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Alexandre Alles

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Inhomogeneous cosmology:
an answer to the Dark Matter and Dark Energy problems?

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Additional thanks to the composers, musicians and singers who offered to the world their art. As said Nietzsche: “Without music, life would be a mistake”

Abstract

The standard model of cosmology describes the formation of large scale structures in the late Universe within a quasi–Newtonian theory. This model requires the presence of unknown compounds of the Universe, Dark Matter and Dark Energy, to properly fit the observations. These two quantities, according to the Standard Model, represent almost 95% of the content of the Universe.

Although the dark components are searched for by the scientific community, there exist several alternatives which try to deal with the problem of the large scale structures. Inhomogeneous theories describe the impact of the kinematical fluctuations on the global behaviour of the Universe. Or some theories proposed to go beyond general relativity.

During my Ph.D. thesis, I developed key–elements of a fully relativistic Lagrangian theory of structure formation. Assuming a specific space–time slicing, I solved the first order system of equations to obtain solutions which describe the matter evolution within the perturbed geometry, and I developed higher order schemes and their correspondences with the Lagrangian perturbation solutions in the Newtonian approach. I also worked on some applications of these results like the description of a silent Universe or the Weyl curvature hypothesis and the problem of gravitational entropy. Further objectives are the description of physical observables and the development of direct applications. Next step of the development is an interaction between theoretical and numerical approaches, a study which would require strong cooperation with observers.

Résumé

Le Modèle Standard de la cosmologie décrit la formation des structures à grande échelle dans l'Univers récent dans un cadre quasi–newtonien. Ce modèle requiert la présence de composantes inconnues, la Matière Noire et l'Énergie Noire, afin de vérifier correctement les observations. Ces deux quantités représentent à elles seules près de 95% du contenu de l'Univers.

Bien que ces composantes sombres soient activement recherchées par la communauté scientifique, il existe plusieurs alternatives qui tentent de traiter le problème des structures à grande échelle. Les théories inhomogènes décrivent l'impact des fluctuations cinématiques sur le comportement global de l'Univers. D'autres théories proposent également d'aller au-delà de la relativité générale.

Durant cette thèse, j'ai mis au point des éléments clés d'une théorie lagrangienne totalement relativiste de la formation des structures. Supposant un feuilletage particulier de l'espace–temps j'ai résolu le système d'équations du premier ordre afin d'obtenir des solutions décrivant l'évolution de la matière dans un espace à la géométrie perturbée. J'ai également développé un schéma de résolution pour les ordres supérieurs de perturbation ainsi que leurs équivalent newtoniens. Une autre partie de ce travail de thèse consiste en le développement de quelques applications directes : la description d'un Univers silencieux ou l'hypothèse de courbure de Weyl et le problème de l'entropie gravitationnelle. Les objectifs à plus ou moins court terme seraient d'obtenir la description d'observables physiques and le développement d'autres applications. Cette étape de développement sera une interaction entre approches théorique et numérique et requerra de se rapprocher fortement des observateurs.

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”Non–Euclidean calculus and quantum physics are enough to stretch any brain; and when one mixes them with folklore, and tries to trace a strange background of multi-dimensional reality behind the ghoulish hints of Gothic tales and the wild whispers of the chimney-corner, one can hardly expect to be wholly free from mental tension.”

–H.P. Lovecraft, The Dreams in the Witch House and Other Weird Stories

Chapter I

Introduction

1 A historical overview 13
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1 A historical overview

Most of the innovative theories were often considered as crazy by the thinkers of their time but are often at the origin of philosophical considerations or artistic creations such as literature. Science is an endless inspiration for the imagination. For instance, in *Histoire comique des États et Empires de la Lune* published in the 1650's, Cyrano de Bergerac narrates the travel of a man on the Moon and the Sun using methods based on optics or fluid mechanics. Moreover, this book illustrates the scientific and philosophical context of a key period: the Scientific Revolution. Science was an opened door to imagination during the XIXth century and the development of science fiction and anticipation literature. One of the most famous of these author is Jules Verne whose work is impressive; *Journey to the Centre of the Earth* (1864), *From the Earth to the Moon* (1865) or *Twenty Thousand Leagues Under the Sea* (1870). This tendency to use science in literature is still present nowadays. For example, *The forever War* (1975) of Joe Haldeman uses the time dilatation during fast travels and the "twin paradox" to deal with war syndromes. Also one of my favourite books, *The chronicles of Amber* by Roger Zelazny (1970) is partially inspired by the concept of Multiverses of Hugh Everett.

The Progress in astronomy is strongly dependent on the observational methods and innovative ideas. The development of the observational techniques led to the discovery of a large catalog of objects, such as galaxies. In 1610, Galileo discovered that the Milky Way was composed of an important number of low–luminosity

stars thanks to his refracting telescope. Most of the precursory ideas before the XXth century are purely philosophical constructions, in a general way the postulate of hypothetical entities often made science drastically progressed. In *Allgemeine Naturgeschichte und Theorie des Himmels* (1755), Immanuel Kant introduced the idea of island–universe: a large number of gravitationally–linked stars, the Milky Way being one of these. Kant assumed that the visible nebulæ could be some other island–universe like the Milky Way. This hypothesis was wildly rejected at this time by the scientific community, since such a large Universe was unconceivable.

At the end of the XVIIIth century, Messier built an observational catalog of nebulæ. Its classification is still used in parallel with other catalogs like the NGC (New General Catalog). These two catalogs are the most popular catalogs in amateur astronomy. In 1912, Vesto Slipher measured a spectral redshift of the "Great Nebulæ of Andromeda" [Slipher, 1913]. In 1915, he released his measurement for fourteen more spiral nebulæ. But we had to wait for 1917 to see the concept of island–universe reappear. Herber Curtis determined the distance of the "Great nebula of Andromeda" (M31) thanks to the photography of a supernova [Curtis, 1917]. The luminosity of this exploding star was fainter than usual. The observed luminosity of a star is directly linked to the distance with respect to the emission source because of the dilution of the light flux, a flash of light propagates as an expanding sphere then its luminosity decreases as the square of the distance to the source. The distance was estimated to 150kpc, far beyond the limit of our island–universe. Convinced by the fact that this object could not be in the Milky Way, he decided to enlarge its studies and deal with the extinction of the dust and the spectral redshift. It seemed to confirm his affirmation. Nevertheless, this discovery stayed very controversial for several years, the term of galaxy appeared during this debate period. Finally, Edwin Hubble gave a definitive answer in 1923–1924 thanks to his reflective telescope. He managed to observe several Cepheids [Hubble, 1925a] and the structures of the objects he observed [Hubble, 1925b], which allowed him to measure more precisely the distance between Earth and these "nebulæ".

The description of the Universe and its dynamics is one of the most interesting problem of the XXth and XXIst centuries. This dynamics shaped the sky as we know it. An important parameter is the Hubble factor H measuring the velocity of the Universe expansion in km/s/Mpc. It defines the relation between the radial velocity and distance from the observer; this discovery is awarded to E. Hubble in 1929 [Hubble, 1929], but should be attributed to G. Lemaître who obtained the same conclusions two years before [Lemaître, 1927] ($H_0 \approx 625$km/s at 1Mpc). This discovery was possible because of the observations of the redshift by V. Silpher in 1912 and was in agreement with a prediction of the Einstein equations. The Hubble factor is now a fundamental parameter of every model of cosmology. A small modification of its value triggers a drastically different Universe evolution. A higher Hubble factor would lead to the absence of structures and galaxies because of the fast expansion which makes the structure formation impossible. On

the contrary a lower Hubble factor (thus a slower expansion) would lead to more compact structures, the expansion would not have any dilution effect and all the matter would collapse faster. The sky would be modified in consequence and a different matter collapse could lead to a totally different development of life. Since the birth of relativity in 1916 [Einstein, 1916] and the first modelisation of the evolution of the space–time by Einstein, new observations are available. The general relativity was an essential step in the improvement of our comprehension of the Universe. Several solutions of the Einstein equations are now known and describe some specific cases. During several centuries, the Universe was imagined fixed and geocentric, and until the 1920's most people still thought there was only one galaxy, the Milky Way. The cosmological principle is a key assumption of the standard model of cosmology. Historically, the strong principle assumes Earth does not occupy a specific place in the Universe and implies the global homogeneity and isotropy, this is the formulation used in the Friedmann equations. The weak principle, used today, is based on the existence of a spatial length beyond which the Universe is statistically homogeneous and isotropic. This scale is estimated to be $100h^{-1}$Mpc with the dimensionless factor $h = (H/100)$km/s/Mpc [Amendola and Palladino, 1999; Yadav et al., 2005; Thieberger and Célérier, 2008; Labini, 2011]. This hides an assumed decorrelation between the small–scale structures and the global dynamics. But the improvements of the observation techniques and statistical treatments show the impact of fluctuations on larger and larger scales, [Kerscher et al., 1998, 2001; Hikage et al., 2003] found a scale about $200h^{-1}$Mpc. Moreover a recent paper of Wiegand *et al.* [Wiegand et al., 2013] showed significant deviations from the standard model of cosmology on scale as large as $500h^{-1}$–$700h^{-1}$Mpc. Most of the estimations are based on lower order statistical properties, whereas the larger results are extracted from Minkowski functionals and higher orders of perturbations. Another recent reference also proposes a new method to estimate the homogeneity scale [Scrimgeour et al., 2012]. Anyway, the modern formulation of the cosmological principle is:

- At a sufficiently large scale the Universe has identical properties everywhere, the matter is homogeneously distributed (there is no special point in the Universe).
- At a sufficiently large scale the Universe is isotropic (there is no preferred direction in space).

Nevertheless, the fact is the FLRW is enforced as an average model in most of the description of cosmology. It means that the old version of the cosmological principle is implemented and required a local average and isotropy.

The first cosmological model was based only on visible matter. Observers measured luminosity of galaxies and estimated the associated mass. This luminous mass was assumed to be equal to the dynamical mass. But problematic observations appeared rapidly. In 1933 Fritz Zwicky [Zwicky, 1933] observed an important

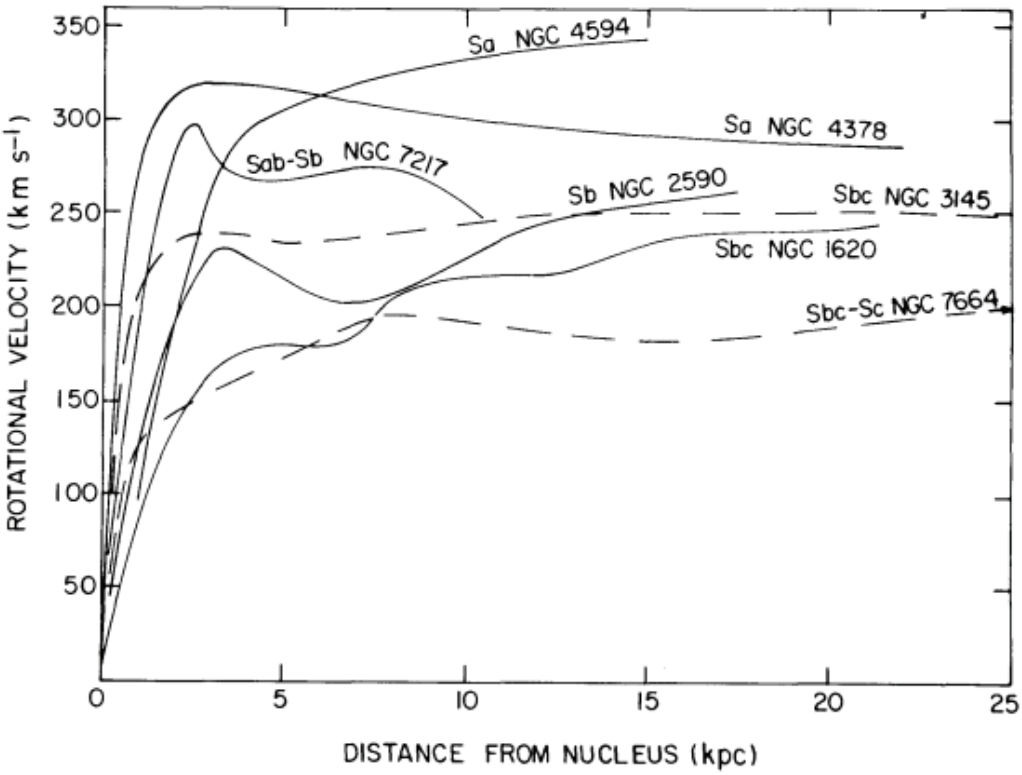


Figure 1.1: Measurement of the rotational velocity of galaxies with respect to the distance from their nucleus [Rubin et al., 1978] figure 3. The expectations of the model were a curve with a similar behaviour for the small distances from the nucleus and a decrease far from the centre of the galaxies. (©The American Astronomical Society)

discrepancy in the dynamics of the Coma cluster between the observations and the predictions based on the luminous matter. The observations of the rotation profiles of galaxies of Vera Rubin and Kent Ford in 1970 [Rubin and Ford, 1970] confirmed the anomaly observed by Zwicky several decades earlier (see figure 1.1 for examples of rotation profiles of galaxy). The model predicted a decreasing velocity far from the nucleus of the galaxy which is clearly not observed. Both observations led to an underestimated velocity of the objects. In 1933, we did not know so much about the importance of dust in galactic disks, their radiative behaviour, or dark objects (as black holes and neutron stars). Even if the observations of Rubin were more precise (improvement of the techniques and knowledge about dust, dark objects...), they show a higher velocity than predicted by the model. They conclude that a fraction of the matter was missed by considering only the visible matter. The Universe had to contain invisible matter: the Dark Matter (DM).

The conundrum of DM stimulates the development of detection experiments spread all over the world: (CMS [CMS Collaboration, 2012], CDMS [CDMS Collaboration, 2013], EDELWEISS [EDELWEISS Collaboration, 2012], XENON100 [XENON100 Collaboration, 2012]...) or in space (PAMELA [PAMELA Collab-

oration, 2013], AMS [AMS Collaboration, 2013] aboard the International Space Station...). All these experiments propose different techniques to identify Dark Matter: direct detection, lack of energy in collision experiments, excess of cosmic radiations. Nevertheless, the community is still waiting for an answer from one of them. The second difficulty is the need of a non–zero cosmological constant for an FLRW metric based on various observations such as the faint galaxy counts, the correlation functions (see [Fukugita et al., 1990; Yoshii and Peterson, 1995]) or the gravitational lensing (see [Fort et al., 1997]). It had been highlight by Adam Riess *et al.* [Riess et al., 1998; Riess et al., 2007] and Saul Perlmutter *et al.* [Perlmutter et al., 1999] who observed the magnitude–redshift relation of several sets of type Ia supernovæ and conclude to the necessity of this non–zero cosmological constant. This is interpreted as an acceleration of the recent Universe scale thanks to the standard model. The commonly accepted cause is the presence of an energy of constant density all across the Universe, several interpretations have been proposed: a negative pressure fluid, the quantum vacuum energy [Rugh and Zinkernagel, 2002], a scalar fluid or quintessence [Zlatev et al., 1999]. The reader may consult the review by S. Caroll [Carroll, 2001]. This invisible energy is called Dark Energy (DE), with reference to Dark Matter. The idea of Dark Energy is more recent and currently there is not any convincing experimental trail either. These two enigmas stimulate actively the modern research. But it is important to note that the interpretations are model dependent, in some Lemaître–Tolman–Bondi (LTB) models the rapid variation of redshift does not lead to the conclusion of an accelerated expansion. For instance, Marie–Noëlle Célérier showed in 2000 [Célérier, 2000], that the parameters of an LTB model could be tune in order to match the value of the matter density Ω_M and cosmological constant Ω_Λ parameters. Thus can fit the supernovæ data such as an Friedmann–Lemaître–Robertson–Walker (FLRW) model with Dark Energy. An alternative explanation to the Dark Energy–like effect is proposed by studying the effect of inhomogeneities on the global dynamical properties (see the reviews by [Buchert, 2008, 2011; Buchert and Räsänen, 2012; Ellis, 2011; Rasanen, 2011; Wiltshire, 2011, 2013]). A recent study by Roukema *et al.* proposes an application (see [Roukema et al., 2013]). They explain this effect by the recent virialisation of matter in the Universe. The matter collapsed violently and triggers a local void which can mimic Dark Energy. Some references illustrate the impact of inhomogeneous model and backreactions on the light propagation (see [Lavinto et al., 2013]).

2 The standard model of cosmology

The standard model of cosmology is the ΛCDM model. Λ is the cosmological constant and represents Dark Energy whereas CDM means Cold Dark Matter. During the last decades this model and its predictions were quite successful. In view of this prescription of the Universe, the observations allow us to constrain the different

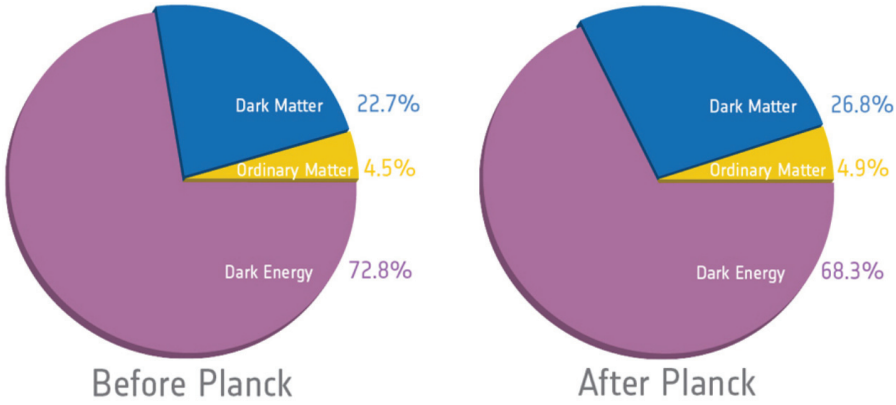


Figure 1.2: Estimation of the Universe budget before and after the Planck mission, the previous data are from WMAP. (http://www.esa.int/spaceinimages/Images/2013/03/Planck_cosmic_recipe, ©ESA and the Planck Collaboration, 2013)

parameters of the model. A power of the standard model is the small numbers of parameters (baryon density Ω_{b}, dark matter density Ω_{DM}, dark energy density Ω_Λ, scalar spectral index n_s, curvature fluctuation amplitude Δ_R^2 and reionisation optical depth τ). The last data released by the satellite Planck [Planck Collaboration, 2013a] show, according to the standard model, a Universe composed of 4.9% of baryonic matter, 26.8% of Dark Matter and 68.3% of Dark Energy (see figure 1.2). Planck also provided a re–estimation of the Hubble constant of 67.8 km/s/Mpc. Two effects contribute to the redshift: the physical motion and the apparent motion. The first effect is due to the physical motion of the source and contributes such as $z = (\sqrt{c+v}/\sqrt{c-v}) - 1$ with c the speed of light and v the velocity of the observed object along the line view, whereas the second effect is due to the expansion of the Universe and is not linked to a physical motion, its contribution is $z = a_0/a - 1$ with a the scale factor and a_0 the current scale factor. Actually, the second effect is much more important on a cosmological scale. For instance, a velocity of 220km/s, as the Sun around the Milky Way center, corresponds to a redshift of $z = 0.0007$ or the Andromeda galaxy has a blueshift of 0.001, whereas the CMB has a redshift of 1,100.

Some details of the observations of the Cosmic Microwave Background (CMB) are provided in section 5. Note that all these numbers are interpreted thanks to the ΛCDM model: a different model could lead to different interpretations of the observations. We are nowadays in the high precision cosmology era. The improvement brought by Planck are very fine tuning, the epoch of the conceptual revolutions, like the introduction of a huge quantity of Dark Matter and Dark Energy or the cosmological constant, seems to be over. The cosmological constant was first introduced by Einstein himself in his equations in order to describe a static space–time [Einstein, 1917], without this constant, the Universe would collapse ir-

reparably. Nevertheless it leads to an unstable static solution: if the Universe gains volume, the cosmological constant represents a more important part of the energy contained in the Universe, then the Universe necessarily grows. On the contrary if the Universe decreases a little, it triggers a loss of energy and the Universe shrinks. Thus Einstein decided to remove it. But in the same time Lemaître and Hubble observed independently a global redshift of the objects which can characterise an expansion of space–time. This result was consistent with the solution derived by Friedmann. It remains during several decades a theoretical and observational debate. But the confirmation of the expansion of the Universe and the recent increasing of the redshift, interpreted as a recent acceleration of the Universe scale, put the cosmological constant into the light. It well describes a Dark Energy.

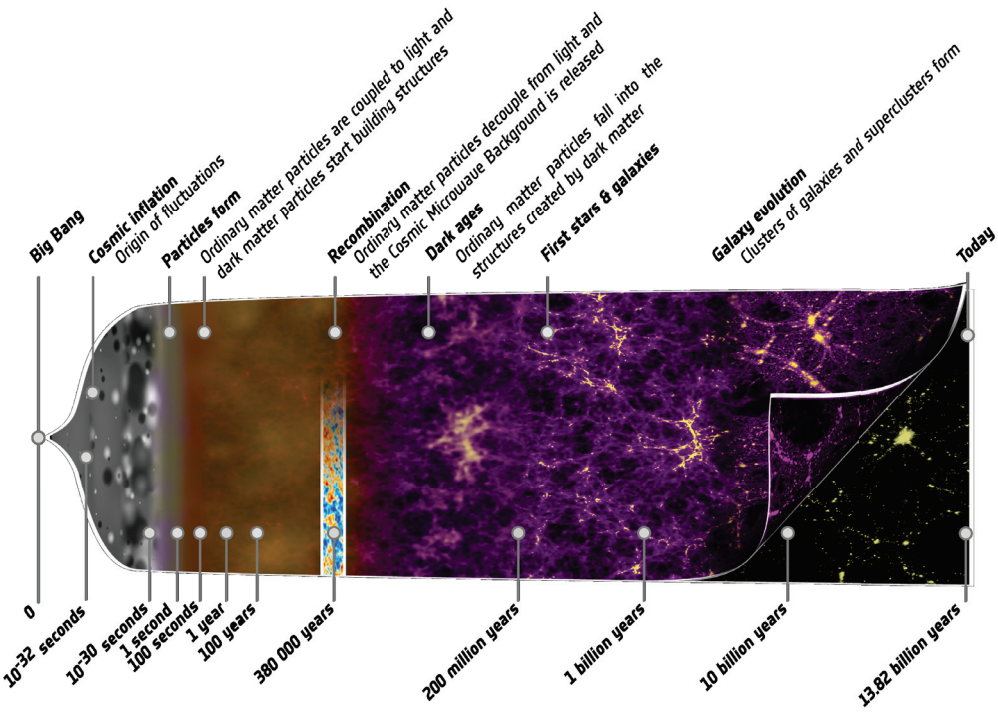


Figure 1.3: Representation of the History of the Universe according to the standard model of the cosmology. (http://sci.esa.int/planck/51560-the-history-of-structure-formation-in-the-universe/, ©ESA - C. Carreau, 2013)

The History of the Universe, as described by the standard model of cosmology, is represented on the figure 1.3. The different predicted epochs and the observations are quite coherent. The standard model of cosmology, through which the observations are interpreted, implies the existence of an initial singularity followed by a cosmic inflation of 10^{-32}s during which the initial inhomogeneities appeared

thanks to quantum fluctuations. During this period of inflation the Universe was composed of very hot and dense plasma, the first hadrons were formed after 1μs from quarks forced to be confined. After 1s the neutrinos were decoupled from the rest of the matter and the leptons were formed between 1s and 10s after the Big Bang. Thus, the Universe continued to expand itself and became less and less dense, after 0.01s the primordial nucleosynthesis could begin and heavier atoms appeared (helium, lithium essentially by the nuclear reactions: $p + n \longrightarrow D^+ + \gamma$, $D^+ + D^+ \longrightarrow {}^3He^{2+} + \gamma$, ${}^3He^{2+} + D^+ \longrightarrow {}^4He^{2+} + p$ ...) and it stopped approximately 3min after the Big Bang. The Universe was so dense that the light could not travel freely until the recombination because of the high cross section of the scattering processes like Compton scattering $e^- + \gamma \longrightarrow e^- + \gamma$ and Coulomb scattering $e^- + p \longrightarrow e^- + p$. The recombination occurred 380,000 years after the Big Bang, it corresponds to the instant when electrons and protons start to combine and form neutral atoms of hydrogen $e^- + p \longrightarrow H + \gamma$. Then the mean free path of light became important, light could travel. Today, it is technically impossible to have a direct observations before the recombination, because the light could not go through space on a sufficiently large distance. Nevertheless, gravitational waves could be a tracer of the pre–recombination Universe. Moreover, in theory the neutrinos are decoupled from the bath before the recombination and should bring information about this epoch but it is impossible to obtain statistical data from them nowadays. The first emitted light is called the Cosmic Microwave Background (CMB). Inhomogeneities in the temperature of this first radiation are observed since, it is directly linked to the inhomogeneities of density in the early Universe: a denser region is obviously hotter and emit a more energetic light. But also the propagation of gravitational waves could modify the polarisation of light. Nevertheless, even after the recombination, the Universe was still too dense to the appearance of important structures and objects. We have to wait until the end of the Dark ages to see the first stars and galaxies appear. Except the thermal radiations present in the sky there was no other light because there was no star, this is the reason why this epoch is named Dark Ages. The oldest objects observed seems to appear 500 million years after the Big Bang.

The Dark Matter and Dark energy enigmas are among the hot topics of modern physics. Additional probes have been developed to estimate the effects of Dark Matter and Dark Energy, and constrain their value:

- The gravitational lenses: according to the theory of general relativity, the presence of important masses in the Universe deviates the photons and then can distort the observations. The distortions manifest themselves through shear and magnification of the images. The study of these distortions permits to reconstruct the repartition of mass between the source and the observer. Some recent results tend to highlight the gravitational lensing triggered by filaments [Jauzac et al., 2012] and confirm the cosmic filaments of matter predicted by the numerical simulations.

- The Baryon Acoustic Oscillations (BAO): it consists of pressure waves in the early Universe that propagate through space–time and induce the outbreak of specific structures in the matter repartition observed nowadays. It is one of the standard ruler in modern cosmology. The SDSS catalog of galaxies (redshift $0.16 \leqslant z \leqslant 0.47$) shows a clear peak in the correlation function around $100h^{-1}$ Mpc [Eisenstein et al., 2005]. This number is compatible with BAO predictions.

- The Sachs–Wolfe effect (SW) [Sachs and Wolfe, 1967]: the denser regions in the early Universe increase the redshift effect according to general relativity, light has to cede energy to leave a potential well. On the one hand this decrease of energy manifests itself by the redshift of the photons (they are colder because less energetic). On the other hand a denser region is hotter, then the photons should be hotter but it is not the case: the redshift overwhelms the density effect. Then regions that appear cold in observations of the CMB correspond to hotter regions according to the Sachs–Wolfe effect. This effect is the main source of fluctuations in the CMB radiations. Two different Sachs–Wolfe effects exist, the non–integrated and integrated SW effect. The first is directly due to the inhomogeneities in the CMB, whereas the second one is due to the regions crossed by the photons during their propagation until the observer.

- Another method, which is a hot topic at the moment is the gravitational waves. The gravitational waves are the second kind of inhomogeneities in the early Universe with the matter repartition. These waves should be able to generate special polarisation in the Cosmic Microwave Background. Two modes exist, the E–mode and the B–mode in reference to electrostatics and magnetostatics: the E–modes are curl–free electromagnetic waves and the B–modes are divergence–free electromagnetic waves. In other words, the E–modes have radial or tangential polarisations with respect to their emitted region, whereas B–modes have a non zero vorticity polarisation vectors with respect to their emitted region. Very recently, the south pole ground based telescope BICEP2 seems to show the existence of B–modes [BICEP2 Collaboration, 2014] which are gravitational waves tracers. Such a discovery could confirm some scenarios of inflations and restrict the possible models. A confirmation is expected from the Planck consortium for september or october 2014.

In parallel to the developments of new cosmological probes, the research of Dark Matter and Dark Energy also stimulates the inventiveness to develop alternative models of the gravitation, such as MOND [Milgrom, 1983, 2011], its relativistic version TEVES [Bekenstein, 2005] or the f(R) theory [Starobinsky, 1980] for instance, but many other models exist. Nevertheless, consistent proofs are still lacking to conclude if one of these theories is more valid than the others or if Dark Matter and Dark Energy exist in the quantities predicted by the ΛCDM model. Without

going so far in alternative theory, it is possible to study general relativity more carefully and describe more precise and/or inhomogeneous model of space–time. In 2008, Krzysztof Bolejko and Lars Andersson explained that the acceleration along the light–cone and the space acceleration could be different [Bolejko and Andersson, 2008], whereas in 2010, Célérier *et al.* showed that an LTB model with a central overdensity can describe a space–time in accelerated expansion as well as an underdense FLRW model [Célérier et al., 2010]. More recently, Bolejko *et al.* described the effect of voids on the light propagation [Bolejko et al., 2013]. A void triggers an anti–lensing effect, the light does not travel along a straight line but is deflected by a negative effect.

According to the cosmological principle, it is impossible for the ΛCDM model to describe any structure under a scale between $100h^{-1}$ and $700h^{-1}$Mpc. The restrictions introduced by this principle should be examined with a critical eye. The homogeneity leads to a total decorrelation between the local and the global dynamics. In some other fields of physics, where we consider local and global equations, effective terms arise and link the local and global representation, like in Electromagnetism. A similar concept exists in gravitation: it is called backreaction, a problem that was raised earlier by Ellis in 1984 [Ellis, 1984] with a solution given by T. Buchert in 2000 [Buchert, 2000a]. This backreaction is additional effective terms which appear if the averaging process is applied carefully to the local equations of gravitation. This leads to a set of average equations which depends on the local behaviour. Some studies conclude that the backreaction is small [Peebles and Ratra, 2003; Ishibashi and Wald, 2006], but the regular approach is based on a perturbative development which consider only small perturbations of density. The reader may refer to [Bardeen, 1982; Mukhanov et al., 1992; Kodama and Sasaki, 1984; Durrer, 1988] for the standard perturbation theory. Nevertheless, the Universe is highly inhomogeneous nowadays, it means the local behaviour could have a very important impact on the global dynamics. Under the typical scale of $100h^{-1}$ Mpc the Universe is highly inhomogeneous, the reader may refer to any recent galactic survey, figure 1.4 shows the results from the SDSS and 2dFGRS. The parameter which best describes the inhomogeneities is the contrast of density $\delta = (\varrho - \varrho_H)/\varrho_H$ with the density $\varrho(\vec{X}, t)$ and the average density $\varrho_H(t)$ where the position is denoted by $\vec{X}$ and the time by t. The index $_H$ denotes the average over the whole space whereas the bracket $\langle\rangle_{\mathcal{D}}$ denotes the average over a specific domain $\mathcal{D}$. On the one hand, some objects present a very high density contrast: galaxies clusters $\langle\delta_{cluster}\rangle \sim 10$, galaxies $\langle\delta_{Galaxy}\rangle \sim 5\cdot 10^5$, stars $\langle\delta_{Star}\rangle \sim 7\cdot 10^{28}$ or even planets $\langle\delta_{\oplus}\rangle \sim 2.8\cdot 10^{29}$. The average density contrast on a region of 10Mpc is $\langle\delta_{10Mpc}\rangle \sim 1$. On the other hand, there exist voids which present a very low density contrast. Even the isotropy of the Universe is debatable, since the last data of the CMB seem to show the existence of a radiation dipole in the early Universe [Planck Collaboration, 2013d].

My Ph.D. thesis proposes a fully–relativistic perturbation theory. I developed

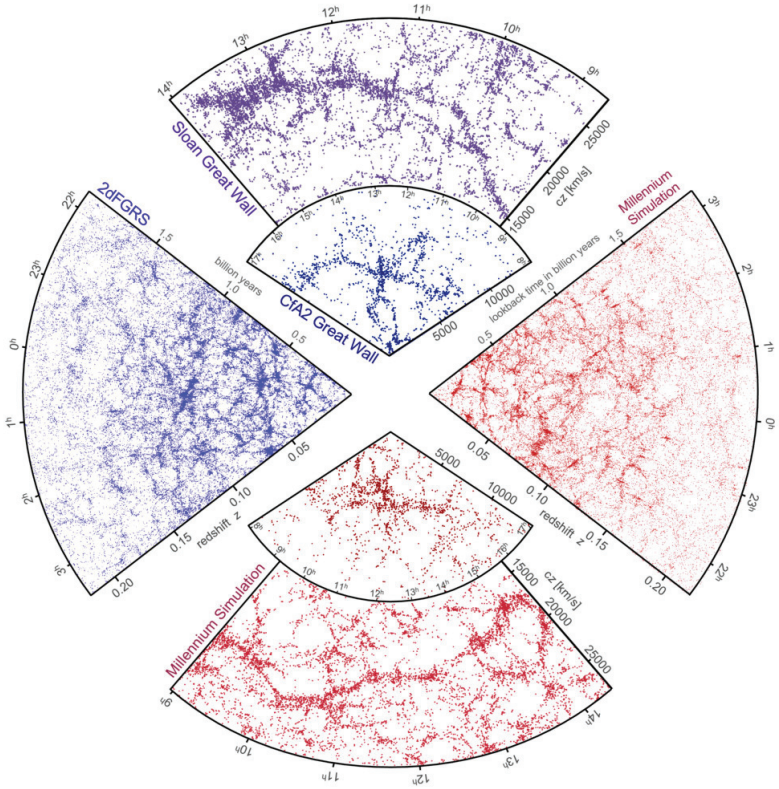


Figure 1.4: Comparison between Sky Survey programs (Sloan Digital Sky Survey SDSS, Two degree Field Galaxy Redshift Survey 2dFGRS) and Numerical Simulations (Millennium) from [Springel et al., 2006] figure 1. (©Nature Publishing Group)

a relativistic approach inspired by the Lagrange–Newton point of view of fluid mechanics. In a similar way, the system of relativistic equations can be reduced to a system of only one variable. Moreover, the density is no longer a variable of the problem and does not have to be perturbed. It allows us to work with a high density contrast computable at any time with respect to its initial value. This higher density contrasts can lead to more important structures formation and backreaction on small scales. In my opinion, some points in the ΛCDM model could fail to describe the formation of structures in the late Universe because of its over–restrictive assumptions. It is very important to work in a spirit of constructive criticism in front of models and theories, and verify the details where an invalid assumption could be hidden. I do not question the efficiency of the ΛCDM to deal with the early stages of the Universe, when it is reasonable to assume that the Universe was fairly homogeneous with very small perturbations. The mean temperature of the CMB is 2.72548K [Fixsen, 2009] whereas the anisotropies reach $\pm 200\mu$K (results from WMAP).

In the sections 3 to 5, I will present the ΛCDM model, some of its classical results and the classical references. The chapter II details the Newtonian perturbation theory, this is the starting point of our formalism. All along my Ph.D. thesis I

kept in mind this Newtonian model and identified the improvements due to the relativistic considerations. Understanding the Newtonian approach, the relativistic perturbation theory is presented in chapters III and IV. The first chapter deals with exact description of the first–order solutions and comparison with the standard perturbation theory, whereas the second chapter deals with a general resolution scheme. It highlights the difference between my approach and the usual way to deal with relativistic perturbations, which is quasi–Newtonian in reality. Chapter IV also presents the general n order gravitoelectric scheme and specific solutions until second–order. Then chapters V and VI present some direct applications and prospectives I began to study during my Ph.D. thesis.

3 Homogeneous cosmology: Friedmann equations

The starting point of the ΛCDM model is the equation of general relativity, made by Albert Einstein in 1916 [Einstein, 1916]. This equation describes the space–time structure. When Newton considered matter in a absolute and fixed space, the Einstein theory does not differentiate the space–time from its content: the matter moves in the space–time and the space–time is deformed by the matter. The Einstein equations describe the evolution of a mathematical object which is interpreted as our space–time, a manifold $(\mathcal{M}, g)$. A manifold is a space which is locally Euclidean, and described by a tensor called the metric g,

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \; ; \tag{1.1}$$

with $G_{\mu\nu}$ the Einstein tensor, based on $\mathcal{R}_{\mu\nu}$ the Ricci tensor, $\mathcal{R}$ its trace, $g_{\mu\nu}$ the 4–dimensional space–time metric tensor, $T_{\mu\nu}$ the energy–momentum tensor, G the constant of gravitation, Λ the cosmological constant and the convention $c = 1$ for the speed of light. Indices on the 4–dimension space–time are noted with Greek letters, and indices on the 3–dimension space with Latin letters. The metric can be defined as a function which allows us to write an infinitesimal length element, where the Einstein notation is assumed, such as:

$$ds^2 = g_{\mu\nu} \cdot dx^\mu \; dx^\nu \; . \tag{1.2}$$

The application of general relativity to describe the evolution of the Universe appeared a few years after the original paper of Einstein. The standard model of cosmology considers the evolution of small perturbations over a homogeneous and isotropic background. Thus, the first essential step is to determine this background behaviour by looking for a solution of the homogeneous Einstein equations. Since the original paper of Einstein some particular solutions were found, we can cite the Schwarzchild metric which is the solution for a spherical massive object [Schwarzschild, 1916] *(translation)*, the Lemaitre–Tolman–Bondi (LTB) metric for a spherical symmetric and inhomogeneous repartition of matter [Lemaître,

1933; Tolman, 1934] or the Friedmann–Lemaitre–Robertson–Walker (FLRW) metric which describes a homogeneous and constant curved space [Friedmann, 1999] *(translation)*.

The standard cosmological model, based on the cosmological principle, is a special case of the Einstein equations. A sufficiently large scale is considered, and the results are assumed to be true for any scale. Thus, the cosmological principle implies a complete decoupling between the global and local dynamics and also the isotropy of the Universe. In this case the energy–momentum tensor reads,

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon_H & 0 & 0 & 0 \\ 0 & -p_H & 0 & 0 \\ 0 & 0 & -p_H & 0 \\ 0 & 0 & 0 & -p_H \end{pmatrix} ; \tag{1.3}$$

where ε_H is the homogeneous energy (which is reduced to the homogeneous density of matter ϱ_H for dust matter), p_H the pressure of fluid content of the Universe and c the light velocity. This matrix is diagonal because of the cosmological principle and the assumption of a perfect fluid. In this approximation, the equations of general relativity are reduced to independent scalar formula, the Friedmann equations and the energy conservation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varepsilon_H + 3p_H\right) + \frac{\Lambda}{3} ; \tag{1.4}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon_H + \frac{\Lambda}{3} - \frac{k}{a^2} ; \tag{1.5}$$

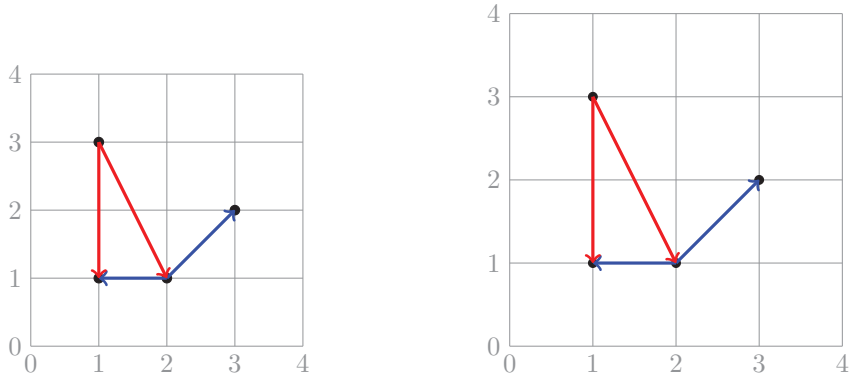
$$\dot{\varepsilon}_H = -3\frac{\dot{a}}{a}\left(\varepsilon_H + p_H\right) ; \tag{1.6}$$

with $a(t)$ the scalar factor, $\varepsilon_H(t)$ the average energy, $p_H(t)$ the average pressure and k a constant curvature. The Hubble parameter is linked to the scale factor by the relation $\mathrm{H}(t) = (\dot{a}/a)(t)$. Three different cases can be distinguished, according to the sign of k: the FLRW solutions. Since the homogeneity and isotropy of the space–time have been assumed, the metric reads $ds^2 = -dt^2 + dr^2 + S^2(k)d\Omega^2$, with dt the time elementary element, dr the radial elementary distance and $d\Omega$ the orthoradial elementary distance. According to the sign of k, the solutions are given by:

$$S(k) = \begin{cases} \dfrac{1}{\sqrt{k}}\sin(r\sqrt{k}) & k > 0 . \\ r & k = 0 . \\ \dfrac{1}{\sqrt{k}}\sinh(r\sqrt{k}) & k < 0 . \end{cases} \tag{1.7}$$

Note the solution for $k = 0$ is a parametrisation of the Euclidean flat space. Nevertheless, it is very important to know that in a strict FLRW model there are no

inhomogeneities: matter does not collapse and no astrophysical objects are formed and then life cannot appear. The standard model does not stop here, the next step is to consider small inhomogeneities, perturbations.



(a) Time t. (b) Time t'>t.

Figure 1.5: Illustration of the homogeneous expansion, any point seems to be the centre of the expansion if we are focused on it but in reality there is no centre.

Any of the three FLRW cases allows us to define the scale factor $a(t)$. This function characterises the relative size of the Universe. Most of the solutions lead to an expanding Universe. But what does it mean? The Universe is in expansion in what? Are the objects pulled away from one another? The correct description is illustrated by the figure 1.5. Let us put objects on the nodes of a grid. A Universe in expansion means the grid is stretched along the time, but every object is fixed to its node and does not move. The stretching of the grid creates an apparent increasing of the distance between the objects. This phenomenon is a homogeneous expansion, it means there is no privileged point as suggested by the cosmological principle. For instance, let us focus on the node $(2, 1)$ and two of its neighbours $(1, 1)$ and $(3, 2)$. Between t and t' the apparent distances changed by a certain factor and every object seems further than before; but the same conclusions rise if the point $(1, 3)$ and its neighbours are considered, all the distance are increased independently of where you are. Moreover, we can see that the objects are still at the same place on the grid, they occupy the same nodes at each time. It is possible to rescale the grid if the distances are divided by the scale factor. The coordinates system rescaled by a factor $a(t)$ is called the comoving coordinates and denotes the proper distance between objects. In this space only the physical movement is described, the expansion is hidden in the scale factor. It is possible to apply the factor at any time in order to come back to the non–comoving (physical) coordinates.

4 Large–scale structure formation

I will not detail the standard perturbation theory here but the curious reader may refer himself to the following references. A large amount of papers deal with the standard perturbation theory and the formation of large scale structures. It is possible to find the perturbative resolution of the equations for an irrotational and pressureless fluid of Dark Matter in [Bouchet et al., 1992; McDonald, 2007; Matsubara, 2008; Scoccimarro, 2001; Taylor and Hamilton, 1996], [Bouchet et al., 1995] which offers a comparison with the results of N–body simulations, the Lagrangian perturbation theory with vorticity in the Zel'dovich Approximation sub–case [Buchert, 1992], whereas the standard perturbation theory is presented in [Bernardeau et al., 2002].

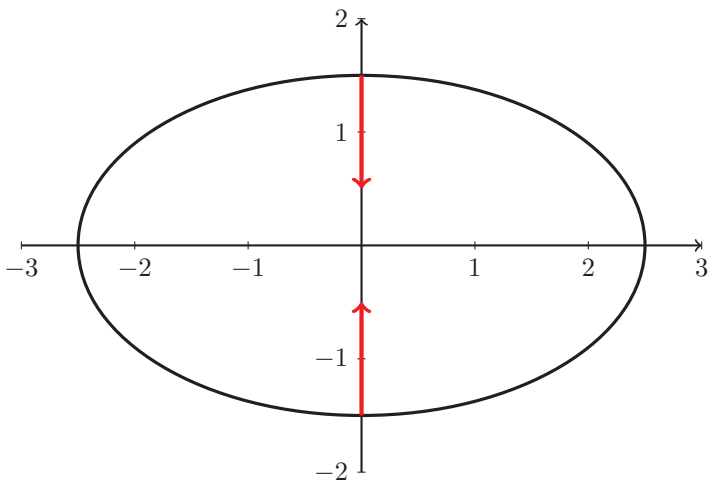


Figure 1.6: Privileged direction of collapse of an overdensity. The smallest axis is the collapse direction, flat structures appears: pancakes.

The Zel'dovich Approximation has been developed in [Zel'dovich, 1970a,b]. This is the assumption that overdensities have a preferred direction of collapse: the shortest axis. It leads to the formation of flattened structures called the Zel'dovich Pancake (or Blini, or Crêpe... it depends of your native language). So after a sufficiently long time, the dynamics of a collapsing structure is driven by only one direction (see figure 1.6). In such a dynamics, objects collapse into a quasi 2–dimensional structure: a pancake. This dynamics leads to a property of colinearity between the velocity and acceleration of particles during the collapse of an overdensity. This approximation allows us to describe a subclass of solutions, which can be found in the Lagrangian papers. Resolutions without vorticity are available in [Buchert, 1989; Buchert and Ehlers, 1993; Buchert, 1994; Vanselow, 1995] respectively for the first, second, third and fourth–order. The full Lagrangian perturbation theory is proposed in [Ehlers and Buchert, 1997], some papers with additional effects exist: the first–order with vorticity [Buchert, 1992], cosmologi-

cal constant [Bildhauer et al., 1992] or the Lagrangian perturbation theory with pressure [Adler and Buchert, 1999].

5 Observations and initial data

The Cosmic Microwave Background (CMB) provides the main probe to constrain the parameters of the ΛCDM model. It allows us to measure a large amount of characteristics and fit the parameters to the chosen model. The CMB is an ambient thermal radiation of 2.7255K emitted at the decoupling of the photons. It happened just after the recombination, 378,000 years after the Big Bang. It is the first light of the Universe, or the older "photography" of the Universe which is possible to take (see figure 1.7). The position of the emissions is called the last scattering surface. This radiation appears naturally in the model of the Big Bang and was predicted by R. Alpher and R. Herman in 1948 [Alpher and Herman, 1948] to be 5K. This prediction was accidentally confirmed in 1965 by A. Penzias and R. Wilson [Penzias and Wilson, 1965] at the Holmdel Horn Antenna of the Bell Telephone laboratories. Since this discovery, several programs of observation have created maps of the CMB. The first complete mapping of the CMB over the sky was obtained thanks to the satellite COBE in the 1990's. The satellites WMAP and Planck refined the map in 2001 and 2009. Moreover, there exist also ground–based radio telescopes. The development of the detectors and cryogenics methods is improving day after day the precision of the measurements. Figure 1.8 illustrates the evolution of the precision of the detectors.

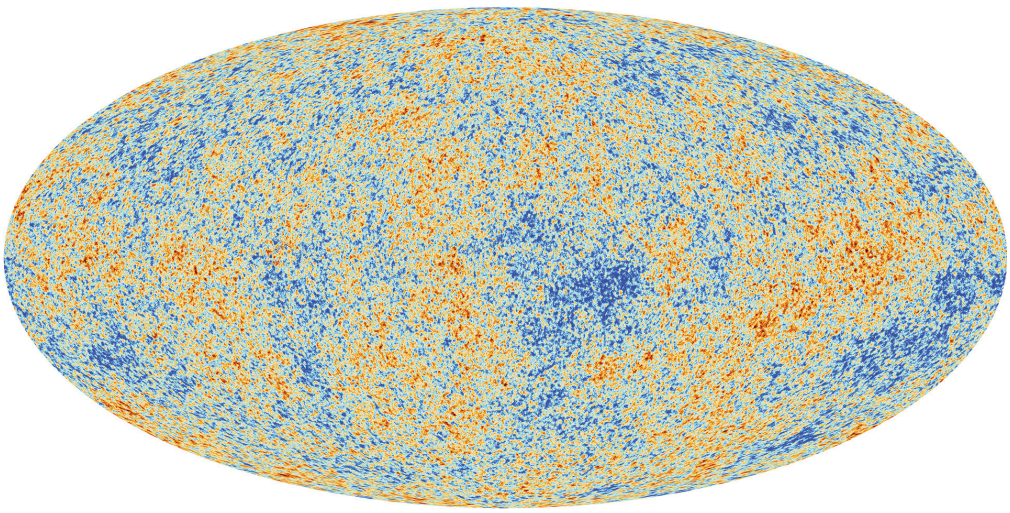


Figure 1.7: Map of the CMB radiation by Planck, released in 2013. (http://www.esa.int/spaceinimages/Images/2013/04/Planck_CMB_black_background, ©ESA and the Planck Collaboration, 2013)

During its early stages, the Universe was so dense and so hot that its matter content was entirely composed of electrically charged particles. Thus the photons were submitted to strong scattering processes and their mean free path was very

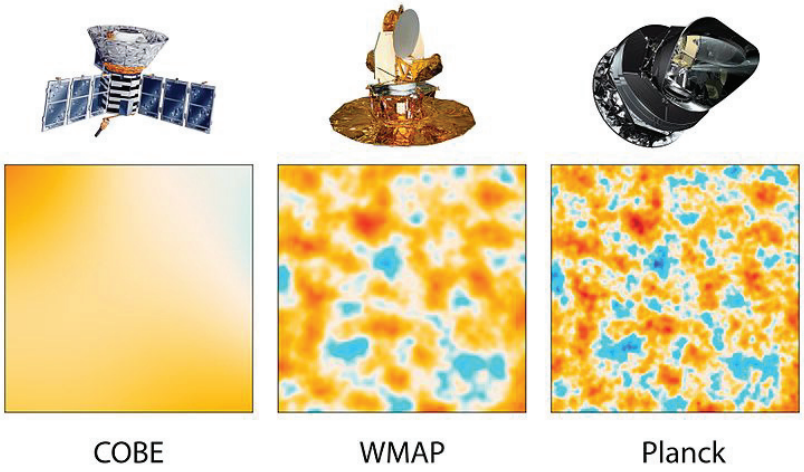


Figure 1.8: Precision in the capture of the CMB radiation by COBE, WMAP and Planck. (http://photojournal.jpl.nasa.gov/catalog/PIA16874, ©NASA/JPL-Caltech/ESA, 2013)

short. The recombination corresponds to the time when electrons and protons started to combine to form neutral atoms of hydrogen. Thus photons could start propagating freely in the Universe (it is the decoupling), before the light propagation was negligible. Using the conservation of density number (function which represents the density of particles depending on the number of quantum degrees of freedom and the distribution function of the considered species) it is possible to estimate the temperature of the decoupling T_{dec} and also the epoch of the decoupling (thanks to the redshift z_{dec} and the time t_{dec}):

$$\begin{cases} T_{dec} \sim 3{,}000 \text{ K} . \\ z_{dec} \sim 1{,}091 \pm 1 . \\ t_{dec} \sim 379 \pm 5 \ 10^3 \text{ years} . \end{cases} \tag{1.8}$$

The Universe is assumed to be in expansion and adiabatic, it means its entropy is conserved. Then the temperature is directly linked to the scale factor $T \propto a^{-1}$. The current CMB temperature can be estimated to 2.72548 ± 0.00057K [Fixsen, 2009] which is in agreement with the Planck results [Planck Collaboration, 2013b]. The last measures of WMAP revealed an average temperature of the CMB radiations of 2.72548 ± 0.00057K. The CMB is quasi homogeneous but some anisotropies exist and are observable.

The inhomogeneities in this radiative background give information about the primordial inhomogeneities in the matter repartition. As mentioned earlier, we have to be careful with some contributions such as the Sachs–Wolfe effect. The treatment of the Sachs–Wolfe effect permits to obtain the original map of density fluctuations: a denser (or hotter) region presents a decrement of temperature according to the mentioned effect. This first image of the Universe is used to

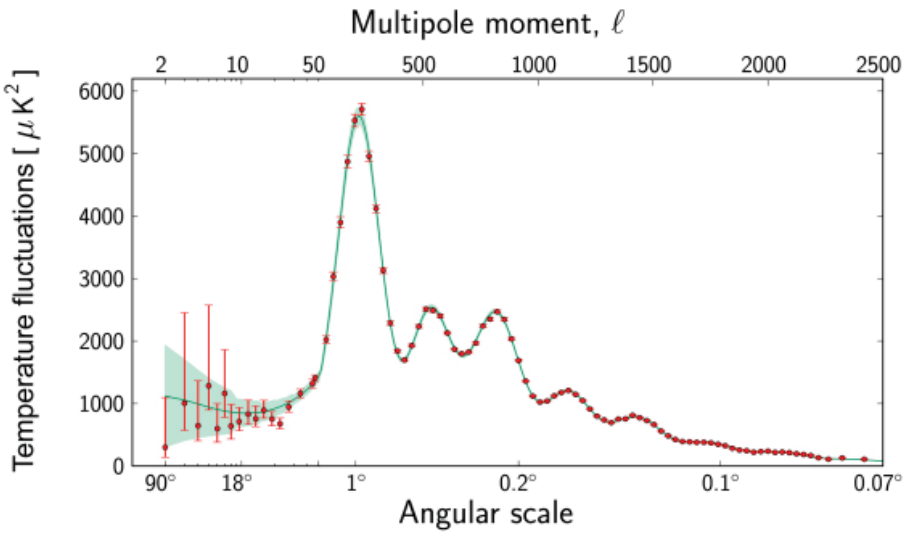


Figure 1.9: Power Spectrum of the CMB radiation by Planck. (http://sci.esa.int/planck/51555-planck-power-spectrum-of-temperature-fluctuations-in-the-cosmic-microwave-background, ©ESA and the Planck Collaboration, 2013)

constrain the initial data. Statistical studies about the CMB anisotropies lead to the power spectrum of the initial perturbations. It characterises the temperature fluctuations with respect to the angular scale of the anisotropies. Figure 1.9 shows the agreement between CMB statistical observations and the initial fluctuations predicted by the ΛCDM model after the parameters tuning. The power spectrum is the statistical representation of the "initial" perturbations in the Universe. The word "initial" is used crudely here, it is in fact the oldest measurement of the perturbations of the Universe accessible to us. It represents a density distribution of inhomogeneities in angle and temperature. Using the power spectrum and assuming gaussian inhomogeneities, sets of initial data can be generated and used as input in numerical simulations. These simulations describe the formation of the large–scale structures and show the outbreak of filaments (see figure 1.10). There exist two types of simulations, but most of them are based on the ΛCDM model.

- Most of the modern simulations describe the components of the Universe as a self–gravitating N–body ensemble of "particles" of Dark Matter and hydrodynamic baryonic matter. The physical effects are computed during the collapse of Dark Matter, the star formation for instance or the effect of the pressure are described in real time for the hydrodynamic baryons. The baryons can be described by two methods : a strictly hydrodynamic fluid distributed on a grid (RAMSES [Teyssier, 2002]) or a kind of N–body description with additional interactions called "Smoothed–particles hydrodynamics" (GADGET, GADGET–2 [Springel et al., 2001; Springel, 2005] and GASOLINE [Wadsley et al., 2004]. Matter evolves in a box of a given size.

- Semi–Analytical simulation are performed in two time. The first step is an N–body simulation of self–gravitating "particles" of Dark Matter. Then Dark Matter halos are identified and their evolution history built. These halos are now considered as potential well in which baryonic matter evolves. This second step is not an N–body simulation, only the quantities are known but matter does not evolve on a grid. Informations such as the density for each galaxy are only a number and the effects are added by hand, there is no gravitational description of the baryons (GalICS [Hatton et al., 2003]).

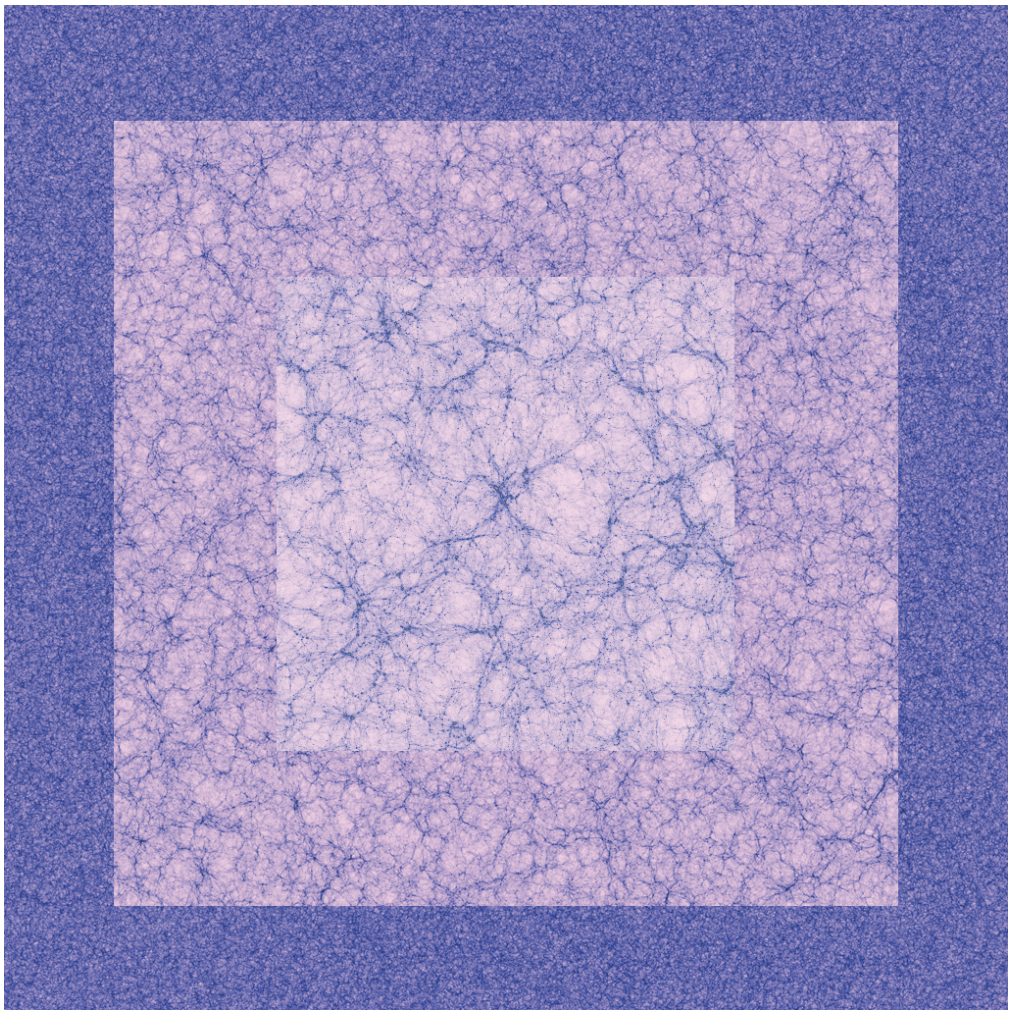


Figure 1.10: Example of results of a simulation, large–scale structures and filaments from Horizon. (http://www.projet-horizon.fr/article323.html, D. Aubert, S. Colombi, J. Devriendt, P. Ocvirk, C. Pichon, S. Prunet, R. Teyssier, 2008)

Chapter II

Newtonian perturbation theory

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1 Introduction

The Newtonian perturbation theory is at the heart of the standard model and its applications. It is important to well understand it to be able to go further and develop a fully relativistic perturbation theory. The Newtonian description of gravitation, and the associated perturbation theory presented here are well–known. I

do not provide drastic modifications of the theory but some new tools to write down the perturbations and a framework which allow us to build by analogy a powerful relativistic theory presented in chapter III. Usually only dust matter (i.e. pressureless fluid) without vorticity on a Friedmannian background is considered. During the last decades several analytic methods have been developed to deal with the inhomogeneities and the formation of large–scale structures [Buchert, 1989; Bouchet et al., 1992, 1995; Buchert, 1992; McDonald, 2007; Matsubara, 2008; Scoccimarro, 2001; Taylor and Hamilton, 1996]. The gravitational equations, determined by analogy with the Maxwell equations, and the equation of fluid mechanics describe the dynamics of a self–gravitating fluid: the Euler–Newton system. The more natural way to describe the system is the Euler point of view of fluid mechanics. This is the standard description [Bernardeau et al., 2002], but it is possible to translate these equations in the Lagrangian coordinates system to obtain the so–called Lagrange–Newton system. This is the approach I decided to follow, the Newtonian approach developed here is based on the Lagrangian perturbative theory of Buchert *et al.* [Buchert, 1992; Buchert and Ehlers, 1993; Buchert, 1994; Ehlers and Buchert, 1997]. These papers deal with perturbations up to third order or the presentation of a general Newtonian scheme. Other papers present a fourth order perturbations description [Vanselow, 1995; Rampf and Buchert, 2012]; for a tutorial see [Buchert, 1996]. Both Eulerian and Lagrangian systems are closed and admit analytical solutions. In this chapter, a subclass of the Lagrangian solutions up to the second order will be presented: a generalisation of the Zel'dovich approximation [Zel'dovich, 1970a]. Moreover, some of these works describe a dynamics with vorticity [Buchert, 1992, 1989; Bildhauer et al., 1992] or pressure [Adler and Buchert, 1999]. For further comprehensive studies of Lagrangian perturbations, relevant to the formal development of the theory (see [Nadkarni-Ghosh and Chernoff, 2011; Nadkarni-Ghosh and Chernoff, 2012; Salopek et al., 1994; Rampf and Rigopoulos, 2012]).

The Newtonian mechanics describes the dynamics of a mechanical system, here dust matter without vorticity, placed in a fixed space. The term"fixed" means that the dynamics of our space is independent of its content, unlike in general relativity where space–time and matter are merged in a unique concept (the dynamics of space and matter impact on each other). Dust matter is a pressureless and collisionless continuum of matter. This assumption is relevant for the large scale structures formation because the collision rate at the considered scale is low, a particle which travel in such a domain has an important mean free path. Nevertheless, dust matter is not always collisionless, the pressureless assumption is also important. So it is necessary to do not have a significant velocity dispersion. Thus the dust model is not appropriate to describe the inner galactic dynamics but can be used to the large scale structure formation. The Newtonian space–time is generally flat, even if it is possible to consider Newtonian mechanics in a constant curved space. Its size is characterised by the scalar factor $a(t)$ which is a tracer of

the relative expansion of the Universe. Even if the space is constantly curved, it is possible to make matter perturbations grow thanks to the Newtonian perturbation theory.

The following is then a summary of known works cited just before, essentially [Buchert, 1992; Buchert and Ehlers, 1993; Buchert, 1994; Rampf and Buchert, 2012; Ehlers and Buchert, 1997]. The relativistic approach which will be presented in chapter III and IV is viewed as an analogy to the Newtonian development. I will adapt and extend the formal results from [Ehlers and Buchert, 1997] in order to build an extrapolation to the relativistic description of the problem.

The whole Newtonian perturbations theory is described in this chapter structured as follows. Section 2, justifies the use of the Lagrangian approach of the fluid mechanics and presents the differential forms formalism used for its compactness and mathematical useful properties, whereas section 3 details the solution scheme of the Newtonian perturbative framework and exhibits the solutions in the Einstein–de Sitter case. This chapter is ended by some concluding remarks, section 4, and a motivation to develop a relativistic description for the large–scale structures formation, section 5.

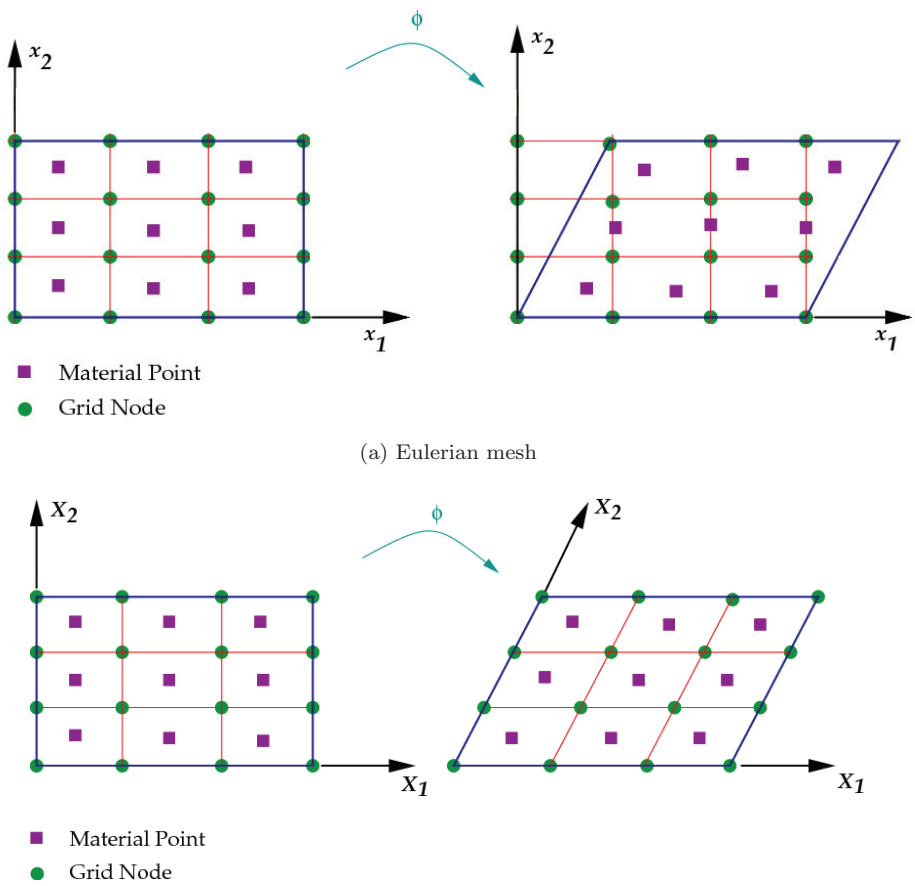
2 Newtonian model expressions

Two frameworks exist to describe the evolution of a fluid continuum in a Newtonian space–time, the Eulerian and Lagrangian points of view. This discrepancy is essential to understand the motivations of the following work.

On the one hand, the Eulerian coordinates x_i describe the fluid particle i like a stream in the space–time (see figure 2.1a). In other words, it consists in the study of a moving fluid in a frame of reference at rest. The Eulerian picture can also be seen as the motion of a fluid for an external observer. At the zeroth order the fluid is in an equilibrium state (i.e. at rest with respect to the frame). A perturbation of this state is a small deviation from this "equilibrium", namely from the rest position x_i.

On the other hand, Lagrangian coordinates X_i only index the fluid particle i (see figure 2.1b). It consists of the study of the deformation of the fluid in its own frame. At zeroth order the fluid is in the same equilibrium state than in the Eulerian point of view. Nevertheless, in this picture the Lagrangian frame (which is equivalent to the resting fluid) is in motion with respect to the Eulerian frame. In this approach, the displacement of the particle i with respect to its rest position is a deviation along a Lagrangian trajectory. A Lagrangian perturbation described inhomogeneities in the fluid, but in the same time a non–perturbative deviation from the Eulerian rest position x_i because the displacement can be important.

The figure 2.2 illustrates the scheme of the trajectories. The left panel 2.2a represents an Eulerian description of the fluid mechanics. A frame of reference is defined and the movement of the fluid with respect to this frame of reference



(a) Eulerian mesh

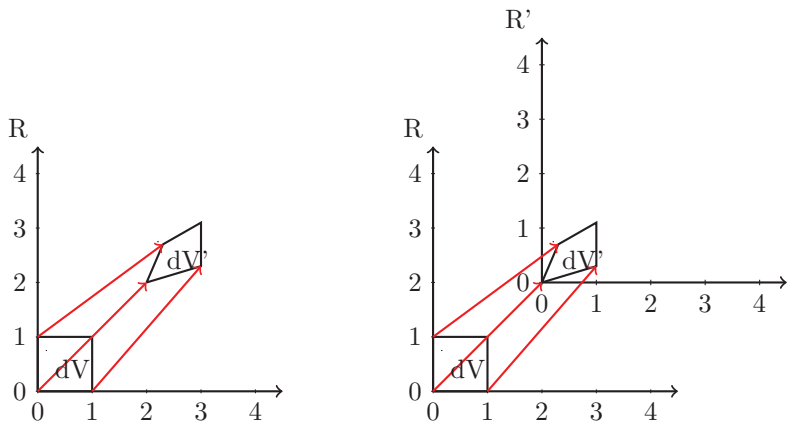
(b) Lagrangian mesh

Figure 2.1: Eulerian and Lagrangian points of view in fluid mechanics. (en.wikiversity.org/wiki/File:LagrangianMesh.png, http://en.wikiversity.org/wiki/File:EulerianMesh.png, cc-by-sa, Biswajit Banerjee, 2007)

is studied, there are two perturbative variables: the velocity field v_i and density field ϱ. The major problems of this approach are the impossibility to describe high density contrast because the density field is a perturbed variable. The right panel 2.2b represents a Lagrangian description. The frame of reference is attached to the fluid element and it consists of studying the deformation of this volume in its own frame of reference. Since the density is not a perturbative variable anymore, high density contrasts can be described by this method. The only perturbed variable is the position field, but the fluid is studied in its own frame of reference which can be in motion. Thus this approach can describe high density contrast of fluid particles during significative motion.

We note $\vec{f}$ the position field and J the Jacobian of the Eulerian–to–Lagrangian coordinates transformation (it is also the fluid element volume deformation), defined by the following one–parameter diffeomorphism Φ_t:

$$\Phi_t : \vec{X} \longmapsto \vec{x} = \vec{f}(\vec{X}, t) \; ; \tag{2.1}$$



(a) Eulerian (b) Lagrangian

Figure 2.2: Comparison of the Eulerian and Lagrangian approach in terms of trajectories for a volume element.

$$J = \det\left(\frac{\partial f_i}{\partial X_k}(X,t)\right) := \frac{1}{6}\epsilon_{ijk}\epsilon^{lmn} f^i_{|l} f^j_{|m} f^k_{|n} \; ; \tag{2.2}$$

where ϵ_{ijk} is the Levi-Civita pseudo–tensor, equal to -1 (if indices are not ordered as the direct basis), 0 (if at least two indices are identical) or 1 (if indices are ordered as the direct basis). The position field translates the Lagrangian coordinates into Eulerian coordinates. It is the essential field of the theory and the only unknown field of the Lagrangian point of view. Every other field can be written as a function of this one. The Newtonian space is Euclidean, so the inverse map transformation is defined such as $\vec{X} = \vec{h}(\vec{x},t)$ whose Jacobian matrix is

$$h^i{}_{,j} = \frac{1}{2J}\epsilon_{jkl}\epsilon^{ipq} f^k{}_{|p} f^l{}_{|q} \; . \tag{2.3}$$

In the following developments the Eulerian derivative will be denoted by a coma ",", the Lagrangian derivative by a vertical stroke "|", the nabla with respect to Eulerian coordinates by "$\vec{\nabla}$", the nabla with respect to Lagrangian coordinates by "$\vec{\nabla}_0$" and the total derivative by "d/dt."

