = m_2$ in the estimation, however, if the seasonal/cyclical frequency ω is either very close to zero or π , the equality of m_1 and m_2 is likely to contaminate or fail to yield estimation results.

Theorem 3.3.1. *Under the assumptions of Theorem 3.2.7, $p \geq s + 1$, $p = 2, 3, \dots$, $\{y_t\}$ Gaussian and for $i = 1, 2$,*

$$\frac{(m_i \eta)^{1/2}}{l^{\max\{1, \beta\}}} + \frac{m_i^{1/(2p-1)}}{\eta} + \frac{(m_i \eta)^{1+1/2\beta}}{n} + \frac{l(\log n)^2}{m_i \eta} \rightarrow 0, \quad n \rightarrow \infty, \quad (3.8)$$

we obtain for the estimators $\hat{d}_{LPSE,i}^p$, $i = 1, 2$ defined in Eq.(3.6) and Eq.(3.7),

$$m_i^{1/2}(\hat{d}_{LPSE,i}^p - d) \rightarrow_P N(0, \frac{\pi^2}{24}).$$

Since the asymptotic results in [103] only involve in the error the estimation of the covariance of ω_p^T at the frequencies near the spectral peak and the Gaussianity of $\{y_t\}$, the demonstration idea can readily be applied to v_p^T and hence the argument in [103] goes through in this case. For details see [103] and [117].

We go on to present a Gaussian semi-parametric estimation method in the next subsection.

3.3.2 Gaussian Semi-parametric Estimator

The Gaussian semi-parametric estimator (GSE), one of the most widely-used memory parameter estimators, should surely not be ignored in this paper. The GSE for SCLM series is proposed based on the estimator of [118] for LM series, and in particular, our GSE is applicable to the non-stationary SCLM series, i.e., $d > \frac{1}{2}$. Like the LPSE, for SCLM series, the memory parameter d can be estimated utilizing two-sided spectral information near the seasonal/cyclical frequency ω , $\omega \in (0, \pi)$, while for LM series, only one-sided information is available to the estimation. Therefore consider the two objective functions, for $\omega \in (0, \pi)$,

$$\begin{aligned} Q_1(G, d) &= \frac{1}{m_1 - l} \sum_{j=l+1}^{m_1} \left\{ \log G \lambda_{jp}^{-2d} + \frac{\lambda_{jp}^{2d}}{G} I_p^T(\omega + \lambda_{jp}) \right\}, \\ Q_2(G, d) &= \frac{1}{m_2 - l} \sum_{j=l+1}^{m_2} \left\{ \log G \lambda_{jp}^{-2d} + \frac{\lambda_{jp}^{2d}}{G} I_p^T(\omega - \lambda_{jp}) \right\}, \end{aligned}$$

and the memory parameter d_i is the second ordinate of the doublets $(\hat{G}_i, \hat{d}_i)$ such that $(\hat{G}_i, \hat{d}_i)$ minimizes the objective function Q_i , $i = 1, 2$. Concentrating out G to simplify the numerical computation, we have the estimators

$$\hat{d}_i = \operatorname{argmin}_{d \in \Theta} R_i(d), \quad i = 1, 2,$$

where

$$\begin{aligned} R_i(d) &= \log \hat{G}_i(d) - \frac{2d}{m_i - l} \sum_{j=l+1}^{m_i} \log \lambda_{jp}, \\ \hat{G}_1(d) &= \frac{1}{m_1 - l} \sum_{j=l+1}^{m_1} \lambda_{jp}^{2d} I_p^T(\omega + \lambda_{jp}), \\ \hat{G}_2(d) &= \frac{1}{m_2 - l} \sum_{j=l+1}^{m_2} \lambda_{jp}^{2d} I_p^T(\omega - \lambda_{jp}), \end{aligned}$$

where Θ is a closed interval of $\mathbb{R}$. The interval Θ , like the bandwidths m_1 , m_2 and the frequency ordinate trimming number l , is chosen according to user's experience and preference. It still needs to be underlined that the GSE, like the LPSE, only utilizes a grid of Fourier frequencies with step of p .

Theorem 3.3.2. *Under the Assumptions 3.2.4, 3.2.5 and 3.2.6 with $\min \Theta > -\frac{1}{2}$ and $p \geq \lfloor \max \Theta + \frac{1}{2} \rfloor + 1$ such that $d \in \Theta$, $p = 2, 3, \dots$,*

$$\frac{1}{m_i} + \frac{m_i}{n} \rightarrow 0, \quad n \rightarrow \infty,$$

we have $\hat{d}_{GSE, i}^p \rightarrow_P d$, $i = 1, 2$.

In light of the convolution representation of $I_p^T(\cdot)$ in [8], the proof of [104] can be adapted to this case. For details, see [8], [103], [104], [117] and [118].

3.4 Monte Carlo Study for Seasonal/Cyclical Long-Memory Processes

In the beginning of this chapter, we have briefly mentioned the k -factor Gegenbauer process as an particular example of SCLM series, and in this section we first formally introduce this type of process and give its statistical processes.

However, due to the problems of simulating a non-stationary k -factor Gegenbauer process, we only employ 1-factor GG processes in this Monte Carlo experiment. Then the 1-factor Gegenbauer process has the representation as follows:

$$(I - 2\nu B + B^2)^d x_t = \varepsilon_t, \quad \varepsilon_t \sim i.i.d.N(0, 1), \quad t = 1, \dots, n,$$

where the spectral peak $\arccos \nu$ is called the Gegenbauer frequency, denoted as Gfreq. In this section we carry out a Monte Carlo experiment and compute the bias and MSE of the estimates by simulating 1000 realizations of a Gegenbauer process (denoted as $GG(\nu, d)$) to evaluate the FSP of the semi-parametric estimators proposed.

In the former sections we only focus on SCLM series with $d \geq \frac{1}{2}$, however, in this experiment we evaluate the estimators' FSP for both positive and negative memory parameters and d varies from -1.95 to 1.95 with a step of $.1$, but due to the limited space, we report the estimation results with a step of $.2$. Furthermore, we let ν in $GG(\nu, d)$ be $\cos(\pi/6)$ and $\cos(\pi/3)$; the sample size is set to $n = 512$ (medium sample size) and $n = 1024$ (large sample size); the bandwidth m takes on the values in $\{\lfloor n^{.5} \rfloor, \lfloor n^{.65} \rfloor, \lfloor n^{.8} \rfloor\}$ and three taper orders $p = 1, 2, 3$. The $GG(\nu, d)$ processes are simulated by authors' own R codes making use of a R package 'waveslim'. For reference on the wavelet synthesis of long-memory processes, see e.g. [?], [122]. It also needs to be noted that the Kolmogorov-Zurbenko tapers utilized in this experiment could eliminate the impact on the estimation of a polynomial time trend's of order no bigger than $p - 1$ because of specific orthogonality properties of the tapers. From this experiment, the effect that the bandwidth, the frequency-ordinate trimming number and the location of the seasonal/cyclical frequency have on the estimation results is also investigated and hence an optimal determination of the user-chosen parameters is practically implied.

Remarque 3.4.1. *The LPSE as $\hat{d}_{LPSE}^p = (\hat{d}_{LPSE,1} + \hat{d}_{LPSE,2})/2$, and the GSE in this Monte Carlo experiment is defined as $\hat{d}_{GSE}^p = \arg\min_{\Theta} \{R_1(d) + R_2(d)\}$*

According to this Monte Carlo experiment, we first separately discuss the FSP of the LPSE and GSE, and then compare the FSP of these two estimators.

First, take a look at the FSP of LPSE.

- **Consistency:** The consistency is numerically revealed through this experiment since both bias and MSE decrease as sample size increases, which is revealed in Tables 6.48-6.55.
- **Taper order impact:** The impact the taper order p has on the estimation results is illustrated in Fig.3.1 and Fig.3.2, where the LPSE is plotted against the varying bandwidth numbers and the horizontal black long-dashed line represents the true value of d . The time series in the top-left of Fig.3.1 is $GG(\cos(\frac{\pi}{3}), .15)$ with zero mean and sample size $n = 1024$, and we integrate it once, twice and thrice thus obtain $GG(\cos(\frac{\pi}{3}), 1.15)$, $GG(\cos(\frac{\pi}{3}), 2.15)$ and $GG(\cos(\frac{\pi}{3}), 3.15)$ series, respectively. (The time series in the top-left of Fig.3.2 is $GG(\cos(\frac{\pi}{3}), -.15)$ with zero mean and sample size $n = 1024$, and we difference it once, twice and thrice thus obtain $GG(\cos(\frac{\pi}{3}), -1.15)$, $GG(\cos(\frac{\pi}{3}), -2.15)$ and $GG(\cos(\frac{\pi}{3}), -3.15)$ series, respectively.) The bandwidth values m are ranged from about 20 to $\lfloor \frac{n}{2} \rfloor$ with an increment of 10, and for every realization, the actual frequency ordinates involved in each estimation are m/p , where p is the taper order. We find that consistent with Thm.3.3.1, a bigger taper order numerically allows a wider range of admissible interval of d to be estimated, and a taper order bigger than a sufficient one cannot improve the results, only to inflate MSE. Furthermore, if a taper order is not sufficiently chosen when $d > 0$, i.e., $p < s$ (s , the difference order as in Eq.(3.3)), the estimates converge to the value of taper order p , which guides us to use a bigger taper order when an estimate with a smaller taper order yields a value close to this taper order for a practical utilization.
- **Bandwidth impact:** As is known, the bandwidth is theoretically a key element to the estimation performance, which is numerically illustrated through this Monte Carlo study. For the LPSE, the MSEs decrease as the bandwidth m moves from $\lfloor n^{.5} \rfloor$ up to $\lfloor n^{.8} \rfloor$, but sometimes a smaller MSE is achieved at the expense of an increase in bias. Therefore, a very narrow bandwidth $m < \lfloor n^{.5} \rfloor$ together with a very wide bandwidth $m > \lfloor n^{.8} \rfloor$, is not favored in the estimation for a practical utilization.
- **Frequency ordinates trimming impact:** From a pragmatic perspective, for the LPSE, it is theoretically suggested that trimming the frequency ordinates closest to the right and left-hand sides of the spectral peak/zero help improve the estimation. However, according to this Monte Carlo experiment, trimming is not necessary at all since this manipulation only leads to a comparable FSP with respect to the untrimmed estimator, see e.g. Tab.3.1 for an illustration.
- **Spectral peak location impact:** The location of the seasonal/cyclical spectral peak/zero exerts a tangible impact on the LPSE's performance. In this experiment

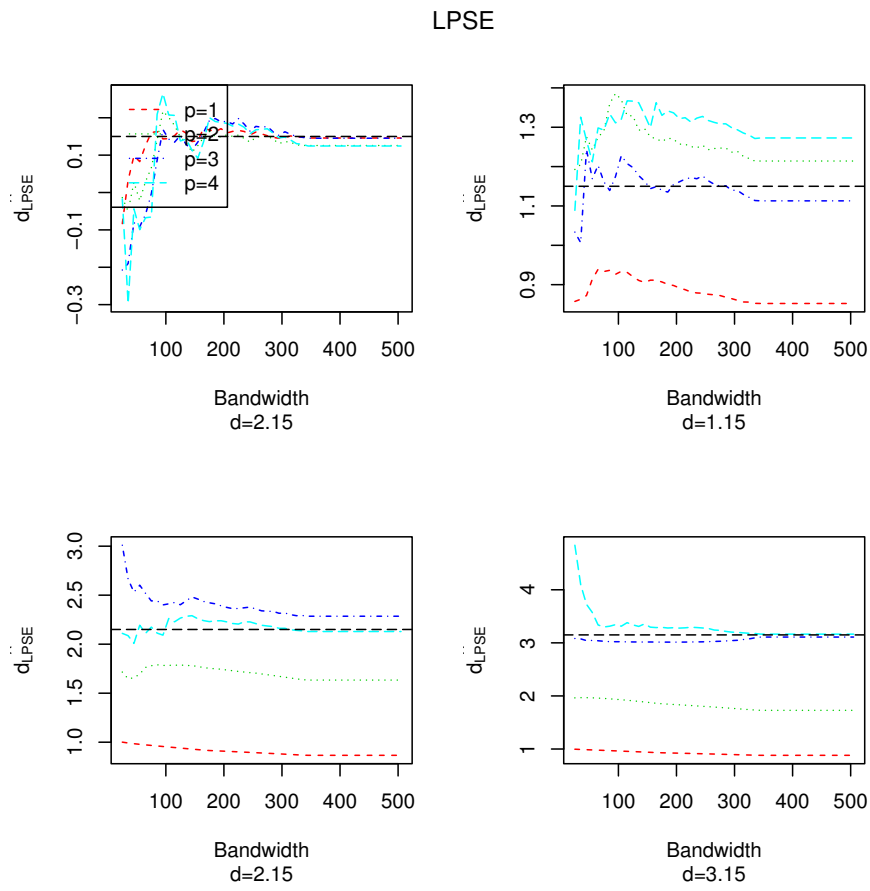


Figure 3.1: Stationary and non-stationary analysis. The LPSE are estimated with Kolmogorov-Zurbenko tapers of orders $p = 1, 2, 3, 4$.

LPSE

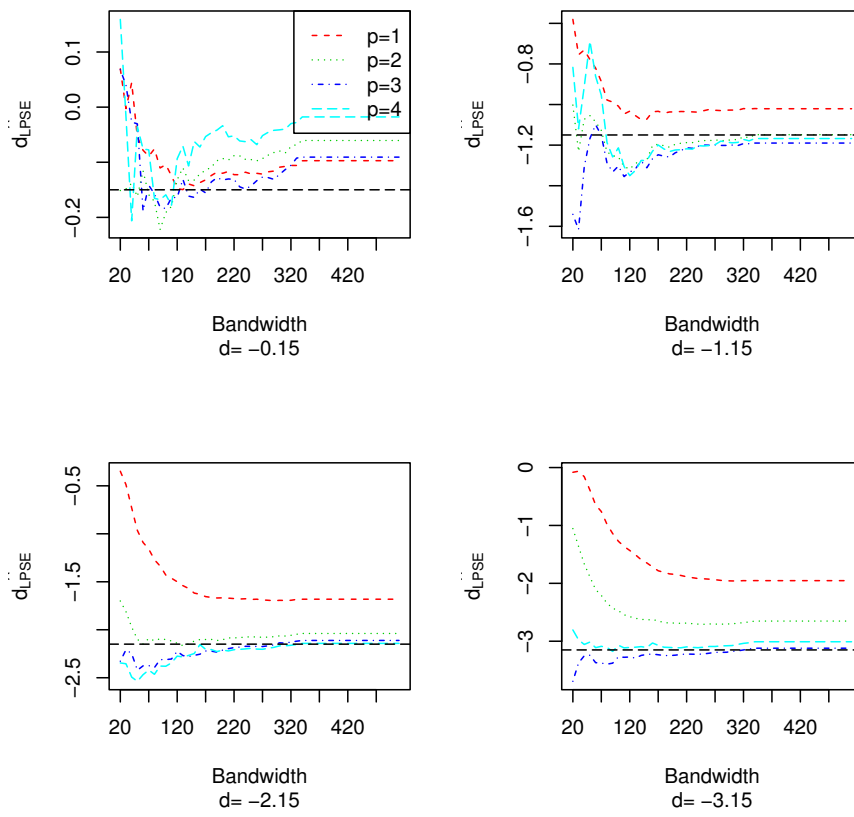


Figure 3.2: Differencing analysis. The LPSE are estimated with Kolmogorov-Zurbenko tapers of orders $p = 1, 2, 3, 4$.

we find that if the Gfreq is substantially away from the bounds of the interval $(0, \pi)$ (i.e. $\text{Gfreq} = \frac{\pi}{3}$ in this experiment), the estimation results evidently exceed those of the processes with Gfreq near the origin (i.e. $\text{Gfreq} = \frac{\pi}{6}$ in this experiment) in terms of both bias and MSE, which is revealed in Tab.3.2. We assume that this phenomenon is due to the inadequate amount of frequency ordinates which could be utilized in the estimation on the left-hand side of the spectral peak/zero for the $\text{Gfreq} = \frac{\pi}{6}$ case.

Table 3.1: Monte Carlo Results for LPSE for the Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0697 (0.0647)	0.0847 (0.0814)	-0.0007 (-0.0109)	0.0212 (0.0241)	0.0076 (0.0064)	0.0091 (0.0081)
		1024	0.0790 (0.0842)	0.0497 (0.0493)	-0.0012 (-0.0095)	0.0131 (0.0122)	-0.0177 (-0.0157)	0.0047 (0.0051)
	-1.55	512	-0.0429 (-0.0577)	0.0735 (0.0739)	0.0116 (0.0030)	0.0208 (0.0249)	-0.0165 (-0.0130)	0.0086 (0.0081)
		1024	-0.0010 (0.0063)	0.0493 (0.0447)	0.0005 (0.0024)	0.0127 (0.0134)	-0.0340 (-0.0336)	0.0053 (0.0055)
	-1.15	512	0.0023 (0.0074)	0.0733 (0.0790)	-0.0502 (-0.0491)	0.0250 (0.0246)	-0.0063 (-0.0091)	0.0086 (0.0079)
		1024	0.0051 (0.0084)	0.0455 (0.0433)	-0.0416 (-0.0398)	0.0147 (0.0145)	-0.0215 (-0.0239)	0.0047 (0.0050)
	-.75	512	-0.0358 (-0.0249)	0.0804 (0.0744)	0.0016 (0.0022)	0.0216 (0.0231)	-0.0189 (-0.0143)	0.0095 (0.0086)
		1024	-0.0187 (-0.0181)	0.0502 (0.0446)	-0.0056 (-0.0034)	0.0126 (0.0134)	-0.0133 (-0.0148)	0.0046 (0.0045)
	-.35	512	0.0118 (0.0382)	0.0761 (0.0738)	-0.0521 (-0.0493)	0.0253 (0.0253)	0.0079 (0.0109)	0.0081 (0.0091)
		1024	0.0407 (0.0357)	0.0446 (0.0485)	-0.0282 (-0.0276)	0.0123 (0.0123)	-0.0000 (0.0001)	0.0043 (0.0043)
	.05	512	-0.0041 (-0.0143)	0.0722 (0.0831)	0.0097 (0.0027)	0.0217 (0.0213)	0.0047 (0.0003)	0.0089 (0.0080)
		1024	-0.0017 (0.0025)	0.0449 (0.0433)	0.0050 (0.0020)	0.0130 (0.0115)	-0.0010 (0.0026)	0.0041 (0.0043)
	.45	512	-0.0231 (-0.0262)	0.0721 (0.0735)	0.0808 (0.0763)	0.0286 (0.0292)	0.0084 (0.0052)	0.0085 (0.0090)
		1024	0.0060 (0.0167)	0.0466 (0.0460)	0.0532 (0.0492)	0.0158 (0.0159)	0.0084 (0.0087)	0.0041 (0.0048)
	.85	512	0.0659 (0.0576)	0.0785 (0.0825)	0.0305 (0.0346)	0.0238 (0.0227)	0.0272 (0.0290)	0.0091 (0.0093)
		1024	0.0437 (0.0416)	0.0519 (0.0495)	0.0176 (0.0306)	0.0131 (0.0131)	0.0249 (0.0251)	0.0045 (0.0049)
	1.25	512	0.0766 (0.0895)	0.0871 (0.0818)	0.1021 (0.1089)	0.0345 (0.0352)	0.0384 (0.0368)	0.0097 (0.0099)
		1024	0.0756 (0.0600)	0.0539 (0.0478)	0.0871 (0.0826)	0.0204 (0.0194)	0.0412 (0.0423)	0.0058 (0.0062)
	1.65	512	0.1982 (0.2207)	0.1208 (0.1264)	0.1034 (0.1003)	0.0370 (0.0342)	0.0757 (0.0744)	0.0159 (0.0159)
		1024	0.1942 (0.2034)	0.0865 (0.0873)	0.1113 (0.1027)	0.0273 (0.0254)	0.0807 (0.0796)	0.0120 (0.0114)

^a The unbracketed numbers are computed by LPSE with trimming order $l = 0$ and the bracketed numbers with and trimming order $l = 2$.

^b The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 3.2: Monte Carlo Results for LPSE with tapering order $p = 2$ and trimming order $l = 0$

Gfreq	d	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$		MSE
		n	bias	MSE	bias	MSE	bias	
$\pi/6$	-1.75	512	-0.0003	0.0754	-0.1581	0.0517	-0.1951	0.0514
		1024	0.0234	0.0495	-0.1456	0.0344	-0.2010	0.0475
	-1.15	512	-0.0745	0.0830	-0.1179	0.0388	-0.1487	0.0352
		1024	-0.0386	0.0456	-0.1064	0.0241	-0.1464	0.0276
	-.55	512	-0.0142	0.0770	-0.1025	0.0345	-0.0866	0.0215
		1024	-0.0203	0.0468	-0.0965	0.0222	-0.0723	0.0113
	.05	512	0.0017	0.0814	0.0032	0.0236	0.0055	0.0137
		1024	0.0094	0.0436	0.0009	0.0125	0.0061	0.0062
	.65	512	0.0803	0.0760	0.1153	0.0393	0.1146	0.0262
		1024	0.0811	0.0505	0.1091	0.0244	0.1051	0.0169
	1.25	512	0.1734	0.1112	0.1795	0.0588	0.1987	0.0538
		1024	0.1260	0.0606	0.1352	0.0323	0.1833	0.0400
	1.85	512	0.2408	0.1273	0.2320	0.0808	0.1326	0.0446
		1024	0.1924	0.0849	0.2133	0.0606	0.1556	0.0387
$\pi/3$	-1.75	512	0.0162	0.0825	-0.0059	0.0211	-0.0092	0.0085
		1024	0.0193	0.0488	-0.0026	0.0123	-0.0256	0.0048
	-1.15	512	0.0023	0.0733	-0.0502	0.0250	-0.0063	0.0086
		1024	0.0051	0.0455	-0.0416	0.0147	-0.0215	0.0047
	-.55	512	-0.0680	0.0850	0.0301	0.0226	-0.0112	0.0089
		1024	-0.0072	0.0451	0.0184	0.0133	-0.0176	0.0048
	.05	512	-0.0041	0.0722	0.0097	0.0217	0.0047	0.0089
		1024	-0.0017	0.0449	0.0050	0.0130	-0.0010	0.0041
	.65	512	0.0856	0.0775	-0.0096	0.0207	0.0339	0.0091
		1024	0.0778	0.0536	0.0058	0.0122	0.0298	0.0051
	1.25	512	0.0766	0.0871	0.1021	0.0345	0.0384	0.0097
		1024	0.0756	0.0539	0.0871	0.0204	0.0412	0.0058
	1.85	512	0.1831	0.1102	0.1258	0.0387	0.0232	0.0151
		1024	0.1678	0.0739	0.1041	0.0244	0.0491	0.0098

^a The estimates are computed for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ and $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$, respectively.

^b The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Second, take a look at the FSP of the GSE.

- **Consistency:** The consistency is numerically revealed in Tables 6.56-6.59, since both bias and MSE decrease as sample size increases.
- **Taper order impact:** The simulation and plot setting in Fig.3.3 and Fig.3.4 is the same with the LPSE case. Like LPSE, we find that a bigger taper order allows a wider range of admissible interval of d to be estimated. If a taper order is not sufficiently chosen $p < s$ when $d > 0$, the estimate approaches the horizontal line with an intercept of p , which is revealed in Fig.3.3; if $p < \lfloor d + .5 \rfloor$ when $d < 0$, it is observed that the GSE is close to the negative value of taper order p ; see two bottom plots in Fig.3.4. According to our Monte Carlo study, it should be underlined that

GSE

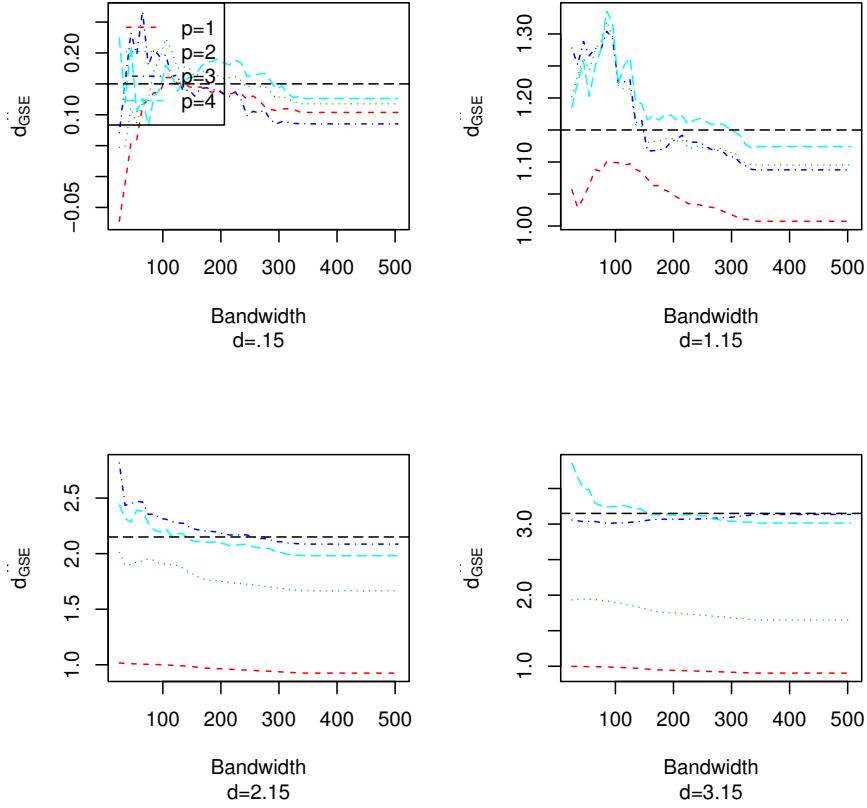


Figure 3.3: Stationary and non-stationary analysis. The GSE are estimated with Kolmogorov-Zurbenko tapers of orders $p = 1, 2, 3, 4$.

whatever taper order p is chosen, the GSE for $d < -\frac{1}{2}$ works badly in terms of MSE.

- Bandwidth impact:** For the GSE, if $d \in (-\frac{1}{2}, \frac{1}{2})$, the bias and MSE generally diminish with the increment of the bandwidth number m from $\lfloor n^{.5} \rfloor$ up to $\lfloor n^{.8} \rfloor$. However, this finding does not imply that for d in this interval, the wider the bandwidth is the better estimation result we can obtain, which is reflected in Fig.3.3 and Fig.3.4. Outside this stationary and invertible interval of d , the GSE exhibits a superior performance in terms of bias and MSE with $m = \lfloor n^{.65} \rfloor$ to that with $m = \lfloor n^{.5} \rfloor$, but the choice of m with $\lfloor n^{.65} \rfloor$ or $\lfloor n^{.8} \rfloor$ has its own merits, respectively. Therefore in accordance with the Monte Carlo study displayed in Tables 6.56-6.59 and in Fig.3.3 and Fig.3.4, a narrow bandwidth $m \leq \lfloor n^{.5} \rfloor$ together with a very wide bandwidth $m \geq \lfloor n^{.8} \rfloor$, is not preferred in the estimation.
- Spectral peak location impact:** The same finding is obtained as for the LPSE case, see Tab.3.3 for a direct impression.

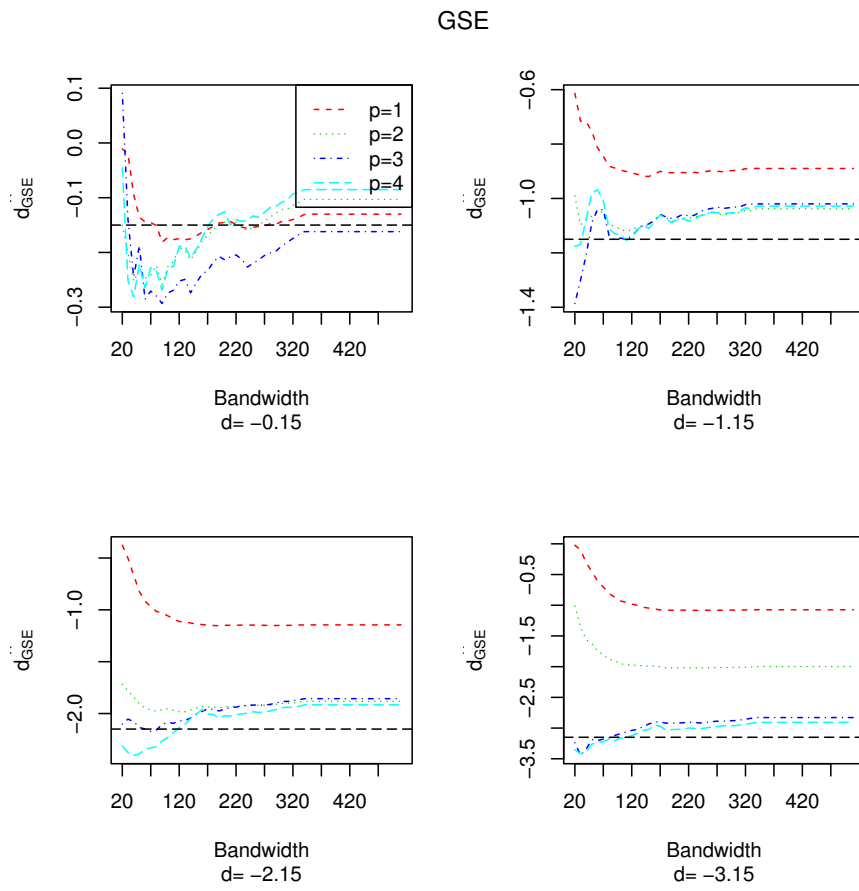


Figure 3.4: Differencing analysis. The GSE are estimated with Kolmogorov-Zurbenko tapers of orders $p = 1, 2, 3, 4$.

Table 3.3: The GSE with tapering order $p = 2$

Greq	d	$m = \lfloor n^{.5} \rfloor$			$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
		n	bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.75	512	0.1807	0.2335	0.1811	0.2369	0.1728	0.3839
		1024	0.1210	0.0551	0.1402	0.1117	0.1359	0.2183
	-1.15	512	0.0468	0.1126	0.0717	0.0659	0.0311	0.0268
		1024	0.0306	0.0518	0.0737	0.0434	0.0252	0.0213
	-.55	512	-0.0051	0.0772	0.0037	0.0467	0.0324	0.0823
		1024	-0.0002	0.0277	-0.0079	0.0296	0.0347	0.0666
	.05	512	-0.0196	0.0272	-0.0056	0.0087	-0.0021	0.0050
		1024	-0.0133	0.0148	-0.0068	0.0039	-0.0029	0.0024
	.65	512	0.0312	0.0277	0.0076	0.0086	0.0282	0.0059
		1024	0.0266	0.0162	0.0130	0.0047	0.0228	0.0028
	1.25	512	0.0459	0.0313	-0.0043	0.0101	0.0060	0.0075
		1024	0.0405	0.0177	-0.0285	0.0053	0.0438	0.0042
	1.85	512	0.0900	0.0342	0.0068	0.0119	-0.0426	0.0173
		1024	0.0928	0.0242	-0.0085	0.0059	-0.0150	0.0085
$\pi/3$	-1.75	512	0.1082	0.3010	0.1604	0.3540	0.2439	0.4775
		1024	0.0791	0.1542	0.0971	0.2101	0.1818	0.2677
	-1.15	512	0.0278	0.0619	0.0660	0.1462	0.1311	0.0860
		1024	0.0163	0.0153	0.0271	0.0219	0.1053	0.0456
	-.55	512	-0.0172	0.0885	0.0389	0.0587	0.0353	0.0545
		1024	-0.0122	0.0152	0.0070	0.0165	0.0235	0.0146
	.05	512	-0.0188	0.0284	-0.0041	0.0070	-0.0104	0.0033
		1024	-0.0091	0.0149	-0.0052	0.0039	-0.0074	0.0016
	.65	512	0.0349	0.0271	-0.0131	0.0072	-0.0142	0.0035
		1024	0.0457	0.0191	0.0052	0.0043	-0.0151	0.0018
	1.25	512	0.0458	0.0278	0.0021	0.0078	-0.0893	0.0114
		1024	0.0343	0.0178	-0.0026	0.0043	-0.0751	0.0073
	1.85	512	0.1109	0.0380	0.0205	0.0089	-0.1118	0.0192
		1024	0.1034	0.0269	0.0349	0.0061	-0.0858	0.0107

^a The estimates are computed for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ and $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$, respectively.

^b The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Finally, we compare the FSP of LPSE and GSE. The GSE numerically has a superior efficiency to the LPSE when $d > -\frac{1}{2}$. Explicitly, the GSE prevails in terms of MSE compared to the LPSE and furthermore, the GSE with $m = \lfloor n^{.65} \rfloor$ surpasses the LPSE with all three bandwidth choices. Hence the GSE is favored for $d > -\frac{1}{2}$ and for this case it still remains a question that whether to choose the GSE with $m = \lfloor n^{.65} \rfloor$ or $m = \lfloor n^{.8} \rfloor$, and it can actually be determined based on user's preference on better performance of bias or MSE. Meanwhile as is already mentioned, GSE does not yield as good estimation results as LPSE does for $d \leq -\frac{1}{2}$, which can be clearly discovered in Tables 6.48-6.59. Therefore the LPSE is favored for the $d \leq -\frac{1}{2}$ case and the way to choose the bandwidth

number can be done as for the GSE in the $d > -\frac{1}{2}$ case.

In sum, after the introduction of a definition for a non-stationary/non-invertible SCLM time series, we propose the LPSE and the GSE operative for SCLM series with $d \in \mathbb{R}$ and $d > -\frac{1}{2}$, respectively. The statistical inference on these estimators are readily to be presented in the light of the unified theories of the estimators in [8], [117] and [118]. From our Monte Carlo experiment, the FSP of the estimators is illustrated that the GSE excels the LPSE for $d > -\frac{1}{2}$ and the LPSE yields reliable estimates for $d < -\frac{1}{2}$ while the GSE can not. We also find that the location of spectral peak also has a substantial impact on the estimation result, and furthermore from a pragmatic perspective, we cast a light on the optimal selection of parameter values, i.e., the bandwidth, the taper order, for both estimators. It is left for the future investigation that how these estimators work if the SCLM series presents heteroscedasticity.

Chapter 4

Applications

4.1 The Chinese Housing Market Outlook Based on the US House Price Analysis

House price is always one of the hottest public issues on people's livelihood in China and it is closely related to national economic environment. For common households, the continuously rising house prices can provide homeowners with more sense of security and it can also boost the consumer confidence and expand the domestic demand in a way. However, the house prices, greatly deviating from the intrinsic value, will inevitably incur the oversupply in the housing market, the formation of stagflation and finally the house price plummeting. Moreover, if the price precipitately drops, it will have a tremendous negative impact on the real estate business and its upstream and downstream industries. Along with the housing market, the credit crisis and the growth of unemployment rate are what we are unwilling to see. From the U.S. house price evolution history, we see that since the end of 1993, the U.S. housing market began to experience a more-than-12-year rising period; nevertheless, from the beginning of 2006 the housing market bubble burst and resulted in a severe economic recession caused by the subprime mortgage crisis which lasts until today and the house prices still seem to be in a downward channel. The tragic lesson we learned from the U.S. house price history can guide our policymakers for the time being to stabilize the Chinese housing market. Moreover, establishing appropriate regulations on the housing market now becomes a pressing matter for the Chinese policymakers and also a stern test of the ability on the government authority.

As a developed country, the United States has completed its urbanization process, while China is just on the way of its urbanization, hence there is much more demand for new housing, compared to the U.S. Therefore it implies that we cannot directly apply the U.S. house price curve to the Chinese housing market to draw the conclusion. However, although there exist many differences between the housing markets of China and the U.S., the people's reaction to the house price volatility is somewhat the same, which means that the Chinese policymakers can obtain some beneficial insights into the Chinese housing market by taking a look at the U.S. house-price history. Specifically, the memory property of the U.S. all-transactions house price indexes (afterwards abbreviated to HPI) series (The reason why we choose the U.S. data is that the Chinese HPI series is only traced

back to 2008, i.e. less than 50 observations are available, which is not suitable for the identification of long memory feature in the data.). is studied in this chapter, and the 'memory' refers to the degree of the influence of the past price oscillations on the future price.

