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Université Pierre et Marie Curie

Ecole doctorale Sciences Mathématiques de Paris Centre (386)

Tropical intersection theory, and real inflection points of real algebraic curves

Par Cristhian Emmanuel Garay-López

Thèse de doctorat de Mathématiques

Dirigée par Erwan Brugallé et Jean-Jacques Risler

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À ma femme.

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Introduction

This thesis is divided into two main themes. We first study the relationships between intersection theories in tropical geometry and algebraic geometry. In the last two chapters of this manuscript, we tackle the question of possible distributions of real inflection points of real linear series on real algebraic curves.

0.1.– Tropical and algebraic intersection theories

We refer to Chapter 1 for the definitions of basic objects in tropical geometry. Let $A \subset \mathbb{R}^n$ be an effective tropical k-cycle, and let B_1 and B_2 be two tropical cycles in A of dimension ℓ_1 and ℓ_2 respectively. The tropical intersection $B_1 \cdot B_2$ of B_1 and B_2 in A has been defined in the following situations:

- $A = \mathbb{R}^n$, see [TRGS05] and [Mik06];
- $B_1 \cap B_2$ lies inside the set of simple points of A (i.e. points contained in a facet of weight 1 of A), see [Sha13].
- either B_1 or B_2 is an affine tropical Cartier divisor, see [AR09];
- A is a smooth tropical manifold (i.e. locally matroidal), see [Sha13];

The first two cases are treated by means of the so-called stable intersection product in $\mathbb{R}^n$. The last two situations use tropical modifications, a tool introduced by G. Mikhalkin. In all the four above cases, the tropical intersection $B_1 \cdot B_2$ is a tropical $(\ell_1 + \ell_2 - k)$-cycle in A.

Let $\mathbb{K}$ be the Mal'cev-Neumann field $F((t^{\mathbb{R}}))$, where F is an algebraically closed field of characteristic zero. Let $X \subset (\mathbb{K}^*)^n$ be an algebraic variety of dimension k, and let $Y_1, Y_2 \subset X$ be subvarieties of dimension ℓ_1 and ℓ_2 respectively. Let $Y_1 \cap Y_2$ be the intersection scheme of Y_1 and Y_2 in X, and suppose that the tropical intersection product $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)$ of $\mathrm{Trop}(Y_1)$ and $\mathrm{Trop}(Y_2)$ in $\mathrm{Trop}(X)$ is defined. One important problem in tropical geometry is the following.

Question: What is the relationship between $\mathrm{Trop}(Y_1 \cap Y_2)$ and $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)$?

Up to now, only a few partial answers to this problem are known. Let us briefly discuss three of them.

We say that $\mathrm{Trop}(Y_1)$ and $\mathrm{Trop}(Y_2)$ meet properly at a point p in $\mathrm{Trop}(X)$ if $\mathrm{Trop}(Y_1) \cap \mathrm{Trop}(Y_2)$ has pure dimension $\ell_1 + \ell_2 - k$ in a neighborhood of p. If $\mathrm{Trop}(Y_1)$ and $\mathrm{Trop}(Y_2)$ meet properly at a simple point p of $\mathrm{Trop}(X)$, and if U is a facet of $\mathrm{Trop}(Y_1) \cap \mathrm{Trop}(Y_2)$ containing p, then the tropical intersection multiplicity $w_{\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)}(U)$ of U can be defined using the stable intersection product in $\mathbb{R}^n$. See [OP11].

The following result by Osserman-Payne relates the tropicalization of the intersection scheme $Y_1 \cap Y_2$ of Y_1 and Y_2 with the stable intersection product $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)$ in the case of proper intersection at simple points of $\mathrm{Trop}(X)$. In next theorem, we denote by $w_{Y_1 \cap Y_2}(U)$ the weight of the facet U of $\mathrm{Trop}(Y_1 \cap Y_2)$.

Theorem ([OP11]). *Let $X \subset (\mathbb{K}^*)^n$ be an algebraic variety, and let Y_1 and $Y_2 \subset X$ be subvarieties. Suppose that $\mathrm{Trop}(Y_1)$ and $\mathrm{Trop}(Y_2)$ meet properly at a facet U of $\mathrm{Trop}(Y_1) \cap \mathrm{Trop}(Y_2)$ that contains a simple point of $\mathrm{Trop}(X)$. Then $U \subset \mathrm{Trop}(Y_1 \cap Y_2)$, $w_{Y_1 \cap Y_2}(U) \geq w_{\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)}(U)$ and*

$$w_{\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)}(U) = \sum_Z i(Z, Y_1 \cdot Y_2; X) w_Z(U), \tag{0.1}$$

where $i(Z, Y_1 \cdot Y_2; X)$ is the intersection multiplicity of Y_1 and Y_2 along Z, and the sum is taken over all components $Z \subset Y_1 \cap Y_2$ such that $U \subset \mathrm{Trop}(Z)$.

In particular, if $X \subset (\mathbb{K}^*)^n$ is smooth, $\mathrm{Trop}(Y_1)$ and $\mathrm{Trop}(Y_2)$ meet properly in $\mathrm{Trop}(X)$, and simple points of $\mathrm{Trop}(X)$ are dense in $\mathrm{Trop}(Y_1) \cap \mathrm{Trop}(Y_2)$, then the stable intersection product $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)$ in $\mathrm{Trop}(X)$ is defined and is equal to $\mathrm{Trop}(Y_1 \cdot Y_2)$, where $Y_1 \cdot Y_2$ is the refined intersection product of Y_1 and Y_2 in X. Finally, when Y_1 and Y_2 are Cohen-Macaulay subvarieties, the algebraic cycle $Y_1 \cdot Y_2$ coincides with the fundamental cycle $[Y_1 \cap Y_2]$ associated to the closed subscheme $Y_1 \cap Y_2 \subset X$, and in particular we have $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2) = \mathrm{Trop}(Y_1 \cap Y_2)$. See Corollaries 5.1.2 and 5.1.3 in [OP11].

E. Brugallé and K. Shaw considered in [BS15] the case of tropicalizations of constant families of planar curves. Let $P \subset (\mathbb{C}^*)^n$ be a non-degenerate plane, and let $C_1, C_2 \subset P$ be two algebraic curves. Then $\mathrm{Trop}(P)$ is a matroidal fan and $\mathrm{Trop}(C_1)$ and $\mathrm{Trop}(C_2)$ are tropical fan 1-cycles in $\mathrm{Trop}(P)$. The following result relates the algebraic intersection number $\overline{C_1} \cdot \overline{C_2}$ of the compactification of C_1 and C_2 in a suitable compactification $\overline{P}$ of P, and the tropical intersection product $\mathrm{Trop}(C_1) \cdot \mathrm{Trop}(C_2)$ of $\mathrm{Trop}(C_1)$ and $\mathrm{Trop}(C_2)$ in $\mathrm{Trop}(P)$.

Theorem ([BS15]). *Let $P \subset (\mathbb{C}^*)^n$ be a non-degenerate plane and let $C_1, C_2 \subset P$ be two algebraic curves. Then*

$$\overline{C_1} \cdot \overline{C_2} = \mathit{Trop}(C_1) \cdot \mathit{Trop}(C_2),$$

where $\overline{C_i}$ is the compactification of C_i in a suitable toric compactification $\overline{P}$ of P.

Finally E. Brugallé and L. López de Medrano considered stable intersections in $\mathbb{R}^2$ to cover the case of two curves C_1 and C_2 in $(\mathbb{K}^*)^2$ with proper intersection.

Theorem ([BL12]). *Let C_1 and C_2 be two algebraic curves in $(\mathbb{K}^*)^2$, and let E be a connected component of $\mathit{Trop}(C_1) \cap \mathit{Trop}(C_2)$. Then we have*

$$\sum_{\mathit{Trop}(x) \in E} i(x, C_1 \cdot C_2; (\mathbb{K}^*)^2) \leq \sum_{p \in E} w_{\mathit{Trop}(C_1) \cdot \mathit{Trop}(C_2)}(p). \tag{0.2}$$

Equality is attained if E is compact.

Each connected component E of $\mathrm{Trop}(C_1) \cap \mathrm{Trop}(C_2)$ has either dimension zero or dimension one. If $E = \{p\}$, then $\mathrm{Trop}(C_2)$ meets properly $\mathrm{Trop}(C_2)$ at p and Equations (0.1) and (0.2) coincide.

The last theorem is proved using algebraic modifications of a subvariety $X \subset (\mathbb{K}^*)^n$ along a non-zero regular function $f \in \mathcal{O}_X(X)$, which we shall describe briefly. The graph $\Gamma_f(X_f) = \{(x, f(x)) \,:\, x \in X_f\}$ is a closed subscheme of the product $(\mathbb{K}^*)^n \times \mathbb{K}^*$, and the projection $\pi : (\mathbb{K}^*)^n \times \mathbb{K}^* \longrightarrow (\mathbb{K}^*)^n$ induces an open embedding $\pi : \Gamma_f(X_f) \longrightarrow X$. The tropicalization $\mathrm{Trop}(\pi) : \mathrm{Trop}(\Gamma_f(X_f)) \longrightarrow \mathrm{Trop}(X)$ is by definition the algebraic modification of X along f.

0.2.– Real inflection points of real linear series on real algebraic curves

A linear series on a non-singular complex algebraic curve X is a pair $Q = (V, L)$, where L is a line bundle defined on X with $H^0(X, L) \neq 0$ and $V \subset H^0(X, L)$ is a linear subspace distinct from $\{0\}$. The degree of Q is the degree of L and the rank of Q is $\dim_{\mathbb{C}}(V) - 1$. If Q is a linear series of degree d and rank r, we also say that Q is a g^r_d on X. We say that $x \in X$ is an inflection point of Q if there exists $s \in (V \setminus \{0\})$ such that $\mathrm{ord}_x(s) > r$. If X has genus g, then any g^r_d on X has exactly $(r+1)(d + r(g-1))$ inflection points (counted with multiplicity). An inflection point of the complete canonical series on X is called a Weierstrass point of X.