2.1 Euler–Newton system

In Eulerian coordinates the dynamics of a fluid is described by the Euler–Newton system composed of:

- the equation of continuity (also named mass conservation):

$$\frac{d\varrho}{dt} = -\varrho\vec{\nabla}\cdot\vec{v} \; ; \tag{2.4}$$

- the Euler equation:

$$\frac{d\vec{v}}{dt} = \vec{g}\ ; \tag{2.5}$$

- the Newtonian field equations:

$$\begin{cases} \vec{\nabla} \times \vec{g} = \vec{0}\ ; \\ \vec{\nabla} \cdot \vec{g} = \Lambda - 4\pi G \varrho\ ; \end{cases} \tag{2.6}$$

with G the gravitational constant and Λ the cosmological constant. The variables are the velocity field $\vec{v}$, the acceleration field (or gravitational strength) $\vec{g}$ and the density field ϱ. They depend on the Eulerian position $\vec{x}$ and Newtonian time t:

$$\left\{ \vec{v}(\vec{x},t),\ \vec{g}(\vec{x},t),\ \varrho(\vec{x},t) \right\}\ . \tag{2.7}$$

Only the velocity and the density fields are independent variables. Thus a quick counting leads to a number of 4 independent variables (the density ϱ and the components of the velocity field v_i) and 5 equations (the mass conservation, the divergence equation and the three components of the curl equation). Since we have more functions than equations, we can affirm the system is closed and solvable. The standard approach is a perturbative resolution of the system of equations and imposes to perturb the velocity and the density fields. On the contrary, the Eulerian perturbation theory has the important restriction we discussed earlier: only small displacement from the equilibrium state can be described. Thus the contrast of density $\delta = (\varrho - \varrho_H)/\varrho_H$ has to remain small with respect to the average density ϱ_H. Thus, this approach presents obvious shortcomings when treating the problem of highly inhomogeneous large–scale structures which are clearly observable in the late Universe.

2.2 Lagrange–Newton system

To obtain the Lagrange–Newton system, the map transformation from Eulerian to Lagrangian coordinates (2.8) is introduced. It admits the following initial conditions $\vec{f}(\vec{X},t_{\mathbf{i}}) = \vec{X}$ with the index $_{\mathbf{i}}$ which denotes the initial time. One of the main advantages of this system is that the density field is not a variable anymore. The only function which has to be perturbed is the position field $\vec{f}$. Moreover its gradient $f^i{}_{|j}$ measures the deformation of the fluid element.

$$\left\{ \vec{x} = \vec{f}(\vec{X},t),\ \vec{v} = \dot{\vec{f}}(\vec{X},t),\ \vec{g} = \ddot{\vec{f}}(\vec{X},t) \right\}\ . \tag{2.8}$$

The over–dot denotes the time derivative. Several formalisms exist to express the equations of cosmology. The most intuitive is the functional (or vectorial) formulation (see paragraph 2.2.1). The second way to express the system, is the

differential forms formalism (see paragraph 2.2.2). This approach allows us for an easy generalisation of the Newtonian equations to a subsystem of the relativistic equations (the gravitoelectric equations (see chapter IV).

2.2.1 Functional determinants

In the following "[.]" will denote the anti–symmetrisation operator and δ^{ij} the Kronecker delta. The functional determinant is defined by:

$$\mathcal{J}\left(f^k; f^p; f^q\right) = \epsilon^{lmn} f^k_{|l} f^p_{|m} f^q_{|n} \ . \tag{2.9}$$

This last function allows us to write the curl and the divergence of the gravitational field in a Lagrangian form.

$$\begin{aligned}\left(\vec{\nabla}\times\vec{g}\right)_i &:= 2\ g_{[k,j]} = 2\ g_{[k|l}\ h^l_{,j]} \\ &= J^{-1}\ \epsilon^{lmn}\epsilon_{pq[j}\ddot{f}_{k]|l} f^p_{\ |m} f^q_{\ |n} \\ &= J^{-1}\ \epsilon_{pq[j}\mathcal{J}\left(\ddot{f}_{r]}; f^p; f^q\right) = 0\ ;\end{aligned} \tag{2.10}$$

$$\begin{aligned}\vec{\nabla}\cdot\vec{g} &:= g^i_{,i} = g^i_{\ |l} h^l_{,i} \\ &= \frac{1}{2J}\epsilon^{lmn}\epsilon_{ijk}\ddot{f}^i_{\ |l} f^j_{\ |m} f^k_{\ |n} \\ &= \frac{1}{2J}\epsilon_{ijk}\mathcal{J}\left(\ddot{f}^i; f^j; f^k\right) = \Lambda J - 4\pi G\varrho_{\mathbf{i}} J_{\mathbf{i}}\ .\end{aligned} \tag{2.11}$$

I used in these derivations the expression of the inverse of the position field (2.3). Note the curl equation can also be written in a quadratic form,

$$\delta_{ij}\ddot{f}^i_{\ |[p} f^j_{\ |k]} = 0\ . \tag{2.12}$$

The four remaining equations of the Euler–Newton system are the continuity equation (or mass conservation) and the Euler equation which are respectively reduced to the definition of the density and acceleration. The first defines the density at any time according to its initial value and the Jacobian of the Eulerian–to–Lagrangian coordinates transformation, the three others are reduced to the definition of the gravitational field with respect to the velocity field.

$$\begin{cases} \varrho J = \varrho_{\mathbf{i}} J_{\mathbf{i}}\ ; \\ \vec{g}(t) = \ddot{\vec{f}}(t) = \dfrac{d\dot{\vec{v}}}{dt}(t)\ . \end{cases} \tag{2.13}$$

The four equations ((2.10),(2.11)) form the Lagrange–Newton system. Its associated unknowns are the three trajectory functions f^i. We count 3 functions for 4 equations (the Newtonian field equations), this system is closed and solvable, such as the Euler–Newton system.

2.2.2 Differential forms

Let $\mathcal{M}$ be a Riemannian manifold, i.e. a space locally Euclidean in our case. Note that in Newtonian mechanics, the manifold is globally Euclidean. At any point P of $\mathcal{M}$ it is possible to define a cotangent space whose basis is noted $\{\mathbf{d}X^i\}$. In general relativity the considered four dimensional manifold are semi–Riemannian, it means the space is locally Minkowskian.

Let $\boldsymbol{\phi}^a$ be a spatial *k–differential form* (abbreviated in k–form). For general forms we choose the letters $a, b, c...$ as counter indices, while the letters $i, j, k...$ are reserved for coordinate indices. The coefficients of a k–form are noted $\phi^a{}_{i_1...i_k}$ and can be expressed in the exact basis of the cotangent space at a given point such as:

$$\boldsymbol{\phi}^a = \phi^a{}_{i_1...i_k}\mathbf{d}X^{i_1} \wedge ... \wedge \mathbf{d}X^{i_k} \ . \tag{2.14}$$

The *exterior derivative* of a differential form $\boldsymbol{\phi}^a$ is defined by:

$$\mathbf{d}\boldsymbol{\phi}^a = \phi^a{}_{i_1...i_k|i_p}\mathbf{d}X^{i_1} \wedge ... \wedge \mathbf{d}X^{i_k} \wedge \mathbf{d}X^{i_p} \ . \tag{2.15}$$

A *k–form* is said to be *exact* if and only if there exist a *(k-1)–form* from which the form derives,

$$\boldsymbol{\phi}^a \text{ is exact} \quad \Longleftrightarrow \quad \exists f^a \mid \boldsymbol{\phi}^a = \mathbf{d}f^a \ . \tag{2.16}$$

A major specificity of the Newtonian formalism is that the relevant functions are the gradients of deformation $f^i{}_{|j}$ (9 functions). Since these functions are built from the gradients of the trajectory field f^i (3 functions), they are exact forms. The *Hodge dual* of a *k–form* defined on a n–dimensional space of metric g_{ij} is denoted by a star $*$ and is defined by:

$$*\boldsymbol{\phi}^a = \frac{\sqrt{g}\ \phi^a{}_{i_1...i_k}}{(n-k)!}\epsilon^{i_1...i_k}{}_{j_{k+1}...j_n}\mathbf{d}X^{j_{k+1}} \wedge ... \wedge \mathbf{d}X^{j_n} \ ; \tag{2.17}$$

with $\epsilon^{i_1...i_k}{}_{j_{k+1}...j_n}$ the Levi-Civita pseudo–tensor and $\sqrt{g}$ is the square root of the determinant of the metric. Since the Newtonian case describes a flat space, any tangent space matches the global space, namely any local coordinates are global. Thus the counter indices become coordinate indices, since the functions f^a can be used to define global coordinates $x^i = f^{a\to i}$.

It is possible in this approach to reformulate the previous system of equations. The method is to apply the Hodge star operator to the coefficient equations ((2.10),(2.11)). The Lagrange–Newton system reads,

$$*(2.10) \quad \Longrightarrow \quad \delta_{ij}\mathbf{d}\ddot{f}^i \wedge \mathbf{d}f^j = 0 \ ; \tag{2.18}$$

$$*(2.11) \quad \Longrightarrow \quad \frac{1}{2}\epsilon_{ijk}\mathbf{d}\ddot{f}^i \wedge \mathbf{d}f^j \wedge \mathbf{d}f^k = (\Lambda J - 4\pi G\varrho_{\mathbf{i}}J_{\mathbf{i}})\,\mathbf{d}^3X \ ; \tag{2.19}$$

with $\mathbf{d}^3X$ the Lagrangian volume 3–form which allows us to define the Eulerian volume 3–form $\mathbf{J}$,

$$\begin{aligned} & \mathbf{J} := \frac{\epsilon_{ijk}}{6} J \mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = \frac{\epsilon_{ijk}}{6} \mathbf{d}f^i \wedge \mathbf{d}f^j \wedge \mathbf{d}f^k \\ \Longleftrightarrow \quad & \mathbf{J} := J\mathbf{d}^3X = \mathbf{d}^3 f \; ; \end{aligned} \tag{2.20}$$

2.2.3 Tidal force tensor

The tidal force tensor is an alternative to express the previous equations in a compact way. It is built from the derivatives of the gravitational field and is defined by:

$$\mathcal{E}^i{}_j = g^i{}_{,j} - \frac{1}{3}\delta^i{}_j g^k{}_{,k} \tag{2.21}$$

$$= \frac{1}{2J}\epsilon_{jkl} J \left(\ddot{f}^i; f^k; f^l\right) - \frac{1}{3}\left(\Lambda - 4\pi G\varrho\right)\delta^i{}_j \; . \tag{2.22}$$

The tidal tensor is built in order to generate the Lagrange–Newton system,

$$(2.18) \quad \Longleftrightarrow \quad \delta^{ir}\mathcal{E}_{[rj]} = 0 \; ; \tag{2.23}$$

$$(2.19) \quad \Longleftrightarrow \quad \mathcal{E}^k{}_k = 0 \; . \tag{2.24}$$

This approach is also compatible with the differential forms formalism through:

$$\mathcal{E}^i = \mathbf{d}\ddot{f}^i - \frac{1}{3}(\Lambda - 4\pi G\varrho)\mathbf{d}f^i \; ; \tag{2.25}$$

$$(2.18) \quad \Longleftrightarrow \quad \delta^j_i \mathcal{E}^i \wedge \mathbf{d}f^j = \mathbf{0} \; ; \tag{2.26}$$

$$(2.19) \quad \Longleftrightarrow \quad \epsilon_{ijk}\mathcal{E}^i \wedge \mathbf{d}f^j \wedge \mathbf{d}f^k = 0 \; . \tag{2.27}$$

An analytical equation of evolution for the tidal tensor can be computed to write the dynamical formulation of the Newtonian gravitational problem. This approach is used to introduce the silent Universe in chapter V.

3 Perturbative development of the Lagrange–Newton system

Without any additional assumption or restriction, looking for an analytical solution is clearly utopian. The standard method is to perform a perturbative development and choose a specific class of solution as a spherical symmetric matter distribution or Einstein–de Sitter (EdS) for a Universe whose average density is equal to the critical density (the density of a homogeneous flat FLRW model of Universe) and a vanishing Cosmological constant. I recall here the perturbation theory for the Newtonian dynamics. Previously, the system in tensorial and differential forms formalism are presented. I chose to perform the computation with differential forms and to project the solutions into the tensorial formalism in a second time in order to check the consistency with the standard results.

3.1 Perturbation scheme

To solve this system a general perturbation (2.28) has to be introduced. The gradient of deformation field $f^i{}_{|j}$ is assumed to be the superposition of a homogeneous background and inhomogeneous perturbations,

$$\begin{cases} \mathbf{d}f^i(\vec{X},t) = a(t)\left(\mathbf{d}X^i + \mathbf{d}P^i\right) \ ; \\ f^i{}_{|j}(\vec{X},t) = a(t)\left(\delta^i{}_j + P^i{}_{|j}(\vec{X},t)\right) \ ; \end{cases} \tag{2.28}$$

where $a(t)$ is the usual scale factor, $\delta^i{}_j$ the Kronecker symbol, $P^i{}_{|j}$ the perturbation of the gradient of deformation and $t_{\mathbf{i}}$ the initial time which is set to the CMB epoch. A rescaling by the factor $a(t)$ allows us to define the comoving quantities (i.e. quantities which are not influenced by the expansion of the Universe). The quantities denoted by a capital letter are comoving whereas the others are non–comoving. It is also interesting to identify properly the homogeneous and perturbed quantities: $\left\{\mathbf{d}f^i,\ f^i{}_{|j}\right\}$ are the complete functions; it means they contain the homogeneous $\left\{\mathbf{d}X^i,\ \delta^i{}_j\right\}$ and the perturbed $\left\{\mathbf{d}P^i,\ P^i{}_{|j}\right\}$ quantities. According to this decomposition of the position field, it is possible to write the Jacobian of the Eulerian–to–Lagrangian coordinates transformation such as,

$$\begin{aligned} J &= \frac{a^3}{6}\epsilon_{ijk}\epsilon^{lmn}\left(\delta^i{}_l + P^i{}_l\right)\left(\delta^j{}_m + P^j{}_m\right)\left(\delta^k{}_n + P^k{}_n\right) \\ &= a^3\left(1 + P + \frac{1}{2}\left(P^2 - P^i{}_jP^j{}_i\right) + \frac{1}{6}\left(P^3 + 2P^i{}_jP^j{}_kP^k{}_i - 3PP^i{}_jP^j{}_i\right)\right) \ ; \end{aligned} \tag{2.29}$$

where the trace quantities are noted without index $P^i{}_{|i} = P$. The Newtonian initial data are defined without loss of generality by,

$$\begin{cases} P^i{}_{|j}(\vec{X},t_{\mathbf{i}}) = 0 \ ; \\ \dot{P}^i{}_{|j}(\vec{X},t_{\mathbf{i}}) = U^i{}_{|j}(\vec{X}) \ ; \\ \ddot{P}^i{}_{|j}(\vec{X},t_{\mathbf{i}}) = W^i{}_{|j}(\vec{X}) - 2H_{\mathbf{i}}U^i{}_{|j}(\vec{X}) \ . \end{cases} \tag{2.30}$$

The initial data are the initial gradient of deformation, velocity and acceleration respectively noted $P^i{}_{|j}(\vec{X},t_{\mathbf{i}})$, $U^i{}_{|j}$ and $W^i{}_{|j}$. The last relation about the initial data comes naturally from the equation of motion (2.49). Since the equations are 2nd order differential equations, two independent initial data are required. Then the third initial data is a free choice: a vanishing initial deformation is chosen. The value of P^i only encodes the distortion of the grid with respect to the initially regular grid (see figure 2.1b). Even without initial deviation in the position field, the contrast of density implies an inhomogeneous gravity field thanks to the Poisson equations and put the fluid elements into motion. The Poisson equation can be written with differential forms. On the one hand the divergence of the gravitational

field respects:

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{g} := a^{-1}\ddot{f}^i{}_i = \frac{\ddot{a}}{a}\left(3+P^i{}_{|i}\right) + 2\frac{\dot{a}}{a}\dot{P}^i{}_{|i} + \ddot{P}^i{}_{|i} = \Lambda - 4\pi G\varrho \\
\Longrightarrow\quad & 3\frac{\ddot{a}}{a} = \Lambda - 4\pi G\varrho_H \quad ; \quad 2H\dot{P}^i{}_{|i} + \ddot{P}^i{}_{|i} = -4\pi G\delta\varrho \\
\Longrightarrow\quad & 3\frac{\ddot{a}}{a} = \Lambda - 4\pi G\varrho_H \quad ; \quad W^i{}_{|i} = -4\pi G\delta\varrho_{\mathbf{i}}\ .
\end{aligned} \tag{2.31}$$

The two relations obtained here are respectively the derivative of the Friedmann equation and the Poisson equation. On the other hand, the curled of the gravitational field respects:

$$\begin{aligned}
&\vec{\nabla}\times\vec{g} := a^{-1}\ddot{f}_{[i|j]} = 0 \\
\Longrightarrow\quad & W_{[i|j]} = 0\ ;
\end{aligned} \tag{2.32}$$

The constraints of the initial acceleration gradients can be summarised as,

$$\begin{cases} W^i{}_{|i} = *\dfrac{1}{2}\epsilon_{ijk}\mathbf{d}W^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k = -4\pi G\delta\varrho_{\mathbf{i}}\ ; \\ W_{[i|j]} = 0\ . \end{cases} \tag{2.33}$$

The density is also split between a homogeneous and inhomogeneous part:

$$\varrho(\vec{X},t) = \varrho_H(t) + \delta\varrho(\vec{X},t) = \varrho_H(t)\left(1+\delta(\vec{X},t)\right)\ ; \tag{2.34}$$

where ϱ is the density, ϱ_H is the homogeneous density, $\delta\varrho$ the inhomogeneous density and $\delta = (\varrho-\varrho_H)/\varrho_H$ the density contrast. Nevertheless, in the Lagrangian approach, the density is not a perturbative variable of the system, unlike in the Eulerian case. The density is exactly known at any time (2.13) with respect to its initial data and the Jacobian of the Eulerian–to–Lagrangian coordinates transformation. The definition of the perturbation (2.34) has to be developed to set up relations between the density and its initial value $\varrho J = \varrho_{\mathbf{i}}$. Using the expression of the Jacobian (2.29), the relation leads on the background to,

$$\varrho_H a^3 = \varrho_{H\mathbf{i}}\ ; \tag{2.35}$$

and on the perturbed level to,

$$\begin{aligned}
&(\varrho_H+\delta\varrho)\,a^3\Big[P + \frac{1}{2}\left(P^2 - P^i{}_{|j}P^j{}_{|i}\right) + \frac{1}{6}\left(P^3 + 2P^i{}_{|j}P^j{}_{|k}P^k{}_{|i} - 3PP^j{}_{|k}P^k{}_{|j}\right)\Big] \\
&\qquad = \varrho_{H\mathbf{i}} + \delta\varrho_{\mathbf{i}}\ .
\end{aligned} \tag{2.36}$$

It could be disturbing to affirm the density is not a perturbative variable and write it down with respect to the perturbations. The density is not a perturbed variable of the system of equation, this is clear but it is possible to write it as a functional of

the perturbations. Thus, its expression is nonlinear; in such a case the density can grow relatively fast and triggers high density contrasts despite its perturbations dependent expression.

The general perturbation introduced earlier (2.28), allows us to split the Lagrange–Newton system of equations between a homogeneous and a perturbed system. Only one of the equations do not vanish when we consider the homogeneous gradient of deformation, the divergence of the acceleration field (2.19):

$$\frac{1}{2}\epsilon_{ijk}\ddot{a}\delta^i{}_l\mathbf{d}X^l\wedge a\delta^j{}_m\mathbf{d}X^m\wedge a\delta^k{}_n\mathbf{d}X^n=\epsilon_{ijk}\left(\Lambda-4\pi G\varrho_H\right)\frac{a^3}{6}\epsilon_{ijk}\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k$$

$$\Longrightarrow\quad \epsilon_{ijk}\,3\frac{\ddot{a}}{a}\,\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k=\epsilon_{ijk}\left(\Lambda-4\pi G\varrho_H\right)\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k. \tag{2.37}$$

We can recognise similarities with the Friedmann equation. Indeed, using the Hodge star operator this equation leads to the derivative of the Friedmann equation:

$$3\frac{\ddot{a}}{a}=\Lambda-4\pi G\varrho_H\ . \tag{2.38}$$

On the other hand, the perturbed equations obtained from ((2.18),(2.19)) read,

$$\begin{aligned}
(2.18)\quad &\Longrightarrow\quad \delta_{ij}\mathbf{d}\dot{f}^i\wedge\mathbf{d}f^j=const.=\delta_{ij}\left(\dot{a}_{\mathbf{i}}\mathbf{d}X^i+\mathbf{d}P^i_{\mathbf{i}}\right)\wedge\mathbf{d}X^j\\
&\Longrightarrow\quad \delta_{ij}\left(\dot{a}\mathbf{d}\tilde{f}^i+a\mathbf{d}\dot{\tilde{f}}^i\right)\wedge a\mathbf{d}\tilde{f}^j=\delta_{ij}\mathbf{d}U^i\wedge\mathbf{d}X^j\\
&\Longrightarrow\quad \delta_{ij}a\mathbf{d}\dot{\tilde{f}}^i\wedge a\mathbf{d}\tilde{f}^j=\delta_{ij}\mathbf{d}U^i\wedge\mathbf{d}X^j\\
&\Longrightarrow\quad \delta_{ij}\mathbf{d}\dot{P}^i\wedge\left(\mathbf{d}X^j+\mathbf{d}P^j\right)=\delta_{ij}a^{-2}\mathbf{d}U^i\wedge\mathbf{d}X^j\ ;
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
(2.19)\quad &\Longrightarrow\quad \frac{1}{2}\epsilon_{ijk}\Big(\ddot{a}\mathbf{d}X^i+\ddot{a}\mathbf{d}P^i+2\dot{a}\mathbf{d}\dot{P}^i+a\mathbf{d}\ddot{P}^i\Big)\wedge a\big(\mathbf{d}X^j+\mathbf{d}P^j\big)\wedge a\big(\mathbf{d}X^k+\mathbf{d}P^k\big)\\
&\qquad=\left(\Lambda-4\pi G\varrho\right)\frac{1}{6}\epsilon_{ijk}a\left(\mathbf{d}X^i+\mathbf{d}P^i\right)\wedge a\left(\mathbf{d}X^j+\mathbf{d}P^j\right)\wedge a\left(\mathbf{d}X^k+\mathbf{d}P^k\right)\\
&\Longrightarrow\quad \frac{1}{2}\epsilon_{ijk}\Big[a^3\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k+3\ddot{a}a^2\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\\
&\qquad\qquad+2a^3\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k+3\ddot{a}a^2\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k\\
&\qquad\qquad+a^3\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k+\ddot{a}a^2\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big]\\
&\qquad=\left(\Lambda-4\pi G\varrho\right)\frac{a^3}{6}\epsilon_{ijk}\Big[\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k+3\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\\
&\qquad\qquad\qquad+3\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k+\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big]\\
&\Longrightarrow\quad \epsilon_{ijk}\Big[\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k+2\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k+\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{4\pi G}{3}\delta\varrho\epsilon_{ijk}\Big[\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k + 3\mathbf{d}P^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\
&\qquad + 3\mathbf{d}P^i \wedge \mathbf{d}P^j \wedge \mathbf{d}X^k + \mathbf{d}P^i \wedge \mathbf{d}P^j \wedge \mathbf{d}P^k\Big] \\
\Longrightarrow\quad & \epsilon_{ijk}\Big[\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k + 2\mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k + \mathcal{D}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big] \\
&= 4\pi G\varrho_H a^3\epsilon_{ijk}\Big[\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k + \mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k + \frac{1}{3}\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big] \\
&\quad - \frac{4\pi G}{3}\delta a_{\mathbf{i}}{}^{-3}\varrho\epsilon_{ijk}\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\
\Longrightarrow\quad & \epsilon_{ijk}\Big[\Big(\mathcal{D} - 4\pi G\varrho_H\Big)\mathbf{d}P^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k \\
&\qquad + \Big(2\mathcal{D} - 4\pi G\varrho_H\Big)\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}X^k \\
&\qquad + \Big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\Big)\mathbf{d}P^i\wedge\mathbf{d}P^j\wedge\mathbf{d}P^k\Big] \\
&= -\epsilon_{ijk}\frac{4\pi G}{3}\delta\varrho_{\mathbf{i}}a^{-3}\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\ ;
\end{aligned}
\tag{2.40}
$$

To simplify the expression of the first equation the integration of (2.18) has been performed before the perturbative development. I also define the comoving position field by $\tilde{f}^i = a^{-1}f^i$. Moreover, we have the time operator $\mathcal{D}$ defined by $\mathcal{D} := d^2/dt^2 + 2H \cdot d/dt$ with the Hubble factor $H(t) = (\dot{a}/a)(t)$.

To solve a perturbative development, the perturbations have to be decomposed order by order. It means we consider a first perturbation which is a small displacement from the homogeneous behaviour and then we add more and more precise perturbations to describe more and more details. Let me decompose the perturbations order by order:

$$
\begin{cases}
\mathbf{d}P^i(\vec{X},t) = \displaystyle\sum_{m=1}^{\infty}\mathbf{d}P^{i(m)}(\vec{X},t)\ ; \\
P^i{}_{|j}(\vec{X},t) = \displaystyle\sum_{m=1}^{\infty}P^i{}_{|j}^{\ (m)}(\vec{X},t)\ .
\end{cases}
\tag{2.41}
$$

Some other conventions used a pre factor $1/m!$ or ϵ^m to insure it is possible to neglect the higher orders. Here the assumption that any term of order n can be neglected with respect to any term of order $n-1$ is made. Without loss of generality it is possible to constrain the initial data. The following set, with only first order initial velocity and acceleration, is chosen.

$$
\begin{cases}
\forall k\ P^i{}_{|j}^{\ (k)}(\vec{X},t_{\mathbf{i}}) = 0\ ; \\
\dot{P}^i{}_{|j}(\vec{X},t_{\mathbf{i}}) = \dot{P}^i{}_{|j}^{\ (1)}(\vec{X},t_{\mathbf{i}}) = U^i{}_{|j}(\vec{X})\ ; \\
\ddot{P}^i{}_{|j}(\vec{X},t_{\mathbf{i}}) = \ddot{P}^i{}_{|j}^{\ (1)}(\vec{X},t_{\mathbf{i}}) = W^i{}_{|j}(\vec{X}) - 2H_{\mathbf{i}}U^i{}_{|j}(\vec{X})\ .
\end{cases}
\tag{2.42}
$$

Each order requires to set two initial data, here vanishing initial data for the orders $n > 1$ have been chosen. The decomposition of the perturbations and the initial data set lead to a system of equations at any order. In this development quadratic and cubic terms in P will appears, since the equations will be written order by order these terms can be considered as sources. Thus the generic n^{th} order system of equation will be written with an implicit summation over the order of perturbations of the source terms:

$$\begin{cases} A^{(p)}B^{(q)} = \sum\limits_{p+q=n} A^{(p)}B^{(q)} \; ; \\ A^{(r)}B^{(s)}C^{(t)} = \sum\limits_{r+s+t=n} A^{(r)}B^{(s)}C^{(t)} \; . \end{cases} \tag{2.43}$$

Then, it is possible to develop from the first to a general n^{th} order the equations ((2.39),(2.40)). For the first order of perturbation, the equations are reduced to,

$$(2.39) \quad \Longrightarrow \quad \delta_{ij}\, \mathbf{d}\dot{P}^{i(1)} \wedge \mathbf{d}X^j = \delta_{ij}\; a^{-2}\mathbf{d}U^i \wedge \mathbf{d}X^j \; ; \tag{2.44}$$

$$\begin{aligned} (2.40) \quad \Longrightarrow \quad & \epsilon_{ijk}\left(\mathcal{D} - 4\pi G\varrho_H\right)\mathbf{d}P^{i(1)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ & = a^{-3}\epsilon_{ijk}\; \mathbf{d}W^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \; ; \end{aligned} \tag{2.45}$$

and the n order, for $n > 1$, Lagrange–Newton system reads,

$$(2.39) \quad \Longrightarrow \quad \delta_{ij}\mathbf{d}\dot{P}^{i(n)} \wedge \mathbf{d}X^j = -\delta_{ij}\mathbf{d}\dot{P}^{i(p)} \wedge \mathbf{d}P^{j(q)} \; ; \tag{2.46}$$

$$\begin{aligned} (2.40) \quad \Longrightarrow \quad & \epsilon_{ijk}\Big[\big(\mathcal{D} - 4\pi G\varrho_H\big)\mathbf{d}P^{i(n)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ & + \big(2\mathcal{D} - 4\pi G\varrho_H\big)\mathbf{d}P^{i(p)} \wedge \mathbf{d}P^{j(q)} \wedge \mathbf{d}X^k \\ & + \big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)\mathbf{d}P^{i(r)} \wedge \mathbf{d}P^{j(s)} \wedge \mathbf{d}P^{k(t)}\Big] = 0 \; . \end{aligned} \tag{2.47}$$

At any order a closed system of equations is obtained, then any order admits an analytical solution. All of these equations contain a linear time operator applied to the higher order perturbation and non–linear source terms described by lower order perturbations. To check the differential forms approach it is useful to perform the projection onto tensorial equations using the Hodge star operator,

$$\begin{aligned} *(2.44) \quad \Longleftrightarrow \quad & \delta_{ij}\epsilon^{ljk}\dot{P}^{i(1)}_{\;\;|l}\mathbf{d}X_k = \delta_{ij}\epsilon^{ljk}a^{-2}U^i_{\;\;|l}\mathbf{d}X_k \\ \Longrightarrow \quad & \delta_{i[j}\dot{P}^{i(1)}_{\;\;|l]} = \delta_{i[j}a^{-2}U^i_{\;\;|l]} \\ \Longleftrightarrow \quad & \dot{P}^{(1)}_{[j|l]} = a^{-2}U_{[j|l]} \; ; \end{aligned} \tag{2.48}$$

$$\begin{aligned} *(2.45) \quad \Longleftrightarrow \quad & \epsilon_{ijk}\epsilon^{ljk}\left(\mathcal{D} - 4\pi G\varrho_H\right)P^{i(1)}_{\;\;|l} = a^{-3}\epsilon_{ijk}\epsilon^{ljk}\; \mathbf{d}W^i_{\;\;|l} \\ \Longleftrightarrow \quad & 2\delta_i^{\;l}\left(\mathcal{D} - 4\pi G\varrho_H\right)P^{i\;(1)}_{\;\;|l} = 2\delta_i^{\;l}a^{-3}W^i_{\;\;|l} \end{aligned}$$

$$\iff \quad \ddot{P}^{i\ (1)}_{\ |i} + 2H\dot{P}^{i\ (1)}_{\ |i} - 4\pi G\varrho_{H\mathbf{i}}\, a^{-3}P^{i\ (1)}_{\ |i} = a^{-3}W^{i}_{\ |i}\ ; \tag{2.49}$$

$$\begin{aligned}
*(2.46) \quad &\iff \quad \delta_{ij}\epsilon^{ljk}\dot{P}^{i(n)}_{\ |l}\mathbf{d}X_k = \delta_{ij}\epsilon^{lpk}\dot{P}^{i(p)}_{\ |l}P^{j(q)}_{\ |p}\mathbf{d}X^k \\
&\iff \quad \delta_{ip}\epsilon^{lpk}\dot{P}^{i(n)}_{\ |l}\mathbf{d}X_k = \delta_{ij}\epsilon^{lpk}\dot{P}^{i(p)}_{\ |l}P^{j(q)}_{\ |p}\mathbf{d}X^k \\
&\implies \quad \delta_{i[p}\dot{P}^{i(n)}_{\ |l]} = \delta_{ij}\dot{P}^{i(p)}_{\ |[l}P^{j(q)}_{\ |p]} \\
&\iff \quad \dot{P}^{(n)}_{[p|l]} = \dot{P}^{(p)}_{j|[l}P^{j(q)}_{\ |p]}\ ;
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
*(2.47) \quad &\iff \quad \epsilon_{ijk}\epsilon^{lpq}\big(\mathcal{D} - 4\pi G\varrho_H\big)P^{i(n)}_{\ |l}\delta^j_{\ p}\delta^k_{\ q} \\
&= -\epsilon_{ijk}\epsilon^{lpq}\Big[\big(2\mathcal{D} - 4\pi G\varrho_H\big)P^{i(p)}_{\ |l}P^{j(q)}_{\ |p}\delta^k_{\ q} + \big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |l}P^{j(s)}_{\ |p}P^{k(t)}_{\ |q}\Big] \\
&\iff \quad 2\big(\mathcal{D} - 4\pi G\varrho_H\big)P^{i(n)}_{\ |i} = -\big(2\mathcal{D} - 4\pi G\varrho_H\big)P^{i(p)}_{\ |i}P^{j(q)}_{\ |j} \\
&\qquad + \big(2\mathcal{D} - 4\pi G\varrho_H\big)P^{i(p)}_{\ |j}P^{j(q)}_{\ |i} - \big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |i}P^{j(s)}_{\ |j}P^{k(t)}_{\ |k} \\
&\qquad - 2\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |j}P^{j(s)}_{\ |k}P^{k(t)}_{\ |i} + 3\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |i}P^{j(s)}_{\ |k}P^{k(t)}_{\ |j} \\
&\iff \quad \ddot{P}^{i(n)}_{\ |i} + 2H\dot{P}^{i(n)}_{\ |i} - 4\pi G\varrho_H P^{i(n)}_{\ |i} = -\big(\mathcal{D} - 2\pi G\varrho_H\big)P^{i(p)}_{\ |i}P^{j(q)}_{\ |j} \\
&+ \big(\mathcal{D} - 2\pi G\varrho_H\big)P^{i(p)}_{\ |j}P^{j(q)}_{\ |i} - \frac{1}{2}\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |i}P^{j(s)}_{\ |j}P^{k(t)}_{\ |k} \\
&- \big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |j}P^{j(s)}_{\ |k}P^{k(t)}_{\ |i} + \frac{3}{2}\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big)P^{i(r)}_{\ |i}P^{j(s)}_{\ |k}P^{k(t)}_{\ |j}\ .
\end{aligned} \tag{2.51}$$

We have to admit the nth order system of equations is not user–friendly at all. The equations (2.47) and (2.51) are exactly the same. This highlights one of the main advantages of the differential forms formalism: the compactness of the equations.

3.2 Solution scheme

First, I present a general solution scheme inspired by [Ehlers and Buchert, 1997]. The difference is that in our approach the perturbation gradients are considered whereas Ehlers and Buchert used the integrated form of the equations (which is possible, since the basic equations are vectorial) of the Newtonian dynamics. Like any matrix, the perturbations and the initial data can be decomposed into trace, symmetric traceless and antisymmetric traceless parts. The symmetric and antisymmetric parts of any matrix are defined by

$$V_{ij} = V_{(ij)} + V_{[ij]} = V^{\Sigma}_{ij} + V^{S}_{ij} + V^{A}_{ij} = \frac{1}{3}V\delta_{ij} + V^{S}_{ij} + V^{A}_{ij}\ . \tag{2.52}$$

V^{Σ} denotes the trace part, V^{S} the symmetric traceless part and V^{A} the antisymmetric traceless part. The reader is referred to Appendix A for expressions of the

symmetric and antisymmetric parts in term of differential forms. Thus the gradient of deformation, of the initial velocity and acceleration are:

$$\begin{cases} \mathbf{d}P_i{}^{(n)} = \mathbf{d}P_i^{\Sigma(n)} + \mathbf{d}P_i^{S(n)} + \mathbf{d}P_i^{A(n)} = P_{(i|j)}^{(n)}\mathbf{d}X^j + P_{[i|j]}^{(n)}\mathbf{d}X^j \ ; \\ \mathbf{d}U_i = \mathbf{d}U_i^{\Sigma} + \mathbf{d}U_i^{S} + \mathbf{d}U_i^{A} = U_{(i|j)}\mathbf{d}X^j + U_{[i|j]}\mathbf{d}X^j \ ; \\ \mathbf{d}W_i = \mathbf{d}W_i^{\Sigma} + \mathbf{d}W_i^{S(1)} = W_{(i|j)}\mathbf{d}X^j \ . \end{cases} \tag{2.53}$$

I will denote by $^{\Sigma}$, ^{S} and ^{A} the two indices objects $\mathbf{d}P_i$, $P_{i|j}$, i.e. the operator is applied on the whole object.

The Lagrange–Newton system is composed of the curl (2.48) and the divergence (2.49) of the gravitational field. The curl of the gravitational field strength only involves antisymmetric terms and vanishes. Thus the initial peculiar acceleration gradient is symmetric, whereas the divergence of the gravitational field strength only involves trace quantities. Using the definition of the symmetric and antisymmetric parts, we can show that the general first order solution verifies:

$$(2.44) \quad \Longrightarrow \quad \delta_{ij}\mathbf{d}P^{i^A(1)} \wedge \mathbf{d}X^j = \mathbf{d}U^{i^A} \wedge \mathbf{d}X^j \int_{t_\mathbf{i}}^{t} a^{-2}dt' \ ; \tag{2.54}$$

$$\begin{aligned} (2.45) \quad \Longrightarrow \quad & \epsilon_{ijk}\Big(\mathcal{D} - 4\pi G\varrho_H\Big)\mathbf{d}P^{i^\Sigma(1)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ & = \epsilon_{ijk}\ a^{-3}\mathbf{d}W^{i^\Sigma} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \ ; \end{aligned} \tag{2.55}$$

uniquely determined by the initial data $\mathbf{d}U^{i^A}$ and $\mathbf{d}W^{i^\Sigma}$. In a similar way, the symmetric/antisymmetric separation of the gradients applied to the general n^{th} order leads to:

$$(2.46) \quad \Longrightarrow \quad \delta_{ij}\ \mathbf{d}P^{i^A(n)} \wedge \mathbf{d}X^j = {}^N\mathcal{S}^{(n)} \ ; \tag{2.56}$$

$$(2.47) \quad \Longrightarrow \quad \epsilon_{ijk}\,(\mathcal{D} - 4\pi G\varrho_H)\,\mathbf{d}P^{i^\Sigma(n)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = {}^N\mathcal{T}^{(n)} \ ; \tag{2.57}$$

uniquely determined by the following source terms:

$${}^N\mathcal{S}^{(n)} = -\,\delta_{ij}\int_{t_\mathbf{i}}^{t}\Big(\mathbf{d}\dot{P}^{i(p)} \wedge \mathbf{d}P^{j(q)}\Big)^A dt' \ ; \tag{2.58}$$

$$\begin{aligned} {}^N\mathcal{T}^{(n)} = & -\,\epsilon_{ijk}\Big[\Big(2D - 4\pi G\varrho_H\Big)\mathbf{d}P^{i(p)} \wedge \mathbf{d}P^{j(q)} \wedge \mathbf{d}X^k \\ & + \Big(D - \frac{4\pi G}{3}\varrho_H\Big)\mathbf{d}P^{i(r)} \wedge \mathbf{d}P^{j(s)} \wedge \mathbf{d}P^{k(t)}\Big]^\Sigma \ . \end{aligned} \tag{2.59}$$

Be careful, these sources contain non–trivial terms according to the solution scheme, the source of the nth order curl equation is $\delta_{ij}(\mathbf{d}\dot{P}^{i(p)} \wedge \mathbf{d}P^{j(q)})^A$. This involves the coefficient $\dot{P}_{i|[j}^{(p)}P^{i\ (q)}_{|k]}$ which cannot be expressed properly with the solutions of the lower order equations. The traceless antisymmetric (curl) and trace (divergence) equations are not enough because the mentioned source term contains also a trace-free symmetric part which is not described by our first order system. It could be

a serious problem to go further in the resolution but I recall that the Newtonian equations are vectorial. Thus the solutions can entirely be determined using this alternative formulation. It is crucial to note that in the relativistic approach, the vectorial expression does no exist, this technique cannot be used. Fortunately, the relativistic system of equations furnishes additional constraints which give informations about the symmetric tracefree part.

The differential forms formalism is more convenient in some cases, for instance we have seen that the nth order divergence is much more compact. Nonetheless, the tensorial formalism is more frequently encountered. Our equations are expressed with differential forms, in order to check the consistency of our results I compute here their tensorial analog in order to compare them with the usual results.

$$(2.44) \quad \Longrightarrow \quad P^{(1)}_{[i|j]} = U_{[i|j]} \int_{t_{\mathbf{i}}}^{t} a^{-2} dt' \; ; \tag{2.60}$$

$$(2.45) \quad \Longrightarrow \quad \ddot{P}^{i(1)}_{\;\;|i} + 2H\dot{P}^{i(1)}_{\;\;|i} - 4\pi G\rho_{H\mathbf{i}} a^{-3} P^{i(1)}_{\;\;|i} = a^{-3} W^{i}_{\;\;|i} \; ; \tag{2.61}$$

$$(2.46) \quad \Longrightarrow \quad P^{(n)}_{[i|j]} = \int_{t_{\mathbf{i}}}^{t} \dot{P}^{k(p)}_{\;\;|[i} P^{(q)}_{k|j]} dt' \; ; \tag{2.62}$$

$$\begin{aligned}(2.47) \quad \Longrightarrow \quad & \ddot{P}^{i(n)}_{\;\;|i} + 2H\dot{P}^{i(n)}_{\;\;|i} - 4\pi G\varrho_H P^{i(n)}_{\;\;|i} = -\big(\mathcal{D} - 2\pi G\varrho_H\big) P^{i(p)}_{\;\;|i} P^{j(q)}_{\;\;|j} \\ & + \big(\mathcal{D} - 2\pi G\varrho_H\big) P^{i(p)}_{\;\;|j} P^{j(q)}_{\;\;|i} - \frac{1}{2}\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big) P^{i(r)}_{\;\;|i} P^{j(s)}_{\;\;|j} P^{k(t)}_{\;\;|k} \\ & - \big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big) P^{i(r)}_{\;\;|j} P^{j(s)}_{\;\;|k} P^{k(t)}_{\;\;|i} + \frac{3}{2}\big(\mathcal{D} - \frac{4\pi G}{3}\varrho_H\big) P^{i(r)}_{\;\;|i} P^{j(s)}_{\;\;|k} P^{k(t)}_{\;\;|j} \; .\end{aligned} \tag{2.63}$$

Then the sources are determined by the lower orders of perturbations with the trick that the solutions from the Newtonian vectorial equations give all the informations (trace, symmetric traceless and antisymmetric traceless parts).

3.3 Explicit solutions scheme

In this section I precise the explicit equations for the homogeneous, first and second order Lagrange–Newton system. The differential forms and tensorial equations have been both computed previously. I will continue the developments in parallel with the two approaches. The Newtonian resolution of this system has been proposed in the papers of Buchert *et al.* [Buchert, 1992; Buchert and Ehlers, 1993; Buchert, 1994; Rampf and Buchert, 2012; Ehlers and Buchert, 1997]. The last paper develops the differential forms formalism for the Newtonian dynamics, whereas the four others deal with tensorial first, second, third and fourth order of perturbations.

3.3.1 Homogeneous solution

The homogeneous Lagrange–Newton system is reduced to the derivative of the Friedmann equation,

$$\epsilon_{ijk}\, 3\frac{\ddot{a}}{a}\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = \epsilon_{ijk}\left(\Lambda - 4\pi G\varrho_H\right)\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k\,. \tag{2.64}$$

Using the Hodge star operator, the homogeneous differential forms equation leads to the derivative of the Friedmann equation $3\,(\ddot{a}/a) = \Lambda - 4\pi G\varrho_{H\mathbf{i}}\, a^{-3}$. Then the homogeneous behaviour of our model follows the Friedmann equation, and could be assumed to have one of the behaviour previously mentioned (FLRW, EdS, spherical symmetry...). The Friedmann equation is easily obtained and is $3H^2 = 8\pi G\varrho_{H\mathbf{i}}\, a^{-3} - \Lambda$. The explicit solutions for an Einstein–de Sitter background will be presented later.

3.3.2 First order solution

The first order Lagrange–Newton system has to be compared with existing first order Newtonian perturbative models [Buchert, 1992]:

$$(2.54) \quad \Longleftrightarrow \quad \delta_{ij}\mathbf{d}P^{i(1)} \wedge \mathbf{d}X^j = \delta_{ij}\mathbf{d}U^{i(1)} \int_{t_\mathbf{i}}^{t} a^{-2}dt' \wedge \mathbf{d}X^j\,; \tag{2.65}$$

$$\begin{aligned}(2.55) \quad \Longleftrightarrow \quad & \epsilon_{ijk}\left(\mathcal{D} - 4\pi G\varrho_H\right)\mathbf{d}P^{i(1)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ & = \epsilon_{ijk}a^{-3}\mathbf{d}W^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k\,;\end{aligned} \tag{2.66}$$

and the associated coefficient equations are:

$$(2.60) \quad \Longleftrightarrow \quad P^{(1)}_{[i|j]} = U_{[i|j]} \int_{t_\mathbf{i}}^{t} a^{-2}dt'\,; \tag{2.67}$$

$$(2.61) \quad \Longleftrightarrow \quad \ddot{P}^i{}_{|i} + 2H\dot{P}^i{}_{|i} - 4\pi G\varrho_{\mathbf{i}H}a^{-3}P^i{}_{|i} = a^{-3}W^i{}_{|i}\,. \tag{2.68}$$

The two equations are composed of a linear operator applied to $P^{(1)}$ and source terms which only depends on the initial data. The system is closed and leads to solutions for the antisymmetric traceless and the trace part.

3.3.3 Second order solution

The second order has to be compared with existing second order Newtonian perturbative models [Buchert and Ehlers, 1993]. The second order Lagrange–Newton system is obtained by a restriction of the nth order perturbative equations,

$$(2.56) \quad \Longleftrightarrow \quad \delta_{ij}\mathbf{d}P^{i(2)} \wedge \mathbf{d}X^j = -\delta_{ij}\int_{t_\mathbf{i}}^{t}\left[\mathbf{d}\dot{P}^{i(1)} \wedge \mathbf{d}P^{j(1)}\right]dt'\,; \tag{2.69}$$

$$
\begin{aligned}
(2.57) \quad \Longleftrightarrow \quad & \epsilon_{ijk}\left(\mathcal{D} - 4\pi G\varrho_H\right)\mathbf{d}P^{i(2)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\
& = \epsilon_{ijk}\left[\left(4\pi G\varrho_H - 2\mathcal{D}\right)\mathbf{d}P^{i(1)} \wedge \mathbf{d}P^{j(1)} \wedge \mathbf{d}X^k\right]^{\Sigma} . \qquad (2.70)
\end{aligned}
$$

On the other hand, in the tensorial formalism, the equations read:

$$
(2.62) \quad \Longleftrightarrow \quad P^{(2)}_{[i|j]} = -\int_{t_{\mathbf{i}}}^{t}\left[\dot{P}^{(1)}_{k|[j}P^{k(1)}_{|i]}\right]dt' \; ; \qquad (2.71)
$$

$$
\begin{aligned}
(2.63) \quad \Longleftrightarrow \quad & \ddot{P}^{i\ (2)}_{|i} + 2H\dot{P}^{i\ (2)}_{|i} - 4\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (2)}_{|i} \\
& = \left(\ddot{P}^{i\ (1)}_{|j} + 2H\dot{P}^{i\ (1)}_{|j} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (1)}_{|j}\right)P^{j\ (1)}_{|i} \\
& \quad - \left(\ddot{P}^{i\ (1)}_{|i} + 2H\dot{P}^{i\ (1)}_{|i} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (1)}_{|i}\right)P^{j\ (1)}_{|j} . \qquad (2.72)
\end{aligned}
$$

At first sight this system is not closed because of the expression of the source terms that required the symmetric traceless part. But, as mentioned earlier, the Newtonian equations admit a vectorial formulation, in which the curl and the divergence equations described completely the perturbations. Then the source terms are known thanks to the lower orders ((2.65)–(2.68)) and the second order system is closed.

3.4 Einstein–de Sitter explicit solutions

The equations of cosmology cannot be solved in the general case. Here, an irrotational dust fluid in an Einstein–de Sitter space–time has been chosen. It means our equations do not contain vorticity and pressure. The EdS background is the solution for the Friedmann equation for an average density equal to the critical density $\varrho_H = \varrho_c$ and a vanishing cosmological constant $\Lambda = 0$. For this solution the scale factor behaves as,

$$
a(t) = \left(\frac{t}{t_{\mathbf{i}}}\right)^{2/3} . \qquad (2.73)
$$

The derivatives of the scale factor and the Hubble factor are frequently encountered quantities, then let us write them with respect to time:

$$
\begin{cases}
\dot{a} = \dfrac{2}{3t_{\mathbf{i}}}a^{-1/2} \quad ; \quad \ddot{a} = -\dfrac{2}{9{t_{\mathbf{i}}}^2}a^{-2} \; ; \\
H = \dfrac{2}{3}t^{-1} \quad ; \quad \dfrac{\ddot{a}}{a} = -\dfrac{2}{9}t^{-2} .
\end{cases} \qquad (2.74)
$$

3.4.1 First order

The first order system of equations has been computed previously in this manuscript. The divergence equation (2.68) is solved in a first time.

It is frequently assumed that the solution admits a space–time splitting

$$
P^{i\ (n)}_{|j} = \xi^{(n)}(t)z^{i\ (n)}_{|j}(\vec{X}) . \qquad (2.75)
$$

This method can be applied to a partial differential equation. It could give only a subclass of the solutions, but it seems to fit pretty well reality. Thus the divergence equation (2.68) can be split between its spatial and time parts as follows.

$$\begin{aligned}
&\left(\ddot{\xi}^{(1)} + 2\frac{\dot{a}}{a}\dot{\xi}^{(1)} - 4\pi G\rho_{H\mathbf{i}}a^{-3}\xi^{(1)}\right) z^{i}{}_{|i}^{(1)} = a^{-3}W^{i}{}_{|i} \; ; \\
\Longrightarrow \quad & \begin{cases} \ddot{\xi}^{(1)} + 2\frac{\dot{a}}{a}\dot{\xi}^{(1)} + 3\frac{\ddot{a}}{a}\xi^{(1)} = a^{-3} = G^{(1)} \; ; \\ z^{i}{}_{|i}^{(1)} = W^{i}{}_{|i} \; . \end{cases}
\end{aligned} \tag{2.76}$$

Henceforth, $G^{(1)}$ will denote the right–hand side of the first order equation (2.76). On the one hand, the spatial equation is simply a definition of the spatial part of the solution with respect to the initial data. On the other hand, the time differential equation admits an analytical solution which can be computed analytically. Since the equation (2.76) is a linear second order differential equation, its solution is composed of two homogeneous (without source term) solutions ${}^{\mathbf{h}}\xi(t)$ and a particular solution ${}^{\mathbf{p}}\xi(t)$. These three solutions could admit three different constant coefficients, thus I define ${}^{\mathbf{1}}\mathcal{C}^{(1)}$, ${}^{\mathbf{2}}\mathcal{C}^{(1)}$ and ${}^{\mathbf{p}}\mathcal{C}^{(1)}$ these coefficient. The complete solution (superposition of the two homogeneous and the particular solution) is factorised by the spatial function $z^{i}{}_{|i}(\vec{X}) = W^{i}{}_{|i}$ then I define also spatial coefficients for the space–time solutions by ${}^{\mathbf{i}}C^{i}{}_{|i}^{(1)} = {}^{\mathbf{i}}\mathcal{C}^{(1)}\; W^{i}{}_{|i}$.

$$\begin{aligned}
P^{i}{}_{|i}^{(1)} &= {}^{\mathbf{1}}C^{i}{}_{|i}^{(1)}\; {}^{\mathbf{1}}\xi^{(1)}(t) + {}^{\mathbf{2}}C^{i}{}_{|i}^{(1)}\; {}^{\mathbf{2}}\xi^{(1)}(t) + {}^{\mathbf{p}}C^{i}{}_{|i}^{(1)}\; {}^{\mathbf{p}}\xi^{(1)}(t) \\
&= \left({}^{\mathbf{1}}\mathcal{C}^{(1)}\; {}^{\mathbf{1}}\xi^{(1)}(t) + {}^{\mathbf{2}}\mathcal{C}^{(1)}\; {}^{\mathbf{2}}\xi^{(1)}(t) + {}^{\mathbf{p}}\mathcal{C}^{(1)}\; {}^{\mathbf{p}}\xi^{(1)}(t)\right) W^{i}{}_{|i} \; .
\end{aligned} \tag{2.77}$$

A first test is to look for specific solutions such as exponential or power law. In fact, one of these special solutions is correct: the power law solution. The homogeneous solution is assumed to be of the form $\xi(t) = a^{n}$, its derivatives are $\dot{\xi} = na^{n-1}\dot{a}$ and $\ddot{\xi} = n(n-1)a^{n-2}\dot{a}^{2} + na^{n-1}\ddot{a}$. To be a homogeneous solution of the equation (2.76), the power n has to verify the relation:

$$\begin{aligned}
& \left(n(n-1)a^{n-2}\dot{a}^{2} + na^{n-1}\ddot{a}\right) + 2\dot{a}a^{-1}\left(na^{n-1}\dot{a}\right) + 3\ddot{a}a^{-1}a^{n} = 0 \\
\Longleftrightarrow \quad & \frac{2}{9{t_{\mathbf{i}}}^{2}}\left[2(n^{2}-n) - n + 4n - 3\right]a^{n-3} = 0 \\
\Longleftrightarrow \quad & 2n^{2} + n - 3 = 0 \\
\Longleftrightarrow \quad & \left\{n_{1} = 1,\; n_{2} = -\frac{3}{2}\right\} \; .
\end{aligned} \tag{2.78}$$

As expected, the assumption of a power law solution leads to the existence of two homogeneous solutions for time part of the first order divergence equation. These two functions are the homogeneous solutions,

$$ {}^{\mathbf{1}}\xi^{(1)} = a \quad ; \quad {}^{\mathbf{2}}\xi^{(1)} = a^{-3/2} \; . \tag{2.79}$$

Hence the first order perturbations are composed of a homogeneous solution that increases with the scale factor, and another that increases with the inverse of the scale factor.

$$ {}^{\mathbf{h}}P^{i\ (1)}_{\ |i} = {}^{\mathbf{1}}P^{i\ (1)}_{\ |i} + {}^{\mathbf{2}}P^{i\ (1)}_{\ |i} = {}^{\mathbf{1}}C^{i\ (1)}_{\ |i}\ a + {}^{\mathbf{2}}C^{i\ (1)}_{\ |i}\ a^{-3/2}\ . \tag{2.80} $$

It exists a general method to find the particular solution of an equation of the form of (2.76). This particular solution can be written,

$$ \begin{aligned} {}^{\mathbf{p}}\xi^{(1)}(t) = \tilde{\mathcal{C}}^{(1)} \Bigg(& {}^{\mathbf{1}}\xi^{(1)}(t) \int a^2 G^{(1)}(t')\, {}^{\mathbf{2}}\xi^{(1)}(t') dt' \\ & - {}^{\mathbf{2}}\xi^{(1)}(t) \int a^2 G^{(1)}(t')\, {}^{\mathbf{1}}\xi^{(1)}(t') dt' \Bigg)\ ; \end{aligned} \tag{2.81} $$

with $G^{(1)} = a^{-3}$. Injecting the homogeneous solutions presented earlier in the paragraph, the particular solution reads,

$$ {}^{\mathbf{p}}\xi^{(1)}(t) = -\frac{5}{2} t_{\mathbf{i}} \tilde{\mathcal{C}}^{(1)}\ . \tag{2.82} $$

The coefficient $\tilde{\mathcal{C}}^{(1)}$ can be determined if the function ${}^{\mathbf{p}}\xi$ is injected in the equation (2.76). For this we need to compute the time derivatives of this function. Fortunately, the particular solution ${}^{\mathbf{p}}\xi(t)$ is constant in time, then its derivatives vanish. In another case, these derivatives could be non zero. When the function ${}^{\mathbf{p}}\xi(t)$ is injected, the time differential equation reads,

$$ \begin{aligned} & 3 \left(\frac{-2}{9 {t_{\mathbf{i}}}^2} \right) a^{-3} \left(\frac{-5}{2} \right) t_{\mathbf{i}}\ \tilde{\mathcal{C}}^{(1)} = a^{-3} \\ \Longleftrightarrow\quad & \frac{5}{3} {t_{\mathbf{i}}}^{-1}\ \tilde{\mathcal{C}}^{(1)} = 1 \\ \Longleftrightarrow\quad & \tilde{\mathcal{C}}^{(1)} = \frac{3}{5} t_{\mathbf{i}}\ . \end{aligned} \tag{2.83} $$

This last results give all the informations about the constant factor of the particular solution, which reads:

$$ {}^{\mathbf{p}}\xi^{(1)}(t) = {}^{\mathbf{p}}\mathcal{C}^{(1)} = -\frac{3}{2} {t_{\mathbf{i}}}^2\ . \tag{2.84} $$

Nevertheless, in the case of our comoving equation the particular solution can be found trivially. Before the substitution of the Friedmann equation, the non derived term of the equation (2.76) ($-4\pi G \varrho_{\mathbf{i}}\ a^{-3}\ \xi^{(1)}$) and the source ($G^{(1)} = a^{-3}$) have the same $a(t)$ dependence. Moreover an EdS model was chosen, then $-4\pi G \varrho_{\mathbf{i}}\ a^{-3} = 3\ (\ddot{a}/a) = -(2/3)\ {t_{\mathbf{i}}}^{-2} a^{-3}$. Then a constant is obviously a particular solution.

$$ 3 \frac{\ddot{a}}{a}\ {}^{\mathbf{p}}\xi(t) = a^{-3} \quad \Longleftrightarrow \quad {}^{\mathbf{p}}\xi(t) = {}^{\mathbf{p}}\mathcal{C}^{(1)} = -\frac{3}{2} {t_{\mathbf{i}}}^2\ . \tag{2.85} $$

In fact, the result of the expression of ${}^{\mathbf{p}}\xi$ gives a little more information than the time dependence: the expression of ${}^{\mathbf{p}}\mathcal{C}^{(1)}$ is obtained. This expression will be confirmed by the further calculations about the initial data. At first order it is possible to put aside these informations and work with a simple constant which will be evaluated thanks to the initial data, but it is not the case for higher orders. The first order solution is now described by the functions,

$$\begin{aligned} P^{i(1)}_{|i}(\vec{X},t) &= \left({}^{\mathbf{1}}\mathcal{C}^{(1)}(\vec{X})a(t) + {}^{\mathbf{2}}\mathcal{C}^{(1)}(\vec{X})a^{-3/2}(t) + {}^{\mathbf{p}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i}(\vec{X}) \\ &= \left({}^{\mathbf{1}}\mathcal{C}^{(1)}(\vec{X})a(t) + {}^{\mathbf{2}}\mathcal{C}^{(1)}(\vec{X})a^{-3/2}(t) - \frac{3}{2}{t_{\mathbf{i}}}^2\right) z^{i\ (1)}_{\ |i}(\vec{X}) \ ; \end{aligned} \tag{2.86}$$

where the spatial part is $z^{i\ (1)}_{\ |i} = W^{i}_{\ |i}$ and ${}^{\mathbf{1}}\mathcal{C}^{(1)}(\vec{X})$, ${}^{\mathbf{2}}\mathcal{C}^{(1)}(\vec{X})$ are undetermined coefficients. The initial data allow us to express these coefficients through a system of three equations: the initial gradient of deformation, the initial velocity gradient and the initial acceleration gradient. Moreover, these constraints offer a consistency check of the expression of the spatial part. I remind that the scale factor $a(t)$ is defined by (2.73).

$$\begin{cases} P^{i(1)}_{|i}(\vec{X},t_{\mathbf{i}}) = \left({}^{\mathbf{1}}\mathcal{C}^{(1)} + {}^{\mathbf{2}}\mathcal{C}^{(1)} + {}^{\mathbf{p}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i} = 0 \\ \dot{P}^{i(1)}_{|i}(\vec{X},t_{\mathbf{i}}) = \left(\dfrac{2{t_{\mathbf{i}}}^{-1}}{3}\ {}^{\mathbf{1}}\mathcal{C}^{(1)} - {t_{\mathbf{i}}}^{-1}\ {}^{\mathbf{2}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i} = U^{i}_{\ |i} \\ \ddot{P}^{i(1)}_{|i}(\vec{X},t_{\mathbf{i}}) = \left(-\dfrac{2{t_{\mathbf{i}}}^{-2}}{9}\ {}^{\mathbf{1}}\mathcal{C}^{(1)} + 2{t_{\mathbf{i}}}^{-2}\ {}^{\mathbf{2}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i} = W^{i}_{\ |i} - \dfrac{4{t_{\mathbf{i}}}^{-1}}{3}U^{i}_{\ |i} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \left({}^{\mathbf{1}}\mathcal{C}^{(1)} + {}^{\mathbf{2}}\mathcal{C}^{(1)} + {}^{\mathbf{p}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i} = 0 \\ \left(-\dfrac{5}{3}{t_{\mathbf{i}}}^{-1}\ {}^{\mathbf{2}}\mathcal{C}^{(1)} - \dfrac{2}{3}{}^{\mathbf{p}}\mathcal{C}^{(1)}{t_{\mathbf{i}}}^{-1}\right) z^{i\ (1)}_{\ |i} = U^{i}_{\ |i} \\ \left(\dfrac{20}{9}{t_{\mathbf{i}}}^{-2}\ {}^{\mathbf{2}}\mathcal{C}^{(1)} + \dfrac{2}{9}{}^{\mathbf{p}}\mathcal{C}^{(1)}{t_{\mathbf{i}}}^{-2}\right) z^{i\ (1)}_{\ |i} = W^{i}_{\ |i} - \dfrac{4{t_{\mathbf{i}}}^{-1}}{3}U^{i}_{\ |i} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \left({}^{\mathbf{1}}\mathcal{C}^{(1)} + {}^{\mathbf{2}}\mathcal{C}^{(1)} + {}^{\mathbf{p}}\mathcal{C}^{(1)}\right) z^{i\ (1)}_{\ |i} = 0 \\ \left(-\dfrac{5}{3}{t_{\mathbf{i}}}^{-1}\ {}^{\mathbf{2}}\mathcal{C}^{(1)} - \dfrac{2}{3}{}^{\mathbf{p}}\mathcal{C}^{(1)}{t_{\mathbf{i}}}^{-1}\right) z^{i\ (1)}_{\ |i} = U^{i}_{\ |i} \\ -\dfrac{2}{3}{}^{\mathbf{p}}\mathcal{C}^{(1)}{t_{\mathbf{i}}}^{-2} z^{i\ (1)}_{\ |i} = W^{i}_{\ |i} \end{cases}$$

$$\iff \begin{cases} {}^{\mathbf{1}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = \dfrac{3}{5}\left(U^{i}_{\ |i} t_{\mathbf{i}} - W^{i}_{\ |i} t_{\mathbf{i}}{}^{2}\right) + \dfrac{3}{2} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} \ ; \\ -\dfrac{5}{3}\,{}^{\mathbf{2}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = U^{i}_{\ |i} t_{\mathbf{i}} - W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} \ ; \\ {}^{\mathbf{P}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = -\dfrac{3}{2} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} \ . \end{cases}$$

This system contains three equations and three functions. The Gaussian elimination leads to the following constraints:

$$\begin{cases} {}^{\mathbf{1}}C^{i\ (1)}_{\ |i} = {}^{\mathbf{1}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = \dfrac{9}{10} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} + \dfrac{3}{5} U^{i}_{\ |i} t_{\mathbf{i}} \ ; \\ {}^{\mathbf{2}}C^{i\ (1)}_{\ |i} = {}^{\mathbf{2}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = \dfrac{3}{5} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} - \dfrac{3}{5} U^{i}_{\ |i} t_{\mathbf{i}} \ ; \\ {}^{\mathbf{P}}C^{i\ (1)}_{\ |i} = {}^{\mathbf{P}}\mathcal{C}^{(1)} z^{i\ (1)}_{\ |i} = -\dfrac{3}{2} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} \ . \end{cases} \tag{2.87}$$

The first order equation is now completely determined thanks to the initial velocity gradient, initial acceleration gradient and the scale factor.

$$P^{i\ (1)}_{\ |i} = \left(\frac{9}{10} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} + \frac{3}{5} U^{i}_{\ |i} t_{\mathbf{i}}\right) a + \left(\frac{3}{5} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2} - \frac{3}{5} U^{i}_{\ |i} t_{\mathbf{i}}\right) a^{-3/2} - \frac{3}{2} W^{i}_{\ |i} t_{\mathbf{i}}{}^{2}. \tag{2.88}$$

The second equation is the curl of the gravitation field strength at first order (2.67). It trivially gives the expression of the curl of the first order perturbations with respect to the initial velocity gradient,

$$P^{(1)}_{[i|j]} = -3 t_{\mathbf{i}} U_{[i|j]} a^{-1/2} \ . \tag{2.89}$$

According to the vanishing of the initial perturbations, the relation is reduced to:

$$U_{[i|j]} = 0 \ . \tag{2.90}$$

3.4.2 The Zel'dovich Approximation

The Zel'dovich Approximation (see [Zel'dovich, 1970a,b]) is assumed to be true for our initial data, the CMB epoch. This assumption has already been discussed previously (see chapter I section 4). After a sufficiently large amount of time, assumed to be reached at the CMB epoch (i.e. our initial data), the dynamics is led by the growing mode solution. Our initial time is sufficiently large, the growing mode solution is:

$$P^{i(1)}_{\ |i} = {}^{\mathbf{1}}C^{i(1)}_{\ |i} a \ . \tag{2.91}$$

The associated initial data and the time behaviour of the scale factor (2.73) lead to:

$$\begin{cases} \dot{P}^{i(1)}_{\ |i}(t_{\mathbf{i}}) = \dfrac{2}{3t_{\mathbf{i}}}{}^{\mathbf{1}}C^{i(1)}_{\ |i} = U^{i}_{\ |i}\ ; \\ \ddot{P}^{i(1)}_{\ |i}(t_{\mathbf{i}}) = -\dfrac{2}{9{t_{\mathbf{i}}}^{2}}{}^{\mathbf{1}}C^{i(1)}_{\ |i} = W^{i}_{\ |i} - 2H_{\mathbf{i}}U^{i}_{\ |i}\ . \end{cases} \tag{2.92}$$

The initial data set leads to a relation between the initial velocity gradient and the initial acceleration gradient: the *slaving condition*.

$$\begin{aligned} & W^{i}_{\ |i} = -\frac{1}{3}\left(\frac{2}{3t_{\mathbf{i}}}{}^{\mathbf{1}}C^{i(1)}_{\ |i}\right){t_{\mathbf{i}}}^{-1} + \frac{4}{3}{t_{\mathbf{i}}}^{-1}U^{i}_{\ |i} \\ \Longleftrightarrow \quad & W^{i}_{\ |i} = U^{i}_{\ |i}{t_{\mathbf{i}}}^{-1} = \frac{3}{2}H_{\mathbf{i}}U^{i}_{\ |i}\ . \end{aligned} \tag{2.93}$$

This additional constraint brought by the Zel'dovich Approximation allows us to write the solution of the first order Lagrange–Newton system with only one of the two initial data: the initial velocity gradient or the initial acceleration gradient. Thus, in the framework of this approximation the first order solution reads,

$$P^{i\ (1)}_{\ |i} = \frac{3}{2}W^{i}_{\ |i}{t_{\mathbf{i}}}^{2}\,(a-1)\ . \tag{2.94}$$

3.4.3 Second order

Here, the resolution of the divergence comoving equation (2.72) is proposed. A similar space–time splitting is assumed for the second order solutions, the associated equations are:

$$\begin{aligned} & \ddot{P}^{i\ (2)}_{\ |i} + 2H\dot{P}^{i\ (2)}_{\ |i} - 4\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (2)}_{\ |i} \\ & \quad = \left(\ddot{P}^{i\ (1)}_{\ |j} + 2H\dot{P}^{i\ (1)}_{\ |j} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (1)}_{\ |j}\right)P^{j\ (1)}_{\ |i} \\ & \qquad - \left(\ddot{P}^{i\ (1)}_{\ |i} + 2H\dot{P}^{i\ (1)}_{\ |i} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i\ (1)}_{\ |i}\right)P^{j\ (1)}_{\ |j}\ ; \\ \Longrightarrow \quad & \begin{cases} \ddot{\xi}^{(2)} + 2\dfrac{\dot{a}}{a}\dot{\xi}^{(2)} + 3\dfrac{\ddot{a}}{a}\xi^{(2)} = \dfrac{3}{4}{t_{\mathbf{i}}}^{2}\left(a^{-1} - a^{-3}\right) = G^{(2)}\ ; \\ C^{i\ (2)}_{\ |i} = W^{i}_{\ |j}W^{j}_{\ |i} - W^{i}_{\ |i}W^{j}_{\ |j}\ . \end{cases} \end{aligned} \tag{2.95}$$

It is important to note that the first order solution gives information only about the trace. So what does $W^{i}_{\ |j}$ mean? Do we chose an arbitrary expression? This problem is more or less treated implicitly thanks to the vectorial formalism. Here, the Zel'dovich solution (2.94) is generalised such as $P^{i(1)}_{\ |j} = (3/2)\ W^{i}_{\ |j}{t_{\mathbf{i}}}^{2}\,(a-1)$. This extrapolation is correct and proved in chapter IV section 4.4. The time differential equation is quite similar to the first order equation. The homogeneous equation is identical to the first order: the time operator is exactly the same. The

difference resides in the source $G^{(2)}$ which is quite different but rather simple thanks to the Zel'dovich Approximation. This equation admits the same homogeneous solutions:

$$^{\mathbf{h}}P^{i\ (2)}_{\ |i} = {}^{\mathbf{1}}C^{i\ (2)}_{\ |i}\ a + {}^{\mathbf{2}}C^{i\ (2)}_{\ |i}\ a^{-3/2}\ . \tag{2.96}$$

Nevertheless the source term is a bit different. The differential equations respect some very useful theorems. The superposition theorem assures that a superposition of two particular solutions for two different sources is a particular solution for the whole source: for all linear systems, the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually. It is also possible and fairly straightforward to use the general method described in the first order paragraph in order to compute the particular solution. Anyway, both methods lead to:

$$^{\mathbf{p}}\xi = \frac{9}{8}t_{\mathbf{i}}{}^{4}\left(1+\frac{3}{7}a^{2}\right)\ . \tag{2.97}$$

Thus, the second order solution can be written such as,

$$P^{i\ (2)}_{\ |i} = \left({}^{\mathbf{1}}\mathcal{C}^{(2)}a + {}^{\mathbf{2}}\mathcal{C}^{(2)}a^{-3/2} + \frac{9}{8}t_{\mathbf{i}}{}^{4}\left(1+\frac{3}{7}a^{2}\right)\right) z^{i\ (2)}_{\ |i}\ . \tag{2.98}$$

A specific set of initial data has been chosen in which the initial velocity gradient and the initial acceleration gradient are first order quantities. Moreover the initial gradient of deformation is null; thus the initial data set leads to,

$$\begin{cases} P^{i(2)}_{\ |i}(\vec{X},t_{\mathbf{i}}) = \left({}^{\mathbf{1}}\mathcal{C}^{(2)} + {}^{\mathbf{2}}\mathcal{C}^{(2)} + \frac{45}{28}t_{\mathbf{i}}{}^{4}\right) z^{i\ (2)}_{\ |i} = 0\ ; \\ \dot{P}^{i(2)}_{\ |i}(\vec{X},t_{\mathbf{i}}) = \left(\frac{2t_{\mathbf{i}}{}^{-1}}{3}\ {}^{\mathbf{1}}\mathcal{C}^{(2)} - t_{\mathbf{i}}{}^{-1}\ {}^{\mathbf{2}}\mathcal{C}^{(2)}\right) z^{i\ (2)}_{\ |i} = 0\ ; \\ \ddot{P}^{i(2)}_{\ |i}(\vec{X},t_{\mathbf{i}}) = \left(-\frac{2t_{\mathbf{i}}{}^{-2}}{9}\ {}^{\mathbf{1}}\mathcal{C}^{(2)} + 2t_{\mathbf{i}}{}^{-2}\ {}^{\mathbf{2}}\mathcal{C}^{(2)}\right) z^{i\ (2)}_{\ |i} = 0\ . \end{cases} \tag{2.99}$$

This system of equations constraints the constant of the second order solution. The following system is obtained thanks to a Gaussian elimination:

$$\begin{cases} {}^{\mathbf{1}}C^{(2)} = {}^{\mathbf{1}}\mathcal{C}^{(2)}z^{i\ (2)}_{\ |i} = -\frac{27}{20}t_{\mathbf{i}}{}^{4}z^{i\ (2)}_{\ |i}\ ; \\ {}^{\mathbf{2}}C^{(2)} = {}^{\mathbf{2}}\mathcal{C}^{(2)}z^{i\ (2)}_{\ |i} = -\frac{4}{35}t_{\mathbf{i}}{}^{4}z^{i\ (2)}_{\ |i}\ . \end{cases} \tag{2.100}$$

Note the first order influences the second order dynamics through the source term $G^{(2)}$. We assumed the slaving condition for the initial data, it leads to a simpler expression for the first order solution. Then the Zel'dovich Approximation second order solution is:

$$P^{i\ (2)}_{\ |i} = \frac{9}{4}t_{\mathbf{i}}{}^{4}\left(\frac{3}{14}a^{2} - \frac{3}{5}a + \frac{1}{2} - \frac{4}{35}a^{-3/2}\right) z^{i\ (2)}_{\ |i}\ . \tag{2.101}$$

3.4.4 Third order

I do not present here the explicit calculations but the third order solution is available and can be computed with the same method (see [Buchert, 1994]).

$$P^{i}{}_{|i}^{(3)} = \frac{9}{4}\left(\frac{1}{126}a^3 - \frac{3}{7}a^2 + \frac{1}{10}a - \frac{1}{6} + \frac{4}{35}a^{-1/2} - \frac{4}{315}a^{-3/2}\right) z^{i}{}_{|i}^{(3)} \,. \tag{2.102}$$

4 Concluding remarks

The perturbative solution of this problem up to a certain order is obtained from the summation of the perturbative solutions until the chosen order. The perturbation up to second order is then described by the following function,

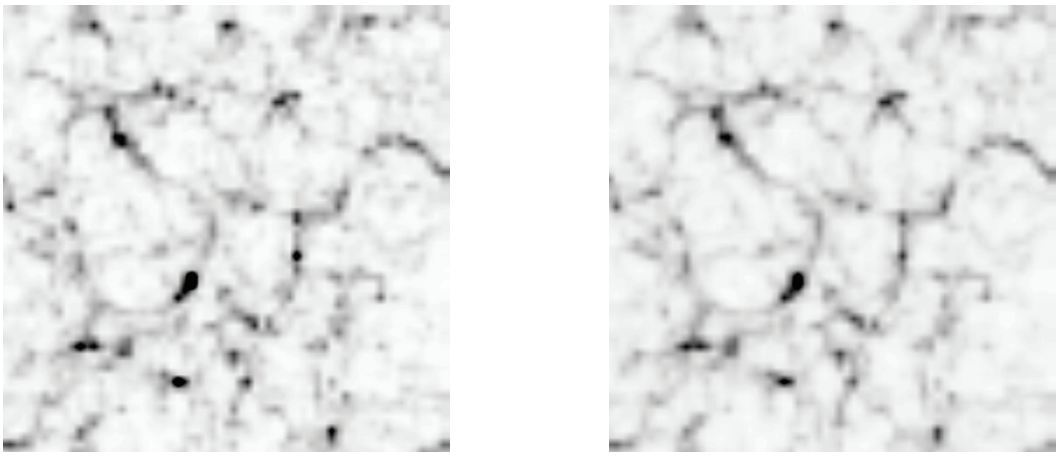
$$P = \frac{3}{2}t_{\mathbf{i}}{}^2\,(a-1)\,z^{i}{}_{|i}^{(1)} + \frac{9}{4}t_{\mathbf{i}}{}^4\left(\frac{3}{14}a^2 - \frac{3}{5}a + \frac{1}{2} - \frac{4}{35}a^{-3/2}\right) z^{i}{}_{|i}^{(2)} \,; \tag{2.103}$$

$$\begin{cases} z^{i}{}_{|i}^{(1)} = W^{i}{}_{|i} \,; \\ z^{i}{}_{|i}^{(2)} = W^{i}{}_{|j}W^{j}{}_{|i} - W^{i}{}_{|i}W^{j}{}_{|j} \,. \end{cases} \tag{2.104}$$

These results extracted from the comoving equations are naturally coherent with the results from Buchert *et al.* [Buchert, 1992; Buchert and Ehlers, 1993; Buchert, 1994; Ehlers and Buchert, 1997]. Most of the applications to large–scale structure formation are based on these equations and resolutions. One of the main advantages of the Lagrangian formulation of gravitation is the possibility to describe high density contrast which logically appears during the large–scale structures formation. Many statistical studies exist. A beautiful image of the power of the Lagrange–Newton perturbative approach was shown in [Weiss et al., 1996]. The two points of view of the fluids mechanics are compared in this paper. For the same kind of initial data two simulations are performed. The fig. 2.3 shows the density contrast in a box of $(200\ h^{-1}\mathrm{Mpc})^3$ box.