This chapter is targeted at providing Chinese housing-market policymakers with a useful reference and suggestions so as to help to stabilize the housing market. [69] qualitatively confirmed the existence of the Chinese housing market bubble and stated its perniciousness, hence a stable and healthy housing market is of great importance. [78] reasoned the long memory property of house price from a psychological angle, i.e. the 'crowd mentality', while this chapter quantitatively proves the existence of house price's long memory property and finds the linkage between the long memory property of the HPI and the geographic locations, which in turn verifies that the housing-market regulations vary from regions to regions.

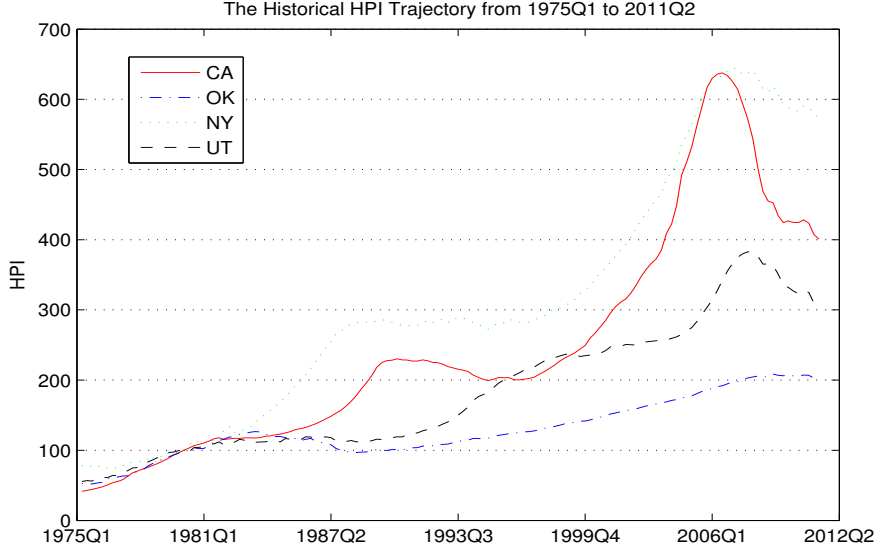
The remaining of this chapter is structured as follows. The model and empirical study of U.S. single-family HPI is documented in Sec.4.1.2, which includes the memory parameter estimations with respect to the HPI of 50 states and Washington D.C. and a cluster analysis based on the memory parameter estimates. We find that the state HPI memory parameters are closely linked to their geographic locations as well as their population density. The policy implications for the Chinese property market are contained in Sec.4.1.3. The chapter is concluded in Sec.4.1.4.

4.1.1 Semi-parametric Estimation Procedure

This chapter is interested in the values of memory parameter of the U.S. 50-state HPI series and explores the reasons behind the big difference in the memory parameter estimates across the country. We employ the fractionally integrated $I(d)$ process as stated in Ch.2. Therefore we first introduce a semi-parametric estimation procedure in order to estimate the memory parameter since there is no need to estimate the short memory components embedded in the HPI series. For the selected data, we employ the fully-extended local polynomial Whittle estimation (fextLPWE) procedure, which is documented in [51] and studied in Ch.2. Due to the limited space, we don't detail this procedure in this chapter and interested readers can refer to the works [2] and [88]. However, four points need to be emphasized in the following:

1. Semi-parametric estimation procedures, to a certain extent, eliminate the possibility to wrongly model the short memory components of the time series, i.e. $u_t, t = 1, 2, \dots$;
2. The semi-parametric procedure is computationally less burdened than the parametric one;
3. The fextLPWE procedure is applicable to the $I(d)$ process with $d \in (-3/2, \infty)$, not just limited to the stationary time series for which $d \in [-1/2, -1/2)$;
4. If the Assumption A and B in [2] are satisfied, then for $X_t \sim I(d)$, the fextLPWE procedure is invariant to the polynomial $a_0 + a_1t + \dots + a_pt^p$ where $d = p + d_\xi$,

Figure 4.1: Four representative HPI series of the states in U.S.



$\xi \in (-1/2, 1/2)$, $p \in \mathbb{Z}$. Let

$$Y_t = X_t + a_0 + a_1 t + \dots + a_p t^p, \quad t \geq 1,$$

and we have

$$\omega_Y(\lambda; d) = (1 - \exp^{i\lambda_j})^{-p} \omega_{\nabla^p Y}(\lambda_j) = (1 - \exp^{i\lambda_j})^{-p} \omega_\xi(\lambda_j) = \omega_X(\lambda; d),$$

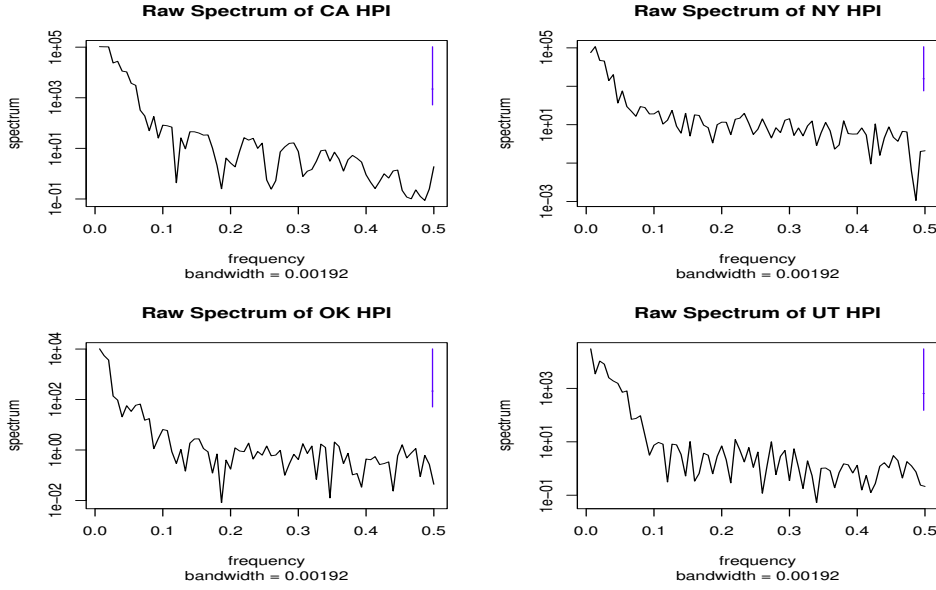
where $\Delta^p X_t = \xi_t$, $t = 1-p, 2-p, \dots$, and the extended discrete Fourier transforms ω_Y , ω_ξ and ω_X are defined in [2]. Hence the estimate of the fextLPWE procedure also won't be affected by the level shift in X_t .

4.1.2 Empirical Study

Data Description

The HPI series (from 1975Q1 to 2011Q2) of the 50 states and Washington D.C. in U.S. were selected for this empirical study. The HPI series downloaded from the Federal Housing Finance Agency (FHFA) website (<http://www.fhfa.gov/Default.aspx?Page=87>) has 146 observations for each area. The selected data, used to measure the single-family house price oscillations in the U.S., is quarterly released by FHFA based on the transaction data of two mortgage companies, Fannie Mae and Freddie Mac. The state HPI can be traced back to 1975, and the 1980Q1 index is set as the benchmark 100. It is displayed in Figure 4.1 that the magnitude of the HPI series differs markedly across the U.S., but as an economic environment indicator, the HPI curves exhibit a similar pattern.

Figure 4.2: The Spectrum of Four Representative HPI Series of the States in U.S.



Estimation of Memory Parameter and Cluster Analysis

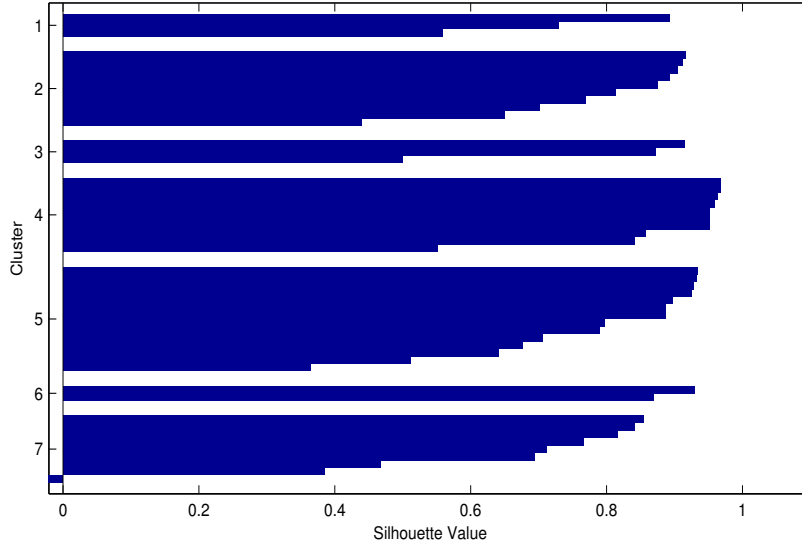
From the periodogram of the HPI series in Fig.4.2, a peak is observed at the zero frequency and obviously there is no cyclical components in HPI data, so it is reasonable to adopt the $I(d)$ model to estimate memory parameter d . In order to avoid the information loss of the raw data, we do not smooth the data before estimation (usual smoothing techniques including differencing and detrending the linear time component) and we directly apply the fextLWE procedure to the HPI series of the 50 states and Washington D.C. in U.S. We employ this procedure because, according to the Monte Carlo results in Ch.2, through the finite sample estimation performance comparison we find this procedure yields the best estimate in terms of mean square error (MSE) with respect to other semi-parametric estimation procedures applicable to the non-stationary time series. The estimation result is displayed in Tab.4.1.

From Tab.4.1, it is observed that the memory parameter estimates of the HPI exhibit a big difference across the country. For example, the d_{ND} of North Dakota is 0.8331, and it is the minimum of all 51 memory parameter estimates, which means that among all the states the past house-price shocks have the smallest degree of influence on the future house prices in ND, while the d_{OR} of Oregon is 2.0935 and the past house-price shocks have the largest degree of influence on the future house prices in OR. Moreover, as is displayed in Tab.4.1, the HPI memory parameter estimates are generally within the interval $[1, 2]$. We then perform the K -means clustering procedure via Matlab for the memory parameter estimates d of the 50 states and Washington D.C. Once the clustering procedure is decided, we need to consider how many clusters we should choose. In this

Table 4.1: Memory Parameter Estimates of the U.S. HPI Series

Alaska (AK)	1.3516	Alabama (AL)	1.5670
Arkansas (AR)	1.4689	Arizona (AZ)	1.8869
California (CA)	2.0380	Colorado (CO)	1.7193
Connecticut (CT)	1.8214	Delaware (DE)	2.0713
Florida (FL)	1.9072	Georgian (GA)	1.6904
Hawaii (HI)	1.5973	Iowa (IA)	1.1902
Idaho (ID)	1.7894	Illinois (IL)	1.6874
Indiana (IN)	1.3896	Kansas (KS)	1.5224
Kansas (KY)	1.3685	Louisiana(LA)	1.7435
Massachusetts (MA)	1.8410	Maryland (MD)	1.9848
Maine (ME)	1.3349	Michigan (MI)	1.5085
Minnesota (MN)	1.5342	Missouri (MO)	1.2887
Mississippi (MS)	1.3119	Montana (MT)	1.7248
North Carolina (NC)	1.7747	North Dakota (ND)	0.8331
Nebraska (NE)	1.2679	New Hampshire (NH)	1.5000
New Jersey (NJ)	1.9277	New Mexico (NM)	1.9988
Nevada (NV)	1.8910	New York (NY)	1.7270
Ohio (OH)	1.5161	Oklahoma (OK)	1.7696
Oregon (OR)	2.0935	Pennsylvania (PA)	1.7174
Rhode Island (RI)	1.9061	South Carolina (SC)	1.6707
South Dakota (SD)	0.9620	Tennessee (TN)	1.6459
Texas (TX)	1.5000	Utah(UT)	1.7721
Virginia (VA)	1.8957	Vermont (VT)	1.5000
Washington (WA)	1.9474	Wisconsin (WI)	1.3739
West Virginia (WV)	1.2553	Wyoming (WY)	1.6697
Wyoming (DC)	1.9195		

Figure 4.3: Silhouette Index Classification Plot



chapter we use the Silhouette index to determine the number of clusters and the Silhouette index is defined as follows:

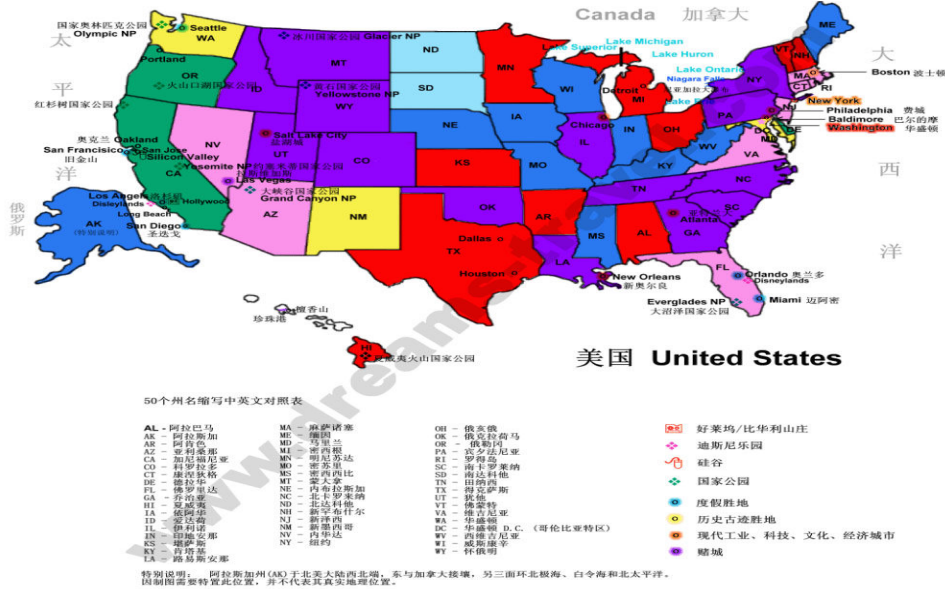
$$s(i) = \frac{(b(i) - a(i))}{\max[a(i), b(i)]}, \quad i = 1, 2, \dots, k,$$

where $a(i)$ is the average distance from the i^{th} point to the other points in its cluster, and $b(i)$ is the smallest average distance from the i^{th} point to points in another cluster. As we do not know the actual number of clusters, we try every integer in $[3, 10]$ for k , and find that when $k = 7$, the Silhouette index enjoys the maximum mean, so the 51 memory parameter estimates are grouped into 7 clusters. In Fig.4.3 is displayed the Silhouette index classification, according to which we color the U.S. map with different colors to illustrate different clusters, one color representing one cluster, thus we obtain Fig.4.4.

Inference on Clustering Analysis

Most existing literature on the house-price trend description and forecast is through a qualitative analysis, while this chapter quantitatively analyzes the memory parameter estimates of the HPI series and the advantage of the quantitative analysis over the qualitative one is that it yields an direct numerical result complementing the deficiency of the text-description result obtained from the qualitative analysis. The memory parameter of HPI series helps the housing-market policymakers to comprehend the degree of influence of the past price oscillations on the future house price, and moreover, the cluster analysis result indicates that the degrees of the house-price memory are comparable in every

Figure 4.4: The U.S. Map Illustrating the Cluster Analysis Result



respective region (a cluster of states) so that it gives the policymakers a useful resource to determine the tightness of the regional policies based on the values of memory parameter. In addition, China and the U.S. share many similarities such as the vast land, large populations, the imbalance of economic development level across the country and the big difference of population density between different regions, so we expect the following conclusion for the U.S. HPI empirical study can provide some useful references to Chinese housing-market researchers. On the whole, it is displayed in Tab.4.1 that the HPI memory parameter estimate spans from 0.8331 to 2.0935 among the 50 states and Washington D.C.. It can be observed in Fig.4.4 that the memory parameter estimates exhibit a certain regional aggregation characteristics. For example, the HPI memory parameter estimates of the 5 northwest adjacent states (Montana, Idaho, Wyoming, Utah and Colorado) are in the same cluster; and except Illinois, those of 8 U.S. central states belong to the blue region. Thus we have a reason to believe that this kind of regional aggregation may be due to the company and/or resident mobility, because commodity, service, capital and other resources in the adjacent areas enjoy a similar configuration. To sum up, we find the following facts:

- First Cluster (Light Blue) including North Dakota and South Dakota;
- Second Cluster (Dark Blue) including the states mainly located in the middle north-east;
- Third Cluster (Red) including the states mainly located in the middle east;
- Fourth Cluster (Purple) including the states mainly located in the west and east;
- Fifth Cluster (Pink) including the Washington D.C. and densely populated states;
- Sixth Cluster (Yellow) including the states loosely scattered in the country;

- Seventh Cluster (Green) including Oregon and California on the west coast as well as Delaware on the east coast.

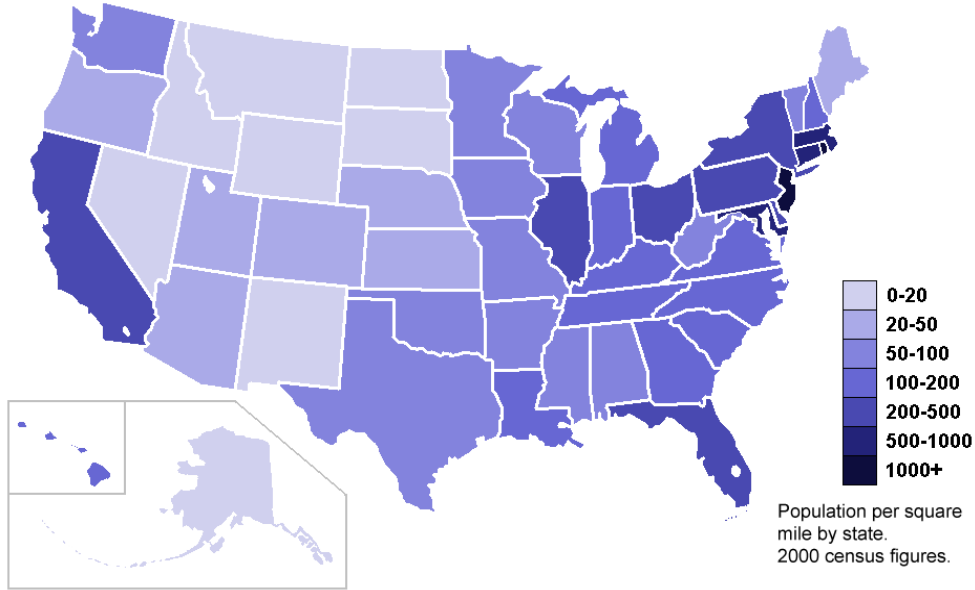
Now taking a closer look at the Fig.4.4, compared to the U.S. population density distribution in Fig.4.5, we find that the distribution of HPI memory parameter estimates have some relevance with that of U.S. state population densities. For example, except Maryland, the northeast coast states and densely populated Washington D.C. belong to the same cluster; Among the northeast states with a similar high population density, New York, Pennsylvania and Illinois are also within the same cluster; in addition, although California and Delaware, having similar population density, are situated along west coast and east coast respectively, their HPI memory parameters are grouped into the same cluster. Then to observe the HPI memory parameter estimates of the regions with a relatively high population density (here we only take into account the regions with more than 200 people per square mile), we find that the estimates are often large and usually bigger than 1.7. Exploring the reasons behind this fact that d is usually large for a densely populated region, we think that the densely populated regions commonly have a high economic level, which means that they possess better welfare, better public facilities, better medical facilities and greater security compared with respect to other regions and as a consequence, there will be an influx of population into these regions, and this phenomenon will be reflected in the housing market where the demand for housing exceeds supply, hence house prices will climb. Furthermore, when people have a good expectation for the housing market, crowd mentality often plays a role in fueling the rise in prices, resulting in irrational house purchases and home flipping. Then the housing bubble forms and the bubble will abruptly bloat. Nevertheless, as time goes by, the housing market bubble will inevitably burst and at this time the people's crowd mentality in turn could lead to the collapse of the house price. This is reflected in the big values of d . In the sparsely populated regions, the housing market turnover is small and the transaction is not active, therefore, generally speaking, with the exception of Oregon, dramatic fluctuations in the house price will not occur in these regions as in the densely populated regions, hence the values of d are relatively smaller.

In a word, we find that the distribution of the HPI memory parameter estimates exhibit a certain extent of correlation with the house locations and the region's population density.

4.1.3 Policy Implications

As is known, over the past three decades, the world economic crises were all caused by asset bubbles. Therefore it is important to stabilize the housing markets in order to avoid economic chaos, however, the overall economic environment will in turn affect the housing markets. Hence first taking a look at the Chinese economic and financial environment, we find that the domestic economic dynamics becomes weaker, the producer price index (PPI) declines, 5 trillion Yuan evaporated from the stock market in 2011 and consumer confidence is becoming lower due to the depressing economic expectation, all of which indicates that Chinese economy seems to have entered into a downward channel.

Figure 4.5: The Distribution of Population Density in the U.S.



Under this macroeconomic environment, we ought to handle the housing market issue more carefully because the house price is not only an economic problem, but also an important social and political problem as to people's livelihood. The burst of housing market bubble will lead to a sharp decrease of residents' purchasing power, a rapid increase of non-performing loans (see [69]) and even more, residents' discontent and social unrest which the U.S. has experienced since the subprime crisis in 2008 and Japan underwent after the burst of its housing market bubble. As for the international economic and financial environment, the world financial and economic crisis has evolved into social and political disorders (for example, the European debt crisis, the protest movement 'Occupy Wall Street', to name just a few). Therefore we believe that a precipitate decline of the house prices is more pernicious than the stabilizing prices with some bubbles.

Due to the lack of the historical data for the Chinese housing market and the similarities that China and the U.S. share, we thus take the conclusion made for the U.S. HPI series as a useful and important reference for the Chinese housing-market researchers and policymakers. We herein present some suggestions for the housing market policymakers on how to stabilize house prices in China:

1. In line with the result in Sec.4.1.2, we propose a similar degree of tightness of housing market regulations in the adjacent areas since the estimates of the HPI memory parameter show a regional aggregation phenomenon.
2. It is not necessary to introduce more strict housing market regulations in 2012: from the housing market data of the fourth quarter in 2011, after a series of property regulatory interventions, the housing market bubble is gradually shrinking. Moreover, for the first-tier cities, like Shanghai and Beijing, we believe that they are endowed with a big value of HPI memory parameter, i.e. $d > 1/2$, since they are all densely populated. When $d > 1/2$, the HPI series is no longer a mean-reverting process so

even a small oscillation in house price will have a tremendous impact on the future house price for a very long period, for example, when the price falls, the downward movement will continue for a long time and the magnitude of resulting decline will usually be larger.

3. Population control in densely populated regions: Since the introduction of incentive regulations for housing market in 1998, houses in China became commercial ones instead of the allocated ones and the housing supply is relatively rigid while the demand constantly increases. Therefore the housing market experienced a 5-year rapid growth, inducing high house prices, unreasonable structure and social contradictions. However, the yearly regulations aimed to rein in house prices from 2003 to 2009 didn't work effectively. This situation continued until 2010 when the implementation of housing market regulation 'New National 10 terms', and this regulation seems to produce the positive effect, which is well reflected in the 2011 housing market data. In addition, it should be emphasized that the high house prices of China's first-tier cities are not a result of housing speculations or investment behaviors, but due to the contradiction between the relatively small house supply (less than 3% of the city/town households' demand) and the house demand for urbanization and the large marriage-age population, as well as the house shortage resulting from the rural residents' influx into the cities (National Bureau of statistics data shows that the urban population for the first time surpassed the rural population in 2011), which are all rigid demands. Therefore, judging from the inference of the cluster analysis in Sec.4.1.2 – i.e. the close linkage between the distribution of the HPI memory parameter estimates across the country and that of population density as well as the significant influence of people's crowd mentality on the house prices, we suggest that the policymakers handle the housing market bubble issue from the perspective of population control in densely populated regions. For example, we ought to still go on carrying out the regulations to limit non-registered population's access to house purchase and not reform the rigid household registration system hastily in metropolitan areas, which is conducive to achieving stable, reasonable house prices.

4.1.4 Discussion

The estimation of the HPI memory parameters of the 50 states and Washington D.C. in the U.S. is conducted, and we do not have the data differenced before the estimation, which effectively avoids the information loss of the raw data. Therefore it motivates us to use the fextLPWE procedure which is suitable for non-stationary time series and invariant to a linear time trend. Then according to the estimates, d , the K-means clustering method is employed and it divides 50 states and Washington D.C. into 7 clusters. It should be noted that clustering analysis can be applied widely and is easy to perform for most researchers. Furthermore, it is observed that the estimates of HPI memory parameter possess a regional aggregation characteristic and this kind of aggregation is associated with the population density. Finally, by taking the result of the quantitative analysis of the U.S. HPI series as a useful reference, we provide some suggestions for the Chinese

housing-market policymakers: there is no need to introduce more strict property control policies in 2012 and it is still necessary to control the population in metropolitan areas. We expect that our suggestions will help to reduce the housing market bubble and stabilize housing market.

4.2 Exchange Rate Time Series: U.S. Dollar v.s. Euro

In this section, we consider the monthly exchange rate time series, Euro v.s. U.S. Dollar (USD), whose time span is from January 1999 to March 2012. It is not seasonally adjusted and we denote this series as $\{X_t\}$ in this application. The trajectory of this series is displayed in Fig.4.6. The series can be downloaded from the Federal Reserve Bank of St. Louis web site <http://research.stlouisfed.org/fred2/categories/95>.

The presence of the long memory in the exchange rates has been documented in many articles, for example, [16], [14], [118] and [76]. However, the study on the long memory property in non-stationary real-life data commonly discontinued at the estimation stage, that is, up to our knowledge, not much effort was made in terms of the prediction of non-stationary long memory time series and moreover, especially of non-stationary seasonal/cyclical long memory time series.

Therefore in this contribution, our target is to forecast the exchange rates from April 2012 to March 2013. In order to achieve this target, we first need to determine what models we should use to fit the exchange rate trajectory. To take a look at Fig.4.7, we observe that the sample autocorrelation doesn't decay exponentially, and there exists one spectrum peak in the periodogram. Hence this observation clearly shows a long range dependence in $\{X_t\}$. Most practitioners and researchers treated the peak as at the origin, which we cannot be hundred percent sure about. Actually, the peak is possibly located near the zero frequency, not necessarily at the origin. Therefore, in this application we employ two candidate processes to model the Euro v.s. USD exchange rate series: (i) ARFIMA(p, d, q) and (ii) 1-factor GARMA(p, d, ν, q) (due to the fact that only one peak is detected in the periodogram).

We evaluated the forecasting ability of ARFIMA(p, d, q) and GARMA(p, d, ν, q) in this case via performance of their 1-step ($h = 1$), 6-step ($h = 6$) and 12-step ($h = 12$) prediction results, where h is the prediction horizon. In this application we restrict p and q within the set $\{0, 1, 2\}$, and the choice of p and q are based on the Akaike information criterion, that is, the p and q with smallest AIC value are chosen.

Now we clarify the estimation methods used for ARFIMA model and then present its forecasting performance as follows. **ARFIMA(p, d, q):**

- For Model4.1, the parameters are obtained by using the exact maximum likelihood estimation method, which is detailed in [113].

$$(1 - 0.9860B)(1 - B)^{-0.0411}(X_t - 1.1920) = (1 + 0.3809B)\varepsilon_t. \quad (4.1)$$

Figure 4.6: The Monthly Exchange Rate Trajectory: USD v.s. Euro

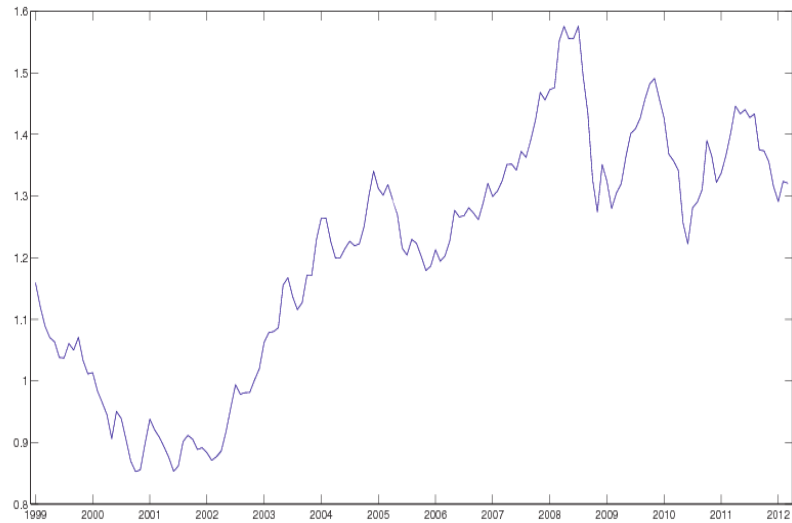
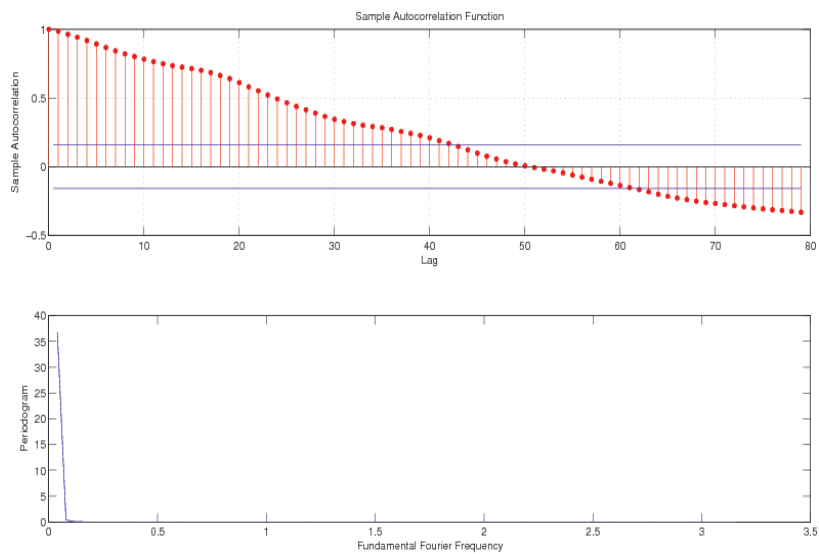


Figure 4.7: Sample Autocorrelation and Periodogram of the Monthly Exchange Rate Trajectory: USD v.s. Euro



- For Model4.2, the parameters are obtained by using Fourier-domain Whittle method.

$$(1 - 0.8176B)(1 - B)^{0.4979}(X_t - 1.2067) = \varepsilon_t. \quad (4.2)$$

The forecasting performance according to the ARFIMA model is displayed in Tab.4.2. We find that for both two models suffers a big MAPE and RMSE for a medium-run prediction horizon ($h = 6$). Moreover, the $d = -0.0411$ in Model(4.1) indicates the underlying data is anti-long-memory, which is opposite to our a priori impression, however, it exhibits a superior quality of forecasting to Model(4.2) in terms of MAPE (mean absolute percentage error) and RMSE.

Table 4.2: Exchange Rate Forecasting from ARFIMA Model

Model	Criterion	h=1	h=6	h=12
(4.1)	MAPE	0.0150	0.4301	0.0236
	RMSE	0.0235	0.8425	0.0389
(4.2)	MAPE	0.0160	0.4381	0.0261
	RMSE	0.0246	0.8477	0.0425

^a Model (4.1) is estimated by time-domain maximum likelihood method.

^b Model (4.1) is estimated by Fourier-domain Whittle method.

^c The forecast error is the difference between the actual value and the forecast value for the corresponding period. $E_t = X_t - F_t$, where E_t is the forecast error at period t , X_t is the actual value at period t , and F_t is the forecast for period t . Then $MAPE =$

$$\sum_{s=1}^h \frac{|E_{t+s}/Y_{t+s}|}{h} \text{ and } RMSE = \sqrt{\sum_{s=1}^h \frac{E_{t+s}^2}{h}}.$$

We now turn to the GARMA model to fit and forecast the exchange rate series. The parameters are estimated by the pseudo-maximum Whittle likelihood method ([76]). **GARMA**(p, d, ν, q): Specifically, we use GARMA(0, $d, \nu, 0$) and GARMA(1, $d, \nu, 0$) processes to fit the exchange rate series and they have the representations as follows:

- GARMA(0, $d, \nu, 0$):

$$(1 - 2 * 0.9992B + B^2)^{.5481}(X_t - 1.2067) = \varepsilon_t. \quad (4.3)$$

- GARMA(1, $d, \nu, 0$):

$$(1 - 2 * 0.9992B + B^2)^{.5664}(1 + 0.0553B)(X_t - 1.2067) = \varepsilon_t. \quad (4.4)$$

The forecasting results according to the GARMA model are displayed in Tab.4.3. Like the ARFIMA model, we find that for both models suffers a bad performance for 6-step

ahead prediction. Two models yield comparable results, and the Gegenbauer process, i.e. GARMA(0, $d, \nu, 0$), has a bit better performance in forecasting in terms of MAPE and RMSE.

Table 4.3: Exchange Rate Forecasting from GARMA Model

Model	Criterion	h=1	h=6	h=12
(4.3)	MAPE	0.0987	0.4675	0.0984
	RMSE	0.1320	0.8513	0.1315
(4.4)	MAPE	0.1012	0.4689	0.1006
	RMSE	0.1351	0.8515	0.1344

^a The G-frequency is estimated by maximizing the periodogram for both models.

^b The other parameters are simultaneously estimated by the Whittle method.

^c The forecast error is the difference between the actual value and the forecast value for the corresponding period. $E_t = X_t - F_t$, where E_t is the forecast error at period t , X_t is the actual value at period t , and F_t is the forecast for period t . Then $MAPE =$

$$\sum_{s=1}^h \frac{|E_{t+s}/Y_{t+s}|}{h} \text{ and } RMSE = \sqrt{\sum_{s=1}^h \frac{E_{t+s}^2}{h}}.$$

To conclude this chapter, when trying to forecast the US Dollar v.s. Euro exchange rate series using long-memory models, we find even for the same underlying model, different estimation methods may result in quite different model's detailed structure. Moreover, although GARMA process enjoys more flexibility over ARFIMA process, from a forecasting point of view, the latter process outperforms the former one in this case.

Chapter 5

Conclusion

In this dissertation, we have studied non-stationary long-memory time series. We have investigated their probabilistic properties, parameter estimation methods and the applications to the real econometric data are also contained.

5.1 Overview of Contribution

Our main contributions are listed below:

- In discrete-time context, we have reviewed the different definitions of long memory and clarified the relationship between them. We also review the the models of long memory and present their statistical properties. As for the memory parameter estimation methods for the stationary and invertible ARFIMA process, we give a brief summary in Chapter 1.
- We review the preliminaries of Fourier and wavelet analysis. Then we present popular semi-parametric memory parameter estimators applicable to non-stationary $I(d)$ processes and their statistical properties are examined. Moreover, the estimators' consistency is numerically verified via a Monte Carlo experiment under the ARFIMA(p, d, q) setting. Meanwhile the FSP of these estimators is carefully compared in terms of bias and RMSE. The FSP comparison reveals that fextLPWF estimator generally enjoys the smallest RMSE from a pragmatic perspective; and implies an instruction to the determination of user-chosen parameters, like taper order, difference order, bandwidth and upper/lower wavelet scale trimming number.
- Since there exists no theoretical results for $I(d)$ processes with heteroscedasticity and regime switching in levels, which are often detected in real-life data. Therefore we also carry out a simulation experiment to check the estimators' robustness to these situation. We conclude from the experiment results that the FSP of the estimators in the heteroscedastic case are essentially the as the non-heteroscedastic case; the regime switching in levels seemingly contaminates the estimation results and the log-regression wavelet estimator's FSP with small sample size is really lousy.

