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Modeling, optimization and estimation for the on-line control of trading algorithms in limit-order markets

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Soutenue le 10 Septembre 2015, devant le jury composé de :

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à ma muse des mathématiques:
loyale, malgré mon esprit brouillon.

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Chapter 1

Introduction

1.1 Industrial context

Trading is a search problem: buyers and sellers of financial assets must find a counterpart to trade the quantity they desire at a price that seems fair to them [77]. Financial markets are the place where supply and demand for financial instruments meet. We say that markets are *liquid* if participants are able to trade when they want to and without incurring excessive trading costs. One of the main roles of the financial industry is to facilitate access for *liquidity* to investors, the fundamental drivers of supply and demand of assets. Two intermediaries help investors to accomplish their goals:

- Brokers: acting on behalf of investors by buying (or selling) their orders on the market. They try to meet a contractual *benchmark price* which measures execution performance. Brokers make their profits through commissions.

- Market-Makers (or *dealers*): acting as the counterpart for brokers, they are simultaneously buyers and sellers of assets. They make profit by buying at a lower price than the price they sell at or by receiving incentives in order to animate markets.

We say that brokers *consume liquidity* whereas market-makers *provide liquidity.*

Because of advances in information technologies, computational performances and the fact that the market has become essentially electronic; financial agents rely today on the use of automated strategies (i.e. *algorithmic trading*) to control costs, manage risks and interact with markets [89]. Thus, technology has become the business bottom-line of the financial industry; whose *raison d'être* is the optimal allocation of liquidity.

The present study provides focuses on algorithmic trading for European equities from the point of view of a *sell-side* firm (i.e. brokers and dealers *selling* services to investors).

The following section provides the necessary background in order to understand how equity markets and their auction mechanisms work. In addition, we present the types of problems confronted by practitioners. In this way, we will be able to put our technical results in perspective thus demonstrating the need for a quantitative approach in this field, such as that elaborated in this study.

1.1.1 Market microstructure

Traditionally, most equity trading was centralized in the local primary exchanges (Euronext Paris, London Stock Exchange, Deutsche Börse, etc). Because of technology, globalization and increasing competition, regulators have been led to liberalize the exchange-markets, giving participants the choice among different exchanges, each one with its own matching rules, fee-schemes and degrees of transparency [102]. This is called *market fragmentation*. As concerns Europe, this regulation initially took place in 2007 under the name of Markets in Financial Instrument Directive, or MiFID [47].

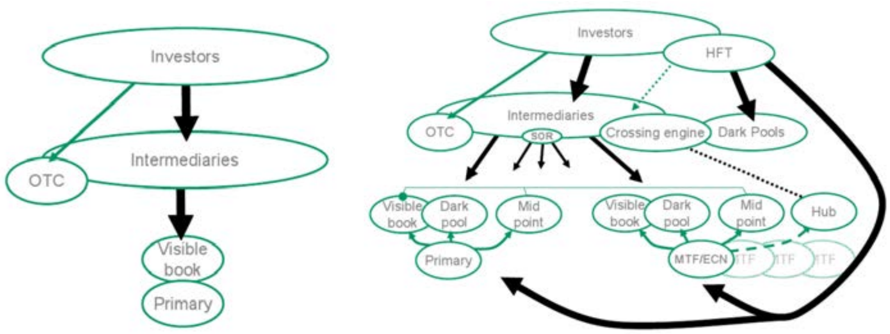

Figure 1.1: Organization of European equity-markets before MiFID took place in 2007 (left figure) then a few years after MiFID (right figure).

For our purposes, we will pay attention to the following classes of venues as they represent the main places where equity trading is performed today:

- Primary Exchange: Traditional exchanges being the main place for trading before MiFID. They are essentially *pre-trade transparent*, meaning that the information about the existing liquidity is available to participants before they trade. They are still the main places to trade and they fulfill special roles like serving as reference for prices for other type of exchanges, for example dark-pools (see below).
- Multilateral Trading Facilities (or MTFs): Places directly competing with primary-exchanges as the trading services they offer and their trading mechanism are roughly the same. In the case of European stocks the main MTFs are Chi-X, BATS and Turquoise. They account for around 20%-30% of the total market share.
- Dark Pools: Exchanges that do not provide pre-trade transparency; this means they do not communicate about their available liquidity. The value provided by dark-pools is to allow investors to execute large orders avoiding information leakage or being gamed by opportunistic agents, for example high-frequency traders.

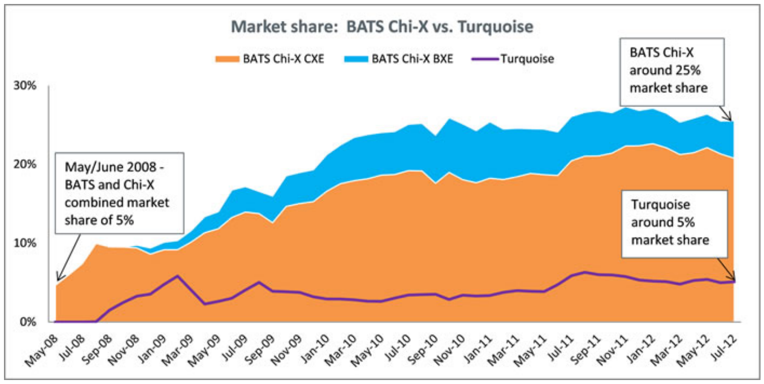

Figure 1.2: Evolution of the market share for the larger European MTFs trading the main indexes. The remaining is mostly concentrated on the primary exchange.

1.1.2 Auction mechanisms

It is not only the long-term supply/demand forces (i.e. investors) which drive the evolution of prices, but also how market-makers, brokers and other players interact on exchanges. They also influence the process of price formation [26, 27, 75, 96]. The auction mechanism is the set of matching rules that defines how this interaction is carried out. For primary exchanges, two main auction phases exist during the trading day: *fixing auctions* (orders are matched after being accumulated on a book during a certain period) and *continuous auctions* (orders are matched continuously as they appear on the market). Most of the volume is traded during the continuous phase (indeed, besides exceptional situations, fixing auctions only happens at the very beginning and at the end of the day). Here, we focus solely on the mechanics of continuous auctions.

The limit-order book

The virtual place where offer meets demand during a continuous auction is called *limit-order book*. This is the way of functioning for primary-exchanges and MTFs. There are two main ways to send trading intentions to the order book in order to buy or sell assets:

- Market orders: The agent specifies a quantity (to buy or sell) to be immediately executed. i.e. the agent consumes liquidity at the (best) available price.
- Limit orders: The agent specifies a quantity and the price to trade. Then she waits until a market order arrives as counterpart at this price. Limit orders tend to improve execution prices. However, as a downside, the agent bears the risk of never getting a counterpart and hence executing at a worse price in the future.

We say that market orders are *aggressive* trades while limit orders are *passive*. If a participant who sends a limit order is no longer interested in keeping that order in the book he can cancel his order before it gets executed.

The order book can be divided into two different sides: the *bid-side* (passive buyers) and the *ask-side* (passive sellers). The highest proposed bid-price is called *best-bid* and the lowest proposed ask-price is called *best-ask*. By design, the best bid-price is always lower than the best ask-price. If were not the case, a trade would have already occurred (i.e. the seller would have already matched the buyer).

The difference between the best-ask and the best-bid is called *bid-ask spread*, and it is one of the main indicators to measure liquidity. We also define the *mid-price* as the average between the best-bid and the best-ask (in practice, there is no definition for 'the price' of an asset; when people talk about price, they usually refer to the mid-price or to the price of the last trade). Another important concept is the *market-depth* defined as the available liquidity in the order book. Bid-ask spread and market-depth are the two main measures of how much the consumption of liquidity costs.

There are some important practical features to be taken into account when dealing with limit order books:

- The minimal unit of price is called the tick size. The difference between two prices of a given asset cannot be smaller than 1 tick.
- Orders arrive at random times in the order book. In particular, mid-price changes occur at random times in a discrete way.
- On the equity market, limit orders are executed first by price priority, then, for orders at the same price level, by time priority.

Thus, the short-term (intraday) price formation process is mainly driven by the way in which different agents choose their strategies (that is, their trading algorithms) to trade liquidity in the order book, and also by the different features of the auction mechanisms on a given exchange. In the long-term, investors' objectives are prominent. Thus, classical measures such as volatility or trend, should be taken with caution as their interpretation necessarily depends on a particular time-scale (milliseconds, day, years etc.).

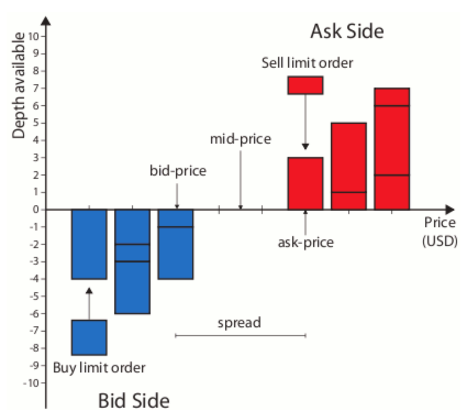

Figure 1.3: Graphical representation of the Limit Order Book.

Dark liquidity-pools

The other important marketplaces involved in the price-formation process, outside the limit-order book, are the dark-liquidity pools. As previously mentioned, dark pools are venues without pre-trade transparency, i.e. in which it is possible to send large orders that can only be discovered by someone trading as counterpart. The price at which orders are matched in the dark pool correspond to the current mid-price in the primary exchange. Thus, dark pools do not directly participate in the price formation process, even if they indirectly participate by attracting part of the liquidity from *lit exchanges* (exchanges with visible liquidity e.g. primary and MTFs) hence reducing their market-depth.

High-frequency trading

Besides algorithmic trading, another by-product of recent technological advances has been what it is called *high-frequency trading* (HFT) [12, 25, 31, 50, 117, 86, 112], a style of trading (mostly opportunistic), accounting for a large share of the liquidity in the market, in which the strategies are short-term and based on sophisticated statistical models or on structural advantages. HFT relies on technology, speed and fast data processing when accessing electronic markets (e.g. taking advantage of latency by being close to the exchange or by earning liquidity-provision rebates). After the *flash crash* (figure below) and because of its reputation of introducing toxic liquidity into the market, HFT is source of controversy (detailed analysis of HFT exceeds the scope of the present study).

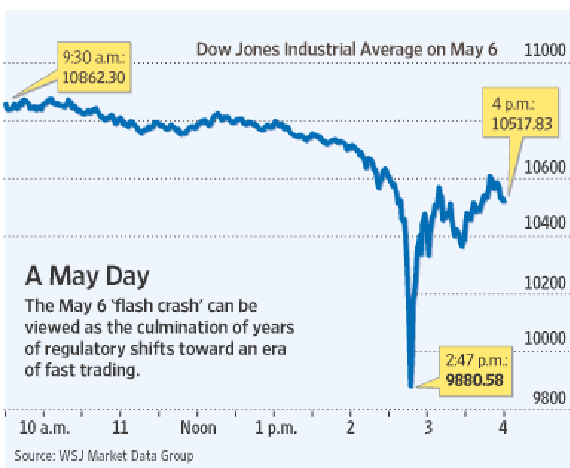

Figure 1.4: On May 6th, in 2010, the Dow Jones index experienced a fall of more than 10% in just few minutes. One of the amplification factors that motivated this phenomenon was the chain-reaction of different automated-trading algorithms on related families of assets. This event raised several regulatory questions.

1.1.3 Optimal trading

One of the main consequences of the evolution in trading automation is the increasing need for a quantitative approach to its analysis and optimization. *Optimal trading* refers to the design of trading strategies and tactics by means of quantitative methods. This is particularly important for strategies whose performance is measured on a statistical basis (e.g. algorithmic trading). The following is an overview of optimal trading in the context of the two main areas of concern for the present study: brokerage and market-making.

Brokerage

After investors and brokers agree on a benchmark to measure execution performance, brokers face the problem of controlling all the transaction costs related to the liquidation of large orders. Among these costs, *market impact* (i.e. fluctuations in stock price and changes in supply and demand, caused by the execution of a trading strategy) deserves particular attention as it represents the part of transaction costs more prone to optimization. In order to reduce market impact, brokers split their *parent orders* into small chunks (*child orders*) which are executed throughout the day. Their goal becomes to find the optimal rate at which to liquidate the order: if too fast, they face higher market-impact costs; if too slow, they risk market prices moving in an unfavorable direction, resulting in a worse-than-expected execution price at the end of the day.

Two problems emerge at this point, a *strategic* one, meaning, how to define the right trading rate in order to meet the benchmark constraints, and a *tactical* one, meaning, how to interact with the market in order to execute the strategy at the lowest cost.

The mathematical framework to solve this problem will be discussed in the next section. What it is important to understand here is: once the benchmark and the market-impact model are defined, the optimization gives as a result an *optimal scheduling* (i.e. the strategy layer) represented by a trading curve defining the liquidation pace during the day. To decide this optimal pace, practitioners take as input the benchmark, their proprietary market impact model and intraday volatility patterns as well as the intraday volume patterns, among other parameters [24, 87, 88, 90, 91, 103, 104, 105].

It remains now to solve the problem of interacting with the market (tactical layer) in order to execute these smaller orders in a given span of time. Two main problems arise at this stage which can impact the performance of the trading algorithm:

- *Order placement*: choice between consuming liquidity immediately and sending passive orders at an specified price. The more aggressive the order placement, the faster the execution; the more passive, the better the price, but with some uncertainty regarding the execution.

- *Order routing*: Two aspects should be considered: first of all, choosing if the order will be sent to a dark pool or to a lit exchange (e.g. primary or MTF). Secondly, how to trade in each venue, i.e. to take into account the execution probabilities, spreads and fee-schemes of each one of the exchanges.

Market-Making

Market-makers provide liquidity to the market by simultaneously proposing buying and selling (passive) orders [116, 81]. In order for their activity to be profitable, the price at which they propose to buy has to be lower than the price at which they propose to sell. The gap between both prices is called market-marker's spread (not to be confused with bid-ask spread). The risk they bear is to have an unbalanced inventory if only one side of their trade is executed (ideally, they would end the day with no inventory). In order to accomplish their goals, market-makers continuously update the prices at which they post their buying and selling orders during the trading day. The way they choose their prices (their spreads and position with respect to the mid-price) is subject to several trade-offs:

- *Probability of execution and spread*: Varying the spread can have both positive and negative effects on the profits made by the market-maker. Indeed, profit increases as the spread becomes larger, however, larger spreads represent prices that are less appealing for aggressive traders, thus inducing a smaller probability of execution.
- *Inventory risk, prices and spread*: The market-maker centers his quotes around prices that increase the probability that execution will occur on the side on which he has an excess of inventory. The impact of an unbalanced inventory is more pronounced as the day ends, as he needs his inventory to vanish more rapidly. The spread also plays a second order effect in the inventory risk, as executing too many orders increases the variance of the inventory, inducing another source of risk.
- *Adverse selection*: an important concern for market-makers involves cases in which an opportunistic trader, with more accurate information about the price than the market-maker, buys (or sells) shares knowing that the price will increase (or decrease). This creates inventory unbalance for the market maker, who will be forced to re-balance it in the future at an unfavorable price; this is called *adverse selection*.
- *Rebates*: market-makers have a supplementary incentive to trade (and to propose closer spreads) because they are paid by the exchange as liquidity providers. This is one of the main sources of (high-frequency) market-makers' profits.

Final remarks

It should be mentioned that, in the practice of implementation of trading strategies and tactics, three separate stages are involved [92]:

- Pre-trade estimation of costs: calibrate the market-impact model by using historical data and estimate the cost of trading through backtests .
- Control of the execution process: real-time control of the algorithm interacting with exchanges taking into account current market conditions [14].
- Transaction cost analysis: ex-post performance analysis through a breakdown of the different costs affecting the resulting strategy's PnL.

1.2 Models for price and liquidity

In the next three sections we highlight the reasoning behind the modeling choices made in this dissertation, in particular to introduce the Avellaneda-Stoikov model, and also give a brief presentation of the state-of-art in quantitative trading from the point of view of our research. This step is important not only to situate this work within the academic literature on the subject but also to underscore the project's relevance for the financial industry. We also present two approaches that can be used in order to solve the multistage optimization problems resulting from the modeling, in particular, dynamic-programming (through the HJB equation) and on-line learning (through stochastic approximation methods).

1.2.1 Descriptive and statistical modeling

The main difference between the quantitative models used in algorithmic-trading and the models used in the classical areas of mathematical finance (e.g. portfolio theory, derivatives pricing etc.) is that, in the case of algorithmic-trading, not only the price process needs to be modeled, but also the liquidity (i.e. the order book) and the impact of our own strategy as it affects supply and demand (market impact). Moreover, the different features of price and liquidity for short time-horizons such as the tick-size, the spread, order book depth, price-impact and the discrete arrival of orders, should be taken into account as their effects are no more secondary.

Several approaches to include microstructure aspects in the quantitative models have been proposed in the literature in recent years [1, 32]. One way to classify the different models of price and liquidity is to consider their granularity when modeling the price. In that view, *microstructure-noise models* [6, 124, 125, 127, 136], the coarser in granularity, take as a starting point a diffusive price in which microstructure effects are added as noise. These models were originally devised for statistical purposes (estimation of volatility using high-frequency data); finer-granularity models devised for the same purposes are studied in [15] by modeling the price as a difference of Hawkes processes in such a way that the statistical local and asymptotic behaviors can be characterized using spectral analysis.

Another way to look at price and liquidity is to take as a starting point the finer-granularity mechanics of the order book, studying the behavior of the price, liquidity and other market quantities as the macroscopic resultants of the microscopic interplay of limit and market orders. In this direction, we can count the *zero-intelligence models* [53, 132, 43], allowing us to obtain statistical relations between macroscopic quantities by taking as a starting point the intensities of the different events arising in the order book (seen as an array of queues). The drawback of these models is that, even if they are interesting from theoretical and simulation viewpoints, they are difficult to handle if we are interested in a simple mathematical characterization for the asymptotics of the price and the market impact. Approaches still relying on the order book dynamics, but limiting the focus only to the first levels (or using a continuous representation of liquidity), can provide closed-form relations between microscopic and macroscopic quantities [40, 82, 106]. Approaches aiming for the asymptotic behavior of a complete order book are in [2].

Finally, we can mention the econophysics approach, which attempts to devise microscopic model so that the resulting behavior for macroscopic quantities is consistent with observations of large sets of data [51, 110, 52, 74, 29]. The caveat with all the models just mentioned is that they were originally devised to solve problems such as parameter estimation, or simply describe mechanically the relations between price and liquidity. These goals are not necessarily in line with the optimization of the trading process which is more focused on the analysis of the impact of trades on price, the probabilities of capturing liquidity and measuring price-risk. Besides some articles in econophysics which study the relation between orders and their market-impact [30, 133, 135], this class of models does not take as a main point of view an algorithm, so they do not naturally integrate the logic of a trading algorithm.

1.2.2 Market-impact models for trading

As mentioned above, one of the most important aspects of modeling of algorithmic trading strategies is to take into account the impact of the algorithm on the dynamics of supply and demand (and hence the price at which the algorithms buy or sell). This effect is called *market impact* and it can be mechanical (movement of the price due to the volume of our orders) or informational (changes in the underlying supply/demand views on the price). The following is a survey of the two main families of models of price and liquidity that take this factor into account: resilience models and black-box models.

Resilience models

This family of models, inspired by the works of Obizhaeva and Wang [115] and Gatheral [62], models the impact of trades in the order book by considering that a trade first moves the price by the mechanical liquidity consumption, before the order book reacts by re-filling again (this is called resilience). Modeling this resiliency of the book allows for optimal-trading solutions. This line of research has been explored by Alfonsi et al. in [7]. They model the shape of the order book and derive from it market-impact functions. The latter is interesting from a theoretical point of view as it yields non-arbitrage relationships characterizing prices and liquidity. However, from a trading perspective these models are difficult to calibrate by using the available data.

Black-box models

A more pragmatic line of research has been opened by the seminal article of Almgren and Chriss [10] and its follow-up [8, 9, 11]. They introduce a discrete black-box model for market-impact which make it possible to obtain the optimal solution of the trading problem via classical optimization techniques (variational calculus in the continuous version and stochastic control in the adaptive case). The advantage is that the impact is characterized by a small-dimensional set of parameters which are calibrated from real-data. This makes the model flexible, adaptive for practical applications and easily generalizable.

By following this framework, we obtain an optimal scheduling curve that traders should follow throughout the day in order to meet the benchmark. This optimal scheduling curve is obtained by balancing the trade-off between market impact and market risk after defining a risk-aversion parameter. This framework, intended originally for implementation-shortfall, can be applied to a larger range of benchmarks. This approach has three main advantages:

- Defining an optimal scheduling curve naturally separates the problem of splitting the parent-order into child-orders from the micro-structural aspects of trading (interaction with the order book or a dark pool). It simplifies the modeling, the resolution of the problem as well as the computational design of the trading algorithm (strategy versus tactic).
- It is mathematically solvable and can be adapted to more complicated problems (e.g. traders with views on the price, varying volatility, varying liquidity, different market-impact functions etc.)
- It is flexible as the model is represented by parameters that can be computed from real data without needing too much granularity in the data-set (performing regressions to obtain the market-impact parameter is sufficient), by contrast with models relying on the detailed behavior of the order book.

1.2.3 Trading in limit-order books

Models such as the Almgren-Chris model make it possible to solve the strategic layer of the algorithm, however the granularity of the model is not fine enough to take into account microstructure effect. In order to optimize the interaction with the limit-order book or with a dark pool (i.e. the tactic) we need to model how liquidity is captured as a function of the posting of orders in the book (keeping in mind that we still want to keep control over issues related to price-risk, i.e. volatility and asymptotic behavior of the price). In the present study we are concerned with this kind of tactical problems in the cases of optimal liquidation and high-frequency market-making.

High-frequency market-making

From a mathematical point of view, the market making problem corresponds to the choice of optimal quotes (*i.e.* the bid and ask prices) throughout the trading session, in order to optimize a given utility function, taking into account the cash and inventory dynamics of the trader. To provide a good model, we need to take into account two features of the prices:

- The probability of getting orders executed as a function of where they are posted by the market-maker.
- Measuring market-risk (i.e. cost of liquidating the inventory at an unfavorable price at the end of the day).

A successful approach to treating this problem (and that will be detailed in the next section) is that of Avellaneda and Stoikov [13], which proposes an innovative optimization framework when trading in a limit order book. In their approach, rooted in a paper by Ho and Stoll [80], the market is modeled using a reference price or *fair price* S_t following a Brownian motion with standard deviation σ, and the arrival of a buy or sell liquidity-consuming order at a distance δ from the reference price S_t by a point process with intensity $A\exp(-k\delta)$, A and k being two positive real constants which characterize statistically the liquidity of the stock.

Since the publication of Avellaneda-Stoikov's article, other authors have analyzed market-making models in a similar way. We can cite here studies by Guéant et al. [65] which fully solved the Avellaneda-Stoikov problem, hence obtaining a detailed analysis of the solutions. Other contributions, for example Cartea et al. [33], which extend the model to different utility functions, more sophisticated market dynamics and switching regimes among other approaches. In a similar line, [73] studies the problem of trading with limit and market orders, whereas in [72] the same authors treat the problem of pro-rata limit-order books.

Optimal liquidation tactics

The literature on optimal liquidation with limit-orders is very close to market-making as we can consider that liquidation, at a tactical level, is a one-sided market-making (i.e. we propose limit orders just in one side of the spread). Two papers in particular have pioneered research in this field, one by Guéant et al. [64] and one by Bayraktar et al. [18]. The modeling is roughly the same as in Avellaneda-Stoikov's approach.

Further advances in that line of research were proposed by Guéant et al. [63] by continuing the study of the problem of optimal liquidation with limit orders in the case of more general shapes for the intensity function (taken as exponential in the original model), then introducing research in two new areas. First, optimal liquidation as a *pricing problem* [71, 67, 66] in which brokers propose a guaranteed benchmark to clients and the strategy is to meet this benchmark at lower risk (similar to the Black-Scholes framework for derivatives). Second, studying liquidation from the point of view of complex products, such as derivatives or ASR [68, 69, 70], in order to study how transaction costs impact the optimal hedging strategy.

A third direction of research, which we will treat in the last two chapters of this dissertation, was proposed by Laruelle et al. [101] which, taking as a starting point a modified version of the Avellaneda-Stoikov model, solves the problem by a stochastic approximation approach which is not based on the (backwards) dynamic-programming principle but in a (forward) trial-and-error optimization. This approach is original as it provides an iterative way to look at the optimization of trading tactics which is more flexible in terms of defining price dynamics and the prior knowledge of the model-parameters than in the HJB approach.

Trading in dark-pools

Trading in dark-pools is closed to trading in limit-order books since the routing across liquidity venues is done in terms of the overall algorithmic framework, at the same level as tactical liquidation problems. This topic has been treated less in the literature and it goes beyond the scope of this dissertation. Among the work in this field, we can mention Kratz and Schöneborn [94] and Laruelle et al. [100].

1.3 Avellaneda-Stoikov model

As mentioned before, the Avellaneda-Stoikov model [13] is a successful attempt at a mathematically tractable model that integrates simultaneously price and the liquidity from the perspective of a trading algorithm which liquidates orders passively in the order book. The model, originally intended for high-frequency market-making has been studied in [65] and extended to the case of optimal liquidation with limit orders [64]. We present this model as it will be central in all of the following chapters in this dissertation. The main goal of the Avellaneda-Stoikov model is to consider, in a unified framework, the two aspects that are important for an algorithm interacting with the order book: controlling the short-term probability of execution (for a market maker, the spread and, for a broker, how passive an order can be), as well as the price-risk (measured by the volatility of long-term price movements) when liquidating the remaining inventory.

1.3.1 The reference price

The first brick of the model is to consider a *reference price* S_t (for example the mid-price) evolving as a Brownian motion (a reasonable hypothesis as we will consider primarily short time horizons):

$$dS_t = \sigma dW_t. \tag{1.1}$$

In practice, the specific choice of what the 'reference price' means has to be thought from the point of view of the applications. We are not trying to find some ideal 'fair price'; the reference price is just the reference point to which we will measure the distance where we place orders in the order book. So, this reference price will usually be a sampling of the mid-price (for market-making) or the best-opposite price (for optimal liquidation). The main features we ask for the reference price is to have a volatility that serves as a proxy for market risk (in the context of liquidation at the end of a period) and a calibration of the liquidity parameters (see below) consistent with the model.

1.3.2 View on liquidity

In order to model the liquidity, the idea is to consider that if we place a passive order at a distance δ from the reference price, it will get executed with a probability given by the intensity:

$$\lambda(\delta) = Ae^{-k\delta}. \tag{1.2}$$

By an abuse of terminology, we mean that the probability for such an order to be executed between t and $t+dt$ is equal to $Ae^{-k\delta}dt$ up to the second order, independently of the past.

In particular, a trader who continuously posts orders (for example selling) at distances δ_t from the reference price will execute a flow given by the compound Poisson process $(N_t^{(\delta_t)})_{t\geq 0}$ with compensator process:

$$\Lambda_t = \int_0^T Ae^{-k\delta_t}dt. \tag{1.3}$$

So, for orders set at the ask, the realized gain of the trader evolves following the evolution equation

$$dX_t = (S_t + \delta_t)dN_t^{(\delta_t)}. \tag{1.4}$$

For orders set at the bid, the quantity of inventory bought is given by:

$$dX_t = (S_t - \delta_t)dN_t^{(\delta_t)}. \tag{1.5}$$

Depending on the application (market-making or optimal liquidation), we can define the payoff by using the final values of X_T, S_T and the inventory at the end of the trading session. We will begin by using a CARA utility function.

The in-depth discussion about the mathematical formulation of the model (formal relation between price and liquidity) will be made in each of the chapters as their specifics are slightly different depending on the optimization model.

1.3.3 Advantages of the model

First, liquidity (i.e. the order book) appears as a *blackbox*-like statistical model with low-dimensionality; the parameters of the model are estimated from real-time data under the hypothesis that these parameters characterize intraday features of the stock liquidity on a given exchange. Secondly, the price-process behaves as a Brownian motion which keeps consistency with classical models. Finally, the functions involved in the model allow a straightforward analysis, and even closed formulas, when approaching the problem through optimization techniques (e.g. stochastic control or stochastic approximation).

Several advantages plead in favor of this approach: dealing with a low-dimensional statistical-model where liquidity capturing is viewed through a blackbox makes this approach flexible. It can be adapted to any kind of market regime if we update often enough the parameter estimation. Moreover, it highlights the fact that our point of view is the trading algorithm and a focus on what concerns its performance; it is not an attempt to describe, in general, the processes arising in the market.

Another advantage is that a simple stylized model providing explicit numerical methods, or even closed formulas, makes it possible to perform a *comparative static* analysis; that is understanding the dependency between the different parameters of the model and its relations with the optimal solution. Also it suggests clues and heuristics to perform the mathematical (and numerical) analysis in more complex settings. Finally it helps to achieve a qualitative understanding of the solution before its numerical computation.

Last but not least, the use of parametric models and functional representations leading to explicit methods and solutions is helpful as far as implementation is concerned. The flow in a financial firm involves several people with different kinds of backgrounds and technical skills. A quantitative model where intuitions can be communicated along the different modules (IT, statisticians, quants, etc) working at the development of an algorithm allows a faster deployment than a more sophisticated model that probably cannot be implemented, due to its inability to be internally communicated inside the firm.

1.4 Optimization approaches

Once a mathematical model for prices and liquidity is defined, we are in a position to define the optimization problem to be solved. In this dissertation we will focus on two situations: tactics for algorithmic-brokerage and high-frequency market-making. In both cases the structure of the problem is roughly the same:

1. The goal of the tactic is to find the optimal price at which to passively post a given volume in the limit-order book, with the aim of providing or capturing liquidity.
2. The optimal quotes are updated throughout the day in successive steps (or continuously), taking into account both our increasing knowledge of the market dynamics and the state of our algorithm. Thus, the problem is said to be *multistage*.
3. The mathematical goal is to maximize a utility function depending on the dynamics of the algorithm, which by interacting with the market, is subject to exogenous random factors; these can represent *risk* (we know the statistical laws governing the randomness) or, otherwise, when the statistical laws of the noise are not addressed, we talk about *uncertainty*.

As we will see, two types of reasoning can be adopted in order to solve problems of this kind: by backwards reasoning, using the dynamic-programming principle and, by forward reasoning, leading to online-learning techniques.

1.4.1 Exploitation-versus-exploration problem

When solving multistage optimization problems in the presence of randomness, i.e. when a given variable is updated at each stage in order to control some system with the aim of optimizing its performance, two types of problems occur:

- Exploitation: finding a rule giving as output, at each moment, the next-stage optimal action by using the information at hand.
- Exploration: learning the underlying nature of the randomness governing the system by exploring the different states of the world while performing the actions.

As mentioned above, two ways of reasoning are possible to this end: a *backwards* and a *forwards* one. In a nutshell, the idea of the backwards approach is to characterize the randomness through a previously devised – parametric or non-parametric – statistical model (estimated beforehand using information about past performances, via classical inference techniques). Hence, the expectation of the function involved in the optimization can be considered known (indeed, they can be calculated explicitly or obtained through simulation). In this case we can define (e.g. by means of the dynamic-programming principle) a deterministic equation for the running expectation of the (optimal) final payoff. Doing so, we can directly apply the dynamic-programming principle which yields a backward relation between the (optimal) expected final-payoff at one stage, in relation to that of the preceding stage.

Note that in this approach, the function relating past information to future action is deterministic, even if the trajectory of the system is not. The exploration phase is performed before launching the algorithm and the optimization methodology only tackles the exploitation problem. This is a situation similar to option-pricing where the *control of the system* (i.e. the Δ of the portfolio) can be obtained by means of a PDE.

On the other hand, in the *forward* approach, the exploration and exploitation problems are solved simultaneously during the run-time of the algorithm, i.e. there is no *a priori* specification about the laws (or the parameters) driving the randomness of the system: historical data do not totally define, beforehand, our view of the dynamics of the system.

Thus, the optimal control of the system is chosen in an iterative way. It is updated by two factors: first, the current state of the system, and second, the innovations (incoming information). In this case, the downside is that it is more difficult to have, from the beginning, an overall picture of the system until the end of the strategy; the function relating past information to future choices is identified on-line. In this approach the dynamics of the algorithm choosing actions based on past information is stochastic.

It is clear that the best approach depends on the nature of the system; for a system in which the laws of nature are similar across the days, the backward approach is more suitable as an on-line exploration phase is less prioritary. However, when the uncertainty in the system has a different nature from day to day, a more adaptive approach is needed and it is more suitable to use online-learning techniques. Ideally, we would like to combine both approaches as is the goal in some approaches for reinforcement learning [84] or dynamic policy programming [4], but this goes beyond the scope of this work.

In the next subsections, we give a brief introduction to the rationale behind the mathematical techniques we will use in the following chapters. These are, for the backward approach, the HJB equation, and for the forward approach, online learning.

1.4.2 Dynamic-programming and the HJB equation

The dynamic-programming principle, developed in Bellman's seminal work [19] (see also [128]) is one of the most influential results in Mathematics in the 20th century and relies on the simple observation that the optimal strategy for a multistage problem with $N+1$

stages, starting at a point $x \in \mathcal{X}$ ($\mathcal{X}$ being the state-space) can be decomposed in two parts: (1) the next step and (2) a new multistage problem, with N stages starting at a point $y \in \mathcal{X}$ (which is the state of the system after applying the optimal control during the first stage). This idea is the core of the backward approach.

Assume, for example, a discrete, deterministic dynamic system which evolves according to the equation

$$x_{n+1} = x_n + f(n, x_n, u_n), \quad n \in \{0, \dots, N\}, \quad x_n \in \mathbb{R}^d, \quad u_n \in \mathbb{R}^p.$$

where u_n is our control strategy at time n.

Let us consider that the goal of our control problem is to minimize a cost function of the form:

$$C(x_0, (u_n)) = \phi(x_N) + \sum_{n=0}^{N-1} R(n, x_n, u_n).$$

The problem can be seen as a classical optimization problem where we want to find $u \in \mathbb{R}^{(N+1)m}$. However, if N and m are large, the problem will suffer from the *curse of dimensionality*. The idea proposed by Bellman is to define a backward recursive procedure to tackle the problem in a way where the numerical complexity grows only linearly in the number of steps. The key point is to represent the control problem (of dimension $\mathbb{R}^{(N+1)m}$) as a recursion of $N+1$ simpler problems defined on $\mathbb{R}^m$. This has the advantage of making the numerical problem more tractable. It also gives us qualitative ideas about the structure of the solutions. To understand how the principle works, the key element to introduce is the *optimal cost-to-go function*:

$$J(n, x_n) = \min_{(u_k)_{k=n,\dots,N-1}} \left[\phi(x_N) + \sum_{k=n}^{N-1} R(k, x_k, u_k) \right].$$

This function represents the optimal value of the cost function if we consider that the problem starts at time n with the system being in the state x_n.

We can easily verify the relationship:

$$\begin{aligned} J(N, x) &= \phi(x) \\ J(n, x) &= \min_{u \in \mathbb{R}^p} \left[R(n, x, u) + J(n+1, x + f(n, x, u)) \right]. \end{aligned}$$

The dynamic programming algorithm works in the following way:

- Step 1: We set the final condition $J(N, x) = \phi(x)$.
- Step 2: We compute (by backward induction on n)

$$u_n^*(x) = \operatorname{argmin} \left[R(n, x, u) + J(n+1, x + f(n, x, u)) \right]$$

$$J(n, x) = \min \left[R(n, x, u_n^*(x)) + J(n+1, x + f(n, x, u_n^*(x))) \right]$$

- Step 3: Using the initial condition we get: $x^*_{n+1} = x^*_n + f(n, x^*_n, u^*_n)$ and the optimal control is $u^*_n = u^*_n(x^*_n)$.

A virtue of the dynamic programming principle is that the framework works in the very same way even if the system has a stochastic evolution. For example, when the controlled system dynamics follows an Itô diffusion of the form:

$$dX_s = b(X_s, u_s)ds + \sigma(X_s, u_s)dW_s$$

and the performance criterion is defined by:

$$J(X, \alpha) = \mathbb{E}\left[\int_0^T f(X_t, u_t)dt + g(X_T)\right].$$

The cost-to-go function (also called value function) is defined by:

$$J(t, x, \alpha) = \inf_{u \in \mathcal{A}} \mathbb{E}\left[\int_t^T f\left(X_s^{t,x}, u_s\right) ds + g\left(X_T^{t,x}, u_s\right)\right].$$

The process $X_s^{t,x}$ represents a diffusion that follows the dynamics given by the above SDE starting from x at time t.

By virtue of the Itô formula we have (informally):

$$\mathbb{E}[dJ(t, X_t, \alpha)] = \partial_t J(t, X_t, \alpha)dt + \partial_x J(t, X_t, \alpha)b(X_t, t)dt + \frac{1}{2}\sigma^2(X_t, t)\partial_{xx} J(t, X_t, \alpha)dt.$$

Hence, the backward recursion leads to the Hamilton-Jacobi-Bellman equation:

$$(HJB) \qquad -\partial_t J(t, x) = \inf_{u \in \mho}\left[R(t, x, u) + b(x, u, t)\partial_x J(x, t) + \frac{1}{2}\sigma^2(x, u, t)\partial_{xx} J(x, t)\right].$$

It is important to notice that the infimum over the cost-to-go function is taken over the space of admissible controls (infinite dimensional). The infimum in the HJB equation is taken over $\mathbb{R}$. The extension to processes incorporating jumps (which we will use in chapters 2 and 3) follows the same logic. Mathematical details of this theory are beyond the scope of this work and can be found in [54, 126, 17, 134].

1.4.3 Stochastic approximation and on-line learning

Stochastic approximation is the theory behind our forward approach. The idea is to represent the optimal control as a Markov process where the stochastic part is a function of the incoming new information so that the current control is updated in order to converge to the optimal control, which can be represented as the zero of a critical point of a function writen as $h(\theta) = \mathbb{E}[H(\theta, Y)]$, where h is difficult to compute but $H(\theta, Y)$ can be simulated (or observed after the application of the control θ) at a reasonable cost.

Simply put, if θ_n represents the current control, we would like to devise a dynamical system (also known as *stochastic algorithm*) reading

$$\underbrace{\theta_{n+1}}_{\text{new value}} = \underbrace{\theta_n}_{\text{old value}} - \underbrace{\gamma_{n+1}}_{\text{step (weight)}} \times \underbrace{H(\theta_n, \overbrace{Y_{n+1}}^{\text{innovation}})}_{\text{correction using incoming information}}, \tag{1.6}$$

in such a way that this sequence is converging to the target value we are looking for.

A caveat must be stated at this point: in order to implement this kind of approach we do not need to specify the stochastic dynamics of innovations. However it is important that these innovations are i.i.d., stationary and ergodic. Throughout this work we will focus in the i.i.d. situation, however there are more general results on the convergence of stochastic algorithms, e.g. the works of Laruelle [99] (averaging innovations) and the work of Schreck et al. [131] (innovations are allowed to be controlled Markov chains).

Because this powerful class of methods are less known in the financial community, we explore them in more detail in chapter 5.

1.5 Outline and contributions

The goal of this work is to expand the Avellaneda-Stoikov model, originally devised for market-making, in different directions. We also study the model from different angles, discussing along the way issues that are interesting from a mathematical and practical standpoint. We also want to highlight throughout this work the two optimization paradigms (dynamic programming vs. online learning), the importance of black-box models which make it possible to characterize the market via a small set of parameters estimated on-line and the importance of mathematical results that not only provide 'a number' as a result but also provide closed formulas in some important cases which permit the study of the nature of the solutions, comparative statics and approximations.

Concretely, this dissertation is structured as follows:

- The next chapter provides a full resolution of the Avellaneda-Stoikov problem in its original setting by formulating via the dynamic programming principle, a non-linear HJB equation characterizing the market-maker optimal quotes. We show that we can transform by a non-trivial change of variable the non-linear PDE into a linear system of ODE. This allows us to provide an in-depth analysis of the optimal quotes, asymptotic solutions and comparative statics. We also extend the analysis to more involved situations like a price dynamics with a trend or adverse selection effects. We conclude by illustrating backtests of the algorithm's implementation.

- Chapter 3 provides an original result, not only from the point of view of mathematics but also one of the first quantitative models of optimal liquidation through limit-orders. The idea is to take as starting point the results from chapter 2 and consider optimal-liquidation with passive orders as a *one-sided market-making*. Again, the solutions via a similar change of variable allow a complete analysis of the solutions. Results from chapter 2 and 3 have been published in [64, 65].

- In chapter 4, we study the problem from a different angle. We are concerned with the interpretation and the calibration of the model by using real data. We provide a mathematical analysis of the convergence and the efficiency of two estimators for the intensity of the Poisson process $N^{(\delta_t)}$. One of the estimators we will present takes as input data the waiting time until execution. This is interesting to study execution probabilities, not only in our context but also in situations where liquidity is dark so that we can only observe our own deals. We provide examples from real data comparing the parameters of the model to different market quantities, showing that the Avellaneda-Stoikov model is a good representation of liquidity and prices.

- In the last two chapters (5 and 6), we study the optimization problem in algorithmic trading (with market-making as guiding problem) from the point of view of stochastic approximation as an alternative (or complement) to the dynamic programming approach. First, in chapter 5, we give a overview of the main mathematical results from the theory of recursive stochastic algorithm for optimization problems. Then, in chapter 6, we start from a modified version of the Avellaneda-Stoikov model like in the work of Laruelle et al. [101] in order to find the market-maker optimal quotes. In this chapter we emphasize the closed formulas obtained in the Brownian situation, then we use it as a benchmark case to understand the method.

Throughout this study, we focus on how to integrate the model with practical applications and test the results on both numerical simulations and real data. We conclude with a discussion on possible areas for future research as well as providing the bibliographical references in the final pages of this study.

Chapter 2

Dynamic Programming Approach to Market-Making

2.1 Introduction

In this chapter we present an optimization framework for market-making based on the theory of stochastic-control (through the dynamic programing principle in the form of the Hamilton-Jacobi-Bellman equation). This approach was originally introduced by Ho and Stoll [80] and formalized mathematically by Avellaneda and Stoikov in their seminal paper [13]. The idea is to represent the stock's reference price by a process S_t following a centered Brownian motion with standard deviation σ and to consider the distances to this price as controls which are updated throughout the day in order to maximize an utility function. This version of the dynamic programming principle defines an non-linear PDE for the value function, which allows us to characterize the optimal quotes.

We will show, by using a non-trivial change of variables, that the Hamilton-Jacobi-Bellman equations associated to the problem boil down to a system of linear ordinary differential equations. This change of variables (i) simplifies the computation of a solution since numerical approximation of the PDE is now unnecessary, and (ii) allows to study the asymptotic behavior of the optimal quotes. In addition, we use results from spectral analysis to provide an approximation of the optimal quotes in closed-form and provide a comparative-statics analysis. All these results were presented in [65].

We start this chapter by providing a description of the model, the associated stochastic-control problem and we introduce our change of variables, leading to the analytical solution of the HJB equation. The latter allows to study the asymptotic behavior of the optimal quotes, yielding good approximations in closed-form and this leading to a comparative-statics analysis. We provide two generalizations of the model: (i) the introduction of a drift in the price dynamics and (ii) the introduction of 'passive' market-impact (that may also be regarded as adverse selection). At the end of the chapter we present backtests of the model and the proof of the main mathematical results.

2.2 Setup of the model

2.2.1 Price and liquidity

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume that all random variables and stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We consider a high-frequency market maker operating on a single stock. We suppose that the mid-price of this stock, or more generally a reference price of the stock, moves as an arithmetic Brownian motion:

$$dS_t = \sigma dW_t.$$

The market maker under consideration will continuously propose bid and ask prices denoted respectively S_t^b and S_t^a and will hence buy and sell shares according to the rate of arrival of market orders at the quoted prices. His inventory q, that is the (signed) quantity of shares he holds, is given by

$$q_t = N_t^b - N_t^a,$$

where N^b and N^a are the point processes (independent of $(W_t)_t$) giving the number of shares the market maker respectively bought and sold (we assume that transactions are of constant size, scaled to 1). Arrival rates obviously depend on the prices S_t^b and S_t^a quoted by the market maker and we assume, in accordance with the model proposed by Avellaneda and Stoikov [13], that intensities λ^b and λ^a associated respectively to N^b and N^a depend on the difference between the quoted prices and the reference price (*i.e.* $\delta_t^b = S_t - S_t^b$ and $\delta_t^a = S_t^a - S_t$) and are of the following form:

$$\begin{aligned} \lambda^b(\delta^b) &= Ae^{-k\delta^b} = A\exp(-k(s - s^b)), \\ \lambda^a(\delta^a) &= Ae^{-k\delta^a} = A\exp(-k(s^a - s)), \end{aligned}$$

where A and k are positive constants that characterize the liquidity of the stock. In particular, this specification means – for positive δ^b and δ^a – that the closer to the reference price an order is posted, the faster it will be executed.

As a consequence of his trades, the market maker has an amount of cash evolving according to the following dynamics:

$$dX_t = (S_t + \delta_t^a)dN_t^a - (S_t - \delta_t^b)dN_t^b.$$

To this original setting introduced by Avellaneda and Stoikov (itself following partially Ho and Stoll [80]), we add a bound Q to the inventory that a market maker is authorized to have. In other words, we assume that a market maker with inventory Q ($Q > 0$ depending in practice on risk limits) will never set a bid quote and symmetrically that a market maker with inventory $-Q$, that is a short position of Q shares in the stock under consideration, will never set an ask quote. This realistic restriction may be read as a risk limit and allows to solve rigorously the problem.

2.2.2 The objective function

Now, coming to the objective function, the market maker has a time horizon T and his goal is to optimize the expected utility of his P&L at time T. In line with [13], we will focus on CARA utility functions and we suppose that the market maker optimizes:

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T)\right)\right]$$

where $\mathcal{A}$ is the set of predictable processes bounded from below, γ is the absolute risk aversion coefficient characterizing the market maker, X_T is the amount of cash at time T and $q_T S_T$ is the evaluation of the (signed) remaining quantity of shares in the inventory at time T (liquidation at the reference price S_T[1]).