- On the first plot, a numerical resolution of the Euler–Newton equations. The code is built to numerically approach the solution of the system. For each time–step the code solves locally the Poisson equation by a fast Fourier transform and re–evaluates the density in each cell of the simulation. This process is repeated step by step in order to obtain the final repartition of matter. It is not a perturbative development and it demands a certain amount of calculus time. At each step the resolution of the Poisson equation has to be performed. The left panel 2.3a shows the density map produce by this approach.

- On the second plot, a numerical plotting of the second order solutions of the Lagrange–Newton system. The code is fed with the solution, just like these we computed earlier, and CMB initial data. Then the trajectories are evaluated and at the ending time a regular grid is applied and according to the number



(a) Eulerian (all orders) numerical resolution.

(b) Lagrangian (order 2) analytical solution.

Figure 2.3: Density contrast in a simulated box of size $(200h^{-1}\text{Mpc})^3$. The left panel fig.2.3a is the result of an Eulerian numerical resolution of the equation, the right panel fig.2.3b is the result of a second order Lagrangian analytical realisation. (credits: A. Weiss *et al.* see [Weiss et al., 1996])

of trajectories which end in a cell the density is computed and plotted. An advantage of this method is to avoid the numerical resolution step and gain calculus time. The right panel 2.3b shows the numerical plotting of the second order Lagrange–Newton analytical solutions.

The difference between the two maps is slight. But the first method requires a numerical resolution whereas the second is just the plotting of analytical functions. Both simulations are performed with the same set of initial data. It is really impressive to see that a second order Lagrange–Newton analytical solution is enough to fit the whole Eulerian numerical resolution. It is one of the best example of the power of the Lagrange approach.

Other comparisons between the Lagrangian Perturbation theory results and N–body simulations are available in [Buchert et al., 1994; Melott et al., 1995; Pauls and Melott, 1995; Sahni and Shandarin, 1996; Buchert et al., 1997; Karakatsanis et al., 1997] and and more recently in [Nadkarni-Ghosh and Chernoff, 2011].

5 The motivation of a relativistic perturbation theory

The Newtonian matter evolves on a flat Euclidean space. But we know since [Einstein, 1916] that the gravitation is not so simple as suggested by Newton in the XVIIth century in *Philosophiæ Naturalis Principia Mathematica*. The space and its content cannot be considered independently: "Space–time tells matter how to move; matter tells space–time how to curve" (John Archibald Wheeler in Geons, Black Holes, and Quantum Foam, p. 235).

- Most of the models, even the standard model of cosmology, are so restrictive that they are quasi–Newtonian or even fully Newtonian. The standard model requires Dark Matter and Dark Energy to fit the observations. Since several decades the scientific community is looking for them. Could this model be too restrictive? Does the cosmological principle, in its commonly admitted formulation, intrinsically lead to a Newtonian dynamics? Why does the standard model ignore the impact of geometrical inhomogeneities on average properties of the Universe model?

- These open–ended questions motivate the necessity to deal with richer descriptions in order to go beyond the standard model. Perhaps the Universe is naturally homogeneous and the global dynamics decorrelated from the local dynamics. I am very optimistic with the relativistic perturbations and the impact of important contrasts of density on the geometry, the dynamics and the content of the Universe. Meanwhile a huge amount of works has been dedicated to this subject, references may be found in [Buchert et al., 2000; Kolb et al., 2006; Räsänen, 2006b,c; Buchert, 2008; Buchert and Carfora, 2008; Larena et al., 2009; Kolb, 2011; Buchert and Ostermann, 2012; Buchert et al., 2013].

Chapter III

Lagrangian relativistic perturbation theory

This chapter is based on a paper in preparation, [Alles et al., 2014].

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1 Introduction

In the previous work of the series of papers [Buchert and Ostermann, 2012; Buchert et al., 2013], the authors laid down the foundations of the Lagrangian perturbation theory by writing Einstein equations in the $3+1$ form for a single dynamical variable. They investigated its first order solutions for the trace and antisymmetric parts, extrapolated this solution in the spirit of Zel'dovich Approximation in Newtonian cosmology, and provided a definition of a non–perturbative scheme of structures formation [Buchert and Ostermann, 2012]. Then, they studied the average properties of this latter in relation to the Dark Energy and Dark Matter problems in [Buchert et al., 2013]. Here, we proceed with the presentation of the general first order solution that cover the full Newtonian Lagrangian schemes through a restriction procedure that we shall define. As in previous works we restrict our attention to irrotational dust continua for simplicity. The generalisation to more general matter models and general foliations of space–time is scheduled.

The reader may wonder why we investigate the problem of perturbation solutions in general relativity, while this has already been done by a plethora of works (for a selection of key–references on standard perturbation theory see [Bardeen, 1982; Mukhanov et al., 1992; Kodama and Sasaki, 1984; Durrer, 1988]). The reason lies in the conceptual difference of our framework in comparison with the standard one. We are perturbing a single dynamical variable which, intuitively, is the square root of the spatial metric. Since we are looking for an intrinsic description of the evolution of the perturbation fields, we define perturbations on the perturbed space, not on a global background.

A similar point of view has also been taken in previous works, the pioneering works by Kasai presents a relativistic generalisation of Zel'dovich Approximation, and follow–up works with his collaborators presents a class of second order perturbation solutions [Kasai, 1995; Russ et al., 1996]; see also the earlier papers by Tomita [Tomita, 1972, 1975, 1993; Tomita and Deruelle, 1994; Tomita, 2008], as well as the series of papers by Matarrese, Pantano and Saez [Matarrese et al., 1994a,b, 1998]. All these works are in a wider sense concerned with the relativistic Lagrangian perturbation theory. Still the present work goes beyond these latter works through the following elements:

- We consider, as in [Buchert and Ostermann, 2012; Buchert et al., 2013], a formalism that allows us to write the Einstein equations with a single dynamical variable comprising the Cartan coframe fields. These furnish the conceptual generalisation of the Lagrangian deformation gradient being the single dynamical variable in the Newtonian theory. One advantage of this approach is that only perturbations of this dynamical variable are considered, which entitles us to express all other physical quantities as functionals of this variable. Thus, we inject the deformation solutions at a given order of expansion of the Einstein equations into the functional definitions of these fields, without

a posteriori expanding the functional expressions. This provides highly non-linear approximations for structure formation (e.g., the density field is known through an exact integral of the perturbation variable; the metric as a bilinear form maintains its role as a measure of distance, i.e. as a quadratic expression, *etc.*);

- We provide construction rules to derive relativistic solutions from known Newtonian solutions: we have to additionally study the traceless symmetric part of the equations having no obvious Newtonian analog, and which is fundamentally linked to the traceless Ricci tensor and the physics of gravitational waves;
- By furnishing this Newtonian analogy we employ a division of the system of equations into gravitoelectric and gravitomagnetic parts;
- We perform a strictly intrinsic derivation, i.e. without reference to an external background space. In previous works, although starting with the Cartan formalism, the non–integrability of the Cartan deformations is given up for the building of solutions, hence implicitly introducing a reference background space—see section 4.2;
- We extend the perturbation and solution schemes to any order of the perturbations for the gravitoelectric part of the Lagrange–Einstein system. These schemes will allow us to construct the trace–parts of relativistic solutions at any order.

Before we start, let us recall our strategy (for details the reader is directed to [Buchert and Ostermann, 2012]). In the Newtonian theory the Lagrangian picture of fluid motion allows us to represent Newton equations in terms of a single dynamical variable, the deformation gradient built from the trajectory field. For this system the general perturbation and solution schemes at any order are provided in [Ehlers and Buchert, 1997]. Einstein equations can be formulated in terms of equations for the electric and magnetic parts of the Weyl tensor. Subjecting the electric subsystem of equations to a "*Minkowski Restriction*", i.e. by sending the Cartan coframes to exact forms, directly yields the Newtonian system with a finite light velocity in Lagrangian form [Buchert and Ostermann, 2012] and [Buchert, 2011]: Sect. 7.1. In the following, the term Newtonian means Newtonian with a finite speed velocity.

Furthermore, the reverse process, i.e. the transposition from integrable to non–integrable deformations, enables us to construct a gravitoelectric subclass of the relativistic perturbation and solution schemes that directly corresponds to the Newtonian perturbation and solution schemes. While the Newtonian system furnishes a vector theory, where the gravitational field strength is determined by its divergence and its curl (the trace and antisymmetric parts of the deformation gradient), the so generalised schemes deliver nontrivial solutions for the tracefree symmetric

part that is connected to the gravitomagnetic part of the Weyl tensor, having no obvious Newtonian counterpart.

This chapter is structured as follows. Section 2 recalls the equations of relativistic cosmology for an irrotational dust model. We highlight the equivalence between the Newtonian and the gravitoelectric sets of equations in the *Minkowski Restriction* in section 2.4.1 and 2.4.2. Section 3 provides the first order perturbation and solution schemes, with a discussion of the initial data and constraints. Section 4 investigates tracks of application to the linear theory of gravitational waves and performs a comparison with the standard perturbation theory, which includes a discussion of related work. Finally, section 5 sums up and concludes.

2 Equations of motion and constraints

In this section, after setting notations, we recall the Einstein equations, written in the $3+1$ form and expressed through a single dynamical variable, represented by Cartan coframe fields as functions of local coordinates in the $3-$hypersurfaces, the $3+1$ space–time foliation is described in the appendix D. This recalls the parts of [Buchert and Ostermann, 2012] relevant to this chapter.

2.1 Notations and technicalities

We first employ the differential forms formalism for its compactness and antisymmetric properties, and its diffeomorphism invariance. We then project to the common coefficient formalism in which we work out the solutions.

We consider a set of a (any number) differential $k-$forms $\mathbf{k}^a$. The coefficients of these fields can be expressed in the exact basis $\{\mathbf{d}X^i\}$ of the cotangent space at a given point, $\mathbf{k}^a = k^a{}_{i_1\dots i_k}\mathbf{d}X^{i_1}\wedge\dots\wedge\mathbf{d}X^{i_k}$, where $\wedge$ is the wedge product (antisymmetrisation of the tensorial product $A\wedge B = A\otimes B - B\otimes A$). Their exterior derivative yields $\mathbf{d}\mathbf{k}^a = k^a{}_{i_1\dots i_k|i_p}\mathbf{d}X^{i_1}\wedge\dots\wedge\mathbf{d}X^{i_k}\wedge\mathbf{d}X^{i_p}$. For general forms we choose the letters $a,b,c\cdots$ as counter indices (they refer to the non–exact basis—see below), while the letters $i,j,k\cdots$ are reserved for coordinate indices (they refer to the exact basis). The Hodge dual is denoted by a star and defined in $n-$dimensional space by:

$$*\mathbf{k}^a = \frac{\sqrt{g}\; k^a{}_{i_1\dots i_k}}{(n-k)!}\epsilon^{i_1\dots i_k}{}_{j_{k+1}\dots j_n}\mathbf{d}X^{j_{k+1}}\wedge\dots\wedge\mathbf{d}X^{j_n}\;; \tag{3.1}$$

with $\epsilon^{i_1,\dots,i_k}{}_{j_{k+1},\dots,j_n}$ the Levi-Civita pseudo–tensor. In standard perturbation theory the bilinear metric form is considered as the dynamical variable. In this chapter, we consider the matter model "irrotational dust" and employ a $3+1$ flow–orthogonal foliation of space–time, for which the 4– and 3–metrics read:

$${}^{(4)}\mathbf{g} = -\mathbf{d}t\otimes\mathbf{d}t + {}^{(3)}\mathbf{g} \quad \text{with} \quad {}^{(3)}\mathbf{g} = g_{ij}\,\mathbf{d}X^i\otimes\mathbf{d}X^j\;; \tag{3.2}$$

where X^i are Gaussian normal coordinates, equivalent to the Newtonian Lagrangian coordinates. It results in the $3+1$ equations, equivalent to the ADM equations (Arnowitt–Deser–Misner), composed of six equations of motion and four constraint equations. In this foliation the 4–forms can be restricted to a $t-$parametrisation of three spatial 1–form fields.

The description of the fluid continuum in terms of vector–valued trajectories is impossible if we do not move to a higher dimensional representation since it needs an embedding vector space. To describe the fluid intrinsically, it is necessary to introduce non–exact forms, known as the Cartan spatial coframe fields $\boldsymbol{\eta}^a = \eta^a{}_i \mathbf{d}X^i$, with $a = 1 \cdots 3$, known as the Cartan spatial coframe fields. This approach is called the Cartan (or tetrad or vielbein) formalism [Kasai, 1995]. The Cartan formalism permits to switch between a non–exact basis and the coordinate basis. A key–element is the freedom of choice of the normalisation of the non–exact basis. In order to obtain equations that are closer to the Newtonian ones, we choose the orthogonal basis, as explained in Appendix E. The reader is directed to [Buchert and Ostermann, 2012; Buchert et al., 2013] for additional informations and implications related to this choice. Formally, this means that the metric is decomposed as:

$$^{(3)}\mathbf{g} = G_{ab}\boldsymbol{\eta}^a \otimes \boldsymbol{\eta}^b \quad ; \tag{3.3}$$

where G_{ab} is constant in time: $G_{ab} = G_{ab}(\mathbf{X})$. Note that for exact forms, $\boldsymbol{\eta}^a = \mathbf{d}f^a$, the counter indices become coordinate indices, since the functions f^a can be used to define global coordinates $x^i = f^{a\to i}$. In this case,

$$^{(3)}\mathbf{g} = G_{ij}\mathbf{d}x^i \otimes \mathbf{d}x^j \; ; \tag{3.4}$$

which defines a flat space–time with x^i the global (Eulerian) coordinates.

Whatever the basis choice is, the exact functional for the density is given as in the Newtonian approach: $\varrho J = \varrho_{\mathbf{i}} J_{\mathbf{i}}$, where the index $_{\mathbf{i}}$ marks the initial conditions and J is defined as coefficient function of the $3-$volume form:

$$\begin{aligned} &\mathbf{J} := \frac{\epsilon_{abc}}{6}\boldsymbol{\eta}^a \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c = \frac{\epsilon_{ijk}}{6} J \mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ \Longleftrightarrow \quad &\mathbf{J} := J\mathbf{d}^3X = {}^3\boldsymbol{\eta} \quad ; \quad J = \sqrt{g} \; ; \end{aligned} \tag{3.5}$$

with $\mathbf{d}^3X$ the volume $3-$form of the exact basis, and $g := \det(g_{ij})$.

We first recall the basic systems of equations governing an irrotational dust continuum in the Lagrangian formulation of the Newtonian theory. Then, after presenting Einstein theory formulated in the Lagrangian frame, we list the counterpart of the gravitoelectric equations in the latter theory.

2.2 Newtonian theory

In the Lagrangian picture of self–gravitating fluids a family of trajectories, $x^i = f^i(X^k, t)$, labelled by their Lagrangian coordinates $\{X^i,\ i = 1, 2, 3\}$, is introduced. It furnishes a one–parameter family of diffeomorphisms, parametrised

by the Newtonian time t, between the Eulerian $\{x^i\}$ and Lagrangian coordinates $\{X^i\}$. Regular solutions of the Lagrange–Newton system of equations have to obey four evolution equations. The three components of the trajectory field (also position field) $f^i(X^k, t)$ are the only dynamical variables. Other fields are conceived to be represented as functionals of the trajectory field like the velocity, acceleration, density and vorticity fields, *etc.*,

$$\begin{aligned} v^i &:= \dot{f}^i \quad ; \quad a^i := \ddot{f}^i \quad ; \quad \varrho = \varrho_{\mathbf{i}}(J/J_{\mathbf{i}})^{-1} \; ; \\ \omega^i &:= (\omega_{\mathbf{i}}{}^k f^i{}_{|k})(J/J_{\mathbf{i}})^{-1} \; ; \end{aligned} \tag{3.6}$$

where the over–dot denotes time–derivative along the trajectories. $J/J_{\mathbf{i}} = \det(f^i{}_{|k})$ is the Jacobian of the coordinate transformation, normalised to the initial Jacobian. The spatial derivatives with respect to Lagrangian coordinates are abbreviated by a stroke $|$. The acceleration field a^i is identified with the gravitational field strength g^i, respecting equivalence of inertial and gravitational mass. Once a given field is represented as a functional of the deformation, it can be represented in the Eulerian frame by inserting the inverse of the transformation f^i. Note that $J_{\mathbf{i}}$ can be set to 1 if we require $x^i = X^i$ at initial time. The closed Lagrange–Newton system is defined by the nonlinear gravitational evolution equations ((3.7),(3.8)) for the deformation gradient (see [Ehlers and Buchert, 1997] and chapter II section 2.2.2 equations ((2.18),(2.19))):

$$\delta_{ij}\mathbf{d}\ddot{f}^i \wedge \mathbf{d}f^j = \mathbf{0} \; ; \tag{3.7}$$

$$\frac{1}{2}\epsilon_{ijk}\mathbf{d}\ddot{f}^i \wedge \mathbf{d}f^j \wedge \mathbf{d}f^k = (\Lambda - 4\pi G\varrho)\,\mathbf{d}^3 f \; ; \tag{3.8}$$

with Λ the cosmological constant, G the gravitational constant, $\mathbf{d}^3 f = J\mathbf{d}^3 X$ the 3−volume form. Equation (3.7) corresponds to the Eulerian curl, whereas equation (3.8) corresponds to the Eulerian divergence of the gravitational field strength. In the above equations the exact integral for the density (third equation of (3.6)) has to be inserted to reduce the number of variables. For $\Lambda = 0$ the system does not explicitly contain the Jacobian, provided $J \neq 0$. Regular solutions are characterised by $J > 0$. Since, in the Newtonian theory, the Cartan coframe fields are exact forms, equation (3.5) reads:

$$\begin{aligned} &\mathbf{J} := \frac{\epsilon_{ijk}}{6}\mathbf{d}f^i \wedge \mathbf{d}f^j \wedge \mathbf{d}f^k = \frac{\epsilon_{ijk}}{6} J\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ \Longleftrightarrow \quad &\mathbf{J} := J\mathbf{d}^3 X = \mathbf{d}^3 f \quad ; \quad J = \det(f^k{}_{|i}) \; ; \end{aligned} \tag{3.9}$$

where $\mathbf{d}^3 X$ is the Lagrangian volume 3−form. The coefficients of these equations are equivalent to the usual coefficient equations for the deformation gradient $\mathbf{d}f^i$ in Lagrangian coordinates:

$$\delta_{ij}\ddot{f}^i{}_{|[p}f^j{}_{|q]} = 0 \; ; \tag{3.10}$$

$$\frac{1}{2}\epsilon_{ijk}\epsilon^{pqr}\ddot{f}^{i}{}_{|p}f^{j}{}_{|q}f^{k}{}_{|r} = \Lambda J - 4\pi G\varrho J \ . \tag{3.11}$$

An alternative to express the curl in the Lagrangian approach is:

$$\delta^{kr}\epsilon_{pq[j}\epsilon^{ilm}\ddot{f}_{r]|i}f^{p}{}_{|l}f^{q}{}_{|m} = 0 \ . \tag{3.12}$$

2.3 Einstein theory in the Lagrangian frame

We here formulate the Einstein equations in the Lagrangian frame using Cartan coframe fields. The system of equations we obtain will be called the Lagrange–Einstein system (LES).

With a similar method as previously with ((3.7),(3.8)) , the relativistic equations can be derived. In terms of these Cartan coframe fields and in the orthogonal basis representation, the irrotational dust continuum is assumed and governed by the following evolution and constraint equations:

$$G_{ab}\,\ddot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b} = \mathbf{0} \ ; \tag{3.13}$$

$$\frac{1}{2}\epsilon_{dbc}\left(\dot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b}\wedge\boldsymbol{\eta}^{c}\right)^{\cdot} = \left(-\mathcal{R}^{a}{}_{d} + \left(4\pi G\varrho + \Lambda\right)\delta^{a}{}_{d}\right)J\mathbf{d}^{3}X \ ; \tag{3.14}$$

$$\epsilon_{abc}\,\dot{\boldsymbol{\eta}}^{a}\wedge\dot{\boldsymbol{\eta}}^{b}\wedge\boldsymbol{\eta}^{c} = \left(16\pi G\varrho + 2\Lambda - \mathcal{R}\right)J\mathbf{d}^{3}X \ ; \tag{3.15}$$

$$\epsilon_{abc}\left(\mathbf{d}\dot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b} + \boldsymbol{\omega}^{a}{}_{d}\wedge\dot{\boldsymbol{\eta}}^{d}\wedge\boldsymbol{\eta}^{b}\right) = \mathbf{0} \ ; \tag{3.16}$$

with $\mathcal{R}^{a}{}_{d}$ then Ricci tensor curvature and $\mathcal{R}$ the Ricci scalar curvature. The equations are respectively the symmetry condition (3.13), the equation of motion (or equation of evolution of the fluid expansion tensor) (3.14), the energy constraint (3.15) and the momentum constraints (3.16). They are called the $3+1$ equations and are equivalent to the ADM system of equations obtained thanks to an Hamiltonian formulation of general relativity. The combination of the trace of the equation of motion and the energy constraint straightforwardly leads to the Raychaudhuri equation:

$$\begin{aligned}
&\frac{1}{2}\epsilon_{abc}\ddot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b}\wedge\boldsymbol{\eta}^{c} + \epsilon_{abc}\dot{\boldsymbol{\eta}}^{a}\wedge\dot{\boldsymbol{\eta}}^{b}\wedge\boldsymbol{\eta}^{c} = \left(-\mathcal{R}^{a}{}_{a} + \left(4\pi G\varrho + \Lambda\right)\delta^{a}{}_{a}\right)J\mathbf{d}^{3}X \\
\Longrightarrow\quad &\frac{1}{2}\epsilon_{abc}\ddot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b}\wedge\boldsymbol{\eta}^{c} + \epsilon_{abc}\dot{\boldsymbol{\eta}}^{a}\wedge\dot{\boldsymbol{\eta}}^{b}\wedge\boldsymbol{\eta}^{c} = \\
&\quad \epsilon_{abc}\dot{\boldsymbol{\eta}}^{a}\wedge\dot{\boldsymbol{\eta}}^{b}\wedge\boldsymbol{\eta}^{c} - \left(16\pi G\varrho + 2\Lambda\right)J\mathbf{d}^{3}X + 3\left(4\pi G\varrho + \Lambda\right)J\mathbf{d}^{3}X \\
\Longrightarrow\quad &\frac{1}{2}\epsilon_{abc}\ddot{\boldsymbol{\eta}}^{a}\wedge\boldsymbol{\eta}^{b}\wedge\boldsymbol{\eta}^{c} = \left(\Lambda - 4\pi G\varrho\right)J\mathbf{d}^{3}X \ .
\end{aligned} \tag{3.17}$$

To express the above equations we have implicitly used the Cartan 1–form connection $\omega^{a}{}_{b}$ and the 2–form curvature $\Omega^{a}{}_{b}$ that we do not explicitly need:

$$\boldsymbol{\omega}^{a}{}_{b} = \gamma^{a}{}_{cb}\boldsymbol{\eta}^{c} \ ; \tag{3.18}$$

$$\mathbf{\Omega}^a{}_d = \frac{1}{2}\mathcal{R}^a{}_{bcd}\boldsymbol{\eta}^c \wedge \boldsymbol{\eta}^d \; ; \tag{3.19}$$

with the connection and curvature coefficients in the non–exact basis $\gamma^a{}_{cb}$ and $\mathcal{R}^a{}_{bcd}$. The 3–Ricci tensor can be expressed with the 2–form curvature:

$$\mathcal{R}^a{}_d\, \boldsymbol{\eta}^d \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c = \delta^{db}\mathbf{\Omega}^a{}_d \wedge \boldsymbol{\eta}^c - \delta^{dc}\mathbf{\Omega}^a{}_d \wedge \boldsymbol{\eta}^b \; . \tag{3.20}$$

We use the Hodge star operator to obtain the coefficients of the $3+1$ equations in the exact basis $\mathbf{d}X^i$. Since $\varrho = \varrho_{\mathbf{i}} J^{-1}$, $J \geqslant 0$, we get:

$$\begin{aligned}
*(3.13) \quad &\Longrightarrow \quad G_{ab}\ddot{\eta}^a{}_i\eta^b{}_j\epsilon^{ij}{}_k\mathbf{d}X^k = 0 \\
&\Longrightarrow \quad G_{ab}\,\ddot{\eta}^a_{[i}\eta^b{}_{j]} = 0 \; ; \qquad (3.21)\\
*(3.14) \quad &\Longrightarrow \quad \frac{1}{2}\epsilon_{dbc}\epsilon^{ijk}\left(\dot{\eta}^a{}_i\eta^b{}_j\eta^c{}_k\right)^{\cdot} = \left(-\mathcal{R}^a{}_d + (4\pi G\varrho + \Lambda)\,\delta^a{}_d\right) J \\
&\Longrightarrow \quad \frac{1}{J}\left(J\dot{\eta}^a{}_k e_d{}^k\right)^{\cdot} e_a{}^i\eta^d{}_j = \left(-\mathcal{R}^a{}_d + (4\pi G\varrho + \Lambda)\,\delta^a{}_d\right) e_a{}^i\eta^d{}_j \\
&\Longrightarrow \quad \frac{1}{J}\Big(\dot{J}e_d{}^k\dot{\eta}^a{}_k e_a{}^i\eta^d{}_j + J\left(\dot{e}_d{}^k\dot{\eta}^a{}_k + e_d{}^k\ddot{\eta}^a{}_k\right) e_a{}^i\eta^d{}_j\Big) \\
&\qquad\qquad = \left(-\mathcal{R}^i{}_j + (4\pi G\varrho + \Lambda)\,\delta^i{}_j\right) J \\
&\Longrightarrow \quad \frac{\dot{J}}{J}e_a{}^i\dot{\eta}^a{}_j + \left(\dot{e}_d{}^k\dot{\eta}^a{}_k e_a{}^i\eta^d{}_j + e_d{}^k\ddot{\eta}^a{}_k e_a{}^i\eta^d{}_j + e_d{}^k\dot{\eta}^a{}_k\dot{e}_a{}^i\eta^d{}_j - e_d{}^k\dot{\eta}^a{}_k\dot{e}_a{}^i\eta^d{}_j\right) \\
&\qquad\qquad = \left(-\mathcal{R}^i{}_j + (4\pi G\varrho\Lambda)\,\delta^i_{\mathrm{j}}\right) J \\
&\Longrightarrow \quad \frac{\dot{J}}{J}\Theta^i{}_j + \left(\dot{e}_d{}^k\dot{\eta}^a{}_k e_a{}^i\eta^d{}_j + e_d{}^k\ddot{\eta}^a{}_k e_a{}^i\eta^d{}_j + e_d{}^k\dot{\eta}^a{}_k\dot{e}_a{}^i\eta^d{}_j + e_d{}^k\dot{\eta}^a{}_k e_a{}^i\dot{\eta}^d{}_j\right) \\
&\qquad\qquad = \left(-\mathcal{R}^i{}_j + (4\pi G\varrho\Lambda)\,\delta^i_{\mathrm{j}}\right) J \\
&\Longrightarrow \quad \frac{1}{J}\Big(\dot{J}e_a{}^i\dot{\eta}^a{}_j + J\left(e_d{}^k\dot{\eta}^a{}_k e_a{}^i\eta^d{}_j\right)^{\cdot}\Big) = \left(-\mathcal{R}^i{}_j + (4\pi G\varrho\Lambda)\,\delta^i_{\mathrm{j}}\right) J \\
&\Longrightarrow \quad \frac{1}{J}\Big(\dot{J}e_a{}^i\dot{\eta}^a{}_j + J\left(\dot{e}_a{}^i\dot{\eta}^a{}_j + e_a{}^i\ddot{\eta}^a{}_j\right)\Big) = \left(-\mathcal{R}^i{}_j + (4\pi G\varrho\Lambda)\,\delta^i_{\mathrm{j}}\right) J \\
&\Longrightarrow \quad \frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\left(\dot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l\right)^{\cdot} = -\mathcal{R}^i{}_j + (4\pi G\varrho + \Lambda)\,\delta^i{}_j \; ; \qquad (3.22)\\
*(3.15) \quad &\Longrightarrow \quad \epsilon_{abc}\epsilon^{ijk}\dot{\eta}^a{}_i\dot{\eta}^b{}_j\eta^c{}_k = (16\pi G\varrho + 2\Lambda - \mathcal{R})\, J \\
&\Longrightarrow \quad \frac{1}{2J}\epsilon_{abc}\epsilon^{mjk}\dot{\eta}^a{}_m\dot{\eta}^b{}_j\eta^c{}_k = -\frac{\mathcal{R}}{2} + (8\pi G\varrho + \Lambda) \; ; \qquad (3.23)\\
*(3.16) \quad &\Longrightarrow \quad \left(\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l\right)_{||i} = \left(\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\eta^b{}_k\eta^c{}_l\right)_{|j} \; ; \qquad (3.24)
\end{aligned}$$

where a double stroke $_{||}$ is the covariant derivative with respect to the 3–metric and the symmetric connection. The last equivalence is not easy to show in the general case, but the perturbative equations match well. As for the differential forms system of equations, the trace of the equation of motion and the energy

constraint (3.23) leads to the Raychaudhuri equation:

$$*(3.17) \quad \Longrightarrow \quad \frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_i\eta^b{}_k\eta^c{}_l = \Lambda - 4\pi G\varrho \ . \tag{3.25}$$

The system ((3.21)–(3.24)) consists of 13 equations, where the first corresponds to the symmetry condition (3 equations), the second to the symmetric evolution equations (6 equations), subjected to four constraint equations (1 equation for the energy constraint, and 3 equations for the momentum constraints). Thus, the first nine equations furnish evolution equations for the nine coefficient functions of the three Cartan coframe fields $\boldsymbol{\eta}^a$. Note that the relativistic system requires nine functions, $\eta^a{}_i(X^k, t)$, to be determined, whereas the Newtonian system just requires three functions, $f^i(X^k, t)$. These latter are fully determined by the Lagrange–Newton system.

As we will see in the next section, the first three equations together with Raychaudhuri equation are generated by the spatially projected gravitoelectric part of the Weyl tensor and yield the Lagrangian form of Newton equations in the *Minkowski Restriction* [Buchert and Ostermann, 2012]. For irrotational matter flows, as is assumed throughout this chapter, the first equation of both system ((3.13) and (3.21)) can be replaced by:

$$G_{ab}\,\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = \mathbf{0} \quad ; \quad G_{ab}\,\dot{\eta}^a_{[i}\eta^b{}_{j]} = 0 \ . \tag{3.26}$$

The above system of equations ((3.21)–(3.24)), i.e. the coefficients of the differential forms equations, is equivalent to the results developed in [Buchert and Ostermann, 2012] in a different basis. In the first paper the choice of the orthonormal basis has been made, whereas since the second paper [Buchert et al., 2013] the choice of an orthogonal basis is preferred for reasons of a formally closer Newtonian analogy.

2.4 Gravitoelectric set of equations in the Minkowski Restriction

We here define the *Minkowski Restriction* that will prove useful when we will compare the Lagrangian perturbation theory with the standard one. We will also show that, in the *Minkowski Restriction*, the relativistic and the Newtonian gravitoelectric systems coincide.

2.4.1 Definition of the Minkowski Restriction

Let $\boldsymbol{\eta}^\alpha$ be a set of four 1–form fields indexed by α (Greek letters running in $\{0, 1, 2, 3\}$) in a 4−dimensions manifold. This set of forms $\boldsymbol{\eta}^\alpha$ is said to be *exact*, if there exist functions f^α such that $\boldsymbol{\eta}^\alpha = \mathbf{d}f^\alpha$ (see definition (2.16)), where $\mathbf{d}$ denotes the exterior derivative operator, acting on forms and functions.

The *Minkowski Restriction* (henceforth *MR*) consists in the replacement of the non–integrable coefficients by integrable ones, $\eta^\alpha{}_\nu \rightarrow f^{\alpha\rightarrow\mu}_{|\nu}$, keeping the speed of light c finite. With this restriction, the Cartan coframes become deformation

gradient, and the local tangent spaces all become identical and form the global Minkowski space–time. The *Newtonian limit* can be defined as an *MR* of Einstein theory with the additional property to send c to infinity. In the flow–orthogonal foliation, employed in this chapter, the 4−dimensional coframes reduce to $\boldsymbol{\eta}^\alpha = (\mathbf{d}t, \boldsymbol{\eta}^a)$, and their *MR* reads $\mathbf{d}f^\alpha = (\mathbf{d}t, \mathbf{d}f^{a\to i})$. Note that c is set to 1 throughout this chapter. Inverting the *MR* provides us with a rule to construct relativistic Lagrangian solutions from known Newtonian solutions, as will be detailed in this chapter.

2.4.2 Equivalence of the gravitoelectric equations

As we have seen in section 2.3, the Cartan coframe fields obey the symmetry condition (3.13) and Raychaudhuri equation (3.17). These equations make up the gravitoelectric part of Einstein equations in the orthogonal basis [Buchert and Ostermann, 2012] equations (62-63). The name "gravitoelectric part" is motivated by the fact that these equations are generated by the spatially projected gravitoelectric part of the Weyl tensor. This tensor is an analogy to the Faraday tensor in electromagnetism, and the gravitoelectric part of the Weyl tensor is analog to the electric part of the electromagnetic tensor. In the orthogonal basis we have $J_{\mathbf{i}} = \sqrt{G}$, so $\varrho J = \varrho_{\mathbf{i}}$. We already saw similarities between the gravitoelectric part of the relativistic equations ((3.13),(3.14)) and the Newtonian equations ((3.7),(3.8)). It is possible to go further by considering only the trace part of the relativistic equation (3.13),

$$\begin{aligned}
& \frac{1}{2}\epsilon_{abc}\left(\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c\right)^{\cdot} = \left(-\mathcal{R} + 12\pi G\varrho + 3\Lambda\right) J\mathbf{d}^3X \\
\Longleftrightarrow \quad & \frac{1}{2}\epsilon_{abc}\left(\ddot{\boldsymbol{\eta}}^a \wedge \dot{\boldsymbol{\eta}}^b \wedge \dot{\boldsymbol{\eta}}^c + 2\dot{\boldsymbol{\eta}}^a \wedge \dot{\boldsymbol{\eta}}^b \wedge \boldsymbol{\eta}^c\right) = \\
& \epsilon_{abc}\dot{\boldsymbol{\eta}}^a \wedge \dot{\boldsymbol{\eta}}^b \wedge \boldsymbol{\eta}^c + \left(\Lambda - 4\pi G\varrho\right) J\mathbf{d}^3X \\
\Longleftrightarrow \quad & \frac{1}{2}\epsilon_{abc}\ddot{\boldsymbol{\eta}}^a \wedge \dot{\boldsymbol{\eta}}^b \wedge \dot{\boldsymbol{\eta}}^c = \left(\Lambda - 4\pi G\varrho\right) J\mathbf{d}^3X \ .
\end{aligned} \tag{3.27}$$

This expression leads to a strong result: the gravitoelectric system ((3.13),(3.27)) is formally equivalent to the Newtonian system ((3.7),(3.8)). It is pretty natural to imagine their solutions are also formally equivalent. According to the irrotational flow constraint, implied by the flow–orthogonal foliation of space–time, the first equation (3.13) can be replaced by $G_{ab}\,\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = \mathbf{0}$, indeed the 3+1 foliation triggers a null vorticity and the time derivative of this last relation leads to the symmetric condition (3.13),

$$\left(G_{ab}\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b\right)^{\cdot} = G_{ab}\left(\ddot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b + \dot{\boldsymbol{\eta}}^a \wedge \dot{\boldsymbol{\eta}}^b\right) = G_{ab}\ddot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = 0 \ ; \tag{3.28}$$

because of the antisymmetry of the wedge product $\wedge$ and the symmetry of the initial metric G_{ab}. Nevertheless, we consider the double time–derivative expression

for two reasons: (i) in the Newtonian limit we want to reproduce the field equations, which involve a second time–derivative; (ii) for a general system with vorticity, irrotationality does not hold, whereas the second time–derivative equation does. This equation is always true because of the conservation of the vorticity 2−form, $\boldsymbol{\omega} = G_{ab}\,\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = \boldsymbol{\omega}_{\mathbf{i}}$; for the Newtonian case, $\boldsymbol{\omega} = \mathbf{d}\left(v_i \mathbf{d}X^i\right) = \boldsymbol{\omega}_{\mathbf{i}}$ (see Appendix D). Note, however, that the presence of vorticity will require a $1+3$ threading of space–time (see appendix D for details). A projection of the differential forms system above, using the Hodge star operator, yields the coefficient formulation of the *gravitoelectric system* ((3.21),(3.25)):

$$G_{ab}\ddot{\eta}^a{}_{[i}\eta^b{}_{j]} = 0\ ; \tag{3.29}$$

$$\frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_i\eta^b{}_k\eta^c{}_l = \Lambda J - 4\pi G\varrho J\ . \tag{3.30}$$

Sending the spatial Cartan coframes $\eta^a{}_i$ to exact forms, we obtain the coefficients of the Newtonian deformation gradient $f^a{}_{|i}$ (see [Buchert and Ostermann, 2012] section III.A). The system of equations ((3.29),(3.30)) corresponding to the gravitoelectric part of the Lagrange–Einstein system, is then closed. A consideration of the *MR* for the remaining equations, yielding nontrivial Newtonian analogs, will not be needed in this chapter but will be the subject of a forthcoming work.

Considering only the gravitoelectric equations is not enough to determine the nine functions of the coframe coefficients. The relativistic aspects contained in the gravitomagnetic equations will lead to a richer structure of the solutions and also constraints on these gravitoelectric solutions. To conclude, the Lagrange–Einstein gravitoelectric equations are (up to non–integrability) equivalent to their Newtonian analogs, whereas the gravitomagnetic equations have no trivial Newtonian counterpart.

3 General first order perturbation and solution schemes

3.1 First order perturbation scheme

As in standard perturbation theories and the previous Newtonian chapter, we decompose the perturbed quantity into a Friedmann–Lemaître–Robertson–Walker (FLRW) solution and deviations which are expanded up to a chosen order n of the perturbations. Contrary to the standard perturbation theory we do not perturb the metric but the Cartan coframes locally and can build in a second time a nonlinear metric as a functional of the perturbations. A comparison with the standard perturbations of the metric is furnished in section 4.2,

$$\boldsymbol{\eta}^a(\vec{X},t) = \eta^a{}_i(\vec{X},t)\ \mathbf{d}X^i = a(t)\left(\delta^a{}_i + \sum_n P^{a(n)}_i(\vec{X},t)\right)\mathbf{d}X^i\ . \tag{3.31}$$

We made another choice here, the scale factor is assumed to be only a function of time. Thus this ansatz describes a Universe with the same scale factor everywhere,

it can be an FLRW background for instance. Nevertheless, it is still possible to describe regions with different scale factor and build a Universe with several regions of different scale factor, such as a Swiss Cheese model which is globally Friedmannian but contains several inhomogeneous LTB regions. The local metric coefficients can be calculated from this ansatz:

$$g_{ij}(\vec{X},t) = G_{ab}\ \eta^a{}_i(\vec{X},t)\ \eta^b{}_j(\vec{X},t) \quad ; \quad G_{ij} := g_{ij}(\vec{X},t_{\mathbf{i}})\ . \tag{3.32}$$

We arbitrarily choose to perturb a zero–curvature FLRW model but it is possible to encode a background curvature (constant or not) in the coefficient functions G_{ab}. For details on the orthogonal metric and its link to the orthonormal quantities, the reader is directed to Appendix E.

From now on, we only consider first order deviations:

$$\eta^a{}_i(\vec{X},t) = a(t)\left(\delta^a{}_i + P^a_i(\vec{X},t)\right)\ . \tag{3.33}$$

Injecting the ansatz (3.33) into the metric tensor (3.32), we get:

$$g_{ij}(\vec{X},t) = a^2(t)\left(G_{ij} + 2\,P_{(ij)}(\vec{X},t) + P_{ai}(\vec{X},t)P^a_j(\vec{X},t)\right)\ ; \tag{3.34}$$

where we have defined:

$$P_{ij}(\vec{X},t) := G_{ai}P^a_j(\vec{X},t)\ . \tag{3.35}$$

From here we will generally omit the variables in the metric, coframes and perturbations. The first order resolution for the Cartan coframes allows us to write higher order expressions for the non–perturbed variables and quantities. These evaluated functions can be highly nonlinear and trigger a very complex dynamics unlike other approach limited to a linear metric by their first order resolution. For instance, the generalisation of the Jacobian of the coordinates transformation is written such as,

$$\begin{aligned} J &= \frac{a^3}{6}\epsilon_{abc}\epsilon^{ikl}\left(\delta^a{}_i + P^a_i\right)\left(\delta^b{}_k + P^b_k\right)\left(\delta^c{}_l + P^c_l\right) \\ &= \frac{a^3}{6}\epsilon_{abc}\epsilon^{ikl}\left(\delta^a{}_i\delta^b{}_k\delta^c{}_l + 3P^a_i\delta^b{}_k\delta^c{}_l + 3P^a_iP^b_k\delta^c{}_l + P^a_iP^b_kP^c_k\right) \\ &= a^3\left(1 + P + \frac{1}{2}\left(P^2 - P^i{}_jP^j_i\right) + \frac{1}{6}\left(P^3 + 2P^i{}_jP^j{}_kP^k_i - 3PP^j{}_kP^k_j\right)\right). \end{aligned} \tag{3.36}$$

The aim of the next subsections is to calculate the first order deviation fields. Note that only the zero order metric tensor will appear in the linearised equations. Once the first order deformation solution is computed, we are entitled to inject it into the functional expressions of the other fields, such as the metric tensor above, without truncating to first order the expansion of the functionals.

3.1.1 Homogeneous equations

At zero order in the perturbation field, the metric coefficients and their inverse lead to the flat Friedmannian metric:

$$G^{(0)}_{ij} = \delta_{ij} \quad ; \quad g^{(0)}_{ij} = a^2\delta_{ij} \quad ; \quad g^{ij(0)} = a^{-2}\delta^{ij} \ . \tag{3.37}$$

We remind the reader that the perturbations are defined not through the metric, as in standard perturbation approaches, but through the coframes. At least, the perturbations are built on this flat space and then create an inhomogeneous "background" for the higher order perturbations. As a matter of fact, the physical space in which the perturbations propagate is described by the perturbed metric.

The homogeneous (i.e. zero order) equations read:

$$3\frac{\ddot{a}}{a} = \Lambda - 4\pi G\varrho_{H\mathbf{i}}a^{-3} \ ; \tag{3.38}$$

$$3H^2 = 8\pi G\varrho_{H\mathbf{i}}a^{-3} + \Lambda \ ; \tag{3.39}$$

with the Hubble function $H := \dot{a}/a$. These equations are the well–known expansion and acceleration laws: the flat Friedmann equations. As a consequence of the expression of the orthogonal coframes, the zero order spatial Christoffel symbols and scalar Ricci curvature are trivially zero:

$$\Gamma^{i\ (0)}_{\ jk} = 0 \quad ; \quad \mathcal{R}^{(0)} = 0 \ . \tag{3.40}$$

3.1.2 General initial data setting for the perturbations

We choose initial data in formal correspondence with Newtonian cosmology and generalise these initial fields to the relativistic stage. This has obvious advantages with regard to the aim to give construction rules that translate the known Newtonian solutions to general relativity; see subsection 2.4.1 defining the *Minkowski Restriction*. For the initial data setting in the Newtonian case see [Ehlers and Buchert, 1997].

Let the three 1–form fields $\mathbf{U}^a = U^a_{\ i}\mathbf{d}X^i$ be the initial 1–form generalisation of the Newtonian peculiar–velocity gradient and, accordingly, $\mathbf{W}^a = W^a_{\ i}\mathbf{d}X^i$ the initial 1–form generalisation of the Newtonian peculiar–acceleration gradient. Our solutions will be written in terms of these initial data. We summarise them together with the constraints at the end of this paragraph ((3.50)–(3.52)).

By generalising the Newtonian initial data and using the relativistic trace equation of motion, the initial data for the comoving perturbation read (we denote $P^a_{\ i}(t_i) := \mathcal{P}^a_{\ i}$, and correspondingly for its time–derivatives):

$$\begin{cases} \mathcal{P}^a_{\ i} = 0 \ ; \\ \dot{\mathcal{P}}^a_{\ i} = U^a_{\ i} \ ; \\ \ddot{\mathcal{P}}^a_{\ i} = W^a_{\ i} - 2H_{\mathbf{i}}U^a_{\ i} \ . \end{cases} \tag{3.41}$$

The relation about the initial second time–derivative of the perturbations is a generalisation of the relation obtained for the trace. We assume, without loss of generality [Ehlers and Buchert, 1997], that the initial data are first order. It is possible to show from the first order Raychaudhuri equation (3.25) that the initial density contrast $\delta_{\mathbf{i}} := (\varrho_{\mathbf{i}} - \varrho_{H\mathbf{i}})/\varrho_{H\mathbf{i}}$ satisfies the equality:

$$
\begin{aligned}
-4\pi G\delta\varrho_{\mathbf{i}} a^{-3} &= \frac{a^{-3}}{2}\epsilon_{abc}\epsilon^{ijk}\left[\left(\ddot{a}P^a_{\ i} + 2\dot{a}\dot{P}^a_{\ i} + a\ddot{P}^a_{\ i}\right)a^2\delta^b_{\ j}\delta^c_{\ k} + 2\ddot{a}\delta^a_{\ i}a^2P^b_{\ j}\delta^c_{\ k}\right] \\
&= 3\frac{\ddot{a}}{a}P^a_{\ i}\delta_a^{\ i} + 2H\dot{P}^a_{\ i}\delta_a^{\ i} + \ddot{P}^a_{\ i}\delta_a^{\ i} \\
\Longleftrightarrow \quad -4\pi G\delta\varrho_{\mathbf{i}} &= \ddot{\mathcal{P}}^a_{\ i}\delta_a^{\ i} + 2H_{\mathbf{i}}\dot{\mathcal{P}}^a_{\ i}\delta_a^{\ i} \\
&= W^a_{\ i}\delta_a^{\ i} \\
\Longleftrightarrow \quad \delta\varrho_{\mathbf{i}}^{(1)} = \delta\varrho_{\mathbf{i}} &= \varrho_{H\mathbf{i}}\delta_{\mathbf{i}} = -\frac{1}{4\pi G}\delta^k_{\ a}W^a_{\ k}\,. \qquad (3.42)
\end{aligned}
$$

In view of the flow–orthogonal foliation, we have the irrotationality constraint:

$$
\boldsymbol{\omega} = G_{ab}\dot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = \mathbf{0} \quad \Longrightarrow \quad \delta_{ab}\mathbf{U}^a \wedge \delta^b_{\ j}\mathbf{d}X^j = \mathbf{0}\,. \qquad (3.43)
$$

This implies for the coefficient functions: $U_{[ij]} = 0$.

3.1.3 Relativistic "Poisson equation" and consequences for $\mathbf{W}^a$

In the Newtonian approach the initial peculiar–acceleration and the inhomogeneities are linked by the Poisson equation. In order to generalise this equation to the relativistic case, we note the following relativistic generalisation of the Newtonian field strength gradient that follows from inspection of the Lagrange–Einstein system (for details the reader can always consult [Buchert and Ostermann, 2012]):

$$
\mathcal{F}^i_{\ j} := \dot{\Theta}^i_{\ j} + \Theta^i_{\ k}\Theta^k_{\ j} = -\mathcal{R}^i_{\ j} - \Theta\Theta^i_{\ j} + (4\pi G\varrho + \Lambda)\,\delta^i_{\ j} + \Theta^i_{\ k}\Theta^k_{\ j}\,; \qquad (3.44)
$$

with the 3−Ricci tensor coefficients $\mathcal{R}_{ij}$ whose trace is the Ricci scalar $\mathcal{R}$ and $\Theta^i_{\ j}$ the fluid expansion tensor. According to the energy constraint $\mathcal{R} + \Theta^2 - \Theta^k_{\ l}\Theta^l_{\ k} = 16\pi G\varrho + 2\Lambda$, the symmetry of the fluid expansion tensor and Ricci curvature, it is straightforward to show that the relativistic gravitational field coefficients $\mathcal{F}^i_{\ j}$ respect the following field equations:

$$
\mathcal{F}^k_{\ k} = -\mathcal{R} - \theta^2 + 12\pi G\varrho + 3\Lambda + \Theta^l_{\ k}\Theta^k_{\ l} \quad \Longleftrightarrow \quad \mathcal{F}^k_{\ k} = \Lambda - 4\pi G\varrho\,; \qquad (3.45)
$$

$$
\mathcal{F}_{[ij]} = \Theta_{[ik}\Theta^k_{\ j]} \quad \Longleftrightarrow \quad \mathcal{F}_{[ij]} = 0\,. \qquad (3.46)
$$

In terms of the coframe fields, the relativistic gravitational field can be written as follows:

$$
\mathcal{F}^i_{\ j} = \left(e_a^{\ i}\dot{\eta}^a_{\ j}\right)^{\cdot} + e_a^{\ i}\dot{\eta}^a_{\ k}e_b^{\ k}\dot{\eta}^b_{\ j}
$$

$$
\begin{aligned}
&= \left(e_a{}^i \dot{\eta}^a{}_j\right)^{\cdot} - \dot{e}_a{}^i \dot{\eta}^a{}_j \\
&= \frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l + \frac{1}{J}\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_j\dot{\eta}^b{}_k\eta^c{}_l - \frac{\dot{J}}{2J^2}\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l \\
&\quad - \frac{1}{J}\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_j\dot{\eta}^b{}_k\eta^c{}_l + \frac{\dot{J}}{2J^2}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l \\
&= \frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l \; ;
\end{aligned}
\tag{3.47}
$$

hence, inserting the coframe perturbations and evaluating this expression at initial time, we get the following identity (note that the zero order field trivially satisfies the second constraint):

$$
\begin{aligned}
\mathcal{F}^k{}_k(t_{\mathbf{i}}) &= \Lambda - 4\pi G\varrho_{\mathbf{i}} = \Lambda - 4\pi G\varrho_{H\mathbf{i}}(1+\delta_{\mathbf{i}}) = \Lambda - 4\pi G\varrho_{H\mathbf{i}}(1-\delta^k{}_a W^a{}_k) \; ; \\
\mathcal{F}_{[ij]}(t_{\mathbf{i}}) &= \delta^b{}_{[i}G_{ba}W^a{}_{j]} = W_{[ij]} = 0 \; .
\end{aligned}
\tag{3.48}
$$

Thus, using the zero order Friedmannian relation in the first equation, the deviation 1–form fields $\mathbf{W}^a$ obey the following equations that generalise the Poisson equation for the inhomogeneous deviations off the zero order solution:

$$
\begin{aligned}
{}_*\frac{1}{2}\epsilon_{abc}\mathbf{W}^a \wedge \delta^b{}_j\mathbf{d}X^j \wedge \delta^c{}_k\mathbf{d}X^k = -4\pi G\delta\varrho_{\mathbf{i}} \quad &\Longrightarrow \quad W^i{}_i = -4\pi G\delta\varrho_{\mathbf{i}} \; ; \\
\delta_{ab}\mathbf{W}^a \wedge \delta^b{}_j\mathbf{d}X^j = \mathbf{0} \quad &\Longrightarrow \quad W_{[ij]} = 0 \; ;
\end{aligned}
\tag{3.49}
$$

implying for the coefficient functions: $W_{[ij]} = 0$.

3.1.4 Summary

We summarise the set of initial data, determined by our choice of the basis and subjected to the constraints (recall that we assume, without loss of generality [Ehlers and Buchert, 1997], that the initial data are first order ((3.50)–(3.52)), we decide to drop the index $^{(1)}$ for reasons of clarity). We set some constraints about:

- the generalisations of the acceleration and velocity, and the initial deformation,

$$
\begin{cases}
\mathbf{W}^{a(1)} = \mathbf{W}^a, \; W_{[ij]} = 0 \; ; \\
\mathbf{U}^{a(1)} = \mathbf{U}^a, \; U_{[ij]} = 0 \; ; \\
\forall n \boldsymbol{\mathcal{P}}^{a(n)} = \mathbf{0} \; ;
\end{cases}
\tag{3.50}
$$

- about the initial time derivatives of the deformation,

$$
\begin{cases}
\dot{\mathcal{P}}^a{}_i = U^a{}_i \; ; \\
\ddot{\mathcal{P}}^a{}_i = W^a{}_i - 2H_{\mathbf{i}}U^a{}_i \; .
\end{cases}
\tag{3.51}
$$

- and additional initial constraints are obtained thanks to definition of the metric and constraint equations evaluated at initial time,

$$\begin{cases} G_{ij} = G_{ab}\,\eta^a{}_i(t_{\mathbf{i}})\eta^b{}_j(t_{\mathbf{i}}) = G_{ab}\delta^a{}_i\delta^b{}_j\ ; \\ \dot{a}(t_{\mathbf{i}})\ \dot{\eta}^a{}_m(t_{\mathbf{i}})\delta_a{}^m = -\dfrac{\mathcal{R}_{\mathbf{i}}}{2} + 8\pi G\varrho_{\mathbf{i}} + \Lambda\ ; \\ \left(\dot{\eta}^a{}_j(t_{\mathbf{i}})\delta_a{}^i\right)_{||i} = \left(\dot{\eta}^a{}_i(t_{\mathbf{i}})\delta_a{}^i\right)_{|j}\ ; \end{cases} \tag{3.52}$$

with the initial density contrast $\delta_{\mathbf{i}} := (\varrho_{\mathbf{i}} - \varrho_{H\mathbf{i}})/\varrho_{H\mathbf{i}}$. The metric coefficients are by definition symmetric $G_{[ij]} = 0$.

In the next section, we will express the initial Ricci curvature tensor, $\mathcal{R}^{(1)}_{ij}(t_i) := \mathcal{R}^{(1)}_{ij} = \mathcal{R}_{ij}$, as a function of this set of initial data. We henceforth use the following abbreviations for the trace expressions: $\delta^k{}_a U^a{}_k =: U$, $\delta^k{}_a W^a{}_k =: W$.

3.2 First order equations

In this part, we expand the Lagrange–Einstein system for the Cartan coframes to find the first order deformation fields. Recall that, once we found the solutions, we are entitled to consider all other variables as functional of the deformation, injected at the given order. The strength of this approach lies in taking the definition of the actual curvature, metric and others that is produced by the deformation at first order but we do not truncate the expression to first order. This way we are able to furnish nonlinear approximations that will improve iteratively by going to higher order deformations. Linearising the Lagrange–Einstein system implies that the first order deformation field only "sees" the first order contribution from the curvature.

The deformation coefficients P^a_i only appear summed over the non–coordinate index in the equations, so we introduce the following tensor coefficients and their trace:

$$P^i{}_j := \delta_a{}^i P^a_j \quad ; \quad P := P^k_k = \delta^k{}_a P^a{}_k\ ; \tag{3.53}$$

and use this notation throughout the remaining part of this chapter.

The first order Lagrange–Einstein equations, in the orthogonal basis, read (we omit the index $^{(1)}$ for the deformation field, but keep it for the Ricci curvature that is inserted according to its definition and expanded for linearising the equations):

$$\begin{aligned} (3.21) \quad \Longrightarrow \quad & G_{ab}\left(\dot{a}\delta^a{}_{[i} + \dot{a}P^a{}_{[i} + a\dot{P}^a{}_{[i}\right) a\left(\delta^b{}_{j]} + P^b{}_{j]}\right) \\ & \qquad = G_{ab}\left(\dot{a}\delta^a{}_{[i} + a\dot{P}^a{}_{[i}(t_{\mathbf{i}})\right) a\delta^b{}_{j]} \\ \Longrightarrow \quad & G_{ab}\left(\dot{a}a\delta^a{}_{[i}P^b{}_{j]} + \dot{a}aP^a{}_{[i}\delta^b{}_{j]} + a^2\dot{P}^a{}_{[i}\delta^b{}_{j]}\right) = G_{ab}a^2U^a{}_{[i}\delta^b{}_{j]} \end{aligned}$$

$$\Longrightarrow \quad a^2 G_{a[j} P^a_{\ i]} = a^2 G_{a[j} U^a_{\ i]}$$

$$\Longrightarrow \quad \dot{P}_{[ij]} = U_{[ij]} a^{-2} = 0 \ ; \tag{3.54}$$

(3.22) $$\Longrightarrow \quad \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\Big[\Big(\ddot{a}\delta^a_{\ j} + \ddot{a}P^a_{\ j} + 2\dot{a}\dot{P}^a_{\ j} + a\ddot{P}^a_{\ j}\Big)\, a\left(\delta^b_{\ k} + P^b_{\ k}\right) a\left(\delta^c_{\ l} + P^c_{\ l}\right)$$

$$+2\left(\dot{a}\delta^a_{\ j} + a\dot{P}^a_{\ j} + \dot{a}P^a_{\ j}\right)\left(\dot{a}\delta^a_{\ k} + \dot{a}P^a_{\ k} + a\dot{P}^a_{\ k}\right) a\left(\delta^c_{\ l} + P^c_{\ l}\right)\Big]$$

$$= \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\Big[\Big(2\ddot{a}a^2\delta^a_{\ j}P^b_{\ k}\delta^c_{\ l} + \ddot{a}a^2 P^a_{\ j}\delta^b_{\ k}\delta^c_{\ l} + 2\dot{a}a^2\dot{P}^a_{\ j}\delta^b_{\ k}\delta^c_{\ l} + a^3\ddot{P}^a_{\ j}\delta^b_{\ k}\delta^c_{\ l}\Big)$$

$$+\ 2\left(2\dot{a}^2 a\delta^a_{\ j}P^b_{\ k}\delta^c_{\ l} + \dot{a}a^2\delta^a_{\ j}\dot{P}^b_{\ k}\delta^c_{\ l} + \dot{a}a^2\dot{P}^a_{\ j}\delta^b_{\ k}\delta^c_{\ l} + \dot{a}^2 a P^a_{\ j}\delta^b_{\ k}\delta^c_{\ l}\right)\Big]$$

$$= \ \ddot{a}a^2\left(P\delta^i_{\ j} - P^i_{\ j}\right) + \ddot{a}a^2 P^i_{\ j} + 2\dot{a}a^2\dot{P}^i_{\ j} + a^3\ddot{P}^i_{\ j}$$

$$+\ 2\dot{a}^2 a\left(P\delta^i_{\ j} - P^i_{\ j}\right) + \dot{a}a^2\left(\dot{P}\delta^i_{\ j} - \dot{P}^i_{\ j}\right) + 2\dot{a}a^2\dot{P}^i_{\ j} + 2\dot{a}^2 a P^i_{\ j}$$

$$= -\mathcal{R}^i_{\ j}a^3 + \left(4\pi G\varrho_H + \Lambda\right)\delta^i_{\ j}a^3 P + 4\pi G\delta\varrho a^3$$

$$\Longrightarrow \quad \ddot{P}^i_{\ j} + 3H\dot{P}^i_{\ j} + \left(2H^2 + \frac{\ddot{a}}{a}\right)P\delta^i_{\ j} + H\dot{P}\delta^i_{\ j}$$

$$= -\mathcal{R}^i_{\ j} + \left(4\pi G\varrho_H + \Lambda\right)P\delta^i_{\ j} + 4\pi G\delta\varrho$$

$$\Longrightarrow \quad \ddot{P}^i_{\ j} + 3H\dot{P}^i_{\ j} + H\dot{P}\delta^i_{\ j} = -\mathcal{R}^i_{\ j} + 4\pi G\delta\varrho$$

$$= -\mathcal{R}^i_{\ j} + 4\pi G\delta\varrho_{\mathbf{i}}a^{-3} - 4\pi G\varrho_{H\mathbf{i}}a^{-3}P \ ; \tag{3.55}$$

(3.23) $$\Longrightarrow \quad \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\left(\dot{a}\delta^a_{\ i} + \dot{a}P^a_{\ i} + a\dot{P}^a_{\ i}\right)\left(\dot{a}\delta^b_{\ k} + \dot{a}P^b_{\ k} + a\dot{P}^b_{\ k}\right) a\left(\delta^c_{\ l} + P^c_{\ l}\right)$$

$$= \frac{1}{2}\epsilon_{abc}\epsilon^{ikl}\left(3\dot{a}^2 a P^a_{\ i}\delta^b_{\ k}\delta^c_{\ l} + 2\dot{a}a^2\dot{P}^a_{\ i}\delta^b_{\ k}\delta^c_{\ l}\right)$$

$$\Longleftrightarrow \quad 3\dot{a}^2 aP + 2\dot{a}a^2\dot{P} = -\frac{\mathcal{R}}{2}a^3 + \left(8\pi G\varrho_H + \Lambda\right)a^3 P + 8\pi G\delta\varrho a^3$$

$$\Longrightarrow \quad H\dot{P} + \frac{3}{2}H^2 P = -\frac{\mathcal{R}}{4} + \left(4\pi G\varrho_H + \frac{\Lambda}{2}\right)P + 4\pi G\delta\varrho$$

$$\Longrightarrow \quad H\dot{P} = -\frac{\mathcal{R}}{4} + 4\pi G\delta\varrho_{\mathbf{i}}a^{-3} - 4\pi G\varrho_{H\mathbf{i}}a^{-3}$$

$$\Longrightarrow \quad H\dot{P} + 4\pi G\varrho_{H\mathbf{i}}a^{-3}P = -\frac{\mathcal{R}^{(1)}}{4} - a^{-3}W \ ; \tag{3.56}$$

(3.24) $$\Longleftrightarrow \quad \Theta^i_{\ j||i} = \Theta^i_{\ i||j}$$

$$\Longleftrightarrow \quad \left(e_a^{\ i}\dot{\eta}^a_{\ j}\right)_{||j} = \left(e_a^{\ i}\dot{\eta}^a_{\ i}\right)_{||j}$$

$$\Longrightarrow \quad \left(\dot{a}a^{-1}\delta_a^{\ i}\delta^a_{\ j} + a^{-1}\delta_a^{\ i}\left(\dot{a}P^a_{\ j} + a\dot{P}^a_{\ j}\right) - a^{-1}P^i_{\ a}\dot{a}\delta^a_{\ j}\right)_{||i}$$

$$= \left(\dot{a}a^{-1}\delta_a^{\ i}\delta^a_{\ i} + a^{-1}\dot{a}\delta_a^{\ i}P^a_{\ i} + \delta_a^{\ i}\dot{P}^a_{\ i} - a^{-1}\dot{a}P^i_{\ a}\delta^a_{\ i}\right)_{||j}$$

$$\begin{aligned}
&\Longrightarrow \quad \left(\dot{a}a^{-1}\delta^i{}_j + \dot{P}^i{}_j\right)_{||j} = \left(3\dot{a}a^{-1} + \dot{P}^i{}_i\right)_{|j} \\
&\Longrightarrow \quad \dot{P}^i{}_{i|j} = \dot{P}^i{}_{i|j} \\
&\Longrightarrow \quad \dot{P}^i{}_{[i|j]} = 0 \ . \qquad (3.57)
\end{aligned}$$

The equation of evolution (3.55) can be rewritten thanks to the energy constraint (3.56) such as,

$$\ddot{P}_{ij} + 3H\dot{P}_{ij} = -a^{-2}\left(\mathcal{R}^{(1)}_{ij} - \frac{\mathcal{R}^{(1)}}{4}a^2\delta_{ij}\right) \ . \qquad (3.58)$$

Note the last calculation (3.57) required to know the expression of the first order inverse Cartan coframe $e_a{}^i$,

$$\begin{aligned}
& e_a{}^i\eta^a{}_j = e_a{}^{i(0)}a\delta^a{}_j + e_a{}^{i(1)}a\delta^a{}_j + e_a{}^{i(0)}aP^a{}_j = \delta^i{}_j \\
\Longrightarrow \quad & e_a{}^{i(0)}\delta^a{}_j\delta^j{}_b = a^{-1}\delta^i{}_j\delta^j{}_b \quad ; \quad e_a{}^{i(1)}\delta^a{}_j\delta^j{}_b = -e_a{}^{i(0)}P^a{}_j\delta^j{}_b \\
\Longrightarrow \quad & e_b{}^{i(0)} = a^{-1}\delta^i{}_b \quad ; \quad e_b{}^{i(1)} = -a^{-1}P^i{}_b \ . \qquad (3.59)
\end{aligned}$$

We sum up the first order Lagrange–Einstein system,

$$\dot{P}_{[ij]} = U_{[ij]}a^{-2} = 0 \ ; \qquad (3.60)$$

$$\ddot{P}_{ij} + 3H\dot{P}_{ij} = -a^{-2}\left(\mathcal{R}^{(1)}_{ij} - \frac{\mathcal{R}^{(1)}}{4}a^2\delta_{ij}\right) \ ; \qquad (3.61)$$

$$H\dot{P} + 4\pi G\varrho_{H\mathbf{i}}a^{-3}P = -\frac{\mathcal{R}^{(1)}}{4} - a^{-3}W \ ; \qquad (3.62)$$

$$\dot{P}^i{}_{[i|j]} = 0 \ . \qquad (3.63)$$

Evaluating the first order equation of motion (3.61) at initial time, we can express the initial value of the curvature tensor as a function of the initial data set:

$$\mathcal{R}^{(1)}_{ij}(t_{\mathbf{i}}) =: \mathcal{R}_{ij} = -(W_{ij} + H_{\mathbf{i}}U_{ij}) - \delta_{ij}(W + H_{\mathbf{i}}U) \ . \qquad (3.64)$$

Raychaudhuri equation can be extracted from the previous system of equations:

$$\ddot{P} + 2H\ddot{P} - 4\pi G\varrho_{Hi}a^{-3}P = a^{-3}W \ . \qquad (3.65)$$

A more transparent representation of these equations is obtained by introducing the decomposition of the deformation field into its trace, tracefree symmetric, and antisymmetric parts:

$$P_{ij} = P_{(ij)} + P_{[ij]} = \frac{1}{3}P\delta_{ij} + \Pi_{ij} + \mathfrak{P}_{ij} \ ; \qquad (3.66)$$

where we define $\Pi_{ij} := P_{(ij)} - (1/3)P \cdot G_{ij}$, $\mathfrak{P}_{ij} := P_{[ij]}$, and we introduce the trace–free symmetric part of the Ricci tensor, $\tau^{(1)}_{ij} := \mathcal{R}^{(1)}_{ij} - (1/3)\,\mathcal{R}^{(1)}g_{ij}$. The first order system for the deformation coefficients now reads:

$$(3.60) \quad \Longrightarrow \quad \frac{1}{3}\dot{P}\delta_{[ij]} + \Pi_{[ij]} + \mathfrak{P}_{[ij]} = U_{[ij]}a^{-2}$$

$$\Longrightarrow \quad \mathfrak{P}_{[ij]} = U_{[ij]} a^{-2} \; ; \tag{3.67}$$

$$\begin{aligned}
(3.61) \quad \Longrightarrow \quad & \frac{1}{3}\left(\ddot{P} + 3H\dot{P}\right)\delta_{ij} + \ddot{\Pi}_{ij} + 3H\dot{\Pi}_{ij} + \ddot{\mathfrak{P}}_{ij} + 3H\dot{\mathfrak{P}}_{ij} \\
& = -\, a^{-2}\left(\tau^{(1)}_{ij} + \frac{1}{3}\mathcal{R}^{(1)} a^2 \delta_{ij} - \frac{1}{4}\mathcal{R}^{(1)} a^2 \delta_{ij}\right) \\
\Longrightarrow \quad & \ddot{P} + 3H\dot{P} = -\frac{\mathcal{R}^{(1)}}{4} \quad ; \quad \ddot{\Pi}_{ij} + 3H\dot{\Pi}_{ij} = -a^{-2}\tau^{(1)}_{ij} \; ;
\end{aligned} \tag{3.68}$$

$$(3.62) \quad \Longrightarrow \quad H\dot{P} + 4\pi G \varrho_{H\mathrm{i}} a^{-3} P = -\frac{\mathcal{R}^{(1)}}{4} - a^{-3} W \; ; \tag{3.69}$$

$$\begin{aligned}
(3.63) \quad \Longrightarrow \quad & \frac{1}{2}\dot{P}^{i}{}_{i|j} - \frac{1}{3}\frac{1}{2}\dot{P}^{k}{}_{k|i)}\delta^{i}{}_{j} - \frac{1}{2}\dot{\Pi}^{i}{}_{j|i} \\
\Longrightarrow \quad & \frac{1}{3}\dot{P}^{i}{}_{i|j} - \frac{1}{2}\dot{\Pi}^{i}{}_{j|i} = 0 \; .
\end{aligned} \tag{3.70}$$

In order to solve the first order trace and traceless symmetric equations, it is necessary to express the first order scalar curvature and the traceless Ricci tensor ${}^{(1)}\tau_{ij}$. To do so, we inject the metric and its inverse, truncated to first order:

$$g_{ij} = a^2\left(\delta_{ij} + G^{(1)}_{ij} + 2P_{(ij)}\right) \; ; \tag{3.71}$$

$$g^{ij} = a^{-2}\left(\delta^{ij} - G^{ij(1)} - 2P^{(ij)}\right) \; ; \tag{3.72}$$

into the definitions of the spatial Christoffel symbol $\Gamma^{k}{}_{ij}$ and the spatial Ricci tensor $\mathcal{R}_{ij}$:

$$\begin{aligned}
\Gamma^{k}{}_{ij}{}^{(1)} &= \frac{1}{2} g^{kl(0)}\left(g^{(1)}_{li|j} + g^{(1)}_{lj|i} - g^{(1)}_{ij|l}\right) \\
&= \frac{1}{2}\delta^{kl}\left(G^{(1)}_{li|j} + G^{(1)}_{lj|i} - G^{(1)}_{ij|l}\right) + \delta^{kl}\left(P_{(li)|j} + P_{(lj)|i} - P_{(ij)|l}\right) \; ;
\end{aligned} \tag{3.73}$$

$$\begin{aligned}
\mathcal{R}^{(1)}_{ij} &= \Gamma^{k}{}_{ij|l} - \Gamma^{k}{}_{il|j} + \Gamma^{k}{}_{jm}\Gamma^{m}{}_{il} - \Gamma^{k}{}_{lm}\Gamma^{m}{}_{ij} = \Gamma^{k}{}_{ij|k}{}^{(1)} - \Gamma^{k}{}_{ik|j}{}^{(1)} \\
&= G^{(1)}_{k(i|j)}{}^{|k} - \frac{1}{2}\left(G^{(1)}_{ij|k}{}^{|k} + G^{(1)k}{}_{k|ij}\right) + P_{k(i|j)}{}^{|k} + P_{(ik|j)}{}^{|k} - P_{(ij)|k}{}^{|k} + P^{k}_{i|ij} \\
&= G^{(1)}_{i[k|j]}{}^{|k} + G^{k(1)}_{[j|k]i} + P_{i[k|j]}{}^{|k} + P_{j[k|i]}{}^{|k} \; ;
\end{aligned} \tag{3.74}$$

$$\mathcal{R}^{(1)} = 2a^{-2} G^{l(1)}_{[k|l]}{}^{|k} := a^{-2}\mathcal{R} \; . \tag{3.75}$$

Using the splitting into parts with different symmetries, we express the curvature through the functions P, Π_{ij} and $\mathfrak{P}_{ij}$. The momentum constraints (3.70) allow us to rewrite some terms in the curvature to obtain:

$$\begin{aligned}
\mathcal{R}^{(1)}_{ij} &= \mathcal{R}_{ij} + \frac{1}{2}\left(P_{ik|j}{}^{|k} - P_{ij|k}{}^{|k} + P_{jk|i}{}^{|k} - P_{ji|k}{}^{|k}\right) \\
&= \mathcal{R}_{ij} + \frac{1}{2}\frac{1}{3}\left(P_{|jk}\delta_{i}{}^{k} - P^{|k}_{|k}\delta_{ij} + P_{|ik}\delta_{j}{}^{k} - P^{|k}_{|k}\delta_{ij}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\Pi_{ik|j}{}^{|k}-\Pi_{ij|k}{}^{|k}+\Pi_{jk|i}{}^{|k}-\Pi_{ji|k}{}^{|k}\right)\\
&=\mathcal{R}_{ij}+\frac{1}{3}\left(P_{|ij}-P^{|k}{}_{|k}\delta_{ij}\right)+\frac{1}{2}\left(\Pi_{ik|j}{}^{|k}+\Pi_{jk|i}{}^{|k}-2\Pi_{ji|k}{}^{|k}\right)\\
&=\mathcal{R}_{ij}+\frac{1}{3}\left(P_{|ij}-P^{|k}{}_{|k}\delta_{ij}\right)+\frac{2}{3}P^{k}{}_{k|ij}-\Pi_{ji|k}{}^{|k}\\
&=\mathcal{R}_{ij}+P_{|ij}-\frac{1}{3}P^{|k}{}_{|k}\delta_{ij}-\Pi_{ij}{}^{|k}{}_{|k}\,. \qquad (3.76)
\end{aligned}
$$

We can now write the first order traceless part of the Ricci curvature tensor as:

$$
\tau^{(1)}_{ij}=\mathcal{R}^{(1)}_{ij}-\frac{1}{3}\mathcal{R}^{(1)}\delta_{ij}=\mathcal{T}_{ij}+P_{|ij}-\frac{1}{3}P^{|k}{}_{|k}\delta_{ij}-\Pi_{ij}{}^{|k}{}_{|k}\;; \qquad (3.77)
$$

where we defined $\mathcal{T}_{ij}:=\tau_{ij}(t_i):=\mathcal{R}_{ij}-(1/3)\,\mathcal{R}\delta_{ij}$.

