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Université Pierre et Marie Curie - Paris 6

École Doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Karam FAYAD

Semi-simplicity of ℓ-adic representations with applications to Shimura varieties

---

dirigée par Jan Nekovář

Soutenue le 29 septembre 2015 devant le jury composé de :

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M. Jean-François Dat	Professeur, Université Paris 6	Examinateur
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Dans la vie il n'y a pas de solutions. Il y a des forces en marche: il faut les créer et les solutions suivent.

Antoine de Saint-Exupéry - Vol de nuit

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Ces deux pages, qui constituent à peine 3% du manuscrit, seront peut-être la partie la plus lue. J'en profite alors pour dire Merci du fond du cœur à tous ceux qui ont participé, de près ou de loin, chacun à sa manière, à rendre cette thèse possible.

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Semi-simplicité des représentations ℓ-adiques et applications aux variétés de Shimura

Résumé

On étudie dans un cadre abstrait des critères de semi-simplicité pour des représentations ℓ-adiques de groupes profinis. On applique les résultats obtenus pour montrer que les relations d'Eichler-Shimura généralisées entraînent la semi-simplicité de certaines représentations galoisiennes non triviales qui apparaissent dans la cohomologie des variétés de Shimura unitaires. Les résultats les plus intéressants sont obtenus pour les variétés de Shimura unitaires de signature $(n,0)^a \times (n-1,1)^b \times (1,n-1)^c \times (0,n)^d$.

Mots-clefs

Variétés de Shimura, groupes unitaires, relations d'Eichler-Shimura, représentations galoisiennes, critères de semi-simplicité, représentations induites.

Abstract

We prove several abstract criteria for semi-simplicity of ℓ-adic representations of profinite groups. As an application, we show that generalised Eichler-Shimura relations imply the semi-simplicity of a non-trivial subspace of middle cohomology of unitary Shimura varieties. The most complete results are obtained for unitary Shimura varieties of signature $(n,0)^a \times (n-1,1)^b \times (1,n-1)^c \times (0,n)^d$.

Keywords

Shimura varieties, unitary groups, Eichler-Shimura relations, Galois representations, semi-simplicity criteria, induced representations.

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Introduction

We begin to state, in an abstract context, semi-simplicity criteria for finite-dimensional representations of (profinite) groups, then we switch to the geometric context, where we construct the abstract objects apprearing before, and use our results to prove the semi-simplicity of certain Galois representations appearing in middle cohomology of unitary Shimura varieties.

0.1 Algebraic part

For an endomorphism u of a finite-dimensional vector space over a field k, the characteristic polynomial of u will be denoted by $P_u(X) \in k[X]$.
For $i = 1, \ldots, r$, let $\rho_i : \Gamma \longrightarrow \mathrm{Aut}_k(W_i)$ be an irreducible finite-dimensional representation of a group Γ with coefficients in a field k, and let $\rho : \Gamma \longrightarrow \mathrm{Aut}_k(V)$ be a finite-dimensional representation such that

$$\forall\ g \in \Gamma, \quad P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0. \tag{1}$$

In [Nek], Nekovář considers the following two questions:

(Q1) Under condition (1), is it true that

$$\rho^{\mathrm{ss}} \subseteq ((\rho_1 \otimes \ldots \otimes \rho_r)^{\oplus m})^{\mathrm{ss}} \tag{2}$$

for some integer $m \geq 1$?
Note that $\rho_1 \otimes \ldots \otimes \rho_r$ is automatically semi-simple if k has characteristic zero.

(Q2) If (2) holds, under what additional assumptions is ρ semi-simple?

Boston, Lenstra and Ribet [BLR91] showed that both questions (Q1) and (Q2) have positive answers if $r = 1$ and ρ_1 is a two-dimensional absolutely irreducible representation of Γ. Their result states that ρ is isomorphic to a direct sum of copies of ρ_1.

Dimitrov [Dim05, Lemma 6.5] considered a variant of question (Q1) for certain two-dimensional representations $\rho_1, \ldots, \rho_r : \Gamma \longrightarrow GL_2(\mathbb{F}_q)$.

Emerton and Gee [EG12] showed that for $r = 1$, (Q1) has a positive answer for certain higher-dimensional representations ρ_1 which have a sufficiently large image. More precisely, their result states that if ρ is irreducible, and ρ_1 is irreducible of dimension n such that for some subfield k' of k, $\mathrm{SL}_n(k') \subset \rho_1(\Gamma) \subset k^\times \mathrm{GL}_n(k')$, then under assumption (1), ρ_1 is isomorphic to ρ.

We consider questions (Q1) and (Q2) for continuous representations of a profinite group Γ with coefficients in $\overline{\mathbb{Q}}_\ell$.

Using abstract results on Lie algebra representations, Nekovář shows that (Q1) has a positive answer if the images of ρ_i are sufficiently large, more precisely if the following assumption holds:

(A1) Each W_i is a direct sum of simple modules for the $\overline{\mathbb{Q}}_\ell$-Lie algebra $\overline{\mathbb{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma))$, where each of these simple modules is one-dimensional or minuscule;

and that (Q2) has a positive answer under the following two assumptions:

(A2) The restriction of each ρ_i to any open subgroup of Γ is irreducible.

(A3) The image of ρ contains sufficiently many semi-simple elements. More precisely, there exists a dense subset $\Sigma \subset \Gamma$ such that for each $g \in \Sigma$, $\rho(g)$ is a semi-simple element of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$.

In the case where for $i = 1, \ldots, r$, ρ_i is isomorphic to an induction $\mathrm{Ind}_{\Gamma_n}^{\Gamma}(\alpha_i)$, with Γ_n a normal subgroup of Γ such that $\Gamma/\Gamma_n = \langle \sigma \rangle \simeq \mathbb{Z}/n\mathbb{Z}$, and $\alpha_i : \Gamma_n \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ a character, (Q1) has a positive answer [Nek]. Assumption (A2) does not hold in this case, however (Q2) has a positive answer under (A3) and the additional assumption:

(A4) n is a prime number.

Variants of (1) and (A3): The condition (1) on the characteristic polynomial is satisfied if there exists a dense subset $\Sigma \subset \Gamma$ with the following property:

(P) For each $g \in \Sigma$, there exist pairwise commuting elements $u_1, \ldots, u_r \in \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ such that $\rho(g) = u_1 \ldots u_r$ and $\forall\ i = 1, \ldots, r$, $P_{\rho_i(g)}(u_i) = 0$;

and assumption (A3) holds if in the previous property, the polynomials $P_{\rho_i(g)}(X)$ are without multiple roots.
In our context, the morphisms u_i appear naturally as partial Frobenius morphisms, the relations $P_{\rho_i(g)}(u_i) = 0$ are generalisation of Eichler-Shimura relations, and the fact that $P_{\rho_i(g)}(X)$ are without multiple roots is a consequence of the following assumption on the Lie algebras $\mathfrak{g}_i = \mathrm{Lie}(\rho_i(\Gamma))$:

(A5) For $i = 1, \ldots, r$, a Cartan subalgebra of $\mathfrak{g}_i$ acts on W_i without multiplicities.

In the case where $\rho_i \simeq \mathrm{Ind}_{\Gamma_n}^{\Gamma}(\alpha_i)$, the fact that the $P_{\rho_i(g)}(X)$ are without multiple roots is a consequence of the following assumption on the characters α_i:

(A6) For $i = 1, \ldots, r$ and $j = 1, \ldots, n-1$, the character $\alpha_i/\alpha_i^{\sigma^j}$ is of infinite order.

In the present work, we give three main improvements to [Nek].

Abstract results

• First, we prove the semi-simplicity of ρ independently of (Q1). More precisely, we show that under assumptions (P), (A2) and (A5), the semi-simplicity of ρ holds.
(c.f. Theorem 1 of the outline).

• Second, we generalise the case of induced characters to several normal cyclic subgroups $\Gamma_1, \ldots, \Gamma_s$ of Γ, where for $i = 1, \ldots, s$, $j = 1, \ldots, m$ (here $r = sm$), $\Gamma/\Gamma_i = \langle \sigma_i \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ with p a prime number, $\alpha_{ij} : \Gamma_i \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is a character of Γ_i, and $\rho_{ij} \simeq \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$. More precisely, denoting $\Gamma_1 \cap \ldots \cap \Gamma_s$ by Γ_0, we show that if $[\Gamma : \Gamma_0] = p^s$, then (Q2) has a positive answer under (a variant of) (A3), also implied by (P) and (A6).
(Q1) has a positive answer in the case $s = 1$ [Nek]. The case $s > 1$ is not proven yet.
(c.f. Theorem 2 and Proposition 1 of the outline).

• Third, we show that when $r = 1$, we can merely assume that $[\Gamma : \Gamma_1]$ is a power of a prime number. More precisely, we show that if $r = 1$ and $[\Gamma : \Gamma_1] = p^v$ for p a prime number and v an integer, then under assumption (A6), (Q1) and (Q2) have positive answers. This is a generalisation of the case $v = 1$ in [Nek].
(c.f. Theorem 3 of the outline).

0.2 Geometric part

The representation ρ

Let $(G, \mathscr{X})$ be a pure Shimura datum, and for an open compact subgroup $K \subset G(\widehat{\mathbb{Q}})$, where $\widehat{\mathbb{Q}}$ is the ring of finite adèles of $\mathbb{Q}$, consider the analytic Shimura variety

$$Sh_K^{an}(G, \mathscr{X}) = G(\mathbb{Q}) \setminus (\mathscr{X} \times (G(\widehat{\mathbb{Q}})/K)).$$

Baily and Borel [BB66] proved that Sh_K^{an} is the analytic space attached to a quasi-projective complex algebraic variety Sh_K. This algebraic variety is defined over a number field E, the reflex field of the Shimura datum (Shimura, Deligne, Milne, Shih and Borovoi).

A complex algebraic representation $\xi : G_{\mathbb{C}} \longrightarrow GL(V_\xi)$ whose restriction to the center satisfies appropriate conditions gives rise, for $K \subset G(\widehat{\mathbb{Q}})$ sufficiently small, to a locally constant sheaf $\mathscr{L}_\xi$ of complex vector spaces on the analytic Shimura variety $Sh_K^{an}(G, \mathscr{X})$. We will assume that the derived group G^{der} is anisotropic over $\mathbb{Q}$. We fix a prime number ℓ and an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, and we consider the $\overline{\mathbb{Q}}_\ell$-vector space

$$H^i_{et} := \varinjlim_K H^i_{et}(Sh_K \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell}) \xrightarrow{\sim} \varinjlim_K H^i(Sh_K^{an}, \mathscr{L}_\xi)_\ell,$$

where the subscript ℓ denotes the base change from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$. Matsushima's formula [BW00, VII 5.2], combined with several cohomology theories, leads to the following decomposition of the ℓ-adic étale cohomology as a representation of $G(\widehat{\mathbb{Q}}) \times \Gamma_E$, indexed by irreducible representations π^∞ of $G(\widehat{\mathbb{Q}})$:

$$H^i_{et} = \bigoplus_{\pi^\infty} \left(\pi^\infty \otimes V^i(\pi^\infty)\right),$$

where π^∞ comes from an irreducible automorphic representation $\pi = \pi_\infty \otimes \pi^\infty$ of $G(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{R}) \times G(\widehat{\mathbb{Q}})$. The canonical action of $\Gamma_E := \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on H^i_{et} induces a continuous $\overline{\mathbb{Q}}_\ell$-linear action of Γ_E on the space $V^i(\pi^\infty)$. Our main objects of interest will the Galois representations $V^i(\pi^\infty)$, which will be denoted by ρ:

$$\rho : \Gamma_E \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V^i(\pi^\infty))$$

and we will be interested in studying the semi-simplicity of such representations.

We will consider algebraic groups G of the form $G = R_{F^+/\mathbb{Q}}(H)$, where R denotes the restriction of scalars; F^+ is a totally real field of degree r, whose r infinite primes $F^+ \hookrightarrow \mathbb{R}$ are denoted $v_1, \ldots, v_r$; and H is a reductive algebraic group over F^+, defined by means of a finite-dimensional simple $\mathbb{Q}$-algebra B with positive involution $*$ of the second kind, and with centre F, a CM field of degree two over F^+. The group H will be defined in a way that

$$G_{\mathbb{R}} \simeq \prod_{i=1}^{r} GU(a_i, b_i),$$

where $a_i + b_i = n \geq 2$. In this case, $G_{\mathbb{C}} \simeq (GL(n)_{\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}})^r$, and we have an explicit description of irreducible representations $\xi : G_{\mathbb{C}} \longrightarrow GL(V_\xi)$ and of the representations $\rho_1, \ldots, \rho_r$ defined in the next paragraph.

The representations $\rho_1, \ldots, \rho_r$

The irreducible automorphic representations π of $G(\mathbb{A}_{\mathbb{Q}})$ considered in our results are those who admit a "very weak base change" (which we will define in detail in Chapter 3). Roughly speaking, we consider representations π to which we can attach irreducible automorphic representations (Π, ψ) of $GL_n(\mathbb{A}_F) \times \mathbb{A}_F^\times$ satisfying certain conditions. Thanks to a big industry based on the work of Clozel, Harris, Taylor, Labesse and many other mathematicians, we can attach to Π a Galois representation [HT01, VII 1.9], [CH13]

$$\rho_{\Pi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$$

which will play an important role in the sequel, since it allows us to define the representations $\rho_1, \ldots, \rho_r$ appearing, in an abstract sense, in Chapters 1-2. More precisely, for $i = 1, \ldots, r$, ρ_i will be - up to a Tate twist - of the form

$$\rho_i = {}^{\sigma_i}\left(\left(\bigwedge^{a_i} \rho_{\Pi,\ell}^\vee\right) \otimes \rho_{\psi,\ell}^\vee\right),$$

where $\rho_{\psi,\ell}$ is a character attached to ψ, which appears because of the factors $\mathbb{G}_{m,\mathbb{C}}$ in $G_{\mathbb{C}}$, and $\sigma_i : F \hookrightarrow \mathbb{C}$ is a fixed embedding inducing v_i on F^+. The superscript σ_i in the previous formula denotes the conjugation, i.e. the left composition of a representation by the interior automorphism associated to σ_i^{-1}.

The relation between ρ and $\rho_1, \ldots, \rho_r$

The last ingredient of this part, leading us to study semi-simplicity criteria for representations of profinite groups over $\overline{\mathbb{Q}}_\ell$ in an abstract context, is a known generalisation of the Eichler-Shimura relations for modular curves to some Shimura varieties, also known by congruence relations, conjectured by Blasius and Rogawski [BR94], and proved separately by Wedhorn [Wed00] and Moonen [Moo04] in some cases. It states that the Frobenius correspondence on étale cohomology of the Shimura variety is a root of a Hecke polynomial defined starting from the Shimura datum $(G, \mathscr{X})$.
Recall that ρ denotes the Galois representation $V^i(\pi^\infty)$ of Γ_E in which we are interested, possibly after restriction to a suitably chosen open subgroup $\Gamma' \subset \Gamma_E$. In our context, the generalised Eichler-Shimura relations will be translated by identities

$$P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0$$

for g in a dense subset of Γ'. This is condition (1) in Section 0.1, which is implied by assumption (P) where, as mentioned before, the morphisms u_i will be given by partial Frobenius morphisms. We now understand better the motivation behind the abstract context, and are ready to combine both algebraic and geometric parts (c.f. Theorem 4 of the outline) to prove, in favourable cases, the semi-simplicity of the representation ρ.

0.3 Outline

0.3.1 Chapter 1

Let Γ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$, $\rho_i : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(W_i)$ continuous representations of Γ. Let $\mathfrak{g} = \mathrm{Lie}(\rho(\Gamma))$ and for each $i = 1, \ldots, r$, let $\mathfrak{g}_i = \mathrm{Lie}(\rho_i(\Gamma))$ and $\overline{\mathfrak{g}}_i = \mathfrak{g}_i \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.
After showing some results on Lie algebras and Lie algebra representations, we exhibit criteria for the semi-simplicity of ρ, independently of whether property (2) holds or not.

Theorem 1. *With the previous notations, assume that:*

1. *For each $i = 1, \ldots, r$, the restriction of ρ_i to any open subgroup of Γ is irreducible. (This implies that $\overline{\mathfrak{g}}_i$ is a reductive Lie algebra and each element of its center acts on W_i by a scalar).*
2. *For each $i = 1, \ldots, r$, a fixed Cartan subalgebra $\overline{\mathfrak{h}}_i \subset \overline{\mathfrak{g}}_i$ acts on W_i without multiplicities (that is the weight subspaces of W_i with respect to the action of $\overline{\mathfrak{h}}_i$ are one-dimensional).*
3. *There exists a dense subset $\Sigma \subset \Gamma$ such that for each $g \in \Sigma$, there exist pairwise commuting elements $u_1, \ldots, u_r \in Aut_{\overline{\mathbb{Q}}_\ell}(V)$ satisfying $\rho(g) = u_1 \ldots u_r$ and $P_{\rho_i(g)}(u_i) = 0 \in End_{\overline{\mathbb{Q}}_\ell}(V)$, $\forall\ i = 1, \ldots, r$.*

Then ρ is semi-simple.

0.3.2 Chapter 2

Let Γ be a profinite group, and $\rho : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ be a continuous finite-dimensional representation of Γ. For $i = 1, \ldots, s$, $j = 1, \ldots, m$, let $\Gamma_i \lhd \Gamma$ be an open normal subgroup such that $\Gamma/\Gamma_i = \langle \sigma_i \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ with p a prime number, $\alpha_{ij} : \Gamma_i \to \overline{\mathbb{Q}}_\ell^*$ be a continuous character of Γ_i, and $\rho_{ij} = \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$ be the induced representation. Let $\Gamma_0 = \Gamma_1 \cap \ldots \cap \Gamma_s$, and denote by $\mathrm{pr} : \Gamma \to \Gamma/\Gamma_0$ the canonical projection.

Theorem 2. *With the previous notations, assume that:*

1. *There exist dense subsets $\Sigma_0 \subset \Gamma_0$ and $\Sigma_1 \subset pr^{-1}(\sigma_1 \ldots \sigma_s)$ such that $\rho(g)$ is a semi-simple element of $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ for each $g \in \Sigma_0 \cup \Sigma_1$.*
2. *There exists an open subgroup $U \subset \Gamma$ such that $pr(U) = \Gamma/\Gamma_0$ and $\rho^{\mathrm{ss}}|_U$ is isomorphic to a subrepresentation of $(\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij})|_U^{\oplus n}$ for an integer $n \geq 1$.*
3. *The canonical injection $\Gamma/\Gamma_0 \hookrightarrow \prod_{i=1}^{s} \Gamma/\Gamma_i \simeq (\mathbb{Z}/p\mathbb{Z})^s$ is an isomorphism.*

Then ρ is semi-simple.

Proposition 1. *Assume that:*

1. *There exists a dense subset $\Sigma \subset \Gamma$ such that for each $g \in \Sigma$, there exist sm pairwise commuting elements $(u_{ij})_{i=1,\ldots,s,\ j=1,\ldots,m}$ in $Aut_{\overline{\mathbb{Q}}_l}(V)$ such that*

$$\rho(g) = \prod_{i=1}^{s} \prod_{j=1}^{m} u_{ij}$$

   and $P_{\rho_{ij}(g)}(u_{ij}) = 0$, $\forall\ (i,j) \in [\![1,s]\!] \times [\![1,m]\!]$.
2. *$\forall\ (i,j,k) \in [\![1,s]\!] \times [\![1,m]\!] \times [\![1,p-1]\!]$, the character $\alpha_{ij}/\alpha_{ij}^{\sigma_i^k}$ is of infinite order.*

3. *The canonical injection* $\Gamma/\Gamma_0 \hookrightarrow \prod_{i=1}^{s} \Gamma/\Gamma_i \simeq (\mathbb{Z}/p\mathbb{Z})^s$ *is an isomorphism.*

Then

1. *Assumption (1) of the previous theorem holds.*
2. *If* $s = 1$*, assumption (2) of the previous theorem holds (hence* ρ *is semi-simple).*

The case $r = 1$**.** Let Γ_1 be an open normal subgroup of Γ such that $\Gamma/\Gamma_1 = \langle\sigma\rangle \simeq \mathbb{Z}/n\mathbb{Z}$, with $n = p^v$. Let $\alpha : \Gamma_1 \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character of Γ_1 and $\rho_1 = \mathrm{Ind}_{\Gamma_1}^{\Gamma}(\alpha)$ be the induced representation.

Theorem 3. *With the previous notations, assume that:*

1. *There exists a dense subset* $\Sigma \subset \Gamma$ *such that* $P_{\rho_1(g)}(\rho(g)) = 0$ *for all* $g \in \Sigma$*.*
2. *For each* $i = 1, \ldots, n-1$*, the character* α/α^{σ^i} *is of infinite order.*

Then

1. *There exist dense subsets* $\Sigma_0 \subset \Gamma_1$ *and* $\Sigma_1 \subset pr^{-1}(\sigma)$ *such that* $\rho(g)$ *is a semi-simple element of* $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ *for each* $g \in \Sigma_0 \cup \Sigma_1$*.*
2. *There exists an open subgroup* $U \subset \Gamma$ *satisfying* $pr(U) = \Gamma/\Gamma_1$ *such that* $\rho|_U$ *is isomorphic to a subrepresentation of* $(\rho_1|_U)^{\oplus m}$ *for an integer* $m \geq 1$*.*
3. ρ *is semi-simple.*

0.3.3 Chapter 3

Let F^+, F, B be as in Section 0.2. To define our algebraic group H over F^+, we need to consider a finitely generated non-zero left B-module V, with a non-degenerate F^+-bilinear, $*$-Hermitian form. This data defines an involution $\#$ on $\mathrm{End}_{F^+}(V)$ which restricts to an involution $\#$ on $C := \mathrm{End}_B(V)$ (details in Section 3.2.1).
We define the algebraic group H by setting, for any F^+-algebra S, $H(S)$ equal to the set of $f \in (C \otimes_{F^+} S)^\times$ such that $ff^\# \in S^\times$, and we let $G = R_{F^+/\mathbb{Q}}(H)$.

Let $\xi : G_{\mathbb{C}} \longrightarrow GL(V_\xi)$ be an irreducible algebraic representation (of integral weight in the sense of Definition 3.2.1) which gives rise to $\mathscr{L}_\xi$ as in Section 0.2, and π be an irreducible automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ which admits a very weak base change as in Section 0.2.

Theorem 4. *In the situation described above, assume that a variant of the generalised Eichler-Shimura relations holds, in the same sense of the variant (P) of relation (1) of Section 0.1. Then*

1. *If the Galois representation* $\rho_{\Pi,\ell} : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ *attached to* Π *is irreducible and not induced from a proper subgroup of* Γ_F*, and if* $G_{\mathbb{R}}$ *has only signatures* $(n,0), (n-1,1), (1,n-1)$ *or* $(0,n)$*, then* $V^{dim(Sh)}(\pi^\infty)$ *is semi-simple.*
2. *If the* $\overline{\mathbb{Q}}_\ell$*-Lie algebra* $\overline{\mathfrak{g}} = \overline{\mathbb{Q}}_\ell \cdot Lie(\rho_{\Pi,\ell}(\Gamma_F)) \subset \mathfrak{gl}(n, \overline{\mathbb{Q}}_\ell)$ *contains* $\mathfrak{sl}(n, \overline{\mathbb{Q}}_\ell)$*, then* $V^{dim(Sh)}(\pi^\infty)$ *is semi-simple.*

Remark: Assuming an analogue of condition (2) of Section 0.1, we also prove some semi-simplicity criteria in the case where $\rho_{\Pi,\ell}$ is induced from a proper subgroup of Γ_F. To do this, we will need some of the abstract results exhibited in Chapter 2.

Chapter 1

Representations of profinite groups and semi-simplicity criteria

1.1 Preliminaries and notations

1.1.1 Split semi-simple Lie algebras

Let $\mathfrak{g}$ be a split semi-simple Lie algebra over a field $k \supset \mathbb{Q}$, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

1. We have the root decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha,$$

   where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} |\ [h, x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\},$$
$$R = \{\alpha \in \mathfrak{h}^* \smallsetminus \{0\} |\ \mathfrak{g}^\alpha \neq 0\}.$$

2. The set R is called the root system of $\mathfrak{g}$, and the subspaces $\mathfrak{g}^\alpha$ are called the root subspaces. The root lattice is $Q = \sum_{\alpha \in R} \mathbb{Z}\alpha \subset \mathfrak{h}^*$.
3. For a root α, let $\alpha^\vee$ denote the corresponding co-root. Let $R^\vee$ be the set of co-roots.
4. Denote by $\langle\ ,\ \rangle : \mathfrak{h} \times \mathfrak{h}^* \to k$ the duality pairing. For any root α, we have $\langle \alpha^\vee, \alpha \rangle = 2$, and $\langle R^\vee, R \rangle \subset \mathbb{Z}$.
5. Given a root α, there is a reflection s_α of $\mathfrak{h}^*$ defined by

$$s_\alpha(u) = u - \langle \alpha^\vee, u \rangle \alpha.$$

   It verifies $s_\alpha(R) \subset R$.
6. Let W be the Weyl group of the root system. It is generated by the reflections s_α for $\alpha \in R$, with the relations

$$s_\alpha^2 = \mathrm{id} \qquad \text{and} \qquad s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}, \qquad \forall \alpha, \beta \in R.$$

7. A choice of a Weyl chamber $C \subset \mathfrak{h}^*$ is equivalent to a decomposition $R = R_+ \sqcup R_-$ where

$$R_+ = \{\alpha \in R |\ \langle \alpha^\vee, u \rangle \geq 0\ \forall\ u \in C\}$$

   is the set of positive roots, and $R_- = -R_+$ is the set of negative roots.

8. The decomposition $R = R_+ \sqcup R_-$ gives a decomposition
$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \qquad \text{where} \qquad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}^\alpha.$$
9. Denote by $P \subset \mathfrak{h}^*$ the lattice of integral weights, given by
$$P = \{\lambda \in \mathfrak{h}^* |\ \lambda(\alpha^\vee) \in \mathbb{Z}, \forall\ \alpha \in R^\vee\}.$$
Define the set of dominant weights
$$P_{++} = P \cap \overline{C} \subset P_+ = P \cap \sum_{\alpha \in R_+} \mathbb{Q}_{\geqslant 0}\alpha,$$
where $\overline{C}$ is the closure of C. Let $Q_+ = P_+ \cap Q$ and $Q_{++} = P_{++} \cap Q$.

1.1.2 Representations of split semi-simple Lie algebras

1. Let $\mathfrak{g}$ be a split semi-simple Lie algebra, and let V be a finite-dimensional representation of $\mathfrak{g}$. We have a decomposition of V, called the weight decomposition with respect to the action of $\mathfrak{h}$,
$$V = \bigoplus_{\alpha \in P(V)} V(\alpha),$$
where
$$V(\alpha) = \{v \in V |\ hv = \alpha(h)v, \forall h \in \mathfrak{h}\}$$
$$P(V) = \{\alpha \in \mathfrak{h}^* |\ V(\alpha) \neq 0\}.$$
2. $P(V)$ is the set of weights of V, and is contained in P; $V(\alpha)$ is the weight subspace. Note that the roots of $\mathfrak{g}$ are the weights of its adjoint representation.
3. A non-zero representation V of $\mathfrak{g}$ is called a highest weight representation if it is generated by a vector $v \in V(\lambda)$ such that $\mathfrak{n}_+ \cdot v = 0$. In this case, v is called the highest weight vector, and λ the highest weight of V.
In other words, a highest weight vector of V is a non-zero vector which is both an eigenvalue for the action of $\mathfrak{h}$ and in the kernel of $\mathfrak{g}^\alpha$ for all $\alpha \in R_+$.
4. For a highest weight representation V, of highest weight λ, $P(V) \subset \lambda - Q_+$.
5. Every finite-dimensional irreducible representation of $\mathfrak{g}$ is a highest weight representation.

1.2 Some Lie algebra properties

Definition 1.2.1. Let V and W be finite-dimensional vector spaces over a field of characteristic zero, and let $f : V \to W$ be a polynomial application. f is said to be dominant if $f(V)$ is dense in W for the Zariski topology.

Definition 1.2.2. Let $\mathfrak{g}$ be a Lie algebra over k. We call elementary automorphism group and denote by $\mathrm{Aut}_e(\mathfrak{g})$ the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the elements $e^{\mathrm{ad}_\mathfrak{g} x}$ with $\mathrm{ad}_\mathfrak{g} x$ nilpotent.