2.3 Solution to the Avellaneda-Stoikov problem

2.3.1 Characterization of the optimal quotes

The optimization problem set up in the preceding section can be solved using the classical tools of stochastic optimal control. The first step of our reasoning is therefore to introduce the Hamilton-Jacobi-Bellman (HJB) equation associated to the problem. More exactly, we introduce a system of Hamilton-Jacobi-Bellman partial differential equations which consists of the following equations indexed by $q \in \{-Q, \dots, Q\}$ for $(t, s, x) \in [0, T] \times \mathbb{R}^2$:

For $|q| < Q$:

$$\begin{aligned}\partial_t u(t,x,q,s) &+ \frac{1}{2}\sigma^2 \partial^2_{ss} u(t,x,q,s)\\ &+ \sup_{\delta^b} \lambda^b(\delta^b)\left[u(t,x-s+\delta^b,q+1,s) - u(t,x,q,s)\right]\\ &+ \sup_{\delta^a} \lambda^a(\delta^a)\left[u(t,x+s+\delta^a,q-1,s) - u(t,x,q,s)\right] = 0.\end{aligned}$$

For $q = Q$:

$$\begin{aligned}\partial_t u(t,x,Q,s) &+ \frac{1}{2}\sigma^2 \partial^2_{ss} u(t,x,Q,s)\\ &+ \sup_{\delta^a} \lambda^a(\delta^a)\left[u(t,x+s+\delta^a,Q-1,s) - u(t,x,Q,s)\right] = 0.\end{aligned}$$

For $q = -Q$:

$$\begin{aligned}\partial_t u(t,x,-Q,s) &+ \frac{1}{2}\sigma^2 \partial^2_{ss} u(t,x,-Q,s)\\ &+ \sup_{\delta^b} \lambda^b(\delta^b)\left[u(t,x-s+\delta^b,-Q+1,s) - u(t,x,-Q,s)\right] = 0\end{aligned}$$

with the final condition:

$$\forall q \in \{-Q, \dots, Q\}, \qquad u(T,x,q,s) = -\exp\left(-\gamma(x+qs)\right).$$

[1] Our results would be *mutatis mutandis* the same if we added a penalization term $-b(|q_T|)$ for the shares remaining at time T. The rationale underlying this point is that price risk prevents the trader from having important exposure to the stock. Hence, q_t should naturally mean-revert around 0.

To solve these equations we will use a change of variables based on two different ideas. First, the choice of a CARA utility function allows to factor out the Mark-to-Market value of the portfolio $(x + qs)$. Then, the exponential decay for the intensity functions λ^b and λ^a allows to reduce the Hamilton-Jacobi-Bellman (HJB) equations associated to our control problem to a linear system of ordinary differential equations:

Proposition 1 (Change of variables for (HJB))**.** *Let us consider a family* $(v_q)_{|q|\leq Q}$ *of positive functions solution of:*

$$\begin{aligned}
\dot{v}_q(t) &= \alpha q^2 v_q(t) - \eta\left(v_{q-1}(t) + v_{q+1}(t)\right), \quad q \in \{-Q+1, \ldots, Q-1\}, && (2.1)\\
\dot{v}_Q(t) &= \alpha Q^2 v_Q(t) - \eta v_{Q-1}(t), && (2.2)\\
\dot{v}_{-Q}(t) &= \alpha Q^2 v_{-Q}(t) - \eta v_{-Q+1}(t) && (2.3)
\end{aligned}$$

with $\forall q \in \{-Q, \ldots, Q\}, v_q(T) = 1$*, where* $\alpha = \frac{k}{2}\gamma\sigma^2$ *and* $\eta = A(1 + \frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.
Then, $u(t, x, q, s) = -\exp(-\gamma(x + qs))v_q(t)^{-\frac{\gamma}{k}}$ *is solution of (HJB).*

Then, the following proposition proves that there exists such a family of positive functions:

Proposition 2 (Solution of the ordinary differential equations)**.** *Let us introduce the matrix M defined by:*

$$M = \begin{pmatrix}
\alpha Q^2 & -\eta & 0 & \cdots & \cdots & \cdots & 0\\
-\eta & \alpha(Q-1)^2 & -\eta & 0 & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & 0 & -\eta & \alpha(Q-1)^2 & -\eta\\
0 & \cdots & \cdots & \cdots & 0 & -\eta & \alpha Q^2
\end{pmatrix}$$

where $\alpha = \frac{k}{2}\gamma\sigma^2$ *and* $\eta = A(1 + \frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.

Let us define

$$\begin{aligned}
v(t) &= (v_{-Q}(t), v_{-Q+1}(t), \ldots, v_0(t), \ldots, v_{Q-1}(t), v_Q(t))'\\
&= \exp(-M(T-t)) \times (1, \ldots, 1)'
\end{aligned}$$

Then, $(v_q)_{|q|\leq Q}$ *is a family of positive functions solution of the system of ordinary differential equations of Proposition 1.*

Using the above change of variables and a verification approach, we are now able to solve the stochastic control problem, that is to find the value function of the problem and the optimal quotes:

Theorem 1 (Solution of the control problem). *Let consider $(v_q)_{|q|\leq Q}$ as in Proposition 2.*

Then $u(t,x,q,s) = -\exp(-\gamma(x+qs))v_q(t)^{-\frac{\gamma}{k}}$ is the value function of the control problem.

Moreover, the optimal quotes are given by:

$$s - s^{b*}(t,q,s) = \delta^{b*}(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q+1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right), \quad q \neq Q$$

$$s^{a*}(t,q,s) - s = \delta^{a*}(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right), \quad q \neq -Q$$

and the resulting bid-ask spread quoted by the market maker is given by:

$$\psi^*(t,q) = -\frac{1}{k}\ln\left(\frac{v_{q+1}(t)v_{q-1}(t)}{v_q(t)^2}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right), \quad |q| \neq Q$$

2.3.2 Asymptotic behavior and approximation of the optimal quotes

To exemplify our findings and in order to motivate the asymptotic approximations that we shall provide, we plotted on Figure 2.1 and Figure 2.2 the behavior as a function of time and the inventory of the optimal quotes. The resulting bid-ask spread quoted by the market maker is plotted on Figure 2.3.

We clearly see that the optimal quotes are almost independent of t, as soon as t is far from the terminal time T.

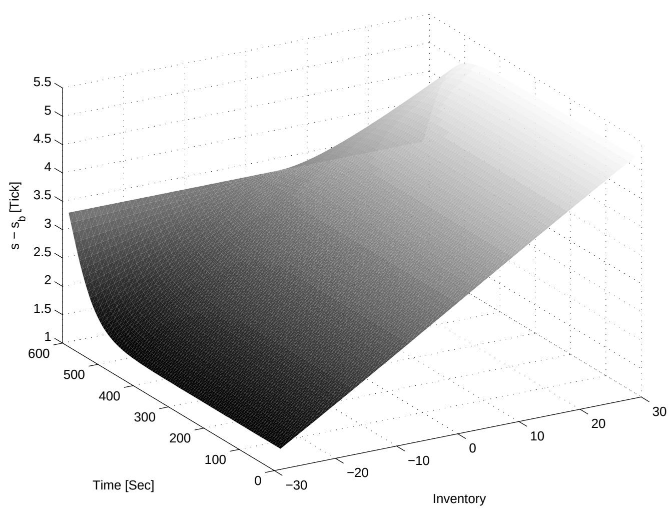

Figure 2.1: Behavior of the optimal bid quotes with time and inventory. $\sigma = 0.3 \quad \text{Tick} \cdot \text{s}^{-1/2}$, $A = 0.9 \quad \text{s}^{-1}$, $k = 0.3 \quad \text{Tick}^{-1}$, $\gamma = 0.01 \quad \text{Tick}^{-1}$, $T = 600 \quad s$.

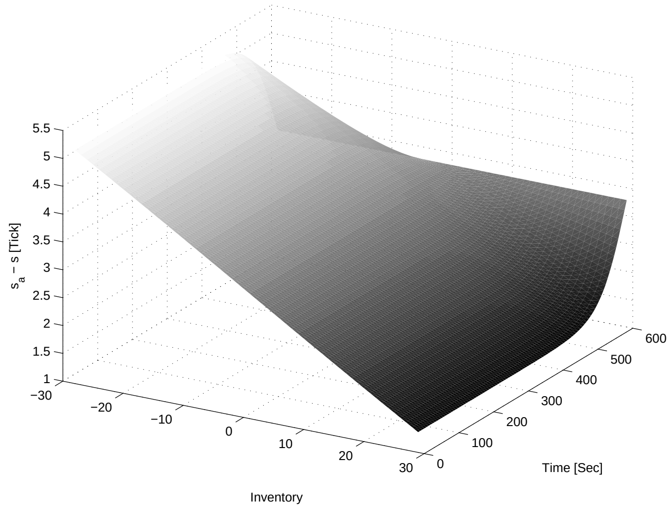

Figure 2.2: Behavior of the optimal ask quotes with time and inventory. $\sigma = 0.3 \quad \mathrm{Tick} \cdot \mathrm{s}^{-1/2}$, $A = 0.9 \quad \mathrm{s}^{-1}$, $k = 0.3 \quad \mathrm{Tick}^{-1}$, $\gamma = 0.01 \quad \mathrm{Tick}^{-1}$, $T = 600 \quad s$.

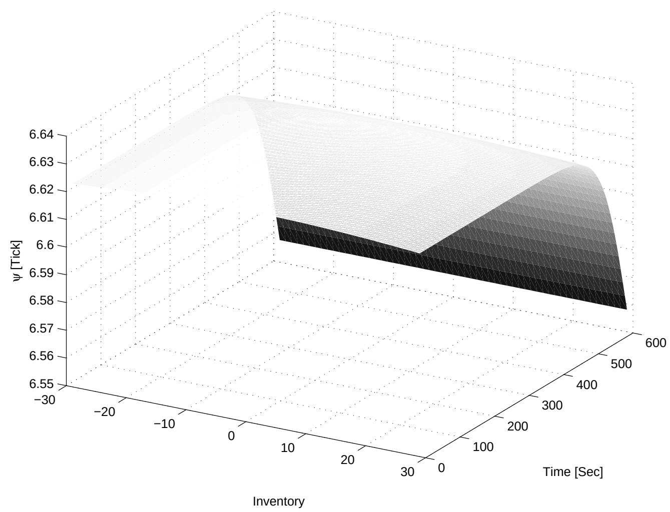

Figure 2.3: Behavior of the resulting bid-ask spread with time and inventory. $\sigma = 0.3 \quad \mathrm{Tick} \cdot \mathrm{s}^{-1/2}$, $A = 0.9 \quad \mathrm{s}^{-1}$, $k = 0.3 \quad \mathrm{Tick}^{-1}$, $\gamma = 0.01 \quad \mathrm{Tick}^{-1}$, $T = 600 \quad s$.

Theorem 2 (Asymptotics for the optimal quotes)**.** *The optimal quotes have asymptotic limits*

$$\lim_{T \to +\infty} \delta^{b*}(0, q) = \delta_{\infty}^{b*}(q)$$

$$\lim_{T \to +\infty} \delta^{a*}(0, q) = \delta_{\infty}^{a*}(q)$$

that can be expressed as:

$$\delta_\infty^{b*}(q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+1}^0}\right) \quad \delta_\infty^{a*}(q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-1}^0}\right)$$

where $f^0 \in \mathbb{R}^{2Q+1}$ *is an eigenvector corresponding to the smallest eigenvalue of the matrix* M *introduced in Proposition 2 and characterized (up to a multiplicative constant) by:*

$$f^0 \in \underset{f\in\mathbb{R}^{2Q+1}, \|f\|_2=1}{\operatorname{argmin}} \sum_{q=-Q}^{Q} \alpha q^2 {f_q}^2 + \eta \sum_{q=-Q}^{Q-1} (f_{q+1}-f_q)^2 + \eta {f_Q}^2 + \eta {f_{-Q}}^2.$$

The resulting bid-ask spread quoted by the market maker is asymptotically:

$$\psi_\infty^*(q) = -\frac{1}{k}\ln\left(\frac{f_{q+1}^0 f_{q-1}^0}{{f_q^0}^2}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right).$$

The above result, along with the example of Figure 2.1, Figure 2.2 and Figure 2.3, encourages to approximate the optimal quotes and the resulting bid-ask spread by their asymptotic value. These asymptotic values depend on f^0 and we shall provide a closed-form approximation for f^0.

The above characterization of f^0 corresponds to an eigenvalue problem in $\mathbb{R}^{2Q+1}$ and we propose to replace it by a similar eigenvalue problem in $L^2(\mathbb{R})$ for which a closed-form solution can be computed. More precisely we replace the criterion

$$f^0 \in \underset{f\in\mathbb{R}^{2Q+1}, \|f\|_2=1}{\operatorname{argmin}} \sum_{q=-Q}^{Q} \alpha q^2 {f_q}^2 + \eta \sum_{q=-Q}^{Q-1} (f_{q+1}-f_q)^2 + \eta {f_Q}^2 + \eta {f_{-Q}}^2$$

by the following criterion for $\tilde{f}^0 \in L^2(\mathbb{R})$:

$$\tilde{f}^0 \in \underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{+\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta \tilde{f}'(x)^2\right) dx$$

The introduction of this new criterion is rooted to the following proposition which provides (up to its sign) the expression for $\tilde{f}^0$ in closed form:

Proposition 3. *Let us consider*

$$\tilde{f}^0 \in \underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{\mathbb{R}} \left(\alpha x^2 \tilde{f}(x)^2 + \eta \tilde{f}'(x)^2\right) dx.$$

Then:

$$\tilde{f}^0(x) = \pm\frac{1}{\pi^{\frac{1}{4}}}\left(\frac{\alpha}{\eta}\right)^{\frac{1}{8}} \exp\left(-\frac{1}{2}\sqrt{\frac{\alpha}{\eta}}x^2\right).$$

From the above proposition, we expect f_q^0 to behave, up to a multiplicative constant, as $\exp\left(-\frac{1}{2}\sqrt{\frac{\alpha}{\eta}}q^2\right)$. This heuristic viewpoint induces an approximation of the optimal quotes and the resulting optimal bid-ask-spread:

$$
\begin{aligned}
\delta^{b*}_{\infty}(q) &\simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)+\frac{1}{2k}\sqrt{\frac{\alpha}{\eta}}(2q+1) \\
&\simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)+\frac{2q+1}{2}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}. \\
\delta^{a*}_{\infty}(q) &\simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)-\frac{1}{2k}\sqrt{\frac{\alpha}{\eta}}(2q-1) \\
&\simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)-\frac{2q-1}{2}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}.
\end{aligned}
$$

$$
\psi^{*}_{\infty}(q) \simeq \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)+\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}.
$$

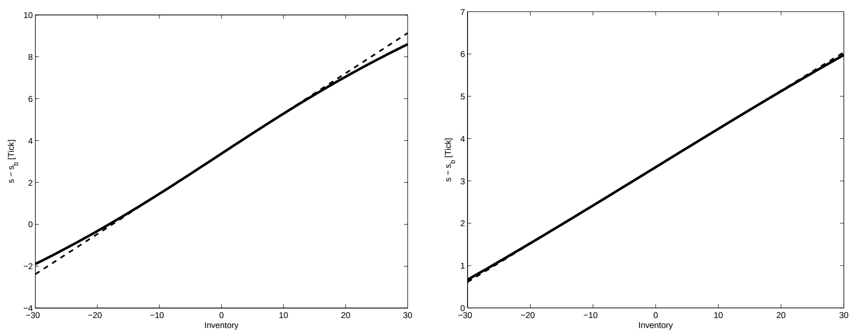

Figure 2.4: Asymptotic behavior of optimal bid quote (bold line). Approximation (dotted line). Left: $\sigma = 0.4\text{Tick}\cdot\text{s}^{-1/2}$, $A = 0.9\text{s}^{-1}$, $k = 0.3\text{Tick}^{-1}$, $\gamma = 0.01\text{Tick}^{-1}$, $T = 600s$. Right: $\sigma = 1.0\text{Tick}\cdot\text{s}^{-1/2}$, $A = 0.2\text{s}^{-1}$, $k = 0.3\text{Tick}^{-1}$, $\gamma = 0.01\text{Tick}^{-1}$, $T = 600s$.

We exhibit on Figure 2.4 and Figure 2.5 the values of the optimal quotes, along with their associated approximations. Empirically, these approximations for the quotes are satisfactory in most cases and are always very good for small values of the inventory q. In fact, even though f^0 appears to be well approximated by the Gaussian approximation, we cannot expect a very good fit for the quotes when q is large because we are approximating expressions that depend on ratios of the form $\frac{f^0_q}{f^0_{q+1}}$ or $\frac{f^0_q}{f^0_{q-1}}$.

2.3.3 The case of a trend in the price dynamics

So far, the reference price was supposed to be a Brownian motion. In what follows we extend the model to the case of a trend in the price dynamics:

$$dS_t = \mu dt + \sigma dW_t.$$

In that case we have the following proposition (the proof is not repeated):

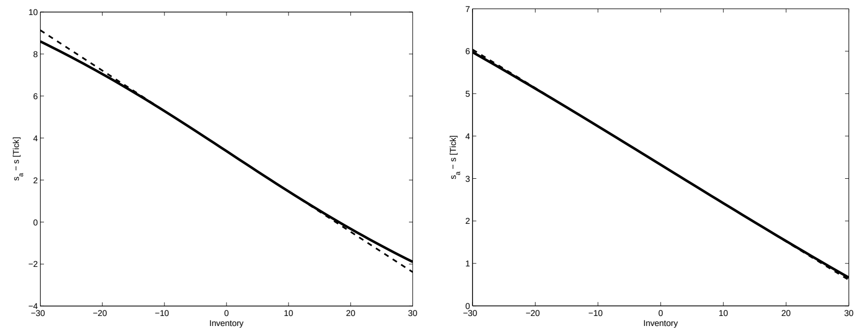

Figure 2.5: Asymptotic behavior of optimal ask quote (bold line). Approximation (dotted line). Left: $\sigma = 0.4\text{Tick}\cdot\text{s}^{-1/2}$, $A = 0.\text{s}^{-1}$, $k = 0.3\text{Tick}^{-1}$, $\gamma = 0.01\text{Tick}^{-1}$, $T = 600s$. Right: $\sigma = 1.0\text{Tick}\cdot\text{s}^{-1/2}$, $A = 0.2\text{s}^{-1}$, $k = 0.3\text{Tick}^{-1}$, $\gamma = 0.01\text{Tick}^{-1}$, $T = 600s$.

Proposition 4 (Solution with a drift)**.** *Let us consider a family of functions* $(v_q)_{|q|\leq Q}$ *solution of the linear system of ODEs that follows:*

$$\begin{aligned}
\dot{v}_q(t) &= (\alpha q^2 - \beta q)v_q(t) - \eta\left(v_{q-1}(t) + v_{q+1}(t)\right), \quad \forall q \in \{-Q+1,\dots,Q-1\}, && (2.4)\\
\dot{v}_Q(t) &= (\alpha Q^2 - \beta Q)v_Q(t) - \eta v_{Q-1}(t), && (2.5)\\
\dot{v}_{-Q}(t) &= (\alpha Q^2 + \beta Q)v_{-Q}(t) - \eta v_{-Q+1}(t), && (2.6)
\end{aligned}$$

with $\forall q \in \{-Q,\dots,Q\}, v_q(T) = 1$, *where* $\alpha = \frac{k}{2}\gamma\sigma^2$, $\beta = k\mu$ *and* $\eta = A(1+\frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.

Then, $u(t,x,q,s) = -\exp(-\gamma(x+qs))v_q(t)^{-\frac{\gamma}{k}}$ *is the value function of the control problem.*

The optimal quotes are given by:

$$s - s^{b*}(t,q,s) = \delta^{b*}(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q+1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right),$$

$$s^{a*}(t,q,s) - s = \delta^{a*}(t,q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

and the resulting bid-ask spread of the market maker is :

$$\psi^*(t,q) = -\frac{1}{k}\ln\left(\frac{v_{q+1}(t)v_{q-1}(t)}{v_q(t)^2}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right).$$

Moreover,

$$\lim_{T\to+\infty}\delta^{b*}(0,q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+1}^0}\right),$$

$$\lim_{T\to+\infty}\delta^{a*}(0,q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-1}^0}\right),$$

$$\lim_{T\to+\infty} \psi^*(0,q) = -\frac{1}{k}\ln\left(\frac{f^0_{q+1}f^0_{q-1}}{{f^0_q}^2}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right).$$

where f^0 is an eigenvector corresponding to the smallest eigenvalue of:

$$\begin{pmatrix}
\alpha Q^2 - \beta Q & -\eta & 0 & \cdots & \cdots & \cdots & 0 \\
-\eta & \alpha(Q-1)^2-\beta(Q-1) & -\eta & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & -\eta & \alpha(Q-1)^2-\beta(Q-1) & -\eta \\
0 & \cdots & \cdots & \cdots & 0 & -\eta & \alpha Q^2 - \beta Q
\end{pmatrix}.$$

In addition to this theoretical result, we can consider an approximation similar to the approximation used for the initial model with no drift. We then obtain the following approximations for the optimal quotes and the bid-ask spread:

$$\delta^{b*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \left[-\frac{\mu}{\gamma\sigma^2} + \frac{2q+1}{2}\right]\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}},$$

$$\delta^{a*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \left[\frac{\mu}{\gamma\sigma^2} - \frac{2q-1}{2}\right]\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}},$$

$$\psi^*_\infty(q) \simeq \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}.$$

2.3.4 Comparative statics

We argued that the value of the optimal quotes was almost independent of t for t sufficiently far from the terminal time T and we characterized the asymptotic value of the optimal quotes. We also provided approximations for the asymptotic value of the optimal quotes. The latter allows us to obtain good approximations about the influence of the different parameters on the optimal quotes.

Dependence on σ^2

The dependence of optimal quotes on σ^2 depends on the sign of the inventory. More precisely, we observe numerically, in accordance with the approximations, that:

$$\begin{cases}
\frac{\partial\delta^{b*}_\infty}{\partial\sigma^2} < 0, & \frac{\partial\delta^{a*}_\infty}{\partial\sigma^2} > 0, \quad \text{if } q < 0, \\
\frac{\partial\delta^{b*}_\infty}{\partial\sigma^2} > 0, & \frac{\partial\delta^{a*}_\infty}{\partial\sigma^2} > 0, \quad \text{if } q = 0, \\
\frac{\partial\delta^{b*}_\infty}{\partial\sigma^2} > 0, & \frac{\partial\delta^{a*}_\infty}{\partial\sigma^2} < 0, \quad \text{if } q > 0.
\end{cases}$$

For the bid-ask spread, we obtain:

$$\frac{\partial \psi_\infty^*}{\partial \sigma^2} > 0.$$

An increase of σ^2 increases inventory risk. Hence, to reduce this risk, a market maker that has a long position will try to reduce his exposure and hence ask less for his stocks (to get rid of some of them) and accept to buy at a lower price (to avoid buying new stocks). Similarly, an algorithm with a short position tries to buy stocks, and hence increases its bid quote, while avoiding short selling new stocks, and increasing its ask quote to that purpose. Overall, due to the increase in price risk, the bid-ask spread widens as it is well instanced in the case of a market maker with a flat position (this one wants indeed to earn more per trade to compensate the increase in inventory risk).

Dependence on μ

The dependence of optimal quotes on the drift μ is straightforward and corresponds to the intuition. If the agent expects the price to increase (resp. decrease) he will post orders with higher (resp. lower) prices. Hence we have:

$$\frac{\partial \delta_\infty^{b*}}{\partial \mu} < 0, \quad \frac{\partial \delta_\infty^{a*}}{\partial \mu} > 0.$$

Dependence on A

Because of the form of the system of equations that defines v, the dependence on A must be the exact opposite of the dependence on σ^2:

$$\begin{cases} \frac{\partial \delta_\infty^{b*}}{\partial A} > 0, & \frac{\partial \delta_\infty^{a*}}{\partial A} < 0, \quad \text{if } q < 0, \\ \frac{\partial \delta_\infty^{b*}}{\partial A} < 0, & \frac{\partial \delta_\infty^{a*}}{\partial A} < 0, \quad \text{if } q = 0, \\ \frac{\partial \delta_\infty^{b*}}{\partial A} < 0, & \frac{\partial \delta_\infty^{a*}}{\partial A} > 0, \quad \text{if } q > 0. \end{cases}$$

For the bid-ask spread, we obtain:

$$\frac{\partial \psi_\infty^*}{\partial A} < 0.$$

The rationale behind these results is that an increase of A reduces the inventory risk, since it increases the frequency of trades and hence reduces the risk of being stuck with a large inventory (in absolute value). For this reason, an increase in A should have the same effect as a decrease in σ^2.

Dependence on γ

Using the closed-form approximations, we see that the dependence on γ is ambiguous. The market maker faces indeed two different risks that contribute to inventory risk: (i) trades occur at random times and (ii) the reference price is stochastic. But if risk aversion increases, the market maker will mitigate the two risks: (i) he may set his quotes closer to one another to reduce the randomness in execution and (ii) he may widen his spread to reduce price risk. The tension between these two roles played by γ explains the different behaviors we may observe on Figure 2.6 and Figure 2.7 for the bid-ask spread resulting from the asymptotic optimal quotes:

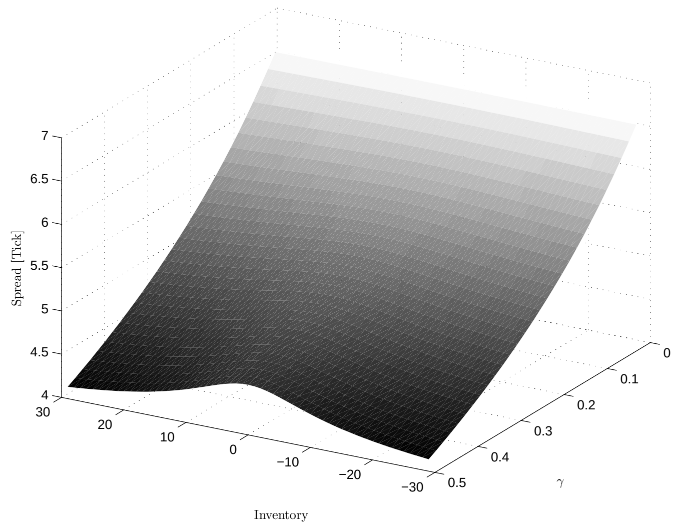

Figure 2.6: Bid-ask spread resulting from the asymptotic optimal quotes for different inventories and different values for the risk aversion parameter γ. $\sigma = 0.3 \quad \mathrm{Tick} \cdot \mathrm{s}^{-1/2}$, $A = 0.9 \quad \mathrm{s}^{-1}$, $k = 0.3 \quad \mathrm{Tick}^{-1}$, $T = 600 \quad s$.

Dependence on k

From the closed-form approximations, we expect δ_∞^{b*} to be decreasing in k for q greater than some negative threshold. Below this threshold, we expect it to be increasing. Similarly, we expect δ_∞^{a*} to be decreasing in k for q smaller than some positive threshold. Above this threshold we expect it to be increasing.

Eventually, as far as the bid-ask spread is concerned, the closed-form approximations indicate that the resulting bid-ask spread should be a decreasing function of k.

$$\frac{\partial \psi_\infty^*}{\partial k} < 0.$$

In fact several effects are in interaction. On one hand, there is a “no-volatility” effect that is completely orthogonal to any reasoning on the inventory risk: when k increases, in a situation where δ^b and δ^a are positive, trades occur closer to the reference price S_t. For this reason, and in absence of inventory risk, the optimal bid-ask spread has to shrink. However, an increase in k also affects the inventory risk since it decreases the probability to be executed (for $\delta^b, \delta^a > 0$). Hence, an increase in k is also, in some aspects, similar to a decrease in A. These two effects explain the expected behavior.

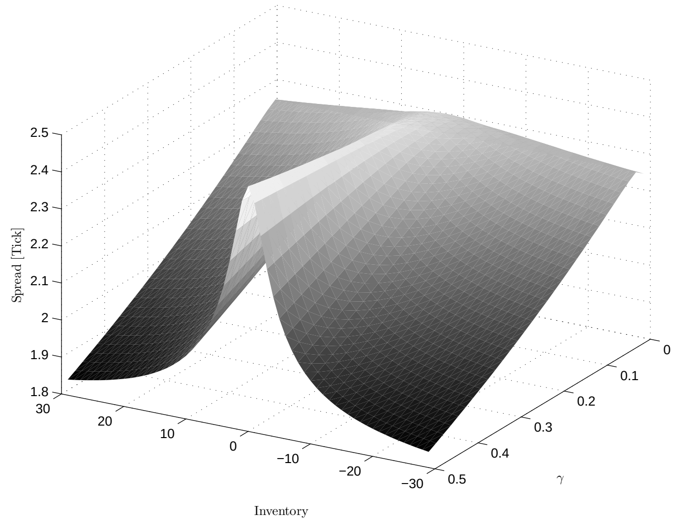

Figure 2.7: Bid-ask spread resulting from the asymptotic optimal quotes for different inventories and different values for the risk aversion parameter γ. $\sigma = 0.6 \quad \text{Tick} \cdot \text{s}^{-1/2}$, $A = 0.9 \quad \text{s}^{-1}$, $k = 0.9 \quad \text{Tick}^{-1}$, $T = 600 \quad s$.

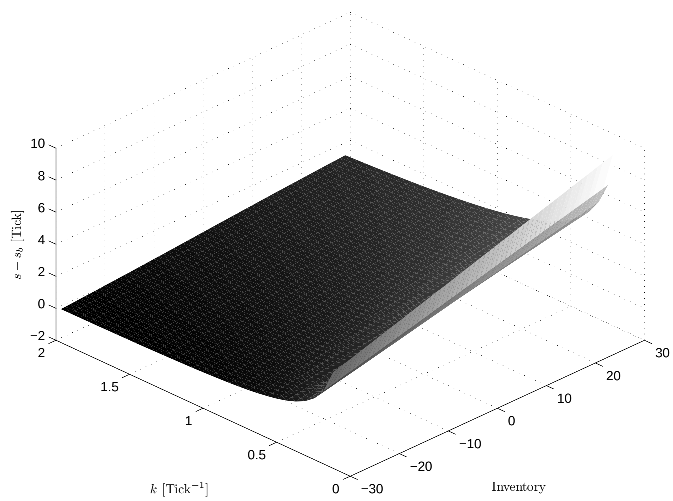

Figure 2.8: Asymptotic optimal bid quotes for different inventories and different values of k. $\sigma = 0.3 \quad \text{Tick} \cdot \text{s}^{-1/2}$, $A = 0.9 \quad \text{s}^{-1}$, $\gamma = 0.01 \quad \text{Tick}^{-1}$, $T = 600 \quad s$.

Numerically, we observed that the "no-volatility" effect dominates for the values of the inventory under consideration (see Figure 2.8 for the case of the bid quote[2]).

[2]The case of the ask quote is obviously similar.

2.4 Including market-impact

Another extension of the model consists in introducing market impact. The simplest way to proceed is to consider the following dynamics for the price:

$$dS_t = \sigma dW_t + \xi dN_t^a - \xi dN_t^b, \qquad \xi > 0.$$

When a limit order on the bid side is filled, the reference price decreases. On the contrary, when a limit order on the ask side is filled, the reference price increases. This is in line with the classical modeling of market impact for market orders, ξ being a constant since the limit orders posted by the market maker are all supposed to be of the same size. Adverse selection is another way to interpret the interaction we consider between the price process and the point processes modeling execution: trades on the bid side are often followed by a price decrease and, conversely, trades on the ask side are often followed by a price increase. In this framework, the problem can be solved using a change of variables that is slightly more involved than the one presented above but the method is exactly the same. We have the following result (the proof is not repeated):

Proposition 5 (Solution with market impact)**.** *Let us consider a family of functions $(v_q)_{|q|\le Q}$ solution of the linear system of ODEs that follows:*

$$\begin{aligned}
\dot{v}_q(t) &= \alpha q^2 v_q(t) - \eta e^{-\frac{k}{2}\xi}\left(v_{q-1}(t) + v_{q+1}(t)\right), \quad \forall q \in \{-Q+1, \dots, Q-1\}, && (2.7)\\
\dot{v}_Q(t) &= \alpha Q^2 v_Q(t) - \eta e^{-\frac{k}{2}\xi} v_{Q-1}(t), && (2.8)\\
\dot{v}_{-Q}(t) &= \alpha Q^2 v_{-Q}(t) - \eta e^{-\frac{k}{2}\xi} v_{-Q+1}(t), && (2.9)
\end{aligned}$$

with $\forall q \in \{-Q, \dots, Q\}, v_q(T) = \exp(-\frac{1}{2}k\xi q^2)$, where $\alpha = \frac{k}{2}\gamma\sigma^2$ and $\eta = A(1+\frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.

Then, $u(t, x, q, s) = -\exp(-\gamma(x + qs + \frac{1}{2}\xi q^2))v_q(t)^{-\frac{\gamma}{k}}$ is the value function of the control problem and the optimal quotes are given by:

$$s - s^{b*}(t, q, s) = \delta^{b*}(t, q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q+1}(t)}\right) + \frac{\xi}{2} + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right)$$

$$s^{a*}(t, q, s) - s = \delta^{a*}(t, q) = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q-1}(t)}\right) + \frac{\xi}{2} + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right)$$

and the resulting bid-ask spread of the market maker is :

$$\psi^*(t, q) = -\frac{1}{k}\ln\left(\frac{v_{q+1}(t)v_{q-1}(t)}{v_q(t)^2}\right) + \xi + \frac{2}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right).$$

Moreover,

$$\lim_{T\to+\infty} \delta^{b*}(0, q) = \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) + \frac{\xi}{2} + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+1}^0}\right)$$

$$\lim_{T\to+\infty} \delta^{a*}(0, q) = \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right) + \frac{\xi}{2} + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-1}^0}\right)$$

$$\lim_{T\to+\infty}\psi^*(0,q) = -\frac{1}{k}\ln\left(\frac{f^0_{q+1}f^0_{q-1}}{{f^0_q}^2}\right) + \xi + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

where f^0 is an eigenvector corresponding to the smallest eigenvalue of:

$$\begin{pmatrix}
\alpha Q^2 & -\eta e^{-\frac{k}{2}\xi} & 0 & \cdots & \cdots & \cdots & 0 \\
-\eta e^{-\frac{k}{2}\xi} & \alpha(Q-1)^2 & -\eta e^{-\frac{k}{2}\xi} & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & -\eta e^{-\frac{k}{2}\xi} & \alpha(Q-1)^2 & -\eta e^{-\frac{k}{2}\xi} \\
0 & \cdots & \cdots & \cdots & 0 & -\eta e^{-\frac{k}{2}\xi} & \alpha Q^2
\end{pmatrix}.$$

In addition to this theoretical result, we can consider an approximation similar to the approximation used for the initial model. We then obtain the following approximations for the optimal quotes and the bid-ask spread:

$$\delta^{b*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{\xi}{2} + \frac{2q+1}{2}e^{\frac{k}{4}\xi}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}},$$

$$\delta^{a*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{\xi}{2} - \frac{2q-1}{2}e^{\frac{k}{4}\xi}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}},$$

$$\psi^*_\infty(q) \simeq \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \xi + e^{\frac{k}{4}\xi}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}.$$

The market impact introduced above, has two effects on the optimal quotes. In the absence of price risk, given the functional form of the execution intensities, the direct effect of ξ is approximately to add $\frac{\xi}{2}$ the each optimal quote: the market maker approximately maintains his profit per round trip on the market but the probability of occurrence of a trade is reduced. This adverse selection effect has a side-effect linked to inventory risk: since adverse selection gives the market maker an incentive to post orders deeper in the book, it increases the risk of being stuck with a large inventory for a market maker holding such an inventory. As a consequence, for a trader holding a positive (resp. negative) inventory, there is a second effect inciting to buy and sell at lower (resp. higher) prices. These two effects are clearly highlighted by the closed-form approximations:

$$\delta^{b*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \underbrace{\frac{\xi}{2}}_{adverse\ selection} + \frac{2q+1}{2}\underbrace{e^{\frac{k}{4}\xi}}_{side-effect}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}},$$

$$\delta^{a*}_\infty(q) \simeq \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \underbrace{\frac{\xi}{2}}_{adverse\ selection} - \frac{2q-1}{2}\underbrace{e^{\frac{k}{4}\xi}}_{side-effect}\sqrt{\frac{\sigma^2\gamma}{2kA}\left(1+\frac{\gamma}{k}\right)^{1+\frac{k}{\gamma}}}.$$

2.5 Backtests

Before using the above model on historical data, we need to discuss some features of the model that need to be adapted before any backtest attempt.

First of all, the model is continuous in both time and space while the real control problem is intrinsically discrete in space, because of the tick size, and in time, because orders have a certain priority and changing position too often reduces the actual chance to be reached by a market order. Hence, the model has to be reinterpreted in a discrete way. In terms of prices, quotes must not be between two ticks and we decided to round the optimal quotes to the nearest tick. In terms of time, an order of size ATS[3] is sent to the market and is not canceled nor modified for a given period of time Δt, unless a trade occurs and, though perhaps partially, fills the order. Now, when a trade occurs and changes the inventory or when an order stayed in the order book for longer than Δt, then the optimal quote is updated .Concerning the parameters, σ, A and k can be calibrated on trade-by-trade limit order book data while γ has to be chosen; we decided in our backtests to assign γ an arbitrary value for which the inventory stayed between -10 and 10 during the day under consideration (the unit being the ATS).

Turning to the backtests, they were carried out with trade-by-trade data and we assumed that our orders were entirely filled when a trade occurred at or above the ask price quoted by the agent. Our goal here is just to exemplify the use of the model and we considered the case of the French stock France Telecom on March 15^{th} 2012. We first plot the price of the stock France Telecom on March 15^{th} 2012 on Figure 2.9, the evolution of the inventory on Figure 2.10 and the associated P&L on Figure 2.11.

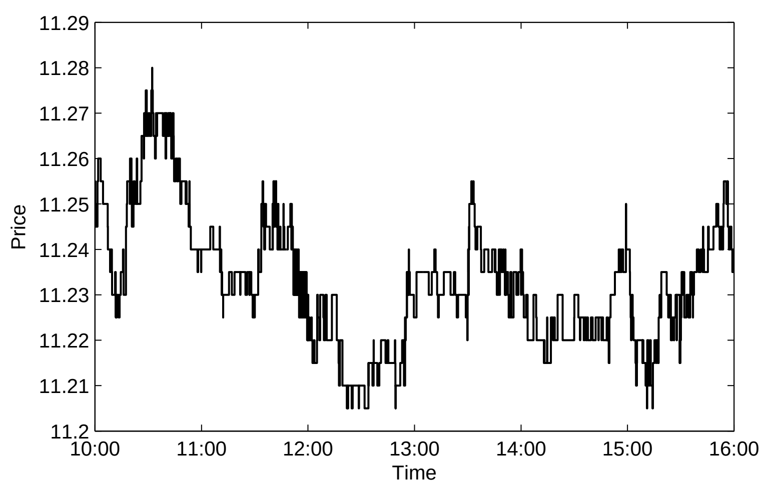

Figure 2.9: Price of the stock France Telecom on 15/03/2012, from 10:00 to 16:00.

[3]ATS is the average trade size.

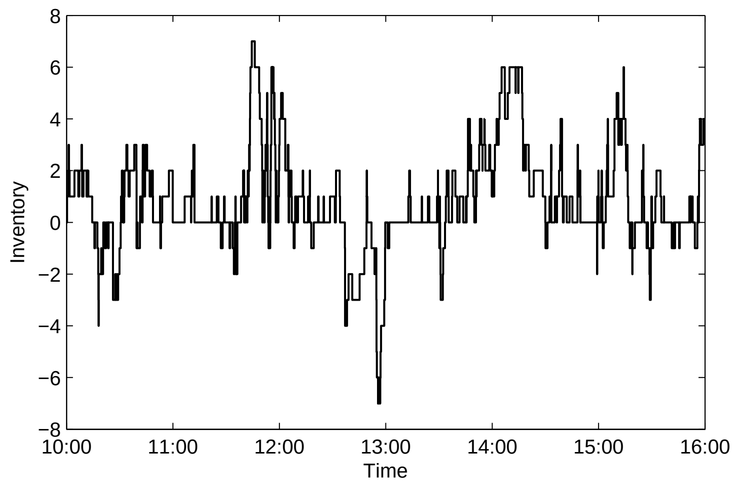

Figure 2.10: Inventory (in ATS) when the strategy is used on France Telecom (15/03/2012) from 10:00 to 16:00.

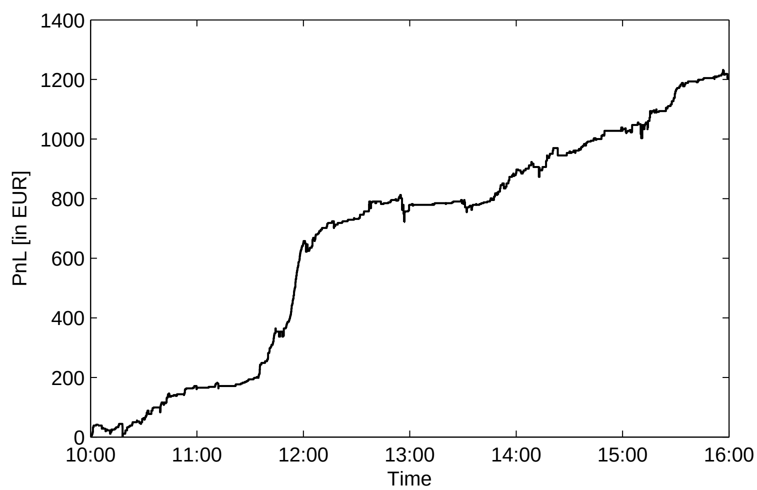

Figure 2.11: P&L when the strategy is used on France Telecom (15/03/2012) from 10:00 to 16:00.

This P&L can be compared to the P&L of a naive trader (Figure 2.12) who only posts orders at the first limit of the book on each side, whenever he is asked to post orders – that is when one of his orders has been executed or after a period of time Δt with no execution. To understand the details of the strategy, we focused on a subperiod of 1 hour and we plotted the state of the market along with the quotes of the market maker (Figure 2.13). Trades involving the market maker are signalled by a dot.

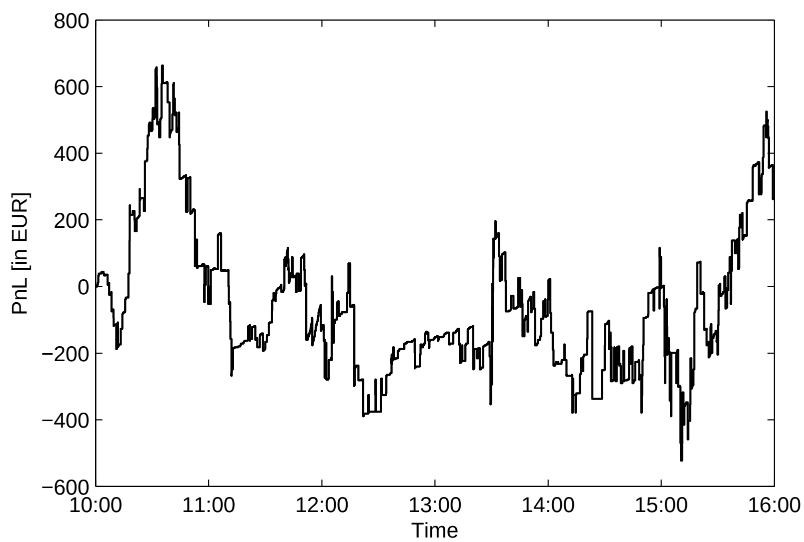

Figure 2.12: P&L of a naive market maker on France Telecom (15/03/2012) from 10:00 to 16:00.

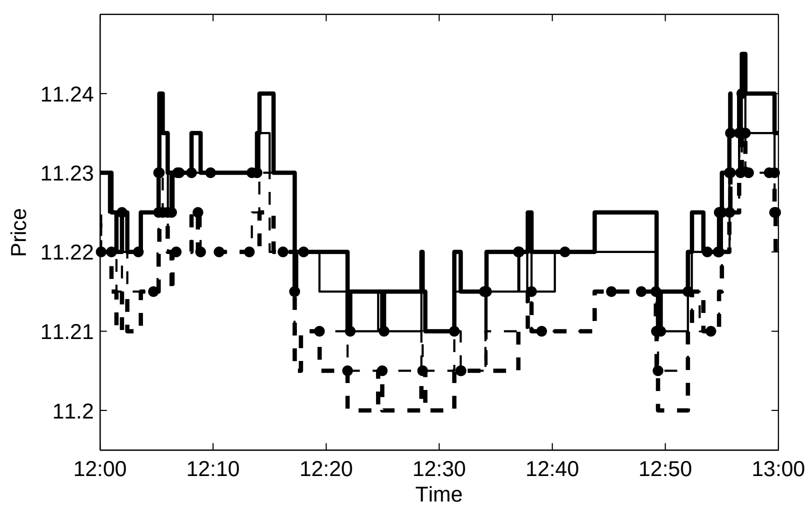

Figure 2.13: Details for the quotes and trades when the strategy is used on France Telecom (15/03/2012). Thin lines represent the market while bold lines represent the quotes of the market maker. Dotted lines are associated to the bid side while plain lines are associated to the ask side. Black points represent trades in which the market maker is involved.

2.6 Conclusion

In this chapter we present a model for the optimal quotes of a market maker. Starting from a model in line with Avellaneda and Stoikov [13] we introduce a change of variables that allows to transform the HJB equation into a system of linear ODEs. This yields the optimal quotes, characterize their asymptotic behavior and obtain comparative statics. Potential extensions of this work are the generalization of the model to any intensity function (see Guéant et al. [63]) and the quantitative modeling of "passive market impact" (*i.e.* the perturbations of the price process due to liquidity provision).

Appendix: Proofs of the results

Proof of Proposition 1, Proposition 2 and Theorem 1:

Let us consider a family $(v_q)_{|q|\le Q}$ of positive functions solution of the system of ODEs introduced in Proposition 1 and let us define $u(t,x,q,s) = -\exp\left(-\gamma(x+qs)\right) v_q(t)^{-\frac{\gamma}{k}}$.