Now that we have the first order scalar curvature and the traceless Ricci tensor, we can inject them into the first order system for the deformation coefficients ((3.60)–(3.63)). We also perform the time–integration of the antisymmetric part (3.60) and the momentum constraints (3.63), and now make use of the constraints on initial data ((3.50)–(3.52)):

$$
\mathfrak{P}_{ij}=\mathfrak{P}_{ij}(t_{\mathbf{i}})\quad;\quad \mathfrak{P}_{ij}(t_{\mathbf{i}})=0\;; \qquad (3.78)
$$

$$
\ddot{P}+3H\dot{P}=-\frac{a^{-2}\mathcal{R}}{4}\;; \qquad (3.79)
$$

$$
\ddot{\Pi}_{ij}+3H\dot{\Pi}_{ij}-a^{-2}\Pi_{ij}{}^{|k}{}_{|k}=-a^{-2}\left(\mathcal{T}_{ij}+P_{|ij}-\frac{1}{3}P^{|k}{}_{|k}\delta_{ij}\right)\;; \qquad (3.80)
$$

$$
H\dot{P}+4\pi G\varrho_{Hi}a^{-3}P=-\frac{a^{-2}\mathcal{R}}{4}-a^{-3}W\;; \qquad (3.81)
$$

$$
\frac{2}{3}P_{|j}=\Pi^{i}{}_{j|i}\,. \qquad (3.82)
$$

We remark that the evolution equation for the tracefree symmetric part containing gravitational waves is sourced by the scalar perturbations. Physically this is expected, since the scalar perturbations are linked to the matter source and gravitational waves are scattered at the sources. Further manipulations of this system of equations will aim at formally separating the gravitational wave tensor part from the scalar perturbations.

3.3 Solution building

3.3.1 Space–time splitting

Due to the linearity of the system, we make a superposition ansatz for the separable part of the solutions, it reads:

$$\begin{cases} \mathfrak{P}_{ij} = 0 \; ; \\ \quad P = {}^{\mathbf{1}}C(X^i)\ {}^{\mathbf{1}}\xi(t) + {}^{\mathbf{2}}C(X^i)\ {}^{\mathbf{2}}\xi(t) + {}^{\mathbf{p}}C(X^i)\ {}^{\mathbf{p}}\xi(t) \; ; \\ \Pi^E_{ij} = {}^{\mathbf{1}}C_{ij}(X^i)\ {}^{\mathbf{1}}\xi^S(t) + {}^{\mathbf{2}}C_{ij}(X^i)\ {}^{\mathbf{2}}\xi^S(t) + {}^{\mathbf{p}}C_{ij}(X^i)\ {}^{\mathbf{p}}\xi^S(t) \; . \end{cases} \tag{3.83}$$

Here we note $\Pi_{ij} = \Pi^E_{ij} + \Pi^H_{ij}$, ^{E} denotes the separable modes and ^{H} the non–separable modes. The non-separable solution Π^H_{ij} is currently studied and should be released in [Alles et al., 2014]. Nevertheless, we have to be very careful: the symmetric traceless equation is a wave equation. The associated solution should be a sum of wave modes, at least the homogeneous part of the solution. In reality, this solution should be a discrete sum or an integral of separable models. This will be discussed in section 3.3.3.

The terms of the previous decomposition verify the following ordinary differential equations. For the trace part of the solution,

$$\begin{cases} {}^{\mathbf{h}}\ddot{\xi} + 2H\ {}^{\mathbf{h}}\dot{\xi} - 4\pi G\varrho_{H\mathbf{i}}a^{-3\mathbf{h}}\xi = 0 \; ; \\ {}^{\mathbf{p}}C\left({}^{\mathbf{p}}\ddot{\xi} + 2H\ {}^{\mathbf{p}}\dot{\xi} - 4\pi G\varrho_{H\mathbf{i}}a^{-3\mathbf{p}}\xi\right) = a^{-3}W \; ; \end{cases} \tag{3.84}$$

and for the symmetric traceless part of the perturbations,

$$\begin{cases} {}^{\mathbf{h}}C_{ij}\left({}^{\mathbf{h}}\ddot{\xi}^S + 3H\ {}^{\mathbf{h}}\dot{\xi}^S\right) - a^{-2}\ {}^{\mathbf{h}}C_{ij}{}^{|k}_{\ |k}\ {}^{\mathbf{h}}\xi^S = 0 \; ; \\ {}^{\mathbf{p}}C_{ij}\left({}^{\mathbf{p}}\ddot{\xi}^S + 3H\ {}^{\mathbf{p}}\dot{\xi}^S\right) - a^{-2}\ {}^{\mathbf{p}}C_{ij}{}^{|k}_{\ |k}\ {}^{\mathbf{p}}\xi^S = -a^{-2}\left(\mathcal{T}_{ij} + {}^{\boldsymbol{\alpha}}\xi\left({}^{\boldsymbol{\alpha}}C_{|ij} - \frac{1}{3}{}^{\boldsymbol{\alpha}}C^{|k}_{\ |k}G^{(0)}_{ij}\right)\right). \end{cases} \tag{3.85}$$

The index ${}^{\mathbf{h}}$ denotes the homogeneous solutions, ${}^{\mathbf{p}}$ the particular solutions and ${}^{\boldsymbol{\alpha}}$ both of them. We have to be careful, we work with local coordinates X^i, and a locally flat manifold so a simple Laplacian can describe the local dynamics. Nevertheless, the Fourier transform is built thanks to an integral over the whole space so cannot be defined (see below for further explanations). An important difference here is the presence of a wave equation: the equation of evolution of the symmetric part of the perturbations. Thus the symmetric solution is a superposition of propagating modes. It is already possible to extract informations from these equations. The ${}^{\mathbf{p}}\xi(t)$ term and the source of the second equation (the particular trace equation) have the same time dependent coefficient a^{-3}. Thus a constant ${}^{\mathbf{p}}\xi(t) = const.$ is an obvious solution.

3.3.2 Initial data evaluation

We are now going to determine the spatial coefficient functions in term of the restricted initial data.

The initial data for the antisymmetric part have been defined earlier and are:

$$\mathfrak{P}_{ij}(t_i) = 0 \quad ; \quad U_{[ij]} = 0 \quad ; \quad W_{[ij]} = 0 \, . \tag{3.86}$$

The initial trace solution, more specifically their spatial coefficients, and the initial data are related as follows:

$$\begin{cases} {}^{1}C\ {}^{1}\xi_{\mathbf{i}} + {}^{2}C\ {}^{2}\xi_{\mathbf{i}} + {}^{\mathrm{p}}C = 0 \\ {}^{1}C\ {}^{1}\dot{\xi}_{\mathbf{i}} + {}^{2}C\ {}^{2}\dot{\xi}_{\mathbf{i}} = U \\ {}^{1}C\ {}^{1}\ddot{\xi}_{\mathbf{i}} + {}^{2}C\ {}^{2}\ddot{\xi}_{\mathbf{i}} = W - 2H_{\mathbf{i}}U \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} {}^{1}C\ {}^{1}\xi_{\mathbf{i}} + {}^{2}C\ {}^{2}\xi_{\mathbf{i}} + {}^{\mathrm{p}}C = 0 \\ {}^{1}C\ {}^{1}\dot{\xi}_{\mathbf{i}} + {}^{2}C\ {}^{2}\dot{\xi}_{\mathbf{i}} = U \\ {}^{2}C \left({}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}} \right) = \left(W - 2H_{\mathbf{i}}U\right) {}^{1}\dot{\xi}_{\mathbf{i}} - U\ {}^{1}\ddot{\xi}_{\mathbf{i}} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} {}^{1}C\ {}^{1}\xi_{\mathbf{i}} + {}^{2}C\ {}^{2}\xi_{\mathbf{i}} + {}^{\mathrm{p}}C = 0 \\ {}^{1}C\ {}^{1}\dot{\xi}_{\mathbf{i}} + \dfrac{\left(W - 2H_{\mathbf{i}}U\right) {}^{1}\dot{\xi}_{\mathbf{i}} - U\ {}^{1}\ddot{\xi}_{\mathbf{i}}}{\left({}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}} \right)} {}^{2}\dot{\xi}_{\mathbf{i}} = U \dfrac{{}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}}}{{}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}}} \\ {}^{2}C = \dfrac{\left(W - 2H_{\mathbf{i}}U\right) {}^{1}\dot{\xi}_{\mathbf{i}} - U\ {}^{1}\ddot{\xi}_{\mathbf{i}}}{\left({}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}} \right)} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} {}^{1}C\ {}^{1}\xi_{\mathbf{i}} + {}^{2}C\ {}^{2}\xi_{\mathbf{i}} + {}^{\mathrm{p}}C = 0 \\ {}^{1}C\ {}^{1}\dot{\xi}_{\mathbf{i}} = -\dfrac{\left(W - 2H_{\mathbf{i}}U\right) {}^{1}\dot{\xi}_{\mathbf{i}}\ {}^{2}\dot{\xi}_{\mathbf{i}} - U\ {}^{1}\ddot{\xi}_{\mathbf{i}}\ {}^{2}\dot{\xi}_{\mathbf{i}} - U\ {}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} + U\ {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}}}{\left({}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} - {}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}} \right)} \\ {}^{2}C = \dfrac{-W\ {}^{1}\dot{\xi}_{\mathbf{i}} + \left({}^{1}\ddot{\xi}_{\mathbf{i}} + 2H_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} \right) U}{{}^{2}\dot{\xi}_{\mathbf{i}}\ {}^{1}\ddot{\xi}_{\mathbf{i}} - {}^{2}\ddot{\xi}_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}}} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} {}^{1}C = \dfrac{{}^{2}\dot{\xi}_{\mathbf{i}} W - \left(2H_{\mathbf{i}}\ {}^{2}\dot{\xi}_{\mathbf{i}} - {}^{2}\ddot{\xi}_{\mathbf{i}} \right) U}{{}^{1}\ddot{\xi}_{\mathbf{i}}\ {}^{2}\dot{\xi}_{\mathbf{i}} - {}^{1}\dot{\xi}_{\mathbf{i}}\ {}^{2}\ddot{\xi}_{\mathbf{i}}} \ ; \\ {}^{2}C = \dfrac{-{}^{1}\dot{\xi}_{\mathbf{i}} W + \left({}^{1}\ddot{\xi}_{\mathbf{i}} + 2H_{\mathbf{i}}\ {}^{1}\dot{\xi}_{\mathbf{i}} \right) U}{{}^{1}\ddot{\xi}_{\mathbf{i}}\ {}^{2}\dot{\xi}_{\mathbf{i}} - {}^{1}\dot{\xi}_{\mathbf{i}}\ {}^{2}\ddot{\xi}_{\mathbf{i}}} \ ; \\ {}^{\mathrm{p}}C = -{}^{1}C\ {}^{1}\xi_{\mathbf{i}} - {}^{2}C\ {}^{2}\xi_{\mathbf{i}} \, . \end{cases} \tag{3.87}$$

It requires the condition ${}^{\mathbf{1}}\ddot{\xi}_{\mathbf{i}}\ {}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}} - {}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}\ {}^{\mathbf{2}}\ddot{\xi}_{\mathbf{i}} \neq 0$. For the remaining traceless symmetric coefficient functions it is useful to define the following quantity in order to express the spatial coefficients with respect to the initial data:

$$\Delta^{\boldsymbol{\alpha}}_{qr} := {}^{\mathbf{q}}\dot{\xi}_{\mathbf{i}}{}^{\boldsymbol{\alpha}}\ {}^{\mathbf{r}}\xi_{\mathbf{i}}{}^{\boldsymbol{\alpha}} - {}^{\mathbf{q}}\xi_{\mathbf{i}}{}^{\boldsymbol{\alpha}}\ {}^{\mathbf{r}}\dot{\xi}_{\mathbf{i}}{}^{\boldsymbol{\alpha}}\ .$$

The symmetric trace free part is built as a superposition of independent modes, because the symmetric equation is a wave equation. So each mode has to respect the relations:

$$\begin{cases} {}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{p}}C^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega} = 0 \\ {}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{p}}C^{\omega}_{ij}\ {}^{\mathbf{p}}\dot{\xi}_{\mathbf{i}}{}^{\omega} = U^{\omega}_{ij} \\ {}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\ddot{\xi}_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\ddot{\xi}_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{p}}C^{\omega}_{ij}\ {}^{\mathbf{p}}\ddot{\xi}_{\mathbf{i}}{}^{\omega} = -\mathcal{T}^{\omega}_{ij} - 3H_{\mathbf{i}}U^{\omega}_{ij} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} {}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{p}}C^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega} = 0 \\ {}^{\mathbf{1}}C^{\omega}_{ij}\ \Delta^{\omega}_{1p} + {}^{\mathbf{2}}C^{\omega}_{ij}\ \Delta^{\omega}_{2p} = U^{\omega}_{ij}{}^{\mathbf{p}}\xi_{\mathbf{i}} \\ {}^{\mathbf{1}}C^{\omega}_{ij}\ \dot{\Delta}^{\omega}_{1p} + {}^{\mathbf{p}}C_{ij}\ \dot{\Delta}^{\omega}_{2p} = -\left(\mathcal{T}^{\omega}_{ij} + 3H_{\mathbf{i}}U^{\omega}_{ij}\right){}^{\mathbf{p}}\xi_{\mathbf{i}} \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} {}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\xi_{\mathbf{i}}{}^{\omega} + {}^{\mathbf{p}}C^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega} = 0 \\ \dfrac{-\left(\mathcal{T}^{\omega}_{ij} + 3H_{\mathbf{i}}U^{\omega}_{ij}\right){}^{\mathbf{p}}\xi_{\mathbf{i}}\ \Delta^{\omega}_{2p} - U^{\omega}_{ij}{}^{\mathbf{p}}\xi_{\mathbf{i}}\ \dot{\Delta}^{\omega}_{2p}}{\dot{\Delta}^{\omega}_{1p}\ \Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\ \dot{\Delta}^{\omega}_{2p}}\Delta^{\omega}_{1p} + {}^{\mathbf{2}}C^{\omega}_{ij}\ \Delta^{\omega}_{2p} = U^{\omega}_{ij}\ {}^{\mathbf{2}}\xi_{\mathbf{i}} \\ {}^{\mathbf{1}}C^{\omega}_{ij}\left(\dot{\Delta}^{\omega}_{1p}\ \Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\ \dot{\Delta}^{\omega}_{2p}\right) = -\left(\mathcal{T}^{\omega}_{ij} + 3H_{\mathbf{i}}U^{\omega}_{ij}\right){}^{\mathbf{p}}\xi_{\mathbf{i}}\ \Delta^{\omega}_{2p} - U^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}\ \dot{\Delta}^{\omega}_{2p} \end{cases}$$

$$\Longrightarrow \quad \begin{cases} {}^{\mathbf{1}}C^{\omega}_{ij} = -\dfrac{\mathcal{T}^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\Delta^{\omega}_{2p} + U^{\omega}_{(ij)}{}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\left(3H_{\mathbf{i}}\Delta^{\omega}_{2p} + \dot{\Delta}^{\omega}_{2p}\right)}{\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p}}\ ; \\ {}^{\mathbf{2}}C^{\omega}_{ij} = \dfrac{\mathcal{T}^{\omega}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\Delta^{\omega}_{1p} + U^{\omega}_{(ij)}{}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\left(3H_{\mathbf{i}}\Delta^{\omega}_{1p} + \dot{\Delta}^{\omega}_{1p}\right)}{\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p}}\ ; \\ {}^{\mathbf{p}}C^{\omega}_{ij} = -{}^{\mathbf{1}}C^{\omega}_{ij}\ {}^{\mathbf{1}}\xi_{\mathbf{i}}{}^{\omega} - {}^{\mathbf{2}}C^{\omega}_{ij}\ {}^{\mathbf{2}}\xi_{\mathbf{i}}{}^{\omega}\ ; \end{cases} \tag{3.88}$$

with the following condition which has to be fulfilled: $\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p} \neq 0$. Finally, the constrained initial data lead to the following relations for the trace and tracefree symmetric part of the Ricci curvature. The first one is given by the initial evaluation of the energy constraint which can be identified as the definition of the Ricci scalar, and the second is given by the definition of the traceless Ricci tensor:

$$\frac{-\mathcal{R}}{4} = H_{\mathbf{i}}\left({}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}{}^{\mathbf{1}}C + {}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}C\right) + W = H_{\mathbf{i}}U\frac{{}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\ddot{\xi}_{\mathbf{i}} + {}^{\mathbf{1}}\ddot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}}}{{}^{\mathbf{1}}\ddot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}} - {}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\ddot{\xi}_{\mathbf{i}}} + W; \tag{3.89}$$

$$\mathcal{T}_{ij} = G^{(1)}_{i[k|j]}{}^{|k} + G^{(1)k}{}_{[j|k]i} - \frac{2}{3}G^{(1)l}{}_{[k|l]}{}^{|k}\delta_{ij}$$

$$= \frac{1}{2\pi G \varrho_{H\mathbf{i}}} \left(W_{i[k|j]}{}^{|k} + W^{k}{}_{[j|k]i} - \frac{2}{3} W^{l}{}_{[k|l]}{}^{|k} \delta_{ij} \right) . \tag{3.90}$$

The last equality is obtained according to the equivalence between orthonormal and orthogonal basis (see appendix F).

3.3.3 Example: solutions at an EdS background

We are going to examine the example of the Einstein–de Sitter (EdS) model of Universe, it corresponds to an FLRW model with an energy density equal to the critical density. The relation between the age of the universe and its expansion is $t_{\mathbf{i}} = 2/(3H_{\mathbf{i}})$, then the expression of the scalar factor of the Universe is $a(t) = (t/t_{\mathbf{i}})^{2/3}$. This specific solution leads to the following expressions for the perturbations:

$$P = {}^{\mathbf{1}}C \left(\frac{t}{t_{\mathbf{i}}}\right)^{2/3} + {}^{\mathbf{2}}C \left(\frac{t}{t_{\mathbf{i}}}\right)^{-1} + {}^{\mathbf{p}}C \; ; \tag{3.91}$$

$$\Pi^{E}_{ij} = {}^{\mathbf{1}}C_{ij} \left(\frac{t}{t_{\mathbf{i}}}\right)^{2/3} + {}^{\mathbf{2}}C_{ij} \left(\frac{t}{t_{\mathbf{i}}}\right)^{-1} + {}^{\mathbf{p}}C_{ij} \; ; \tag{3.92}$$

$$\mathfrak{P}_{ij} = 0 \; ; \tag{3.93}$$

The trace solution is straightforward according to its formal equivalence with Newtonian equation. There exists a part of the solution, which has the same time–dependence as the trace solution, but with the trace-free coefficients: it corresponds of what we call Π^{E}_{ij}, the scalar part of the standard perturbation theory (the expression is derived in chapter IV). The antisymmetric part is a constant thanks to the constraints. Other modes seem to exist,

$$\begin{aligned} {}^{\mathbf{h}}\Pi^{H}_{ij}(\omega) =& A_{ij}(\omega) \, \frac{t^{-1} t_{\mathbf{i}}{}^{-2}}{\omega^3} \Big[\sin\left(3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega - k^l X_l\right) \\ & \qquad - 3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega \cos\left(3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega - k^l X_l\right) \Big] \\ & + B_{ij}(\omega) \, \frac{t^{-1} t_{\mathbf{i}}{}^{-2}}{\omega^3} \Big[\cos\left(3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega - k^l X_l\right) \\ & \qquad + 3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega \sin\left(3 t^{1/3} t_{\mathbf{i}}{}^{2/3} \omega - k^l X_l\right) \Big] \; ; \end{aligned} \tag{3.94}$$

where A_{ij} and B_{ij} are space coefficients which are mode–dependent, X^l the position and k^l the associated wave vector. The symmetric homogeneous solution, Π_{ij}, is obtained thanks to the homogeneous equation $\ddot{\Pi}_{ij} + 2H\dot{\Pi}_{ij} - a^{-2}\Pi_{ij}{}^{|k}_{|k} = 0$. A space–time splitting $\Pi_{ij} = \xi(t) C_{ij}(\vec{X})$ is assumed to solve this equation and reads $a^2(\ddot{\xi} + 3H\dot{\xi})/\xi = c_l \left(C_{ij}{}^{|k}_{|k} / C_{ij} \right) = -\omega^2$ with a positive constant ω. Thus the symmetric homogeneous solution is a combination of modes which depend on the angular frequency ω (which can be related to velocity of the wave c_l and the wave

vector k by the following relation $c_l = \omega/k$, in the case of the gravitational waves the velocity c_l is the light velocity). Each of these modes admits a space–time split. This solution is a candidate for a monochromatic wave and corresponds to the part we call Π^H_{ij}. The complete particular solution for the symmetric traceless equation is the superposition of different modes, it depends on the topology of the space–time considered: it can be a discrete or a continuous summation. Moreover, both space and time parts of a mode are sinusoidal functions and can be gathered in a single function. This symmetric solution includes only separable modes, as mentioned earlier additional developments are in progress to deal with the non–separable mode and should be available in the incoming paper [Alles et al., 2014]. In an EdS space–time, the initial trace data read,

$$\begin{cases} {}^{\mathbf{1}}C = \dfrac{3}{5}Ut_{\mathbf{i}} + \dfrac{9}{10}Wt_{\mathbf{i}}{}^2 \ ; \\ {}^{\mathbf{2}}C = -\dfrac{3}{5}Ut_{\mathbf{i}} + \dfrac{3}{5}Wt_{\mathbf{i}}{}^2 \ ; \\ {}^{\mathbf{p}}C = -\dfrac{3}{2}Wt_{\mathbf{i}}{}^2 \ . \end{cases} \tag{3.95}$$

Moreover, the required condition ${}^{\mathbf{1}}\dot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\ddot{\xi}_{\mathbf{i}} - {}^{\mathbf{1}}\ddot{\xi}_{\mathbf{i}}{}^{\mathbf{2}}\dot{\xi}_{\mathbf{i}} = -(2/3)\ t_{\mathbf{i}}{}^{-3} \neq 0$ is verified. From equation (3.89) we can also find the scalar curvature in the EdS case to be:

$$\frac{-\mathcal{R}}{4} = -\frac{1}{2}G^{l(1)\ |k}_{[k|l]} = W + \frac{2}{3t_i}U \ . \tag{3.96}$$

The traceless initial data are,

$$\begin{cases} {}^{\mathbf{1}}C_{ij} = -\dfrac{\mathcal{T}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\Delta^{\omega}_{2p} + U_{(ij)}{}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\left(3H_{\mathbf{i}}\Delta^{\omega}_{2p} + \dot{\Delta}^{\omega}_{2p}\right)}{\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p}} \ ; \\ {}^{\mathbf{2}}C_{ij} = \dfrac{\mathcal{T}_{ij}\ {}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\Delta^{\omega}_{1p} + U_{(ij)}{}^{\mathbf{p}}\xi_{\mathbf{i}}{}^{\omega}\left(3H_{\mathbf{i}}\Delta^{\omega}_{1p} + \dot{\Delta}^{\omega}_{1p}\right)}{\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p}} \ ; \\ {}^{\mathbf{p}}C_{ij} = -{}^{\mathbf{1}}C_{ij}{}^{\mathbf{1}}\xi_{\mathbf{i}}{}^{\omega} - {}^{\mathbf{2}}C_{ij}{}^{\mathbf{2}}\xi_{\mathbf{i}}{}^{\omega} \ ; \end{cases} \tag{3.97}$$

The following conditions has to be fulfilled: $\dot{\Delta}^{\omega}_{1p}\Delta^{\omega}_{2p} - \Delta^{\omega}_{1p}\dot{\Delta}^{\omega}_{2p} \neq 0$. Currently, we have only a homogeneous symmetric solution. But in chapter IV section 4 we propose a solution of the complete equation thanks to a generalisation of the Newtonian solutions under some restrictions.

4 Comparison with other works

The aim of this section is to build a formalism for gravitational wave propagation in the Lagrangian description. We will then compare our approach to the one developed in other works in the comoving synchronous gauge, which is the closest to our approach.

4.1 Solving the first order Lagrange–Einstein system in terms of scalar and tensor modes

Complementary to our solutions derived in the previous section, we here consider a decomposition of the linearised Lagrangian equations into scalar and tensor modes. We do this for the sake of comparison with other works in the literature, and to elucidate the physics of propagation of gravitational waves. We will then be able to straightforwardly compare our results to the others [Matarrese et al., 1998]. Inspired by the standard perturbation theory written in the synchronous comoving gauge and neglecting the vector modes with respect the scalar and tensor modes, we express the comoving metric perturbation (3.71) as follows:

$$G^{(1)}_{ij} + 2P_{(ij)} = -2\phi^{(1)}_{\mathrm{S}}\delta_{ij} + \mathrm{D}_{ij}\chi^{(1)\parallel}_{\mathrm{S}} + \chi^{(1)\top}_{ij} \ ; \tag{3.98}$$

with the traceless derivative $D_{ij} := \partial_i\partial_j - (1/3)\,\Delta\delta_{ij}$. $\phi^{(1)}_{\mathrm{S}}$ and $\chi^{(1)\parallel}_{\mathrm{S}}$ are the two scalar fields, the symbol $^{\parallel}$ denotes a function whose curl vanishes. $\chi^{(1)\top}_{ij}$ is the symmetric traceless tensor field that will encode the propagation of gravitational waves, the symbol $^{\top}$ is used to denote a function whose divergence vanishes ($\partial^i\chi^{(1)\top}_{ij} = 0$). They are related to our previously used variables in the following way:

$$P = -3\,\phi^{(1)}_{\mathrm{S}} - \frac{1}{2}G^{(1)} \ ; \tag{3.99}$$

$$\Pi_{ij} = \frac{1}{2}\left(\mathrm{D}_{ij}\chi^{(1)\parallel}_{\mathrm{S}} + \chi^{(1)\top}_{ij} - G^{(1)tl}_{ij}\right) \ ; \tag{3.100}$$

$G^{(1)tl}_{ij}$ being the traceless part of $G^{(1)}_{ij}$. We do not have so–called vector modes, since our foliation and constraints trigger a comoving synchronous gauge where these modes are set to zero. In the comoving local coordinate system (in the tangent space at a point in the Riemannian manifold) there exist no vector fields stricto sensu but, correspondingly, an antisymmetric tensor part, which vanishes for the irrotational case.

We are now looking for a reformulation of the linearised system in terms of these modes extracted from the Lagrangian approach. In the first subsection, we will deal with the scalar mode of the solutions, whereas in the second subsection, we will treat the tensorial model of the solutions.

4.1.1 Scalar equations and solutions

Combining the trace of the evolution equation (3.79) with the energy constraint (3.81) we get the Raychaudhuri equation in terms of $\phi^{(1)}_{\mathrm{S}}$:

$$\ddot{\phi}^{(1)}_{\mathrm{S}} + 2H\dot{\phi}^{(1)}_{\mathrm{S}} - 4\pi G\varrho_{H\mathbf{i}}a^{-3}\phi^{(1)}_{\mathrm{S}} = \frac{1}{a^3}\left(W - 2\pi G\varrho_{H\mathbf{i}}G^{(1)}\right). \tag{3.101}$$

The momentum constraints (3.82) can be written in terms of the scalar potentials:

$$\left(6\,\dot{\phi}^{(1)}_{\mathrm{S}} + \Delta\dot{\chi}^{(1)\parallel}_{\mathrm{S}}\right)_{|i} = 0 \ . \tag{3.102}$$

The traceless part of the evolution equation gives an equation for $\chi_{\rm S}^{(1)\parallel}$:

$$\mathrm{D}_{ij}\left(\ddot{\chi}_{\rm S}^{(1)\parallel}+3H\dot{\chi}_{\rm S}^{(1)\parallel}-\frac{1}{a^2}\left[\Delta\chi_{\rm S}^{(1)\parallel}(t_{\mathbf{i}})+6\phi_{\rm S}^{(1)}(t_{\mathbf{i}})\right]\right)=0\ . \tag{3.103}$$

Combining the time–derivative of the last equation with the momentum constraints (3.102), we get:

$$\mathrm{D}_{ij}\left(\dddot{\chi}_{\rm S}^{(1)\parallel}+5H\ddot{\chi}_{\rm S}^{(1)\parallel}+4\pi G\varrho_{H\mathbf{i}}a^{-3}\dot{\chi}_{\rm S}^{(1)\parallel}\right)=0\ . \tag{3.104}$$

The scalar mode solutions are separable: they are the product of a spatial and a time–dependent functions. The time–dependence of the two scalar modes is the same. In fact, since the deformation field is initially null, the general solution ansatz reads (assuming an Einstein–de Sitter background for simplicity):

$$\phi_{\rm S}^{(1)}={}^{\mathbf{1}}C_{\phi}(\mathbf{X})\left(\left(\frac{t}{t_{\mathbf{i}}}\right)^{2/3}-1\right)+{}^{\mathbf{2}}C_{\phi}(\mathbf{X})\left(\left(\frac{t}{t_{\mathbf{i}}}\right)^{-1}-1\right)-\frac{G^{(1)}(\mathbf{X})}{6}\quad ; \tag{3.105}$$

$$\mathrm{D}_{ij}\chi_{\rm S}^{(1)\parallel}=\mathrm{D}_{ij}{}^{\mathbf{1}}C_{\chi}(\mathbf{X})\left(\left(\frac{t}{t_{\mathbf{i}}}\right)^{2/3}-1\right)+\mathrm{D}_{ij}{}^{\mathbf{2}}C_{\chi}(\mathbf{X})\left(\left(\frac{t}{t_{\mathbf{i}}}\right)^{-1}-1\right)+G_{ij}^{(1)tl}(\mathbf{X})\ ; \tag{3.106}$$

where, up to a time–dependent function,

$$\Delta^{\boldsymbol{\alpha}}C_{\chi}=-6\,{}^{\boldsymbol{\alpha}}C_{\phi}\ ; \tag{3.107}$$

for $\alpha=\{1,2\}$. The space–dependent coefficients can be determined by taking the first and second time–derivatives of $\phi_{\rm S}^{(1)}$ and $\chi_{\rm S}^{(1)\parallel}$, and then evaluating them at the initial time $t_{\mathbf{i}}$. We obtain:

$${}^{\mathbf{1}}C_{\phi}(\mathbf{X})=Wt_{\mathbf{i}}{}^2+\frac{2}{9}Ut_{\mathbf{i}}\quad ;\quad {}^{\mathbf{2}}C_{\phi}(\mathbf{X})=\frac{2}{3}Wt_{\mathbf{i}}{}^2-\frac{5}{27}Ut_{\mathbf{i}}\ ; \tag{3.108}$$

and $\Delta^{\boldsymbol{\alpha}}C_{\chi}$ can be deduced from (3.107).

4.1.2 Tensor equation and solution

The tensor part satisfies the following Lagrangian propagation equation:

$$\ddot{\chi}_{ij}^{(1)\top}+3H\dot{\chi}_{ij}^{(1)\top}-\frac{\Delta_g}{a^2}\chi_{ij}^{(1)\top}=0\ ; \tag{3.109}$$

where Δ_g denotes the Laplace–Beltrami operator taken with respect to the linearised metric in local coordinates $\mathbf{X}$. This equation, which is linear in terms of the perturbation field, has no trivial solution due to the Laplace–Beltrami operator, and standard Fourier transformation techniques in flat space cannot be applied. We may solve locally for Lagrangian $\mathbf{K}-$modes, being the eigenmodes of the

Laplace–Beltrami operator, which would form a normal bundle in the Riemannian space section: the $\mathbf{K}-$space corresponding to each point with local coordinates $\mathbf{X}$ forms a fibre, and the corresponding mode propagates in this fibre. The backward, global solution cannot be obtained straightforwardly, since it would need the specification of a three–dimensional section in the fibre bundle. The solution $\chi_{ij}^{\top(1)}(\mathbf{X},t)$ is thus a superposition of wave type solutions. If we assume that gravitational waves propagate on a flat space, i.e. replacing the Laplace–Beltrami operator by an ordinary Laplacian, we can solve this equation and get the solution:

$$\chi_{ij}^{\top(1)}(\mathbf{X},t) = \frac{1}{(2\pi)^3}\int d^3\mathbf{K}\ \exp(i\mathbf{K}\cdot\mathbf{X})\ \chi_{\sigma}^{(1)}(\mathbf{K},t)\ \epsilon_{ij}^{\sigma}(\hat{\mathbf{K}})\ ; \tag{3.110}$$

where $\epsilon_{ij}^{\sigma}(\hat{\mathbf{K}})$ is the local polarisation tensor of the gravitational wave, σ is ranging over the polarisation components $+,\times$, and $\chi_{\sigma}^{(1)}(\mathbf{K},t)$ are their corresponding amplitudes, which follow the solution:

$$\chi_{\sigma}^{(1)}(\mathbf{K},t) = A(K)a_{\sigma}(\mathbf{K})\left(\frac{3j_1\left(3Kt_i^{2/3}t^{1/3}\right)}{3Kt_i^{2/3}t^{1/3}}\right). \tag{3.111}$$

$j_1(y) = \sin y/y^2 - \cos y/y$ is the first spherical Bessel function, $K := |\mathbf{K}|$, and $a_{\sigma}(\mathbf{K})$ is a zero mean random variable. $A(K)$ encodes the form of the Lagrangian spectrum of hypothetical primordial gravitational waves.

Nevertheless, we have to be very careful. Note that the superposition depends of the topology of the background space–time: for instance a periodic geometry will reduce the integral to a discrete sum. More generally, a spectrum on a compact manifold is discrete whereas the spectrum of an infinite manifold can be either discrete or continuum according to the compactness of the considered manifold (see [Aurich and Steiner, 2001]). For the assumed space–time topology of [Matarrese et al., 1998] (a 3–torus for the space part [Ellis and Schreiber, 1986; Stevens et al., 1993] thus a space–time whose topology is $\mathbb{T}^3\times\mathbb{R}$), which is widely assumed as a constraint for the numerical simulation, it should be a discrete summation. Recently, several works had been realised in order to check the compatibility of the observations with a torus topology [Aurich, 2008; Planck Collaboration, 2013c; Roukema et al., 2014].

4.2 Comparison with other works

As we said previously, other works on perturbation theory consider metrical perturbations instead of coframe perturbations. More importantly, they define quantities as functions of global coordinates of a flat background, invoking gauge transformations that are different for different orders of the perturbations. The assumption of existence of global coordinates for the perturbations, which in our approach propagate in the perturbed space and are by construction invariant by

diffeomorphism, has strong implications when it comes to determining the solutions of the equations. In fact, in [Russ et al., 1996; Matarrese et al., 1998], the Newtonian cosmological Poisson equation is used instead of the true relativistic counterpart of Poisson equation that we established in 3.1.3. Such an assumption implies integrability of the coframes, i.e. working on a flat space. Note that the assumption of integrability of the coframes immediately implies the vanishing of the spatial Ricci tensor in the perturbed space, even if its representation in terms of global coordinates of the background is non–zero (see [Buchert, 2011] Sect. 7). In [Kasai, 1995], section III.A defines the ADM equations with tetrad. Nevertheless in section III.B, only the scalar perturbations are assumed to be relevant, then the peculiar deformation tensor is fully integrable. The integrable condition reduces the problem to the Newtonian framework. In this section, we will enlighten the consequences of these assumptions for the physics of scalar perturbations and gravitational wave propagation.

Let us first compare the scalar solutions we obtained in section 4.1.1 to the ones established in [Matarrese et al., 1998] equations (4.11)–(4.23). The comoving perturbed metric is split into modes as in (3.98) where quantities are defined on the flat, global background, and are functions of the global $\mathbf{x}$ coordinate system. The solutions given in [Matarrese et al., 1998] are restricted to the growing mode only. A scalar potential, φ, is related to the initial density contrast $\delta_{\mathbf{i}}$, by a Newtonian cosmological Poisson equation:

$$\Delta\varphi(\mathbf{x}) = \frac{2}{3t_i^2}\delta_i(\mathbf{x}) \ . \tag{3.112}$$

In addition, the authors use the residual gauge freedom of the synchronous gauge to fix $\chi_i^{(1)\parallel}$ such that $\Delta\chi_{\mathrm{S}i}^{(1)\parallel} = -2\delta_i$. Then, their solutions for the scalar sector are (for an Einstein–de Sitter background),

$$\mathrm{D}_{ij}\chi_{\mathrm{S}}^{(1)\parallel} = -3t_i^2\left(\frac{t}{t_i}\right)^{\frac{2}{3}}\mathrm{D}_{ij}\varphi \ ; \tag{3.113}$$

and,

$$\phi_{\mathrm{S}}^{(1)}(\mathbf{x},t) = \frac{5}{3}\varphi(\mathbf{x}) + \frac{1}{2}t_i^2\left(\frac{t}{t_i}\right)^{\frac{2}{3}}\Delta\varphi(\mathbf{x}) \ . \tag{3.114}$$

To compare these results to ours, we relax some of the restrictions imposed in [Matarrese et al., 1998]: we include the decaying mode and use a different gauge choice, which is $\Delta\chi_{\mathrm{S}i}^{(1)\parallel} = 0$ instead of $\Delta\chi_{\mathrm{S}i}^{(1)\parallel} = -2\delta_i$. The potential is no longer independent of time since the density contrast depends on space and time. φ can now be decomposed into two components ψ_1 and ψ_2 that satisfy $\varphi = -(2/3t_i^2)\ \psi$ and $\psi\,(\mathbf{x},t) = \psi_1(\mathbf{x}) + a^{-5/2}\psi_2\,(\mathbf{x})$. They are solutions of the following equation:

$$\Delta\psi_1(\mathbf{x}) + a^{-5/2}\Delta\psi_2\,(\mathbf{x}) = -\frac{1}{a}\delta(\mathbf{x},t) \ . \tag{3.115}$$

The resulting metric perturbations read:

$$\mathrm{D}_{ij}\chi_{\mathrm{S}}^{(1)\parallel} = 2\left(\left(\frac{t}{t_i}\right)^{\frac{2}{3}} - 1\right)\mathrm{D}_{ij}\psi_1 + 2\left(\left(\frac{t}{t_i}\right)^{-1} - 1\right)\mathrm{D}_{ij}\psi_2 \; ; \tag{3.116}$$

$$\phi_{\mathrm{S}}^{(1)}(\mathbf{x},t) = -\frac{10}{9t_i^2}\psi_1(\mathbf{x}) - \frac{1}{3}\left(\left(\frac{t}{t_i}\right)^{\frac{2}{3}} - 1\right)\Delta\psi_1 - \frac{1}{3}\left(\left(\frac{t}{t_i}\right)^{-1} - 1\right)\Delta\psi_2 \; . \tag{3.117}$$

To conclude, once we changed the gauge condition for $\chi_{\mathrm{S}}^{(1)\parallel}$ and considered the decaying mode, we obtained solutions that are formally equivalent to the ones we obtained in the intrinsic Lagrangian approach. But, the main difference between our approach and the one developed in [Matarrese et al., 1998] lies in the fact that a Newtonian cosmological Poisson equation cannot be used in our Lagrangian approach since an intrinsic description of the perturbation fields implies non–integrability.

In the following subsection, we explicitly consider the *MR* of our Lagrangian solutions, and we will enlighten the shortcomings implied by considering perturbations as functions of global coordinates of a flat background. Moreover, we will be able to highlight which innovations our approach introduces compared to standard perturbation theory.

4.3 *MR*: recovering the standard solutions for scalar perturbations

4.3.1 *MR* of the perturbed Cartan coframes

In the *MR*, the Cartan coframes become deformation gradient, as we said in section 2.4.1. In the case of the first order perturbation scheme, the deformation fields defined in (3.33) become exact forms with integrable coefficients:

$$P^a_{\ i} = P^a_{\ |i} \; . \tag{3.118}$$

From (3.78) and the Helmholtz theorem, we can conclude that there exists a scalar field $\mathcal{A}$ from which P_{ij} derives:

$$P_{ij} = \mathcal{A}_{|ij} \; . \tag{3.119}$$

The Helmholtz theorem states that a vector field can be split in the sum of an irrotational (curl–free) and a solenoidal (divergence–free) vector fields. Let us F^i be a vector field, there exist Φ and A^i a scalar and a vectorial potential such as:

$$F^i = -\Phi_{|i} + \epsilon^{ijk}A_{k|j} \; ; \tag{3.120}$$

The *MR* restricts the relativistic perturbation functions to Newtonian perturbations. Thus the spatial geometry is Euclidean. The Cartan coframes naturally

contain nine degrees of freedom. The symmetry condition constrains three of them and the trace reduce the three diagonal degrees of freedom to one (i.e. constraints two degrees of freedom). Therefore, the four remaining degrees of freedom that were encoded in the Cartan coframe coefficients can be generated by only one scalar potential. We now consider the *MR* of the scalar and tensor modes to understand how they are modified in this limit and which information is kept. From (3.100) and since $\Pi_{ij} = \mathrm{D}_{ij}\mathcal{A}$, there must exist a scalar field $\tilde{G}$ such that $G_{ij}^{(1)tl} = \mathrm{D}_{ij}\tilde{G}$. Then,

$$\mathrm{D}_{ij}\chi_{\mathrm{S}}^{(1)\parallel} = \mathrm{D}_{ij}\left(2\,\mathcal{A} + \tilde{G}\right) \quad . \tag{3.121}$$

Moreover, from (3.99), we can conclude that:

$$\phi_{\mathrm{S}}^{(1)} = -\frac{1}{6}\Delta\left(2\,\mathcal{A} + \tilde{G}\right) \quad . \tag{3.122}$$

Hence, no tensor mode survives, and the two scalar modes both derive from a single scalar potential: $2\,\mathcal{A} + \tilde{G}$. $\chi_{\mathrm{S}}^{(1)\parallel}$ and $\phi_{\mathrm{S}}^{(1)}$ then automatically satisfy the momentum constraints.

4.3.2 *MR* of the equations

There is no longer a tensor equation in the *MR*. The Poisson equation in the *MR* is obtained from the Raychaudhuri equation for $\mathcal{A}$ which can be derived from the gravitational field definition (3.47) and its relation to the divergence (3.45):

$$\begin{aligned}
\delta\mathcal{F}^k{}_k &= \frac{1}{2a^3}\left(1-P\right)\left(6\ddot{a}a^2 + \ddot{a}a^2 2P + 2\dot{a}a^2 2\dot{P} + a^3 2\ddot{P} + 4\ddot{a}a^2 P\right) \\
&= \frac{1}{2}\left(-6\frac{\ddot{a}}{a}P + 2\frac{\ddot{a}}{a}P + 4H\dot{P} + 2\ddot{P} + 4\frac{\ddot{a}}{a}P\right) \\
&= \ddot{P} + 2H\dot{P}\ ; \\
\delta\mathcal{F}^k{}_k &= \Delta\left(\ddot{\mathcal{A}} + 2H\dot{\mathcal{A}}\right) = -4\pi G\delta\rho\ \ ;
\end{aligned} \tag{3.123}$$

where:

$$\delta\rho = -\frac{\rho_{H_{\mathrm{i}}}}{a^3}\left(\Delta\mathcal{A} + \frac{W}{4\pi G\rho_{H_{\mathrm{i}}}}\right)\ ; \tag{3.124}$$

is the density contrast and $\delta\mathcal{F}^k{}_k$ is the first order trace of the generalisation of the Newtonian field strength gradient defined in (3.44). We can thus conclude that we get the Newtonian cosmological Poisson equation used in literature. We determine, up to a harmonic field, the expression for $\mathcal{A}(\mathbf{X}, t)$ from (3.123):

$$\mathcal{A}(\mathbf{X}, t) = {}^{\mathbf{1}}\mathcal{A}(\mathbf{X})\left(\left(\frac{t}{t_i}\right)^{\frac{2}{3}} - 1\right) + {}^{\mathbf{2}}\mathcal{A}(\mathbf{X})\left(\left(\frac{t}{t_i}\right)^{-1} - 1\right) \quad . \tag{3.125}$$

We conclude that the scalar perturbations discussed in the standard relativistic perturbation theory literature are strictly equivalent to the Newtonian Lagrangian perturbations up to a redefinition of the spatial variables, the time dependance is exactly the same because of the choice of an EdS space–time $a = (t/t_{\mathbf{i}})^{3/2}$, see the first order Newtonian soluion (2.88). The perturbation is built thanks to the second spatial derivatives of the function $\mathcal{A}$ which can be identifies to the Newtonian spatial initial data $U^i{}_{|j}$ and $W^i{}_{|j}$.

Thus the Newtonian perturbations obtained in this section are integrable and admit the same time dependency that presented earlier in chapter II and reminded during the derivation of the analogy at the beginning of this chapter. The metric can be written by replacing our perturbations or the usual perturbations, both results are consistent,

$$\begin{aligned} g^{(1)}_{ij} &= G^{(1)}_{ij} + 2P_{(ij)} = G^{(1)}_{ij} + 2\mathcal{A}_{|ij} \\ &= -2\phi^{(1)}_{\mathrm{S}}\delta_{ij} + D_{ij}\chi^{(1)\parallel}_{\mathrm{S}} = \widetilde{G}_{|ij} + 2\mathcal{A}_{|ij} \ . \end{aligned} \tag{3.126}$$

5 Concluding remarks

As in the Newtonian chapter, it has been decided to restrict our problem to an irrotational dust continuum of matter. The basis of the perturbative scheme had been defined in this chapter in analogy to the Newtonian perturbative resolution. A major specificity is that we decided to work with differential forms which is not usual in the literature. The system of equations and the first order resolution for the Newtonian and a subpart of the relativistic equations were developed in parallel. It allowed us to highlight the formal equivalence between the Newtonian approach and the electric part of the relativistic equations (i.e. the symmetry condition and the trace part of the equation of evolution). Moreover, the *Minkowski Restriction* described in this chapter consists in assuming the Cartan coframe to be integrable. According to this additional hypothesis, the relativistic equations naturally led to the Newtonian system of equations.

In this chapter, the first order solution scheme is available for a general resolution and explicit solutions are given for a specific solution: we assumed the relativistic generalisation of the Zel'dovich Approximation (RZA) which consists in the slaving condition and an Einstein–de Sitter space–time. The trace and antisymmetric part of the perturbations were reduced to very simple functions. As expected according to the formal equivalence of the Newtonian equations and the electric part of the $3+1$ equations, the trace and antisymmetric solution are non–integrable generalisation of their Newtonian analog. Nevertheless, the relativistic system of equations also describes the traceless symmetric part of the perturbations. Moreover, the associated equation is in reality a wave equation. Thus the symmetric traceless part has to behave as a wave: a gravitational wave.

In the last section, a comparison with the standard perturbation theory was proposed. This usual approach introduces perturbations of the metric which are

assumed to be integrable. We discussed this assumption in this chapter, it appears that such solutions are quasi–Newtonian and neglect some aspects of the general relativity. Most of the papers which deal with relativistic perturbations seem to reduce the problem to integrable quantities and then quasi–Newtonian dynamics.

The next chapter will present the resolution scheme for a generic order n and a technique to obtain the complete first order solution in a specific case.

Chapter **IV**

Construction schemes for relativistic perturbations and solutions at any order

This chapter is based on a paper in preparation, [Alles et al., 2014].

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In this chapter we furnish construction rules for the perturbations and the solution schemes at any order n. The sections 1, 2 and 3 successively develop through the formally described analogy the electric part of the $3+1$ equations, the resolution scheme and generic solutions. Since our relativistic approach is built on this analogy, we focus on the electric part of the set of equations and consider the remaining magnetic equations as constraints. Section 3 presents the electric perturbative equations at any order n, this system is strongly similar to the Newtonian expression of [Ehlers and Buchert, 1997].

We propose in section 4 to build a relativistic solution for the complete perturbations (trace, symmetric traceless and antisymmetric part) up to first order. Assuming a particular background cosmology, it is possible to extrapolate the Newtonian solution even for the symmetric traceless part. In section 5 we manage to prove that in this case the first order solution has an identical time dependency for any of its parts and we describe the second order trace solution for the same hypothesis.

1 Gravitoelectric equations

In this section a particular n^{th} order relativistic resolution scheme is presented. Only a subpart of the Lagrange–Einstein system will be studied: the gravitoelectric equations. This allows us to furnish inhomogeneous models for large–scale structures formation in the Universe. The successful Lagrangian perturbation theory in Newtonian cosmology is well–developed. Similarities between the Newtonian system and the gravitoelectric equations will allow us to translate the Newtonian solutions to the relativistic context by an *Minkowski Restriction* (henceforth *MR*), as introduced in Chapter III. We will here generalise the perturbation and solution schemes of Newtonian cosmology given in the review [Ehlers and Buchert, 1997], whose essential steps will be recalled in this section, followed by their relativistic counterparts.

As before, all schemes are applied to the matter model "irrotational dust." It is possible to extend the present schemes by employing the framework for more general fluids in a Lagrangian description, developed in [Roy and Buchert, 2014]. Most of the known representations are focused on writing equations in terms of tensor or form coefficients. Our investigation will be guided by the differential forms formalism as before. However, we will also project to the coefficient form in parallel to ease reading.

We are going to give construction rules for building relativistic Lagrangian perturbation and solution schemes from the known Newtonian schemes at any order. These rules are based on the formal analogy of a part of Einstein equations, identified as *gravitoelectric part*, with the Newtonian equations ((3.7),(3.8)). The name "gravitoelectric part" is motivated by the fact that these equations are generated by the electric part of the spatially projected Weyl tensor, represented by the three 1−form fields $\mathbf{E}^a$. The reader can find more details about the associated calculations in chapter V. This tensor is tracefree and generates the electric equations [Buchert and Ostermann, 2012] equations (A23, A25–26):

$$\mathbf{E}^a = -\ddot{\boldsymbol{\eta}}^a + \frac{1}{3}(\Lambda - 4\pi G\varrho)\boldsymbol{\eta}^a \; ; \tag{4.1}$$

$$G_{ab}\,\mathbf{E}^a \wedge \boldsymbol{\eta}^b = \mathbf{0} \tag{4.2}$$

$$\Longrightarrow \quad -G_{ab}\ddot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b + \frac{1}{3}G_{ab}\,(\Lambda - 4\pi G\varrho)\,\boldsymbol{\eta}^a \wedge \boldsymbol{\eta}^b = 0$$

$$\Longrightarrow \quad G_{ab}\ddot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b = 0\,;$$

$$\epsilon_{abc}\,\mathbf{E}^a \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c = \mathbf{0} \tag{4.3}$$

$$\Longrightarrow \quad -\,\epsilon_{abc}\ddot{\boldsymbol{\eta}}^a \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c + \frac{1}{3}\epsilon_{abc}\,(\Lambda - 4\pi G\varrho)\,\boldsymbol{\eta}^a \wedge \boldsymbol{\eta}^b \wedge \boldsymbol{\eta}^c = 0\,.$$

Thus the electric part of the Weyl tensor $\mathbf{E}^a$ describes completely the symmetry condition and the evolution equation of the trace. A projection of the equations ((4.2),(4.3)) using the Hodge star operator yields their coefficient representations, which are equivalent to the symmetry condition (3.29) and the trace equation of motion (3.30):

$$E^i{}_j = -\frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\ddot{\eta}^a{}_j\eta^b{}_k\eta^c{}_l + \frac{1}{3}\,(\Lambda - 4\pi G\varrho)\,\delta^i{}_j; \tag{4.4}$$

$$E_{[ij]} = 0\,; \tag{4.5}$$

$$E^k{}_k = 0\,. \tag{4.6}$$

Sending the Cartan coframes to exact forms is defined above as *MR*: imposing this restriction, the previous equations are closed and form the Lagrange–Newton system ((3.7),(3.8)) or ((3.10),(3.11)). Note that the remaining relativistic equations, called the gravitomagnetic equations, vanish in the *MR*. We recall here that the relativistic system requires nine functions, $\eta^a{}_i(X^k,t)$, to be determined, whereas the Newtonian system just requires three functions, $f^i(X^k,t)$. These latter are fully determined by the Lagrange–Newton system.

As a consequence we will obtain the full hierarchy of relativistic perturbations at any order n.

2 Gravitoelectric Perturbation Scheme

We now recall the general Lagrangian perturbation scheme of Newtonian cosmology and generalise it into a gravitoelectric scheme in relativistic cosmology. By construction, this latter will already contain the known Lagrangian perturbation scheme at any order in the geometrical limit of exact deformation 1−forms.

2.1 Recap: Newtonian Theory

The general perturbation scheme has been fully developed in [Ehlers and Buchert, 1997] and chapter II. Our approach only slightly differs in terms of the initial data: we formulate them such that they are formally closer to the relativistic approach. We proceed by defining a perturbative ansatz to solve the Lagrange–Newton system on an FLRW background cosmology through the introduction of three comoving perturbation gradients $\mathbf{d}P^i$, i.e. nine functions which depend on the three components of the comoving vector perturbation fields $P^i(X^i,t)$:

$$\mathbf{d}f^i(\vec{X},t) =: a(t)\;\mathbf{d}F^i(\vec{X},t) = a(t)\left(\mathbf{d}X^i + \mathbf{d}P^i(\vec{X},t)\right)\,. \tag{4.7}$$

It is, of course, possible to consider the perturbations of the position fields f^i, because the Newtonian equation can be expressed in a vectorial form. The relativistic equations are tensorial and cannot be written with vectors. Therefore, we consider the representation in term of the gradient of deformation. The homogeneous Lagrange–Newton system is reduced to the nontrivial divergence equation (Friedmann equation):

$$\begin{aligned} &\epsilon_{ijk}\, 3\frac{\ddot{a}}{a}\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = \epsilon_{ijk}\left(\Lambda - 4\pi G\varrho_H\right)\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \\ \Longrightarrow\quad & 3\frac{\ddot{a}}{a} = \Lambda - 4\pi G\varrho_H\ . \end{aligned} \tag{4.8}$$

Let us decompose the gradient of the perturbations on this background solution order by order:

$$\mathbf{d}P^i = \sum_{m=1}^{\infty}\mathbf{d}P^{i(m)}\ . \tag{4.9}$$

In the Newtonian theory this can be done in integral form, using Gauss theorem and respecting boundary conditions (which is uniquely possible by imposing a 3−torus architecture on the perturbations [Buchert and Ehlers, 1997]). Integration over a compact spatial domain $\mathcal{M}$ implies the following vector ansatz, if integrated over a domain with empty boundary, according to the Stokes or Divergence theorem:

$$\int_{\mathcal{M}}\mathbf{d}P^i = \int_{\partial\mathcal{M}} P^i = 0 \quad\Longrightarrow\quad P^i = \sum_{m=1}^{\infty} P^{i(m)}. \tag{4.10}$$

We will not use the integral form in what follows for the reason explained above. Indeed the Stokes theorem requires integrable quantities, in other words we have to be able to write a vectorial formalism to use this theorem. Nevertheless the discrete decomposition is available also in the relativistic approach.

Recall now that $\mathbf{U}^i = \mathbf{d}U^i = U^i{}_{|j}\mathbf{d}X^j$ and $\mathbf{W}^i = \mathbf{d}W^i = W^i{}_{|j}\mathbf{d}X^j$ are the initial 1–form peculiar–velocity gradients and initial 1–form peculiar–acceleration gradients. The fields W^i are determined nonlocally by the following set of equations, equivalent to Poisson equation:

$$\begin{aligned} W^i{}_{|i} &= *\frac{1}{2}\epsilon_{ijk}\mathbf{d}W^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = -4\pi G\delta\varrho_{\mathbf{i}}\ ; \\ \delta_{ij}\mathbf{d}W^i \wedge \mathbf{d}X^j &= \delta_{ij}\mathbf{d}\left(W^i\mathbf{d}X^j\right) = 0\ . \end{aligned} \tag{4.11}$$

In view of the restriction to irrotational flows, we additionally impose the irrotationality constraint:

$$\delta_{ij}\mathbf{d}\dot{f}^i \wedge \mathbf{d}f^j = 0 \quad\Longrightarrow\quad \delta_{ij}\mathbf{d}U^i \wedge \mathbf{d}X^j = *\mathbf{d}\left(U^i\mathbf{d}X^i\right) = 0\ . \tag{4.12}$$

Without loss of generality, we can choose the following general set of initial data that can be obtained in the Newtonian theory or, else, from the *Minkowski Restriction* of ((3.50)–(3.52)):

- the initial deformation, velocity and acceleration,

$$\begin{cases} \mathbf{d}W^{i(1)} = \mathbf{d}W^i, \; W_{[i|j]} = 0 \; ; \\ \mathbf{d}U^{i(1)} = \mathbf{d}U^i, \; U_{[i|j]} = 0 \; ; \\ \forall n \mathbf{d}\mathcal{P}^{i(n)} = \mathbf{0} \; ; \end{cases} \tag{4.13}$$

- and additional initial constraints are obtained thanks to definition of the metric and constraint equations evaluated at initial time,

$$\begin{cases} \delta\varrho_{\mathbf{i}}{}^{(1)} = \delta\varrho_{\mathbf{i}} = \varrho_{H\mathbf{i}}\delta_{\mathbf{i}} = -\dfrac{1}{4\pi G}\delta_i{}^k W^i{}_{|k} \; ; \\ g_{ij} = \delta_{kl} f^k{}_{|i} f^l{}_{|j} \; ; \\ \dot{a}(t_{\mathbf{i}})\dot{f}^i{}_{|i}(t_{\mathbf{i}}) = -\dfrac{\mathcal{R}_{\mathbf{i}}}{2} + 8\pi G\varrho_{\mathbf{i}} + \Lambda \; . \end{cases} \tag{4.14}$$

The metric is Euclidean, and in terms of differential forms the constraints about the initial velocity and acceleration are:

$$\begin{cases} \delta_{ij}\mathbf{d}U^i \wedge \mathbf{d}X^j = 0 \; ; \\ \delta_{ij}\mathbf{d}W^i \wedge \mathbf{d}X^j = 0 \; . \end{cases} \tag{4.15}$$

The full hierarchy of the perturbation equations reads:

$$\delta_{ij}\, \mathbf{d}\dot{P}^i \wedge \left(\mathbf{d}X^j + \mathbf{d}P^j\right) = \delta_{ij} a^{-2}\mathbf{d}U^i \wedge \mathbf{d}X^j \; ; \tag{4.16}$$

$$\epsilon_{ijk}\Bigg[\Big(D - 4\pi G\varrho_H\Big)\mathbf{d}P^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k + \Big(2D - 4\pi G\varrho_H\Big)\mathbf{d}P^i \wedge \mathbf{d}P^j \wedge \mathbf{d}X^k$$
$$+ \Big(D - \frac{4\pi G}{3}\varrho_H\Big)\mathbf{d}P^i \wedge \mathbf{d}P^j \wedge \mathbf{d}P^k\Bigg] = -\epsilon_{ijk}\frac{4\pi G}{3}\delta\varrho_{\mathbf{i}}\; a^{-3}\mathbf{d}X^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \; ; \tag{4.17}$$

where the operator D equals $(d^2/dt^2) + 2H\ (d/dt)$. The reader may note a difference with the reference [Ehlers and Buchert, 1997] in the numerical coefficients. Indeed, the mentioned paper presents errata in these coefficients. It is possible to straightforwardly compute the coefficient equations, using the Hodge operator:

$$\begin{aligned} *(4.16) \quad &\Longrightarrow \quad \delta_{ij}\dot{P}^i{}_{|k}\mathbf{d}X_l\; \epsilon^{kjl} + \delta_{ij}\dot{P}^i{}_{|k}P^j{}_m\mathbf{d}X_l\; \epsilon^{kml} = 0 \\ &\Longrightarrow \quad \dot{P}_{j|k}\mathbf{d}X_l\; \epsilon^{kjl} + \dot{P}_{m|k}P^m_j\mathbf{d}X_l\; \epsilon^{kjl} = 0 \\ &\Longrightarrow \quad P_{[i|j]} = \int_{t_{\mathbf{i}}}^t \dot{P}_{m|[i}P^m{}_{|j]}dt' \; ; \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
*(4.17) \Longrightarrow \quad & \epsilon_{ijk}\Big[\left(D-4\pi G\varrho_H\right)P^i{}_{|l}\epsilon^{ljk}+\left(2D-4\pi G\varrho_H\right)P^i{}_{|l}P^j{}_{|m}\epsilon^{lmk} \\
& +\left(D-\frac{4\pi G}{3}\varrho_H\right)P^i{}_{|l}P^j{}_{|m}P^k{}_{|n}\epsilon^{lmn}\Big]=-\epsilon_{ijk}\frac{4\pi G}{3}\delta\varrho_{\mathbf{i}}\ a^{-3}\epsilon^{ijk} \\
\Longrightarrow \quad & 2\left(D-4\pi G\varrho_H\right)P^i{}_{|i}+\left(2D-4\pi G\varrho_H\right)\left(P^i{}_{|i}P^j{}_{|j}-P^i{}_{|j}P^j{}_{|i}\right) \\
& +\left(D-\frac{4\pi G}{3}\varrho_H\right)\epsilon_{ijk}\epsilon^{lmn}P^i{}_{|l}P^j{}_{|m}P^k{}_{|n}=-8\pi G\delta\varrho_{\mathbf{i}}\ a^{-3} \\
\Longrightarrow \quad & \left(D-4\pi G\varrho_H\right)P^i{}_{|i}=-4\pi G\delta\varrho_{\mathbf{i}}a^{-3}-\left(D-2\pi G\varrho_H\right)P^i{}_{|i}P^j{}_{|j} \\
& -\frac{1}{2}\epsilon_{ijk}\epsilon^{lmn}\left(D-\frac{4\pi G}{3}\varrho_H\right)P^i{}_{|l}\ P^j{}_{|m}P^k{}_{|n}\ . \qquad (4.19)
\end{aligned}
$$

After splitting the system ((4.16),(4.17)) order by order, we obtain n sets of equations. A perturbative development naturally makes product of lower indices appeared as source terms. These terms are clearly identifiable in the equation (4.19) as the quadratic and cubic terms in P. The resolution of a perturbative development is performed order by order. The generic n^{th} order system of equations will be written in the next paragraph with an implicit summation over the order of perturbations in the source terms:

$$
\begin{cases}
A^{(p)}B^{(q)}=\displaystyle\sum_{p+q=n}A^{(p)}B^{(q)}\ ; \\
A^{(r)}B^{(s)}C^{(t)}=\displaystyle\sum_{r+s+t=n}A^{(r)}B^{(s)}C^{(t)}\ .
\end{cases}
\qquad (4.20)
$$

Here, we only have to truncate the previous equations at a given order. At first order we get:

$$
\delta_{ij}\mathbf{d}\dot{P}^{i(1)}\wedge\mathbf{d}X^j=\mathbf{0}\ ; \qquad (4.21)
$$

$$
\epsilon_{ijk}\left(D-4\pi G\varrho_H\right)\mathbf{d}P^{i(1)}\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k=a^{-3}\epsilon_{ijk}\mathbf{d}W^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\ ; \qquad (4.22)
$$

which is in coefficient form:

$$
P^{(1)}_{[i|j]}=0\ ; \qquad (4.23)
$$

$$
\ddot{P}^{i(1)}{}_{|i}+2H\dot{P}^{i(1)}{}_{|i}-4\pi G\varrho_{h\mathbf{i}}a^{-3}P^{i(1)}_{|i}=a^{-3}W^i{}_{|i}\ ; \qquad (4.24)
$$

i.e. a set of linear equations. We recall the Poisson equation $W^i{}_{|i}=-4\pi G\delta\varrho_{\mathbf{i}}$ which is used to replace the inhomogeneous density by the initial acceleration gradient. At any order $n>1$, the perturbation equations read:

$$
\delta_{ij}\mathbf{d}\dot{P}^{i(n)}\wedge\mathbf{d}X^j=-\delta_{ij}\mathbf{d}\dot{P}^{i(p)}\wedge\mathbf{d}P^{j(q)}\ ; \qquad (4.25)
$$

$$
\begin{aligned}
\epsilon_{ijk}\Big[&\left(D-4\pi G\varrho_H\right)\mathbf{d}P^{i(n)}\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k-\left(2D-4\pi G\varrho_H\right)\mathbf{d}P^{i(p)}\wedge\mathbf{d}P^{j(q)}\wedge\mathbf{d}X^k \\
&+\left(D-\frac{4\pi G}{3}\varrho_H\right)\mathbf{d}P^{i(r)}\wedge\mathbf{d}P^{j(s)}\wedge\mathbf{d}P^{k(t)}\Big]=0\ ; \qquad (4.26)
\end{aligned}
$$

and in coefficient form:

$$P^{(n)}_{[i|j]} = \int_{t_{\mathbf{i}}}^{t} \dot{P}^{(p)}_{m|[i} P^{m(q)}_{|j]} dt' \; ; \tag{4.27}$$

$$\begin{aligned}
&\ddot{P}^{i(n)}_{|i} + 2H\dot{P}^{i(n)}_{|i} - 4\pi G \varrho_{H\mathbf{i}} a^{-3} P^{i(n)}_{|i} \\
&\quad = -\frac{1}{2}\epsilon_{ijk}\epsilon^{lmn} P^{j(s)}_{|m} P^{k(t)}_{|n} \left(\ddot{P}^{i(r)}_{|l} + 2H\dot{P}^{i(r)}_{|l} - \frac{4\pi G}{3}\varrho_{H\mathbf{i}} a^{-3} P^{i(r)}_{|l} \right) \\
&\qquad - \left(\ddot{P}^{i(p)}_{|i} + 2H\dot{P}^{i(p)}_{|i} - 2\pi G \varrho_{H\mathbf{i}} a^{-3} P^{i(p)}_{|i} \right) P^{j(q)}_{|j} \\
&\qquad + \left(\ddot{P}^{i(p)}_{|j} + 2H\dot{P}^{i(p)}_{|j} - 2\pi G \varrho_{H\mathbf{i}} a^{-3} P^{i(p)}_{|j} \right) P^{j(q)}_{|i} \; .
\end{aligned} \tag{4.28}$$

The reader may consult the review [Ehlers and Buchert, 1997] and references therein for further details.