- We also consider non-stationary Gegenbauer processes and proposed two memory parameter semi-parametric estimators. The consistency of the estimators is proved and their robustness is verified via Monte Carlo simulations. Moreover, in light this simulation study, we conclude that GSE excels the LPSE for $d > -\frac{1}{2}$ and the LPSE yields reliable estimates for $d < -\frac{1}{2}$ while the GSE can not. We also find that the location of spectral peak also has a substantial impact on the estimation result, and furthermore from a pragmatic perspective, we cast a light on the optimal selection of parameter values, i.e., the bandwidth, the taper order, for both estimators.
- Our first application is concerned with the house price index's memory parameters of the 50 states and Washington D.C. in the U.S. According to the estimates, ds , the K-means clustering method is employed and it divides 50 states and Washington D.C. into 7 clusters. Then it is observed that the estimates of HPI memory parameter possess a regional aggregation characteristic and this kind of aggregation is associated with the population density. Finally, by taking the result of the quantitative analysis of the U.S. HPI series as a useful reference, we provide some suggestions for the Chinese housing-market policymakers: there is no need to introduce more strict property control policies in 2012 and it is still necessary to control the population in metropolitan areas.
- When trying to forecast the US Dollar v.s. Euro exchange rate series using long-memory models, we find even for the same underlying model, different estimation methods may result in quite different model's detailed structure. Moreover, although GARMA process enjoys more flexibility over ARFIMA process, from a forecasting point of view, the latter process outperforms the former one in this case.

5.2 Possible Directions for Future Research

As for the future research, there are several problems of interest listed below.

- In this dissertation the 'non-stationarity' signifies that the constant memory parameter d is bigger than $\frac{1}{2}$. However, the constant restriction on d can be relaxed, for example, d can be treated as a function of t , i.e., $d = d(t)$ and $d(t)$ is not constrained with the interval $(-\frac{1}{2}, \frac{1}{2})$, which implies the locally stationary requirement is also loosened. As is already stated, the wavelet estimators are applicable to the non-stationary $I(d)$ (where d is a constant and $d \in \mathbb{R}$) process, but they only yield an inferior estimation performance to the Fourier estimators. The power of wavelets to localize data in both time and frequency is obviously not fully exploited. We assume the advantage of wavelet methods can be made good use of to estimate $d(t)$.
- In the existing literature, there are three popular ways to simulate a Gegenbauer time series. i) The moving average truncation simulation method proposed in [59] is very

computationally burdened and it depends on the convergence of the process; ii) The simulation method (usually called *Hosking method*) proposed in [125] is involved in computing autocovariance sequence so it's also not an ideal method; iii) A wavelet-based simulation method proposed in [122] is faster than the two former simulation methods, but it doesn't practically function for some Gegenbauer frequencies, for example $\text{Greq} = \pi/2$. Therefore this reality urges us to come up with a new method of simulating Gegenbauer processes or more generally speaking, seasonal/cyclical time series.

- The wavelet-based methods may address the memory parameter estimation problem for non-stationary seasonal/cyclical long-memory process, i.e., the case $d > \frac{1}{2}$.
- According to our empirical findings in Sec.2.4.3, the distinction between the appearance of long memory in data and regime switching is also of great interest.

Chapter 6

Appendix

6.1 Notations

- $\stackrel{\text{def}}{=}$: defined as;
- i.i.d. $\varepsilon_t \sim N(0, \sigma^2)$: independent and identically distributed $\varepsilon_t \sim N(0, \sigma^2)$, where σ is a constant;
- $\sim$: ratio of the left and right hand sides tending to one in the limit
- $f(x) = \mathcal{O}_P(g(x))$: There is a sufficiently large constant M such that for all sufficiently large values of x , $f(x)$ is at most M multiplied by $g(x)$ in absolute value in probability;
- $\asymp$: asymptotically equal;
- K : trimming number of the highest wavelet scales;
- J : trimming number of the lowest wavelet scales;
- $\bar{\cdot}$: complex conjugation;
- B : lag operator, i.e., $BX_t = X_{t-1}$;
- $\lambda_j = \frac{2\pi j}{n}$, $j = 1, \dots, n$, $n \in \mathbb{N}$: fundamental Fourier frequency;
- $\lfloor x \rfloor$: greatest integer less than x ;
- $f(x) = \mathcal{O}_P(g(x))$: There is a sufficiently large constant M such that for all sufficiently large values of x , $f(x)$ is at most M multiplied by $g(x)$ in absolute value in probability.

The following are the abbreviations of the estimation procedures:

- feLWF estimation: feasible local Whittle Fourier estimation;

- fextLPWF estimation: fully-extended local polynomial Whittle estimation;
- GPHF estimation: tapered and pooled log-regression Fourier estimation;
- modLWF estimation: modified local Whittle Fourier estimation;
- tLWF estimation: tapered local Whittle Fourier estimation;
- LRW estimation: log-regression wavelet estimation;
- LWW estimation: local Whittle wavelet estimation.

6.2 Tables for Monte Carlo Results

6.2.1 Results of Fourier Estimators for Long-Memory Processes

Table 6.1: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 0) Model with $p = 1$

d	n	τ	$d - \delta \in [-.5, .5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	1	-0.0009	0.3897	0.0016	0.2337	0.0006	0.1518
	300	1	0.0028	0.3067	-0.0025	0.1698	0.0001	0.1055
	500	1	-0.0038	0.2478	-0.0021	0.1384	-0.0020	0.0842
	1000	1	-0.0034	0.2083	0.0034	0.1049	0.0002	0.0622
.7	150	1	0.0034	0.3842	-0.0048	0.2362	-0.0048	0.1529
	300	1	-0.0072	0.3123	-0.0022	0.1697	-0.0019	0.1060
	500	1	-0.0078	0.2512	0.0009	0.1389	-0.0013	0.0840
	1000	1	-0.0003	0.2005	-0.0015	0.1062	0.0005	0.0612
.9	150	1	-0.0076	0.3818	-0.0037	0.2362	-0.0040	0.1518
	300	1	0.0025	0.3038	-0.0016	0.1697	-0.0010	0.1074
	500	1	0.0023	0.2476	0.0015	0.1362	0.0006	0.0846
	1000	1	0.0052	0.2004	-0.0008	0.1060	0.0005	0.0622
1.1	150	1	-0.0056	0.3821	0.0076	0.2348	0.0030	0.1484
	300	1	0.0013	0.3056	0.0024	0.1702	-0.0014	0.1051
	500	1	0.0027	0.2450	-0.0006	0.1380	-0.0009	0.0831
	1000	1	0.0021	0.2011	0.0025	0.1060	0.0012	0.0617
1.3	150	1	0.0126	0.3879	0.0107	0.2378	0.0039	0.1499
	300	1	0.0076	0.3051	0.0037	0.1676	0.0024	0.1082
	500	1	0.0091	0.2490	0.0025	0.1402	0.0006	0.0820
	1000	1	0.0070	0.2018	0.0053	0.1086	0.0004	0.0613
1.5	150	2	-0.0130	0.5035	0.0025	0.2970	0.0009	0.1891
	300	2	0.0003	0.4127	-0.0013	0.2141	-0.0007	0.1332
	500	2	0.0035	0.3206	0.0010	0.1757	-0.0017	0.1037
	1000	2	-0.0009	0.2491	-0.0019	0.1342	-0.0004	0.0766
1.7	150	2	-0.0115	0.4963	0.0030	0.2975	-0.0092	0.1871
	300	2	0.0088	0.4109	-0.0086	0.2199	-0.0025	0.1319
	500	2	0.0041	0.3225	0.0008	0.1752	-0.0022	0.1036
	1000	2	0.0001	0.2525	-0.0003	0.1320	0.0006	0.0767
1.9	150	2	-0.0066	0.5024	-0.0014	0.3025	0.0004	0.1863
	300	2	-0.0002	0.4218	0.0032	0.2123	0.0014	0.1341
	500	2	0.0077	0.3217	0.0019	0.1739	0.0019	0.1033
	1000	2	-0.0053	0.2544	-0.0032	0.1327	-0.0003	0.0758

^a The p is the pooling order; The τ is the tapering order;

Table 6.2: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 0) Model with $p = 1$

d	n	τ	$d - \delta \in [-1.5, -.5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	2	0.0767	0.5003	0.0501	0.2997	0.0224	0.1871
	300	2	0.0597	0.4359	0.0284	0.2166	0.0132	0.1355
	500	2	0.0367	0.3270	0.0217	0.1796	0.0120	0.1047
	1000	2	0.0347	0.2520	0.0177	0.1338	0.0058	0.0756
.7	150	2	0.0431	0.4967	0.0333	0.3009	0.0173	0.1885
	300	2	0.0402	0.4089	0.0198	0.2181	0.0134	0.1314
	500	2	0.0301	0.3273	0.0163	0.1788	0.0076	0.1027
	1000	2	0.0273	0.2544	0.0086	0.1294	0.0059	0.0772
.9	150	2	0.0394	0.5090	0.0251	0.3058	0.0138	0.1879
	300	2	0.0215	0.4167	0.0138	0.2135	0.0087	0.1313
	500	2	0.0273	0.3204	0.0151	0.1747	0.0041	0.1030
	1000	2	0.0166	0.2551	0.0038	0.1337	0.0018	0.0746
1.1	150	2	0.0223	0.4945	0.0152	0.3047	0.0079	0.1869
	300	2	0.0229	0.4197	0.0096	0.2177	-0.0002	0.1346
	500	2	0.0181	0.3188	0.0066	0.1767	0.0022	0.1035
	1000	2	0.0088	0.2546	0.0021	0.1330	0.0026	0.0760
1.3	150	2	0.0170	0.4961	0.0085	0.2946	0.0027	0.1875
	300	2	0.0113	0.4166	0.0068	0.2206	0.0011	0.1319
	500	2	0.0071	0.3312	0.0050	0.1765	-0.0000	0.1030
	1000	2	0.0062	0.2556	0.0031	0.1363	0.0016	0.0758
1.5	150	3	0.0744	0.6352	0.0454	0.3592	0.0234	0.2266
	300	3	0.0571	0.4847	0.0314	0.2529	0.0193	0.1541
	500	3	0.0387	0.4029	0.0202	0.2039	0.0121	0.1213
	1000	3	0.0412	0.3193	0.0193	0.1543	0.0082	0.0886
1.7	150	3	0.0378	0.6265	0.0290	0.3592	0.0190	0.2251
	300	3	0.0514	0.4894	0.0258	0.2512	0.0126	0.1577
	500	3	0.0316	0.4083	0.0202	0.2082	0.0101	0.1190
	1000	3	0.0338	0.3241	0.0140	0.1549	0.0069	0.0882
1.9	150	3	0.0410	0.6284	0.0203	0.3646	0.0064	0.2280
	300	3	0.0325	0.4834	0.0134	0.2534	0.0099	0.1572
	500	3	0.0238	0.4065	0.0142	0.2025	0.0050	0.1194
	1000	3	0.0165	0.3212	0.0060	0.1552	0.0047	0.0880

^a The p is the pooling order; The τ is the tapering order;

Table 6.3: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 0) Model with $p = 3$

d	n	τ	$d - \delta \in [-.5, .5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	1	-0.0097	0.3511	-0.0010	0.2006	-0.0031	0.1245
	300	1	-0.0054	0.2733	0.0006	0.1396	-0.0034	0.0869
	500	1	-0.0079	0.2304	-0.0017	0.1153	0.0007	0.0667
	1000	1	0.0020	0.1789	-0.0007	0.0851	0.0000	0.0495
.7	150	1	-0.0091	0.3499	-0.0011	0.1979	-0.0041	0.1246
	300	1	-0.0148	0.2668	-0.0027	0.1401	-0.0003	0.0869
	500	1	-0.0071	0.2249	-0.0020	0.1126	-0.0021	0.0663
	1000	1	-0.0026	0.1779	-0.0018	0.0828	-0.0016	0.0493
.9	150	1	-0.0082	0.3432	0.0012	0.1965	-0.0033	0.1231
	300	1	-0.0061	0.2674	-0.0012	0.1379	-0.0013	0.0850
	500	1	-0.0097	0.2249	-0.0027	0.1122	-0.0015	0.0665
	1000	1	-0.0030	0.1723	-0.0012	0.0851	-0.0006	0.0482
1.1	150	1	0.0025	0.3470	0.0043	0.1970	0.0017	0.1253
	300	1	0.0056	0.2677	0.0054	0.1372	0.0030	0.0857
	500	1	0.0038	0.2231	0.0024	0.1093	0.0016	0.0647
	1000	1	0.0042	0.1751	0.0004	0.0852	0.0003	0.0485
1.3	150	1	0.0307	0.3389	0.0206	0.1950	0.0141	0.1252
	300	1	0.0259	0.2670	0.0146	0.1388	0.0100	0.0853
	500	1	0.0200	0.2244	0.0110	0.1104	0.0065	0.0666
	1000	1	0.0171	0.1750	0.0060	0.0844	0.0033	0.0488
1.5	150	2	-0.0039	0.5700	-0.0022	0.2438	-0.0018	0.1500
	300	2	-0.0071	0.3685	0.0015	0.1771	0.0004	0.1056
	500	2	-0.0064	0.2925	-0.0029	0.1432	-0.0006	0.0832
	1000	2	-0.0033	0.2112	-0.0015	0.1083	-0.0013	0.0594
1.7	150	2	-0.0157	0.5661	-0.0033	0.2445	-0.0039	0.1490
	300	2	-0.0070	0.3615	0.0008	0.1736	-0.0012	0.1048
	500	2	-0.0133	0.2881	0.0005	0.1400	-0.0020	0.0836
	1000	2	-0.0031	0.2138	-0.0011	0.1058	-0.0024	0.0600
1.9	150	2	-0.0111	0.5581	-0.0055	0.2451	-0.0010	0.1530
	300	2	-0.0063	0.3659	-0.0011	0.1726	-0.0009	0.1041
	500	2	0.0035	0.2864	0.0017	0.1404	-0.0023	0.0827
	1000	2	0.0040	0.2062	-0.0036	0.1063	-0.0014	0.0584

^a The p is the pooling order; The τ is the tapering order;

Table 6.4: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 0) Model with $p = 3$

d	n	τ	$d - \delta \in [-1.5, -.5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	2	0.0598	0.2625	0.0582	0.2617	0.0369	0.1659
	300	2	0.0460	0.1874	0.0401	0.1842	0.0227	0.1121
	500	2	0.0325	0.1489	0.0313	0.1495	0.0169	0.0860
	1000	2	0.0241	0.1173	0.0215	0.1157	0.0109	0.0624
.7	150	2	0.0458	0.2646	0.0400	0.2621	0.0259	0.1614
	300	2	0.0338	0.1854	0.0203	0.1839	0.0180	0.1096
	500	2	0.0243	0.1478	0.0240	0.1479	0.0112	0.0871
	1000	2	0.0145	0.1108	0.0142	0.1114	0.0091	0.0619
.9	150	2	0.0292	0.2576	0.0257	0.2571	0.0156	0.1546
	300	2	0.0178	0.1820	0.0205	0.1833	0.0124	0.1094
	500	2	0.0173	0.1432	0.0122	0.1452	0.0066	0.0852
	1000	2	0.0095	0.1110	0.0108	0.1106	0.0055	0.0614
1.1	150	2	0.0083	0.2504	0.0152	0.2492	0.0091	0.1507
	300	2	0.0117	0.1763	0.0075	0.1754	0.0040	0.1057
	500	2	0.0037	0.1451	0.0100	0.1429	0.0037	0.0832
	1000	2	0.0056	0.1098	0.0045	0.1107	0.0020	0.0599
1.3	150	2	-0.0020	0.2510	-0.0012	0.2474	0.0021	0.1532
	300	2	-0.0014	0.1802	0.0023	0.1749	0.0031	0.1083
	500	2	0.0004	0.1425	-0.0024	0.1436	0.0009	0.0845
	1000	2	0.0017	0.1090	0.0027	0.1095	-0.0007	0.0596
1.5	150	3	0.0494	0.3245	0.0523	0.3228	0.0322	0.1893
	300	3	0.0439	0.2364	0.0395	0.2339	0.0228	0.1318
	500	3	0.0295	0.1752	0.0299	0.1760	0.0166	0.1004
	1000	3	0.0232	0.1358	0.0217	0.1327	0.0122	0.0729
1.7	150	3	0.0468	0.3174	0.0329	0.3159	0.0186	0.1877
	300	3	0.0277	0.2341	0.0298	0.2329	0.0175	0.1300
	500	3	0.0211	0.1727	0.0231	0.1745	0.0108	0.0993
	1000	3	0.0160	0.1320	0.0138	0.1334	0.0064	0.0714
1.9	150	3	0.0199	0.3174	0.0283	0.3125	0.0176	0.1839
	300	3	0.0233	0.2296	0.0188	0.2322	0.0099	0.1307
	500	3	0.0126	0.1712	0.0116	0.1763	0.0072	0.0973
	1000	3	0.0101	0.1314	0.0108	0.1296	0.0034	0.0717

^b The p is the pooling order; The τ is the tapering order;

Table 6.5: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $p = 1$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	0.0510	0.3773	0.1244	0.2683	0.2823	0.3196
		300	1	0.0207	0.3057	0.0885	0.1911	0.2327	0.2554
		500	1	0.0183	0.2495	0.0648	0.1531	0.2033	0.2193
		1000	1	0.0025	0.1997	0.0364	0.1124	0.1675	0.1784
	.7	150	1	0.0438	0.3781	0.1283	0.2695	0.2745	0.3126
		300	1	0.0158	0.3095	0.0889	0.1904	0.2281	0.2521
		500	1	0.0093	0.2472	0.0639	0.1539	0.2017	0.2181
		1000	1	0.0044	0.2006	0.0389	0.1145	0.1692	0.1801
	.9	150	1	0.0427	0.3841	0.1225	0.2652	0.2749	0.3131
		300	1	0.0167	0.3065	0.0886	0.1951	0.2298	0.2526
		500	1	0.0125	0.2465	0.0612	0.1517	0.2037	0.2202
		1000	1	0.0089	0.2007	0.0419	0.1136	0.1671	0.1781
	1.1	150	1	0.0558	0.3883	0.1288	0.2723	0.2753	0.3148
		300	1	0.0244	0.3141	0.0905	0.1912	0.2288	0.2527
		500	1	0.0124	0.2529	0.0644	0.1547	0.2039	0.2204
		1000	1	0.0150	0.1979	0.0400	0.1150	0.1689	0.1799
	1.3	150	1	0.0577	0.3826	0.1340	0.2715	0.2763	0.3153
		300	1	0.0367	0.3121	0.0930	0.1931	0.2296	0.2526
		500	1	0.0204	0.2487	0.0666	0.1554	0.2068	0.2230
		1000	1	0.0187	0.2006	0.0402	0.1106	0.1696	0.1808
	1.5	150	2	0.0567	0.5105	0.1216	0.3265	0.2804	0.3377
		300	2	0.0127	0.4235	0.0899	0.2318	0.2298	0.2650
		500	2	0.0202	0.3323	0.0628	0.1882	0.2040	0.2293
		1000	2	0.0189	0.2555	0.0413	0.1373	0.1699	0.1863
	1.7	150	2	0.0476	0.5067	0.1396	0.3251	0.2848	0.3407
		300	2	0.0133	0.4147	0.0788	0.2342	0.2260	0.2618
		500	2	0.0158	0.3278	0.0601	0.1887	0.2020	0.2266
		1000	2	0.0027	0.2598	0.0363	0.1386	0.1697	0.1855
	1.9	150	2	0.0452	0.5053	0.1321	0.3202	0.2778	0.3353
		300	2	0.0210	0.4156	0.0870	0.2308	0.2268	0.2626
		500	2	0.0040	0.3264	0.0564	0.1878	0.2064	0.2307
		1000	2	0.0059	0.2504	0.0364	0.1369	0.1683	0.1850

^a The p is the pooling order; The τ is the tapering order;

Table 6.6: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $p = 1$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	0.1171	0.5159	0.1822	0.3506	0.3014	0.3536
		300	2	0.0673	0.4311	0.1197	0.2480	0.2478	0.2805
		500	2	0.0643	0.3271	0.0808	0.1933	0.2171	0.2410
		1000	2	0.0441	0.2518	0.0524	0.1426	0.1748	0.1902
	.7	150	2	0.0898	0.5112	0.1547	0.3384	0.2973	0.3507
		300	2	0.0561	0.4269	0.1069	0.2432	0.2392	0.2743
		500	2	0.0466	0.3234	0.0743	0.1952	0.2161	0.2392
		1000	2	0.0283	0.2587	0.0520	0.1432	0.1737	0.1897
	.9	150	2	0.0752	0.5020	0.1511	0.3321	0.2913	0.3469
		300	2	0.0478	0.4217	0.0990	0.2382	0.2368	0.2716
		500	2	0.0394	0.3257	0.0663	0.1875	0.2108	0.2345
		1000	2	0.0250	0.2597	0.0469	0.1403	0.1736	0.1888
	1.1	150	2	0.0660	0.5131	0.1432	0.3353	0.2852	0.3410
		300	2	0.0323	0.4155	0.0939	0.2356	0.2345	0.2685
		500	2	0.0190	0.3273	0.0655	0.1889	0.2109	0.2340
		1000	2	0.0099	0.2556	0.0428	0.1391	0.1688	0.1857
	1.3	150	2	0.0607	0.4995	0.1408	0.3249	0.2833	0.3408
		300	2	0.0315	0.4117	0.0868	0.2329	0.2347	0.2693
		500	2	0.0193	0.3246	0.0601	0.1857	0.2059	0.2309
		1000	2	0.0068	0.2584	0.0399	0.1383	0.1706	0.1860
	1.5	150	3	0.1265	0.6317	0.1722	0.3947	0.3069	0.3818
		300	3	0.0756	0.5004	0.1231	0.2805	0.2446	0.2891
		500	3	0.0670	0.4194	0.0861	0.2178	0.2209	0.2505
		1000	3	0.0420	0.3197	0.0560	0.1624	0.1775	0.1981
	1.7	150	3	0.1161	0.6309	0.1671	0.3970	0.2922	0.3690
		300	3	0.0585	0.4857	0.1058	0.2736	0.2440	0.2887
		500	3	0.0490	0.4028	0.0839	0.2149	0.2166	0.2472
		1000	3	0.0308	0.3187	0.0548	0.1642	0.1743	0.1956
	1.9	150	3	0.0747	0.6257	0.1541	0.3854	0.2932	0.3684
		300	3	0.0455	0.4845	0.1094	0.2767	0.2372	0.2845
		500	3	0.0391	0.4079	0.0737	0.2180	0.2108	0.2427
		1000	3	0.0252	0.3165	0.0498	0.1620	0.1742	0.1956

^a The p is the pooling order; The τ is the tapering order;

Table 6.7: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $p = 3$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	0.0535	0.3614	0.1368	0.2449	0.2790	0.3046
		300	1	0.0180	0.2666	0.0892	0.1667	0.2304	0.2461
		500	1	0.0169	0.2281	0.0619	0.1300	0.2063	0.2166
		1000	1	0.0057	0.1783	0.0374	0.0948	0.1684	0.1755
	.7	150	1	0.0454	0.3486	0.1305	0.2388	0.2797	0.3062
		300	1	0.0186	0.2711	0.0862	0.1651	0.2302	0.2453
		500	1	0.0037	0.2205	0.0633	0.1288	0.2047	0.2151
		1000	1	0.0025	0.1760	0.0369	0.0927	0.1699	0.1766
	.9	150	1	0.0496	0.3463	0.1348	0.2381	0.2789	0.3048
		300	1	0.0202	0.2673	0.0903	0.1643	0.2327	0.2475
		500	1	0.0131	0.2254	0.0659	0.1299	0.2061	0.2165
		1000	1	0.0046	0.1747	0.0371	0.0938	0.1685	0.1753
	1.1	150	1	0.0597	0.3468	0.1446	0.2426	0.2829	0.3096
		300	1	0.0270	0.2667	0.0933	0.1675	0.2334	0.2490
		500	1	0.0213	0.2212	0.0669	0.1288	0.2081	0.2180
		1000	1	0.0086	0.1718	0.0410	0.0941	0.1701	0.1770
	1.3	150	1	0.0917	0.3523	0.1672	0.2590	0.2950	0.3194
		300	1	0.0507	0.2684	0.1108	0.1768	0.2404	0.2553
		500	1	0.0377	0.2245	0.0746	0.1347	0.2133	0.2231
		1000	1	0.0222	0.1737	0.0470	0.0962	0.1731	0.1800
	1.5	150	2	0.0326	0.5642	0.1477	0.2898	0.3001	0.3355
		300	2	0.0186	0.3659	0.0926	0.1974	0.2401	0.2623
		500	2	0.0176	0.2893	0.0625	0.1536	0.2050	0.2206
		1000	2	0.0062	0.2128	0.0373	0.1138	0.1721	0.1822
	1.7	150	2	0.0465	0.5580	0.1520	0.2904	0.2955	0.3321
		300	2	0.0127	0.3736	0.0908	0.1949	0.2371	0.2594
		500	2	0.0116	0.2904	0.0631	0.1545	0.2038	0.2203
		1000	2	-0.0003	0.2121	0.0363	0.1133	0.1721	0.1818
	1.9	150	2	0.0428	0.5577	0.1485	0.2889	0.2982	0.3347
		300	2	0.0250	0.3644	0.0950	0.1972	0.2404	0.2623
		500	2	0.0173	0.2892	0.0617	0.1539	0.2050	0.2208
		1000	2	-0.0006	0.2080	0.0396	0.1137	0.1707	0.1804

^a The p is the pooling order; The τ is the tapering order;

Table 6.8: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $p = 3$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	0.1771	0.6083	0.2090	0.3372	0.3323	0.3681
		300	2	0.0922	0.4016	0.1398	0.2314	0.2654	0.2864
		500	2	0.0844	0.3163	0.0969	0.1770	0.2198	0.2364
		1000	2	0.0504	0.2306	0.0622	0.1275	0.1833	0.1930
	.7	150	2	0.1328	0.5943	0.1986	0.3281	0.3248	0.3607
		300	2	0.0811	0.3977	0.1243	0.2214	0.2585	0.2804
		500	2	0.0574	0.3064	0.0865	0.1685	0.2181	0.2340
		1000	2	0.0364	0.2251	0.0568	0.1253	0.1797	0.1898
	.9	150	2	0.0976	0.5820	0.1770	0.3098	0.3131	0.3505
		300	2	0.0541	0.3741	0.1127	0.2133	0.2526	0.2746
		500	2	0.0456	0.3012	0.0778	0.1640	0.2112	0.2279
		1000	2	0.0258	0.2222	0.0482	0.1216	0.1788	0.1889
	1.1	150	2	0.0661	0.5775	0.1616	0.2985	0.3031	0.3402
		300	2	0.0376	0.3828	0.1059	0.2064	0.2469	0.2685
		500	2	0.0275	0.2930	0.0732	0.1625	0.2075	0.2238
		1000	2	0.0193	0.2130	0.0447	0.1181	0.1743	0.1846
	1.3	150	2	0.0688	0.5810	0.1573	0.2948	0.3008	0.3370
		300	2	0.0225	0.3705	0.1027	0.2053	0.2419	0.2651
		500	2	0.0152	0.2962	0.0670	0.1602	0.1724	0.1829
		1000	2	0.0010	0.2122	0.0412	0.1176	0.2050	0.2210
	1.5	150	3	0.1571	0.6300	0.2057	0.3752	0.3353	0.3828
		300	3	0.1106	0.6182	0.1293	0.2657	0.2585	0.2904
		500	3	0.0780	0.4090	0.0982	0.2023	0.2287	0.2489
		1000	3	0.0518	0.2655	0.0620	0.1450	0.1813	0.1949
	1.7	150	3	0.1194	0.6046	0.1955	0.3748	0.3264	0.3752
		300	3	0.0749	0.6067	0.1149	0.2543	0.2514	0.2842
		500	3	0.0681	0.4062	0.0852	0.1950	0.2240	0.2442
		1000	3	0.0449	0.2630	0.0547	0.1421	0.1795	0.1927
	1.9	150	3	0.1139	0.6013	0.1720	0.3582	0.3131	0.3634
		300	3	0.0498	0.5903	0.1048	0.2481	0.2449	0.2768
		500	3	0.0409	0.3953	0.0783	0.1895	0.2194	0.2399
		1000	3	0.0226	0.2593	0.0480	0.1411	0.1753	0.1886

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.9: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $p = 1$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	-0.0098	0.3742	-0.0361	0.2375	-0.1566	0.2191
		300	1	-0.0051	0.3048	-0.0200	0.1694	-0.1094	0.1542
		500	1	0.0021	0.2488	-0.0111	0.1396	-0.0875	0.1214
		1000	1	-0.0045	0.1996	-0.0075	0.1051	-0.0626	0.0873
	.7	150	1	-0.0173	0.3763	-0.0436	0.2453	-0.1548	0.2171
		300	1	-0.0096	0.3090	-0.0243	0.1716	-0.1060	0.1512
		500	1	-0.0069	0.2471	-0.0159	0.1390	-0.0890	0.1209
		1000	1	-0.0026	0.2004	-0.0088	0.1070	-0.0634	0.0879
	.9	150	1	-0.0177	0.3837	-0.0290	0.2456	-0.1556	0.2173
		300	1	-0.0089	0.3062	-0.0153	0.1700	-0.1076	0.1516
		500	1	-0.0037	0.2463	-0.0121	0.1391	-0.0884	0.1214
		1000	1	0.0018	0.2006	-0.0086	0.1084	-0.0621	0.0870
	1.1	150	1	-0.0056	0.3849	-0.0351	0.2413	-0.1493	0.2111
		300	1	-0.0012	0.3127	-0.0200	0.1709	-0.1092	0.1527
		500	1	-0.0039	0.2527	-0.0119	0.1371	-0.0845	0.1186
		1000	1	0.0079	0.1975	-0.0077	0.1069	-0.0606	0.0859
	1.3	150	1	-0.0031	0.3771	-0.0251	0.2368	-0.1503	0.2111
		300	1	0.0112	0.3104	-0.0114	0.1709	-0.1056	0.1503
		500	1	0.0044	0.2477	-0.0073	0.1386	-0.0833	0.1178
		1000	1	0.0115	0.2005	-0.0055	0.1063	-0.0622	0.0868
	1.5	150	2	-0.0092	0.5091	-0.0314	0.2949	-0.1622	0.2486
		300	2	-0.0118	0.4233	-0.0235	0.2159	-0.1037	0.1691
		500	2	0.0046	0.3315	-0.0108	0.1776	-0.0869	0.1354
		1000	2	0.0115	0.2551	-0.0085	0.1326	-0.0637	0.1000
	1.7	150	2	-0.0175	0.5043	-0.0367	0.3049	-0.1589	0.2451
		300	2	-0.0117	0.4148	-0.0232	0.2232	-0.1072	0.1740
		500	2	0.0001	0.3273	-0.0137	0.1773	-0.0881	0.1364
		1000	2	-0.0046	0.2596	-0.0082	0.1365	-0.0626	0.0983
	1.9	150	2	-0.0193	0.5037	-0.0353	0.2973	-0.1626	0.2497
		300	2	-0.0036	0.4148	-0.0201	0.2176	-0.1100	0.1710
		500	2	-0.0115	0.3264	-0.0169	0.1812	-0.0880	0.1365
		1000	2	-0.0015	0.2505	-0.0089	0.1331	-0.0626	0.0995

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.10: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $p = 1$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.0444	0.5128	0.0058	0.3020	-0.1354	0.2307
		300	2	0.0563	0.4262	0.0068	0.2163	-0.0901	0.1611
		500	2	0.0358	0.3325	0.0155	0.1754	-0.0791	0.1307
		1000	2	0.0368	0.2583	0.0081	0.1339	-0.0552	0.0939
	.7	150	2	0.0282	0.5148	-0.0083	0.3096	-0.1430	0.2379
		300	2	0.0396	0.4232	0.0011	0.2168	-0.0961	0.1637
		500	2	0.0292	0.3340	0.0037	0.1767	-0.0790	0.1303
		1000	2	0.0264	0.2544	0.0026	0.1315	-0.0589	0.0969
	.9	150	2	0.0133	0.5099	-0.0135	0.3027	-0.1464	0.2383
		300	2	0.0270	0.4167	-0.0094	0.2115	-0.1031	0.1699
		500	2	0.0163	0.3247	-0.0015	0.1764	-0.0831	0.1333
		1000	2	0.0111	0.2534	0.0009	0.1320	-0.0593	0.0975
	1.1	150	2	-0.0026	0.5038	-0.0205	0.3018	-0.1535	0.2422
		300	2	0.0131	0.4177	-0.0111	0.2153	-0.1039	0.1669
		500	2	0.0117	0.3225	-0.0086	0.1778	-0.0864	0.1326
		1000	2	0.0124	0.2566	-0.0051	0.1327	-0.0610	0.0965
	1.3	150	2	0.0001	0.4900	-0.0362	0.2989	-0.1575	0.2437
		300	2	0.0063	0.4132	-0.0194	0.2209	-0.1054	0.1703
		500	2	-0.0057	0.3272	-0.0137	0.1789	-0.0854	0.1344
		1000	2	0.0048	0.2520	-0.0063	0.1331	-0.0620	0.0979
	1.5	150	3	0.0516	0.6302	0.0040	0.3631	-0.1266	0.2530
		300	3	0.0535	0.4823	0.0061	0.2496	-0.0900	0.1810
		500	3	0.0396	0.4032	0.0034	0.2040	-0.0771	0.1444
		1000	3	0.0332	0.3161	0.0109	0.1565	-0.0551	0.1052
	1.7	150	3	0.0394	0.6269	-0.0011	0.3613	-0.1353	0.2617
		300	3	0.0339	0.4921	0.0009	0.2589	-0.0935	0.1806
		500	3	0.0371	0.3966	0.0035	0.1987	-0.0808	0.1450
		1000	3	0.0312	0.3184	0.0028	0.1549	-0.0549	0.1036
	1.9	150	3	0.0177	0.6361	-0.0189	0.3607	-0.1412	0.2607
		300	3	0.0093	0.4875	-0.0057	0.2458	-0.0988	0.1851
		500	3	0.0195	0.4023	-0.0040	0.2051	-0.0832	0.1449
		1000	3	0.0261	0.3089	0.0021	0.1547	-0.0588	0.1059