Let (X, σ) be a real algebraic curve. If $L_{\mathbb{R}}$ is an algebraic line bundle on (X, σ) defined over $\mathbb{R}$, then the space of real sections $H^0(X, L_{\mathbb{R}})$ is a real vector space. In [GH81], B. Gross and J. Harris showed that these line bundles are precisely those induced by a σ-invariant divisor on (X, σ). Furthermore, they showed that the divisor classes $\mathrm{Pic}^+(X)(\mathbb{R})$ of algebraic line bundles defined over $\mathbb{R}$ are the real points $\mathrm{Pic}(X)(\mathbb{R})$ of the Picard variety $\mathrm{Pic}(X)$ of X when $X(\mathbb{R}) \neq \emptyset$. See Proposition 2.2 in [GH81].

We introduce the concept of real linear series on a real algebraic curve (X, σ) as a pair $Q = (V_{\mathbb{R}}, L_{\mathbb{R}})$, where $L_{\mathbb{R}}$ is an algebraic line bundle defined over $\mathbb{R}$ with $H^0(X, L_{\mathbb{R}}) \neq 0$ and $V \subset H^0(X, L)$ is a real linear subspace distinct from $\{0\}$. We say that $x \in X$ is an inflection point of Q if and only if it is an inflection point of $Q_{\mathbb{C}}$, where $Q_{\mathbb{C}}$ is the linear series $(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, L_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ induced on the complex curve X. An inflection point x of Q is said to be real if $x \in X(\mathbb{R})$.

Up to our knowledge, the study of real inflection points of real linear series defined on real algebraic curves has yet focused in two cases. The first one is the study of the real roots of the Wronskian associated to a real linear series on $\mathbb{CP}^1$. The second one is the study of real inflection points of real plane algebraic curves. We briefly discuss them in the next two sections.

0.2.1 The case of genus zero

The concept of inflection point of a linear series defined on the curve $X = \mathbb{CP}^1$ admits a formulation in terms of the so-called Wronskian map. The source for all the assertions in this part is [Pur09].

Let us endow X with projective coordinates $[z : w]$ and set $K_X = -2 \cdot \infty$, where $\infty = [1 : 0]$. Let $d \geq 0$ and consider the divisor $D = d \cdot \infty$, then the space $H^0(X, D)$ can be identified with the space $\mathbb{C}[z]_{\leq d}$ of complex polynomials of degree at most d in the variable z. It follows that the Grassmannian $\mathrm{Gr}(r+1, \mathbb{C}[z]_{\leq d})$ can be considered as the space of linear series $Q = (V, L)$ of degree d and rank r on X.

Suppose that the polynomial $f \in \mathbb{C}[z]_{\leq d}$ factors as $\lambda \prod_i (z - a_i)^{n_i}$. When f is regarded as an element of $H^0(X, D)$ we use instead the homogeneous polynomial $F(z, w) = \lambda w^{d-\deg(f)} \prod_i (z - a_i w)^{n_i}$ of degree d.

Given $f_0, \ldots, f_r \in \mathbb{C}[z]_{\leq d}$ with $r \leq d$, their Wronskian $\mathrm{Wr}(f_0, \ldots, f_r)$ is the following polynomial in the variable z :

$$\mathrm{Wr}(f_0, \ldots, f_r)(z) = \det \begin{pmatrix} f_0(z) & \cdots & f_r(z) \\ f_0'(z) & \cdots & f_r'(z) \\ \vdots & \ddots & \vdots \\ f_0^{(r)}(z) & \cdots & f_r^{(r)}(z) \end{pmatrix}.$$

We have that $\mathrm{Wr}(f_0, \ldots, f_r)(z)$ belongs to $H^0(X, (r+1)D + \frac{r(r+1)}{2} K_X)$ if and only if the polynomials $f_0, \ldots, f_r$ are linearly independent. Since $(r+1)D + \frac{r(r+1)}{2} K_X = (r+1)(d-r) \cdot \infty$, we can identify $H^0(X, (r+1)D + \frac{r(r+1)}{2} K_X)$ with the space $\mathbb{C}[z]_{\leq (r+1)(d-r)}$. We also have that a family $g_0, \ldots, g_r \in \mathbb{C}[z]_{\leq d}$ span the same linear subspace as the f_i's if and only if $\mathrm{Wr}(g_0, \ldots, g_r)(z) = \lambda \mathrm{Wr}(f_0, \ldots, f_r)(z)$ for some $\lambda \in \mathbb{C}^*$. We have the following result.

Theorem 0.2.1 (Eisenbud-Harris). *The Wronskian* $\mathrm{Wr} : \mathrm{Gr}(r+1, \mathbb{C}[z]_{\leq d}) \longrightarrow \mathbb{P}(\mathbb{C}[z]_{\leq (r+1)(d-r)})$ *is a flat and finite morphism of schemes.*

Let $V \in \mathrm{Gr}(r+1, \mathbb{C}[z]_{\leq d})$ so that $Q = (V, \mathcal{L}(D))$ is a linear series of degree d and rank r on X. A point $x \in \mathbb{CP}^1 \setminus \{\infty\}$ will not be an inflection point of Q if for any $i = 0, \ldots, r$ there exists $f \in V$ such that $\mathrm{ord}_p(f) = i$. In other words, a point $x \in \mathbb{CP}^1 \setminus \{\infty\}$ is an inflection point of Q if $\mathrm{Wr}(f_0, \ldots, f_r)(x) = 0$, where $\{f_0, \ldots, f_r\}$ is a basis for V. See [Mir95].

Let us interpret $\mathrm{Wr}(f_0, \ldots, f_r)$ as an element of $H^0(X, (r+1)D + \frac{r(r+1)}{2} K_X)$, and let $F(z, w) = \prod_i (a_i z + b_i w)^{n_i}$ be the unique homogeneous polynomial of degree $(r+1)(d-r)$ such that $\mathrm{Wr}(f_0, \ldots, f_r)(z) = F(z, 1)$. The roots of the polynomial $F(z, w)$ do not depend on the representative $\mathrm{Wr}(f_0, \ldots, f_r)(z)$ for $\mathrm{Wr}(Q)$. If $[z_0 : w_0]$ is a root of multiplicity n of $F(z, w)$, then we say that $[z_0 : w_0]$ is an inflection point of multiplicity n of the linear series Q.

We now take $X = \mathbb{CP}^1$ endowed with the standard real structure $\sigma([z : w]) = [\overline{z} : \overline{w}]$ so that $X(\mathbb{R}) = \mathbb{RP}^1$. Since $D = d \cdot \infty$ is defined over $\mathbb{R}$, we have $\mathbb{C}[z]_{\leq d} = \mathbb{R}[z]_{\leq d} \otimes_{\mathbb{R}} \mathbb{C}$ and thus $\mathrm{Gr}(r+1, \mathbb{R}[z]_{\leq d})$ is a parameter space for the real linear series of degree d and rank r on (X, σ).

Let $V_{\mathbb{R}} \in \mathrm{Gr}(r+1, \mathbb{R}[z]_{\leq d})$ so that $Q = (V_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}(D))$ is a real g_d^r on (X, σ). If $\{f_0, \ldots, f_r\}$ a basis for $V_{\mathbb{R}}$, then $\mathrm{Wr}(f_0, \ldots, f_r)$ is in $\mathbb{P}(\mathbb{R}[z]_{\leq (r+1)(d-r)})$. The following result is the solution given by Mukhin, Tarasov and Varchenko to a part of the so-called Shapiro-Shapiro conjecture on the reality of the fibers of the map Wr over $\mathbb{P}(\mathbb{R}[z]_{\leq (r+1)(d-r)})$.

Theorem 0.2.2 (Mukhin, Tarasov and Varchenko). *Let* $g \in \mathbb{R}[z]_{\leq (r+1)(d-r)}$ *be a polynomial with* $(r+1)(d-r)$ *different real roots. Then the fiber* $\mathrm{Wr}^{-1}([g])$ *is reduced and every point in the fiber is real.*

We can interpret the previous theorem as follows. Let $E = \sum_{i=1}^{(r+1)(d-r)} p_i$ be an effective divisor supported on $\mathbb{RP}^1 \setminus \{\infty\}$ such that $p_i \neq p_j$ for $i \neq j$, then every linear series $Q \in \mathrm{Gr}(r+1, \mathbb{C}[z]_{\leq d})$ of degree d and rank r defined on $(\mathbb{CP}^1, \sigma)$ with inflection divisor (i.e. its set of infection points) supported on E is real. In particular, for any $1 \leq r \leq d$, there exists real g_d^r on $(\mathbb{CP}^1, \sigma)$ whose inflection points are all real.

0.2.2 The case of dimension two

Let $C \subset \mathbb{CP}^2$ be an irreducible plane algebraic curve of degree $d > 1$. Given a regular point $p \in C$, let T_pC be the tangent line to C at p. Recall that the regular point $p \in C$ is an inflection point of C if $i(p, C \cdot T_pC; \mathbb{CP}^2) > 2$.

We say that the curve $C \subset \mathbb{CP}^2$ has traditional singularities if it only possesses:

1. nodes and cusps as singularities,
2. inflection points of multiplicity one, and
3. bi-tangents as multitangents.

The closure in $\mathbb{CP}^{2*}$ of the set of tangent lines to C at regular points $p \in C$ is the dual curve C^* of C, which is an irreducible algebraic curve. If C has traditional singularities, then its dual curve C^* has traditional singularities too, and under the projective duality $p \mapsto T_pC$, the nodes of one curve correspond to the bi-tangents of the other, and the regular inflection points of one curve correspond to the cusps of the other.

Suppose that C has traditional singularities. We denote by $\delta(C)$ its number of nodes, and by $\kappa(C)$ its number of cusps. According to Plücker formulas, we have

$$\begin{aligned} \kappa(C^*) &= 3d(d-2) - 6\delta(C) - 8\kappa(C), \\ \deg(C^*) &= d(d-1) - 2\delta(C) - 3\kappa(C). \end{aligned} \tag{0.3}$$

When the curve C is non-singular (but C^* still has traditional singularities), it follows from Equations (0.3) that the number $w(C)$ of inflection points of C is equal to $\kappa(C^*) = 3d(d-2)$ and that $\deg(C^*) = d(d-1)$. In the case when C is real, F. Klein gave in [Kle76] a linear formula that relates different real elements of C.