2.2 Einstein Theory

Assuming the previous perturbation ansatz for the coframes, and using the operator D as defined above, the analogous expansion is performed. The zero order leads to the Friedmann equation (3.39), and the general perturbation scheme reads:

$$G_{ab}\dot{\mathbf{P}}^a \wedge \delta^b{}_j \mathbf{d}X^j + G_{ab}\dot{\mathbf{P}}^a \wedge \mathbf{P}^b = \mathbf{0} \; ; \tag{4.29}$$

$$\begin{aligned}
&\epsilon_{abc}\Big[\Big(D - 4\pi G\varrho_H\Big)\mathbf{P}^a \wedge \delta^b{}_j \mathbf{d}X^j \wedge \delta^c{}_k \mathbf{d}X^k + \Big(2D - 4\pi G\varrho_H\Big)\mathbf{P}^a \wedge \mathbf{P}^b \wedge \delta^c{}_k \mathbf{d}X^k \\
&+ \left(D - \frac{4\pi G}{3}\varrho_H\right)\mathbf{P}^a \wedge \mathbf{P}^b \wedge \mathbf{P}^c \Big] = \epsilon_{abc}\frac{Wa^{-3}}{3}\delta^a{}_i \mathbf{d}X^i \wedge \delta^b{}_j \mathbf{d}X^j \wedge \delta^c{}_k \mathbf{d}X^k \; .
\end{aligned} \tag{4.30}$$

A comparison with the Newtonian equations ((4.16),(4.17)) is very convincing: the gravitoelectric equations are formally similar to the Newtonian equations. Thanks to the Hodge operator their coefficient form reads:

$$\begin{aligned}
*(4.29) \implies\; & G_{ab}\dot{P}^a{}_k \delta^b{}_j \mathbf{d}X_l \; \epsilon^{kjl} + G_{ab}\dot{P}^a{}_k P^b{}_m \mathbf{d}X_l \; \epsilon^{kml} = 0 \\
\implies\; & \dot{P}_{bk}\delta^b{}_j \mathbf{d}X_l \; \epsilon^{kjl} + \dot{P}_{bk}P^b{}_j \mathbf{d}X_l \; \epsilon^{kjl} = 0 \\
\implies\; & \dot{P}_{[jk]} + \dot{P}_{b[k}P^b{}_{j]} = 0 \\
\implies\; & P_{[ij]} = \int_{t_{\mathbf{i}}}^{t} \dot{P}_{a[i}P^a{}_{j]} dt' \; ;
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
*(4.30) \implies\; & \epsilon_{abc}\Big[\Big(D - 4\pi G\varrho_H\Big)P^a{}_l \epsilon^{lbc} + \Big(2D - 4\pi G\varrho_H\Big)P^a{}_l P^b{}_m \epsilon^{lmc} \\
& + \Big(D - \frac{4\pi G}{3}\varrho_H\Big)P^a{}_l P^b{}_m P^c{}_n \epsilon^{lmn} \Big] = \epsilon_{abc}\frac{Wa^{-3}}{3}\epsilon^{abc}
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\quad & 2\Big(D-4\pi G\varrho_H\Big)P^i{}_i+\Big(2D-4\pi G\varrho_H\Big)\left(P^i{}_iP^j{}_j-P^i{}_jP^j{}_i\right)\\
&+\epsilon_{ijk}\epsilon^{lmn}\Big(D-\frac{4\pi G}{3}\varrho_H\Big)P^i{}_lP^j{}_mP^k{}_n=2Wa^{-3}\\
\Longrightarrow\quad & \left(D-4\pi G\varrho_H\right)P^i{}_i=-\left(D-2\pi G\varrho_H\right)\left(P^i{}_iP^j{}_j-P^i{}_jP^j{}_i\right)\\
&-\frac{1}{2}\epsilon_{ijk}\epsilon^{lmn}\left(D-\frac{4\pi G}{3}\varrho_H\right)P^i{}_lP^j{}_mP^k{}_n+Wa^{-3}\ . \qquad (4.32)
\end{aligned}$$

These two equations are comparable to their Newtonian analogs ((4.18),(4.19)) and the analogy is even clearer when we look at the integrable and non integrable perturbations $P^i{}_{|j}$ and $P^i{}_j$. Let us now introduce, as before, the six 1–form fields $\mathbf{U}^a=U^a{}_i\mathbf{d}X^i$ and $\mathbf{W}^a=W^a{}_i\mathbf{d}X^i$. We insert the ordered perturbation coefficients of the coframe fields and obtain the following first order gravitoelectric equations:

$$G_{ab}\dot{\mathbf{P}}^{a(1)}\wedge\mathbf{d}X^b=\mathbf{0}\ ; \qquad (4.33)$$

$$\begin{aligned}
\epsilon_{abc}\big(D-4\pi G\varrho_H\big)\mathbf{P}^{a(1)}\wedge\delta^b{}_j\mathbf{d}X^j\wedge\delta^c{}_k\mathbf{d}X^k&\\
=a^{-3}\epsilon_{abc}\mathbf{W}^a\wedge\delta^b{}_j\mathbf{d}X^j\wedge\delta^c{}_k\mathbf{d}X^k\ ;& \qquad (4.34)
\end{aligned}$$

which are comparable to ((4.21),(4.22)) whose coefficient expression is:

$$P^{(1)}_{[ij]}=0\ ; \qquad (4.35)$$

$$\ddot{P}^{i(1)}_i+2H\dot{P}^{i(1)}_i-4\pi G\varrho_{h\mathbf{i}}a^{-3}P^{i(1)}_i=Wa^{-3}\ . \qquad (4.36)$$

These two first order tensorial equations have to be compared with the first order Newtonian equations ((4.23),(4.24)). The general gravitoelectric scheme n^{th} order, with $n>1$, leads to the following set of nonlinear equations:

$$G_{ab}\dot{\mathbf{P}}^{a(n)}\wedge\delta^b{}_j\mathbf{d}X^j=-G_{ab}\dot{\mathbf{P}}^{a(p)}\wedge\mathbf{P}^{b(q)}\ ; \qquad (4.37)$$

$$\begin{aligned}
\epsilon_{abc}\big(D-4\pi G\varrho_H\big)\mathbf{P}^{a(n)}\wedge\delta^b{}_j\mathbf{d}X^j\wedge\delta^c{}_k\mathbf{d}X^k&\\
=-\epsilon_{abc}\Big[\left(2D-4\pi G\varrho_H\right)\mathbf{P}^{a(p)}\wedge\mathbf{P}^{b(q)}\wedge\delta^c{}_k\mathbf{d}X^k&\\
+\left(D-\frac{4\pi G}{3}\varrho_H\right)\mathbf{P}^{a(r)}\wedge\mathbf{P}^{b(s)}\wedge\mathbf{P}^{c(t)}\Big]\ ;& \qquad (4.38)
\end{aligned}$$

which have to be compared with the Newtonian equations ((4.25),(4.26)) and leads to the following tensorial expressions:

$$P^{(n)}_{[ij]}=\int_{t_{\mathbf{i}}}^{t}\dot{P}^{(p)}_{m[i}P^{m(q)}_{j]}dt'\ ; \qquad (4.39)$$

$$
\begin{aligned}
&\ddot{P}^{i(n)}_{i} + 2H\dot{P}^{i(n)}_{i} - 4\pi G\varrho_{H\mathbf{i}}a^{-3}P^{i(n)}_{i} \\
&\quad = -\frac{1}{2}\epsilon_{ijk}\epsilon^{lmn}P^{j(s)}_{m}P^{k(t)}_{n}\left(\ddot{P}^{i(r)}_{l} + 2H\dot{P}^{i(r)}_{l} - \frac{4\pi G}{3}\varrho_{H\mathbf{i}}a^{-3}P^{i(r)}_{l}\right) \\
&\qquad - \left(\ddot{P}^{i(p)}_{i} + 2H\dot{P}^{i(p)}_{i} - 2\pi G\varrho_{H\mathbf{i}}a^{-3}P^{i(p)}_{i}\right)P^{j(q)}_{j} \\
&\qquad + \left(\ddot{P}^{i(p)}_{j} + 2H\dot{P}^{i(p)}_{j} - 2\pi G\varrho_{H\mathbf{i}}a^{-3}P^{i(p)}_{j}\right)P^{j(q)}_{i} \; ;
\end{aligned}
\tag{4.40}
$$

These equations provide solutions for the perturbation fields at any order n from solutions of order $n-1$. The reader can notice the very close expressions for the Newtonian equations ((4.27),(4.28)) and the relativistic gravitoelectric equations. Thus, a direct analogy can be set up between the two sets of equations. This analogy suggests to construct the gravitoelectric part of the relativistic perturbation scheme by inversion of the *Minkowski Restriction*:

$$
\mathbf{d}P^{i} = P^{i}{}_{|j}\mathbf{d}X^{j} \mapsto P^{a}{}_{j}\mathbf{d}X^{j} = \mathbf{P}^{a} \; . \tag{4.41}
$$

The initial data respect the following generalisation rule:

$$
\mathbf{d}U^{i} = U^{i}{}_{|j}\mathbf{d}X^{j} \mapsto U^{a}{}_{j}\mathbf{d}X^{j} = \mathbf{U}^{a} \; ; \tag{4.42}
$$

$$
\mathbf{d}W^{i} = W^{i}{}_{|j}\mathbf{d}X^{j} \mapsto W^{a}{}_{j}\mathbf{d}X^{j} = \mathbf{W}^{a} \; . \tag{4.43}
$$

According to these results, the "Newtonian–relativistic gravitoelectric" formal equivalence is obvious at any order of the perturbations. It follows that the trace and traceless antisymmetric solutions are also formally equivalent. However, note already here that the inversion of the *Minkowski Restriction* can produce a symmetric traceless component as part of the relativistic gravitational wave solutions even if there does not exist a Newtonian equation for the symmetric traceless part.

3 Gravitoelectric solution scheme

3.1 Recap: Newtonian Theory

We first recall the general solution scheme given in [Ehlers and Buchert, 1997], written for the perturbation gradient only. We decompose the perturbations and the initial data into their trace $\mathbf{d}P$, symmetric traceless $\mathbf{d}\Pi^{i}$ and antisymmetric $\mathbf{d}\mathfrak{P}^{i}$ parts. Thus, the perturbations and the initial data can be written as:

$$
\begin{cases}
\mathbf{d}P^{i(n)} = \mathrm{tr}\left(\mathbf{d}P^{i(n)}\right) + \mathbf{d}\Pi^{i(n)} + \mathbf{d}\mathfrak{P}^{i(n)} \\
\qquad\qquad = \dfrac{1}{3}P^{k}{}_{|k}\delta^{i}{}_{j}\mathbf{d}X^{j} + \Pi^{i(n)}{}_{|j}\mathbf{d}X^{j} + \mathfrak{P}^{i(n)}{}_{|j}\mathbf{d}X^{j} \; ; \\
\mathbf{d}U^{i} = U^{i}{}_{|j}\mathbf{d}X^{j} \quad ; \quad U_{[i|j]} = 0 \; ; \\
\mathbf{d}W^{i} = W^{i}{}_{|j}\mathbf{d}X^{j} \quad ; \quad W_{[i|j]} = 0 \; .
\end{cases}
\tag{4.44}
$$

According to the split of the perturbations, the general first order system of equations can be written as follows:

$$\delta_{ij}\mathbf{d}\mathfrak{P}^{i(1)} \wedge \mathbf{d}X^j = \mathbf{0} \; ; \tag{4.45}$$

$$\epsilon_{ijk}\Big(D - 4\pi G\varrho_H\Big)\mathbf{d}\Pi^{i(1)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = \epsilon_{ijk}a^{-3}\mathbf{d}W^i \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k \; ; \tag{4.46}$$

uniquely determined by the constraint initial data for $\mathbf{d}U^i$ and $\mathbf{d}W^i$. The general n^{th} order solution scheme, with $n > 1$, reads:

$$\delta_{ij}\mathbf{d}\mathfrak{P}^{i(n)} \wedge \mathbf{d}X^j = {}^N\boldsymbol{\mathcal{S}}^{(n)} \; ; \tag{4.47}$$

$$\epsilon_{ijk}\left(D - 4\pi G\varrho_H\right)\mathbf{d}P^{i(n)} \wedge \mathbf{d}X^j \wedge \mathbf{d}X^k = {}^N\boldsymbol{\mathcal{T}}^{(n)}; \tag{4.48}$$

uniquely determined by the source terms:

$$\begin{aligned} {}^N\boldsymbol{\mathcal{S}}^{(n)} &= -\,\delta_{ij}\int_{t_{\mathbf{i}}}^{t}\mathbf{d}\dot{P}^{i(p)} \wedge \mathbf{d}P^{j(q)}dt' \; ; \qquad (4.49)\\ {}^N\boldsymbol{\mathcal{T}}^{(n)} &= -\,\epsilon_{ijk}\Big[\big(2D - 4\pi G\varrho_H\big)\mathbf{d}P^{i(p)} \wedge \mathbf{d}P^{j(q)} \wedge \mathbf{d}X^k \\ &\quad + \big(D - \frac{4\pi G}{3}\varrho_H\big)\mathbf{d}P^{i(r)} \wedge \mathbf{d}P^{j(s)} \wedge \mathbf{d}P^{k(t)}\Big] \; . \qquad (4.50) \end{aligned}$$

We have demonstrated earlier in section 2 the formal equivalence between the Newtonian equations and the relativistic gravitoelectric equations. The generalisation of the Newtonian solution scheme to obtain the corresponding relativistic scheme is now straightforward.

3.2 Einstein Theory

Splitting the coefficient matrices into their trace, symmetric tracefree and antisymmetric parts, the perturbations are written accordingly:

$$\mathbf{P}^a = \frac{1}{3}P\delta^a{}_j\mathbf{d}X^j + \mathbf{\Pi}^{a(n)} + \mathfrak{\mathbf{P}}^{a(n)} \; . \tag{4.51}$$

The perturbative gravitoelectric Lagrange–Einstein system composed of the symmetry condition and the trace part equation of motion leads to the first order gravitoelectric equations:

$$G_{ab}\mathfrak{\mathbf{P}}^{a(1)} \wedge \mathbf{d}X^b = \mathbf{0} \; ; \tag{4.52}$$

$$\begin{aligned} \epsilon_{abc}\big(D - 4\pi G\varrho_H\big)\mathbf{P}^{a(1)} \wedge \delta^b{}_j\mathbf{d}X^j \wedge \delta^c{}_k\mathbf{d}X^k \\ = -\epsilon_{abc}a^{-3}\mathbf{W}^a \wedge \delta^b{}_j\mathbf{d}X^j \wedge \delta^c{}_k\mathbf{d}X^k \; ; \qquad (4.53) \end{aligned}$$

uniquely determined by G_{ab} and the corresponding constraint initial data for $\mathbf{U}^a$ and $\mathbf{W}^a$. The nth order gravitoelectric solution scheme, with $n > 1$, reads:

$$G_{ab}\mathfrak{\mathbf{P}}^{a(n)} \wedge \delta^b{}_j\mathbf{d}X^j = \boldsymbol{\mathcal{S}}^{(n)} \; ; \tag{4.54}$$

$$\epsilon_{abc}\big(D - 4\pi G\varrho_H\big)\mathbf{P}^{a(n)} \wedge \delta^b{}_j \mathbf{d}X^j \wedge \delta^c{}_k \mathbf{d}X^k = \boldsymbol{\mathcal{T}}^{(n)} \; ; \tag{4.55}$$

and is uniquely determined by the source terms:

$$\boldsymbol{\mathcal{S}}^{(n)} = G_{ab} \int_{t_{\mathrm{i}}}^{t} \left(-\dot{\mathbf{P}}^{a(p)} \wedge \mathbf{P}^{b(q)}\right) dt' \; ; \tag{4.56}$$

$$\begin{aligned}\boldsymbol{\mathcal{T}}^{(n)} = &- \epsilon_{abc}\Big(\Big(2D - 4\pi G\varrho_H\Big)\mathbf{P}^{a(p)} \wedge \mathbf{P}^{b(q)} \wedge \delta^c{}_k \mathbf{d}X^k \\ &+ \left(D - \frac{4\pi G}{3}\varrho_H\right)\mathbf{P}^{a(r)} \wedge \mathbf{P}^{b(s)} \wedge \mathbf{P}^{c(t)}\Big) \; .\end{aligned} \tag{4.57}$$

the relativistic solution scheme presents in this section is very similar to the Newtonian solutions scheme as mentioned several time earlier. The definition of the relativistic sources can be compared with their Newtonian analogs ((4.49),(4.50)).

3.3 Space–time splitting implications

At any order, the solutions are composed of several modes which admit different time dependencies. A space–time splitting is assumed for each of these modes l at any order n:

$$P^{i(n)}_{\;\;j} = \sum_l \xi^{(n,l)}(t) P^{i}_{\;\;j}{}^{(n)l}(\vec{X}) \; . \tag{4.58}$$

This decomposition allows us to rewrite the gravitoelectric system ((4.18),(4.19)) as,

$$\sum_l \xi^{(n,l)} P^{(n)l}_{[ij]} = \sum_{l,m} P^{(p)l}_{r[i} P^{r(q)m}_{\;\;\;\;\;\;j]} \int_{t_{\mathrm{i}}}^{t} \dot{\xi}^{(p,l)} \xi^{(q,m)} dt' \; ; \tag{4.59}$$

$$\begin{aligned}&\sum_l P^{(n)l}\left[\ddot{\xi}^{(n,l)} + 2H\dot{\xi}^{(n,l)} - 4\pi G\varrho_H\, \xi^{(n,l)}\right] = \\ &- \sum_{l,m}\left[P^{(p)l}P^{(q)m} - P^{i}_{\;\;j}{}^{(p)l} P^{j}_{\;\;i}{}^{(q)m}\right]\xi^{(q,m)}\left[\ddot{\xi}^{(p,l)} + 2H\dot{\xi}^{(p,l)} - 2\pi G\varrho_H\, \xi^{(p,l)}\right] \\ &- \frac{1}{2}\epsilon_{ijk}\epsilon^{uvw}\sum_{l,m,p} P^{i}_{\;\;u}{}^{(r)l} P^{j}_{\;\;v}{}^{(s)m} P^{k}_{\;\;w}{}^{(t)p} \xi^{(r,l)}\xi^{(s,m)}\left[\ddot{\xi}^{(t,p)} + 2H\dot{\xi}^{(t,p)} - \frac{4\pi G}{3}\varrho_H\, \xi^{(t,p)}\right] \; .\end{aligned} \tag{4.60}$$

The first equation (the symmetry condition) is, as usual, only a definition of the antisymmetric traceless part of the perturbations, whereas the second equation describes the evolution of the trace part, the usual way to solve such an equation is to solve term after term thanks to the superposition theorem.

The homogeneous solution, denoted by $l = 0$, respects:

$$\ddot{\xi}^{(n,0)} + 2H\dot{\xi}^{(n,0)} - 4\pi G\varrho_H\, \xi^{(n,0)} = 0 \; . \tag{4.61}$$

The particular solution can be computed for a subpart of the source, each $P^{(n)l}\xi^{(n,l)}$ is a particular solution for a time mode in the source after the development of the sums over l, m and l, m, p. Notice that the second order and higher order solutions also require symmetric tracefree source terms.

In the Newtonian approach we can integrate the equations for the trace and antisymmetric traceless parts and solve the vectorial system to complete the solution, but in the relativistic case we have to employ the gravitomagnetic equations in order to fully determine the symmetric tracefree part of $\mathbf{P}^a$. We shall therefore explain the construction of relativistic solutions from Newtonian ones in the following examples and formulate the construction rule thereafter.

The system of equations and the source terms are formally similar. This equivalence is a very strong result and advantage of the approach developed here. Nevertheless, we have to remain very careful because general relativity brings additional constraints. The Newtonian coordinates are naturally global thanks to the descriptions of the space–time this theory is based on. General relativity does not allow us to describe the space–time so easily with a global coordinates system. All our results are developed in a local space tangent to a non–Euclidean manifold. Thus the equations obtained are in reality local equations and it is impossible, a priori, to travel through the manifold continuously without changing the exact basis system $\mathbf{d}X^i$. Fortunately, it is possible to derived some results relatively easily even with this "problem" of local description and coordinates. One of these results are highlighted in the following sections and chapters.

4 Example 1: recovering parts of the general first order solution

4.1 Generalisation of the Newtonian solution

In this subsection we are going to exploit the above–developed schemes for the explicit construction of the relativistic solution from the Newtonian solution [Buchert, 1989; Bildhauer et al., 1992; Buchert, 1992], restricting attention to an Einstein–de Sitter background for simplicity. According to [Ehlers and Buchert, 1997], all solutions of the Lagrange–Newton system can be written in terms of longitudinal and transverse components of a vector; the Newtonian first order solution reads:

$$P^{i(1)} = 3t_{\mathbf{i}}\left(\left(\frac{1}{5}U^{i^L} + \frac{3}{10}W^{i^L}t_{\mathbf{i}}\right)a - \left(\frac{1}{5}U^{i^L} - \frac{1}{5}W^{i^L}t_{\mathbf{i}}\right)a^{-3/2} - \frac{1}{2}W^{i^L}t_{\mathbf{i}} - U^{i^T}\left(a^{-1/2} - 1\right)\right) . \tag{4.62}$$

The longitudinal and transverse parts are denoted by an index ^{L} and ^{T}. Henceforth we drop the transverse part due to the irrotationality constraint in the relativistic scheme. The Newtonian equations are the divergence and the curl of the gravitation

field, so it reads:

$$\mathfrak{P}^{(1)}_{i|j} = 0 \; ; \tag{4.63}$$

$$P^{i(1)}_{\;\;|i} = \left(\frac{3}{5}U^{i}{}_{|i}t_{\mathbf{i}} + \frac{9}{10}W^{i}{}_{|i}t_{\mathbf{i}}{}^{2}\right)a - \left(\frac{3}{5}U^{i}{}_{|i}t_{\mathbf{i}} - \frac{3}{5}W^{i}{}_{|i}t_{\mathbf{i}}{}^{2}\right)a^{-3/2} - \frac{3}{2}W^{i}{}_{|i}t_{\mathbf{i}}{}^{2} \; . \tag{4.64}$$

These derivatives can be generalised to the relativistic tensor solution by dropping the integrability of the perturbations i.e. sending $P^{i(1)}_{\;\;|j} \rightarrow P^{a(1)}_{i}$:

$$\mathfrak{P}^{a(1)}_{i} = 0 \; ; \tag{4.65}$$

$$P^{i(1)}_{\;\;i} = \left(\frac{3}{5}U^{i}{}_{i}t_{\mathbf{i}} + \frac{9}{10}W^{i}{}_{i}t_{\mathbf{i}}{}^{2}\right)a - \left(\frac{3}{5}U^{i}{}_{i}t_{\mathbf{i}} - \frac{3}{5}W^{i}{}_{i}t_{\mathbf{i}}{}^{2}\right)a^{-3/2} - \frac{3}{2}W^{i}{}_{i}t_{\mathbf{i}}{}^{2} \; . \tag{4.66}$$

In view of what has been previously said, this solution solves the gravitoelectric part of the corresponding relativistic equations and can be obtained by the resolution of the system of equations. Indeed, it corresponds to the results obtained in chapter III. However, with this generalisation we can also produce a symmetric tracefree solution. Here we are going to complete the Newtonian gradient solution by the symmetrisation of the derivative of the vectorial solution and generalised it to its non–integrable analog. The question is, can we extract any information about the symmetric solution from this relativistic generalisation? It remains to be checked, whether this part furnishes at least few modes of the relativistic solution. A specific notation is introduced to identity the generalisations of the longitudinal and transverse parts,

$$F^{i^L}_{\;\;|j} \mid F^{L}_{[i|j]} = 0 \longrightarrow F^{a^1}_{j} \mid F^{1}_{[ij]} = 0 \;\; ; \quad F^{i^T}_{\;\;|j} \mid F^{i^T}_{\;\;|i} = 0 \longrightarrow F^{a^2}_{j} \mid F^{j^2}_{j} = 0 \; . \tag{4.67}$$

Note that a longitudinal and a transverse parts cannot be defined in general relativity with the Newtonian derivative, it could be possible with the Hodge–Helmholtz decomposition. The indexes ^{1} and ^{2} only denote functions which are obtained by the non–integrable generalisation of the Newtonian longitudinal and transverse parts. Nevertheless, these non–integrable quantities respect the generalisation of the constraints for the longitudinal and transverse parts $F_{(ij)} = F^{1}_{(ij)} + F^{2}_{(ij)}$, $F^{i}{}_{i} = F^{i^1}_{\;\;i}$ and $F_{[ij]} = F^{2}_{[ij]}$. This technique leads to the following solution,

$$\mathfrak{P}^{a(1)}_{i} = 0 \; ; \tag{4.68}$$

$$P^{a(1)}_{i}\delta_{a}{}^{i} = \left(\frac{3}{5}U^{a}{}_{i}t_{\mathbf{i}} + \frac{9}{10}W^{a}{}_{i}t_{\mathbf{i}}{}^{2}\right)\delta_{a}{}^{i}a - \left(\frac{3}{5}U^{a}{}_{i}t_{\mathbf{i}} - \frac{3}{5}W^{a}{}_{i}t_{\mathbf{i}}{}^{2}\right)\delta_{a}{}^{i}a^{-3/2} - \frac{3}{2}W^{a}{}_{i}\delta_{a}{}^{i}t_{\mathbf{i}}{}^{2} \; ; \tag{4.69}$$

$$\begin{aligned}\Pi^{(1)}_{ij} = &\left(\frac{3}{5}\left(U^{1}_{(ij)} - \frac{1}{3}U^{1}\delta_{ij}\right)t_{\mathbf{i}} + \frac{9}{10}\left(W^{1}_{(ij)} - \frac{1}{3}W^{1}\delta_{ij}\right){t_{\mathbf{i}}}^{2}\right)a \\ &- \left(\frac{3}{5}\left(U^{1}_{(ij)} - \frac{1}{3}U^{1}\delta_{ij}\right)t_{\mathbf{i}} - \frac{3}{5}\left(W^{1}_{(ij)} - \frac{1}{3}W^{1}\delta_{ij}\right){t_{\mathbf{i}}}^{2}\right)a^{-3/2} \\ &- \frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{1}\delta_{ij}\right){t_{\mathbf{i}}}^{2} - 3U^{2}_{(ij)}t_{\mathbf{i}}\left(a^{-1/2} - 1\right). \end{aligned} \tag{4.70}$$

We will define the following functions in order to lighten the next calculations,

$$\begin{aligned}\Pi_{ij} &= \mathcal{A}_{ij}(U,W)a + \mathcal{B}_{ij}(U,W)a^{-3/2} + \mathcal{C}_{ij}(U,W)(a^{-1/2} - 1) + \mathcal{D}_{ij}(U,W)\ ; \\ P^{a}_{\ i}\delta_{a}^{\ i} &= A(U,W)a + B(U,W)a^{-3/2} + C(U,W)\ .\end{aligned}$$

The antisymmetric and trace part solutions are directly generalised from the Newtonian solutions. Note the solutions we get for these two parts are exactly the same solutions we get in the general relativistic approach, chapter III section 3.3.3.

4.2 Validity of the generalised solution

The validity of the generated symmetric solutions has to be verified by injecting it in the relativistic traceless symmetric equation. Performing the integration of the Newtonian trace solution. We remind that the symmetric relativistic equation is written,

$$\ddot{\Pi}^{(1)}_{ij} + 3H\dot{\Pi}^{(1)}_{ij} - a^{-2}\Pi_{ij\ |k}^{\ |k(1)} = -a^{-2}\left(\mathcal{T}_{ij} + P^{(1)}_{|ij} - \frac{1}{3}P^{|k(1)}_{\ |k}\delta_{ij}\right)\ . \tag{4.71}$$

We introduced the generalised solution into the relativistic equation in order to check if this solution is correct,

$$\begin{aligned}&\mathcal{A}_{ij}(U,W)\left(\ddot{a} + 3H\dot{a}\right) + \mathcal{B}_{ij}(U,W)\left(-\frac{3}{2}\ddot{a}a^{-5/2} + \frac{15}{4}\dot{a}^{2}a^{-7/2} - \frac{9}{2}H\dot{a}a^{-5/2}\right) \\ &+ \mathcal{C}_{ij}(U,W)\left(-\frac{1}{2}\ddot{a}a^{-3/2} + \frac{3}{4}\dot{a}^{2}a^{-5/2} - \frac{3}{2}H\dot{a}a^{-3/2}\right) - \mathcal{A}_{ij\ |k}^{\ |k}(U,W)a^{-1} \\ &- \mathcal{B}_{ij\ |k}^{\ |k}(U,W)a^{-7/2} - \mathcal{C}_{ij\ |k}^{\ |k}(U,W)(a^{-5/2} - a^{-2}) - \mathcal{D}_{ij\ |k}^{\ |k}(U,W)a^{-2} \\ =&- a^{-2}\mathcal{T}_{ij} - A_{|ij}(U,W)a^{-1} - B_{|ij}(U,W)a^{-7/2} - C_{|ij}(U,W)a^{-2} \\ &+ \frac{1}{3}A^{|k}_{\ |k}(U,W)\delta_{ij}a^{-1} + \frac{1}{3}B^{|k}_{\ |k}(U,W)\delta_{ij}a^{-7/2} + \frac{1}{3}C^{|k}_{\ |k}(U,W)\delta_{ij}a^{-2}\end{aligned}$$

$$
\begin{aligned}
\Longrightarrow \quad & \mathcal{A}_{ij}(U,W)\frac{10}{9}{t_{\mathbf{i}}}^{-2}a^{-2} - \mathcal{C}_{ij}(U,W)\frac{2}{9}{t_{\mathbf{i}}}^{-2}a^{-7/2} - \mathcal{A}_{ij}{}^{|k}{}_{|k}(U,W)a^{-1} \\
& - \mathcal{B}_{ij}{}^{|k}{}_{|k}(U,W)a^{-7/2} - \mathcal{C}_{ij}{}^{|k}{}_{|k}(U,W)(a^{-5/2}-a^{-2}) - \mathcal{D}_{ij}{}^{|k}{}_{|k}(U,W)a^{-2} \\
= & - a^{-2}\mathcal{T}_{ij} - A_{|ij}(U,W)a^{-1} - B_{|ij}(U,W)a^{-7/2} - C_{|ij}(U,W)a^{-2} \\
& + \frac{1}{3}A^{|k}{}_{|k}(U,W)\delta_{ij}a^{-1} + \frac{1}{3}B^{|k}{}_{|k}(U,W)\delta_{ij}a^{-7/2} + \frac{1}{3}C^{|k}{}_{|k}(U,W)\delta_{ij}a^{-2}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad & \left(\mathcal{A}_{ij}(U,W)\frac{10}{9}{t_{\mathbf{i}}}^{-2} + \mathcal{C}_{ij}{}^{|k}{}_{|k}(U,W) - \mathcal{D}_{ij}{}^{|k}{}_{|k}(U,W)\right)a^{-2} \\
& - \left(\mathcal{C}_{ij}(U,W)\frac{2}{9}{t_{\mathbf{i}}}^{-2} + \mathcal{B}_{ij}{}^{|k}{}_{|k}(U,W)\right)a^{-7/2} - \mathcal{A}_{ij}{}^{|k}{}_{|k}(U,W)a^{-1} - \mathcal{C}_{ij}{}^{|k}{}_{|k}(U,W)a^{-5/2} \\
= & - \left(\mathcal{T}_{ij} + C_{|ij}(U,W) - \frac{1}{3}C^{|k}{}_{|k}(U,W)\delta_{ij}\right)a^{-2} - \left(A_{|ij}(U,W) - \frac{1}{3}A^{|k}{}_{|k}(U,W)\delta_{ij}\right)a^{-1} \\
& - \left(B_{|ij}(U,W) - \frac{1}{3}B^{|k}{}_{|k}(U,W)\delta_{ij}\right)a^{-7/2}\,. \qquad (4.72)
\end{aligned}
$$

Four modes appear in the equation: a^{-2}, a^{-1}, $a^{-7/2}$ and $a^{-5/2}$. The equation has to be true at any time, thus each mode must cancel itself. It leads to the following constraints for the initial data respectively extracted from the a^{-2}, a^{-1}, $a^{-7/2}$ and $a^{-5/2}$ modes:

$$
\begin{cases}
\frac{2}{3}\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}{}_{l}\delta_{ij}\right){t_{\mathbf{i}}}^{-1} + \left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}{}_{l}\delta_{ij}\right) \\
+\frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}{}_{l}\delta_{ij}\right)^{|k}_{|k}{t_{\mathbf{i}}}^{2} - 3U^{2}_{(ij)}{}^{|k}{}_{|k}t_{\mathbf{i}} \\
\quad = -\mathcal{T}_{ij} + \frac{3}{2}\left(W^{l^1}{}_{l|ij} - \frac{1}{3}W^{l^1|k}{}_{l}{}_{|k}\delta_{ij}\right){t_{\mathbf{i}}}^{2} \\
\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}{}_{l}\delta_{ij}\right)^{|k}_{|k} + \frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}{}_{l}\delta_{ij}\right)^{|k}_{|k}t_{\mathbf{i}} \\
\quad = \left(U^{l^1}{}_{l|ij} - \frac{1}{3}U^{l^1|k}{}_{l}{}_{|k}\delta_{ij}\right) + \frac{3}{2}\left(W^{l^1}{}_{l|ij} - \frac{1}{3}W^{l^1|k}{}_{l}{}_{|k}\delta_{ij}\right)t_{\mathbf{i}} \\
\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}{}_{l}\delta_{ij}\right)^{|k}_{|k}t_{\mathbf{i}} - \left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}{}_{l}\delta_{ij}\right)^{|k}_{|k}{t_{\mathbf{i}}}^{2} + \frac{10}{9}U^{2}_{(ij)}{t_{\mathbf{i}}}^{-1} \\
\quad = \left(U^{l^1}{}_{l|ij} - \frac{1}{3}U^{l^1|k}{}_{l}{}_{|k}\delta_{ij}\right)t_{\mathbf{i}} - \left(W^{l^1}{}_{l|ij} - \frac{1}{3}W^{l^1|k}{}_{l}{}_{|k}\delta_{ij}\right){t_{\mathbf{i}}}^{2} \\
U^{2}_{(ij)}{}^{|k}{}_{|k}t_{\mathbf{i}} = 0
\end{cases}
$$

$$
\Longleftrightarrow \quad \begin{cases}
\frac{2}{3}\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}_{\ l}\delta_{ij}\right) t_{\mathbf{i}}{}^{-1} + \left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right) \\
+\frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} t_{\mathbf{i}}{}^{2} - 3U^{2\ |k}_{(ij)\ |k} t_{\mathbf{i}} \\
\quad = -\mathcal{T}_{ij} + \frac{3}{2}\left(W^{l^1}_{\ l|ij} - \frac{1}{3}W^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) t_{\mathbf{i}}{}^{2} \\
\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} + \frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} t_{\mathbf{i}} \\
\quad = \left(U^{l^1}_{\ l|ij} - \frac{1}{3}U^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) + \frac{3}{2}\left(W^{l^1}_{\ l|ij} - \frac{1}{3}W^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) t_{\mathbf{i}} \\
\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} - \left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} t_{\mathbf{i}} + \frac{10}{9}U^{2}_{(ij)} t_{\mathbf{i}}{}^{-1} \\
\quad = \left(U^{l^1}_{\ l|ij} - \frac{1}{3}U^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) - \left(W^{l^1}_{\ l|ij} - \frac{1}{3}W^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) t_{\mathbf{i}} \\
U^{2\ |k}_{(ij)\ |k} = 0
\end{cases}
$$

$$
\Longleftrightarrow \quad \begin{cases}
\frac{2}{3}\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}_{\ l}\delta_{ij}\right) t_{\mathbf{i}}{}^{-1} + \left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right) = -\mathcal{T}_{ij} \\
-\frac{5}{2}\left(W^{1}_{(ij)} - \frac{1}{3}W^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} + \frac{10}{9}U^{2}_{(ij)} t_{\mathbf{i}}{}^{-2} = -\frac{5}{2}\left(W^{l^1}_{\ l|ij} - \frac{1}{3}W^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) \\
\left(U^{1}_{(ij)} - \frac{1}{3}U^{l^1}_{\ l}\delta_{ij}\right)^{|k}_{|k} = \left(U^{l^1}_{\ l|ij} - \frac{1}{3}U^{l^1|k}_{\ \ l\ \ |k}\delta_{ij}\right) \\
U^{2\ |k}_{(ij)\ |k} = 0
\end{cases}
$$

$$
\Longleftrightarrow \quad \begin{cases}
-\mathcal{T}_{ij} = \frac{2}{3t_{\mathbf{i}}}\left(U^{1}_{(ij)} - \frac{1}{3}U\delta_{ij}\right) + \left(W^{1}_{(ij)} - \frac{1}{3}W\delta_{ij}\right) \\
\qquad\qquad + \frac{3t_{\mathbf{i}}{}^{2}}{2}\left(W^{1|k}_{(ij)\ |k} - W^{k}_{\ k|ij}\right) \ ; \\
U^{1|k}_{(ij)\ |k} - \frac{2}{3}U^{2}_{(ij)} t_{\mathbf{i}}{}^{-2} = U^{k}_{\ k|ij} \ ; \\
W^{1|k}_{(ij)\ |k} - \frac{4}{9}U^{2}_{(ij)} t_{\mathbf{i}}{}^{-3} = W^{k}_{\ k|ij} \ ; \\
U^{2|k}_{(ij)\ |k} = 0 \ .
\end{cases} \tag{4.73}
$$

Thus, the symmetric traceless relativistic generalisation of the Newtonian solu-

tion proposed earlier is correct if it verifies the constraints (4.73). This brings the following informations about the initial data: (i) a relation between the initial traceless Ricci curvature $\mathcal{T}_{ij}$ and the generalised longitudinal initial velocity gradient $\overset{1}{U}_{ij}$ and acceleration gradient $\overset{1}{W}_{ij}$, (ii) constraints between components of the generalised velocity and acceleration gradients, and (iii) the generalised initial symmetric transverse velocity gradient $\overset{2}{U}_{(ij)}$ is harmonic.

These last constraints already give a lot of informations. Nevertheless, it is possible to go further. The Relativistic Zel'dovich Approximation (RZA) can be assumed (see [Buchert and Ostermann, 2012; Buchert et al., 2013]), some details about this approximation in the Newtonian case are available in chapter II. RZA is equivalent to the slaving conditions $U^k_{\ k} = W^k_{\ k} t_{\mathbf{i}}$. Moreover, the generalisation of the Poisson equation is also known as follows: $W^k_{\ k} = -4\pi G\delta\varrho_{\mathbf{i}}$. The slaving condition and the Poisson equation allow us to constraint the components of the initial velocity and acceleration gradients:

$$U^k_{\ k} = W^k_{\ k} t_{\mathbf{i}} = -4\pi G\delta\varrho_{\mathbf{i}} t_{\mathbf{i}} \ ; \tag{4.74}$$

$$U^{1|k}_{(ij)\ |k} - \frac{3}{2}\left(W^{1|k}_{(ij)\ |k} - W^k_{\ k|ij}\right) t_{\mathbf{i}} = U^k_{\ k|ij}$$

$$\Longrightarrow \quad U^{1|k}_{(ij)|k} t_{\mathbf{i}} - \frac{3}{2} W^{1|k}_{(ij)|k} {t_{\mathbf{i}}}^2 = -\frac{1}{2} W^k_{\ k|ij} {t_{\mathbf{i}}}^2 = \frac{1}{2} 4\pi G\delta\varrho_{\mathbf{i}|ij} {t_{\mathbf{i}}}^2 \ . \tag{4.75}$$

All these results are relativistic. It is possible to go further with additional constrains. This is what we proposed in the next section.

4.3 Integration of the initial data relation

The usual method to solve this last equation is to use the Fourier transformation and some of its properties. In general, we do not have a global coordinates system to describe the whole relativistic space–time. Thus it is impossible to define the Fourier transform. Here we work in the tangent spaces of the manifold and use the exact basis $\mathbf{d}X^i$ associated to these tangent spaces. If we want to define the Fourier transform with respect to these coordinates, they have to be global. Assuming these coordinates to be global reduces the problem to the Newtonian problem because all the tangent spaces coincide with the whole space–time. We choose to respect this assumption in this section to show the Newtonian aspect of most of the "relativistic" papers. The following convention is chosen for the Fourier transformation,

$$\begin{cases} \widetilde{F} = \mathrm{TF}\left[F\right] = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int_{\mathbb{R}^3} F(\vec{X}) e^{-i\vec{K}\cdot\vec{X}} d^3X \ ; \\ F = \mathrm{TF}^{-1}\left[\widetilde{F}\right] = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int_{\mathbb{R}^3} \widetilde{F}(\vec{K}) e^{i\vec{K}\cdot\vec{X}} d^3K \ ; \end{cases} \tag{4.76}$$

the convolution product is defined such as,

$$F(\vec{X}) * G(\vec{X}) = \int_{\mathbb{R}^3} F(\vec{X} - \vec{Y}) G(\vec{Y}) d^3Y \ ; \tag{4.77}$$

and the relations between convolution product and Fourier transform are,

$$\begin{cases} \mathrm{TF}\left[F \cdot G\right] = \dfrac{1}{(2\pi)^{3/2}} \ \mathrm{TF}\left[F\right] * \mathrm{TF}\left[G\right] \ ; \\ \mathrm{TF}^{-1}\left[F \cdot G\right] = \dfrac{1}{(2\pi)^{3/2}} \ \mathrm{TF}^{-1}\left[F\right] * \mathrm{TF}^{-1}\left[G\right] \ ; \\ \mathrm{TF}\left[F * G\right] = (2\pi)^{3/2} \ \mathrm{TF}\left[F\right] \cdot \mathrm{TF}\left[G\right] \ ; \\ \mathrm{TF}^{-1}\left[F * G\right] = (2\pi)^{3/2} \ \mathrm{TF}^{-1}\left[F\right] \cdot \mathrm{TF}^{-1}\left[G\right] \ . \end{cases} \tag{4.78}$$

Thus the Fourier transformation of the equation (4.75) is,

$$-K^2 \left(\widetilde{U}^1_{(ij)} t_{\mathbf{i}} - \frac{3}{2} \widetilde{W}^1_{(ij)} {t_{\mathbf{i}}}^2 \right) = -2\pi G {t_{\mathbf{i}}}^2 K_i K_j \widetilde{\delta \varrho_{\mathbf{i}}} \ . \tag{4.79}$$

This leads to the following relation,

$$\begin{aligned} U^1_{(ij)} t_{\mathbf{i}} - \frac{3}{2} W^1_{(ij)} {t_{\mathbf{i}}}^2 &= -2\pi G {t_{\mathbf{i}}}^2 \ \mathrm{TF}^{-1}\left[\mathrm{TF}\left[\delta \varrho_{\mathbf{i}|ij}\right] ||K||^{-2}\right](\vec{X}) \\ &= (2\pi)^{-1/2} \, G {t_{\mathbf{i}}}^2 \ \delta \varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_i K_j}{K^2}\right](\vec{X}) \ . \end{aligned} \tag{4.80}$$

The reader may remark that the trace leads to the Fourier transformation of a constant, which is a Dirac delta,

$$\mathrm{TF}^{-1}\left[1\right] = \int_{\mathbb{R}^3} e^{i\vec{K}\cdot\vec{X}} d^3K = (2\pi)^{3/2} \ \delta(\vec{X}) \ ; \tag{4.81}$$

and the convolution of a function with a Dirac Delta is simply the function

$$\delta \varrho_{\mathbf{i}}(\vec{X}) * \delta(\vec{X}) = \int_{\mathbb{R}^3} \delta \varrho_{\mathbf{i}}(\vec{X} - \vec{Y}) \delta(\vec{Y}) d^3Y = \delta \varrho_{\mathbf{i}}(\vec{X}) \ . \tag{4.82}$$

Thus, the trace of (4.79) leads to the generalised Poisson equation $W {t_{\mathbf{i}}}^2 = -4\pi G \delta \varrho_{\mathbf{i}} {t_{\mathbf{i}}}^2$. According to these last constraints, the traceless Ricci tensor can be write as follows:

$$\begin{aligned} -\mathcal{T}_{ij} = 2W^1_{(ij)} - \frac{5}{9} W \delta_{ij} + \frac{3}{2} W^{1|k}_{(ij)\ \ |k} {t_{\mathbf{i}}}^2 - \frac{3}{2} W_{|ij} {t_{\mathbf{i}}}^2 \\ + \frac{2}{3} \left((2\pi)^{-1/2} \, G {t_{\mathbf{i}}}^2 \ \delta \varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_i K_j}{K^2}\right] \right) \ . \end{aligned} \tag{4.83}$$

4.4 The RZA solution

The previous assumption allows us to perform additional treatments. We can also include the Relativistic Zel'dovich Approximation. The RZA generalised first order solution is,

$$\mathfrak{P}^a{}_i = 0 \; ; \tag{4.84}$$

$$\delta_a{}^i P^a_i = \frac{3}{2}\delta_a{}^i W^a{}_i t_{\mathbf{i}}{}^2 \left(a-1\right) \; ; \tag{4.85}$$

$$\begin{aligned}\Pi_{ij} = &\left(\frac{3}{5}U^1_{(ij)}t_{\mathbf{i}} + \frac{9}{10}W^1_{(ij)}t_{\mathbf{i}}{}^2 - \frac{1}{2}W t_{\mathbf{i}}{}^2\delta_{ij}\right)a - \frac{3}{5}\left(U^1_{(ij)}t_{\mathbf{i}} - W^1_{(ij)}t_{\mathbf{i}}{}^2\right)a^{-3/2} \\ &- \frac{3}{2}\left(W^1_{(ij)} - \frac{1}{3}W\delta_{ij}\right)t_{\mathbf{i}}{}^2 - 3U^2_{(ij)}t_{\mathbf{i}}\left(a^{-1/2}-1\right) \; ;\end{aligned} \tag{4.86}$$

which can be written in its complete form such as:

$$\begin{aligned}P_{ij} = &\frac{9}{5}W^1_{(ij)}t_{\mathbf{i}}{}^2 a - \frac{3}{10}W^1_{(ij)}t_{\mathbf{i}}{}^2 a^{-3/2} - \frac{3}{2}W^1_{(ij)}t_{\mathbf{i}}{}^2 - 3U^2_{(ij)}t_{\mathbf{i}}\left(a^{-1/2}-1\right) \\ &+ \frac{3}{5}\left(2\pi\right)^{-1/2} G t_{\mathbf{i}}{}^2 \; \delta\varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_i K_j}{K^2}\right]\left(a - a^{-3/2}\right) \; ;\end{aligned} \tag{4.87}$$

It is still possible to constraint the coefficients of the solution by injecting the RZA generalised solution into the traceless symmetric equation. As previously several modes appear and have to cancel themselves. Here there is only three different modes in the calculation, $a^{-7/2}$, a^{-2} and a^{-1}. This last mode leads to:

$$\begin{aligned}&-\frac{3}{5}U_{ij}{}^{|k}_{|k}t_{\mathbf{i}} - \frac{9}{10}W_{ij}{}^{|k}_{|k}t_{\mathbf{i}}{}^2 + \frac{1}{2}W^l{}_l{}^{|k}_{|k}t_{\mathbf{i}}{}^2\delta_{ij} = -\frac{3}{2}\left(W^k{}_{k|ij} - \frac{1}{3}W^l{}_l{}^{|k}_{|k}\delta_{ij}\right)t_{\mathbf{i}}{}^2 \\ \Longrightarrow \quad &U_{ij}{}^{|k}_{|k} + \frac{3}{2}W_{ij}{}^{|k}_{|k}t_{\mathbf{i}} = \frac{5}{2}W^k{}_{k|ij}t_{\mathbf{i}} \; .\end{aligned} \tag{4.88}$$

If we combine this last relation with the equation obtained earlier which constraints the initial data (4.75), we get:

$$U_{ij}{}^{|k}_{|k} = W_{ij}{}^{|k}_{|k}t_{\mathbf{i}} = W^k{}_{k|ij}t_{\mathbf{i}} \; . \tag{4.89}$$

Moreover, the relation (4.75) can be expressed in a more compact way, and the previous calculations, using the Fourier transform, allow us to obtain a relation between $W^1_{(ij)}$ and the initial density:

$$U^1_{(ij)}t_{\mathbf{i}}{}^{-1} = W^1_{(ij)} = -2(2\pi)^{-1/2}\; G\; \delta\varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_i K_j}{K^2}\right](\vec{X}) \; . \tag{4.90}$$

Thanks to this last relation the mode $a^{-7/2}$ leads to:

$$U^2_{(ij)} = 0 \; . \tag{4.91}$$

The last remaining mode leads to the RZA definition of the traceless Ricci tensor,

$$-\mathcal{T}_{ij} = \frac{5}{3}\left(W^{1}_{(ij)} - \frac{1}{3}W\delta_{ij}\right) \ . \tag{4.92}$$

The RZA Newtonian generalised first order solution is finally,

$$\begin{cases} \mathfrak{P}^{a}{}_{i} = 0 \ ; \\ \delta_{a}{}^{i}P^{a}_{i} = \frac{3}{2}\delta_{a}{}^{i}W^{a}_{i}t_{\mathbf{i}}{}^{2}\,(a-1) \ ; \\ \Pi_{ij} = \frac{3}{2}\left(W^{1}_{(ij)} - \frac{1}{3}\delta_{a}{}^{i}W^{a}_{i}\delta_{ij}\right)t_{\mathbf{i}}{}^{2}\,(a-1) \ ; \end{cases} \tag{4.93}$$

$$\Longleftrightarrow \quad P^{a}_{i} = \frac{3}{2}W^{a}_{i}t_{\mathbf{i}}{}^{2}\,(a-1) \ . \tag{4.94}$$

This can be written with respect the the initial density inhomogeneities such as,

$$P_{ij} = -\frac{3}{2}\left(2\,(2\pi)^{-1/2}\,Gt_{\mathbf{i}}{}^{2}\ \delta\varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_iK_j}{K^2}\right]\right)(a-1) \ . \tag{4.95}$$

The initial traceless Ricci tensor can also be expressed with respect to the initial density inhomogeneities,

$$\begin{aligned} -\mathcal{T}_{ij} &= \frac{5}{3}\left(W^{1}_{(ij)} - \frac{1}{3}W\delta_{ij}\right) \\ &= -\frac{10}{3}\,(2\pi)^{-1/2}\,Gt_{\mathbf{i}}{}^{2}\ \delta\varrho_{\mathbf{i}}(\vec{X}) * \mathrm{TF}^{-1}\left[\frac{K_iK_j}{K^2}\right] + \frac{20\pi Gt_{\mathbf{i}}{}^{2}}{9}\delta\varrho_{\mathbf{i}}\delta_{ij} \ . \end{aligned} \tag{4.96}$$

Then we can show the trace of these two relations leads to the well-known trace results,

$$\begin{aligned} P &= -\frac{3}{2}\frac{2}{\sqrt{2\pi}}Gt_{\mathbf{i}}{}^{2}\delta\varrho_{\mathbf{i}} * \mathrm{TF}^{-1}\,[1]\,(a-1) \\ &= -\frac{3}{2}\frac{2(2\pi)^{3/2}}{\sqrt{2\pi}}Gt_{\mathbf{i}}{}^{2}\delta\varrho_{\mathbf{i}} * \delta(\vec{X})\,(a-1) \\ &= -\frac{3}{2}4\pi Gt_{\mathbf{i}}{}^{2}\delta\varrho_{\mathbf{i}}(\vec{X})\,(a-1) = \frac{3}{2}Wt_{\mathbf{i}}{}^{2}\,(a-1) \ ; \end{aligned} \tag{4.97}$$

$$\begin{aligned} -\mathcal{T} &= -\frac{10}{3\sqrt{2\pi}}Gt_{\mathbf{i}}{}^{2}\delta\varrho_{\mathbf{i}} * \mathrm{TF}^{-1}\,[1] + \frac{20\pi Gt_{\mathbf{i}}{}^{2}}{3}\delta\varrho_{\mathbf{i}} \\ &= -\frac{10(2\pi)^{3/2}}{3\sqrt{2\pi}}Gt_{\mathbf{i}}{}^{2}\delta\varrho_{\mathbf{i}}(\vec{X}) + \frac{20\pi Gt_{\mathbf{i}}{}^{2}}{3}\delta\varrho_{\mathbf{i}} = 0 \ . \end{aligned} \tag{4.98}$$

The solution (4.94) is valid if the exact coordinates are supposed to be global, i.e. the space–time is reduced to a Newtonian space–time. Moreover, this solution is

identical to most of the usual solutions proposed in literature. For some authors, it corresponds to the scalar part of the perturbations. We can conclude that the scalar solution of the standard perturbation theory is in reality a Newtonian solution, it should exist a diffeomorphism which reduce the associated metric to an Euclidean metric.

4.5 Functional evaluation of the volume element

The other quantities are computed as functional evaluations with respect to the perturbations. Here, we propose to write down the expression of the volume element J:

$$
\begin{aligned}
J &= a^3\left[1+P+\frac{1}{2}\left(PP-P^i{}_jP^j{}_i\right)+\frac{1}{6}\left(PPP+2P^i{}_jP^j{}_kP^k{}_i-3PP^j{}_kP^k{}_j\right)\right]\\
&= a^3\left[1+\frac{3}{2}Wt_{\mathbf{i}}{}^2\left\{a-1\right\}+\frac{9}{8}t_{\mathbf{i}}{}^4\left(W^2-W^i{}_jW^j{}_i\right)\left\{a^2-2a+1\right\}\right.\\
&\quad\left.+\frac{9}{16}t_{\mathbf{i}}{}^6\left(W^3+2W^i{}_jW^j{}_kW^k{}_i-3WW^i{}_jW^j{}_i\right)\left\{a^3-3a^2+3a-1\right\}\right]\\
&= a^3\Bigg[1-6\pi G\delta\varrho_{\mathbf{i}}\ t_{\mathbf{i}}{}^2\left\{a-1\right\}\\
&\quad+\frac{9}{2}G^2t_{\mathbf{i}}{}^4\left\{4\pi^2\delta\varrho_{\mathbf{i}}{}^2-\frac{1}{2\pi}\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^iK_j}{K^2}\right]\right)\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^jK_i}{K^2}\right]\right)\right\}\\
&\qquad\times\left\{a^2-2a+1\right\}\\
&\quad-9G^3t_{\mathbf{i}}{}^6\left\{4\pi^3\delta\varrho_{\mathbf{i}}{}^3-\frac{3}{2}\delta_{\mathbf{i}}\varrho\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^iK_j}{K^2}\right]\right)\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^jK_i}{K^2}\right]\right)\right.\\
&\quad\left.+\frac{1}{(2\pi)^{3/2}}\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^iK_j}{K^2}\right]\right)\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^jK_k}{K^2}\right]\right)\left(\delta\varrho_{\mathbf{i}}*\mathrm{TF}^{-1}\left[\frac{K^kK_i}{K^2}\right]\right)\right\}\\
&\qquad\times\left\{a^3-3a^2+3a-1\right\}\Bigg]\\
&= a^3\Bigg[1-6\pi G\delta\varrho_{\mathbf{i}}\ t_{\mathbf{i}}{}^2\left\{a-1\right\}\\
&\quad+\frac{9}{2}G^2t_{\mathbf{i}}{}^4\left\{4\pi^2\delta\varrho_{\mathbf{i}}{}^2-\frac{1}{2\pi}(2\pi)^3\ \mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^i{}_j}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^j{}_i}\right]\right\}\left\{a^2-2a+1\right\}\\
&\quad-9G^3t_{\mathbf{i}}{}^6\left\{4\pi^3\delta\varrho_{\mathbf{i}}{}^3+(2\pi)^3\ \mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^i{}_j}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^j{}_k}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^k{}_i}\right]\right.\\
&\qquad\left.-\frac{3}{2}\delta\varrho_{\mathbf{i}}(2\pi)^3\ \mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^i{}_j}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^j{}_i}\right]\right\}\left\{a^3-3a^2+3a-1\right\}\Bigg]\,. \qquad (4.99)
\end{aligned}
$$

where $\widetilde{\Delta\varrho_{\mathbf{i}}{}^{i}{}_{j}} = \widetilde{\delta\varrho_{\mathbf{i}}}\left(K^i K_j/K^2\right)$. It is possible to make appear correlation functions and autocorrelation functions in the expression of the volume element:

$$\delta\varrho_{\mathbf{i}}{}^2 = \mathrm{TF}^{-1}\left[\widetilde{\delta\varrho_{\mathbf{i}}}\right]\mathrm{TF}^{-1}\left[\widetilde{\delta\varrho_{\mathbf{i}}}\right] = \frac{1}{(2\pi)^{3/2}}\mathrm{TF}^{-1}\left[\widetilde{\delta\varrho_{\mathbf{i}}} * \widetilde{\delta\varrho_{\mathbf{i}}}\right] \ ; \tag{4.100}$$

$$\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^{i}{}_{j}}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^{j}{}_{i}}\right] = \frac{1}{(2\pi)^{3/2}}\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^{i}{}_{j}} * \widetilde{\Delta\varrho_{\mathbf{i}}{}^{j}{}_{i}}\right] \ . \tag{4.101}$$

5 Example 2: constructing second order solutions for 'slaved initial data'

The second order equations have been computed before in sections 2.2 and 3.2. Here, we propose to solve the trace equation of motion. A similar space–time separation is assumed for the second order solutions and the associated equations are:

$$\begin{aligned}
&\ddot{P}^{(2)} + 2H\dot{P}^{(2)} - 4\pi G\varrho_{\mathbf{i}H}a^{-3}P^{(2)} \\
&\quad = \left(\ddot{P}^{i}{}_{j}{}^{(1)} + 2H\dot{P}^{i}{}_{j}{}^{(1)} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{i}{}_{j}{}^{(1)}\right)P^{j}{}_{i}{}^{(1)} \\
&\quad - \left(\ddot{P}^{(1)} + 2H\dot{P}^{(1)} - 2\pi G\varrho_{\mathbf{i}H}a^{-3}P^{(1)}\right)P^{(1)} \ ; \\
\Longrightarrow \quad & \begin{cases} \ddot{\xi}^{(2)} + 2\dfrac{\dot{a}}{a}\dot{\xi}^{(2)} + 3\dfrac{\ddot{a}}{a}\xi^{(2)} = \dfrac{3}{4}t_{\mathbf{i}}{}^2\left(a^{-1} - a^{-3}\right) \ ; \\ C^{(2)} = W^{1i}{}_{j}W^{1j}{}_{i} - WW \ . \end{cases}
\end{aligned} \tag{4.102}$$

Given the identical linear time operator D at every order of the hierarchy (4.40), the homogeneous part of the time differential equation is the same as for the first order equation. Only the source $G^{(2)} = (3/4)\ t_{\mathbf{i}}{}^2\left(a^{-1} - a^{-3}\right)$ is different, but still quite simple thanks to the slaving assumption. The homogeneous solution is then the same as before:

$$P^{(2)} = {}^{\mathbf{1}}C^{(2)}a + {}^{\mathbf{2}}C^{(2)}a^{-3/2} \ . \tag{4.103}$$

The superposition theorem assure that a superposition of two particular solutions is a particular solution. Thus, we can look for the solution for one part of the source and then the other part to obtain the complete particular solution. Moreover, as for the first order, the trace equation of evolution is formally similar to the divergence of the gravitational field equation and thus admit similar solution, it leads to:

$${}^{\mathbf{p}}\xi = \frac{9}{8}t_{\mathbf{i}}{}^4\left(1 + \frac{3}{7}a^2\right) \ . \tag{4.104}$$

Thus, the second order solution can be written such as,

$$P^{(2)} = {}^{\mathbf{1}}C^{(2)}a + {}^{\mathbf{2}}C^{(2)}a^{-3/2} + \frac{9}{8}t_{\mathbf{i}}{}^4\left(1 + \frac{3}{7}a^2\right)C^{(2)} \ . \tag{4.105}$$

The coefficient $C^{(2)}$ is an unknown spatial coefficient which has to be determined. To find the spatial coefficients of the solution, we need to relate them to the initial data. The initial values for the coframe and its time derivative have been chosen to be zero for any order n, with $n > 1$, in the hierarchy of solutions of equation (4.40). Therefore, we find the following system:

$$\begin{aligned} P^{(2)}(t_{\mathbf{i}}) &= {}^{\mathbf{1}}C^{(2)} + {}^{\mathbf{2}}C^{(2)} + \frac{45}{28}C^{(2)}{t_{\mathbf{i}}}^4 \ ; \\ \dot{P}^{(2)}(t_{\mathbf{i}}) &= \frac{2}{3t_{\mathbf{i}}}{}^{\mathbf{1}}C^{(2)} - \frac{1}{t_{\mathbf{i}}}{}^{\mathbf{2}}C^{(2)} + \frac{9}{14t_{\mathbf{i}}}C^{(2)}{t_{\mathbf{i}}}^4 \ . \end{aligned} \tag{4.106}$$

After some straightforward computations, this system solution is,

$$\begin{aligned} & \begin{cases} {}^{\mathbf{1}}C^{(2)} + {}^{\mathbf{2}}C^{(2)} + \frac{45}{28}C^{(2)}{t_{\mathbf{i}}}^4 = 0 \\ {}^{\mathbf{1}}C^{(2)} - \frac{3}{2}{}^{\mathbf{2}}C^{(2)} + \frac{27}{28}C^{(2)}{t_{\mathbf{i}}}^4 = 0 \end{cases} \\ \Longleftrightarrow \quad & \begin{cases} {}^{\mathbf{1}}C^{(2)} + {}^{\mathbf{2}}C^{(2)} + \frac{45}{28}C^{(2)}{t_{\mathbf{i}}}^4 = 0 \\ -\frac{5}{2}{}^{\mathbf{2}}C^{(2)} + \frac{9}{14}C^{(2)}{t_{\mathbf{i}}}^4 = 0 \end{cases} \\ \Longleftrightarrow \quad & \begin{cases} {}^{\mathbf{1}}C^{(2)} + {}^{\mathbf{2}}C^{(2)} + \frac{45}{28}C^{(2)}{t_{\mathbf{i}}}^4 = 0 \\ {}^{\mathbf{2}}C^{(2)} = -\frac{9}{35}C^{(2)}{t_{\mathbf{i}}}^4 \end{cases} \\ \Longleftrightarrow \quad & \begin{cases} {}^{\mathbf{1}}C^{(2)} = -\frac{27}{20}C^{(2)}{t_{\mathbf{i}}}^4 \ ; \\ {}^{\mathbf{2}}C^{(2)} = -\frac{9}{35}C^{(2)}{t_{\mathbf{i}}}^4 \ . \end{cases} \end{aligned}$$

Thus the RZA Newtonian generalised second order trace solution is,

$$\begin{aligned} P^{(2)} &= \frac{9}{4}{t_{\mathbf{i}}}^4\left(\frac{3}{14}a^2 - \frac{3}{5}a + \frac{1}{2} - \frac{4}{35}a^{-3/2}\right)\left(W^{1}{}_{j}^{i}W^{1}{}_{i}^{j} - WW\right) \\ &= \frac{9}{4}{t_{\mathbf{i}}}^4\left(\frac{3}{14}a^2 - \frac{3}{5}a + \frac{1}{2} - \frac{4}{35}a^{-3/2}\right) \\ &\qquad \times 16\pi^2G^2{t_{\mathbf{i}}}^4\left(\ \mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^{i}{}_{j}}\right]\mathrm{TF}^{-1}\left[\widetilde{\Delta\varrho_{\mathbf{i}}{}^{j}{}_{i}}\right] - \delta\varrho_{\mathbf{i}}{}^2\right) \ . \end{aligned} \tag{4.107}$$

The first equality is true in the relativistic case and we recover a generalisation of the Newtonian solution proposed in chapter II. The second equality is more restrictive, in order to write it like this we have to assume the existence of the Fourier transform. Thus the second relation is true only in the Newtonian case.

6 Concluding remarks

We developed in this chapter the resolution scheme for any order of perturbation. We focused ourself on the gravitoelectric equations and considered the remaining gravitomagnetic equations as constraints. Thus we can see the electric part of the relativistic solution is effectively formally close to the Newtonian solution available in [Ehlers and Buchert, 1997]. Nevertheless, there is still an equation of evolution: the symmetric traceless part of the $3+1$ equation of evolution. There is no obvious equivalent equation in the Newtonian description of a gravitational system (see, however, Chapter V). The three first sections lead to relativistic solutions for the trace and the antisymmetric part of the perturbations.

The general results are strongly dependent of our hypothesis of course. We developed a relativistic perturbation theory in order to describe the dynamics of the space–time. Thus in a general case the space–time is curved and can be mathematically described thanks to non–Euclidean manifolds. The equations are written in the tangent space at a given point of the general manifold. Thus this system of coordinates is local and cannot describe the dynamics through the whole space–time. This local aspect prevent us to use some tools or properties which required a global system of coordinates, such as the Fourier transform. If we make the assumption that we can build a global system of coordinates, it is possible to use the Fourier transform. The development which followed this assumption led to a solution generalised from the Newtonian vectorial solution. Nevertheless these results are identical to the usual results presented in the standard perturbation theory papers. These solutions are obtained thanks to the assumption of the existence of a global coordinates system, i.e. in a flat (or constantly curved) space–time in section 4. The results are quasi–Newtonian because in our approach global coordinates do not exist. We managed to show that it is possible to build a relativistic solution from the Newtonian approach for the symmetric traceless part too. Moreover, assuming the relativistic generalisation of the Zel'dovich Approximation (equivalent to the slaving condition), an Einstein–de Sitter space–time and the existence of global coordinates (i.e. a Newtonian restriction in our case) the solution takes a simple expression: every part of the first order solution have an identical time dependence in $(a-1)$ which correspond to the scalar solution of the standard perturbation theory. This was assumed in several papers, for instance [Buchert et al., 2013], but I did not find conclusive explanation in literature. It finally led to a generalisation of the slaving condition for the traceless part of the initial velocity and acceleration, but also to a relation between these terms and functions which depend of the initial density fluctuations.

This strong restriction of the first order solution will be used in chapter VI to study the Weyl curvature hypothesis in a simpler case. Nevertheless it will describe a more complex case than what can be found in literature. Indeed our development will contain the magnetic part of the Weyl tensor, even if the solution seems to be

only Newtonian (i.e. the scalar part of the standard perturbations).

We are also developing improvement to these results by the consideration of separable and non–separable modes in the perturbations. It could lead to a redefinition of the electromagnetic separation of the dynamics. The electric part would be linked to the scalar perturbations whereas the magnetic part would be linked to the tensorial perturbations.

Chapter V

Silent Universe

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1 Introduction

In the context of inhomogeneous cosmology, several models provide special solutions to the Einstein equations. Moreover, some of them allow us to avoid the addition of a cosmological constant while describing an accelerated expansion of the Universe. Nevertheless, controversial and strong hypothesis and constraints are involved. We can cite for instance the LTB (Lemaître–Tolman–Bondi) model which describes a spherical dust cloud in expansion or collapse (see [Enqvist, 2008] for an LTB overview). The LTB solutions, for an inhomogeneous cosmology are able to trigger a Universe in accelerated expansion without Dark Energy by considering an under–dense region of 300Mpc. This effect is due to the negative curvature associated with an under–dense region. Recent studies highlight the effect of voids and their negative pressure, which even have an antilensing effect [Bolejko et al., 2013]. Different classes of solutions and applications are presented in [Bolejko et al., 2011] and the inhomogeneities effects on the Universe dynamics are described in

[Clarkson et al., 2011].