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.11: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $p = 3$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	-0.0170	0.3580	-0.0456	0.2061	-0.1536	0.1992
		300	1	-0.0110	0.2663	-0.0231	0.1439	-0.1077	0.1374
		500	1	0.0016	0.2275	-0.0168	0.1130	-0.0914	0.1135
		1000	1	-0.0011	0.1783	-0.0076	0.0872	-0.0633	0.0802
	.7	150	1	-0.0253	0.3469	-0.0420	0.2042	-0.1523	0.1954
		300	1	-0.0105	0.2708	-0.0283	0.1436	-0.1108	0.1416
		500	1	-0.0116	0.2208	-0.0198	0.1126	-0.0903	0.1123
		1000	1	-0.0043	0.1761	-0.0107	0.0856	-0.0641	0.0809
	.9	150	1	-0.0212	0.3435	-0.0411	0.2028	-0.1582	0.2016
		300	1	-0.0089	0.2668	-0.0234	0.1397	-0.1070	0.1373
		500	1	-0.0022	0.2251	-0.0118	0.1146	-0.0891	0.1107
		1000	1	-0.0022	0.1747	-0.0090	0.0848	-0.0637	0.0804
	1.1	150	1	-0.0107	0.3418	-0.0334	0.1976	-0.1497	0.1933
		300	1	-0.0022	0.2652	-0.0211	0.1396	-0.1034	0.1354
		500	1	0.0060	0.2203	-0.0118	0.1141	-0.0876	0.1093
		1000	1	0.0018	0.1716	-0.0091	0.0863	-0.0624	0.0784
	1.3	150	1	0.0207	0.3403	-0.0152	0.1988	-0.1402	0.1866
		300	1	0.0216	0.2643	-0.0044	0.1370	-0.1004	0.1317
		500	1	0.0224	0.2224	-0.0060	0.1106	-0.0824	0.1050
		1000	1	0.0154	0.1729	-0.0025	0.0853	-0.0586	0.0762
	1.5	150	2	-0.0276	0.5645	-0.0396	0.2466	-0.1745	0.2317
		300	2	-0.0095	0.3657	-0.0260	0.1769	-0.1178	0.1578
		500	2	0.0014	0.2887	-0.0163	0.1436	-0.0876	0.1215
		1000	2	-0.0018	0.2127	-0.0105	0.1081	-0.0648	0.0875
	1.7	150	2	-0.0136	0.5564	-0.0480	0.2513	-0.1783	0.2326
		300	2	-0.0154	0.3735	-0.0243	0.1761	-0.1203	0.1585
		500	2	-0.0045	0.2902	-0.0158	0.1422	-0.0862	0.1187
		1000	2	-0.0082	0.2122	-0.0110	0.1076	-0.0638	0.0868
	1.9	150	2	-0.0177	0.5563	-0.0412	0.2407	-0.1749	0.2279
		300	2	-0.0031	0.3636	-0.0224	0.1751	-0.1162	0.1567
		500	2	0.0013	0.2886	-0.0173	0.1393	-0.0869	0.1198
		1000	2	-0.0086	0.2082	-0.0099	0.1076	-0.0651	0.0883

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.12: Monte Carlo Results of the LPF Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$ and $p = 3$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.0743	0.6031	0.0140	0.2606	-0.1363	0.2094
		300	2	0.0776	0.4008	0.0181	0.1859	-0.0901	0.1425
		500	2	0.0588	0.3096	0.0183	0.1502	-0.0677	0.1100
		1000	2	0.0505	0.2283	0.0166	0.1142	-0.0510	0.0787
	.7	150	2	0.0517	0.5827	-0.0039	0.2523	-0.1508	0.2184
		300	2	0.0387	0.3897	0.0040	0.1794	-0.0988	0.1472
		500	2	0.0460	0.3039	0.0097	0.1455	-0.0752	0.1141
		1000	2	0.0302	0.2194	0.0094	0.1138	-0.0579	0.0848
	.9	150	2	0.0216	0.5902	-0.0231	0.2570	-0.1581	0.2210
		300	2	0.0238	0.3853	-0.0064	0.1783	-0.1070	0.1497
		500	2	0.0267	0.2967	-0.0050	0.1459	-0.0814	0.1180
		1000	2	0.0209	0.2178	0.0012	0.1114	-0.0586	0.0843
	1.1	150	2	0.0122	0.5693	-0.0315	0.2541	-0.1648	0.2250
		300	2	0.0222	0.3791	-0.0134	0.1762	-0.1112	0.1538
		500	2	0.0086	0.2951	-0.0045	0.1410	-0.0823	0.1176
		1000	2	0.0051	0.2135	-0.0037	0.1101	-0.0620	0.0863
	1.3	150	2	-0.0039	0.5606	-0.0442	0.2530	-0.1762	0.2313
		300	2	0.0020	0.3829	-0.0169	0.1750	-0.1163	0.1583
		500	2	-0.0020	0.2919	-0.0108	0.1422	-0.0857	0.1206
		1000	2	0.0003	0.2146	-0.0038	0.1069	-0.0643	0.0874
	1.5	150	3	0.0788	0.6112	0.0190	0.3202	-0.1383	0.2334
		300	3	0.0659	0.6146	0.0218	0.2348	-0.0840	0.1552
		500	3	0.0686	0.4046	0.0186	0.1763	-0.0752	0.1232
		1000	3	0.0434	0.2668	0.0121	0.1335	-0.0508	0.0894
	1.7	150	3	0.0576	0.6089	0.0054	0.3118	-0.1517	0.2412
		300	3	0.0309	0.6091	0.0090	0.2301	-0.0913	0.1597
		500	3	0.0371	0.3972	0.0101	0.1738	-0.0780	0.1230
		1000	3	0.0289	0.2681	0.0098	0.1323	-0.0551	0.0900
	1.9	150	3	0.0190	0.6085	-0.0210	0.3155	-0.1574	0.2444
		300	3	0.0252	0.6080	-0.0006	0.2288	-0.0993	0.1624
		500	3	0.0232	0.3944	-0.0003	0.1687	-0.0849	0.1280
		1000	3	0.0212	0.2559	0.0031	0.1298	-0.0584	0.0916

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.13: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $p = 1$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	-0.0424	0.3874	-0.1255	0.2703	-0.2733	0.3131
		300	1	-0.0199	0.3074	-0.0883	0.1909	-0.2263	0.2502
		500	1	-0.0121	0.2512	-0.0598	0.1492	-0.2032	0.2195
		1000	1	-0.0074	0.2009	-0.0397	0.1146	-0.1678	0.1787
	.7	150	1	-0.0541	0.3901	-0.1238	0.2695	-0.2758	0.3134
		300	1	-0.0206	0.3149	-0.0918	0.1938	-0.2291	0.2524
		500	1	-0.0156	0.2479	-0.0615	0.1502	-0.2014	0.2182
		1000	1	-0.0133	0.1986	-0.0388	0.1152	-0.1680	0.1789
	.9	150	1	-0.0536	0.3804	-0.1208	0.2702	-0.2732	0.3132
		300	1	-0.0331	0.3107	-0.0873	0.1894	-0.2275	0.2520
		500	1	-0.0155	0.2444	-0.0630	0.1522	-0.2031	0.2192
		1000	1	-0.0065	0.1995	-0.0384	0.1137	-0.1689	0.1797
	1.1	150	1	-0.0425	0.3908	-0.1300	0.2692	-0.2736	0.3121
		300	1	-0.0147	0.3079	-0.0877	0.1918	-0.2283	0.2514
		500	1	-0.0119	0.2459	-0.0612	0.1487	-0.2039	0.2210
		1000	1	-0.0053	0.2028	-0.0376	0.1132	-0.1677	0.1786
	1.3	150	1	-0.0397	0.3908	-0.1190	0.2623	-0.2679	0.3066
		300	1	-0.0123	0.3119	-0.0834	0.1901	-0.2260	0.2497
		500	1	-0.0062	0.2472	-0.0574	0.1513	-0.2009	0.2176
		1000	1	-0.0036	0.2018	-0.0344	0.1099	-0.1676	0.1788
	1.5	150	2	-0.0540	0.5106	-0.1281	0.3242	-0.2828	0.3381
		300	2	-0.0214	0.4234	-0.0866	0.2338	-0.2300	0.2660
		500	2	-0.0111	0.3275	-0.0586	0.1870	-0.2045	0.2288
		1000	2	-0.0066	0.2538	-0.0424	0.1391	-0.1684	0.1847
	1.7	150	2	-0.0553	0.4963	-0.1398	0.3300	-0.2803	0.3383
		300	2	-0.0205	0.4144	-0.0922	0.2335	-0.2309	0.2658
		500	2	-0.0142	0.3345	-0.0605	0.1850	-0.2048	0.2296
		1000	2	-0.0064	0.2555	-0.0425	0.1414	-0.1715	0.1873
	1.9	150	2	-0.0501	0.4944	-0.1393	0.3355	-0.2831	0.3397
		300	2	-0.0308	0.4310	-0.0912	0.2363	-0.2298	0.2661
		500	2	-0.0149	0.3264	-0.0632	0.1866	-0.2054	0.2303
		1000	2	-0.0108	0.2600	-0.0383	0.1366	-0.1674	0.1838

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.14: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $p = 1$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	0.0104	0.4992	-0.0925	0.3135	-0.2541	0.3158
		300	2	0.0240	0.4233	-0.0576	0.2264	-0.2143	0.2523
		500	2	0.0343	0.3227	-0.0350	0.1769	-0.1936	0.2197
		1000	2	0.0270	0.2616	-0.0197	0.1340	-0.1647	0.1816
	.7	150	2	-0.0092	0.4942	-0.1013	0.3199	-0.2664	0.3248
		300	2	0.0228	0.4161	-0.0609	0.2236	-0.2197	0.2572
		500	2	0.0103	0.3319	-0.0432	0.1790	-0.1977	0.2230
		1000	2	0.0193	0.2575	-0.0277	0.1363	-0.1639	0.1807
	.9	150	2	-0.0158	0.5068	-0.1145	0.3234	-0.2694	0.3272
		300	2	0.0007	0.4163	-0.0743	0.2312	-0.2237	0.2601
		500	2	0.0021	0.3210	-0.0560	0.1873	-0.1993	0.2236
		1000	2	0.0098	0.2523	-0.0323	0.1350	-0.1662	0.1829
	1.1	150	2	-0.0358	0.4928	-0.1256	0.3275	-0.2696	0.3288
		300	2	-0.0043	0.4237	-0.0819	0.2355	-0.2294	0.2672
		500	2	0.0015	0.3303	-0.0547	0.1810	-0.2033	0.2283
		1000	2	-0.0024	0.2637	-0.0329	0.1369	-0.1669	0.1826
	1.3	150	2	-0.0622	0.5196	-0.1242	0.3281	-0.2747	0.3295
		300	2	-0.0187	0.4131	-0.0762	0.2254	-0.2278	0.2641
		500	2	0.0019	0.3272	-0.0590	0.1880	-0.2059	0.2317
		1000	2	-0.0021	0.2503	-0.0386	0.1387	-0.1682	0.1846
	1.5	150	3	0.0027	0.6139	-0.0922	0.3736	-0.2515	0.3370
		300	3	0.0354	0.4840	-0.0607	0.2582	-0.2119	0.2636
		500	3	0.0372	0.4115	-0.0439	0.2077	-0.1951	0.2291
		1000	3	0.0318	0.3244	-0.0266	0.1566	-0.1633	0.1856
	1.7	150	3	-0.0031	0.6377	-0.1158	0.3792	-0.2594	0.3456
		300	3	0.0104	0.4856	-0.0668	0.2617	-0.2169	0.2660
		500	3	0.0323	0.4080	-0.0481	0.2108	-0.2006	0.2337
		1000	3	0.0267	0.3151	-0.0276	0.1567	-0.1631	0.1863
	1.9	150	3	-0.0352	0.6361	-0.1168	0.3789	-0.2671	0.3475
		300	3	-0.0003	0.4874	-0.0806	0.2661	-0.2256	0.2745
		500	3	-0.0032	0.4081	-0.0503	0.2067	-0.2017	0.2357
		1000	3	0.0049	0.3133	-0.0328	0.1586	-0.1649	0.1872

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.15: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $p = 3$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	-0.0631	0.3633	-0.1425	0.2498	-0.2822	0.3085
		300	1	-0.0222	0.2770	-0.0953	0.1692	-0.2330	0.2482
		500	1	-0.0186	0.2303	-0.0653	0.1315	-0.2084	0.2186
		1000	1	-0.0082	0.1774	-0.0399	0.0949	-0.1711	0.1780
	.7	150	1	-0.0680	0.3532	-0.1497	0.2504	-0.2853	0.3109
		300	1	-0.0295	0.2713	-0.0963	0.1677	-0.2345	0.2502
		500	1	-0.0171	0.2245	-0.0688	0.1330	-0.2094	0.2198
		1000	1	-0.0112	0.1767	-0.0417	0.0948	-0.1696	0.1762
	.9	150	1	-0.0688	0.3532	-0.1426	0.2433	-0.2796	0.3051
		300	1	-0.0305	0.2658	-0.0963	0.1679	-0.2318	0.2475
		500	1	-0.0161	0.2212	-0.0677	0.1316	-0.2082	0.2185
		1000	1	-0.0080	0.1723	-0.0398	0.0938	-0.1706	0.1773
	1.1	150	1	-0.0625	0.3495	-0.1345	0.2382	-0.2802	0.3057
		300	1	-0.0197	0.2650	-0.0896	0.1630	-0.2299	0.2452
		500	1	-0.0056	0.2197	-0.0613	0.1278	-0.2055	0.2162
		1000	1	-0.0003	0.1750	-0.0380	0.0909	-0.1700	0.1768
	1.3	150	1	-0.0192	0.3432	-0.1166	0.2299	-0.2710	0.2979
		300	1	0.0005	0.2701	-0.0786	0.1606	-0.2234	0.2393
		500	1	0.0121	0.2211	-0.0530	0.1242	-0.2019	0.2126
		1000	1	0.0160	0.1726	-0.0315	0.0905	-0.1672	0.1742
	1.5	150	2	-0.0629	0.5596	-0.1559	0.2952	-0.3025	0.3388
		300	2	-0.0287	0.3722	-0.0985	0.1985	-0.2424	0.2634
		500	2	-0.0100	0.2845	-0.0645	0.1536	-0.2065	0.2227
		1000	2	-0.0147	0.2114	-0.0379	0.1136	-0.1731	0.1835
	1.7	150	2	-0.0688	0.5600	-0.1574	0.2919	-0.3039	0.3380
		300	2	-0.0309	0.3733	-0.1001	0.2004	-0.2433	0.2653
		500	2	-0.0197	0.2876	-0.0701	0.1584	-0.2077	0.2238
		1000	2	-0.0129	0.2138	-0.0421	0.1155	-0.1742	0.1842
	1.9	150	2	-0.0719	0.5697	-0.1520	0.2840	-0.2998	0.3346
		300	2	-0.0372	0.3691	-0.0993	0.1994	-0.2409	0.2625
		500	2	-0.0146	0.2860	-0.0682	0.1536	-0.2062	0.2223
		1000	2	-0.0070	0.2108	-0.0408	0.1154	-0.1711	0.1810

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.16: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $p = 3$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	0.0423	0.6023	-0.0940	0.2716	-0.2591	0.3037
		300	2	0.0590	0.3884	-0.0599	0.1944	-0.2191	0.2442
		500	2	0.0597	0.3121	-0.0313	0.1501	-0.1876	0.2067
		1000	2	0.0372	0.2276	-0.0155	0.1127	-0.1606	0.1716
	.7	150	2	0.0132	0.5854	-0.1120	0.2725	-0.2759	0.3164
		300	2	0.0275	0.3861	-0.0692	0.1954	-0.2279	0.2518
		500	2	0.0282	0.3001	-0.0421	0.1527	-0.1918	0.2095
		1000	2	0.0295	0.2186	-0.0221	0.1145	-0.1649	0.1762
	.9	150	2	-0.0054	0.5824	-0.1203	0.2777	-0.2829	0.3229
		300	2	0.0019	0.3822	-0.0751	0.1933	-0.2304	0.2542
		500	2	0.0170	0.3032	-0.0528	0.1553	-0.1967	0.2135
		1000	2	0.0151	0.2184	-0.0250	0.1142	-0.1671	0.1778
	1.1	150	2	-0.0272	0.5816	-0.1412	0.2886	-0.2882	0.3266
		300	2	-0.0053	0.3765	-0.0831	0.1979	-0.2344	0.2578
		500	2	0.0030	0.2954	-0.0573	0.1561	-0.2029	0.2193
		1000	2	-0.0035	0.2149	-0.0356	0.1155	-0.1704	0.1807
	1.3	150	2	-0.0308	0.5784	-0.1461	0.2887	-0.2987	0.3356
		300	2	-0.0260	0.3751	-0.0930	0.2016	-0.2415	0.2628
		500	2	-0.0129	0.2938	-0.0667	0.1594	-0.2066	0.2228
		1000	2	-0.0063	0.2144	-0.0386	0.1149	-0.1725	0.1822
	1.5	150	3	0.0207	0.6073	-0.0929	0.3273	-0.2688	0.3267
		300	3	0.0826	0.6085	-0.0432	0.2363	-0.2136	0.2516
		500	3	0.0523	0.3918	-0.0350	0.1769	-0.1952	0.2175
		1000	3	0.0425	0.2655	-0.0131	0.1341	-0.1595	0.1749
	1.7	150	3	-0.0077	0.6090	-0.1098	0.3368	-0.2779	0.3342
		300	3	0.0588	0.6032	-0.0557	0.2376	-0.2212	0.2570
		500	3	0.0385	0.3906	-0.0407	0.1807	-0.2036	0.2265
		1000	3	0.0288	0.2617	-0.0233	0.1319	-0.1645	0.1791
	1.9	150	3	-0.0313	0.6004	-0.1255	0.3392	-0.2886	0.3418
		300	3	0.0064	0.5973	-0.0694	0.2382	-0.2275	0.2610
		500	3	0.0140	0.3991	-0.0557	0.1796	-0.2066	0.2282
		1000	3	0.0176	0.2579	-0.0274	0.1298	-0.1658	0.1802

^a The p is the pooling order; The τ is the tapering order; The δ is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.17: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $p = 1$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	0.0111	0.3742	0.0312	0.2369	0.1584	0.2174
		300	1	-0.0040	0.3119	0.0208	0.1699	0.1103	0.1529
		500	1	0.0027	0.2487	0.0137	0.1370	0.0861	0.1206
		1000	1	0.0065	0.2022	0.0079	0.1083	0.0619	0.0871
	.7	150	1	0.0038	0.3754	0.0341	0.2384	0.1554	0.2157
		300	1	0.0038	0.3133	0.0218	0.1733	0.1079	0.1528
		500	1	0.0053	0.2489	0.0139	0.1392	0.0887	0.1223
		1000	1	-0.0034	0.2010	0.0083	0.1094	0.0628	0.0876
	.9	150	1	0.0029	0.3822	0.0377	0.2348	0.1527	0.2144
		300	1	0.0002	0.3126	0.0225	0.1714	0.1122	0.1539
		500	1	0.0050	0.2454	0.0123	0.1374	0.0877	0.1209
		1000	1	0.0005	0.1978	0.0098	0.1075	0.0648	0.0895
	1.1	150	1	0.0158	0.3848	0.0362	0.2351	0.1525	0.2149
		300	1	0.0039	0.3061	0.0202	0.1651	0.1114	0.1532
		500	1	0.0044	0.2467	0.0175	0.1418	0.0867	0.1206
		1000	1	0.0053	0.2003	0.0064	0.1080	0.0630	0.0884
	1.3	150	1	0.0178	0.3782	0.0482	0.2391	0.1606	0.2201
		300	1	0.0152	0.3044	0.0245	0.1750	0.1098	0.1544
		500	1	0.0111	0.2484	0.0183	0.1408	0.0898	0.1221
		1000	1	0.0084	0.2034	0.0101	0.1066	0.0640	0.0887
	1.5	150	2	0.0090	0.5011	0.0356	0.3040	0.1618	0.2464
		300	2	-0.0003	0.4162	0.0262	0.2195	0.1079	0.1714
		500	2	0.0014	0.3253	0.0148	0.1780	0.0878	0.1341
		1000	2	-0.0031	0.2553	0.0101	0.1315	0.0638	0.1008
	1.7	150	2	0.0024	0.5041	0.0283	0.3078	0.1598	0.2448
		300	2	0.0042	0.4238	0.0180	0.2161	0.1073	0.1690
		500	2	0.0010	0.3200	0.0116	0.1769	0.0882	0.1357
		1000	2	-0.0006	0.2555	0.0055	0.1302	0.0641	0.0997
	1.9	150	2	0.0028	0.4990	0.0348	0.3041	0.1576	0.2441
		300	2	0.0077	0.4180	0.0167	0.2168	0.1110	0.1715
		500	2	0.0039	0.3206	0.0082	0.1759	0.0868	0.1352
		1000	2	0.0001	0.2530	0.0082	0.1346	0.0614	0.0983

^a The p is the pooling order; The τ is the tapering order;

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.18: Monte Carlo Results of the LPF Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $p = 3$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.1183	0.6126	0.1019	0.2751	0.2141	0.2659
		300	2	0.0848	0.3973	0.0692	0.1943	0.1419	0.1801
		500	2	0.0714	0.3110	0.0471	0.1528	0.1022	0.1336
		1000	2	0.0486	0.2262	0.0322	0.1192	0.0745	0.0966
	.7	150	2	0.0727	0.5924	0.0880	0.2750	0.1967	0.2509
		300	2	0.0493	0.3887	0.0570	0.1910	0.1341	0.1723
		500	2	0.0498	0.3110	0.0389	0.1509	0.0962	0.1275
		1000	2	0.0303	0.2257	0.0226	0.1121	0.0713	0.0942
	.9	150	2	0.0604	0.5725	0.0713	0.2657	0.1908	0.2479
		300	2	0.0475	0.3769	0.0410	0.1857	0.1258	0.1660
		500	2	0.0383	0.2993	0.0320	0.1457	0.0935	0.1260
		1000	2	0.0263	0.2140	0.0176	0.1120	0.0682	0.0907
	1.1	150	2	0.0266	0.5685	0.0504	0.2552	0.1814	0.2378
		300	2	0.0153	0.3792	0.0349	0.1794	0.1191	0.1602
		500	2	0.0228	0.2967	0.0183	0.1462	0.0871	0.1201
		1000	2	0.0111	0.2156	0.0137	0.1105	0.0656	0.0889
	1.3	150	2	0.0166	0.5652	0.0402	0.2542	0.1763	0.2337
		300	2	0.0008	0.3756	0.0218	0.1815	0.1181	0.1586
		500	2	0.0056	0.2938	0.0150	0.1433	0.0883	0.1219
		1000	2	0.0030	0.2131	0.0097	0.1095	0.0649	0.0884
	1.5	150	3	0.0897	0.6100	0.0905	0.3332	0.2069	0.2775
		300	3	0.0811	0.6079	0.0639	0.2409	0.1304	0.1861
		500	3	0.0580	0.4003	0.0440	0.1782	0.1077	0.1446
		1000	3	0.0556	0.2705	0.0313	0.1374	0.0736	0.1037
	1.7	150	3	0.0705	0.6148	0.0880	0.3259	0.2021	0.2735
		300	3	0.0490	0.6071	0.0476	0.2372	0.1204	0.1753
		500	3	0.0490	0.3998	0.0356	0.1751	0.1027	0.1413
		1000	3	0.0361	0.2641	0.0240	0.1333	0.0700	0.1002
	1.9	150	3	0.0558	0.5911	0.0610	0.3226	0.1882	0.2605
		300	3	0.0345	0.5937	0.0432	0.2325	0.1161	0.1751
		500	3	0.0288	0.3902	0.0271	0.1728	0.0975	0.1375
		1000	3	0.0226	0.2547	0.0181	0.1323	0.0678	0.0981

^a The p is the pooling order; The τ is the tapering order; The δ is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-1.5, -.5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.19: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 0) Model

d	n	τ	$d - \delta \in [-.5, .5)$					
			$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	RMSE	bias	RMSE	bias	RMSE
.5	150	1	-0.0271	0.2918	-0.0180	0.1715	-0.0110	0.1088
	300	1	-0.0207	0.2203	-0.0122	0.1237	-0.0061	0.0750
	500	1	-0.0161	0.1849	-0.0088	0.0985	-0.0023	0.0586
	1000	1	-0.0170	0.1447	-0.0079	0.0768	-0.0017	0.0421
.7	150	1	-0.0318	0.2876	-0.0146	0.1688	-0.0100	0.1076
	300	1	-0.0211	0.2211	-0.0096	0.1219	-0.0070	0.0762
	500	1	-0.0178	0.1856	-0.0079	0.1016	-0.0032	0.0593
	1000	1	-0.0142	0.1469	-0.0056	0.0760	-0.0034	0.0428
.9	150	1	-0.0335	0.2824	-0.0145	0.1718	-0.0102	0.1073
	300	1	-0.0215	0.2180	-0.0095	0.1214	-0.0076	0.0762
	500	1	-0.0193	0.1833	-0.0094	0.0983	-0.0050	0.0573
	1000	1	-0.0108	0.1429	-0.0057	0.0753	-0.0018	0.0427
1.1	150	1	-0.0224	0.2932	-0.0148	0.1694	-0.0089	0.1073
	300	1	-0.0148	0.2212	-0.0127	0.1219	-0.0057	0.0766
	500	1	-0.0138	0.1848	-0.0068	0.0987	-0.0060	0.0583
	1000	1	-0.0129	0.1456	-0.0057	0.0749	-0.0037	0.0423
1.3	150	1	-0.0135	0.2852	-0.0080	0.1691	-0.0078	0.1067
	300	1	-0.0106	0.2191	-0.0082	0.1237	-0.0033	0.0750
	500	1	-0.0114	0.1816	-0.0053	0.0997	-0.0022	0.0573
	1000	1	-0.0065	0.1435	-0.0050	0.0742	-0.0014	0.0425
1.5	150	2	-0.0338	0.3710	-0.0210	0.2084	-0.0127	0.1268
	300	2	-0.0231	0.2714	-0.0096	0.1464	-0.0065	0.0883
	500	2	-0.0176	0.2256	-0.0089	0.1190	-0.0053	0.0681
	1000	2	-0.0159	0.1724	-0.0054	0.0878	-0.0036	0.0491
1.7	150	2	-0.0280	0.3630	-0.0254	0.2116	-0.0124	0.1292
	300	2	-0.0297	0.2759	-0.0135	0.1476	-0.0084	0.0890
	500	2	-0.0270	0.2254	-0.0103	0.1183	-0.0057	0.0682
	1000	2	-0.0178	0.1740	-0.0085	0.0878	-0.0029	0.0495
1.9	150	2	-0.0265	0.3616	-0.0166	0.2034	-0.0118	0.1267
	300	2	-0.0279	0.2666	-0.0142	0.1466	-0.0092	0.0888
	500	2	-0.0233	0.2274	-0.0141	0.1171	-0.0034	0.0671
	1000	2	-0.0124	0.1751	-0.0067	0.0887	-0.0037	0.0494

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.20: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	0.0292	0.2975	0.1350	0.2194	0.3050	0.3244
		300	1	0.0037	0.2159	0.0886	0.1531	0.2518	0.2642
		500	1	0.0050	0.1822	0.0624	0.1186	0.2221	0.2306
		1000	1	-0.0083	0.1454	0.0379	0.0833	0.1819	0.1874
	.7	150	1	0.0205	0.2879	0.1355	0.2203	0.3011	0.3218
		300	1	-0.0005	0.2168	0.0868	0.1516	0.2513	0.2641
		500	1	-0.0027	0.1873	0.0587	0.1162	0.2216	0.2299
		1000	1	-0.0083	0.1437	0.0359	0.0836	0.1825	0.1877
	.9	150	1	0.0279	0.2954	0.1332	0.2190	0.3012	0.3222
		300	1	0.0073	0.2257	0.0846	0.1515	0.2514	0.2634
		500	1	0.0007	0.1816	0.0619	0.1167	0.2201	0.2284
		1000	1	-0.0059	0.1432	0.0377	0.0853	0.1828	0.1881
	1.1	150	1	0.0359	0.2922	0.1421	0.2242	0.3047	0.3248
		300	1	0.0029	0.2158	0.0860	0.1504	0.2499	0.2622
		500	1	-0.0024	0.1804	0.0598	0.1162	0.2207	0.2290
		1000	1	-0.0035	0.1458	0.0367	0.0835	0.1822	0.1877
	1.3	150	1	0.0391	0.2914	0.1437	0.2237	0.3073	0.3273
		300	1	0.0200	0.2182	0.0939	0.1554	0.2523	0.2641
		500	1	0.0076	0.1827	0.0617	0.1167	0.2208	0.2294
		1000	1	-0.0035	0.1445	0.0389	0.0846	0.1838	0.1888
	1.5	150	2	0.0475	0.3605	0.1526	0.2613	0.3200	0.3469
		300	2	0.0142	0.2716	0.0939	0.1762	0.2630	0.2789
		500	2	-0.0020	0.2270	0.0620	0.1337	0.2271	0.2383
		1000	2	-0.0079	0.1760	0.0394	0.0983	0.1866	0.1935
	1.7	150	2	0.0372	0.3615	0.1522	0.2593	0.3181	0.3452
		300	2	0.0054	0.2732	0.0916	0.1766	0.2631	0.2785
		500	2	-0.0023	0.2271	0.0647	0.1360	0.2285	0.2394
		1000	2	-0.0074	0.1709	0.0391	0.0967	0.1865	0.1935
	1.9	150	2	0.0505	0.3619	0.1505	0.2557	0.3222	0.3501
		300	2	0.0071	0.2746	0.0972	0.1778	0.2615	0.2776
		500	2	-0.0018	0.2252	0.0668	0.1384	0.2285	0.2396
		1000	2	-0.0080	0.1747	0.0376	0.0965	0.1876	0.1946

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.21: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	0.0939	0.3705	0.1795	0.2766	0.3354	0.3622
		300	2	0.0448	0.2699	0.1102	0.1857	0.2683	0.2837
		500	2	0.0283	0.2195	0.0746	0.1412	0.2322	0.2429
		1000	2	0.0179	0.1761	0.0452	0.0997	0.1897	0.1962
	.7	150	2	0.0752	0.3719	0.1694	0.2685	0.3288	0.3541
		300	2	0.0344	0.2739	0.1079	0.1830	0.2710	0.2866
		500	2	0.0213	0.2241	0.0732	0.1383	0.2315	0.2425
		1000	2	0.0118	0.1716	0.0460	0.0993	0.1898	0.1964
	.9	150	2	0.0649	0.3673	0.1704	0.2708	0.3265	0.3522
		300	2	0.0224	0.2726	0.1034	0.1810	0.2648	0.2799
		500	2	0.0079	0.2264	0.0695	0.1384	0.2296	0.2402
		1000	2	0.0030	0.1759	0.0443	0.0989	0.1894	0.1966
	1.1	150	2	0.0506	0.3655	0.1539	0.2617	0.3203	0.3466
		300	2	0.0174	0.2699	0.0975	0.1768	0.2655	0.2812
		500	2	0.0049	0.2254	0.0648	0.1344	0.2301	0.2409
		1000	2	-0.0038	0.1715	0.0415	0.0979	0.1873	0.1940
	1.3	150	2	0.0405	0.3670	0.1493	0.2582	0.3187	0.3466
		300	2	0.0115	0.2722	0.0926	0.1754	0.2666	0.2824
		500	2	-0.0018	0.2223	0.0677	0.1366	0.2299	0.2409
		1000	2	-0.0078	0.1762	0.0419	0.0983	0.1875	0.1944
	1.5	150	3	0.0995	0.4320	0.1920	0.3128	0.3514	0.3825
		300	3	0.0559	0.3234	0.1163	0.2061	0.2792	0.2977
		500	3	0.0289	0.2646	0.0821	0.1589	0.2399	0.2531
		1000	3	0.0171	0.2012	0.0520	0.1102	0.1940	0.2021
	1.7	150	3	0.0904	0.4405	0.1850	0.3072	0.3436	0.3768
		300	3	0.0468	0.3189	0.1109	0.2056	0.2746	0.2936
		500	3	0.0127	0.2575	0.0797	0.1567	0.2391	0.2522
		1000	3	0.0090	0.2051	0.0484	0.1103	0.1944	0.2028
	1.9	150	3	0.0771	0.4390	0.1808	0.3022	0.3431	0.3749
		300	3	0.0235	0.3176	0.1051	0.2021	0.2730	0.2923
		500	3	0.0155	0.2553	0.0733	0.1560	0.2379	0.2508
		1000	3	0.0010	0.1995	0.0467	0.1096	0.1931	0.2016

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.22: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$

ϕ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	-0.0460	0.2940	-0.0619	0.1786	-0.1898	0.2184
		300	1	-0.0247	0.2245	-0.0401	0.1300	-0.1242	0.1450
		500	1	-0.0198	0.1833	-0.0224	0.1018	-0.0967	0.1136
		1000	1	-0.0159	0.1445	-0.0150	0.0764	-0.0680	0.0803
	.7	150	1	-0.0415	0.2912	-0.0619	0.1814	-0.1873	0.2164
		300	1	-0.0261	0.2232	-0.0356	0.1279	-0.1242	0.1456
		500	1	-0.0208	0.1847	-0.0281	0.1026	-0.0957	0.1122
		1000	1	-0.0195	0.1463	-0.0153	0.0759	-0.0677	0.0803
	.9	150	1	-0.0440	0.2929	-0.0631	0.1790	-0.1869	0.2150
		300	1	-0.0293	0.2230	-0.0354	0.1286	-0.1229	0.1440
		500	1	-0.0243	0.1853	-0.0259	0.1028	-0.0958	0.1118
		1000	1	-0.0126	0.1450	-0.0147	0.0751	-0.0677	0.0801
	1.1	150	1	-0.0303	0.2943	-0.0601	0.1831	-0.1873	0.2153
		300	1	-0.0232	0.2203	-0.0322	0.1266	-0.1233	0.1440
		500	1	-0.0190	0.1829	-0.0242	0.1014	-0.0953	0.1114
		1000	1	-0.0136	0.1447	-0.0129	0.0738	-0.0670	0.0792
	1.3	150	1	-0.0242	0.2862	-0.0527	0.1778	-0.1832	0.2106
		300	1	-0.0151	0.2200	-0.0340	0.1265	-0.1212	0.1425
		500	1	-0.0086	0.1811	-0.0196	0.0998	-0.0946	0.1113
		1000	1	-0.0114	0.1444	-0.0087	0.0752	-0.0678	0.0798
	1.5	150	2	-0.0402	0.3573	-0.0673	0.2190	-0.2059	0.2432
		300	2	-0.0249	0.2715	-0.0383	0.1520	-0.1328	0.1591
		500	2	-0.0235	0.2278	-0.0272	0.1202	-0.1021	0.1229
		1000	2	-0.0175	0.1765	-0.0160	0.0888	-0.0708	0.0870
	1.7	150	2	-0.0507	0.3608	-0.0740	0.2187	-0.2063	0.2420
		300	2	-0.0338	0.2743	-0.0392	0.1514	-0.1358	0.1626
		500	2	-0.0239	0.2279	-0.0285	0.1213	-0.1025	0.1230
		1000	2	-0.0170	0.1714	-0.0165	0.0910	-0.0711	0.0863
	1.9	150	2	-0.0377	0.3581	-0.0691	0.2141	-0.2060	0.2416
		300	2	-0.0321	0.2754	-0.0380	0.1525	-0.1358	0.1626
		500	2	-0.0233	0.2260	-0.0272	0.1221	-0.1012	0.1223
		1000	2	-0.0176	0.1752	-0.0155	0.0886	-0.0707	0.0860

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.23: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$