Lemma 1.2.1. *[Bou72, Ch. VII, §3, Exercise 15] Let k be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a Lie algebra over k, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and A a subset of $\mathfrak{g}$. Suppose that A is Zariski dense in $\mathfrak{g}$ and stable by $Aut_e(\mathfrak{g})$; then $A \cap \mathfrak{h}$ is Zariski dense in $\mathfrak{h}$.*

Proof. Let X be the Zariski closure in $\mathfrak{h}$ of $A \cap \mathfrak{h}$ and $U = \mathfrak{h} \smallsetminus X$. If $U \neq \emptyset$, then it is a dense subset for the Zariski topology on $\mathfrak{h}$. Let $\{\alpha_1, \ldots, \alpha_p\}$ be the root system of $\mathfrak{g}$ corresponding to $\mathfrak{h}$, and consider the map

$$\begin{array}{rccl} F: & \mathfrak{h} \times \mathfrak{g}^{\alpha_1} \times \ldots \times \mathfrak{g}^{\alpha_p} & \longrightarrow & \mathfrak{g} \\ & (h, x_1, \ldots, x_p) & \longmapsto & e^{\mathrm{ad}_{\mathfrak{g}} x_1} \cdots e^{\mathrm{ad}_{\mathfrak{g}} x_p} h. \end{array}$$

Let $B := F(U \times \mathfrak{g}^{\alpha_1} \times \ldots \times \mathfrak{g}^{\alpha_p})$. Since F is a dominant polynomial application [Bou72, Ch. VII, §3, n°2, Lemma 2], it follows that B contains a non-empty open subset of $\mathfrak{g}$ [Bou72, Ch. VII, App. I, Proposition 3]. On the other hand, $B \subset \mathfrak{g} \smallsetminus A$, which contradicts the fact that A is dense in $\mathfrak{g}$. Hence $U = \emptyset$ and $A \cap \mathfrak{h}$ is dense in $\mathfrak{h}$. □

Proposition 1.2.1. *[Bou72, Ch.VII, §3, Exercise 16] Let k be an algebraically closed field of characteristic zero, V a finite-dimensional k-vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra. Suppose that the set of semi-simple elements of $\mathfrak{g}$ is Zariski dense in $\mathfrak{g}$; then the Cartan subalgebras of $\mathfrak{g}$ are commutative and consist of semi-simple elements.*

Proof. Let A denote the set of semi-simple elements of $\mathfrak{g}$. If $x \in \mathfrak{g}$ is nilpotent, and $y \in \mathfrak{g}$, then

$$e^{\mathrm{ad}_{\mathfrak{g}} x}(y) = e^x \cdot y \cdot e^{-x}$$

by [Bou72, Ch. VII, §3, n°1, Lemma 1]. This implies that if y is semi-simple, then $e^{\mathrm{ad}_{\mathfrak{g}} x}(y)$ is also semi-simple. It follows that A is stable under $e^{\mathrm{ad}_{\mathfrak{g}} x}$, hence it is stable under $\mathrm{Aut}_e(\mathfrak{g})$. This means that A satisfies the assumptions of the previous lemma, which implies that for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $A \cap \mathfrak{h}$ is dense in $\mathfrak{h}$. For $x \in A \cap \mathfrak{h}$, $\mathrm{ad}_{\mathfrak{g}} x$ is semi-simple, and $\mathrm{ad}_{\mathfrak{h}} x$ is nilpotent, so $[A \cap \mathfrak{h}, \mathfrak{h}] = 0$, hence $A \cap \mathfrak{h}$ is a subset of the centralizer $C_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$, and since it is dense in $\mathfrak{h}$, then $\mathfrak{h} \subset C_{\mathfrak{g}}(\mathfrak{h})$, which means that $\mathfrak{h}$ is commutative.
$A \cap \mathfrak{h}$ consists of diagonalisable elements, and since it is commutative, then all its elements can be diagonalized simultaneously. This remains true in the subalgebra of $\mathfrak{gl}(V)$ generated by $A \cap \mathfrak{h}$, which coincides with $\mathfrak{h}$, because $A \cap \mathfrak{h}$ is dense in $\mathfrak{h}$. It follows directly that the elements of $\mathfrak{h}$ are semi-simple. □

Definition 1.2.3. [Bou72, Ch. VII, §5, Definition 1] Let V be a finite-dimensional vector space over a field k of characteristic zero, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra. We say that $\mathfrak{g}$ is a *decomposable linear Lie algebra* if both the semi-simple and the nilpotent part of every element of $\mathfrak{g}$ belong to $\mathfrak{g}$.

Proposition 1.2.2. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra.*

1. *If $\mathfrak{g}$ is decomposable, then there exists a Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$, reductive in $\mathfrak{gl}(V)$, such that $\mathfrak{g} = \mathfrak{m} \ltimes \mathfrak{n}_V(\mathfrak{g})$, where $\mathfrak{n}_V(\mathfrak{g})$ is the set of all elements of the radical of $\mathfrak{g}$ that are nilpotent in $\mathfrak{gl}(V)$.*
2. *The following are equivalent:*
   - *(a) $\mathfrak{g}$ is decomposable;*
   - *(b) each Cartan subalgebra of $\mathfrak{g}$ is decomposable;*
   - *(c) a Cartan subalgebra of $\mathfrak{g}$ is decomposable.*
3. *Let $X \subset \mathfrak{g}$ be a subset generating $\mathfrak{g}$ as a k-Lie algebra. Suppose that every element of X is either semi-simple or nilpotent; then $\mathfrak{g}$ is decomposable.*

Proof.

1. [Bou72, Ch. VII, §5, Proposition 7].

2. [Bou72, Ch. VII, §5, Theorem 2].
3. [Bou72, Ch. VII, §5, Theorem 1].

□

Proposition 1.2.3. *[Nek] Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ be simple Lie algebras of finite dimension over a field $k \supset \mathbb{Q}$. Let $\mathfrak{g} \subset \mathfrak{g}_1 \times ... \times \mathfrak{g}_m$ be a Lie subalgebra such that $\forall\ i = 1, \ldots, r$, $p_i(\mathfrak{g}) = \mathfrak{g}_i$, where $p_i : \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_m \to \mathfrak{g}_i$ is the canonical surjection. Then there exist:*
- *a partition of $I = \{1, \ldots, m\}$: $I = I_1 \sqcup \ldots \sqcup I_n$,*
- *for each $j \in J = \{1, \ldots, n\}$, a Lie algebra $\mathfrak{g}^{(j)}$,*
- *for each $j \in J$ and each $i \in I_j$, an isomorphism of Lie algebras $f_{ji} : \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_i$, such that*

$$\mathfrak{g} = \mathrm{Im}\left(\prod_{j\in J}\mathfrak{g}^{(j)} \xrightarrow{\Delta} \prod_{j\in J}\left(\mathfrak{g}^{(j)}\right)^{I_j} \xrightarrow{f} \prod_{j\in J}\prod_{i\in I_j}\mathfrak{g}_i = \prod_{i\in I}\mathfrak{g}_i\right),$$

where $\Delta = (\Delta_j)_{j\in J}$, each $\Delta_j : \mathfrak{g}^{(j)} \longrightarrow \left(\mathfrak{g}^{(j)}\right)^{I_j}$ is the diagonal map, and $f = (f_j)_{j\in J}$, each $f_j : \left(\mathfrak{g}^{(j)}\right)^{I_j} \longrightarrow \prod_{i\in I_j}\mathfrak{g}_i$ whose components are $(f_{ji})_{i\in I_j}$.

Proof. Let $\mathfrak{n} \subset \mathfrak{g}$ be an abelian ideal, then $\forall\ i = 1, \ldots, m$, $p_i(\mathfrak{n})$ is an abelian ideal of $\mathfrak{g}_i$, hence $p_i(\mathfrak{n}) = 0$, which implies that $\mathfrak{n} = 0$, hence $\mathfrak{g}$ is semi-simple. Write $\mathfrak{g} = \mathfrak{g}^{(1)} \times \ldots \times \mathfrak{g}^{(n)}$ with $\mathfrak{g}^{(j)}$ simple Lie algebras. For $j \in J = \{1, \ldots, n\}$, set

$$I_j = \{i \in I |\ p_i(\mathfrak{g}^{(j)}) \neq 0\} = \{i \in I |\ p_i \text{ induces an isomorphism } f_{ji} : \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_i\}.$$

Then $I_j \neq \emptyset$ and if $j \neq j'$ and $i \in I_j \cap I_{j'}$, then $\forall\ (X, X') \in \mathfrak{g}^{(j)} \times \mathfrak{g}^{(j')}$,

$$[f_{ji}(X), f_{j'i}(X')] = p_i([X, X']) = p_i(0) = 0$$

since X and X' commute. This implies that $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$ which is impossible because $\mathfrak{g}_i$ is simple. It follows that $I_j \cap I_{j'} = \emptyset$, and since $I = \bigcup_{j=1}^{n} I_j$, then $I_1, \ldots, I_n$ form a partition of I. The rest follows from the previous discussion. □

Proposition 1.2.4. *[Nek] Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ be finite-dimensional reductive Lie algebras over an algebraically closed field $k \supset \mathbb{Q}$. For each $i \in \{1, \ldots, m\}$, let M_i be a non-zero simple $\mathfrak{g}_i$-module of finite dimension. Let $\mathfrak{g} \subset \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_m$ be a Lie subalgebra such that $\forall\ i \in \{1, \ldots, m\}$, $p_i(\mathfrak{g}) = \mathfrak{g}_i$. Then:*
- *$\mathfrak{g}$ is reductive: $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathscr{D}\mathfrak{g}$,*
- *each element of $\mathfrak{z}(\mathfrak{g})$ acts on $M = M_1 \boxtimes \ldots \boxtimes M_m$ by a scalar,*
- *there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that all weights of $\mathfrak{h}$ occurring in the $\mathfrak{g}$-module M lie in one coset of the root lattice of $(\mathfrak{g}, \mathfrak{h})$.*

Proof. For each i, we have $\mathfrak{g}_i = \mathfrak{z}(\mathfrak{g}_i) \oplus \mathscr{D}\mathfrak{g}_i$, with $\mathscr{D}\mathfrak{g}_i$ semi-simple. Writing each $\mathscr{D}\mathfrak{g}_i = \prod_{t_i=1}^{m_i}\mathfrak{g}_{i,t_i}$ as a product of simple Lie algebras, and applying the previous proposition to

$$\mathscr{D}\mathfrak{g} \subset \prod_{i=1}^{m}\mathscr{D}\mathfrak{g}_i = \prod_{i=1}^{m}\prod_{t_i=1}^{m_i}\mathfrak{g}_{i,t_i},$$

where each $\mathfrak{g}_{i,t_i}$ is a simple Lie algebra, we deduce that $\mathscr{D}\mathfrak{g}$ is semi-simple, hence $\mathfrak{g}$ is reductive and $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathscr{D}\mathfrak{g}$. Since M_i is a simple $\mathfrak{g}_i$-module, then by Schur's lemma, each

element of $\mathfrak{z}(\mathfrak{g}_i) = p_i(\mathfrak{z}(\mathfrak{g}))$ acts on M_i by a scalar, the same holds for the action of each element of $\mathfrak{z}(\mathfrak{g})$ on M.
Again, by applying Proposition 1.2.3 to the latter decomposition of $\mathscr{D}\mathfrak{g}$ into a product of simple Lie algebras $\mathfrak{g}_{i,t_i}$ and the corresponding set

$$I = \{(i,t_i)|\ i \in \{1,\dots,m\} \text{ and } t_i \in \{1,\dots,m_i\}\},$$

we have simple Lie algebras $\mathfrak{g}^{(j)}$ and, for each $(i,t_i) \in I_j$, an isomorphism of Lie algebras

$$f_{j,(i,t_i)} : \mathfrak{g}^{(j)} \xrightarrow{\sim} \mathfrak{g}_{i,t_i}$$

and we can identify $\mathscr{D}\mathfrak{g}$ with $\prod\limits_{j\in J} \mathfrak{g}^{(j)}$ via

$$\mathscr{D}\mathfrak{g} = \operatorname{Im}\left(\prod_{j\in J}\mathfrak{g}^{(j)} \xrightarrow{\Delta} \prod_{j\in J}\left(\mathfrak{g}^{(j)}\right)^{I_j} \xrightarrow{f} \prod_{j\in J}\prod_{(i,t_i)\in I_j}\mathfrak{g}_{i,t_i} = \prod_{(i,t_i)\in I}\mathfrak{g}_{i,t_i}\right)$$

For each $i \in \{1,\dots,m\}$, write $M_i = \boxtimes\limits_{t_i} M_{i,t_i}$ where M_{i,t_i} is a simple $\mathfrak{g}_{i,t_i}$-module, and for each $(i,t_i) \in I_j$, let

$$N_{i,t_i} = f^*_{j,(i,t_i)}(M_{i,t_i})$$

be the corresponding simple $\mathfrak{g}^{(j)}$-module. The identification $\mathscr{D}\mathfrak{g} = \prod\limits_{j\in J}\mathfrak{g}^{(j)}$ implies that the $\mathscr{D}\mathfrak{g}$-module M is isomorphic to $\boxtimes\limits_{j\in J} M^{(j)}$, where

$$M^{(j)} = \bigotimes_{(i,t_i)\in I_j} N_{i,t_i}.$$

For each $j \in J$, denote by $pr_j : \mathscr{D}\mathfrak{g} \to \mathfrak{g}^{(j)}$ the canonical projection; and let $\mathfrak{h}^{(j)}$ be a Cartan subalgebra of $\mathfrak{g}^{(j)}$, $\lambda_{i,t_i} \in \mathfrak{h}^{(j)*}$ the highest weight N_{i,t_i}. Since N_{i,t_i} is a simple $\mathfrak{g}^{(j)}$-module (in particular, it is a highest weight representation of $\mathfrak{g}^{(j)}$), then all weights of $\mathfrak{h}^{(j)*}$ occurring in N_{i,t_i} lie in $\lambda_{i,t_i} + Q^{(j)}$ where $Q^{(j)}$ is the root lattice of $(\mathfrak{g}^{(j)}, \mathfrak{h}^{(j)})$.
Let $\mathfrak{h}' := \prod\limits_{j\in J}\mathfrak{h}^{(j)}$, this is a Cartan subalgebra of $\mathscr{D}\mathfrak{g}$, and

$$\lambda := \sum_{j\in J}\sum_{(i,t_i)} \lambda_{i,t_i}$$

is a weight $\mathfrak{h}'$. It follows from the previous discussion and the identification

$$M = \boxtimes_{j\in J}\left(\bigotimes_{(i,t_i)\in I_j} N_{(i,t_i)}\right)$$

that all weights of $\mathfrak{h}'$ occurring in M lie in the coset $\lambda + Q$ of the root lattice Q of $(\mathscr{D}\mathfrak{g}, \mathfrak{h}')$ which satisfies $pr_j(Q) = Q^{(j)}$ for each $j \in J$.
Finally, since $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathscr{D}\mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g})$ acts on M by a scalar, it follows that all weights of the Cartan subalgebra $\mathfrak{h} := \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}'$ of $\mathfrak{g}$ lie in one coset of the root lattice of $(\mathfrak{g}, \mathfrak{h})$. $\square$

1.3 Representations of profinite groups over $\overline{\mathbb{Q}}_\ell$

Let Γ be a profinite group, V a non-zero vector space of finite dimension over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ a continuous representation; then $\rho(\Gamma)$ is a compact Lie group of finite dimension over $\mathbb{Q}_\ell$, and its Lie algebra $\mathfrak{g} := \mathrm{Lie}(\rho(\Gamma)) \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ is a $\mathbb{Q}_\ell$-Lie algebra of finite dimension.
Let $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ be the $\overline{\mathbb{Q}}_\ell$-Lie algebra obtained by scalar extension, and $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ be the $\overline{\mathbb{Q}}_\ell$-vector subspace of $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ generated by $\mathfrak{g}$. It is a $\overline{\mathbb{Q}}_\ell$-Lie algebra, and we have a canonical surjection $\overline{\mathfrak{g}} \twoheadrightarrow \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$. A Cartan subalgebra of $\overline{\mathfrak{g}}$ will be denoted by default $\overline{\mathfrak{h}}$.

Proposition 1.3.1. *Let Γ be a profinite group, V a non-zero vector space of finite dimension over $\overline{\mathbb{Q}}_\ell$ and $\rho : \Gamma \longrightarrow Aut_{\overline{\mathbb{Q}}_\ell}(V)$ a continuous representation. Let $\mathfrak{g} = Lie(\rho(\Gamma))$. The following properties are equivalent:*

1. *ρ is semi-simple.*
2. *There exists an open subgroup $U \subset \Gamma$ such that $\rho|_U$ is semi-simple.*
3. *V is a semi-simple $\overline{\mathfrak{g}}$-module.*

Proof. We will prove that (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3). Note that (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): First, U being an open subgroup for the profinite topology, it is of finite index, let $m = [\Gamma : U]$, and X be a set of coset representatives for U in Γ.
Let W be a subrepresentation of ρ, it is in particular a subrepresentation of $\rho|_U$, hence there exists W' such that $V = W \oplus W'$ as U-representations. Let $\pi : V \to W$ be the projection on W, and $\varphi : V \to W$ defined by

$$\varphi = \frac{1}{m} \sum_{x \in X} x^{-1} \pi x.$$

The fact that π is an U-homomorphism implies that φ is independent of the choice of coset representatives X.
For $g \in \Gamma$,

$$g\varphi = \frac{1}{m} \sum_{x \in X} g x^{-1} \pi x = \frac{1}{m} \sum_{x \in X} (x g^{-1})^{-1} \pi x g^{-1} g = \varphi g$$

Hence φ is a Γ-homomorphism, and it verifies $\varphi \circ j = \mathrm{id}$, where $j : W \hookrightarrow V$ is the inculsion. This means that $V = W \oplus \mathrm{Ker}(\varphi)$ as Γ-representations; therefore ρ is semi-simple.

(1) $\Leftrightarrow$ (3): It is enough to see that for a subspace $W \subset V$, W is invariant under $\rho(\Gamma)$ if and only if it is invariant under $\mathfrak{g}$. This is a result of [Bou72, Ch. III, §6, n°5 cor. 1]. $\square$

Corollary 1.3.1. *The semi-simplification ρ^{ss} of ρ satisfies $\rho^{\mathrm{ss}}|_U = (\rho|_U)^{\mathrm{ss}}$, for any open subgroup $U \subset \Gamma$.*

Proposition 1.3.2. *Let $\mathfrak{g}$ be a split reductive finite-dimensional Lie algebra over $\mathbb{Q}_\ell$, and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let M be a finite direct sum of finite tensor products of one-dimensional $\mathfrak{g}$-modules, and $N \neq 0$ be a simple $\mathfrak{g}$-module such that each weight of $\mathfrak{h}$ occurring in N occurs in M. Then N is isomorphic to a submodule of M.*

Proof. Write $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}'$ where $\mathfrak{h}'$ is a Cartan subalgebra of $\mathscr{D}\mathfrak{g}$. The Lie algebra $\mathscr{D}\mathfrak{g}$ acts trivially on each one-dimensional $\mathfrak{g}$-module, hence it acts trivially on M, which means that M is a representation of $\mathfrak{g}/\mathscr{D}\mathfrak{g} \simeq \mathfrak{z}(\mathfrak{g})$, and each weight of $\mathfrak{h}'$ occurring in M is zero.

Our assumption implies then that each weight of $\mathfrak{h}'$ occurring in N is zero, hence N is also a representation of the abelian Lie algebra $\mathfrak{z}(\mathfrak{g})$, which means that N is a one-dimensional submodule of M (as $\mathfrak{z}(\mathfrak{g})$-modules, and hence as $\mathfrak{g}$-modules). $\square$

Proposition 1.3.3. *Let Γ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma \to Aut_{\overline{\mathbb{Q}}_\ell}(V)$, $\rho_i : \Gamma \to Aut_{\overline{\mathbb{Q}}_\ell}(W_i)$ continuous representations. Let $\mathfrak{g} = Lie(\rho(\Gamma))$ and for $i = 1, \ldots, r$, $\mathfrak{g}_i = Lie(\rho_i(\Gamma))$. Suppose that:*

1. *Each W_i is a direct sum of one-dimensional $\overline{\mathfrak{g}}_i$-modules.*
2. *There exist an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ (for the profinite topology), such that*
$$\forall\ g \in \Sigma, \quad P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0.$$

Then there exists an open subgroup $U \subset \Gamma'$ such that $\rho^{\mathrm{ss}}|_U = (\rho|_U)^{\mathrm{ss}}$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \ldots \otimes \rho_r)|_U^{\oplus m}$ for an integer $m \geq 1$.

Proof. By Proposition 1.3.1, we can assume that $\Gamma' = \Gamma$. By continuity, assumption (2) implies then that
$$\forall\ g \in \Gamma, \quad P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0.$$
As ρ^{ss} is semi-simple, it is enough to consider any of its simple submodules, hence we can assume that ρ is irreducible. After shrinking Γ if necessary, we can assume that the restriction of ρ to each open subgroup of Γ is irreducible, hence that V is a simple $\overline{\mathfrak{g}}$-module.
For $i = 1, \ldots, r$, $\mathfrak{g}_i \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i)$, hence $\overline{\mathfrak{g}}_i$ acts faithfully on W_i, and by assumption (1), it follows that $\overline{\mathfrak{g}}_i$ is abelian. Since V is a faithful representation of $\mathfrak{g}$, the simple action of $\overline{\mathfrak{g}}$ on V factors through $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$, which is then a reductive Lie algebra over $\overline{\mathbb{Q}}_\ell$. Let
$$\rho_0 = \rho_1 \oplus \ldots \oplus \rho_r : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(W_0)$$
where $W_0 = W_1 \oplus \ldots \oplus W_r$. Assumption (2) implies that $\forall\ g \in \mathrm{Ker}(\rho_0)$, $(\rho(g) - 1)^N = 0$ where $N = \dim(W_1 \otimes \ldots \otimes W_r)$, hence the Lie ideal $\mathfrak{a} = \overline{\mathbb{Q}}_\ell \cdot \mathrm{Lie}(\rho(\mathrm{Ker}(\rho_0))) \subset \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ consists of nilpotent elements, and by [Bou72, Ch. I, §4, n°2, Cor. 3], it is a nilpotent Lie ideal. Since $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ is reductive, $\mathfrak{a} = 0$. It follows that $\rho(\mathrm{Ker}(\rho_0))$ is a finite subgroup of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ consisting of unipotent elements, hence it is trivial, since in characteristic zero, a non-identity unipotent element has infinite order.
The inclusion $\mathrm{Ker}(\rho_0) \subset \mathrm{Ker}(\rho)$ yields a canonical surjection $f : \mathfrak{g}_0 = \mathrm{Lie}(\rho_0(\Gamma)) \twoheadrightarrow \mathfrak{g}$. Let $\overline{f} : \overline{\mathfrak{g}}_0 \to \overline{\mathfrak{g}} \twoheadrightarrow \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ be the map induced by f. We consider V and all W_i as irreducible representations of the abelian Lie algebra $\overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times \ldots \times \overline{\mathfrak{g}}_r$.
Assumption (2), together with the fact that $\mathfrak{g}_0$ is Zariski dense in $\overline{\mathfrak{g}}_0$, imply that
$$\forall\ X = (X_1 \ldots, X_r) \in \overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times \ldots \times \overline{\mathfrak{g}}_r, \quad P_{(X_1, \ldots, X_r)|W_1 \otimes \ldots \otimes W_r}(\overline{f}(X)) = 0,$$
where $(X_1, \ldots, X_r)$ acts on $W_1 \otimes \ldots \otimes W_r$ by $\sum_i 1 \otimes \ldots \otimes 1 \otimes X_i \otimes 1 \otimes \ldots \otimes 1$. It follows that an eigenvalue of $\overline{f}(X)$ for its action on V is an eigenvalue of $(X_1, \ldots, X_r)$ for its action on $W_1 \otimes \ldots \otimes W_r$. This means that each weight of the abelian Lie algebra $\overline{\mathfrak{g}}_0$ occurring in V occurs in $W_1 \otimes \ldots \otimes W_r$. By Proposition 1.3.2, the $\overline{\mathfrak{g}}_0$-module V is isomorphic to a submodule of $W_1 \otimes \ldots \otimes W_r$. This is equivalent to the existence of an open subgroup $U \subset \Gamma$ such that $\rho|_U$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \ldots \otimes \rho_r)|_U$. $\square$

Two elementary lemmas

Lemma 1.3.1. *Let k be a field and $u_1, \dots, u_r$ be pairwise commuting endomorphisms of a k-vector space V such that $\forall\ i = 1, \dots, r$, there exists $A_i \in M_{n_i}(k)$ satisfying $P_{A_i}(u_i) = 0$. Then $P_{A_1 \otimes \dots \otimes A_r}(u_1 \dots u_r) = 0$.*

Proof. By induction, it suffices to prove the lemma for $r = 2$. By Schur's unitary triangularization theorem, for $i = 1, 2$, there exist unitary matrices $U_i \in M_{n_i}(k)$ such that $U_i A_i U_i^{-1} = T_i$ is upper triangular. Then

$$(U_1 \otimes U_2)(A_1 \otimes A_2)(U_1 \otimes U_2)^{-1} = (U_1 A_1 U_1^{-1}) \otimes (U_2 A_2 U_2^{-1}) = T_1 \otimes T_2$$

is upper triangular. Let $\lambda_1, \dots, \lambda_{n_1}$ (resp. $\mu_1, \dots, \mu_{n_2}$) be the eigenvalues of A_1 (resp. A_2) counting multiplicities, then $\lambda_1, \dots, \lambda_{n_1}$ (resp. $\mu_1, \dots, \mu_{n_2}$) are the main diagonal of T_1 (resp. T_2). Since the $n_1 n_2$ products $(\lambda_i \mu_j)_{i=1,\dots,n_1,\ j=1,\dots,n_2}$ are the main diagonal of $T_1 \otimes T_2$ (which is similar to $A_1 \otimes A_2$), then $(\lambda_i \mu_j)_{i=1,\dots,n_1,\ j=1,\dots,n_2}$ are the eigenvalues of $A_1 \otimes A_2$. It remains to show that for $u_1, u_2 \in \mathrm{End}_k(V)$, if $u_1 u_2 = u_2 u_1$, then

$$\prod_{i=1}^{n_1}(u_1 - \lambda_i) = \prod_{j=1}^{n_2}(u_2 - \mu_j) = 0 \overset{?}{\Longrightarrow} \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}(u_1 u_2 - \lambda_i \mu_j) = 0.$$

This is a direct consequence of the following lemma: □

Lemma 1.3.2. *Let k be a field, and consider the ring $R = k[X, Y, (\lambda_i)_i, (\mu_j)_j]$ with variables $X, Y, (\lambda_i)_{i=1,\dots,m}, (\mu_j)_{j=1,\dots,n}$. Then*

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(XY - \lambda_i \mu_j) \in \left\langle \prod_{i=1}^{m}(X - \lambda_i), \prod_{j=1}^{n}(Y - \mu_j) \right\rangle,$$

where for $f, g \in R$, $\langle f, g \rangle$ denotes the ideal generated by f and g.

Proof. For simplicity, for $k = 1, \dots, m$ and $l = 1, \dots, n$, set

$$f_k = \prod_{i=1}^{k}(X - \lambda_i)\ , \qquad g_l = \prod_{j=1}^{l}(Y - \mu_j)\ , \qquad h_{kl} = \prod_{i=1}^{k}\prod_{j=1}^{l}(XY - \lambda_i \mu_j).$$

We have to show then that $h_{mn} \in \langle f_m, g_n \rangle$. We proceed by double induction.
Assume first that $m = 1$.
We proceed by induction on n: the result is true for $n = 1$ since

$$h_{11} = XY - \lambda_1 \mu_1 = (X - \lambda_1)Y + \lambda_1(Y - \mu_1) = Y f_1 + \lambda_1 g_1 \in \langle f_1, g_1 \rangle.$$

Suppose it is true for $n - 1$, then it holds for n since

$$h_{1n} = (XY - \lambda_1 \mu_n) h_{1,n-1} \in \langle f_1, (Y - \mu_n) \rangle . \langle f_1, g_{n-1} \rangle \subset \langle f_1, g_n \rangle.$$

Hence the result holds for $m = 1$ (and n arbitrary).
We now proceed by induction on m. Suppose the result is true for $m - 1$, then it holds for m since

$$h_{mn} = h_{m-1,n}\left(\prod_{j=1}^{n}(XY - \lambda_m \mu_j)\right) \in \langle f_{m-1}, g_n \rangle . \langle (X - \lambda_m), g_n \rangle \subset \langle f_m, g_n \rangle.$$

□

1.4 Semi-simplicity criteria - Main result

Theorem 1.4.1. *Let Γ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\overline{\mathbb{Q}}_\ell$, and $\rho : \Gamma \to Aut_{\overline{\mathbb{Q}}_\ell}(V)$, $\rho_i : \Gamma \to Aut_{\overline{\mathbb{Q}}_\ell}(W_i)$ continuous representations of Γ. Let $\mathfrak{g} = Lie(\rho(\Gamma))$ and for each $i = 1, \ldots, r$, let $\mathfrak{g}_i = Lie(\rho_i(\Gamma))$. Suppose that*

1. *For each $i = 1, \ldots, r$, the restriction of ρ_i to any open subgroup of Γ is irreducible. This implies that $\overline{\mathfrak{g}}_i$ is a reductive Lie algebra and each element of its center acts on W_i by a scalar.*
2. *For each $i = 1, \ldots, r$, a fixed Cartan subalgebra $\overline{\mathfrak{h}}_i \subset \overline{\mathfrak{g}}_i$ acts on W_i without multiplicities (i.e., the weight subspaces of W_i with respect to the action of $\overline{\mathfrak{h}}_i$ are one-dimensional).*
3. *There exist an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$, there exist pairwise commuting elements $u_1, \ldots, u_r \in Aut_{\overline{\mathbb{Q}}_\ell}(V)$ satisfying $\rho(g) = u_1 \ldots u_r$ and $P_{\rho_i(g)}(u_i) = 0 \in End_{\overline{\mathbb{Q}}_\ell}(V)$, $\forall\ i = 1, \ldots, r$.*

Then ρ is semi-simple.

1.4.1 Steps of the proof and preliminaries

As before, we can assume that $\Gamma = \Gamma'$. Thanks to Lemma 1.3.1, assumption (3) implies that $\forall\ g \in \Sigma \subset \Gamma$, $P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0$ in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$. By continuity, this implies

$$\forall\ g \in \Gamma, \quad P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0 \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V). \tag{1.1}$$

As before, let $\rho_0 = \rho_1 \oplus \ldots \oplus \rho_r$ and $\mathfrak{g}_0 = \mathrm{Lie}(\rho_0(\Gamma))$.

We will proceed in five steps. Our strategy is based on [Nek, Theorem 2.6].

1. There exist an open subgroup $\Gamma_0 \subset \Gamma$ and a dense subset $\Sigma_0 \subset \Gamma_0$ (for the profinite topology) such that for each $g \in \Sigma_0$, $\rho(g)$ is a semi-simple element of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$.
2. $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} \subset \mathfrak{gl}(V)$ is a decomposable linear Lie algebra. We write $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} = \overline{\mathfrak{m}} \ltimes \overline{\mathfrak{n}}$, where $\overline{\mathfrak{n}} := \mathfrak{n}_V(\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g})$ and $\overline{\mathfrak{m}} \subset \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ a Lie subalgebra, reductive in $\mathfrak{gl}(V)$ (as in Proposition 1.2.2(1)).
   $\overline{\mathfrak{m}}$ acts semi-simply on V, and there exists a flag $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V$ of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$-submodules such that $\overline{\mathfrak{n}}$ acts trivially (and $\overline{\mathfrak{m}}$ semi-simply) on $\mathrm{gr}(V) := \bigoplus_{i=1}^{s} V_i/V_{i-1}$.
3. For a Cartan subalgebra $\overline{\mathfrak{h}}_0 \subset \overline{\mathfrak{g}}_0$, all weights of $\overline{\mathfrak{h}}_0$ occurring in $W' = W_1 \otimes \ldots \otimes W_r$ lie in one coset of the root lattice of $(\overline{\mathfrak{g}}_0, \overline{\mathfrak{h}}_0)$; and each weight of $\overline{\mathfrak{h}}_0$ occurring in $\mathrm{gr}(V)$ occurs in W'.
4. For a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$, all weights of $\overline{\mathfrak{h}}$ occuring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$.
5. [Nek, Theorem 2.6] applies, which means that $\overline{\mathfrak{n}} = 0$, $\overline{\mathfrak{m}} = \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ is a reductive Lie algebra and V is a semi-simple $\overline{\mathfrak{g}}$-module.