Theorem (Klein). *Let $C \subset \mathbb{CP}^2$ be a real algebraic curve with traditional singularities. Then*

$$deg(C) + i_{\mathbb{R}}(C) + 2t''(C) = deg(C^*) + \kappa_{\mathbb{R}}(C) + 2\delta''(C). \tag{0.4}$$

Where

- $\delta''(C)$ is the number of real solitary nodes[1] of C,
- $\kappa_{\mathbb{R}}(C)$ is the number of real cusps of C,
- $i_{\mathbb{R}}(C)$ is the number of regular real inflection points of C, and
- $t''(C)$ denote the number of real bi-tangents at a pair of complex conjugated points.

When the curve C is real and non-singular, Equation (0.4) gives a linear relation between $i_{\mathbb{R}}(C)$ and the number of real bi-tangents at a pair of complex conjugated points of C, namely

$$i_{\mathbb{R}}(C) + 2t''(C) = d(d-2).$$

Note that in this case, the number $i_{\mathbb{R}}$ is precisely the number of real inflection points of the restriction on $\mathcal{O}_{\mathbb{C}P^2}(1)$ on C.

It follows that a smooth generic real plane algebraic curve C satisfies $i_{\mathbb{R}}(C) \leq d(d-2)$, i.e. at most one third of the inflection points of C may be real. Klein also showed that this bound is sharp by constructing examples of real algebraic curves C for which $t''(C) = 0$ using deformations of algebraic curves. In fact, using Klein's method one can easily prove the following result.

Theorem. *For every $d > 2$, there exists smooth real algebraic curves $C \subset \mathbb{CP}^2$ with*

$$i_{\mathbb{R}}(C) = \begin{cases} d(d-2) - 4k,\ k = 0, \dots, \frac{d(d-2)}{4} & \text{if } d \text{ is even,} \\ d(d-2) - 4k,\ k = 0, \dots, \frac{d(d-2)-3}{4} & \text{if } d \text{ is odd.} \end{cases}$$

A non-singular real plane algebraic curve $C \subset \mathbb{CP}^2$ of degree d is said to be maximally inflected if it possesses $d(d-2)$ real inflection points.

Another method to construct maximally inflected plane real algebraic curves has been proposed by E. Brugallé and L. López de Medrano in [BL12]. Studying tropical limits of inflection points of plane real algebraic curves in $(\mathbb{R}^*)^2$, they showed that Viro's patchworking technique mainly produces maximally inflected real curves in $\mathbb{RP}^2$. For any $d > 0$, we denote by T_d the convex lattice triangle $\mathrm{Conv}\{(0,0),(d,0),(0,d)\}$.

[1] A real node of a real curve is solitary if its branches are complex.

Theorem ([BL12]). *Let C be a non-singular tropical curve in $\mathbb{R}^2$ with Newton polygon the triangle T_d with $d \geq 2$, and defined by the tropical polynomial*

$$\phi(p_1, p_2) = \max_{(i,j)\in T_d\cap\mathbb{Z}^2}(a_{ij} + \langle (i,j), (p_1, p_2)\rangle).$$

Suppose that if v is a vertex of C dual to T_1, then its three adjacent edges have three different length. Then the real algebraic curve defined by the polynomial $P(x, y) = \sum_{(i,j)\in T_d\cap\mathbb{Z}^2} \alpha_{ij} t^{-a_{ij}} x^i y^j$ with $\alpha_{ij} \in \mathbb{R}^*$ has exactly $d(d-2)$ real inflection points in $\mathbb{CP}^2$ for $t > 0$ small enough.

0.3.– Results

0.3.1 Chapter 2

Let $\mathbb{K}$ be the Mal'cev-Neumann field $F((t^{\mathbb{R}}))$, where F is an algebraically closed field of characteristic zero. We say that a subvariety $X \subset (\mathbb{K}^*)^n$ has simple tropicalization if every regular point of $\mathrm{Trop}(X)$ is simple. Examples of such varieties are $(\mathbb{K}^*)^n$ itself, and linear subvarieties of $(\mathbb{K}^*)^n$.

In the class of subvarieties with simple tropicalization, the stable intersection product $\mathrm{Trop}(Y_1).\mathrm{Trop}(Y_2)$ in $\mathrm{Trop}(X)$ of two subvarieties $Y_1, Y_2 \subset X$ can be defined, whenever the tropical cycles $\mathrm{Trop}(Y_1)$, $\mathrm{Trop}(Y_2)$ meet properly in $\mathrm{Trop}(X)$ and regular points of $\mathrm{Trop}(X)$ are dense in $\mathrm{Trop}(Y_1) \cap \mathrm{Trop}(Y_2)$.

We introduce the slightly more general concept of generically integral $\mathbb{K}$-variety as being a $\mathbb{K}$-variety X that admits a closed embedding $g : X \hookrightarrow (\mathbb{K}^*)^n$ such that $g(X)$ has simple tropicalization. Any such variety is necessarily very-affine, so it comes equipped with a particular embedding into an algebraic $\mathbb{K}$-torus, called its intrinsic embedding.

We show that the generically integral $\mathbb{K}$-varieties are characterized by their intrinsic embedding.

Theorem. *Let X be a very affine $\mathbb{K}$-variety with intrinsic embedding $f : X \longrightarrow (\mathbb{K}^*)^m$. Then X is generically integral if and only if $f(X)$ has simple tropicalization.*

Next we generalize Equation (0.2) to the case when X is an arbitrary surface in $(\mathbb{K}^*)^n$ with simple tropicalization, and one of the two curves is a principal divisor.

First we introduce a notion of tropical Cartier divisor ϕ defined on a tropical k-cycle A in $\mathbb{R}^n$. Next, we define an intersection product $Y \cdot \phi$ of ϕ with any ℓ-tropical cycle $Y \subset A$ which generalizes the intersection product introduced by Allermann and Rau in [AR09]. Then we show that when $X \subset (\mathbb{K}^*)^n$ has simple tropicalization, the algebraic modification $\mathrm{Trop}(\pi) : \mathrm{Trop}(\Gamma_f(X_f)) \longrightarrow \mathrm{Trop}(X)$ of X along a non-zero regular function $f \in \mathcal{O}_X(X)$ defines a tropical Cartier divisor $\mathcal{T}(f)$ on $\mathrm{Trop}(X)$. As an application, we prove the following generalization of Equation (0.2).

Theorem. *Let $X \subset (\mathbb{K}^*)^n$ be a non-singular variety with simple tropicalization, $C \subset X$ a purely 1-dimensional closed subscheme, and f a non-zero regular function on X such that C and $\mathrm{div}_X(f)$ intersect properly in X. If E is a connected component of the set $\mathrm{Trop}(C) \cap \mathrm{Trop}(\mathrm{div}_X(f))$, then we have*

$$\sum_{\mathrm{Trop}(x)\in E} \ell(\mathcal{O}_{C\cap \mathrm{div}_X(f),x}) \leq \sum_{p\in E} w_{\mathrm{Trop}(C).\mathcal{T}(f)}(p),$$

where $\mathrm{Trop}(C).\mathcal{T}(f)$ is the tropical intersection product of $\mathrm{Trop}(C)$ with the tropical Cartier divisor $\mathcal{T}(f) : \mathrm{Trop}(X) \longrightarrow \mathbb{R}$. If E is compact, then equality is attained.

Note that if $\mathrm{Trop}(X)$ is non-singular, then $\mathrm{Trop}(C).\mathcal{T}(f)$ can be replaced by $\mathrm{Trop}(C).\mathrm{Trop}(\mathrm{div}_X(f))$.

0.3.2 Chapter 3

We have seen that the study of real inflection points of real linear series of degree d and rank r defined on real algebraic curves of genus g has been thoroughly studied in the cases $g = 0$ or $r = 2$. In this chapter we study the cases $g = 1$ or $r = 3$.

First, we classify all possible distributions of real inflection points of a real complete linear series of degree $d \geq 2$ on a real elliptic curve (X, σ).

Theorem. *Let $X = (X, \sigma)$ be a real algebraic curve of genus 1 with $X(\mathbb{R}) \neq \emptyset$, and let Q be a real complete linear series of degree $d \geq 2$. Then Q has exactly d^2 complex inflection points. Moreover Q has exactly either 0, d, or $2d$ real inflection points according to the following cases:*

- *if $X(\mathbb{R})$ is connected, then Q has d real inflection points;*

- *if $X(\mathbb{R})$ has two connected components and d is odd, then Q has d real inflection points; these points are located on the connected component of $X(\mathbb{R})$ on which Q has odd degree;*
- *if $X(\mathbb{R})$ has two connected components and d is even, then*
  - *if Q has even degree in both connected components, then Q has exactly d real inflection points on each connected component (hence Q has $2d$ real inflection points);*
  - *if Q has odd degree in both connected components, then Q has no real inflection point.*

Let $C \subset \mathbb{CP}^3$ be a real, smooth, non-hyperelliptic curve of genus four and degree six, and let $I_C = \{(x, H) \in \mathbb{CP}^3 \times \mathbb{CP}^{3*} \ : \ x \in C \cap H\}$ be its incidence variety. If C is 4-simple (i.e. the multiplicity function mult_{C^*} takes values in $\{1, 2, 3\}$), then we give the following expression for the number $w_{\mathbb{R}}(C)$ of real inflection points of C.

Theorem. *Let $C \subset \mathbb{CP}^3$ be a real, smooth, non-hyperelliptic curve of genus four and degree six, and let $C^*(\mathbb{R}) \subset \mathbb{RP}^{3*}$ be the real part of its dual variety. If C is 4-simple, then*

$$w_{\mathbb{R}}(C) = -\chi(\pi_2^{-1}(C^*(\mathbb{R}))), \tag{0.5}$$

where $\pi_2 : I_C \longrightarrow \mathbb{CP}^{3*}$ is the projection $(x, H) \mapsto H$.