More detailed models of the Universe propose to study a collection of LTB regions embedded in a Friedmannian background Universe: the Swiss Cheese model. For the model to be coherent, it is essential to consider additional constraints: the boundary conditions and the tidal forces between regions for instance. The LTB model assumes the following metric form:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R'^2(t,r)}{1+2E(r)}\mathrm{d}r^2 - R^2(t,r)\mathrm{d}\Omega^2 \ . \tag{5.1}$$

with ds the line element, dt the infinitesimal time element, dr the infinitesimal radial coordinate, dΩ the infinitesimal solid angle element, $R(t,r)$ a radial function, $R'(t,r) = (\partial R/\partial r)(t,r)$ and $E(r)$ a local parameter identifiable to the energy per unit mass of the dust particles at the radial position r. This metric describes spherically symmetric inhomogeneous repartition of matter. An important implication of the LTB solution is the vanishing of the Weyl tensor magnetic part $H_{\mu\nu}$. Thus we decided to look for specific solutions which respect the approximation $H_{\mu\nu} = 0$. The class of solutions provided by this restriction is called "silent Universe." The designation "silent" refers to the vanishing of the magnetic part of the Weyl tensor, which means waves cannot propagate through space–time: the Universe is silent. For previous studies about the silent Universe class of solutions the reader can refer to [Bruni et al., 1995, 1996; Sopuerta, 1997; van Elst and Uggla, 1997]. Other papers deal with the solutions of this type and also explain that the first order FLRW solution leads to a silent Universe solution [Matarrese et al., 1993; Matarrese et al., 1994a,b]. A key result is the co–diagonalisability of the shear and electric part of the Weyl tensor showed by Barnes [Barnes and Rowlingson, 1989]. In section 2, I present the Newtonian approach of the silent Universe and introduce the dynamical description of gravitation. Then, I perform the same work in the relativistic framework and I will take a look to the Einstein equations, which provide a closed system of Ordinary Differential Equations (ODE), in section 3. Both sections also treat the problem of the average process. I proposed a small discussion about closure relations in section 4. It consists of finding an additional relation in order to close the average system of equations. In section 5, I present qualitative results obtained from a simple numerical simulation based on the silent Universe hypothesis. And I finally gather concluding remarks in the last section 6.

2 Newtonian approach

2.1 Newtonian dynamical equations

The vectorial Euler–Newton system (ENS) is composed of:

- the continuity equation (or mass conservation),

$$\dot{\varrho} = -\theta\varrho \ ; \tag{5.2}$$

- the Euler equation,

$$\frac{d\vec{v}}{dt} = \vec{g} \; ; \tag{5.3}$$

- and the Newtonian field equations (inspired by the Maxwell equations) and describing the Newtonian dynamics of a self–gravitating system,

$$\vec{\nabla} \times \vec{g} = \vec{0} \; ; \tag{5.4}$$

$$\vec{\nabla} \cdot \vec{g} = \Lambda - 4\pi G \varrho \; ; \tag{5.5}$$

where the variables are $\varrho(t, \vec{x})$ the density field, $\vec{v}(t, \vec{x})$ the velocity field, $\vec{g}(t, \vec{x})$ the acceleration field, $\theta(t, \vec{x})$ the expansion rate and the constants Λ, G respectively the cosmological constant and the gravitational constant. The dynamical formulation of the equations is a coordinate description. It means that the form of the equations depends on the coordinates (here inertial non–rotating coordinates). This description of the equations is useful to develop the silent Universe approach and reads:

$$\begin{cases} \dot{\varrho} = -\theta \varrho \; ; \\ \dfrac{dv_i}{dt} = g_i \; ; \\ g_{[i,j]} = 0 \; ; \\ g_{i,i} = \Lambda - 4\pi G \varrho \; . \end{cases} \tag{5.6}$$

The combination of the spatial derivative of the gravitation field $g_{i,j}$ and the Euler equation allows us to write the total derivative of the velocity gradient, defined by $v_{i,j} = h_{k,i} \dot{f}^k{}_{|j}$:

$$\begin{aligned} g_{i,j} &= \left(\frac{dv_i}{dt} \right)_{,j} = \left(\frac{\partial v_i}{\partial t} + v_k v_{i,k} \right)_{,j} \\ &= \frac{\partial v_{i,j}}{\partial t} + v_{k,j} v_{i,k} + v_k v_{i,kj} \\ &= \frac{dv_{i,j}}{dt} + v_{k,j} v_{i,k} \; . \end{aligned} \tag{5.7}$$

This last relation is the equation of evolution of the gradient of velocity along a trajectory (or flow line) $\vec{f}$. It is very convenient to decompose the velocity gradient matrix into its symmetric and antisymmetric part (any matrix admits such a decomposition):

$$v_{i,j} = v_{(i,j)} + v_{[i,j]}$$

$$
\begin{aligned}
&= \Theta_{ij} + \omega_{ij} \\
&= \left(\frac{1}{3}\theta\delta_{ij} + \sigma_{ij}\right) + \omega_{ij} \; ;
\end{aligned} \tag{5.8}
$$

where Θ_{ij} is the expansion tensor, ω_{ij} the vorticity tensor, θ the rate of expansion (trace of the expansion tensor) and σ_{ij} the shear (the tracefree part of the expansion tensor). For reasons of simplicity only irrotational dust matter is considered, thus the vorticity tensor is zero. This decomposition allows us to split the equation of evolution of the velocity gradient into its trace and tracefree parts, which are respectively the equation of evolution of the expansion rate and shear tensor.

$$
\begin{cases}
\dot{\varrho} = -\theta\varrho \; ; \\
\dot{\theta} = -\dfrac{1}{3}\theta^2 - 2\sigma^2 + (\Lambda - 4\pi G\varrho) \; ; \\
\dot{\sigma}_{ij} = -\dfrac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} + \dfrac{2}{3}\sigma^2\delta_{ij} + \varepsilon_{ij} \; ;
\end{cases} \tag{5.9}
$$

where $\sigma_{ij}\sigma^{ji} = 2\sigma^2$. The tidal force tensor is the tracefree symmetric part of the acceleration gradient defined by $\varepsilon_{ij} = g_{i,j} - (1/3)\; g_{k,k}\; \delta_{ij}$. The second equation is the Raychaudhuri equation. The obtained system is only composed of five ordinary differential equations but is not closed because of the eight variables involved; an evolution equation for ε_{ij} is missing. Fortunately, it can be described by the gravitation equation (which are partial equations). Note that it is impossible to have $\dot{\theta} > 0$ without a positive cosmological constant: it could be an argument in favour to Dark Energy but these equations are a local description of the dynamics whereas the expansion is a non–local behaviour. As we shall see, the non–local average behaviour can produces a positive contribution.

Equations of evolution for the tidal force tensor allow us to close the silent system of equations (see [Kofman and Pogosyan, 1995]). Note there exists a more general description: the Bertschinger–Hamilton theory (see [Bertschinger and Hamilton, 1994; Ellis and Dunsby, 1997]). They take the following form:

$$
\frac{d\varepsilon_{ij}}{dt} + \frac{\dot{a}}{a}\varepsilon_{ij} - \nabla_k \epsilon^{kl}{}_{(i}H_{j)l} + \theta\varepsilon_{ij} - \delta_{ij}\sigma^{kl}\varepsilon_{kl} - 3\sigma^k{}_{(i}\varepsilon_{j)k} = -4\pi G a^2 \varrho\; \sigma_{ij} \; ; \tag{5.10}
$$

$$
\frac{dH_{ij}}{dt} + \frac{\dot{a}}{a}H_{ij} + \nabla_k \epsilon^{kl}{}_{(i}\varepsilon_{j)l} + \theta H_{ij} - \delta_{ij}\sigma^{kl}H_{kl} - 3\sigma^k{}_{(i}H_{j)k} = 0 \; ; \tag{5.11}
$$

with the tidal force tensor ε_{ij}, a magnetic–like part H_{ij}, $a(t)$ the scale factor, δ_{ij} the kronecker symbol and ϵ^{ilk} the Levi-Civita pseudo–tensor. In the case of a silent Universe $H_{ij} = 0$ and these equations get simpler:

$$
\frac{d\varepsilon_{ij}}{dt} + \frac{\dot{a}}{a}\varepsilon_{ij} + \theta\varepsilon_{ij} - \delta_{ij}\sigma^{kl}\varepsilon_{kl} - 3\sigma^k{}_{(i}\varepsilon_{j)k} = -4\pi G a^2 \varrho\sigma_{ij} \; ; \tag{5.12}
$$

$$
\nabla_k \epsilon^{kl}{}_{(i}\varepsilon_{j)l} = 0 \; . \tag{5.13}
$$

Thus, the equations extracted from the Kofman–Pogosyan approach and the assumption of a silent Universe leads to a closed Newtonian system of equations: it contains eight equations and eight variable functions. Note these equations are naturally local. One of the aim of a model is to describe the global (or non–local) behaviour of the Universe in order to extract informations about its History. The local system of equations has to be extrapolated to obtain the non–local evolution of the Universe. It can be achieved by considering the spatial average of the equations and variables. The next section intends to describe properly the Newtonian averaging process.

2.2 Newtonian averaging process

Let us define the average of a scalar function $A(\vec{x}, t)$ over a domain $\mathcal{D}$ of volume $V_{\mathcal{D}}$ by:

$$\langle A \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} A \; d^3x \quad \text{with} \quad V_{\mathcal{D}} = \int\limits_{\mathcal{D}} d^3x \; . \tag{5.14}$$

The time derivative operator and the averaging operator do not necessary commute, but a precise commutation makes naturally appear additional effective terms:

$$\begin{aligned}
\frac{d\langle A \rangle_{\mathcal{D}}}{dt} &= \frac{d}{dt}\frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} A \; d^3x + \frac{1}{V_{\mathcal{D}}}\frac{d}{dt} \int\limits_{\mathcal{D}} A \; d^3x \\
&= -\frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}_0} \dot{J} \; d^3X \cdot \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} A \; d^3x + \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}_0} \left(\frac{dA}{dt} J + A \frac{dJ}{dt} \right) d^3X \\
&= -\frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} \theta \; d^3x \cdot \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} A \; d^3x + \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} \frac{dA}{dt} d^3x + \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}_0} A\theta \; d^3x \\
&= -\langle \theta \rangle_{\mathcal{D}} \langle A \rangle_{\mathcal{D}} + \left\langle \frac{dA}{dt} \right\rangle_{\mathcal{D}} + \langle A\theta \rangle_{\mathcal{D}} \; ;
\end{aligned} \tag{5.15}$$

with $\mathcal{D}_0$ the Lagrangian (or covariant) domain, x the Eulerian coordinates, X the Lagrangian coordinates and the Jacobian J which respects the relation $\dot{J} = J\theta$. The mass conservation and evolution equation of the expansion rate are scalar equations, thus it is possible to apply the average operator to them:

$$\frac{d\langle \varrho \rangle_{\mathcal{D}}}{dt} = -\langle \theta \rangle_{\mathcal{D}} \langle \varrho \rangle_{\mathcal{D}} \; ; \tag{5.16}$$

$$\begin{aligned}
\frac{d\langle \theta \rangle_{\mathcal{D}}}{dt} &= \frac{2}{3} \left\langle \theta^2 \right\rangle_{\mathcal{D}} - 2 \left\langle \sigma^2 \right\rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 + \Lambda - 4\pi G \langle \varrho \rangle_{\mathcal{D}} \\
&= \mathcal{Q}_{\mathcal{D}} - \frac{1}{3} \langle \theta \rangle_{\mathcal{D}}^2 + \Lambda - 4\pi G \langle \varrho \rangle_{\mathcal{D}} \; .
\end{aligned} \tag{5.17}$$

The second equation is the average Raychaudhuri equation. The quantity $\mathcal{Q}_{\mathcal{D}} = (2/3)\left(\left\langle\theta^2\right\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right) - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}}$ is called the kinematical backreaction and measures the discrepancy with respect the FLRW model. The sign of this term depends explicitly on the standard deviation of the expansion rate and the squared shear which are related to the inhomogeneities. Without any other constraint about these quantities, the kinematical backreaction can be positive or negative. A positive backreaction would mimic the effect of Dark Energy, whereas a negative one would mimic the effect of Dark Matter. It is possible to go further in the interpretation and obtain an equation for the average scale factor. Let us define an effective scale factor by the volume of a domain $\mathcal{D}$ in a time–hypersurface, normalised by the volume of the initial domain D_i:

$$a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}_i}}\right)^{1/3} . \tag{5.18}$$

The equation of evolution of the average expansion rate can be modified to become an equation for the effective scale factor. It is due to the fact that the averaged rate of expansion $\langle\theta\rangle_{\mathcal{D}}$ can be related to the volume $V_{\mathcal{D}}$ and the scale factor:

$$\begin{aligned}
\langle\theta\rangle_{\mathcal{D}} &= \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \theta\ d^3x = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}_0} \theta J\ d^3X = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}_0} \dot{J}\ d^3X \\
&= \frac{1}{V_{\mathcal{D}}}\frac{d}{dt}\int_{\mathcal{D}_0} J\ d^3X = \frac{1}{V_{\mathcal{D}}}\frac{d}{dt}\int_{\mathcal{D}_0} d^3x = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} \\
&= 3\frac{\dot{a}_{\mathcal{D}} a_{\mathcal{D}}^2}{a_{\mathcal{D}}^3} = 3H_{\mathcal{D}}\ ;
\end{aligned} \tag{5.19}$$

and the total derivative of this averaged expansion rate is,

$$\dot{\langle\theta\rangle}_{\mathcal{D}} = 3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} - 3\frac{\dot{a}_{\mathcal{D}}^2}{a_{\mathcal{D}}^2} = 3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} - \frac{1}{3}\langle\theta\rangle_{\mathcal{D}}^2 . \tag{5.20}$$

Now by a simple substitution of the average fluid expansion rate and its derivative, the equation of evolution for the average expansion rate (5.17) reads,

$$\begin{aligned}
&\dot{\langle\theta\rangle}_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}} - \frac{1}{3}\langle\theta\rangle_{\mathcal{D}}^2 + \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}} \\
\Longleftrightarrow\quad & 3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} - \frac{1}{3}\langle\theta\rangle_{\mathcal{D}}^2 = \mathcal{Q}_{\mathcal{D}} - \frac{1}{3}\langle\theta\rangle_{\mathcal{D}}^2 + \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}} \\
\Longleftrightarrow\quad & 3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = \mathcal{Q}_{\mathcal{D}} + \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}} .
\end{aligned} \tag{5.21}$$

After the integration of this expansion law, the following Friedmann–like equation is obtained,

$$3H_{\mathcal{D}}^2 + 3\frac{k_{\mathcal{D}}}{a_{\mathcal{D}}^2} - 8\pi G\langle\varrho\rangle_{\mathcal{D}} - \Lambda = \frac{1}{a_{\mathcal{D}}^2}\int_{t_{\mathrm{i}}}^{t} \mathcal{Q}_{\mathcal{D}}\frac{da_{\mathcal{D}}^2}{dt'}dt' \; ; \tag{5.22}$$

with the effective Hubble factor $H_{\mathcal{D}} = \dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ and a constant parameter of curvature $k_{\mathcal{D}}$. These two equations ((5.21),(5.22)) are known as the Buchert equations [Buchert, 2000a]. Without any additional constraint, the kinematical backreaction $\mathcal{Q}_{\mathcal{D}}$ can be negative or positive. Then the cosmological constant is not required anymore to trigger a positive $\langle\theta\rangle^{\cdot}_{\mathcal{D}}$. The backreaction is directly linked to the inhomogeneities through the standard deviation of the expansion rate and the opposite of the shear scalar. Such a positive backreaction is worth considering as a substitute to Dark Energy.

3 Relativistic approach

3.1 Relativistic dynamical equations

Our relativistic approach is based on the $3+1$ space–time foliation, the space is reduced to an hypersurface orthogonal to time as described in chapter III. The hypersurfaces are built to be associated to a specific time: the proper time. Thus it is quite easy to generalise the Newtonian quantities to their relativistic analogs. The exact Newtonian gradient of deformation becomes the Cartan coefficients, as well as their inverse:

$$\begin{cases} f^i_{|j} \longrightarrow \eta^a_j \; ; \\ h_i{}^{|j} \longrightarrow e_a{}^j \; . \end{cases} \tag{5.23}$$

The concepts of Eulerian and Lagrangian frames are no longer available in general relativity but it is possible to set their relativistic analogs. Let us x^i be the Newtonian Eulerian coordinates and X^i the Newtonian Lagrangian coordinates, linked by the gradient of deformation $dx^i = f^i{}_{|k}dX^k$ in the Newtonian description of the gravitation. Their relativistic generalisations are the non–exact basis $\boldsymbol{\eta}^a$ and the exact basis $\mathbf{d}X^i$, linked by the generalisation of the gradient of deformation which are the coefficients of the exact basis, the Cartan coframes,

$$\begin{cases} x^i \longrightarrow \boldsymbol{\eta}^a \; ; \\ X^i \longrightarrow \delta^a{}_i\mathbf{d}X^i \; ; \\ dx^i = f^i{}_{|k}dX^k \longrightarrow \boldsymbol{\eta}^a = \eta^a{}_k\mathbf{d}X^k \; . \end{cases} \tag{5.24}$$

The inverse of the Cartan coframe is defined by $\mathbf{e}_a = e_a{}^k(\partial/\partial X^k)$. According to this generalisation of the coordinates, it is direct to write the relativistic analogs

of the velocity gradient $v_{i,j} = h_{k,i}\dot{f}^k_{|j}$ and metric,

$$\begin{cases} g_{ij} = \delta_{ab}\eta^a{}_i\eta^b{}_j \ ; \\ \Theta^i{}_j = e_a{}^i\dot{\eta}^a{}_j \ ; \\ \Theta_{ij} = \delta_{ab}\eta^a{}_i\dot{\eta}^b{}_j = \dfrac{1}{2}\dot{g}_{ij} \ . \end{cases} \tag{5.25}$$

Here I choose to work in the orthonormal basis. A work in the orthogonal basis will lead to the same result, since I perform some preliminary calculations not mentioned here, a discussion about the basis choice is available in [Buchert et al., 2013]. In general relativity, we do not speak about velocity gradient anymore since the gradient is not defined. Note in the 4–dimensions description the expansion tensor also describes an expansion in time. As for the Newtonian approach, the dynamical decomposition of the expansion tensor is available (a 2–tensor is a matrix and admit a decomposition between its symmetric and antisymmetric parts): $\Theta_{ij} = (1/3)\ \theta g_{ij} + \sigma_{ij} + \omega_{ij}$. With the trace of the expansion tensor (or expansion rate) θ, the symmetric traceless shear tensor σ_{ij} and the antisymmetric traceless vorticity tensor ω_{ij} (zero for an irrotational fluid model). The last thing is to write the relativistic generalisation of the Newtonian field strength gradient: $\dot{\Theta}^i{}_j + \Theta^i{}_k\Theta^k{}_j = \mathcal{F}^i{}_j$. This last quantity respects the following equations in order to be in agreement with the Einstein equations:

$$\begin{cases} \mathcal{F}^k{}_k = \dot{\Theta}^k{}_k + \Theta^k{}_l\Theta^l{}_k = \Lambda - 4\pi G\varrho \ ; \\ \mathcal{F}_{[ij]} = \dot{\Theta}_{[ij]} = 0 \ . \end{cases} \tag{5.26}$$

Using the decomposition of the expansion tensor the previous system of equations allows us to write the following equations:

- the first equation is the trace expression of the tidal force tensor which leads to an equation of evolution for the expansion rate (the Raychaudhuri equation),

$$\begin{aligned} \mathcal{F}^k{}_k &= \dot{\theta} + \left(\frac{1}{3}\theta\delta^k{}_l + \sigma^k{}_l\right)\left(\frac{1}{3}\theta\delta^l{}_k + \sigma^l{}_k\right) \\ &= \dot{\theta} + \frac{1}{3}\theta^2 + \sigma^k{}_l\sigma^l{}_k + \frac{2}{3}\theta\sigma^k{}_l\delta^l{}_k = 0 \\ \Longleftrightarrow \quad \dot{\theta} &= -\frac{1}{3}\theta^2 - 2\sigma^2 + \Lambda - 4\pi G\varrho \ ; \end{aligned} \tag{5.27}$$

- and the second is the dynamical expression of the electric part of the Weyl tensor (or tracefree part of the field strength tensor) which leads to the equation of evolution for the traceless shear tensor,

$$-E^i{}_j = \mathcal{F}^i{}_j - \frac{1}{3}\mathcal{F}^k{}_k\delta^i{}_j$$

$$
\begin{aligned}
&= \frac{1}{3}\dot{\theta}\delta^i{}_j + \dot{\sigma}^i{}_j + \left(\frac{1}{3}\theta\delta^i{}_k + \sigma^i{}_k\right)\left(\frac{1}{3}\theta\delta^k{}_j + \sigma^k{}_j\right) \\
&\quad - \frac{1}{3}\dot{\theta}\delta^i{}_j - \frac{1}{3}\left(\frac{1}{3}\theta\delta^k{}_l + \sigma^k{}_l\right)\left(\frac{1}{3}\theta\delta^l{}_k + \sigma^l{}_k\right)\delta^i{}_j \\
&= \dot{\sigma}^i{}_j + \frac{1}{9}\theta^2\delta^i{}_j + \frac{2}{3}\theta\sigma^i{}_j + \sigma^i{}_k\sigma^k{}_j - \frac{1}{9}\theta^2\delta^i{}_j - \frac{2}{27}\theta\sigma^k{}_l\delta^l{}_k - \frac{1}{3}\sigma^k{}_l\sigma^l{}_k\delta^i{}_j \\
\Longleftrightarrow \quad & \dot{\sigma}^i{}_j = -\frac{2}{3}\theta\sigma^i{}_j - \sigma^i{}_k\sigma^k{}_j + \frac{2}{3}\sigma^2\delta^i{}_j - E^i{}_j \,. \qquad (5.28)
\end{aligned}
$$

where $E^i{}_j$ is the electric part of the Weyl tensor, defined as the generalisation of the Newtonian tidal force tensor: $-E^i{}_j = \mathcal{F}^i{}_j - (1/3)\,\mathcal{F}^k{}_k\delta^i{}_j$.

3.2 Weyl tensor

The Weyl tensor is defined as the tracefree part of the 4–Riemann curvature (see [Hawking, 1966; Ellis, 1973; Bertschinger and Hamilton, 1994] for more details). This tensor describes the curvature of space–time and respects laws similar to the electromagnetism. Indeed, this tensor is built as an analogy of the Faraday tensor which allows us to describe completely the electromagnetism. Let us define the Weyl tensor by the traceless part of the 4–Riemann tensor curvature:

$$
C^{\mu\kappa}{}_{\nu\lambda} = {}^{(4)}R^{\mu\kappa}{}_{\nu\lambda} - 2\delta^{[\mu}_{[\nu}\,{}^{(4)}R^{\kappa]}{}_{\lambda]} + \frac{1}{3}\delta^{[\mu}_{[\nu}\delta^{\kappa]}_{\lambda]}\,{}^{(4)}R \,. \qquad (5.29)
$$

This tensor admits an irreducible decomposition into an electric–like and a magnetic–like tensor, respectively called the electric part and the magnetic part of the Weyl tensor still according to the analogy with the electromagnetism:

$$
E_{\mu\nu} = C_{\mu\kappa\nu\lambda}u^\kappa u^\lambda \;; \qquad (5.30)
$$

$$
H_{\mu\nu} = \frac{\sqrt{-{}^{(4)}g}}{2}\epsilon_{\varrho\tau\kappa(\mu}C^{\varrho\tau}{}_{\nu)\lambda}u^\kappa u^\lambda \;; \qquad (5.31)
$$

where the diagonal matrix $u^\kappa = u^\lambda = \mathrm{diag}(1,0,0,0)$. Then the only non zero components are $u^0 = 1$. Nevertheless, since I decided to work in a $3+1$ space–time foliation, as justified earlier, the 4–metric is reduced to:

$$
g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & . & . & . \\ 0 & . & g_{ij} & . \\ 0 & . & . & . \end{pmatrix} . \qquad (5.32)
$$

Moreover, in comoving coordinates the determinant of the metric $\det(g_{ij}) = g$ respects:

$$
g = \det(g_{ij}) = \frac{1}{6}\epsilon^{ijk}\epsilon^{lmn}g_{il}g_{jm}g_{kn}
$$

$$\Longleftrightarrow \quad \epsilon_{ijk} = g^{-1}\,\epsilon^{lmn} g_{il} g_{jm} g_{kn} \quad \text{with} \quad \sqrt{g} = J\ . \tag{5.33}$$

I recall the expression of the 4–Riemann tensor curvature with respect to the Christoffel connection and the expression of the Christoffel connection with respect to the metric:

$$^{(4)}R^{\mu}{}_{\nu\kappa\lambda} = \Gamma^{\mu}{}_{\lambda\nu|\kappa} - \Gamma^{\mu}{}_{\kappa\nu|\lambda} + \Gamma^{\mu}{}_{\kappa\eta}\Gamma^{\eta}{}_{\lambda\nu} - \Gamma^{\mu}{}_{\lambda\eta}\Gamma^{\eta}{}_{\kappa\nu}\ ; \tag{5.34}$$

$$\Gamma^{\mu}{}_{\nu\kappa} = \frac{1}{2} g^{\mu\eta}(g_{\nu\eta|\kappa} + g_{\kappa\eta|\nu} - g_{\nu\kappa|\eta})\ . \tag{5.35}$$

Now the basic tools are set it is possible to compute the spatial parts of the Weyl tensor in our $3+1$ space–time foliation,

$$\begin{aligned}
E^{i}{}_{j} =& C^{i}{}_{\kappa j\lambda} u^{\kappa} u^{\lambda} = C^{i}{}_{0j0} = g_{\eta 0} C^{i\eta}{}_{j0} \\
=& g_{0\eta}\left({}^{(4)}R^{i\eta}{}_{j0} - 2\delta^{[i}{}_{[j}\,{}^{(4)}R^{\eta]}{}_{0]} + \frac{1}{3}\delta^{[i}{}_{[j}\delta^{\eta]}{}_{0]}\,{}^{(4)}R \right) \\
=& {}^{(4)}R^{i}{}_{0j0} - \frac{1}{2} g_{0\eta}\left(\delta^{i}{}_{j}\,{}^{(4)}R^{\eta}{}_{0} - \delta^{i}{}_{0}\,{}^{(4)}R^{\eta}{}_{j} - \delta^{\eta}{}_{j}\,{}^{(4)}R^{i}{}_{0} + \delta^{\eta}{}_{0}\,{}^{(4)}R^{i}{}_{j} \right) \\
&+ \frac{1}{12} g_{0\eta}\left(\delta^{i}{}_{j}\delta^{\eta}{}_{0} - \delta^{i}{}_{0}\delta^{\eta}{}_{j} - \delta^{\eta}{}_{j}\delta^{i}{}_{0} + \delta^{\eta}{}_{0}\delta^{i}{}_{j} \right) {}^{(4)}R \\
=& {}^{(4)}R^{i}{}_{0j0} - \frac{1}{2}\left({}^{(4)}R_{00}\delta^{i}{}_{j} + g_{00}\,{}^{(4)}R^{i}{}_{j} \right) + \frac{1}{6}\delta^{i}{}_{j} g_{00}\,{}^{(4)}R \\
=& {}^{(4)}R^{i}{}_{0j0} - \frac{1}{2}\,{}^{(4)}R_{00}\delta^{i}{}_{j} + \frac{c^2}{2}\,{}^{(4)}R^{i}{}_{j} - \frac{c^2}{6}\,{}^{(4)}R\ \delta^{i}{}_{j}\ ;
\end{aligned} \tag{5.36}$$

$$\begin{aligned}
H^{i}{}_{j} =& g^{ik} H_{kj} = g^{ik}\frac{\sqrt{g}}{2}\epsilon_{\varrho\tau 0(k} C^{\varrho\tau}{}_{j)0} = \frac{\sqrt{g}}{2}\epsilon_{rt(k}\ C^{rt}{}_{j)0} \\
=& g^{ik}\frac{\sqrt{g}}{2}\epsilon_{rt(k}\ \left({}^{(4)}R^{rt}{}_{j)0} - 2\delta^{[r}{}_{[j)}\,{}^{(4)}R^{t]}{}_{0]} + \frac{1}{3}\delta^{[r}{}_{[j)}\delta^{t]}{}_{0]}\,{}^{(4)}R \right) \\
=& g^{ik}\frac{\sqrt{g}}{2}\epsilon_{rt(k}\ \Big({}^{(4)}R^{rt}{}_{j)0} - \frac{1}{2}\delta^{r}{}_{j)}\,{}^{(4)}R^{t}{}_{0} + \frac{1}{2}\delta^{r}{}_{0}\,{}^{(4)}R^{t}{}_{j)} + \frac{1}{2}\delta^{t}{}_{j)}\,{}^{(4)}R^{r}{}_{0} - \frac{1}{2}\delta^{t}{}_{0}\,{}^{(4)}R^{r}{}_{j)} \\
&+ \frac{1}{12}\delta^{r}{}_{j)}\delta^{t}{}_{0}\,{}^{(4)}R - \frac{1}{12}\delta^{r}{}_{0}\delta^{t}{}_{j)}\,{}^{(4)}R - \frac{1}{12}\delta^{t}{}_{j)}\delta^{r}{}_{0}\,{}^{(4)}R + \frac{1}{12}\delta^{t}{}_{0}\delta^{r}{}_{j)}\,{}^{(4)}R \Big) \\
=& g^{ik}\frac{\sqrt{g}}{2}\epsilon_{rt(k}\ g^{ts}\ {}^{(4)}R^{r}{}_{sj)0} \\
=& g^{ik}\frac{\epsilon^{lmn}}{2\sqrt{g}} g_{lr} g_{mt} g_{n(k}\ g^{ts}\ {}^{(4)}R^{r}{}_{sj)0} \\
=& g^{ik}\frac{\epsilon^{lsn}}{2J} g_{lr} g_{n(k}\ {}^{(4)}R^{r}{}_{sj)0}\ .
\end{aligned} \tag{5.37}$$

The electric and magnetic parts of the Weyl tensor can be written thanks to the components of the Riemann tensor curvature. The expression of these components are known with respect to the Christoffel connection and then the metric. It

also leads to a formulation with respect to the expansion tensor. The required 4–Riemann curvature tensor components are:

$$
\begin{aligned}
{}^{(4)}R^{i}{}_{0j0} &= \Gamma^{i}{}_{00|j} - \Gamma^{i}{}_{j0|0} + \Gamma^{i}{}_{j\eta}\Gamma^{\eta}{}_{00} - \Gamma^{i}{}_{0\eta}\Gamma^{\eta}{}_{j0} \\
&= -\Gamma^{i}{}_{j0|0} - \Gamma^{i}{}_{0\eta}\Gamma^{\eta}{}_{j0} \\
&= -\frac{1}{2}\left[g^{ik}\left(g_{kj|0} + g_{k0|j} - g_{j0|k}\right)\right]_{|0} \\
&\quad -\frac{1}{4}\left[g^{ik}\left(g_{k\eta|0} + g_{k0|\eta} - g_{\eta 0|k}\right)\right]\left[g^{\eta k}\left(g_{kj|0} + g_{k0|j} - g_{j0|k}\right)\right] \\
&= -\frac{1}{2}\left(g^{ik}\dot{g}_{kj}\right)_{|0} - \frac{1}{4}g^{ik}\ \dot{g}_{kl}\ g^{lk}\ \dot{g}_{kj} \\
&= -\dot{\Theta}^{i}{}_{j} - \Theta^{i}{}_{k}\Theta^{k}{}_{j}\ ;
\end{aligned}
\tag{5.38}
$$

$$
{}^{(4)}R_{00} = \mathrm{tr}\left({}^{(4)}R^{i}{}_{0j0}\right) = -\dot{\theta} - \Theta^{l}{}_{k}\Theta^{k}{}_{l} = 4\pi G\varrho - \Lambda\ ; \tag{5.39}
$$

$$
\begin{aligned}
c^{2}{}^{(4)}R^{i}{}_{j} &= c^{2}{}^{(4)}R^{\kappa i}{}_{\kappa j} = c^{2}\left({}^{(4)}R^{ki}{}_{kj} - {}^{(4)}R^{0i}{}_{0j}\right) \\
&= c^{2}{}^{(4)}R^{ki}{}_{kj} - c^{2}g^{00}\ {}^{(4)}R^{i}{}_{0j0} \\
&= c^{2}\mathcal{R}^{i}{}_{j} + \dot{\Theta}^{i}{}_{j} + \theta\Theta^{i}{}_{j} = (4\pi G\varrho + \Lambda)\delta^{i}{}_{j}\ ;
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
c^{2}{}^{(4)}R &= c^{2}{}^{(4)}R^{\kappa}{}_{\kappa} = c^{2}{}^{(4)}R^{k}{}_{k} - c^{2}{}^{(4)}R^{0}{}_{0} \\
&= \mathrm{tr}\left(c^{2}{}^{(4)}R^{i}{}_{j}\right) - c^{2}g^{00}\ {}^{(4)}R_{00} \\
&= 3\left(4\pi G\varrho + \Lambda\right) - \left(4\pi G\varrho - \Lambda\right)\ ;
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
{}^{(4)}R^{r}{}_{sj0} &= \Gamma^{r}{}_{0s|j} - \Gamma^{r}{}_{js|0} + \Gamma^{r}{}_{j\eta}\Gamma^{\eta}{}_{0s} - \Gamma^{r}{}_{0\eta}\Gamma^{\eta}{}_{js} \\
&= \Theta^{r}{}_{s|j} - \Gamma^{r}{}_{js|0} + \Theta^{m}{}_{s}\Gamma^{r}{}_{mj} - \Theta^{r}{}_{m}\Gamma^{m}{}_{sj} \\
&= \Theta^{r}{}_{s||j} - \Gamma^{r}{}_{js|0} \\
&= \Theta^{r}{}_{s||j} + \Theta^{rm}\left(g_{mj|s} + g_{ms|j} - g_{sj|m}\right) - g^{rm}\left(\Theta_{mj|s} + \Theta_{ms|j} - \Theta_{sj|m}\right) \\
&= \Theta^{r}{}_{s||j} + \Theta^{r}{}_{m}2\Gamma^{m}{}_{sj} - g^{rp}\left(\Theta_{pj|s} + \Theta_{ps|j} - \Theta_{sj|p}\right) \\
&= \Theta^{r}{}_{s|j} + \Theta^{m}{}_{s}\Gamma^{r}{}_{mj} + \Theta^{r}{}_{m}\Gamma^{m}{}_{sj} - g^{rm}\left(\Theta_{mj|s} + \Theta_{ms|j} - \Theta_{sj|m}\right)\ .
\end{aligned}
\tag{5.42}
$$

Finally, according to these expressions for the components of the 4–Riemann curvature, the electric part of the Weyl tensor leads directly to,

$$
\begin{aligned}
E^{i}{}_{j} &= {}^{(4)}R^{i}{}_{0j0} - \frac{1}{2}{}^{(4)}R_{00}\delta^{i}{}_{j} + \frac{c^{2}}{2}{}^{(4)}R^{i}{}_{j} - \frac{c^{2}}{6}{}^{(4)}R\ \delta^{i}{}_{j} \\
&= -\dot{\Theta}^{i}{}_{j} - \Theta^{i}{}_{k}\Theta^{k}{}_{j} - \frac{1}{3}(4\pi G\varrho - \Lambda)\delta^{i}{}_{j}\ .
\end{aligned}
\tag{5.43}
$$

The magnetic part however needs some additional calculations linked to the anti-symmetry of the Levi-Civita pseudo–tensor ϵ^{ipq} and the symmetry of the expansion

tensor Θ_{pq}:

$$
\begin{aligned}
H^i{}_j &= g^{ik}\frac{\epsilon^{lsn}}{2J}g_{lr}g_{n(k}\left[\Theta^r{}_{s||j)} - \Gamma^r{}_{j)s|0}\right] \\
&= g^{ik}\frac{\epsilon^{lsn}}{2J}g_{n(k}\left[\Theta_{ls||j)} - g_{lr}\Gamma^r{}_{j)s|0}\right] \\
&= g^{ik}\frac{\epsilon^{lsn}}{2J}g_{n(k}\left[\Theta_{ls|j)} - \Theta_{ms}\Gamma^m{}_{lj)} - \Theta_{lm}\Gamma^m{}_{sj)} + 2\Theta_{lm}\Gamma^m{}_{sj)} - \left(\Theta_{lj)|s} + \Theta_{ls|j)} - \Theta_{sj)|l}\right)\right] \\
&= -g^{ik}\frac{\epsilon^{lsn}}{J}g_{n(k}\left[\Theta_{lj)|s} - \Theta_{lm}\Gamma^m{}_{sj)}\right] \\
&= -g^{ik}\frac{\epsilon^{lsn}}{J}g_{n(k}\left[\Theta_{lj)||s} + \Theta_{mj)}\Gamma^m{}_{ls}\right] \\
&= -g^{ik}\frac{\epsilon^{lsn}}{J}g_{n(k}\Theta_{lj)||s} \ . \qquad (5.44)
\end{aligned}
$$

The relativistic Bertschinger–Hamilton theory developed by the eponymous authors in [Bertschinger and Hamilton, 1994], assures the existence of two equations of evolution, one for the electric part and the other for the magnetic part of the Weyl tensor:

$$
\begin{cases}
\dot{E}^i{}_j = -2\theta E^i{}_j + \Theta^k{}_j E^i{}_k + \Theta^k{}_l E^l{}_k \delta^i{}_j + \epsilon^{ikl} H_{jl||k} - 4\pi G \varrho \sigma^i{}_j \ ; \\
\dot{H}^i{}_j = -2\theta H^i{}_j + \Theta^k{}_j H^i{}_k + \Theta^k{}_l H^l{}_l \delta^i{}_j - \epsilon^{ikl} E_{jl||k} - \dfrac{4\pi G}{3}\dfrac{J_{\mathbf{i}}}{J}\varrho_{\mathbf{i}|k}\epsilon^{ikl} g_{jl} \ .
\end{cases}
\qquad (5.45)
$$

These two equations involve covariant and spatial derivative, without additional treatment or constraint the relativistic dynamical system will not be composed of ordinary differential equations. Fortunately, the silent Universe hypothesis has not been used yet. This assumption is the same as in the Newtonian theory, it is simply the vanishing of the magnetic part of the Weyl tensor $H^i{}_j = 0$. The first equation leads to an evolution equation of the electric part of the Weyl tensor, whereas the second equation is reduced to a constraint about the electric part. It is also interesting to split these equations into their symmetric and antisymmetric parts. The antisymmetric parts are:

$$
\begin{cases}
E^k{}_{i||k} - g_{ik}\epsilon^{kmn}\Theta_{ml}H^l{}_n = \dfrac{8\pi G}{3}\dfrac{J_{\mathbf{i}}}{J}\varrho_{\mathbf{i}|i} \ ; \\
H^k{}_{i||k} + g_{ik}\epsilon^{kmn}\Theta_{ml}E^l{}_n = 0 \ ;
\end{cases}
\qquad (5.46)
$$

whereas the symmetric parts are:

$$
\begin{cases}
\dot{E}_{ij} + 2\theta E_{ij} - 3\Theta_{k(i}E^k{}_{j)} - \Theta^k{}_l E^l{}_k g_{ij} - g_{m(i}\epsilon^{mkl}H_{j)l||k} = -4\pi G\dfrac{J_{\mathbf{i}}}{J}\varrho_{\mathbf{i}}\ \sigma_{ij} \ ; \\
\dot{H}_{ij} + 2\theta H_{ij} - 3\Theta_{k(i}H^k{}_{j)} - \Theta^k{}_l H^l{}_k g_{ij} + g_{m(i}\epsilon^{mkl}E_{j)l||k} = 0 \ .
\end{cases}
\qquad (5.47)
$$

Assuming the silent Universe hypothesis, $H^i{}_j = 0$, the three pairs of equations (the full equations of evolution, the antisymmetric and symmetric equations of evolution for the electric and magnetic parts of the Weyl tensor) read,

$$\begin{cases} \dot{E}^i{}_j = -2\theta E^i{}_j + \Theta^k{}_j E^i{}_k + \Theta^k{}_l E^l{}_k \delta^i{}_j - 4\pi G \varrho \sigma^i{}_j \ ; \\ \epsilon^{ikl} E_{jl||k} = -\dfrac{4\pi G}{3} \dfrac{J_{\mathbf{i}}}{J} \varrho_{\mathbf{i}|k} \epsilon^{ikl} g_{jl} \ ; \end{cases} \tag{5.48}$$

$$\begin{cases} E^k{}_{i||k} = \dfrac{8\pi G}{3} \dfrac{J_{\mathbf{i}}}{J} \varrho_{\mathbf{i}|i} \ ; \\ g_{ik} \epsilon^{kmn} \Theta_{ml} E^l{}_n = 0 \ ; \end{cases} \tag{5.49}$$

$$\begin{cases} \dot{E}_{ij} + 2\theta E_{ij} - 3\Theta_{k(i} E^k{}_{j)} - \Theta^k{}_l E^l{}_k g_{ij} = -4\pi G \dfrac{J_{\mathbf{i}}}{J} \varrho_{\mathbf{i}} \ \sigma_{ij} \ ; \\ g_{m(i} \epsilon^{mkl} E_{j)l||k} = 0 \ . \end{cases} \tag{5.50}$$

The equation of evolution and the constraints for $E^i{}_j$ are easy to identify. Finally, the silent Universe assumption allows us to reduce the local relativistic system of equations to a set of eight ordinary differential equations with eight unknowns:

$$\begin{cases} \dot{\varrho} = -\theta \varrho \ ; \\ \dot{\theta} = -\dfrac{1}{3}\theta^2 - 2\sigma^2 + \Lambda - 4\pi G \varrho \ ; \\ \dot{\sigma}^i{}_j = -\dfrac{2}{3}\theta \sigma^i{}_j - \sigma^i{}_k \sigma^k{}_j + \dfrac{2}{3}\sigma^2 \delta^i{}_j - E^i{}_j \ ; \\ \dot{E}^i{}_j = -2\theta E^i{}_j + \Theta^k{}_j E^i{}_k + \Theta^k{}_l E^l{}_k \delta^i{}_j - 4\pi G \varrho \sigma^i{}_j \ . \end{cases} \tag{5.51}$$

The local relativistic system is closed and composed of ordinary differential equations: it can be solved numerically without difficulty. It is very promising to see that in our less–restrictive case. Most of the models, like LTB for instance, are a subclass of this solution.

Nevertheless, these equations are local and, as in the Newtonian approach, a cosmological constant is required to trigger a local accelerated expansion. Fortunately, such a behaviour is only the effect the local attractiveness of the gravitation. It is natural to require the addition of an artificial negative force (or internal energy) to overcome this effect.

3.3 Relativistic averaging process and averaged relativistic equations

As in the Newtonian approach, the average of a scalar function $\Psi(t, \vec{x})$ over a domain $\mathcal{D}$ of volume $V_{\mathcal{D}}$ is defined by:

$$\langle \Psi \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \Psi \sqrt{g} \ d^3X \quad \text{with} \quad V_{\mathcal{D}} = \int_{\mathcal{D}_0} \sqrt{g} \ d^3X \ . \tag{5.52}$$

Since the determinant of the metric and the relativistic generalisation of the Jacobian are linked ($\sqrt{g} = J$), it can be proved that the evolution operator and average operator do not necessary commute in the relativistic approach either:

$$\begin{aligned}
\frac{d\langle\Psi\rangle_{\mathcal{D}}}{dt} &= \frac{d}{dt}\left(\frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\Psi J d^3X\right) \\
&= -\frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}^2}\int\limits_{\mathcal{D}_0}\Psi J\ d^3X + \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\left(\dot{\Psi}J + \Psi\dot{J}\right)d^3X \\
&= -\frac{1}{V_{\mathcal{D}}^2}\int\limits_{\mathcal{D}_0}\dot{J}\ d^3X\int\limits_{\mathcal{D}_0}\Psi J\ d^3X + \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\dot{\Psi}J\ d^3X + \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\Psi\dot{J}\ d^3X \\
&= -\frac{1}{V_{\mathcal{D}}^2}\int\limits_{\mathcal{D}_0}J\theta\ d^3X\int\limits_{\mathcal{D}_0}\Psi J\ d^3X + \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\dot{\Psi}J\ d^3X + \frac{1}{V_{\mathcal{D}}}\int\limits_{\mathcal{D}_0}\Psi J\theta\ d^3X \\
&= -\langle\theta\rangle_{\mathcal{D}}\langle\Psi\rangle_{\mathcal{D}} + \left\langle\frac{d\Psi}{dt}\right\rangle_{\mathcal{D}} + \langle\Psi\theta\rangle_{\mathcal{D}}\ . \qquad (5.53)
\end{aligned}$$

As in the Newtonian framework, it is impossible to define the average of a tensor in general relativity without the addition of extra mathematical structure [Zalaletdinov, 1992, 1993; Mars and Zalaletdinov, 1997; Korzyński, 2010]. Nevertheless, the two tensorial equations of the system (5.51) (the evolution of the shear and the evolution of the electric part of the Weyl tensor) can be reduced to eigenvalues equations. The shear σ^i_j and the electric part of the Weyl tensor E^i_j are diagonalisable in the same basis. This property is due to the silent Universe hypothesis [Barnes and Rowlingson, 1989] and is very useful to write the relativistic system of equations with only scalar ordinary differential equations. The reduction to scalar equations allows us to write the whole averaged relativistic system. Let us take the second equation of the system (5.49): the antisymmetric part of the equation of evolution of the magnetic part of the Weyl tensor for a silent Universe, using the decomposition of the expansion tensor $\Theta_{ij} = (1/3)\ \theta g_{ij} + \sigma_{ij}$, which is justified by the choice of an irrotational flows. Note the Levi-Civita pseudo–tensor ϵ^{kmn} is antisymmetric, while E_{mn} is symmetric. Then the relation reads:

$$\begin{aligned}
g_{ik}\epsilon^{kmn}E^l{}_n\sigma_{ml} &= g_{ik}\left(g^{pk}g^{qm}g^{rn}g\epsilon_{pqr}\right)E^l{}_n\sigma_{ml} \\
&= \delta_i{}^p\left(g\epsilon_{pqr}\right)E^{lr}\sigma^q{}_l \\
&= \delta_i{}^p\left(g\epsilon_{per}\right)E^r{}_l\sigma^{ql} = 0\ ; \\
\Longrightarrow \quad E^{[n}{}_l\sigma^{m]l} &= 0 \text{ for a given i.} \qquad (5.54)
\end{aligned}$$

This condition about the matrices E_{ij} and σ_{ij} proves they commute with each other, so they are diagonalisable in a common basis. Thus there exists a basis where E_{ij} and σ_{ij} are diagonal, but these two matrices are tracefree, i.e. their

eigenvalues σ_α and E_α respect:

$$\sum_{\alpha=1...3} \sigma_\alpha = \sum_{\alpha=1...3} E_\alpha = 0 \ . \tag{5.55}$$

Arbitrarily, I arbitrarily chose to express the third eigenvalue with respect to the two others, $\sigma_3 = -\sigma_1 - \sigma_2$ and $E_3 = -E_1 - E_2$. Thus the tensorial equations of evolution for the shear and the electric part of the Weyl tensor can be written in an eigenvalue form:

$$\dot{\sigma}_1 = \frac{2}{3}\sigma_2(\sigma_1 + \sigma_2) - \frac{1}{3}(\sigma_1)^2 - \frac{2}{3}\theta\sigma_1 - E_1 \ ; \tag{5.56}$$

$$\dot{\sigma}_2 = \frac{2}{3}\sigma_1(\sigma_1 + \sigma_2) - \frac{1}{3}(\sigma_2)^2 - \frac{2}{3}\theta\sigma_2 - E_2 \ ; \tag{5.57}$$

$$\dot{E}_1 = E_1(\sigma_1 - \sigma_2) - E_2(\sigma_1 + 2\sigma_2) - \theta E_1 - \frac{1}{2}\varrho\sigma_1 \ ; \tag{5.58}$$

$$\dot{E}_2 = E_2(\sigma_2 - \sigma_1) - E_1(\sigma_2 + 2\sigma_1) - \theta E_2 - \frac{1}{2}\varrho\sigma_2 \ . \tag{5.59}$$

The equations about the third eigenvalues (σ_3, E_3) are simply a linear combination of the equations about the first and second eigenvalues. By construction, the eigenvalue equations are scalar equation. In the basis which diagonalises the shear and electric part of the Weyl tensor, the scalar σ^2 can be rewritten:

$$2\sigma^2 = \sigma^{ij}\sigma_{ji} = A\tilde{\sigma}^{ij}A^{-1}\ A\tilde{\sigma}_{ij}A^{-1} = A\sum_\alpha \sigma_\alpha^2 A^{-1} = 2\left(\sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2\right) \ . \tag{5.60}$$

The average operator can be applied to them and straightforwardly leads to the average relativistic system of equations:

$$\frac{d\langle\varrho\rangle_{\mathcal{D}}}{dt} = -\langle\theta\rangle_{\mathcal{D}}\langle\varrho\rangle_{\mathcal{D}} \ ; \tag{5.61}$$

$$\frac{d\langle\theta\rangle_{\mathcal{D}}}{dt} = \frac{2}{3}\left\langle\theta^2\right\rangle_{\mathcal{D}} - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2 + \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}} \ ; \tag{5.62}$$

$$\begin{aligned}\frac{d\langle\sigma_1\rangle_{\mathcal{D}}}{dt} &= -\langle\theta\rangle_{\mathcal{D}}\langle\sigma_1\rangle_{\mathcal{D}} + \frac{2}{3}\langle\sigma_2\sigma_1\rangle_{\mathcal{D}} + \frac{2}{3}\left\langle(\sigma_2)^2\right\rangle_{\mathcal{D}} \\ &\quad - \frac{1}{3}\left\langle(\sigma_1)^2\right\rangle_{\mathcal{D}} + \frac{1}{3}\langle\theta\sigma_1\rangle_{\mathcal{D}} - \langle E_1\rangle_{\mathcal{D}} \ ;\end{aligned} \tag{5.63}$$

$$\begin{aligned}\frac{d\langle\sigma_2\rangle_{\mathcal{D}}}{dt} &= -\langle\theta\rangle_{\mathcal{D}}\langle\sigma_2\rangle_{\mathcal{D}} + \frac{2}{3}\langle\sigma_2\sigma_1\rangle_{\mathcal{D}} + \frac{2}{3}\left\langle(\sigma_1)^2\right\rangle_{\mathcal{D}} \\ &\quad - \frac{1}{3}\left\langle(\sigma_2)^2\right\rangle_{\mathcal{D}} + \frac{1}{3}\langle\theta\sigma_2\rangle_{\mathcal{D}} - \langle E_2\rangle_{\mathcal{D}} \ ;\end{aligned} \tag{5.64}$$

$$\frac{d\langle E_1\rangle_{\mathcal{D}}}{dt} = -\langle\theta\rangle_{\mathcal{D}}\langle E_1\rangle_{\mathcal{D}} + \langle E_1\sigma_1\rangle_{\mathcal{D}} - \langle E_1\sigma_2\rangle_{\mathcal{D}} - \langle E_2\sigma_1\rangle_{\mathcal{D}} - 2\langle E_2\sigma_2\rangle_{\mathcal{D}} - \frac{1}{2}\langle\varrho\sigma_1\rangle_{\mathcal{D}} \; ; \tag{5.65}$$

$$\frac{d\langle E_2\rangle_{\mathcal{D}}}{dt} = -\langle\theta\rangle_{\mathcal{D}}\langle E_2\rangle_{\mathcal{D}} + \langle E_2\sigma_2\rangle_{\mathcal{D}} - \langle E_2\sigma_1\rangle_{\mathcal{D}} - \langle E_1\sigma_2\rangle_{\mathcal{D}} - 2\langle E_1\sigma_1\rangle_{\mathcal{D}} - \frac{1}{2}\langle\varrho\sigma_2\rangle_{\mathcal{D}} \; . \tag{5.66}$$

This system contains six equations but eighteen variables. This is clearly not a closed system, the averaging process drastically increases the number of unknowns. Nevertheless, it is still composed of ordinary differential equations. Note that the average expansion rate can be accelerated because of the terms triggered by the noncommutation rule: the kinematical backreaction. This term defined in the Newtonian theory can also be defined in the relativistic theory and has the same formal expression $\mathcal{Q}_{\mathcal{D}} = (2/3)\left(\langle\theta^2\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2\right) - 2\langle\sigma^2\rangle_{\mathcal{D}}$. In the same way, without additional constraint the value of this backreaction term can be either positive or negative. It translates the impact of local inhomogeneous behaviour onto the global dynamics. It can mimic the effects of Dark Energy.

4 Closure relation

Both the Newtonian and relativistic approaches lead to a local and closed system of ordinary differential equations in a first time. Then the average operator allows us to transcribe this system in order to describe the global dynamics of space–time. Unfortunately, in the two approaches the non–local system of equations is not closed anymore: the average operator increases drastically the number of unknowns. A way out is the idea of a closure relation, the implementation of an additional relation could reduce the number of unknowns. Such a relation could be extracted from the extrapolation of a relation obtained in the perturbation theory or a state equation.

In the paper [Buchert and Ostermann, 2012], the perturbative development gives a first order relation between the electric part of the Weyl tensor and the shear tensor, $E_{ij}^{(1)} = -\dot{\sigma}_{ij}^{(1)} - 2H\sigma_{ij}^{(1)}$. The extrapolation of this relation to any order allows us to reduce the eigenvalues equations, but is not sufficient because it introduces only two additional equations.

$$2\langle\sigma_1\sigma_2\rangle_{\mathcal{D}} + 2\langle\sigma_2^2\rangle_{\mathcal{D}} - \langle\sigma_1^2\rangle_{\mathcal{D}} + 4\langle\theta\sigma_1\rangle_{\mathcal{D}} + 2H\langle\sigma_1\rangle_{\mathcal{D}} = 0 \; ; \tag{5.67}$$

$$2\langle\sigma_1\sigma_2\rangle_{\mathcal{D}} + 2\langle\sigma_1^2\rangle_{\mathcal{D}} - \langle\sigma_2^2\rangle_{\mathcal{D}} + 4\langle\theta\sigma_2\rangle_{\mathcal{D}} + 2H\langle\sigma_2\rangle_{\mathcal{D}} = 0 \; . \tag{5.68}$$

This closure relation strongly constrains the system, the equations of evolution of the electric part of the Weyl tensor are not required anymore to close the local system. It also reduces the number of equations and variables of the non–local set

of equations: four equations and ten unknowns. Another proposition of closure relations can be mentioned $E_\alpha = \varrho\sigma_\alpha$. It reduces the system to eight equations and seventeen unknowns which is slightly better.

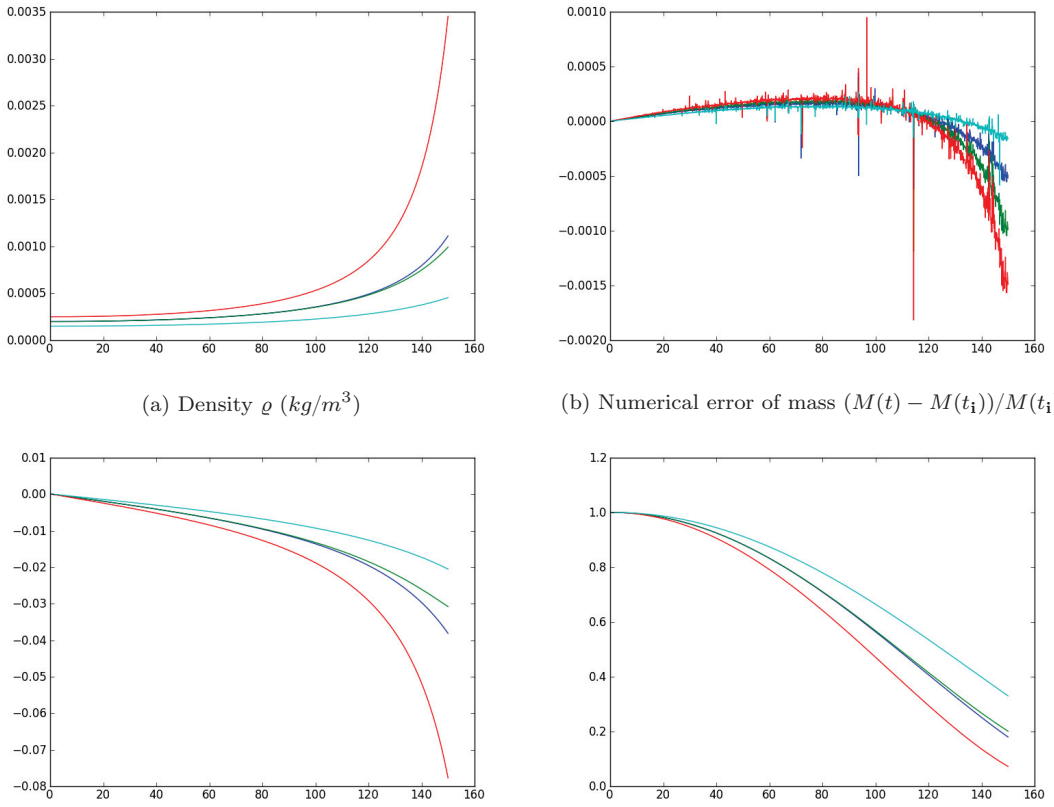
5 Numerical insight

Even if the idea of introducing closure relations does not give any great result, it is possible at least to study the impact of the average process numerically. I made a Python code to compute the evolution of homogeneous regions and perform at any time–step the spatial average in order to highlight the discrepancy between the standard homogeneous model and an average model. The code numerically solves the following local system of equations:

$$\begin{cases} \dot{\varrho} = -\theta\varrho \; ; \\ \dot{\theta} = -\dfrac{1}{3}\theta^2 - 2\sigma^2 + \Lambda - \dfrac{1}{2}\varrho \; ; \\ \dot{\sigma}_1 = \dfrac{2}{3}\sigma_2(\sigma_1 + \sigma_2) - \dfrac{1}{3}(\sigma_1)^2 - \dfrac{2}{3}\theta\sigma_1 - E_1 \; ; \\ \dot{\sigma}_2 = \dfrac{2}{3}\sigma_1(\sigma_1 + \sigma_2) - \dfrac{1}{3}(\sigma_2)^2 - \dfrac{2}{3}\theta\sigma_2 - E_2 \; ; \\ \dot{E}_1 = E_1(\sigma_1 - \sigma_2) - E_2(\sigma_1 + 2\sigma_2) - \theta E_1 - \dfrac{1}{2}\varrho\sigma_1 \; ; \\ \dot{E}_2 = E_2(\sigma_2 - \sigma_1) - E_1(\sigma_2 + 2\sigma_1) - \theta E_2 - \dfrac{1}{2}\varrho\sigma_2 \; . \end{cases} \tag{5.69}$$

This system of equations is valid for irrotational dust matter in a silent Universe. They are purely local equations, I did not implement the global averaged equations but in the following the code computes the average quantities. Nevertheless I performed the spatial average for each time step and we can see on the following plots that there is an effect of the average process with respect to the standard homogeneous model, the backreaction effect appears naturally. It shows that the impact of the backreaction cannot be neglected. This study is purely qualitative and not quantitative.

The figures 5.1 shows the impact of the backreaction on the non–local behaviour. The red curves represent an overdense region, the light blue curves an underdense region, the green curve the average of these two regions and the dark blue curve the standard model with no backreaction. The initial data are set such as the average region and the standard region are identical at the beginning. Moreover, these results are computed for vanishing initial shears and initial electric part of the Weyl tensor. Thus the system of equations is drastically reduced, an initial shear and electric part of the Weyl tensor ensure that these quantities remain null at any time. Then the results present in the figure 5.1 take into account only the effects of the backreaction. It is clear that the overdense region (the red



(a) Density ϱ (kg/m^3)

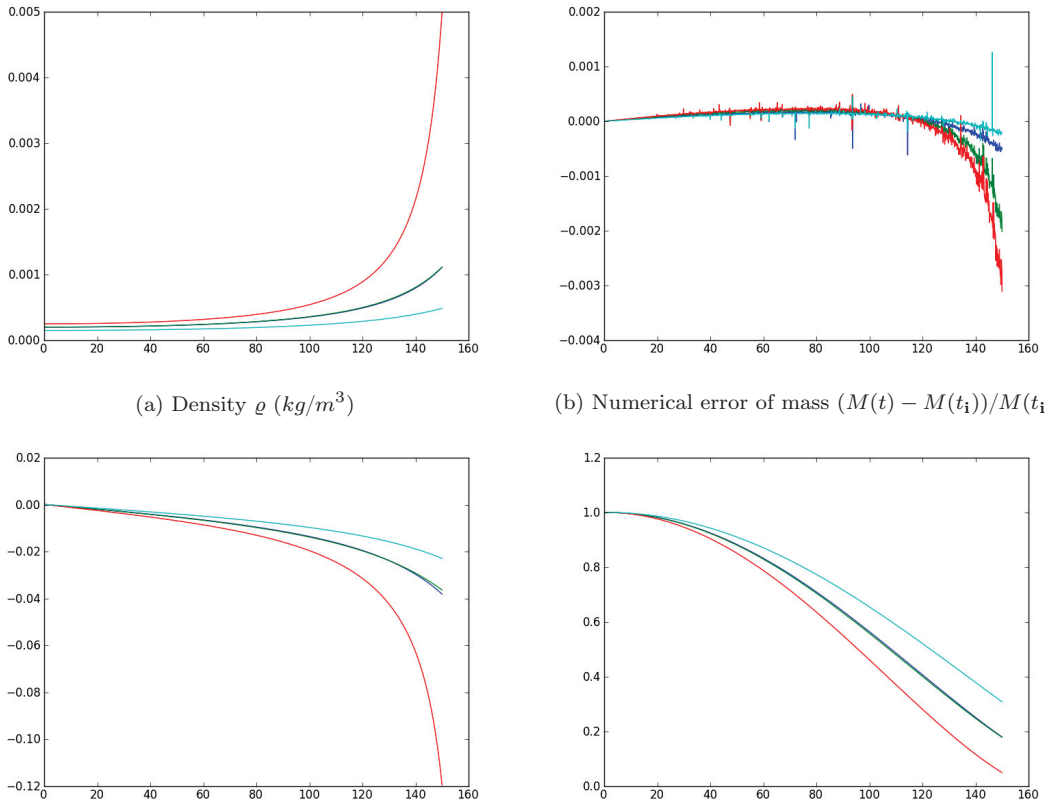
(b) Numerical error of mass $(M(t) - M(t_{\mathbf{i}}))/M(t_{\mathbf{i}})$

(c) Rate of expansion θ (s^{-1})

(d) Volume V (m^3)

Figure 5.1: These plots compare the numerical results for an overdense region in red, an underdense region in light blue, the average on these two regions in green and the "standard" model with no backreaction in blue.

curves) collapses faster than the other: its density increases, its rate of expansion is the most negative and its volume decreases faster than the others; whereas the underdense region (light blue curves) collapses more slowly: the density increases slightly, its expansion rate is slightly negative and its volume decreases slowly. The most interesting result is the comparison between the average of these two regions and an initially equivalent region in the standard model. The two models (the average model: green curves, the standard model: dark blue curves) are fairly close but discrepancies appear during the evolution. The region collapse is restrained in the average model, the backreaction is opposed to the gravitation: it acts like an internal energy. Indeed, for this set of initial data, the backreaction acts as Dark Energy and slows down the collapse of the global region. I do not study a large variety of initial data until now, but it could be very interesting to try to describe different cases and observe the effects of the backreaction with respect to the standard model.



(a) Density ϱ (kg/m^3)

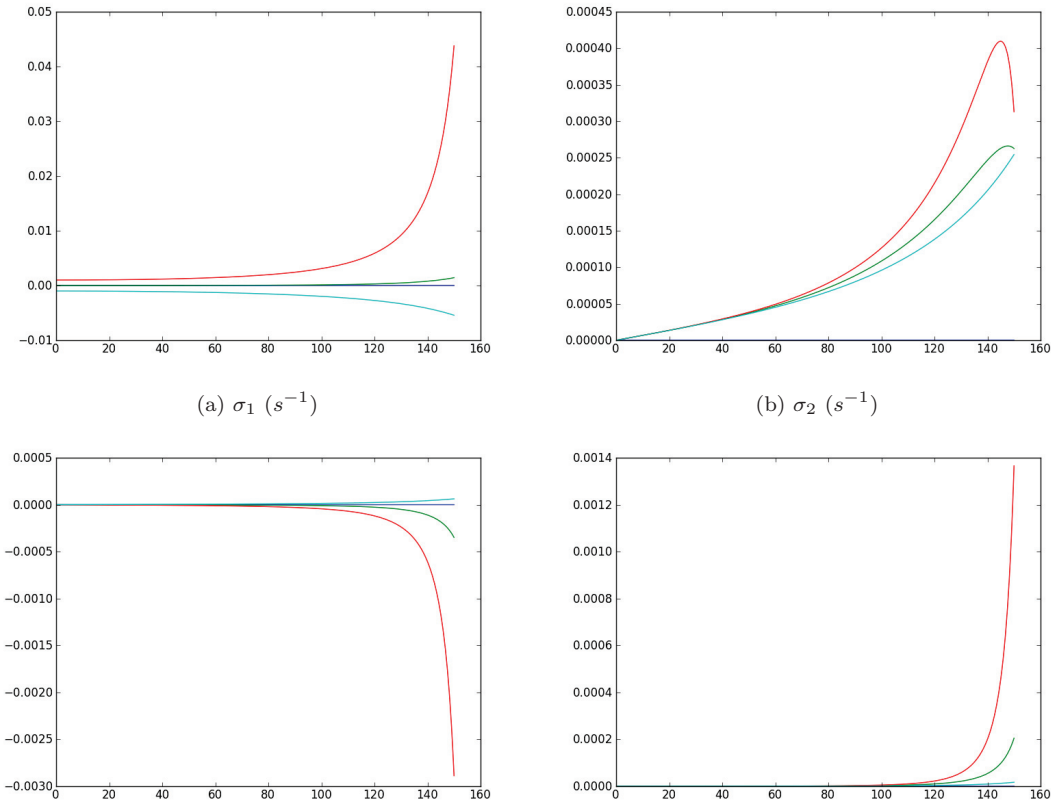
(b) Numerical error of mass $(M(t) - M(t_\mathbf{i}))/M(t_\mathbf{i})$

(c) Rate of expansion θ (s^{-1})

(d) Volume V (m^3)

Figure 5.2: These plots show the effect of the shear for an over–dense region in red, an under–dense region in light blue, the average on these two regions in green and the "standard" model with no backreaction in blue.

The second parameter I studied with this numerical simulation is the impact of the eigenvalues of the shear and electric part of the Weyl tensor. I set an initial value to σ_1 which is zero on average. Then the density, numerical error of mass, the rate of expansion and the volume are plotted in the figure 5.2; the eigenvalues of the shear and electric part of the Weyl tensor in the figure 5.3. The impacts of a small initial shear on the density, error of mass, rate of expansion and volume are slight. At least we can remark the modification of the average behaviour which is closer to the standard homogeneous dynamics. We can conclude to the existence of an effect due to the shear and electric part of the Weyl tensor. Nevertheless, since I do not consider a large variety of initial shear we cannot conclude to the quality of this effect: it could be Dark Matter or Dark Energy–like. I only set an initial value for σ_1 and the coupling between the equations of the system triggers a dynamics for the three other eigenvalues. The behaviour triggered by the second scalar shear σ_2 is identical because of the symmetry of the ordinary differential



(a) σ_1 (s^{-1}) (b) σ_2 (s^{-1})

(c) E_1 (s^{-2}) (d) E_2 (s^{-2})

Figure 5.3: These plots compare the evolution of the shear and electric scalar for an over–dense region in red, an under–dense region in light blue, the average on these two regions in green and the "standard" model with no backreaction in blue.

equations that composed the system. Moreover, the same kind of plots can be produced for an initial eigenvalue E_α.

Nevertheless, we have to be careful with the part of the plot for large times. This numerical simulation clearly has some limitations. For instance, it cannot describe properly the large time behavior because of the absence of pressure or vorticity to avoid the appearing of singularities. For a large time, the averaged rate of expansion (green curve) begins to diverge from the standard model results (dark blue curve) but is it a physical result or is it attributable to the singularity formation? On the other hand, the evolution of the shear and electric part of the Weyl tensor are non zero and could have an effect on a long term simulation without singularity formation.

6 Concluding remarks

The silent Universe approach leads to a very convenient local system of equations composed of a limited number of ordinary differential equations and unknowns. Moreover some famous models are sub–cases of it, like the LTB model. I also reviewed the averaging process and the associated backreaction and applied it to this description of space–time. Nevertheless, this operator introduces a large number of new unknowns in the problem but does not increase the number of equations. The theoretical research of a closure relation is not a piece of cake, but some can be extrapolated or intuited. They could allow us to reduce the number of unknowns and/or increase the number of equations.

My Python code gives a numerical resolution of the local system of equations and performs the average of the variables. Even if it is not a very powerful code it showed a qualitative difference between the average of the local equations and the assumed average standard model. The backreaction can have a non–negligible effect on the global dynamics, a universe which on average is FLRW clearly does not behave as a strictly FLRW universe even if it has the same initial average properties as the latter. I also showed the existence of a qualitative effect due to the shear and electric part of the Weyl tensor. I know that this code is widely perfectible: the incorporation of the pressure and vorticity could fight the apparition of singularities for the large time, increase the number of regions and realise an average over a more important space or more resolved space. Moreover, this numerical simulation giving the evolution for all the local unknowns could allow us to fit a closure relation.

Chapter VI

Weyl curvature hypothesis

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Penrose conjectured a relation between the growth of the Weyl curvature and the growth of entropy in the course of structure formation in the Universe ([Penrose, 1979, 2006, 2010]). It is based on the Weyl curvature hypothesis which assumes that the Universe starts its evolution with a vanishing Weyl curvature. This relation will be set by employing the information theoretical measures of entropy and the spatially averaged the Weyl curvature invariants calculated for

an irrotational dust model, using a relativistic Lagrangian perturbation approach to first order with an extrapolation in the spirit of Zel'dovich approximation. I show that, at leading order, the relation between the average information entropy, the average Weyl curvature invariant, and the kinematical backreaction functional found in a previous work ([Li et al., 2012]) is recovered. I extend these results by also including the magnetic part of the Weyl tensor, and discuss the results in the non–perturbative regime using the proposed extrapolation.