1.4.2 Proof of the theorem

First step: We show that there exist an open subgroup $\Gamma_0 \subset \Gamma$ and a dense subset $\Sigma_0 \subset \Gamma_0$ (for the profinite topology) such that for each $g \in \Sigma_0$, $\rho(g)$ is a semi-simple

element of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$.

For each $i = 1, \dots, r$, denote by m_i the dimension of W_i and write

$$W_i = \bigoplus_{j=1}^{m_i} W_i(\alpha_{ij})$$

where $(\alpha_{ij})_{j=1,\dots,m_i}$ are the weights of $\overline{\mathfrak{h}}_i$ occurring in W_i, and the $W(\alpha_{ij})$ are the weight spaces, which are one-dimensional by assumption (2). For each $j = 1, \dots, m_i$, fix a generator w_{ij} of $W_i(\alpha_{ij})$. Then the action of $H_i \in \overline{\mathfrak{h}}_i$ on W_i, with respect to the basis $(w_{ij})_{j=1,\dots,m_i}$ of W_i, is given by the diagonal matrix whose m_i diagonal terms are $\alpha_{i1}(H_i), \dots, \alpha_{im_i}(H_i)$. Let

$$\Delta_i : \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i) \longrightarrow \overline{\mathbb{Q}}_\ell$$

be the polynomial function given by the discriminant of the characteristic polynomial. For each i, fix $H_i \in \overline{\mathfrak{h}}_i$ such that $\alpha_{ij}(H_i) \neq \alpha_{ij'}(H_i)$ for all $j \neq j'$ in $[\![1, m_i]\!]$. Then $\Delta_i(H_i) \neq 0$, and

$$(H_1, \dots, H_r) \in \bigcap_{i=1}^{r} p_i^{-1}(\Delta_i^{-1}(\overline{\mathbb{Q}}_\ell^\times))$$

which is then a non-empty open subset of $\overline{\mathfrak{g}}_0$, and hence is Zariski dense in $\overline{\mathfrak{g}}_0$. Its intersection U with $\mathfrak{g}_0$ is a dense Zariski open in $\mathfrak{g}_0$. Let $\Gamma_0 \subset \Gamma$ be an open subgroup such that the exponential map $\mathfrak{g}_0 \longrightarrow \rho_0(\Gamma)$ induces a homeomorphism

$$\exp : \exp^{-1}(\rho_0(\Gamma_0)) \longrightarrow \rho_0(\Gamma_0).$$

Let $T_0 = \exp^{-1}(\rho_0(\Gamma_0))$, then

$$U_0 := (\rho_0|_{\Gamma_0})^{-1}(\exp(T_0 \cap U))$$

is open and dense in Γ_0. For each dense subset $\Sigma \subset \Gamma$, $\Sigma_0 := \Sigma \cap U_0$ is dense in Γ_0 and the polynomials $P_{\rho_i(g)}(X)$ are without multiple roots for each $g \in \Sigma_0$, by definition of U (hence of U_0).
In particular, taking Σ as in assumption (3), for each $g \in \Sigma_0$, $\rho(g) = u_1 \dots u_r$ with u_i pairwise commuting elements of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$, and each u_i is annihilated by the polynomial $P_{\rho_i(g)}(X)$, which does not have multiple roots, hence each u_i is a semi-simple element of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$, and so is their product $\rho(g)$.

Second step: We show that $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} \subset \mathfrak{gl}(V)$ is a decomposable linear Lie algebra.

By the first step, the set of elements of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ that are semi-simple in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ is Zariski dense in $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$. It follows, thanks to Proposition 1.2.1, that the Cartan subalgebras of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ consist of semi-simple elements, hence they are decomposable (by Proposition 1.2.2(3) applied to Cartan subalgebras of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$), and so is $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ (by Proposition 1.2.2(2)). We write then $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} = \overline{\mathfrak{m}} \ltimes \overline{\mathfrak{n}}$, where $\overline{\mathfrak{n}} := \mathfrak{n}_V(\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g})$ and $\overline{\mathfrak{m}} \subset \overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ a Lie subalgebra, reductive in $\mathfrak{gl}(V)$ (as in Proposition 1.2.2(1)).

Since $\overline{\mathfrak{m}}$ is a reductive Lie algebra in $\mathfrak{gl}(V)$, the restriction to $\overline{\mathfrak{m}}$ of the adjoint representation of $\mathfrak{gl}(V)$ is then semi-simple [Bou72, Ch. I, §6, n°6, Cor. 1], and each element of the center $\mathfrak{z}(\overline{\mathfrak{m}})$ acts semi-simply on V, hence V is a semi-simple $\overline{\mathfrak{m}}$-module.
Since $\overline{\mathfrak{n}}$ is a nilpotent Lie algebra, there exists a flag $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{s-1} \subsetneq V_s = V$

of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$-submodules such that $\overline{\mathfrak{n}}$ acts trivially on $\mathrm{gr}(V) := \bigoplus_{i=1}^{s} V_i/V_{i-1}$. For example, take $V_0 = \{0\}$, and $V_{i+1} = \{v \in V \mid \overline{\mathfrak{n}} \cdot v \subset V_i\}$.

Third step: We show that for a Cartan subalgebra $\overline{\mathfrak{h}}_0 \subset \overline{\mathfrak{g}}_0$, all weights of $\overline{\mathfrak{h}}_0$ occurring in $W' = W_1 \otimes \ldots \otimes W_r$ lie in one coset of the root lattice of $(\overline{\mathfrak{g}}_0, \overline{\mathfrak{h}}_0)$; and each weight of $\overline{\mathfrak{h}}_0$ occurring in $\mathrm{gr}(V)$ occurs in W'.

We have $\mathfrak{g}_0 \subset \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_r$ with $p_i(\mathfrak{g}_0) = \mathfrak{g}_i$, where $p_i : \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_r \twoheadrightarrow \mathfrak{g}_i$ is the canonical surjection. The previous inclusion induces

$$\overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times \ldots \times \overline{\mathfrak{g}}_r$$

with $p_i(\overline{\mathfrak{g}}_0) = \overline{\mathfrak{g}}_i$.

Let $\mathfrak{s} = \mathrm{Lie}(\rho^{ss}(\Gamma)) \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathrm{gr}(V))$. The semi-simple action of $\overline{\mathfrak{g}}$ on $\mathrm{gr}(V)$ factors through $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{s}$, hence $\mathrm{gr}(V)$ is a faithful, semi-simple, finite-dimensional representation of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{s}$ which is then a reductive Lie algebra over $\overline{\mathbb{Q}}_\ell$.

As in Proposition 1.3.3, the property (1.1) implies that $\forall\, g \in \mathrm{Ker}(\rho_0)$, $(\rho(g)-1)^{\dim(W')} = 0$, which means in particular that the Lie ideal $\mathfrak{a} = \overline{\mathbb{Q}}_\ell \cdot \mathrm{Lie}(\rho^{\mathrm{ss}}(\mathrm{Ker}(\rho_0))) \subset \overline{\mathbb{Q}}_\ell \cdot \mathfrak{s}$ consists of nilpotent elements, hence is a nilpotent Lie ideal, and therefore $\mathfrak{a} = 0$ because $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{s}$ is reductive. This means that $\rho^{\mathrm{ss}}(\mathrm{Ker}(\rho_0))$ is a finite subgroup of $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathrm{gr}(V))$ consisting of unipotent elements, hence it is trivial.
The inclusion $\mathrm{Ker}(\rho_0) \subset \mathrm{Ker}(\rho^{\mathrm{ss}})$ yields a surjection

$$f : \mathfrak{g}_0 = \mathrm{Lie}(\rho_0(\Gamma)) \twoheadrightarrow \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma)) = \mathfrak{s}.$$

Assumption (1) implies that for each $i = 1, \ldots, r$, W_i is a simple $\overline{\mathfrak{g}}_i$-module, hence we can apply Proposition 1.2.4 to $\overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times ... \times \overline{\mathfrak{g}}_r$ and the W_i's. We get an isomorphism

$$h : \mathfrak{g}^{(1)} \times \ldots \times \mathfrak{g}^{(t)} \xrightarrow{\sim} \mathscr{D}\overline{\mathfrak{g}}_0$$

where each $\mathfrak{g}^{(i)}$ is a simple $\overline{\mathbb{Q}}_\ell$-Lie algebra. Fix Cartan subalgebras $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$ and $\overline{\mathfrak{h}}_0 \subset \overline{\mathfrak{g}}_0$ such that $\overline{\mathfrak{h}}_0 \cap \mathscr{D}\overline{\mathfrak{g}}_0 = h(\mathfrak{h}^{(1)} \times \ldots \times \mathfrak{h}^{(t)})$. This is a Cartan subalgebra of $\mathscr{D}\overline{\mathfrak{g}}_0$; Proposition 1.2.4 implies that all weights of $\overline{\mathfrak{h}}_0$ occurring in the $\overline{\mathfrak{g}}_0$-module W' lie in one coset $\lambda_0 + Q_0$ of the root lattice Q_0 of $(\overline{\mathfrak{g}}_0, \overline{\mathfrak{h}}_0)$.

Denote by $\overline{f} : \overline{\mathfrak{g}}_0 \twoheadrightarrow \overline{\mathfrak{s}}$ the surjection induced by f. In particular, $\mathrm{gr}(V)$ is a $\overline{\mathfrak{g}}_0$-module. Each $X = (X_1, \ldots, X_r) \in \overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times ... \times \overline{\mathfrak{g}}_r$ acts on W' by $\sum_{i=1}^{r} 1 \otimes \ldots \otimes 1 \otimes X_i \otimes 1 \otimes \ldots \otimes 1$, which means that W' is a $\overline{\mathfrak{g}}_0$-module. The property (1.1) implies that

$$\forall\, X \in \overline{\mathfrak{g}}_0, \quad P_{X|W'}(\overline{f}(X)) = 0 \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathrm{gr}(V)).$$

The previous property applied to elements $X \in \overline{\mathfrak{h}}_0$ implies that each weight of $\overline{\mathfrak{h}}_0$ occurring in $\mathrm{gr}(V)$ occurs in W'.

Fourth step: We show that for a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$, all weights of $\overline{\mathfrak{h}}$ occuring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$.

The Lie algebra $\overline{\mathfrak{g}}_0$ being reductive, its adjoint representation is semi-simple, hence its Lie ideal $\mathrm{Ker}(\overline{f})$ has a supplementary Lie ideal $\mathfrak{a}$, and $\overline{\mathfrak{g}}_0$ identifies with $\mathrm{Ker}(\overline{f}) \times \mathfrak{a}$. This implies that $\mathfrak{a} \simeq \overline{\mathfrak{g}}_0/\mathrm{Ker}(\overline{f}) \simeq \mathrm{Im}(\overline{f}) = \overline{\mathfrak{s}}$, hence

$$\overline{\mathfrak{g}}_0 = \mathrm{Ker}(\overline{f}) \times \overline{\mathfrak{s}}.$$

Let pr_1 and pr_2 be the projections

$$pr_1 : \overline{\mathfrak{g}}_0 \twoheadrightarrow \mathrm{Ker}(\overline{f}) \qquad\qquad pr_2 : \overline{\mathfrak{g}}_0 \twoheadrightarrow \overline{\mathfrak{s}}.$$

It follows that $\overline{\mathfrak{h}}_0$ is of the form

$$\overline{\mathfrak{h}}_0 = \overline{\mathfrak{h}}_1 \times \overline{\mathfrak{h}}_2$$

where $\overline{\mathfrak{h}}_1 = pr_1(\overline{\mathfrak{h}}_0) \subset \mathrm{Ker}(\overline{f})$ and $\overline{\mathfrak{h}}_2 = pr_2(\overline{\mathfrak{h}}_0) \subset \overline{\mathfrak{s}}$ are Cartan subalgebras. The same decomposition of the duals implies that we can decompose $\lambda_0 \in \overline{\mathfrak{h}}_0^*$ into

$$(\lambda_1, \lambda_2) \in \overline{\mathfrak{h}}_1^* \times \overline{\mathfrak{h}}_2^*.$$

Let $Q_1 = pr_1(Q_0)$, $Q_2 = pr_2(Q_0)$ the corresponding root lattices of $(\mathrm{Ker}(\overline{f}), \overline{\mathfrak{h}}_1)$ and $(\overline{\mathfrak{s}}, \overline{\mathfrak{h}}_2)$, respectively.

We have then the following:

$$\begin{aligned}
\alpha \text{ is a weight of } \overline{\mathfrak{h}}_2 \text{ occurring in } \mathrm{gr}(V) &\overset{1}{\Rightarrow} pr_2^*(\alpha) = (0, \alpha) \text{ occurs in } \mathrm{gr}(V) \text{ as a weight of } \overline{\mathfrak{h}}_0 \\
&\overset{2}{\Rightarrow} (0, \alpha) \text{ occurs in } W' \text{ as a weight of } \overline{\mathfrak{h}}_0 \\
&\overset{3}{\Rightarrow} (0, \alpha) \text{ lies in } \lambda_0 + Q_0 \\
&\overset{4}{\Rightarrow} \alpha \text{ lies in } \lambda_2 + Q_2.
\end{aligned}$$

The implications (1) and (4) are obvious, using the projection pr_2; and the implications (2) and (3) are direct consequences of the third step. We deduce that all weights of $\overline{\mathfrak{h}}_2$ occurring in $\mathrm{gr}(V)$ belong to $\lambda_2 + Q_2$.

Finally, the action of $\overline{\mathfrak{s}}$ on $\mathrm{gr}(V)$ factors through $\overline{\mathfrak{t}} := \mathrm{Im}(\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} \longrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\mathrm{gr}(V)))$ which is a quotient of the reductive Lie algebra $(\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g})/\overline{\mathfrak{n}} \simeq \overline{\mathfrak{m}}$. Denoting by $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$ a Cartan subalgebra of $\overline{\mathfrak{m}}$ corresponding to $\overline{\mathfrak{h}}_2 \subset \overline{\mathfrak{s}}$ (i.e. $f_1(\overline{\mathfrak{h}}) = f_2(\overline{\mathfrak{h}}_2)$ where $\overline{\mathfrak{s}} \xrightarrow{f_2} \overline{\mathfrak{t}} \xleftarrow{f_1} \overline{\mathfrak{m}}$), we deduce that the weights of $\overline{\mathfrak{h}}$ occuring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$ (since the weights of $\overline{\mathfrak{h}}_2$ occuring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{s}}, \overline{\mathfrak{h}}_2)$, as proved before).

Fifth step: We check that [Nek, Theorem 2.6] applies for the $\overline{\mathbb{Q}}_\ell$-Lie subalgebra $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$.

Recall the assumptions of [Nek, Theorem 2.6]:
(H1-ZAR) $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ contains a Zariski dense set of elements that are semi-simple in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$.
(H2) for $\overline{\mathfrak{m}}$ as in the second step, all weights of a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$ occuring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$.

By the first step, the set of elements of $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$ that are semi-simple in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ is Zariski dense in $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$, which means that (H1-ZAR) holds. The fourth step implies that (H2) holds. Hence [Nek, Theorem 2.6] applies, it follows that $\overline{\mathfrak{n}} = 0$, $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g} = \overline{\mathfrak{m}}$ and V is a semi-simple $\overline{\mathbb{Q}}_\ell \cdot \mathfrak{g}$-module.

Chapter 2

The case of induced characters

2.1 Some properties of induced representations

For $i = 1, 2$, let G_i be a group, H_i, K_i two subgroups of G_i, π_i a representation of H_i, and ρ_i a representation of G_i. We have the following properties:

1. If $G_1 = G_2$, $\rho_1 \otimes \rho_2 \simeq (\rho_1 \boxtimes \rho_2)|_{\Delta(G_1 \times G_1)}$, where Δ denotes the diagonal.
2. $\mathrm{Ind}_{H_1}^{G_1}(\pi_1) \boxtimes \mathrm{Ind}_{H_2}^{G_2}(\pi_2) \simeq \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\pi_1 \boxtimes \pi_2)$.
3. $\rho_1 \otimes \mathrm{Ind}_{H_1}^{G_1}(\pi_1) \simeq \mathrm{Ind}_{H_1}^{G_1}(\mathrm{Res}_{G_1}^{H_1}(\rho_1) \otimes \pi_1)$.
4. If $H_1 \triangleleft G_1$, there is a natural right action of G_1/H_1 on the set of isomorphism classes of representations of H_1. Let $\mathrm{pr} : G_1 \to G_1/H_1$ be the canonical projection. The action is given by
$$\pi_1^{\mathrm{pr}(g)}(h) = \pi_1(ghg^{-1}), \qquad \forall\ g \in G_1,\ h \in H_1.$$
5. If W is the representation space of π_1, then the representation space of $\mathrm{Ind}_{H_1}^{G_1}(\pi_1)$ is given by
$$\{f : G_1 \longrightarrow W |\ f(hg) = \pi_1(h)f(g)\ \forall\ h \in H_1,\ \forall\ g \in G_1\},$$
and the action of G_1 on $\mathrm{Ind}_{H_1}^{G_1}(\pi_1)$ is $(g_1 \cdot f)(g) = f(gg_1)$.
6. Mackey's restriction formula:
$$\mathrm{Res}_{G_1}^{K_1}(\mathrm{Ind}_{H_1}^{G_1}(\pi_1)) \simeq \bigoplus_{g \in K_1 \backslash G_1/H_1} \mathrm{Ind}_{gH_1g^{-1} \cap K_1}^{K_1}(\pi_1^g),$$
where π_1^g is the representation of $gH_1g^{-1} \cap K_1$ given by $\pi_1^g(x) = \pi_1(g^{-1}xg)$, and $K_1 \setminus G_1/H_1$ is the set of double cosets K_1gH_1 for $g \in G_1$.
7. If H_1 is abelian and of finite index in G_1, then every irreducible representation of G_1 is a subrepresentation of $\mathrm{Ind}_{H_1}^{G_1}(\beta)$ for some character β of H_1.
Let S be the stabilizer of β under the action of G_1/H_1, and set $H_\beta = \mathrm{pr}^{-1}(S) \subset G_1$, then the decomposition of $\mathrm{Ind}_{H_1}^{G_1}(\beta)$ into irreducible representations is
$$\mathrm{Ind}_{H_1}^{G_1}(\beta) = \bigoplus_{\tilde{\beta}} \mathrm{Ind}_{H_\beta}^{G_1}(\tilde{\beta})$$
where $\tilde{\beta}$ runs through all characters $\tilde{\beta}$ of H_β such that $\tilde{\beta}|_{H_1} = \beta$.

2.2 Semi-simplicity criteria - Main result

Notations. Let Γ be a profinite group and $\rho : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ a continuous finite-dimensional representation of Γ. For $i = 1, \dots, s$, $j = 1, \dots, m$, let $\Gamma_i \lhd \Gamma$ be an open normal subgroup such that $\Gamma/\Gamma_i = \langle \sigma_i \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ with p a prime number, $\alpha_{ij} : \Gamma_i \to \overline{\mathbb{Q}}_\ell^\times$ a continuous character of Γ_i, and $\rho_{ij} = \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$ the induced representation. For $i = 1, \dots, s$, let $\tilde{\sigma}_i$ be a lift of σ_i in Γ. Set

$$\Gamma_0 := \bigcap_{i=1}^{s} \Gamma_i$$

and denote by $\mathrm{pr} : \Gamma \to \Gamma/\Gamma_0$ the canonical projection.
Let $J = (1, \dots, 1) \in (\mathbb{Z}/p\mathbb{Z})^s$, and for $I = (i_1, \dots, i_s) \in (\mathbb{Z}/p\mathbb{Z})^s$, set

$$\tilde{\sigma}_I := \tilde{\sigma}_1^{\tilde{i_1}} \dots \tilde{\sigma}_s^{\tilde{i_s}} \in \Gamma \quad \text{and} \quad \sigma_I = \mathrm{pr}(\tilde{\sigma}_I) \in \Gamma/\Gamma_0,$$

where $\forall\ k = 1, \dots, s$, $\tilde{i_k} \in [\![0, p-1]\!]$ is the representative of i_k.
The following theorem is a generalisation of a result of [Nek] where the case $s = 1$ is studied.

Theorem 2.2.1. *In the situation described above, assume that:*

1. *There exist dense subsets $\Sigma_0 \subset \Gamma_0$ and $\Sigma_1 \subset pr^{-1}(\sigma_J)$ such that $\rho(g)$ is a semi-simple element of $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ for each $g \in \Sigma_0 \cup \Sigma_1$.*
2. *There exists an open subgroup $U \subset \Gamma$ such that $pr(U) = \Gamma/\Gamma_0$ and $\rho^{\mathrm{ss}}|_U$ is isomorphic to a subrepresentation of $(\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij})|_U^{\oplus n}$ for an integer $n \geq 1$.*
3. *The canonical injection $\Gamma/\Gamma_0 \hookrightarrow \prod_{i=1}^{s} \Gamma/\Gamma_i \simeq (\mathbb{Z}/p\mathbb{Z})^s$ is an isomorphism.*

Then ρ is semi-simple.

2.2.1 Steps of the proof and preliminaries

We will proceed in five steps:

1. We reduce to the case where ρ sits in an exact sequence of representations

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0$$

   for suitable characters α and β of Γ_0, where for a character α, $I(\alpha)$ denotes the induced representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\alpha)$.

2. We identify the representation spaces $I(\alpha)$ and $I(\beta)$ with $\overline{\mathbb{Q}}_\ell^{p^s}$. After choosing a splitting of the previous exact sequence of vector spaces, an element $g \in \Gamma$ acts on V by a matrix

$$\rho(g) = \begin{pmatrix} \eta_2(g) & c(g)\eta_1(g) \\ 0 & \eta_1(g) \end{pmatrix},$$

   where η_1 and η_2 denote the representations $I(\alpha)$ and $I(\beta)$, respectively, and $c \in Z^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta)))$ is a 1-cocycle. Then we make explicit the characteristic polynomials of η_1 and η_2 for elements of Γ of the form $h\tilde{\sigma}_J$ for $h \in \Gamma_0$.

3. We show that for $h \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, the main diagonal of $c(h)$ vanishes.

4. We show that for h as in the previous step, $c(h) = 0$.
5. The exact sequence of representations in the first step splits, which means that $\rho \simeq I(\alpha) \oplus I(\beta)$ and hence ρ is semi-simple.

Lemma 2.2.1 (elementary)**.** *Let K be a field and $A = (A_{ij}) \in M_n(K)$ be a monomial matrix, that is a matrix whose only non-zero entries are $A_{i,\tau(i)}$ for some permutation $\tau \in \mathfrak{S}_n$. Let $O_1, \dots, O_r$ denote the orbits of τ, and set n_i the cardinality of O_i. Then the characteristic polynomial of A is*

$$P_A(X) = \prod_{i=1}^{r} \left(X^{n_i} - \prod_{j \in O_i} A_{j,\tau(j)} \right).$$

Lemma 2.2.2 (elementary)**.** *Let $A = (A_{ij}), B = (B_{ij}) \in GL_n(K)$ be monomial matrices with the same associated permutation $\tau \in \mathfrak{S}_n$, and let $C = (C_{ij}) \in M_n(K)$. For $l \in \mathbb{Z}$, denote by $(A^l)_{ij}$ the entries of A^l, then for $l \geq 0$,*

$$(A^l)_{ij} = \begin{cases} \prod_{k=0}^{l-1} A_{\tau^k(i),\tau^{k+1}(i)} & \text{if } j = \tau^l(i) \\ 0 & \text{if } j \neq \tau^l(i) \end{cases}$$

and

$$(A^{-l})_{ij} = \begin{cases} \left(\prod_{k=0}^{l-1} A_{\tau^k(j),\tau^{k+1}(j)} \right)^{-1} & \text{if } i = \tau^l(j) \\ 0 & \text{if } i \neq \tau^l(j) \end{cases}$$

Let D be the matrix $A^l C B^{-l}$, then for $i, j = 1, \dots, n$,

$$D_{ij} = \prod_{k=0}^{l-1} \frac{A_{\tau^k(i)\tau^{k+1}(i)}}{B_{\tau^k(j)\tau^{k+1}(j)}} C_{\tau^l(i)\tau^l(j)}.$$

Lemma 2.2.3 (Sah's lemma)**.** *Let G be a group, M a G-module and g in the center of G. Then $H^1(G, M)$ is killed by the endomorphism $x \mapsto gx - x$ of M. In particular, if this endomorphism is an automorphism, then $H^1(G, M) = 0$.*

Lemma 2.2.4. *Let M be a module of finite length over a ring R such that any subquotient of M of length 2 is a semi-simple R-module, then M is a semi-simple R-module.*

Proof. We will proceed by induction on the length l of M. If $l = 1$, there is nothing to prove. Assume that $l > 2$ and the lemma is proved for R-modules of length $< l$. Let $N \subset M$ be a simple non-zero R-submodule. Consider a Jordan-Hölder series $0 \subset N \subset M_2 \subset \cdots \subset M_l = M$ of M, then $0 \subset M_2/N \subset \cdots \subset M_l/N$ is a Jordan-Hölder series of M/N. This means that M/N has length $l - 1$, hence it is a semi-simple R-module, by induction. We write $M/N = N_1 \oplus \cdots \oplus N_{l-1}$ with each N_i a simple R-module.
Fix $i = 0, \dots, l-1$ and denote by M^i the kernel of the projection

$$M \longrightarrow M/N \longrightarrow \bigoplus_{j \neq i} N_j.$$

Then M^i is a subquotient of M of length 2, since it sits in an exact sequence

$$0 \longrightarrow N \longrightarrow M^i \longrightarrow N_i \longrightarrow 0.$$

It follows, thanks to our assumption, that M^i is a semi-simple R-module, which means that its class in $\mathrm{Ext}^1_R(N_i, N)$ is zero. As a result, the class of M in

$$\mathrm{Ext}^1_R(M/N, N) = \mathrm{Ext}^1_R\left(\bigoplus_i N_i, N\right) = \bigoplus_i \mathrm{Ext}^1_R(N_i, N)$$

is also zero, hence $M \simeq M/N \oplus N$ as R-modules, and M is a semi-simple R-module.

□

Lemma 2.2.5. *Let k be a field of characteristic zero, and V be a k-vector space of dimension $2n$ over k. Let $f \in End_k(V)$ such that, with respect to a fixed basis of V over k, f acts by a matrix*

$$F = \begin{pmatrix} D & A \\ 0 & D \end{pmatrix},$$

where $D = diag(d_1, \dots, d_n)$ is a diagonal matrix, and $A = (a_{ij})_{i,j} \in M_n(k)$.
If f is a semi-simple endomorphism of V, then the main diagonal of A vanishes.

Proof. The characteristic polynomial of F is $P_F(X) = (X-d_1)^2 \dots (X-d_n)^2$. The minimal polynomial of F divides $P_F(X)$ and is without multiple roots (since f is a semi-simple endomorphism), hence it divides $Q(X) = (X - d'_1) \cdots (X - d'_m)$ where $d'_1, \dots, d'_m$ are the pairwise distinct d_i's, $m \leq n$. Hence, we have in $M_{2n}(k)$,

$$(F - d'_1 I_{2n}) \cdots (F - d'_m I_{2n}) = 0.$$

For $i = 1, \dots, m$, set $D_i := D - d'_i I_n$, the previous equation is

$$\begin{pmatrix} D_1 & A \\ 0 & D_1 \end{pmatrix} \cdots \begin{pmatrix} D_m & A \\ 0 & D_m \end{pmatrix} = 0 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

where

$$* = \sum_{i=1}^{m} D_1 \cdots D_{i-1} A D_{i+1} \cdots D_m.$$

For $l = 1, \dots, n$, the lth diagonal entry of the previous matrix $*$ is $a_{ll} \prod_{\substack{j=1 \\ d'_j \neq d_l}}^{m} (d_l - d'_j)$. Since $* = 0$, we deduce that $a_{ll} = 0$. □

Corollary 2.2.1. *Let k be a field of characteristic zero, V a k-vector space of dimension $2n$ over k, $\nu \in \mathfrak{S}_n$ a permutation, and $f_\nu \in End_k(V)$ such that, with respect to a fixed basis of V over k, f_ν acts by a matrix*

$$F_\nu = \begin{pmatrix} D & A \\ 0 & D_\nu \end{pmatrix},$$

where $D = diag(d_1, \dots, d_n)$, $D_\nu = diag(d_{\nu(1)}, \dots, d_{\nu(n)})$ are diagonal matrices, and $A = (a_{ij})_{i,j} \in M_n(k)$.
If f_ν is semi-simple, then $\forall\ l = 1, \dots, n$, $a_{\nu(l),l} = 0$.

Proof. Let P_ν be the permutation matrix associated to ν, i.e. the only non-zero entries of P_ν are $(P_\nu)_{i,\nu(i)} = 1$, then $D_\nu = P_\nu D P_{\nu^{-1}}$. It suffices to apply the previous lemma to the semi-simple endomorphism f given by

$$F = \begin{pmatrix} I_n & 0 \\ 0 & P_\nu \end{pmatrix}^{-1} \cdot F_\nu \cdot \begin{pmatrix} I_n & 0 \\ 0 & P_\nu \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & P_{\nu^{-1}} \end{pmatrix} \cdot \begin{pmatrix} D & A \\ 0 & D_\nu \end{pmatrix} \cdot \begin{pmatrix} I_n & 0 \\ 0 & P_\nu \end{pmatrix} = \begin{pmatrix} D & AP_\nu \\ 0 & D \end{pmatrix}.$$

□

2.2.2 Proof of the theorem

First step: We reduce to the case where ρ sits in an exact sequence

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0$$

for suitable characters α and β of Γ_0.