Note that $I_C(\mathbb{R}) \subset \pi_2^{-1}(C^*(\mathbb{R}))$, however the inclusion might be strict as $C^*(\mathbb{R})$ is singular in general.

0.3.3 Chapter 4

The main result of this chapter is the construction of examples of non-singular, non-hyperelliptic real curves of degree six and genus four in $\mathbb{RP}^3$ such that exactly 30 of its 60 complex inflection points are real. Our result (see Theorem 4.5.1) can be stated as follows.

Theorem. *There exist non-singular, non-hyperelliptic real algebraic curves of genus four having 30 real Weierstrass points.*

These examples are constructed using Viro's Patchworking method in the dense torus of the normal toric surface $\mathbb{CP}(2, 1, 1)$. In particular, we thoroughly study all possible distribution of real inflection points of real algebraic curves in $(\mathbb{C}^*)^2$ having some particular "small" Newton polygon. A generic algebraic curve C with Newton polygon the parallelogram with vertices $(0, 2)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$ is non-singular of genus one. The tautological embedding of the normal toric surface $\mathbb{CP}(2, 1, 1) \hookrightarrow \mathbb{CP}^3$ defines a complete linear series Q of degree 4 and rank 3 on C by restricting $\mathcal{O}_{\mathbb{CP}^3}(1)$. Thus Q has 16 inflection points with at most 8 of them real. The above construction relies on the following proposition.

Proposition 0.3.6. *The complete linear series Q of degree 4 and rank 3 on a non-singular real algebraic curve in $\mathbb{CP}(2, 1, 1)$ defined by a polynomial $f(X, Y) = u_{02}Y^2 + (u_{01} + u_{11}X + u_{21}X^2)Y + u_{20}X^2$ satisfying $u_{11} \neq 0$, $u_{21}u_{01} < 0$ and $u_{20}u_{02} < 0$ has eight real inflection points.*

Chapter 1

Preliminaries

All rings considered in this work will be commutative, Noetherian rings with unit. If R is a ring, we will denote by R^* its multiplicative group of units. If M is an R-module of finite length, its length will be denoted by $\ell_R(M)$, and by $\ell(R)$ when $M = R$.

We mark the end of a Proof by a black square ■ and the end of an Example by a star $*$.

1.1.– Glossary of algebraic geometry

Unless otherwise stated, in this work *we will always use an algebraically closed base field of characteristic zero* $\mathbb{K}$. Most of the following conventions can be found in [Ful84].

By **scheme** we mean an algebraic[1] scheme over $\mathbb{K}$. If $(X, \mathcal{O}_X)$ is a scheme, we will usually denote it by X. By algebraic **variety** we mean a reduced and irreducible (i.e. integral) scheme. If X is a variety, we denote by $\mathbb{K}(X)$ its **field of rational functions**. By **curve** (respectively **surface**) we mean a variety of dimension one (respectively of dimension two).

Let X be a scheme.

- By a **subscheme** of X we mean a locally closed subscheme (i.e. the intersection of an open subscheme with a closed subscheme of X). If Y is a subscheme of X, we denote by $\mathrm{Supp}(Y)$ its **support**.
- By **subvariety** of X we mean an integral closed subscheme of X. If Y is a subvariety of X, its **local ring** $\mathcal{O}_{X,Y} = (\mathcal{O}_{X,Y}, \mathfrak{m}_{X,Y})$ will be denoted $\mathcal{O}_{X,Y}$.

1.1.1 Example (Affine schemes): If $X = \mathrm{Spec}(R)$ is an affine scheme, we will denote by $\mathbb{K}[X] = R$ its ring of regular functions. If $I \subset \mathbb{K}[X]$ is an ideal, the closed subscheme of X defined by I will be denoted $V(I)$. We use the next conventions:

1. we will denote by $T^n_{\mathbb{K}}$ the algebraic n-torus $\mathrm{Spec}(\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$, and by $(\mathbb{K}^*)^n = T^n_{\mathbb{K}}(\mathbb{K})$ its set of $\mathbb{K}$-points;
2. we will denote by $\mathbb{A}^n_{\mathbb{K}}$ the affine n-space $\mathrm{Spec}(\mathbb{K}[x_1, \ldots, x_n])$, and by $\mathbb{K}^n = \mathbb{A}^n_{\mathbb{K}}(\mathbb{K})$ its set of $\mathbb{K}$-points. $*$

By **embedding** of schemes we mean a locally closed embedding of schemes. If $i : X' \hookrightarrow X$ is an embedding of schemes, by its **closure** we mean the scheme-theoretical image of i, this is, the smallest closed subscheme of X containing the image of i.

Remark 1.1.2 (Some properties of the closure of an embedding): *Let $i : X' \hookrightarrow X$ be an embedding of schemes. The closure of i is supported on the topological closure of its image. Furthermore,*

1. *if X' is reduced, then the closure of $i : X' \hookrightarrow X$ is also reduced;*
2. *if X is affine, then the ideal $Ker(\mathbb{K}[X] \xrightarrow{i^*} \mathbb{K}[X'])$ defines the closure of $i : X' \hookrightarrow X$.*

By point x of a scheme X we mean a closed point, and we say that $x \in X$ is **regular** if $\mathcal{O}_{X,x}$ is a regular local ring. We set $X_{\mathrm{Smooth}} = \{x \in X \ : \ x \text{ is regular}\}$, $X_{\mathrm{Sing}} = X \setminus X_{\mathrm{Smooth}}$ and say that X is **smooth** or **non-singular** if $X_{\mathrm{Sing}} = \emptyset$.

Definition: Let X be a scheme. A subset of X is **constructible** if it can be expressed as a finite disjoint union of locally closed subsets. A function $f : X \longrightarrow \mathbb{Z}$ is **constructible** if there exists a stratification of X consisting of disjoint constructible sets such that f is constant on each stratum.

[1] A scheme of finite type over $\mathbb{K}$. In particular, all schemes will be Noetherian.

The set $\mathcal{F}(X)$ of all constructible functions $X \longrightarrow \mathbb{Z}$ is an abelian group. If $f : X \longrightarrow \mathbb{Z}$ is constructible, we will denote by $\mathrm{Supp}(f)$ its support.

Let F be a sheaf on a scheme X. We will denote by F_x its **stalk** at the point $x \in X$, and if $U \subset X$ is an open subset, we will denote by s_x the image in F_x of an element $s \in F(U)$. If G is another sheaf defined on X and $\eta : F \longrightarrow G$ is a morphism of sheaves, we denote by $\eta_x : F_x \longrightarrow G_x$ the morphism on stalks induced by η. A sheaf F on X is said to be **invertible** if it is locally free of rank one.

Definition: Let X be a scheme. The group of (algebraic) k-**cycles** on X is

$$Z_k(X) := \left\{ \sum_{i \in I} n_i [Y_i] \ : \ I \text{ finite}, n_i \in \mathbb{Z} \text{ and } Y_i \subset X \text{ is a } k\text{-dimensional subvariety for all } i \in I \right\}.$$

The group of (algebraic) **cycles** of X is $Z_*(X) = \bigoplus_k Z_k(X)$. If X is a variety of dimension k, then $Z_{k-1}(X)$ is the group of **Weil divisors** on X.

Definition: Let X be a scheme, $Y \subset X$ a subvariety and $f \in \mathbb{K}(Y)^*$. The **Weil divisor** on Y associated to f is the cycle

$$[\mathrm{div}_Y(f)] = \sum_{W \subset Y} \mathrm{ord}_W(f)[W], \tag{1.1}$$

where the sum is taken over all codimension one subvarieties W of Y and $\mathrm{ord}_W(f)$ is the **order** of vanishing of f along W.

Definition: A k-cycle α in X is **rationally equivalent to zero** if there exist a finite number of $(k+1)$-dimensional subvarieties $W_1, \ldots, W_s \subset X$ such that $\alpha = \sum_{j=1}^{s} [\mathrm{div}_{W_j}(f_j)]$ for some $f_j \in \mathbb{K}(W_j)^*$.

The set of k-cycles which are rationally equivalent to zero form the subgroup $\mathrm{Rat}_k(X)$ of $Z_k(X)$, and the group $Z_k(X)/\mathrm{Rat}_k(X)$ of k-cycles on X modulo rational equivalence is denoted by $A_k(X)$.

Definition: Let $X_1, \ldots, X_s$ be the irreducible components of a scheme X, then

1. the **geometric multiplicity** of X_i in X is $\ell(\mathcal{O}_{X,X_i})$;
2. the **fundamental cycle** [X] of X is the cycle $\sum_{i=1}^{s} \ell(\mathcal{O}_{X,X_i})[X_i]$.

The scheme X is said to be **pure dimensional** if $X_1, \ldots, X_s$ have the same dimension.

Any closed subscheme $Y \subset X$ defines a closed embedding $Y \hookrightarrow X$. We will denote also by $[Y]$ the cycle that Y defines in $Z_*(X)$.

Definition: An **effective Cartier divisor** on X is a closed subscheme D of X whose ideal sheaf is locally generated by one function which is a non-zero divisor.

If D is an effective Cartier divisor D on X, we will denote by $\mathrm{Supp}(D)$ its support and by $[D] \in Z_*(X)$ the cycle that it defines in X.

1.1.1 Intersection theory on varieties

In this part, X will be a variety of dimension k over an algebraically closed field of characteristic zero $\mathbb{K}$. We will denote by:

1. $\mathrm{Div}(X)$ the group of Cartier divisors on X, and by $D \mapsto [D]$ the usual morphism $\mathrm{Div}(X) \longrightarrow Z_{k-1}(X)$;
2. div_X the usual morphism $\mathbb{K}(X)^* \longrightarrow \mathrm{Div}(X)$.

The set $\mathrm{div}_X(\mathbb{K}(X)^*)$ is the group of **principal Cartier divisors**.