Then:

$$\partial_t u + \frac{1}{2}\sigma^2 \partial^2_{ss} u = -\frac{\gamma}{k}\frac{\dot v_q(t)}{v_q(t)} u + \frac{\gamma^2\sigma^2}{2} q^2 u.$$

Now, concerning the hamiltonian parts, we have for the bid part ($q \neq Q$):

$$\sup_{\delta^b} \lambda^b(\delta^b)\left[u(t,x-s+\delta^b,q+1,s) - u(t,x,q,s)\right]$$

$$= \sup_{\delta^b} A e^{-k\delta^b} u(t,x,q,s)\left[\exp(-\gamma\delta^b)\left(\frac{v_{q+1}(t)}{v_q(t)}\right)^{-\frac{\gamma}{k}} - 1\right].$$

The first order condition of this problem corresponds to a maximum (because u is negative) and writes:

$$(k+\gamma)\exp(-\gamma\delta^{b*})\left(\frac{v_{q+1}(t)}{v_q(t)}\right)^{-\frac{\gamma}{k}} = k.$$

Hence:

$$\delta^{b*} = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q+1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

and

$$\begin{aligned}\sup_{\delta^b} \lambda^b(\delta^b)\left[u(t,x-s+\delta^b,q+1,s) - u(t,x,q,s)\right] &= -\frac{\gamma}{k+\gamma} A \exp(-k\delta^{b*}) u(t,x,q,s) \\ &= -\frac{\gamma A}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{-\frac{k}{\gamma}} \frac{v_{q+1}(t)}{v_q(t)} u(t,x,q,s).\end{aligned}$$

Similarly, the maximizer for the ask part (for $q \neq -Q$) is:

$$\delta^{a*} = \frac{1}{k}\ln\left(\frac{v_q(t)}{v_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

and

$$\sup_{\delta^a} \lambda^a(\delta^a)\left[u(t,x+s+\delta^a,q-1,s) - u(t,x,q,s)\right]$$
$$= -\frac{\gamma}{k+\gamma}A\exp(-k\delta^{a*})u(t,x,q,s) = -\frac{\gamma A}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{-\frac{k}{\gamma}}\frac{v_{q-1}(t)}{v_q(t)}u(t,x,q,s).$$

Hence, putting the terms altogether we get for $|q| < Q$:

$$\partial_t u(t,x,q,s) + \frac{1}{2}\sigma^2\partial^2_{ss}u(t,x,q,s)$$
$$+\sup_{\delta^b}\lambda^b(\delta^b)\left[u(t,x-s+\delta^b,q+1,s) - u(t,x,q,s)\right]$$
$$+\sup_{\delta^a}\lambda^a(\delta^a)\left[u(t,x+s+\delta^a,q-1,s) - u(t,x,q,s)\right]$$
$$= -\frac{\gamma}{k}\frac{\dot{v}_q(t)}{v_q(t)}u + \frac{\gamma^2\sigma^2}{2}q^2u - \frac{\gamma A}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{\frac{k}{\gamma}}\left[\frac{v_{q+1}(t)}{v_q(t)} + \frac{v_{q-1}(t)}{v_q(t)}\right]u$$
$$= -\frac{\gamma}{k}\frac{u}{v_q(t)}\left[\dot{v}_q(t) - \frac{k\gamma\sigma^2}{2}q^2v_q(t) + A\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)}\left(v_{q+1}(t)+v_{q-1}(t)\right)\right] = 0.$$

For $q = -Q$ we have:

$$\partial_t u(t,x,q,s) + \frac{1}{2}\sigma^2\partial^2_{ss}u(t,x,q,s)$$
$$+\sup_{\delta^b}\lambda^b(\delta^b)\left[u(t,x-s+\delta^b,q+1,s) - u(t,x,q,s)\right]$$
$$= -\frac{\gamma}{k}\frac{\dot{v}_q(t)}{v_q(t)}u + \frac{\gamma^2\sigma^2}{2}q^2u - \frac{\gamma A}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{\frac{k}{\gamma}}\frac{v_{q+1}(t)}{v_q(t)}u$$
$$= -\frac{\gamma}{k}\frac{u}{v_q(t)}\left[\dot{v}_q(t) - \frac{k\gamma\sigma^2}{2}q^2v_q(t) + A\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)}v_{q+1}(t)\right] = 0.$$

Similarly, for $q = Q$ we have:

$$\partial_t u(t,x,q,s) + \frac{1}{2}\sigma^2\partial^2_{ss}u(t,x,q,s)$$
$$+\sup_{\delta^a}\lambda^a(\delta^a)\left[u(t,x-s+\delta^a,q+1,s) - u(t,x,q,s)\right]$$
$$= -\frac{\gamma}{k}\frac{\dot{v}_q(t)}{v_q(t)}u + \frac{\gamma^2\sigma^2}{2}q^2u - \frac{\gamma A}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{\frac{k}{\gamma}}\frac{v_{q-1}(t)}{v_q(t)}u$$
$$= -\frac{\gamma}{k}\frac{u}{v_q(t)}\left[\dot{v}_q(t) - \frac{k\gamma\sigma^2}{2}q^2v_q(t) + A\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)}v_{q-1}(t)\right] = 0.$$

Now, noticing that the terminal condition for v_q is consistent with the terminal condition for u, we get that u verifies (HJB) and this proves Proposition 1.

The positivity of the functions $(v_q)_{|q|\leq Q}$ was essential in the definition of u. Hence we need to prove that the solution to the above linear system of ordinary differential equations $v(t) = \exp(-M(T-t)) \times (1,\dots,1)'$ (where M is given in Proposition 2), defines a family $(v_q)_{|q|\leq Q}$ of positive functions.

In fact, we are going to prove that:

$$\forall t \in [0,T], \forall q \in \{-Q,\dots,Q\}, \quad v_q(t) \geq e^{-(\alpha Q^2-\eta)(T-t)}.$$

If this was not true, then there would exist $\varepsilon > 0$ such that:

$$\min_{t\in[0,T],|q|\leq Q} e^{-2\eta(T-t)}\left(v_q(t) - e^{-(\alpha Q^2-\eta)(T-t)}\right) + \varepsilon(T-t) < 0.$$

But this minimum is achieved at some point (t^*, q^*) with $t^* < T$ and hence:

$$\frac{d}{dt}\, e^{-2\eta(T-t)}\left(v_{q^*}(t) - e^{-(\alpha Q^2-\eta)(T-t)}\right)\Big|_{t=t^*} \geq \varepsilon.$$

This gives:

$$2\eta e^{-2\eta(T-t^*)}\left(v_{q^*}(t^*) - e^{-(\alpha Q^2-\eta)(T-t^*)}\right)$$
$$+e^{-2\eta(T-t^*)}\left(v'_{q^*}(t^*) - (\alpha Q^2-\eta)e^{-(\alpha Q^2-\eta)(T-t^*)}\right) \geq \varepsilon.$$

Hence:

$$2\eta v_{q^*}(t^*) + v'_{q^*}(t^*) - (\eta+\alpha Q^2)e^{-(\alpha Q^2-\eta)(T-t^*)} \geq \varepsilon e^{2\eta(T-t^*)}.$$

Now, if $|q^*| < Q$, this gives:

$$\alpha {q^*}^2 v_{q^*}(t^*) - \eta(v_{q^*+1}(t^*) - 2v_{q^*}(t^*) + v_{q^*-1}(t^*))$$
$$-(\eta+\alpha Q^2)e^{-(\alpha Q^2-\eta)(T-t^*)} \geq \varepsilon e^{2\eta(T-t^*)}.$$

Thus:

$$\alpha {q^*}^2\left(v_{q^*}(t^*) - e^{-(\alpha Q^2-\eta)(T-t^*)}\right) - \eta(v_{q^*+1}(t^*) - 2v_{q^*}(t^*) + v_{q^*-1}(t^*))$$
$$-(\eta+\alpha(Q^2-{q^*}^2))e^{-(\alpha Q^2-\eta)(T-t^*)} \geq \varepsilon e^{2\eta(T-t^*)}.$$

All the terms on the left hand side are nonpositive by definition of (t^*, q^*) and this gives a contradiction.

If $q^* = Q$, we have:

$$(\alpha Q^2+\eta)v_Q(t^*) - \eta(v_{Q-1}(t^*) - v_Q(t^*)) - (\eta+\alpha Q^2)e^{-(\alpha Q^2-\eta)(T-t^*)} \geq \varepsilon e^{2\eta(T-t^*)}.$$

Thus:

$$-\eta(v_{Q-1}(t^*) - v_Q(t^*)) + (\eta + \alpha Q^2)\left(v_Q(t^*) - e^{-(\alpha Q^2-\eta)(T-t^*)}\right) \geq \varepsilon e^{2\eta(T-t^*)}.$$

All the terms on the left hand side are nonpositive by definition of $(t^*, q^*) = (t^*, Q)$ and this gives a contradiction.

Similarly, if $q^* = -Q$, we have:

$$(\alpha Q^2 + \eta)v_{-Q}(t^*) - \eta(v_{-Q+1}(t^*) - v_Q(t^*)) - (\eta + \alpha Q^2)e^{-(\alpha Q^2-\eta)(T-t^*)} \geq \varepsilon e^{2\eta(T-t^*)}$$

$$-\eta(v_{-Q+1}(t^*) - v_{-Q}(t^*)) + (\eta + \alpha Q^2)\left(v_{-Q}(t^*) - e^{-(\alpha Q^2-\eta)(T-t^*)}\right) \geq \varepsilon e^{2\eta(T-t^*)}$$

All the terms on the left hand side are nonpositive by definition of $(t^*, q^*) = (t^*, -Q)$ and this gives a contradiction.

As a consequence, $v_q(t) \geq e^{-(\alpha Q^2-\eta)(T-t)} > 0$ and this completes the proof of Proposition 2.

Combining the above results, we see that u, as defined in Theorem 1, is a solution of (HJB). Then, we are going to use a verification argument to prove that u is the value function of the optimal control problem under consideration and prove subsequently that the optimal controls are as given in Theorem 1.

Let us consider processes (ν^b) and $(\nu^a) \in \mathcal{A}$. Let $t \in [0, T)$ and let us consider the following processes for $\tau \in [t, T]$:

$$\begin{aligned}
dS_\tau^{t,s} &= \sigma dW_\tau, \qquad S_t^{t,s} = s, \\
dX_\tau^{t,x,\nu} &= (S_\tau + \nu_\tau^a)dN_\tau^a - (S_\tau - \nu_\tau^b)dN_\tau^b, \qquad X_t^{t,x,\nu} = x, \\
dq_\tau^{t,q,\nu} &= dN_\tau^b - dN_\tau^a, \qquad q_t^{t,q,\nu} = q.
\end{aligned}$$

where the point process N^b has intensity $(\lambda_\tau^b)_\tau$ with $\lambda_\tau^b = Ae^{-k\nu_\tau^b}1_{q_{\tau-}<Q}$ and where the point process N^a has intensity $(\lambda_\tau^a)_\tau$ with $\lambda_\tau^a = Ae^{-k\nu_\tau^a}1_{q_{\tau-}>-Q}$ [4].

Now, since u is smooth, let us write Itô's formula for u, between t and t_n where $t_n = T \wedge \inf\{\tau > t, |S_\tau - s| \geq n \text{ or } |N_\tau^a - N_t^a| \geq n \text{ or } |N_\tau^b - N_t^b| \geq n\}$ $(n \in \mathbb{N})$:

$$\begin{aligned}
u(t_n, X_{t_n-}^{t,x,\nu}, q_{t_n-}^{t,q,\nu}, S_{t_n}^{t,s}) &= u(t,x,q,s) \\
&+ \int_t^{t_n} \left(\partial_\tau u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \frac{\sigma^2}{2}\partial_{ss}^2 u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) d\tau \\
&+ \int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau^a, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) \lambda_\tau^a d\tau \\
&+ \int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} - S_\tau^{t,s} + \nu_\tau^b, q_{\tau-}^{t,q,\nu} + 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) \lambda_\tau^b d\tau \\
&+ \int_t^{t_n} \sigma \partial_s u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) dW_\tau
\end{aligned}$$

[4]These intensities are bounded since ν^b and ν^a are bounded from below.

$$+\int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau^a, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) dM_\tau^a$$
$$+\int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} - S_\tau^{t,s} + \nu_\tau^b, q_{\tau-}^{t,q,\nu} + 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) dM_\tau^b$$

where M^b and M^a are the compensated processes associated respectively to N^b and N^a for the intensity processes $(\lambda_\tau^b)_\tau$ and $(\lambda_\tau^a)_\tau$.

Now, because each v_q is continuous and positive on a compact set, it has a positive lower bound and $v_{q_\tau}(\tau)^{-\frac{\gamma}{k}}$ is bounded along the trajectory, independently of the trajectory. Also, because ν^b and ν^a are bounded from below, and because of the definition of t_n, all the terms in the above stochastic integrals are bounded and, local martingales being in fact martingales, we have:

$$\mathbb{E}\left[u(t_n, X_{t_n-}^{t,x,\nu}, q_{t_n-}^{t,q,\nu}, S_{t_n}^{t,s})\right] = u(t,x,q,s)$$
$$+\mathbb{E}\left[\int_t^{t_n} \left(\partial_\tau u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \frac{\sigma^2}{2}\partial_{ss}^2 u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) d\tau\right.$$
$$+\int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau^a, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) \lambda_\tau^a d\tau$$
$$\left.+\int_t^{t_n} \left(u(\tau, X_{\tau-}^{t,x,\nu} - S_\tau^{t,s} + \nu_\tau^b, q_{\tau-}^{t,q,\nu} + 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right) \lambda_\tau^b d\tau\right]$$

Using the fact that u solves (HJB), we then have that

$$\mathbb{E}\left[u(t_n, X_{t_n-}^{t,x,\nu}, q_{t_n-}^{t,q,\nu}, S_{t_n}^{t,s})\right] \leq u(t,x,q,s)$$

with equality when the controls are taken equal the maximizers of the hamiltonians (these controls being in $\mathcal{A}$ because v is bounded and has a positive lower bound).

Now, if we prove that

$$\lim_{n\to\infty} \mathbb{E}\left[u(t_n, X_{t_n-}^{t,x,\nu}, q_{t_n-}^{t,q,\nu}, S_{t_n}^{t,s})\right] = \mathbb{E}\left[u(T, X_T^{t,x,\nu}, q_T^{t,q,\nu}, S_T^{t,s})\right]$$

we will have that for all controls in $\mathcal{A}$:

$$\mathbb{E}\left[-\exp\left(-\gamma(X_T^{t,x,\nu} + q_T^{t,q,\nu} S_T^{t,s})\right)\right] = \mathbb{E}\left[u(T, X_T^{t,x,\nu}, q_T^{t,q,\nu}, S_T^{t,s})\right] \leq u(t,x,q,s)$$

with equality for $\nu_t^b = \delta^{b*}(t, q_{t-})$ and $\nu_t^a = \delta^{a*}(t, q_{t-})$. Hence:

$$\sup_{(\nu_t^a)_t, (\nu_t^b)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T^{t,x,\nu} + q_T^{t,q,\nu} S_T^{t,s})\right)\right] = u(t,x,q,s)$$
$$= \mathbb{E}\left[-\exp\left(-\gamma(X_T^{t,x,\delta^*} + q_T^{t,q,\delta^*} S_T^{t,s})\right)\right]$$

and this will give the result.

It remains to prove that

$$\lim_{n\to\infty} \mathbb{E}\left[u(t_n, X_{t_n-}^{t,x,\nu}, q_{t_n-}^{t,q,\nu}, S_{t_n}^{t,s})\right] = \mathbb{E}\left[u(T, X_T^{t,x,\nu}, q_T^{t,q,\nu}, S_T^{t,s})\right].$$

First, we have, almost surely, that $u(t_n, X^{t,x,\nu}_{t_n-}, q^{t,q,\nu}_{t_n-}, S^{t,s}_{t_n})$ tends towards $u(T, X^{t,x,\nu}_{T-}, q^{t,q,\nu}_{T-}, S^{t,s}_{T})$. Then, in order to prove that the sequence is uniformly integrable we will bound it in L^2. However, because of the uniform lower bound on v already used early, it is sufficient to bound $\exp(-\gamma(X^{t,x,\nu}_{t_n-} + q^{t,q,\nu}_{t_n-} S^{t,s}_{t_n}))$ in L^2.

But,

$$X^{t,x,\nu}_{t_n-} + q^{t,q,\nu}_{t_n-} S^{t,s}_{t_n} = \int_t^{t_n} \nu^a_\tau dN^a_\tau + \int_t^{t_n} \nu^b_\tau dN^b_\tau + \sigma \int_t^{t_n} q^{t,q,\nu}_\tau dW_\tau$$
$$\geq -\|\nu^a_-\|_\infty N^a_T - \|\nu^b_-\|_\infty N^b_T + \sigma \int_t^{t_n} q^{t,q,\nu}_\tau dW_\tau.$$

Hence

$$\begin{aligned}
& \mathbb{E}\left[\exp(-2\gamma(X^{t,x,\nu}_{t_n-} + q^{t,q,\nu}_{t_n-} S^{t,s}_{t_n}))\right] \\
\leq\; & \mathbb{E}\left[\exp\left(2\gamma\|\nu^a_-\|_\infty N^a_T\right)\exp\left(2\gamma\|\nu^a_-\|_\infty N^b_T\right)\exp\left(-2\gamma\sigma\int_t^{t_n} q^{t,q,\nu}_\tau dW_\tau\right)\right] \\
\leq\; & \mathbb{E}\left[\exp\left(6\gamma\|\nu^a_-\|_\infty N^a_T\right)\right]^{\frac{1}{3}} \mathbb{E}\left[\exp\left(6\gamma\|\nu^b_-\|_\infty N^b_T\right)\right]^{\frac{1}{3}} \\
& \times\mathbb{E}\left[\exp\left(-6\gamma\sigma\int_t^{t_n} q^{t,q,\nu}_\tau dW_\tau\right)\right]^{\frac{1}{3}}.
\end{aligned}$$

Now, since the intensity of each point process is bounded, the point processes have a Laplace transform and the first two terms of the product are finite (and independent of n). Concerning the third term, because $|q^{t,q,\nu}_\tau|$ is bounded by Q, we know (for instance applying Girsanov's theorem) that:

$$\mathbb{E}\left[\exp\left(-6\gamma\sigma\int_t^{t_n} q^{t,q,\nu}_\tau dW_\tau\right)\right]^{\frac{1}{3}} \leq \mathbb{E}\left[\exp\left(3\gamma^2\sigma^2(t_n-t)Q^2\right)\right]^{\frac{1}{3}}$$
$$\leq \exp\left(\gamma^2\sigma^2 Q^2 T\right).$$

Hence, the sequence is bounded in L^2, then uniformly integrable and we have:

$$\lim_{n\to\infty} \mathbb{E}\left[u(t_n, X^{t,x,\nu}_{t_n-}, q^{t,q,\nu}_{t_n-}, S^{t,s}_{t_n})\right] = \mathbb{E}\left[u(T, X^{t,x,\nu}_{T-}, q^{t,q,\nu}_{T-}, S^{t,s}_{T})\right]$$
$$= \mathbb{E}\left[u(T, X^{t,x,\nu}_{T}, q^{t,q,\nu}_{T}, S^{t,s}_{T})\right].$$

We have proved that u is the value function and that δ^{b*} and δ^{a*} are optimal controls.

□

Proof of Theorem 2:

Let us first consider the matrix $M+2\eta I$. This matrix is symmetric and it is therefore diagonalizable. Its smallest eigenvalue λ is characterized by:

$$\lambda = \inf_{x\in\mathbb{R}^{2Q+1}\setminus\{0\}} \frac{x'(M+2\eta I)x}{x'x}$$

and the associated eigenvectors $x \neq 0$ are characterized by:

$$\lambda = \frac{x'(M+2\eta I)x}{x'x}$$

It is straightforward to see that:

$$x'(M+2\eta I)x = \sum_{q=-Q}^{Q} \alpha q^2 {x_q}^2 + \eta \sum_{q=-Q}^{Q-1} (x_{q+1}-x_q)^2 + \eta {x_Q}^2 + \eta {x_{-Q}}^2$$

Hence, if x is an eigenvector of $M+2\eta I$ associated to λ:

$$\begin{aligned}
\lambda &\leq \frac{|x|'(M+2\eta I)|x|}{|x|'|x|} \\
&= \frac{1}{|x|'|x|}\left[\sum_{q=-Q}^{Q} \alpha q^2 |x_q|^2 + \eta \sum_{q=-Q}^{Q-1} (|x_{q+1}|-|x_q|)^2 + \eta|x_Q|^2 + \eta|x_{-Q}|^2\right] \\
&\leq \frac{1}{|x|'|x|}\left[\sum_{q=-Q}^{Q} \alpha q^2 |x_q|^2 + \eta \sum_{q=-Q}^{Q-1} (x_{q+1}-x_q)^2 + \eta|x_Q|^2 + \eta|x_{-Q}|^2\right] = \lambda.
\end{aligned}$$

This proves that $|x|$ (componentwise) is also an eigenvector and that necessarily x_{q+1} and x_q are of the same sign (*i.e.* $x_q x_{q+1} \geq 0$).

Now, let $x \geq 0$ (componentwise) be an eigenvector of $M+2\eta I$ associated to λ.

If for some q with $|q| < Q$ we have $x_q = 0$ then:

$$0 = \lambda x_q = \alpha q^2 x_q - \eta(x_{q+1} - 2x_q + x_{q-1}) = -\eta(x_{q+1}+x_{q-1}) \leq 0$$

Hence, because $x \geq 0$, both x_{q+1} and x_{q-1} are equal to 0. By immediate induction $x = 0$ and this yields a contradiction.

Now, if $x_Q = 0$, then $0 = \lambda x_Q = \alpha Q^2 x_Q - \eta(-2x_Q + x_{Q-1}) = -\eta x_{Q-1} \leq 0$ and hence $x_{Q-1} = 0$. Then, by the preceding reasoning we obtain a contradiction.

Similarly if $x_{-Q} = 0$, then $0 = \lambda x_{-Q} = \alpha Q^2 x_{-Q} - \eta(x_{-Q+1} - 2x_{-Q}) = -\eta x_{-Q+1} \leq 0$ and hence $x_{-Q+1} = 0$. Then, as above, we obtain a contradiction.

This proves that any eigenvector $x \geq 0$ of $M+2\eta I$ associated to λ verifies in fact $x > 0$.

Now, if the eigenvalue λ was not simple, there would exist two eigenvectors x and y of $M+2\eta I$ associated to λ such that $|x|'y = 0$. Hence, y must have positive coordinates and negative coordinates and since $y_q y_{q+1} \geq 0$, we know that there must exist q such that $y_q = 0$. However, this contradicts our preceding point since $|y| \geq 0$ should also be an eigenvector of $M+2\eta I$ associated to λ and it cannot have therefore coordinates equal to 0.

As a conclusion, the eigenspace of $M+2\eta I$ associated to λ is spanned by a vector $f^0 > 0$ and we scaled its $\mathbb{R}^{2Q+1}$-norm to 1.

Now, because M is a symmetric matrix, we can write $v(0) = \exp(-MT) \times (1,\ldots,1)'$ as:

$$v_q(0) = \sum_{i=0}^{2Q} \exp(-\lambda^i T)\langle g^i, (1,\ldots,1)'\rangle g^i_q, \qquad \forall q \in \{-Q,\ldots,Q\}$$

where $\lambda^0 \leq \lambda^1 \leq \ldots \leq \lambda^{2Q}$ are the eigenvalues of M (in increasing order and repeated if necessary) and $(g^i)_i$ an associated orthonormal basis of eigenvectors. Clearly, we can take $g^0 = f^0$. Then, both f^0_q and $\langle f^0, (1,\ldots,1)'\rangle$ are positive and hence different from zero. As a consequence:

$$v_q(0) \sim_{T\to+\infty} \exp(-\lambda^0 T)\langle f^0, (1,\ldots,1)'\rangle f^0_q, \qquad \forall q \in \{-Q,\ldots,Q\}$$

Then, using the expressions for the optimal quotes, we get:

$$\lim_{T\to+\infty} \delta^{b*}(0,q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f^0_q}{f^0_{q+1}}\right)$$

$$\lim_{T\to+\infty} \delta^{a*}(0,q) = \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right) + \frac{1}{k}\ln\left(\frac{f^0_q}{f^0_{q-1}}\right)$$

Turning to the characterization of f^0 stated in Theorem 2, we just need to write the Rayleigh ratio associated to the smallest eigenvalue of $M+2\eta I$:

$$f^0 \in \underset{f\in\mathbb{R}^{2Q+1},\|f\|_2=1}{\operatorname{argmin}} f'(M+2\eta I)f$$

Equivalently:

$$f^0 \in \underset{f\in\mathbb{R}^{2Q+1},\|f\|_2=1}{\operatorname{argmin}} \sum_{q=-Q}^{Q} \alpha q^2 {f_q}^2 + \eta \sum_{q=-Q}^{Q-1} (f_{q+1}-f_q)^2 + \eta {f_Q}^2 + \eta {f_{-Q}}^2$$

□

Proof of Proposition 3:

Let us first introduce $H = \{u \in L^1_{loc}(\mathbb{R})/x \mapsto xu(x) \in L^2(\mathbb{R}) \text{ and } u' \in L^2(\mathbb{R})\}$. H equipped with the norm $\|u\|_H = \sqrt{\int_{\mathbb{R}} (\alpha x^2 u(x)^2 + \eta u'(x)^2)\, dx}$ is an Hilbert space.

Step 1: $H \subset L^2(\mathbb{R})$ with continuous injection.

Let us consider $u \in H$ and $\varepsilon > 0$.

We have:

$$\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} u(x)^2 dx \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} x^2 u(x)^2 dx < +\infty.$$

Hence because $u' \in L^2(\mathbb{R})$, we have $u \in H^1(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ with a constant C_ε independent of u such that $\|u\|_{H^1(\mathbb{R}\setminus[-\varepsilon,\varepsilon])} \leq C_\varepsilon \|u\|_H$. In particular u is continuous on $\mathbb{R}^*$.

Now, if $\varepsilon = 1$, $\forall x \in (0,1), u(x) = u(1) - \int_x^1 u'(t)dt$ and then $|u(x)| \leq |u(1)| + \sqrt{1-x}\|u'\|_{L^2((0,1))}$.

Because the injection of $H^1((1,+\infty))$ in $C([1,+\infty))$ is continuous, we know that there exists a constant C independent of u such that $|u(1)| \leq C\|u\|_{H^1((1,+\infty))}$. Hence, there exists a constant C' such that $|u(1)| \leq C'\|u\|_H$ and eventually a constant C'' such that $\|u\|_{L^\infty((0,1))} \leq C''\|u\|_H$. Similarly, we obtain $\|u\|_{L^\infty((-1,0))} \leq C''\|u\|_H$.

Combining the above inequalities we obtain a new constant K so that $\|u\|_{L^2(\mathbb{R})} \leq K\|u\|_H$.

□

A consequence of this first step is that $H \subset H^1(\mathbb{R}) \subset C(\mathbb{R})$.

Step 2: The injection $H \hookrightarrow L^2(\mathbb{R})$ is compact.

Let us consider a sequence $(u_n)_n$ of functions in H with $\sup_n \|u_n\|_H < +\infty$.

Because $H \subset H^1(\mathbb{R})$, $\forall m \in \mathbb{N}^*$, we can extract from $(u_n)_n$ a sequence that converges in $L^2((-m,m))$. Using then a diagonal extraction, there exists a subsequence of $(u_n)_n$, still denoted $(u_n)_n$, and a function $u \in L^2_{loc}(\mathbb{R})$ such that $u_n(x) \to u(x)$ for almost every $x \in \mathbb{R}$ and $u_n \to u$ in the $L^2_{loc}(\mathbb{R})$ sense.

Now, by Fatou's lemma:

$$\int_{\mathbb{R}} x^2 u(x)^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} x^2 u_n(x)^2 dx \leq \frac{\sup_n \|u_n\|_H^2}{\alpha}.$$

Hence, there exists a constant C such that $\forall m \in \mathbb{N}^*$:

$$\begin{aligned}\int_{\mathbb{R}} |u(x) - u_n(x)|^2 dx &\leq \int_{-m}^{m} |u(x) - u_n(x)|^2 dx + \frac{1}{m^2} \int_{\mathbb{R}\setminus[-m,m]} x^2 |u(x) - u_n(x)|^2 dx \\ &\leq \int_{-m}^{m} |u(x) - u_n(x)|^2 dx + \frac{C}{m^2}.\end{aligned}$$

Hence $\limsup_{n\to\infty} \int_{\mathbb{R}} |u(x) - u_n(x)|^2 dx \leq \frac{C}{m^2}$.

Sending m to $+\infty$ we get:

$$\limsup_{n\to\infty} \int_{\mathbb{R}} |u(x) - u_n(x)|^2 dx = 0.$$

Hence $(u_n)_n$ converges towards u in the $L^2(\mathbb{R})$ sense.

□

Now, we consider the equation $-\eta u''(x) + \alpha x^2 u(x) = f(x)$ for $f \in L^2(\mathbb{R})$ and we define $u = Lf$ the weak solution of this equation, *i.e.*:

$$\forall v \in H, \int_{\mathbb{R}} \left(\alpha x^2 u(x)v(x) + \eta u'(x)v'(x)\right) dx = \int_{\mathbb{R}} f(x)v(x)dx.$$

Step 3: $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well defined linear operator, compact, positive and self-adjoint.

For $f \in L^2(\mathbb{R})$, $v \in H \mapsto \int_{\mathbb{R}} f(x)v(x)dx$ is a continuous linear form on H because the injection $H \hookrightarrow L^2(\mathbb{R})$ is continuous. Hence, by Lax-Milgram or Riesz's representation theorem, there exists a unique $u \in H$ weak solution of the above equation and L is a well defined linear operator.

Now, $\|Lf\|_H^2 = \langle f, Lf\rangle \le \|f\|_{L^2(\mathbb{R})}\|Lf\|_{L^2(\mathbb{R})}$. Hence, since the injection $H \hookrightarrow L^2(\mathbb{R})$ is continuous, there exists a constant C such that $\|Lf\|_H^2 \le C\|f\|_{L^2(\mathbb{R})}\|Lf\|_H$, which in turn gives $\|Lf\|_H \le C\|f\|_{L^2(\mathbb{R})}$. Since the injection $H \hookrightarrow L^2(\mathbb{R})$ is compact, we obtain that L is a compact operator.

L is a positive operator because $\langle f, Lf\rangle = \|Lf\|_H^2 \ge 0$.

Eventually, L is self-adjoint because $\forall f, g \in L^2(\mathbb{R})$:

$$\begin{aligned}
\langle f, Lg\rangle &= \int_{\mathbb{R}} \left(\alpha x^2 Lf(x)Lg(x) + \eta(Lf)'(x)(Lg)'(x)\right) dx \\
&= \int_{\mathbb{R}} \left(\alpha x^2 Lg(x)Lf(x) + \eta(Lg)'(x)(Lf)'(x)\right) dx = \langle g, Lf\rangle
\end{aligned}$$

□

Now, using the spectral decomposition of L and classical results on Rayleigh ratios we know that the eigenfunctions f corresponding to the largest eigenvalue λ^0 of L satisfy:

$$\frac{1}{\lambda^0} = \frac{\|f\|_H}{\|f\|_{L^2(\mathbb{R})}} = \inf_{g\in H\setminus\{0\}} \frac{\|g\|_H}{\|g\|_{L^2(\mathbb{R})}}.$$

Hence, our problem boils down to proving that the largest eigenvalue of L is simple and that $g : x \mapsto \exp\left(-\frac{1}{2}\sqrt{\frac{\alpha}{\eta}}x^2\right)$ is an eigenfunction corresponding to this eigenvalue (it is straightforward that $g \in H$).

Step 4: Any positive eigenfunction corresponds to the largest eigenvalue of L.

By definition of $\|\cdot\|_H$, $\forall f \in H$, $\frac{\||f|\|_H}{\||f|\|_{L^2(\mathbb{R})}} = \frac{\|f\|_H}{\|f\|_{L^2(\mathbb{R})}}$. Hence, if f is an eigenfunction of L corresponding to the eigenvalue λ^0, then $|f|$ is also an eigenfunction of L corresponding to the eigenvalue λ^0. Now, if $\tilde{f}$ is an eigenfunction of L corresponding to an eigenvalue

$\lambda \neq \lambda^0$, $\langle |f|, \tilde{f} \rangle = 0$. Therefore $\tilde{f}$ cannot be positive.

$\square$

Step 5: g spans the eigenspace corresponding to the largest eigenvalue of L.

Differentiating g twice, we get $g''(x) = -\sqrt{\frac{\alpha}{\eta}}g(x) + \frac{\alpha}{\eta}x^2 g(x)$.

Hence $-\eta g''(x) + \alpha x^2 g(x) = \sqrt{\alpha\eta}g(x)$ and g is a positive eigenfunction, necessarily associated to the eigenvalue λ^0 that is therefore equal to $\frac{1}{\sqrt{\alpha\eta}}$.

Now, if we look for an eigenfunction $f \in C^\infty(\mathbb{R}) \cap H$ – because any eigenfunction of L is in $C^\infty(\mathbb{R})$ – we can look for f of the form $f = gh$. This gives:

$$\begin{aligned} 0 &= -\eta f''(x) + \alpha x^2 f(x) - \sqrt{\alpha\eta} f(x) \\ &= -\eta\left(g''(x)h(x) + 2g'(x)h'(x) + g(x)h''(x)\right) + \alpha x^2 g(x)h(x) - \sqrt{\alpha\eta}g(x)h(x). \end{aligned}$$

Hence:

$$\begin{aligned} 0 &= 2g'(x)h'(x) + g(x)h''(x) = -2x\sqrt{\frac{\alpha}{\eta}}g(x)h'(x) + g(x)h''(x) \\ &\Rightarrow\ h''(x) = 2x\sqrt{\frac{\alpha}{\eta}}h'(x) \\ &\Rightarrow\ \exists K_1, \quad h'(x) = K_1 \exp\left(\sqrt{\frac{\alpha}{\eta}}x^2\right) \\ &\Rightarrow\ \exists K_1, K_2, \quad h(x) = K_1 \int_0^x \exp\left(\sqrt{\frac{\alpha}{\eta}}t^2\right) dt + K_2 \\ &\Rightarrow\ \exists K_1, K_2, \quad f(x) = K_1 g(x) \int_0^x \exp\left(\sqrt{\frac{\alpha}{\eta}}t^2\right) dt + K_2 g(x). \end{aligned}$$

Now,

$$\begin{aligned} g(x)\int_0^x \exp\left(\sqrt{\frac{\alpha}{\eta}}t^2\right) dt &\geq \exp\left(-\frac{1}{2}\sqrt{\frac{\alpha}{\eta}}x^2\right)\int_{\frac{x}{\sqrt{2}}}^x \exp\left(\sqrt{\frac{\alpha}{\eta}}t^2\right) dt \\ &\geq x\left(1 - \frac{1}{\sqrt{2}}\right). \end{aligned}$$

Hence, for f to be in H, we must have $K_1 = 0$. Thus, g spans the eigenspace corresponding to the largest eigenvalue of L and Proposition 3 is proved.

$\square$

Chapter 3

Application: Optimal Liquidation with Limit-Orders

3.1 Introduction

This chapter addresses portfolio liquidation using a new angle. Instead of focusing only on the scheduling aspect like Almgren and Chriss in [10], or only on the liquidity-consuming orders like Obizhaeva and Wang in [115], we link the optimal trade-schedule to the price of the limit orders that have to be sent to the limit order book to optimally liquidate a portfolio. The idea is to use our results from the precedent chapter, and consider the optimal-liquidation problem as a *one-sided market-making*. From a practical standpoint this approach is not necessarily contradictory with optimal-scheduling, and it can be seen as representing the tactical part (interaction with the market), while optimal-scheduling represent the strategic part (intraday pace in order to meet the benchmark). We want to emphasize that this work was, to our knowledge (and in parallel by Bayraktar and Ludkovski [18] in a risk-neutral model) the first attempt to solve the liquidation problem through limit-orders published in the literature. The result was published in the article in collaboration with O. Guéant and C-A. Lehalle [64].

In our framework (inspired from the Avellaneda-Stoikov model), the flow of trades "hitting" a passive order at a distance δ^a_t from a reference price S_t – modeled by a Brownian motion – follows an adapted point process of intensity $A\exp(-k\delta^a_t)$. It means that the further away from the "fair price" an order is posted, the less transactions it will obtain. In practice, if the limit order price is far above the best ask price, the trading gain may be high but execution is far from being guaranteed and the broker is exposed to the risk of a price decrease. On the contrary, if the limit order price is near the best ask price, or even reduces the market bid-ask spread, gains will be small but the probability of execution will be higher, resulting in faster trading and less price risk. As in the precedent chapter, by defining a HJB equation and then a suitable change of variable, we obtain analytical formulas enabling an in-depth analysis of the solutions.

The remainder of this chapter is organized as follows. We start by presenting the setting of the model and its main hypotheses. Then, via a change of variables, we solve the HJB partial differential equation arising from the control problem. We study three special cases: (i) the time-asymptotic case, (ii) the absence of price risk and the risk-neutral case, and (iii) a limiting case in which the trader has a large incentive to liquidate before the end. These special cases provide simple closed-form formulae allowing us to better understand the forces at stake. As in the market-making situation, we carry out comparative statics and discuss the way optimal strategies depend on the model parameters. Finally we show how our approach can be used in practice for optimal liquidation, both on a long period of time, to solve the entire liquidation problem, and on slices of 5 minutes, when one wants to follow a predetermined trading curve. We provide the proof of the main mathematical results at the very end of the chapter.

3.2 Setup of the model

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume that all random variables and stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We consider a trader who has to liquidate a portfolio containing a large quantity q_0 of a given stock. We suppose that the reference price of the stock (which can be considered the mid-price or the best bid quote for example) moves as a brownian motion with a drift:

$$dS_t = \mu dt + \sigma dW_t.$$

The trader under consideration will continuously propose an ask quote[1] denoted $S_t^a = S_t + \delta_t^a$ and will hence sell shares according to the rate of arrival of aggressive orders at the prices he quotes.

His inventory q, that is the quantity he holds, is given by $q_t = q_0 - N_t^a$ where N^a is the jump process counting the number of shares he sold[2]. We assume that jumps are of unitary size and it is important to notice that 1 share may be understood as 1 bunch of shares, each bunch being of the same size. Arrival rates obviously depend on the price S_t^a quoted by the trader and we assume that intensity λ^a associated to N^a is of the following form:

$$\lambda^a(\delta^a) = A\exp(-k\delta^a) = A\exp(-k(s^a - s)).$$

This means that the lower the order price, the faster it will be executed.

Thus, the trader has an amount of cash whose dynamics is given by:

$$dX_t = (S_t + \delta_t^a)dN_t^a.$$

[1] In what follows, we will often call δ_t^a the quote instead of S_t^a.
[2] Once the whole portfolio is liquidated, we assume that the trader remains inactive.

The trader has a time horizon T to liquidate the portfolio and his goal is to optimize the expected utility of his P&L at time T. We will focus on CARA utility functions and we suppose that the trader optimizes:

$$\sup_{(\delta_t^a)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T(S_T - b))\right)\right]$$

where $\mathcal{A}$ is the set of predictable processes on [0,T], bounded from below, where γ is the absolute risk aversion characterizing the trader, where X_T is the amount of cash at time T, where q_T is the remaining quantity of shares in the inventory at time T and where b is a cost (per share) one has to incur to liquidate the remaining quantity at time T.

3.3 Optimal quotes

3.3.1 Hamilton-Jacobi-Bellman equation

The optimization problem set up in the preceding section can be solved using classical Bellman tools. To this purpose, we introduce the Hamilton-Jacobi-Bellman equation associated to the optimization problem, where u is an unknown function that is going to be the value function of the control problem:

$$\text{(HJB)} \qquad \partial_t u(t,x,q,s) + \mu \partial_s u(t,x,q,s) + \frac{1}{2}\sigma^2 \partial_{ss}^2 u(t,x,q,s)$$
$$+ \sup_{\delta^a} \lambda^a(\delta^a)\left[u(t, x+s+\delta^a, q-1, s) - u(t,x,q,s)\right] = 0$$

with the final condition:

$$u(T,x,q,s) = -\exp\left(-\gamma(x + q(s-b))\right)$$

and the boundary condition:

$$u(t,x,0,s) = -\exp\left(-\gamma x\right)$$

To solve the Hamilton-Jacobi-Belmann equation, we will use a change of variables that transforms the PDEs in a system of linear ODEs.

Proposition 1 (A system of linear ODEs)**.** *Let us consider a family of functions $(w_q)_{q\in\mathbb{N}}$ solution of the linear system of ODEs $(\mathcal{S})$ that follows:*

$$\forall q \in \mathbb{N}, \dot{w}_q(t) = (\alpha q^2 - \beta q) w_q(t) - \eta w_{q-1}(t)$$

with $w_q(T) = e^{-kqb}$ and $w_0 = 1$, where $\alpha = \frac{k}{2}\gamma\sigma^2$, $\beta = k\mu$ and $\eta = A(1 + \frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.

Then $u(t,x,q,s) = -\exp(-\gamma(x+qs)) w_q(t)^{-\frac{\gamma}{k}}$ is solution of (HJB).

The change of variables used in the proposition above is based on two different ideas. First, the choice of a CARA utility function allows to factor out the Mark-to-Market value of the portfolio $(x + qs)$. Then, the exponential decay for the intensity allows to introduce $w_q(t)$ and to end up with a linear system of ODEs.

Now, using this system of ODEs, we can find the optimal quotes through a verification theorem:

Theorem 1 (Verification theorem and optimal quotes)**.** *Let us consider the solution w of the system $(\mathcal{S})$ of Proposition 1.*
Then, $u(t, x, q, s) = -\exp(-\gamma(x+qs))w_q(t)^{-\frac{\gamma}{k}}$ is the value function of the control problem and the optimal ask quote can be expressed as:

$$\delta^{a*}(t, q) = \frac{1}{k}\ln\left(\frac{w_q(t)}{w_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right)$$

3.3.2 Numerical example

Proposition 1 and Theorem 1 provide a way to solve the Hamilton-Jacobi-Bellman equation and to derive the optimal quotes for a trader willing to liquidate a portfolio. To exemplify these results, we compute the optimal quotes when a quantity $q = 6$ has to be sold within 5 minutes (Figure 3.1).

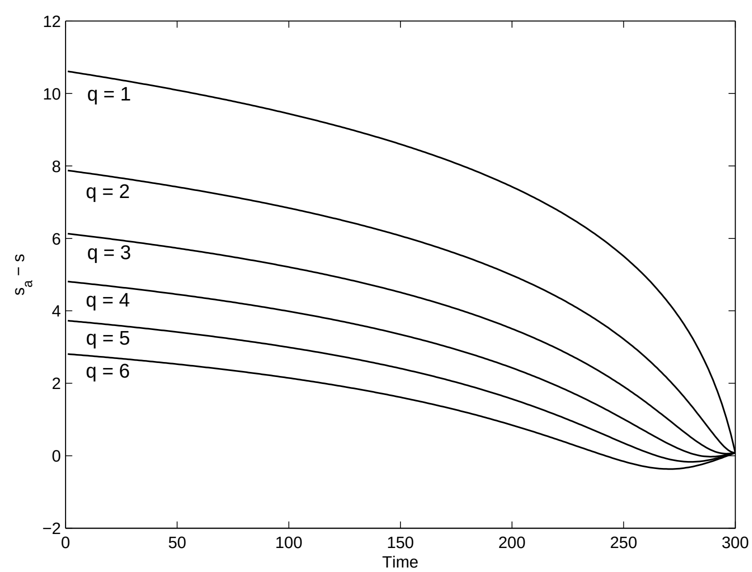

Figure 3.1: Optimal strategy $\delta^{a*}(t, q)$ (in Ticks) for an agent willing to sell a quantity up to $q = 6$ within 5 minutes ($\mu = 0$ (Tick.s^{-1}), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick))

We clearly see that the optimal quotes depend on inventory in a monotonic way. Indeed, a trader with a lot of shares to liquidate need to trade fast to reduce price risk and will therefore propose a low price. On the contrary a trader with only a few shares in his portfolio may be willing to benefit from a trading opportunity and will send an order with a higher price because the risk he bears allows him to trade more slowly.

Now, coming to the time-dependence of the quotes, a trader with a given number of shares will, *ceteris paribus*, lower his quotes as the time horizon gets closer, except near the final time T because a certain maximum discount b is guaranteed. At the limit, when t is close to the time horizon T, the optimal quotes tend to the same value that depend on the liquidation cost b: $\delta^{a*}(T,q) = -b + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$.

As on the first figure, negative quotes may appear. They appear when the quantity to liquidate is large compared to the remaining time, especially when (i) there is a real need to liquidate before time T because the liquidation cost b is high and/or (ii) when risk aversion and volatility are high, because price risk is then an important consideration. When this happens, it means that there is a need to reduce the number of shares at hand. In that case, a model involving both limit orders and market orders would be better suited. Also, if we consider the above optimal strategy on a longer time window (see Figure 3.2), we see that optimal quotes have an asymptotic behavior as the time horizon increases. The associated limiting case will be studied in the next section.

Finally, the average number of shares at each point in time, called trading curve by analogy with the deterministic trading curves of Almgren and Chriss, can be obtained by Monte-Carlo simulations as exemplified on Figure 3.3 when the trader uses the optimal strategy. In particular, because $b = 3$, the trader has a weak incentive to liquidate strictly before time T and there are cases for which liquidation is not complete before time T. This is the reason why we do not have $\mathbb{E}[q_T] = 0$ on Figure 3.3. We will study below the limiting case $b \to +\infty$ that "forces" liquidation before time T.

3.4 Special cases

The above equations can be solved explicitly for w and hence for the optimal quotes using the above verification theorem. However, the resulting closed-form expressions are not really tractable and do not provide any intuition on the behavior of the optimal quotes. Three special cases are now considered for which simpler closed-form formulae can be derived. We start with the limiting behavior of the quotes when the time horizon T tends to infinity. We then consider a case in which there is no price risk and a case where the agent is risk-neutral to both price risk and non-execution risk. We finally consider, by analogy with the classical literature, the behavior of the solution as the liquidation cost b increases. All these special cases allow to comment on the role of the parameters, before we carry out comparative statics in the next section.