For each i, fix a lift $\tilde{\sigma}_i$ of σ_i in Γ such that the image of $\tilde{\sigma}_i$ is $(0, \ldots, 0 \overbrace{,1,}^{i\text{-th component}} 0, \ldots, 0)$ under the composite map

$$\Gamma \twoheadrightarrow \Gamma/\Gamma_0 \xrightarrow{\sim} \prod_{i=1}^{s} \Gamma/\Gamma_i \simeq (\mathbb{Z}/p\mathbb{Z})^s.$$

Set $H := \prod_{i=1}^{s} \Gamma_i$, and for $n \in \mathbb{N}$ denote by $\Delta(\Gamma^n)$ the diagonal of Γ^n. For a fixed $i = 1, \ldots, s$,

$$\begin{aligned}
\bigotimes_{j=1}^{m} \rho_{ij} = \rho_{i1} \otimes \ldots \otimes \rho_{im} &\simeq (\rho_{i1} \boxtimes \ldots \boxtimes \rho_{im})|_{\Delta(\Gamma^m)} \\
&\simeq \left(\mathrm{Ind}_{\Gamma_i^m}^{\Gamma^m}(\alpha_{i1} \boxtimes \ldots \boxtimes \alpha_{im})\right)|_{\Delta(\Gamma^m)} \\
&\simeq \bigoplus_{v} \mathrm{Ind}_{\Delta(\Gamma_i^m)}^{\Delta(\Gamma^m)} \left((\alpha_{i1}^{v_1} \boxtimes \ldots \boxtimes \alpha_{im}^{v_m})|_{\Delta(\Gamma_i^m)}\right) \\
&\simeq \bigoplus_{v} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{i1}^{v_1} \ldots \alpha_{im}^{v_m}),
\end{aligned}$$

where

$$v = (v_1, \ldots, v_m) \in \Delta(\Gamma^m) \setminus \Gamma^m / \Gamma_i^m.$$

In other words, the direct sum is taken over the double cosets $\Delta(\Gamma^m)(v_1, \ldots, v_m)\Gamma_i^m$; and $\forall\ h \in \Gamma_i,\ \forall\ j = 1, \ldots, m,\ \alpha_{ij}^{v_j}(h) = \alpha_{ij}(v_j^{-1} h v_j)$.
Note that the number of summands in the previous direct sum is p^{m-1}, which is the cardinality of the set $\Delta(\Gamma^m) \setminus \Gamma^m/\Gamma_i^m$ of double cosets (or by a dimension argument). Hence $\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij}$ is of the form

$$\begin{aligned}
\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij} &\simeq \bigotimes_{i=1}^{s} \left(\bigoplus_{v} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{i1}^{v_1} \ldots \alpha_{im}^{v_m}) \right) \\
&\simeq \bigoplus_{k} \left(\bigotimes_{i=1}^{s} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\chi_{ik}) \right),
\end{aligned}$$

where χ_{ik} denotes a character of Γ_i of the form $\alpha_{i1}^{\sigma_i^{k_{i1}}} \ldots \alpha_{im}^{\sigma_i^{k_{im}}}$. Note that there are $p^{s(m-1)}$ summands in the previous direct sum.
On the other hand, for each k

$$\begin{aligned}
\bigotimes_{i=1}^{s} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\chi_{ik}) &\simeq \left(\mathrm{Ind}_{\Gamma_1}^{\Gamma}(\chi_{1k}) \boxtimes \ldots \boxtimes \mathrm{Ind}_{\Gamma_s}^{\Gamma}(\chi_{sk})\right)|_{\Delta(\Gamma^s)} \\
&\simeq \mathrm{Ind}_{H}^{\Gamma^s}(\chi_{1k} \boxtimes \ldots \boxtimes \chi_{sk})|_{\Delta(\Gamma^s)} \\
&\simeq \bigoplus_{u} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\chi_{1k}^{u_1} \ldots \chi_{sk}^{u_s}|_{\Gamma_0}),
\end{aligned}$$

where $u = (u_1, \dots, u_s) \in \Delta(\Gamma^s) \setminus \Gamma^s/H$, and $\forall\ h \in \Gamma_0$, $\forall\ i = 1, \dots, s$, $\chi_{ik}^{u_i}(h) = \chi_{ik}(u_i^{-1} h u_i)$.
Since by assumption 3, $[\Gamma : \Gamma_0] = \prod_{i=1}^{s} [\Gamma : \Gamma_i] = p^s$, then there is one summand in the previous direct sum (which is equivalent to the fact that, under assumption 3, $\Delta(\Gamma^s) \setminus \Gamma^s/H$ is trivial); it follows that

$$\bigotimes_{i=1}^{s} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\chi_{ik}) \simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\chi_{1k} \cdots \chi_{sk}|_{\Gamma_0}).$$

Finally

$$\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij} \simeq \bigoplus_{k} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\chi_{1k} \cdots \chi_{sk}|_{\Gamma_0}). \tag{2.1}$$

We are now ready to prove the first step. First, thanks to Proposition 1.3.1, we will assume that $U = \Gamma$.
The assumptions of the theorem are satisfied by any subquotient of V; by Lemma 2.2.4, it is enough to treat the case where V is a $\overline{\mathbb{Q}}_\ell[\Gamma]$-module of length 2, in other words, ρ sits in an exact sequence of representations

$$0 \longrightarrow Y \longrightarrow \rho \longrightarrow X \longrightarrow 0$$

with X and Y irreducible subrepresentations of $\bigotimes_{i=1}^{s} \bigotimes_{j=1}^{m} \rho_{ij}$ (thanks to assumption 2). Hence there exist representations X' and Y' such that $X \oplus X' = I(\alpha)$ and $Y \oplus Y' = I(\beta)$ where $I(\alpha)$ and $I(\beta)$ denote $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\alpha)$ and $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)$ respectively, with

$$\alpha = \chi_{1k} \cdots \chi_{sk}|_{\Gamma_0} \qquad \text{and} \qquad \beta = \chi_{1l} \cdots \chi_{sl}|_{\Gamma_0} \tag{2.2}$$

for some indices k and l (which are fixed from now on), thanks to (2.1). Finally, after replacing ρ by $\rho \oplus X' \oplus Y'$, we can reduce to the case where

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0.$$

Second step: We identify the representation spaces $I(\alpha)$ and $I(\beta)$ with $\overline{\mathbb{Q}}_\ell^{p^s}$ and make explicit their characteristic polynomials for particular elements in Γ.

The representation space

$$I(\alpha) = \{f : \Gamma \longrightarrow \overline{\mathbb{Q}}_\ell |\ f(hg) = \alpha(h) f(g),\ \forall\ h \in \Gamma_0,\ \forall\ g \in \Gamma\}$$

is identified with $\overline{\mathbb{Q}}_\ell^{(\mathbb{Z}/p\mathbb{Z})^s} \simeq \overline{\mathbb{Q}}_\ell^{p^s}$ via

$$\begin{array}{ccc} I(\alpha) & \longrightarrow & \overline{\mathbb{Q}}_\ell^{(\mathbb{Z}/p\mathbb{Z})^s} \\ f & \longmapsto & (f(\tilde{\sigma}_I))_{I \in (\mathbb{Z}/p\mathbb{Z})^s} \end{array} \tag{2.3}$$

Let η_1 and η_2 denote the representations $I(\alpha)$ and $I(\beta)$, repectively. We will now make explicit, with the previous identification of $I(\alpha)$ with $\overline{\mathbb{Q}}_\ell^{p^s}$, the action of elements $h \in \Gamma_0$, and the action of $\tilde{\sigma}_J \in \Gamma$ on $I(\alpha)$. In other words, we will write down the entries of $\eta_1(h)$ and $\eta_1(\tilde{\sigma}_J) \in M_{p^s}(\overline{\mathbb{Q}}_\ell)$. The same will hold for η_2, after replacing α by β.
For $h \in \Gamma_0$ and $I \in (\mathbb{Z}/p\mathbb{Z})^s$,

$$(\eta_1(h) \cdot f)(\tilde{\sigma}_I) = f(\tilde{\sigma}_I h) = \alpha^{\sigma_I}(h) f(\tilde{\sigma}_I),$$

hence an element $h \in \Gamma_0$ acts on $I(\alpha)$ by a diagonal matrix, whose diagonal entries are

$$\eta_1(h)_{II} = \alpha^{\sigma_I}(h).$$

Recall that $J = (1, \dots, 1) \in (\mathbb{Z}/p\mathbb{Z})^s$, and $\tilde{\sigma}_J = \tilde{\sigma}_1 \cdots \tilde{\sigma}_s \in \Gamma$. For $I \in (\mathbb{Z}/p\mathbb{Z})^s$,

$$(\eta_1(\tilde{\sigma}_J) \cdot f)(\tilde{\sigma}_I) = f(\tilde{\sigma}_I \tilde{\sigma}_J) = \alpha(h_I) f(\tilde{\sigma}_{I+J})$$

where $h_I \in \Gamma_0$ is such that $\tilde{\sigma}_I \tilde{\sigma}_J = h_I \tilde{\sigma}_{I+J}$. Hence $\tilde{\sigma}_J$ acts on $I(\alpha)$ by a monomial matrix, whose only non-zero entries are

$$\eta_1(\tilde{\sigma}_J)_{I,I+J} = \alpha(h_I).$$

After choosing a splitting $\kappa : I(\alpha) \longrightarrow V$ of the previous exact sequence of vector spaces, with the previous identification of $I(\alpha)$ and $I(\beta)$ with $\overline{\mathbb{Q}}_\ell^{p^s}$, we may identify V with $\overline{\mathbb{Q}}_\ell^{p^s} \oplus \overline{\mathbb{Q}}_\ell^{p^s}$ and assume that an element $g \in \Gamma$ acts on V by a matrix

$$\rho(g) = \begin{pmatrix} \eta_2(g) & c(g)\eta_1(g) \\ 0 & \eta_1(g) \end{pmatrix}$$

where c is a map

$$c : \Gamma \longrightarrow \mathrm{Hom}(I(\alpha), I(\beta)).$$

Writing down the relation $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in \Gamma$, we see that

$$c(gh)\eta_1(g)\eta_1(h) = \eta_2(g)c(h)\eta_1(h) + c(g)\eta_1(g)\eta_1(h)$$

or equivalently

$$c(gh) = \eta_2(g)c(h)\eta_1(g)^{-1} + c(g) = g.c(h) + c(g).$$

Hence $c \in Z^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta)))$ is a 1-cocycle attached to the splitting κ.

Lemma 2.2.6. *With the previous identifications, the characteristic polynomial of $\eta_1(h\tilde{\sigma}_J)$ is*

$$P_{\eta_1(h\tilde{\sigma}_J)}(X) = (X^p - \alpha((h\tilde{\sigma}_J)^p))^{p^{s-1}}.$$

Idem for $P_{\eta_2(h\tilde{\sigma}_J)}(X)$ after replacing α by β.

Proof. For $i = 0, \dots, p-1$, let H_i be the element of Γ_0 given by

$$\tilde{\sigma}_{I+iJ}\tilde{\sigma}_J = H_i \tilde{\sigma}_{I+(i+1)J}.$$

For $I \in (\mathbb{Z}/p\mathbb{Z})^s$, let P_I be the product of the entries of $\eta_1(h\tilde{\sigma}_J)$ corresponding to the orbit of I under the bijection $I \mapsto I + J$ of $(\mathbb{Z}/p\mathbb{Z})^s$. Note that this is a permutation of $(\mathbb{Z}/p\mathbb{Z})^s$ of order p. It follows that the permutation associated to the monomial matrix $\eta_1(\tilde{\sigma}_J)$ (and $\eta_1(h\tilde{\sigma}_J)$), in the sense of Lemma 2.2.1, is a product of p^{s-1} disjoint p-cycles. We have

$$\begin{aligned} P_I :&= \prod_{i=0}^{p-1} \eta_1(h\tilde{\sigma}_J)_{I+iJ,I+(i+1)J} \\ &= \prod_{i=0}^{p-1} \eta_1(h)_{I+iJ,I+iJ} \prod_{i=0}^{p-1} \eta_1(\tilde{\sigma}_J)_{I+iJ,I+(i+1)J} \\ &= \prod_{i=0}^{p-1} \alpha^{\sigma_{I+iJ}}(h) \prod_{i=0}^{p-1} \alpha(H_i) \end{aligned}$$

A calculation shows that

$$\prod_{i=0}^{p-1} \alpha(H_i) = \alpha^{\sigma_I}(\tilde{\sigma}_J^p) \quad \text{and} \quad \prod_{i=0}^{p-1} \alpha^{\sigma_{I+iJ}}(h) = \alpha^{\sigma_I}((h\tilde{\sigma}_J)^p \tilde{\sigma}_J^{-p}).$$

Hence

$$P_I = \alpha^{\sigma_I}((h\tilde{\sigma}_J)^p).$$

Finally, recall that $\alpha = \chi_{1k} \dots \chi_{sk}|_{\Gamma_0}$ where for $i = 1, \dots, s$, χ_{ik} is a character of Γ_i. Since for $i \neq j \in [\![1, s]\!]$, σ_i acts trivially on χ_{jk}, since $\tilde{\sigma}_i \in \Gamma_j$, then for $I = (i_1, \dots, i_s)$

$$\chi_{jk}^{\sigma_I}((h\tilde{\sigma}_J)^p) = \chi_{jk}^{\sigma_J^{i_j}}((h\tilde{\sigma}_J)^p) = \chi_{jk}^{\mathrm{pr}((h\tilde{\sigma}_J)^{i_j})}((h\tilde{\sigma}_J)^p) = \chi_{jk}((h\tilde{\sigma}_J)^p).$$

Hence

$$P_I = \alpha((h\tilde{\sigma}_J)^p)$$

does not depend on I (which means that it remains the same for all the orbits of the permutation associated to the monomial matrix $\eta_1(h\tilde{\sigma}_J)$); and thanks to Lemma 2.2.1, we deduce the characteristic polynomial of $\eta_1((h\tilde{\sigma}_J)^p)$.

□

Lemma 2.2.7. $P_{\eta_1(h\tilde{\sigma}_J)}(X) = P_{\eta_2(h\tilde{\sigma}_J)}(X)$.

Proof. Thanks to (2.2), we have

$$\frac{\beta}{\alpha} = \prod_{i=1}^{s} \left(\frac{\chi_{il}}{\chi_{ik}}\right)|_{\Gamma_0} = \prod_{i=1}^{s} (\varphi_i^{\sigma_i - 1})|_{\Gamma_0} = \prod_{i=1}^{s} (\varphi_i^{\sigma_J - 1})|_{\Gamma_0},$$

where for each $i = 1, \dots, s$, φ_i is a suitable character of Γ_i; and the last equality holds because, as before, $\forall\ i \neq j \in [\![1, s]\!]$, σ_i acts trivially on φ_j. Note that $(h\tilde{\sigma}_J)^p = h^* \tilde{\sigma}_J^p$, where

$$h^* = h(\tilde{\sigma}_J h \tilde{\sigma}_J^{-1}) \dots (\tilde{\sigma}_J^{p-1} h \tilde{\sigma}_J^{1-p}).$$

Hence

$$(\beta/\alpha)((h\tilde{\sigma}_J)^p) = (\beta/\alpha)(h^* \tilde{\sigma}_J^p) = \prod_{i=1}^{s} \varphi_i^{\sigma_J - 1}(h^*) \varphi_i^{\sigma_J - 1}(\tilde{\sigma}_J^p) = 1,$$

since $\forall\, i = 1, \dots, s$, $\varphi_i^{\sigma_J - 1}(h^*) = \varphi_i^{\sigma_J - 1}(\tilde{\sigma}_J^p) = 1$. It follows that $P_{\eta_1(h\tilde{\sigma}_J)}(X) = P_{\eta_2(h\tilde{\sigma}_J)}(X)$.

□

Third step: For $h \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, the main diagonal of $c(h)$ vanishes.

The previous lemma implies that $\forall\ h \in \Gamma_0$,

$$P_{\rho(h\tilde{\sigma}_J)}(X) = (X^p - \alpha((h\tilde{\sigma}_J)^p))^{2p^{s-1}},$$

hence

$$\{g \in \mathrm{pr}^{-1}(\sigma_J) |\ \rho(g) \text{ is semi-simple}\} = \{g \in \mathrm{pr}^{-1}(\sigma_J) |\ \rho(g)^p \in \overline{\mathbb{Q}}_\ell^\times . \mathrm{id}_V\}$$

and thanks to assumption 1, the previous set, which contains Σ_1, is equal to $\mathrm{pr}^{-1}(\sigma_J)$. In particular, $\tilde{\sigma}_J$ acts semi-simply on V and we can choose our $\overline{\mathbb{Q}}_\ell$-splitting κ to be $\tilde{\sigma}_J$-equivariant, hence the attached cocycle c satisfies $c(\tilde{\sigma}_J) = 0$. It follows that

$$\forall\ h \in \Gamma_0, \quad 0 = c((h\tilde{\sigma}_J)^p) = \sum_{l=0}^{p-1} (h\tilde{\sigma}_J)^l c(h) = \sum_{l=0}^{p-1} \eta_2(h\tilde{\sigma}_J)^l c(h)\eta_1(h\tilde{\sigma}_J)^{-l}. \tag{2.4}$$

The previous equation is equivalent, for each $I, I' \in (\mathbb{Z}/p\mathbb{Z})^s$, to

$$\sum_{l=0}^{p-1} \psi_{_{II'l}} \cdot c(h)_{I+lJ,I'+lJ} = 0, \tag{2.5}$$

where, as in Lemma 2.2.2,

$$\psi_{_{II'l}} = \prod_{k=0}^{l-1} \frac{\eta_2(h\tilde{\sigma}_J)_{I+kJ,I+(k+1)J}}{\eta_1(h\tilde{\sigma}_J)_{I'+kJ,I'+(k+1)J}} = \underbrace{\prod_{k=0}^{l-1} \frac{\eta_2(h)_{I+kJ,I+kJ}}{\eta_1(h)_{I'+kJ,I'+kJ}}}_{\gamma_{_{II'l}}(h)} \underbrace{\prod_{k=0}^{l-1} \frac{\eta_2(\tilde{\sigma}_J)_{I+kJ,I+(k+1)J}}{\eta_1(\tilde{\sigma}_J)_{I'+kJ,I'+(k+1)J}}}_{K_{_{II'l}}}$$

Knowing that $\eta_1(h)_{II} = \alpha^{\sigma_I}(h)$ and $\eta_2(h)_{II} = \beta^{\sigma_I}(h)$, the equation (2.5) is equivalent to

$$\sum_{l=0}^{p-1} \gamma_{_{II'l}}(h) K_{_{II'l}} c(h)_{I+lJ,I'+lJ} = 0, \tag{2.6}$$

where $\gamma_{_{II'l}}$ is the character given by

$$\gamma_{_{II'l}} := \prod_{k=1}^{l} \frac{\beta^{\sigma_{I+(k-1)J}}}{\alpha^{\sigma_{I'+(k-1)J}}}$$

with $\gamma_{_{II'0}} = 1$. Note that $K_{_{II'l}}$ is of the form $\prod_{k=0}^{l-1} \beta(h_I)\alpha(h_{I'})$ for suitable elements $h_I, h_{I'} \in \Gamma_0$. However, we only need to know that $K_{_{II'l}} \in \overline{\mathbb{Q}}_\ell^\times$ and is independent from h.

First case: $\alpha = \beta$
Assumption 1 implies that $\forall\ h \in \Sigma_0$, $\rho(h)$ is a semi-simple element of $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$. Since

$$\rho(h) = \begin{pmatrix} \eta_1(h) & c(h)\eta_1(h) \\ 0 & \eta_1(h) \end{pmatrix}$$

is semi-simple, then Lemma 2.2.5 implies that the main diagonal of $c(h)\eta_1(h)$, and hence the main diagonal of $c(h)$, vanish. By continuity, this remains true $\forall\ h \in \Gamma_0$.

Second case: $\alpha \neq \beta$
Let $h_0, h_1 \in \Gamma_0$ be such that h_0 acts trivially on $I(\alpha) \oplus I(\beta)$ and set $h_2 = h_0h_1$; then $c(h_2) = c(h_1) + c(h_0)$. Substracting the equation (2.6) for h_2 and h_1, we get,

$$\sum_{l=0}^{p-1} K_{_{II'l}}(\gamma_{_{II'l}}(h_2)c(h_2)_{I+lJ,I'+lJ} - \gamma_{_{II'l}}(h_1)c(h_1)_{I+lJ,I'+lJ}) = 0. \tag{2.7}$$

Knowing that $\gamma_{_{II'l}}(h_2) = \gamma_{_{II'l}}(h_0h_1) = \gamma_{_{II'l}}(h_1)$ and $c(h_2) = c(h_1) + c(h_0)$, we get from equation (2.7), in particular when $I = I'$,

$$\forall\ h_1 \in \Gamma_0, \quad \sum_{l=0}^{p-1} \gamma_{_{IIl}}(h_1) K_{_{IIl}} c(h_0)_{I+lJ,I+lJ} = 0.$$

For simplicity, we write γ_l instead of γ_{III}. Linear independence of the distinct characters $(\gamma_l)_{l=0,\dots,p-1}$ (which we prove later in Lemma 2.2.8) implies that for $h_0 \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, the main diagonal of $c(h_0)$ vanishes.

In both cases, for $h \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, the main diagonal of $c(h)$ vanishes.

Fourth step: For $h \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, $c(h) = 0$.

Fix $I_0 \in (\mathbb{Z}/p\mathbb{Z})^s$. Let α' denote the character $\alpha^{\sigma_{I_0}}$ of Γ_0, and η_1' denote the representation $I(\alpha')$. Hence there exists $M \in GL_{p^s}(\overline{\mathbb{Q}}_\ell)$ such that

$$\forall\ g \in \Gamma, \quad \eta_1'(g) = M\eta_1(g)M^{-1}.$$

Let c' be the cocycle associated to the exact sequence

$$0 \longrightarrow I(\beta) \longrightarrow \rho \longrightarrow I(\alpha') \longrightarrow 0.$$

The isomorphism $I(\alpha) \simeq I(\alpha')$ depends on the lift of σ_{I_0} in Γ. For the lift $\tilde{\sigma}_{I_0} \in \Gamma$, this isomorphism is given by

$$\begin{array}{ccc} I(\alpha) & \xrightarrow{\sim} & I(\alpha') \\ f & \longmapsto & f' \end{array}$$

where $f' : \Gamma \longrightarrow \overline{\mathbb{Q}}_\ell$ is defined by

$$f'(g) = f(\tilde{\sigma}_{I_0} g) \quad \forall\ g \in \Gamma.$$

Via our identification of $I(\alpha)$ (resp. $I(\alpha')$) with $\overline{\mathbb{Q}}_\ell^{p^s}$, the matrix M is given by

$$(f'(\tilde{\sigma}_I))_I = M(f(\tilde{\sigma}_I))_I.$$

For $I \in (\mathbb{Z}/p\mathbb{Z})^s$,

$$f'(\tilde{\sigma}_I) = f(\tilde{\sigma}_{I_0}\tilde{\sigma}_I) = f(h_I\tilde{\sigma}_{I+I_0}) = \alpha(h_I)f(\tilde{\sigma}_{I+I_0}),$$

where $h_I \in \Gamma_0$ is such that $\tilde{\sigma}_{I_0}\tilde{\sigma}_I = h_I\tilde{\sigma}_{I+I_0}$. Therefore, M is a monomial matrix whose only non-zero entries are $M_{I,I+I_0} = \alpha(h_I) \in \overline{\mathbb{Q}}_\ell^\times$. Note the the h_I's depend on the isomorphism $I(\alpha) \simeq I(\alpha')$, but the form of M (monomial matrix with only non-zero entries $M_{I,I+I_0}$) does not.
The cocycle c' is given by

$$c' : \Gamma \xrightarrow{c} \mathrm{Hom}(I(\alpha), I(\beta)) \xrightarrow{\sim} \mathrm{Hom}(I(\alpha'), I(\beta))$$

where the right map is composition with $M^{-1} : I(\alpha') \xrightarrow{\sim} I(\alpha)$. Hence

$$\forall\ g \in \Gamma, \quad c'(g) = c(g)M^{-1}.$$

Note that $c(\tilde{\sigma}_J) = 0$ implies that $c'(\tilde{\sigma}_J) = 0$. Applying the third step for $I(\alpha') \simeq I(\alpha)$, we get for $h \in \Gamma_0$ acting trivially on $I(\alpha') \oplus I(\beta)$ (equivalently on $I(\alpha) \oplus I(\beta)$), $\forall\ I$,

$$0 = c'(h)_{II} = c(h)_{I,I+I_0}(M^{-1})_{I+I_0,I} = c(h)_{I,I+I_0}\alpha(h_I)^{-1}.$$

Hence $c(h)_{I,I+I_0} = 0$. This is true $\forall\ I_0 \in (\mathbb{Z}/p\mathbb{Z})^s$, we deduce then that for $h \in \Gamma_0$ acting trivially on $I(\alpha) \oplus I(\beta)$, $c(h) = 0$.

Fifth step (End of the proof): Our exact sequence of representations splits.

Consider the Γ-module $\mathrm{Hom}(I(\alpha), I(\beta))$, where the action of an element $g \in \Gamma$ on $f \in \mathrm{Hom}(I(\alpha), I(\beta))$ is given by $g \cdot f = \eta_2(g) f \eta_1(g)^{-1}$. There is an isomorphism of Γ-modules

$$\mathrm{Hom}(I(\alpha), I(\beta)) \simeq I(\alpha)^\vee \otimes I(\beta) \simeq I(\alpha^{-1}) \otimes I(\beta) \simeq \bigoplus_{I' \in (\mathbb{Z}/p\mathbb{Z})^s} I(\beta/\alpha^{\sigma_{I'}}),$$

where the summand $I(\beta/\alpha^{\sigma_{I'}})$ corresponds to the matrices $X \in M_{p^s}(\overline{\mathbb{Q}}_\ell)$ whose only nonzero entries are $X_{I,I+I'}$ for $I \in (\mathbb{Z}/p\mathbb{Z})^s$. Let $p_{I'}$ denote the projection

$$p_{I'} : \mathrm{Hom}(I(\alpha), I(\beta)) \twoheadrightarrow I(\beta/\alpha^{\sigma_{I'}})$$

which induces a projection, also denoted $p_{I'}$, on the 1-cocycles:

$$p_{I'} : Z^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta))) \twoheadrightarrow Z^1(\Gamma, I(\beta/\alpha^{\sigma_{I'}})).$$

Let c_0 denote the restriction of c to Γ_0. The previous step implies that $p_{I'}(c_0)$ lies in the image of the inflation map

$$Z^1(A, I(\beta/\alpha^{\sigma_{I'}})) \to Z^1(\Gamma_0, I(\beta/\alpha^{\sigma_{I'}})),$$

where A is the image of $\rho(\Gamma_0)$ in $\mathrm{Aut}(I(\alpha) \oplus I(\beta))$, which is an abelian profinite group.

First case: $\beta = \alpha^{\sigma_{I'}}$
The semi-simplicity of $\rho(h)$ for $h \in \Sigma_0$ implies, thanks to Corollary 2.2.1, that $p_{I'}(c(h)) = 0$. Hence, by continuity,

$$p_{I'}(c_0) = 0.$$

Second case: $\beta \neq \alpha^{\sigma_{I'}}$
The image Y of X under the endomorphism in Sah's lemma (Lemma 2.2.3)

$$\begin{array}{ccc} I(\beta/\alpha^{\sigma_{I'}}) & \longrightarrow & I(\beta/\alpha^{\sigma_{I'}}) \\ X & \longmapsto & \eta_2(h) X \eta_1(h)^{-1} - X \end{array}$$

is given by

$$Y_{IL} = \begin{cases} \left(\dfrac{\beta^{\sigma_I}(h)}{\alpha^{\sigma_L}(h)} - 1 \right) X_{IL} & \text{if } L = I + I' \\ 0 & \text{otherwise} \end{cases}$$

Since $\beta \neq \alpha^{\sigma_{I'}}$, the previous endomorphism is an automorphism. It follows, by Sah's lemma, that $H^1(A, I(\beta/\alpha^{\sigma_{I'}})) = 0$.

In both cases, $\forall\ I' \in (\mathbb{Z}/p\mathbb{Z})^s$, $p_{I'}(c_0) = 0 \in H^1(A, I(\beta/\alpha^{\sigma_{I'}}))$. We recall the inflation-restriction sequence

$$0 \to H^1(\Gamma/\Gamma_0, \mathrm{Hom}(I(\alpha), I(\beta))^{\Gamma_0}) \overset{\mathrm{Inf}}{\to} H^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta))) \overset{\mathrm{Res}}{\to} H^1(\Gamma_0, \mathrm{Hom}(I(\alpha), I(\beta)))$$

By the previous discussion, the cohomology class of c lies in

$$\mathrm{Ker}(\mathrm{Res}) \simeq H^1(\Gamma/\Gamma_0, \mathrm{Hom}(I(\alpha), I(\beta))^{\Gamma_0})$$

which is trivial because $\mathrm{Hom}(I(\alpha), I(\beta))^{\Gamma_0}$ is a $\mathbb{Q}$-vector space and Γ/Γ_0 is finite. Hence $c = 0 \in H^1(\Gamma, \mathrm{Hom}(I(\alpha), I(\beta)))$, and $\rho \simeq I(\alpha) \oplus I(\beta)$ is semi-simple.

Lemma 2.2.8. *When $\alpha \neq \beta$, for each $I \in (\mathbb{Z}/p\mathbb{Z})^s$, the p characters $(\gamma_l)_{l=0,\dots,p-1}$ defined in the third step by*

$$\gamma_l := \prod_{k=1}^{l} \frac{\beta^{\sigma_{I+(k-1)J}}}{\alpha^{\sigma_{I+(k-1)J}}}$$

are distinct.

Proof. To compare the characters $(\gamma_l)_{l=0,\dots,p-1}$, after replacing the pair (α, β) by $(\alpha^{\sigma_{-I}}, \beta^{\sigma_{-I}})$, we can always reduce to compare, for $l < p$,

$$\alpha\alpha^{\sigma_J}\alpha^{\sigma_{2J}}\dots\alpha^{\sigma_{(l-1)J}} \qquad \text{and} \qquad \beta\beta^{\sigma_J}\beta^{\sigma_{2J}}\dots\beta^{\sigma_{(l-1)J}}.$$

In other words, we have to show that

$$\prod_{k=0}^{l-1}(\beta/\alpha)^{\sigma_{kJ}} \neq 1.$$

Since p is prime and $l < p$ then there exists an integer a such that $al \equiv 1 \bmod p$. Let N be such that $al - 1 = pN$. Suppose that $\prod_{k=0}^{l-1}(\beta/\alpha)^{\sigma_{kJ}} = 1$. Then by applying σ_J^l successively $(a-1)$ times to the previous identity, we get a identities

$$\prod_{k=0}^{l-1}(\beta/\alpha)^{\sigma_{kJ}} = 1, \qquad \prod_{k=l}^{2l-1}(\beta/\alpha)^{\sigma_{kJ}} = 1, \qquad \dots, \qquad \prod_{k=(a-1)l}^{pN}(\beta/\alpha)^{\sigma_{kJ}} = 1.$$

The product of the previous a products is then

$$\prod_{k=0}^{pN}(\beta/\alpha)^{\sigma_{kJ}} = 1. \tag{2.8}$$

Recall from the proof of Lemma 2.2.7 that $\beta/\alpha = \prod_{i=1}^{s}(\varphi_i^{\sigma_J - 1})|_{\Gamma_0}$ where φ_i is a character of Γ_i, hence

$$\prod_{k=0}^{p-1}(\beta/\alpha)^{\sigma_{kJ}} = \prod_{i=1}^{s}\varphi_i^{(\sigma_J-1)(1+\sigma_J+\dots+\sigma_{(p-1)J})} = \prod_{i=1}^{s}\varphi_i^{\sigma_{pJ}-1} = 1.$$

Combined with (2.8), this gives that $\beta/\alpha = 1$, whence the contradiction.