ϕ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.0077	0.3609	-0.0421	0.2108	-0.1891	0.2278
		300	2	0.0003	0.2673	-0.0197	0.1481	-0.1235	0.1520
		500	2	0.0069	0.2235	-0.0133	0.1176	-0.0924	0.1148
		1000	2	0.0051	0.1744	-0.0050	0.0865	-0.0647	0.0817
	.7	150	2	-0.0067	0.3566	-0.0469	0.2126	-0.1960	0.2344
		300	2	-0.0074	0.2745	-0.0248	0.1504	-0.1267	0.1551
		500	2	-0.0038	0.2240	-0.0178	0.1174	-0.0965	0.1179
		1000	2	0.0001	0.1740	-0.0100	0.0880	-0.0675	0.0836
	.9	150	2	-0.0192	0.3580	-0.0543	0.2123	-0.1944	0.2317
		300	2	-0.0134	0.2644	-0.0300	0.1494	-0.1304	0.1581
		500	2	-0.0117	0.2207	-0.0188	0.1184	-0.0971	0.1186
		1000	2	-0.0048	0.1762	-0.0104	0.0878	-0.0698	0.0858
	1.1	150	2	-0.0349	0.3604	-0.0565	0.2131	-0.2043	0.2408
		300	2	-0.0232	0.2754	-0.0337	0.1506	-0.1306	0.1577
		500	2	-0.0136	0.2251	-0.0190	0.1170	-0.1003	0.1217
		1000	2	-0.0112	0.1769	-0.0135	0.0878	-0.0699	0.0853
	1.3	150	2	-0.0394	0.3719	-0.0654	0.2149	-0.2047	0.2402
		300	2	-0.0275	0.2722	-0.0362	0.1523	-0.1312	0.1583
		500	2	-0.0169	0.2253	-0.0254	0.1187	-0.1015	0.1228
		1000	2	-0.0167	0.1758	-0.0151	0.0887	-0.0703	0.0859
	1.5	150	3	0.0026	0.4197	-0.0447	0.2390	-0.2057	0.2506
		300	3	0.0094	0.3152	-0.0241	0.1705	-0.1317	0.1642
		500	3	0.0089	0.2614	-0.0145	0.1339	-0.0990	0.1243
		1000	3	0.0049	0.2011	-0.0077	0.0985	-0.0669	0.0864
	1.7	150	3	-0.0204	0.4144	-0.0538	0.2443	-0.2105	0.2552
		300	3	-0.0113	0.3114	-0.0264	0.1710	-0.1350	0.1682
		500	3	-0.0002	0.2570	-0.0170	0.1362	-0.1017	0.1266
		1000	3	-0.0004	0.2028	-0.0079	0.0976	-0.0703	0.0895
	1.9	150	3	-0.0196	0.4193	-0.0644	0.2398	-0.2180	0.2617
		300	3	-0.0145	0.3194	-0.0359	0.1739	-0.1366	0.1691
		500	3	-0.0150	0.2633	-0.0225	0.1336	-0.1023	0.1273
		1000	3	-0.0120	0.2005	-0.0118	0.0980	-0.0718	0.0898

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.24: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	1	-0.0987	0.3040	-0.1669	0.2378	-0.3018	0.3207
		300	1	-0.0494	0.2278	-0.1082	0.1650	-0.2389	0.2507
		500	1	-0.0290	0.1830	-0.0767	0.1250	-0.2076	0.2163
		1000	1	-0.0218	0.1483	-0.0487	0.0886	-0.1708	0.1763
	.7	150	1	-0.1011	0.3037	-0.1676	0.2388	-0.3010	0.3199
		300	1	-0.0470	0.2292	-0.1097	0.1643	-0.2416	0.2536
		500	1	-0.0376	0.1841	-0.0760	0.1249	-0.2085	0.2171
		1000	1	-0.0210	0.1460	-0.0485	0.0881	-0.1696	0.1751
	.9	150	1	-0.0871	0.3040	-0.1674	0.2384	-0.2998	0.3183
		300	1	-0.0506	0.2259	-0.1050	0.1617	-0.2413	0.2529
		500	1	-0.0329	0.1807	-0.0762	0.1248	-0.2080	0.2164
		1000	1	-0.0209	0.1453	-0.0489	0.0892	-0.1706	0.1762
	1.1	150	1	-0.0810	0.2944	-0.1592	0.2334	-0.2980	0.3171
		300	1	-0.0508	0.2262	-0.1057	0.1615	-0.2419	0.2539
		500	1	-0.0290	0.1846	-0.0746	0.1227	-0.2071	0.2153
		1000	1	-0.0180	0.1444	-0.0462	0.0867	-0.1708	0.1763
	1.3	150	1	-0.0693	0.2874	-0.1596	0.2295	-0.2941	0.3131
		300	1	-0.0395	0.2222	-0.1017	0.1582	-0.2386	0.2507
		500	1	-0.0265	0.1800	-0.0706	0.1210	-0.2060	0.2141
		1000	1	-0.0105	0.1436	-0.0466	0.0882	-0.1683	0.1739
	1.5	150	2	-0.1074	0.3665	-0.1807	0.2732	-0.3251	0.3487
		300	2	-0.0561	0.2784	-0.1154	0.1881	-0.2584	0.2737
		500	2	-0.0299	0.2246	-0.0847	0.1455	-0.2197	0.2305
		1000	2	-0.0227	0.1755	-0.0517	0.1008	-0.1781	0.1850
	1.7	150	2	-0.1077	0.3730	-0.1872	0.2755	-0.3228	0.3468
		300	2	-0.0489	0.2752	-0.1190	0.1892	-0.2560	0.2708
		500	2	-0.0384	0.2297	-0.0834	0.1437	-0.2184	0.2293
		1000	2	-0.0223	0.1760	-0.0544	0.1041	-0.1781	0.1850
	1.9	150	2	-0.0971	0.3715	-0.1842	0.2741	-0.3258	0.3496
		300	2	-0.0514	0.2796	-0.1195	0.1886	-0.2580	0.2729
		500	2	-0.0344	0.2263	-0.0842	0.1446	-0.2217	0.2320
		1000	2	-0.0255	0.1774	-0.0525	0.1020	-0.1781	0.1848

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.25: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	2	-0.0490	0.3561	-0.1569	0.2579	-0.3049	0.3306
		300	2	-0.0214	0.2715	-0.0972	0.1732	-0.2430	0.2588
		500	2	-0.0063	0.2231	-0.0680	0.1352	-0.2102	0.2213
		1000	2	-0.0001	0.1741	-0.0433	0.0972	-0.1715	0.1789
	.7	150	2	-0.0790	0.3657	-0.1684	0.2662	-0.3122	0.3375
		300	2	-0.0279	0.2721	-0.1068	0.1798	-0.2489	0.2645
		500	2	-0.0197	0.2200	-0.0742	0.1381	-0.2137	0.2246
		1000	2	-0.0089	0.1731	-0.0455	0.0983	-0.1724	0.1797
	.9	150	2	-0.0837	0.3673	-0.1691	0.2648	-0.3144	0.3387
		300	2	-0.0466	0.2714	-0.1135	0.1863	-0.2497	0.2646
		500	2	-0.0260	0.2223	-0.0766	0.1407	-0.2159	0.2270
		1000	2	-0.0189	0.1745	-0.0465	0.0995	-0.1738	0.1810
	1.1	150	2	-0.0822	0.3662	-0.1781	0.2718	-0.3204	0.3444
		300	2	-0.0473	0.2732	-0.1164	0.1861	-0.2547	0.2705
		500	2	-0.0345	0.2288	-0.0781	0.1408	-0.2169	0.2276
		1000	2	-0.0160	0.1727	-0.0507	0.1005	-0.1760	0.1830
	1.3	150	2	-0.0956	0.3719	-0.1826	0.2778	-0.3235	0.3481
		300	2	-0.0512	0.2743	-0.1178	0.1888	-0.2557	0.2712
		500	2	-0.0322	0.2255	-0.0827	0.1436	-0.2169	0.2275
		1000	2	-0.0223	0.1755	-0.0507	0.1013	-0.1771	0.1842
	1.5	150	3	-0.0555	0.4233	-0.1766	0.2955	-0.3250	0.3546
		300	3	-0.0295	0.3133	-0.1140	0.2019	-0.2610	0.2796
		500	3	-0.0116	0.2588	-0.0748	0.1509	-0.2200	0.2330
		1000	3	-0.0041	0.1997	-0.0443	0.1082	-0.1780	0.1862
	1.7	150	3	-0.0933	0.4409	-0.1849	0.3017	-0.3322	0.3627
		300	3	-0.0394	0.3182	-0.1190	0.2038	-0.2636	0.2823
		500	3	-0.0241	0.2604	-0.0784	0.1537	-0.2234	0.2364
		1000	3	-0.0078	0.2002	-0.0488	0.1114	-0.1790	0.1870
	1.9	150	3	-0.0881	0.4291	-0.1941	0.3062	-0.3353	0.3654
		300	3	-0.0465	0.3230	-0.1152	0.2017	-0.2638	0.2818
		500	3	-0.0288	0.2607	-0.0827	0.1564	-0.2240	0.2371
		1000	3	-0.0189	0.1994	-0.0531	0.1117	-0.1809	0.1894

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.26: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$

θ	d	n	τ	$d - \delta \in [-.5, .5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	1	-0.0148	0.2876	0.0271	0.1718	0.1687	0.2025
		300	1	-0.0196	0.2230	0.0103	0.1225	0.1143	0.1380
		500	1	-0.0139	0.1866	0.0043	0.0988	0.0888	0.1069
		1000	1	-0.0144	0.1474	0.0052	0.0740	0.0635	0.0770
	.7	150	1	-0.0178	0.2839	0.0267	0.1749	0.1661	0.1991
		300	1	-0.0213	0.2189	0.0094	0.1240	0.1138	0.1372
		500	1	-0.0183	0.1849	0.0046	0.0998	0.0888	0.1069
		1000	1	-0.0096	0.1431	0.0023	0.0728	0.0636	0.0770
	.9	150	1	-0.0119	0.2931	0.0254	0.1705	0.1677	0.2010
		300	1	-0.0185	0.2240	0.0145	0.1221	0.1139	0.1379
		500	1	-0.0154	0.1822	0.0059	0.0998	0.0894	0.1079
		1000	1	-0.0155	0.1448	0.0046	0.0747	0.0638	0.0772
	1.1	150	1	-0.0123	0.2919	0.0243	0.1706	0.1706	0.2031
		300	1	-0.0151	0.2237	0.0176	0.1247	0.1136	0.1377
		500	1	-0.0133	0.1833	0.0069	0.1007	0.0901	0.1076
		1000	1	-0.0127	0.1438	0.0047	0.0744	0.0641	0.0771
	1.3	150	1	0.0019	0.2877	0.0324	0.1712	0.1732	0.2043
		300	1	-0.0139	0.2112	0.0177	0.1243	0.1157	0.1387
		500	1	-0.0084	0.1816	0.0094	0.1000	0.0899	0.1077
		1000	1	-0.0037	0.1422	0.0065	0.0749	0.0654	0.0786
	1.5	150	2	-0.0157	0.3590	0.0345	0.2113	0.1844	0.2276
		300	2	-0.0206	0.2741	0.0134	0.1474	0.1220	0.1519
		500	2	-0.0109	0.2270	0.0059	0.1171	0.0923	0.1146
		1000	2	-0.0157	0.1774	0.0025	0.0864	0.0660	0.0824
	1.7	150	2	-0.0213	0.3653	0.0278	0.2060	0.1833	0.2274
		300	2	-0.0220	0.2715	0.0136	0.1495	0.1199	0.1507
		500	2	-0.0164	0.2247	0.0085	0.1218	0.0930	0.1165
		1000	2	-0.0151	0.1756	0.0022	0.0884	0.0650	0.0815
	1.9	150	2	-0.0188	0.3556	0.0320	0.2080	0.1807	0.2235
		300	2	-0.0165	0.2717	0.0128	0.1498	0.1217	0.1529
		500	2	-0.0187	0.2253	0.0055	0.1191	0.0925	0.1158
		1000	2	-0.0149	0.1758	0.0043	0.0889	0.0669	0.0832

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.27: Monte Carlo Results of the Tapered Local Whittle Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$

θ	d	n	τ	$d - \delta \in [-1.5, -.5)$					
				$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
				bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	2	0.0397	0.3546	0.0584	0.2162	0.1986	0.2380
		300	2	0.0142	0.2708	0.0297	0.1479	0.1297	0.1591
		500	2	0.0134	0.2228	0.0211	0.1179	0.1002	0.1219
		1000	2	0.0095	0.1709	0.0123	0.0886	0.0707	0.0867
	.7	150	2	0.0188	0.3568	0.0510	0.2148	0.1913	0.2322
		300	2	0.0078	0.2732	0.0300	0.1501	0.1265	0.1554
		500	2	0.0072	0.2246	0.0186	0.1206	0.0965	0.1189
		1000	2	0.0055	0.1732	0.0081	0.0880	0.0682	0.0841
	.9	150	2	0.0059	0.3661	0.0436	0.2120	0.1894	0.2306
		300	2	-0.0034	0.2706	0.0210	0.1482	0.1274	0.1560
		500	2	-0.0025	0.2227	0.0114	0.1182	0.0949	0.1178
		1000	2	-0.0029	0.1779	0.0059	0.0886	0.0679	0.0847
	1.1	150	2	-0.0010	0.3565	0.0332	0.2121	0.1865	0.2284
		300	2	-0.0048	0.2692	0.0188	0.1471	0.1227	0.1531
		500	2	-0.0158	0.2310	0.0089	0.1187	0.0956	0.1183
		1000	2	-0.0043	0.1748	0.0048	0.0875	0.0671	0.0834
	1.3	150	2	-0.0147	0.3602	0.0295	0.2080	0.1858	0.2280
		300	2	-0.0157	0.2724	0.0159	0.1495	0.1216	0.1515
		500	2	-0.0144	0.2244	0.0097	0.1195	0.0942	0.1172
		1000	2	-0.0098	0.1737	0.0017	0.0868	0.0672	0.0839
	1.5	150	3	0.0326	0.4161	0.0606	0.2531	0.2116	0.2589
		300	3	0.0223	0.3176	0.0276	0.1725	0.1343	0.1677
		500	3	0.0131	0.2561	0.0187	0.1356	0.1026	0.1286
		1000	3	0.0080	0.1992	0.0127	0.0988	0.0727	0.0914
	1.7	150	3	0.0275	0.4202	0.0456	0.2456	0.2044	0.2527
		300	3	-0.0011	0.3156	0.0281	0.1726	0.1310	0.1665
		500	3	0.0043	0.2643	0.0157	0.1342	0.1010	0.1270
		1000	3	0.0017	0.2031	0.0097	0.0997	0.0700	0.0886
	1.9	150	3	0.0205	0.4300	0.0457	0.2484	0.2058	0.2563
		300	3	-0.0020	0.3205	0.0190	0.1720	0.1297	0.1648
		500	3	-0.0022	0.2575	0.0127	0.1379	0.1006	0.1281
		1000	3	-0.0032	0.1987	0.0084	0.0988	0.0699	0.0888

^a The τ is the tapering order. The *delta* is the difference order.

^b The δ , the difference order, in the expression $d - \delta \in [-.5, .5)/d - \delta \in [-1.5, -.5)$ is such that for the true value d , select a δ satisfying $-.5 \leq d - \delta < .5/-1.5 \leq d - \delta < -.5$;

Table 6.28: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 0) Model with $m = \lfloor n^{.5} \rfloor$

d	n	fextLWF		fextLPWF		ModLWF		feLWF		dtr-feLWF	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	0.0196	0.2188	-0.1419	0.4167	0.0213	0.2183	0.0097	0.2073	-0.0877	0.2560
	300	0.0099	0.1748	-0.0887	0.3012	0.0254	0.1745	0.0055	0.1675	-0.0661	0.2027
	500	0.0155	0.1496	-0.0612	0.2395	0.0264	0.1526	0.0084	0.1442	-0.0473	0.1669
	1000	0.0115	0.1185	-0.0481	0.1809	0.0297	0.1234	0.0080	0.1184	-0.0318	0.1320
.7	150	-0.0078	0.2064	-0.1389	0.4265	-0.0134	0.2181	0.0027	0.1963	-0.0759	0.2611
	300	-0.0034	0.1627	-0.0988	0.3148	-0.0108	0.1703	0.0003	0.1544	-0.0603	0.2005
	500	-0.0086	0.1430	-0.0728	0.2554	-0.0019	0.1408	0.0005	0.1236	-0.0445	0.1604
	1000	-0.0018	0.1154	-0.0448	0.1961	-0.0005	0.1125	-0.0022	0.1025	-0.0334	0.1251
.9	150	-0.0276	0.2147	-0.1417	0.4336	-0.0288	0.2216	-0.0138	0.2034	-0.0652	0.2612
	300	-0.0192	0.1688	-0.0969	0.3329	-0.0227	0.1726	-0.0133	0.1608	-0.0463	0.1954
	500	-0.0164	0.1453	-0.0620	0.2664	-0.0153	0.1437	-0.0148	0.1411	-0.0354	0.1598
	1000	-0.0097	0.1147	-0.0378	0.2020	-0.0100	0.1105	-0.0104	0.1127	-0.0193	0.1258
1.1	150	-0.0288	0.2259	-0.1358	0.4490	-0.0343	0.2236	-0.0193	0.2177	-0.0470	0.2550
	300	-0.0263	0.1752	-0.0823	0.3341	-0.0233	0.1671	-0.0211	0.1742	-0.0224	0.1814
	500	-0.0194	0.1425	-0.0675	0.2733	-0.0129	0.1445	-0.0175	0.1459	-0.0177	0.1466
	1000	-0.0063	0.1111	-0.0337	0.2010	-0.0120	0.1118	-0.0112	0.1131	-0.0135	0.1146
1.3	150	-0.0199	0.2309	-0.1066	0.4408	-0.0303	0.2184	-0.0293	0.2335	-0.0186	0.2292
	300	-0.0158	0.1792	-0.0616	0.3230	-0.0180	0.1725	-0.0203	0.1782	-0.0157	0.1709
	500	-0.0057	0.1498	-0.0381	0.2620	-0.0129	0.1410	-0.0152	0.1481	-0.0105	0.1420
	1000	-0.0052	0.1187	-0.0356	0.1958	-0.0086	0.1146	-0.0106	0.1192	-0.0074	0.1121
1.5	150	-0.0099	0.2211	-0.0889	0.4241	-0.0227	0.2215	-0.0420	0.2511	-0.0093	0.2186
	300	0.0003	0.1755	-0.0563	0.3104	-0.0047	0.1695	-0.0317	0.2036	-0.0060	0.1647
	500	0.0055	0.1488	-0.0408	0.2503	-0.0004	0.1451	-0.0265	0.1687	-0.0027	0.1388
	1000	0.0123	0.1202	-0.0437	0.1864	0.0027	0.1158	-0.0115	0.1294	-0.0020	0.1101
1.7	150	-0.0238	0.2119	-0.0820	0.4154	-0.0114	0.2162	-0.0293	0.2348	0.0021	0.2091
	300	-0.0121	0.1669	-0.0496	0.2998	0.0032	0.1727	-0.0245	0.1827	0.0019	0.1661
	500	-0.0064	0.1403	-0.0367	0.2397	0.0072	0.1432	-0.0164	0.1471	0.0054	0.1386
	1000	-0.0047	0.1118	-0.0781	0.2017	0.0152	0.1159	-0.0100	0.1162	0.0042	0.1107
1.9	150	-0.0330	0.2143	-0.0757	0.4113	-0.0333	0.2104	-0.0274	0.2315	0.0085	0.2045
	300	-0.0171	0.1676	-0.0712	0.3087	-0.0104	0.1607	-0.0255	0.1750	0.0050	0.1587
	500	-0.0142	0.1416	-0.0551	0.2518	0.0006	0.1356	-0.0194	0.1431	0.0087	0.1341
	1000	-0.0104	0.1142	-0.0574	0.2168	0.0065	0.1091	-0.0116	0.1109	0.0065	0.1103

^a The τ is the tapering order. The *delta* is the difference order.

Table 6.29: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 0) Model with $m = \lfloor n^{.65} \rfloor$

d	n	fextLWF		fextLPWF		ModLWF		feLWF		dtr-feLWF	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.0008	0.1318	-0.0477	0.2229	0.0115	0.1342	0.0194	0.1367	-0.0290	0.1516
	300	0.0005	0.0978	-0.0230	0.1502	0.0137	0.1015	0.0168	0.1024	-0.0138	0.1118
	500	0.0012	0.0793	-0.0190	0.1249	0.0138	0.0840	0.0150	0.0869	-0.0126	0.0898
	1000	0.0018	0.0621	-0.0057	0.0895	0.0133	0.0660	0.0129	0.0691	-0.0059	0.0680
.7	150	-0.0168	0.1318	-0.0566	0.2333	-0.0114	0.1304	0.0106	0.1204	-0.0275	0.1471
	300	-0.0096	0.0989	-0.0375	0.1693	-0.0067	0.0974	0.0029	0.0897	-0.0218	0.1024
	500	-0.0043	0.0798	-0.0270	0.1454	-0.0039	0.0789	0.0012	0.0719	-0.0142	0.0816
	1000	-0.0034	0.0612	-0.0154	0.1057	-0.0002	0.0604	0.0001	0.0558	-0.0073	0.0625
.9	150	-0.0253	0.1334	-0.0649	0.2496	-0.0218	0.1291	-0.0000	0.1322	-0.0163	0.1460
	300	-0.0150	0.0956	-0.0303	0.1712	-0.0135	0.0947	-0.0023	0.0986	-0.0101	0.1055
	500	-0.0131	0.0781	-0.0257	0.1436	-0.0126	0.0796	-0.0030	0.0798	-0.0061	0.0844
	1000	-0.0080	0.0588	-0.0148	0.0973	-0.0058	0.0598	-0.0024	0.0603	-0.0048	0.0604
1.1	150	-0.0272	0.1338	-0.0474	0.2625	-0.0306	0.1344	-0.0027	0.1345	0.0003	0.1323
	300	-0.0185	0.0984	-0.0279	0.1715	-0.0193	0.0953	-0.0015	0.0965	0.0013	0.0944
	500	-0.0147	0.0802	-0.0222	0.1368	-0.0134	0.0799	-0.0036	0.0772	-0.0043	0.0776
	1000	-0.0077	0.0597	-0.0126	0.0986	-0.0085	0.0597	-0.0006	0.0586	-0.0021	0.0593
1.3	150	-0.0275	0.1364	-0.0353	0.2626	-0.0286	0.1328	0.0027	0.1379	0.0059	0.1317
	300	-0.0141	0.0998	-0.0115	0.1732	-0.0161	0.0967	0.0003	0.0980	0.0026	0.0966
	500	-0.0093	0.0797	-0.0042	0.1418	-0.0100	0.0783	0.0011	0.0779	0.0002	0.0772
	1000	-0.0036	0.0608	-0.0018	0.1038	-0.0053	0.0596	0.0013	0.0591	0.0020	0.0594
1.5	150	-0.0164	0.1301	-0.0343	0.2652	-0.0280	0.1345	-0.0043	0.1504	0.0104	0.1272
	300	-0.0062	0.1007	-0.0115	0.1606	-0.0139	0.0985	-0.0013	0.1063	0.0070	0.0933
	500	0.0035	0.0818	-0.0037	0.1273	-0.0079	0.0802	0.0030	0.0848	0.0054	0.0769
	1000	0.0088	0.0634	0.0021	0.0945	-0.0007	0.0602	0.0043	0.0622	0.0055	0.0591
1.7	150	-0.0392	0.1313	-0.0447	0.2575	-0.0383	0.1380	-0.0058	0.1366	0.0162	0.1290
	300	-0.0188	0.0959	-0.0170	0.1584	-0.0118	0.0976	0.0015	0.0975	0.0117	0.0968
	500	-0.0111	0.0776	-0.0132	0.1331	0.0012	0.0794	-0.0005	0.0794	0.0099	0.0787
	1000	-0.0057	0.0596	-0.0079	0.0977	0.0071	0.0634	0.0014	0.0598	0.0080	0.0606
1.9	150	-0.0510	0.1387	-0.0604	0.2767	-0.0746	0.1467	-0.0002	0.1325	0.0167	0.1262
	300	-0.0280	0.1006	-0.0290	0.1643	-0.0385	0.0991	-0.0028	0.0968	0.0133	0.0925
	500	-0.0188	0.0797	-0.0176	0.1317	-0.0205	0.0779	-0.0017	0.0778	0.0122	0.0762
	1000	-0.0113	0.0606	-0.0140	0.0956	-0.0031	0.0570	-0.0009	0.0589	0.0093	0.0594

^a The data generation is $(1 - L)^d X_t = \varepsilon_t, \varepsilon_t \sim N(0, 1)$;

Table 6.30: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 0) Model with $m = \lfloor n^{.8} \rfloor$

d	n	fextLWF		fextLPWF		ModLWF		feLWF		dtr-feLWF	
		bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.5	150	-0.0224	0.0839	-0.0064	0.1275	-0.0175	0.0839	0.0785	0.1162	0.0571	0.1098
	300	-0.0158	0.0599	-0.0029	0.0889	-0.0090	0.0610	0.0545	0.0846	0.0371	0.0772
	500	-0.0133	0.0471	0.0046	0.0698	-0.0063	0.0486	0.0414	0.0672	0.0312	0.0618
	1000	-0.0098	0.0353	0.0037	0.0507	-0.0034	0.0368	0.0291	0.0493	0.0210	0.0447
.7	150	-0.0419	0.0903	-0.0139	0.1323	-0.0445	0.0912	0.0634	0.1021	0.0540	0.1039
	300	-0.0277	0.0641	-0.0077	0.0969	-0.0290	0.0638	0.0390	0.0701	0.0330	0.0712
	500	-0.0223	0.0503	-0.0048	0.0750	-0.0227	0.0504	0.0294	0.0545	0.0238	0.0551
	1000	-0.0166	0.0378	-0.0004	0.0544	-0.0168	0.0381	0.0200	0.0397	0.0174	0.0394
.9	150	-0.0594	0.0989	-0.0214	0.1322	-0.0625	0.1009	0.0598	0.1043	0.0568	0.1032
	300	-0.0421	0.0709	-0.0119	0.0950	-0.0432	0.0717	0.0369	0.0708	0.0368	0.0708
	500	-0.0320	0.0547	-0.0085	0.0735	-0.0326	0.0551	0.0288	0.0552	0.0279	0.0551
	1000	-0.0236	0.0412	-0.0031	0.0521	-0.0239	0.0413	0.0193	0.0391	0.0200	0.0399
1.1	150	-0.0706	0.1053	-0.0208	0.1345	-0.0744	0.1083	0.0611	0.1035	0.0605	0.1024
	300	-0.0493	0.0754	-0.0117	0.0942	-0.0508	0.0765	0.0387	0.0712	0.0376	0.0705
	500	-0.0408	0.0601	-0.0092	0.0723	-0.0416	0.0607	0.0280	0.0544	0.0280	0.0551
	1000	-0.0288	0.0443	-0.0037	0.0518	-0.0291	0.0445	0.0194	0.0388	0.0204	0.0401
1.3	150	-0.0798	0.1131	-0.0124	0.1352	-0.0848	0.1170	0.0640	0.1076	0.0629	0.1048
	300	-0.0550	0.0792	-0.0072	0.0961	-0.0573	0.0809	0.0410	0.0730	0.0389	0.0712
	500	-0.0441	0.0630	-0.0023	0.0737	-0.0453	0.0638	0.0289	0.0549	0.0301	0.0553
	1000	-0.0329	0.0474	0.0006	0.0524	-0.0334	0.0478	0.0205	0.0398	0.0207	0.0397
1.5	150	-0.0857	0.1188	-0.0152	0.1211	-0.1036	0.1323	0.0651	0.1100	0.0649	0.1057
	300	-0.0551	0.0821	-0.0063	0.0845	-0.0672	0.0894	0.0422	0.0750	0.0411	0.0720
	500	-0.0410	0.0628	-0.0010	0.0677	-0.0502	0.0680	0.0333	0.0583	0.0328	0.0560
	1000	-0.0287	0.0457	0.0026	0.0502	-0.0353	0.0490	0.0231	0.0413	0.0231	0.0411
1.7	150	-0.1234	0.1439	-0.0425	0.1449	-0.1358	0.1607	0.0625	0.1071	0.0696	0.1083
	300	-0.0824	0.1015	-0.0183	0.1028	-0.0844	0.1050	0.0385	0.0715	0.0471	0.0762
	500	-0.0629	0.0781	-0.0084	0.0786	-0.0617	0.0777	0.0302	0.0558	0.0369	0.0601
	1000	-0.0432	0.0545	-0.0028	0.0537	-0.0395	0.0527	0.0213	0.0404	0.0275	0.0441
1.9	150	-0.1469	0.1684	-0.0405	0.1462	-0.2170	0.2406	0.0621	0.1043	0.0689	0.1055
	300	-0.0953	0.1122	-0.0131	0.0960	-0.1537	0.1724	0.0354	0.0687	0.0481	0.0750
	500	-0.0723	0.0853	-0.0094	0.0722	-0.1211	0.1359	0.0294	0.0546	0.0377	0.0594
	1000	-0.0508	0.0610	-0.0036	0.0514	-0.0865	0.0976	0.0202	0.0395	0.0293	0.0457

^a The data generation is $(1 - L)^d X_t = \varepsilon_t, \varepsilon_t \sim N(0, 1)$;

^b The m is the number of the frequency ordinates involved in the estimation.

Table 6.31: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $m = \lfloor n^{.5} \rfloor$

ϕ	d	n	fextLWF		fextLPWF		ModLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	0.0531	0.2165	-0.1395	0.4262	0.0531	0.2257	0.0588	0.2184	-0.0381	0.2571
		300	0.0307	0.1746	-0.0790	0.3024	0.0323	0.1776	0.0348	0.1701	-0.0384	0.1945
		500	0.0223	0.1464	-0.0561	0.2423	0.0278	0.1479	0.0221	0.1438	-0.0278	0.1626
		1000	0.0137	0.1155	-0.0398	0.1752	0.0244	0.1197	0.0157	0.1181	-0.0259	0.1318
	.7	150	0.0347	0.2131	-0.1275	0.4308	0.0287	0.2208	0.0525	0.2072	-0.0282	0.2565
		300	0.0151	0.1690	-0.0888	0.3148	0.0101	0.1713	0.0206	0.1513	-0.0331	0.1916
		500	0.0047	0.1412	-0.0599	0.2557	0.0083	0.1422	0.0141	0.1267	-0.0295	0.1576
		1000	0.0049	0.1127	-0.0453	0.1935	0.0026	0.1140	0.0020	0.1014	-0.0230	0.1206
	.9	150	0.0184	0.2130	-0.1355	0.4467	0.0217	0.2195	0.0285	0.2085	-0.0096	0.2565
		300	0.0030	0.1666	-0.0925	0.3319	0.0040	0.1673	0.0043	0.1626	-0.0165	0.1945
		500	-0.0041	0.1420	-0.0648	0.2652	-0.0041	0.1416	-0.0043	0.1405	-0.0170	0.1578
		1000	-0.0079	0.1139	-0.0415	0.2012	-0.0021	0.1124	-0.0052	0.1120	-0.0150	0.1241
	1.1	150	0.0211	0.2244	-0.1206	0.4530	0.0176	0.2254	0.0203	0.2223	0.0137	0.2428
		300	0.0042	0.1700	-0.0721	0.3365	0.0018	0.1754	0.0079	0.1704	-0.0037	0.1748
		500	-0.0076	0.1419	-0.0571	0.2764	-0.0040	0.1431	-0.0015	0.1406	-0.0041	0.1464
		1000	-0.0056	0.1136	-0.0351	0.2013	-0.0060	0.1134	-0.0031	0.1125	-0.0052	0.1134
	1.3	150	0.0328	0.2318	-0.0699	0.4401	0.0223	0.2198	0.0222	0.2365	0.0255	0.2302
		300	0.0125	0.1801	-0.0553	0.3297	0.0085	0.1691	0.0037	0.1828	0.0109	0.1699
		500	0.0085	0.1508	-0.0350	0.2641	0.0015	0.1424	-0.0043	0.1511	0.0050	0.1410
		1000	0.0008	0.1179	-0.0366	0.1978	-0.0013	0.1138	-0.0002	0.1145	-0.0029	0.1113
	1.5	150	0.0309	0.2193	-0.0752	0.4243	0.0262	0.2191	0.0098	0.2449	0.0391	0.2204
		300	0.0185	0.1736	-0.0371	0.3080	0.0129	0.1720	-0.0028	0.1875	0.0177	0.1668
		500	0.0153	0.1450	-0.0294	0.2495	0.0091	0.1424	-0.0103	0.1588	0.0102	0.1388
		1000	0.0155	0.1192	-0.0445	0.1824	0.0073	0.1151	-0.0069	0.1249	0.0036	0.1126
	1.7	150	0.0221	0.2135	-0.0686	0.4117	0.0135	0.2061	0.0227	0.2283	0.0487	0.2186
		300	0.0098	0.1650	-0.0490	0.2971	0.0165	0.1625	0.0050	0.1736	0.0220	0.1716
		500	0.0052	0.1409	-0.0387	0.2387	0.0204	0.1401	0.0015	0.1418	0.0206	0.1388
		1000	0.0014	0.1116	-0.0757	0.1998	0.0183	0.1164	-0.0019	0.1118	0.0102	0.1109
	1.9	150	0.0126	0.2140	-0.0733	0.4236	-0.0355	0.1602	0.0223	0.2259	0.0448	0.2107
		300	-0.0027	0.1697	-0.0590	0.3026	-0.0178	0.1274	0.0017	0.1730	0.0230	0.1586
		500	0.0005	0.1406	-0.0493	0.2572	-0.0088	0.1114	0.0001	0.1409	0.0172	0.1344
		1000	-0.0070	0.1141	-0.0481	0.2126	0.0001	0.0921	-0.0067	0.1132	0.0082	0.1103

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.32: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $m = \lfloor n^{.65} \rfloor$

ϕ	d	n	fextLWF		fextLPWF		ModLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	0.1226	0.1814	-0.0237	0.2261	0.1264	0.1832	0.1483	0.1933	0.1124	0.1813
		300	0.0866	0.1325	-0.0085	0.1579	0.0933	0.1357	0.1021	0.1386	0.0768	0.1281
		500	0.0627	0.1034	-0.0077	0.1271	0.0700	0.1066	0.0778	0.1110	0.0567	0.1038
		1000	0.0416	0.0749	-0.0005	0.0918	0.0500	0.0797	0.0530	0.0837	0.0373	0.0775
	.7	150	0.1125	0.1734	-0.0208	0.2256	0.1116	0.1721	0.1346	0.1882	0.1169	0.1897
		300	0.0776	0.1240	-0.0178	0.1624	0.0765	0.1221	0.0868	0.1296	0.0746	0.1297
		500	0.0561	0.0965	-0.0201	0.1421	0.0562	0.0968	0.0596	0.0988	0.0507	0.0997
		1000	0.0372	0.0708	-0.0086	0.1005	0.0380	0.0706	0.0391	0.0714	0.0330	0.0717
	.9	150	0.1029	0.1678	-0.0285	0.2515	0.1038	0.1680	0.1335	0.1896	0.1292	0.1903
		300	0.0696	0.1189	-0.0186	0.1681	0.0704	0.1194	0.0869	0.1310	0.0831	0.1309
		500	0.0527	0.0926	-0.0170	0.1371	0.0513	0.0927	0.0615	0.1002	0.0616	0.1009
		1000	0.0334	0.0691	-0.0097	0.0969	0.0317	0.0676	0.0376	0.0702	0.0362	0.0704
	1.1	150	0.1052	0.1685	-0.0168	0.2628	0.0981	0.1635	0.1333	0.1887	0.1323	0.1878
		300	0.0672	0.1167	-0.0133	0.1700	0.0656	0.1163	0.0882	0.1314	0.0844	0.1282
		500	0.0481	0.0914	-0.0167	0.1361	0.0514	0.0935	0.0591	0.0975	0.0607	0.0995
		1000	0.0315	0.0672	-0.0100	0.0992	0.0302	0.0676	0.0376	0.0707	0.0388	0.0715
	1.3	150	0.1120	0.1770	-0.0029	0.2642	0.0990	0.1622	0.1376	0.1924	0.1404	0.1908
		300	0.0779	0.1297	0.0051	0.1758	0.0670	0.1167	0.0893	0.1329	0.0855	0.1281
		500	0.0539	0.1010	-0.0005	0.1455	0.0498	0.0928	0.0648	0.1020	0.0649	0.1014
		1000	0.0333	0.0704	0.0004	0.1039	0.0321	0.0680	0.0405	0.0706	0.0425	0.0728
	1.5	150	0.1015	0.1641	-0.0044	0.2556	0.0854	0.1576	0.1340	0.1940	0.1443	0.1927
		300	0.0806	0.1250	0.0096	0.1573	0.0689	0.1181	0.0885	0.1348	0.0939	0.1341
		500	0.0616	0.0991	0.0065	0.1247	0.0539	0.0965	0.0654	0.1039	0.0665	0.1020
		1000	0.0443	0.0748	0.0058	0.0948	0.0376	0.0722	0.0425	0.0739	0.0447	0.0740
	1.7	150	0.0825	0.1534	-0.0167	0.2445	0.0662	0.1447	0.1347	0.1900	0.1445	0.1920
		300	0.0638	0.1148	-0.0079	0.1566	0.0643	0.1151	0.0880	0.1307	0.0964	0.1350
		500	0.0491	0.0921	-0.0084	0.1310	0.0549	0.0951	0.0605	0.0993	0.0718	0.1060
		1000	0.0335	0.0678	-0.0045	0.0977	0.0453	0.0763	0.0396	0.0724	0.0471	0.0760
	1.9	150	0.0757	0.1500	-0.0252	0.2613	-0.0189	0.1125	0.1355	0.1902	0.1261	0.1744
		300	0.0577	0.1113	-0.0148	0.1624	0.0082	0.0807	0.0844	0.1292	0.0888	0.1268
		500	0.0445	0.0894	-0.0165	0.1352	0.0171	0.0686	0.0597	0.0978	0.0653	0.0976
		1000	0.0288	0.0654	-0.0101	0.0971	0.0248	0.0583	0.0390	0.0711	0.0439	0.0730