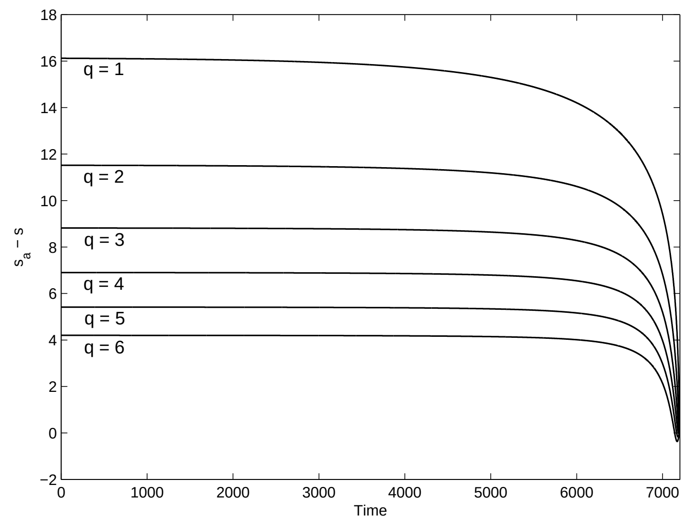

Figure 3.2: Optimal strategy $\delta^{a*}(t,q)$ (in Ticks) for $q = 1,\dots,6$ and $T = 2$ hours ($\mu = 0$ (Tick.s^{-1}), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick))

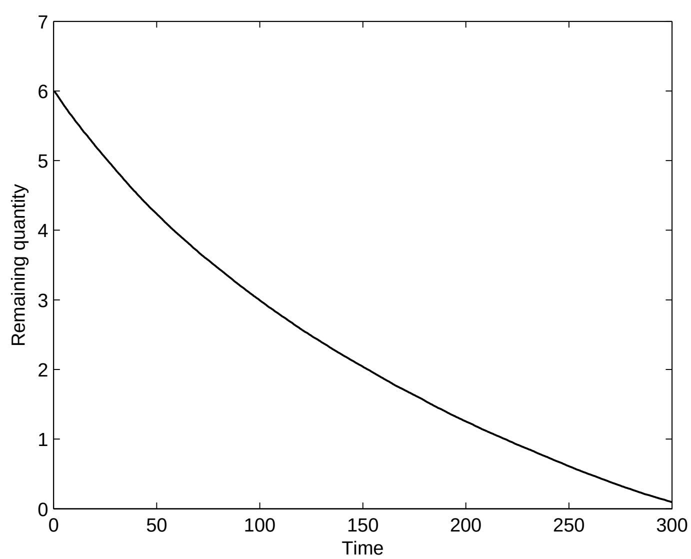

Figure 3.3: Trading curve for an agent willing to sell a quantity of shares $q = 6$ within 5 minutes ($\mu = 0$ (Tick.s^{-1}), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick))

3.4.1 Asymptotic behavior as $T \to +\infty$

We have seen on Figure 3.2 that the optimal quotes seem to exhibit an asymptotic behavior. We are in fact going to prove that $\delta^{a*}(0,q)$ tends to a limit as the time horizon T tends to infinity, when the inequality $\mu < \frac{1}{2}\gamma\sigma^2$ is satisfied[3].

Proposition 2 (Asymptotic behavior of the optimal quotes)**.** *Let us suppose that*[4] $\mu < \frac{1}{2}\gamma\sigma^2$.
Let us consider the solution w of the system $(\mathcal{S})$ of Proposition 2.1. Then:

$$\lim_{T\to+\infty} w_q(0) = \frac{\eta^q}{q!}\prod_{j=1}^{q}\frac{1}{\alpha j - \beta}$$

The resulting asymptotic behavior for the optimal ask quote of Theorem 2.2 is:

$$\lim_{T\to+\infty} \delta^{a*}(0,q) = \frac{1}{k}\ln\left(\frac{A}{1+\frac{\gamma}{k}}\frac{1}{\alpha q^2 - \beta q}\right) = \frac{1}{k}\ln\left(\frac{A}{k+\gamma}\frac{1}{\frac{1}{2}\gamma\sigma^2 q^2 - \mu q}\right)$$

This first closed-form formula deserves a few comments. First of all, the asymptotic quote is obviously a decreasing function of the number of shares in the portfolio.

Coming to the parameters, we can analyze how the asymptotic quote depends on μ, σ, A, k, and γ.

As μ increases, the trader increases his asymptotic quote to slow down the execution process and benefit from the price increase. As far as volatility is concerned, an increase in σ corresponds to an increase in price risk and this provides the trader with an incentive to speed up the execution process. Therefore, it is natural that the asymptotic quote is a decreasing function of σ. Now, as A increases, the asymptotic quote increases. This result is natural because if the rate of arrival of liquidity-consuming orders increases, the trader is more likely to liquidate his shares faster and posting deeper into the book allows for larger profits. Coming to k, the result depends on the sign of the asymptotic quote. If the asymptotic quote is positive – this is the only interesting case since we shall not use our model when market orders are required from the start[5] –, then the asymptotic quote is a decreasing function of k. The mechanism at play is the same as for A: a decrease in k increases the probability to be executed at a given price (when $\delta^a > 0$[6]) and this gives an incentive to post orders deeper into the order book. Finally, the asymptotic quote decreases as the risk aversion increases. An increase in risk aversion forces indeed the trader to reduce both price risk and non-execution risk and this leads to posting orders with lower prices.

One also has to notice that the asymptotic quote does not depend on the liquidation cost b.

[3]In particular, when $\mu = 0$, this means that the result is true as soon as $\gamma > 0$ (and $\sigma > 0$). As we will see below, there is no asymptotic value in the risk-neutral case.

[4]This condition is the same as $\alpha > \beta$.

[5]In all cases, increasing k brings the asymptotic quote closer to 0.

[6]The issue surrounding negative quotes is that $k \mapsto Ae^{-k\delta^a}$ is a decreasing function for $\delta^a > 0$ and an increasing function for $\delta^a < 0$. Subsequently, the intuition we have about k in the usual case $\delta^a > 0$ is reversed for negative quotes.

3.4.2 Absence of price risk and risk-neutrality

The above result on asymptotic behavior does not apply when $\mu = \sigma = 0$. We now concentrate on this case in which there is no drift ($\mu = 0$) and no volatility ($\sigma = 0$). In this case, the trader bears no price risk because $\sigma = 0$ and the only risk he faces is linked to the non-execution of his orders.

We now derive tractable formulae for w and for the optimal quotes:

Proposition 3 (The no-drift/no-volatility case)**.** *Assume that $\sigma = 0$ and that there is no drift ($\mu = 0$).*
Let us define:

$$w_q(t) = \sum_{j=0}^{q} \frac{\eta^j}{j!} e^{-kb(q-j)} (T-t)^j$$

Then w defines a solution of the system ($\mathcal{S}$) and the optimal quote is:

$$\delta^{a*}(t,q) = -b + \frac{1}{k} \ln\left(1 + \frac{\frac{\eta^q}{q!}(T-t)^q}{\sum_{j=0}^{q-1} \frac{\eta^j}{j!} e^{-kb(q-j)} (T-t)^j}\right) + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right)$$

In this no-drift/no-volatility case, the optimal quote still is an increasing function of A and a decreasing function of γ[7]. If the above closed-form formula does not shed any particular light on the dependence on k, it highlights the role played by the liquidation cost b. Differentiating the above formula with respect to b, we indeed get a negative sign and therefore that optimal quote is a decreasing function of b. Since b is the cost to pay for each share remaining at time T, an increase in b gives an incentive to speed up execution and hence to lower the quotes.

We also see that the optimal quote is bounded from below by $-b + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right)$. Since execution is guaranteed at price $s - b$ at time T, it is in particular natural in the absence of price risk, that quotes never go below $-b$.

Now, if one wants to remove risk aversion with respect to both price risk and non-execution risk, one can consider the limit of the above solution when γ tends to 0[8].

One then obtains:

$$\delta^{a*}(t,q) = -b + \frac{1}{k} \ln\left(1 + \frac{\frac{A^q}{e^q q!}(T-t)^q}{\sum_{j=0}^{q-1} \frac{A^j}{e^j j!} e^{-kb(q-j)} (T-t)^j}\right) + \frac{1}{k}$$

and this is the result of Bayraktar and Ludkovski [18] in the case $b = 0$, because they do not consider any liquidation cost. In particular, the optimal quote of [18] does not converge to a limit value as T tends to $+\infty$, but rather increases with no upper bound. This is an important difference between the risk-neutral case and our risk-adverse framework.

---

[7]Recall that $\eta = A(1 + \frac{\gamma}{k})^{-(1+\frac{k}{\gamma})}$.
[8]The same result holds if one sends γ to 0 for any value of the volatility parameter σ.

3.4.3 Limiting behavior as $b \to +\infty$

Let us now consider the limiting case $b \to +\infty$. Sending b to infinity corresponds to a situation in which a very high incentive is given to the trader for complete liquidation before time T. If we look at the Almgren-Chriss-like literature on optimal execution, the authors are often assuming that $q_T = 0$[9]. Hence, if one writes the value functions associated to most liquidity-consuming optimal strategies, it turns out that they are equal to $-\infty$ at the time horizon T except when the inventory is equal to nought (hence $b = +\infty$, in our framework). However, here, due to the uncertainty on execution, we cannot write a well-defined control problem when b is equal to $+\infty$. Rather, we are interested in the limiting behavior when $b \to +\infty$, *i.e.* when the incentive to liquidate before time T is large.

By analogy with the initial literature on optimal liquidation [10], we can also have some limiting results on the trading curve.

Hereafter we denote $w_{b,q}(t)$ the solution of the system $(\mathcal{S})$ for a given liquidation cost b, $\delta_b^{a*}(t,q)$ the associated optimal quote and $q_{b,t}$ the resulting process modeling the number of stocks in the portfolio.

Proposition 4 (Form of the solutions, trading intensity and trading curve)**.** *The limiting solution* $\lim_{b\to+\infty} w_{b,q}(t)$ *is of the form* $A^q v_q(t)$ *where* v *does not depend on* A.

The limit of the trading intensity $\lim_{b\to+\infty} Ae^{-k\delta_b^{a*}}$ *does not depend on* A.

Consequently, the trading curve $V_b(t) := \mathbb{E}[q_{b,t}]$ *verifies that* $V(t) = \lim_{b\to+\infty} V_b(t)$ *is independent of* A.

More results can be obtained in the no-volatility case:

Proposition 5 (no-volatility case, $b \to +\infty$)**.** *Assume that* $\sigma = 0$ *and consider first the case* $\mu \neq 0$. *We have:*

$$\lim_{b\to+\infty} w_{b,q}(t) = \frac{\eta^q}{q!}\left(\frac{e^{\beta(T-t)}-1}{\beta}\right)^q$$

The limit of the optimal quote is:

$$\delta_\infty^{a*}(t,q) = \lim_{b\to+\infty} \delta_b^{a*}(t,q) = \frac{1}{k}\ln\left(\frac{A}{1+\frac{\gamma}{k}}\frac{1}{q}\frac{e^{\beta(T-t)}-1}{\beta}\right)$$

The limit of the associated trading curve is $V(t) = q_0\left(\frac{1-e^{-\beta(T-t)}}{1-e^{-\beta T}}\right)^{1+\frac{\gamma}{k}}$.

[9]The authors most often consider target problems in which the target can always be attained.

Now, in the no-volatility/no-drift case ($\sigma = \mu = 0$), similar results can be obtained, either directly or sending μ to 0 in the above formulae:

$$\lim_{b \to +\infty} w_{b,q}(t) = \frac{\eta^q}{q!}(T-t)^q$$

The limit of the optimal quote is given by:

$$\delta_\infty^{a*}(t,q) = \lim_{b \to +\infty} \delta_b^{a*}(t,q) = \frac{1}{k} \ln \left(\frac{A}{1+\frac{\gamma}{k}} \frac{1}{q}(T-t) \right)$$

The limit of the associated trading curve is $V(t) = q_0 \left(1 - \frac{t}{T}\right)^{1+\frac{\gamma}{k}}$.

This third limiting case confirms the monotonicity results we discussed above: the optimal quote is an increasing function of A and μ[10] and it is a decreasing function of γ (and of the number of shares). Concerning the shape of the trading curve, the role played by the risk aversion parameter γ is the same as in Almgren-Chriss: an increase in γ forces the trader to speed up the execution process and therefore steepens the slope of the trading curve. The role of μ is also interesting because a positive trend goes against the naturally convex shape of the trading curve. Since a trader slows down the execution process to benefit from a positive trend, there is a trade-off between positive trend on one hand and price risk on the other, and the trading curve may turn out to be concave when the upward trend is sufficiently important to compensate the effect of risk aversion.

Coming to k, this third limiting case is particularly interesting because there is no lower bound to the optimal quotes and we have seen above that the occurrence of negative quotes was a problem to interpret the parameter k. Hence, the limiting case $b \to +\infty$ appears to be a worst case.

In normal circumstances, we expect the optimal quote to be a decreasing function of k. However, straightforward computations give (in the no-drift case) that

$$\frac{d\delta_\infty^{a*}(t,q)}{dk} = -\frac{1}{k}\delta_\infty^{a*}(t,q) + \frac{1}{k^2}\frac{\gamma}{\gamma+k}.$$

The sign of this expression being negative if and only if $\delta_\infty^{a*}(t,q)$ is above a certain positive threshold, the dependence on k may be reversed even for positive (but low) quotes. In the case of the asymptotic (and constant) quote discussed above, the threshold was 0. Here, in the dynamic case under consideration, the high probability of negative optimal quotes in the future may break the monotonicity on k and that is the reason why the threshold is positive.

Although this limiting case is rather extreme, it illustrates well the issues of the model when execution is too slow and would ideally require market orders. It is noteworthy that in the comparative statics we carry out in the next section, the usual monotonicity property is only broken for extreme values of the parameters. Also, in most reasonable cases we considered in practice, the quotes were decreasing in k.

[10] Recall that $\beta = k\mu$.

3.5 Comparative statics

We discussed above the role played by the different parameters in particular limiting cases. We now consider the general case and carry out comparative statics on optimal quotes. The tables we obtain confirm the intuitions we developed in the preceding section.

Influence of the drift μ:

As far as the drift is concerned, quotes are naturally increasing with μ. If indeed the trader expects the price to move down, he is going to send orders at low prices to be executed fast and to reduce the impact of the decrease in price on the P&L. On the contrary, if he expects the price to rise, he is going to post orders deeper in the book in order to slow down execution and benefit from the price increase. This is well exemplified by Table 3.1.

q	$\mu = -0.01$ (Tick.s^{-1})	$\mu = 0$ (Tick.s^{-1})	$\mu = 0.01$ (Tick.s^{-1})
1	9.2252	10.6095	12.2329
2	6.581	7.8737	9.3921
3	4.92	6.1299	7.5507
4	3.6732	4.8082	6.1391
5	2.6607	3.728	4.9765
6	1.8012	2.8073	3.9806

Table 3.1: Dependence on μ of $\delta^{a*}(0, q)$ with $T = 5$ (minutes), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick)

Influence of the volatility σ:

Now, coming to volatility, the optimal quotes depend on σ in a monotonic way. If there is an increase in volatility, then price risk increases. In order to reduce this additional price risk the trader will send orders at lower price. This is what we observe numerically on Table 3.2.

q	$\sigma = 0$ (Tick.s$^{-\frac{1}{2}}$)	$\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$)	$\sigma = 0.6$ (Tick.s$^{-\frac{1}{2}}$)
1	10.9538	10.6095	9.6493
2	8.6482	7.8737	6.0262
3	7.3019	6.1299	3.6874
4	6.3486	4.8082	1.9455
5	5.6109	3.728	0.55671
6	5.0097	2.8073	-0.59773

Table 3.2: Dependence on σ of $\delta^{a*}(0, q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick)

Influence of the intensity scale parameter A:

Now, coming to A, we observe numerically, and as expected, that the optimal quote is an increasing function of A (see Table 3.3). If A increases, the probability to be executed indeed increases and the trader will then increase his quotes to obtain transactions at higher prices.

q	$A = 0.05$ (s^{-1})	$A = 0.1$ (s^{-1})	$A = 0.15$ (s^{-1})
1	8.4128	10.6095	11.9222
2	5.6704	7.8737	9.1898
3	3.9199	6.1299	7.4491
4	2.5917	4.8082	6.1302
5	1.5051	3.728	5.0525
6	0.57851	2.8073	4.1341

Table 3.3: Dependence on A of $\delta^{a*}(0, q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $k = 0.3$ (Tick^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick)

Influence of the intensity shape parameter k:

Now, as far as k is concerned, the dependence of the optimal quote on k is ambiguous because the interpretation of k depends on the optimal quote itself. An increase in k should correspond indeed to a decrease in the probability to be executed at a given price in most cases the model is used. However, due to the exponential form of the execution intensity, the very possibility to use negative quotes may reverse the reasoning (see the discussions in section 4 for the asymptotic quotes and in the extreme case $b \to +\infty$).

In the first case we consider, which only leads to positive optimal quotes, an increase in k forces the trader to decrease the price of the orders he sends to the market, as exemplified by Table 3.4. However, if price risk is really important (high volatility and/or large quantity to liquidate) the optimal quotes may be negative and, in that case, the role of k is reversed. This is the case when σ takes (unrealistically) high values, as exemplified on Table 3.5.

q	$k = 0.2$ (Tick^{-1})	$k = 0.3$ (Tick^{-1})	$k = 0.4$ (Tick^{-1})
1	15.8107	10.6095	7.941
2	11.9076	7.8737	5.7972
3	9.4656	6.1299	4.4144
4	7.6334	4.8082	3.3618
5	6.1436	3.728	2.5011
6	4.8761	2.8073	1.7688

Table 3.4: Dependence on k of $\delta^{a*}(0, q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $\sigma = 0.3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick)

q	$k = 0.2$ (Tick^{-1})	$k = 0.3$ (Tick^{-1})	$k = 0.4$ (Tick^{-1})
1	2.8768	0.79631	-0.031056
2	-4.0547	-3.8247	-3.4968
3	-8.1093	-6.5278	-5.5241
4	-10.9861	-8.4457	-6.9625
5	-13.2176	-9.9333	-8.0782
6	-15.0408	-11.1488	-8.9899

Table 3.5: Dependence on k of $\delta^{a*}(0,q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $\sigma = 3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $\gamma = 0.05$ (Tick^{-1}) and $b = 3$ (Tick)

Influence of the liquidation cost b:

Finally, the influence of the liquidation cost b is straightforward. If b increases, then the need to sell strictly before time T is increased because the value of any remaining share at time T decreases. Hence, the optimal quotes must be decreasing in b and this is what we observe on Table 3.6.

q	$b = 0$ (Tick)	$b = 3$ (Tick)	$b = 20$ (Tick)
1	10.7743	10.6095	10.4924
2	8.0304	7.8737	7.7685
3	6.278	6.1299	6.0353
4	4.9477	4.8082	4.7229
5	3.859	3.728	3.6509
6	2.9301	2.8073	2.7374

Table 3.6: Dependence on b of $\delta^{a*}(0,q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $\sigma = 3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}) and $\gamma = 0.05$ (Tick^{-1})

Influence of the risk aversion γ:

Turning to the risk aversion parameter γ, two effects are at stake that go in the same direction. The risk aversion is indeed common for both price risk and non-execution risk. Hence if risk aversion increases, the trader will try to reduce both price risk and non-execution risk, thus selling at lower price. We indeed see on Table 3.7 that optimal quotes are decreasing in γ.

The next table highlights the importance of risk aversion for the optimal liquidation problem. When the number of shares to liquidate is not too small, we indeed see that the optimal quotes depend strongly on γ. In particular, our reference case $\gamma = 0.05$ is really different from the case $\gamma = 0.01$ and therefore very different from the risk-neutral case of [18].

q	$\gamma = 0.01$ (Tick^{-1})	$\gamma = 0.05$ (Tick^{-1})	$\gamma = 0.5$ (Tick^{-1})
1	11.2809	10.6095	9.84
2	8.8826	7.8737	6.7461
3	7.4447	6.1299	4.7262
4	6.4008	4.8082	3.189
5	5.5735	3.728	1.9384
6	4.8835	2.8073	0.88139

Table 3.7: Dependence on γ of $\delta^{a*}(0, q)$ with $T = 5$ (minutes), $\mu = 0$ (Tick.s^{-1}), $\sigma = 3$ (Tick.s$^{-\frac{1}{2}}$), $A = 0.1$ (s^{-1}), $k = 0.3$ (Tick^{-1}), and $b = 3$ (Tick)

3.6 Historical simulations

Before using the above model in reality, we need to discuss some features of the model that need to be adapted before any backtest is possible.

First of all, the model is continuous in both time and space while the real control problem under consideration is intrinsically discrete in space, because of the tick size, and discrete in time, because orders have a certain priority and changing position too often reduces the actual chance to be reached by a market order. Hence, the model has to be reinterpreted in a discrete way. In terms of prices, quotes must not be between two ticks and we decided to round the optimal quotes to the nearest tick[11]. In terms of time, an order is sent to the market and is not canceled nor modified for a given period of time Δt, unless a trade occurs and, though perhaps partially, fills the order. Now, when a trade occurs and changes the inventory or when an order stayed in the order book for longer than Δt, then the optimal quote is updated and, if necessary, a new order is inserted.

Now, concerning the parameters, σ, A and k can be calibrated on trade-by-trade limit order book data while γ has to be chosen. Thus, we simply chose to calibrate A and k from real data (see next chapter). As far as γ is concerned, a choice based on a Value at Risk limit is possible but requires the use of Monte-Carlo simulations. We decided in our backtests to assign γ a value that makes the first quote δ^{a*} equal to 1 for typical values of A and k.

Turning to the backtests, they were carried out with trade-by-trade data and we assumed that our orders were entirely filled when a trade occurred at or above the ask price quoted by the agent. Our goal here is just to provide examples in various situations and, to exemplify the practical use of this model, we carried out several backtests[12] on the French stock AXA, either on very short periods (slices of 5 minutes) or on slightly longer periods of a few hours. Armed with our experience of the model, we believe that it is particularly suited to optimize liquidation within slices of a global trading curve, be it a TWAP, a VWAP, or an Implementation Shortfall trading curve.

[11] We also, alternatively, randomized the choice with probabilities that depend on the respective proximity to the neighboring quotes.

[12] No drift in prices is assumed in the strategy used for backtesting.

The first two examples (Figures 3.4 and 3.5) consist in liquidating a quantity of shares equal to 3 times the ATS[13]. The periods have been chosen to capture the behavior in both bullish and bearish markets.

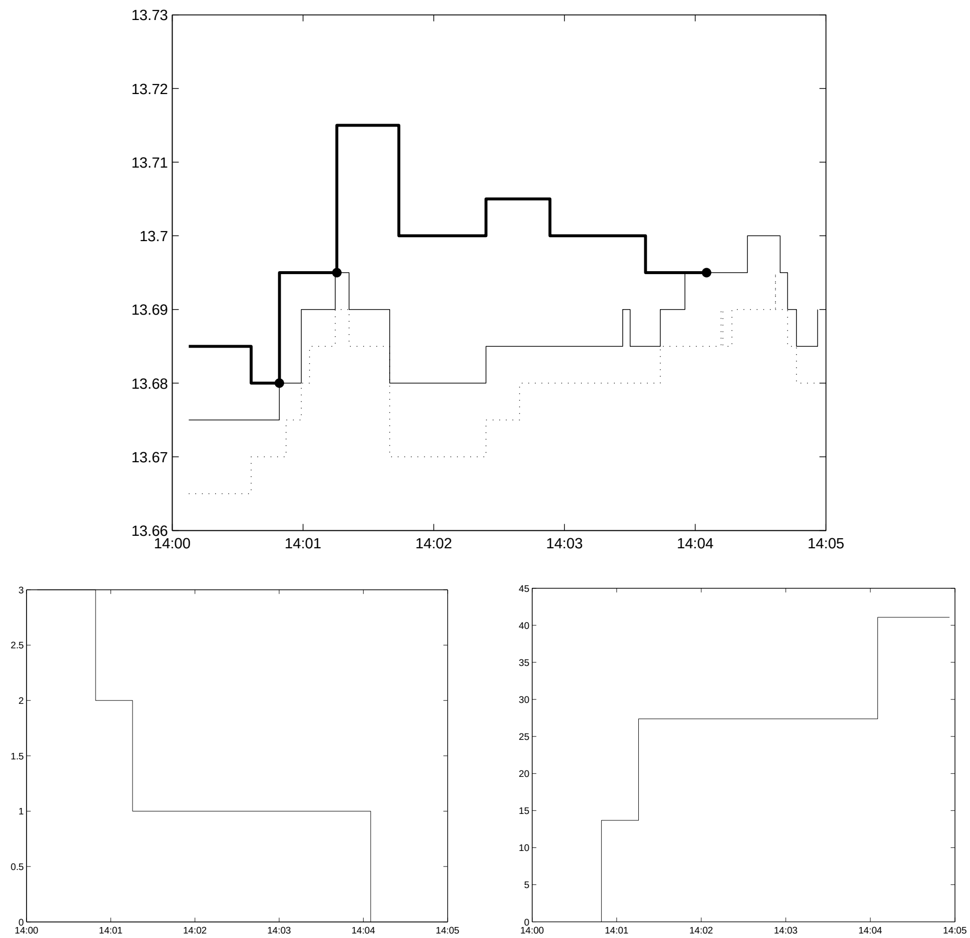

Figure 3.4: Backtest example on AXA (November 5^{th} 2010). The strategy is used to sell a quantity of shares equal to 3 times the ATS within 5 minutes. Top: quotes of the trader (bold line), market best bid and ask quotes (thin lines). Trades are represented by dots. Bottom left: evolution of the inventory. Bottom right: cash at hand.

On Figure 3.4, we see that the first order is executed after 50 seconds. Then, since the trader has only 2 times the ATS left in his inventory, he sends an order at a higher price. Since the market price moves up, the second order is executed in the next 30 seconds, in advance on the average schedule. This is the reason why the trader places a new order far above the best ask. Since this order is not executed within the time window Δt, it

[13] In the backtests we do not deal with quantity and priority issues in the order books and supposed that our orders were always entirely filled.

is canceled and new orders are successively inserted with lower prices. The last trade happens less than 1 minute before the end of the period. Overall, on this example, the strategy works far better than a market order (even ignoring execution costs).

On Figure 3.5, we see the use of the strategy in a bearish period. The first order is executed rapidly and since the market price goes down, the trader's last orders are only executed at the end of the period when prices of orders are lowered substantially as it becomes urgent to sell. Practically, this obviously raises the question of linking a trend detector to these optimal liquidation algorithms.

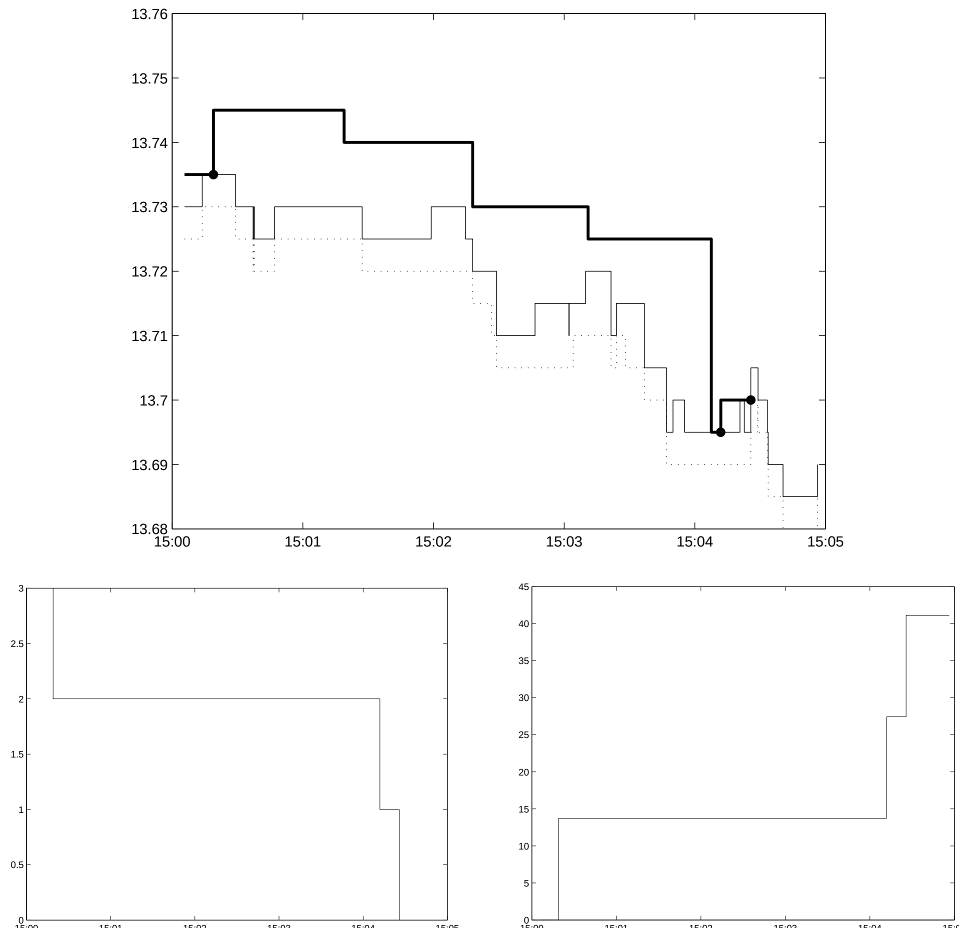

Figure 3.5: Backtest example on AXA (November 5^{th} 2010). The strategy is used to sell a quantity of shares equal to 3 times the ATS within 5 minutes. Top: quotes of the trader (bold line), market best bid and ask quotes (thin lines). Trades are represented by dots. Bottom left: evolution of the inventory. Bottom right: cash at hand.

Finally, the model can also be used on longer periods and we exhibit the use of the algorithm on a period of two hours, to sell a quantity of shares equal to 20 times the ATS, representing here around 5% of the volume during that period (Figure 3.6).

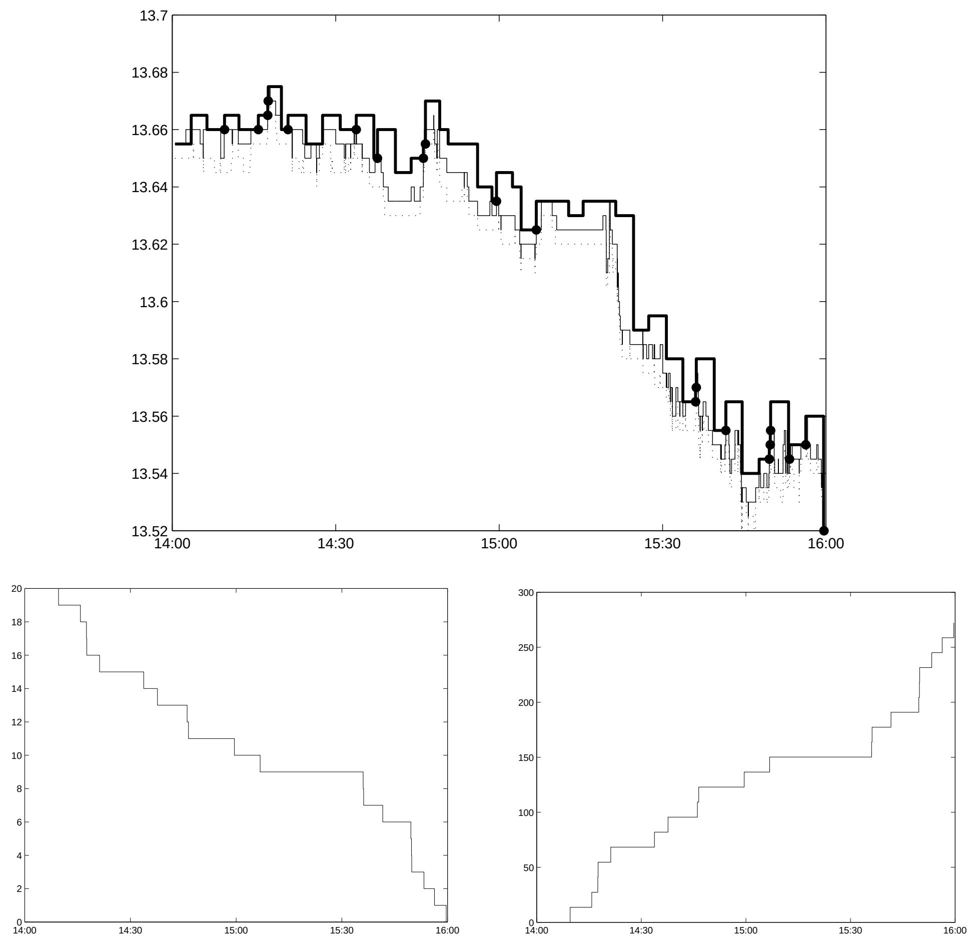

Figure 3.6: Backtest example on AXA (November 8^{th} 2010). The strategy is used to sell a quantity of shares equal to 20 times the ATS within 2 hours. Top: quotes of the trader (bold line), market best bid and ask quotes (thin lines). Trades are represented by dots. Bottom left: evolution of the inventory. Bottom right: cash at hand.

3.7 Conclusion

As claimed in the introduction, this work is, to authors' knowledge, the first proposal to optimize the trade scheduling of large orders with small passive orders when price risk and non-execution risk are taken into account. The classical approach to optimal liquidation, following the Almgren-Chriss framework, consisted in a trade-off between price risk and execution cost/market impact. In the case of liquidity-providing orders, this trade-off disappears but a new risk is borne by the agent: non-execution risk.

The problem is then a new stochastic control problem and an innovative change of variables allows to reduce the 4-variable Hamilton-Jacobi-Bellman equation to a system of linear ordinary differential equations. Practically, the optimal quote can therefore be found in two steps: (1) solve a linear system of ODEs, (2) deduce the optimal price of the order to be sent to the market.

We studied various limiting cases that allowed to find the asymptotic behavior of the optimal strategy and to find the result obtained in parallel by Bayraktar and Ludkovski [18], taking the risk-neutral limit. This also allowed us to confirm our intuition about the role played by the parameters.

Numerical experiments and backtests have been carried out and the results are promising. However, two possible improvements are worth the discussion.

First, no explicit model of what could be called "passive market impact" (*i.e.* the perturbations of the price formation process by liquidity provision) is used here. Interestingly, Jaimungal, Cartea and Ricci [33] recently introduced market impact in a similar model, the market impact occurring when execution takes place. We may consider introducing a similar effect in future versions of the model. Also, thanks to very promising and recent studies of the multi-dimensional point processes governing the arrival of orders (see for instance the link between the imbalance in the order flow and the moves of the price studied in [40] or [42], or interesting properties of Hawkes-like models in [15]), we can hope for obtaining new models with passive market impact in the near future. The authors will try to embed them into the HJB framework used here.

Second, the separation of the variables (x, s) and (t, q) is a property associated to the use of a CARA utility function and to the brownian dynamic of the price and is independent of the exponential decay for the arrival of orders. An on-going work aims at generalizing the above model to a general function $\lambda^a(\cdot)$ using this separation of variables.

Appendix: Proofs of the results

Proposition 1 and Theorem 1

Proof. First, let us remark that a solution $(w_q)_q$ of $(\mathcal{S})$ exists and is unique and that, by immediate induction, its components are strictly positive for all times. Then, let us introduce $u(t,x,q,s) = -\exp\left(-\gamma(x+qs)\right) w_q(t)^{-\frac{\gamma}{k}}$.

We have:

$$\partial_t u + \mu \partial_s u + \frac{1}{2}\sigma^2 \partial^2_{ss} u = -\frac{\gamma}{k}\frac{\dot{w}_q(t)}{w_q(t)} u - \gamma q \mu u + \frac{\gamma^2\sigma^2}{2} q^2 u$$

Now, concerning the non-local part of the equation, we have:

$$\sup_{\delta^a} \lambda^a(\delta^a) \left[u(t, x+s+\delta^a, q-1, s) - u(t,x,q,s)\right]$$

$$= \sup_{\delta^a} A e^{-k\delta^a} u(t,x,q,s) \left[\exp\left(-\gamma\delta^a\right) \left(\frac{w_{q-1}(t)}{w_q(t)}\right)^{-\frac{\gamma}{k}} - 1\right]$$

The first order condition of this problem corresponds to a maximum and writes:

$$(k+\gamma)\exp\left(-\gamma\delta^a\right)\left(\frac{w_{q-1}(t)}{w_q(t)}\right)^{-\frac{\gamma}{k}} = k$$

Hence we introduce the candidate δ^{a*} for the optimal control:

$$\delta^{a*} = \frac{1}{k}\ln\left(\frac{w_q(t)}{w_{q-1}(t)}\right) + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

and

$$\sup_{\delta^a} \lambda^a(\delta^a) \left[u(t, x+s+\delta^a, q-1, s) - u(t,x,q,s)\right]$$

$$= -\frac{\gamma}{k+\gamma} A \exp(-k\delta^{a*}) u(t,x,q,s)$$

$$= -A\frac{\gamma}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{-\frac{k}{\gamma}} \frac{w_{q-1}(t)}{w_q(t)} u(t,x,q,s)$$

Hence, putting the three terms together we get:

$$\partial_t u(t,x,q,s) + \mu\partial_s u(t,x,q,s) + \frac{1}{2}\sigma^2\partial^2_{ss} u(t,x,q,s)$$

$$+\sup_{\delta^a} \lambda^a(\delta^a) \left[u(t, x+s+\delta^a, q-1, s) - u(t,x,q,s)\right]$$

$$= -\frac{\gamma}{k}\frac{\dot{w}_q(t)}{w_q(t)} u - \gamma\mu q u + \frac{\gamma^2\sigma^2}{2} q^2 u - A\frac{\gamma}{k+\gamma}\left(1+\frac{\gamma}{k}\right)^{-\frac{k}{\gamma}} \frac{w_{q-1}(t)}{w_q(t)} u$$

$$= -\frac{\gamma}{k}\frac{u}{w_q(t)}\left[\dot{w}_q(t) + k\mu q w_q(t) - \frac{k\gamma\sigma^2}{2} q^2 w_q(t) + A\left(1+\frac{\gamma}{k}\right)^{-\left(1+\frac{k}{\gamma}\right)} w_{q-1}(t)\right] = 0$$

Now, noticing that the boundary and terminal conditions for w_q are consistent with the conditions on u, we get that u verifies (HJB).

Now, we need to verify that u is indeed the value function associated to the problem and to prove that our candidate $(\delta_t^{a*})_t$ is indeed the optimal control. To that purpose, let us consider a control $\nu \in \mathcal{A}$ and let us consider the following processes for $\tau \in [t, T]$:

$$dS_\tau^{t,s} = \mu d\tau + \sigma dW_\tau, \qquad S_t^{t,s} = s$$
$$dX_\tau^{t,x,\nu} = (S_\tau + \nu_\tau) dN_\tau^a, \qquad X_t^{t,x,\nu} = x$$
$$dq_\tau^{t,q,\nu} = -dN_\tau^a, \qquad q_t^{t,q,\nu} = q$$

where the point process has stochastic intensity $(\lambda_\tau)_\tau$ with $\lambda_\tau = Ae^{-k\nu_\tau} 1_{q_{\tau-} \geq 1}$[14].

Now, let us write It's formula for u since u is smooth:

$$u(T, X_{T-}^{t,x,\nu}, q_{T-}^{t,q,\nu}, S_T^{t,s}) = u(t, x, q, s)$$
$$+ \int_t^T \left(\partial_\tau u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \mu \partial_s u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \frac{\sigma^2}{2} \partial_{ss}^2 u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) \right) d\tau$$
$$+ \int_t^T \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) \right) \lambda_\tau d\tau$$
$$+ \int_t^T \sigma \partial_s u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) dW_\tau$$
$$+ \int_t^T \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) \right) dM_\tau^a$$

where M^a is the compensated process associated to N^a for the intensity process $(\lambda_\tau)_\tau$.

Now, we have to ensure that the last two integrals consist of martingales so that their mean is 0. To that purpose, let us notice that $\partial_s u = -\gamma q u$ and hence we just have to prove that:

$$\mathbb{E}\left[\int_t^T u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})^2 d\tau\right] < +\infty$$

$$\mathbb{E}\left[\int_t^T \left|u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_\tau, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s})\right| \lambda_\tau d\tau\right] < +\infty$$

and

$$\mathbb{E}\left[\int_t^T \left|u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right| \lambda_\tau d\tau\right] < +\infty$$

[14]This intensity being bounded since ν is bounded from below.

Now, remember that the process $q^{t,q,\nu}$ takes values between 0 and q and that $t \in [0,T]$. Hence, $\exists \varepsilon > 0$, $w_q(t) > \varepsilon$ for the values of t and q under scrutiny and:

$$
\begin{aligned}
u(\tau, X_\tau^{t,x,\nu}, q_\tau^{t,q,\nu}, S_\tau^{t,s})^2 &\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(X_\tau^{t,x,\nu} + q_\tau^{t,q,\nu} S_\tau^{t,s})\right) \\
&\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(x - q\|\nu^-\|_\infty + 2q \inf_{\tau\in[t,T]} S_\tau^{t,s} 1_{\inf_{\tau\in[t,T]} S_\tau^{t,s} < 0})\right) \\
&\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(x - q\|\nu^-\|_\infty)\right) \left(1 + \exp\left(-2\gamma q \inf_{\tau\in[t,T]} S_\tau^{t,s}\right)\right)
\end{aligned}
$$

Hence:

$$
\mathbb{E}\left[\int_t^T u(\tau, X_\tau^{t,x,\nu}, q_\tau^{t,q,\nu}, S_\tau^{t,s})^2 d\tau\right] = \mathbb{E}\left[\int_t^T u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})^2 d\tau\right]
$$

$$
\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(x - q\|\nu^-\|_\infty)\right)(T-t)\left(1 + \mathbb{E}\left[\exp\left(-2\gamma q \inf_{\tau\in[t,T]} S_\tau^{t,s}\right)\right]\right)
$$

$$
\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(x - q\|\nu^-\|_\infty)\right)(T-t)\left(1 + \mathbb{E}\left[\exp\left(-2\gamma q \inf_{\tau\in[t,T]} S_\tau^{t,s}\right)\right]\right)
$$

$$
\le \varepsilon^{-\frac{2\gamma}{k}} \exp\left(-2\gamma(x - q\|\nu^-\|_\infty)\right)(T-t)\left(1 + e^{-2\gamma q s}\mathbb{E}\left[\exp\left(2\gamma q \sigma \sqrt{T-t}|Y|\right)\right]\right) < +\infty
$$

where the last inequalities come from the reflection principle with $Y \sim \mathcal{N}(0,1)$ and the fact that $\mathbb{E}\left[e^{C|Y|}\right] < +\infty$ for any $C \in \mathbb{R}$.

Now, the same argument works for the second and third integrals, noticing that ν is bounded from below and that λ is bounded.

Hence, since we have, by construction[15]

$$
\partial_\tau u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \mu \partial_s u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s}) + \frac{\sigma^2}{2}\partial_{ss}^2 u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})
$$

$$
+ \left(u(\tau, X_{\tau-}^{t,x,\nu} + S_\tau^{t,s} + \nu_t, q_{\tau-}^{t,q,\nu} - 1, S_\tau^{t,s}) - u(\tau, X_{\tau-}^{t,x,\nu}, q_{\tau-}^{t,q,\nu}, S_\tau^{t,s})\right)\lambda_\tau \le 0
$$

we obtain that

$$
\mathbb{E}\left[u(T, X_T^{t,x,\nu}, q_T^{t,q,\nu}, S_T^{t,s})\right] = \mathbb{E}\left[u(T, X_{T-}^{t,x,\nu}, q_{T-}^{t,q,\nu}, S_T^{t,s})\right] \le u(t,x,q,s)
$$

and this is true for all $\nu \in \mathcal{A}$. Since for $\nu = \delta^{a*}$ we have an equality in the above inequality we obtain that:

$$
\sup_{\nu\in\mathcal{A}} \mathbb{E}\left[u(T, X_T^{t,x,\nu}, q_T^{t,q,\nu}, S_T^{t,s})\right] \le u(t,x,q,s) = \mathbb{E}\left[u(T, X_T^{t,x,\delta^{a*}}, q_T^{t,q,\delta^{a*}}, S_T^{t,s})\right]
$$

This proves that u is the value function and that δ^{a*} is optimal. $\square$

[15] This inequality is also true when the portfolio is empty because of the boundary conditions.

Proposition 2

Proof. We have that

$$\forall q \in \mathbb{N}, \dot{w}_q(t) = (\alpha q^2 - \beta q) w_q(t) - \eta w_{q-1}(t)$$

Hence if we consider for a given $Q \in \mathbb{N}$ the vector $w(t) = \begin{bmatrix} w_0(t) \\ w_1(t) \\ \vdots \\ w_Q(t) \end{bmatrix}$ we have that $w'(t) = M w(t)$ where:

$$M = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -\eta & \alpha - \beta & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\eta & \alpha(Q-1)^2 - \beta(Q-1) & 0 \\ 0 & \cdots & \cdots & 0 & -\eta & \alpha Q^2 - \beta Q \end{bmatrix}$$

with $w(T) = \begin{bmatrix} 1 \\ e^{-kb} \\ \vdots \\ e^{-kbQ} \end{bmatrix}$. Hence we know that, if we consider a basis $(f_0, \ldots, f_Q)$ of eigenvectors (f_j being associated to the eigenvalue $\alpha j^2 - \beta j$), there exists $(c_0, \ldots, c_Q) \in \mathbb{R}^{Q+1}$ independent of T such that:

$$w(t) = \sum_{j=0}^{Q} c_j e^{-(\alpha j^2 - \beta j)(T-t)} f_j$$

Consequently, since we assumed that $\alpha > \beta$, we have that $w^\infty := \lim_{T \to +\infty} w(0) = c_0 f_0$. Now, w^∞ is characterized by:

$$(\alpha q^2 - \beta q) w_q^\infty = \eta w_{q-1}^\infty, q > 0 \qquad w_0^\infty = 1$$

As a consequence we have:

$$w_q^\infty = \frac{\eta^q}{q!} \prod_{j=1}^{q} \frac{1}{\alpha j - \beta}$$

The resulting asymptotic behavior for the optimal ask quote is:

$$\lim_{T \to +\infty} \delta^{a*}(0, q) = \frac{1}{k} \ln \left(\frac{A}{1 + \frac{\gamma}{k}} \frac{1}{\alpha q^2 - \beta q} \right)$$

□

Proposition 3

Proof. The result of Proposition 3.2 is obtained by induction. For $q = 0$ the result is obvious.
Now, if the result is true for some q we have that:

$$\dot{w}_{q+1}(t) = -\sum_{j=0}^{q} \frac{\eta^{j+1}}{j!} e^{-kb(q-j)} (T-t)^j$$

Hence:

$$w_{q+1}(t) = e^{-kb(q+1)} + \sum_{j=0}^{q} \frac{\eta^{j+1}}{(j+1)!} e^{-kb(q-j)} (T-t)^{j+1}$$

$$w_{q+1}(t) = e^{-kb(q+1)} + \sum_{j=1}^{q+1} \frac{\eta^{j}}{j!} e^{-kb(q-j+1)} (T-t)^{j}$$

$$w_{q+1}(t) = \sum_{j=0}^{q+1} \frac{\eta^{j}}{j!} e^{-kb(q+1-j)} (T-t)^{j}$$

This proves the results for w and then the result follows for the optimal quote. □

Proposition 4

Proof. Because the solutions depend continuously on b, we can directly get interested in the limiting equation:

$$\forall q \in \mathbb{N}, \dot{w}_q(t) = (\alpha q^2 - \beta q) w_q(t) - \eta w_{q-1}(t)$$

with $w_q(T) = 1_{q=0}$ and $w_0 = 1$.