Definition: Let X be a variety. The **intersection scheme** of two embeddings $i : Y \hookrightarrow X$ and $j : W \hookrightarrow X$ is the object representing the fibered product of the diagram $Y \hookrightarrow X \hookleftarrow W$.

If we denote the intersection scheme of $Y \hookrightarrow X \hookleftarrow W$ by $Y \cap W$, then we have a Cartesian square:

$$\begin{tikzcd} Y \cap W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & W \arrow[d, hook] \\ Y \arrow[r, hook] & X. \end{tikzcd}$$

Definition: Let $i : Y \hookrightarrow X$ and $j : W \hookrightarrow X$ be embeddings with Y and W purely dimensional. We say that Y and W **meet properly** at an irreducible component Z of $Y \cap W$ if $\dim(Z) = \dim(Y) + \dim(W) - k$. We say that Y and W **intersect properly** in X if $Y \cap W$ has pure dimension $\dim(Y) + \dim(W) - k$.

1.1.3 Example (Intersecting closed subschemes): If $Y \hookrightarrow X$ and $W \hookrightarrow X$ are the closed embeddings associated to the closed subschemes $Y, W \subset X$, then $Y \cap W$ is the closed subscheme of X defined by the sum of the ideal sheaves of Y and W. The cycle $[Y \cap W]$ associated to the intersection scheme of Y and W in X has the form

$$[Y \cap W] = \sum_Z \ell(\mathcal{O}_{Y \cap W, Z})[Z],$$

where the sum is taken over the irreducible components Z of $Y \cap W$. If Y and W have pure dimension ℓ_1 and ℓ_2 respectively and have proper intersection in X, then $[Y \cap W] \in Z_{\ell_1+\ell_2-k}(X)$. $*$

Consider the following particular situation: let $W \subset X$ be a closed subscheme of pure dimension ℓ and let $Y = D$ be an effective Cartier divisor on X such that W and D intersect properly. Then D induces an effective Cartier divisor $D' = D \cap W$ on W. Let $[W] = \sum_i n_i[W_i]$ be the fundamental cycle of W, then D' defines an effective cartier divisor $D'_i = D' \cap W_i$ on each irreducible component $W_i \subset W$. Then it follows from Lemma 1.7.2 on [Ful84] that

$$[D \cap W] = [D'] = \sum_i n_i[D'_i]. \tag{1.2}$$

In this case, the right-hand side of Equation (1.2) coincides with the intersection product $D \cdot [W]$ of the Cartier divisor D and the ℓ-cycle $[W]$ on X, which is a well-defined intersection class in $A_{\ell-1}(\mathrm{Supp}(Y) \cap \mathrm{Supp}(D))$. See [Ful84], pp.28, 33.

1.1.4 Example (Intersecting with an effective principal Cartier divisor): Let $W \subset X$ be a closed subscheme of pure dimension ℓ and fundamental cycle $[W] = \sum_i n_i[W_i]$, and let $D = \mathrm{div}_X(f)$ be an effective principal Cartier divisor on X such that W and D intersect properly. Let us denote also by f the restriction $f|_W$ of the function f to W, then $D'_i = D' \cap W_i = \mathrm{div}_{W_i}(f)$ for every irreducible component $W_i \subset W$. It follows from Equation (1.2) that

$$[\mathrm{div}_X(f) \cap W] = \sum_i n_i[\mathrm{div}_{W_i}(f)] = D \cdot [W], \tag{1.3}$$

where each $[\mathrm{div}_{W_i}(f)]$ is as in Equation (1.1). $*$

Suppose that X is a non-singular variety and let $Y, W \subset X$ be two closed subschemes of pure dimension ℓ_1 and ℓ_2 respectively. Then one can construct a refined intersection product $Y \cdot W \in A_{\ell_1+\ell_2-k}(Y \cap W)$ as follows. Let N_Δ be the normal bundle associated to the diagonal embedding $\Delta : X \hookrightarrow X \times X$ and consider the Cartesian square

$$\begin{tikzcd} Y \cap W \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & Y \times W \arrow[d, hook] \\ X \arrow[r, hook, "\Delta"] & X \times X. \end{tikzcd}$$

Let T be the restriction of N_Δ to $Y \cap W$. Then the normal cone C associated to the closed embedding $Y \cap W \hookrightarrow Y \times W$ is a $(\ell_1 + \ell_2)$-dimensional closed subscheme of T. The following definition is found in [Ful84], Section 8.1.

Definition: The **refined intersection product** $Y \cdot W$ is the intersection of $[C]$ with the zero section of T.

The only situation in which we will consider refined intersection products will be when Y and W intersect properly. In this case, the intersection class $Y \cdot W$ is a well-defined $(\ell_1 + \ell_2 - k)$-cycle $Y \cdot W = \sum_Z n_Z[Z]$, where the sum is taken over the irreducible components Z of $Y \cap W$. The coefficient $n_Z = i(Z, Y \cdot W; X)$ is the **intersection multiplicity** of Z in $Y \cdot W$. See [Ful84], p.137 for a proof of the following statement.

Proposition 1.1.5. *If Y, W are closed subschemes of pure dimension on a non-singular algebraic variety X with proper intersection, then for every irreducible component Z of $Y \cap W$, we have*

1. $1 \leq i(Z, Y \cdot W; X) \leq \ell(\mathcal{O}_{Y \cap W, Z})$, *and*
2. *if* $\mathcal{O}_{Y \times W, Z}$ *is Cohen-Macaulay, then* $i(Z, Y \cdot W; X) = \ell(\mathcal{O}_{Y \cap W, Z})$.

1.1.6 Example (Intersection theory on a smooth surface): Let X be a smooth surface and let $Y, W \subset X$ be two closed subschemes of pure dimension one with proper intersection. If Y and W are reduced, then in particular they are Cohen-Macaulay schemes, and it follows from [Ful84], Example 8.2.7, that $[Y \cap W] = Y \cdot W$ in $Z_0(X)$. $*$

1.2.– Glossary on convex geometry

We will denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, endowed with the standard inner product $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and the Euclidean volume form vol. Every time that we make reference to topological concepts or arguments in $\mathbb{R}^n$, we assume that they refer to the Euclidean topology.

By a **polyhedron** in $\mathbb{R}^n$ we mean an intersection of finitely many sets of type $\{p \in \mathbb{R}^n \ : \ \langle p, u \rangle \leq c\}$ with $u \neq 0$. If Δ is a polyhedron in $\mathbb{R}^n$, its **relative interior** $\operatorname{relint}(\Delta)$ is interior of Δ with respect to its affine hull $\operatorname{Aff}(\Delta)$.

Definition: Let $\Delta = \cap_{i=1}^m \{p \in \mathbb{R}^n \ : \ \langle p, u_i \rangle \leq c_i\}$ be a polyhedron and let Γ be a subgroup of $(\mathbb{R}, +)$. We say that Δ is Γ-**rational** if $u_1, \ldots, u_m \in \mathbb{Z}^n$ and $c_1, \ldots, c_m \in \Gamma$. If Δ is $\{0\}$-rational, we say that is a (rational polyhedral) **cone**. If Δ is $\mathbb{R}$-rational, then we say that Δ is **rational**.

Remark 1.2.1: *Let* Γ *be a subgroup of* $(\mathbb{R}, +)$ *and let* $G = \{\lambda \in \mathbb{R} \ : \ \exists n \in (\mathbb{N} \setminus \{0\})$ *such that* $m\lambda \in \Gamma\}$. *Then a polyhedron* $\Delta \subset \mathbb{R}^n$ *is* Γ*-rational if and only if the affine hull Aff*(Δ') *of every face* $\Delta' \subset \Delta$ *is of the form* $L_{\Delta'} + p$, *with* $L_{\Delta'}$ *a rational linear space and* $p \in G^n$. *See [Gub11], p. 42.*

Definition: Let $\Delta \subset \mathbb{R}^n$ be a Γ-rational polyhedron and let Δ' be a face of Δ. If $\operatorname{Aff}(\Delta') = L_{\Delta'} + p$, then we set $\Lambda_{\Delta'} := L_{\Delta'} \cap \mathbb{Z}^n$.

Let $\Delta \subset \mathbb{R}^n$ be a Γ-rational polyhedron and let $\Delta'' \subset \Delta' \subset \Delta$ be a chain of faces of Δ such that $\dim(\Delta'') = \dim(\Delta') - 1$. There exists a unique vector $s(\Delta', \Delta'') \in \Lambda_{\Delta'}$ such that

1. the class $[s(\Delta', \Delta'')]$ generates the quotient $\Lambda_{\Delta'}/\Lambda_{\Delta''}$, which is isomorphic to $\mathbb{Z}$.
2. $s(\Delta', \Delta'')$ is orthogonal to $\Lambda_{\Delta''}$,
3. $s(\Delta', \Delta'')$ points in the direction of Δ'.

In particular, the vector $s(\Delta', \Delta'')$ is primitive. We call it the **primitive integer vector** orthogonal to Δ'' generating Δ'.

Definition: A Γ-**rational polyhedral complex** in $\mathbb{R}^n$ is a finite set of Γ-rational polyhedra $P = \{\Delta_i\}_i$ such that

1. for every $\Delta \in P$, if Δ' is a face of Δ, then $\Delta' \in P$, and
2. if $\Delta, \Delta' \in P$, then $\Delta \cap \Delta'$ is a face of both Δ and Δ'.

An element Δ of P is **maximal** if it is not contained in any other polyhedron of P. We say that P is **purely dimensional** if all the maximal elements have the same dimension; in this case, a maximal polyhedron of P is called a **facet**.

A $\{0\}$-rational polyhedral complex is called a (rational polyhedral) **fan**. An $\mathbb{R}$-rational polyhedral complex is called a rational polyhedral complex.

Definition: Let $P = \{\Delta_i\}_i$ be a Γ-rational polyhedral complex in $\mathbb{R}^n$.

1. The **support** $|P|$ of P is the set $|P| = \bigcup_i \Delta_i$.
2. A point $p \in |P|$ is **regular** if there is a polytope $\Delta \subset |P|$ such that $\operatorname{relint}(\Delta)$ is a neighborhood of p in $|P|$.