□

2.3 Variant of the theorem

2.3.1 Variant of assumption (1)

We prove first that assumption (1) in Theorem 2.2.1 is a consequence of other assumptions:

Proposition 2.3.1. *In the situation of Section 2.2, assume that :*

1. *There exists a dense subset $\Sigma \subset \Gamma$ such that for each $g \in \Sigma$, there exist sm pairwise commuting elements $(u_{ij})_{i=1,\dots,s,\ j=1,\dots,m}$ in $Aut_{\overline{\mathbb{Q}}_l}(V)$ such that*

$$\rho(g) = \prod_{i=1}^{s}\prod_{j=1}^{m} u_{ij}$$

and $P_{\rho_{ij}(g)}(u_{ij}) = 0$, $\forall\ (i,j) \in [\![1,s]\!] \times [\![1,m]\!]$.

2. *$\forall\ (i,j,k) \in [\![1,s]\!] \times [\![1,m]\!] \times [\![1,p-1]\!]$, the character $\alpha_{ij}/\alpha_{ij}^{\sigma_i^k}$ is of infinite order.*

3. *The canonical injection $\Gamma/\Gamma_0 \hookrightarrow \prod_{i=1}^{s} \Gamma/\Gamma_i \simeq (\mathbb{Z}/p\mathbb{Z})^s$ is an isomorphism.*

Then there exist dense subsets $\Sigma_0 \subset \Gamma_0$ and $\Sigma_1 \subset pr^{-1}(\sigma_J)$ such that $\rho(g)$ is a semi-simple element of $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ for each $g \in \Sigma_0 \cup \Sigma_1$.

Proof. First, for fixed i,j, we exhibit a simplied form of the characteristic polynomial of $\rho_{ij}(h\tilde{\sigma}_J)$, as in Lemma 2.2.6. After identifying the representation space $\mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$ with $\overline{\mathbb{Q}}_\ell^p = \bigoplus_{i=1}^{p} \overline{\mathbb{Q}}_\ell \cdot e_i$ (particular case of (2.3)) via

$$\begin{array}{ccc} \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij}) & \longrightarrow & \overline{\mathbb{Q}}_\ell^p \\ f & \longmapsto & (f(1), f(\tilde{\sigma}_i), \ldots, f(\tilde{\sigma}_i^{p-1})) \end{array}$$

the action of an element $h \in \Gamma_i$ on $\mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$ is given by

$$\forall\ k = 1, \ldots, p, \qquad h(e_k) = \alpha_{ij}^{\sigma_i^{k-1}}(h) e_k,$$

hence

$$P_{\rho_{ij}(h)}(X) = \prod_{k=0}^{p-1} \left(X - \alpha_{ij}^{\sigma_i^k}(h) \right).$$

In the previous formula, we will replace σ_i^k by σ_J^k, since for $i \neq j$, $\tilde{\sigma}_j \in \Gamma_i$, hence acts trivially on α_{ij}.
On the other hand, the action of $\tilde{\sigma}_J = \tilde{\sigma}_1 \ldots \tilde{\sigma}_s$ is given by

$$\tilde{\sigma}_J(e_k) = \begin{cases} H_1 e_p & \text{for } k = 1 \\ H_k e_{k-1} & \text{for } k = 2, \ldots, p. \end{cases}$$

where $H_1 = \alpha_{ij}(\tilde{\sigma}_i^{p-1}\tilde{\sigma}_J)$, and for $k = 2, \ldots, p$, $H_k = \alpha_{ij}^{\sigma_i^{k-1}}(\tilde{\sigma}_i^{-1}\tilde{\sigma}_J)$.
In fact, for $k = 2, \ldots, p$, if $f_k \in \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij})$ corresponds to $e_k \in \overline{\mathbb{Q}}_\ell^p$ under $\mathrm{Ind}_{\Gamma_i}^{\Gamma}(\alpha_{ij}) \simeq \overline{\mathbb{Q}}_\ell^p$, i.e. $f_k(\tilde{\sigma}_i^{k-1}) = 1$ and $f_k(\tilde{\sigma}_i^l) = 0\ \forall\ l \neq k-1$, then

$$(\tilde{\sigma}_J \cdot f_k)(\tilde{\sigma}_i^{k-2}) = f_k(\tilde{\sigma}_i^{k-2}\tilde{\sigma}_J) = f_k(h_k\tilde{\sigma}_i^{k-1}) = \alpha_{ij}(h_k) f_k(\tilde{\sigma}_i^{k-1}) = (\alpha_{ij}(h_k) f_{k-1})(\tilde{\sigma}_i^{k-2})$$

where $h_k \in \Gamma_i$ is such that $\tilde{\sigma}_i^{k-2}\tilde{\sigma}_J = h_k\tilde{\sigma}_i^{k-1}$; and $(\tilde{\sigma}_J \cdot f_k)(\tilde{\sigma}_i^l) = 0\ \forall\ l \neq k-2$. Hence for $k = 2, \ldots, p$, $\tilde{\sigma}_J \cdot f_k = \alpha_{ij}(h_k) f_{k-1} = H_k f_{k-1}$.

Finally, since $H_1 \ldots H_p = \alpha_{ij}(\tilde{\sigma}_J^p)$, and

$$\alpha_{ij}(\tilde{\sigma}_J^p) \prod_{k=0}^{p-1} \alpha_{ij}^{\sigma_J^k}(h) = \alpha_{ij}((h\tilde{\sigma}_J)^p),$$

we deduce that $\forall\ h \in \Gamma_i$,

$$P_{\rho_{ij}(h\tilde{\sigma}_J)}(X) = X^p - \alpha_{ij}((h\tilde{\sigma}_J)^p).$$

For $g \in \Sigma_1 = \Sigma \cap \mathrm{pr}^{-1}(\sigma_J)$ which is a dense subset of $\mathrm{pr}^{-1}(\sigma_J)$, we have $\forall\ i = 1, \ldots, s$, $\forall\ j = 1, \ldots, m$,

$$0 = P_{\rho_{ij}(g)}(u_{ij}) = u_{ij}^p - \alpha_{ij}((h\tilde{\sigma}_J)^p).$$

It follows that the minimal polynomial of each u_{ij}, which divides $X^p - \alpha_{ij}((h\tilde{\sigma}_J)^p)$, does not have multiple factors, hence each u_{ij} is semi-simple, so is their product $\rho(g)$.

On the other hand, $\forall\ g \in \Gamma_0$, the assumption on the characters $\alpha_{ij}/\alpha_{ij}^{\sigma_i^k}$ implies that for all elements g in a suitable open dense subset $\Gamma_0' \subset \Gamma_0$, $P_{\rho_{ij}(g)}(X)$ has distinct roots, which implies as before that $\rho(g)$ is semi-semi-simple $\forall\ g \in \Sigma_0 = \Gamma_0' \cap \Sigma$ which is a dense subset of Γ_0. □

2.3.2 Special case $s = 1$

We are now going to recall from [Nek] that if in Theorem 2.2.1 we consider only one open normal subgroup $\Gamma_1 \lhd \Gamma$ (i.e. $s = 1$), then assumption (2) of Theorem 2.2.1 is a consequence of assumption (1) of Proposition 2.3.1. Note that in this case assumption (3) is trivial. More precisely, we are in the following context:

Let Γ be a profinite group, and $\rho : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ be a continuous finite-dimensional representation of Γ. Let $\Gamma_1 \lhd \Gamma$ be an open normal subgroup such that $\Gamma/\Gamma_1 \simeq \mathbb{Z}/p\mathbb{Z}$ with p a prime number. Fix a generator σ. Denote by $\mathrm{pr} : \Gamma \to \Gamma/\Gamma_1$ the canonical projection, and by $\tilde{\sigma}$ a lift of σ. For $j = 1, \ldots, m$, let $\alpha_j : \Gamma_1 \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character of Γ_1, and $\rho_j = \mathrm{Ind}_{\Gamma_1}^{\Gamma}(\alpha_j)$ be the induced representation.

Proposition 2.3.2. *[Nek] With the notations above (here we do not need to know that $[\Gamma : \Gamma_1]$ is a prime number), assume that there exist an open subgroup $\Gamma' \subset \Gamma$ such that $pr(\Gamma') = \Gamma/\Gamma_1$, and a dense subset $\Sigma \subset \Gamma'$ such that*

$$\forall\ g \in \Sigma \qquad P_{(\rho_1 \otimes \ldots \otimes \rho_m)(g)}(\rho(g)) = 0. \tag{2.9}$$

Then there exists an open subgroup $U \subset \Gamma'$ such that $pr(U) = \Gamma/\Gamma_1$ and $\rho^{\mathrm{ss}}|_U$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \ldots \otimes \rho_m)|_U^{\oplus n}$ for an integer $n \geq 1$.

Proof. See [Nek, Proposition 4.6]. □

Theorem 2.3.1. *[Nek] In the situation of Section 2.3, assume that*

1. *There exist an open subgroup $\Gamma' \subset \Gamma$ such that $pr(\Gamma') = \Gamma/\Gamma_1$, and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$, there exist m pairwise commuting elements $u_1, \ldots, u_m$ in $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ such that $\rho(g) = u_1 \ldots u_m$ and $P_{\rho_j(g)}(u_j) = 0$, $\forall\ j = 1, \ldots, m$.*
2. *$\forall\ (j,k) \in [\![1, m]\!] \times [\![1, p-1]\!]$, the character $\alpha_j/\alpha_j^{\sigma^k}$ is of infinite order.*

Then

1. *There exist dense subsets $\Sigma_0 \subset \Gamma' \cap \Gamma_1$ and $\Sigma_1 \subset \Gamma' \cap pr^{-1}(\sigma)$ such that $\rho(g)$ is a semi-simple element of $Aut_{\overline{\mathbb{Q}}_\ell}(V)$ for each $g \in \Sigma_0 \cup \Sigma_1$.*
2. *There exists an open subgroup $U \subset \Gamma'$ such that $pr(U) = \Gamma/\Gamma_1$ and $\rho^{\mathrm{ss}}|_U$ is isomorphic to a subrepresentation of $(\rho_1 \otimes \ldots \otimes \rho_m)|_U^{\oplus n}$ for an integer $n \geq 1$.*
3. *ρ is semi-simple.*

Proof. We may assume that $\Gamma = \Gamma'$.

(1) It suffices to apply Proposition 2.3.1.

(2) By Lemma 1.3.1, we have

$$\forall\ g \in \Sigma, \quad P_{(\rho_1 \otimes \ldots \otimes \rho_m)(g)}(\rho(g)) = 0,$$

hence Proposition 2.3.2 applies.

(3) Thanks to the previous steps, Theorem 2.2.1 applies, hence ρ is semi-simple.

□

2.4 Special case $r = 1$

In this section we prove that for $r = 1$, under the assumptions of Theorem 2.3.1, if we merely assume that Γ/Γ_1 is cyclic of order a power of a prime number p, then the semi-simplicity of ρ holds; more precisely:

Let Γ be a profinite group, and $\rho : \Gamma \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ a continuous representation of Γ. Let Γ_1 be an open normal subgroup of Γ such that Γ/Γ_1 is cyclic of order $n = p^v$ with p a prime number and $v \in \mathbb{N}$. Fix a generator σ. Denote by $\mathrm{pr} : \Gamma \to \Gamma/\Gamma_1$ the canonical projection, and by $\tilde{\sigma}$ a lift of σ. Let $\alpha : \Gamma_1 \to \overline{\mathbb{Q}}_\ell^\times$ be a continuous character of Γ_1 and $\rho_1 = \mathrm{Ind}_{\Gamma_1}^{\Gamma}(\alpha)$ the induced representation.
The following theorem is a generalisation of a result of [Nek], where the case $v = 1$ is studied.

Theorem 2.4.1. *In the situation described above, assume that*

1. *There exist an open subgroup $\Gamma' \subset \Gamma$ such that $pr(\Gamma') = \Gamma/\Gamma_1$, and a dense subset $\Sigma \subset \Gamma'$ such that $P_{\rho_1(g)}(\rho(g)) = 0$ for all $g \in \Sigma$.*
2. *For each $i = 1, \ldots, n-1$, the character α/α^{σ^i} is of infinite order.*

Then

1. *There exist dense subsets $\Sigma_0 \subset \Gamma' \cap \Gamma_1$ and $\Sigma_1 \subset \Gamma' \cap pr^{-1}(\sigma)$ such that $\rho(g)$ is semi-simple for each $g \in \Sigma_0 \cup \Sigma_1$.*
2. *There exists an open subgroup $U \subset \Gamma'$ satisfying $pr(U) = \Gamma/\Gamma_1$ such that $\rho|_U$ is isomorphic to a subrepresentation of $(\rho_1|_U)^{\oplus m}$ for an integer $m \geq 1$.*
3. *ρ is semi-simple.*

2.4.1 Steps of the proof and preliminaries

The proof of (1) and (2) is a particular case of Theorem 2.3.1. To prove that ρ is semi-simple, we proceed as in Theorem 2.2.1

1. We reduce to the case where ρ sits in an exact sequence of representations
$$0 \longrightarrow I(\alpha) \longrightarrow \rho \longrightarrow I(\alpha) \longrightarrow 0$$
2. We identify the representation space $I(\alpha)$ with $\overline{\mathbb{Q}}_\ell^n$, hence an element $g \in \Gamma$ acts on V by a matrix
$$\rho(g) = \begin{pmatrix} \rho_1(g) & c(g)\rho_1(g) \\ 0 & \rho_1(g) \end{pmatrix}$$
$c \in Z^1(\Gamma, \mathrm{End}(I(\alpha)))$ is a 1-cocycle.
3. We show that for $h \in \Gamma_1$ acting trivially on $I(\alpha)$, the main diagonal of $c(h)$ vanishes.
4. We show that for h as in the previous step, $c(h) = 0$.
5. The exact sequence of representations in the first step splits, which means that $\rho \simeq I(\alpha) \oplus I(\alpha)$ and hence ρ is semi-simple.

The steps 1,2,3,5 are a particular case of Section 2.2.1. Note that in step 3, we are in the first case, hence we do not need Lemma 2.2.8 (where we used the fact that $[\Gamma : \Gamma_1]$ is a prime number).
The rest of the proof is slightly different. More precisely, in step 4, we use our assumption

that $[\Gamma : \Gamma_1]$ is a power of a prime number p.

We begin with the following elementary lemma whose corollary will be useful later in the proof.

Lemma 2.4.1. *Let K be an algebraically closed field of characteristic zero and $n = p^v$ be a power of a prime number p, then for each $j, k < n$, there exist $g \in K[T]$ and $r < n$ such that*

$$(T^j - 1).\frac{T^k - 1}{T - 1} g \equiv (T^r - 1) \ mod \ (T^n - 1).$$

Proof. For $m \in \mathbb{N}$, define

$$f_m = \frac{T^m - 1}{T - 1}.$$

The desired congruence is then equivalent to

$$f_j f_k g \equiv f_r \ \mathrm{mod} f_n.$$

Since $\gcd(f_j, f_n) = (f_{\gcd(j,n)})$, we can assume that j and k divide $n = p^v$. Then $j \mid k$ or $k \mid j$. Suppose that $j \mid k$, write $k = jq$, $n = kq' = jqq'$, and let μ_n be a primitive n-th root of unity, then $\mu_n^{q'}$ (resp. $\mu_n^{qq'}$) is a primitive k-th (resp. j-th) root of unity, and

$$\gcd(f_j, f_n/f_k) = \gcd\left(\prod_{\substack{a=1 \\ a\equiv 0[qq']}}^{n-1} (T - \mu_n^a), \prod_{\substack{a=1 \\ a\not\equiv 0[q']}}^{n-1} (T - \mu_n^a)\right) = (1),$$

hence there exists $g \in K[T]$ such that

$$f_j g \equiv 1 \ \mathrm{mod}(f_n/f_k)$$

so $f_j f_k g \equiv f_k \ \mathrm{mod} f_n$. □

Corollary 2.4.1. *Let α be a character of Γ_1 such that $\forall\ i = 1, \ldots, n-1$, $\alpha \neq \alpha^{\sigma^i}$. Fix $k \in [\![0, n-2]\!]$, and for $j \in [\![0, n-1]\!]$, let χ_j be the character*

$$\chi_j = \prod_{i=j}^{j+k} \alpha^{\sigma^i}.$$

Then the n characters $\chi_0, \chi_1, \ldots, \chi_{n-1}$ of Γ_1 are pairwise disctinct.

Proof. After shrinking Γ if necessary, we can assume that for each $j = 0, \ldots, n-1$, $\mathrm{Im}(\chi_j)$ is contained in $1 + D_\ell$, where

$$D_\ell = \{x \in \mathbb{Q}_\ell |\ |x|_\ell < \ell^{-1/(\ell-1)}\}.$$

This is an additive subgroup of $\mathbb{Z}_\ell$, and $1 + D_\ell$ is a multiplicative subgroup of $\mathbb{Z}_\ell$; the ℓ-adic logarithm induces an isomorphism of groups $\log_\ell : 1 + D_\ell \xrightarrow{\sim} D_\ell$.
Suppose that there exist two indices $j < j'$ such that $\chi_j = \chi_{j'}$. After applying a suitable power of σ, we can reduce to the case where $\chi_0 = \chi_j$ for some index $j \in [\![1, n-1]\!]$. Applying σ successively to the previous identity, we get

$$\forall\ a = 0, \ldots, n-1, \qquad \chi_a = \chi_{a+j}.$$

After taking the ℓ-adic logarithm in the previous n identities, we get

$$\forall\ h \in \Gamma_1,\ \forall\ a = 0, \dots, n-1,\quad \sum_{i=a}^{a+k} \log_\ell(\alpha^{\sigma^i}(h)) = \sum_{i=a+j}^{a+j+k} \log_\ell(\alpha^{\sigma^i}(h)). \tag{2.10}$$

For $h \in \Gamma_1$ and $i \geq 0$, set $a_i = \log_\ell(\alpha^{\sigma^i}(h))$, and let

$$S(h,T) := \sum_{i \geq 0} a_i T^i \in K[T]/(T^n - 1) \simeq K[\langle \sigma \rangle]$$

be the generating function associated to the sequence $(a_i)_{i \geq 0}$. Thanks to (2.10), we have

$$\left(\sum_{i=0}^{k} T^i - \sum_{i=j}^{j+k} T^i \right) \cdot S(h,T) = 0.$$

Equivalently

$$(T^j - 1) \frac{T^{k+1} - 1}{T - 1} \cdot S(h,T) = 0.$$

Taking r as in the previous lemma (r depends on j and k, not on $h \in \Gamma_1$), this means that $(T^r - 1) \cdot S(h,T) = 0 \in K[T]/(T^n - 1)$, hence $a_0 = a_r$, and $\forall\ h \in \Gamma_1$, $\alpha(h) = \alpha^{\sigma^r}(h)$ which contradicts the fact that $\alpha \neq \alpha^{\sigma^r}$. □

2.4.2 Proof of the theorem

All we have to prove is the fourth step. We show that $c(h_0) = 0$ for every $h_0 \in \Gamma_1$ which acts trivially on $I(\alpha)$. To do this, we write explicitly the equation $c((h\tilde{\sigma})^n) = 0$ in $\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(I(\alpha)) \simeq M_n(\overline{\mathbb{Q}}_\ell)$. As before, since

$$P_{\rho(h\tilde{\sigma})}(X) = X^n - \alpha((h\tilde{\sigma})^n),$$

then

$$\{g \in \mathrm{pr}^{-1}(\sigma) |\ \rho(g) \text{ is semi-simple}\} = \{g \in \mathrm{pr}^{-1}(\sigma) |\ \rho(g)^n \in \overline{\mathbb{Q}}_\ell.\mathrm{id}_V\},$$

and by density, the previous set is equal to $\mathrm{pr}^{-1}(\sigma)$. In particular, $\tilde{\sigma}$ acts semi-simply on V, $c(\tilde{\sigma}) = 0$, and $\forall\ h \in \Gamma_1$, $\rho((h\tilde{\sigma})^n)$ is a diagonal matrix, it follows that $c((h\tilde{\sigma})^n) = 0$ in $M_n(\overline{\mathbb{Q}}_\ell)$. More explicitly,

$$\sum_{l=0}^{n-1} \rho_1(h\tilde{\sigma})^l c(h) \rho_1(h\tilde{\sigma})^{-l} = 0. \tag{2.11}$$

The only non-zero terms of $\rho_1(h\tilde{\sigma}) = (a_{ij})$ are $a_{i,\tau(i)}$ where τ is the n-cycle $(1\ 2 \dots\ n)$, more precisely:

$$\begin{cases} a_{i,i+1} & = \alpha^{\sigma^{i-1}}(h) \qquad \forall\ i < n, \\ a_{n1} & = \alpha^{\sigma^{n-1}}(h)\alpha(\tilde{\sigma}^n). \end{cases}$$

Equation (2.11) in $M_n(\overline{\mathbb{Q}}_\ell)$ is equivalent to the following equations, for $i, j = 1, \dots, n$:

$$\sum_{l=0}^{n-1} \gamma_{ijl}(h) K_{ijl} c(h)_{\tau^l(i)\tau^l(j)} = 0, \tag{2.12}$$

where

$$\gamma_{ijl} = \prod_{r=0}^{l-1} \frac{\alpha^{\sigma^{r+i-1}}}{\alpha^{\sigma^{r+j-1}}}$$

with $\gamma_{ij0} = 1$, and K_{ijl} is either 1 or $\alpha(\tilde{\sigma}^n)$ or $\alpha(\tilde{\sigma}^n)^{-1}$. Now fix $h_0 \in \Gamma_1$ which acts trivially on $I(\alpha)$, and for $h_1 \in \Gamma_1$, let $h_2 = h_0 h_1$. We have then $c(h_2) = c(h_0) + c(h_1)$. Then substracting equation (2.12) for h_2 and h_1, we get $\forall\ h_1 \in \Gamma_1$,

$$\sum_{l=0}^{n-1} \gamma_{ijl}(h_1) K_{ijl} c(h_0)_{\tau^l(i)\tau^l(j)} = 0. \tag{2.13}$$

For $i \neq j$, suppose that there exist $l < l'$ in $[\![0, n-1]\!]$ such that $\gamma_{ijl} = \gamma_{ijl'}$. This means that

$$\prod_{r=l+i-1}^{l'+i-2} \alpha^{\sigma^r} = \prod_{r=l+j-1}^{l'+j-2} \alpha^{\sigma^r},$$

which is impossible, by Corollary 2.4.1.
Hence, for $i \neq j$, the characters $(\gamma_{ijl})_{l=0,\dots,n-1}$ are distinct, and linear independence of characters applied to equation (2.13) implies that

$$\forall\ i \neq j, \qquad c(h_0)_{\tau^l(i),\tau^l(j)} = 0.$$

Together with the previous step, this step implies that $\forall\ h_0 \in \Gamma_1$ acting trivially on $I(\alpha)$, we have $c(h_0) = 0$, which ends the proof.

Chapter 3

Applications to cohomology of unitary Shimura varieties

3.1 Introduction

3.1.1 Shimura data

Let $(G, \mathscr{X})$ be a pure Shimura datum: G is a connected reductive group over $\mathbb{Q}$ and $\mathscr{X}$ is a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow G_{\mathbb{R}}$ satisfying the minimal system of axioms (2.1.1.1)-(2.1.1.3) of [Del79]. The set $\mathscr{X} \simeq G(\mathbb{R})/K_h$, where K_h is the stabilizer of a fixed base point $h \in \mathscr{X}$, has a natural complex structure [Del79, 1.1.14], and its connected components are hermitian symmetric spaces. For example, for $G = GL(2)_{\mathbb{Q}}$ and

$$h : x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

we have $K_h = SO(2)\mathbb{R}_+^\times$ and $\mathscr{X} = \mathbb{C} \smallsetminus \mathbb{R}$.
For each open compact subgroup $K \subset G(\widehat{\mathbb{Q}})$, the analytic Shimura variety

$$Sh_K^{an}(G, \mathscr{X}) = G(\mathbb{Q}) \setminus (\mathscr{X} \times (G(\widehat{\mathbb{Q}})/K))$$

is a complex analytic space (in fact, a complex manifold if K is small enough). Baily and Borel [BB66] proved that Sh_K^{an} is the analytic space attached to a quasi-projective complex algebraic variety Sh_K. Theory of canonical models initiated by Shimura (and developed by Deligne, Milne, Shih and Borovoi) shows that each Sh_K is defined over a number field $E := E(G, \mathscr{X})$, called the reflex field of the Shimura datum.

3.1.2 Cohomology groups attached to a local system - Known results

Let $\xi : G_{\mathbb{C}} \longrightarrow GL(V_\xi)$ be an algebraic representation, where V_ξ is a complex vector space. If the restriction of ξ to the center $Z \subset G$ satisfies appropriate conditions ensuring that $\xi(Z(\mathbb{Q}) \cap K) = \{1\}$ for small enough K, one can attach to ξ a local system

$$\mathscr{L}_\xi = G(\mathbb{Q}) \setminus (\mathscr{X} \times (G(\widehat{\mathbb{Q}})/K) \times V_\xi)$$

of complex vector spaces on Sh_K^{an} (for K small enough).
The group $G(\widehat{\mathbb{Q}})$ acts on the projective system $\{Sh_K^{an}\}_K$ by right multiplication:

$$[\cdot g] : Sh_K^{an} \longrightarrow Sk_{g^{-1}Kg}^{an}.$$

Combined with canonical morphisms $[\cdot g]^* : \mathscr{L}_\xi \longrightarrow \mathscr{L}_\xi$ (see [HT01, III.2]), this defines a canonical left action of $G(\widehat{\mathbb{Q}})$ on the analytic cohomology groups

$$H^i(Sh^{an}, \mathscr{L}_\xi) = \varinjlim_K H^i(Sh_K^{an}, \mathscr{L}_\xi).$$

One can describe the de Rham complex of Sh_K^{an} in purely group-theoretical term [BW00, VII 2.5] which yields a canonical $G(\widehat{\mathbb{Q}})$-equivariant isomorphism with relative Lie algebra cohomology [BW00, I]

$$H^i(Sh^{an}, \mathscr{L}_\xi) \xrightarrow{\sim} H^i(\mathfrak{g}, K_h; C^\infty(G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}))_0 \otimes \mathscr{L}_\xi), \tag{3.1}$$

where $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$ and H_0 denotes the subspace of K_h-finite vectors in a $G(\mathbb{R})$-module H.

$G(\widehat{\mathbb{Q}})$ acts on the space of functions $\{f : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}\}$ by right translations: $(g \cdot f)(g') = f(g'g)$. Given a character $\chi : Z(\mathbb{Q}) \setminus Z(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}^\times$, fix a character $\chi_0 : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{R}_+^\times$ whose restriction to the center is equal to $|\chi|$, and define $L^2(G, \chi)$ to be the set of measurable functions $f : G(\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}$ such that

(a) $\forall\ z \in Z(\mathbb{A}_\mathbb{Q})$, $z \cdot f = \chi(z) f$

(b) the L^2-norm (with respect to the Haar measure dg on $G(\mathbb{A}_\mathbb{Q})$) is finite:

$$\| f \|^2 = \int_{G(\mathbb{Q})Z(\mathbb{A}_\mathbb{Q}) \setminus G(\mathbb{A}_\mathbb{Q})} |\chi_0^{-1}(g) f(g)|^2 dg < \infty.$$

From now on, we will only consider the case where the derived group G^{der} is anisotropic [1] over $\mathbb{Q}$, so that each Sh_K^{an} is compact (and Sh_K is projective), all automorphic forms on $G(\mathbb{A}_\mathbb{Q})$ are cuspidal and each space $L^2(G, \chi)$ decomposes as a discrete Hilbert sum $\widehat{\bigoplus_\pi}\ m(\pi)\pi$ (with finite multiplicities $m(\pi)$) of irreducible automorphic representations $\pi = \pi_\infty \otimes \pi^\infty$ of $(\mathfrak{g}, K_h) \times G(\widehat{\mathbb{Q}})$ with central characher χ. Together with (3.1), this leads to Matsushima's formula [BW00, VII 5.2]

$$H^i(Sh^{an}, \mathscr{L}_\xi) = \bigoplus_{\pi^\infty} \left(\pi^\infty \otimes \left(\bigoplus_{\pi_\infty} m(\pi_\infty \otimes \pi^\infty) H^i(\mathfrak{g}, K_h; \pi_\infty \otimes \xi) \right) \right). \tag{3.2}$$

Moreover, if ξ is irreducible, then the only (finitely many) automorphic representations π contributing to (3.2) are those for which $H^i(\mathfrak{g}, K_h; \pi_\infty \otimes \xi) \neq 0$ ("π_∞ is cohomological in degree i for ξ"). This condition implies that the central and infinitesimal characters of π_∞ coincide with those of the dual $\xi^\vee$ of ξ ([BW00, VII 6.1]).

3.1.3 Towards the algebraic context - Motivation

We now switch to the algebraic context. Denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Fix a prime number ℓ and an isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. For every algebraic or geometric object defined over a subfield of $\mathbb{C}$, we use the subscript ℓ to denote its base change to $\overline{\mathbb{Q}}_\ell$. For a subfield $k \subset \overline{\mathbb{Q}}$, let $\Gamma_k = \mathrm{Gal}(\overline{\mathbb{Q}}/k)$ be its absolute Galois group.
Recall that each $Sh_K(G, \mathscr{X})$ is an algebraic variety defined over the reflex field $E = E(G, \mathscr{X}) \subset \overline{\mathbb{Q}} \subset \mathbb{C}$. Moreover, each ξ as in Section 3.1.2 gives rise (for K small enough)

1. In the general case, one considers the intersection cohomology groups of the Baily-Borel compactification. Similar results are expected. [BB66], [Mor10]

to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,\ell}$ on Sh_K. ([HT01, III.2]).
The comparison isomorphism between Betti and étale cohomology groups gives a $G(\widehat{\mathbb{Q}})$-equivariant isomorphism

$$H^i_{et} := \varinjlim_K H^i_{et}(Sh_K \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell}) \xrightarrow{\sim} \varinjlim_K H^i(Sh_K^{an}, \mathscr{L}_\xi)_\ell = H^i(Sh^{an}, \mathscr{L}_\xi)_\ell.$$

Combined with (3.2), this yields a decomposition

$$H^i_{et} = \bigoplus_{\pi^\infty} \left(\pi^\infty \otimes V^i(\pi^\infty)\right) \tag{3.3}$$

indexed by irreducible representations π^∞ of $G(\widehat{\mathbb{Q}})$, in which

$$\dim_{\overline{\mathbb{Q}}_\ell} V^i(\pi^\infty) = \sum_{\pi_\infty} m(\pi_\infty \otimes \pi^\infty) \cdot \dim_{\mathbb{C}} H^i(\mathfrak{g}, K_h; \pi_\infty \otimes \xi).$$

($\pi = \pi_\infty \otimes \pi^\infty$ automorphic representation of $G(\mathbb{A}_\mathbb{Q})$).
Moreover, the canonical action of Γ_E on H^i_{et} induces a continuous $\overline{\mathbb{Q}}_\ell$-linear action of Γ_E on $V^i(\pi^\infty)$. One of the main goals of the subject is to determine the Galois representation $V^i(\pi^\infty)$.