1 Introduction

I concentrate on possible relations between concepts of entropy and the Weyl curvature. According to Penrose [Penrose, 1979, 2006, 2010] the growth of structure in the Universe may be associated with the growth of the Weyl curvature, given the Weyl curvature hypothesis. This assumes that the initial state of the Universe is characterised by a highly symmetric state with a vanishing Weyl curvature, e.g. by the class of FLRW (Friedmann–Robertson–Walker–Lemaître) solutions. This latter hypothesis may be questioned, since we may contemplate initial chaotic and highly inhomogeneous conditions, e.g. in the context of chaotic inflation (see the recent paper on inhomogeneous inflation in the context of my Ph.D. thesis: [Buchert and Obadia, 2011]). However, the relation of the growth of structures with the growth of the Weyl curvature is highly plausible in a perturbative picture that assumes the Universe being close to homogeneous at some early stage. Gravitational instability leads to the growth of structures leading to the highly inhomogeneous density distribution we observe today. In such a perturbative picture we may assume the Weyl curvature hypothesis at the CMB epoch. With inflation in a homogeneous situation it is possible that the Universe was even Weyl curvature dominated at the Big Bang time ($t = 0$), a scenario that has been suggested in [Buchert and Obadia, 2011]. Since Penrose did not assume an inflation period then there is no problem with the Weyl curvature hypothesis at the Big Bang epoch. In a perturbative picture, starting with the CMB epoch, we are naturally led to consider the geometrical evolution of spatial field variables (the spatial Ricci tensor, the extrinsic curvature, but also the spatial parts of the Weyl tensor, representing the gravitational degrees of freedom) that all emerge during the structure formation process from a quasi–homogeneous Universe.

The present work is inspired by a recent paper [Li et al., 2012], which reports a relation between a spatially averaged information theoretical measure for entropy and the spatial average of one of the Weyl curvature invariants in second order standard perturbation theory. These two measures of inhomogeneity are proportional up to the kinematical backreaction term that necessarily arises from the fact that the spatial average on restmass preserving spatial domains and time–evolution are non–commutative.

I propose here an extension of this work by moving to the Lagrangian pertur-

bative framework, developed in [Buchert and Ostermann, 2012; Buchert et al., 2013; Alles et al., 2014] and chapters III and IV. The relativistic Lagrangian approach is constructed in the spirit of a relativistic analogy with the successful Newtonian Lagrangian perturbation theory. For the present considerations a dust matter model (a pressureless fluid without vorticity) in a comoving–synchronous coordinate system is assumed. I will concentrate on scalar spatial variables that are volume–averaged on a compact spatial domain $\mathcal{D}$.

I proceed by presenting first, in section 2, the variables of interest to this problem and outlining the Lagrangian scheme employed. Then the variables are functionally evaluated in terms of a first order deformation (developed in chapter IV) up to their leading order of contribution, in section 3. The results are discussed in relation to the Penrose conjecture in section 4 and I conclude in section 5.

2 Averaged variables and perturbation scheme

The evolution of structure in the matter–dominated epoch, restricting attention to an irrotational dust matter model, is considered. Assuming the metric signature $(-,+,+,+)$ and $c=1$, in the comoving and synchronous gauge, we write the line element as,

$$ds^2 = -dt^2 + g_{ij}\, dX^i dX^j \ ; \tag{6.1}$$

where X^i denote Lagrangian (Gaussian normal) coordinates, $i, j, k...$ run through $1, 2, 3...$, and g_{ij} are the spatial metric coefficients. The spatial average of any scalar function $A(t, \vec{X})$ over a compact domain $\mathcal{D}$ of volume $V_{\mathcal{D}}$ within the space–like hypersurfaces is defined as in the previous chapter:

$$\langle A \rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} A J d^3X \quad \text{with} \quad V_{\mathcal{D}} = \int\limits_{\mathcal{D}} J d^3X \ ; \tag{6.2}$$

where $V_{\mathcal{D}}$ is the Riemannian volume of the domain $\mathcal{D}$ and $J := \sqrt{\det(g_{ij})}$. A key property of this operation is the non–commutativity with the time evolution expressed through the rule (see (5.15) for the details of the calculation):

$$\frac{d}{dt}\langle A \rangle_{\mathcal{D}} - \left\langle \frac{\partial A}{\partial t} \right\rangle_{\mathcal{D}} = \langle \theta A \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}} \langle A \rangle_{\mathcal{D}} \ ; \tag{6.3}$$

where θ denotes the trace of the fluid expansion tensor defined by $\theta = \Theta^i{}_i = e_a{}^i \dot{\eta}^a{}_i$. For later convenience we introduce a formal (Euclidean–like) average, normalised by the initial volume,

$$\langle A \rangle_{\mathcal{I}} := \frac{1}{V_{\mathcal{D}_{\rm i}}} \int\limits_{\mathcal{D}} A \ d^3X \quad \text{with} \quad V_{\mathcal{D}_{\rm i}} = \int\limits_{\mathcal{D}_{\rm i}} d^3X \ . \tag{6.4}$$

Hence we have,

$$\langle A\rangle_{\mathcal{D}} = \frac{V_{\mathcal{D}_{\rm i}}}{V_{\mathcal{D}_{\rm i}}}\frac{1}{\int\limits_{\mathcal{D}} J d^3X}\int\limits_{\mathcal{D}} AJ\ d^3X = \frac{\langle AJ\rangle_{\mathcal{I}}}{\langle J\rangle_{\mathcal{I}}}\ . \tag{6.5}$$

2.1 Entropy

I choose to represent the entropy of structure formation by the Tsallis one–parameter family of relative entropy measures [Tsallis, 1988], applied to a dust continuum as proposed and discussed in [Hosoya et al., 2004]),

$$\mathcal{F}^{\alpha}_{\mathcal{D}}\left\{\varrho||\langle\varrho\rangle_{\mathcal{D}}\right\} := \frac{\langle\varrho\rangle_{\mathcal{D}}}{\alpha}\left[\left\langle\left(\frac{\varrho}{\langle\varrho\rangle_{\mathcal{D}}}\right)^{\alpha+1}\right\rangle_{\mathcal{D}} - 1\right]\ . \tag{6.6}$$

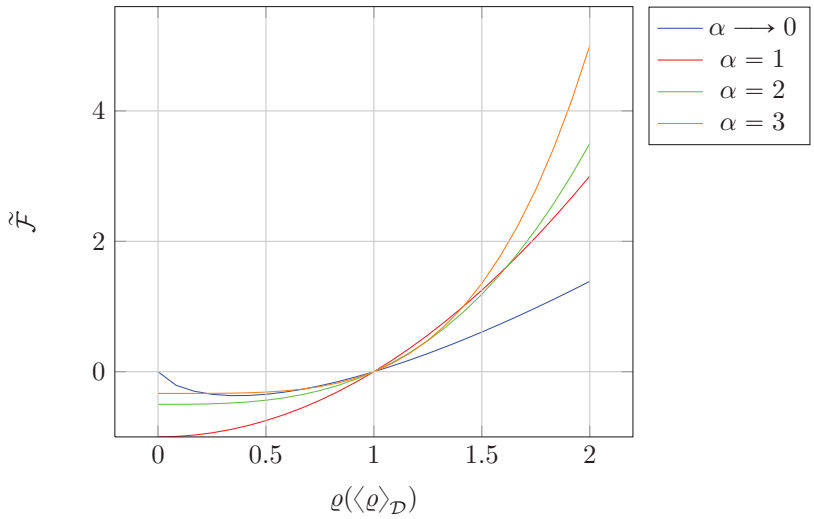

Figure 6.1: Dependence of the argument of the Tsallis one–parameter family of relative entropy measures with respect to the density.

$\widetilde{\mathcal{F}}$ denotes the argument of the Tsallis one–parameter family of relative entropy measures such as,

$$\mathcal{F}^{\alpha}_{\mathcal{D}}\left\{\varrho||\langle\varrho\rangle_{\mathcal{D}}\right\} = \left\langle\widetilde{\mathcal{F}}\right\rangle_{\mathcal{D}} = \left\langle\frac{\langle\varrho\rangle_{\mathcal{D}}}{\alpha}\left[\left(\frac{\varrho}{\langle\varrho\rangle_{\mathcal{D}}}\right)^{\alpha+1} - 1\right]\right\rangle_{\mathcal{D}}\ . \tag{6.7}$$

The figure 6.1 represent the evolution of the argument inside the average brackets for a set of parameters. For these cases, the function $\widetilde{\mathcal{F}}$ is zero when the density equal the average density, is negative for an underdense region and positive for an over dense region. Moreover, the "entropy" $\widetilde{\mathcal{F}}$ associated to an overdensity grows faster than the absolute value associated to an underdensity. The choice of the

strictly positive parameter α produces different entropy measures that all compare the actual density distribution with the average distribution on the domain $\mathcal{D}$. In [Li et al., 2012] the Kullback–Leibler functional (obtained in the limit $\alpha \to 0$) is considered as the volume entropy,

$$\mathcal{F}_{\mathcal{D}}^{\alpha\to 0} = \frac{S\left\{\varrho || \langle\varrho\rangle_{\mathcal{D}}\right\}}{V_{\mathcal{D}}} = \left\langle \varrho \ln \frac{\varrho}{\langle\varrho\rangle_{\mathcal{D}}} \right\rangle_{\mathcal{D}} ; \tag{6.8}$$

with the entropy $S\left\{\varrho || \langle\varrho\rangle_{\mathcal{D}}\right\}$. According to insights that will be presented in a forthcoming work, we here rather consider the density variance, normalised by the average density, as the entropy measure [1] which arises from the choice $\alpha = 1$,

$$\mathcal{F}_{\mathcal{D}}^{1} = \langle\varrho\rangle_{\mathcal{D}} \left[\left\langle \frac{\varrho^2}{\langle\varrho\rangle_{\mathcal{D}}^2} \right\rangle_{\mathcal{D}} - 1 \right] = \frac{\langle\varrho^2\rangle_{\mathcal{D}}}{\langle\varrho\rangle_{\mathcal{D}}} - \langle\varrho\rangle_{\mathcal{D}} = \frac{\langle\varrho^2\rangle_{\mathcal{D}} - \langle\varrho\rangle_{\mathcal{D}}^2}{\langle\varrho\rangle_{\mathcal{D}}} . \tag{6.9}$$

It has been conjectured in [Hosoya et al., 2004] and confirmed for perturbative and spherically symmetric solutions in [Morita et al., 2010] that, for sufficiently large times, information entropy is an increasing function of time due to the fact that a self–gravitating system occupying the domain $\mathcal{D}$ represents an open system with negative feedback (i.e. structure inhomogeneities increase). At least in the early nonlinear regime of structure formation the inhomogeneities increase. So the quantity $\mathcal{F}^1$ can be safely assumed to be a growing entropy measure for structures in the context of this work. This latter remark is not fully obvious for the Weyl curvature invariants to which we turn now, but a proportionality—up to kinematical backreaction—of a Weyl curvature invariant to the Kullback–Leibler relative entropy has been found for the second order in standard perturbation theory [Li et al., 2012].

2.2 Weyl scalar invariants

Two nontrivial scalar invariants of the Weyl tensor are defined in [Ehlers, 1961] (see the translation [Ehlers, 1993]). I will begin by study them and in a second time I will also propose to look at the Bel–Robinson invariant. These three invariants depend on the components of the tracefree Weyl tensor $C_{\mu\nu\kappa\lambda}$ defined through the 4−Riemann tensor, the 4−Ricci tensor and the 4−scalar curvature by:

$$C^{\mu\nu}{}_{\kappa\lambda} := {}^{(4)}R^{\mu\nu}{}_{\kappa\lambda} - 2\delta^{[\mu}_{[\kappa}\,{}^{(4)}R^{\nu]}{}_{\lambda]} + \frac{1}{3}\delta^{[\mu}_{[\kappa}\delta^{\nu]}_{\lambda]}\,{}^{(4)}R . \tag{6.10}$$

[1] As mentioned by David L. Wiltshire during the review of this thesis, this family of entropy is not a measure in the standard mathematical sense, $\mathcal{F}^{\alpha}$ are not extensive. Indeed, if we consider $S = \mathcal{F}^1$ and a domain D composed of disjoint elements D_i, $D = \cup_{i=1}^{n} D_i$, then:

$$S_D - \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} \langle\varrho^2\rangle_i \, \delta\varrho_i / \langle\varrho\rangle_i \neq 0 ;$$

where $\delta\varrho_i = (\langle\varrho\rangle_i - \langle\varrho\rangle_{\mathcal{D}})/\langle\varrho\rangle_{\mathcal{D}}$. Each term S_i represents the information of the region D_i and S_D the global information. Thus the difference between the global information and the sum of the regional informations can be interpreted such as the structure information. This observation was not made by Hosoya [Hosoya et al., 2004] but by H.Yao (University of Canterbury undergraduate project report, 2006)

There exists another relation between the coefficients of the trace free Weyl tensor and the electric and magnetic parts of the Weyl tensor,

$$C_{\mu\nu\kappa\lambda} = \left(g_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta} - \varepsilon_{\mu\nu\alpha\beta}\ \varepsilon_{\kappa\lambda\gamma\delta}\right) u^{\alpha}u^{\gamma}E^{\beta\delta} + \left(\varepsilon_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta} + g_{\mu\nu\alpha\beta}\ \varepsilon_{\kappa\lambda\gamma\delta}\right) u^{\alpha}u^{\gamma}H^{\beta\delta}\ ; \tag{6.11}$$

with $g_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}$ and $\varepsilon_{\mu\nu\alpha\beta} = J\epsilon_{\mu\nu\alpha\beta}$. Moreover $\varepsilon_{\mu\nu\alpha\beta}u^{\alpha} = \varepsilon_{\mu\nu\beta}$, the upper indices Weyl tensor reads,

$$\begin{aligned} C^{\mu\nu\kappa\lambda} &= g^{\mu\xi}g^{\nu\chi}g^{\kappa\phi}g^{\lambda\psi}\ C_{\xi\chi\phi\psi} \\ &= \left(g^{\mu\nu\alpha\beta}\ g^{\kappa\lambda\gamma\delta} - J^{-2}\epsilon^{\mu\nu\alpha\beta}\ \epsilon^{\kappa\lambda\gamma\delta}\right) u_{\alpha}u_{\gamma}E_{\beta\delta} \\ &\quad + J^{-1}\left(\epsilon^{\mu\nu\alpha\beta}\ g^{\kappa\lambda\gamma\delta} + g^{\mu\nu\alpha\beta}\ \epsilon^{\kappa\lambda\gamma\delta}\right) u_{\alpha}u_{\gamma}H_{\beta\delta}\ . \end{aligned} \tag{6.12}$$

We denote by E_{ij} the coefficients of the purely spatial electric part of the Weyl tensor, and by H_{ij} the coefficients of the purely spatial magnetic part of the Weyl tensor. These two quantities are defined by (where each second line is the expression employed in this work, [Buchert and Ostermann, 2012]):

$$E_{\mu\nu} := C_{\mu\kappa\nu\lambda}u^{\kappa}u^{\lambda}\ ; \tag{6.13}$$

$$E^{i}{}_{j} = -\dot{\Theta}^{i}{}_{j} - \Theta^{i}{}_{k}\Theta^{k}{}_{j} - \frac{1}{3}\left(4\pi G\varrho - \Lambda\right)\delta^{i}{}_{j}\ ; \tag{6.14}$$

$$H_{\mu\nu} := {}^{*}C_{\mu\nu\kappa\lambda}u^{\kappa}u^{\lambda} = \frac{\sqrt{-{}^{(4)}g}}{2}\epsilon_{\varrho\tau\kappa(\mu}C^{\varrho\tau}{}_{\nu)\lambda}u^{\kappa}u^{\lambda}\ ; \tag{6.15}$$

$$H^{i}{}_{j} = -\epsilon^{lsn}J^{-1}g^{ik}g_{n(k}\Theta_{j)||s}\ ; \tag{6.16}$$

where the 4−velocity of dust is $u^{\kappa} = (1, 0, 0, 0)$, the determinant of the 4−metric ${}^{(4)}g$, the double stroke $_{||}$ denotes the spatial covariant derivative and the star * denotes the dual operator.

2.2.1 First scalar invariant

First, the average Weyl scalar invariant C^2 is defined by a suitable choice of the constant, and represent it in the present foliation through the spatially projected electric and magnetic parts (see [Buchert and Ostermann, 2012]):

$$C^2 = \frac{1}{32}C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda} \tag{6.17}$$

$$= \frac{1}{4}\left(E^{i}{}_{j}E^{j}{}_{i} + H^{i}{}_{j}H^{j}{}_{i}\right)\ ; \tag{6.18}$$

In the following the derivation of the second equality is presented. This computation is very involved, and is simplified by the preliminary steps:

$$\begin{aligned}
g_{\mu\nu 0\beta}\ g^{\mu\nu 0\beta'} &= g_{\mu\nu\alpha\beta}\ g^{\mu\nu\alpha'\beta'} u^{\alpha} u_{\alpha'} \\
&= \left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}\right) \left(g^{\mu\alpha'} g^{\nu\beta'} - g^{\mu\beta'} g^{\nu\alpha'}\right) u^{\alpha} u_{\alpha'} \\
&= \Big(g_{\mu\alpha} g^{\mu\alpha'} g_{\nu\beta} g^{\nu\beta'} - g_{\mu\alpha} g^{\mu\beta'} g_{\nu\beta} g^{\nu\alpha'} \\
&\quad - g_{\mu\beta} g^{\mu\alpha'} g_{\nu\alpha} g^{\nu\beta'} + g_{\mu\beta} g^{\mu\beta'} g_{\nu\alpha} g^{\nu\alpha'}\Big) u^{\alpha} u_{\alpha'} \\
&= 2 u^{\alpha} u_{\alpha} \delta^{\beta'}_{\ \beta} - 2 u^{\beta'} u_{\beta} = 2\left(\delta^{\beta'}_{\ \beta} - u^{\beta'} u_{\beta}\right) \ ; \qquad (6.19)
\end{aligned}$$

$$\begin{aligned}
g_{\mu\nu\alpha\beta}\ \epsilon^{\mu\nu\beta'} &= \left(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}\right) \epsilon^{\mu\nu\sigma} g^{\rho\beta'} g_{\rho\sigma} \\
&= J^2 \left(\epsilon_{\alpha\beta\rho} - \epsilon_{\beta\alpha\rho}\right) g^{\rho\beta'} \\
&= 2 J^2 \epsilon_{\alpha\beta\rho}\ g^{\rho\beta'} \ ; \qquad (6.20)
\end{aligned}$$

$$\begin{aligned}
g^{\mu\nu\alpha'\beta'} \epsilon_{\mu\nu\beta} &= \left(g^{\mu\alpha'} g^{\nu\beta'} - g^{\mu\beta'} g^{\nu\alpha'}\right) \epsilon_{\mu\nu\sigma} g_{\rho\beta} g^{\rho\sigma} \\
&= J^{-2} \left(\epsilon^{\alpha'\beta'\rho} - \epsilon^{\beta'\alpha'\rho}\right) g_{\rho\beta} \\
&= 2 J^{-2} \epsilon^{\alpha'\beta'\rho} g_{\rho\beta} \ ; \qquad (6.21)
\end{aligned}$$

Note that only the purely spatial parts of the Weyl tensor are non zero because of the choice of the $3+1$ foliation, in particular $u_{\beta} E^{\beta\delta} = u_{\beta} H^{\beta\delta} = 0$. Thus the first scalar invariant of the Weyl tensor can be derived from the following calculation,

$$\begin{aligned}
C^2 &= \frac{1}{32} C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda} \\
&= \frac{1}{32} \Big(g_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta}\ g^{\mu\nu\alpha'\beta'}\ g^{\kappa\lambda\gamma'\delta'} u_{\alpha} u_{\gamma} u^{\alpha'} u^{\gamma'} - g_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta} J^{-2}\ \epsilon^{\mu\nu\beta'}\ \epsilon^{\kappa\lambda\delta'} u^{\alpha} u^{\gamma} \\
&\qquad - J^2\ \epsilon_{\mu\nu\beta}\ \epsilon_{\kappa\lambda\delta}\ g^{\mu\nu\alpha'\beta'}\ g^{\kappa\lambda\gamma'\delta'} u_{\alpha'} u_{\gamma'} + \epsilon_{\mu\nu\beta}\ \epsilon_{\kappa\lambda\delta}\ \epsilon^{\mu\nu\beta'}\ \epsilon^{\kappa\lambda\delta'}\Big) E^{\beta\delta} E_{\beta'\delta'} \\
&\quad + \frac{1}{32} \Big(g_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta}\ J^{-1} \epsilon^{\mu\nu\beta'}\ g^{\kappa\lambda\gamma'\delta'} u^{\alpha} u^{\gamma} u_{\gamma'} + g_{\mu\nu\alpha\beta}\ g_{\kappa\lambda\gamma\delta}\ g^{\mu\nu\alpha'\beta'} J^{-1} \epsilon^{\kappa\lambda\delta'} u^{\alpha} u^{\gamma} u_{\alpha'} \\
&\qquad - J \epsilon_{\mu\nu\beta}\ \epsilon_{\kappa\lambda\delta}\ \epsilon^{\mu\nu\beta'}\ g^{\kappa\lambda\gamma'\delta'} u_{\gamma'} - J \epsilon_{\mu\nu\beta}\ \epsilon_{\kappa\lambda\delta}\ g^{\mu\nu\alpha'\beta'} \epsilon^{\kappa\lambda\delta'} u_{\alpha'}\Big) E^{\beta\delta} H_{\beta'\delta'} \\
&\quad + \frac{1}{32} \Big(g^{\mu\nu\alpha'\beta'}\ g^{\kappa\lambda\gamma'\delta'} J \epsilon_{\mu\nu\beta}\ g_{\kappa\lambda\gamma\delta}\ u_{\alpha'} u_{\gamma'} u^{\gamma} + g^{\mu\nu\alpha'\beta'}\ g^{\kappa\lambda\gamma'\delta'}\ g_{\mu\nu\alpha\beta}\ J \epsilon_{\kappa\lambda\delta} u_{\alpha'} u_{\gamma'} u^{\alpha} \\
&\qquad - J^{-1} \epsilon^{\mu\nu\beta'}\ \epsilon^{\kappa\lambda\delta'} \epsilon_{\mu\nu\beta}\ g_{\kappa\lambda\gamma\delta}\ u^{\gamma} - J^{-1} \epsilon^{\mu\nu\beta'}\ \epsilon^{\kappa\lambda\delta'}\ g_{\mu\nu\alpha\beta}\ \epsilon_{\kappa\lambda\delta}\ u^{\alpha}\Big) E_{\beta'\delta'} H^{\beta\delta} \\
&\quad + \frac{1}{32} \Big(\epsilon_{\mu\nu\beta}\ g_{\kappa\lambda\gamma\delta}\ \epsilon^{\mu\nu\beta'}\ g^{\kappa\lambda\gamma'\delta'} u^{\gamma} u_{\gamma'} + \epsilon_{\mu\nu\beta}\ g_{\kappa\lambda\gamma\delta}\ g^{\mu\nu\alpha'\beta'} \epsilon^{\kappa\lambda\delta'} u^{\gamma} u_{\alpha'} \\
&\qquad + g_{\mu\nu\alpha\beta}\ \epsilon_{\kappa\lambda\delta}\ \epsilon^{\mu\nu\beta'}\ g^{\kappa\lambda\gamma'\delta'} u^{\alpha} u_{\gamma'} + g_{\mu\nu\alpha\beta}\ \epsilon_{\kappa\lambda\delta}\ g^{\mu\nu\alpha'\beta'} \epsilon^{\kappa\lambda\delta'} u^{\alpha} u_{\alpha'}\Big) H^{\beta\delta} H_{\beta'\delta'}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left(\delta^{\beta'}{}_{\beta}\delta^{\delta'}{}_{\delta} - J^2\epsilon_{\alpha\beta\rho}\epsilon_{\gamma\delta\sigma}g^{\rho\beta'}g^{\sigma\delta'} - J^{-2}\epsilon^{\alpha'\beta'\rho}\epsilon^{\gamma'\delta'\sigma}g_{\rho\beta}g_{\sigma\delta} + \delta^{\beta'}{}_{\beta}\delta^{\delta'}{}_{\delta}\right)E^{\beta\delta}E_{\beta'\delta'} \\
&\quad + \frac{1}{8}\left(J\delta^{\delta'}{}_{\delta}\epsilon_{\alpha\beta\rho}g^{\rho\beta'} + J\delta^{\beta'}{}_{\beta}\epsilon_{\gamma\delta\rho}g^{\rho\delta'} - J^{-1}\delta^{\beta'}{}_{\beta}\epsilon^{\gamma'\delta'\rho}g_{\rho\delta} - J^{-1}\delta^{\delta'}{}_{\delta}\epsilon^{\alpha'\beta'\rho}g_{\rho\beta}\right)E^{\beta\delta}H_{\beta'\delta'} \\
&\quad + \frac{1}{8}\left(J^{-1}\delta^{\delta'}{}_{\delta}\epsilon^{\alpha'\beta'\rho}g_{\rho\beta} + J^{-1}\delta^{\beta'}{}_{\beta}\epsilon^{\gamma'\delta'\rho}g_{\rho\delta} - J\delta^{\beta'}{}_{\beta}\epsilon_{\gamma\delta\rho}g^{\rho\delta'} - J\delta^{\delta'}{}_{\delta}\epsilon_{\alpha\beta\rho}g^{\rho\beta'}\right)E_{\beta'\delta'}H^{\beta\delta} \\
&\quad + \frac{1}{8}\left(\delta^{\beta'}{}_{\beta}\delta^{\delta'}{}_{\delta} + \epsilon^{\alpha'\beta'\rho}g_{\rho\beta}\epsilon_{\gamma\delta\sigma}g^{\sigma\delta'} + \epsilon^{\gamma'\delta'\rho}g_{\rho\delta}\epsilon_{\alpha\beta\sigma}g^{\sigma\beta'} + \delta^{\beta'}{}_{\beta}\delta^{\delta'}{}_{\delta}\right)H^{\beta\delta}H_{\beta'\delta'} \\
&= \frac{1}{8}\Big(2E^{\beta\delta}E_{\beta\delta} - J^2\epsilon_{\alpha\beta\rho}\epsilon_{\gamma\delta\sigma}E^{\beta\delta}E^{\rho\sigma} - J^{-2}\epsilon^{\alpha'\beta'\rho}\epsilon^{\gamma'\delta'\sigma}E_{\rho\sigma}E_{\beta'\delta'} \\
&\qquad + 4J\epsilon_{\alpha\beta\rho}E^{\beta\delta}H^{\rho}{}_{\delta} - 4J^{-1}\epsilon^{\gamma'\delta'\rho}E^{\beta}{}_{\rho}H_{\beta\delta'} + 2\epsilon^{\alpha'\beta'\rho}\epsilon_{\gamma\delta\sigma}H^{\delta}{}_{\rho}H^{\sigma}{}_{\beta'} + 2H^{\beta\delta}H_{\beta\delta}\Big) \\
&= \frac{1}{4}\left(E^{ij}E_{ij} + H^{ij}H_{ij}\right) \ . \qquad (6.22)
\end{aligned}
$$

The result is only dependent of the spatial parts of the Weyl tensor E_{ij} and H_{ij} because of the chosen space–time foliation: in the $3+1$ formalism the time parts of the Weyl tensor vanish.

2.2.2 Second scalar invariant

Another nontrivial invariant is the left–dual Weyl scalar invariant that we define accordingly with a convenient factor:

$$
{}^*C^2 = \frac{1}{32}{}^*C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda} = 0 \ ; \qquad (6.23)
$$

where the dual of the Weyl tensor is defined by ${}^*C_{\mu\nu\kappa\lambda} := (1/2)\ \epsilon_{\mu\nu\alpha\beta}C^{\alpha\beta}{}_{\kappa\lambda}$. The computation of the dual of the Weyl tensor requires some other identities which are:

$$
\begin{aligned}
\epsilon_{\mu\nu\xi\chi}\ g^{\xi\chi\alpha\beta}u_\alpha &= \epsilon_{\mu\nu\xi\chi}\left(g^{\xi\alpha}g^{\chi\beta} - g^{\xi\beta}g^{\chi\alpha}\right)u_\alpha \\
&= \epsilon_{\mu\nu\xi\chi}\left(u^{\xi}g^{\chi\beta} - u^{\chi}g^{\xi\beta}\right) \\
&= 2\epsilon_{\mu\nu\xi}\ g^{\beta\xi} \ ; \qquad (6.24) \\
\epsilon_{\mu\nu\xi\chi}\ \epsilon^{\xi\chi\beta} &= \epsilon_{\mu\nu\xi\chi}\ \epsilon^{\xi\chi\alpha\beta}u_\alpha \\
&= 2\left(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu}\right)u_\alpha \\
&= 2\left(u_\mu\delta^{\beta}{}_{\nu} - u_\nu\delta^{\beta}{}_{\mu}\right) \ . \qquad (6.25)
\end{aligned}
$$

Then I start here the details of the calculations for the dual of the Weyl tensor. It is not an easy calculation but the method is straightforward,

$$
\begin{aligned}
{}^*C_{\mu\nu\kappa\lambda} = \frac{1}{2}J\epsilon_{\mu\nu\xi\chi}g^{\xi\phi}g^{\chi\psi}&\left(\left(g_{\phi\psi\alpha\beta}g_{\kappa\lambda\gamma\delta} - J^2\epsilon_{\phi\psi\alpha\beta}\epsilon_{\kappa\lambda\gamma\delta}\right)u^\alpha u^\gamma E^{\beta\delta}\right. \\
&\left. + \left(J\epsilon_{\phi\psi\alpha\beta}g_{\kappa\lambda\gamma\delta} + g_{\phi\psi\alpha\beta}J\epsilon_{\kappa\lambda\gamma\delta}\right)u^\alpha u^\gamma H^{\beta\delta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} J \epsilon_{\mu\nu\xi\chi} \left(\left(g^{\xi\chi\alpha\beta} g_{\kappa\lambda\gamma\delta} u_\alpha u^\gamma - J^2 \epsilon^{\xi\chi\beta} \epsilon_{\kappa\lambda\delta} \right) E_\beta{}^\delta \right. \\
&\quad \left. + J \left(\epsilon^{\xi\chi\beta} g_{\kappa\lambda\gamma\delta} u^\gamma + J g^{\xi\chi\alpha\beta} \epsilon_{\kappa\lambda\delta} u_\alpha \right) H_\beta{}^\delta \right) \\
&= \left(J \epsilon_{\mu\nu\xi} g^{\beta\xi} g_{\kappa\lambda 0\delta} - J^3 \left(u_\mu \delta^\beta{}_\nu - u_\nu \delta^\beta{}_\mu \right) \epsilon_{\kappa\lambda\delta} \right) E_\beta{}^\delta \\
&\quad + \left(J^2 \left(u_\mu \delta^\beta{}_\nu - u_\nu \delta^\beta{}_\mu \right) g_{\kappa\lambda 0\delta} + J^2 \epsilon_{\mu\nu\xi}\, g^{\beta\xi} \epsilon_{\kappa\lambda\delta} \right) H_\beta{}^\delta \\
&= \left(J \epsilon_{\mu\nu\xi} g_{\kappa\lambda 0\delta} E^{\xi\delta} - J^3 \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \epsilon_{\kappa\lambda\delta} \right) \\
&\quad + \left(J^2 \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) g_{\kappa\lambda 0\delta} + J^2 \epsilon_{\mu\nu\xi} \epsilon_{\kappa\lambda\delta} H^{\xi\delta} \right) \ . \qquad (6.26)
\end{aligned}
$$

Then the multiplication of the previous expression by the Weyl tensor with upper indices reads,

$$
\begin{aligned}
{}^*C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda} &= \Big[J \epsilon_{\mu\nu\xi} g_{\kappa\lambda\gamma\delta} u^\gamma E^{\xi\delta} - J^3 \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \epsilon_{\kappa\lambda\delta} \\
&\qquad + J^2 \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) g_{\kappa\lambda\gamma\delta} u^\gamma + J^2 \epsilon_{\mu\nu\xi} \epsilon_{\kappa\lambda\delta} H^{\xi\delta} \Big] \\
&\quad \times \Big[g^{\mu\nu\alpha\beta} g^{\kappa\lambda\gamma'\delta'} u_\alpha u_{\gamma'} E_{\beta\delta'} - J^{-2} \epsilon^{\mu\nu\beta} \epsilon^{\kappa\lambda\delta'} E_{\beta\delta'} \\
&\qquad + J^{-1} \epsilon^{\mu\nu\beta} g^{\kappa\lambda\gamma'\delta'} u_{\gamma'} H_{\beta\delta'} + J^{-1} g^{\mu\nu\alpha\beta} \epsilon^{\kappa\lambda\delta'} u_\alpha H_{\beta\delta'} \Big] \\
&= \Big[4 J^{-1} \epsilon^{\alpha\beta\rho} g_{\rho\xi} \delta^{\delta'}{}_\delta u_\alpha E^{\xi\delta} E_{\beta\delta'} - 4 J \delta^\beta{}_\xi \epsilon_{\gamma\delta\rho} g^{\rho\delta'} u^\gamma E^{\xi\delta} E_{\beta\delta'} \\
&\quad - 2J \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \epsilon^{\gamma'\delta'\rho} g_{\rho\delta} g^{\mu\nu\alpha\beta} u_\alpha u_{\gamma'} E_{\beta\delta'} + 2J^3 \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \epsilon^{\mu\nu\beta} \delta^{\delta'}{}_\delta E_{\beta\delta'} \Big] \\
&\quad + \Big[4 \delta^\beta{}_\xi \delta^{\delta'}{}_\delta E^{\xi\delta} H_{\beta\delta'} + 4 \epsilon^{\alpha\beta\rho} g_{\xi\rho} \epsilon_{\gamma\delta\sigma} g^{\delta'\sigma} u^\gamma u_\alpha E^{\xi\delta} H_{\beta\delta'} \\
&\quad - 2 \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \epsilon^{\gamma'\delta'\rho} g_{\rho\delta} \epsilon^{\mu\nu\beta} u_{\gamma'} H_{\beta\delta'} - 2J^2 \left(u_\mu E_\nu{}^\delta - u_\nu E_\mu{}^\delta \right) \delta^{\delta'}{}_\delta g^{\mu\nu\alpha\beta} u_\alpha H_{\beta\delta'} \\
&\quad + 2J^2 \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) g^{\mu\nu\alpha\beta} u_\alpha \delta^{\delta'}{}_\delta E_{\beta\delta'} - 2J \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) u^\gamma \epsilon^{\mu\nu\beta} \epsilon_{\gamma\delta\rho} g^{\rho\delta'} E_{\beta\delta'} \\
&\quad + 4 J^{-2} \epsilon^{\alpha\beta\rho} \epsilon^{\gamma'\delta'\sigma} g_{\rho\xi} g_{\sigma\delta} u_\alpha u_{\gamma'} H^{\xi\delta} E_{\beta\delta'} - 4 J \delta^\beta{}_\xi \delta^{\delta'}{}_\delta H^{\xi\delta} E_{\beta\delta'} \Big] \\
&\quad + \Big[2J \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) \epsilon^{\mu\nu\beta} \delta^{\delta'}{}_\delta H_{\beta\delta'} + 2J^4 \left(u_\mu H_\nu{}^\delta - u_\nu H_\mu{}^\delta \right) g^{\mu\nu\alpha\beta} \epsilon_{\gamma\delta\rho} g^{\rho\delta'} u_\alpha H_{\beta\delta'} \\
&\quad + 4 J^{-1} \delta^\beta{}_\xi \epsilon^{\gamma'\delta'\rho} g_{\rho\delta} H^{\xi\delta} u_{\gamma'} H_{\beta\delta'} + 4 J^{-1} \epsilon^{\alpha\beta\rho} g_{\rho\xi} \delta^{\delta'}{}_\delta H^{\xi\delta} u_\alpha H_{\beta\delta'} \Big] \\
&= 4 E^{\beta\delta} H_{\beta\delta} - 2J^2 \left(g^{\alpha\nu\gamma\beta} E_\nu{}^\delta - g^{\mu\alpha\gamma\alpha} E_\mu{}^\delta \right) u_\alpha u_\gamma H_{\beta\delta} \\
&\quad + 2J^2 \left(g^{\alpha\nu\gamma\beta} H_\nu{}^\delta - g^{\mu\alpha\gamma\beta} H_\mu{}^\delta \right) u_\alpha u_\gamma E_{\beta\delta} - 4 H^{\beta\delta} E_{\beta\delta} \\
&= 4 E^{\beta\delta} H_{\beta\delta} - 4 J^2 E_\mu{}^\delta H^\mu{}_\delta + 4 J^2 H^{\mu\delta} E_{\beta\delta} - 4 H^{\beta\delta} E_{\beta\delta} \\
&= 0 \ . \qquad (6.27)
\end{aligned}
$$

The dual invariant ${}^*C^2$ is effectively vanishing and brings no information about the dynamics or properties of the space–time, unlike the scalar invariant C^2 whose

expression is interestingly similar to the expression of the electromagnetism scalar of the Faraday tensor.

2.2.3 Another scalar invariant

A third possibility was used by T. Clifton [Clifton et al., 2013]. He proposed to use the symmetric and tracefree Bel–Robinson tensor [Bel, 1958, 1959, 1962, 2000; Robinson, 1958] defined as follows:

$$T_{\mu\nu\kappa\lambda} := \frac{1}{4}\left(C_{\alpha\mu\nu\beta}C^{\alpha}{}_{\kappa\lambda}{}^{\beta} + {}^{*}C_{\alpha\mu\nu\beta}{}^{*}C^{\alpha}{}_{\kappa\lambda}{}^{\beta}\right) \ ; \tag{6.28}$$

whose associated scalar invariant is $W := T_{\mu\nu\kappa\lambda}u^{\mu}u^{\nu}u^{\kappa}u^{\lambda}$ [Clifton et al., 2013; Pelavas and Lake, 2000; Pelavas and Coley, 2006] and reads:

$$W = \frac{1}{4}\left(E^{ij}E_{ij} + H^{ij}H_{ij}\right) \ . \tag{6.29}$$

The expression of this last invariant is easily obtained thanks to the definition of the electric and magnetic part of the Weyl tensor $E_{\mu\nu} = C_{\mu\nu\kappa\lambda}u^{\kappa}u^{\lambda}$ and $H_{\mu\nu} = {}^{*}C_{\mu\nu\kappa\lambda}u^{\kappa}u^{\lambda}$:

$$\begin{aligned} W &= \frac{1}{4}\left(C_{\alpha\mu\nu\beta}C^{\alpha}{}_{\kappa\lambda}{}^{\beta} + {}^{*}C_{\alpha\mu\nu\beta}{}^{*}C^{\alpha}{}_{\kappa\lambda}{}^{\beta}\right)u^{\mu}u^{\nu}u^{\kappa}u^{\lambda} \\ &= \frac{1}{4}\left(C_{\alpha\mu\nu\beta}u^{\mu}u^{\nu}\ C^{\alpha}{}_{\kappa\lambda}{}^{\beta}u^{\kappa}u^{\lambda} + {}^{*}C_{\alpha\mu\nu\beta}u^{\mu}u^{\nu *}\ C^{\alpha}{}_{\kappa\lambda}{}^{\beta}u^{\kappa}u^{\lambda}\right) \\ &= \frac{1}{4}\left(E_{\alpha\beta}E^{\alpha\beta} + H_{\alpha\beta}H^{\alpha\beta}\right) \\ &= \frac{1}{4}\left(E_{ij}E^{ij} + H_{ij}H^{ij}\right) \ . \end{aligned} \tag{6.30}$$

This third choice of Weyl scalar invariant leads to the same results that the first proposition. In the following, this expression will be assumed to be a good choice of a scalar invariant which described the geometrical properties of space–time.

2.3 Kinematical backreaction

The non–commutativity rule (6.3) triggers additional terms in the average evolution equations. It drives deviations from a homogeneous–isotropic acceleration law and leads to a kinematical backreaction term [Buchert and Ehlers, 1997; Buchert, 2000a,b] (see chapter V):

$$\mathcal{Q}_{\mathcal{D}} = \frac{2}{3}\left(\left\langle\theta^2\right\rangle_{\mathcal{D}} - \left\langle\theta\right\rangle_{\mathcal{D}}^2\right) - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}} \ ; \tag{6.31}$$

where θ is the previously mentioned fluid expansion rate, and σ the shear scalar defined by the tracefree part of the fluid expansion tensor Θ_{ij}:

$$\sigma^{i}{}_{j} = \Theta^{i}{}_{j} - \frac{1}{3}\theta\delta^{i}{}_{j} \ ; \tag{6.32}$$

$$2\sigma^2 := \sigma^i{}_j\sigma^j{}_i = \Theta^i{}_j\Theta^j{}_i - \frac{1}{3}\theta^2 \ . \tag{6.33}$$

The backreaction term is the main difference between the Buchert equations and the Friedmann equations. Both equations are obtained by a similar method but the Buchert equation takes into account the non–commutativity rule (6.3). The starting point is the $3+1$ formalism which leads to the evolution of the fluid expansion tensor and the energy constraint (where $\mathcal{R}^i{}_j$ is the Ricci curvature):

$$\dot{\Theta}^i{}_j + \theta\Theta^i{}_j = (4\pi G\varrho + \Lambda)\,\delta^i{}_j - \mathcal{R}^i{}_j \ ; \tag{6.34}$$

$$\theta^2 + \mathcal{R} - \Theta^k{}_l\Theta^l{}_k = 16\pi G\varrho + 2\Lambda \ . \tag{6.35}$$

Then the trace of the equation of evolution is used to substitute the Ricci scalar in the energy constraint and, according to the definitions of the quadratic scalar of the shear tensor, the energy constraint leads to the Raychaudhuri equation:

$$\dot{\theta} = \Lambda - 4\pi G\varrho + 2\left(\frac{1}{3}\theta^2 - \sigma^2\right) - \theta^2 \ . \tag{6.36}$$

Expressed through the principal scalar invariants of the fluid expansion tensor,

$$\begin{cases} \mathrm{I} := & \theta \ ; \\ \mathrm{II} := & \dfrac{1}{3}\theta^2 - \sigma^2 \ ; \end{cases} \tag{6.37}$$

and according to the non–commutativity rule (6.3), the averaged Raychaudhuri equation reads:

$$\langle\theta\rangle_{\mathcal{D}}^{\cdot} = \Lambda - 4\pi G\,\langle\varrho\rangle_{\mathcal{D}} + 2\,\langle\mathrm{II}\rangle_{\mathcal{D}} - \langle\mathrm{I}\rangle_{\mathcal{D}}^2 \ . \tag{6.38}$$

Using the definition of the averaged volume $V_{\mathcal{D}}$, the previous averaged equation can be written with an effective scale factor $a_{\mathcal{D}}$ (see (5.18)):

$$3\frac{\ddot{a}_D}{a_D} = \Lambda - 4\pi G\,\langle\varrho\rangle_{\mathcal{D}} + \mathcal{Q}_D \ . \tag{6.39}$$

with the kinematical backreaction,

$$\mathcal{Q}_{\mathcal{D}} = 2\,\langle\mathrm{II}\rangle_{\mathcal{D}} - \frac{2}{3}\langle\mathrm{I}\rangle_{\mathcal{D}}^2 \ . \tag{6.40}$$

2.4 Relativistic Lagrangian perturbation schemes

The relativistic Lagrangian perturbation scheme is built on the $3+1$ foliation of space–time and allows us to express all the unknowns in terms of a single dynamical variable, which is represented by the three spatial coframe fields $\boldsymbol{\eta}^a = \eta^a_i\,\mathbf{d}X^i$ as a function of the Lagrangian coordinates X^i [Buchert and Ostermann, 2012],

where the indices $a, b, c...$ count the Cartan 1–form fields and $i, j, k...$ the coordinate indices. The non–integrable coefficients $\eta^a_i(X^i, t)$ generalise the integrable deformation gradient of the Newtonian theory presents in chapter II. The any order relativistic perturbation and solution schemes are given in [Alles et al., 2014] and chapter IV of this manuscript. An average model for the first order scheme, known as the relativistic Zel'dovich approximation [Kasai, 1995; Buchert and Ostermann, 2012] was developed in [Buchert et al., 2013].

2.4.1 General considerations

The perturbation scheme is defined on a reference background that is taken to be part of the class of the homogeneous–isotropic FLRW solutions. Note that all the variables—the relative entropies, the Weyl curvature invariants and the kinematical backreaction—vanish if the dynamics are restricted to the background behaviour. Thus they are therefore strictly related to the inhomogeneities (and they are gauge invariants in a standard perturbative setting). The FLRW background solutions are represented through three homogeneous–isotropic deformation 1–forms, expressed in the local exact coordinate basis by:

$$\boldsymbol{\eta}^a_H = \eta^a_{H\mathbf{i}}\mathbf{d}X^i := a(t)\boldsymbol{\eta}^a_H(t_\mathbf{i}) \quad ; \quad \eta^a_{H\mathbf{i}} := a(t)\delta^a_{\ i} \ ; \tag{6.41}$$

where $a(t)$ is a solution of the Friedmann differential equation:

$$H^2 = \frac{8\pi G\varrho_H(t) + \Lambda}{3} - \frac{k}{a^2(t)} \ ; \tag{6.42}$$

with the non–normalised curvature constant k, the Hubble function $H(t) := \dot{a}/a$, the homogeneous density $\varrho_H(t) = \varrho_{H\mathbf{i}}/a^3(t)$, Λ the cosmological constant and the scale factor respects $a(t_\mathbf{i}) = 1$. For the full deformation 1–forms the following superposition is prescribed:

$$\boldsymbol{\eta}^{\,a} = \boldsymbol{\eta}_H^{\ a} + a(t)\mathbf{P}^{\,a} \ ; \tag{6.43}$$

with the inhomogeneous deformation 1–forms $\mathbf{P}^{\,a}(t, X^k) = P^a_{\ i}\,\mathbf{d}X^i$ that may be developed into a perturbation series as in [Alles et al., 2014] and chapter III. Since the 1–form coframes are the only dynamical variables, only these perturbations are to be set up to perform a perturbative resolution of the problem. The remaining fields are expressed as functions of the coframes, called in the following functional evaluation. For example, the metric expression calculated by this method reads:

$$g_{ij}(t, X^k) = a^2(t)\left\{G_{ij} + G_{aj}P^a_{\ i} + G_{ib}P^b_{\ j} + G_{ab}P^a_{\ i}P^b_{\ j}\right\}, \tag{6.44}$$

with the initial data $g_{ij}(t_\mathbf{i}) = G_{ij}$. Note that the orthogonal coframes (the term orthogonal basis is equally used) are employed as in chapter III. For a discussion of the difference between the orthonormal and orthogonal coframes see [Buchert et al., 2013]. General initial data for the inhomogeneous deformation at initial

time $t_{\mathbf{i}}$ are composed of the coefficients of the initial 1–form generalisation of the Newtonian peculiar–velocity and acceleration gradient:

$$\begin{cases} P^a{}_i(X^i, t_i) = \mathcal{P}^a{}_i \; ; \\ \dot{\mathcal{P}}^i{}_i = U^i{}_j \; ; \\ \ddot{\mathcal{P}}^i{}_i = W^i{}_j - 2H_{\mathbf{i}} U^i{}_j \; . \end{cases} \tag{6.45}$$

The density is split into a homogeneous density $\varrho_H(t)$ and deviations written in terms of the density contrast δ or the inhomogeneous density $\delta\varrho = \varrho_H \delta$,

$$\varrho = \varrho_H + \delta\varrho \quad ; \qquad \varrho_{\mathbf{i}} = \varrho_{H\mathbf{i}} + \delta\varrho_{\mathbf{i}} \quad ; \quad 4\pi G \delta\varrho_{\mathbf{i}} = -W^k{}_k \; . \tag{6.46}$$

The last relation between the initial inhomogeneities of density and the initial trace of the acceleration gradient is the relativistic generalisation of the Poisson equation. The obtained expressions below depend on the principal scalar invariants, e.g. written for the initial condition matrix $\dot{\mathcal{P}}^a{}_i$, they are given by:

$$\begin{aligned} \mathrm{I}_{\mathbf{i}} = \mathrm{I}\left(\dot{\mathcal{P}}^a{}_i\right) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\dot{\mathcal{P}}^a{}_i \delta^b{}_j \delta^c{}_k = \dot{\mathcal{P}} = U^k{}_k \; ; \\ \mathrm{II}_{\mathbf{i}} = \mathrm{II}\left(\dot{\mathcal{P}}^a{}_i\right) &:= \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\dot{\mathcal{P}}^a{}_i \dot{\mathcal{P}}^b{}_j \delta^c{}_k \; ; \\ \mathrm{III}_{\mathbf{i}} = \mathrm{III}\left(\dot{\mathcal{P}}^a{}_i\right) &:= \frac{1}{6}\epsilon_{abc}\epsilon^{ijk}\dot{\mathcal{P}}^a{}_i \dot{\mathcal{P}}^b{}_j \dot{\mathcal{P}}^c{}_k \; . \end{aligned} \tag{6.47}$$

A relation can be set between these invariants and the invariants used to define the kinematical backreaction term, $\mathrm{I} = \mathrm{I}\left(\Theta^i{}_j{}^{(1)}\right)$ and $\mathrm{II} = \mathrm{II}\left(\Theta^i{}_j{}^{(1)}\right)$. Moreover, according to the definition of the initial data and the Relativistic Zel'dovich Approximation (RZA) used in the following parts, we have the following relation $\mathrm{I}_{\mathbf{i}} = \dot{\mathcal{P}} = U^k{}_k = W^k{}_k t_{\mathbf{i}}$.

2.4.2 First order deformations and Relativistic Zel'dovich Approximation

For the evaluation of any functional of the coframes the first order deformations are used. It is furthermore convenient to restrict the initial data by the "slaving condition":

$$U^i{}_j = W^i{}_j t_{\mathbf{i}} \; . \tag{6.48}$$

This result is an extrapolation of the slaving condition for a subclass of solution in an Einstein–de Sitter space–time. This condition is asymptotically reached in the general first order approximation. This restriction of the first order deformation is known as the Relativistic Zel'dovich approximation (RZA) [Kasai, 1995]. The RZA including this extrapolation is precisely defined in [Buchert and Ostermann, 2012; Buchert et al., 2013; Alles et al., 2014] and chapter IV. For more details, explicit solutions for different backgrounds may be found in [Buchert, 1989], those

including a cosmological constant are available in [Bildhauer et al., 1992].

Here, the explicit solutions in the orthogonal basis in an FLRW space–time is considered, with no Λ and assuming RZA solutions, which can be reduced to the extrapolated EdS solution presented in chapter IV:

$$P^i{}_j = \xi(t) t_{\mathbf{i}} \ddot{\mathcal{P}}^i{}_j = \xi(t) \dot{\mathcal{P}}^i{}_j = \frac{3}{2} W^i{}_j t_{\mathbf{i}}{}^2 \left(a - 1\right) \ . \tag{6.49}$$

To simplify the further calculations, a peculiar–volume deformation relative to the background deformation is introduced:

$$\mathrm{J} = a^3(t) \mathcal{J} \ ; \tag{6.50}$$

which, specified to the RZA deformation, reads (we drop the identifier "RZA" for this variable):

$$\mathcal{J}(t, X^k) := 1 + \xi(t) \mathrm{I}_{\mathbf{i}} + \xi^2(t) \mathrm{II}_{\mathbf{i}} + \xi^3(t) \mathrm{III}_{\mathbf{i}} \ ; \tag{6.51}$$

$$\mathrm{I}_{\mathbf{i}} := \mathrm{I}\left(\dot{\mathcal{P}}^a{}_i\right) \ ; \ \mathrm{II}_{\mathbf{i}} := \mathrm{II}\left(\dot{\mathcal{P}}^a{}_i\right) \ ; \ \mathrm{III}_{\mathbf{i}} := \mathrm{III}\left(\dot{\mathcal{P}}^a{}_i\right) \ . \tag{6.52}$$

3 Functional evaluation of the averaged variables

In the following I present the functionals that arise from the substitution of the coframes by their perturbative development (3.31) in the entropy measure, the Weyl scalar and the kinematical backreaction. It is possible to evaluate these functionals for any given order of the perturbed coframe fields. In the following the explicit form of these functionals for the first order coframe fields will be given.

3.1 Entropy

The chosen definition for the entropy measure is the density variance normalised by the average density,

$$\mathcal{F}^1_{\mathcal{D}} = \frac{\langle \varrho^2 \rangle_{\mathcal{D}} - \langle \varrho \rangle^2_{\mathcal{D}}}{\langle \varrho \rangle_{\mathcal{D}}} \ . \tag{6.53}$$

An alternative expression is available using the perturbative decomposition of the density $\varrho = \varrho_H + \delta\varrho$ where the homogeneous density and the average density can be identified,

$$\begin{aligned} \mathcal{F}^1_{\mathcal{D}} &= \frac{\langle \varrho_H^2 + 2\varrho_H \delta\varrho + \delta\varrho^2 \rangle_{\mathcal{D}} - \varrho_H^2 - 2\varrho_H \langle \delta\varrho \rangle_{\mathcal{D}} - \langle \delta\varrho \rangle^2_{\mathcal{D}}}{\langle \varrho \rangle_{\mathcal{D}}} \\ &= \frac{\langle \delta\varrho^2 \rangle_{\mathcal{D}} - \langle \delta\varrho \rangle^2_{\mathcal{D}}}{\langle \varrho \rangle_{\mathcal{D}}} \ . \end{aligned} \tag{6.54}$$

The density inhomogeneities are expected to generically increase in the regime where our approximation is applied. The formal average (6.4) applied to the mass

conservation (6.46) with the "slaving condition" (6.48) leads to the following expression of the density:

$$\varrho = \frac{\varrho_H(t)\chi(X^i)}{\mathcal{J}(X^i,t)} \quad ; \quad \chi := 1 - \frac{W^k_{\ k}}{4\pi G \varrho_{H_{\mathbf{i}}}} \quad ; \quad W^k_{\ k} = \mathrm{I}_{\mathbf{i}} t_i^{-1} \ . \tag{6.55}$$

The last equality is obtained from the definition of the scalar invariant and in the RZA framework. A rapid calculation allows us to check the density expression defined previously,

$$\begin{aligned} & \varrho = \frac{\varrho_H \chi}{\mathcal{J}} = \frac{\varrho_H}{a^3 J}\left(1 - \frac{W^k_{\ k}}{4\pi G \varrho_{H\mathbf{i}}}\right) \\ \Longleftrightarrow \quad & \varrho J = \varrho_H a^{-3} + \varrho_H a^{-3} \frac{\delta \varrho_{\mathbf{i}}}{\varrho_{H\mathbf{i}}} \\ \Longleftrightarrow \quad & \varrho_{\mathbf{i}} = \varrho_{H\mathbf{i}} + \delta \varrho_{\mathbf{i}} \ . \end{aligned} \tag{6.56}$$

I recall the definition of the average normalised by the initial volume, $\langle A \rangle_{\mathcal{D}} = \langle AJ \rangle_{\mathcal{I}} / \langle J \rangle_{\mathcal{I}}$. The substitution of the density in the expressions of the entropy measure (6.53) and (6.54) respectively leads to:

$$\begin{aligned} {}^{\mathrm{RZA}}\mathcal{F}^1_{\mathcal{D}} &= \frac{\langle \varrho^2 \rangle_{\mathcal{D}} - \langle \varrho \rangle^2_{\mathcal{D}}}{\langle \varrho \rangle_{\mathcal{D}}} \\ &= \left(\left\langle \left(\frac{\varrho_H \chi}{\mathcal{J}} \right)^2 \right\rangle_{\mathcal{D}} - \left\langle \frac{\varrho_H \chi}{\mathcal{J}} \right\rangle^2_{\mathcal{D}} \right) \left(\left\langle \frac{\varrho_H \chi}{\mathcal{J}} \right\rangle_{\mathcal{D}} \right)^{-1} \\ &= \varrho_H \left(\left\langle \frac{\chi^2}{\mathcal{J}^2} \right\rangle_{\mathcal{D}} - \left\langle \frac{\chi}{\mathcal{J}} \right\rangle^2_{\mathcal{D}} \right) \left(\left\langle \frac{\chi}{\mathcal{J}} \right\rangle_{\mathcal{I}} \right)^{-1} \\ &= \frac{\varrho_H}{a^3} \left(\left\langle \frac{\chi^2}{\mathcal{J}} \right\rangle_{\mathcal{I}} \frac{1}{\langle \mathcal{J} \rangle_{\mathcal{I}}} - \frac{\langle \chi \rangle^2_{\mathcal{I}}}{\langle \mathcal{J} \rangle^2_{\mathcal{I}}} \right) \left(\frac{\langle \chi \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right)^{-1} \\ &= \frac{1}{a^3} \frac{\varrho_{H_{\mathbf{i}}}}{\langle \chi \rangle_{\mathcal{I}}} \left(\left\langle \frac{\chi^2}{\mathcal{J}} \right\rangle_{\mathcal{I}} - \frac{\langle \chi \rangle^2_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right) \ ; \end{aligned} \tag{6.57}$$

or,

$$\begin{aligned} {}^{\mathrm{RZA}}\mathcal{F}^1_{\mathcal{D}} &= \frac{\langle \delta \varrho^2 \rangle_{\mathcal{D}} - \langle \delta \varrho \rangle^2_{\mathcal{D}}}{\langle \varrho \rangle_{\mathcal{D}}} \\ &= \left(\frac{\langle \delta \varrho^2 \mathcal{J} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} - \frac{\langle \delta \varrho \mathcal{J} \rangle^2_{\mathcal{I}}}{\langle \mathcal{J} \rangle^2_{\mathcal{I}}} \right) \left(\frac{\langle \varrho \mathcal{J} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right)^{-1} \\ &= \frac{1}{\langle \varrho_{\mathbf{i}} \mathcal{J} J^{-1} \rangle_{\mathcal{I}}} \left(\langle \delta \varrho^2 \mathcal{J} \rangle_{\mathcal{I}} - \frac{\langle \delta \varrho \mathcal{J} \rangle^2_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right) \end{aligned}$$

$$= \frac{a^3}{\langle \varrho_{\mathbf{i}} \rangle_{\mathcal{I}}} \left(\left\langle \delta\varrho^2 \mathcal{J} \right\rangle_{\mathcal{I}} - \frac{\langle \delta\varrho \mathcal{J} \rangle_{\mathcal{I}}^2}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right) . \tag{6.58}$$

These two expressions are equivalent, and it is possible to go from one to another thanks to the relation between the inhomogeneities of density, the homogeneous density and the full density,

$$\delta\varrho = \varrho - \langle \varrho \rangle_{\mathcal{D}} = \varrho - \langle \varrho \mathcal{J} \rangle_{\mathcal{I}} / \langle \mathcal{J} \rangle_{\mathcal{I}} \ ; \tag{6.59}$$

where we used equation (6.5) for the second equality. Nevertheless, the second entropy measure expression clearly shows that the leading order of the entropy is quadratic in $\delta\varrho$ which can be identified as a first order perturbation. The following quantities are useful to simplify the expression I am about to calculate:

$$\xi = 1 + \frac{\delta\varrho_{\mathbf{i}}}{\varrho_{H\mathbf{i}}} \quad ; \quad \delta\varrho_{\mathbf{i}} = -\frac{\mathrm{I}_{\mathbf{i}} t_{\mathbf{i}}{}^{-1}}{4\pi G} . \tag{6.60}$$

Then the leading order (denotes by the arrow $^{\uparrow}$) of the entropy functional is quadratic with respect to the initial data:

$$\begin{aligned}
{}^{RZA}\mathcal{F}_{\mathcal{D}}^{1\uparrow} &\approx a^{-3}\varrho_{H\mathbf{i}} \left[1 - \frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} \right] \left[\left\langle \left(1 + 2\frac{\delta\varrho_{\mathbf{i}}}{\varrho_{H\mathbf{i}}} + \frac{\delta\varrho_{\mathbf{i}}{}^2}{\varrho_{H\mathbf{i}}^2} \right) \left(1 - \xi \mathrm{I}_{\mathbf{i}} - \xi^2 \mathrm{II}_{\mathbf{i}} + \xi^2 \mathrm{I}_{\mathbf{i}}^2 \right) \right\rangle_{\mathcal{I}} \right. \\
&\qquad \left. - \left(1 + 2\frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} + \frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}^2}{\varrho_{H\mathbf{i}}^2} \right) \left(1 - \xi \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} - \xi^2 \langle \mathrm{II}_{\mathbf{i}} \rangle_{\mathcal{I}} + \xi^2 \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \right) \right] \\
&= a^{-3} \left[\varrho_{H\mathbf{i}} - \langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}} \right] \\
&\qquad \times \left[1 - \xi \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} - \xi^2 \langle \mathrm{II}_{\mathbf{i}} \rangle_{\mathcal{I}} + \xi^2 \left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} + 2\frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} - 2\xi \frac{\langle \delta\varrho_{\mathbf{i}} \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} + \frac{\langle \delta\varrho_{\mathbf{i}}{}^2 \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}^2} \right. \\
&\qquad \left. - 1 + \xi \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} + \xi^2 \langle \mathrm{II}_{\mathbf{i}} \rangle_{\mathcal{I}} - \xi^2 \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} - 2\frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} + 2\xi \frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} - \frac{\langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}^2}{\varrho_{H\mathbf{i}}} \right] \\
&= a^{-3} \varrho_{H\mathbf{i}} \left[\xi^2 \left(\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \right) - 2\xi \frac{\langle \delta\varrho_{\mathbf{i}} \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} - \langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}}}{\varrho_{H\mathbf{i}}} + \frac{\langle \delta\varrho_{\mathbf{i}}{}^2 \rangle_{\mathcal{I}} - \langle \delta\varrho_{\mathbf{i}} \rangle_{\mathcal{I}}^2}{\varrho_{H\mathbf{i}}^2} \right] \\
&= \frac{a^{-3}}{4\pi G} \left[\frac{t_{\mathbf{i}}{}^{-2}}{4\pi G \varrho_{H\mathbf{i}}} + 2t_{\mathbf{i}}{}^{-1}\xi + \xi^2 \right] \left[\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \langle \mathrm{I}_{\mathbf{i}} \rangle_{\mathcal{I}} \right] .
\end{aligned} \tag{6.61}$$

Then the leading order of the entropy measure depends on the first scalar invariant, the scale factor and the solution time dependence $\xi(t)$. This solution is correct at least in an Einstein–de Sitter space–time but more generally in any model which admits a space–time splitting of the solution such as $P^a_{\ i} = \xi(t)\dot{\mathcal{P}}^a_{\ i}$.