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.33: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$ and $m = \lfloor n^{.8} \rfloor$

ϕ	d	n	fextLWF		fextLPWF		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	0.2468	0.2596	0.1209	0.1793	0.2464	0.2593	0.3741	0.3857	0.3725	0.3842
		300	0.2173	0.2255	0.0898	0.1284	0.2162	0.2241	0.2965	0.3041	0.2921	0.3002
		500	0.1959	0.2015	0.0700	0.1020	0.1945	0.2002	0.2547	0.2598	0.2527	0.2583
		1000	0.1661	0.1698	0.0499	0.0736	0.1660	0.1698	0.2052	0.2085	0.2041	0.2076
	.7	150	0.2286	0.2424	0.1127	0.1741	0.2230	0.2372	0.3746	0.3860	0.3722	0.3836
		300	0.2010	0.2096	0.0805	0.1239	0.2007	0.2091	0.2945	0.3015	0.2944	0.3014
		500	0.1831	0.1887	0.0635	0.0965	0.1825	0.1882	0.2544	0.2593	0.2545	0.2592
		1000	0.1565	0.1602	0.0454	0.0697	0.1568	0.1606	0.2058	0.2089	0.2059	0.2092
	.9	150	0.2130	0.2279	0.1044	0.1707	0.2061	0.2216	0.3697	0.3806	0.3699	0.3810
		300	0.1896	0.1982	0.0745	0.1191	0.1873	0.1962	0.2936	0.3007	0.2928	0.3000
		500	0.1743	0.1804	0.0617	0.0933	0.1736	0.1796	0.2536	0.2585	0.2537	0.2587
		1000	0.1505	0.1543	0.0430	0.0683	0.1505	0.1545	0.2052	0.2084	0.2066	0.2097
	1.1	150	0.1978	0.2134	0.1115	0.1748	0.1920	0.2082	0.3721	0.3832	0.3719	0.3834
		300	0.1811	0.1905	0.0746	0.1182	0.1788	0.1883	0.2929	0.2999	0.2937	0.3009
		500	0.1657	0.1722	0.0586	0.0924	0.1639	0.1702	0.2521	0.2572	0.2530	0.2577
		1000	0.1454	0.1492	0.0419	0.0674	0.1445	0.1486	0.2061	0.2094	0.2059	0.2089
	1.3	150	0.1811	0.1973	0.1113	0.1681	0.1743	0.1930	0.3726	0.3836	0.3740	0.3846
		300	0.1753	0.1851	0.0843	0.1263	0.1688	0.1787	0.2944	0.3014	0.2942	0.3012
		500	0.1639	0.1717	0.0660	0.1021	0.1583	0.1649	0.2555	0.2605	0.2540	0.2589
		1000	0.1444	0.1494	0.0453	0.0718	0.1392	0.1435	0.2070	0.2101	0.2068	0.2100
	1.5	150	0.1495	0.1711	0.0894	0.1616	0.1279	0.1584	0.3748	0.3861	0.3730	0.3837
		300	0.1545	0.1661	0.0743	0.1211	0.1422	0.1557	0.2957	0.3025	0.2946	0.3014
		500	0.1499	0.1573	0.0603	0.0949	0.1433	0.1512	0.2560	0.2610	0.2567	0.2615
		1000	0.1365	0.1407	0.0467	0.0719	0.1325	0.1369	0.2072	0.2104	0.2071	0.2101
	1.7	150	0.1191	0.1490	0.0785	0.1729	0.0450	0.1277	0.3709	0.3823	0.3630	0.3734
		300	0.1385	0.1511	0.0710	0.1265	0.0879	0.1216	0.2935	0.3007	0.2932	0.2998
		500	0.1376	0.1454	0.0609	0.0981	0.1027	0.1202	0.2529	0.2578	0.2548	0.2597
		1000	0.1270	0.1316	0.0457	0.0694	0.1097	0.1170	0.2063	0.2094	0.2086	0.2117
	1.9	150	0.0982	0.1348	0.0918	0.1721	-0.1106	0.1996	0.3704	0.3814	0.3188	0.3321
		300	0.1237	0.1381	0.0732	0.1192	-0.0537	0.1435	0.2917	0.2986	0.2603	0.2687
		500	0.1256	0.1343	0.0609	0.0941	-0.0226	0.1150	0.2530	0.2579	0.2284	0.2339
		1000	0.1190	0.1240	0.0431	0.0674	0.0068	0.0879	0.2053	0.2084	0.1910	0.1943

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.34: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $m = \lfloor n^{.5} \rfloor$

θ	d	n	fextLWF		fextLPWF($r=1$)		ModLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	-0.0066	0.2324	-0.1552	0.3987	0.0134	0.2273	-0.0379	0.2216	-0.1360	0.2777
		300	0.0128	0.1900	-0.1059	0.2894	0.0333	0.1843	-0.0192	0.1753	-0.0920	0.2162
		500	0.0194	0.1651	-0.0775	0.2335	0.0394	0.1619	-0.0089	0.1456	-0.0681	0.1737
		1000	0.0228	0.1351	-0.0513	0.1758	0.0428	0.1328	-0.0002	0.1165	-0.0414	0.1354
	.7	150	-0.0430	0.2112	-0.1696	0.4261	-0.0495	0.2246	-0.0426	0.2027	-0.1335	0.2856
		300	-0.0253	0.1685	-0.1135	0.3162	-0.0204	0.1711	-0.0262	0.1567	-0.0823	0.2074
		500	-0.0101	0.1414	-0.0822	0.2532	-0.0121	0.1443	-0.0193	0.1308	-0.0639	0.1676
		1000	-0.0042	0.1143	-0.0530	0.1980	-0.0017	0.1142	-0.0128	0.1039	-0.0430	0.1275
	.9	150	-0.0697	0.2225	-0.1480	0.4339	-0.0722	0.2310	-0.0638	0.2068	-0.1274	0.2811
		300	-0.0439	0.1733	-0.1076	0.3316	-0.0383	0.1792	-0.0381	0.1635	-0.0756	0.2076
		500	-0.0275	0.1451	-0.0774	0.2701	-0.0300	0.1450	-0.0353	0.1409	-0.0505	0.1659
		1000	-0.0159	0.1117	-0.0348	0.1993	-0.0162	0.1131	-0.0202	0.1132	-0.0323	0.1287
	1.1	150	-0.0798	0.2359	-0.1505	0.4435	-0.0839	0.2355	-0.0786	0.2242	-0.0926	0.2641
		300	-0.0464	0.1713	-0.0911	0.3343	-0.0437	0.1728	-0.0479	0.1763	-0.0528	0.1887
		500	-0.0263	0.1461	-0.0623	0.2718	-0.0327	0.1442	-0.0334	0.1440	-0.0360	0.1522
		1000	-0.0184	0.1129	-0.0301	0.1986	-0.0179	0.1150	-0.0223	0.1139	-0.0195	0.1153
	1.3	150	-0.0700	0.2346	-0.1371	0.4337	-0.0693	0.2298	-0.0825	0.2394	-0.0693	0.2373
		300	-0.0377	0.1813	-0.0752	0.3242	-0.0376	0.1749	-0.0474	0.1869	-0.0408	0.1727
		500	-0.0231	0.1483	-0.0482	0.2554	-0.0241	0.1428	-0.0306	0.1519	-0.0270	0.1432
		1000	-0.0129	0.1191	-0.0359	0.1938	-0.0118	0.1143	-0.0152	0.1156	-0.0142	0.1115
	1.5	150	-0.0607	0.2307	-0.1136	0.4192	-0.0552	0.2270	-0.1154	0.2892	-0.0616	0.2227
		300	-0.0194	0.1777	-0.0691	0.3036	-0.0304	0.1751	-0.0738	0.2373	-0.0342	0.1679
		500	-0.0018	0.1559	-0.0468	0.2472	-0.0143	0.1430	-0.0537	0.1968	-0.0175	0.1422
		1000	0.0086	0.1270	-0.0486	0.1775	-0.0026	0.1139	-0.0225	0.1524	-0.0114	0.1104
	1.7	150	-0.0615	0.2186	-0.0957	0.4150	-0.0543	0.2307	-0.0876	0.2606	-0.0505	0.2216
		300	-0.0286	0.1702	-0.0654	0.3016	-0.0171	0.1733	-0.0461	0.1897	-0.0235	0.1652
		500	-0.0154	0.1402	-0.0491	0.2446	-0.0068	0.1446	-0.0325	0.1548	-0.0124	0.1420
		1000	-0.0037	0.1134	-0.0831	0.2005	0.0058	0.1145	-0.0177	0.1169	-0.0045	0.1102
	1.9	150	-0.0829	0.2306	-0.0947	0.4047	-0.0658	0.2143	-0.0774	0.2414	-0.0382	0.2115
		300	-0.0462	0.1725	-0.0611	0.3075	-0.0243	0.1642	-0.0492	0.1791	-0.0191	0.1658
		500	-0.0291	0.1438	-0.0554	0.2556	-0.0105	0.1346	-0.0343	0.1466	-0.0100	0.1380
		1000	-0.0170	0.1145	-0.0587	0.2153	0.0038	0.1065	-0.0191	0.1161	0.0004	0.1133

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.35: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $m = \lfloor n^{.65} \rfloor$

θ	d	n	fextLWF		fextLPWF(r=1)		ModLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	-0.1228	0.1846	-0.0804	0.2221	-0.0885	0.1692	-0.1120	0.1772	-0.1742	0.2286
		300	-0.0839	0.1326	-0.0446	0.1577	-0.0492	0.1236	-0.0745	0.1280	-0.1116	0.1546
		500	-0.0533	0.0995	-0.0297	0.1243	-0.0255	0.0951	-0.0508	0.1006	-0.0778	0.1175
		1000	-0.0326	0.0721	-0.0149	0.0888	-0.0082	0.0723	-0.0326	0.0737	-0.0482	0.0814
	.7	150	-0.1328	0.1844	-0.0953	0.2414	-0.1298	0.1847	-0.1006	0.1586	-0.1567	0.2129
		300	-0.0899	0.1334	-0.0589	0.1733	-0.0844	0.1303	-0.0688	0.1114	-0.1033	0.1445
		500	-0.0667	0.1070	-0.0414	0.1541	-0.0582	0.0982	-0.0515	0.0874	-0.0758	0.1090
		1000	-0.0391	0.0747	-0.0225	0.1122	-0.0363	0.0705	-0.0349	0.0640	-0.0495	0.0774
	.9	150	-0.1517	0.1986	-0.0910	0.2531	-0.1485	0.1963	-0.1262	0.1753	-0.1606	0.2184
		300	-0.0994	0.1386	-0.0475	0.1805	-0.0990	0.1381	-0.0890	0.1322	-0.1084	0.1548
		500	-0.0733	0.1079	-0.0352	0.1512	-0.0683	0.1048	-0.0648	0.1041	-0.0729	0.1142
		1000	-0.0454	0.0745	-0.0143	0.0992	-0.0453	0.0754	-0.0430	0.0756	-0.0442	0.0764
	1.1	150	-0.1567	0.2049	-0.0816	0.2727	-0.1582	0.2050	-0.1311	0.1876	-0.1374	0.1973
		300	-0.1040	0.1425	-0.0420	0.1749	-0.1024	0.1405	-0.0855	0.1302	-0.0902	0.1340
		500	-0.0760	0.1088	-0.0279	0.1333	-0.0739	0.1080	-0.0653	0.1012	-0.0646	0.1027
		1000	-0.0460	0.0743	-0.0164	0.0996	-0.0491	0.0762	-0.0417	0.0712	-0.0412	0.0724
	1.3	150	-0.1558	0.2033	-0.0712	0.2758	-0.1556	0.2035	-0.1337	0.1919	-0.1213	0.1788
		300	-0.1021	0.1410	-0.0273	0.1716	-0.1004	0.1383	-0.0871	0.1320	-0.0832	0.1275
		500	-0.0702	0.1048	-0.0144	0.1384	-0.0703	0.1049	-0.0614	0.0993	-0.0625	0.1001
		1000	-0.0421	0.0729	-0.0059	0.1014	-0.0443	0.0742	-0.0396	0.0709	-0.0386	0.0713
	1.5	150	-0.1436	0.1977	-0.0769	0.2709	-0.1515	0.2007	-0.1518	0.2216	-0.1170	0.1726
		300	-0.0870	0.1363	-0.0308	0.1590	-0.0940	0.1364	-0.0947	0.1551	-0.0798	0.1250
		500	-0.0558	0.1041	-0.0180	0.1253	-0.0659	0.1026	-0.0627	0.1137	-0.0576	0.0967
		1000	-0.0277	0.0757	-0.0062	0.0917	-0.0388	0.0718	-0.0324	0.0739	-0.0359	0.0691
	1.7	150	-0.1540	0.1972	-0.0851	0.2834	-0.1493	0.1989	-0.1338	0.1969	-0.1064	0.1686
		300	-0.0977	0.1345	-0.0350	0.1673	-0.0892	0.1334	-0.0838	0.1319	-0.0703	0.1181
		500	-0.0704	0.1056	-0.0200	0.1346	-0.0581	0.0999	-0.0608	0.1007	-0.0488	0.0910
		1000	-0.0432	0.0744	-0.0106	0.0994	-0.0318	0.0708	-0.0395	0.0720	-0.0297	0.0668
	1.9	150	-0.1776	0.2179	-0.0850	0.2793	-0.1638	0.2045	-0.1346	0.1917	-0.1013	0.1629
		300	-0.1097	0.1454	-0.0421	0.1684	-0.1022	0.1382	-0.0870	0.1293	-0.0641	0.1145
		500	-0.0801	0.1115	-0.0268	0.1336	-0.0674	0.1011	-0.0636	0.1013	-0.0437	0.0901
		1000	-0.0494	0.0766	-0.0139	0.0980	-0.0344	0.0679	-0.0411	0.0713	-0.0241	0.0651

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.36: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$ and $m = \lfloor n^{.8} \rfloor$

θ	d	n	fextLWF		fextLPWF(r=1)		ModLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	-0.2747	0.2877	-0.1244	0.1818	-0.2434	0.2617	-0.2140	0.2308	-0.2492	0.2651
		300	-0.2249	0.2337	-0.0753	0.1194	-0.2005	0.2141	-0.1837	0.1937	-0.2104	0.2201
		500	-0.1994	0.2055	-0.0569	0.0925	-0.1734	0.1836	-0.1649	0.1718	-0.1834	0.1902
		1000	-0.1647	0.1687	-0.0381	0.0646	-0.1463	0.1533	-0.1416	0.1460	-0.1534	0.1577
	.7	150	-0.2740	0.2867	-0.1333	0.1826	-0.2836	0.2970	-0.1831	0.2089	-0.2304	0.2528
		300	-0.2279	0.2347	-0.0951	0.1357	-0.2313	0.2396	-0.1593	0.1759	-0.1914	0.2055
		500	-0.2019	0.2067	-0.0717	0.1070	-0.2047	0.2107	-0.1449	0.1564	-0.1682	0.1782
		1000	-0.1724	0.1759	-0.0480	0.0780	-0.1701	0.1740	-0.1237	0.1307	-0.1392	0.1456
	.9	150	-0.3037	0.3139	-0.1432	0.1989	-0.3069	0.3178	-0.1964	0.2084	-0.2248	0.2402
		300	-0.2492	0.2567	-0.0952	0.1355	-0.2501	0.2571	-0.1815	0.1890	-0.1964	0.2055
		500	-0.2166	0.2219	-0.0694	0.0998	-0.2190	0.2244	-0.1678	0.1736	-0.1776	0.1846
		1000	-0.1807	0.1840	-0.0482	0.0719	-0.1797	0.1831	-0.1453	0.1495	-0.1490	0.1536
	1.1	150	-0.3221	0.3324	-0.1359	0.1904	-0.3209	0.3318	-0.2138	0.2291	-0.2209	0.2387
		300	-0.2613	0.2685	-0.0944	0.1313	-0.2602	0.2673	-0.1888	0.1984	-0.1928	0.2028
		500	-0.2243	0.2295	-0.0729	0.1021	-0.2263	0.2314	-0.1711	0.1778	-0.1731	0.1799
		1000	-0.1860	0.1893	-0.0496	0.0723	-0.1848	0.1882	-0.1455	0.1494	-0.1461	0.1504
	1.3	150	-0.3261	0.3367	-0.1269	0.1861	-0.3321	0.3429	-0.2162	0.2328	-0.2153	0.2316
		300	-0.2632	0.2704	-0.0878	0.1281	-0.2661	0.2732	-0.1874	0.1973	-0.1862	0.1962
		500	-0.2292	0.2343	-0.0681	0.1011	-0.2287	0.2336	-0.1687	0.1757	-0.1686	0.1755
		1000	-0.1871	0.1905	-0.0473	0.0711	-0.1872	0.1906	-0.1442	0.1484	-0.1442	0.1484
	1.5	150	-0.3336	0.3450	-0.1233	0.1809	-0.3387	0.3495	-0.2202	0.2402	-0.2110	0.2282
		300	-0.2647	0.2723	-0.0764	0.1206	-0.2698	0.2771	-0.1899	0.2018	-0.1834	0.1934
		500	-0.2308	0.2361	-0.0577	0.0911	-0.2299	0.2356	-0.1685	0.1767	-0.1645	0.1715
		1000	-0.1872	0.1908	-0.0380	0.0648	-0.1881	0.1917	-0.1406	0.1458	-0.1410	0.1455
	1.7	150	-0.3272	0.3400	-0.1470	0.1906	-0.3440	0.3555	-0.2152	0.2332	-0.2011	0.2190
		300	-0.2576	0.2656	-0.0979	0.1381	-0.2697	0.2776	-0.1873	0.1975	-0.1769	0.1882
		500	-0.2243	0.2289	-0.0727	0.1073	-0.2295	0.2359	-0.1686	0.1755	-0.1584	0.1662
		1000	-0.1915	0.1941	-0.0450	0.0716	-0.1852	0.1895	-0.1437	0.1481	-0.1360	0.1408
	1.9	150	-0.3652	0.3714	-0.1575	0.2105	-0.3685	0.3782	-0.2155	0.2319	-0.1848	0.2046
		300	-0.2959	0.3020	-0.0978	0.1365	-0.2868	0.2934	-0.1887	0.1983	-0.1650	0.1770
		500	-0.2547	0.2595	-0.0718	0.1024	-0.2447	0.2496	-0.1704	0.1772	-0.1478	0.1567
		1000	-0.2049	0.2080	-0.0486	0.0711	-0.1961	0.1995	-0.1454	0.1496	-0.1257	0.1325

^a The data generating process is $(1 - L)^d X_t = (1 - .4L)\varepsilon_t, \varepsilon_t \sim N(0, 1)$;

^b The m is the number of the frequency ordinates involved in the estimation.

Table 6.37: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $m = \lfloor n^{.5} \rfloor$

θ	d	n	fextLWF		fextLPWF(r=1)		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	0.0144	0.2134	-0.1394	0.4275	0.0210	0.2196	0.0195	0.2092	-0.0730	0.2522
		300	0.0147	0.1732	-0.0897	0.3033	0.0261	0.1752	0.0112	0.1670	-0.0577	0.2005
		500	0.0152	0.1458	-0.0579	0.2400	0.0223	0.1500	0.0108	0.1446	-0.0432	0.1684
		1000	0.0119	0.1171	-0.0449	0.1780	0.0247	0.1197	0.0090	0.1190	-0.0292	0.1298
	.7	150	0.0007	0.2048	-0.1452	0.4348	-0.0123	0.2222	0.0073	0.2017	-0.0705	0.2616
		300	-0.0028	0.1634	-0.0916	0.3168	-0.0034	0.1717	0.0005	0.1545	-0.0488	0.1974
		500	-0.0008	0.1378	-0.0693	0.2542	0.0004	0.1449	-0.0024	0.1264	-0.0384	0.1598
		1000	-0.0040	0.1148	-0.0491	0.1949	-0.0054	0.1127	0.0006	0.1020	-0.0280	0.1199
	.9	150	-0.0172	0.2165	-0.1253	0.4378	-0.0189	0.2183	-0.0088	0.2023	-0.0473	0.2555
		300	-0.0140	0.1691	-0.0978	0.3317	-0.0155	0.1666	-0.0083	0.1607	-0.0374	0.1961
		500	-0.0098	0.1378	-0.0662	0.2685	-0.0128	0.1422	-0.0129	0.1384	-0.0292	0.1619
		1000	-0.0134	0.1136	-0.0409	0.2005	-0.0098	0.1134	-0.0117	0.1136	-0.0207	0.1272
	1.1	150	-0.0121	0.2244	-0.1203	0.4453	-0.0243	0.2225	-0.0169	0.2169	-0.0298	0.2461
		300	-0.0137	0.1736	-0.0777	0.3374	-0.0213	0.1709	-0.0188	0.1677	-0.0227	0.1831
		500	-0.0146	0.1434	-0.0601	0.2706	-0.0110	0.1413	-0.0149	0.1440	-0.0172	0.1498
		1000	-0.0073	0.1133	-0.0350	0.2023	-0.0123	0.1113	-0.0137	0.1146	-0.0118	0.1148
	1.3	150	0.0003	0.2296	-0.0980	0.4464	-0.0175	0.2191	-0.0203	0.2320	-0.0126	0.2289
		300	-0.0039	0.1795	-0.0601	0.3292	-0.0135	0.1682	-0.0185	0.1839	-0.0091	0.1723
		500	-0.0043	0.1488	-0.0364	0.2658	-0.0104	0.1443	-0.0125	0.1505	-0.0082	0.1432
		1000	-0.0047	0.1175	-0.0322	0.1981	-0.0067	0.1128	-0.0087	0.1148	-0.0068	0.1131
	1.5	150	0.0006	0.2236	-0.0911	0.4256	-0.0082	0.2172	-0.0308	0.2424	0.0057	0.2143
		300	0.0097	0.1755	-0.0470	0.3088	-0.0035	0.1695	-0.0245	0.1926	0.0011	0.1669
		500	0.0078	0.1435	-0.0358	0.2500	-0.0028	0.1448	-0.0149	0.1615	-0.0016	0.1370
		1000	0.0112	0.1186	-0.0425	0.1823	0.0037	0.1135	-0.0138	0.1303	-0.0016	0.1117
	1.7	150	-0.0163	0.2138	-0.0908	0.4239	-0.0117	0.2029	-0.0191	0.2361	0.0135	0.2134
		300	-0.0060	0.1652	-0.0551	0.3005	0.0063	0.1673	-0.0124	0.1774	0.0070	0.1644
		500	-0.0034	0.1394	-0.0422	0.2395	0.0100	0.1401	-0.0121	0.1478	0.0072	0.1388
		1000	-0.0028	0.1124	-0.0747	0.2021	0.0149	0.1139	-0.0065	0.1141	0.0077	0.1098
	1.9	150	-0.0217	0.2171	-0.0786	0.4116	-0.0575	0.1759	-0.0102	0.2277	0.0162	0.2056
		300	-0.0166	0.1669	-0.0635	0.3057	-0.0278	0.1320	-0.0182	0.1750	0.0085	0.1576
		500	-0.0121	0.1437	-0.0469	0.2529	-0.0161	0.1129	-0.0127	0.1440	0.0099	0.1327
		1000	-0.0092	0.1116	-0.0547	0.2154	-0.0031	0.0938	-0.0091	0.1133	0.0049	0.1089

^a The data generating process is $(1 - L)^d X_t = (1 + .4L)\varepsilon_t, \varepsilon_t \sim N(0, 1)$;

^b The m is the number of the frequency ordinates involved in the estimation.

Table 6.38: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $m = \lfloor n^{.65} \rfloor$

θ	d	n	fextLWF		fextLPWF(r=1)		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	0.0329	0.1334	-0.0473	0.2269	0.0400	0.1394	0.0578	0.1427	0.0135	0.1480
		300	0.0180	0.1000	-0.0176	0.1541	0.0277	0.1028	0.0393	0.1076	0.0104	0.1076
		500	0.0126	0.0810	-0.0146	0.1278	0.0245	0.0855	0.0308	0.0895	0.0070	0.0893
		1000	0.0083	0.0620	-0.0015	0.0910	0.0196	0.0663	0.0221	0.0713	0.0027	0.0690
	.7	150	0.0221	0.1311	-0.0501	0.2322	0.0198	0.1302	0.0418	0.1301	0.0127	0.1447
		300	0.0124	0.0984	-0.0353	0.1647	0.0167	0.1003	0.0237	0.0930	0.0017	0.1007
		500	0.0085	0.0802	-0.0229	0.1440	0.0088	0.0784	0.0141	0.0744	0.0007	0.0818
		1000	0.0073	0.0612	-0.0140	0.1038	0.0066	0.0602	0.0083	0.0564	-0.0019	0.0619
	.9	150	0.0143	0.1285	-0.0530	0.2464	0.0105	0.1281	0.0386	0.1371	0.0266	0.1442
		300	0.0074	0.0943	-0.0290	0.1701	0.0038	0.0959	0.0174	0.1016	0.0135	0.1050
		500	0.0012	0.0790	-0.0259	0.1422	0.0027	0.0786	0.0113	0.0794	0.0103	0.0837
		1000	0.0024	0.0596	-0.0131	0.0984	-0.0001	0.0586	0.0050	0.0600	0.0059	0.0605
	1.1	150	0.0068	0.1312	-0.0501	0.2659	0.0024	0.1293	0.0404	0.1388	0.0374	0.1393
		300	0.0021	0.0934	-0.0274	0.1737	0.0036	0.0954	0.0187	0.0994	0.0211	0.0979
		500	0.0007	0.0789	-0.0226	0.1347	0.0011	0.0770	0.0124	0.0778	0.0128	0.0797
		1000	-0.0001	0.0592	-0.0135	0.0981	0.0019	0.0587	0.0067	0.0596	0.0067	0.0610
	1.3	150	0.0123	0.1356	-0.0334	0.2639	0.0064	0.1298	0.0412	0.1413	0.0436	0.1385
		300	0.0072	0.0993	-0.0081	0.1752	0.0044	0.0955	0.0229	0.1021	0.0212	0.1003
		500	0.0053	0.0805	-0.0078	0.1416	0.0026	0.0784	0.0127	0.0789	0.0178	0.0800
		1000	0.0033	0.0600	-0.0010	0.1043	0.0029	0.0592	0.0084	0.0595	0.0095	0.0592
	1.5	150	0.0170	0.1305	-0.0315	0.2636	0.0018	0.1292	0.0352	0.1492	0.0462	0.1362
		300	0.0163	0.0995	-0.0081	0.1557	0.0055	0.0972	0.0229	0.1073	0.0293	0.0980
		500	0.0140	0.0819	-0.0002	0.1292	0.0082	0.0783	0.0128	0.0850	0.0184	0.0808
		1000	0.0150	0.0629	0.0050	0.0926	0.0077	0.0606	0.0112	0.0618	0.0124	0.0603
	1.7	150	-0.0053	0.1289	-0.0470	0.2576	-0.0062	0.1316	0.0363	0.1415	0.0522	0.1355
		300	0.0002	0.0947	-0.0210	0.1567	0.0062	0.0972	0.0227	0.1019	0.0304	0.0980
		500	0.0038	0.0771	-0.0126	0.1303	0.0122	0.0806	0.0150	0.0807	0.0255	0.0819
		1000	0.0033	0.0592	-0.0073	0.1006	0.0136	0.0642	0.0080	0.0599	0.0170	0.0630
	1.9	150	-0.0138	0.1277	-0.0530	0.2618	-0.0646	0.1343	0.0358	0.1385	0.0475	0.1305
		300	-0.0093	0.0966	-0.0240	0.1648	-0.0276	0.0915	0.0196	0.0984	0.0317	0.0970
		500	-0.0065	0.0785	-0.0201	0.1306	-0.0135	0.0736	0.0140	0.0791	0.0240	0.0768
		1000	-0.0015	0.0579	-0.0121	0.0983	0.0020	0.0553	0.0078	0.0593	0.0171	0.0606

^a The m is the number of the frequency ordinates involved in the estimation.

Table 6.39: Monte Carlo Results of the Local Whittle Fourier Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$ and $m = \lfloor n^{.8} \rfloor$

θ	d	n	fextLWF		fextLPWF(r=1)		modLWF		feLWF		dtr-feLWF	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	0.1354	0.1579	-0.0002	0.1301	0.1313	0.1537	0.2447	0.2599	0.2365	0.2543
		300	0.0937	0.1121	0.0002	0.0897	0.0961	0.1131	0.1629	0.1730	0.1555	0.1670
		500	0.0749	0.0896	0.0075	0.0710	0.0771	0.0902	0.1296	0.1371	0.1255	0.1337
		1000	0.0542	0.0656	0.0051	0.0510	0.0566	0.0667	0.0961	0.1021	0.0915	0.0980
	.7	150	0.1160	0.1423	-0.0118	0.1277	0.1108	0.1366	0.2401	0.2565	0.2415	0.2580
		300	0.0799	0.0988	-0.0069	0.0932	0.0775	0.0969	0.1584	0.1701	0.1586	0.1706
		500	0.0645	0.0788	-0.0054	0.0731	0.0624	0.0767	0.1247	0.1338	0.1201	0.1296
		1000	0.0460	0.0568	-0.0011	0.0527	0.0451	0.0560	0.0870	0.0935	0.0856	0.0927
	.9	150	0.0974	0.1248	-0.0229	0.1310	0.0940	0.1236	0.2390	0.2551	0.2395	0.2550
		300	0.0675	0.0890	-0.0132	0.0939	0.0655	0.0866	0.1576	0.1693	0.1571	0.1690
		500	0.0543	0.0702	-0.0099	0.0737	0.0532	0.0699	0.1226	0.1316	0.1225	0.1316
		1000	0.0392	0.0511	-0.0040	0.0521	0.0387	0.0506	0.0871	0.0936	0.0865	0.0932
	1.1	150	0.0828	0.1142	-0.0230	0.1349	0.0782	0.1110	0.2406	0.2567	0.2386	0.2548
		300	0.0582	0.0811	-0.0135	0.0945	0.0568	0.0811	0.1571	0.1684	0.1564	0.1684
		500	0.0468	0.0648	-0.0106	0.0724	0.0448	0.0639	0.1218	0.1310	0.1213	0.1305
		1000	0.0345	0.0475	-0.0045	0.0519	0.0338	0.0473	0.0860	0.0926	0.0862	0.0927
	1.3	150	0.0754	0.1098	-0.0137	0.1365	0.0640	0.1043	0.2414	0.2573	0.2425	0.2576
		300	0.0526	0.0786	-0.0093	0.0962	0.0488	0.0755	0.1599	0.1706	0.1589	0.1703
		500	0.0426	0.0618	-0.0039	0.0735	0.0395	0.0610	0.1240	0.1332	0.1228	0.1316
		1000	0.0296	0.0447	-0.0004	0.0522	0.0307	0.0457	0.0875	0.0941	0.0874	0.0939
	1.5	150	0.0509	0.0906	-0.0128	0.1207	0.0329	0.0940	0.2431	0.2587	0.2413	0.2570
		300	0.0422	0.0706	-0.0050	0.0847	0.0329	0.0685	0.1610	0.1727	0.1618	0.1730
		500	0.0364	0.0579	-0.0009	0.0671	0.0308	0.0565	0.1263	0.1350	0.1257	0.1344
		1000	0.0279	0.0441	0.0033	0.0502	0.0262	0.0430	0.0894	0.0961	0.0892	0.0958
	1.7	150	0.0182	0.0885	-0.0424	0.1450	-0.0247	0.1037	0.2423	0.2581	0.2409	0.2564
		300	0.0234	0.0630	-0.0189	0.1026	0.0019	0.0668	0.1590	0.1705	0.1643	0.1752
		500	0.0220	0.0511	-0.0093	0.0783	0.0109	0.0514	0.1229	0.1317	0.1289	0.1374
		1000	0.0181	0.0380	-0.0037	0.0537	0.0161	0.0388	0.0870	0.0937	0.0927	0.0991
	1.9	150	-0.0034	0.0856	-0.0371	0.1479	-0.1520	0.2067	0.2418	0.2575	0.2207	0.2352
		300	0.0077	0.0605	-0.0126	0.0955	-0.0994	0.1425	0.1569	0.1684	0.1514	0.1617
		500	0.0106	0.0471	-0.0099	0.0724	-0.0754	0.1103	0.1224	0.1311	0.1212	0.1292
		1000	0.0108	0.0350	-0.0042	0.0514	-0.0500	0.0775	0.0871	0.0937	0.0895	0.0954

^b The m is the number of the frequency ordinates involved in the estimation.