There is a big industry (based on pioneering works of Ihara, Langlands and Kottwitz) devoted to computing the alternating sum of traces

$$\sum_{i\geq 0} (-1)^i \mathrm{Tr}\left(\mathrm{Fr}(v) | V^i(\pi^\infty)\right) \tag{3.4}$$

for all but finitely many primes v of E.

Thanks to the Čebotarev density theorem, the knowledge of these traces is equivalent to the knowledge of the virtual representation

$$\sum_{i\geq 0} (-1)^i [V^i(\pi^\infty)^{ss}]$$

in the Grothendieck group $G_0(\overline{\mathbb{Q}}_\ell[\Gamma_E])$.
The calculation of (3.4) requires a large machinery: a group-theoretical description of points over $\overline{\mathbb{F}}_p$ of canonical integral models of Sh_K; this gives rise, via the Lefschetz formula, to an expression for (3.4) in terms of suitable (twisted) orbital integrals; stabilization of this expression using the fundamental lemma and other results of the formalism of stable trace formula; decomposition of the result in terms of automorphic representations of various endoscopy groups of G.
Note that this method gives information about the semi-simplification of $V^i(\pi^\infty)$, not about the Galois representations $V^i(\pi^\infty)$ themselves.

Nekovář [Nek] observed that in certain simple, but non-trivial cases (when $G = B^\times$, for a quaternion algebra B over a totally real number field), a much more pedestrian approach shows that the representations $V^i(\pi^\infty)$ are, in fact, semi-simple. His method combined generalised Eichler-Shimura relations for partial Frobenius morphisms with certain abstract semi-simplicity crietria. Some of his general machinery was generalised in Chapters 1-2. We now show that under suitable assumptions, these results imply the semi-simplicity

of certain $V^i(\pi^\infty)$ occurring in middle cohomology of suitable unitary Shimura varieties (=essentially PEL Shimura varieties of type (A)) for which

$$G(\mathbb{R}) = \prod GU(a_v, b_v)$$

($a_v + b_v = n \geq 2$). Our assumptions ((A1) and (A2)) will be formulated in an axiomatic way. This will clarify both the strength and the limitations of the method.

Note that the semi-simplicity of $V^i(\pi^\infty)$ is predicted by general geometric conjectures ("the category of pure motives is semi-simple"). For example, if $(G, \mathscr{X})$ is a PEL Shimura datum, then $H^i_{et}(Sh_K \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell})$ is a direct summand of $H^k_{et}(A^m \otimes_E \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)(n)$, for suitable k, m, n (above, A^m is the m-fold fiber product of the universal abelian variety A over Sh_K, (see [HT01, III.2]), and A^m is a smooth projective variety over E).

3.2 Unitary groups

3.2.1 Definition of G and G^*

Suppose we are given the following data:

- a finite-dimensional simple $\mathbb{Q}$-algebra B
- a $\mathbb{Q}$-linear positive involution $* : B \longrightarrow B$ ($\forall\, b \neq 0$, $\mathrm{Tr}_{B/\mathbb{Q}}(bb^*) > 0$).

The center $F = Z(B)$ is a number field and $*|_F$ is a positive involution on F. Let $F^+ = F^{*=\mathrm{id}}$, i.e. the subfield of F fixed by the involution $*$.
Assume that $F^+ \neq F$.
This implies that F is a CM field with maximal totally real subfield F^+. The non-trivial automorphism of F/F^+ (and of $\mathbb{C}/\mathbb{R}$) will be denoted by c. Moreover, for every embedding $\sigma : F \hookrightarrow \mathbb{C}$, the map $* \otimes \mathrm{id}_\mathbb{R}$ is a positive involution on

$$B \otimes_{F^+,\sigma} \mathbb{R} = B \otimes_{F,\sigma} \mathbb{C} \simeq M_N(\mathbb{C}),$$

hence is conjugate, by [Kot92, Lemma 2.11], to the standard involution $M \mapsto {}^t\overline{M}$ on $M_N(\mathbb{C})$.

The additional data are given by:

- a finitely generated non-zero left B-module V
- a non-degenerate alternating F^+-bilinear form

$$\langle\ ,\ \rangle : V \times V \longrightarrow F^+$$

  satisfying $\langle bx, y\rangle = \langle x, b^*y\rangle$, $\forall\, x, y \in V,\ b \in B$.

This defines an involution $\#$ on $\mathrm{End}_{F^+}(V)$ given by $\langle f(x), y\rangle = \langle x, f^\#(y)\rangle$. Let $C = \mathrm{End}_B(V)$, which is a simple algebra of center F, and $\#$ restricts to an involution on C such that $\#|_F = *|_F$. Define a reductive algebraic group $H = GSp_B(V, \langle\ ,\ \rangle) = GU(C, \#)$ over F^+: for every F^+-algebra S,

$$\begin{aligned} H(S) &= \{f \in (C \otimes_{F^+} S)^\times |\ \nu(f) = ff^\# \in S^\times\} \\ &= \{f \in (C \otimes_{F^+} S)^\times |\ \exists\, \nu(f) \in S^\times;\ \forall\, x, y \in V \otimes_{F^+} S,\ \langle f(x), f(y)\rangle = \nu(f)\langle x, y\rangle\}. \end{aligned}$$

The multiplier $\nu : H \longrightarrow \mathbb{G}_{m,F^+}$ is a morphism of algebraic groups. If S is an F-algebra, then ([HT01, I.6])

$$C \otimes_{F^+} S = (C \otimes_{F^+} F) \otimes_F S = (C \otimes_F S) \times (C \otimes_{F,*|_F} S)$$

and $\# \times \mathrm{id}$ interchanges the two factors, thus

$$H(S) = \{f = (x, \lambda.(x^{\#})^{-1}) |\ x \in (C \otimes_F S)^\times,\ \lambda = \nu(f) \in S^\times\}.$$

In other words, we obtain isomorphisms

$$\begin{array}{rccc} \alpha : & H(S) & \xrightarrow{\sim} & (C \otimes_F S)^\times \times S^\times \\ & \alpha(f) & \longmapsto & (\mathrm{can}(f), \nu(f)) \end{array} \tag{3.5}$$

where $\mathrm{can}(f)$ is the image of $f \in (C \otimes_{F^+} S)^\times$ in $(C \otimes_F S)^\times$; and

$$\alpha : H \otimes_{F^+} F \xrightarrow{\sim} \underline{C}^\times \times \mathbb{G}_{m,F} \tag{3.6}$$

The group $G = R_{F^+/\mathbb{Q}}(H)$ does not give rise to a PEL Shimura variety, it has too big a center. We must pass to a subgroup $G^* \subset G$ defined by a cartesian diagram

$$\begin{tikzcd} G^* \arrow[r] \arrow[d] & G \arrow[d, "\nu"] \\ \mathbb{G}_{m,\mathbb{Q}} \arrow[r] & R_{F^+/\mathbb{Q}}(\mathbb{G}_{m,F^+}) \end{tikzcd}$$

For every $\mathbb{Q}$-algebra R,

$$G^*(R) = \{f \in (C \otimes_{\mathbb{Q}} R)^\times |\ \exists\ \nu(f) \in R^\times;\ \forall\ x, y \in V \otimes_{\mathbb{Q}} R,\ \langle f(x), f(y)\rangle_{\mathbb{Q}} = \nu(f)\langle x, y\rangle_{\mathbb{Q}}\}$$

where $\langle\ ,\ \rangle_{\mathbb{Q}} = (\mathrm{Tr}_{F^+/\mathbb{Q}}) \circ \langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{Q}$.

The final piece of data is given by a CM type Φ of F: for each infinite prime $v|\infty$ of F^+, we fix an embedding $\sigma_v : F \hookrightarrow \mathbb{C}$ inducing v.
Recall that $\overline{\mathbb{Q}} \subset \mathbb{C}$, which means that we identify σ_v with an embedding $\sigma_v : F \hookrightarrow \overline{\mathbb{Q}}$.
The choice of Φ induces isomorphisms

$$B \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} (B \otimes_{F^+,v} \mathbb{R}) = \prod_{v|\infty} (B \otimes_{F,\sigma_v} \mathbb{C}) \xrightarrow{\sim} \prod_{v|\infty} M_N(\mathbb{C})$$

and $V \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty}(V \otimes_{F^+,v} \mathbb{R})$, hence, by Morita equivalence, $V \otimes_{F^+,v} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^N \otimes_{\mathbb{C}} W_v$, where $n := \dim_{\mathbb{C}}(W_v) = \dfrac{1}{2N}\dim_{F^+}(V)$, and $B \otimes_{F^+,v} \mathbb{R}$ acts on the first factor $\mathbb{C}^N$. Note that

$$C \otimes_{F^+} \mathbb{R} \simeq \prod_{v|\infty} C \otimes_{F^+} F_v^+ \simeq \prod_{v|\infty} M_n(\mathbb{C}).$$

The skew-symmetric form $\langle\ ,\ \rangle \otimes \mathrm{id}$ on $V \otimes_{F^+,v} \mathbb{R}$ comes from one on W_v, which can be written as the imaginary part of a hermitian pairing $(\ ,\ ) : W_v \times W_v \longrightarrow \mathbb{C}$, of signature (a_v, b_v), with $a_v + b_v = n$. If we replace σ_v by $\overline{\sigma}_v$, then (a_v, b_v) is replaced by (b_v, a_v).
It follows that

$$G_{\mathbb{R}} \xrightarrow{\sim} \prod_{v|\infty} GU(a_v, b_v), \qquad G^*_{\mathbb{R}} \xrightarrow{\sim} G(\prod_{v|\infty} U(a_v, b_v)).$$

3.2.2 Unitary Shimura data

The above data define canonical Shimura data $(G, \mathscr{X})$ and $(G^*, \mathscr{X}^*)$ arising from the morphism

$$h : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \longrightarrow G^*_{\mathbb{R}} \subset G_{\mathbb{R}}$$

given by

$$h(z) = (zI_{a_v}, \overline{z}I_{b_v})_{v|\infty}\,.$$

Here, $\nu(h(z)) = z\overline{z}$.
The induced cocharacter $\mu = \mu_h$ $(= h(z,1))$ is given by

$$\mu : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G(\prod_{v|\infty} U(a_v, b_v))_{\mathbb{C}} = \left(\prod_{v|\infty} GL(n)_{\mathbb{C}}\right) \times \mathbb{G}_{m,\mathbb{C}};$$

$$\mu(z) = \Big((zI_{a_v}, I_{b_v})_{v|\infty}, z\Big)\,.$$

The weight morphism wt: $\mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbb{S}$, $\mathrm{wt}(t) = t^{-1}$, composed with h is equal to $h \circ \mathrm{wt} : t \mapsto (t^{-1} \cdot I_n)_{v|\infty}$.

The morphism h defines a Hodge decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$ into $B \otimes_{\mathbb{Q}} \mathbb{R}$-submodules, with $V^{p,q} = \{x |\ \forall\ z,\ h(z)x = z^{-p}\overline{z}^{-q}x\}$. In concrete terms,

$$V^{-1,0} \xrightarrow{\sim} \bigoplus_{v|\infty} \Big((\mathbb{C}^N)^{a_v} \oplus (\overline{\mathbb{C}}^N)^{b_v}\Big)\,,$$

with $B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \bigoplus_{v|\infty} M_N(\mathbb{C})$ acting by the standard action (resp. by its complex conjugate) on $\mathbb{C}^N$ (resp. $\overline{\mathbb{C}}^N$).

The reflex field $E = E(G, \mathscr{X}) = E(G^*, \mathscr{X}^*)$ is the field of definition of the isomorphism class of the complex representation of B on $V^{-1,0}$: it is the number field generated by the coefficients of the polynomial

$$\det(X_1\alpha_1 + \ldots + X_t\alpha_t | V^{-1,0}) \in \mathbb{C}[X_1, \ldots, X_t]$$

for any basis $\alpha_1, \ldots, \alpha_t$ of B over $\mathbb{Q}$. Explicitly,

$$E = \mathbb{Q}\left(\sum_{v|\infty}(a_v\sigma_v(\beta) + b_v\overline{\sigma}_v(\beta))\,\middle|\, \beta \in F\right) \subset F^{gal} \subset \overline{\mathbb{Q}} \subset \mathbb{C}.$$

For example, if the signature is parallel, i.e. if $\forall\ v$, $(a_v, b_v) = (a, b)$, then

$$E = \mathbb{Q}\left((a-b)\sum_{v|\infty}\sigma_v(\beta)\,\middle|\, \beta \in F\right) = \begin{cases} \mathbb{Q} & \text{if } a = b \\ \text{the reflex field of the CM type } \Phi \text{ of } F & \text{if } a \neq b \end{cases}$$

The inclusion of the Shimura data $(G^*, \mathscr{X}^*) \hookrightarrow (G, \mathscr{X})$ induces an isomorphism on the connected components containing $(h, 1)$ (after passing to the projective limit over all open compact subgroups $K^* \subset G^*(\widehat{\mathbb{Q}})$ and $K \subset G(\widehat{\mathbb{Q}})$):

$$Sh(G^*, \mathscr{X}^*)^0 \xrightarrow{\sim} Sh(G, \mathscr{X})^0. \tag{3.7}$$

This is a special case of [Del79, 2.1.8]; see [Nek] for an explicit argument in a slightly different context.

3.2.3 Representations of $G_{\mathbb{C}}$

We are now going to describe irreducible algebraic representations ξ of $G_{\mathbb{C}}$ giving rise to local systems on $Sh_K(G, \mathscr{X})$ (for small enough K). Recall from Section 3.2.1 an isomorphism

$$\alpha : G_{\mathbb{C}} \xrightarrow{\sim} \prod_{v|\infty} (GL(n)_{\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}})$$

(its conjugacy class depends on the fixed CM type Φ). This means that

$$\xi = \underline{\xi} \otimes \xi_0; \qquad \text{where} \quad \underline{\xi} = \bigotimes_{v|\infty} \underline{\xi}_v \quad \text{and} \quad \xi_0 = \bigotimes_{v|\infty} \xi_{0,v}$$

with $\underline{\xi}_v$ an irreducible algebraic representation of $GL(n)_{\mathbb{C}}$, and $\xi_{0,v} : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{C}}$ a character. In concrete terms, the highest weight of ξ_v is

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{m_{1,v}} \ldots t_n^{m_{n,v}}$$

for $(m_{1,v} \geq \ldots \geq m_{n,v}) \in (\mathbb{Z}^n)_+$, and $\xi_{0,v}(t) = t^{m_{0,v}}$; $m_{0,v} \in \mathbb{Z}$.

The restriction of ξ to the center of $G_{\mathbb{C}}$ can be described as follows. We have $Z_H = R_{F/F^+}(\mathbb{G}_{m,F})$ and $Z_G = R_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$. There is a natural algebraic subgroup $Z_G^+ = R_{F^+/\mathbb{Q}}(\mathbb{G}_{m,F^+}) \subset Z_G$ and the inclusion $(Z_G^+)_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}$ is given by

$$\prod_{v|\infty} (\mathbb{G}_{m,\mathbb{C}} \hookrightarrow GL(n)_{\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}})$$

where for each v, the map is given by $t \mapsto (t \cdot I_n, t^2)$.
In particular, the restriction of $(\underline{\xi}_v, \xi_{0,v})$ to the component at v of $(Z_G^+)_{\mathbb{C}} \xrightarrow{\sim} \prod_{v|\infty} \mathbb{G}_{m,\mathbb{C}}$ is equal to $t \mapsto \underline{\xi}_v(t \cdot I_n)\xi_{0,v}(t^2) = t^{-w_v} \cdot \mathrm{id}$, where

$$-w_v = 2m_{0,v} + \sum_{i=1}^{n} m_{i,v} \in \mathbb{Z}.$$

Definition 3.2.1. We say that the representation ξ has integral weight $w \in \mathbb{Z}$ if $\forall\ v|\infty$ in F^+, $w_v = w$.

Proposition 3.2.1. *The following properties are equivalent:*

1. *ξ has integral weight $w \in \mathbb{Z}$*
2. *the morphism $\mathbb{G}_{m,\mathbb{R}} \xrightarrow{wt} \mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \xrightarrow{\xi} \underline{GL}(V_\xi)$ is given by $t \mapsto t^w \cdot id$*
3. *$\xi(Z_G(\mathbb{Q}) \cap K) = \{1\}$, for all small enough open compact subgroups $K \subset G(\widehat{\mathbb{Q}})$.*

Proof. Note that $Z_G(\mathbb{Q}) = F^\times$, which means that for small enough K, $Z_G(\mathbb{Q}) \cap K$ is contained in the group of totally positive units of F^+ (and has finite index in them). For every such a unit u, $\xi(u) = \prod_{v|\infty} \sigma_v(u)^{w_v}$, which means that ξ vanishes on $Z_G(\mathbb{Q}) \cap K$ if and only if all w_v coincide. $\square$

As a result, we can attach to every $\xi = \underline{\xi} \otimes \xi_0$ with integral weight $w \in \mathbb{Z}$ a local system $\mathscr{L}_\xi$ of complex vector spaces on $Sh_K^{an}(G, \mathscr{X})$ (resp. a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,\ell}$ on $Sh_K(G, \mathscr{X})$), if K is small enough.

If $U \subset G(\widehat{\mathbb{Q}})$ is the subgroup stabilising the subset $Sh(G^*, \mathscr{X}^*) \subset Sh(G, \mathscr{X})$, then (3.7) above implies that

$$Sh(G, X) \simeq Sh(G^*, \mathscr{X}^*) \times_U G(\widehat{\mathbb{Q}}).$$

In particular, there is an isomorphism of $G(\widehat{\mathbb{Q}}) \times \Gamma_E$-modules

$$H^i_{et}(Sh(G, \mathscr{X}) \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell}) \simeq \mathrm{Ind}_U^{G(\widehat{\mathbb{Q}})} H^i_{et}(Sh(G^*, \mathscr{X}^*) \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell}). \tag{3.8}$$

As a result, semi-simplicity of various constituents of the L.H.S. will imply the same for the R.H.S.
Note that one can construct $\mathscr{L}_{\xi,\ell}|_{Sh(G^*,\mathscr{X}^*)}$ in terms of the universal abelian scheme over the PEL Shimura variety $Sh(G^*, \mathscr{X}^*)$, as in [HT01, III.2]. This implies that the sheaf $\mathscr{L}_{\xi,\ell}$ is pure of weight $w[F^+ : \mathbb{Q}]$.

3.3 Weak transfer from GU to $GL(n) \times GL(1)$

There is a close relation between automorphic representations π of $G(\mathbb{A}_\mathbb{Q}) = H(\mathbb{A}_{F^+})$ and suitably self-dual automorphic representations (Π, ψ) of $GL_n(\mathbb{A}_F) \times \mathbb{A}_F^\times = H_0(\mathbb{A}_F) = G_0(\mathbb{A}_\mathbb{Q})$, where

$$H_0 = GL(n)_F \times \mathbb{G}_{m,F}, \qquad G_0 = R_{F/\mathbb{Q}}(H_0).$$

Very strong results in this direction were proved in [Clo91], [Clo93], [Lab99], [HL04], [LS90], [HT01, VI. 2.1] for unitary groups of a special kind. We need instead a very weak form of this base change, but without additional assumptions on the group H.

In fact, in [loc. cit.] one establishes a base change from $G(\mathbb{A}_\mathbb{Q})$ to $(C \otimes_F \mathbb{A}_F)^\times \times \mathbb{A}_F^\times$, which is then transferred to $G_0(\mathbb{A}_\mathbb{Q})$ via the generalised Jacquet-Langlands correspondance.

Recall from (3.5) the isomorphism (for S an F-algebra)

$$\alpha : H(S) \xrightarrow{\sim} (C \otimes_F S)^\times \times S^\times.$$

In the special case when $S = S^+ \otimes_{F^+} F$ for an F^+-algebra S^+, the "complex conjugation" $c = \mathrm{id}_{S^+} \otimes c : H(S) \to H(S)$ corresponds, via α, to the map

$$\begin{array}{ccccccc} (C \otimes_{F^+} S^+)^\times & \times & (S^+ \otimes_{F^+} F)^\times & \longrightarrow & (C \otimes_{F^+} S^+)^\times & \times & (S^+ \otimes_{F^+} F)^\times \\ (A & , & b) & \longmapsto & (c(b)(A^\#)^{-1} & , & c(b)). \end{array} \tag{3.9}$$

Note that, for $S^+ = \mathbb{R} \otimes_\mathbb{Q} F^+ = \prod_{v|\infty} F_v^+ = \mathbb{R}^{\mathrm{Hom}(F^+,\mathbb{R})}$, the choice of the CM type $\Phi = \{\sigma_v\}$ of F gives an isomorphism

$$S^+ \otimes_{F^+} F = \mathbb{R} \otimes_\mathbb{Q} F \xrightarrow{\sim} \prod_{v|\infty} F_{\sigma_v} = \mathbb{C} \otimes_\mathbb{Q} F^+$$

under which $\mathrm{id} \otimes c$ corresponds to $c \otimes \mathrm{id}$.
As a result, we obtain from Φ and α isomorphisms

$$G \otimes_\mathbb{Q} \mathbb{C} \simeq \prod_{v|\infty} (H \otimes_{F^+} F) \otimes_{F,\sigma_v} \mathbb{C} \simeq \prod_{v|\infty} (\underline{C}^\times \times \mathbb{G}_{m,F}) \times_{F,\sigma_v} \mathbb{C} \simeq \prod_{v|\infty} (GL(n)_F \times \mathbb{G}_{m,F}) \otimes_{F,\sigma_v} \mathbb{C}.$$

Moreover,

$$G_0 \otimes_\mathbb{Q} \mathbb{C} \simeq \prod_{v|\infty} \left((GL(n)_F \times \mathbb{G}_{m,F}) \otimes_{F,\sigma_v} \mathbb{C} \times (GL(n)_F \times \mathbb{G}_{m,F}) \otimes_{F,c\sigma_v} \mathbb{C} \right).$$

Let $\xi : G_{\mathbb{C}} \longrightarrow \underline{GL}(V_\xi)$ be an irreducible algebraic representation of integral weight $w \in \mathbb{Z}$ in the sense of Definition 3.2.1. It induces representations:

– $\xi_{\mathbb{C}} : G(\mathbb{C}) \longrightarrow GL(V_\xi)$,
– $\xi_{\mathbb{C}}^c : G(\mathbb{C}) \xrightarrow{c} G(\mathbb{C}) \xrightarrow{\xi_{\mathbb{C}}} GL(V_\xi)$,
– $\xi_{\mathbb{C}} \otimes \xi_{\mathbb{C}}^c : G_0(\mathbb{C}) \longrightarrow GL(V_\xi \otimes V_\xi)$

Recall that $Z_H = R_{F/F^+}(\mathbb{G}_{m,F})$, $Z_G = R_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$, $Z_{G_0} = R_{F/\mathbb{Q}}(\mathbb{G}_{m,F} \times \mathbb{G}_{m,F})$.

Definition 3.3.1. Let π be an irreducible automorphic representation of $G(\mathbb{A}_{\mathbb{Q}}) = H(\mathbb{A}_{F^+})$ such that π_∞ is cohomological (in some degree) for ξ. We say that an irreducible automorphic representation (Π, ψ) of $G_0(\mathbb{A}_{\mathbb{Q}}) = GL_n(\mathbb{A}_F) \times \mathbb{A}_F^\times$ is a *very weak base change* of π if

1. $(\Pi_\infty, \psi_\infty)$ is cohomological (in some degree) for $\xi_{\mathbb{C}} \otimes \xi_{\mathbb{C}}^c$.
2. Π is self-dual: $\Pi^\vee \simeq \Pi^c$.
3. For all but finitely many finite primes x of F^+ at which F/F^+ and C are split (so $x = y\overline{y}$ in F), the local representation π_x of $H(F_x^+) = H(F_y)$ is isomorphic, via $\alpha : H(F_y) \xrightarrow{\sim} (C \otimes_F F_y)^\times \times F_y^\times \simeq GL_n(F_y) \times F_y^\times = H_0(F_y)$, to the local representation $(\Pi, \psi)_y$ of $H_0(F_y)$.

In fact, (Π, ψ) should have the following additional properties, but we are not going to use them [2]

4. The central characters of π and Π satisfy $\omega_\pi = \psi^c$ and $\omega_\Pi = \psi^c/\psi$.
5. An appropriate relation between π_x and Π_y if x is inert in F/F^+ and y is the unique prime of F above x (for all but finitely many x).

Let us make some comments on the conditions (1)-(3) and their consistency with (4) (cf [HT01, VI 2.1]).

• Firstly, $\xi_{\mathbb{C}}$ corresponds, via α, to $\underline{\xi} \otimes \xi_0$, where $\underline{\xi}$ is an irreducible algebraic representation of $(C \otimes_{F^+} \mathbb{R})^\times \simeq GL_n(F \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \prod_{v|\infty} GL_n(\mathbb{C})$, and ξ_0 is an algebraic character of $(F \otimes_{\mathbb{Q}} \mathbb{R})^\times \simeq \prod_{v|\infty} \mathbb{C}^\times$. If we denote by

$$\omega_{\underline{\xi}} : \prod_{v|\infty} \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$$

the central character of $\underline{\xi}$, then we obtain from (3.9) that

$$\xi_{\mathbb{C}}^c(A, b) = \xi_{\mathbb{C}}(c(b)(A^\#)^{-1}, c(b)) = \underline{\xi}(c(b)(A^\#)^{-1})\xi_0(c(b)) = \underline{\xi}((A^\#)^{-1})(\omega_{\underline{\xi}}^c \xi_0^c)(b).$$

Thus, $\xi_{\mathbb{C}}^c$ corresponds to $\underline{\xi}^\# \otimes \omega_{\underline{\xi}}^c \xi_0^c$, where $\underline{\xi}^\#(A) = \underline{\xi}((A^\#)^{-1})$. As the involution $\#$ on each factor $GL_n(F_{\sigma_v}) = GL_n(\mathbb{C})$ is conjugate to the standard involution $A \longmapsto {}^t\overline{A}$, we have $\underline{\xi}^\# = (\underline{\xi}^c)^\vee$. To sum up,

$$\xi_{\mathbb{C}} = \underline{\xi} \otimes \xi_0, \qquad \xi_{\mathbb{C}}^c = (\underline{\xi}^c)^\vee \otimes \omega_{\underline{\xi}}^c \xi_0^c.$$

• Secondly, the fact that π_∞ is ξ-cohomological implies that $\omega_{\pi_\infty}^{-1} = \omega_{\xi_{\mathbb{C}}}|_{Z_G(\mathbb{R})}$. However, the maps $Z_G(\mathbb{R}) \hookrightarrow Z_G(\mathbb{C}) \xrightarrow{\alpha} \prod_{v|\infty}(\mathbb{C}^\times \times \mathbb{C}^\times)$ are given by

$$\begin{array}{ccccc} (F \otimes_{\mathbb{Q}} \mathbb{R})^\times = \prod_{v|\infty} \mathbb{C}^\times & \hookrightarrow & \prod_{v|\infty}(\mathbb{C}^\times \times \mathbb{C}^\times) & \xrightarrow{\alpha} & \prod_{v|\infty}(\mathbb{C}^\times \times \mathbb{C}^\times) \\ z & \mapsto & (z, \overline{z}) & & \\ & & (z_1, z_2) & \longmapsto & (z_1, z_1 z_2) \end{array}$$

2. Moreover, we will use (3) only for x of degree one, when $F_x^+ = \mathbb{Q}_{p(x)}$, where $p(x)$ is the residue characteristic of F_x^+.

which implies that $\omega_{\xi_{\mathbb{C}}}(z,\overline{z}) = \omega_{\underline{\xi}}(z)\xi_0(z\overline{z})$, hence $\omega_{\pi_\infty}^{-1} = \omega_{\underline{\xi}}\xi_0\xi_0^c$.

• Thirdly, $(\Pi_\infty, \psi_\infty)$ being $\xi_{\mathbb{C}} \otimes \xi_{\mathbb{C}}^c$-cohomological is equivalent to the fact that Π_∞ is $\underline{\xi} \otimes (\underline{\xi}^c)^\vee$-cohomological and

$$\psi_\infty^{-1} = \omega_{\underline{\xi}}^c \xi_0 \xi_0^c \ (= (\omega_{\pi_\infty}^c)^{-1}). \tag{3.10}$$

The first condition implies that $\omega_{\Pi_\infty}^{-1} = \omega_{\underline{\xi}}(\omega_{\underline{\xi}}^c)^{-1} = \psi_\infty/\psi_\infty^c$. Thus (1) implies that $\omega_{\pi_\infty} = \psi_\infty^c$ and $\omega_{\Pi_\infty} = \psi_\infty^c/\psi_\infty$, in line with (4).