For any regular point $p \in |P|$ there exists an unique (maximal) $\Delta \in P$ such that $p \in \Delta$; we set $\Lambda_p = \Lambda_\Delta$.

Remark 1.2.2: *Let P be a Γ-rational polyhedral complex. The set $\{p \in |P| \ : \ p \text{ is regular}\}$ is open in $|P|$.*

Let $p \in |P|$ be a regular point, and suppose that the polyhedron $\Delta \in P$ containing p has dimension k, then by the theorem of structure of free abelian groups, there exits a basis $\{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ for $\mathbb{Z}^n$ and natural numbers $d_1 | \cdots | d_k$ such that $\{d_1 v_1, \dots, d_k v_k\}$ is a basis for Λ_Δ. Since L_Δ is a rational linear space, we have that $\{v_1, \dots, v_k\}$ is a basis for L_Δ.

We conclude that for any regular point $p \in |P|$ lying on a k-dimensional polyhedron $\Delta \in P$, there exists a basis $\{v_1, \dots, v_k\}$ for Λ_p which can be extended to a basis $\{v_1, \dots, v_n\}$ for $\mathbb{Z}^n$.

A polyhedron Δ is a **polytope** if it is bounded. This condition is equivalent to the existence of a finite number of points $i_1, \dots, i_m \in \mathbb{R}^n$ such that Δ is the convex hull $\mathrm{Conv}\{i_1, \dots, i_m\}$ of $i_1, \dots, i_m$.

If Δ is a polytope, then there is a unique minimal set $\mathrm{Vert}(\Delta)$ such that $\Delta = \mathrm{Conv}(\mathrm{Vert}(\Delta))$. We call $\mathrm{Vert}(\Delta)$ the set of **vertices** of Δ.

Definition: We say that the polytope $\Delta = \mathrm{Conv}(\mathrm{Vert}(\Delta))$ is **convex lattice** if $\mathrm{Vert}(\Delta) \subset \mathbb{Z}^n$. If Δ is a convex lattice polytope, its set of **inner lattice points** is $\mathrm{relint}(\Delta) \cap \mathbb{Z}^n$.

Let Δ be a convex lattice polytope in $\mathbb{R}^n$. The **lattice volume** $\mathrm{vol}_{\mathbb{Z}}(\Delta)$ of Δ is defined to be $n!\mathrm{vol}(\Delta)$. We say that Δ is **primitive** if $\mathrm{vol}_{\mathbb{Z}}(\Delta) = 1$.

Let F be a field of characteristic zero and let $f = \sum_{i \in A} \alpha_i x^i$ with $A \neq \emptyset$, $\alpha_i \in F^*$, be a polynomial in $F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Definition: The **Newton polytope** $\mathrm{New}(f)$ of f is defined to be $\mathrm{Conv}(A)$.

We will also call $\mathrm{New}(f)$ the Newton polytope of the closed subscheme $V(f)$ of T_F^n.

Definition: If Ω is a face of $\mathrm{New}(f)$, the **truncation** f^Ω of f to Ω is $f^\Omega := \sum_{i \in \Omega \cap \mathbb{Z}^n} \alpha_i x^i$. We say that f is **completely non-degenerate** (with respect to its Newton polygon) if for any face Ω of $\mathrm{New}(f)$, we have that $V(f^\Omega)$ is non-singular in T_F^n.

Being completely non-degenerate is a generic property for polynomials having the same Newton polygon.

Definition: Let Δ be a convex lattice polytope in $\mathbb{R}^n$ and let $\{\Delta\}_k$ be a polyhedral subdivision of Δ consisting of convex lattice polytopes. We say that $\{\Delta\}_k$ is **regular** (or **coherent**) if there exists a continuous, convex, piecewise-linear function $\varphi : \Delta \longrightarrow \mathbb{R}$ which is affine linear on every simplex of $\{\Delta\}_k$. If all the n-dimensional polytopes of the subdivision $\{\Delta_k\}_k$ are primitive, we say that the polyhedral subdivision is **unimodular**.

1.2.3 Example (The regular subdivision on $\boldsymbol{\Delta}$ associated to a function $\boldsymbol{\nu : \Delta \cap \mathbb{Z}^n \longrightarrow \mathbb{R}}$): Let Δ be a convex lattice polytope in $\mathbb{R}^n$ and let $\nu : \Delta \cap \mathbb{Z}^n \longrightarrow \mathbb{R}$ be a function. We denote by $\Delta(\nu)$ the convex hull of the graph of ν, i.e., $\Delta(\nu) := \mathrm{Conv}(\{(i, \nu(i)) \in \mathbb{R}^{n+1} \mid i \in \Delta \cap \mathbb{Z}^n\})$. Let $\{\Delta\}_k$ be the polyhedral subdivision of Δ induced by projecting the union of the lower faces of $\Delta(\nu)$ onto the first n coordinates; then $\{\Delta_k\}_k$ is a regular polyhedral subdivision of Δ.

1.3.– Glossary of tropical geometry

The source for the following material is [Gub11].

Definition: Let $(F, ||\cdot||)$ be a non-Archimedean field. The set $\Gamma := \log ||F^*||$ is a subgroup of $(\mathbb{R}, +)$ known as the **value group** of $(F, ||\cdot||)$. If $\Gamma = \{0\}$, we say that F is **trivially valued**, or that $||\cdot|| = ||\cdot||_0$ is the trivial absolute value.

Let $(F, ||\cdot||)$ be a non-Archimedean field and let X be a closed subscheme of the algebraic n-torus T_F^n. Suppose that $(F, ||\cdot||)$ is complete, then consider the set X^{an} of all multiplicative seminorms on the ring of regular functions $F[X]$ of X extending the absolute value $||\cdot||$, i.e., functions $\rho : F[X] \longrightarrow \mathbb{R}_{\geq 0}$ satisfying:

1. $\rho(fg) = \rho(f)\rho(g)$ and $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in F[X]$;

2. $\rho(1) = 1$ and $\rho(a) = ||a||$ for all $a \in F$.

In this case, the (non-Archimedean) **amoeba** $\mathcal{A}(X)$ of X is the set

$$\mathcal{A}(X) := \{(\log(\rho(x_1)), \ldots, \log(\rho(x_n))) \in \mathbb{R}^n \ : \ \rho \in X^{\mathrm{an}}\},$$

where $x_1 \ldots, x_n \in F[X]$ denote the image of the coordinate functions $x_i \in F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ under the isomorphism $F[X] \cong F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/I(X)$.

Consider now an arbitrary non-Archimedean field $(F, ||\cdot||)$ and let $(\hat{F}, ||\cdot||_{\hat{F}})$ be its completion with respect to its absolute value $||\cdot||$. Let X be a closed subscheme of T_F^n, then its base change $X_{\hat{F}}$ to $\hat{F}$ is a closed subscheme of the torus $T_{\hat{F}}^n$.

Definition: Let $(F, ||\cdot||)$ be a non-Archimedean field and let X be a closed subscheme of T_F^n. The **amoeba** of X is the set $\mathcal{A}(X) = \mathcal{A}(X_{\hat{F}})$.

The following important result describes one of the main combinatorial features of the set $\mathcal{A}(X)$.

Theorem 1.3.1 (Bieri-Groves). *Let $(F, ||\cdot||)$ be a non-Archimedean field with value group Γ. Then $\mathcal{A}(X)$ is a finite union of Γ-rational polyhedra in $\mathbb{R}^n$. If X is pure k-dimensional, then all these polyhedra may be chosen to be k-dimensional.*

Remark 1.3.2: *The amoeba $\mathcal{A}(X)$ of a closed subscheme $X \subset T_F^n$ is more than a finite union of Γ-rational polyhedra in $\mathbb{R}^n$. It turns out that $\mathcal{A}(X)$ can be endowed with the structure of a Γ-rational polyhedral complex in $\mathbb{R}^n$, i.e., there exists a Γ-rational polyhedral complex P such that $|P| = \mathcal{A}(X)$. See [Gub11], p.1.*

The next result gives a characterization of the amoeba $\mathcal{A}(X)$ of a closed subscheme $X \subset T_F^n$ in terms of the set of L-valued points $X(L)$ of X for a particular extension L of F. A proof of it can be consulted in [Gub11], Proposition 3.7.

Theorem 1.3.3 (Gubler). *Let $(L, ||\cdot||_L)$ be a valued extension of $(F, ||\cdot||)$ with L algebraically closed and $||\cdot||_L$ non-trivial. Then $\mathcal{A}(X)$ equals the closure of the set*

$$Log\ ||X(L)||_L = \{(\log ||p_1||_L, \ldots, \log ||p_n||_L) \in \mathbb{R}^n \ : \ (p_1, \ldots, p_n) \in X(L)\}, \tag{1.4}$$

in $\mathbb{R}^n$.

Remark 1.3.4: *Let $(F, ||\cdot||)$ be a non-Archimedean field. If F is algebraically closed and $||\cdot||$ is non-trivial, then $\mathcal{A}(X) = \overline{Log||X(F)||}$, so $\mathcal{A}(X)$ depends only on the set of closed points $X(F) \subset (F^*)^n$ of X. If we have in addition that $\Gamma = \mathbb{R}$, then $\mathcal{A}(X) = Log||X(F)||$.*

1.3.5 Example (Amoebas over a trivially valued field): Let $(F, ||\cdot||_0)$ with F algebraically closed of characteristic zero. Let $(L, ||\cdot||_L)$ be the **field of Puiseux series** with coefficients in F endowed with the **order valuation**:

$$L = \bigcup_{n \geq 1} F((t^{1/n})), \quad \log|| \sum_{i \geq i_0} a_i t^i ||_L = -\mathrm{ord}(\sum_{i \geq i_0} a_i t^i) = -i_0.$$

Then $(L, ||\cdot||_L)$ is a valued extension of $(F, ||\cdot||)$ with L algebraically closed and $||\cdot||_L$ non-trivial (details might be consulted in [Poo93]). In this case, if X is a closed subscheme of T_F^n, then $\mathcal{A}(X) = \overline{\mathrm{Log}||X(L)||}$ can be endowed with the structure of a rational polyhedral fan in $\mathbb{R}^n$ by Remark 1.3.2. This approach has been used, for example, in [ST07], to study problems in elimination theory for subvarieties of $(F^*)^n$. *

1.3.1 A note about non-Archimedean base fields

Although tropical geometry can be worked out over arbitrary non-Archimedean fields, here we introduce a particular type of fields which will facilitate our work. We refer the reader to [Poo93] for more information on this subject.