Then, if we define $v_q(t) = \lim_{b \to +\infty} \frac{w_{b,q}(t)}{A^q}$, v solves:

$$\forall q \in \mathbb{N}, \dot{v}_q(t) = (\alpha q^2 - \beta q) v_q(t) - \tilde{\eta} v_{q-1}(t)$$

with $v_q(T) = 1_{q=0}$ and $v_0 = 1$, where $\tilde{\eta} = \frac{\eta}{A}$ is independent of A.

Hence $v_q(t)$ is independent of A.

Now, for the trading intensity we have:

$$\begin{aligned} \lim_{b \to +\infty} A \exp\left(-k \delta_b^{a*}(t,q)\right) &= \lim_{b \to +\infty} \frac{A w_{b,q-1}(t)}{w_{b,q}(t)} \left(1 + \frac{\gamma}{k}\right)^{-\frac{k}{\gamma}} \\ &= \frac{v_{q-1}(t)}{v_q(t)} \left(1 + \frac{\gamma}{k}\right)^{-\frac{k}{\gamma}} \end{aligned}$$

and this does not depend on A.

Eventually, since the limit of the trading intensity does not depend on A, the resulting trading curve does not depend on A either. □

Proposition 5

Proof. Using the preceding proposition, we can now reason in terms of v and look for a solution of the form $v_q(t) = \frac{h(t)^q}{q!}$.
Then,

$$\forall q \in \mathbb{N}, \dot{v}_q(t) = -\beta q v_q(t) - \tilde{\eta} v_{q-1}(t), \quad v_q(T) = 1_{q=0}, \quad v_0 = 1$$

$$\iff h'(t) = -\beta h(t) - \tilde{\eta} \quad h(T) = 0$$

Hence, if $\beta = k\mu \neq 0$, the solution writes $v_q(t) = \frac{\tilde{\eta}^q}{q!}(\frac{\exp(\beta(T-t))-1}{\beta})^q$.
From Theorem 2.2, we obtain the limit of the optimal quote:

$$\lim_{b \to +\infty} \delta_b^{a*}(t,q) = \left(\frac{1}{k} \ln\left(\frac{\eta}{q} \frac{\exp(\beta(T-t)) - 1}{\beta}\right) + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right)\right)$$

Using the expression for $\tilde{\eta}$, this can also be written:

$$\frac{1}{k} \ln\left(\frac{A}{1 + \frac{\gamma}{k}} \frac{1}{q} \frac{e^{\beta(T-t)} - 1}{\beta}\right)$$

Now, the for the trading intensity we get:

$$\lim_{b \to +\infty} A \exp\left(-k\delta_b^{a*}(t,q)\right) = \left(1 + \frac{\gamma}{k}\right) q \frac{\beta}{e^{\beta(T-t)} - 1}$$

Hence, because the limit of the intensity is proportional to q, the limit $V(t)$ of the trading curve is characterized by the following ODE:

$$V'(t) = -\left(1 + \frac{\gamma}{k}\right) V(t) \frac{\beta}{e^{\beta(T-t)} - 1}, \qquad V(0) = q_0$$

Solving this equation, we get:

$$\begin{aligned}
V(t) &= q_0 \exp\left(-\left(1 + \frac{\gamma}{k}\right) \int_0^t \frac{\beta}{e^{\beta(T-s)} - 1} ds\right) \\
&= q_0 \exp\left(-\left(1 + \frac{\gamma}{k}\right) \int_{e^{\beta(T-t)}}^{e^{\beta T}} \frac{1}{\xi(\xi - 1)} d\xi\right) \\
&= q_0 \exp\left(-\left(1 + \frac{\gamma}{k}\right) \left[\ln\left(1 - \frac{1}{\xi}\right)\right]_{e^{\beta(T-t)}}^{e^{\beta T}}\right) \\
&= q_0 \left(\frac{1 - e^{-\beta(T-t)}}{1 - e^{-\beta T}}\right)^{1 + \frac{\gamma}{k}}
\end{aligned}$$

When $\beta = 0$ ($i.e.$ $\mu = 0$) we proceed in the same way or by a continuity argument. $\square$

Chapter 4

Calibration issues

4.1 Introduction

In the last two chapters we took as a starting point the Avellaneda-Stoikov model in both market-making and brokerage frameworks. Here we present a framework for its calibration. As previously discussed, the model, introduced in [13] and expanded on [65, 64, 101], successfully unifies in a unique model the SDE-driven nature of the dynamics of long-term price movement and the point-process nature of the liquidity process. The main feature of this model is to take as primary point of view a liquidity capturing algorithm to control price-risk while choosing optimal quotes in real-time. In particular, the goal of the model is not only to describe markets but to act as input for a trading algorithm.

In a nutshell, the model is characterized by three parameters: σ, the volatility of the price process, which allows the algorithm to quantify price risk, and two parameters accounting for the liquidity: a parameter A related to the trading intensity and a parameter k related to the market depth, the bid-ask spread and the order book shape.

Because of the statistical nature of the model, interpreting the different variables when facing real-data is far from being a trivial issue; this is due to the fact that the model does not give a mechanical tick-by-tick representation of the price formation process at the order book level, but only an approximation through the model parameters.

In this chapter we study the calibration of the model by using Level I order book data[1], and the statistical issues concerning the parameter estimation. We will also apply the framework on data from liquid European stocks, relating the parameters to physical market quantities such as the spread, the market depth and the trading intensity.

The main results we will present are, first, a framework to calibrate the model by using real data. We prove mathematical results related to the convergence and efficacy of the estimators. Finally we present examples using real data by relating the model parameters to other market metrics used by practitioners.

[1] Information about deals and liquidity at best bid/ask level.

4.2 Model: interpretation and extensions

As said before and presented in previous chapters, the idea is to define a statistical model where price dynamics and liquidity are decomposed into two different processes: a continuous one for price, and a discrete one for liquidity. Mathematically, the model considers two different objects, the reference price and the liquidity process.

The reference-price is modeled by a Brownian motion:

$$S_t - S_0 = \sigma W_t, \quad t \in [0, T]. \tag{4.1}$$

As for liquidity we proceed as follows: for an order placed at a price $S_t \pm \delta$, the instantaneous probability for this order to be executed between t and $t + dt$ is $\lambda(\delta)dt$ where the intensity $\lambda(\delta)$ is given by:

$$\lambda(\delta) = Ae^{-k\delta}, \tag{4.2}$$

(here dt represents an infinitesimal lapse of time).

In order to interpret the model in the light of real-data, we will look at markets from the point of view of a liquidity-capturing algorithm placing passive orders in order books during a time window of length $\Delta T \approx$ few seconds), then updating its orders throughout the trading session, defined by the interval $[0, T]$. In other words, in the model, the order book (conditional to reference price and $\delta > 0$) is a black-box characterized by the parameters σ, k and A.

For instance, assume at time t that the algorithm posts passive orders at prices $S_t + \delta$, and that these orders remain unchanged over the whole time-window $[t, t + \Delta T]$. From this point of view, the captured flow will be a Poisson variable with intensity:

$$\Lambda^+(\delta, t, t + \Delta T) = \int_t^{t+\Delta T} Ae^{-k\delta + k(S_u - S_t)} du. \tag{4.3}$$

Symmetrically, for an order below the reference price at $S_t - \delta$, the intensity of the Poisson variable representing the captured flow is given by:

$$\Lambda^-(\delta, t, t + \Delta T) = \int_t^{t+\Delta T} Ae^{-k\delta - k(S_u - S_t)} du. \tag{4.4}$$

In practice, the reference price S_t, $t \in [0, T]$, represents usually the mid-price (for market-making problems) but it can also represent the best-opposite price (for optimal liquidation). The required features we ask for a reference price are:

- To approximate the asymptotic behavior of the price (its volatility represents price risk).
- It serves as reference point to place the orders (so, ideally, a price inside the spread).
- Parameters as ΔT are chosen such as the exponential form consistent with the actual probabilities of observing trades at prices $S_t \pm \delta$ over a time-window $[t, t + \Delta T]$.

Before moving forward, it is important to notice that two processes are involved in the model: the price S and a generalized Poisson-process with compensator $\Lambda^{-}(\delta, t, t+s)$, $s \in [0, \Delta T]$. So, at a fixed time, the instantaneous probability of observing a trade at distance δ from the reference price is independent of the reference price itself. However, the total captured flow on $s \in [0, \Delta T]$ is indeed dependent upon the realized trajectory of the reference price.

In fact, the captured flow depends indeed on the volatility, the trajectory of the price, and moreover, the captured flow gives us partial information about the movement of the reference price. In particular, our model, though symmetrical at its core, is not contradictory with the empirical relationship that can be found between captured flow and price imbalances. We will discuss this point at the end of this chapter.

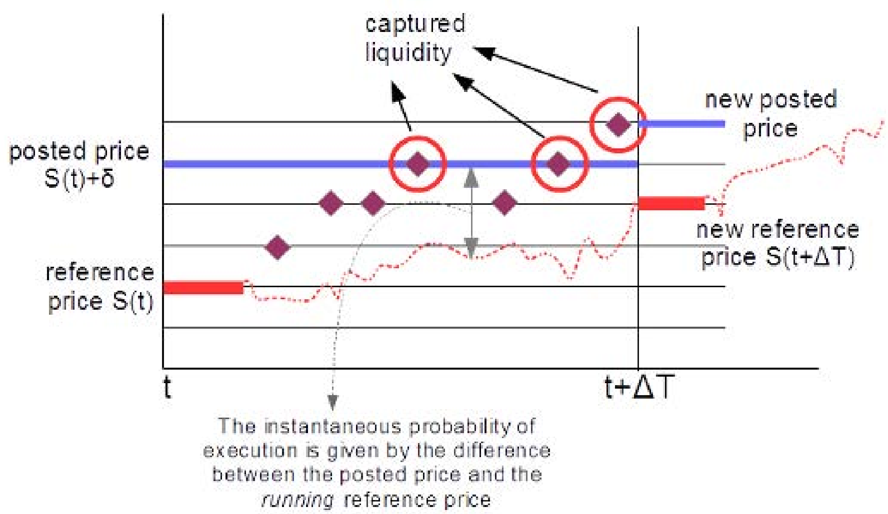

Figure 4.1: Scheme of our interpretation of the Avellaneda-Stoikov model. The dark diamonds represent trades. Captured liquidity are all those trades above the posted price. The instantaneous probability of execution is a function of the running reference-price (dashed line) and the posted price.

4.2.1 Extension to several types of price processes

At an instantaneous level, the reference price represents the part of the order flow dynamics that contains the information about the long-term price movements. It is natural then, for practical trading purposes (hence, an interest in considering the price from the point of view of its historical probability and not a risk-neutral one), to study more general models than the Brownian case proposed above. This allows to consider situations where the algorithm has views about future price movements (because of new information, endogenous factors or economical factors).

In this work, we will show that we can adapt the Avellaneda-Stoikov model to other reference prices than a Brownian motion, for example: mean-reverting price or the presence of trends. These settings represent baseline market regimes that are important for practitioners. At the same time, these regimes can be approximated mathematically by simple expressions that are straightforward to handle (like linear trends or Ornstein-Ulhenbeck processes).

From an economical standpoint, what justifies studying different regimes is that, depending on information, expectations, liquidity, volatility or traders' psychology, the price trajectories can behave in different ways. For example:

- *Mean-reverting prices* (the prices get stabilized around a certain consolidated level while market participants expect some news or the price sticks around a psychological threshold).
- *Trends* (hedge-funds executing a trading algorithm after a good economical news).
- *Diffusive behavior* (uncertainty about the price as the main actors had already taken their profits after a trend).

In terms of mathematical model, we will consider the following cases:

- *Diffusive prices* (modeled by a Brownian motion):

$$S_t = S_0 + \sigma W_t, \quad t \in [0, T]. \tag{4.5}$$

- *Mean-reverting price* (modeled by an Ornstein-Uhlembeck process):

$$S_t = S_0 + \sigma \int_0^t e^{-\theta(t-s)} dW_s, \quad t \in [0, T]. \tag{4.6}$$

- *Trends.* Those can be added in three different ways:

  1. A trend over a diffusive process:

$$S_t = S_0 + \mu t + \sigma W_t, \quad t \in [0, T]. \tag{4.7}$$

  2. A trend around a mean reverting process:

$$S_t = S_0 + \mu t + \sigma \int_0^t e^{-\theta(t-s)} dW_s, \quad t \in [0, T]. \tag{4.8}$$

  3. As a mean-reverting process 'converging' to a fixed level μ:

$$S_t = S_0 + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s, \quad t \in [0, T]. \tag{4.9}$$

4.3 Calibration of parameters A and k

For simplicity, we will only focus here in the one-sided situation, i.e. optimal liquidation with limit orders. Without loss generality, we consider an algorithm posting orders in the ask-side (i.e. a selling algorithm).

Assume we know for each $\delta \in \mathbb{N}$ and $t \in [0,T]$ the value for $\Lambda(\delta, t, t+\Delta T) := \Lambda^+(\delta, t, t+\Delta T)$. We recall from (4.3)

$$\Lambda(\delta, t, t+\Delta T) = \int_t^{t+\Delta T} A e^{-k\delta + k(S_u - S_t)} du, \tag{4.10}$$

which yields to

$$\log\left(\mathbb{E}\left[\Lambda(\delta, t, t+\Delta T)\right]\right) = \log(A) - k\delta + \log\left(\int_t^{t+\Delta T} \mathbb{E}\left[e^{k(S_u - S_t)}\right] du\right). \tag{4.11}$$

With that in mind, we can define the following calibration procedure:

Definition 1. *(Calibration of A and k) Let us consider a reference price process with stationary increments and existing Laplace transform. So, we note*

$$\varphi(k, \xi) = \mathbb{E}\left[e^{k(S_\xi - S_0)}\right].$$

Let $\widehat{\Lambda}(\delta)$ be an estimate of $\mathbb{E}\left[\Lambda(\delta, 0, \Delta T)\right]$, for $\delta \in \{\nu, \ldots, i_{\max}\nu\}$ (ν is the tick-size). Then the calibrated values of A and k correspond to the minimizer of

$$r(A,k) = \sum_{\delta=1}^{\delta_{\max}} \left(\log\left(\widehat{\Lambda}(\delta)\right) + k\delta - \log(A) - \log\left(\int_0^{\Delta T} \varphi(k,\xi) d\xi\right)\right)^2.$$

For practical purposes this can be achieved by a least-squares approach.

Three problems need to be solved in order to successfully perform in the calibration:

- The computation of the quantity $\int_0^{\Delta T} \varphi(k,\xi) d\xi$ for different price dynamics.
- The estimation of $\mathbb{E}\left[\Lambda(\delta, 0, \Delta T)\right]$, for $\delta \in \{1, \ldots, \delta_{\max}\}$ ($\delta = i\nu$).
- The estimation of the parameter involved in the evolution of the reference-price process (e.g. the volatility σ).

Finally, the complete framework will be the following:

1. Estimation of the intensity $\Lambda(\delta)$ for each level δ: i.e. a distance δ-ticks from the reference price.
2. Specify a theoretical model for the continuous version of the reference price (Brownian, Ornstein-Ulhembeck, etc).
3. Estimate A and k using the computed values for $\Lambda(\delta)$ and the estimates for the volatility σ.

The following proposition provides formulas $\int_0^{\Delta T} \varphi(k,\xi)d\xi$ for different types of price processes. In the following sections we will study the estimation of $\Lambda(\delta)$ for different values of δ and a brief survey on how to estimate the volatility σ.

We obtain the following result:

Proposition 6. *(Formulas for $\int_0^{\Delta T} \varphi(k,\xi)d\xi$ for different types of price-process)*

1. *Brownian diffusion:* $S_t - S_0 = \sigma W_t$.

$$\int_0^{\Delta T} \varphi(k,\xi)d\xi = \frac{2}{k^2\sigma^2}\left(e^{\frac{k^2\sigma^2\Delta T}{2}} - 1\right).$$

2. *Brownian diffusion with trend:* $S_t - S_0 = \mu t + \sigma W_t$.

$$\begin{aligned}\int_0^{\Delta T} \varphi(k,\xi)d\xi &= \frac{1}{k\mu + \frac{1}{2}k^2\sigma^2}\left(e^{\left(k\mu+\frac{1}{2}k^2\sigma^2\right)\Delta T} - 1\right) \qquad \text{if } k\mu + \frac{1}{2}k^2\sigma^2 \neq 0,\\ \int_0^{\Delta T} \varphi(k,\xi)d\xi &= \Delta T \qquad \text{if } k\mu + \frac{1}{2}k^2\sigma^2 = 0.\end{aligned}$$

3. *Mean-reverting process:* $S_t - S_0 = \mu(1-e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)}dW_s$.

$$\int_0^{\Delta T} \varphi(k,\xi)d\xi = \int_0^{\Delta T} e^{k\mu(1-e^{-\theta\xi})} e^{\frac{k^2\sigma^2}{2}\left(\frac{1-e^{-2\theta\xi}}{2\theta}\right)}d\xi.$$

4. *Mean-reverting process with trend:* $S_t - S_0 = \mu t + \sigma \int_0^t e^{-\theta(t-s)}dW_s$.

$$\int_0^{\Delta T} \varphi(k,\xi)d\xi = \int_0^{\Delta T} e^{k\mu\xi} e^{\frac{k^2\sigma^2}{2}\left(\frac{1-e^{-2\theta\xi}}{2\theta}\right)}d\xi.$$

Proof. These results are obtained by straightforward computations:

- Brownian diffusion: It follows from a direct integration.

$$\int_0^{\Delta T} \varphi(k,\xi)d\xi = \int_0^{\Delta T} \mathbb{E}\left(e^{k\sigma W_\xi}\right)d\xi = \frac{2}{k^2\sigma^2}\left(e^{\frac{k^2\sigma^2\Delta T}{2}} - 1\right).$$

- Brownian diffusion with trend:

$$\int_0^{\Delta T} \varphi(k,\xi)d\xi = \int_0^{\Delta T} e^{k\mu\xi}\mathbb{E}\left(e^{k\sigma W_\xi}\right)d\xi = \int_0^{\Delta T} e^{\left(k\mu+\frac{1}{2}k^2\sigma^2\right)\xi}d\xi.$$

  In the case $k\mu + \frac{1}{2}k^2\sigma^2 = 0$ we obtain $\int_0^{\Delta T} \varphi(k,\xi)d\xi = \Delta T$,

$$\text{otherwise,} \quad \int_0^{\Delta T} \varphi(k,\xi)d\xi = \frac{1}{k\mu + \frac{1}{2}k^2\sigma^2}\left(e^{\left(k\mu+\frac{1}{2}k^2\sigma^2\right)\Delta T} - 1\right).$$

- Mean reverting process:

$$\int_0^{\Delta T} \varphi(k,\xi) d\xi = \int_0^{\Delta T} e^{k\mu(1-e^{-\theta\xi})} \mathbb{E}\left(e^{k\sigma \int_0^\xi e^{-\theta(\xi-s)} dW_s} \right) d\xi.$$

  As $\int_0^\xi e^{-\theta(\xi-s)} dW_s$ is a Wiener integral, we obtain:

$$k\sigma \int_0^\xi e^{-\theta(\xi-s)} dW_s \sim \mathcal{N}\left(0, k^2\sigma^2 \left(\frac{1-e^{-2\theta\xi}}{2\theta}\right)\right),$$

  so that

$$\int_0^{\Delta T} \varphi(k,\xi) d\xi = \int_0^{\Delta T} e^{k\mu(1-e^{-\theta\xi})} e^{\frac{k^2\sigma^2}{2}\left(\frac{1-e^{-2\theta\xi}}{2\theta}\right)} d\xi.$$

- Mean-reverting process with trend:

  Using the last formula we obtain:

$$\int_0^{\Delta T} \varphi(k,\xi) d\xi = \int_0^{\Delta T} e^{k\mu\xi} e^{\frac{k^2\sigma^2}{2}\left(\frac{1-e^{-2\theta\xi}}{2\theta}\right)} d\xi.$$

□

4.4 Estimation of $\Lambda(\delta)$ for fixed δ

As mentioned above, the first step of the calibration is to compute the intensity of the Poisson process defining the number of orders arriving at a distance δ from the price S_t in an interval $[t, t+\Delta T]$. Under the hypothesis that the events taking place in each interval are independent from the events taking place on the next one, the problem reduces to estimate the intensity of a Poisson process. Notice this is not true in the O-U situation and heavily relies on the independent increments hypothesis, however results can be extended in cases where auto-correlations decrease as a function of the time difference between two events. Our results can be adapted as far as we remains in the Markov situation where the law of the increment can be explicitly computed knowing the past.

There are two ways to interpret this problem:

- As an external observer: we take as data-set the observations of all the deals made during the time window $[(k-1)\Delta T, k\Delta T[$, for $k \in \{1, \dots, [\frac{T}{\Delta T}]\}$, from historical data. Here, we suppose that the probability of being executed at a distance δ that we observe from the historical data (trades of other participants), is the same as if we were participating in the market (i.e. our order does not influence the market dynamics; this is a reasonable hypothesis for small passive orders).

- As a participant: we can only observe whether our order is executed or not. In practice we observe one or zero executions. The most informative variable we have at hand is the time we wait until execution. Other wise said we want to estimate $\Lambda(\delta)$ from a set of observations $X_n = \tau_n \wedge \Delta T$, where τ_n is the time we wait until execution when posting an order at a distance δ from the reference price (or ΔT if the order is never executed).

These two viewpoints define the two estimators for $\lambda = \Lambda(\delta)$: an estimator counting the number of events, and another estimator computing the waiting time to get orders executed. We analyze both in the following subsections.

4.4.1 Estimating the intensity by counting trades

A way to estimate the intensity for a fixed δ is simply to count how many deals are made at that price level during the period. Under our assumptions, the number of trades over the period $[k\Delta T, (k+1)\Delta T]$ follows a Poisson distribution with intensity $\Lambda(\delta)$. Hence, by using a sample $X_1, X_2, \dots, X_n$ where X_k represents an observation of the number of deals during the period. In such framework the natural estimator is simply to consider:

$$\widehat{\Lambda}_n(\delta) = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

This estimator is unbiased and, by the strong law of large numbers, it converges almost surely towards $\Lambda(\delta)$. Moreover, by the CLT its asymptotic variance is $\Lambda(\delta)$.

4.4.2 Estimating the intensity by using the waiting times

The second, and more interesting, approach to estimate the intensity $\widehat{\Lambda}$ is to consider the waiting-time until execution. This means, instead of observing the number of events, we observe the set of variables $\min(\tau_n, T)$ where τ_n are exponential random variables with parameter $\lambda > 0$ which we want to estimate (the relationship between μ, λ and T will be explored further on) and T the size of the time window.

The advantage of this approach is that can also be implemented in situations where the other trades on the exchange are not observable and we can only observe our trades.

Proposition 7. *(Maximum likelihood estimator) Let $\tau_1, \dots, \tau_n$ be n i.i.d. exponential random variables with parameter λ. Let $X_n = \min(\tau_n, T)$ for some $T > 0$. The maximum likelihood estimator (MLE) for μ using $(X_1, \dots, X_n)$ is given by the formula:*

$$\hat{\lambda}_n = \frac{\sum 1_{X_i < T}}{\sum X_i}$$

Proof. If we suppose that the times τ_n are exponentially distributed, $\lambda\Delta T = \Lambda(\delta, t, t + \Delta T)$ and $a = \Delta T$ the distribution function of each of this variables is given by:

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0,a]}(x)dx + e^{-\lambda a} \delta_a(dx)$$

Let us consider the sample of $(X_1, \dots, X_n)$ such as $\mathrm{Card}\{i | X_i = a\} = k$. The likelihood function for this event is given by

$$\mathcal{L}(\lambda; X_1, \dots, X_n) = \lambda^{n-k} \exp\left(-\lambda \sum_{i=1}^{n} X_i\right).$$

The maximum likelihood estimator for λ is

$$\widehat{\lambda}_n = \frac{\sum_{i=1}^n \mathbf{1}_{X_i<a}}{\sum_{i=1}^n X_i}. \tag{4.12}$$

□

Theorem 2. *(A.s. convergence) The estimator $\widehat{\lambda}_n$ converges a.s. towards λ.*

Proof. It follows from the strong law of large numbers

$$\widehat{\lambda}_n = \frac{\sum_{i=1}^n \mathbf{1}_{X_i<a}}{\sum_{i=1}^n X_i} = \frac{n^{-1}\sum_{i=1}^n \mathbf{1}_{X_i<a}}{n^{-1}\sum_{i=1}^n X_i} \xrightarrow{a.s.} \frac{\mathbb{P}(X<a)}{\mathbb{E}(X)} = \lambda.$$

□

Proposition 8. *(Bias) The estimator $\widehat{\lambda}_n$ is biased i.e. $\mathbb{E}\left(\widehat{\lambda}_n\right) > \lambda$.*

Proof. By using a n-dimensional co-monotony principle[2] (see [119, 118]) and Jensen's inequality we get:

$$\mathbb{E}\left(\widehat{\lambda}_n\right) \geq \mathbb{E}\left(\sum_{i=1}^n \mathbf{1}_{X_i<a}\right)\mathbb{E}\left(\frac{1}{\sum_{i=1}^n X_i}\right) \geq \lambda. \tag{4.13}$$

□

Theorem 3. *(L^p-convergence) The estimator $\hat{\lambda}$ converges towards λ in L^p-norm.*

Proof. Directly from propositions 4 and 5 and the fact that a.s. convergence and L^p-bounded yields L^p convergence (by an uniform integrability argument). □

Proposition 9. *(L^p-bounds) The estimator $\hat{\lambda}$ is L^p-bounded*

Proof. This is a consequence of the next proposition. □

Proposition 10. *Let X be a positive random variable such as there exists a real constant $C>0$ such as $\mathbb{P}(X\leq\varepsilon)\leq C\varepsilon$ (in particular, this is the case for any positive random variable with bounded density).*

Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence with X_n having the same law as X, $\forall n\geq 1$. Then, the following inequality holds:

$$\mathbb{P}\left(\frac{1}{2n}\sum_{k=1}^{2n} X_k \leq \varepsilon\right) \leq C^n 8^n \varepsilon^n. \tag{4.14}$$

Moreover, for every $n\geq 1$, the inverse of the empirical mean satisfies for every $p>1$

$$\mathbb{E}\left[\left(\frac{2n}{\sum_{k=1}^{2n} X_k}\right)^p\right] \leq 2C^p 8^p, \quad \forall n > p. \tag{4.15}$$

[2] The multi-dimensional co-monotony principle states that if $f(x_1,\ldots,x_n)$ and $g(x_1,\ldots,x_n)$ are both increasing (resp. decreasing) in each variable, and (X_n) a vector of independent random-variables such as $f(X)$, $g(X)$ and $f(X)g(X)$ are in L^1 and not $a.s.$ constant, then $\mathbb{E}(f(X)g(X)) > \mathbb{E}[f(X)]\mathbb{E}[g(X)]$.

Proof. If a sum of $2n$ positive terms is smaller than $2n\theta$, it means that at least the half of the terms are smaller than 2θ. Hence:

$$\mathbb{P}\left(\frac{1}{2n}\sum_{k=1}^{2n} X_k \leq \varepsilon\right) \leq \binom{2n}{n}\mathbb{P}(X \leq 2\varepsilon)^n \leq \binom{2n}{n} C^n 2^n \varepsilon^n \leq 8^n C^n \varepsilon^n.$$

First of all we have the identity

$$\mathbb{E}\left(\left(\frac{2n}{\sum_{k=1}^{2n} X_k}\right)^p\right) = \int_{\frac{1}{a}}^{+\infty} y^{p-1}\mathbb{P}\left(\frac{\sum_{k=1}^{2n} X_k}{2n} \leq \frac{1}{y}\right) dy.$$

We can decompose this integral into two intervals separated at $y = 8C$.

$$\mathbb{E}\left(\left(\frac{2n}{\sum_{k=1}^{2n} X_k}\right)^p\right) \leq 8^p C^p + \int_{8C}^{+\infty} y^{p-1}\mathbb{P}\left(\frac{\sum_{k=1}^{2n} X_k}{2n} \leq \frac{1}{y}\right) dy.$$

At this point we can use the first inequality to get for every $n > p$,

$$\begin{aligned}
\mathbb{E}\left(\left(\frac{2n}{\sum_{k=1}^{2n} X_k}\right)^p\right) &\leq 8^p C^p + 8^n C^n \int_{8C}^{+\infty} \frac{1}{y^{n-p+1}} dy \\
&= 8^p C^p + 8^p C^p \int_{1}^{+\infty} \frac{1}{z^{n-p+1}} dz \leq 2C^p 8^p \\
&= 8^p C^p \left(1 + \frac{1}{n-p}\right).
\end{aligned}$$

□

The latter allows to easily prove that the estimator is bounded in the L^p norm. As a immediate corollary we have the L^p convergence as we already shown that the estimator converges almost surely.

Theorem 4. *The estimator $\hat{\lambda}$ is asymptotically normal. Indeed,*

$$\sqrt{n}(\widehat{\lambda}_n - \lambda) \longrightarrow \mathcal{N}\left(0, \frac{\lambda^2}{1 - e^{-\lambda a}}\right).$$

Proof. Let us write the quantity $\sqrt{n}(\widehat{\lambda}_n - \lambda)$.

$$\begin{aligned}
\sqrt{n}(\widehat{\lambda}_n - \lambda) &= \sqrt{n}\left(\frac{\sum_{i=1}^n \mathbf{1}_{X_i < a}}{\sum_{i=1}^n X_i} - \lambda\right) \\
&= \sqrt{n}\left(\frac{\sum_{i=1}^n \left(\mathbf{1}_{X_i < a} - \lambda X_i\right)}{\sum_{i=1}^n X_i}\right) \\
&= \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\mathbf{1}_{X_i < a} - \lambda X_i\right)}{\frac{1}{n}\sum_{i=1}^n X_i}.
\end{aligned}$$

The denominator converges almost-surely towards $\frac{1-e^{-\lambda a}}{\lambda}$.

The numerator converges, by the central-limit theorem, to a centered, normal random-variable with variance:

$$s^2 := \mathbb{E}\left[(\mathbf{1}_{X_1<a} - \lambda X_1)^2\right].$$

Straightforward calculations yield:

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{X_1<a}^2] &= 1 - e^{-\lambda a} \\
\mathbb{E}[\lambda \mathbf{1}_{X_1<a} X_1] &= 1 - e^{-\lambda a} - \lambda a e^{-\lambda a} \\
\mathbb{E}[\lambda^2 X_1^2] &= -2a\lambda e^{-\lambda a} + 2(1 - e^{-\lambda a}).
\end{aligned}$$

Which finally leads to:

$$s^2 = 1 - e^{-\lambda a}$$

We conclude the proof by applying the Slutsky theorem.

□

4.5 Estimation of the volatility σ

We briefly discuss the problem of estimating the parameter σ in the intraday case. This problem is far from being trivial and has been widely studied in [15, 124, 136] is beyond the scope of this work.

The main issue arising when estimating the volatility using price data sampled at a high-frequency basis is that the classical volatility estimator, that is

$$\widehat{\sigma^2}_n = \frac{1}{t_n - t_0} \sum_{k=1}^{n} (S_{t_k} - S_{t_{k-1}})^2,$$

does not converges to what we would like to (a proxy of price risk) as the noise due to microstructure effects bias the estimator [136].

Several solutions have been proposed in the literature: sub-sampling, a composite estimator using sub-sampling at different frequencies, modeling of the microstructure noise or building the price by using Hawkes processes, among others. Because our interest is to avoid using high-frequency oscillations in the computation of the volatility, for our purposes, two approaches can be used: subsampling at a large enough frequency compared with ΔT.

A second approach is to use the estimator proposed by Garman and Klass [60] (originally devised for volatility across different days) adapted to the case of a time window of several minutes. The interest of this estimator is that it is based on the long-term oscillations of the price. Such features are exactly what we want to measure (since the local behavior of orders is contained in parameters A and k).

4.6 Confronting the model to market data

4.6.1 Introduction

In this section our goal is to explore real market data with in mind to see whether the parameters A and k can be related to the market quantities commonly used as measures of liquidity (depth, bid-ask spread and trading intensity).

The idea is to compute the variables A and k for a large number of stocks and for a large number of days. If the model is capturing the market behavior, we expect (intuitively) to obtain clear relationships between A and the trading intensity, on the one hand, between k and the spread and the market depth on the other hand. We show in this section that this hypothesis is satisfied. Thus, our model (and the way we interpret it) does capture the features of the market behavior that we want to highlight.

In what follows, we present our methodology and results.

4.6.2 Data scope

Our dataset focuses on a set of 120 European liquid stocks (from the DJ STOXX 600) between July 2012 and June 2013. We include Level I tick data, that is: price, volume and time of each deal, and a snapshot of the corresponding sizes and prices for the best-bid and best-ask levels.

Deal-data and order book data are not perfectly synchronized; this means in practice that determining whether the initiator of a trade was a buyer or a seller is not trivial. We compare the deal-price to best-bid and best-ask prices to decide whether a trade was initiated by a buyer or a seller. Timestamps are not exact (our precision is around 200 ms), moreover, if an aggressive order takes liquidity from two different participants, then two different trades are reported.

These noise sources emphasize the importance of a model based on a statistical approach over a tick-by-tick mechanical one.

4.6.3 Methodology

We consider $T = 60s$, we estimate $\Lambda(\delta)$ by using the waiting-times estimator (4.12) over the whole day. We estimate k and A using a regression method. We suppose that the underlying price is a Brownian diffusion.

4.6.4 Results I: k, depth and spread

In the model, the parameter k modulates the probability of being executed far from the mid-price; a larger k means that it is unlikely to be executed if we place orders with large δ, a smaller k means the opposite. Thus, intuitively, we expect the following:

- The larger (in Average Traded Sizes – ATS) the sizes on the first levels of the order book are, the less likely for orders far away from the reference price to be executed is. This means that large sizes should be characterized by large values of k.

- The wider the bid-ask spread (in basis points) is, the more likely for orders to be executed far away from the reference price (small k) is. This means, that we expect an inverse relationship between bid-ask spread and k.

Relation between k and market depth.

The following figure shows that our intuition is correct; there is a positive relation between the sizes on the first levels of the order book and our parameter k. By performing a linear regression after a log-log transformation, we obtain $k \sim \text{depth}^{\gamma}$ with $\gamma \approx 0.4$.

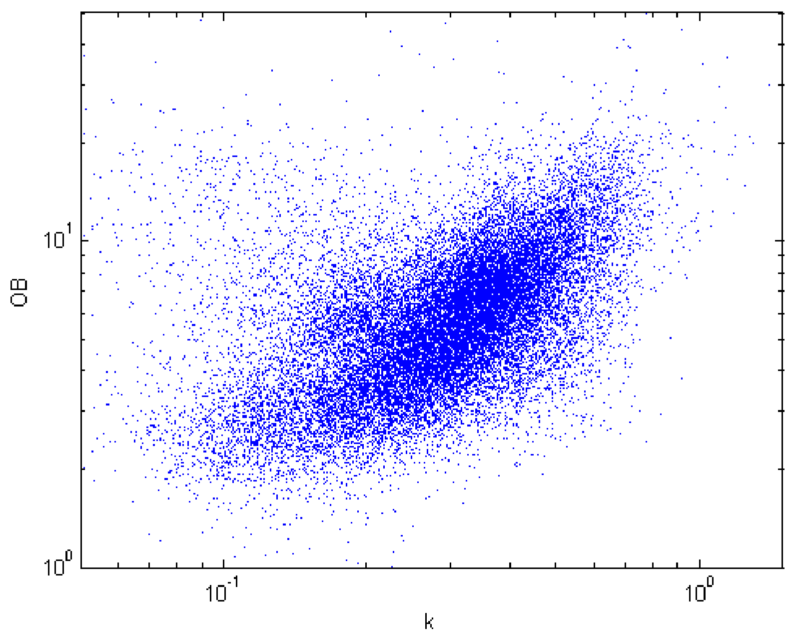

Figure 4.2: Positive relationship between k and market depth. This means that a higher value of k is representative of an order book where it is more difficult to observe trades far from the reference price; this is consistent with the initial guess.

Relation between k and bid ask spread.

The following plot shows again that the intuition is correct; there is a negative relation between the bid-ask spread (**in ticks**) and our parameter k. Indeed, by performing a linear regression after a log-log transformation, we obtain $k \sim \text{spread}^{-\gamma}$ with $\gamma \approx 0.4$.

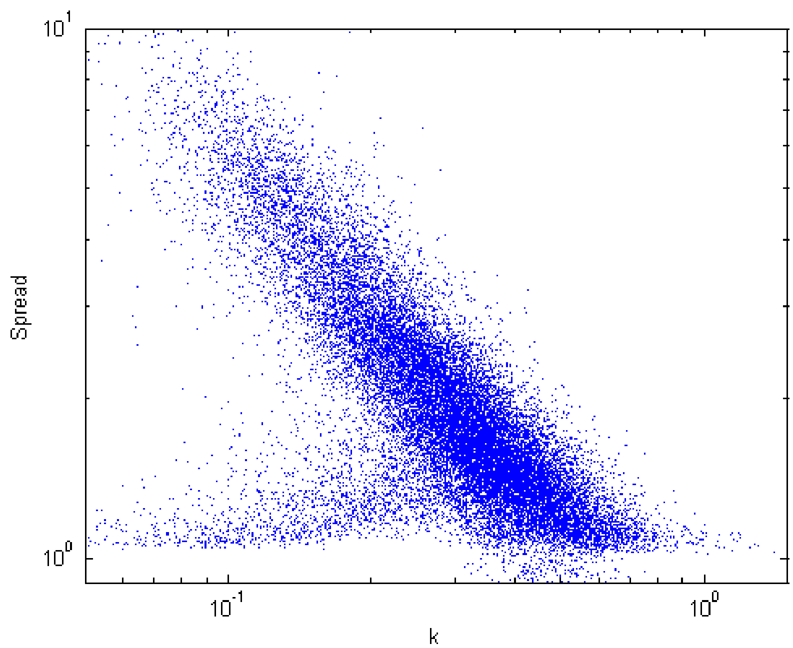

Figure 4.3: Negative relationship between k and the spread. This means that a higher value of k is representative of an order book with small spread, so trades tend to increase near the reference price; this is consistent with intuition. A caveat has to be made, as spreads gets narrower (close to one tick), this relation is no more valid as the tick-size effects are prominent (in this case we expect that depth – and priority – are the determinants of k).

k versus spread and depth

From the latter results it seems reasonable to consider now a relation of the form

$$k \sim \left(\text{depth} \times \text{spread}^{-1}\right)^{\gamma} \quad , \gamma \approx 0.4.$$

4.6.5 Results II: A, intensity and volatility

The parameter A is related to the trading activity. Intuitively, we expect that A will be positively related with the trading intensity. We also expect that A will be slightly related to the volatility σ, however, we would like this relationship not to be very pronounced since, in the model, A is supposed to capture the short term oscillations and σ to capture the amplitude of price movement in the long run, regardless of the intensity.

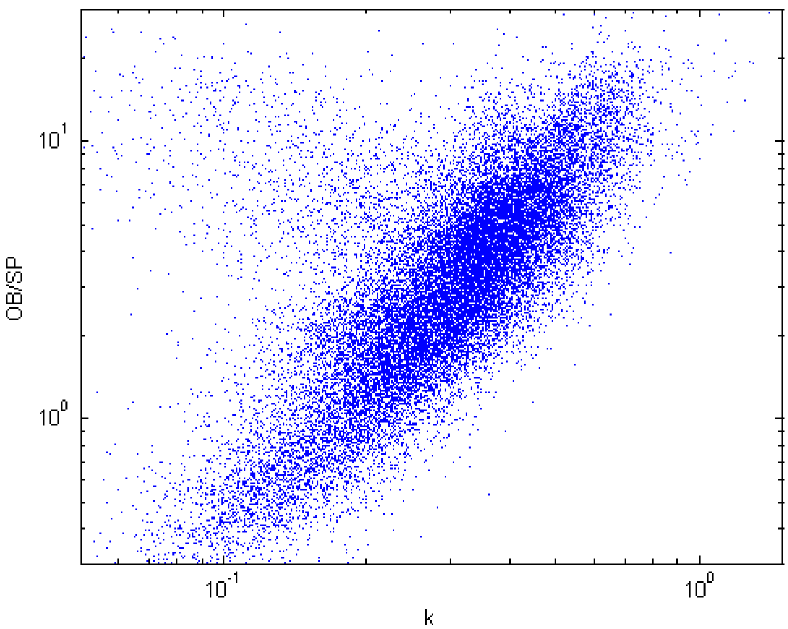

Figure 4.4: Data show that, as intuition predicts, k characterizes the market quantities related with the probability of execution at different levels of δ (spread and depth).

Relation between A and trading intensity

Relation between A and volatility

4.6.6 Practical issues

Some issues remains to be discussed, among others, how integrating into the model variability of volume sizes, the time priority in the order book, and how to choose the time-scale ΔT.

Variability of traded-sizes

The model supposes that the traded-sizes are unitary. This is not true in practice; traded-sizes are variable, meaning that sometimes our orders can be only partially consumed. When implementing the solution for the optimization-problem or when performing the parameter estimation, there is not a unique obvious way to interpret this fact.

A way to conciliate variable-sizes and the model is to think in terms of average-traded-sizes (ATS) as the unit of measurement for the volumes (thus, if we are using the waiting-times method to estimate intensities, we compute the expected-time in order to execute some fixed number of ATS). The caveat with this method is that traded volumes have skewed distributions; taking the median is sometimes better than the average.

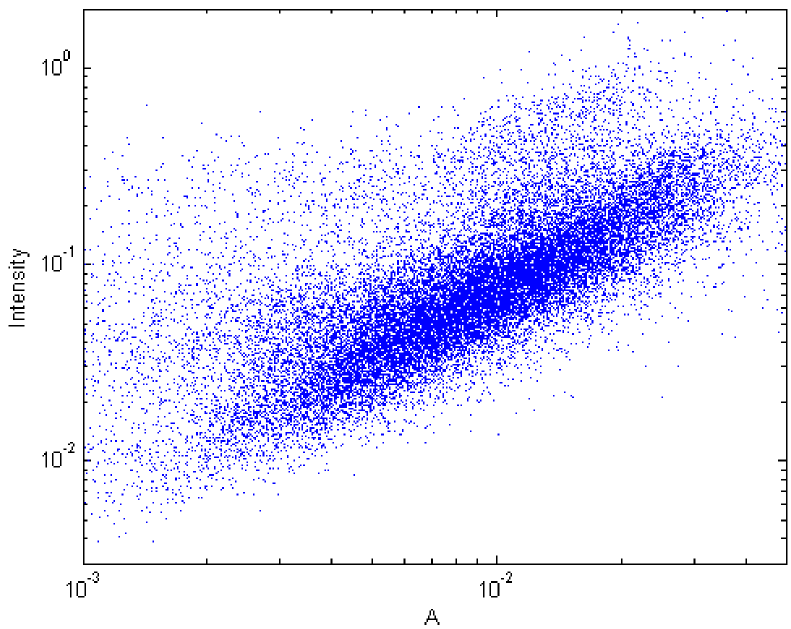

Figure 4.5: Again, as intuition predicts, the parameter A is positively related to the trading intensity.

Time priority

If two limit orders are at the same level in the order book we know that they will be executed according to which order arrived at first in the order book. This feature is statistically incorporated in the model through the parameter k.

Choice of ΔT

The parameter ΔT is chosen in order to decompose the microstructure effects from the asymptotic behavior of the price. We also want this to be accomplished in a consistent way with the exponential form for the probability of execution. Concretely, choosing a ΔT which is to small will create an overlap between the effects of A (trading intensity) and σ: we will consider short term oscillations in the price as price risk, which is not desirable. On the other hand, choosing a too large ΔT, will create an overlap between the effects of k (probability of execute orders far from the reference price) and the volatility effects (price movements are driven orders being executed far from the reference price).

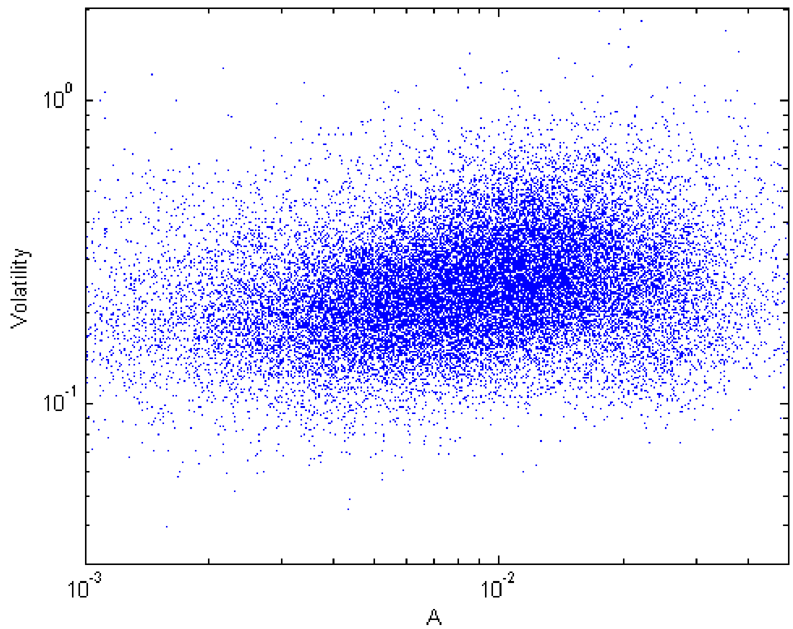

Figure 4.6: We do not find convincing evidence that volatility and A are related (in practice the value of the parameter ΔT should be calibrated in a way that the relationship between A and σ is as weak as possible, i.e. volatility measures price risk and it is not contaminated with local oscillations due to trading intensity).