• Finally, in the situation of (3), the isomorphism $GL_n(F_y) \times F_y^\times = H_0(F_y) \simeq H(F_x^+) \simeq H_0(F_{\overline{y}})$ is given by $(A,b) \mapsto (\overline{b}(A^\#)^{-1}, \overline{b})$, where $A^\# = J\ {}^t\overline{A}J^{-1}$ and $b \mapsto \overline{b}$ is the isomophism $F_y \xrightarrow{\sim} F_{\overline{y}}$ induced by $c : F \longrightarrow F$. We require the representation π_x to be isomorphic to both $(\Pi,\psi)_y$ and $(\Pi,\psi)_{\overline{y}}$. These two conditions are consistent if we admit that $\omega_\Pi = \psi^c/\psi$, since the representation

$$(A,b) \mapsto (\Pi,\psi)_{\overline{y}}(\overline{b}(A^\#)^{-1}, \overline{b}) = (\omega_\Pi^c \psi^c)_{\overline{y}}(b)\Pi_{\overline{y}}(J\ {}^t\overline{A}J^{-1})$$

of $H_0(F_y) = GL_n(F_y) \times F_y^\times$ will be isomorphic to

$$(A,b) \mapsto (\omega_\Pi^c \psi^c)_y(b)\Pi_{\overline{y}}({}^t\overline{A}^{-1}) = (\omega_\Pi^c \psi^c)_y(b)(\Pi_y^\vee)^c(A) = ((\Pi^\vee, \omega_\Pi\psi)^c)_y(A,b) = (\Pi,\psi)_y(A,b).$$

Compatibility with the notation of Taylor, Harris and others [BLGGT14]

Definition 3.3.2. For $\tau : F \hookrightarrow \mathbb{C}$, we let

$$a_{\tau_0} = \begin{cases} m_{v,0} & \text{if } \tau = \sigma_v \\ -m_{v,0} & \text{if } \tau = c\sigma_v = \overline{\sigma}_v, \end{cases}$$

and for $1 \le i \le n$,

$$a_{\tau,i} = \begin{cases} m_{v,i} & \text{if } \tau = \sigma_v \\ -m_{v,n+1-i} & \text{if } \tau = \overline{\sigma}_v. \end{cases}$$

The representation $\underline{\xi} \otimes (\underline{\xi}^c)^\vee = \Xi_a$ of $GL_n(F \otimes_{\mathbb{Q}} \mathbb{C}) = G_0(\mathbb{C}) \simeq \prod_{\tau : F \hookrightarrow \mathbb{C}} GL_n(\mathbb{C})$ has weight $a = (a_{\tau,i}) \in (\mathbb{Z}^n)_0^{\mathrm{Hom}(F,\mathbb{C}),+}$ in the notation of [BLGGT14], hence Π is of weight a in the language of [BLGGT14].
The infinity type ψ_∞ of the character $\psi : \mathbb{A}_F^\times/F^\times \longrightarrow \mathbb{C}^\times$ is given by the formula (3.10). As

$$(\omega_{\underline{\xi}})_v : t \mapsto t^{m_{v,1}+\ldots+m_{v,n}} = t^{-w-2m_{v,0}}, \qquad \xi_{0,v} : t \mapsto t^{m_{v,0}},$$

we see that

$$\psi_\infty^{-1} : (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \simeq \prod_{v|\infty} F_{\sigma_v}^\times = \prod_{v|\infty} \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$$

is given by

$$\psi_\infty^{-1}((t_v)_v) \mapsto \prod_{v|\infty} (\overline{t}_v)^{-w-2m_{v,0}}(t_v\overline{t}_v)^{m_{v,0}} = \prod_{v|\infty} t_v^{m_{v,0}}(\overline{t}_v)^{-w-m_{v,0}}.$$

Set

$$[\psi] := \sum_{v|\infty} (m_{v,0}(\sigma_v) - (w + m_{v,0})(\overline{\sigma}_v)) \in \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$$

and define the algebraic Hecke character attached to ψ by

$$\psi_{alg} : \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times, \quad \psi_{alg}(a) = \psi(a)\psi_\infty(a_\infty)^{-1}.$$

It satisfies $\psi_{alg}|_{(F\otimes\mathbb{R})^\times} = 1$, $\psi_{alg}|_{F^\times} : \alpha \mapsto \alpha^{[\psi]}$, and it takes values in $\overline{\mathbb{Q}}^\times \subset \mathbb{C}^\times$. The corresponding ℓ-adic Hecke character

$$\mathbb{A}_F^\times/F^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times, \quad a \mapsto \psi_{alg}(a)a_\ell^{-[\psi]}$$

(each $\sigma : F \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ induces an embedding $F \hookrightarrow \overline{\mathbb{Q}}_\ell$ via the fixed isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, hence a map $F \otimes \mathbb{Q}_\ell \longrightarrow \overline{\mathbb{Q}}_\ell$) factors as

$$\mathbb{A}_F^\times/F^\times \xrightarrow{rec_F} \Gamma_F^{ab} \xrightarrow{\rho_{\psi,\ell}} \overline{\mathbb{Q}}_\ell^\times.$$

The reciprocity map rec_F is normalised by letting uniformisers at finite primes of F correspond to geometric Frobenius elements.
The Galois representation $\rho_{\psi,\ell}$ attached to ψ is pure of weight $-w$: if $v \nmid \ell$ is a finite prime of F at which $\rho_{\psi,\ell}$ is unramified, then every eigenvalue α of $\rho_{\psi,\ell}(\mathrm{Fr}(v)_{geom})$ is an algebraic number such that $|\iota(\alpha)| = (Nv)^{-w/2}$, for every embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Recall that our goal is an analysis of the semi-simplicity of the Galois representation $V^i(\pi^\infty)$ of Γ_E in the decomposition (3.3). The methods developed in [Nek] and in Chapters 1 and 2 require us to impose the following fundemental assumption (which implies that π_∞ will be cohomological only in the medium degree $i = \dim(Sh)$):

Assumption (A1): π admits a very weak base change (Π, ψ) in the sense of Definition 3.3.1, with Π cuspidal.

This cuspidality condition implies, thanks to a collective effort of many mathematicians (Clozel, Harris, Taylor, Labesse, Shin...), that there exists a Galois representation

$$\rho_{\Pi,\ell} : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell),$$

which is compatible with Π in a very strong sense ([CH13], [BLGGT14]). We need the following properties of $\rho_{\Pi,\ell}$:

- $\rho_{\Pi,\ell}$ is semi-simple.[3]
- $\rho_{\Pi,\ell}^\vee \simeq \rho_{\Pi,\ell}^c(n-1)$.
- $\rho_{\Pi,\ell}$ is unramified at all finite primes $y \nmid \ell$ of F at which Π is unramified.
- For all but finitely many finite primes $y \nmid \ell$ of F

$$L_y(\Pi, s)^{-1} = L_y\left(\rho_{\Pi,\ell}, s + \frac{n-1}{2}\right)^{-1}$$

  (equality in $\mathbb{C}[(Nv)^{-s}] \simeq \overline{\mathbb{Q}}_\ell[(Nv)^{-s}]$). We will need this only for y of degree one, when $F_y = \mathbb{Q}_{p(y)}$.
- For each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ the restriction of $\rho_{\Pi,\ell}$ to $\Gamma_{F_\tau} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/F_\tau)$ is a Hodge-Tate representation with n distinct Hodge-Tate weights (which can be made explicit in terms of $(a_{\tau,i})_{1\le i\le n}$, but we do not need the exact formula).

3. In fact, $\rho_{\Pi,\ell}$ is expected to be irreducible. Hopefully, this will be established in a foreseeable future, but at present the irreducibility is known only if the weight of Π is extremely regular and ℓ belongs to a set of primes of density one ([BLGGT14]). See also [CG13] for irreducibility over F^+.

– $\rho_{\Pi,\ell}$ is pure of weight $n-1$.

Proposition 3.3.1. *1. For every extension F'/F, the restriction $\rho_{\Pi,\ell}|_{\Gamma_{F'}}$ is a direct sum of distinct irreducible representations (without multiplicities).*

2. Each irreducible constituent of $\rho_{\Pi,\ell}$ is of the form $\mathrm{Ind}_{\Gamma_{F'}}^{\Gamma_F}(\sigma)$, where F'/F is a finite extension and $\sigma : \Gamma_{F'} \longrightarrow GL_m(\overline{\mathbb{Q}}_\ell)$ is a representation for which $\sigma|_{\Gamma_{F''}}$ is irreducible for every finite extension F''/F'.

3. It is equivalent: $\rho_{\Pi,\ell}|_{\Gamma_{F'}}$ is irreducible for every finite extension F'/F ($\rho_{\Pi,\ell}$ is "strongly irreducible") $\Leftrightarrow$ $\rho_{\Pi,\ell}$ is irreducible and $\rho_{\Pi,\ell}$ is not induced from a proper open subgroup of Γ_F.

Proof.

1. $\rho_{\Pi,\ell}|_{\Gamma_{F'}}$ is semi-simple (since $\rho_{\Pi,\ell}$ is) and multiplicity free since the Hodge-Tate weights at a fixed $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$ have no multiplicities.
2. see [CG13, Lemma 4.3].
3. follows from (2).

□

Proposition 3.3.2. *Let $\overline{\mathfrak{g}} = \overline{\mathbb{Q}}_\ell \cdot Lie(\rho_{\Pi,\ell}(\Gamma_F)) \subset \mathfrak{gl}(\mathfrak{n}, \overline{\mathbb{Q}}_\ell)$ be the $\overline{\mathbb{Q}}_\ell$-Lie algebra generated by the Lie algebra of the image of $\rho_{\Pi,\ell}$.*

1. $\overline{\mathfrak{g}}$ is reductive and its center acts semi-simply on $\overline{\mathbb{Q}}_\ell^n$.

2. It is equivalent: $\overline{\mathbb{Q}}_\ell^n$ is an irreducible $\overline{\mathfrak{g}}$-module $\Leftrightarrow$ $\rho_{\Pi,\ell}$ is strongly irreducible.

3. A Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ acts on $\overline{\mathbb{Q}}_\ell^n$ by n distinct weights (without multiplicities).

Proof. (1) and (2) are standard. For (3), Sen's theory [Sen73, Theorem 1] implies that $\overline{\mathfrak{g}}$ contains a semi-simple element whose eigenvalues are the n distinct Hodge-Tate weights of $\rho_{\Pi,\ell}|_{\Gamma_{F_\tau}}$ (for some $\tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell$), this element (being semi-simple) is contained in a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ which then has the required property. □

Remark: Property (3) was used in a crucial way in the construction of $\rho_{\Pi,\ell}$ ([HT01, VII 1.8]).

Corollary 3.3.1. *Denote by* Std *the standard representation of $\overline{\mathfrak{g}} \subset \mathfrak{gl}(n, \overline{\mathbb{Q}}_\ell)$ on $\overline{\mathbb{Q}}_\ell$.*

1. $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ acts without multiplicities on $\bigwedge^i \mathrm{Std}$ for $i = 0, 1, n-1, n$.

2. If $\overline{\mathfrak{g}} \supset \mathfrak{sl}(n, \overline{\mathbb{Q}}_\ell)$ (which implies that $\overline{\mathfrak{g}} = \mathfrak{gl}(n, \overline{\mathbb{Q}}_\ell)$, since $\det(\rho_{\Pi,\ell}) : \Gamma_F \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ is a character of infinite order), then $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ acts without multiplicities on the irreducible representations $\bigwedge^i \mathrm{Std}$ for all $i = 0, \ldots, n$.

3.4 Expected properties of $V^i(\pi^\infty)$

A general conjecture describing the semi-simplification $V^i(\pi^\infty)^{ss}$ of the Galois representation $V^i(\pi^\infty)$ of Γ_E in terms of π is described in [BR94, Conjecture 5.2]; we need only the main "naive" building block appearing in the conjecture, given in (3.11) below.
Consider a general pure Shimura datum $(G, \mathscr{X})$. A point $h \in \mathscr{X}$ represented by a morphism $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ gives rise to a cocharacter $\mu = \mu_h : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$ ($\mu(z) = h_{\mathbb{C}}(z, 1)$) whose conjugacy class $[\mu]$ depends only on $\mathscr{X}$, not on h. The set of conjugacy classes of such cocharacters $\mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})/\mathrm{int}(G(\mathbb{C})) = \mathrm{Hom}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G_{\overline{\mathbb{Q}}})/\mathrm{int}(G(\overline{\mathbb{Q}}))$ is the same

over $\mathbb{C}$ and over $\overline{\mathbb{Q}}$. The Galois group $\Gamma_{\mathbb{Q}}$ acts on the latter set and the stabiliser of $[\mu]$ is equal to $(\Gamma_{\mathbb{Q}})_{[\mu]} = \Gamma_E$, where $E = E(G, \mathscr{X})$ is the reflex field of $(G, \mathscr{X})$.

One can view μ (resp. μ^{-1}) as a character of a maximal torus $\widehat{T}$ of the dual group $\widehat{G}$, positive with respect to a suitable Borel subgroup $\widehat{B} \supset \widehat{T}$ (resp. $\widehat{B}_- \supset \widehat{T}$). The complex representation of $\widehat{G}$ with highest weight μ (resp. μ^{-1}) with respect to $\widehat{B}$ (resp. $\widehat{B}_-$) can be extended in a canonical way to a representation r_μ (resp. $r_{-\mu}$) of the L-group ${}^LG_E = \widehat{G} \rtimes \Gamma_E$ acting on the same space ([BR94, 5.1]).

If π is an irreducible automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ such that π_∞ is cohomological with respect to ξ from Section 3.1.2, it is expected that one can attach to π a semi-simple ℓ-adic representation (possibly after modifying the center of $\widehat{G}$; see [BG14])

$$\varphi_{\pi,\ell} : \Gamma_{\mathbb{Q}} \longrightarrow {}^LG_\ell = \widehat{G}_\ell \rtimes \Gamma_{\mathbb{Q}}$$

(with $\widehat{G}_\ell = \widehat{G} \otimes_{\mathbb{C}} \overline{\mathbb{Q}}_\ell$) such that $L(r \circ \varphi_{\pi,\ell}, s) = L(\pi, r, s - s_r)$, for every irreducible representation r of ${}^LG$.
The composite ℓ-adic representation

$$r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_E} : \Gamma_E \longrightarrow \widehat{G}_\ell \rtimes \Gamma_E \longrightarrow GL_{N_\mu}(\overline{\mathbb{Q}}_\ell) \tag{3.11}$$

should be - up to a Tate twist - the main building block [4] of $V^i(\pi^\infty)$, at least in the case when π is tempered.

In this case, one expects [BR94, Conjecture 5.2] $V^i(\pi^\infty)^{ss}$ to be a direct sum of Tate twists of certain direct summands of (3.11), with appropriate multiplicities. In the most favourable case when π is not endoscopic, this boils down to the simplest possible relation

$$V^i(\pi^\infty)^{ss} \stackrel{?}{=} (r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_E})^{\oplus m}(j), \tag{3.12}$$

for suitable $m \geq 0$ and $j \in \mathbb{Z}$.

A (still conjectural) generalisation of the Eichler-Shimura relation [BR94, §6] states that a suitably renormalised characteristic polynomial of $r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_E}$ kills $V^i(\pi^\infty)$. For compact PEL Shimura varieties, this is known after restriction to $\Gamma_{E'}$, for a suitable finite extension E' of E ([Wed00], [Moo04]).

The following example of $r_{\pm\mu}$ (see [BR94, 5.1(b)]) will be relevant for the subsequent discussion. If $G_{\mathbb{R}} = GU(a, b)$, then $G(\mathbb{C}) \simeq GL_n(\mathbb{C}) \times \mathbb{C}^\times$ (via α), and $\widehat{G} \simeq GL_n(\mathbb{C}) \times \mathbb{C}^\times$. The morphism $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ given by $h(z) = (zI_a, \overline{z}I_b)$ gives rise to $h_{\mathbb{C}}(z, w) = ((zI_a, wI_b), zw)$ and $\mu(z) = ((zI_a, I_b), z)$. Therefore

$$\widehat{\mu} : (\mathrm{diag}(t_1, \ldots, t_n), t_0) \mapsto (t_1 \ldots t_a)t_0$$

(for the standard upper triangular Borel subgroup) and, for w_0 the longest element of the Weyl group,

$${}^{w_0}(\widehat{\mu}^{-1}) : (\mathrm{diag}(t_1, \ldots, t_n), t_0) \mapsto (t_{b+1} \ldots t_n)^{-1} t_0^{-1}.$$

4. [BR94] and several other authors use r_μ instead of $r_{-\mu}$. This is compatible with the Galois action on H^0 given by Deligne's reciprocity law [Del79], but the sign in Deligne's formula was corrected by Milne in [Mil92, 1.10]. However, this issue of signs is not important for us, since if one changes $-\mu$ to μ, one must replace π by a suitable twist of $\pi^\vee$.

It follows that

$$\begin{aligned} r_\mu|_{\widehat{G}} &= \textstyle\bigwedge^a \mathrm{Std}_n \otimes \mathrm{Std}_1 \\ r_{-\mu}|_{\widehat{G}} &= \Big((\textstyle\bigwedge^b \mathrm{Std}_n) \otimes (\bigwedge^n \mathrm{Std}_n)^{-1}\Big) \otimes \mathrm{Std}_1^\vee = \bigwedge^a \mathrm{Std}_n^\vee \otimes \mathrm{Std}_1^\vee, \end{aligned} \tag{3.13}$$

where Std_m is the standard representation $\mathbb{C}^m$ of $GL_m(\mathbb{C})$.

3.5 Back to $GU(C,\#)$

We now specialise the previous general discussion to the group $G = R_{F^+/\mathbb{Q}}(H)$, where $H = GU(C,\#) = GSp_B(V, \langle\ ,\ \rangle)$ from Section 3.2. We have ${}^LH = \widehat{H} \rtimes \Gamma_{F^+}$ with Γ_F acting trivially on $\widehat{H}$ (since H is split over F, by (3.6)). As ${}^LG = \widehat{G} \rtimes \Gamma_\mathbb{Q} = \mathrm{Ind}_{\Gamma_{F^+}}^{\Gamma_\mathbb{Q}}({}^LH)$, it follows that the subgroup

$$\Gamma_{E'} = \{\gamma \in \Gamma_\mathbb{Q} |\ \forall\ v|\infty \text{ in } F^+,\ \gamma\sigma_v = \sigma_v\} \subset \Gamma_E$$

acts trivially on $\widehat{G}$. This means that $r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_{E'}}$ can be written in a very explicit form, using (3.13) and the fact that ${}^LG = \mathrm{Ind}_{\Gamma_{F^+}}^{\Gamma_\mathbb{Q}}({}^LH)$:

$$r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_{E'}} \simeq \bigotimes_{v|\infty} {}^{\sigma_v}\Big(\Big(\bigwedge^{a_v} \mathrm{Std}_n^\vee \otimes \mathrm{Std}_1^\vee\Big) \circ \varphi_{\pi,\ell}|_{\Gamma_F}\Big),$$

where $\varphi_{\pi,\ell}|_{\Gamma_F} : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell) \times \overline{\mathbb{Q}}_\ell^\times$ and ${}^{(\sigma_v)}\rho$ denotes, for any representation $\rho : \Gamma_F \longrightarrow GL(V_\rho)$, the representation

$$\Gamma_{E'} \longrightarrow \Gamma_F \xrightarrow{\rho} GL(V_\rho), \tag{3.14}$$

where the left map is conjugation by $\widetilde{\sigma}_v^{-1}$, for any lift $\widetilde{\sigma}_v \in \Gamma_\mathbb{Q}$ of $\sigma_v : F \hookrightarrow \overline{\mathbb{Q}}$, i.e. $\widetilde{\sigma}_v|_F = \sigma_v$.
If assumption (A1) is satisfied, then $\varphi_{\pi,\ell}|_{\Gamma_F} = (\rho_{\Pi,\ell}, \rho_{\psi,\ell})$ and

$$r_{-\mu} \circ \varphi_{\pi,\ell}|_{\Gamma_{E'}} \simeq \bigotimes_{v|\infty} {}^{\sigma_v}\Big(\Big(\bigwedge^{a_v} \rho_{\Pi,\ell}^\vee\Big) \otimes \rho_{\psi,\ell}^\vee\Big).$$

In addition, π_∞ is cohomological only in medium degree $i = \dim(Sh) = \sum_{v|\infty} a_v(n - a_v)$; and π is not endoscopic (since Π is cuspidal), hence the expected relation (3.12) states that

$$V^{\dim(Sh)}(\pi^\infty)^{ss}|_{\Gamma_{E'}} \overset{?}{=} \left(\bigotimes_{v|\infty} {}^{\sigma_v}\Big(\Big(\bigwedge^{a_v} \rho_{\Pi,\ell}^\vee\Big) \otimes \rho_{\psi,\ell}^\vee(t_v)\Big)\right)^{\bigoplus m}, \tag{3.15}$$

where $t_v := \dfrac{1}{2}a_v(a_v+1) - a_v n$ is the Tate twist. The value of t_v is obtained by comparing the weights of both sides: $V^{\dim(Sh)} \subset H^{\dim(Sh)}$ is pure of weight $\dim(Sh) + [F^+ : \mathbb{Q}]w = \sum_{v|\infty}(a_v(n-a_v) + w)$, while $(\bigwedge^{a_v} \rho_{\Pi,\ell}^\vee) \otimes \rho_{\psi,\ell}^\vee$ is pure of weight $-a_v(n-1) + w$.

In order to simplify the notation, let us number the infinite primes of F^+: $\{v_1, \ldots, v_r\}$, where $r = [F^+ : \mathbb{Q}]$, and set

$$\boxed{\rho_i = {}^{\sigma_{v_i}}\Big(\Big(\bigwedge^{a_{v_i}} \rho_{\Pi,\ell}^\vee\Big) \otimes \rho_{\psi,\ell}^\vee(t_{v_i})\Big) : \Gamma_{E'} \longrightarrow GL_{n_i}(\overline{\mathbb{Q}}_\ell)} \tag{3.16}$$

for $1 \leq i \leq r$, where n_i is the dimension of ρ_i; and

$$\boxed{\rho = V^{\dim(Sh)}(\pi^\infty)|_{\Gamma_{E'}} : \Gamma_{E'} \longrightarrow GL_M(\overline{\mathbb{Q}}_\ell),} \tag{3.17}$$

where M is the dimension of ρ. The expected relation (3.15) then predicts that, for some $m \geq 0$,

$$\rho^{ss} \stackrel{?}{=} (\rho_1 \otimes \ldots \otimes \rho_r)^{\oplus m}.$$

Proposition 3.5.1. *Let F^{gal} be the Galois closure of F. There exists a dense subset $\Sigma \subset \Gamma_{F^{gal}}$ such that $\forall\ g \in \Sigma$, $P_{(\rho_1 \otimes \ldots \otimes \rho_r)(g)}(\rho(g)) = 0$.*

Proof. Σ is the set of geometric Frobenius elements $\mathrm{Fr}(p)$ for primes p that split completely in $F/\mathbb{Q}$ and at which B and π are unramified. For such p, the group G is split over $\mathbb{Q}_p$ and the statement for $g = \mathrm{Fr}(p)$ is the generalised Eichler-Shimura relation proved in [Wed00], [Moo04] [5]. □

Assumption (A2): The dense subset $\Sigma \subset \Gamma_{F^{gal}}$ defined in the proof of Proposition 3.5.1 has the following property: for each $g \in \Sigma$, there exist pairwise commuting elements $u_1, \ldots, u_r \in GL_M(\overline{\mathbb{Q}}_\ell)$ such that $\rho(g) = u_1 \ldots u_r$ and $P_{\rho_i(g)}(u_i) = 0\ \forall\ i = 1, \ldots, r$.

The validity of Assumption (A2) will be explained in the following section.

3.6 Partial Frobenius morphisms and Assumption (A2)

3.6.1. In this section, we define an integral model of Sh_K and the partial Frobenius maps on its special fiber (see [Rei97, 2.14] and [TX13, 2.3-2.8, 4.6] for special cases).

Let p be a prime number which splits completely in $F/\mathbb{Q}$, and such that $B_{\mathbb{Q}_p} \simeq \prod_i M_{n_i}(\mathbb{Q}_p)$. We also assume that π_p is unramified. Fix a $*$-stable $O_{F^+} \otimes \mathbb{Z}_{(p)}$-order $O_B \subset B$ such that $O_B \otimes \mathbb{Z}_p$ is a maximal order in $B_{\mathbb{Q}_p}$ and an $O_{F^+} \otimes \mathbb{Z}_p$-stable lattice $\Lambda \subset V_{\mathbb{Q}_p}$ which is self-dual under a suitable scalar multiple of $\langle\ ,\ \rangle$; then $K_p = \{g \in G(\mathbb{Q}_p) \mid g(\Lambda) = \Lambda\} \subset G(\mathbb{Q}_p)$ and verifies $K_p \simeq \prod(GL_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times)$.
Moreover, the group G splits over $\mathbb{Q}_p$ as $G_{\mathbb{Q}_p} \simeq \prod(GL(n)_{\mathbb{Q}_p} \times \mathbb{G}_{m,\mathbb{Q}_p})$, and $V(\pi^\infty)^K \neq 0$ for $K = K_p K^p$, and $K^p \subset G(\widehat{\mathbb{Q}}^{(p)})$ small enough. Let $v|p$ be the prime of E defined by a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Fix a (finite) set $\{\alpha_i\}$ of representatives of double cosets

$$(\widehat{F}^{+,(p)})^\times = \coprod_i (O_{F^+} \otimes \mathbb{Z}_{(p)})^\times_+ \alpha_i (\widehat{O}^{(p)}_{F^+})^\times.$$

As above, fix a sufficiently small open compact subgroup K^p of $G(\widehat{\mathbb{Q}}^{(p)})$.

Consider a set-valued functor $M_{\alpha_i,K}$ from the category of (say, locally noetherian) schemes over O_{E_v} which associates to a scheme S the set of isomorphism classes of quadruples $(A, \iota, \lambda, \overline{(\eta, u)})$, where

- A is an object of $(AV/S) \otimes \mathbb{Z}_{(p)}$ (an abelian scheme over S up to prime-to-p isogeny);
- $\lambda : A \longrightarrow \widehat{A}$ is a $\mathbb{Z}_{(p)}$-polarisation of degree prime to p;

5. They work with the PEL Shimura variety $Sh(G^*, \mathscr{X}^*)$, but this does not affect $V^i(\pi^\infty)$, by (3.8).

• $\iota : O_B \longrightarrow \mathrm{End}(A)$ is a $*$-morphism (with respect to $*$ on the LHS and the Rosati involution attached to λ on the RHS);

• for a fixed geometric point s of (every connected component of) S, $\overline{(\eta, u)}$ is a $\pi_1(S, s)$-invariant K^p-level structure. By the latter we mean a K^p-orbit of pairs (η, u), where $u \in (\widehat{O}_{F^+}^{(p)})^\times$ and $\eta : V \otimes \widehat{\mathbb{Q}}^{(p)} \xrightarrow{\sim} V_s^{(p)}(A)$ is an $O_B \otimes \widehat{\mathbb{Q}}^{(p)}$-linear isomorphism such that the Weil pairing $\langle\ ,\ \rangle_\lambda : V_s^{(p)}(A) \times V_s^{(p)}(A) \longrightarrow \widehat{\mathbb{Q}}^{(p)}$ satisfies

$$\langle \eta(x), \eta(y)\rangle_\lambda = Tr_{F^+/\mathbb{Q}}(\alpha_i u\langle x, y\rangle).$$

An element $g \in K^p$ acts on (η, u) by $(\eta, u)g = (\eta \circ g, u\,\nu(g))$.

In addition, we require the Kottwitz determinant condition to hold.

An isomorphism between $(A, \iota, \lambda, \overline{(\eta, u)})$ and $(A', \iota', \lambda', \overline{(\eta', u')})$ is given by an isomorphism $f : A \xrightarrow{\sim} A'$ in $(AV/S) \otimes \mathbb{Z}_{(p)}$ such that $\lambda = \widehat{f} \circ \lambda' \circ f$, $f \circ \iota = \iota' \circ f$ and $\overline{(\eta', u')} = \overline{(f \circ \eta, u)}$.

Note that the degree of λ is determined by the above conditions. Moreover, $(A, \iota, \lambda, \overline{(\eta, u)})$ has no non-trivial automorphism if K^p is small enough ([Rei97, 2.13]), which we assume, from now on.

As in [Kot92] and [Rei97], it follows that $M_{\alpha_i,K}$ is represented by a smooth quasi-projective scheme over O_{E_v}, which will also be denoted by $M_{\alpha_i,K}$.

The group of totally positive units $O_{F^+,+}^\times$ acts on $M_{\alpha_i,K}$ by the formula

$$\varepsilon \cdot (A, \iota, \lambda, \overline{(\eta, u)}) = (A, \iota, \iota(\varepsilon)\lambda, \overline{(\eta, \varepsilon u)}).$$

If $\varepsilon = N_{F/F^+}(\varepsilon')$ for $\varepsilon' \in O_F^\times \cap K^p \subset F^\times = Z_G(\mathbb{Q}) \subset G(\widehat{\mathbb{Q}}^{(p)})$, then multiplication by $\iota(\varepsilon')$ on A defines an isomorphism

$$[\iota(\varepsilon')] : (A, \iota, \lambda, \overline{(\eta, u)}) \xrightarrow{\sim} \varepsilon^{-1} \cdot (A, \iota, \lambda, \overline{(\eta, u)}).$$

As a result, one obtains an action of a finite abelian group

$$\Delta = O_{F^+,+}^\times / N_{F/F^+}(O_F^\times \cap K^p)$$

on $M_{\alpha_i,K}$. After replacing K^p by a suitable open subgroup, the group Δ will act on $M_{\alpha_i,K}$ freely by permuting its connected components; this is proved in [Kis10] in general and in (3.6.7) below in the case when n is even. As a result, the quotient scheme $M_{\alpha_i,K}/\Delta$ exists and is again quasi-projective over O_{E_v}.

3.6.2. We are now going to define a map

$$M_{\alpha_i,K}(\mathbb{C}) \longrightarrow Sh_K(G, \mathscr{X})(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathscr{X} \times G(\widehat{\mathbb{Q}}^{(p)})/K^p \times G(\mathbb{Q}_p)/K_p)$$

([TX13, 2.4], [Mil05, 6.3, 6.9]). Let $(A, \lambda, \iota, \overline{(\eta, u)})$ be a quadruple representing an element of $M_{\alpha_i,K}(\mathbb{C})$. The group $H = H_1(A, \mathbb{Q})$ is a B-module via ι equipped with a skew-hermitian pairing $\langle\ ,\ \rangle_{H,\lambda} : H \times H \longrightarrow F^+$ such that $Tr_{F^+/\mathbb{Q}} \circ \langle\ ,\ \rangle_{H,\lambda}$ is attached to λ.

The arguments in [Kot92, p.338, 339] show that the skew-hermitian modules H and V are locally isomorphic. The isomorphism of $H \otimes \mathbb{Q}_l$ with $V \otimes \mathbb{Q}_l$ for $l \neq p, \infty$ follows from the existence of η. For $l = p$, the Tate module $T_p(A) = H_1(A, \mathbb{Z}_p) \subset H \otimes \mathbb{Q}_p$ is a self-dual $O_B \otimes \mathbb{Z}_p$-lattice, to which a variant of [Kot92, Lemma 7.2] applies. For $l = \infty$ one applies [Kot92, Lemma 4.2] (and the determinant condition).

The validity of the Hasse principle for G (see 3.6.5 below) implies that there is a B-linear isomorphism $a : H \xrightarrow{\sim} V$ (which we fix) sending $\langle\ ,\ \rangle_{H,\lambda}$ to a multiple of $\langle\ ,\ \rangle$ by an non-zero element of F^+. This isomorphism is unique up to left multiplication by $G(\mathbb{Q})$.