Definition: Let F be a field and let Γ be an ordered abelian group. The **Mal'cev-Neumann field** $F((t^\Gamma))$ is defined as the set of formal sums $\alpha = \sum_{i \in I} a_i t^i$, where $I \subset \Gamma$ is a well-ordered subset of Γ and $a_i \in F^*$.

The set $F((t^\Gamma))$ can be endowed with natural operations of addition and multiplication so that it becomes a field of the same characteristic as F.

For $\alpha \in F((t^\Gamma))$ as above, we define $\mathrm{ord}(\alpha) = \min(I)$ if $\alpha \neq 0$ and $\mathrm{ord}(0) = +\infty$. Then $\mathrm{ord} : F \longrightarrow \Gamma \cup \{+\infty\}$ is a valuation with value group Γ. The **valuation ring** of $(F((t^\Gamma)), \mathrm{ord})$ is $R = \{\alpha \in F((t^\Gamma)) \mid \mathrm{ord}(\alpha) \geq 0\}$, which is a local ring with maximal ideal $\mathfrak{m} = \{\alpha \in F((t^\Gamma)) \mid \mathrm{ord}(\alpha) > 0\}$. The **residue field** of $F((t^\Gamma))$ is $R/\mathfrak{m}$. We will use the following result (see [Poo93], Proposition 6 on p. 94).

Proposition 1.3.6 (Poonen). *If a Mal'cev-Neumann field $F((t^\Gamma))$ has divisible value group Γ and algebraically closed residue field $R/\mathfrak{m}$, then it is algebraically closed.*

When $\Gamma \subseteq \mathbb{R}$, the function $||\cdot|| : F((t^\Gamma)) \longrightarrow \mathbb{R}_{\geq 0}$ given by $||\alpha|| := e^{-\mathrm{ord}(\alpha)}$ defines a non-Archimedean absolute value on $F((t^\Gamma))$. In this case we have that the residue field $R/\mathfrak{m}$ is isomorphic to F, which is itself contained in $F((t^\Gamma))$ as the image of the map $a \mapsto at^0$. Observe that the function $||\cdot|| : F((t^\Gamma)) \longrightarrow \mathbb{R}_{\geq 0}$ restricts to the trivial absolute value on this copy of F, thus $(F((t^\Gamma)), ||\cdot||)$ is a valued extension of $(F, ||\cdot||_0)$. We summarize the properties of fields of type $\mathbb{K} = (F((t^\Gamma)), ||\cdot||)$ in the following Theorem.

Theorem 1.3.7 (Poonen). *If F is an algebraically closed field of characteristic zero and $\Gamma \subseteq \mathbb{R}$ is a divisible subgroup, then $\mathbb{K} = (F((t^\Gamma)), ||\cdot||)$ is a complete, non-Archimedean, algebraically closed field of characteristic zero extending $(F, ||\cdot||_0)$.*

1.3.8 Example (Field of generalized Puiseux series): Let F be an algebraically closed of characteristic zero, then we have basically three choices for a divisible abelian group $\Gamma \subset \mathbb{R}$, namely $\{0\}, \mathbb{Q}$ or $\mathbb{R}$. When $F = \mathbb{C}$ and $\Gamma = \mathbb{R}$, we can construct the field of generalized Puiseux series which are locally convergent near zero

$$\mathbb{C}\{t^{\mathbb{R}}\} = \{\alpha = \sum_{i \in I} a_i t^i \in \mathbb{C}((t^{\mathbb{R}})) \ : \ \alpha(\varepsilon) = \sum_{i \in I} a_i \varepsilon^i \text{ is convergent for } \varepsilon > 0 \text{ small enough}\}.$$

The field $\mathbb{C}\{t^{\mathbb{R}}\}$ is also algebraically closed and of characteristic zero. A scheme X over $\mathbb{C}\{t^{\mathbb{R}}\}$ can be interpreted as a 1-parametric family $\{X_\varepsilon\}_{\varepsilon > 0}$ of complex schemes X_ε. $*$

Remark 1.3.9: *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Unless otherwise stated, if we assume that $\mathbb{K}$ is non-Archimedean, then $\mathbb{K}$ will be a Mal'cev-Neumann field $\mathbb{K} = (F((t^{\mathbb{R}})), ||\cdot||)$, where F is an algebraically closed field of characteristic zero.*

Let $\mathbb{K} = F((t^{\mathbb{R}}))$ and consider $\alpha \in \mathbb{K}^*$. We set $\tilde{\alpha} = \alpha / t^{\mathrm{ord}(\alpha)}$, then $\tilde{\alpha} \in R^*$ and we can write $\tilde{\alpha} = a_0 t^0 + \alpha'$ with $a_0 \in F^*$ and $\alpha' \in \mathfrak{m}$. We deduce then an unique expression $t^{\mathrm{ord}(a)}(a_0 + \alpha')$ for α, and the assignment $\alpha \mapsto a_0$ gives us a function $\mathrm{ic} : \mathbb{K}^* \longrightarrow F^*$ which is the **initial coefficient** function.

Definition: Let $\mathbb{K} = F((t^{\mathbb{R}}))$. For $\alpha \in \mathbb{K}$ we define $\mathrm{val}(\alpha) \in \mathbb{R} \cup \{-\infty\}$ as $\mathrm{val}(\alpha) := \log ||\alpha|| = -\mathrm{ord}(\alpha)$, and we denote as $\mathrm{Val} : \mathbb{K}^n \longrightarrow (\mathbb{R} \cup \{-\infty\})^n$ the function $(\alpha_1, \ldots, \alpha_n) \mapsto (\mathrm{val}(\alpha_1), \ldots, \mathrm{val}(\alpha_n))$.

In this case the function Val has a section $\mathbb{R}^n \longrightarrow (\mathbb{K}^*)^n$ given by $r = (r_1, \ldots, r_n) \mapsto t^{-r} = (t^{-r_1}, \ldots, t^{-r_n})$, and the fiber $\mathrm{Val}^{-1}(r)$ over $r \in \mathbb{R}^n$ is the translated torus $t^{-r} \cdot \{||\alpha|| = 1\}^n$.

If X is a closed subscheme of $(\mathbb{K}^*)^n$, then it follows from Remark 1.3.4 that its amoeba $\mathcal{A}(X)$ coincides with $\mathrm{Val}(X) = \mathrm{Val}(X(\mathbb{K}))$.

1.3.2 Tropicalization of a closed subscheme of $(\mathbb{K}^*)^n$

Let $\mathbb{K} = F((t^{\mathbb{R}}))$ and let X be a closed subscheme of $(\mathbb{K}^*)^n$. We want to define a tropical multiplicity function $m_X : \mathbb{R}^n \longrightarrow \mathbb{Z}_{\geq 0}$ associated to X. To do so we define the initial degeneration $\mathrm{in}_p(I)$ of an ideal $I \subset \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ at a point $p \in \mathbb{R}^n$, which will be an ideal $F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition: Let $f = \sum_{i \in A} \alpha_i x^i$ be a polynomial in $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $p \in \mathbb{R}^n$.

1. The **tropicalization** $\mathrm{Trop}(f)$ of f is the function $\mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $p \mapsto \max_{i \in A}\{\mathrm{val}(\alpha_i) + \langle p, i \rangle\}$.
2. The **initial polynomial** $\mathrm{in}_p(f)$ of f at p is the polynomial in $F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by

$$\mathrm{in}_p(f) = \sum_{\substack{i \in A \\ \mathrm{val}(\alpha_i) + \langle i, p \rangle = \mathrm{Trop}(f)(p)}} \mathrm{ic}(\alpha_i) x^i,$$

where $\mathrm{ic}(\alpha)$ is the initial coefficient of $\alpha \in \mathbb{K}^*$.

Definition: Let $X = V(I)$ be the closed subscheme of $(\mathbb{K}^*)^n$ defined by the ideal $I \subset \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

1. The **initial ideal** $\mathrm{in}_p(I)$ of I at p is the ideal $\langle \mathrm{in}_p(f) \mid f \in I \rangle \subset F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.
2. The **initial degeneration** $\mathrm{in}_p(X)$ of X at p is the closed subscheme $\mathrm{in}_p(X) := V(\mathrm{in}_p(I))$ of $(F^*)^n$.

Definition: Let X be a closed subscheme of $(\mathbb{K}^*)^n$. The **tropical multiplicity** $m_X(p)$ at $p \in \mathbb{R}^n$ is the sum of the geometric multiplicities of the irreducible components of $\mathrm{in}_p(X)$.

1.3.10 Example (Tropical multiplicity function of a closed point): Let $x = (\alpha_1, \dots, \alpha_n) \in (\mathbb{K}^*)^n$ with $\alpha_i = t^{-b_i}(a_{i,0} + \alpha_i')$ for $i = 1, \dots, n$. Let $f_i = x_i - \alpha_i$ for $i = 1, \dots, n$ and set $I = \langle f_1, \dots, f_n \rangle$, then since I is generated by linear forms, it follows from Theorem 2.6 of [TRGS05] that $\mathrm{in}_p(I) = \langle \mathrm{in}_p(f_1), \dots, \mathrm{in}_p(f_n) \rangle$ for all $p \in \mathbb{R}^n$.