4.6.7 Volume imbalances and price movement

It can be argued that the model is not realistic as it is symmetric in terms of probability of observing orders on both sides of the reference price, while real data show daily imbalances that are in fact correlated with the price movements [93]. This is indeed an important point to be considered for the interpretation of the Avellaneda-Stoikov model.

- If the model is interpreted in a naive 'instantaneous' way, i.e. by considering the mid-price *at each instant* as the reference price and by simply understanding the parameter k as a proxy of the spread and the time priority, we force a frontal contradiction since empirical data show that there is a direct relationship between volume imbalance and price movements.

- On the other hand, if we interpret the model in the way we did throughout this chapter, there is no contradiction between the relation imbalance/price and the symmetry of the model. This is due to the fact that of fixing the reference price over a time window ΔT creates a dependency between the trajectory of the price and the realized flow.

Roughly speaking, the observed imbalance during a time window ΔT given the reference price, will be

$$I_{\Delta T} = \frac{\int_0^{\Delta T} e^{k(S_t - S_0)} dt - \int_0^{\Delta T} e^{-k(S_t - S_0)} dt}{\int_0^{\Delta T} e^{k(S_t - S_0)} dt + \int_0^{\Delta T} e^{-k(S_t - S_0)} dt}.$$

That is, the model, even if symmetric, naturally relates the observed imbalance with the direction of the price.

4.7 Conclusion

In this chapter, we studied the Avellaneda-Stoikov model from the point of view of its calibration by using real-data. Several issues where discussed: the interpretation of the model, how to extend the model to a wider range of price dynamics, the calibration for parameters A and k, an estimator of the intensity of a Poisson process by using the waiting times and, finally, we confront the approach to real data and analyze the different issues (volatility estimation, volume imbalances, clean the data) arising in practice.

Chapter 5

On-line Learning and Stochastic Approximation

5.1 Introduction

In this chapter we will present the main results of the theory of stochastic approximation that we will be used in the next chapter. We will principally focus on how stochastic approximation plays a role in the design of recursive optimization algorithms in problems where an agent faces uncertainty; this is particularly useful when the source of randomness can be simulated numerically or obtained from a historical dataset. This is also a powerful tool when we want to optimize, iteratively and in real-time, the behavior of a system; in that case stochastic approximation methods are also known as *on-line learning*.

Designing recursive optimization algorithms through the theory of stochastic approximation is not a new phenomenon; a large part of the methods were devised and developed early in the second half of the 20th century (Robbins and Monro [123], Kiefer and Wolfowitz [85]). Because we nowadays experience an increase in the availability and velocity of data, computer speed but also because the performance of industrial and financial systems are mainly measured statistically, these methods adapt particularly well in today's real-life applications and are getting increasing attention from the industry.

In a nutshell, the idea of stochastic approximation is to find a value of interest (it can be a parameter we want to estimate, the critical point of a function or the zero of a vector field) in an iterative way by updating the *'current best-guess'* with the new (noisy) information arriving on a regular basis. This information can be a stream of real-time data, historical data or a set of simulated data. Simply put, let θ_n represent the process giving us the *current best-guess*: we would like to define a dynamical system (also known as *stochastic algorithm*) reading

$$\underbrace{\theta_{n+1}}_{\text{new value}} = \underbrace{\theta_n}_{\text{old value}} - \underbrace{\gamma_{n+1}}_{\text{step (weight)}} \times \underbrace{H(\theta_n, Y_{n+1})}_{\text{correction using incoming information}} \tag{5.1}$$

in such a way that this sequence is converging towards the true 'best-guess' θ^* we are looking for. Mathematically, this means that if the incoming information Y_n (or a transformation of it) satisfies a stationary ergodic property, then θ_n converges towards θ^*. The simplest setting is $(Y_n)_{n\geq 1}$ is i.i.d. In that sense, stochastic approximation can be seen as a generalization of the law of large numbers (indeed, the law of large numbers can be written recursively in the form of the equation (5.1)).

From a mathematical viewpoint, analyzing the convergence of this type of procedure relies heavily upon both the theory of martingales and the theory of the stability of ODEs. References on the mathematical theory can be found in the books by Duflo [49], Benveniste et al. [22] and Kushner et al. [95]. Also, as we will show, the way these procedures are designed is inspired by two deterministic methods widely known in numerical analysis: the Newton-Rhapson zero-search procedure and the gradient-descent method.

5.2 Examples of stochastic algorithms

5.2.1 Recursive estimation of the average

For some applications, how to choose the function $H(\cdot,\cdot)$ appearing in equation (5.1) emerges naturally from the nature of the problem. As an elementary example, we can consider a recursive algorithm to compute the average of a stream of data $(Y_n)_{n\geq 0}$ with Y_n i.i.d. and in L^1. Let us denote θ_n the estimated value at time n by the following procedure: every time a new value arrives, we weight the current estimated value by $(1-\gamma_n)$ and we sum the new information, that is Y_{n+1}, weighted by γ_n, $\gamma_n \in]0,1[$. Then

$$\theta_{n+1} = (1-\gamma_n)\theta_n + \gamma_n Y_{n+1}. \tag{5.2}$$

In this example, the function $H(\cdot,\cdot)$, from equation (5.1), correspond to:

$$H(\theta, y) = \theta - y. \tag{5.3}$$

Indeed, equation (5.1) can be rewritten as

$$\theta_{n+1} = \theta_n - \gamma_n(\theta_n - Y_{n+1}). \tag{5.4}$$

Notice that, if γ_n is constant, this is equivalent to an exponential moving average, and if $\gamma_n = (n+1)^{-1}$, we are in the situation of the law of large numbers. Indeed, it is easy to check that in that case the algorithm reads

$$\theta_n = \frac{1}{n}\sum_{k=1}^{n} Y_k$$

.

Moreover, the convergence of θ_n towards $\theta^* = \mathbb{E}[Y_1]$ can be also proved in the general case $\sum_n \gamma_n = +\infty$ and $\sum_n \gamma_n^2 < +\infty$, for $\gamma_n > 0$.

5.2.2 The two-armed bandit algorithm

A more interesting example is to use stochastic approximation in order to continuously learn how to optimally allocate a budget between two agents.

For instance, let us consider an investor hesitating between two ventures A and B. He decides to invest a share θ of his total budget in venture A and $(1-\theta)$ in venture B, with $\theta \in (0,1)$. Every time venture A report a good result (e.g. a sale or a new client) the investor re-allocates a share $\gamma \in (0,1)$ from venture B to the venture A (and vice-versa). For example, we can imagine at time $n=0$ the investor starts with an allocation $\theta_0 = \frac{1}{2}$.

Let us consider this quantity γ decreases as time advances (time is measured as the number of total good-results reported for both the two ventures) because the opinion of the investor becomes sounder so that he reduces the impact of the re-allocation. Thus, the evolution of the share for the venture A evolves by the following equation

$$\theta_{n+1} = (1-\gamma_{n+1})\theta_n + \gamma_{n+1}\mathbf{1}_{A_{n+1}}.$$

where A_n is the event: the $n+1$ positive report was from venture A (or as an stochastic-algorithm we can write $\theta_{n+1} = \theta_n - \gamma_{n+1}\left(\theta_n - \mathbf{1}_{A_{n+1}}\right)$).

This is a probabilistic version of the so-called *bandit-algorithm* and its convergence was studied in-depth in [97, 98]. It can be shown that under suitable conditions on γ_n the algorithm always converges toward the best venture, regardless of initial conditions.

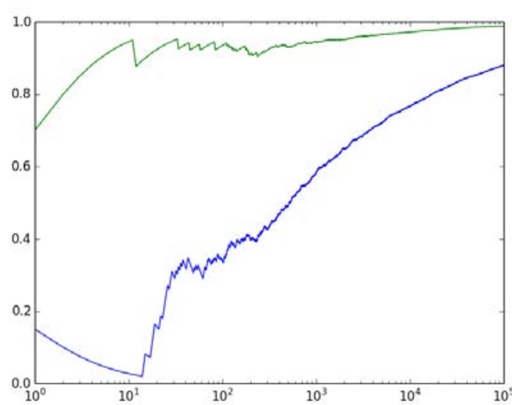

Figure 5.1: Evolution of the bandit algorithm with different initial conditions. The x-axis represent the number of iterations. The y-axis represents how much budget is allocated to the best venture (here A).

In the two preceding examples, the update function arises heuristically from the nature of the real-life problem, and a mathematical analysis would be intended just to prove that the procedure converges towards the expected value. So is not always the case: for some applications, the way in which the function $H(\cdot,\cdot)$ is defined is far from being trivial. However, two deterministic methods will help us to know how to define the function $H(\cdot,\cdot)$, depending on the underlying application: these two methods are the Newton-Rhapson method for zero-search and the gradient-descent method for optimization problems, as mentioned above. We will see that several parallels can be drawn between these deterministic methods and stochastic approximation; in terms of the shape of the algorithm and, also, on how the proof of the convergence is structured.

5.2.3 The Robbins-Monro algorithm

Another example (in line with what follows in the next section) is to consider the following zero-search procedure. Assume an input $\theta \in \mathbb{R}$ creates a random output measured by $H(\theta, Y)$ where Y is a random variable and H is either a known function or observations from real-data for which we only know the control we apply. Our goal is to find θ^* such as $\mathbb{E}[H(\theta^*, Y)] = \alpha$, where $h(\cdot) := \mathbb{E}[H(\cdot, Y)]$ is increasing and unknown.

Let us recall that, when the function $h(\cdot)$ is known, we can find the solution of the equation $h(x) = \alpha$ via the Newton-Rhapson algorithm which reads:

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \left(h(\theta_n) - \alpha \right).$$

The idea of the Robbins-Monro algorithm is to apply a similar procedure by taking, instead of h, the continuous realization of the (unknown) function $H(\theta, Y)$. This is, to find θ^* by applying the algorithm:

$$\theta_{n+1} = \theta_n - \gamma_{n+1}(H(\theta_n, Y_{n+1}) - \alpha), \quad \gamma_{n+1} \geq 0.$$

It is important to note that this can be seen from two perspectives:

- We do not know the function H but we have access to the observations $H(\theta_n, Y_{n+1})$ when we control the input (classical Robbins-Monro setting).

- We know the function H but we do not know the statistical laws driving the noise other than qualitative properties such as i.i.d., stationary or being a controlled Markov chain. This is the case most interesting for us, as in financial applications we count with a parametric model of how the control affects the outcomes.

Again, it can be shown (by the techniques we will present in this chapter) that in a wide range of situations, this procedure converges towards the target point.

5.2.4 Search of extrema in stochastic settings

A final example, and the most important for the purposes of this study, is the search for the critical points of a function. Two situations are considered: when the function is represented as the expectation of an expression that we can differentiate (stochastic gradient), or when we want to search for the maximum of a function for which we only have access via trial-and-error observations (Kiefer-Wolfowitz algorithm).

Stochastic gradient

The goal is to find the minimum of a function of the form $f(\theta) = \mathbb{E}\left[F(\theta, Y)\right]$.

A widely used method in numerical analysis is the well-known gradient-descent algorithm

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \nabla f(\theta_n)$$

which converges towards the critical point of the function f, where $f(\theta) = \mathbb{E}[F(\theta, Y)]$.

The idea is to apply a similar procedure to the function F, i.e.

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \nabla_\theta F(\theta_n, Y_{n+1}).$$

As we will show in this chapter, under certain conditions, this procedure converges towards the target point. For us, this is the most interesting case as in general the goal is usually to find the value which maximizes (or minimizes) a function defined by an expectation.

The Kiefer-Wolfowitz algorithm

Let us consider, in the Robbins-Monro situation, an unknown function to which we apply an input θ and obtain a noisy output $F(\theta, Y)$. Instead of wanting to reach a given level, we would like to maximize the average output when the function $f(\theta) = \mathbb{E}[F(\theta, Y)]$ is concave. The difference between the procedure presented below and the stochastic gradient is that, in the Kiefer-Wolfowitz situation we do not know the function, we only observe its effects, and thus computing the local gradient is not possible.

Because in a concave framework, the maximization of a function amounts to finding the zero of its gradient, the idea of the Kiefer-Wolfowitz algorithm is to apply the Robbins-Monro procedure to a discrete version of the derivative, i.e. the algorithm reads

$$\theta_{n+1} = \theta_n + \frac{\gamma_{n+1}}{2c_n} \left(f(\theta_n + c_n, Y_{n+1}) - f(\theta_n - c_n, Y_{n+1}) \right).$$

The convergence of this algorithm is studied in [85].

5.2.5 Caveat: convergence and implementation

When implementing stochastic algorithms as in the previous examples, usually there are some differences between the hypothesis under which the theorem holds and the real life scenario.

Constant-step algorithms

It is interesting in practical applications to define algorithms with constant step $\gamma_n = \gamma$. The idea is to have an estimator that can take advantage of regime changes along the life of the algorithm. This is similar to the case where a moving-average is preferred to the estimator of the average in order to have an estimator take into account regime changes. In this study we do not present the formal analysis of the convergence of constant-step algorithm, however in the implementation of real-life processes it could be used.

Ruppert-Polyak algorithm

As we will see in the following sections, asymptotically the fastest convergence speed is reached through a Central Limit Theorem of the form in which the optimal variance depends on the rate of decrease of the step-size.

When $\gamma_n = Cn^{-a}$ for an exponent $\frac{1}{2} < a \leq 1$ it can be shown that the fastest convergence is reached for $a = 1$. However, smallest values for a are interesting too as they allow the algorithm to explore the space more during the early stages. The Ruppert-Polyak approach (presented in the last section) is a way to get the best of two worlds: fast asymptotic convergence and exploration of the space during early stages.

The idea is to implement the algorithm with step $a < 1$ but take as 'solution' for each period the current average of the values taken by the stochastic algorithm. Indeed, it can be proven that the sequence:

$$\bar{\theta}_n = \frac{1}{n}\sum_{k=1}^{n}\theta_k$$

converges optimally towards the limit point of the algorithm (in terms of the associated CLT). Details are provided in the last section of this chapter.

Dependence in the innovations

A strong assumption we carry throughout the mathematical results we will present in this chapter, is that the innovations Y_n in (5.1) are i.i.d. This is not always the case in real applications as it is common to face problems in which the dynamics of the innovations have dependence features. However, some of the results we will present are still valid in this situation (see the works of Doukhan et al. [48] and, recently, Laruelle [99]). Results also exist when the innovations are controlled Markov chains [131].

Algorithms with projections

Another (usual) situation in which the hypothesis of our results can differ from the real-life context is in the case when the domain in which the solutions of the algorithm should reside is a given set, and the dynamic of the algorithm should not explore points outside this zone (e.g. an algorithm which is constrained to live in a surface).

To solve this kind of situation, variations can be derived from the form (5.1), by projecting after each iteration the value of θ_{n+1} on the set within which we want the solution to reside. This kind of approach has been studied in [108, 39].

Choice of constants

The choice of constants in the stochastic algorithm can impact the variance obtained in the CLT. It can be shown than for an algorithm of the form

$$\theta_{n+1} = \theta_n - \gamma_{n+1}H(\theta_n, Y_{n+1}), \quad \gamma_n = \frac{c}{n},$$

with $h(\theta) = \mathbb{E}[H(\theta, Y)]$, that the optimal speed of convergence is obtained, in the one-dimensional case, for

$$c^* = \frac{1}{h'(\theta^*)}$$

which leads to an asymptotic variance of

$$\frac{\mathrm{Var}(H(\theta^*, Z))}{h'(\theta^*)^2}.$$

In order to control the step at the beginning of the learning procedure (when n is small we can give too much weight to the initial information), we can use a second constant $b > 0$ and choose as step

$$\gamma_n = \frac{1}{h'(\theta^*)(b+n)}.$$

This is mainly a theoretical result as in practice the value $h'(\theta^*)$ is usually unknown. A better way to obtain an optimal convergence speed is through the Ruppert-Polyak procedure.

5.3 Convergence of stochastic algorithms

5.3.1 Outline of the proofs

Roughly speaking, stochastic approximation can be presented as a probabilistic extension of Newton-Rhapson-like zero search recursive procedures of the form

$$\forall n \geq 0, y_{n+1} = y_n - \gamma_{n+1} h(y_n), \quad 0 < \gamma_n < \gamma_0$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field satisfying a sub-linear growth assumption at infinity.

Mean-reverting assumption

The convergence of this class of methods relies on a mean-reverting assumption. In one dimension, this can be obtained by a nondecreasing assumption made on the function h, or more simply by assuming that $h(y)(y - y^*) > 0$ for every $y \neq y^*$. In higher dimensions, this assumption becomes $\langle h(y), y - y^* \rangle > 0$ and will be extensively called upon in the following pages.

More generally mean-reversion follows from the existence of a so-called *Lyapunov function.* To introduce this notions, let us make a connection with Ordinary Differential Equations.

The Lyapunov function

Let us consider the dynamical system $ODE_h \equiv \dot{y} = -h(y)$. A Lyapunov function for ODE_h is a function $L : \mathbb{R}^d \to \mathbb{R}_+$ such that any solution $t \mapsto x(t)$ of the equation satisfies that $t \mapsto L(x(t))$ is nonincreasing as t increases. If L is differentiable this is mainly equivalent to the condition $\langle \nabla L, h \rangle \geq 0$ since

$$\frac{d}{dt} L(y(t)) = \langle \nabla L(y(t)), \dot{y}(t) \rangle = -\langle \nabla L, h \rangle (y(t))$$

If such a Lyapunov function does exist (which is not always the case), the system is said to be dissipative.

The two frameworks

There are two situations regarding the role of the Lyapunov function on the analysis of stochastic algorithms:

- The function L is identified *a priori*, for example in a stochastic-gradient procedure, is the function itself we want to minimize which can be the Lyapunov function of the system (the ODE being defined by its gradient – or proportional to its gradient).
- The function of interest h arises naturally from the problem (e.g. Robbins-Monro, bandit algorithms, etc.) and one has to search for a Lyapunov function L (which may not exist). This usually requires a deep understanding of the problem from a dynamical point of view.

Martingale analysis

One method to prove the convergence of the stochastic approximation procedure (as we will see in the next sections) is by studying the convergence of the sequence θ_n evaluated on some function L serving as Lyapunov function. Using a super-martingale property, the mean-reverting condition and a the divergence of the series γ_n, we prove the convergence of the algorithm. This approach is essentially of a probabilistic nature.

The ODE method

Another way to prove the convergence (that will be called *the ODE method*) is by considering the algorithm as a perturbed version of the Euler method of the related ODE. The idea is to prove, via topological and functional analysis arguments, that the limit points of the dynamical system without the perturbation are the same as the limit points of the stochastic algorithm.

5.3.2 Convergence of deterministic algorithms

Let us first consider the deterministic situation. In this case we take a vector field $h : \mathbb{R}^d \to \mathbb{R}^d$ and its associated zero-search algorithm:

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n), \quad \theta_0 \in \mathbb{R}^d, \tag{5.5}$$

where γ_n is a (strictly) positive sequence.

The next theorems shows that under a set of hypothesis, the convergence set of this algorithm, is a point θ^* such as $h(\theta^*) = 0$.

Theorem 5. *Assume h is continuous, increasing at a linear rate (i.e. $|h(\theta)| \leq C(1-|\theta|)$) and satisfies the following mean-reversion condition:*

$$\langle h(\theta), \theta - \theta^* \rangle > 0. \tag{5.6}$$

If furthermore γ_n satisfies

$$\forall n \geq 1, \quad \gamma_n > 0, \quad \sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty. \tag{5.7}$$

then the algorithm (5.5) *converges towards the only zero of h.*

Proof. Because of equation (5.5), we have

$$\begin{aligned} |\theta_{n+1} - \theta^*|^2 &= |\theta_n - \theta^*|^2 - 2\gamma_{n+1}\langle h(\theta_n)|\theta_n - \theta^*\rangle + \gamma_{n+1}^2 |h(\theta_n)|^2 \\ &= |\theta_n - \theta^*|^2 - 2\gamma_{n+1}\langle h(\theta_n)|\theta_n - \theta^*\rangle + C\gamma_{n+1}^2(1 + |\theta_n - \theta^*|^2). \end{aligned}$$

This implies that the sequence

$$s_n = \frac{|\theta_n - \theta^*|^2 + \sum_{k=1}^n \gamma_k \langle h(\theta_{k-1}|\theta_n - \theta^*\rangle + C\sum_{k\geq n+1} \gamma_k^2}{\prod_{k=1}^n (1 - C\gamma_k^2)}$$

is positive and decreasing, then, it converges towards a finite limit s_∞. The latter implies

$$|\theta_n - \theta^*|^2 \to l_\infty \quad \text{and} \quad \sum_{k=1}^n \gamma_k \langle h(\theta_{k-1})|\theta_{k-1} - \theta^*\rangle < +\infty.$$

As $\sum_{n\geq 1} \gamma_n = +\infty$ the second inequality yields

$$\liminf \langle h(\theta_n)|\theta_n - \theta^*\rangle = 0.$$

There exists a subsequence $\theta_{\varphi(n)}$ such that $\langle h(\theta_{\varphi(n)})|\theta_{\varphi(n)} - \theta^*\rangle \to 0$. Up to a new extraction there exists a subsequence $\theta_{\phi(n)}$ such as $\theta_{\phi(n)} \to \theta^*$, since h is continuous.

Then $|\theta_{\phi(n)} - \theta^*|^2 \to 0$ which implies that $\theta_n \to \theta^*$. □

In order to extend the idea to optimization algorithms, first notice that the precedent scheme can be interpreted as finding the minimum of the function $L : \theta \mapsto |\theta - \theta^*|^2$.

Theorem 6. *Let us consider a function $L : \mathbb{R}^d \to \mathbb{R}_+$, $L \in \mathcal{C}^1$, ∇L Lipschitz, and satisfying the following conditions controlling its growth:*

$$\begin{aligned} |\nabla L|^2 &\leq C(1 + L), \quad (5.8) \\ \lim_{|\theta|\to+\infty} L(\theta) &= +\infty. \quad (5.9) \end{aligned}$$

If h is continouous with $\sqrt{L}$-linear growth, that is

$$|h| \leq C\sqrt{1 + L}$$

and satisfies the following mean-reversion condition:

$$\langle \nabla L, h \rangle > 0 \quad \text{on } \{h \neq 0\}.$$

Then the algorithm converges towards the zero of h. If L is convex, then $\{\nabla L = 0\}$ is convex and the algorithm converges towards $\{\nabla L = 0\}$, that coincides with the argument of the minimum of L.

In particular, we can take $h = \nabla L$ if our goal is to minimize L. In that case the algorithm is called a gradient-descent method.

Proof. See [119]. □

5.3.3 Convergence through ODE analysis

A way to study recursive procedures is to relate equation (5.5) to the discrete-time approximation of the ODE $\equiv \dot{\theta} = -h(\theta)$. Indeed, if we set $\theta(\Gamma_n) = \theta_n$, where $\Gamma_n = \gamma_1 + \ldots + \gamma_n$, $n \geq 1$, then we have the equation

$$\frac{\theta_{n+1} - \theta_n}{\gamma_{n+1}} = -h(\theta_n),$$

which correspond to an approximation of $\dot{\theta}_n$, otherwise said, the algorithm can be seen as the Euler scheme (with decreasing step) of the ODE.

Theorem 7. *Let $L : \mathbb{R}^d \to \mathbb{R}_+$ essentially quadratic, i.e. such as $L \in \mathcal{C}^1$, ∇L Lipschitz,*

$$|\nabla L|^2 \leq C(1 + L) \quad \text{and} \quad \lim_{|\theta| \to \infty} L(\theta) = +\infty.$$

If h is continous such as $|h| \leq C\sqrt{1 + L}$ and $\theta \mapsto \langle \nabla L, h \rangle$ is non-negative and lower semi-continuous, then, the set of limiting points of the sequence $(\theta_n)_{n \geq 0}$ denoted Θ^∞, is a connected component of $\{L = l_\infty\} \cap \{\langle \nabla L, h \rangle = 0\}$.

Moreover, if h is continuous, then Θ^∞ is a connected, compact subset, stable by both ODEs $\dot{\theta} = -h(\theta)$ and $\dot{\theta} = h(\theta)$.[1]

Proof. By using the proof of the direct approach, we have:

- $L(\theta_n) \longrightarrow l_\infty \in [0, \infty[$ then $(\theta_n)_{n \geq 0}$ is bounded and $\theta_{n+1} - \theta_n \to 0$.
- $\sum \gamma_{n-1} \langle \nabla L, h \rangle(\theta_n) < +\infty$.

Thus, the set of limit points Θ^∞ of the sequence $(\theta_n)_{n \geq 0}$ is a connected, compact subset of $\{L = l_\infty\} \cap \{\langle \nabla L, h \rangle = 0\}$ (as it is well-chained).

[1] If we suppose beforehand that $(\theta_n)_{n \geq 0}$ is bounded, then we can weaken the step condition to

$$\gamma_n \to 0 \quad \text{and} \quad \sum \gamma_n = +\infty.$$

Let us set:

$$\theta^{(0)}_{\Gamma_n} := \theta_n, \quad \theta^{(0)}_t := \theta_0 - \int_0^t h(\theta^{(0)}_{\underline{s}})ds$$

where, by linear interpolation, $\underline{s} := \Gamma_n$ on $[\Gamma_n, \Gamma_{n+1})$.

The idea of the proof is to study the asymptotic behavior starting from Γ_n as $n \to +\infty$, by taking time as continuous, i.e. instead of studying $(\theta_n)_{n\geq 0}$ we will be interested in the sequence of functions

$$\theta^{(n)}_t := \theta^{(0)}_{\Gamma_n+t}, \quad t \geq 0.$$

Let $K = \overline{\{\theta_n, n \geq 1\}} \subset \mathbb{R}_+$. This is a compact set and for every $s, t \in [0, T]$, $s \leq t$,

$$|\theta^{(n)}_t - \theta^{(n)}_s| \leq \int_{\Gamma_n+s}^{\Gamma_n+t} |h(\theta^{(0)}_{\underline{u}})|du \leq |t-s| \sup_{x\in K} |h(x)|.$$

Hence the sequence $(\theta^{(n)})$ is uniformly relatively-compact for the topology of the uniform convergence on the compacts of $\mathbb{R}_+$ (noted U_K) by the Arzela-Ascoli theorem.

Let $\theta_\infty \in \Theta^\infty$, there exists a sub-sequence $(\varphi(n))_{n\geq 1}$ such that

$$\theta_{\varphi(n)} \longrightarrow \theta_\infty \quad \text{and} \quad \theta_{\varphi(n)} \xrightarrow{U_K} \theta^{(\infty)} \quad \text{with} \quad \theta^{(\infty)}_0 = \theta_\infty.$$

On the other hand,

$$\sum_{n\geq 1} \gamma_{n+1} \langle \nabla L | h\rangle(\theta_n) = \int_0^\infty \langle \nabla L, h\rangle(\theta^{(0)}_t)dt < +\infty$$

and

$$\int_0^\infty \langle \nabla L, h\rangle(\theta^{(n)}_t)dt = \int_{\Gamma_n}^\infty \langle \nabla L, h\rangle(\theta^{(0)}_t)dt = \sum_{k\geq 0} \gamma_k \langle \nabla L, h\rangle(\theta_k) \longrightarrow 0.$$

Hence, the lower semi-continuous condition and the Fatou lemma yields:

$$\begin{aligned} 0 \leq \int_0^\infty \langle \nabla L, h\rangle(\theta^{(\infty)}_t)dt &\leq \int_0^\infty \liminf \langle \nabla L | h\rangle(\theta^{(\varphi(n))}_t)dt & (5.10)\\ &\leq \liminf \int_0^\infty \langle \nabla L, h\rangle(\theta^{(\varphi(n))}_t)dt & (5.11)\\ &= 0. & (5.12)\end{aligned}$$

Thus,

$$\langle \nabla L, h\rangle(\theta^\infty_t) = 0 \quad dt - a.e.$$

and hence, as $\langle \nabla L, h\rangle$ is lower semi-continuous.

$$\langle \nabla L, h\rangle(\theta_\infty) = 0.$$

We conclude that Θ^∞ is a connected component of $\{L = l_\infty\} \cap \{\langle \nabla L, h\rangle = 0\}$.

If h is continuous, we conclude from

$$\theta_t^{(n)} := \theta_n - \int_0^t h(\theta_{\overline{s}}^{(n)})ds, \quad \overline{s} := \underline{s + \Gamma_n} - \Gamma_n \approx s$$

that every limit point of (θ_n) satisfies the ODE and take its values on the set Θ^∞. Thus Θ^∞ is a connected, compact subset, invariant for the ODE $\dot{\theta} = -h(\theta)$.

The case of $\dot{\theta} = h(\theta)$ can be handled likewise by considering $(\theta_{T-t}^{(n)})_{t\in[0,T]}$ for every $T > 0$, for n large enough.

□

5.4 Convergence of stochastic algorithms

In this section we will outline the mathematical analysis of the convergence of stochastic algorithms. We will focus on the Markov setting with i.i.d. innovations, that is, when the algorithm has the form:

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1}), \tag{5.13}$$

with $(Y_n)_{n\in\mathbb{N}}$ an i.i.d. sequence of μ-distributed $\mathbb{R}^q$-valued random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Roughly speaking, the analysis of the convergence of the algorithm is studied by rewriting the equation (6.44) as:

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \times \left(\underbrace{h(\theta_n)}_{\text{deterministic drift}} + \underbrace{\Delta M_{n+1}}_{\text{martingale difference (noise)}} \right). \tag{5.14}$$

Intuitively, we expect the effect of the deterministic drift to dominate the effect of the noise so that the algorithm behaves like the deterministic procedure under consideration in the former section. In particular, if we take for example $\gamma_n = n^{-1}$, we will have on the one hand that $\sum \gamma_n$ diverges, while the weighted sum of martingale differences is expected to be controlled, since if we do the analogy with a i.i.d. noise, the variance should be proportional to $\sum_n \gamma_n^2$.

5.4.1 Main theorem (Robbins-Zygmund)

In order to prove our main result, let us consider a random vector Y taking values in $\mathbb{R}^q$ with distribution μ and a Borel function $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$.

Following what precedes, we introduce into our analysis the following function:

$$h : \theta \mapsto \mathbb{E}\left[H(\theta, Y)\right]. \tag{5.15}$$

And, for this function to be well defined, we add the following condition:

$$\forall \theta \in \mathbb{R}^d, \quad \mathbb{E}\left[|H(\theta, Y)|\right] < +\infty. \tag{5.16}$$

Theorem 8. *(Robbins-Zygmund Lemma) Let us consider the following hypothesis.*

1. *Mean reverting assumption: There exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}_+$ satisfying:*
$$\langle \nabla L, h \rangle \geq 0. \tag{5.17}$$

2. *Hypothesis about the step-size:*
$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < \infty. \tag{5.18}$$

Furthermore assume the following technical assumptions:

1. *Controlled growth for L (quadratic at most):*
$$|\nabla L|^2 \leq C(1+L). \tag{5.19}$$

2. *Controlled growth for H (pseudo-linear):*
$$\forall \theta \in \mathbb{R}^d, \quad \|H(\theta, Y)\|_2 \leq C\sqrt{1+L(\theta)}. \tag{5.20}$$

3. *The initial condition $\theta_0 : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$ is independent of $(Y_n)_{n\geq 1}$ and $\mathbb{E}[L(\theta_0)] < +\infty$.*

Then, the stochastic algorithm satisfies: $\theta_n - \theta_{n-1} \to 0$ ($\mathbb{P}$-a.s. and in $L^2(\mathbb{P})$), $L(\theta_n)$ is $L^1(\mathbb{P})$-bounded, $L(\theta_n) \to L_\infty$ and
$$\sum_{n\geq 1} \gamma_n \langle \nabla L, h \rangle(\theta_{n-1}) < +\infty \tag{5.21}$$

Proof. Let us set $\mathcal{F}_n := \sigma(\theta_0, Y_1, \ldots, Y_n)$, $n \geq 1$, and to simplify the notations, $\Delta\theta_n := \theta_n - \theta_{n-1}$, $n \geq 1$. By a first order Taylor-expansion on L we know that there exists $\xi_{n+1} \in (\theta_n, \theta_{n+1})$ (as geometrical interval) such as

$$\begin{aligned}
L(\theta_{n+1}) &= L(\theta_n) + \langle \nabla L(\xi_{n+1}) | \Delta\theta_{n+1} \rangle \\
&\leq L(\theta_n) + \langle \nabla L(\theta_n) | \Delta\theta_{n+1} \rangle + [\nabla L]_{Lip} |\Delta\theta_{n+1}|^2 \\
&= L(\theta_n) - \gamma_{n+1} \langle \nabla L(\theta_n) | H(\theta_n, Y_{n+1}) \rangle + [\nabla L]_{Lip} \gamma_{n+1}^2 |H(\theta_n, Y_{n+1})|^2 \\
&= L(\theta_n) - \gamma_{n+1} \langle \nabla L(\theta_n) | h(\theta_n) \rangle - \gamma_{n+1} \langle \nabla L(\theta_n) | \Delta M_{n+1} \rangle \\
&\quad + [\nabla L]_{Lip} \gamma_{n+1}^2 |H(\theta_n, Y_{n+1})|^2
\end{aligned}$$

where
$$\Delta M_{n+1} = H(\theta_n, Y_{n+1}) - \mathbb{E}\left[H(\theta_n, Y_{n+1})\right] = H(\theta_n, Y_{n+1}) - h(\theta).$$

Our goal is to show that ΔM_{n+1} is an $\mathcal{F}_n$-adapted martingale difference, belonging to L^2 and satisfying $\mathbb{E}[\Delta M_{n+1}^2] \leq C(1 + L(\theta_n)$ for an appropriate constant $C > 0$.

Let us notice that for every $n \geq 0$, $L(\theta_n) \in L^1(\mathbb{P})$ and $H(\theta_n, Y_{n+1}) \in L^2(\mathbb{P})$, this comes from the latter inequality and from

$$\mathbb{E}[\langle \nabla L(\theta_n) | H(\theta_n, Y_{n+1}) \rangle] \leq \frac{1}{2} \left(\mathbb{E}[|\nabla L(\theta_n)|^2] + \mathbb{E}[|H(\theta_n, Y_{n+1})|^2] \right) \leq C(1 + \mathbb{E}[L(\theta_n)]).$$

As θ_n is $\mathcal{F}_n$-measurable and Y_{n+1} is independent of $\mathcal{F}_n$, we have

$$\mathbb{E}[H(\theta_n, Y_{n+1}) | \mathcal{F}_n] = \mathbb{E}[H(x, Y_1)]_{|x=\theta_n} = h(\theta_n).$$

Consequently, $\mathbb{E}[\Delta M_{n+1} | \mathcal{F}_n] = 0$. The inequality for $\mathbb{E}\left[|\Delta M_{n+1}|^2 | \mathcal{F}_n\right]$ comes from the fact that $|\Delta M_{n+1}|^2 \leq 2\left(|H(\theta_n, Y_{n+1})|^2 + |h(\theta_n)|^2\right)$ and hypothesis (5.20).

We can deduce from the hypothesis made on the step size that we can find a constant $C_L > 0$ (Lipschitz constant) such as

$$S_n = \frac{L(\theta_n) + \sum_{k=0}^{n-1} \gamma_{k+1} \langle \nabla L(\theta_k) | h(\theta_k) \rangle + C_L \sum_{k \geq n+1} \gamma_k^2}{\Pi_{k=1}^n (1 + C_L \gamma_k^2)}$$

is a non-negative supermartingale starting at $S_0 = L(\theta_0) \in L^1(\mathbb{P})$. This uses that $\langle \nabla L | h \rangle \geq 0$. The latter implies that S_n is converging a.s. toward an integrable r.v. S_∞. Consequently, by using that the tail of the series on γ_{n+1}^2 converges toward zero, we obtain

$$L(\theta_n) + \sum_{k=0}^{n-1} \gamma_{k+1} \langle \nabla L, h \rangle (\theta_k) \longrightarrow S_\infty \prod (1 + C_L \gamma_n^2) \in L^1(\mathbb{P}).$$

The non-negative supermartingale (S_n) being $L^1(\mathbb{P})$-bounded, we deduce likewise that $L(\theta_n)$ is L^1-bounded as

$$L(\theta_n) \leq \Pi_{k=1}^n (1 + C_L \gamma_k^2) S_n, \quad n \geq 0.$$

Owing to the mean-reversion assumption (which ensures that every term on the following series is positive), we deduce that

$$\sum_{n \geq 0} \gamma_{n+1} \langle \nabla L | h \rangle (\theta_n) < +\infty.$$

It follows that $L(\theta_n) \to L_\infty$, which is integrable as $(L(\theta_n))$ is L^1-bounded. Finally,

$$\sum_{n \geq 1} \mathbb{E}(|\Delta \theta_n|^2) \leq \sum \gamma_n^2 \mathbb{E}[|H(\theta_{n-1}, Y_n)|^2] \leq C \sum \gamma_n^2 (1 + \mathbb{E}(L(\theta_{n-1}))) < +\infty.$$

Hence $\sum_{n \geq 1} \mathbb{E}[|\Delta \theta_n|^2] \leq +\infty$, which implies $\theta_n - \theta_{n-1} \to 0$ a.s.

□

5.4.2 Two important corollaries

The Robbins-Monro algorithm

Corollary 1. *Assume that the mean function h of the algorithm is continuous and satisfies*

$$\forall y \in \mathbb{R}^d, y \neq y^*, \quad \langle y - y^*, h(y) \rangle > 0 \tag{5.22}$$

(which implies that $\{h = 0\} = \{y^*\}$). Suppose furthermore that $Y_0 \in L^2(\mathbb{P})$ and that H satisfies

$$\forall y \in \mathbb{R}^d, \quad \mathbb{E}\left[\|H(y, Z)\|^2\right] \leq C(1 + \|y\|^2) \tag{5.23}$$

If the step sequence γ_n satisfies the conditions of the Robbins-Zygmund lemma, we have:

$$Y_n \to y^*, \quad \mathbb{P} - a.s. \tag{5.24}$$

and in every $L^p(\mathbb{P})$, $p \in (0, 2]$.

Proof. The Robbins-Monro case, correspond to the Robbins-Zygmund lemma when the Lyapunov function is given by $L(\theta) = \frac{1}{2}|\theta - \theta^*|^2$. It follows that

$$|\theta_n - \theta^*| \to L_\infty \in L^1(\mathbb{P}) \quad \text{and} \quad \sum_{n \geq 1} \gamma_n \langle \theta_n - \theta^* | h(\theta_{n-1}) \rangle < +\infty \quad a.s.$$

Moreover, $(|\theta_n - \theta^*|^2)_{n \geq 1}$ is $L^1(\mathbb{P})$-bounded.

Let $\omega \in \Omega$ be such as $(|\theta_n(\omega) - \theta^*|^2)_{n \geq 1}$ converges towards $l_\infty(\omega) \in \mathbb{R}^+$ and

$$\sum_{n \geq 1} \gamma_n \langle \theta_{n-1}(\omega) - \theta^* | h(\theta_{n-1}(\omega)) \rangle < +\infty.$$

Because of the last inequality, we obtain

$$\liminf \langle \theta_{n-1}(\omega) - \theta^* | h(\theta_{n-1}(\omega)) \rangle = 0,$$

then, up to successive extractions, there exsists a subsequence $\varphi(n)$ such that

$$\langle \theta_{\varphi(n)}(\omega) - \theta^* | h(\theta_{\varphi(n)}(\omega)) \rangle \to 0 \quad \text{and} \quad \theta_{\varphi(n)}(\omega) \to \theta_\infty(\omega), \quad \text{as } n \to \infty.$$

As h is continuous, it follows that $\langle \theta_\infty(\omega) - \theta^* | h(\theta_\infty(\omega)) \rangle = 0$, which implies $\theta_\infty(\omega) = \theta^*$ owing to (6.47). We conclude that $l_\infty = 0$ so that

$$\lim |\theta_n(\omega) - \theta^*|^2 = \lim |\theta_{\varphi(n)}(\omega) - \theta^*|^2 = 0.$$

The L^p-convergence, $p \in (0, 2)$, follows from the fact that the sequences $|\theta_n - \theta^*|^p$ are $L^{2/p}$ bounded, then, uniformly integrable. Consequently these sequences are L^1 convergent, and hence $\theta_n \to \theta^*$ in L^p. □

The stochastic-gradient algorithm

Corollary 2. *Let $L : \mathbb{R}^d \to \mathbb{R}_+$ be a differentiable function satisfying ∇L Lipschitz continuous and $|\nabla L|^2 \leq C(1 + L)$, $\lim_{|y| \to \infty} L(y) = +\infty$, and $\{\nabla L = 0\} = \{\theta^*\}$. Assume the mean function of the algorithm is given by $h = \nabla L$ and that H satisfies:*

$$\mathbb{E}(|H(y, Z)|^2) \leq C(1 + L(y)), \quad \forall y \in \mathbb{R}^d. \tag{5.25}$$

Assume $L(\theta_0) \in L^1(\mathbb{P})$. Assume that the sequence γ_n satisfies the conditions of the Robbins-Zygmund lemma.

Then $L(y^*) = \min_{\mathbb{R}} L$ and

$$\theta_n \to \theta^* \quad a.s. \tag{5.26}$$

and $\|\nabla L(\theta_n)\|$ converges to zero in every $L^p(\mathbb{P})$, $p \in (0, 2)$.

Proof. In the case of the stochastic-gradient, we apply the Robbins-Zygmund lemma by using L as Lyapunov function as naturally we have $\langle \nabla L, h\rangle = |\nabla L|^2 \geq 0$. Consequently

$$L(\theta_n) \to L_\infty \in L^1(\mathbb{P}) \quad \text{and} \quad \sum_{n\geq 1} \gamma_n |\nabla L(\theta_{n-1})|^2 < +\infty \quad a.s.$$

Let $\omega \in \Omega$ be such as $L(\theta_n(\omega)) \to L_\infty(\omega) \in \mathbb{R}_+$, $\sum_{n\geq 1} \gamma_n |\nabla L(\theta_{n-1}(\omega))|^2 < +\infty$ and $\theta_n(\omega) - \theta_{n-1}(\omega) \to 0$. The same argument as for the Robbins-Monro algorithm shows that

$$\liminf |\nabla L(\theta_n(\omega))|^2 = 0$$

and because of the convergence of $L(\theta_n(\omega))$ towards $L_\infty(\omega)$ and $\lim L(\theta) = +\infty$ as $|\theta| \to \infty$, we derive the boundedness of $\theta_n(\omega)$. Thus, it exists a subsequence $\varphi_\omega(n)$ such that $\theta_{\varphi_\omega(n)}(\omega) \to \theta_\infty(\omega)$. Hence, by continuity of L and ∇L

$$\nabla L(\theta_\infty(\omega)) = 0 \quad \text{and} \quad L(\theta_\infty(\omega)) = L_\infty(\omega).$$

Hence, $\nabla L(\theta_\infty(\omega)) = 0$ implies $\theta_\infty(\omega) = \theta^*$, then, $L_\infty(\omega) = L(\theta^*)$. The function L being positive, differentiable and converging towards $+\infty$ for θ large enough, it implies that L has an unique (global) minimum at θ^*. In particular, the only possible limit value for $\theta_n(\omega)$ is θ^*.

The convergence in L^p, $p \in (0, 2)$, follows from the same uniform-integrability argument as in the proof for the Robbins-Monro algorithm.

□

5.5 The ODE method

Let us consider the following recursive algorithm defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\forall n \geq n_0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1}(\Delta M_{n+1} + r_{n+1})$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continous locally Lipschitz function, θ_{n_0} is an $\mathcal{F}_{n_0}$-measurable, finite, random vector and, for all $n \geq n_0$, ΔM_{n+1} is an $\mathcal{F}_n$-martingale increment and r_n is an $\mathcal{F}_n$-adapted remainder term.

The following theorem emerges from the subtle liens between the asymptotic behavior of stochastic algorithms and that of ODEs (see [20]).

Theorem 9. *Assume that h is locally Lipschitz (or continuous) and that $(\theta_n)_{n\geq 1}$ is bounded. Also assume that*

$$r_n \xrightarrow{a.s.} 0 \quad \text{and} \quad \sup_{n\geq n_0} \mathbb{E}\left[|\Delta M_{n+1}|^2 \middle| \mathcal{F}_n\right] < +\infty \quad a.s., \tag{5.27}$$

and that $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty. \tag{5.28}$$

Then, the limiting points of Θ^∞ of $(\theta_n)_{n\geq n_0}$ is a.s. connected and compact, stable by the flows of both

$$\dot{\theta} = -h(\theta) \quad \text{and} \quad \dot{\theta} = h(\theta).$$

Moreover, if $\theta^* \in \Theta^\infty$ is the unique uniformly-stable[2] *equilibrium of the ODE*

$$\dot{\theta} = -h(\theta),$$

then:

$$\theta_n \xrightarrow{a.s.} \theta^*.$$

Proof. Let us rewrite the algorithm as follows:

$$\theta_n = \theta_0 - \sum_{k=1}^{n} \gamma_k (h(\theta_{k-1}) - \Delta M_k - r_k).$$

The idea is to rewrite the algorithm in a continuous form by setting $\theta^{(0)}_{\Gamma_n} = \theta_n$:

$$\theta^{(0)}_t = \theta_0 - \int_0^\infty h(\theta^{(0)}_{\underline{s}}) ds + \sum_{k=1}^{N_t} \gamma_k \Delta M_k + \sum_{k=1}^{N_t} \gamma_k r_k,$$

where $N_t = \min\{k \geq 0 | \Gamma_{k+1} > t\}$.

For every $n \geq 0$, we define the functions

$$\begin{aligned} \theta^{(n)}_t &= \theta^{(0)}_{\Gamma_n + t} \\ &= \theta_n - \int_{\Gamma_n}^{\Gamma_{n+t}} h(\theta^{(0)}_{\underline{s}}) ds + \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k \Delta M_k + \sum_{k=n+1}^{N(\Gamma_n+t)} \gamma_k r_k. \end{aligned}$$

As $L(\theta_n) \to L_\infty \in L^1(\mathbb{P})$, this implies that $K_\infty(h) = \sup_n |h(\theta_n)| < +\infty$ a.s., we always have

$$\left| \int_{\Gamma_n}^{\Gamma_{n+t}} h(\theta^{(0)}_{\underline{u}}) du - \int_{\Gamma_n}^{\Gamma_{n+s}} h(\theta^{(0)}_{\underline{u}}) du \right| \leq K_\infty(h) |t-s|.$$

On one hand, by using (5.27) and (5.28), we know that $\sum_{n\geq 1} \gamma_n \Delta M_n$ converges a.s. and by using the Cauchy property,

$$\sup_{m\geq n} \left| \sum_{k=n+1}^{m} \gamma_k \Delta M_k \right| \xrightarrow{a.s.} 0 \quad \text{as } n \to \infty.$$[3]

---

[2]For uniformly stable we understand $\sup_{\theta\in\Theta^\infty} |\theta(\theta_0, t) - \theta^*| \longrightarrow 0$ as $t \to +\infty$.