The natural complex structure h_A on $H \otimes \mathbb{R} = \mathrm{Lie}(A)$ defines a complex structure $ah_A = (z \mapsto a \circ h_A(z) \circ a^{-1})$ on V, hence a pure real Hodge structure of weight -1; this Hodge structure lies in $\mathscr{X}$, thanks to [Kot92, Lemma 4.2].

The composite map $a \circ \eta : V \otimes \widehat{\mathbb{Q}}^{(p)} \longrightarrow V \otimes \widehat{\mathbb{Q}}^{(p)}$ is given by an element $g^p \in G(\widehat{\mathbb{Q}}^{(p)})$.

Finally, $a(T_p(A)) \subset V \otimes \mathbb{Q}_p$ is an $O_B \otimes \mathbb{Z}_p$-lattice, self-dual up to a scalar. According to [Kot92, Lemma 7.3] there exists $g_p \in G(\mathbb{Q}_p)$ such that $a(T_p(A)) = g_p\Lambda$.

We assign to $(A, \lambda, \iota, \overline{(\eta, u)})$ the class $[(ah_A, g^p, g_p)]_K \in Sh_K(G, \mathscr{X})(\mathbb{C})$ of the triple (ah_A, g^p, g_p).

Proposition 3.6.3. The above formula defines a bijection

$$\coprod_i (M_{\alpha_i,K}(\mathbb{C})/\Delta) \xrightarrow{\sim} Sh_K(G, \mathscr{X})(\mathbb{C}).$$

Proof. The map is well-defined: firstly, another choice of a results in left multiplication by an element of $G(\mathbb{Q})$. Secondly, replacing (η, u) by $(\eta \circ g, u\,\nu(g))$ $(g \in K^p)$ amounts to multiplying g^p on the right by g. Thirdly, isomorphic quadruples have the same image. Finally, the same holds for two quadruples related by the action of Δ.

The map is surjective: any $h_x \in \mathscr{X}$ defines a Hodge decomposition $V \otimes \mathbb{C} = V_x^{-1,0} \oplus V_x^{0,-1}$. The subgroup $V \cap g_p\Lambda \subset V \otimes \mathbb{Q}_p$ is a $\mathbb{Z}_{(p)}$-lattice stable by O_B and $A = V_x^{-1,0}/V \cap g_p\Lambda$ is an object of $(AV/\mathbb{C}) \otimes \mathbb{Z}_{(p)}$ which is equipped with a canonical map $\iota : O_B \longrightarrow \mathrm{End}(A)$ and for which the determinant condition holds. The group $H = H_1(A, \mathbb{Q})$ is canonically identified with V; we let $\eta = g^p$. Choose a $\mathbb{Z}_{(p)}$-polarisation $\lambda : A \longrightarrow \widehat{A}$ compatible with O_B for which $g_p\Lambda$ is self-dual; then $\langle \eta(x), \eta(y)\rangle_\lambda = Tr_{F^+/\mathbb{Q}}(c\alpha_i u\langle x, y\rangle)$ for some i, $c \in (O_{F^+} \otimes \mathbb{Z}_{(p)})^\times_+$ and $u \in (\widehat{O}^{(p)}_{F^+})^\times$. It follows that the class of (x, g^p, g_p) is equal to the image of $(A, \iota, c^{-1}\lambda, \overline{(\eta, u)})$.

The map is injective: assume that $(A, \iota, \lambda, \overline{(\eta, u)})$ (resp. $(A', \iota', \lambda', \overline{(\eta', u')})$) represents an element of $M_{\alpha_i,K}(\mathbb{C})$ (resp. of $M_{\alpha_{i'},K}(\mathbb{C})$) and that they have the same image $[(ah_A, g^p, g_p)]_K = [(a'h_{A'}, g'^p, g'_p)]_K$ in $Sh_K(G, \mathscr{X})(\mathbb{C})$. After changing $a' : H' = H_1(A', \mathbb{Q}) \xrightarrow{\sim} V$ if necessary, we can assume that $ah_A = a'h_{A'}$; thus $g^pK^p = g'^pK^p$ and $g'_pK_p = g_pK_p$. The construction from the proof of surjectivity together with the fact that $a(T_p(A)) = a'(T_p(A'))$ yield an O_B-linear isomorphism $f : A \xrightarrow{\sim} A'$ in $(AV/\mathbb{C}) \otimes \mathbb{Z}_{(p)}$. As a result, we can assume that $A' = A$ and $\iota' = \iota$. Moreover, we can replace (η, u) by $(\eta, u)(g^p)^{-1}g'^p$, hence asume that $\eta' = \eta$. By definition,

$$\langle \eta(x), \eta(y)\rangle_\lambda = Tr_{F^+/\mathbb{Q}}(\alpha_i u\langle x, y\rangle)$$

and

$$\langle \eta(x), \eta(y)\rangle_{\lambda'} = \langle \eta'(x), \eta'(y)\rangle_{\lambda'} = Tr_{F^+/\mathbb{Q}}(\alpha_{i'} u'\langle x, y\rangle),$$

which implies that λ' is a scalar multiple of λ with the same Rosati involution; thus $\lambda' = \varepsilon\lambda$ for some $\varepsilon \in (O_{F^+} \otimes \mathbb{Z}_{(p)})^\times_+$. In particular, $\varepsilon\alpha_i u = \alpha_{i'}u'$, hence $i = i'$ and $\varepsilon = u'/u \in O^\times_{F^+,+}$. This means that, after replacing $(A, \iota, \lambda, \overline{(\eta, u)})$ by $\varepsilon \cdot (A, \iota, \lambda, \overline{(\eta, u)})$, we can assume that $A' = A$, $\iota' = \iota$, $\eta' = \eta$ and $\lambda' = \lambda$, but this implies that $u' = u$. $\square$

3.6.4. It follows that the quasi-projective O_{E_v}-scheme

$$S_K := \coprod_i (M_{\alpha_i,K}/\Delta)$$

is a model of $Sh_K(G, \mathscr{X})$.

Our assumption that G_{der} is anisotropic implies that $Sh_K(G, \mathscr{X})$ is projective, hence so is every $M_{\alpha_i,K} \otimes_{O_{E_v}} E_v$. It follows from a general result of Lan [Lan11, Theorem. 4.6] that $M_{\alpha_i,K}$ itself is projective ([Kot92, p. 392]) in the case when C is a division algebra), hence so is S_K.

3.6.5. The general results of Kisin [Kis10] imply that S_K is smooth over O_{E_v} whenever K^p is small enough. We give an elementary explanation of this fact in the case when $n = 2k$ is even, based on an analysis of geometric connected components.

Recall that the torus $D = G/G_{der}$ can be described in terms of tori $T = R_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ and $T_+ = R_{F^+/\mathbb{Q}}\mathbb{G}_{m,F^+}$ as follows. The map "determinant" together with ν induce an isomorphism $D \xrightarrow{\sim} \{(a, b) \in T \times T_+ \mid N_{F/F^+}(a) = b^n\}$.

If $n = 2k+1$ (resp. if $n = 2k$), the map $(a, b) \mapsto ab^{-k}$ (resp. $(a, b) \mapsto (ab^{-k}, b)$) defines an isomorphism $\beta : D \xrightarrow{\sim} T$ (resp. $\beta = (\beta_1, \nu) : D \xrightarrow{\sim} \mathrm{Ker}(N_{F/F^+} : T \longrightarrow T_+) \times T_+$). Both tori T T_+ satisfy the Hasse principle (by Hilbert's Theorem 90), hence so does D. It follows that G also satisfies the Hasse principle ([Mil05, Lemma 8.20, 8.21]).

Deligne's description of $\pi_0 Sh_K(G, \mathscr{X})(\mathbb{C})$ quoted in [Mil05, Theorem. 5.17] applies in our case. It yields, via β, isomorphisms (depending on a choice of a connected component of $\mathscr{X}$)

$$\pi_0 Sh_K(G, \mathscr{X})(\mathbb{C}) \xrightarrow{\sim} (O_F \otimes \mathbb{Z}_{(p)})^\times \backslash \widehat{F}^{(p),\times} / \beta(K^p)$$

if $n = 2k + 1$ and

$$\pi_0 Sh_K(G, \mathscr{X})(\mathbb{C}) \xrightarrow{\sim} U_1(K^p) \times U_2(K^p),$$
$$U_1(K^p) = \mathrm{Ker}(N : (O_F \otimes \mathbb{Z}_{(p)})^\times \longrightarrow (O_{F^+} \otimes \mathbb{Z}_{(p)})^\times_+) \backslash \mathrm{Ker}(N : \widehat{F}^{(p),\times} \longrightarrow \widehat{F}^{+,(p),\times}) / \beta_1(K^p)$$
$$U_2(K^p) = (O_{F^+} \otimes \mathbb{Z}_{(p)})^\times_+ \backslash \widehat{F}^{+,(p),\times} / \nu(K^p)$$

if $n = 2k$. In this case

$$\pi_0 M_{\alpha_i,K}(\mathbb{C}) = U_1(K^p) \times (\widehat{O}^{(p)}_{F^+})^\times / \nu(K^p) = U_1(K^p) \times \widehat{O}^\times_{F^+} / \nu(K),$$

$$\pi_0(M_{\alpha_i,K}/\Delta)(\mathbb{C}) = U_1(K^p) \times O^\times_{F^+,+} \backslash \widehat{O}^\times_{F^+} / \nu(K)$$

and the stabiliser in Δ of any connected component of $M_{\alpha_i,K}(\mathbb{C})$ is equal to

$$\Delta_0(K) = \left(O^\times_{F^+,+} \cap \nu(K)\right) / N_{F/F^+}(O^\times_F \cap K)$$

(see [TX13, 2.3, 2.4] in the case of Hilbert modular varieties).

Proposition 3.6.6. [TX13, Lemma 2.5] After shrinking K^p if necessary (i.e., after replacing K^p by a suitable open subgroup) one can achieve $\Delta_0(K) = 0$.

3.6.7. In particular, for a good choice of K^p as in Proposition 3.6.6 the passage from $M_{\alpha_i,K}(\mathbb{C})$ to $M_{\alpha_i,K}(\mathbb{C})/\Delta$ amounts to identifying several connected components. As a result, the action of Δ on the full scheme $M_{\alpha_i,K}$ is free, the map $M_{\alpha_i,K} \longrightarrow M_{\alpha_i,K}/\Delta$ is étale and $M_{\alpha_i,K}/\Delta$ (hence S_K) is smooth over O_{E_v}.

3.6.8. Let S°_K be the special fiber of the model S_K over the residue field $k(v) = \mathbb{F}_p$ of O_{E_v}. Fix a prime $P \mid p$ of F^+. We are going to describe explicitly the partial Frobenius morphism $\varphi_P : S^\circ_K \longrightarrow S^\circ_K$ in terms of the moduli problems defining the schemes $M_{\alpha_i,K}$ (the case of Hilbert modular varieties is treated in [TX, 4.6]).

Fix a totally positive element $c \in F^+$ such that $v_P(c) = 1$ and $v_{P'}(c) = 0$ for all $P' \mid p$, $P' \neq P$.

Let S be a scheme over $k(v)$ and let $(A, \iota, \lambda, \eta, u)$ be a 5-tuple representing an element $a \in M_{\alpha_i,K}(S)$. Denote by $Fr_A : A \longrightarrow A^{(p)}$ the relative Frobenius morphism. We define $A' = A/\mathrm{Ker}(Fr_A)[P]$ and we denote by $f_P : A \longrightarrow A'$ the quotient map. The formula $c\lambda = \widehat{f_P} \circ \lambda' \circ f_P$ defines a $\mathbb{Z}_{(p)}$-polarisation $\lambda' : A' \longrightarrow \widehat{A'}$. There is a canonical morphism $\iota' : O_B \longrightarrow \mathrm{End}(A')$ induced by ι and f_P. We let

$$\varphi_P(A, \iota, \lambda, \eta, u) = (A', \iota', \lambda', \eta', u'),$$

where $\eta' = f_P \circ \eta$ and $c\alpha_i u = \alpha_{i'} u'$. This recipe is compatible with the right K^p-action on the pairs (η, u), with isomorphisms and with the action of Δ. However, it depends on the choice of c.

If we replace c by $\tilde{c}$, then $\tilde{c} = \varepsilon c$ with $\varepsilon \in O^\times_{F^+,+}$ and the 5-tuple $(A', \iota', \lambda', \eta', u')$ is replaced by $(A', \iota', \varepsilon\lambda', \eta', \varepsilon u')$. This implies that the above formula gives rise to a well-defined partial Frobenius morphism

$$\varphi_P : M^\circ_{\alpha_i,K}/\Delta \longrightarrow M^\circ_{\alpha_{i'},K}/\Delta,$$

hence to $\varphi_P : S^\circ_K \longrightarrow S^\circ_K$.

The usual Frobenius morphism $\varphi : S^\circ_K \longrightarrow S^\circ_K$ has a similar description, with A' replaced by $A^{(p)}$ and c by p.

If $P_1, P_2 \mid p$ are two distinct primes in F^+ above p, then

$$\varphi_{P_j}(A, \iota, \lambda, \eta, u) = (A'_j, \iota'_j, \lambda'_j, \eta'_j, u'_j)$$

and

$$\varphi_{P_1} \circ \varphi_{P_2}(A, \iota, \lambda, \eta, u) = (A', \iota', \lambda', \eta', u') = \varphi_{P_2} \circ \varphi_{P_1}(A, \iota, \lambda, \eta, u),$$

where

$$A' = A/(\mathrm{Ker}(Fr_A)[P_1P_2]), \quad (f_{P_1} \circ f_{P_2})^*(\lambda') = c_1c_2\lambda, \quad \eta' = f_{P_1} \circ f_{P_2} \circ \eta, \quad c_1c_2\alpha_i u = \alpha_{i'} u'.$$

These formulas imply that $\prod_{P|p} \varphi_P = \varphi$.

Validity of Assumption (A2)

Assumption (A2) is an abstract version of generalised Eichler-Shimura relations for partial Frobenius morphisms. Let us give below some details following the appendix of [Nek], where a detailed discussion of this topic following [Wed00] is included.

Let $S \supset \{\ell, \infty\}$ be a finite set of primes of $\mathbb{Q}$ containing all primes at which $F/\mathbb{Q}$, H and π are ramified. Let $\mathbb{Q}_S \supset F^{gal}$ be the maximal subextension of $\overline{\mathbb{Q}}/\mathbb{Q}$ which is unramified at all rational primes not dividing elements of S. Recall that $\mathbb{Q} \subset E \subset E' \subset F^{gal} \subset \mathbb{Q}_S \subset \overline{\mathbb{Q}}$. We consider primes P_S of $\mathbb{Q}_S$ not above S such that

$$\mathrm{Fr}_{\mathbb{Q}_S/\mathbb{Q}}(P_S) \in \mathrm{Gal}(\mathbb{Q}_S/F^{gal}).$$

Fix $v|\infty$ in F^+ and denote by

$$(p) = P_S \cap \mathbb{Z}, \qquad P'_v = \tilde{\sigma}_v^{-1} P_S \cap O_F, \qquad P_v = P'_v \cap O_{F^+}.$$

Then p is a prime of $\mathbb{Q}$ that splits completely in $F/\mathbb{Q}$, and P_v is a prime of F^+ that splits in F as $P_v = P'_v\overline{P'_v}$.

The conjugacy class $[\mu]$ of the cocharacter μ_h contains a cocharacter defined over $\mathbb{Q}_p$, which extends to a homomorphism $\mu : \mathbb{G}_{m,\mathbb{Z}_p} \longrightarrow \underline{G}$, where $\underline{G}$ is a reductive model of G over $\mathbb{Z}_p$ given by Λ. We fix such a μ and we let $\Lambda = \Lambda_0 \oplus \Lambda_1$ be the weight decomposition of Λ with respect to μ [Wed00, 5.1]. The decomposition $\Lambda = \prod_{P|p} \Lambda_P$ defines, for each $P|p$ in F^+, a cocharacter $\mu_P : \mathbb{G}_{m,O_{F_p^+}} = \mathbb{G}_{m,\mathbb{Z}_p} \longrightarrow H_P$ where H_P is a reductive model of $H \otimes_{F^+} F_P^+$ over $O_{F_P^+} = \mathbb{Z}_p$.
Let $\underline{M} = \{g \in \underline{G} |\ g(\Lambda_i) = \Lambda_i\}$ be the centraliser of μ in $\underline{G}$, and $L = M \cap K_p$ where $M = \underline{M}(\mathbb{Q}_p)$. Let $M_- = \{m \in M |\ m^{-1}(\Lambda_i) \subset \Lambda_i\}$. We consider the Hecke algebra $\mathcal{H}(M_-, \mathbb{Q})$ which decomposes into tensor product

$$\mathcal{H}(M_-, \mathbb{Q}) = \bigotimes_{P|p} \mathcal{H}(M_{P-}, \mathbb{Q}).$$

Note that the argument behind considering $\mathcal{H}(M_-, \mathbb{Q})$ instead of $\mathcal{H}(M, \mathbb{Q})$ is explained in [Wed00, 1.4] (for considering $\mathcal{H}_0$ instead of $\mathcal{H}$ in the notations of [Wed00]).

On the other hand, let $p - \mathrm{Isog}_{K^p}$ (resp. $p - \mathrm{Isog}_{K^p}^{ord}$) denote the moduli space of p-isogenies between abelian varieties (resp. ordinary abelian varieties) which are points of $M_K := \coprod_i M_{\alpha_i,K}$, and let $pr : M_K \to S_K$ be the canonical map. Let $\mathbb{Q}[\star]$ be the free $\mathbb{Q}$-vector space on the set of connected components of $\star$. We define here another version of the partial Frobenius maps, and the main ingredients are firstly the map

$$\overline{h} : \mathcal{H}(M_-, \mathbb{Q}) \longrightarrow \mathbb{Q}[p - \mathrm{Isog}_{K^p}^{ord} \otimes \mathbb{F}_p]$$

defined in [Nek, A4], analogously as in [Wed00, 5.8] by sending, up to a constant, the characteristic function of a double coset LmL to the union of the connected components on $p - \mathrm{Isog}_{K^p}^{ord}$ classifying ordinary p-isogenies of type LmL in the sense analogous to [Wed00, 5.3]; and secondly, the fact that $p - \mathrm{Isog}_{K^p}^{ord} \otimes \mathbb{F}_p$ is dense in $p - \mathrm{Isog}_{K^p} \otimes \mathbb{F}_p$, provided the group G splits over $\mathbb{Q}_p$ as in [Wed00, §6], hence

$$\mathbb{Q}[p - \mathrm{Isog}_{K^p}^{ord} \otimes \mathbb{F}_p] = \mathbb{Q}[p - \mathrm{Isog}_{K^p} \otimes \mathbb{F}_p].$$

The previous data allows us to define [Nek, A5.2] the partial Frobenii at P (commuting with each other) by

$$\varphi_P = \overline{h}(L_P \mu_P(p)^{-1} L_P)$$

whose product

$$\varphi = \prod_{P|p} \varphi_P = \overline{h}(L\mu(p)^{-1}L)$$

is the Frobenius isogeny, whose action on étale cohomology of $S_K \otimes \overline{\mathbb{F}}_p$ corresponds to the action of $\mathrm{Fr}(P_S)$.

For $v_i|\infty$ in F^+, we consider the partial Frobenii $\varphi_{P_{v_i}}$ for P_v primes of F^+ defined as before. The elements u_i in Assumption (A2) are given by the action of the various $\varphi^*_{P_{v_i}}$ on the $V(\pi^\infty)$-component of

$$H_{et}^{\dim}(S_K \otimes \overline{\mathbb{F}}_p, \mathscr{L}_{\xi,\ell}) \simeq H_{et}^{\dim}(Sh_K \otimes_E \overline{\mathbb{Q}}, \mathscr{L}_{\xi,\ell}) \subset H_{et}^{\dim}(M_K \otimes_E \overline{\mathbb{Q}}, pr^*(\mathscr{L}_{\xi,\ell}))$$

and the set Σ is $\{\mathrm{pr}^{-1}(\mathrm{Fr}(P_S))\}$ where $\mathrm{pr} : \Gamma_{F^{gal}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_S/F^{gal})$ is the canonical projection. The subset $\Sigma \subset \Gamma_{F^{gal}}$ is dense by the Čebotarev density theorem, and the congruence relations

$$P_{\rho_i(\mathrm{Fr}(P_S))}(u_i) = 0 \in \mathrm{End}(V(\pi^\infty))$$

follow from [Nek, A5.10].

It remains to see that this version of the partial Frobenius maps is compatible with the geometric φ_P defined in Section 3.6. In fact, the correspondance $\overline{h}(L\mu(p)^{-1}L)$ (resp. $\overline{h}(L_P\mu_P(p)^{-1}L_P)$) on M_K° is equal to the pullback by $pr \times pr$ of the graph of the Frobenius $\varphi : S_K^\circ \to S_K^\circ$ (resp. of the graph of the partial Frobenius $\varphi_P : S_K^\circ \to S_K^\circ$) defined in Section 3.6. As a result, its action on étale cohomology of $M_K \otimes \overline{\mathbb{F}}_p$ leaves stable the image under pr^* of étale cohomology of $S_K \otimes \overline{\mathbb{F}}_p$ and its action on the latter coincides with the action of $\varphi \otimes \mathrm{id}$ (resp. $\varphi_P \otimes \mathrm{id}$) [Nek, A4.4].

3.7 Semi-simplicity of $V(\pi^\infty)$

We are now ready to combine the previous discussion with the abstract results proved in Chapters 1-2. **In this section, suppose we are in the following situation**:

3.7.0. Let $G = R_{F^+/\mathbb{Q}}(GU(C,\#))$ be as in Section 3.2, $G_\mathbb{R} \simeq \prod_{v|\infty} GU(a_v, b_v)$, where $a_v + b_v = n \geq 2$. Let $\xi : G_\mathbb{C} \longrightarrow \underline{GL}(V_\xi)$ be an irreducible algebraic representation of integral weight in the sense of Definition 3.2.1. Let π be an irreducible automorphic representation satisfying (A1). As explained at the end of Section 3.5, assumption (A2) holds, thanks to [Nek, A5.10]. Let (Π, ψ) be a weak base change of π to $GL_n(\mathbb{A}_F) \times \mathbb{A}_F^\times$.

Theorem 3.7.1. *In the situation of 3.7.0:*

1. *If the Galois representation $\rho_{\Pi,\ell} : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ attached to Π is strongly irreducible (i.e. it is irreducible and not induced from a proper subgroup of Γ_F), and if $G_\mathbb{R}$ has only signatures $(n,0), (n-1,1), (1,n-1)$ or $(0,n)$, then $V^{dim(Sh)}(\pi^\infty)$ is semi-simple.*
2. *If the $\overline{\mathbb{Q}}_\ell$-Lie algebra $\overline{\mathfrak{g}} = \overline{\mathbb{Q}}_\ell \cdot Lie(\rho_{\Pi,\ell}(\Gamma_F)) \subset \mathfrak{gl}(n, \overline{\mathbb{Q}}_\ell)$ contains $\mathfrak{sl}(n, \overline{\mathbb{Q}}_\ell)$, then $V^{dim(Sh)}(\pi^\infty)$ is semi-simple.*

Proof. In either case, the assumptions of Theorem 1.4.1 are satisfied for $\rho|_{\Gamma_{F^{gal}}}$ and $\rho_i|_{\Gamma_{F^{gal}}}$, thanks to (A2) and Corollary 3.3.1. □

Remark: Note that (2) also follows from results of [Nek], by a more indirect route: [Nek, Proposition 3.10] implies that $\rho^{ss}|_{\Gamma_{F^{gal}}} \subseteq (\rho_1 \otimes \ldots \otimes \rho_r)^{\oplus m}|_{\Gamma_{F^{gal}}}$ for some integer $m \geq 1$, which means that [Nek, Theorem 3.12] applies, hence $\rho = \rho^{ss}$.

In the following theorem, we consider a particular case of when $\rho_{\Pi,\ell}$ is not strongly irreducible. More precisely, we study a case when $\rho_{\Pi,\ell}$ is induced by a character of a proper open subgroup of Γ_F (see Proposition 3.3.1).

Theorem 3.7.2. *In the situation of 3.7.0, assume that $n = p$ is prime, G has only signature $(1, p-1)$ (i.e. $(a_v, b_v) = (1, p-1)\ \forall\ v : F^+ \hookrightarrow \overline{\mathbb{Q}}$), $F = F^+\mathbb{Q}(\sqrt{-d})$, $F^+/\mathbb{Q}$ is a Galois extension, the CM type Φ of F is induced from $\mathbb{Q}(\sqrt{-d})$, and the representation $\rho_{\Pi,\ell}$ is of the form*

$$\rho_{\Pi,\ell} = \mathrm{Ind}_{\Gamma_{F_1}}^{\Gamma_F}(\chi)$$

where F_1 is a Galois extension of F of degree p, and χ is a character of Γ_{F_1}.
Fix in $\Gamma_{\mathbb{Q}(\sqrt{-d})}$ lifts $\gamma_1 = id, \gamma_2, \ldots, \gamma_r$ of the r elements of $Gal(F/\mathbb{Q}(\sqrt{-d}))$; and for $i = 1, \ldots, r$, let F_i be the conjugate $\gamma_i(F_1)$ of F_1. These conjugates are not necessarily pairwise distinct; we rearrange the indices so that $F_1, \ldots, F_s$ are the pairwise distinct conjugates ($s \leq r$). Assume that $F_1, \ldots, F_s$ are totally linearly disjoint over F, in the sense that $F_1 \otimes_F \ldots \otimes_F F_s \simeq F_1 \ldots F_s$.

If $\rho = V^{dim(Sh)}(\pi^\infty)$ satisfies $\rho^{ss} \subseteq (\rho_1 \otimes \ldots \otimes \rho_r)^{\oplus a}$ for some integer $a \geq 1$, then ρ is semi-simple.

Proof. First, let us illustrate the situation with the following diagram of field extensions

$$\begin{tikzcd}
 & & \overline{\mathbb{Q}} \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
F_1 \arrow[drr, dash, "p"'] & \cdots \arrow[dr, dash] & F_i \arrow[d, dash, "p"] & \cdots \arrow[dl, dash] & F_s \arrow[dll, dash, "p"] \\
 & & F \arrow[dl, dash, "r"'] \arrow[d, dash, "2"] & & \\
 & \mathbb{Q}(\sqrt{-d}) \arrow[d, dash, "2"'] & F^+ \arrow[dl, dash, "r"] & & \\
 & \mathbb{Q} & & &
\end{tikzcd}$$

Note that the fields F_i do not depend on the choice of the lifts γ_i, since F is a Galois extension of $\mathbb{Q}(\sqrt{-d})$. The group $Gal(F/\mathbb{Q}(\sqrt{-d})) \simeq \Gamma_{\mathbb{Q}(\sqrt{-d})}/\Gamma_F$ acts on the set of Galois extensions of F via lifts of its elements in $\Gamma_{\mathbb{Q}(\sqrt{-d})}$. The integer s is then the cardinality of the orbit of F_1 under this action, hence it is a divisor of $[F : \mathbb{Q}(\sqrt{-d})] = [F^+ : \mathbb{Q}] = r$. For $i = 1, \ldots, r$, let

$$\chi_i : \Gamma_{F_i} = \gamma_i \Gamma_{F_1} \gamma_i^{-1} \longrightarrow \Gamma_{F_1} \xrightarrow{\chi^\vee} \overline{\mathbb{Q}}_\ell^\times$$

where the left map is conjugation by γ_i^{-1}. In our situation, the representations ρ_i defined in (3.16) become then

$$\rho_i = \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F}(\chi_i) \otimes \rho_{\psi,\ell}^\vee(1-p)$$

Since for $i \in \{1, \ldots, r\}$, the F_i's are not necessarily pairwise distinct (unless $s = r$), we number the characters χ_i (resp. the representations ρ_i) by double indices χ_{ij} (resp. ρ_{ij}) in a way that for $i = 1, \ldots, s$, $\chi_{i1}, \ldots, \chi_{im}$ are the characters defined on Γ_{F_i}; hence $sm = r$, and $\forall\ i = 1, \ldots, s$, $\forall\ j = 1, \ldots, m$,

$$\rho_{ij} = \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F}(\chi_{ij}) \otimes \rho_{\psi,\ell}^\vee(1-p) = \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F}(\chi'_{ij}),$$

where $\chi'_{ij} = \chi_{ij} \otimes \rho_{\psi,\ell}^\vee(1-p)|_{\Gamma_{F_i}}$. These notations are coherent with those of Chapter 2. Assumption (A2) tells us then that for each element g in the set Σ defined in Proposition 3.5.1, there exist pariwise commuting elements $(u_{ij})_{i=1,\ldots,s,\ j=1,\ldots,m}$ in $GL_M(\overline{\mathbb{Q}}_\ell)$ such that

$$\rho(g) = \prod_{i=1}^{s} \prod_{j=1}^{m} u_{ij}$$

and $P_{\rho_{ij}(g)}(u_{ij}) = 0$, $\forall\ i, j$.

We show first that Proposition 2.3.1 applies, in fact:
- Assumption (1) of Proposition 2.3.1 holds thanks to assumption (A2).
- Assumption (2) of Proposition 2.3.1 holds since the Hodge-Tate weights of $\rho_{\Pi,\ell}|_{\Gamma_{F_\tau}}$ are distinct.
- Assumption (3) of Proposition 2.3.1 is reflected by the fact that $F_1, \ldots, F_s$ are totally linearly disjoint over F, which means that $\mathrm{Gal}(F_1 \ldots F_s/F) \simeq \prod_i \mathrm{Gal}(F_i/F)$.

The semi-simplicity of ρ is then a direct consequence of Theorem 2.2.1, in fact:
- Assumption (1) of Theorem 2.2.1 holds, thanks to Proposition 2.3.1.
- Assumption (2) of Theorem 2.2.1 holds, by our hypothesis.
- Assumption (3) of Theorem 2.2.1 holds, as already mentioned.

□

Remarks: In the situation of Theorem 3.7.2,

1. If $F_1/\mathbb{Q}(\sqrt{-d})$ is a Galois extension, then $s = 1$, which means that Theorem 2.3.1 [Nek, Theorem 4.7] applies and the semi-simplicity of ρ holds even if we do not assume that $\rho^{ss} \subseteq (\rho_1 \otimes \ldots \otimes \rho_r)^{\oplus a}$ for some $a \geq 1$.
2. If $F^+ = \mathbb{Q}$, then $r = 1$ and $F = \mathbb{Q}(\sqrt{-d})$, which means that Theorem 2.4.1 applies and the semi-simplicity of ρ holds if we merely assume that $[F_1 : F]$ is a power of a prime number.

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