Let us assume the solution describes an Einstein–de Sitter space–time:

$$\begin{cases} \xi(t) = \dfrac{3}{2} t_{\mathbf{i}} \left(a - 1\right) \ ; \\ a(t) = \left(\dfrac{t}{t_{\mathbf{i}}}\right)^{2/3} \quad ; \quad 3\dfrac{\ddot{a}}{a} = -4\pi G \varrho_H \quad ; \quad 4\pi G \varrho_{H\mathbf{i}} = -3\ddot{a}_{\mathbf{i}} = \dfrac{2}{3} t_{\mathbf{i}}{}^{-2} \ . \end{cases}$$

$$\begin{aligned} {}^{\mathrm{EdS}}\mathcal{F}^{1\uparrow}_{\mathcal{D}} &= \frac{a^{-3}}{4\pi G} \frac{2}{3t_{\mathbf{i}}{}^2} \left[\frac{9t_{\mathbf{i}}{}^2}{4} + 3t_{\mathbf{i}} \, {}^{\mathrm{EdS}}\xi + {}^{\mathrm{EdS}}\xi^2 \right] \left(\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \right) \\ &= \frac{3a^{-3}}{8\pi G} \left[1 + \frac{4}{3t_{\mathbf{i}}} \, {}^{\mathrm{EdS}}\xi + \frac{4}{9t_{\mathbf{i}}{}^2} \, {}^{\mathrm{EdS}}\xi^2 \right] \left[\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \right] \\ &= \frac{3a^{-3}}{8\pi G} \left[1 + \frac{4}{3t_{\mathbf{i}}} \frac{3}{2} t_{\mathbf{i}} \left(a - 1\right) + \frac{4}{9t_{\mathbf{i}}{}^2} \frac{9}{4} t_{\mathbf{i}}{}^2 \left(a^2 - 2a + 1\right) \right] \left[\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \right] \\ &= \frac{3a^{-3}}{8\pi G} \left[1 + 2\left(a - 1\right) + \left(a^2 - 2a + 1\right) \right] \left[\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \right] \\ &= \frac{3a^{-1}}{8\pi G} \left[\left\langle \mathrm{I}_{\mathbf{i}}^2 \right\rangle_{\mathcal{I}} - \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \left\langle \mathrm{I}_{\mathbf{i}} \right\rangle_{\mathcal{I}} \right] \ . \end{aligned} \tag{6.62}$$

For an EdS space–time the chosen entropy measure, i.e. the density variance, comes down to a simple formula decreasing in time, but I recall that this quantity is identified as the entropy per volume unit (see [Li et al., 2012]). The volume grows as $a^3_{\mathcal{D}} = (t/t_{\mathbf{i}})^2$ thus the quantity defined as entropy is a growing function. Even with a different approach for the perturbations, the results are coherent with literature:

3.2 Weyl scalar invariants

The spatial part of the Weyl scalar invariant can be evaluated with the fluid expansion tensor and its time and spatial covariant derivatives. The fluid expansion tensor coefficients are expressed in terms of the coframes:

$$\Theta^i{}_j = e_a{}^i \dot{\eta}^a{}_j = \frac{1}{2J} \epsilon_{abc} \epsilon^{ik\ell} \dot{\eta}^a{}_j \eta^b{}_k \eta^c{}_\ell \ ; \tag{6.63}$$

where the inverse (frame) coefficients $e_a{}^i$ have been expressed algebraically through the coframe coefficients. The spatially projected electric and magnetic parts of the Weyl tensor (see chapter V) are:

$$\begin{aligned} E^i{}_j &= -\dot{\Theta}^i{}_j - \Theta^i{}_k \Theta^k{}_j - \frac{1}{3} \left(4\pi G \varrho - \Lambda\right) \delta^i{}_j \ ; \\ H^i{}_j &= -g^{ik} \frac{\epsilon^{lsn}}{J} g_{n(k} \Theta_{lj)||s} \ ; \end{aligned} \tag{6.64}$$

where a double stroke "$||$" denotes spatial covariant derivative which can be expressed through the coframe coefficients. Note that the fluid expansion tensor with

mixed indices is independent of the basis choice. Employing our formal average (6.4) and the expression (6.22), the Weyl scalar invariant is expressed as follows:

$$\left\langle C^2 \right\rangle_{\mathcal{D}} = \frac{\langle C^2 \mathfrak{J} \rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} = \frac{1}{4} \left(\frac{\left\langle E^i{}_j E^j{}_i \mathcal{J} \right\rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} + \frac{\left\langle H^i{}_j H^j{}_i \mathcal{J} \right\rangle_{\mathcal{I}}}{\langle \mathcal{J} \rangle_{\mathcal{I}}} \right) . \tag{6.65}$$

For calculational convenience in manipulating the functional expressions of the Weyl invariant, the following tensors—whose trace provides the scalar invariants—are defined:

$$\mathrm{I}^i{}_j = \frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \dot{\mathcal{P}}^a{}_j \delta^b{}_k \delta^c{}_l = \dot{\mathcal{P}}^i{}_j \; ; \tag{6.66}$$

$$\mathrm{I}^i{}_j{}^{(2)} = \frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \delta^a{}_j \dot{\mathcal{P}}^b{}_k \delta^c{}_l = \frac{1}{2} \left(\mathrm{I_i} \delta^i{}_j - \mathrm{I}^i{}_j \right) \; ; \tag{6.67}$$

$$\mathrm{II}^i{}_j = \frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \dot{\mathcal{P}}^a{}_j \dot{\mathcal{P}}^b{}_k \delta^c{}_l = \frac{1}{2} \left(\mathrm{I_i} \mathrm{I}^i{}_j - \mathrm{I}^i{}_k \mathrm{I}^k{}_j \right) \; ; \tag{6.68}$$

$$\mathrm{II}^i{}_j{}^{(2)} = \frac{1}{2} \epsilon_{abc} \epsilon^{ikl} \delta^a{}_j \dot{\mathcal{P}}^b{}_k \dot{\mathcal{P}}^c{}_l \; ; \tag{6.69}$$

$$\mathrm{III}^i{}_j = \frac{1}{6} \epsilon_{abc} \epsilon^{ikl} \dot{\mathcal{P}}^a{}_j \dot{\mathcal{P}}^b{}_k \dot{\mathcal{P}}^c{}_l \; . \tag{6.70}$$

The following special combinations of these tensors will appear in the functional evaluation too:

$$\mathrm{I}^i{}_j{}' = \mathrm{I}^i{}_j + 2\mathrm{I}^i{}_j{}^{(2)} = \mathrm{I_i} \delta^i{}_j \; ; \tag{6.71}$$

$$\mathrm{II}^i{}_j{}' = 2\mathrm{II}^i{}_j + \mathrm{II}^i{}_j{}^{(2)} = \mathrm{II_i} \delta^i{}_j \; ; \tag{6.72}$$

$$\mathrm{III}^i{}_j{}' = 3\mathrm{III}^i{}_j \; . \tag{6.73}$$

With these auxiliary expressions it is possible to build matrices inspired by the volume deformation expression (the first being defined earlier in this chapter):

$$J = a^3 \mathcal{J} = a^3 \left(1 + \xi \mathrm{I_i} + \xi^2 \mathrm{II_i} + \xi^3 \mathrm{III_i} \right) \; ; \tag{6.74}$$

$$\mathcal{J}^i{}_j := \frac{1}{3} \delta^i{}_j + \xi \mathrm{I}^i{}_j + \xi^2 \mathrm{II}^i{}_j + \xi^3 \mathrm{III}^i{}_j \; ; \tag{6.75}$$

$$3\mathcal{J}^i{}_j{}' := \delta^i{}_j + \xi \mathrm{I}^i{}_j{}' + \xi^2 \mathrm{II}^i{}_j{}' + \xi^3 \mathrm{III}^i{}_j{}' = \mathcal{J} \delta^i{}_j + \xi^3 \left(3\mathrm{III}^i{}_j - \mathrm{III_i} \delta^i{}_j \right) \; . \tag{6.76}$$

Note that the traces of the latter two tensors both yield the peculiar–volume deformation $\mathcal{J}^i{}_i = \mathcal{J}^i{}_i{}' = \mathcal{J}$. Thanks to the expression (6.63), the perturbative development of the coframes (6.49) and the last definitions, the fluid expansion tensor can be written in the compact form:

$$\Theta^i{}_j = \frac{1}{\mathcal{J}} \left(3H \mathcal{J}^i{}_j{}' + \dot{\mathcal{J}}^i{}_j \right) \; . \tag{6.77}$$

The Weyl scalar invariant term generated by the electric part of the Weyl tensor, expressed in terms of the fluid expansion tensor, reads,

$$\begin{aligned} E^i{}_j E^j{}_i &= \dot{\Theta}^i{}_j \dot{\Theta}^j{}_i + \Theta^i{}_k \Theta^k{}_j \Theta^j{}_l \Theta^l{}_i + \frac{1}{3}\left(\dot{\theta}^2 + 2\dot{\theta}\Theta^l{}_k \Theta^k{}_l + \Theta^l{}_k \Theta^k{}_l \Theta^m{}_n \Theta^n{}_m\right) \\ &\quad + \dot{\Theta}^i{}_j \Theta^j{}_k \Theta^k{}_i + \Theta^i{}_k \Theta^k{}_j \dot{\Theta}^j{}_i - \frac{2}{3}\dot{\theta}\left(\dot{\theta} + \Theta^k{}_l \Theta^l{}_k\right) - \frac{2}{3}\Theta^l{}_k \Theta^k{}_l \left(\dot{\theta} + \Theta^m{}_n \Theta^n{}_m\right) \\ &= \dot{\Theta}^i{}_j \dot{\Theta}^j{}_i + \Theta^i{}_k \Theta^k{}_j \Theta^j{}_l \Theta^l{}_i - \frac{1}{3}\dot{\theta}^2 - \frac{2}{3}\dot{\theta}\Theta^l{}_k \Theta^k{}_l - \frac{1}{3}\Theta^l{}_k \Theta^k{}_l \Theta^m{}_n \Theta^n{}_m + 2\dot{\Theta}^i{}_j \Theta^j{}_k \Theta^k{}_i \,. \end{aligned} \tag{6.78}$$

Let us perform some preliminaries calculations to simplify the writing of (6.78). In order to simplify the following developments, the computation is limited to the second order. Assuming this truncation the following quantities are redefined such as:

$$3\mathcal{J}^i{}_j{}' = \mathcal{J}\delta^i{}_j = \left(1 + \xi \mathrm{I_i} + \xi^2 \mathrm{II_i}\right)\delta^i{}_j \,; \tag{6.79}$$

$$\mathcal{J}^i{}_j = \frac{1}{3}\delta^i{}_j + \xi \mathrm{I_i} + \xi^2 \mathrm{II_i} \,. \tag{6.80}$$

Moreover it is important to note that their time derivatives are at least first order quantities,

$$\dot{\mathcal{J}}^i{}_j{}' = \dot{\mathcal{J}}\delta^i{}_j = \left(\dot{\xi}\mathrm{I_i} + 2\dot{\xi}\xi \mathrm{II_i}\right)\delta^i{}_j \,; \tag{6.81}$$

$$\dot{\mathcal{J}}^i{}_j = \dot{\xi}\mathrm{I_i} + 2\dot{\xi}\xi \mathrm{II_i} \,; \tag{6.82}$$

then the terms which involve at least three time derivatives are at least of third order and are not considered. Thus the fluid expansion tensor and its trace read,

$$\Theta^i{}_j = H\delta^i{}_j + \frac{\dot{\mathcal{J}}^i{}_j}{\mathcal{J}} \,; \tag{6.83}$$

$$\theta = 3H + \frac{\dot{\mathcal{J}}}{\mathcal{J}} \,; \tag{6.84}$$

$$\dot{\Theta}^i{}_j = \left(\frac{\ddot{a}}{a} - H^2\right)\delta^i{}_j + \frac{\ddot{\mathcal{J}}^i{}_j}{\mathcal{J}} - \frac{\dot{\mathcal{J}}^i{}_j \dot{\mathcal{J}}}{\mathcal{J}^2} \,; \tag{6.85}$$

$$\dot{\theta} = 3\left(\frac{\ddot{a}}{a} - H^2\right) + \frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} \,; \tag{6.86}$$

Here the results of the computation of the different terms involved in the expression of the electric part of the Weyl scalar invariant up to the second order are presented. The second order is chosen because the leading order of the entropy measure is a second order expression:

$$\Theta^i{}_k \Theta^k{}_j \Theta^j{}_l \Theta^l{}_i = 3H^4 + 2H^2 \frac{\dot{\mathcal{J}}^i{}_j \dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 4H^3 \frac{\dot{\mathcal{J}}}{\mathcal{J}} + 4H^2 \frac{\dot{\mathcal{J}}^i{}_j \dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} \,; \tag{6.87}$$

$$\Theta^k{}_l\Theta^l{}_k\Theta^m{}_n\Theta^n{}_m = 9H^4 + 4H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} + 6H^2\frac{\dot{\mathcal{J}}^k{}_l\dot{\mathcal{J}}^l{}_k}{\mathcal{J}^2} + 12H^3\frac{\dot{\mathcal{J}}}{\mathcal{J}}\ ; \tag{6.88}$$

$$\begin{aligned}\dot{\Theta}^i{}_j\Theta^j{}_k\Theta^k{}_i &= 3H^2\left(\frac{\ddot{a}}{a} - H^2\right) + H^2\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2}\\ &+ \left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 2H\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}}{\mathcal{J}} + 2H\frac{\ddot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2}\ .\end{aligned} \tag{6.89}$$

$$\begin{aligned}\dot{\theta}\Theta^i{}_j\Theta^j{}_i &= 3H^2\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - 3H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} + 9H^2\left(\frac{\ddot{a}}{a} - H^2\right)\\ &+ 3\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 2H\frac{\dot{\mathcal{J}}\ddot{\mathcal{J}}}{\mathcal{J}^2} + 6H\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}}{\mathcal{J}}\ ;\end{aligned} \tag{6.90}$$

$$\dot{\Theta}^i{}_j\dot{\Theta}^j{}_i = 3\left(\frac{\ddot{a}}{a} - H^2\right)^2 + \frac{\ddot{\mathcal{J}}^i{}_j\ddot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 2\left(\frac{\ddot{a}}{a} - H^2\right)\left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2}\right); \tag{6.91}$$

$$\dot{\theta}^2 = \frac{\ddot{\mathcal{J}}^2}{\mathcal{J}^2} + 9\left(\frac{\ddot{a}}{a} - H^2\right)^2 + 6\left(\frac{\ddot{a}}{a} - H^2\right)\left(\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - \frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2}\right)\ ; \tag{6.92}$$

Then the second order electric part of the Weyl scalar invariant is,

$$\begin{aligned}E^i{}_jE^j{}_i{}^{\uparrow} &= 3\left(\frac{\ddot{a}}{a} - H^2\right)^2 + \frac{\ddot{\mathcal{J}}^i{}_j\ddot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2}\\
&+ 3H^4 + 2H^2\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 4H^3\frac{\dot{\mathcal{J}}}{\mathcal{J}} + 4H^2\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} - \frac{1}{3}\frac{\ddot{\mathcal{J}}^2}{\mathcal{J}^2} - 3\left(\frac{\ddot{a}}{a} - H^2\right)^2\\
&- 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\ddot{\mathcal{J}}}{\mathcal{J}} + 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} - 2H^2\frac{\ddot{\mathcal{J}}}{\mathcal{J}} + 2H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} - 6H^2\left(\frac{\ddot{a}}{a} - H^2\right)\\
&- 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} - \frac{4}{3}H\frac{\dot{\mathcal{J}}\ddot{\mathcal{J}}}{\mathcal{J}^2} - 4H\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}}{\mathcal{J}} - 3H^4\\
&- \frac{4}{3}H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} - 2H^2\frac{\dot{\mathcal{J}}^k{}_l\dot{\mathcal{J}}^l{}_k}{\mathcal{J}^2} - 4H^3\frac{\dot{\mathcal{J}}}{\mathcal{J}} + 6H^2\left(\frac{\ddot{a}}{a} - H^2\right) + 2H^2\frac{\ddot{\mathcal{J}}}{\mathcal{J}}\\
&- 2H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} + 2\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 4H\left(\frac{\ddot{a}}{a} - H^2\right)\frac{\dot{\mathcal{J}}}{\mathcal{J}} + 4H\frac{\ddot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2}\\
&= \frac{\ddot{\mathcal{J}}^i{}_j\ddot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} + 4H^2\frac{\dot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2} - \frac{1}{3}\frac{\ddot{\mathcal{J}}^2}{\mathcal{J}^2} - \frac{4}{3}H\frac{\dot{\mathcal{J}}\ddot{\mathcal{J}}}{\mathcal{J}^2} - \frac{4}{3}H^2\frac{\dot{\mathcal{J}}^2}{\mathcal{J}^2} + 4H\frac{\ddot{\mathcal{J}}^i{}_j\dot{\mathcal{J}}^j{}_i}{\mathcal{J}^2}\\
&= \left(\ddot{\xi}^2 + 4H^2\dot{\xi}^2 + 4H\ddot{\xi}\dot{\xi}\right)\mathrm{I}^i{}_j\mathrm{I}^j{}_i - \frac{1}{3}\left(\ddot{\xi}^2 + 4H^2\dot{\xi}^2 + 4H\ddot{\xi}\dot{\xi}\right)\mathrm{I}^2_{\mathbf{i}}\ .\end{aligned} \tag{6.93}$$

In the Einstein–de Sitter case, the time function is replaced by its expression and the leading order of the electric part of the Weyl scalar invariant reads,

$$
\begin{aligned}
{}^{EdS}E^i{}_j E^j{}_i{}^{\uparrow} &= \left(\frac{9}{4}t_{\mathbf{i}}{}^2\ddot{a}^2 + 4\frac{9}{4}t_{\mathbf{i}}{}^2\frac{\dot{a}^4}{a^2} + 4\frac{9}{4}t_{\mathbf{i}}{}^2\frac{\ddot{a}\dot{a}^2}{a}\right)\left(\mathrm{I}^i{}_j\mathrm{I}^j{}_i - \frac{1}{3}\mathrm{I}_{\mathbf{i}}^2\right) \\
&= \left(\frac{1}{9t_{\mathbf{i}}{}^2}a^{-4} + \frac{16}{9t_{\mathbf{i}}{}^2}a^{-4} - \frac{8}{9t_{\mathbf{i}}{}^2}a^{-4}\right)\left(\mathrm{I}^i{}_j\mathrm{I}^j{}_i - \frac{1}{3}\mathrm{I}_{\mathbf{i}}^2\right) \\
&= a^{-4}t_{\mathbf{i}}{}^{-2}\left(\mathrm{I}^i{}_j\mathrm{I}^j{}_i - \frac{1}{3}\mathrm{I}_{\mathbf{i}}^2\right) \; ; \qquad (6.94)
\end{aligned}
$$

and its average is,

$$
\begin{aligned}
{}^{EdS}\left\langle E^i{}_j E^j{}_i\right\rangle_{\mathcal{D}}^{\uparrow} &= {}^{EdS}\left\langle E^i{}_j E^j{}_i \mathcal{J}\right\rangle_{\mathcal{I}}^{\uparrow} / \left\langle \mathcal{J}\right\rangle_{\mathcal{I}}^{\uparrow} \approx {}^{EdS}\left\langle E^i{}_j E^j{}_i\right\rangle_{\mathcal{I}}^{\uparrow} \\
&= a^{-4}t_{\mathbf{i}}{}^{-2}\left(\left\langle \mathrm{I}^i{}_j\mathrm{I}^j{}_i\right\rangle_{\mathcal{I}} - \frac{1}{3}\left\langle \mathrm{I}_{\mathbf{i}}^2\right\rangle_{\mathcal{I}}\right) \; . \qquad (6.95)
\end{aligned}
$$

More of half of the work is done, because the expression of the leading order of the electric part of the Weyl scalar invariant is the most complicated to compute.

Let us now compute the magnetic part of the Weyl scalar invariant. The expression of the generalised Jacobian functions are also truncated at the second order. The functional evaluation of the magnetic part of the Weyl tensor is,

$$
\begin{aligned}
H^i{}_j &= -a^{-3}\frac{\epsilon^{lsn}}{\mathcal{J}}g^{ik}g_{n(k}g_{lm}\Theta^m{}_{j)||s} \\
&= -a^{-3}\frac{\epsilon^{lsn}}{\mathcal{J}}g^{ik}g_{n(k}g_{lm}\left(H\delta^m{}_{j)} + \frac{\dot{\mathcal{J}}^m{}_{j)}}{\mathcal{J}}\right)_{||s} \; . \qquad (6.96)
\end{aligned}
$$

Thus the second order of the magnetic part of the Weyl scalar is,

$$
\begin{aligned}
H^i{}_j H^j{}_i{}^{\uparrow} &= \frac{\epsilon^{lsn}\epsilon^{pqr}}{\mathcal{J}^2}a^{-6}g^{ik}g_{n(k}g_{lm}g^{tj}g_{r(t}g_{pu} \\
&\times\left[\left(H\delta^m{}_{j)}\right)_{||s}\left(H\delta^u{}_{i)}\right)_{||q} + \left(H\delta^m{}_{j)}\right)_{||s}\frac{\dot{\mathcal{J}}^u{}_{i)||q}}{\mathcal{J}} + \frac{\dot{\mathcal{J}}^m{}_{j)||s}}{\mathcal{J}}\left(H\delta^u{}_{i)}\right)_{||q} + \frac{\dot{\mathcal{J}}^m{}_{j)||s}}{\mathcal{J}}\frac{\dot{\mathcal{J}}^u{}_{i)||q}}{\mathcal{J}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\epsilon^{lsn}\epsilon^{pqr}}{\mathcal{J}^2} a^{-6} g^{ik} g_{n(k} g^{tj} g_{r(t} \Bigg[\left(H g_{lj)}\right)_{||s} \left(H g_{pi)}\right)_{||q} + \left(H g_{lj)}\right)_{||s} g_{pu} \frac{\dot{\mathcal{J}}^{u}_{\ i)||q}}{\mathcal{J}} \\
&\qquad + g_{lm} \frac{\dot{\mathcal{J}}^{m}_{\ j)||s}}{\mathcal{J}} \left(H g_{pi)}\right)_{||q} + g_{lm} g_{pu} \frac{\dot{\mathcal{J}}^{m}_{\ j)||s}}{\mathcal{J}} \frac{\dot{\mathcal{J}}^{u}_{\ i)||q}}{\mathcal{J}} \Bigg] \\
&= \frac{\epsilon^{lsn}\epsilon^{pqr}}{4\mathcal{J}^2} a^{-6} g^{ik} g^{tj} \times \\
&\Bigg[\left(g_{nk} g_{lj} g_{rt} g_{pi} + g_{nk} g_{lj} g_{ri} g_{pt} + g_{nj} g_{lk} g_{rt} g_{pi} + g_{nj} g_{lk} g_{ri} g_{pt}\right) H_{||s} H_{||q} \\
&+ \left(\left(g_{nk} g_{lj} g_{rt} + g_{nj} g_{lk} g_{rt}\right) \frac{\dot{\mathcal{J}}^{u}_{\ i||q}}{\mathcal{J}} + \left(g_{nk} g_{lj} g_{ri} + g_{nj} g_{lk} g_{ri}\right) \frac{\dot{\mathcal{J}}^{u}_{\ t||q}}{\mathcal{J}} \right) g_{pu} H_{||s} \\
&+ \left(\left(g_{nk} g_{rt} g_{pi} + g_{nk} g_{ri} g_{pt}\right) \frac{\dot{\mathcal{J}}^{m}_{\ j||s}}{\mathcal{J}} + \left(g_{nj} g_{rt} g_{pi} + g_{nj} g_{ri} g_{pt}\right) \frac{\dot{\mathcal{J}}^{m}_{\ k||s}}{\mathcal{J}} \right) g_{lm} H_{||q} \\
&+ \left(g_{nk} \frac{\dot{\mathcal{J}}^{m}_{\ j||s}}{\mathcal{J}} \left(g_{rt} \frac{\dot{\mathcal{J}}^{u}_{\ i||q}}{\mathcal{J}} + g_{ri} \frac{\dot{\mathcal{J}}^{u}_{\ t||q}}{\mathcal{J}} \right) + g_{nj} \frac{\dot{\mathcal{J}}^{m}_{\ k||s}}{\mathcal{J}} \left(g_{rt} \frac{\dot{\mathcal{J}}^{u}_{\ i||q}}{\mathcal{J}} + g_{ri} \frac{\dot{\mathcal{J}}^{u}_{\ t||q}}{\mathcal{J}} \right) \right) g_{pu} g_{lm} \Bigg] \\
&= \frac{\epsilon^{lsn}\epsilon^{pqr}}{4\mathcal{J}^2} a^{-6} \Bigg[2 \left(g_{rl} g_{pn} + g_{nr} g_{pl}\right) H_{||s} H_{||q} \\
&+ 2 \left(g_{lr} \frac{\dot{\mathcal{J}}^{u}_{\ n||q}}{\mathcal{J}} + g_{rn} \frac{\dot{\mathcal{J}}^{u}_{\ l||q}}{\mathcal{J}} \right) g_{pu} H_{||s} + 2 \left(\frac{\dot{\mathcal{J}}^{m}_{\ r||s}}{\mathcal{J}} g_{pn} + \frac{\dot{\mathcal{J}}^{m}_{\ p||s}}{\mathcal{J}} g_{rn} \right) g_{lm} H_{||q} \\
&+ \left(2 \frac{\dot{\mathcal{J}}^{m}_{\ r||s}}{\mathcal{J}} \frac{\dot{\mathcal{J}}^{u}_{\ n||q}}{\mathcal{J}} + g_{rn} g^{tj} \frac{\dot{\mathcal{J}}^{m}_{\ j||s}}{\mathcal{J}} \frac{\dot{\mathcal{J}}^{u}_{\ t||q}}{\mathcal{J}} + g_{rn} g^{ik} \frac{\dot{\mathcal{J}}^{m}_{\ k||s}}{\mathcal{J}} \frac{\dot{\mathcal{J}}^{u}_{\ i||q}}{\mathcal{J}} \right) g_{pu} g_{lm} \Bigg]; \quad (6.97)
\end{aligned}
$$

and according to the symmetry of the metric g_{ij} and the antisymmetry of the Levi-Civita pseudo–tensor, the magnetic part of the Weyl scalar invariant reads,

$$
\begin{aligned}
H^{i}_{\ j} H^{j}_{\ i}{}^{\uparrow} &= \frac{\epsilon^{lsn}\epsilon^{pqr}}{2\mathcal{J}^4} a^{-6} g_{pu} g_{lm} \left(\dot{\mathcal{J}}^{m}_{\ r||s} \dot{\mathcal{J}}^{u}_{\ n||q} + g_{rn} g^{tj} \dot{\mathcal{J}}^{m}_{\ j||s} \dot{\mathcal{J}}^{u}_{\ t||q} \right) \\
&= \frac{\epsilon^{lsn}\epsilon^{pqr}}{2} a^{-2} \delta_{pu} \delta_{lm} \dot{\xi}^2 \left(\mathrm{I}^{m}_{\ r||s} \mathrm{I}^{u}_{\ n||q} + \delta_{rn} \delta^{tj} \mathrm{I}^{m}_{\ j||s} \mathrm{I}^{u}_{\ t||q} \right) . \quad (6.98)
\end{aligned}
$$

Then, the average of the leading order is in the Einstein–de Sitter case,

$$
\begin{aligned}
\left\langle H^{i}_{\ j} H^{j}_{\ i} \right\rangle^{\uparrow}_{\mathcal{D}} &= {}^{\mathrm{EdS}} \left\langle H^{i}_{\ j} H^{j}_{\ i} \mathcal{J} \right\rangle^{\uparrow}_{\mathcal{I}} / \left\langle \mathcal{J} \right\rangle^{\uparrow}_{\mathcal{I}} \\
&\approx {}^{\mathrm{EdS}} \left\langle H^{i}_{\ j} H^{j}_{\ i} \right\rangle^{\uparrow}_{\mathcal{I}} \\
&= \frac{\epsilon^{lsn}\epsilon^{pqr}}{2} a^{-3} \delta_{pu} \delta_{lm} \left(\left\langle \mathrm{I}^{m}_{\ r||s} \mathrm{I}^{u}_{\ n||q} \right\rangle_{\mathcal{I}} + \delta_{rn} \delta^{tj} \left\langle \mathrm{I}^{m}_{\ j||s} \mathrm{I}^{u}_{\ t||q} \right\rangle_{\mathcal{I}} \right) . \quad (6.99)
\end{aligned}
$$

Thus the leading order of the Weyl scalar invariant (6.65) takes the following form in the Einstein–de Sitter case,

$$
\begin{aligned}
{}^{EdS}\left\langle C^2\mathcal{J}\right\rangle^{\uparrow}_{\mathcal{I}}/\left\langle\mathcal{J}\right\rangle^{\uparrow}_{\mathcal{I}} &= {}^{EdS}\left\langle C^2\right\rangle^{\uparrow}_{\mathcal{I}} \\
&= \frac{1}{4}\left({}^{EdS}\left\langle E^i{}_jE^j{}_i\right\rangle^{\uparrow}_{\mathcal{I}} + {}^{EdS}\left\langle H^i{}_jH^j{}_i\right\rangle^{\uparrow}_{\mathcal{I}}\right) \\
&= \frac{a^{-4}}{4}t_{\mathbf{i}}{}^{-2}\left(\left\langle \mathrm{I}^i{}_j\mathrm{I}^j{}_i\right\rangle_{\mathcal{I}} - \frac{1}{3}\left\langle \mathrm{I}^2_{\mathbf{i}}\right\rangle_{\mathcal{I}}\right) \\
&+ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}a^{-3}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right) \ . \qquad (6.100)
\end{aligned}
$$

3.3 Kinematical backreaction

Employing again our formal average (6.4), the kinematical backreaction (6.31) is written in terms of the expansion matrix and its trace read:

$$
\begin{aligned}
\mathcal{Q}_{\mathcal{D}} &= 2\left\langle \mathrm{II}\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle \mathrm{I}\right\rangle^2_{\mathcal{D}} \\
&= -2\left\langle\sigma^2\right\rangle_{\mathcal{D}} + \frac{2}{3}\left\langle\theta^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle\theta\right\rangle^2_{\mathcal{D}} \\
&= -\left\langle\Theta^i{}_j\Theta^j{}_i\right\rangle_{\mathcal{D}} + \left\langle\theta^2\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle\theta\right\rangle^2_{\mathcal{D}} \\
&= -\frac{\left\langle\Theta^i{}_j\Theta^j{}_iJ\right\rangle_{\mathcal{I}}}{\left\langle J\right\rangle_{\mathcal{I}}} + \frac{\left\langle\theta^2J\right\rangle_{\mathcal{I}}}{\left\langle J\right\rangle_{\mathcal{I}}} - \frac{2}{3}\frac{\left\langle\theta J\right\rangle^2_{\mathcal{I}}}{\left\langle J\right\rangle^2_{\mathcal{I}}} \ . \qquad (6.101)
\end{aligned}
$$

Expressed through the coframe coefficients, and using the identities $\theta J = \dot{J}$ and $\left(e_a{}^i\eta^a{}_j\right)^{\cdot} = e_a{}^i\dot{\eta}^a{}_j + \dot{e}_a{}^i\eta^a{}_j = 0$, we get:

$$
\begin{aligned}
\Theta^i{}_j\Theta^j{}_iJ &= e_a{}^i\dot{\eta}^a{}_je_b{}^j\dot{\eta}^b{}_iJ = -\left(\frac{1}{2J}\epsilon_{abc}\epsilon^{ikl}\eta^b{}_k\eta^c{}_l\right)^{\cdot}\dot{\eta}^a{}_iJ \\
&= -\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\dot{\eta}^b{}_k\eta^c{}_l + \frac{\dot{J}}{2J}\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\eta^b{}_k\eta^c{}_l \\
&= -\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\dot{\eta}^b{}_k\eta^c{}_l + \theta^2J \ . \qquad (6.102)
\end{aligned}
$$

Thus the kinematical backreaction reads:

$$
\mathcal{Q}_{\mathcal{D}} = \frac{1}{\left\langle J\right\rangle^2_{\mathcal{I}}}\left(\left\langle\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\dot{\eta}^b{}_k\eta^c{}_l\right\rangle_{\mathcal{I}}\left\langle J\right\rangle_{\mathcal{I}} - \frac{2}{3}\left\langle\dot{J}\right\rangle^2_{\mathcal{I}}\right) . \qquad (6.103)
$$

The different terms of the kinematical backreaction can be specified with respect to the volume deformation $\mathcal{J}$ as,

$$
\begin{aligned}
\epsilon_{abc}\epsilon^{ikl}\dot{\eta}^a{}_i\dot{\eta}^b{}_k\eta^c{}_l &= a^3\left(6H^2\mathcal{J} + 4H\dot{\mathcal{J}} + 2\dot{\xi}^2\left(\mathrm{II}_{\mathbf{i}} + 3\xi\mathrm{III}_{\mathbf{i}}\right)\right) \ ; \\
\dot{J} = \theta J &= a^3\left(3H\mathcal{J} + \dot{\mathcal{J}}\right) \ . \qquad (6.104)
\end{aligned}
$$

It leads to the final expression, derived and quantified in [Buchert et al., 2013]:

$$\begin{aligned}\mathcal{Q}_{\mathcal{D}} &= \frac{1}{\langle\mathcal{J}\rangle_{\mathcal{I}}^2}\left[2\dot{\xi}^2\left(\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} + 3\xi\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}\right)\langle\mathcal{J}\rangle_{\mathcal{I}} - \frac{2}{3}\left\langle\dot{\mathcal{J}}\right\rangle_{\mathcal{I}}^2\right] \\ &= \frac{\dot{\xi}^2\left(\gamma_1 + \xi\gamma_2 + \xi^2\gamma_3\right)}{\left(1 + \xi\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} + \xi^3\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}}\right)^2}\ ; \end{aligned} \tag{6.105}$$

with the terms:

$$\gamma_1 = 2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2 = \mathcal{Q}_{\mathcal{D}_{\mathbf{i}}}\ ; \tag{6.106}$$

$$\gamma_2 = 6\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}\ ; \tag{6.107}$$

$$\gamma_3 = 2\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}\langle \mathrm{III}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}}^2\ . \tag{6.108}$$

The leading order of the kinematical backreaction is quadratic, as the entropy measure and the Weyl scalar invariant:

$$\mathcal{Q}_{\mathcal{D}}^{\uparrow} \cong \dot{\xi}^2\mathcal{Q}_{\mathcal{D}_{\mathbf{i}}} = \dot{\xi}^2\left(2\langle \mathrm{II}_{\mathbf{i}}\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2\right). \tag{6.109}$$

Moreover, the second scalar invariant can be expressed with the first scalar invariant and its generalisation,

$$2\mathrm{II}_{\mathbf{i}} = \dot{\mathcal{P}}^2 - \mathcal{P}^i{}_j\mathcal{P}^j{}_i = \mathrm{I}_{\mathbf{i}}^2 - \mathrm{I}^i{}_j\mathrm{I}^j{}_i\ . \tag{6.110}$$

According to this relation and assuming an EdS solution, the leading order of the the kinematical backreaction is,

$$\begin{aligned}{}^{\mathrm{EdS}}\mathcal{Q}_{\mathcal{D}}^{\uparrow} &= \frac{9}{4}t_{\mathbf{i}}{}^2\dot{a}^2\left(2\langle \mathrm{I}_{\mathbf{i}}^2\rangle_{\mathcal{I}} - \langle \mathrm{I}^i{}_j\mathrm{I}^j{}_i\rangle_{\mathcal{I}} - \frac{2}{3}\langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2\right) \\ &= \frac{2}{3}\left(\langle \mathrm{I}_{\mathbf{i}}^2\rangle_{\mathcal{I}} - \langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2\right)a^{-1} - \left(\langle \mathrm{I}^i{}_j\mathrm{I}^j{}_i\rangle_{\mathcal{I}} - \frac{1}{3}\langle \mathrm{I}_{\mathbf{i}}^2\rangle_{\mathcal{I}}\right)a^{-1}\ . \end{aligned} \tag{6.111}$$

4 Results

4.1 The leading order relation

The leading order of the entropy measure (the variance of the density fluctuations), the Weyl scalar invariant and the kinematical backreaction have been computed in the previous section. In this part I highlight the relation between these three quantities. From equations (6.62), (6.100) and (6.111) we can write:

$$\begin{aligned}{}^{EdS}\mathcal{F}_{\mathcal{D}}^{1\uparrow} &= \frac{3a^{-1}}{8\pi G}\left[\langle \mathrm{I}_{\mathbf{i}}^2\rangle_{\mathcal{I}} - \langle \mathrm{I}_{\mathbf{i}}\rangle_{\mathcal{I}}^2\right] \\ &= \frac{9a^{-1}}{16\pi G}\left[{}^{EdS}\mathcal{Q}_{\mathcal{D}}^{\uparrow}a + \left(\langle \mathrm{I}^i{}_j\mathrm{I}^j{}_i\rangle_{\mathcal{I}} - \frac{1}{3}\langle \mathrm{I}_{\mathbf{i}}^2\rangle_{\mathcal{I}}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{9a^{-1}}{16\pi G}\left[{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}}a + \left(4a^4 t_{\mathbf{i}}{}^2\ {}^{EdS}\left\langle C^2\right\rangle^{\uparrow}_{\mathcal{I}} - a^4 t_{\mathbf{i}}{}^2\ {}^{EdS}\left\langle H^i{}_j H^j{}_i\right\rangle^{\uparrow}_{\mathcal{I}}\right)\right] \\
&= \frac{9a^{-1}}{16\pi G}\left[{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}}a + 4a^4 t_{\mathbf{i}}{}^2\ {}^{EdS}\left\langle C^2\right\rangle^{\uparrow}_{\mathcal{I}}\right. \\
&\quad \left. -a t_{\mathbf{i}}{}^2\ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right)\right] \\
&= \frac{9}{16\pi G}\left[{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}} + 4t^2\ {}^{EdS}\left\langle C^2\right\rangle^{\uparrow}_{\mathcal{I}}\right. \\
&\quad \left. -t_{\mathbf{i}}{}^2\ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right)\right] . \qquad (6.112)
\end{aligned}
$$

This result is equivalent to the simpler approach of [Li et al., 2012] up to a constant magnetic term. By replacing the scalar invariant by its expression with the Weyl tensor components (6.17), the last result can be written as,

$$
\begin{aligned}
{}^{EdS}\mathcal{F}^{1\uparrow}_{\mathcal{D}} &= \frac{9}{16\pi G}\left[{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}} + \frac{t^2}{8}{}^{EdS}\left\langle C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda}\right\rangle^{\uparrow}_{\mathcal{I}}\right. \\
&\quad \left. -t_{\mathbf{i}}{}^2\ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right)\right] \\
&= 2\frac{S_{\mathcal{D}}}{V_{\mathcal{D}}} - \frac{9t_{\mathbf{i}}{}^2}{32\pi G}\epsilon^{lsn}\epsilon^{pqr}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right) , \qquad (6.113)
\end{aligned}
$$

where $S_{\mathcal{D}}/V_{\mathcal{D}}$ is the definition of the average entropy measure proposed by [Li et al., 2012]. At the initial time, the variance of the density fluctuations reads:

$$
\begin{aligned}
{}^{EdS}\mathcal{F}^{1\uparrow}_{\mathcal{D}\ \mathbf{i}} &= \frac{9}{16\pi G}\left[{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}\ \mathbf{i}} + \frac{t_{\mathbf{i}}{}^2}{8}{}^{EdS}\left\langle C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda}\right\rangle^{\uparrow}_{\mathcal{I}}\right. \\
&\quad \left. -t_{\mathbf{i}}{}^2\ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right)\right] . \qquad (6.114)
\end{aligned}
$$

Thus, the relation can be compacted in the following expression:

$$
{}^{EdS}\mathcal{F}^{1\uparrow}_{\mathcal{D}} = {}^{EdS}\mathcal{F}^{1\uparrow}_{\mathcal{D}\ \mathbf{i}} + {}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}\ \mathbf{i}}\left(t^{-2/3} - t_{\mathbf{i}}{}^{-2/3}\right)t^{2/3} + \frac{t^2 - t_{\mathbf{i}}{}^2}{8}{}^{EdS}\left\langle C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda}\right\rangle^{\uparrow}_{\mathcal{I}} . \qquad (6.115)
$$

4.2 Orthonormal basis leading order

I briefly present in this section the results obtained in the orthonormal basis. Assuming this alternative basis choice,

- the first order equation of evolution and its solution read,

$$
\ddot{P}^i{}_j + 2H\dot{P}^i{}_j - 4\pi G\varrho_H P^i{}_j = -4\pi G\varrho_{H_{\mathbf{i}}}P^i{}_j(t_{\mathbf{i}})a^{-3} + W^i{}_j a^{-3} ; \qquad (6.116)
$$

$$P^i{}_j = \frac{3}{2}U^i{}_j t_{\mathbf{i}}\,(a-1) + 6\pi G\varrho_{H_{\mathbf{i}}}P^i{}_j(t_{\mathbf{i}})t_{\mathbf{i}}{}^2 = \xi(t)\dot{\mathcal{P}}^i{}_j + \zeta\ ; \tag{6.117}$$

- the electric part of the Weyl tensor restricted to its leading order reads,

$$\begin{aligned} E^i{}_j &= -\dot{\Theta}^i{}_j - \Theta^i{}_k\Theta^k{}_j + \frac{1}{3}\left(\dot{\theta} + \Theta^l{}_k\Theta^k{}_l\right)\delta^i{}_j \\ &= -\left(\mathrm{I}^i{}_j - \frac{1}{3}\mathrm{I}_{\mathbf{i}}\delta^i{}_j\right)\left(\ddot{\xi} + 2H\dot{\xi}\right)\ ; \\ {}^{EdS}E^i{}_j &= -\left(\mathrm{I}^i{}_j - \frac{1}{3}\mathrm{I}\delta^i{}_j\right)t_{\mathbf{i}}{}^{-1}a^{-2}\ ; \end{aligned} \tag{6.118}$$

- the leading order of the magnetic part of the Weyl tensor reads,

$$\begin{aligned} H^i{}_j &= -a^{-3}\frac{\epsilon^{lsn}}{\mathcal{J}}g^{ik}g_{n(k}\ g_{lm}\frac{\dot{\mathcal{J}}^m{}_{j)||s}}{\mathcal{J}} \\ &= -a^{-1}\epsilon^{lsn}\delta^{ik}\delta_{n(k}\ \delta_{lm}\mathrm{I}^m{}_{j)||s}\ \dot{\xi}\ ; \\ {}^{EdS}H^i{}_j &= -a^{-1/2}\epsilon^{lsn}\delta^{ik}\delta_{n(k}\ \delta_{lm}\mathrm{I}^m{}_{j)||s}\ . \end{aligned} \tag{6.119}$$

Then, the desired relation in the orthonormal basis is identical to the orthogonal relation (6.112),

$$\begin{aligned} \mathcal{F}^{1\uparrow}_{\mathcal{D}} = \frac{9}{16\pi G}\Big[&{}^{EdS}\mathcal{Q}^{\uparrow}_{\mathcal{D}} + 4t^2\ {}^{EdS}\left\langle C^2\right\rangle^{\uparrow}_{\mathcal{I}} \\ &-t_{\mathbf{i}}{}^2\ \frac{\epsilon^{lsn}\epsilon^{pqr}}{2}\delta_{pu}\delta_{lm}\left(\left\langle \mathrm{I}^m{}_{r||s}\mathrm{I}^u{}_{n||q}\right\rangle_{\mathcal{I}} + \delta_{rn}\delta^{tj}\left\langle \mathrm{I}^m{}_{j||s}\mathrm{I}^u{}_{t||q}\right\rangle_{\mathcal{I}}\right)\Big]\ . \end{aligned} \tag{6.120}$$

5 Concluding remarks

I presented here the possible connections between an inhomogeneous theory of cosmology and the problem of the definition of a gravitational entropy. I chose to define the entropy as the variance of the density fluctuations:

$$\mathcal{F}^1_{\mathcal{D}} = \frac{\left\langle \varrho^2\right\rangle_{\mathcal{D}} - \left\langle \varrho\right\rangle^2_{\mathcal{D}}}{\left\langle \varrho\right\rangle_{\mathcal{D}}} = \frac{\left\langle \delta\varrho^2\right\rangle_{\mathcal{D}} - \left\langle \delta\varrho\right\rangle^2_{\mathcal{D}}}{\left\langle \varrho\right\rangle_{\mathcal{D}}}\ ;$$

which is a sub–case of the Tsallis one–parameter family of relative entropy measures [Tsallis, 1988]. Since this entropy should be a tracer of the quantity and properties of the formed structures, it is natural to try to link the variance of the density fluctuations to a scalar curvature parameter, here the average of the Weyl scalar invariant:

$$\left\langle C^2\right\rangle_{\mathcal{D}} = \frac{1}{32}\left\langle C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda}\right\rangle_{\mathcal{D}} = \frac{1}{4}\left(\left\langle E^i{}_jE^j{}_i\right\rangle_{\mathcal{D}} + \left\langle H^i{}_jH^j{}_i\right\rangle_{\mathcal{D}}\right)\ .$$

The first step of this study was to look for a relation for the leading order, it appeared that this order is composed of quadratic terms of the first order perturbations. Moreover, the leading order of the kinematical backreaction appeared naturally in the expression of the entropy measure. Thus a relation between the gravitational entropy, the Weyl scalar invariant and the kinematical backreaction was obtained.

The Weyl scalar invariant is composed of two parts, respectively triggered by the electric and the magnetic part of the Weyl tensor. The most recent studies (see [Li et al., 2012]) ignored the magnetic part. Here I took into account this term. It led to the same relation up to this magnetic term. Moreover, this term appeared as a constant in the relation between the entropy, the Weyl scalar invariant and the backreaction whereas the electric part grows as t^2. Thus for a sufficiently large time the magnetic term is outdone by the electric term.

The relation obtained is dependent on the chosen average model. Here the solution are obtained for a model whose average is a standard FLRW background. Moreover, the explicit results are shown for an Einstein–de Sitter space–time. The Weyl scalar invariant was the gravitational entropy proposed by Penrose, here I chose another definition but finally there is a relatively simple relation between them. The next step will be to consider higher order in the computations of the different terms and more general background with a spatially dependent scale factor for instance.

Chapter VII

Conclusions and outlooks

I dedicated my Ph.D. thesis to the building of a less restrictive model than the usual description of the Universe in order to confirm, or not, the use of its restrictions such as the cosmological principle or the standard perturbation theory.

In chapter II, I reviewed the Newtonian perturbation theory and justified several choices. I presented the Lagrangian description of a fluid in the framework of cosmology and its advantages: high density contrast, low number of unknowns. Moreover, I chose to write down the equations with differential forms. The forms are a tricky but very efficient tool to formulate the gravitational problem and I showed the coherence of the results obtained thanks to differential forms with respect to the usual tensorial formalism. This chapter did not present new things but was here to remind the reader about the standard Newtonian development; and was used in order to propose a relativistic generalisation of the Lagrangian perturbation theory in the next chapter.

The new theoretical elements provided by my Ph.D. thesis are in chapters III and IV which dealt with the relativistic Lagrangian perturbation theory of the cosmology. Using the Cartan coframes and a reasoning similar to the Newtonian Lagrangian approach of fluid dynamics, the relativistic Lagrangian perturbation theory was built. By a simple generalisation of the Newtonian equations, a part of the relativistic solution can be obtained. Additional equations remain, their solutions allow us to complete and constrain the relativistic solution. The Newtonian system furnishes equations for the trace and the antisymmetric part of the solution, whereas the relativistic system furnishes equations for the symmetric traceless part of the deformations. This last equation furnishes a wave equation that describes the propagation of gravitational waves. Under some restrictions, the Relativistic Zel'dovich Approximation (RZA) and an Einstein–de Sitter (EdS) Universe, it is possible to obtain explicit solutions, as in chapter IV.

The two next chapters presented some applications I performed during my Ph.D. thesis. I presented in chapter V the system of dynamical equations for the Newtonian and relativistic problem in the case of a silent Universe. This restriction means the magnetic part of the Weyl tensor vanishes. This tensor describes the curvature

of the space–time and is the analog of the electromagnetic Faraday tensor. Then, still by analogy, an electric and a magnetic part of the tensor are defined. The term silent is used because without a magnetic part the waves cannot propagate. One of the essential results of this chapter is that the dynamical silent system of equations in the relativistic case is simply composed of a closed system of ordinary differential equations. This result is very strong, it means a numerical resolution of the local system of equations is easy to perform. Nevertheless, the application of the average process creates a lot of new variables and the average (or global) system of equations is not closed anymore. Thus a closure relation is intensively searched. Nevertheless, it is possible to numerically describe the global behaviour with a code that computes the local solutions and performs the average at each time step.

Finally, the concept of gravitational entropy and a relation of this quantity to the space–time characteristics were proposed in chapter VI. The new content of this development is the addition of the magnetic part of the Weyl tensor. After some calculations, I showed that the leading order of the chosen entropy measure (the variance of the fluctuations of density), the Weyl scalar invariant and the kinematical backreaction term are linked by a relatively simple relation close to the preceding results without magnetic part. The impact of the space–time curvature and the kinematical backreaction on the dynamics can be clearly established thanks to this relation.

The natural next step will be an implementation of this theory of perturbation in numerical simulations. The flow–orthogonal foliation, which defines a unique proper time through space, drastically simplifies numerical implementations with respect to fully relativistic simulations. Moreover, the Cartan coframes formulation with a unique variable is relatively close to the Newtonian Lagrangian approach which showed its efficiency in analytical simulations. Weiss *et al.* [Weiss et al., 1996] presented a comparison between Eulerian numerical simulation and Lagrangian analytical simulation in the Newtonian context. An implementation of the relativistic perturbation theory developed during my Ph.D. thesis will provide essential input for the numerical realisation of relativistic large–scale structure formation.

Multiple theoretical applications are also possible. Unlike most of the previous works, this formalism contains the symmetric traceless dynamics. Then, it is possible to describe gravitational waves, radiations and their impact on the matter collapse. At the second order of perturbation these traceless first order terms are involved in the sources and can affect the trace dynamics (the collapse for instance). Thus, most of the usual descriptions of the large–scale structures formation neglect the impact of the gravitational waves and other traceless effects. A specific feature of the Lagrangian description is that even at first order we have a coupling between the symmetric traceless equation and the trace part of the solution, which shows that gravitational wave perturbations cannot be simply dis-

entangled from the matter sources at which they scatter. The developments about the relation between entropy and space–time curvature has to continue in order to find a general relation and not only a leading order relation. Consequences of the thermodynamics of the gravitation attraction and the link between entropy and gravitational waves are still to be studied. Another analytical application is the estimation of the effect of the backreactions: a well–defined spatial average process could bring important effective curvature from the small initial conditions of the Cosmic Microwave Background (see [Buchert, 2008; Räsänen, 2006a]).

Other applications would account for a re–estimation of the quantities of Dark Matter and Dark Energy by including inhomogeneity effects, together with the consequences for observational and experimental projects. It will trigger numerous implications and strong interactions with observers to confront and adjust the parameters of this modelisation. A new description of the light–cone and its effect on the gravitational lensing would also be really interesting to study.

Appendices

A Differential forms, symmetric and antisymmetric parts

Many matrix decompositions exist and we decided to split them between their trace, symmetric and antisymmetric traceless parts. This decomposition is pretty simple when we deal with matrix, but the forms formalism is used in my Ph.D. thesis. Thus it is important to properly define these parts also with differential forms. Let us $\mathbf{V}^a$ be a 1–form in a 3–dimensional space. Its trace part is defined by:

$$\begin{aligned}\frac{1}{2}\epsilon_{abc}\mathbf{V}^a \wedge \delta^b{}_j \mathbf{d}X^j \wedge \delta^c{}_k \mathbf{d}X^k &= \frac{1}{\sqrt{|g|}} * \left(\frac{1}{2}\epsilon_{abc}\epsilon^{ijk}V^a_i \delta^b{}_j \delta^c{}_k\right) \\ &= \frac{1}{\sqrt{|g|}} * \delta_a{}^i V^a_i = \frac{1}{\sqrt{|g|}} V^i_i \; ; \end{aligned} \tag{A1}$$

whereas its antisymmetric part is defined by:

$$\begin{aligned} G_{ab}\mathbf{V}^a \wedge \delta^b{}_j \mathbf{d}X^j &= \frac{3}{\sqrt{|g|}} * \left(G_{ab}\epsilon^{ij}{}_k V^a_i \delta^b{}_j \mathbf{d}X^k\right) \\ \Longrightarrow \quad * \left(G_{ab}V^a_{[i}\delta^b{}_{j]}\mathbf{d}X^k\right) &= * \left(V_{[ji]}\mathbf{d}X^k\right) \text{ for a given k.} \end{aligned} \tag{A2}$$

The symmetric traceless part does not have a proper definition with differential forms. Nevertheless any matrix can be decomposed between its trace, symmetric and antisymmetric traceless part. Then knowing the trace and the antisymmetric parts the remaining symmetric traceless part can be extracted. This is how we manage to write the equation of evolution for the symmetric traceless perturbations $\Pi^i{}_j$ in chapter IV.

There exist equivalent to the usual operators in the differential forms formalism, but it is important to be careful when they are used because of the non–

Euclideanity of the space–time. The divergence can be identified to:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{V} = *\mathbf{d}*\,\mathbf{V} = *\mathbf{d}*\left(V_i\mathbf{d}X^i\right) &= \frac{\sqrt{|g|}}{2}\;*\,\mathbf{d}\left(V_i\epsilon_{ijk}\mathbf{d}X^j\wedge\mathbf{d}X^k\right)\\ &= \frac{\sqrt{|g|}}{2}\;*\left(V_{i|i}\epsilon_{ijk}\mathbf{d}X^i\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\right)\\ &= \frac{|g|}{2}V_{i|i} := \frac{|g|}{2}\vec{\nabla}\cdot\vec{V}\;; \end{aligned}\tag{A3}$$

whereas the curl can be identified to:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{V} = *\mathbf{d}\mathbf{V}\wedge\mathbf{d}X^k &= *\left(\mathbf{d}V_i\mathbf{d}X^i\wedge\mathbf{d}X^k\right)\\ &= *\left(V_{i|j}\mathbf{d}X^j\wedge\mathbf{d}X^i\wedge\mathbf{d}X^k\right)\\ &= \sqrt{|g|}\epsilon^{kji}V_{i|j} := \sqrt{|g|}\vec{\nabla}\times\vec{V}\;. \end{aligned}\tag{A4}$$

It is also possible to define an equivalent to the Laplacian operator. An intermediate operator is required, let $\mathbf{V}$ be a k–form on a n–dimensional space. Thus the codifferential is defined such as,

$$\mathbf{d}^* = \boldsymbol{\delta} = (-1)^{kn+n+1} * \mathbf{d} * \;. \tag{A5}$$

The codifferential allows us to define a differential forms Laplacian operator (the De Rham–Laplace operator). In the case of a 1–form it leads to,

$$\begin{aligned}(\boldsymbol{\delta}\mathbf{d}+\mathbf{d}\boldsymbol{\delta})\,\mathbf{V} &= (*\mathbf{d}*\mathbf{d}-\mathbf{d}*\mathbf{d}*)\,\mathbf{V}^a\\ &= *\mathbf{d}*\left(V^a_{i|j}\mathbf{d}X^j\wedge\mathbf{d}X^i\right)+\frac{\sqrt{|g|}}{2}\mathbf{d}*\left(\mathbf{d}V^a_i\epsilon^i{}_{jk}\mathbf{d}X^j\wedge\mathbf{d}X^k\right)\\ &= \sqrt{|g|}*\left(\mathbf{d}V^a_{i|j}\epsilon^{ji}{}_k\mathbf{d}X^k\right)+\frac{\sqrt{|g|}}{2}\mathbf{d}*\left(V^a_{i|l}\epsilon^i{}_{jk}\mathbf{d}X^l\wedge\mathbf{d}X^j\wedge\mathbf{d}X^k\right)\\ &= \sqrt{|g|}*\left(V^a_{i|jl}\epsilon^{ji}{}_k\mathbf{d}X^l\wedge\mathbf{d}X^k\right)+\frac{|g|}{2}\mathbf{d}V^a_{i|l}\epsilon^i{}_{jk}\epsilon^{ljk}\\ &= |g|\,V^a_{i|jl}\epsilon^{ji}{}_k\epsilon^{lk}{}_p\mathbf{d}X^p+\frac{|g|}{2}V^a_{i|lp}\epsilon^i{}_{jk}\epsilon^{ljk}\mathbf{d}X^p\\ &= |g|\,V^a_{i|jl}\left(g^{il}\delta^j{}_p-g^{jl}\delta^i{}_p\right)\mathbf{d}X^p+|g|\,V^a_{i|lp}g^{il}\mathbf{d}X^p\\ &= -\,|g|\,V^a{}_{i|j}{}^{|j}\mathbf{d}X^i := |g|\,\Delta V^a_i\mathbf{d}X^i\;. \end{aligned}\tag{A6}$$

B Vorticity expression

Eulerian approach

The vorticity can be expressed thanks to the curl of the gravitational field and the Euler equation:

$$\begin{cases} \vec{\nabla} \times \vec{g} = \vec{0} \; ; \\ \dfrac{d\vec{v}}{dt} = \dfrac{\partial \vec{v}}{\partial t} + \left(\vec{v}\ \vec{\nabla}\right) \vec{v} = \vec{g} \; . \end{cases} \tag{B1}$$

Some identities of vectorial analysis are required to make the vorticity appear:

$$\begin{cases} 2(\vec{a}\vec{\nabla})\vec{b} = \vec{\nabla} \times (\vec{b} \times \vec{a}) + \vec{\nabla}(\vec{a}\vec{b}) + \vec{a}(\vec{\nabla}\vec{b}) \\ \qquad -\vec{b}(\vec{\nabla}\vec{a}) - \vec{a} \times (\vec{\nabla} \times \vec{b}) - \vec{b} \times (\vec{\nabla} \times \vec{a}) \; ; \\ \vec{\nabla} \times (\vec{a} \times \vec{b}) = (\vec{b}\vec{\nabla})\vec{a} - (\vec{a}\vec{\nabla})\vec{b} + \vec{a}(\vec{\nabla}\vec{b}) - \vec{b}(\vec{\nabla}\vec{a}) \; . \end{cases} \tag{B2}$$

The introduction of the Euler equation into the curl of the gravity field and the first identity give:

$$\begin{aligned} & \frac{\partial \vec{\nabla} \times \vec{v}}{\partial t} + \vec{\nabla} \times \left(\left(\vec{v}\ \vec{\nabla}\right) \vec{v}\right) = 0 \\ \Longleftrightarrow \quad & \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{v}) = 0 \; . \end{aligned} \tag{B3}$$

The second identity leads to:

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega}\vec{\nabla})\vec{v} - \vec{\omega}(\vec{\nabla}\vec{v}) \; . \tag{B4}$$

This transport equation for the vorticity shows that vorticity changes along streamlines of the flow.

Lagrangian approach

There exist two different ways to write the curl of the gravitation field, from the tensorial expression:

$$\begin{aligned} (\vec{\nabla} \times \vec{g})^i &= \epsilon^{ikj} g_{k,j} = \epsilon^{ijk} \ddot{f}_{k|l}\ h^l{}_{,j} \\ &= \frac{1}{2J} \epsilon^{ijk} \epsilon^{lmn} \epsilon_{jpq} \ddot{f}_{k|l} f^p{}_{|m} f^q{}_{|n} \\ &= \frac{1}{J} \epsilon^{lmn} \delta^i_{[q} \ddot{f}_{p]|l} f^p{}_{|m} f^q{}_{|n} \\ &= \frac{\delta_{pk}}{J} \mathcal{J}(\ddot{f}^k; f^p; f^i) \; ; \end{aligned} \tag{B5}$$

or from the form field:

$$\mathbf{db} = \mathbf{d}\ddot{\mathbf{f}} = \mathbf{d}(\ddot{f}_i \mathbf{d}f^i) = \delta_{ij}\mathbf{d}\ddot{f}^i \wedge \mathbf{d}f^j$$

$$\Longrightarrow \quad \delta_{pk}\ddot{f}^p{}_{|[m}f^k{}_{|n]}\mathbf{d}X^m \wedge \mathbf{d}X^n \ . \tag{B6}$$

From this equation, we can get:

$$\delta_{pk}\left(\dot{f}^p{}_{|[m}f^k{}_{|n]}\right)^{\cdot} - \delta_{pk}\dot{f}^p{}_{|[m}\dot{f}^k{}_{|n]} = 0 \ . \tag{B7}$$

This transport equation for the vorticity shows that vorticity changes along streamlines of the flow.

Differential forms

The components of the 1–form vorticity derive from the components of the tensorial vorticity, $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$:

$$\tilde{\omega}_{ij} = -v_{[i,j]} \ . \tag{B8}$$

Then we have:

$$\boldsymbol{\omega} = -\tilde{\omega}_{ij}\mathbf{d}X^i \wedge \mathbf{d}X^j = v_{[i,j]}\mathbf{d}X^i \wedge \mathbf{d}X^j \tag{B9}$$

$$= v_{i,j}\mathbf{d}X^i \wedge \mathbf{d}X^j = \mathbf{d}\dot{\mathbf{f}} \ ; \tag{B10}$$

and the equivalent of the curl of the gravity field is:

$$\mathbf{d}(\ddot{f}^k \mathbf{d}f^k) = \mathbf{d}\ddot{\mathbf{f}} = \mathbf{0} \ ; \tag{B11}$$

so we get:

$$\mathbf{d}\dot{\mathbf{f}} = \mathbf{const}. \tag{B12}$$

Then the 2–form vorticity is conserved.

C Curvature tensors

Riemann tensor

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \Gamma^\rho{}_{\nu\sigma|\mu} - \Gamma^\rho{}_{\mu\sigma|\nu} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \ ; \tag{C1}$$

$$\begin{cases} \mathcal{R}_{\rho\mu\sigma\nu} = -\mathcal{R}_{\mu\rho\sigma\nu} = -\mathcal{R}_{\rho\mu\nu\sigma} \ ; \\ \mathcal{R}_{\rho\mu\sigma\nu} = \mathcal{R}_{\sigma\nu\rho\mu} \ ; \\ \mathcal{R}_{\rho\mu\sigma\nu} + \mathcal{R}_{\rho\sigma\nu\mu} + \mathcal{R}_{\rho\nu\mu\sigma} = 0 \ ; \\ \mathcal{R}_{\rho\mu\sigma\nu||\lambda} + \mathcal{R}_{\rho\mu\nu\lambda||\sigma} + \mathcal{R}_{\rho\mu\lambda\sigma||\nu} = 0 \ . \end{cases} \tag{C2}$$

Ricci tensor

$$\mathcal{R}^{\rho}{}_{\mu\rho\nu} = \mathcal{R}_{\mu\nu} \; ; \tag{C3}$$

$$g^{\rho\sigma}\mathcal{R}_{\rho\mu\sigma\nu} = g^{\rho\sigma}\mathcal{R}_{\sigma\nu\rho\mu} \quad \Longrightarrow \quad \mathcal{R}_{\mu\nu} = \mathcal{R}_{\nu\mu} \; . \tag{C4}$$

Ricci scalar

$$\mathcal{R}^{\mu}{}_{\mu} = \mathcal{R} \; . \tag{C5}$$

D Space–time foliation

General relativity does not consider the time as a peculiar dimension. It leads to equations which depend explicitly of the 4–dimensional space–time. Nevertheless, the $3+1$ formalism allows us to restrict a 4–dimension relativistic problem to a 3–dimensional spatial relativistic problem in the space $\mathcal{M} \times \mathbb{R}$ where $\mathcal{M}$ is a Riemannian manifold. The equations and the associated properties become simpler thanks to the split between the space manifold and the time dimension. It leads to a decomposition of the space–time between its space and time parts.

Space–time is cut out into hypersurfaces orthogonal to the time flow and have the same proper time. This foliation allows us to write in a more intuitive manner the Einstein equations because the time becomes again a parameter. The first description of this peculiar framework was proposed by Darmois in 1927 [Darmois, 1927]. More general cases followed in the papers of Lichnerowicz in the 30's–50's [Lichnerowicz, 1939, 1944, 1952] and the general case has been developed by Choquet-Bruhat in the 40's and 50's [Choquet-Bruhat, 1948, 1952, 1956]. This formalism known important success thanks to the development of the Hamiltonian formulation of general relativity by Arnowitt, Deser and Misner in 1962 (see republication [Arnowitt et al., 2008]). The $3+1$ space–time foliation also has an essential role in the relativistic numerical simulations. The reader may read the review of Gourgoulhon [Gourgoulhon, 2007] for additional details.

Thus assuming the $3+1$ space–time foliation, we have only to deal with spatial tensors: the 3–dimension metric, the 3–Ricci tensor *etc.* Nevertheless an important restriction of this formalism is the constraint of a vanishing vorticity. There exist alternatives to the $3+1$ formalism which allows us to describe vorticity, for instance a similar approach which is built on the threading of space–time exist, it is the $1+3$ formalism (see the recent paper [Roy, 2014]).

E Orthonormal and orthogonal coframes representation

As detailed in [Buchert and Ostermann, 2012; Buchert et al., 2013], two different basis choices are conceivable to represent the perturbation schemes: (i) an

orthonormal basis, defined by:

$$\begin{cases} \text{The metric coefficients are written as: } g_{ij} = \delta_{ab}\, \eta^{a^N}_{\ i}\, \eta^{b^N}_{\ j} \\ \text{with non–normalised initial coframes: } \eta^{a^N}_{\ k}\,(t_i) \neq \delta^a_{\ k}\ . \end{cases}$$

The other choice is (ii) an *orthogonal basis*, described by:

$$\begin{cases} \text{The metric coefficients are written as: } g_{ij} = G_{ab}\eta^{a^G}_{\ i}\, \eta^{b^G}_{\ j} \\ \text{with normalised initial coframes: } \eta^{a^G}_{\ k}\,(t_i) = \delta^a_{\ k}\ . \end{cases}$$

These choices arise by the request of a non–Euclidean space (e.g. an orthonormal basis together with normalised initial deformations imply an Euclidean space). In the orthogonal basis, the initial functions G_{ab} encode the non–Euclideanity. Whatever the basis choice is, we can express our system of equations in terms of our single variable. We decided to work in the orthogonal basis, because it allows a formally closer analogy between the Newtonian equations and part of the relativistic system. Thus, we expect this approach to be more relevant when it comes to comparing Einstein equations subjected to the *Minkowski Restriction* in the orthogonal basis to the standard perturbation theory.

Since the metric coefficients g_{ij} are defined in the exact basis $\mathbf{d}X^i$, we can either express them using orthogonal or orthonormal coframes,

$$g_{ij} = \delta_{ab}{}^N\eta^a_{\ i}{}^N\eta^b_{\ j} = G_{ab}{}^G\eta^a_{\ i}{}^G\eta^b_{\ j}\ ; \tag{E6}$$

where ^{N} and ^{G} denote the two representations. Then, if we use the perturbation development given by (3.33) for the coframes, we can show:

$$\begin{cases} G^{(0)}_{ij} &= \delta_{ij}\ ; \\ G^{(1)}_{ij} &= 2\ {}^N\mathcal{P}_{ij}\ ; \\ G_{ij} &= g_{ij}\,(t_{\mathbf{i}})\ ; \end{cases} \tag{E7}$$

since the initial perturbation fields are null in the orthogonal representation.

The scale factor $a(t)$ explicitly appears as a prefactor in the decomposition (3.33). The perturbation fields are then defined on the homogeneous comoving space, which has G_{ij} as a metric. This metric will thus be used to raise or lower the indices for the perturbation fields. To act on tensors defined on the physical space such as the Riemann tensor, we shall use the full metric g_{ij}:

$$\begin{cases} P_{ij} = G_{ai}P^a_{\ j}\ ; \\ \mathcal{R}_{ij} = g_{ip}\mathcal{R}^p_{\ j}\ . \end{cases} \tag{E8}$$

F Basis deformation transformation

We here compare the first order orthogonal equations that we obtained above ((3.60)–(3.63)) to the ones obtained in [Buchert and Ostermann, 2012] at the same order but in the orthonormal representation [Buchert and Ostermann, 2012](89), (90), (92), (99) and (101). The orthogonal and the orthonormal representations are equivalent but the initial deviations are handled in a different way as we explained before in this Appendix. In fact, in the orthonormal case, the initial perturbation fields are non null: ${}^N\mathcal{P}^a_{\ i} \neq 0$ and the conformal background metric is flat and equal to δ_{ij}. We here have called "background metric" the part of the metric which does not depend on perturbation fields. In the orthogonal representation, it is the opposite: the initial perturbation fields are null ${}^G\mathcal{P}^a_{\ i} = 0$ but the conformal background metric is not flat at first order and is equal to $\delta_{ij} + G^{(1)}_{ij}$. Thus, there should be a link between the initial deformation fields $U^a_{\ i}$, $W^a_{\ i}$ and $G^{(1)}_{ij}$ that could enable us to go from the orthonormal formulation of Einstein equations to the orthogonal one. The evaluation at initial time of the two sets of first order equations will enable us to identify this transformation. For the orthonormal basis we obtain at initial time:

$$\begin{cases} {}^N\ddot{\mathcal{P}} + 2H_{\mathbf{i}}{}^N\dot{\mathcal{P}} - 4\pi G\varrho_{H\mathbf{i}}{}^N\mathcal{P} = 0\ ; \\ {}^N\ddot{\Pi}_{ij}(t_{\mathbf{i}}) + 3H{}^N\dot{\Pi}_{ij}(t_{\mathbf{i}}) = -{}^N\mathcal{T}_{ij}\ ; \\ H_{\mathbf{i}}{}^N\dot{\mathcal{P}} + 4\pi G\varrho_{H\mathrm{i}}{}^N\mathcal{P} = -\dfrac{{}^N\mathcal{R}}{4}\ . \end{cases} \tag{F9}$$

For the orthogonal basis we correspondingly obtain:

$$\begin{cases} {}^G\ddot{\mathcal{P}} + 2H_{\mathbf{i}}{}^G\dot{\mathcal{P}} - W = 0\ ; \\ {}^G\ddot{\Pi}_{ij}(t_{\mathbf{i}}) + 3H{}^G\dot{\Pi}_{ij}(t_{\mathbf{i}}) = -{}^G\mathcal{T}_{ij}\ ; \\ H_{\mathbf{i}}{}^G\dot{\mathcal{P}} + W = -\dfrac{{}^G\mathcal{R}}{4}\ . \end{cases} \tag{F10}$$

Thus, if we choose $4\pi G\varrho_{H\mathrm{i}}{}^N\mathcal{P}(X^i) = W(X^i)$, the two left hand sides of the systems are the same. It is possible to extrapolate from the last result to a general link between the two representations at first order by defining:

$${}^NP^a_{\ i} = {}^GP^a_{\ i} + \frac{W^a_{\ i}}{4\pi G\varrho_{H\mathrm{i}}}\ . \tag{F11}$$

With this choice, $G^{(1)}_{ij} = W_{(ij)}/2\pi G\varrho_{H\mathrm{i}}$, since:

$$g_{ij} = a^2\left(\delta_{ij} + 2{}^NP_{(ij)}\right) = a^2\left(\delta_{ij} + 2{}^GP_{(ij)} + 2\frac{W_{(ij)}}{4\pi G\varrho_{H\mathrm{i}}}\right). \tag{F12}$$

By construction, the representation transformation is built in such a way that

$${}^NR^{(1)}_{ij}({}^NP^a_{\ i}) = {}^GR^{(1)}_{ij}({}^GP^a_{\ i})\ . \tag{F13}$$

This latter can be easily checked. In particular, the whole set of first order orthogonal Einstein equations can be recovered from the orthonormal ones by injecting the transformation rule (F11). The orthogonal solutions will then be obtained from the orthonormal ones by a translation.

G Notations and abbreviations

Tensorial objects

δ_{ij}	Kronecker symbol ;
ϵ_{ijk}	Levi-Civita pseudo–tensor ;
$\epsilon_{ijk}\epsilon^{pqr}$	$= \delta^p{}_i\delta^q\delta^r{}_k + \delta^q{}_i\delta^r{}_j\delta^p{}_k + \delta^r{}_i\delta^p{}_j\delta^q{}_k - \delta^p{}_i\delta^r{}_j\delta^q{}_k - \delta^r{}_i\delta^q{}_j\delta^p{}_k - \delta^q{}_i\delta^p{}_j\delta^r{}_k$;
$\epsilon_{ijk}\epsilon^{iqr}$	$= \delta^q{}_j\delta^r{}_k - \delta^r{}_j\delta^q{}_k$;
$\epsilon_{ijk}\epsilon^{ijr}$	$= 2\delta^r{}_k$;
$\epsilon_{ijk}\epsilon^{ijk}$	$= 6$;

Differential objects

$\mathcal{M}$	Manifold ;
${}^4g_{\mu\nu}$	4–dimension metric ;
${}^3g_{ij}$	3–dimension spatial metric ;
$\wedge$	wedge product ;
$\mathbf{d}$	differential operator ;
$\mathbf{P}$, $\boldsymbol{\phi}$, $\boldsymbol{\eta}$...	bold symbol denotes differential forms ;
$*$	Hodge operator ;

Notations

$A_{\mathbf{i}}$	index $_{\mathbf{i}}$ denotes initial data ;
$A^{(n)}$	index $^{(n)}$ denotes the nth order perturbation ;
$\dot{A} = \dfrac{dA}{dt}$	time derivative ;
$A_{i,j}$	Eulerian derivative ;
$A_{i\|j}$	Lagrangian derivative ;
$A^i{}_{j\|\|k} = A^i{}_{j\|k} + A^m_j\Gamma^i{}_{mk} - A^i{}_m\Gamma^m_{jk}$	covariant derivative ;
$A_{[ij]} = \dfrac{1}{2}\left(A_{ij} - A_{ji}\right)$	antisymmetrisation ;
$A_{(ij)} = \dfrac{1}{2}\left(A_{ij} + A_{ji}\right)$	symmetrisation ;
$A = A^i{}_i$	trace of the tensor A ;

Constants and variables

G	gravitation constant ;
Λ	cosmological constant ;

a	scale factor ;
$H = \dfrac{\dot{a}}{a}$	Hubble factor ;
$\varrho(\vec{X}, t)$	density ;
$\varrho_H(t) = \varrho_{H\mathbf{i}} a^{-3}$	homogeneous density ;
$\delta\varrho(\vec{X}, t)$	density perturbations ;
$\delta(\vec{X}, t)$	density contrast ;
$\mathcal{Q}_{\mathcal{D}} = 2\langle \mathbf{II} \rangle_{\mathcal{D}} - \dfrac{2}{3}\langle \mathbf{I} \rangle_{\mathcal{D}}^2$	kinematical backreaction ;
$\mathbf{I}(P^a{}_i) = \dfrac{1}{2}\epsilon_{abc}\epsilon^{ijk} P^a{}_i \delta^b{}_j \delta^c{}_k$	first scalar invariant ;
$\mathbf{II}(P^a{}_i) = \dfrac{1}{2}\epsilon_{abc}\epsilon^{ijk} P^a{}_i P^b{}_j \delta^c{}_k$	second scalar invariant ;
$\mathbf{III}(P^a{}_i) = \dfrac{1}{6}\epsilon_{abc}\epsilon^{ijk} P^a{}_i P^b{}_j P^c{}_k$	third scalar invariant ;

Newtonian theory

X^i	Lagrangian position ;
$x^i = f^i(\vec{X}, t)$	Eulerian position or position field or map transformation ;
$v^i = \dot{f}^i(\vec{X}, t)$	Eulerian velocity ;
$g^i = \ddot{f}^i(\vec{X}, t)$	Eulerian acceleration or gravitation field ;
$X^i = h^i(\vec{x}, t) = \dfrac{1}{2J}\epsilon^{ijk}\epsilon_{pqr} f^j{}_{\vert q} f^k{}_{\vert r}$	inverse map transformation ;
$f^i{}_{\vert j}(\vec{X}, t)$	gradient of deformation ;
$\dot{f}^i{}_{\vert j}(\vec{X}, t)$	velocity gradient ;
$\ddot{f}^i{}_{\vert j}(\vec{X}, t)$	acceleration gradient ;
$\varepsilon^i{}_j(\vec{X}, t) = g^i{}_{,j} - \dfrac{1}{3} g^k{}_{,k}\delta^i{}_j$	tidal force tensor ;
$J = \dfrac{1}{6}\epsilon_{ijk}\epsilon^{lmn} f^i{}_{\vert l} f^j{}_{\vert m} f^k{}_{\vert n}$	Jacobian of the Euler–Lagrange coordinates transformation ;
$\mathcal{J}\left(A^i, B^j, C^k\right) = \epsilon^{lmn} A^i{}_{\vert l} B^j{}_{\vert m} C^k{}_{\vert n}$	functional determinant ;
$f^i(\vec{X}, t) = a(t)(X^i + P^i(\vec{X}, t))$	perturbative development of the position field ;
$f^i{}_{\vert j}(\vec{X}, t) = a(t)(\delta^i{}_j + P^i{}_{\vert j}(\vec{X}, t))$	perturbative development of the gradient of deformation ;
$P^i(\vec{X}, t)$	perturbations ;
$U^i(\vec{X})$	initial velocity ;

$U^i{}_{\vert j}(\vec{X})$	initial velocity gradient ;
$W^i(\vec{X})$	initial acceleration ;
$W^i{}_{\vert j}(\vec{X})$	initial acceleration gradient ;

Relativistic theory

X^i	tangential space coordinates or exact coordinates ;
$\eta^a{}_i(\vec{X},t)$	Cartan coframes field ;
$\Theta^i{}_j = e_a{}^i \eta^a{}_j$	fluid expansion tensor ;
$\Gamma^i{}_{kl} = \frac{1}{2} g^{ij}\left(g_{jk\vert l} + g_{jl\vert k} - g_{kl\vert j}\right)$	Christoffel connection ;
$C^{\mu\nu}{}_{\kappa\lambda} = {}^{(4)}R^{\mu\nu}{}_{\kappa\lambda} - 2\delta^{[\mu}_{[\kappa}\,{}^{(4)}R^{\nu]}{}_{\lambda]} + \frac{1}{3}\delta^{[\mu}_{[\kappa}\delta^{\nu]}_{\lambda]}\,{}^{(4)}R$	Weyl tensor ;
$E_{\mu\nu} = C_{\mu\nu\kappa\lambda}u^\kappa u^\lambda$	electric part of the Weyl tensor ;
$H_{\mu\nu} = {}^*C_{\mu\nu\kappa\lambda}u^\kappa u^\lambda$	magnetic part of the Weyl tensor ;
$*$	dual operator ;
$J = \frac{1}{6}\epsilon_{abc}\epsilon^{ijk}\eta^a{}_i\eta^b{}_j\eta^c{}_k$	relativistic equivalent of the Jacobian ;
$\eta^a{}_i(\vec{X},t) = a(t)(\delta^a{}_i + P^a{}_i(\vec{X},t))$	perturbative development of the Cartan coframes field ;
$P^a{}_i(\vec{X},t)$	perturbations ;
$U^a{}_i(\vec{X})$	initial velocity ;
$W^a{}_i(\vec{X})$	initial acceleration ;

Operators

$\mathcal{D} = \frac{d^2}{dt^2} + 2H\frac{d}{dt}$	time–linear operator of the evolution equation ;
$\langle A\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} A\ d^3x$	average of a scalar function on a domain $\mathcal{D}$ of volume $V_{\mathcal{D}}$;
$\langle A\rangle_{\mathcal{I}} = \frac{1}{V_{\mathcal{D}_{\mathrm{i}}}}\int_{\mathcal{D}} A\ d^3X$	average normalised by the initial volume $V_{\mathcal{D}_{\mathrm{i}}}$;

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