[3]Indeed, the convergence of the martingale is not necessary, we just need $\sum \gamma_n = \infty$ and $\gamma_n \to 0$ and

$$\left| \sum_{k=n+1}^{N(\Gamma_n+T)} \gamma_k \Delta M_k \right| \xrightarrow{a.s.} 0, \forall T > 0.$$

These conditions are the result of the two following criteria

- $\sup_{n\geq 0} \mathbb{E}\left[|\Delta M_{n+1}|^q\right] < +\infty$ and $\sum \gamma^{1+q/2} < +\infty$ for $q \geq 2$. (see [114])

On the other hand

$$\sup_{t\in[0,T]}\left|\sum_{k=n+1}^{N_{\Gamma_n+t}}\gamma_k r_k\right| \leq \max_{k\geq n}|r_k|\sum_{k=n+1}^{N_{\Gamma_n+t}}\gamma_k = \max_{k\geq n}|r_k|T \xrightarrow{a.s.} 0.$$

Hence, the sequence of functions $(\theta_t^{(n)})_{t\geq 0}$ is U_K relatively compact, and Θ^∞ is a connected, compact set. Moreover, as h is locally Lipschitz, Θ^∞ is flow invariant.

If $\theta^* \in \Theta^\infty$ is the unique stable equilibrium, then $\Theta^\infty = \{\theta^*\}$ yielding our result.

□

5.6 Rate of convergence: a CLT

In standard settings, stochastic algorithms converges to its target at a $\sqrt{\gamma_n}$ rate. This may suggest at a first glance to use rates of the form $\frac{c}{n}$ for faster convergence rather than $\frac{c}{n^\gamma}$ with $\frac{1}{2} < \gamma < 1$. Moreover, it can be shown that the quantity $\frac{Y_n - y^*}{\sqrt{\gamma_n}}$ converges in law to a normal distribution $\mathcal{N}(0, \alpha\Sigma)$ with Σ a dispersion matrix based on $H(y^*, Z)$ and α a real number satisfying some condition depending on the eigenvalues of $Dh(y^*)$.

Theorem 10. *(see [22]) Let θ^* be a solution of $\{h = 0\}$. Let us suppose that the function h is differentiable at the point θ^* and that all the eigenvalues of $Dh(\theta^*)$ have a positive real part. Assume there exists $\delta > 0$ such as,*

$$\sup_{n\geq n_0}\mathbb{E}\left[|\Delta M_{n+1}|^{2+\delta}\middle|\mathcal{F}_n\right] < +\infty \quad a.s.$$

and

$$\mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t\middle|\mathcal{F}_n\right] \xrightarrow{a.s.} \Gamma$$

where Γ is a deterministic matrix, symmetric, positive.

Suppose that there exists $\varepsilon > 0$ such as

$$\mathbb{E}\left[(n+1)|r_{n+1}|^2\mathbf{1}_{\{|\theta_n-\theta^*|\leq\varepsilon\}}\right] \longrightarrow 0.$$

Let us specify the following step sequence as follows:

$$\forall n \geq 1, \quad \gamma_n = \frac{\alpha}{n+\beta}, \quad \alpha > \frac{1}{2\mathcal{R}e(\lambda_{\min})}$$

where $\lambda_{\min}$ is the eigenvalue of $Dh(\theta^*)$ with smaller real part.

- There exists $\lambda > 0$ such as, for all $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[e^{\langle x,\Delta M_{n+1}\rangle}\middle|\mathcal{F}_n\right] \leq e^{\frac{\lambda}{2}|x|^2}$$

and $\sum e^{-c/\gamma_n} < +\infty$ for all $c > 0$ (see [95]). This framework includes the bounded random variables owing to the Hoeffding inequality.

Then, the a.s. convergence on the set $\{\theta_n \xrightarrow{a.s} \theta^*\}$ *satisfies the following CLT*

$$\sqrt{n}(\theta_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \alpha\Sigma) \quad \text{(stably)}$$

where

$$\Sigma := \int_0^\infty \left(e^{-(Dh(\theta^*)-2^{-1}\alpha^{-1}I_d)u}\right)^t \Gamma e^{-(Dh(\theta^*)-2^{-1}\alpha^{-1}I_d)u} du.$$

In other terms, $\forall A \in \mathcal{F}$ *and* $\forall f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$

$$\mathbb{E}\left[\mathbf{1}_{A\cap\{\theta_k\to\theta^*\}} f(\sqrt{n}(\theta_n - \theta^*))\right] \to \mathbb{E}\left[\mathbf{1}_{A\cap\{\theta_k\to\theta^*\}} f(\sqrt{\alpha\Sigma}Z)\right], \quad Z \sim \mathcal{L}(0; I_d).$$

5.7 Averaging principle (Ruppert-Polyak)

The idea of the averaging principle is to smoothen the trajectory of a convergent stochastic-algorithm by considering the average of all past values rather than just the last computed value. We do so, in the context of an algorithm with slowly decreasing step (see below).

The advantage of this method is to obtain the best of the two world in terms of 'exploration' during the first stages of the algorithm (as the step is slowly decreasing), and an optimal convergence rate asymptotically (effect of the averaging).

In practice, let us consider $(\gamma_n)_{n\geq 1}$ of the form

$$\gamma_n \sim \left(\frac{\alpha}{\beta + n}\right)^a, \quad n \geq 1, \quad a \in (1/2, 1).$$

Then, one implements the standard procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1})$$

and sets

$$\bar{\theta}_n := \frac{\theta_0 + \ldots + \theta_{n-1}}{n}.$$

Under natural assumptions (see [120]), we can show that for the different situations we studied before (stochastic gradient, Robbins-Monro algorithm) we have

$$\bar{\theta} \xrightarrow{a.s.} \theta^*$$

where θ^* is the target of the algorithm.

Moreover

$$\sqrt{n}(\bar{\theta}_n - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma^*_{\min}), \quad \text{as } n \to +\infty,$$

where $\Sigma^*_{\min}$ is the smallest possible asymptotic variance matrix. Thus, if $d = 1$,

$$\Sigma^*_{\min} = \frac{\operatorname{Var}(H(\theta^*, Z))}{h'(\theta^*)^2}.$$

Theorem 11. *(see [45]) We place ourselves on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Let us consider the following stochastic algorithm:

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(h(\theta) + \Delta M_{n+1}\right)$$

where $h : \mathbb{R}^d \to \mathbb{R}$ *is a Borel function, continuous at its only zero* θ^**, and satisfying*

$$\forall \theta \in \mathbb{R}^d, \quad h(\theta) = Dh(\theta^*)(\theta - \theta^*) + \mathcal{O}(|\theta - \theta^*|^2)$$

where all the eigenvalues of $Dh(\theta^*)$ *have a positive real part.*

Moreover, assume that there exists a real constant $C > 0$ *such that*

$$\mathbb{E}[\Delta M_{n+1}|\mathcal{F}_n]\mathbf{1}_{\{|\theta_n - \theta^*| < C\}} = 0 \quad a.s.$$

and that there exists an exponent $\delta > 0$ *such as*

$$\mathbb{E}\left[\Delta M_{n+1}(\Delta M_{n+1})^t \middle| \mathcal{F}_n\right] \xrightarrow{a.s.} \mathbf{1}_{\{|\theta_n - \theta^*| \leq C\}}\Gamma \in \mathcal{S}(d, \mathbb{R}),$$

$$\sup_n \mathbb{E}\left[|\Delta M_{n+1}|^{2+\delta} \middle| \mathcal{F}_n\right]\mathbf{1}_{\{|\theta_n - \theta^*| \leq C\}} < +\infty, \quad a.s..$$

Then, if $\gamma_n = cn^{-\alpha}$*,* $n \geq 1$*,* $1/2 < \alpha < 1$*, the sequence of empirical means*

$$\bar{\theta}_n = \frac{\theta_0 + \ldots + \theta_{n-1}}{n}$$

satisfies a central limit-theorem with optimal variance

$$\mathbb{E}\left[\mathbf{1}_{\{\theta_n \xrightarrow{a.s.} \theta^*\}} f\left(\sqrt{n}(\bar{\theta}_n - \theta^*)\right)\right] \stackrel{\mathcal{L}-\text{stably}}{\longrightarrow} \mathbb{E}\left[\mathbf{1}_{\{\theta_n \xrightarrow{a.s.} \theta^*\}} f(U)\right], \quad \forall A \in \mathcal{F}, \quad \forall f \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R}).$$

where

$$U \sim \mathcal{N}(0, Dh(\theta^*)^{-1}\Gamma Dh(\theta^*)).$$

Chapter 6

Stochastic approximation approach for market-making

6.1 Introduction

In this chapter, we propose an optimization framework for market-making in a limit-order book, based on the theory of stochastic approximation. We consider a discrete-time variant of the Avellaneda-Stoikov model [13] similar to its developent in the article of Laruelle, Lehalle and Pagès [101] in the context of optimal liquidation tactics. The idea is to take advantage of the iterative nature of the process of updating bid and ask quotes in order to make the algorithm optimize its strategy on a trial-and-error basis (i.e. *on-line learning*). An advantage of this approach is that the exploration of the system by the algorithm is performed in run-time, so explicit specifications of the price dynamics are not necessary, as is the case in the stochastic-control approach [65]. As it will be discussed, the rationale of our method can be extended to a wider class of algorithmic-trading tactical problems other than market-making.

One of the main problems in algorithmic and high-frequency trading, is the optimization of tactics whose main role is to interact with the limit-order book, during a short lapse of time, in order to perform a basic task: this can be the optimal posting of a child order in the order book, routing an order across different liquidity pools or a high-frequency market-maker posting bid and ask quotes during a couple of seconds. Among the main features of these tactics is that they are short lived, have a straightforward goal and they are repeated several times during the trading session. Moreover, most of the performance of algorithmic trading tactics depends, not necessarily in financial aspects (e.g. asset-valuation) but in microstructural factors (e.g. auction mechanics, tick-size, short-term liquidity, etc.). Another important aspect is that the performance of these tactics is measured on a statistical basis, as their execution is systematic and, their use, intensive.

A large number of contributions have been published in the recent years in the field of optimizing algorithmic-trading tactics. After the seminal paper of Avellaneda and Stoikov [13] several others have extended its approach (e.g. [18, 64] for optimal liquidation with limit orders and [33, 65] for market-making). However, most of the approaches to analyze these problems have been through the lenses of stochastic-control techniques, which demand to define explicitly the statistical laws governing the price dynamics. The latter makes the model less flexible for applications, where we would like the algorithm learn by itself in order to adapt to the nature of the forces driving the price and liquidity (which at the intraday scale can evolve through different market regimes difficult to anticipate).

Here, we aim to take advantage of the iterative nature of the trading process at short time-scales by proposing an on-line learning framework to analyze the market-making problem (which can be extended to more general trading tactics) based on the theory of stochastic approximation [22, 49]. We follow a similar path that [101] (in the context of optimal liquidation) by considering a modified version of the Avellaneda-Stoikov model [13].

Hence, throughout this work we consider a market-maker trading in an electronic limit order book. In a nutshell, the goal of a market maker is to provide liquidity by setting quotes at the bid and the ask sides of the order book. Each time one side of a bid/ask pair gets executed, the market-maker earns the price-difference between these two orders. Thus, the market-maker's algorithm would like to maximize the number of pairs of buy/sell trades executed, at the larger possible spreads and by holding the smallest possible inventory at the end of the trading session. Hence, the market-maker faces the following trade-off: it is expected that a large spread means a lower probability of execution while a narrower spread will mean a lower gain for each executed trade. On the other hand, if the trading algorithm only executes its orders on one side (because of price movements, for example), then its inventory moves away from zero, bearing the risk to eventually having to execute those shares at the end of the trading session at a worst price. The latter motivates the algorithm to center its orders around a 'fair price' in order to keep the inventory close to zero. Moreover, the more orders are executed, the larger the risk of ending the period with a large unbalanced inventory (variance effect), inducing still another trade-off.

From a modeling standpoint, one iteration of a market-making tactic can be seen as interacting with a black-box to which we apply, as input, the controls δ_a and δ_b (representing the positions in the order-book where to place orders with respect to some reference price) then, obtaining as output the variables N_a and N_b (depending on the controls and exogenous variables) representing the liquidity captured at each side of the spread during a time window of length ΔT (representing the duration for one iteration of the algorithm).

At the end of each iteration the payoff is represented by a random variable

$$\Xi(\delta_a, \delta_b) = \Pi(N_a(\delta_a, \xi), N_b(\delta_b, \xi), \xi)$$

where the ξ represent the exogenous variables influencing the payoff (e.g. price, spread).

In practice, ξ can both be a finite or an infinite dimensional random variable, and it is modeled in a way that it represents an *innovation* i.e. the part of the exogenous variables such as its realizations can be considered to be i.i.d. (or stationary ergodic or weakly dependent), which are the kind of property allowing the optimization procedure presented below, to converge. For example, instead of considering ξ being the price, from a modeling perspective it will be more convenient to consider it to be the price returns.

Here we will want to maximize the expectation $\mathbb{E}\left[\Xi(\delta_a, \delta_b)\right]$. The question is, how, in order to find the solutions $\delta^* = (\delta_a^*, \delta_b^*)$, to exploit the iterative features of the algorithm (e.g. using errors as feedback) and, at the same time, to propose an adaptive framework less depending on an accurate modeling of the market-dynamics.

Our setting naturally joins the framework of stochastic approximation; where we define recursive procedures converging to the critical point of functions represented in the form of an expectation $h(\delta) = \mathbb{E}[H(\delta, \xi)]$, $\delta \in \mathbb{R}^d$, in cases when h cannot be computed, but where $H(\delta, \xi)$ and its derivatives can be evaluated at a reasonable cost and the random variable ξ can be simulated or obtained from a real-time data-stream (in the real-time case we call the procedure *on-line learning*). For our problem, we define $H(\delta_a, \delta_b, \xi) := \mathbb{E}[\Xi|\xi]$.

In order to maximize $h(\cdot)$, a probabilistic extension of the gradient method, namely

$$\delta_{n+1} = \delta_n + \gamma_{n+1} \nabla_\delta H(\delta_n, \xi_n), \quad \xi_n \sim \xi. \tag{6.1}$$

can be shown to converge towards the optimal value $\delta^* = (\delta_a^*, \delta_b^*)$.

As mentioned before, the advantage of the approach introduced and analyzed in this chapter, is its flexibility not only to approach problems where the price follows a Brownian diffusion (key hypothesis in the stochastic control approach) but also to much more general situations where the algorithm continuously extracts information from its environment without needing to further specify its dynamics. Moreover, the recursive aspect makes the procedure naturally adaptive and easily implementable. Notice also, that this framework can be generalized to other types of trading tactics (e.g. dark-pool trading, execution, routing to lit venues. See the works of Laruelle et al. [101, 100]), which can be seen as iterative problems where we control $\delta \in \mathbb{R}^d$, getting as output a 'liquidity captured' vector $N \in \mathbb{R}^p$, usually this N is represented by a vector of Poisson variables. Generically, the goal of a trading tactic is to maximize the expectation of a given functional of N.

In this study we can look at stochastic approximation from two standpoints: first as a numerical procedure, in the sense that the innovation in the algorithm is a simulated random variable, and secondly, as a learning procedure in which the innovations are the observations of real-time data (or a historical data-set). We can also consider two different situations, depending on the way the market-maker valuates the inventory (mark-to-market or by adding a penalization function for the inventory). We will give an special focus on computing closed formulas in the case the price is Brownian, this is useful for several reasons: (i) exhibits numerical results (ii) highlights the relation between the different parameters and the solution and (iii) this can be used to compute 'first guesses' at the optimal solution and set them as a starting point for an algorithm working in a general case. The next section by introduces the model and the optimization problem.

6.2 Problem setting

6.2.1 Optimization problem

Let place ourselves on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We split the trading day $[0, T]$ into N periods of ΔT seconds each. The n-th period correspond to the time-interval $[(n-1)\Delta T, n\Delta T]$. We consider a market-making algorithm updating its bid and ask quotes at the beginning of each period. It posts its quotes at respective distances $\delta_a^{(n-1)}$ and $\delta_b^{(n-1)}$ around a reference price $S_{(n-1)\Delta T}$ fixed at the beginning of the ΔT-seconds period (this can be the mid-price at time $(n-1)\Delta T$, for instance). The orders remain unchanged until the end of the period.

Let us note $N_b^{(n)}$ and $N_a^{(n)}$ the respective number of orders the market-maker executes at the bid and at the ask during this n-th period. The realized payoff of the market-maker during this period is given by the following random-variable:

$$\Xi_n = \underbrace{N_a^{(n)}\left(S_{(n-1)T} + \delta_a^{(n-1)}\right)}_{\text{sold}} - \underbrace{N_b^{(n)}\left(S_{(n-1)T} - \delta_b^{(n-1)}\right)}_{\text{bought}} + \underbrace{\left(N_b^{(n)} - N_a^{(n)}\right) S_{nT}}_{\text{inventory valuation}}.$$

We introduce for each time period, the running price return

$$Y_t^{(n)} = S_{(n-1)\Delta T+t} - S_{(n-1)\Delta T}, \quad t \in [0, \Delta T].$$

In particular $Y_{\Delta T}^{(n)} = S_{nT} - S_{(n-1)T}$.

Now, let us introduce the variables

$$N^{(n)} = N_b^{(n)} + N_a^{(n)} \quad \text{and} \quad Q^{(n)} = N_b^{(n)} - N_a^{(n)}.$$

Also, we define the market-maker' *half-spread* as

$$\psi^{(n)} = \frac{\delta_a^{(n)} + \delta_b^{(n)}}{2}$$

and an *off-center* factor $\theta^{(n)}$ such as

$$\theta^{(n)} = \frac{\delta_a^{(n)} - \delta_b^{(n)}}{2}.$$

The latter variables have an intuitive interpretation: $N^{(n)}$ represents the total number of trades executed by the market-maker and $Q^{(n)}$ is the increase of inventory over the period $[(n-1)\Delta T, n\Delta T]$.

In our new variables, the realized payoff of the market-maker is given by:

$$\Xi_n = \psi^{(n)} N^{(n)} - \left(\theta^{(n)} - Y_{\Delta T}^{(n)}\right) Q^{(n)}.$$

The goal of the market-maker is to maximize the expectation of this quantity.

6.2.2 The one-period problem

Notice that if we suppose that the law of $Y^{(n)}$ (and $N^{(n)}$ and Q^n) does not depends on $S_{(n-1)\Delta T}$, we can always think in terms of a one-period problem as the function to maximize is always the same. i.e.

$$\max_{\psi\geq 0,\theta\in\mathbb{R}} \pi(\psi,\theta) := \mathbb{E}\left[\Xi_1\right]. \tag{6.2}$$

In this study, for the sake of putting the modeling forward, we focus on this *independent price-increments* situation leading to a *one-period problem*.

However, it is important to keep in mind that these hypothesis can be weakened and interpreted as an approximation for more general cases, namely, when price increments are stationnary, weakly dependent or in an ergodic setting in which we want to optimize the sum of future payoffs (from an asymptotic point of view).

In what follows, it will be interesting to consider the payoff conditioned to the trajectory of the price (as it represents an exogenous factor). Hence, for a function y_t, $t \in [0, \Delta T]$, let us define:

$$\Pi(\psi,\theta,(y_t)_{t\in[0,\Delta T]}) = \mathbb{E}_y\left[\Xi_n\right]. \tag{6.3}$$

In particular $\pi(\psi,\theta) = \mathbb{E}\left[\Pi(\psi,\theta,Y)\right]$, with $Y \sim Y^{(1)}_{\Delta T}$.

6.2.3 Stochastic-gradient method

The goal of the market maker is to maximize the expectation of the function $\mathbb{E}\left[\Pi(\psi,\theta,Y)\right]$. In a concave setting, this is equivalent of finding the zero of $\nabla\pi(\psi,\theta) = \nabla_{\psi,\theta}\mathbb{E}\left[\Pi(\psi,\theta,Y)\right]$. After showing that the solution of this zero-searching problem is unique, we can consider applying the Robbins-Monro theorem, which states that the zero of $\nabla\pi$, under a given set of hypothesis, can be found through an algorithm of the form:

$$\psi_{n+1} = \psi_n + \gamma_{n+1}\partial_\psi\Pi(\psi_n,\theta_n,Y_{n+1}) \tag{6.4}$$

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\partial_\theta\Pi(\psi_n,\theta_n,Y_{n+1}). \tag{6.5}$$

We will see that in our setting one of the conditions of the Robbins-Monro theorem will not be satisfied. Namely

$$\mathbb{E}\left[|\nabla_{\psi,\theta}\Pi(\psi,\theta,Y)|^2\right] \leq C(1+\psi^2+\theta^2)$$

however, by modifying the function by a multiplicative factor, as it is proposed in the article by Lemaire and Pagès [109], we can show that a procedure of the form:

$$\psi_{n+1} = \psi_n + \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\psi\Pi(\psi_n,\theta_n,Y_{n+1}) \tag{6.6}$$

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\theta\Pi(\psi_n,\theta_n,Y_{n+1}) \tag{6.7}$$

converges towards the solution (i.e. the point (ψ^*,θ^*) solution of $\nabla\pi(\psi,\theta) = 0$).

Here ρ is a strictly positive function (to be defined), mainly intended to control the behavior of the function Π for large values of θ.

6.2.4 Relating price and liquidity

Thus far we have introduced the optimization problem regardless of the explicit relations between price and liquidity. The interest of introducing a price dynamics is to relate statistically the price with the captured liquidity, i.e. N and Q. The latter is related to the fundamental relations between price direction and order imbalance for example.

In this chapter we will use the variation of the Avellaneda-Stoikov model as in [101].

We consider that, during a period $t \in [0, \Delta T]$ the probability to capture liquidity at a distance δ from the reference price Y_t is defined between t and $t + dt$, up to the second order, by the instantaneous intensity

$$\lambda(\delta)dt = Ae^{-\delta k}dt. \tag{6.8}$$

and independent from the trajectory of Y.

Thus, the liquidity captured by an order placed at $S_0 \pm \delta$, during $[0, \Delta T]$, given the trajectory of the price is $(y_t)_{t\geq 0}$, will be Poisson random variables with intensities:

$$\lambda^-(\delta_b; (y_t)_{t\in[0,\Delta T]}) = \int_0^{\Delta T} Ae^{-k(\delta_b + y_t)}dt \qquad \text{(bought shares)}, \tag{6.9}$$

$$\lambda^+(\delta_a; (y_t)_{t\in[0,\Delta T]}) = \int_0^{\Delta T} Ae^{-k(\delta_a - y_t)}dt \qquad \text{(sold shares)}. \tag{6.10}$$

So, the captured liquidity $N^{(a)}$ and $N^{(b)}$ is modeled by Poisson random variables with intensities given (respectively) by λ^+ and λ^-.

The latter can be seen in two ways: (i) simply define the intensities of the Poisson variables $N^{(a)}$ in that $N^{(b)}$ as functionals of the realized price and simply using the preceding justification as a heuristic, or (ii) formally model how prices and liquidity are related in continuous-time (i.e. thinking in terms of the generalized Poisson processes $N_t^{(a)}$ and $N_t^{(b)}$). In the latter case we need to introduce a standard Poisson process $(N_u)_{u\geq 0}$ independent of Y such that the orders we capture are given by $\tilde{N}_s$ which correspond to a time change of N defined by $ds = Ae^{-k(\delta + Y_u)}du$ or via a thinning operation.

Again, in order to work with symmetric variables, we define for every $y \in \mathcal{C}([0, \Delta T], \mathbb{R})$,

$$\lambda(\psi, \theta; y) = \lambda^- + \lambda^+ = 2Ae^{-k\psi} \int_0^{\Delta T} \cosh(k(\theta - y_t))dt, \tag{6.11}$$

$$\mu(\psi, \theta; y) = \lambda^- - \lambda^+ = 2Ae^{-k\psi} \int_0^{\Delta T} \sinh(k(\theta - y_t))dt. \tag{6.12}$$

with λ^- and λ^+ defined in (6.9) and (6.10). In particular we have $\mathbb{E}_Y[N] = \lambda$ and $\mathbb{E}_Y[Q] = \mu$. Furthermore, because negative spreads do not make sense in our problem, we suppose $\psi \geq 0$ and $\theta \in \mathbb{R}$. Moreover, in practice it does not makes sense to have passive bid quotes on the ask side (and vice-versa), which means we look for solutions on $|\theta| < C + \psi$, where C represents the half bid-ask spread (or an upper bound for it).

In this way we have:

$$\Pi(\psi,\theta;Y) = \psi\lambda(\psi,\theta;Y) - (\theta - Y_{\Delta T})\mu(\psi,\theta;Y). \tag{6.13}$$

Throughout this stutyd it will be interested in comparing with the situation when the price is Brownian, where we can obtain closed formulas for the optimal quotes (see Theorem below). This have several advantages: (i) exhibit numerical results (ii) understand the relation between the different parameters and the solution and (iii) compute 'first guesses of the optimal solution' and set them as the starting point for an algorithm working on a general case. The following theorem gives the optimal quotes for the Brownian situation:

Theorem 12. *In the Brownian case, the optimal solution is given by*[1]*:*

$$\begin{aligned} \psi^* &= \frac{1}{k}\left(\frac{k^2\sigma^2\Delta T - 1 + e^{-\frac{k^2\sigma^2\Delta T}{2}}}{1 - e^{-\frac{k^2\sigma^2\Delta T}{2}}}\right), && (6.14)\\ \theta^* &= 0. && (6.15)\end{aligned}$$

Proof. See Section 6.7.1. □

6.3 Mark-to-Market inventory valuation

6.3.1 Introduction

First, we consider that no market-impact affecting the liquidation of the inventory. That is, the conditional expectation of the profits, given the price trajectory, is defined by:

$$\Pi(\psi,\theta;Y) = \psi\lambda(\psi,\theta;Y) - (\theta - Y_{\Delta T})\mu(\psi,\theta;Y). \tag{6.16}$$

while the function to maximize is defined by:

$$\pi(\psi,\theta) = \mathbb{E}\left[\Pi(\psi,\theta;Y)\right]. \tag{6.17}$$

Let us notice that in this case we can write the target function as

$$\pi(\psi,\theta) = 2Ae^{-k\psi}(a(\theta)\psi - b(\theta)), \tag{6.18}$$

where:

$$\begin{aligned} a(\theta) &= \mathbb{E}\left[\int_0^{\Delta T}\cosh(k(\theta - Y_t))dt\right], && (6.19)\\ b(\theta) &= \mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T}\sinh(k(\theta - Y_t))dt\right]. && (6.20)\end{aligned}$$

[1]This result, as $\Delta T \to 0$, is consistent with the asymptotic solution in the article [65] in the case $\gamma \to 0$.

6.3.2 Existence and uniqueness of the solution

The next theorem states the existence and uniqueness of the maximum and in particular, an unique solution for the equation $\nabla\pi(\psi,\theta)=0$.

Theorem 13. *The function $\pi(\psi,\theta)=e^{-k\psi}(\psi a(\theta)-b(\theta))$ has an unique maximum.*

Proof. See Section 6.7.2. □

With this result, we now need to compute the local derivatives of the function $\Pi(\psi,\theta;Y)$ then verify is the conditions of the Robbins-Monro theorem are satisfied.

6.3.3 Stochastic gradient

In order to define the stochastic algorithm we start by computing the derivatives of the function $\Pi(\psi,\theta;y_t)$ with respect to ψ and θ.

Proposition 11. *The gradient $\nabla_{\psi,\theta}\Pi(\psi,\theta;y_t)$ is given by the equation:*

$$\begin{pmatrix} \frac{\partial}{\partial\psi}\Pi(\psi,\theta;y_t) \\ \frac{\partial}{\partial\theta}\Pi(\psi,\theta;y_t) \end{pmatrix} = \begin{pmatrix} (1-k\psi) & k(\theta-y_{\Delta T}) \\ -k(\theta-y_{\Delta T}) & -(1-k\psi) \end{pmatrix} \begin{pmatrix} \lambda(\psi,\theta;y_t) \\ \mu(\psi,\theta;y_t) \end{pmatrix}. \tag{6.21}$$

Proof. See Section 6.7.3. □

In order to prove the convergence of the stochastic algorithm, we modify the procedure, as in [109], by multiplying the gradient by a factor $\rho(\psi,\theta)$ which help us to guarantee the convergence of the modified stochastic gradient algorithm:

$$\psi_{n+1} = \psi_n - \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\psi\Pi(\psi_n,\theta_n,Y_{n+1}) \tag{6.22}$$
$$\theta_{n+1} = \theta_n - \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\theta\Pi(\psi_n,\theta_n,Y_{n+1}) \tag{6.23}$$

Where $\rho(\psi,\theta)=e^{-k(|\theta|-\psi)}$.

Indeed the function $\rho(\psi,\theta)\nabla\Pi(\psi,\theta,Y)$ satisfies the condition of the Robbins-Monro algorithm as it is proven in the following theorem.

Theorem 14. *Let us consider the function $\rho(\psi,\theta)=e^{-k(|\theta|-\psi)}$, $\theta\in\mathbb{R}$, and, for a i.i.d. sequence of random trajectories Y_{n+1} (with same law as Y), the following recursive algorithm:*

$$\begin{aligned} \psi_{n+1} &= \psi_n + \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\psi\Pi(\psi_n,\theta_n,Y_{n+1}) \\ \theta_{n+1} &= \theta_n + \gamma_{n+1}\rho(\psi_n,\theta_n)\partial_\theta\Pi(\psi_n,\theta_n,Y_{n+1}), \end{aligned}$$

where ψ_0 and θ_0 are in $L^1(\mathbb{P})$, $\sum_n\gamma_n=+\infty$ and $\sum_n\gamma_n^2<+\infty$.

Consider also the following technical conditions

$$\begin{aligned}
\mathbb{E}\left[Y_T^2\int_0^{\Delta T} e^{2k|Y_t|}dt\right] &< +\infty,\\
\mathbb{E}\left[\int_0^{\Delta T} e^{2k|Y_t|}dt\right] &< +\infty.
\end{aligned}$$

Then, the recursive algorithm converges towards the point (ψ^*,θ^*) *which correspond to the maximum of* $\pi(\psi,\theta)=\mathbb{E}[\Pi(\psi_n,\theta_n,Y)]$ *(which is the unique solution of the equation* $\nabla\pi=0$*).*

Proof. See Section 6.7.3. □

A delicate issue that it has not been addressed up to this point is that the innovation is an infinite dimensional object (trajectory of the reference price), while in the Robbins-Monro theorem (see the proof of the last theorem) is set up for finite dimensional innovations. Fortunately, this is not a serious issue as in fact the algorithm only depends on the innovation through a finite dimensional functional of the trajectory (indeed, the innovation is a 3-dimensional object, after a change of variable).

Theorem 15. *The function* $\Pi(\psi,\mu,Y)$ *only depends on the stochastic process* $(Y)_{t\in[0,T]}$ *through the one-dimensional random variables* Y_T *and:*

$$b_k = \frac{1}{2}\log\left(\int_0^T e^{kY_t}dt\int_0^T e^{-kY_t}dt\right), \tag{6.24}$$

$$\rho_k = \frac{1}{2}\log\left(\frac{\int_0^T e^{-kY_t}dt}{\int_0^T e^{kY_t}dt}\right). \tag{6.25}$$

Moreover, we have the formulas

$$\lambda(\psi,\theta) = 2Ae^{-k\psi+b_k}\cosh(k\theta+\rho_k), \tag{6.26}$$

$$\mu(\psi,\theta) = 2Ae^{-k\psi+b_k}\sinh(k\theta+\rho_k). \tag{6.27}$$

Proof. See Section 6.7.3. □

6.4 Numerical examples

6.4.1 Numerical example: Brownian case

We implement the stochastic optimization algorithm in the Brownian case and compare with the explicit solutions we just found. We consider an algorithm with projections in order to be always searching for the solution in the region where the payoff is expected to be positive.

Shape of the function to maximize

We set the following values for the models parameters

$$T = 30, \quad \sigma = 1.2, \quad A = 0.9, \quad k = 0.3.$$

The maximum is reached at:

$$\psi^* = 2.52024 \pm 10^{-6}, \theta^* = 0.0$$

The following figure shows a heatmap of the function:

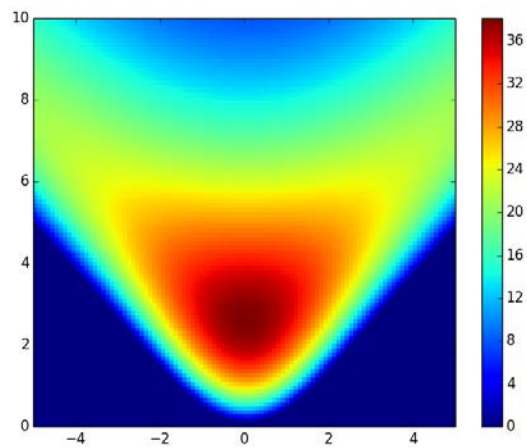

Figure 6.1: Heatmap of the target function in the Brownian situation (the abscissa correspond to θ and the ordinate correspond to ψ).

Three previous results has been helpful to reduce the computing cost:

- Separating variables on the representation of the function
- The innovations are in reality a three-dimensional variable
- Recognizing the admissible region (outside the admissible region the function has a very steeped derivative, creating numerical problems)

6.4.2 Stochastic algorithm

We apply the stochastic algorithm taking as initial point $(8, 1)$ and step $0.3 \times n^{-1}$.

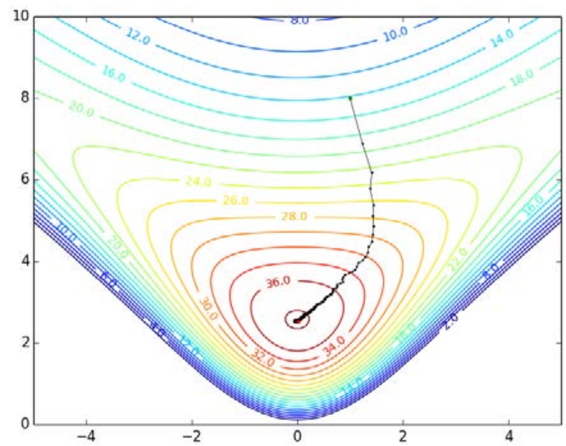

Figure 6.2: Example of the stochastic algorithm converging towards the solution.

Wider steps and Ruppert-Polyak

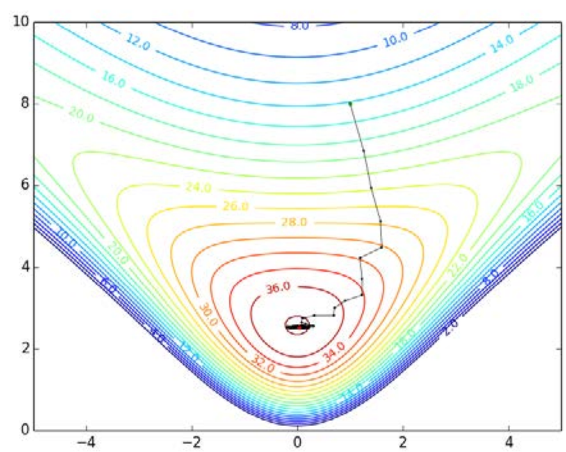

Figure 6.3: Convergence of the algorithm with exponent 0.6.

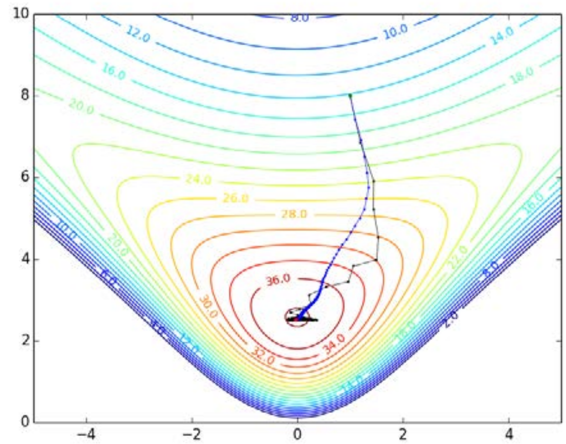

Figure 6.4: Convergence of the algorithm with exponent 0.6 in the Ruppert-Polyak situation.

Algorithm freeze-up

In the examples we just saw, the constant $C = 0.3$ was chosen manually by looking at previous numerical experimentation. One problem we can experience in practice, when the algorithm has a $\mathcal{O}(n^{-1})$ step, is that the algorithm freezes-up, that is, it starts to take too much iterations to converge as the step is gets too small for a large n.

The following figure shows how the algorithm freezes-up if we set $C = 0.1$.

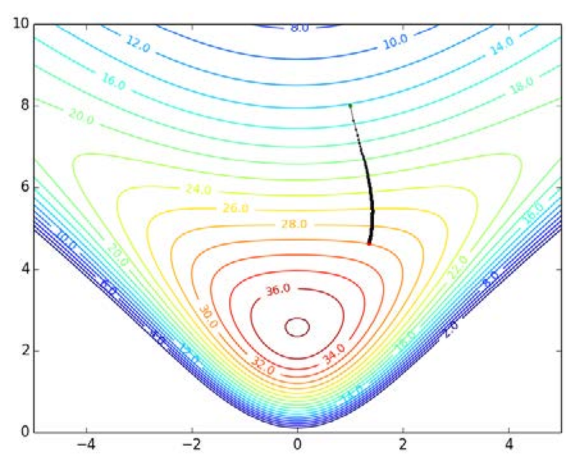

Figure 6.5: Example of a situation when the algorithm freezes up.

One of the advantages of the Ruppert-Polyak approach is that the algorithm gets the best of two worlds: it explores the environment on early stages in order to get closer to the solution, then the averaging improves the convergence once the algorithm is near the solution.

The following figure shows the algorithm with step Cn^{-6} (black path) and the Ruppert-Polyak averaged algorithm (blue path).

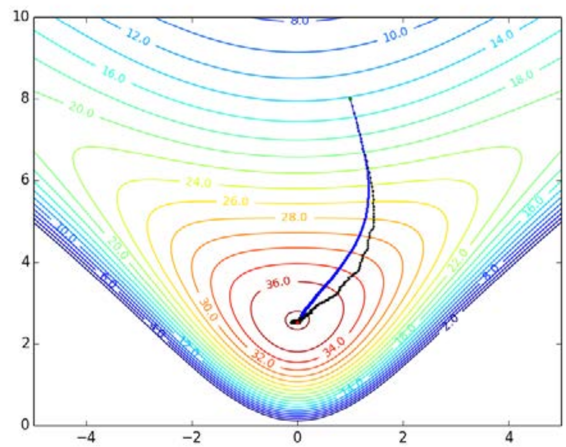

Figure 6.6: Ruppert-Polyak avoids the freeze-up situation.

6.5 Penalizing the inventory

In order to understand the reasons for penalizing the inventory, it is interesting to make a parallel with the stochastic-control approach as in [65].

6.5.1 Relation to the stochastic-control approach

A key hypothesis in our model is that the incoming information have a stationary dynamics and the function to maximize is the same at each iteration. We have already mentioned the advantages of this approach (e.g. adaptability, model-free) however it has the weakness of not having a view on the whole strategy (i.e. inventory risk) which is the strong point on the stochastic-control approach [65] as it is based on the dynamic-programming principle.

However, as it is mentioned in the papers [64, 65] the solutions of the Hamilton-Jacobi-Bellman equation solving the optimal quotes in market-making and optimal liquidation are close to those of the asymptotic regime if we are not close to the end of the trading session. Otherwise said, the hypothesis of stationary innovations is not a constraining one as in the stochastic-control situation; we are solving the 'asymptotic problem'.

Moreover, if we try to apply the dynamic-programming approach on our discrete version of the Avellaneda-Stoikov problem it is easy to see that the utility function we consider in our forward approach correspond to the situation when in the dynamic programming problem there is no risk-aversion (i.e. the market-maker just wants to maximize the PnL).

Otherwise said, the dynamic programming approach solves the inventory risk problem while the on-line learning approach focuses on adapting to changing market dynamics and eventually adverse selection. Adding penalization function on the remaining inventory has as goal add this 'inventory risk' dimension to the problem.

6.5.2 Changes in the structures of the target-function

So far, we have been valuating the inventory at the end of each period mark-to-market. In practice, when liquidating a given quantity in the market, agents incur in liquidation costs. Let us $\Phi(\cdot)$ be the cost of liquidating a remaining inventory Q. We suppose $\Phi(\cdot)$ positive, even and convex. Thus, we define:

$$\kappa(\lambda, \mu) = \mathbb{E}_Y \left[\Phi\left(Q\right)\right]. \tag{6.28}$$

The function to be maximized becomes

$$\pi(\psi, \theta) = \mathbb{E}\left[\psi\lambda - (\theta - Y_T)\mu - \kappa(\lambda, \mu)\right]. \tag{6.29}$$

When we add liquidation costs the dependency in λ and μ is no longer linear. Indeed, in most of the interesting situations there is no closed formula for $\kappa(\lambda, \mu)$.

There are two ways of thinking about the penalization in this context:

- Quantifying costs as if we were liquidating the inventory at the end of each period (to penalize the market-maker payoff), in this way, liquidation costs represent bid-ask spread costs and market impact costs. In that way we set:

$$\Phi(Q) = \underbrace{C|Q|}_{\text{bid}-\text{ask spread}} + \underbrace{\gamma|Q|^{1+\alpha}}_{\text{market impact}}, \quad 0 \leq \alpha < 1. \tag{6.30}$$

  An important situation is $\Phi(Q) = C|Q|$ (i.e. $\gamma = 0$) as it represents the case where the only cost is the bid-ask spread, quantified by the real number $C > 0$. Another situation of interest is $\Phi(Q) = CQ^2$ as it provides a case where we can obtain a closed formula for $\kappa(\lambda, \mu)$. We will look at these two situations in further detail.

- The other way, is to think that the penalization term in the inventory represents a function quantifying costs from the point of view of an (external) algorithm, controlling the inventory risk of the overall strategy. In other words, even though we are considering the 'one-period' problem, in practice the penalization function may be evolving over time (e.g. at a lower frequency than the refreshing of the algorithm).

For example, we can think that early in the trading session we value the inventory in a mark-to-market way, but as we are near the end of the day, we increase the weight we give to the penalization term.

For practical purposes we will address only the first situation here. However we will study general formulas for the expectation of liquidation costs, which can be useful in further studies in this case.

6.5.3 Outline of the stochastic algorithm

The idea is to apply the same recursive optimization procedure than in the precedent section, that is (some variation) of an algorithm of the form:

$$\psi_{n+1} = \psi_n + \gamma_{n+1}\partial_\psi \Pi(\psi_{n+1}, \theta_{n+1}, Y) \tag{6.31}$$

$$\theta_{n+1} = \theta_n + \gamma_{n+1}\partial_\theta \Pi(\psi_{n+1}, \theta_{n+1}, Y) \tag{6.32}$$

The following formula allows to easily compute the derivatives of liquidation costs when $\Phi(\cdot)$ is even and $\Phi(0) = 0$.

Proposition 12.

$$\partial_\lambda \mathbb{E}(\Phi(Q)) = \frac{1}{2}\left(\mathbb{E}(\Phi(Q+1)) - 2\mathbb{E}(\Phi(Q+1)) + \mathbb{E}(\Phi(Q-1))\right) \tag{6.33}$$

$$\partial_\mu \mathbb{E}(\Phi(Q)) = \frac{1}{2}\left(\mathbb{E}(\Phi(Q-1)) - \mathbb{E}(\Phi(Q+1))\right) \tag{6.34}$$

Proof. See Section 6.7.4. □

6.5.4 Closed-formulas for the expectation

The following theorem provides useful closed formulas for $\mathbb{E}_Y[\Phi(Q)]$:

Theorem 16. *Let λ, μ and ε be defined by:*

$$\lambda(\psi, \theta; Y) = 2Ae^{-k\psi}\int_0^T \cosh(k(\theta - Y_t))dt, \tag{6.35}$$

$$\mu(\psi, \theta; Y) = 2Ae^{-k\psi}\int_0^T \sinh(k(\theta - Y_t))dt, \tag{6.36}$$

$$\varepsilon(\theta; Y) := \frac{\lambda(\psi, \theta; Y)}{\mu(\psi, \theta; Y)} = \frac{\int_0^T \sinh(k(\theta - Y_t))dt}{\int_0^T \cosh(k(\theta - Y_t))dt}. \tag{6.37}$$

Then, the following formulas hold:

$$\begin{aligned}
\mathbb{E}_Y[Q^2] &= \lambda + \lambda^2\varepsilon^2, \\
\mathbb{E}_Y[|Q|] &= \int_0^\lambda e^{-s} I_0\left(s\sqrt{1-\varepsilon^2}\right) ds \\
&+ 2|\varepsilon| \sum_{n=1}^{\infty} \sinh\left(\frac{n}{2}\log\left(\frac{1+|\varepsilon|}{1-|\varepsilon|}\right)\right)\int_0^\lambda e^{-s} I_n\left(s\sqrt{1-\varepsilon^2}\right) ds,
\end{aligned}$$

where $I_n(\cdot)$ denotes the modified Bessel function of order n (see [5]).