6.2.2 Results of Wavelet Estimators for Long-Memory Processes

Table 6.40: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$

ϕ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	0.0898	0.2185	-0.0502	0.3995	0.1849	0.2745	0.0728	0.4926
		300	0.1042	0.1547	-0.0110	0.2025	0.1540	0.1891	0.0570	0.2125
		500	0.0775	0.1250	-0.0354	0.1705	0.1362	0.1544	0.0417	0.1314
		1000	0.0967	0.1113	-0.0005	0.0941	0.1266	0.1348	0.0437	0.0862
	.7	150	0.0708	0.2164	-0.0740	0.4126	0.1548	0.2577	0.0563	0.4944
		300	0.0888	0.1460	-0.0191	0.2043	0.1389	0.1785	0.0459	0.2131
		500	0.0653	0.1163	-0.0469	0.1792	0.1179	0.1409	0.0378	0.1330
		1000	0.0812	0.0999	-0.0064	0.0946	0.1134	0.1230	0.0362	0.0840
	.9	150	0.0528	0.2126	-0.0904	0.4132	0.1410	0.2511	0.0380	0.4870
		300	0.0681	0.1355	-0.0289	0.2112	0.1193	0.1643	0.0294	0.2117
		500	0.0513	0.1117	-0.0485	0.1782	0.1027	0.1291	0.0243	0.1328
		1000	0.0706	0.0921	-0.0159	0.1030	0.0983	0.1102	0.0273	0.0839
	1.1	150	0.0291	0.2124	-0.0959	0.4149	0.1130	0.2380	0.0264	0.4741
		300	0.0560	0.1311	-0.0387	0.2165	0.1024	0.1552	0.0181	0.2134
		500	0.0367	0.1072	-0.0551	0.1869	0.0864	0.1185	0.0138	0.1372
		1000	0.0563	0.0833	-0.0215	0.1002	0.0845	0.0989	0.0208	0.0823
	1.3	150	0.0218	0.2080	-0.1127	0.4218	0.0945	0.2215	0.0031	0.4607
		300	0.0396	0.1275	-0.0473	0.2119	0.0839	0.1430	0.0172	0.2089
		500	0.0274	0.1050	-0.0639	0.1869	0.0740	0.1100	0.0054	0.1353
		1000	0.0421	0.0758	-0.0268	0.1042	0.0716	0.0889	0.0136	0.0822
	1.5	150	-0.0029	0.2086	-0.1211	0.4170	0.0761	0.2082	-0.0010	0.4333
		300	0.0264	0.1260	-0.0537	0.2202	0.0657	0.1321	0.0046	0.2015
		500	0.0119	0.1038	-0.0726	0.1934	0.0596	0.1004	0.0008	0.1345
		1000	0.0337	0.0712	-0.0324	0.1077	0.0589	0.0793	0.0082	0.0832
	1.7	150	-0.0154	0.2077	-0.1156	0.4022	0.0578	0.1928	-0.0184	0.4093
		300	0.0133	0.1237	-0.0572	0.2158	0.0494	0.1229	-0.0088	0.1887
		500	0.0016	0.1050	-0.0744	0.1937	0.0457	0.0922	-0.0078	0.1285
		1000	0.0193	0.0665	-0.0356	0.1112	0.0465	0.0701	0.0026	0.0801
	1.9	150	-0.0298	0.2033	-0.1343	0.4015	0.0367	0.1725	-0.0217	0.3555
		300	-0.0008	0.1218	-0.0635	0.2140	0.0336	0.1100	-0.0155	0.1733
		500	-0.0127	0.1078	-0.0789	0.1972	0.0299	0.0848	-0.0095	0.1248
		1000	0.0067	0.0677	-0.0420	0.1158	0.0330	0.0626	-0.0025	0.0821

^a The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.41: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(1, d , 0) Model with $\phi = .4$

ϕ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	-0.2108	0.2928	-0.2142	0.4591	-0.1920	0.2776	-0.1307	0.5062
		300	-0.1593	0.1952	-0.1221	0.2349	-0.1512	0.1852	-0.0817	0.2229
		500	-0.1527	0.1793	-0.1359	0.2204	-0.1275	0.1472	-0.0664	0.1401
		1000	-0.1212	0.1338	-0.0859	0.1286	-0.1063	0.1154	-0.0521	0.0887
	.7	150	-0.2121	0.2948	-0.2044	0.4430	-0.1950	0.2843	-0.1278	0.5056
		300	-0.1563	0.1936	-0.1173	0.2356	-0.1483	0.1841	-0.0800	0.2240
		500	-0.1506	0.1800	-0.1308	0.2131	-0.1247	0.1462	-0.0691	0.1457
		1000	-0.1174	0.1302	-0.0832	0.1281	-0.1044	0.1145	-0.0477	0.0882
	.9	150	-0.2104	0.2957	-0.1962	0.4532	-0.1898	0.2806	-0.1266	0.5075
		300	-0.1549	0.1942	-0.1188	0.2367	-0.1413	0.1818	-0.0777	0.2250
		500	-0.1456	0.1748	-0.1296	0.2168	-0.1192	0.1422	-0.0662	0.1460
		1000	-0.1144	0.1291	-0.0834	0.1292	-0.1010	0.1122	-0.0453	0.0905
	1.1	150	-0.1983	0.2887	-0.1888	0.4419	-0.1789	0.2731	-0.1171	0.5004
		300	-0.1460	0.1901	-0.1113	0.2380	-0.1334	0.1781	-0.0738	0.2226
		500	-0.1408	0.1726	-0.1283	0.2200	-0.1137	0.1393	-0.0640	0.1484
		1000	-0.1072	0.1230	-0.0786	0.1271	-0.0941	0.1070	-0.0435	0.0911
	1.3	150	-0.1886	0.2798	-0.1909	0.4487	-0.1709	0.2678	-0.1143	0.4969
		300	-0.1365	0.1819	-0.1113	0.2373	-0.1232	0.1713	-0.0729	0.2190
		500	-0.1332	0.1694	-0.1201	0.2101	-0.1062	0.1345	-0.0592	0.1477
		1000	-0.1005	0.1183	-0.0779	0.1276	-0.0879	0.1023	-0.0427	0.0918
	1.5	150	-0.1769	0.2783	-0.1999	0.4524	-0.1501	0.2522	-0.0960	0.4573
		300	-0.1279	0.1781	-0.1053	0.2357	-0.1100	0.1610	-0.0672	0.2148
		500	-0.1250	0.1638	-0.1164	0.2136	-0.0956	0.1269	-0.0570	0.1470
		1000	-0.0922	0.1123	-0.0730	0.1260	-0.0786	0.0953	-0.0390	0.0916
	1.7	150	-0.1626	0.2665	-0.1740	0.4282	-0.1341	0.2386	-0.0993	0.4309
		300	-0.1117	0.1681	-0.1024	0.2357	-0.0966	0.1517	-0.0617	0.2027
		500	-0.1111	0.1530	-0.1174	0.2125	-0.0810	0.1161	-0.0487	0.1401
		1000	-0.0831	0.1053	-0.0694	0.1262	-0.0668	0.0864	-0.0371	0.0914
	1.9	150	-0.1433	0.2548	-0.1685	0.4185	-0.1082	0.2120	-0.0909	0.3864
		300	-0.1000	0.1605	-0.1009	0.2312	-0.0786	0.1361	-0.0546	0.1869
		500	-0.1005	0.1464	-0.1102	0.2084	-0.0678	0.1078	-0.0470	0.1348
		1000	-0.0708	0.0974	-0.0666	0.1272	-0.0557	0.0788	-0.0322	0.0874

^a The data generation is $(1 - .4L)(1 - L)^d X_t = \varepsilon_t, \varepsilon_t \sim N(0, 1)$;

^b The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.42: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$

θ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	-0.3558	0.4074	-0.2916	0.5002	-0.3677	0.4202	-0.2388	0.5517
		300	-0.2796	0.3014	-0.1975	0.2826	-0.2890	0.3082	-0.1742	0.2679
		500	-0.2626	0.2796	-0.1928	0.2555	-0.2511	0.2615	-0.1439	0.1908
		1000	-0.2228	0.2299	-0.1366	0.1654	-0.2182	0.2227	-0.1129	0.1351
	.7	150	-0.3500	0.4050	-0.2891	0.5001	-0.3596	0.4136	-0.2431	0.5408
		300	-0.2743	0.2984	-0.1870	0.2774	-0.2884	0.3082	-0.1635	0.2645
		500	-0.2564	0.2740	-0.1856	0.2501	-0.2439	0.2551	-0.1332	0.1847
		1000	-0.2168	0.2245	-0.1302	0.1604	-0.2118	0.2169	-0.1049	0.1292
	.9	150	-0.3383	0.3951	-0.2734	0.4954	-0.3527	0.4080	-0.2282	0.5386
		300	-0.2673	0.2919	-0.1773	0.2720	-0.2766	0.2988	-0.1531	0.2623
		500	-0.2486	0.2671	-0.1777	0.2462	-0.2318	0.2445	-0.1240	0.1789
		1000	-0.2091	0.2172	-0.1228	0.1563	-0.2040	0.2095	-0.0976	0.1250
	1.1	150	-0.3282	0.3903	-0.2655	0.4874	-0.3313	0.3900	-0.2149	0.5309
		300	-0.2531	0.2807	-0.1682	0.2684	-0.2608	0.2858	-0.1405	0.2543
		500	-0.2363	0.2567	-0.1703	0.2413	-0.2245	0.2388	-0.1159	0.1775
		1000	-0.1991	0.2079	-0.1176	0.1556	-0.1926	0.1993	-0.0906	0.1208
	1.3	150	-0.3097	0.3752	-0.2514	0.4717	-0.3130	0.3756	-0.1927	0.5236
		300	-0.2390	0.2687	-0.1642	0.2729	-0.2452	0.2728	-0.1336	0.2497
		500	-0.2228	0.2439	-0.1604	0.2377	-0.2071	0.2224	-0.1062	0.1724
		1000	-0.1855	0.1960	-0.1120	0.1511	-0.1780	0.1858	-0.0823	0.1172
	1.5	150	-0.2865	0.3574	-0.2274	0.4611	-0.2890	0.3575	-0.1752	0.4911
		300	-0.2216	0.2548	-0.1491	0.2583	-0.2233	0.2530	-0.1180	0.2383
		500	-0.2066	0.2306	-0.1529	0.2321	-0.1888	0.2067	-0.0939	0.1641
		1000	-0.1717	0.1832	-0.1026	0.1459	-0.1600	0.1690	-0.0713	0.1093
	1.7	150	-0.2597	0.3330	-0.2228	0.4518	-0.2509	0.3215	-0.1599	0.4634
		300	-0.1968	0.2336	-0.1388	0.2529	-0.1947	0.2283	-0.1008	0.2212
		500	-0.1838	0.2118	-0.1447	0.2293	-0.1645	0.1850	-0.0882	0.1599
		1000	-0.1521	0.1661	-0.0959	0.1428	-0.1409	0.1511	-0.0637	0.1043
	1.9	150	-0.2227	0.3102	-0.2009	0.4358	-0.2153	0.2910	-0.1348	0.4033
		300	-0.1643	0.2090	-0.1308	0.2494	-0.1604	0.1970	-0.0830	0.2009
		500	-0.1631	0.1971	-0.1376	0.2276	-0.1368	0.1608	-0.0682	0.1447
		1000	-0.1293	0.1467	-0.0859	0.1384	-0.1163	0.1295	-0.0545	0.1007

^a The data generation is $(1 - L)^d X_t = (1 - .4L)\varepsilon_t$, $\varepsilon_t \sim N(0, 1)$;

^b The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.43: Monte Carlo Results of the Log-Regression Wavelet Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$

θ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	-0.0335	0.2026	-0.1307	0.4267	0.0377	0.2096	-0.0368	0.4890
		300	-0.0038	0.1140	-0.0709	0.2181	0.0351	0.1143	-0.0174	0.2061
		500	-0.0123	0.0949	-0.0851	0.1877	0.0274	0.0794	-0.0171	0.1255
		1000	0.0069	0.0577	-0.0448	0.1035	0.0307	0.0548	-0.0095	0.0741
	.7	150	-0.0421	0.2082	-0.1399	0.4380	0.0160	0.2038	-0.0384	0.4812
		300	-0.0144	0.1179	-0.0778	0.2227	0.0223	0.1127	-0.0266	0.2091
		500	-0.0238	0.1003	-0.0890	0.1913	0.0186	0.0777	-0.0236	0.1302
		1000	-0.0019	0.0587	-0.0470	0.1058	0.0199	0.0512	-0.0128	0.0771
	.9	150	-0.0552	0.2160	-0.1567	0.4352	0.0106	0.2070	-0.0463	0.4894
		300	-0.0236	0.1205	-0.0878	0.2272	0.0101	0.1136	-0.0293	0.2111
		500	-0.0315	0.1022	-0.0927	0.1951	0.0070	0.0781	-0.0255	0.1355
		1000	-0.0103	0.0602	-0.0565	0.1155	0.0117	0.0507	-0.0160	0.0803
	1.1	150	-0.0642	0.2175	-0.1503	0.4355	-0.0108	0.2086	-0.0522	0.4916
		300	-0.0303	0.1264	-0.0854	0.2256	-0.0005	0.1172	-0.0306	0.2108
		500	-0.0408	0.1087	-0.0973	0.1999	-0.0001	0.0801	-0.0315	0.1377
		1000	-0.0164	0.0622	-0.0563	0.1140	0.0040	0.0510	-0.0181	0.0832
	1.3	150	-0.0695	0.2255	-0.1508	0.4272	-0.0234	0.2122	-0.0602	0.4757
		300	-0.0415	0.1304	-0.0888	0.2294	-0.0107	0.1191	-0.0379	0.2160
		500	-0.0458	0.1112	-0.0999	0.2027	-0.0062	0.0808	-0.0304	0.1385
		1000	-0.0222	0.0652	-0.0565	0.1172	-0.0014	0.0529	-0.0207	0.0840
	1.5	150	-0.0801	0.2300	-0.1587	0.4320	-0.0305	0.2072	-0.0600	0.4471
		300	-0.0443	0.1331	-0.0948	0.2379	-0.0139	0.1180	-0.0363	0.2026
		500	-0.0510	0.1171	-0.1010	0.2053	-0.0129	0.0846	-0.0367	0.1415
		1000	-0.0269	0.0694	-0.0614	0.1196	-0.0054	0.0542	-0.0205	0.0843
	1.7	150	-0.0851	0.2269	-0.1540	0.4214	-0.0288	0.1932	-0.0606	0.4161
		300	-0.0462	0.1339	-0.0884	0.2287	-0.0201	0.1143	-0.0398	0.2045
		500	-0.0549	0.1198	-0.0999	0.2031	-0.0168	0.0843	-0.0325	0.1367
		1000	-0.0290	0.0713	-0.0571	0.1177	-0.0067	0.0542	-0.0227	0.0870
	1.9	150	-0.0820	0.2205	-0.1681	0.4211	-0.0293	0.1802	-0.0639	0.3713
		300	-0.0498	0.1327	-0.0884	0.2254	-0.0226	0.1096	-0.0344	0.1789
		500	-0.0585	0.1228	-0.0990	0.2066	-0.0203	0.0843	-0.0353	0.1326
		1000	-0.0309	0.0737	-0.0587	0.1243	-0.0102	0.0554	-0.0207	0.0841

^a The data generation is $(1 - L)^d X_t = (1 + .4L)\varepsilon_t$, $\varepsilon_t \sim N(0, 1)$;

^b The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.44: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(1, d , 0) Model with $\phi = -.4$

ϕ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	0.1875	0.2556	0.0667	0.3545	0.2052	0.2891	0.0728	0.4926
		300	0.1669	0.1949	0.0751	0.1855	0.1815	0.2096	0.0864	0.2195
		500	0.1604	0.1747	0.0708	0.1331	0.1629	0.1781	0.0764	0.1434
		1000	0.1473	0.1536	0.0643	0.0930	0.1493	0.1561	0.0683	0.0985
	.7	150	0.1643	0.2383	0.0456	0.3611	0.1848	0.2716	0.0563	0.4944
		300	0.1507	0.1816	0.0591	0.1856	0.1613	0.1938	0.0721	0.2152
		500	0.1384	0.1555	0.0598	0.1310	0.1462	0.1635	0.0648	0.1416
		1000	0.1299	0.1377	0.0585	0.0919	0.1339	0.1420	0.0596	0.0940
	.9	150	0.1395	0.2234	0.0325	0.3535	0.1670	0.2617	0.0380	0.4870
		300	0.1307	0.1665	0.0500	0.1889	0.1421	0.1802	0.0548	0.2151
		500	0.1262	0.1454	0.0489	0.1299	0.1301	0.1497	0.0558	0.1366
		1000	0.1165	0.1255	0.0475	0.0864	0.1189	0.1282	0.0526	0.0923
	1.1	150	0.1173	0.2110	0.0169	0.3568	0.1394	0.2435	0.0264	0.4741
		300	0.1121	0.1529	0.0397	0.1818	0.1225	0.1665	0.0434	0.2131
		500	0.1075	0.1314	0.0377	0.1261	0.1125	0.1362	0.0433	0.1363
		1000	0.1009	0.1120	0.0399	0.0851	0.1034	0.1148	0.0418	0.0883
	1.3	150	0.0971	0.2017	-0.0015	0.3317	0.1138	0.2304	0.0031	0.4607
		300	0.0952	0.1438	0.0288	0.1785	0.1019	0.1501	0.0403	0.2083
		500	0.0912	0.1181	0.0289	0.1256	0.0957	0.1228	0.0302	0.1330
		1000	0.0855	0.0984	0.0328	0.0829	0.0898	0.1026	0.0346	0.0865
	1.5	150	0.0765	0.1869	-0.0057	0.3324	0.0916	0.2088	-0.0010	0.4333
		300	0.0795	0.1313	0.0167	0.1749	0.0874	0.1401	0.0206	0.1992
		500	0.0757	0.1066	0.0201	0.1197	0.0795	0.1106	0.0236	0.1271
		1000	0.0726	0.0881	0.0249	0.0799	0.0732	0.0886	0.0272	0.0832
	1.7	150	0.0552	0.1761	-0.0361	0.3179	0.0686	0.1929	-0.0184	0.4093
		300	0.0577	0.1180	0.0005	0.1691	0.0666	0.1244	0.0114	0.1850
		500	0.0587	0.0947	0.0135	0.1182	0.0631	0.0980	0.0170	0.1214
		1000	0.0562	0.0749	0.0172	0.0772	0.0586	0.0765	0.0199	0.0788
	1.9	150	0.0315	0.1579	-0.0423	0.2997	0.0474	0.1690	-0.0217	0.3555
		300	0.0380	0.1065	-0.0133	0.1625	0.0472	0.1104	0.0041	0.1658
		500	0.0395	0.0833	0.0009	0.1151	0.0449	0.0852	0.0059	0.1160
		1000	0.0399	0.0640	0.0072	0.0755	0.0427	0.0653	0.0103	0.0765

^a The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.45: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(1, d , 0)
Model $\phi = .4$

ϕ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	-0.1537	0.2286	-0.0998	0.3643	-0.1823	0.2703	-0.1401	0.5170
		300	-0.1224	0.1552	-0.0577	0.1818	-0.1392	0.1754	-0.0665	0.2151
		500	-0.1104	0.1292	-0.0432	0.1209	-0.1139	0.1339	-0.0484	0.1305
		1000	-0.0968	0.1061	-0.0332	0.0753	-0.0988	0.1079	-0.0355	0.0784
	.7	150	-0.1554	0.2341	-0.0995	0.3599	-0.1780	0.2684	-0.1244	0.5045
		300	-0.1220	0.1569	-0.0568	0.1832	-0.1334	0.1721	-0.0599	0.2111
		500	-0.1077	0.1283	-0.0442	0.1256	-0.1115	0.1326	-0.0414	0.1287
		1000	-0.0935	0.1036	-0.0309	0.0752	-0.0964	0.1068	-0.0338	0.0787
	.9	150	-0.1538	0.2342	-0.0878	0.3592	-0.1743	0.2674	-0.1132	0.4960
		300	-0.1173	0.1565	-0.0518	0.1845	-0.1278	0.1687	-0.0629	0.2190
		500	-0.1012	0.1245	-0.0387	0.1259	-0.1080	0.1304	-0.0402	0.1328
		1000	-0.0891	0.1012	-0.0305	0.0793	-0.0917	0.1032	-0.0312	0.0802
	1.1	150	-0.1423	0.2278	-0.0972	0.3679	-0.1681	0.2663	-0.1048	0.4889
		300	-0.1121	0.1532	-0.0540	0.1886	-0.1184	0.1637	-0.0529	0.2146
		500	-0.0974	0.1228	-0.0380	0.1294	-0.1023	0.1279	-0.0410	0.1336
		1000	-0.0837	0.0971	-0.0288	0.0805	-0.0858	0.0990	-0.0272	0.0820
	1.3	150	-0.1370	0.2254	-0.0932	0.3589	-0.1499	0.2533	-0.1067	0.4909
		300	-0.1009	0.1468	-0.0532	0.1889	-0.1121	0.1597	-0.0560	0.2143
		500	-0.0907	0.1188	-0.0370	0.1302	-0.0926	0.1213	-0.0370	0.1323
		1000	-0.0769	0.0918	-0.0271	0.0810	-0.0775	0.0927	-0.0274	0.0830
	1.5	150	-0.1241	0.2192	-0.0958	0.3534	-0.1420	0.2462	-0.1030	0.4660
		300	-0.0930	0.1423	-0.0452	0.1823	-0.1014	0.1519	-0.0481	0.2071
		500	-0.0804	0.1121	-0.0374	0.1294	-0.0820	0.1147	-0.0369	0.1326
		1000	-0.0670	0.0845	-0.0236	0.0812	-0.0693	0.0861	-0.0241	0.0816
	1.7	150	-0.1128	0.2095	-0.0901	0.3383	-0.1193	0.2283	-0.0937	0.4340
		300	-0.0819	0.1353	-0.0462	0.1808	-0.0877	0.1427	-0.0435	0.1934
		500	-0.0690	0.1043	-0.0343	0.1267	-0.0713	0.1070	-0.0353	0.1284
		1000	-0.0574	0.0772	-0.0213	0.0812	-0.0609	0.0808	-0.0247	0.0820
	1.9	150	-0.0970	0.1963	-0.0978	0.3243	-0.1020	0.2079	-0.0766	0.3838
		300	-0.0690	0.1261	-0.0511	0.1712	-0.0712	0.1294	-0.0452	0.1809
		500	-0.0586	0.0987	-0.0349	0.1231	-0.0594	0.0990	-0.0291	0.1208
		1000	-0.0479	0.0718	-0.0217	0.0797	-0.0481	0.0711	-0.0228	0.0802

^a The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.46: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(0, d , 1) Model with $\theta = -.4$

θ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
-.4	.5	150	-0.3052	0.3488	-0.2001	0.4090	-0.3421	0.3942	-0.2450	0.5413
		300	-0.2539	0.2719	-0.1376	0.2214	-0.2756	0.2946	-0.1562	0.2573
		500	-0.2290	0.2393	-0.1152	0.1617	-0.2389	0.2490	-0.1234	0.1721
		1000	-0.2015	0.2064	-0.0950	0.1159	-0.2093	0.2141	-0.0991	0.1212
	.7	150	-0.3067	0.3528	-0.1966	0.4008	-0.3441	0.3990	-0.2372	0.5350
		300	-0.2481	0.2674	-0.1297	0.2164	-0.2733	0.2935	-0.1484	0.2527
		500	-0.2212	0.2320	-0.1059	0.1563	-0.2339	0.2448	-0.1126	0.1659
		1000	-0.1969	0.2023	-0.0865	0.1117	-0.2035	0.2086	-0.0907	0.1161
	.9	150	-0.2948	0.3437	-0.1820	0.3989	-0.3361	0.3924	-0.2331	0.5478
		300	-0.2394	0.2606	-0.1209	0.2137	-0.2636	0.2866	-0.1410	0.2498
		500	-0.2149	0.2269	-0.0978	0.1545	-0.2233	0.2357	-0.1062	0.1659
		1000	-0.1902	0.1958	-0.0803	0.1084	-0.1948	0.2011	-0.0836	0.1116
	1.1	150	-0.2803	0.3322	-0.1695	0.3903	-0.3195	0.3785	-0.2121	0.5375
		300	-0.2243	0.2483	-0.1094	0.2116	-0.2513	0.2756	-0.1217	0.2399
		500	-0.2024	0.2161	-0.0880	0.1518	-0.2136	0.2270	-0.0984	0.1610
		1000	-0.1793	0.1860	-0.0742	0.1062	-0.1844	0.1913	-0.0739	0.1076
	1.3	150	-0.2616	0.3180	-0.1676	0.3940	-0.3043	0.3665	-0.2038	0.5248
		300	-0.2097	0.2361	-0.1024	0.2067	-0.2315	0.2581	-0.1105	0.2339
		500	-0.1891	0.2041	-0.0804	0.1471	-0.1993	0.2145	-0.0888	0.1574
		1000	-0.1654	0.1728	-0.0647	0.1003	-0.1716	0.1791	-0.0678	0.1027
	1.5	150	-0.2413	0.3038	-0.1468	0.3753	-0.2733	0.3401	-0.1634	0.4880
		300	-0.1949	0.2241	-0.0944	0.2026	-0.2129	0.2425	-0.1035	0.2287
		500	-0.1705	0.1881	-0.0738	0.1449	-0.1792	0.1965	-0.0762	0.1514
		1000	-0.1484	0.1574	-0.0588	0.0972	-0.1535	0.1623	-0.0607	0.1003
	1.7	150	-0.2130	0.2816	-0.1288	0.3492	-0.2440	0.3136	-0.1524	0.4472
		300	-0.1733	0.2056	-0.0865	0.1978	-0.1852	0.2181	-0.0897	0.2159
		500	-0.1493	0.1695	-0.0661	0.1380	-0.1556	0.1753	-0.0684	0.1414
		1000	-0.1316	0.1418	-0.0515	0.0939	-0.1336	0.1437	-0.0512	0.0946
	1.9	150	-0.1841	0.2575	-0.1275	0.3393	-0.2016	0.2738	-0.1290	0.4021
		300	-0.1464	0.1833	-0.0753	0.1861	-0.1538	0.1888	-0.0797	0.1967
		500	-0.1276	0.1513	-0.0606	0.1350	-0.1302	0.1538	-0.0581	0.1326
		1000	-0.1100	0.1232	-0.0434	0.0886	-0.1117	0.1241	-0.0441	0.0894

^a The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

Table 6.47: Monte Carlo Results the Local Whittle Wavelet Estimation for ARFIMA(0, d , 1) Model with $\theta = .4$

θ	d	n	K=0 J=1		K=0 J=2		K=1 J=1		K=1 J=2	
			bias	RMSE	bias	RMSE	bias	RMSE	bias	RMSE
.4	.5	150	0.0536	0.1785	-0.0209	0.3529	0.0661	0.2084	-0.0276	0.4889
		300	0.0494	0.1072	-0.0018	0.1717	0.0554	0.1180	0.0006	0.2013
		500	0.0477	0.0827	0.0042	0.1144	0.0501	0.0862	0.0054	0.1210
		1000	0.0436	0.0618	0.0083	0.0675	0.0447	0.0632	0.0091	0.0698
	.7	150	0.0354	0.1803	-0.0282	0.3424	0.0447	0.2080	-0.0355	0.4813
		300	0.0368	0.1063	-0.0069	0.1725	0.0415	0.1159	-0.0052	0.2029
		500	0.0357	0.0792	0.0031	0.1164	0.0371	0.0821	0.0033	0.1233
		1000	0.0346	0.0563	0.0082	0.0700	0.0355	0.0575	0.0091	0.0725
	.9	150	0.0183	0.1783	-0.0382	0.3536	0.0277	0.2059	-0.0478	0.4919
		300	0.0237	0.1065	-0.0068	0.1783	0.0271	0.1156	-0.0027	0.2073
		500	0.0276	0.0778	-0.0031	0.1212	0.0287	0.0807	-0.0021	0.1284
		1000	0.0257	0.0542	0.0038	0.0728	0.0266	0.0554	0.0042	0.0752
	1.1	150	0.0099	0.1775	-0.0512	0.3622	0.0156	0.2028	-0.0541	0.4863
		300	0.0132	0.1059	-0.0150	0.1830	0.0165	0.1145	-0.0109	0.2080
		500	0.0172	0.0768	-0.0080	0.1219	0.0183	0.0788	-0.0070	0.1274
		1000	0.0175	0.0521	-0.0007	0.0748	0.0182	0.0531	-0.0003	0.0766
	1.3	150	-0.0055	0.1767	-0.0571	0.3519	-0.0012	0.1987	-0.0678	0.4772
		300	0.0075	0.1063	-0.0201	0.1785	0.0096	0.1136	-0.0171	0.2010
		500	0.0080	0.0767	-0.0078	0.1213	0.0090	0.0785	-0.0071	0.1270
		1000	0.0107	0.0508	-0.0017	0.0755	0.0111	0.0516	-0.0010	0.0769
	1.5	150	-0.0142	0.1771	-0.0618	0.3461	-0.0072	0.1928	-0.0664	0.4576
		300	-0.0009	0.1065	-0.0256	0.1786	0.0008	0.1133	-0.0216	0.1989
		500	0.0019	0.0769	-0.0162	0.1229	0.0029	0.0785	-0.0150	0.1267
		1000	0.0069	0.0518	-0.0056	0.0766	0.0073	0.0526	-0.0047	0.0781
	1.7	150	-0.0241	0.1734	-0.0803	0.3325	-0.0178	0.1856	-0.0701	0.4154
		300	-0.0094	0.1052	-0.0293	0.1722	-0.0069	0.1096	-0.0242	0.1857
		500	-0.0025	0.0765	-0.0159	0.1202	-0.0013	0.0773	-0.0140	0.1227
		1000	0.0018	0.0502	-0.0078	0.0772	0.0022	0.0504	-0.0070	0.0776
	1.9	150	-0.0301	0.1658	-0.0748	0.3144	-0.0208	0.1682	-0.0597	0.3705
		300	-0.0137	0.1026	-0.0366	0.1708	-0.0106	0.1019	-0.0307	0.1773
		500	-0.0077	0.0772	-0.0209	0.1198	-0.0062	0.0769	-0.0176	0.1190
		1000	-0.0029	0.0519	-0.0114	0.0755	-0.0025	0.0515	-0.0106	0.0751

^a The K is the trimming number of the highest wavelet scales; and the J is the trimming number of the lowest wavelet scales.

6.2.3 Results of Fourier Estimators for Seasonal/Cyclical Long-Memory Processes

Table 6.48: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 0$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	0.1541	0.1088	-0.0788	0.0330	-0.1541	0.0394
		1024	0.1861	0.0950	-0.0728	0.0231	-0.1614	0.0336
	-1.75	512	-0.0003	0.0754	-0.1581	0.0517	-0.1951	0.0514
		1024	0.0234	0.0495	-0.1456	0.0344	-0.2010	0.0475
	-1.55	512	-0.0316	0.0817	-0.1954	0.0595	-0.2095	0.0572
		1024	-0.0295	0.0462	-0.1784	0.0434	-0.1966	0.0450
	-1.35	512	-0.0753	0.0879	-0.1168	0.0389	-0.1645	0.0399
		1024	-0.0448	0.0498	-0.1069	0.0239	-0.1567	0.0306
	-1.15	512	-0.0745	0.0830	-0.1179	0.0388	-0.1487	0.0352
		1024	-0.0386	0.0456	-0.1064	0.0241	-0.1464	0.0276
	-.95	512	-0.0609	0.0770	-0.1107	0.0365	-0.1257	0.0289
		1024	-0.0421	0.0475	-0.0943	0.0220	-0.1273	0.0222
	-.75	512	-0.0457	0.0801	-0.0960	0.0324	-0.1027	0.0236
		1024	-0.0282	0.0441	-0.1002	0.0232	-0.1041	0.0167
	-.55	512	-0.0142	0.0770	-0.1025	0.0345	-0.0866	0.0215
		1024	-0.0203	0.0468	-0.0965	0.0222	-0.0723	0.0113
	-.35	512	-0.0388	0.0763	-0.0146	0.0228	-0.0294	0.0137
		1024	-0.0110	0.0439	-0.0034	0.0124	-0.0341	0.0080
	-.15	512	-0.0151	0.0697	-0.0124	0.0274	-0.0220	0.0135
		1024	-0.0076	0.0423	-0.0044	0.0127	-0.0173	0.0072
	.05	512	0.0017	0.0814	0.0032	0.0236	0.0055	0.0137
		1024	0.0094	0.0436	0.0009	0.0125	0.0061	0.0062
	.25	512	0.0358	0.0738	0.0188	0.0260	0.0329	0.0143
		1024	0.0199	0.0472	0.0088	0.0129	0.0296	0.0071
	.45	512	0.0767	0.0826	0.0170	0.0229	0.0484	0.0164
		1024	0.0556	0.0479	0.0242	0.0133	0.0675	0.0104
	.65	512	0.0803	0.0760	0.1153	0.0393	0.1146	0.0262
		1024	0.0811	0.0505	0.1091	0.0244	0.1051	0.0169
	.85	512	0.0888	0.0752	0.1284	0.0387	0.1361	0.0317
		1024	0.0748	0.0513	0.1067	0.0244	0.1191	0.0209
	1.05	512	0.1224	0.0948	0.1501	0.0493	0.1680	0.0415
		1024	0.0985	0.0544	0.1220	0.0282	0.1536	0.0299
	1.25	512	0.1734	0.1112	0.1795	0.0588	0.1987	0.0538
		1024	0.1260	0.0606	0.1352	0.0323	0.1833	0.0400
	1.45	512	0.2326	0.1292	0.1899	0.0611	0.2170	0.0608
		1024	0.1839	0.0772	0.1604	0.0388	0.2208	0.0553
	1.65	512	0.2467	0.1388	0.2902	0.1100	0.2396	0.0736
		1024	0.2068	0.0913	0.2558	0.0799	0.2385	0.0655
	1.85	512	0.2408	0.1273	0.2320	0.0808	0.1326	0.0446
		1024	0.1924	0.0849	0.2133	0.0606	0.1556	0.0387

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.49: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 1$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	0.1520	0.1062	-0.0771	0.0332	-0.1526	0.0390
		1024	0.2104	0.0961	-0.0745	0.0190	-0.0143	0.0046
	-1.75	512	0.0165	0.0818	-0.1603	0.0518	-0.2003	0.0529
		1024	0.0181	0.0514	-0.1497	0.0357	-0.0272	0.0053
	-1.55	512	-0.0327	0.0776	-0.1941	0.0632	-0.1964	0.0517
		1024	-0.0294	0.0443	-0.1719	0.0411	-0.0298	0.0047
	-1.35	512	-0.0862	0.0823	-0.1329	0.0429	-0.1674	0.0427
		1024	-0.0303	0.0469	-0.0943	0.0228	-0.0197	0.0046
	-1.15	512	-0.0768	0.0842	-0.1191	0.0376	-0.1431	0.0332
		1024	-0.0609	0.0482	-0.0980	0.0219	-0.0204	0.0050
	-.95	512	-0.0451	0.0734	-0.1147	0.0371	-0.1362	0.0320
		1024	-0.0332	0.0452	-0.1012	0.0229	-0.0226	0.0048
	-.75	512	-0.0465	0.0752	-0.1025	0.0353	-0.1042	0.0235
		1024	-0.0198	0.0422	-0.0956	0.0218	-0.0125	0.0046
	-.55	512	0.0196	0.0763	-0.1047	0.0345	-0.0937	0.0219
		1024	-0.0249	0.0470	-0.0902	0.0203	-0.0124	0.0045
	-.35	512	-0.0187	0.0838	-0.0269	0.0249	-0.0340	0.0148
		1024	0.0044	0.0437	0.0039	0.0131	0.0009	0.0043
	-.15	512	-0.0212	0.0841	-0.0120	0.0235	-0.0187	0.0144
		1024	-0.0121	0.0441	-0.0057	0.0126	-0.0043	0.0042
	.05	512	0.0088	0.0791	0.0015	0.0253	0.0091	0.0133
		1024	-0.0006	0.0441	0.0083	0.0135	0.0025	0.0043
	.25	512	0.0443	0.0768	0.0267	0.0227	0.0325	0.0147
		1024	0.0201	0.0429	0.0156	0.0132	0.0089	0.0047
	.45	512	0.0725	0.0825	0.0249	0.0254	0.0483	0.0154
		1024	0.0489	0.0458	0.0161	0.0131	0.0066	0.0039
	.65	512	0.0751	0.0786	0.1161	0.0382	0.1204	0.0267
		1024	0.0558	0.0487	0.1128	0.0250	0.0253	0.0050
	.85	512	0.0762	0.0850	0.1360	0.0425	0.1369	0.0319
		1024	0.0751	0.0540	0.1146	0.0246	0.0274	0.0051
	1.05	512	0.1383	0.0910	0.1594	0.0486	0.1644	0.0412
		1024	0.1066	0.0560	0.1256	0.0283	0.0340	0.0058
	1.25	512	0.1710	0.0975	0.1864	0.0595	0.2003	0.0530
		1024	0.1199	0.0615	0.1410	0.0331	0.0430	0.0063
	1.45	512	0.2298	0.1338	0.1956	0.0642	0.2116	0.0599
		1024	0.1694	0.0784	0.1577	0.0382	0.0493	0.0070
	1.65	512	0.2566	0.1403	0.2866	0.1079	0.2442	0.0766
		1024	0.2058	0.0866	0.2605	0.0828	0.0781	0.0112
	1.85	512	0.2480	0.1301	0.2256	0.0801	0.1441	0.0445
		1024	0.1796	0.0768	0.2119	0.0602	0.0520	0.0106