More generally, for $\Phi(\cdot)$ even, increasing on $\mathbb{R}^+$ and $\Phi(0) = 0$, we have:

$$\begin{aligned}
\mathbb{E}_Y[\Phi(Q)] &= \Phi(1)\int_0^\lambda e^{-s} I_0\left(s\sqrt{1-\varepsilon^2}\right) ds \\
&+ 2\sum_{n=1}^{\infty} D^2\Phi(n)\cosh\left(\frac{n}{2}\log\left(\frac{1+|\varepsilon|}{1-|\varepsilon|}\right)\right)\int_0^\lambda e^{-s} I_n\left(s\sqrt{1-\varepsilon^2}\right) ds \\
&+ 2|\varepsilon|\sum_{n=1}^{\infty} D\Phi(n)\sinh\left(\frac{n}{2}\log\left(\frac{1+|\varepsilon|}{1-|\varepsilon|}\right)\right)\int_0^\lambda e^{-s} I_n\left(s\sqrt{1-\varepsilon^2}\right) ds.
\end{aligned}$$

Where, for $n \in \mathbb{Z}$, we defined:

$$D^2\Phi(n) = \frac{\Phi(n+1) - 2\Phi(n) + \Phi(n-1)}{2},$$

$$D\Phi(n) = \frac{\Phi(n+1) - \Phi(n-1)}{2}.$$

Proof. See Section 6.7.4. □

6.5.5 Bounds for liquidation costs

Upper bound for the liquidation-costs

Conditionally to Y we can place ourselves in the case where $\lambda(\psi, \theta; Y)$ and $\mu(\psi, \theta; Y)$ are considered as constant. As Q is always an integer, we get:

$$\mathbb{E}(\Phi(Q)) \leq (C+\gamma)\mathbb{E}(Q^2) = (C+\gamma)(\lambda + \mu^2)$$

Lower bound for the liquidation-costs

The first lower bound follows Jensen inequality: $\mathbb{E}(\Phi(Q)) \geq \Phi(\mu)$. However, this bound is not very useful when $\mu = 0$. A better bound can be obtained by observing the following two facts:

- Conditional to N the variable Q can be writen as $Q = 2B - N$ where B follows a binomial law with parameters (N, p).
- Secondly we use the fact that $\Phi(Q) \geq (C+\gamma)|Q|$; We want to find a lower bound for $\mathbb{E}[|Q|]$.

Proposition 13. *We have*

$$\mathbb{E}(|Q|) \geq \frac{\lambda}{2\sqrt{2(\lambda+1)}} \geq \frac{1}{4}\min\left(\lambda, \sqrt{\lambda}\right).$$

Leading to:

$$\mathbb{E}(\Phi(Q)) \geq \left(\frac{C+\gamma}{4}\right)\min\left(\lambda, \sqrt{\lambda}\right).$$

Proof. See Section 6.7.4. □

6.5.6 Existence of a minimum

As in the mark-to-market case, we want to show first that for a fixed θ there is a unique ψ where $\mathbb{E}[\Pi(\psi, \theta; Y)]$ has its maximum; in fact, we can show that this is true every time the function κ has the following form:

$$\kappa(\lambda, \mu) = \sum_{j=1}^{J} \lambda^j g_j\left(\frac{\mu}{\lambda}\right), \quad g_j\left(\frac{\mu}{\lambda}\right) \geq 0$$

Proposition 14. *If the conditional expectation of the liquidation costs is characterized by the function*

$$\kappa(\lambda, \mu) = \sum_{j=1}^{J} \lambda^j g_j\left(\frac{\mu}{\lambda}\right), \quad g_j(x) \geq 0, \forall x \in \mathbb{R}.$$

Then, for any fixed θ_0 such as $a(\theta_0) \geq b(\theta_0)$ we can find an unique ψ_{θ_0} maximizing $\mathbb{E}[\pi(\psi, \theta_0; Y)]$.

Proof. See Section 6.7.4. □

6.6 Conclusion

In this chapter we provided a framework, based on the theory of stochastic approximation, for solving the problem of a market-maker participating on an electronic limit-order book. The idea is to take advantage of the iterative nature of trading tactics when proceeding algorithmically, on a high-frequency basis and when the performances are measured statistically. The advantage of our framework is that it is devised for the type of situations where we aim for a model-free approach in which the algorithm extract information from the environment during its execution, i.e. more adapted to the case of a liquid stock in which the velocity of order-book data allows the algorithm a rapid learning.

The mathematical proofs of the convergence of our learning algorithm are based on the Robbins-Monro theorem, and its formal development is provided in the next section as well as results guaranteeing that the problem is well-posed, i.e. it exists a solution, and this solution is unique.

We also studied the situation in which penalization costs are included in order to control inventory risk (maybe the only weakness of the framework, compared to the stochastic-control approach). We provided mathematical results on the analytical properties of the penalization cost function.

Among the possible future directions of research for this work are:

- Generalize the approach to a wider class of trading-tactics.
- Study in detail the relation with the approaches based on the dynamic programming principle (i.e. obtain the best of two worlds in terms of on-line learning and inventory control).

- Study stochastic algorithms in continuous time, not as a model approximation as in the stochastic-control approach, but, for example, by updating quotes in Poissonian time (as it is the natural way at which the algorithm aggregates information from markets). In this case the optimal quotes can be seen as controls of a Poisson process whose events update the control itself, so adding a source of self-reinforcement.

6.7 Proof of the main results

6.7.1 Closed formulas in the Brownian case

Proof of Theorem 12:

Let us start by defining

$$\begin{aligned} a(\theta) &= \mathbb{E}\left[\int_0^{\Delta T} \cosh(k(\theta - Y_t))dt\right] \\ b(\theta) &= \mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T} \sinh(k(\theta - Y_t))dt\right]. \end{aligned}$$

The function to maximize can be writen as

$$\pi(\psi, \theta) = 2Ae^{-k\psi}(\psi a(\theta) - b(\theta)).$$

By classical calculus we can show that the maximum of this function satisfies

$$\psi = \frac{1}{k} + \frac{b(\theta)}{a(\theta)},$$

which in the Brownian case can be explicitly computed, as it is shown below.

Proposition 15. *If the reference price evolves following the stochastic differential equation $dS_t = \sigma dW_t$, then we have the following closed formulas:*

$$\begin{aligned} a(\theta) &= \frac{2\cosh(k\theta)}{k^2\sigma^2}\left(e^{\frac{k^2\sigma^2\Delta T}{2}} - 1\right), \\ b(\theta) &= \frac{2\theta\sinh(k\theta)}{k^2\sigma^2}\left(e^{\frac{k^2\sigma^2\Delta T}{2}} - 1\right) \\ &+ \frac{2\cosh(k\theta)}{k^3\sigma^2}\left(e^{\frac{k^2\sigma^2\Delta T}{2}}(k^2\sigma^2\Delta T - 2) + 2\right). \end{aligned}$$

In particular, we have the identity

$$\frac{1}{k} + \frac{b(\theta)}{a(\theta)} = \theta\tanh(k\theta) + \frac{1}{k}\left(\frac{k^2\sigma^2\Delta T - 1 + e^{-\frac{k^2\sigma^2\Delta T}{2}}}{1 - e^{-\frac{k^2\sigma^2\Delta T}{2}}}\right) > 0.$$

Proof. Two types of integrals will appear in our computations:

$$I_{\Delta T}(\alpha) = \int_0^{\Delta T} \mathbb{E}\left[e^{\alpha W_t}\right] dt \quad \text{and} \quad J_{\Delta T}(\alpha) = \int_0^{\Delta T} \mathbb{E}\left[W_t e^{\alpha W_t}\right] dt.$$

Let us first compute $I_T(\alpha)$.

$$\begin{aligned} I_{\Delta T}(\alpha) &= \int_0^{\Delta T} \mathbb{E}\left[e^{\alpha W_t}\right] dt \\ &= \int_0^{\Delta T} e^{\frac{\alpha^2 t}{2}} dt \\ &= \frac{2}{\alpha^2}\left(e^{\frac{\alpha^2 \Delta T}{2}} - 1\right). \end{aligned}$$

In order to compute $J_{\Delta T}(\alpha)$, we realise that $J_{\Delta T}(\alpha) = I'_{\Delta T}(\alpha)$, this implies:

$$J_{\Delta T}(\alpha) = \frac{2}{\alpha^3}\left(e^{\frac{\alpha^2 \Delta T}{2}}(\alpha^2 \Delta T - 2) + 2\right).$$

Three other identities will be used:

- If W is a Brownian motion, W and $-W$ have same laws, this yields:

$$I_{\Delta T}(-\alpha) = I_{\Delta T}(\alpha).$$

- By the same token:

$$J_{\Delta T}(-\alpha) = -J_{\Delta T}(\alpha).$$

- Finally, because W has independent increments:

$$\int_0^{\Delta T} \mathbb{E}\left[W_{\Delta T} f(W_t)\right] = \int_0^{\Delta T} \mathbb{E}\left[W_t f(W_t)\right], \quad \forall f \in \mathcal{C}([0, \Delta T])$$

Using the latter, we obtain:

$$\begin{aligned} a(\theta) &= \mathbb{E}\left[\int_0^{\Delta T} \cosh(k(\theta - \sigma W_t)) dt\right] \\ &= \frac{1}{2}\int_0^{\Delta T} \mathbb{E}\left[e^{k\theta} e^{-k\sigma W_t} + e^{-k\theta} e^{+k\sigma W_t}\right] dt \\ &= \frac{e^{k\theta} I_{\Delta T}(-k\sigma) + e^{-k\theta} I_{\Delta T}(k\sigma)}{2} \\ &= \cosh(k\theta) I_{\Delta T}(k\sigma) \end{aligned}$$

and

$$\begin{aligned}
b(\theta) &= \mathbb{E}\left[(\theta-\sigma W_{\Delta T})\int_0^{\Delta T}\sinh(k(\theta-\sigma W_t))dt\right] \\
&= \theta\mathbb{E}\left[\int_0^{\Delta T}\sinh(k(\theta-\sigma W_t))dt\right]-\sigma\mathbb{E}\left[W_T\int_0^{\Delta T}\sinh(k(\theta-\sigma W_t))dt\right] \\
&= \theta k^{-1}a'(\theta)-\sigma\mathbb{E}\left[\int_0^{\Delta T}W_t\sinh(k(\theta-\sigma W_t))dt\right] \\
&= \theta\sinh(k\theta)I_{\Delta T}(k\sigma)-\sigma\left(\frac{e^{k\theta}}{2}J_{\Delta T}(-k\sigma)-\frac{e^{-k\theta}}{2}J_{\Delta T}(k\sigma)\right) \\
&= \theta\sinh(k\theta)I_{\Delta T}(k\sigma)+\sigma\cosh(k\theta)J_{\Delta T}(k\sigma).
\end{aligned}$$

By replacing the values of $I_{\Delta T}$ and $J_{\Delta T}$ we end the proof. □

Proposition 16. *(Brownian motion with trend) If $Y_t = \mu t + \sigma W_t$, we have*

$$\begin{aligned}
a(\theta) &= \int_0^{\Delta T} e^{\frac{k^2\sigma^2 t}{2}}\cosh(k\theta-k\mu t)dt \\
b(\theta) &= (\theta-\mu\Delta T)\int_0^{\Delta T} e^{\frac{k^2\sigma^2 t}{2}}\sinh(k\theta-k\mu t)dt \\
&- \frac{\sigma^2}{k}\int_0^{\Delta T} te^{\frac{k^2\sigma^2 t}{2}}\cosh(k\theta-k\mu t)dt.
\end{aligned}$$

Proof. Same reasoning as in the no-trend situation. □

6.7.2 Proof of existence and uniqueness

Before starting the proof of existence and uniqueness of the maximum for our target function (i.e. Theorem 2). We introduce the following concept which will be key in the proof.

Functional co-monotony

The functional co-monotony principle (see Pagès [118]) is the extension of the classical co-monotony principle for real-valued variables, to some stochastic processes such as Brownian diffusion processes, Processes with independent increments, etc. The classic co-monotony principle is stated as follows:

Proposition 17. *Let X be a real-valued random variable and $f, g : \mathbb{R} \to \mathbb{R}$ two monotone functions sharing the same monotony property. Then, if $f(X)$, $g(X)$ and $f(X)g(X)$ are in $L^1(\mathbb{P})$, the following inequality holds:*

$$\mathbb{E}\left[f(X)g(X)\right] \geq \mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(X)\right]. \tag{6.38}$$

The inequality holds as an inequality if and only if $f(X)$ or $g(X)$ are $\mathbb{P}$-a.s. constant.

In order to extend this idea to functional of stochastic processes, we need first define an order relation between them. In that sense, we will consider that processes are random variables taking values in a path vector subspace, and define a (partial) order by saying that if α and β are two processes, then:

$$\alpha \leq \beta \quad \text{if} \quad \forall t \in [0, \Delta T], \alpha(t) \leq \beta(t).$$

Hence, we say that a functional is monotone if it is non-decreasing or non-increasing with the order relation defined above, and we will say that two functionals are co-monotone if they share the same monotony.

We can state now a functional co-monotony principle for pathwise continuous Markov processes that will be useful to prove various inequalities:

Theorem 17. *(Functional co-monotonicity principle) Let X be a pathwise-continuous Markov process, with a monotony preserving transition probabilities*[2]*. If F and G are two real-valued co-monotone functionals, continuous on $(\mathcal{C}([0,T],\mathbb{R}), \|\cdot\|_{\sup})$*[3]*, then:*

$$\mathbb{E}\left[F(X)G(X)\right] \geq \mathbb{E}[F(X)]\mathbb{E}[G(X)]. \tag{6.39}$$

Proposition 18. *If $(Y_t)_{t\in[0,T]}$ satisfies the precedent conditions, we have:*

$$\mathbb{E}\left[(\theta - Y_{\Delta T})\int_0^{\Delta T}\sinh(k(\theta - Y_t))dt\right] \geq (\theta - \mathbb{E}\left[Y_{\Delta T}\right])\,\mathbb{E}\left[\int_0^{\Delta T}\sinh(k(\theta - Y_t))dt\right].$$

Proof. This comes from the fact that the two functionals involved in the expectation: $F(X) = X_{\Delta T}$, $G(X) = \int_0^{\Delta T}\sinh(kX_t)dt$, have same monotony. □

Existence and uniqueness of the maximum

(We prove Theorem 13 by the following sequence of technical propositions.)

For the following propositions we will define

$$\begin{aligned}
a(\theta) &= \mathbb{E}\left[\int_0^{\Delta T}\cosh(k(\theta - Y_t))dt\right] && (6.40)\\
b(\theta) &= \mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T}\sinh(k(\theta - Y_t))dt\right] && (6.41)\\
c(\theta) &= \mathbb{E}\left[\int_0^{\Delta T}\sinh(k(\theta - Y_t))dt\right] && (6.42)\\
d(\theta) &= \mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T}\cosh(k(\theta - Y_t))dt\right] && (6.43)
\end{aligned}$$

[2] This means, for every monotone function $f:\mathbb{R}\to\mathbb{R}$, Pf is monotone with the same monotony (here, $P_{s,t}(x,dy)$ is the transition probability). In particular, so is the case for one-dimensional diffusions satisfying the comparison theorem e.g. solutions to stochastic differential equations under strong existence and uniqueness assumptions (See [122] for more details).

[3] For the space of càdlàg processes $\mathbb{D}([0,T],\mathbb{R})$ we also consider this topology, which is coarser, rather than the more classical Skorokhod J_1-topology(see [118]).

Proposition 19. *The following assertions are true:*

1. *For every $\theta \in \mathbb{R}$, we have $a(\theta) > |c(\theta)|$.*
2. *$c(\theta)$ is strictly increasing, with limits*
$$\lim_{\theta \to +\infty} c(\theta) = +\infty \quad \text{and} \quad \lim_{\theta \to -\infty} c(\theta) = -\infty.$$
3. *It exists a positive constant C satisfying*
$$b(\theta) > 0, \qquad \forall \theta \text{ such as } |\theta| > C.$$
4. *It exists two real constants C_- and C_+ such as*
$$c(\theta)b(\theta) \leq a(\theta)b(\theta), \qquad \forall \theta \geq C_+$$
and
$$c(\theta)b(\theta) \geq -a(\theta)b(\theta), \qquad \forall \theta \leq C_-.$$

Proof. 1. It follows directly from Jensen inequality and that for every $x \in \mathbb{R}$ we have $\cosh(x) \geq |\sinh(x)|$.

2. It follows directly from the fact that $c'(\theta) \geq k\Delta T > 0$, for every $\theta \in \mathbb{R}$.

3. From the functional co-monotony principle we have
$$b(\theta) \geq (\theta - \mathbb{E}[Y_T])c(\theta).$$
Then, from a direct application of the second result of this proposition, we have that it exists $C' > 0$ such as, for $\theta > C$, both factors of the left are positive. Similarly, it exists $C'' > 0$ such as, for $\theta < -C''$, both factors of the left are negative. We conclude our result by taking $C = \max(C', C'')$.

4. By a similar argument as in the last point, it exists C_+ such that, for $\theta \geq C_+$ both $c(\theta)$ and $b(\theta)$ are positive, so we can use the first result in this proposition which yields to $c(\theta)b(\theta) \leq a(\theta)b(\theta)$.

   The second inequality follows in the same way. We know it exists C_- such that, for $\theta \leq C_-$ we have $c(\theta)$ negative and $b(\theta)$ positive, so we can use the first result in this proposition which yields to $c(\theta)b(\theta) \geq -a(\theta)b(\theta)$.

$\square$

Proposition 20. *Let us define the function*
$$f(\theta) = b(\theta)c(\theta) - a(\theta)d(\theta).$$
The following assertions are true

1. *$f(\cdot)$ is strictly decreasing.*
2. $\lim_{\theta \to +\infty} f(\theta) < 0$.
3. $\lim_{\theta \to -\infty} f(\theta) > 0$.

In particular, this implies that $f(\theta) = 0$ has a unique solution.

Proof. 1. First of all, observe that $a'(\theta) = kc(\theta)$, $c'(\theta) = ka(\theta)$, $b'(\theta) = c(\theta) + kd(\theta)$ and $d'(\theta) = a(\theta) + kb(\theta)$. This leads to

$$f'(\theta) = c^2(\theta) - a^2(\theta)$$

which is strictly negative, due to the first result in the precedent proposition.

Hence, f is strictly decreasing.

2. From the last result in the precedent proposition we have that if $\theta > C_+$ then

$$f(\theta) = b(\theta)c(\theta) - a(\theta)d(\theta) \leq a(\theta)(b(\theta) - d(\theta)).$$

Otherwise said

$$f(\theta) \leq -a(\theta)\mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T} e^{-k(\theta - Y_t)}dt\right] = -a(\theta)e^{-k\theta}\mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T} e^{kY_t}dt\right]$$

where the right side is negative for θ large enough.

3. From the last result in the precedent proposition we have that if $\theta < C_-$ then

$$f(\theta) = b(\theta)c(\theta) - a(\theta)d(\theta) \geq -a(\theta)(b(\theta) + d(\theta)).$$

Otherwise said

$$f(\theta) \geq -a(\theta)\mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T} e^{k(\theta - Y_t)}dt\right] = -a(\theta)e^{k\theta}\mathbb{E}\left[(\theta - Y_T)\int_0^{\Delta T} e^{-kY_t}dt\right]$$

In this case, as $-\theta$ becomes large enough, the right side becomes positive.

□

Proposition 21. *The function g defined by*

$$g(\theta) = a(\theta)\exp\left(-k\frac{b(\theta)}{a(\theta)}\right), \quad \theta \in \mathbb{R},$$

has a unique maximum.

Proof. We have

$$g'(\theta) = k \exp\left(-k\frac{b(\theta)}{a(\theta)}\right)\left(a'(\theta)\left(\frac{1}{k} + \frac{b(\theta)}{a(\theta)}\right) - b'(\theta)\right),$$

which is equivalent to

$$g'(\theta) = k^2 \exp\left(-k\frac{b(\theta)}{a(\theta)}\right)\left(\frac{c(\theta)b(\theta) - d(\theta)a(\theta)}{a(\theta)}\right).$$

Thus, the sign of $g'(\theta)$ is the same as the sign of $f(\theta)$ from the last proposition. Hence, $g'(\theta)$ is strictly decreasing and has a unique point where it becomes zero. Otherwise said, g is strictly concave and has a unique maximum. □

Proposition 22. *The function $\pi(\psi, \theta) = e^{-k\psi}(\psi a(\theta) - b(\theta))$ has an unique maximum.*

Proof. First of all, if we fix θ. The maximum is achieved in

$$\psi^*(\theta) = \frac{1}{k} + \frac{b(\theta)}{a(\theta)}.$$

The value of this maximum is given by

$$\pi(\psi^*(\theta), \theta) = k^{-1}e^{-1}e^{-k\frac{b(\theta)}{a(\theta)}}a(\theta)$$

By the precedent proposition, it exists a unique θ^* maximum of this function.

Thus, the maximum of $\pi(\psi, \theta)$ is achieved at the point $(\psi^*(\theta^*), \theta^*)$.

□

6.7.3 Convergence of the stochastic algorithm

Before proving the convergence of the stochastic algorithm, we recall the hypothesis of the Robbins-Monro theorem (see [119]) which is central to complete the proof.

The Robbins-Monro theorem

Let us consider an algorithm of the form

$$\delta_{n+1} = \delta_n + \gamma_{n+1} H(\delta_n, Y_{n+1}), \tag{6.44}$$

with $(Y_n)_{n\in\mathbb{N}}$ an i.i.d. sequence of ν-distributed $\mathbb{R}^q$-valued random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In order to prove our main result, let us consider a random vector Y taking values in $\mathbb{R}^q$ with distribution ν and a Borel function $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$.

Following what precedes, we introduce into our analysis the following function:

$$h : \delta \mapsto \mathbb{E}\left[H(\delta, Y)\right]. \tag{6.45}$$

And, for this function to be well defined, we add the following condition:

$$\forall \delta \in \mathbb{R}^d, \quad \mathbb{E}\left[|H(\delta, Y)|\right] < +\infty. \tag{6.46}$$

Theorem 18. *Assume that the mean function h of the algorithm is continuous and satisfies*

$$\forall y \in \mathbb{R}^d, \delta \neq \delta^*, \quad \langle \delta - \delta^*, h(\delta) \rangle < 0 \tag{6.47}$$

(which implies that $\{h = 0\} = \{\delta^*\}$). Suppose furthermore that $Y_0 \in L^2(\mathbb{P})$ and that H satisfies

$$\forall \delta \in \mathbb{R}^d, \quad \mathbb{E}\left[\|H(\delta, Y)\|^2\right] \leq C(1 + \|\delta\|^2) \tag{6.48}$$

If the step sequence γ_n satisfies $\sum_n \gamma_n = +\infty$ and $\sum_n \gamma_n^2 < +\infty$, then:

$$\delta_n \to \delta^*, \quad \mathbb{P} - a.s. \tag{6.49}$$

and in every $L^p(\mathbb{P})$, $p \in (0, 2)$.

Computing the local gradient

Proof of Proposition 11:

Proof. By direct calculation we have:

$$\begin{aligned}
\frac{\partial}{\partial \psi} \lambda(\psi, \theta; y_t) &= -k\lambda(\psi, \theta; y_t), \\
\frac{\partial}{\partial \psi} \mu(\psi, \theta; y_t) &= -k\mu(\psi, \theta; y_t), \\
\frac{\partial}{\partial \theta} \lambda(\psi, \theta; y_t) &= k\mu(\psi, \theta; y_t), \\
\frac{\partial}{\partial \theta} \mu(\psi, \theta; y_t) &= k\lambda(\psi, \theta; y_t).
\end{aligned}$$

which applied to

$$\Pi(\psi, \theta; Y) = \psi\lambda(\psi, \theta; Y) - (\theta - Y_{\Delta T})\mu(\psi, \theta; Y).$$

lead directly to the result we want. □

Convergence of the stochastic algorithm

Proof of Theorem 14:

Proof. The main idea here is to apply the Robbins-Monro algorithm. However, as it was said, we cannot apply it directly on the local gradient $\nabla_{\psi,\theta}\Pi(\psi_n, \theta_n, Y)$, because of its behavior for larger values of θ. The latter makes impossible to obtain the Robbins-Monro condition

$$\mathbb{E}\left[\|\nabla_{\psi,\theta}\Pi(\psi_n, \theta_n, Y)\|^2\right] \leq C(1 + \psi^2 + \theta^2).$$

However, by multiplying by the factor $\rho(\psi, \theta) = e^{-k(|\theta| - \psi)}$, which does not impact the other Robbins-Monro conditions, does not changes the point to which the algorithm converges and allow us to retrieve the inequality we are looking for. Moreover, the choice of $\rho(\psi, \theta)$ is rather intuitive; it is based on the observation that the intensities λ and μ

have terms with magnitudes $\cosh(k\theta)$ and $\sinh(k\theta)$ multiplied by a factor $e^{-k\psi}$, so our choice of $\rho(\psi, \theta)$ is quite natural.

Because the function in the algorithm corresponds to the gradient of a well-behaved function reaching a unique maximum, the only Robbins-Monro condition it remains to prove in order to obtain the convergence of the algorithm, is to show it exists a real constant C such as

$$\mathbb{E}\left[\|\rho(\psi,\theta)\nabla_{\psi,\theta}\Pi(\psi,\theta,Y)\|^2\right] \leq C(1+\psi^2+\theta^2).$$

By straightforward computation and regrouping terms, we have

$$\begin{aligned}
\|\nabla_{\psi,\theta}\Pi(\psi,\theta,Y)\|^2 &= \|(\theta - Y_T)\nabla\mu - \psi\nabla\lambda\|^2 + \lambda^2 + \mu^2 \\
&+ 2\psi(\lambda\partial_\psi\lambda - \mu\partial_\theta\lambda) + 2(\theta - Y_T)(\mu\partial_\theta\mu - \lambda\partial_\psi\mu) \\
&= \|(\theta - Y_T)\nabla\mu - \psi\nabla\lambda\|^2 + (\lambda^2+\mu^2)(1-2k\psi) + 4k\lambda\mu(\theta - Y_T) \\
&\leq k^2(|\theta - Y_T|^2 + \psi^2)(\lambda^2+\mu^2) + (\lambda^2+\mu^2)(1-2k\psi) + 4k\lambda\mu(\theta - Y_T) \\
&= (\lambda^2+\mu^2)(k^2|\theta - Y_T|^2 + (1-k\psi)^2) + 4k\lambda\mu(\theta - Y_T) \\
&\leq 2\lambda^2(k^2|\theta - Y_T|^2 + (1-k\psi)^2 + 4k(\theta - Y_T)) \\
&\leq 2\lambda^2(3k^2|\theta - Y_T|^2 + (1-k\psi)^2 + 1) \\
&\leq 12k^2\lambda^2(\theta^2 + \psi^2 + k^{-2} + Y_T^2)
\end{aligned}$$

On the other hand, by Jensen inequality we have

$$\lambda^2 \leq 4A^2 e^{-2k\psi}\int_0^{\Delta T}\cosh^2(k(\theta - Y_t))dt \leq 4A^2 e^{-2k\psi}\int_0^{\Delta T} e^{2k|\theta|}e^{2k|Y_t|}dt.$$

This leads to

$$\|\nabla_{\psi,\theta}\Pi(\psi,\theta,Y)\|^2 \leq 48k^2A^2e^{2k(|\theta|-\psi)}(\theta^2+\psi^2+k^{-2}+Y_T^2)\int_0^{\Delta T} e^{2k|Y_t|}dt \tag{6.50}$$

which is equivalent to

$$\begin{aligned}
\mathbb{E}\left[\|\rho(\psi,\theta)\nabla_{\psi,\theta}\Pi(\psi,\theta,Y)\|^2\right] &\leq 48k^2A^2\mathbb{E}\left[(\theta^2+\psi^2+k^{-2}+Y_T^2)\int_0^{\Delta T} e^{2k|Y_t|}dt\right] &(6.51)\\
&\leq C(1+\theta^2+\psi^2). &(6.52)
\end{aligned}$$

with

$$C = 48k^2A^2\mathbb{E}\left[(1+k^{-2}+Y_T^2)\int_0^{\Delta T} e^{2k|Y_t|}dt\right]$$

which is bounded by hypothesis.

Hence the Robbins-Monro theorem can apply, which concludes the proof.

□

Dimensionality reduction for the innovation

Proof of Theorem 15:

Proof. Besides the end-value Y_T, all the dependency in the process Y is contained in the functions λ and μ (given by the equations (6.11) and (6.12)). To obtain our result, let us rewrite the following integral:

$$\int_0^T e^{k(\theta-Y_t)}dt = e^{k\theta}\sqrt{\frac{\int_0^T e^{-kY_t}dt}{\int_0^T e^{kY_t}dt}}\sqrt{\int_0^T e^{-kY_t}dt\int_0^T e^{kY_t}dt} = e^{k\theta+\rho_k}e^{b_k}. \qquad (6.53)$$

By the same argument, we have

$$\int_0^T e^{-k(\theta-Y_t)}dt = e^{k\theta}\sqrt{\frac{\int_0^T e^{kY_t}dt}{\int_0^T e^{-kY_t}dt}}\sqrt{\int_0^T e^{-kY_t}dt\int_0^T e^{kY_t}dt} = e^{-k\theta-\rho_k}e^{b_k}. \qquad (6.54)$$

The results follows by combining these quantities to obtain $\sinh(\cdot)$ and $\cosh(\cdot)$ as in the formulas for λ and μ ((6.11) and (6.12)). ☐

6.7.4 Penalizing the inventory

Derivatives for the penalization function

Proof of Proposition 12:

Proof. Conditional to λ and μ, the random variable Q is the difference of two independent Poisson random-variables N_b and N_a representing the liquidity capture at the bid and at the ask respectively. The variable N, on the other hand, represent the sum of these two variables. Because of the independence between N_b and N_a we know that conditional to N, the variable Q satisfies:

$$\mathbb{E}\left[\Phi(Q)|N\right] = \mathbb{E}\left[\Phi(2B-N)|N\right], \quad B \sim \mathrm{Bin}\left(N, \frac{\mu+\lambda}{2\lambda}\right).$$

Using that N is a Poisson variable with intensity λ we can write

$$\begin{aligned}
\mathbb{E}\left[\Phi(Q)\right] &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{e^{-\lambda}(\lambda+\mu)^k(\lambda-\mu)^{n-k}}{2^n k!(n-k)!}\Phi(2k-n) && (6.55)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\frac{e^{-\lambda}(\lambda+\mu)^k(\lambda-\mu)^{n+1-k}}{2^{n+1} k!(n+1-k)!}\Phi(2k-n-1) && (6.56)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\frac{e^{-\lambda}(\lambda-\mu)^k(\lambda+\mu)^{n+1-k}}{2^{n+1} k!(n+1-k)!}\Phi(2k-n+1). && (6.57)
\end{aligned}$$

The identity (6.56) arises from the hypothesis $\Phi(0) = 0$. The identity (6.57) arises just by inverting the order of summation.

Let us note $a = \lambda + \mu$ and $b = \lambda - \mu$, from (6.56) we obtain

$$\partial_a \mathbb{E}\left[\Phi(Q)\right] = \frac{1}{2}\mathbb{E}\left[\Phi(Q-1)\right]$$

and from (6.57)

$$\partial_b \mathbb{E}\left[\Phi(Q)\right] = \frac{1}{2}\mathbb{E}\left[\Phi(Q+1)\right].$$

This leads to the following formulas

$$\partial_\lambda \mathbb{E}\left[\Phi(Q)\right] = \frac{1}{2}\mathbb{E}\left[\Phi(Q+1) - \Phi(Q) + \Phi(Q-1)\right] \quad (6.58)$$

$$\partial_\mu \mathbb{E}\left[\Phi(Q)\right] = \frac{1}{2}\mathbb{E}\left[\Phi(Q-1) - \Phi(Q+1)\right]. \quad (6.59)$$

□

Closed-formulas for the penalization function

Proof of Theorem 16:

Proof. First of all, the idea is to exploit the following two facts:

1. The dependency of $\mathbb{E}_Y[\Phi(Q)]$ in the variables ψ and θ is totally contained on the functions λ and μ.

2. Under the information provided by the process Y, the inventory is the difference of two independent Poisson variables N^+ and N^- satisfying the following equations:

$$\mathbb{E}[N^- + N^+] = \lambda, \quad (6.60)$$

$$\mathbb{E}[N^- - N^+] = \mu. \quad (6.61)$$

This implies that we just need to do the analysis considering λ and μ as constants; the results can be immediately transferred to the case of the conditional expectation of $\Phi(Q)$ under the information provided by Y.

Now, let us consider a auxiliary Markov process $Q_s = N_s^{(1)} - N_s^{(2)}$ where $N_s^{(1)}$ and $N_s^{(2)}$ are two independent homogeneous Poisson processes with intensities $\lambda_1 = \frac{1+\varepsilon}{2}$ and $\lambda_2 = \frac{1-\varepsilon}{2}$ respectively. It is immediate that Q and Q_λ have the same law, thus, for any function $\Phi(\cdot)$, the quantities $\mathbb{E}\left[\Phi(Q_\lambda)\right]$ and $\mathbb{E}\left[\Phi(Q)\right]$ have the same value. To compute $\mathbb{E}\left[\Phi(Q_\lambda)\right]$ is done by using the infinitesimal generator and Dynkin's formula [83].

For $q \in \mathbb{Z}$ the infinitesimal generator of Q_s is

$$\mathcal{A}\Phi(q) = -\Phi(q) + \left(\frac{1+\varepsilon}{2}\right)\Phi(q+1) + \left(\frac{1-\varepsilon}{2}\right)\Phi(q-1).$$

By the Dynkin's formula we obtain:

$$\mathbb{E}\left[\Phi(Q)\right] = \Phi(0) + \int_0^\lambda \mathbb{E}\left[D^2\Phi(Q_t)\right]dt + \varepsilon\int_0^\lambda \mathbb{E}\left[D\Phi(Q_t)\right]dt.$$

Where

$$D^2\Phi(q) = \frac{\Phi(q+1) - 2\Phi(q) + \Phi(q-1)}{2},$$

$$D\Phi(q) = \frac{\Phi(q+1) - \Phi(q-1)}{2}.$$

The next step is to evaluate expectations of random variables that are defined as the difference between two Poisson random variables. Indeed, these kind of integer valued random variable are said to follow a Skellam distribution with parameters $\lambda_1, \lambda_2 > 0$ if they can be written as the difference of two independent Poisson random variables, N_1 and N_2, with respective parameters λ_1 and λ_2.

Let Q_s follows a Skellam distribution (see [129]) with mean εs and variance s; the support of Q_s is the whole set $\mathbb{Z}$. Elementary computations show that the distribution of the variable Q_s is defined by the following formula:

$$\mathbb{P}(Q = |q|) = 2e^{-s} \cosh\left(\frac{q}{2} \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\right) I_{|q|}\left(s\sqrt{1-\varepsilon^2}\right) \quad , \quad \forall q \in \mathbb{Z} \setminus \{0\} \tag{6.62}$$

$$\mathbb{P}(Q = 0) = e^{-s} I_0\left(s\sqrt{1-\varepsilon^2}\right). \tag{6.63}$$

At this point we just need to replace these values in the equations obtained through the Dynkin's formula and using the proprerties of function Φ (these are, even, increasing in $\mathbb{R}^+$ and $\Phi(0) = 0$ (in particular $D\Phi(q) = -D\Phi(-q)$).

Finally, the equation for $\mathbb{E}[|Q|]$ is obtained by replacing $\Phi(\cdot)$ by $|\cdot|$.

The formula for $\mathbb{E}[Q^2]$ can be obtained by the same idea or by more elementary computations using the properties of the mean and variance of Poisson random variables. □

Other properties of the penalization function

Proof of Proposition 13:

Proof. Because of the symmetry of the function $\Phi(\cdot)$ we can always suppose $p \leq 1/2$. In particular, we can consider a binomial random variable $\widetilde{B}$ with parameters $(N, 1/2)$ which stochastically dominates B. This is easy to explicitly generate:

$$B = \sum_{k=1}^{N} \mathbf{1}_{U_k \leq p} \leq \sum_{k=1}^{N} \mathbf{1}_{U_k \leq \frac{1}{2}} = \widetilde{B}.$$

We obtain the following:

$$\mathbb{E}(|2B - N|) = 2\mathbb{E}\left(\left|B - \frac{N}{2}\right|\right) \geq 2\mathbb{E}\left(\left|B - \frac{N}{2}\right| \mathbf{1}_{\widetilde{B} \leq \frac{N}{2}}\right).$$

By stochastic domination and symmetry of $\widetilde{B}$ we obtain:

$$\mathbb{E}(|2B-N|) \geq 2\mathbb{E}\left(\left|\widetilde{B}-\frac{N}{2}\right|\mathbf{1}_{\widetilde{B}\leq\frac{N}{2}}\right) = \mathbb{E}\left(\left|\widetilde{B}-\frac{N}{2}\right|\right).$$

Moreover, we can use the following De Moivre's result for the absolute deviation of the Binomial distribution[44]:

$$\mathbb{E}\left(\left|\widetilde{B}-\frac{N}{2}\right|\right) = 2^{-N}\binom{N}{\lceil N/2\rceil}\lceil N/2\rceil.$$

At this point, we consider the following elementary inequality:

$$2^{-N}\binom{N}{\lceil N/2\rceil}\lceil N/2\rceil \geq \frac{\sqrt{N}}{2\sqrt{2}}.$$

Now let us take the expectation on N.

$$\mathbb{E}(\sqrt{N}) \geq \sum_{k=1}^{\infty}\sqrt{k}e^{-\lambda}\frac{\lambda^k}{k!} = \lambda\mathbb{E}\left(\frac{1}{\sqrt{N+1}}\right) \geq \frac{\lambda}{\sqrt{\lambda+1}}.$$

(The last inequality comes from Jensen inequality) □

Proof of Proposition 14:

Proof. First of all, we have

$$\partial_\psi\pi(\psi,\theta) = \mathbb{E}\left[\lambda - k\psi\lambda - (\theta - Y_T)\mu + k\sum_{j=1}^{J} j\lambda^{j-1}g_j\left(\frac{\mu}{\lambda}\right)\right] \tag{6.64}$$

We are interested in the solution of $\partial_\psi\pi(\psi,\theta;Y)=0$:

$$Ae^{-k\psi}(k\psi-1)a(\theta) + Ae^{-k\psi}b(\theta) - k\sum_{j=1}^{J} je^{-k(j-1)\psi}c_j(\theta) = 0 \tag{6.65}$$

Here $c_j(\theta) = \mathbb{E}\left(A^{j-1}g_j\left(\frac{\mu}{\lambda}\right)\right)$.

We can divide by $e^{-k\psi}$ an re-arrange terms, this leads to

$$k\psi a(\theta) + (b(\theta)-a(\theta)) = c_1(\theta) + k\sum_{j=1}^{J-1}(j+1)A^je^{-kj\psi}c_{j+1}(\theta) \tag{6.66}$$

The left side is an increasing linear function starting from a negative point, the left side a decreasing exponential starting from a positive point, so they intersect at some point. □

Chapter 7

Conclusion

In concluding the present study, we highlight in this chapter the main ideas that this work was intended to convey, pointing to the main scientific contributions presented and, finally, presenting some possible directions for future research.

7.1 Perspective of this study

Three ideas represent the main perspective of this study:

- The first idea we wanted to put forward is the decomposition of trading algorithms into strategies and tactics. While strategies have a dominant financial rationale and a larger time-horizon, the optimization logic (and the payoff) of a tactic is based on microstructural aspects of the interaction of the market, and even on strategies whose financial rationale is not the same, the tactics associated can be similar (this is for example the case with market-making and optimal liquidation, which at the tactical level obey the same kind of logic – i.e. a liquidity capturing algorithm interacting with the order book in order to maximize a utility function).

- In this study we also wanted to put forward the use of black-box statistical models, i.e. instead of modeling the order book in an exact way, we characterize liquidity by a low-dimensional set of parameters which are estimated from data; this makes the model more flexible. This is the case of the Avellaneda-Stoikov model which permits a straightforward mathematical treatment of optimal trading problems (via dynamic programming or on-line learning techniques).

- Finally, we compared two optimization approaches for the multistage type of problems we deal with throughout this study: the dynamic programming principle (bellman equation), model-based and with a backward induction method of reasoning, and, on the other hand, on-line learning which tends to be more model-free and allows the algorithm to learn from its interaction with the environment. Both approaches have advantages and disadvantages (e.g. in the case of market-making, dynamic programming is more natural for the control of the inventory whereas online learning represents a trend-following manner of adapting to different market-dynamics).

7.2 Scientific contributions

These are the main scientific contributions of this dissertation:

- First of all, the complete solution of the Avellaneda-Stoikov problem for market-making in a limit order book (Chapter 2) by a non-trivial change of variables which make it possible to transform the non-linear HJB equation into a linear system of ODE, allowing in-detail study of the different properties of the optimal solution (asymptotic solutions, comparative statics, numerical representations).

- Using the same result, we apply it to the case of optimal liquidation with limit orders (regarded as a one-sided market-making situation), which, from a practical standpoint, was one of the first articles in the literature treating this problem quantitatively. Again, as in the market-making situation, we provide an in-depth analysis of the optimal solution.

- We provide an analysis for the calibration of the parameters A and k in the Avellaneda-Stoikov model (Chapter 4) and the mathematical study of the estimators involved. We also compare the estimated parameters with real data.

- After providing an introduction to stochastic approximation methods (Chapter 5), we propose a new way to treat the market-making problem through a recursive approximation procedure. This approach is more natural as trading tactics are short-termed and repeated several times a day, and has the advantage of being less constrained by the modeling of the price (as in the classical Avellaneda-Stoikov model). Moreover, our approach can be extended to other classes of trading tactics (optimal liquidation tactics and smart order routing, among others).

- Throughout this work we not only study the mathematical problems but we put them into their industrial context in order to understand a range of issues, from mathematical modeling to implementation. Backtests and calibration on real-data were also presented.

7.3 Future directions

Several problems can be derived from our study. First of all, the more mundane ones entail extending our results to cases with variable trading sizes, general shapes for the liquidation function on Chapter 6 or applying the same ideas on other types of model (e.g. De Larrard et al. [40]). In this section we want to discuss a more interesting class of problems that, in the author's opinion, would be the natural extension of this study.

7.3.1 A general framework for algorithmic-trading tactics using the stochastic approximation approach

The idea is to generalize the model we propose for market-making to a wider class of trading tactics (optimal liquidation with limit orders, routing across lit and dark pools etc.) by using stochastic approximation (as in Chapter 6) . Some advances in this direction have already been proposed on [101, 100].

Roughly speaking, an algorithmic-trading tactic can be seen as a black-box controlled by an input $\delta \in \mathbb{R}^d$ and giving as output a vector $N \in \mathbb{R}^k$ representing the captured liquidity during one iteration of the algorithm. The realized payoff at the end of each iteration is given by a functional of N and $\xi \in \mathbb{R}^p$ (xi is representing exogenous process that participates in the trader's profit but that we cannot control – e.g. the price). We write this functional by

$$\Xi = \Pi(N^{(\delta)}, \xi)$$

and the trader's goal is to solve

$$\delta^* = \arg\max_{\delta} \mathbb{E}[\Pi(N^{(\delta)}, \xi)].$$

Hence, the goal is to find δ^* by a procedure of the form

$$\delta_{n+1} = \delta_n - \gamma_{n+1} \nabla_\delta \mathbb{E}_{\xi_{n+1}}[\Pi(N^{(\delta_n)}, \xi_{n+1})].$$

(Again, one iteration of the algorithm takes ΔT seconds and they are repeated $[\frac{T}{\Delta T}]$ times during the trading session $[0, T]$.

Three ideas are important here:

- The explicit relation between price and liquidity is a choice that does not change the fundamental form of the algorithm. We can go beyond variations of the Avellaneda-Stoikov model. e.g. centering liquidity on the price at the end of each period, and not depending on the whole trajectory of the price.
- δ is the control of the algorithm and this can represent not only the posting distance on an order-book, but also where to post orders in different order books or even the size of the order we send to each venue.
- The processes N will usually represent Poisson processes but as we are interested in functional of N, it would be of interest to study processes with continuous distributions approximating N. For example in the case of a market-maker's inventory (a difference of Poisson processes – i.e. a Skellam distribution) we can think in Gaussian approximations which can lead to simpler formulas.

7.3.2 On-line learning through self-exciting point processes for optimization algorithms in continuous time

An interesting approach that naturally emerges in this kind of optimization problem is proceed entirely in a 'point processes' way. That is, instead of approximating market by a continuous diffusion (as in the stochastic control approach) or instead of splitting the trading session in laps of fixed length ΔT, we fully use the point-process nature of events in the learning procedure and make δ evolve as a continuous function of time, depending on the behavior of the Poisson processes representing the captured liquidity.

Otherwise said, instead of studying the stochastic algorithms as the convergence of a discrete Markov chain we are led to study the convergence of a Point process whose intensity is controlled by a parameter δ_t which at the same time depends on the Poisson process $N^{(\delta_t)}$. i.e. the analysis of the convergence of the algorithm is related to the convergence of a self-reinforcing point processes (this is closely related to the Hawkes process theory).

Concretely, the optimal control δ will not be computed iteratively in a discrete-time way, but in a continuous time way by a procedure that (informally) can be written as

$$\delta_{t+dt} = \delta_t + A(N^{\delta_t}, \delta_t)dt + \langle B(N^{\delta_t}, \delta_t), dN_t \rangle$$

In a way such as $\delta_t \to \delta^*$, as $t \to \infty$.

7.3.3 Including a dynamic-programming logic (for inventory control) to the on-line learning algorithm

The idea is to take as a starting point the dynamic-programming principle applied to a discrete version of the market-making problem (e.g. as in Chapter 6) in the case where the valuation at the end of the $[\frac{T}{\Delta T}]$ periods is made by using an exponential utility function (e.g. CARA investor).

The advantage of the CARA valuation is that it allows a useful product decomposition when using the dynamic-programming principle. i.e. The Bellman equation will lead (for example in a independent increments situation) to maximizing an exponential functional of the variations of price, inventory and cash, and the current state of the inventory.

Hence, for each state of the inventory and time, the market-maker is lead to compute the maximum of a functional which derives from the Bellman equation, the goal is to make this functional the target function in the on-line learning approach.

With the case treated in this study, we were able to reduce to the one-period problem. This would not be the case if the target function depended on the state of the algorithm (inventory process).

A possible way to tackle this issue is to consider an array of stochastic algorithms indexed by an integer value $q \in \mathbb{Z}$ such that the algorithm computing the sequence $(\delta_n^{(q)})_{n\geq 1}$ converges upon the optimal solution, given that at the beginning of the ΔT period the inventory is equal to q. In this way, at the n-th iteration, if the inventory is q_n, we set our orders at a distance $\delta_n^{(q_n)}$. The feedback for this stage is used to update all the values $\delta_{n+1}^{(q)}$, for $q \in \mathbb{Z}$ and the orders for the next step will be posted at $\delta_{n+1}^{(q_{n+1})}$, where q_{n+1} represents the new value of the inventory.

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