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.50: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 2$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	0.1676	0.1081	-0.0768	0.0344	-0.1551	0.0388
		1024	0.1877	0.0922	-0.0731	0.0208	-0.1725	0.0378
	-1.75	512	0.0048	0.0776	-0.1483	0.0452	-0.1969	0.0529
		1024	0.0289	0.0509	-0.1474	0.0344	-0.1929	0.0441
	-1.55	512	-0.0196	0.0838	-0.1902	0.0589	-0.2026	0.0537
		1024	-0.0199	0.0473	-0.1820	0.0453	-0.1952	0.0441
	-1.35	512	-0.0753	0.0778	-0.1209	0.0369	-0.1666	0.0406
		1024	-0.0407	0.0485	-0.0924	0.0210	-0.1572	0.0306
	-1.15	512	-0.0870	0.0833	-0.1256	0.0403	-0.1516	0.0362
		1024	-0.0517	0.0514	-0.1018	0.0230	-0.1418	0.0261
	-.95	512	-0.0577	0.0807	-0.1112	0.0353	-0.1204	0.0285
		1024	-0.0360	0.0430	-0.1082	0.0240	-0.1264	0.0220
	-.75	512	-0.0359	0.0746	-0.0948	0.0326	-0.1035	0.0249
		1024	-0.0311	0.0452	-0.0941	0.0218	-0.0991	0.0156
	-.55	512	-0.0248	0.0752	-0.1032	0.0350	-0.0843	0.0190
		1024	-0.0099	0.0401	-0.0899	0.0202	-0.0713	0.0114
	-.35	512	-0.0373	0.0789	-0.0188	0.0244	-0.0341	0.0145
		1024	-0.0018	0.0471	0.0018	0.0130	-0.0345	0.0079
	-.15	512	-0.0031	0.0726	-0.0084	0.0250	-0.0175	0.0144
		1024	-0.0071	0.0445	-0.0057	0.0132	-0.0150	0.0062
	.05	512	0.0001	0.0677	0.0107	0.0235	0.0080	0.0128
		1024	0.0028	0.0430	0.0035	0.0135	0.0141	0.0060
	.25	512	0.0403	0.0776	0.0189	0.0248	0.0286	0.0131
		1024	0.0318	0.0438	0.0024	0.0128	0.0313	0.0069
	.45	512	0.0653	0.0845	0.0258	0.0240	0.0606	0.0152
		1024	0.0464	0.0439	0.0168	0.0130	0.0718	0.0113
	.65	512	0.0848	0.0905	0.1074	0.0375	0.1095	0.0254
		1024	0.0657	0.0530	0.1041	0.0242	0.1037	0.0169
	.85	512	0.0932	0.0837	0.1364	0.0434	0.1383	0.0315
		1024	0.0687	0.0527	0.1125	0.0263	0.1252	0.0224
	1.05	512	0.1293	0.0983	0.1471	0.0476	0.1675	0.0403
		1024	0.1026	0.0574	0.1231	0.0275	0.1565	0.0311
	1.25	512	0.1798	0.1101	0.1855	0.0596	0.2071	0.0565
		1024	0.1302	0.0618	0.1459	0.0343	0.1867	0.0413
	1.45	512	0.2084	0.1176	0.1865	0.0593	0.2146	0.0609
		1024	0.1659	0.0771	0.1600	0.0385	0.2195	0.0551
	1.65	512	0.2566	0.1457	0.2923	0.1102	0.2416	0.0749
		1024	0.2245	0.0961	0.2613	0.0819	0.2452	0.0686
	1.85	512	0.2365	0.1319	0.2379	0.0832	0.1330	0.0431
		1024	0.1904	0.0848	0.2134	0.0598	0.1508	0.0375

Table 6.51: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 3$ and trimming order $l = 0$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	-0.1027	0.1503	-0.2361	0.0995	-0.2710	0.0939
		1024	-0.0624	0.0862	-0.2002	0.0609	-0.2602	0.0773
	-1.75	512	-0.1118	0.1518	-0.2328	0.0954	-0.2395	0.0803
		1024	-0.0685	0.0829	-0.2004	0.0595	-0.2376	0.0668
	-1.55	512	-0.0679	0.1566	-0.2482	0.0989	-0.2483	0.0827
		1024	-0.0537	0.0881	-0.1876	0.0543	-0.2032	0.0515
	-1.35	512	-0.0917	0.1489	-0.1383	0.0616	-0.1730	0.0520
		1024	-0.0580	0.0958	-0.1117	0.0327	-0.1697	0.0381
	-1.15	512	-0.0695	0.1512	-0.1293	0.0556	-0.1557	0.0458
		1024	-0.0627	0.0872	-0.1130	0.0336	-0.1589	0.0356
	-.95	512	-0.0757	0.1573	-0.1220	0.0541	-0.1348	0.0396
		1024	-0.0727	0.1002	-0.1045	0.0308	-0.1287	0.0258
	-.75	512	-0.0464	0.1432	-0.1164	0.0515	-0.1177	0.0361
		1024	-0.0245	0.0791	-0.0996	0.0297	-0.1110	0.0225
	-.55	512	-0.0316	0.1373	-0.1189	0.0531	-0.1091	0.0332
		1024	-0.0162	0.0897	-0.0855	0.0267	-0.0692	0.0152
	-.35	512	-0.0102	0.1329	-0.0270	0.0428	-0.0344	0.0219
		1024	0.0061	0.0811	-0.0065	0.0199	-0.0360	0.0114
	-.15	512	-0.0101	0.1398	-0.0119	0.0428	-0.0229	0.0211
		1024	-0.0104	0.0881	-0.0071	0.0203	-0.0163	0.0095
	.05	512	0.0187	0.1400	-0.0022	0.0434	0.0059	0.0212
		1024	0.0053	0.0816	-0.0071	0.0217	0.0042	0.0097
	.25	512	0.0357	0.1566	0.0231	0.0365	0.0349	0.0241
		1024	0.0389	0.0924	0.0126	0.0191	0.0334	0.0109
	.45	512	0.0890	0.1553	0.0193	0.0394	0.0475	0.0229
		1024	0.0393	0.0853	0.0258	0.0210	0.0678	0.0142
	.65	512	0.0410	0.1485	0.1203	0.0525	0.1200	0.0355
		1024	0.0599	0.0919	0.1185	0.0331	0.1056	0.0201
	.85	512	0.1082	0.1499	0.1395	0.0576	0.1279	0.0383
		1024	0.0579	0.0925	0.1115	0.0313	0.1217	0.0238
	1.05	512	0.1196	0.1619	0.1534	0.0604	0.1721	0.0513
		1024	0.0927	0.0940	0.1312	0.0378	0.1552	0.0336
	1.25	512	0.1746	0.1900	0.1790	0.0704	0.2046	0.0642
		1024	0.1220	0.0987	0.1426	0.0401	0.1789	0.0418
	1.45	512	0.3046	0.2468	0.2100	0.0839	0.2415	0.0788
		1024	0.1783	0.1091	0.1672	0.0486	0.2340	0.0648
	1.65	512	0.2954	0.2263	0.3218	0.1408	0.3202	0.1221
		1024	0.2420	0.1484	0.2769	0.0965	0.2783	0.0873
	1.85	512	0.3221	0.2493	0.3630	0.1742	0.3678	0.1549
		1024	0.2553	0.1446	0.2861	0.1022	0.3113	0.1060

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.52: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 0$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0697	0.0847	-0.0007	0.0212	0.0076	0.0091
		1024	0.0790	0.0497	-0.0012	0.0131	-0.0177	0.0047
	-1.75	512	0.0162	0.0825	-0.0059	0.0211	-0.0092	0.0085
		1024	0.0193	0.0488	-0.0026	0.0123	-0.0256	0.0048
	-1.55	512	-0.0429	0.0735	0.0116	0.0208	-0.0165	0.0086
		1024	-0.0010	0.0493	0.0005	0.0127	-0.0340	0.0053
	-1.35	512	0.0353	0.0750	-0.0772	0.0293	-0.0010	0.0078
		1024	0.0422	0.0508	-0.0492	0.0145	-0.0212	0.0050
	-1.15	512	0.0023	0.0733	-0.0502	0.0250	-0.0063	0.0086
		1024	0.0051	0.0455	-0.0416	0.0147	-0.0215	0.0047
	-.95	512	-0.0234	0.0802	-0.0234	0.0229	-0.0131	0.0088
		1024	-0.0154	0.0421	-0.0258	0.0119	-0.0165	0.0046
	-.75	512	-0.0358	0.0804	0.0016	0.0216	-0.0189	0.0095
		1024	-0.0187	0.0502	-0.0056	0.0126	-0.0133	0.0046
	-.55	512	-0.0680	0.0850	0.0301	0.0226	-0.0112	0.0089
		1024	-0.0072	0.0451	0.0184	0.0133	-0.0176	0.0048
	-.35	512	0.0118	0.0761	-0.0521	0.0253	0.0079	0.0081
		1024	0.0407	0.0446	-0.0282	0.0123	-0.0000	0.0043
	-.15	512	0.0178	0.0745	-0.0237	0.0220	-0.0025	0.0085
		1024	0.0102	0.0465	-0.0132	0.0131	0.0002	0.0042
	.05	512	-0.0041	0.0722	0.0097	0.0217	0.0047	0.0089
		1024	-0.0017	0.0449	0.0050	0.0130	-0.0010	0.0041
	.25	512	-0.0194	0.0736	0.0330	0.0226	0.0006	0.0091
		1024	0.0002	0.0431	0.0260	0.0128	0.0071	0.0045
	.45	512	-0.0231	0.0721	0.0808	0.0286	0.0084	0.0085
		1024	0.0060	0.0466	0.0532	0.0158	0.0084	0.0041
	.65	512	0.0856	0.0775	-0.0096	0.0207	0.0339	0.0091
		1024	0.0778	0.0536	0.0058	0.0122	0.0298	0.0051
	.85	512	0.0659	0.0785	0.0305	0.0238	0.0272	0.0091
		1024	0.0437	0.0519	0.0176	0.0131	0.0249	0.0045
	1.05	512	0.0859	0.0851	0.0681	0.0269	0.0265	0.0097
		1024	0.0654	0.0497	0.0526	0.0148	0.0338	0.0058
	1.25	512	0.0766	0.0871	0.1021	0.0345	0.0384	0.0097
		1024	0.0756	0.0539	0.0871	0.0204	0.0412	0.0058
	1.45	512	0.0929	0.0850	0.1624	0.0498	0.0533	0.0118
		1024	0.0937	0.0574	0.1165	0.0274	0.0439	0.0064
	1.65	512	0.1982	0.1208	0.1034	0.0370	0.0757	0.0159
		1024	0.1942	0.0865	0.1113	0.0273	0.0807	0.0120
	1.85	512	0.1831	0.1102	0.1258	0.0387	0.0232	0.0151
		1024	0.1678	0.0739	0.1041	0.0244	0.0491	0.0098

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.53: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 1$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0738	0.0790	-0.0050	0.0206	0.0075	0.0089
		1024	0.0822	0.0557	0.0025	0.0125	-0.0143	0.0046
	-1.75	512	0.0129	0.0697	-0.0117	0.0215	-0.0064	0.0085
		1024	0.0285	0.0480	-0.0120	0.0128	-0.0272	0.0053
	-1.55	512	-0.0671	0.0768	0.0131	0.0228	-0.0163	0.0087
		1024	0.0078	0.0470	0.0071	0.0130	-0.0298	0.0047
	-1.35	512	0.0249	0.0748	-0.0714	0.0311	-0.0025	0.0083
		1024	0.0422	0.0445	-0.0493	0.0146	-0.0197	0.0046
	-1.15	512	0.0008	0.0768	-0.0453	0.0232	-0.0051	0.0088
		1024	-0.0044	0.0458	-0.0391	0.0139	-0.0204	0.0050
	-.95	512	-0.0190	0.0757	-0.0323	0.0237	-0.0090	0.0099
		1024	-0.0028	0.0428	-0.0198	0.0115	-0.0226	0.0048
	-.75	512	-0.0470	0.0806	0.0042	0.0215	-0.0098	0.0085
		1024	-0.0212	0.0477	-0.0055	0.0119	-0.0125	0.0046
	-.55	512	-0.0812	0.0787	0.0291	0.0240	-0.0126	0.0087
		1024	-0.0069	0.0458	0.0209	0.0139	-0.0124	0.0045
	-.35	512	0.0471	0.0753	-0.0479	0.0243	0.0020	0.0098
		1024	0.0446	0.0466	-0.0252	0.0127	0.0009	0.0043
	-.15	512	-0.0051	0.0738	-0.0188	0.0242	0.0000	0.0096
		1024	0.0070	0.0473	-0.0133	0.0132	-0.0043	0.0042
	.05	512	-0.0024	0.0691	0.0004	0.0229	-0.0008	0.0089
		1024	-0.0045	0.0455	0.0088	0.0123	0.0025	0.0043
	.25	512	-0.0107	0.0747	0.0350	0.0223	-0.0033	0.0093
		1024	-0.0129	0.0441	0.0337	0.0126	0.0089	0.0047
	.45	512	-0.0267	0.0759	0.0813	0.0293	0.0047	0.0083
		1024	0.0102	0.0477	0.0509	0.0151	0.0066	0.0039
	.65	512	0.0728	0.0763	-0.0052	0.0221	0.0314	0.0098
		1024	0.0847	0.0536	0.0132	0.0120	0.0253	0.0050
	.85	512	0.0571	0.0809	0.0320	0.0236	0.0260	0.0094
		1024	0.0566	0.0446	0.0282	0.0138	0.0274	0.0051
	1.05	512	0.0831	0.0821	0.0795	0.0283	0.0341	0.0096
		1024	0.0506	0.0436	0.0513	0.0146	0.0340	0.0058
	1.25	512	0.0895	0.0842	0.0992	0.0306	0.0397	0.0102
		1024	0.0633	0.0517	0.0790	0.0187	0.0430	0.0063
	1.45	512	0.0844	0.0785	0.1548	0.0473	0.0536	0.0113
		1024	0.0756	0.0526	0.1101	0.0257	0.0493	0.0070
	1.65	512	0.2341	0.1290	0.1025	0.0366	0.0690	0.0150
		1024	0.2037	0.0872	0.1100	0.0268	0.0781	0.0112
	1.85	512	0.1807	0.1066	0.1178	0.0372	0.0367	0.0152
		1024	0.1570	0.0680	0.1054	0.0253	0.0520	0.0106

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.54: Monte Carlo Results for the LPSE for the Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$ and trimming order $l = 2$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0647	0.0814	-0.0109	0.0241	0.0064	0.0081
		1024	0.0842	0.0493	-0.0095	0.0122	-0.0157	0.0051
	-1.75	512	0.0041	0.0836	-0.0082	0.0231	-0.0108	0.0084
		1024	0.0171	0.0494	-0.0115	0.0125	-0.0276	0.0051
	-1.55	512	-0.0577	0.0739	0.0030	0.0249	-0.0130	0.0081
		1024	0.0063	0.0447	0.0024	0.0134	-0.0336	0.0055
	-1.35	512	0.0256	0.0773	-0.0756	0.0288	-0.0022	0.0086
		1024	0.0412	0.0514	-0.0499	0.0149	-0.0181	0.0046
	-1.15	512	0.0074	0.0790	-0.0491	0.0246	-0.0091	0.0079
		1024	0.0084	0.0433	-0.0398	0.0145	-0.0239	0.0050
	-.95	512	-0.0138	0.0752	-0.0288	0.0237	-0.0130	0.0089
		1024	-0.0097	0.0472	-0.0220	0.0132	-0.0163	0.0042
	-.75	512	-0.0249	0.0744	0.0022	0.0231	-0.0143	0.0086
		1024	-0.0181	0.0446	-0.0034	0.0134	-0.0148	0.0045
	-.55	512	-0.0717	0.0710	0.0236	0.0217	-0.0150	0.0094
		1024	-0.0087	0.0450	0.0222	0.0127	-0.0135	0.0042
	-.35	512	0.0382	0.0738	-0.0493	0.0253	0.0109	0.0091
		1024	0.0357	0.0485	-0.0276	0.0123	0.0001	0.0043
	-.15	512	0.0098	0.0728	-0.0175	0.0230	0.0021	0.0095
		1024	0.0028	0.0444	-0.0172	0.0129	-0.0055	0.0042
	.05	512	-0.0143	0.0831	0.0027	0.0213	0.0003	0.0080
		1024	0.0025	0.0433	0.0020	0.0115	0.0026	0.0043
	.25	512	-0.0228	0.0810	0.0326	0.0228	0.0060	0.0083
		1024	0.0014	0.0426	0.0232	0.0122	0.0056	0.0046
	.45	512	-0.0262	0.0735	0.0763	0.0292	0.0052	0.0090
		1024	0.0167	0.0460	0.0492	0.0159	0.0087	0.0048
	.65	512	0.0734	0.0797	-0.0018	0.0232	0.0280	0.0087
		1024	0.0754	0.0529	0.0112	0.0126	0.0262	0.0052
	.85	512	0.0576	0.0825	0.0346	0.0227	0.0290	0.0093
		1024	0.0416	0.0495	0.0306	0.0131	0.0251	0.0049
	1.05	512	0.0863	0.0862	0.0739	0.0276	0.0310	0.0092
		1024	0.0569	0.0482	0.0506	0.0149	0.0334	0.0054
	1.25	512	0.0895	0.0818	0.1089	0.0352	0.0368	0.0099
		1024	0.0600	0.0478	0.0826	0.0194	0.0423	0.0062
	1.45	512	0.0861	0.0807	0.1590	0.0490	0.0513	0.0117
		1024	0.0991	0.0549	0.1095	0.0256	0.0497	0.0072
	1.65	512	0.2207	0.1264	0.1003	0.0342	0.0744	0.0159
		1024	0.2034	0.0873	0.1027	0.0254	0.0796	0.0114
	1.85	512	0.1956	0.1089	0.1185	0.0370	0.0313	0.0148
		1024	0.1442	0.0667	0.1090	0.0256	0.0541	0.0107

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.55: Monte Carlo Results for the LPSE of the Series $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 3$ and trimming order $l = 0$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0124	0.1611	-0.0562	0.0418	-0.0205	0.0142
		1024	-0.0105	0.0873	-0.0439	0.0204	-0.0375	0.0080
	-1.75	512	-0.0306	0.1422	-0.0165	0.0347	-0.0138	0.0139
		1024	-0.0322	0.0879	-0.0179	0.0192	-0.0318	0.0075
	-1.55	512	-0.1233	0.1669	-0.0004	0.0373	-0.0185	0.0131
		1024	-0.0152	0.0908	-0.0167	0.0187	-0.0304	0.0068
	-1.35	512	0.0084	0.1349	-0.0789	0.0445	-0.0002	0.0134
		1024	0.0224	0.0808	-0.0651	0.0230	-0.0226	0.0072
	-1.15	512	-0.0203	0.1467	-0.0449	0.0352	-0.0029	0.0133
		1024	-0.0136	0.0869	-0.0427	0.0205	-0.0224	0.0071
	-.95	512	-0.0214	0.1436	-0.0270	0.0342	-0.0130	0.0141
		1024	-0.0085	0.0818	-0.0212	0.0201	-0.0175	0.0070
	-.75	512	-0.0573	0.1675	-0.0118	0.0363	-0.0155	0.0145
		1024	-0.0113	0.0845	-0.0031	0.0201	-0.0173	0.0067
	-.55	512	-0.1363	0.1589	0.0365	0.0385	-0.0099	0.0135
		1024	-0.0122	0.0825	0.0189	0.0191	-0.0178	0.0065
	-.35	512	0.0240	0.1393	-0.0650	0.0396	0.0019	0.0139
		1024	0.0419	0.0896	-0.0385	0.0214	-0.0027	0.0065
	-.15	512	0.0029	0.1486	-0.0293	0.0376	0.0058	0.0129
		1024	0.0095	0.0844	-0.0161	0.0200	-0.0030	0.0064
	.05	512	0.0141	0.1383	0.0104	0.0369	0.0091	0.0131
		1024	0.0155	0.0822	0.0090	0.0194	-0.0018	0.0068
	.25	512	-0.0073	0.1496	0.0377	0.0346	-0.0015	0.0139
		1024	0.0043	0.0795	0.0310	0.0220	0.0062	0.0065
	.45	512	-0.0876	0.1516	0.0677	0.0428	0.0059	0.0130
		1024	0.0197	0.0829	0.0403	0.0226	0.0173	0.0069
	.65	512	0.0689	0.1524	-0.0032	0.0310	0.0256	0.0129
		1024	0.0604	0.0881	-0.0004	0.0196	0.0246	0.0070
	.85	512	0.0738	0.1415	0.0276	0.0354	0.0209	0.0135
		1024	0.0403	0.0925	0.0172	0.0202	0.0207	0.0070
	1.05	512	0.0575	0.1517	0.0574	0.0367	0.0258	0.0132
		1024	0.0383	0.0844	0.0573	0.0228	0.0294	0.0075
	1.25	512	0.0603	0.1513	0.1030	0.0449	0.0287	0.0141
		1024	0.0593	0.0908	0.0733	0.0233	0.0404	0.0082
	1.45	512	0.0604	0.1478	0.1594	0.0618	0.0468	0.0158
		1024	0.1211	0.1062	0.1218	0.0339	0.0488	0.0093
	1.65	512	0.2574	0.2047	0.0996	0.0453	0.0866	0.0203
		1024	0.2238	0.1420	0.0924	0.0286	0.0732	0.0120
	1.85	512	0.2782	0.2356	0.1540	0.0602	0.0841	0.0223
		1024	0.2018	0.1288	0.1153	0.0331	0.0763	0.0124

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.56: Monte Carlo Results for the GSE for Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t, i.i.d. \varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$

Greq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	0.3485	0.2501	0.3319	0.3488	0.3080	0.3806
		1024	0.3419	0.2168	0.2919	0.2293	0.2377	0.1599
	-1.75	512	0.1807	0.2335	0.1811	0.2369	0.1728	0.3839
		1024	0.1210	0.0551	0.1402	0.1117	0.1359	0.2183
	-1.55	512	0.0802	0.1724	0.1104	0.1823	0.1179	0.2835
		1024	0.0630	0.0815	0.0581	0.0492	0.0854	0.1127
	-1.35	512	0.0730	0.0675	0.1093	0.0950	0.0657	0.0701
		1024	0.0528	0.0367	0.1109	0.0352	0.0519	0.0257
	-1.15	512	0.0468	0.1126	0.0717	0.0659	0.0311	0.0268
		1024	0.0306	0.0518	0.0737	0.0434	0.0252	0.0213
	-.95	512	0.0232	0.0881	0.0414	0.0429	0.0214	0.0696
		1024	0.0158	0.0472	0.0426	0.0219	0.0169	0.0191
	-.75	512	0.0037	0.0257	0.0464	0.0951	0.0190	0.0630
		1024	0.0213	0.0713	0.0209	0.0190	0.0083	0.0174
	-.55	512	-0.0051	0.0772	0.0037	0.0467	0.0324	0.0823
		1024	-0.0002	0.0277	-0.0079	0.0296	0.0347	0.0666
	-.35	512	0.0180	0.0273	0.0289	0.0096	0.0121	0.0066
		1024	0.0215	0.0165	0.0479	0.0065	0.0118	0.0029
	-.15	512	-0.0130	0.0266	0.0032	0.0087	0.0005	0.0055
		1024	-0.0091	0.0154	0.0136	0.0041	-0.0003	0.0025
	.05	512	-0.0196	0.0272	-0.0056	0.0087	-0.0021	0.0050
		1024	-0.0133	0.0148	-0.0068	0.0039	-0.0029	0.0024
	.25	512	-0.0193	0.0250	-0.0206	0.0088	-0.0002	0.0047
		1024	-0.0144	0.0145	-0.0256	0.0048	0.0014	0.0023
	.45	512	-0.0175	0.0283	-0.0256	0.0096	0.0200	0.0056
		1024	-0.0023	0.0147	-0.0468	0.0065	0.0215	0.0030
	.65	512	0.0312	0.0277	0.0076	0.0086	0.0282	0.0059
		1024	0.0266	0.0162	0.0130	0.0047	0.0228	0.0028
	.85	512	0.0019	0.0263	-0.0033	0.0098	0.0271	0.0054
		1024	0.0188	0.0155	-0.0167	0.0051	0.0177	0.0025
	1.05	512	0.0117	0.0294	-0.0108	0.0090	0.0411	0.0060
		1024	0.0237	0.0173	-0.0236	0.0049	0.0282	0.0031
	1.25	512	0.0459	0.0313	-0.0043	0.0101	0.0060	0.0075
		1024	0.0405	0.0177	-0.0285	0.0053	0.0438	0.0042
	1.45	512	0.0457	0.0319	0.0020	0.0108	0.0725	0.0104
		1024	0.0543	0.0195	-0.0338	0.0067	0.0658	0.0067
	1.65	512	0.1307	0.0449	0.0483	0.0134	0.0571	0.0115
		1024	0.1301	0.0344	0.0401	0.0076	0.0649	0.0078
	1.85	512	0.0900	0.0342	0.0068	0.0119	-0.0426	0.0173
		1024	0.0928	0.0242	-0.0085	0.0059	-0.0150	0.0085

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.57: Monte Carlo Results for the GSE for the Gegenbauer Series of $(I - 2(\cos \frac{\pi}{6})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 3$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/6$	-1.95	512	0.0912	0.2560	0.1304	0.1792	0.0920	0.1703
		1024	0.0752	0.2039	0.1209	0.0709	0.0688	0.0615
	-1.75	512	0.1629	0.6636	0.1358	0.2965	0.0701	0.1328
		1024	0.0292	0.0527	0.1093	0.1282	0.0528	0.0549
	-1.55	512	0.0560	0.2615	0.0739	0.1868	0.0847	0.2031
		1024	0.0232	0.0946	0.0714	0.1141	0.0550	0.0708
	-1.35	512	0.0674	0.1342	0.0840	0.0431	0.0375	0.0354
		1024	0.0444	0.0903	0.1083	0.0374	0.0416	0.0274
	-1.15	512	0.0348	0.1290	0.0719	0.1059	0.0238	0.0484
		1024	0.0262	0.0804	0.0694	0.0292	0.0210	0.0235
	-.95	512	0.0135	0.1158	0.0560	0.0949	0.0296	0.0901
		1024	-0.0012	0.0305	0.0406	0.0085	0.0119	0.0215
	-.75	512	0.0207	0.1231	0.0434	0.1277	0.0026	0.0255
		1024	-0.0021	0.0603	0.0206	0.0215	0.0133	0.0334
	-.55	512	-0.0322	0.1078	0.0065	0.1176	0.0211	0.0606
		1024	-0.0058	0.0437	0.0092	0.0447	0.0192	0.0174
	-.35	512	0.0109	0.0563	0.0265	0.0167	0.0030	0.0103
		1024	0.0127	0.0287	0.0373	0.0081	0.0094	0.0046
	-.15	512	-0.0006	0.0544	-0.0017	0.0160	-0.0053	0.0092
		1024	-0.0112	0.0312	0.0121	0.0063	-0.0040	0.0042
	.05	512	-0.0297	0.0542	-0.0193	0.0152	-0.0064	0.0089
		1024	-0.0153	0.0305	-0.0102	0.0062	-0.0035	0.0038
	.25	512	-0.0405	0.0554	-0.0161	0.0145	-0.0031	0.0083
		1024	-0.0183	0.0307	-0.0287	0.0075	0.0068	0.0037
	.45	512	-0.0535	0.0601	-0.0370	0.0151	0.0205	0.0094
		1024	-0.0128	0.0320	-0.0347	0.0079	0.0248	0.0041
	.65	512	0.0096	0.0548	-0.0007	0.0136	0.0205	0.0080
		1024	0.0055	0.0298	0.0057	0.0069	0.0166	0.0037
	.85	512	-0.0026	0.0540	-0.0098	0.0152	0.0195	0.0075
		1024	-0.0050	0.0295	-0.0245	0.0074	0.0162	0.0037
	1.05	512	0.0122	0.0567	-0.0195	0.0152	0.0273	0.0077
		1024	-0.0030	0.0301	-0.0379	0.0085	0.0210	0.0034
	1.25	512	-0.0019	0.0559	-0.0250	0.0166	0.0495	0.0099
		1024	0.0145	0.0304	-0.0395	0.0095	0.0380	0.0049
	1.45	512	0.0478	0.0645	0.0014	0.0172	0.1029	0.0197
		1024	0.0538	0.0361	-0.0175	0.0094	0.0789	0.0108
	1.65	512	0.1415	0.0796	0.0655	0.0220	0.1110	0.0198
		1024	0.1405	0.0514	0.0338	0.0111	0.0878	0.0116
	1.85	512	0.1604	0.0857	0.0740	0.0251	0.1339	0.0264
		1024	0.1396	0.0548	0.0294	0.0111	0.0992	0.0141

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.58: Monte Carlo Results for the GSE for Gegenbauer Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 2$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.1589	0.2628	0.1597	0.2651	0.2808	0.4603
		1024	0.1647	0.2749	0.1137	0.1355	0.2127	0.2512
	-1.75	512	0.1082	0.3010	0.1604	0.3540	0.2439	0.4775
		1024	0.0791	0.1542	0.0971	0.2101	0.1818	0.2677
	-1.55	512	0.0937	0.3772	0.1341	0.3028	0.2010	0.4107
		1024	0.0424	0.0784	0.0746	0.1712	0.1338	0.1769
	-1.35	512	0.0644	0.1427	0.0632	0.0670	0.1701	0.0684
		1024	0.0629	0.0566	0.0496	0.0250	0.1496	0.0782
	-1.15	512	0.0278	0.0619	0.0660	0.1462	0.1311	0.0860
		1024	0.0163	0.0153	0.0271	0.0219	0.1053	0.0456
	-.95	512	0.0048	0.0402	0.0498	0.0862	0.0826	0.0271
		1024	0.0039	0.0463	0.0187	0.0043	0.0749	0.0374
	-.75	512	0.0192	0.1265	0.0316	0.0222	0.0608	0.0757
		1024	-0.0173	0.0163	0.0122	0.0041	0.0420	0.0033
	-.55	512	-0.0172	0.0885	0.0389	0.0587	0.0353	0.0545
		1024	-0.0122	0.0152	0.0070	0.0165	0.0235	0.0146
	-.35	512	0.0141	0.0272	0.0006	0.0074	0.0596	0.0071
		1024	0.0339	0.0174	0.0139	0.0045	0.0548	0.0047
	-.15	512	-0.0104	0.0261	0.0016	0.0067	0.0231	0.0041
		1024	-0.0028	0.0151	0.0000	0.0041	0.0153	0.0019
	.05	512	-0.0188	0.0284	-0.0041	0.0070	-0.0104	0.0033
		1024	-0.0091	0.0149	-0.0052	0.0039	-0.0074	0.0016
	.25	512	-0.0131	0.0270	-0.0022	0.0071	-0.0385	0.0046
		1024	-0.0097	0.0164	-0.0074	0.0040	-0.0290	0.0025
	.45	512	-0.0349	0.0269	-0.0038	0.0072	-0.0546	0.0064
		1024	-0.0013	0.0164	-0.0146	0.0042	-0.0456	0.0035
	.65	512	0.0349	0.0271	-0.0131	0.0072	-0.0142	0.0035
		1024	0.0457	0.0191	0.0052	0.0043	-0.0151	0.0018
	.85	512	0.0162	0.0278	-0.0220	0.0079	-0.0474	0.0055
		1024	0.0197	0.0154	-0.0073	0.0042	-0.0460	0.0036
	1.05	512	0.0253	0.0298	-0.0103	0.0076	-0.0689	0.0080
		1024	0.0218	0.0164	-0.0103	0.0044	-0.0647	0.0057
	1.25	512	0.0458	0.0278	0.0021	0.0078	-0.0893	0.0114
		1024	0.0343	0.0178	-0.0026	0.0043	-0.0751	0.0073
	1.45	512	0.0541	0.0289	0.0201	0.0087	-0.0957	0.0129
		1024	0.0598	0.0206	0.0018	0.0043	-0.0823	0.0084
	1.65	512	0.1451	0.0476	0.0285	0.0102	-0.0485	0.0067
		1024	0.1399	0.0369	0.0565	0.0086	-0.0356	0.0040
	1.85	512	0.1109	0.0380	0.0205	0.0089	-0.1118	0.0192
		1024	0.1034	0.0269	0.0349	0.0061	-0.0858	0.0107

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

Table 6.59: Monte Carlo Results for the GSE for the Gegenbauer Series of $(I - 2(\cos \frac{\pi}{3})B + B^2)^d = \varepsilon_t$, *i.i.d.* $\varepsilon_t \sim N(0, 1)$ with tapering order $p = 3$

Gfreq	d	n	$m = \lfloor n^{.5} \rfloor$		$m = \lfloor n^{.65} \rfloor$		$m = \lfloor n^{.8} \rfloor$	
			bias	MSE	bias	MSE	bias	MSE
$\pi/3$	-1.95	512	0.0986	0.4240	0.1443	0.3145	0.2631	0.4327
		1024	0.0501	0.1511	0.0651	0.0826	0.1804	0.1727
	-1.75	512	0.0716	0.3686	0.1149	0.1766	0.2478	0.5238
		1024	0.0339	0.0961	0.0530	0.0540	0.1647	0.2208
	-1.55	512	-0.0052	0.2448	0.1035	0.1829	0.1670	0.2667
		1024	0.0156	0.0911	0.0651	0.0917	0.1180	0.0959
	-1.35	512	0.0700	0.1523	0.0553	0.0527	0.1768	0.0729
		1024	0.0335	0.0319	0.0397	0.0076	0.1409	0.0403
	-1.15	512	0.0215	0.0835	0.0411	0.0473	0.1313	0.0717
		1024	0.0127	0.0668	0.0198	0.0064	0.0952	0.0118
	-.95	512	-0.0025	0.0983	0.0490	0.0907	0.0939	0.0742
		1024	0.0036	0.0614	0.0250	0.0380	0.0660	0.0069
	-.75	512	-0.0281	0.0705	0.0482	0.0972	0.0528	0.0498
		1024	-0.0132	0.0434	0.0151	0.0208	0.0445	0.0180
	-.55	512	-0.0473	0.1058	0.0301	0.0372	0.0327	0.0436
		1024	-0.0042	0.0682	0.0231	0.0195	0.0267	0.0279
	-.35	512	0.0125	0.0500	0.0006	0.0121	0.0611	0.0097
		1024	0.0214	0.0295	0.0091	0.0068	0.0505	0.0051
	-.15	512	-0.0081	0.0519	-0.0076	0.0121	0.0206	0.0056
		1024	-0.0097	0.0312	-0.0039	0.0069	0.0150	0.0027
	.05	512	-0.0292	0.0560	-0.0097	0.0124	-0.0140	0.0052
		1024	-0.0141	0.0315	-0.0051	0.0067	-0.0086	0.0026
	.25	512	-0.0301	0.0545	-0.0029	0.0113	-0.0424	0.0066
		1024	-0.0258	0.0285	-0.0077	0.0066	-0.0279	0.0032
	.45	512	-0.0568	0.0533	-0.0100	0.0109	-0.0663	0.0095
		1024	-0.0190	0.0325	-0.0036	0.0063	-0.0471	0.0046
	.65	512	0.0237	0.0557	-0.0242	0.0117	-0.0235	0.0055
		1024	0.0398	0.0317	-0.0024	0.0069	-0.0193	0.0026
	.85	512	-0.0054	0.0542	-0.0322	0.0139	-0.0565	0.0078
		1024	0.0024	0.0300	-0.0166	0.0069	-0.0472	0.0044
	1.05	512	0.0033	0.0522	-0.0266	0.0116	-0.0808	0.0116
		1024	0.0074	0.0299	-0.0139	0.0068	-0.0659	0.0066
	1.25	512	0.0130	0.0529	-0.0204	0.0127	-0.1089	0.0168
		1024	0.0025	0.0304	-0.0113	0.0062	-0.0850	0.0095
	1.45	512	0.0517	0.0564	0.0145	0.0135	-0.1014	0.0167
		1024	0.0640	0.0369	0.0144	0.0076	-0.0881	0.0104
	1.65	512	0.1677	0.0877	0.0145	0.0126	-0.0480	0.0083
		1024	0.1534	0.0575	0.0393	0.0089	-0.0469	0.0051
	1.85	512	0.1815	0.0906	0.0227	0.0158	-0.0767	0.0118
		1024	0.1442	0.0522	0.0337	0.0089	-0.0711	0.0083

¹ The number of frequency ordinates utilized in the estimation is actually equal to $\lfloor m/p \rfloor$.

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