Let $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Observe that $\mathrm{Trop}(f_i)(p) = \max\{p_i, b_i\}$ for $i = 1, \dots, n$, so we have that

$$\mathrm{in}_p(I) = \begin{cases} \langle x_1 - a_{1,0}, \dots, x_n - a_{n,0} \rangle, & \text{if } p = \mathrm{Val}(x) = (b_1, \dots, b_n), \\ \langle 1 \rangle, & \text{otherwise.} \end{cases}$$

It follows that if $X = V(I)$ is a (reduced) point x in $(\mathbb{K}^*)^n$ with $\mathrm{Val}(x) = b$, then $m_X(p) = 1$ if $p = b$, and $m_X(p) = 0$ otherwise. $*$

The following is a list of the main properties of the tropical multiplicity function $m_X : \mathbb{R}^n \longrightarrow \mathbb{Z}_{\geq 0}$. See [Gub11], Section 12 for the corresponding proofs.

Proposition 1.3.11. *Let X be a closed subscheme of $(\mathbb{K}^*)^n$ and let $m_X : \mathbb{R}^n \longrightarrow \mathbb{Z}_{\geq 0}$ be the tropical multiplicity function associated to X. Then*

1. *the function m_X is supported on the amoeba $\mathrm{Val}(X)$ of X;*
2. *if $[X] = \sum n_i [X_i]$ is the fundamental cycle of X, then for any regular point $p \in \mathrm{Val}(X)$ we have that $m_X(p) = \sum_i n_i m_{X_i}(p)$;*
3. *the restriction of m_X to the set of regular points of $\mathrm{Val}(X)$ is locally constant (by Remark 1.3.2, we can talk about regular points of the set $\mathrm{Val}(X)$).*

Definition: Let X be a closed subscheme of $(\mathbb{K}^*)^n$. The **tropicalization** $\mathrm{Trop}(X)$ of X is the pair $(\mathrm{Val}(X), m_X)$.

Let X be a k-dimensional subvariety of $(\mathbb{K}^*)^n$ and let $p \in \mathrm{Val}(X)$ be a regular point. We close this part with an alternative description of the value $m_X(p)$ found in [BL12].

Let $\Lambda_p = L_p \cap \mathbb{Z}^n$, where $L_p + p$ is the affine linear space containing a polytopal neighborhood of p. Let $\{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a basis for $\mathbb{Z}^n$ such that $\{v_1, \dots, v_k\}$ is a basis for Λ_p. Let $B = [v_{k+1}, \dots, v_n]$ be the matrix whose columns are the vectors $\{v_{k+1}, \dots, v_n\}$. This matrix induces a closed embedding $\Phi_B : (\mathbb{K}^*)^{n-k} \longrightarrow (\mathbb{K}^*)^n$, and we let X' be the translation by t^p of $\Phi_B((\mathbb{K}^*)^{n-k})$, i.e., $X' = t^p \cdot \Phi_B((\mathbb{K}^*)^{n-k})$.

It can be shown (see Proposition 2.7.3 and Theorem 4.4.5 in [OP11]) that X and X' meet properly at every point $x \in X$ with $\mathrm{Val}(x) = p$. We have the following relation

$$\sum_{\mathrm{Val}(x)=p} i(x, X \cdot X'; (\mathbb{K}^*)^n) = m_X(p). \tag{1.5}$$

Recall that $i(x, X \cdot X'; (\mathbb{K}^*)^n)$ stands for the intersection multiplicity of X and X' in $(\mathbb{K}^*)^n$ at x.

1.3.3 Tropical cycles in $\mathbb{R}^n$

Definition: A **tropical k-cycle** in $\mathbb{R}^n$ is a pair $A = (A, w)$ consisting of a rational polyhedral complex A of pure dimension k and the assignment of a weight $w(F) \in \mathbb{Z}$ for each facet $F \in A$, such that the equation

$$\sum_{E \subset \overline{F}} w(F) s(F, E) = 0$$

holds for every face $E \subset F$ of codimension one. Here $s(F, E)$ is the primitive integral vector orthogonal to E and generating F.

The set $Z_k(\mathbb{R}^n)$ of tropical k-cycles in $\mathbb{R}^n$ can be endowed with the structure of (additive) abelian group as well as $Z_*(\mathbb{R}^n) := \bigoplus_{k=0}^n Z_k(\mathbb{R}^n)$, which is then the **group of tropical cycles** in $\mathbb{R}^n$. A tropical cycle (A, w) is said to be **effective** if $w(F) > 0$ for any maximal face $F \in A$.

Definition: A **tropical polynomial** in $\mathbb{R}^n$ is a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form $p \mapsto \max_{i\in A}\{a_i + \langle p, i\rangle\}$, where $\emptyset \neq A \subset \mathbb{Z}^n$ is finite and $a_i \in \mathbb{R}$. The Newton polytope $\mathrm{New}(\phi)$ of ϕ is defined to be $\mathrm{Conv}(A)$.

1.3.12 Example (The tropical cycle of a tropical polynomial): Let $\phi(p) = \max_{i\in A}\{a_i + \langle p, i\rangle\}$ be a tropical polynomial in $\mathbb{R}^n$. We define a tropical $(n-1)$-cycle $\mathrm{div}_{\mathbb{R}^n}(\phi) = (S, w_S)$ as follows. Set

$$S = \{p \in \mathbb{R}^n \ : \ \exists\, i \neq j \in A \text{ such that } \phi(p) = a_i + \langle p, i\rangle = a_j + \langle p, j\rangle\}.$$

The function $\nu : A \longrightarrow \mathbb{R}$ given by $\nu(i) = -a_i$ induces a regular convex polyhedral subdivision $\{\Delta_k\}_k$ on $\mathrm{New}(\phi)$. We now define a structure of rational polyhedral complex on S as follows: for any $\Delta \in \{\Delta_k\}_k$, we denote by $\Delta^\vee$ the closure in S of the set

$$\{p \in S \ : \ \phi(p) = a_i + \langle p, i\rangle \ \text{ for all } i \in \Theta\}.$$

We have that $\Delta^\vee$ is a polyhedron in $\mathbb{R}^n$ that satisfies $\Delta^\vee = \emptyset$ if $\dim(\Delta) = 0$, and $\dim(\Delta)+\dim(\Delta^\vee) = n$ if $\dim(\Delta) > 0$. This polyhedral structure on S is said to be **dual** to the polyhedral subdivision $\{\Delta_k\}_k$ of $\mathrm{New}(\phi)$.

In particular, if $\dim(\Delta^\vee) = n-1$, then $\dim(\Delta) = 1$, so there exists $i, j \in A$ such that $\Delta = \mathrm{Conv}\{i, j\}$. If we set $w(\Delta^\vee) = \gcd|i - j|$, then $\mathrm{div}_{\mathbb{R}^n}(\phi) = (S, w_S)$ is a tropical $(n-1)$-cycle.

We will say that the convex polyhedral subdivision $\{\Delta_k\}_k$ of $\mathrm{New}(\phi)$ is the **combinatorial type** of the tropical $(n-1)$-cycle $\mathrm{div}_{\mathbb{R}^n}(\phi)$. $*$

Let $\mathbb{K} = F((t^{\mathbb{R}}))$ and let $Y \subset (\mathbb{K}^*)^n$ be a k-dimensional subvariety, then according to Remark 1.3.2, the amoeba $\mathrm{Val}(Y)$ of Y can be endowed with the structure of a rational polyhedral complex of pure dimension k in $\mathbb{R}^n$. Let us endow $\mathrm{Val}(Y)$ with such a structure and let $F \subset \mathrm{Val}(Y)$ be a facet. If we define $w(F) = m_Y(p)$ for $p \in F$ a regular point, then $\mathrm{Trop}(Y) = (\mathrm{Val}(Y), w)$ becomes a tropical k-cycle in $\mathbb{R}^n$. See [Gub11], Theorem 12.11.

On the other hand, let $A = (A, w)$ be a tropical cycle in $\mathbb{R}^n$ and let $U \subset |A|$ be the set of regular points of the support of A. For any $p \in U$ there exists a maximal face $F \in A$ such that $p \in F$, so we can define a locally constant function $m_A : U \longrightarrow \mathbb{Z}$ by setting $m_A(p) = w(F)$.

Remark 1.3.13: *In what follows and depending on the convenience of the situation, if A is a tropical cycle in $\mathbb{R}^n$ we will consider it either as a pair $A = (A, m)$ of a set $A \subset \mathbb{R}^n$ and a function m defined on the set of regular points of A, or as a pair $A = (A, w)$ of a rational polyhedral complex A and a weight function w defined on the maximal faces of A.*

Let $Y \subset (\mathbb{K}^*)^n$ be a subvariety. We define the group homomorphism $\mathrm{Trop} : Z_*((\mathbb{K}^*)^n) \longrightarrow Z_*(\mathbb{R}^n)$ by extending the assignment $[Y] \mapsto \mathrm{Trop}(Y)$ by linearity. If $X \subset (\mathbb{K}^*)^n$ is any closed subscheme with tropicalization $\mathrm{Trop}(X)$ and fundamental cycle $[X] = \sum_i n_i[X_i]$, then we have the linearity formula (see [Gub11], p.39) :

$$\mathrm{Trop}(X) = \mathrm{Trop}([X]) = \sum n_i \mathrm{Trop}(X_i). \tag{1.6}$$

We have the following important result for the tropicalization of principal effective Cartier divisors in $(\mathbb{K}^*)^n$.

Theorem 1.3.14 (Kapranov's Theorem). *For any $f \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we have that*

$$\mathit{Trop}(V(f)) = \mathit{div}_{\mathbb{R}^n}(\mathit{Trop}(f)). \tag{1.7}$$

Definition: Let $\alpha \in Z_*((\mathbb{K}^*)^n)$ be an effective cycle. We say that a regular point $p \in \mathrm{Val}(\alpha)$ is **simple** if $m_\alpha(p) = 1$. We say that α has **simple tropicalization** if every regular point of $\mathrm{Val}(\alpha)$ is simple.

Suppose that $\alpha = \sum_i n_i[Y_i]$ for some $n_i \geq 0$, then we have that $m_\alpha(p) = \sum_i n_i m_{Y_i}(p)$ for every $p \in \mathrm{Val}(\alpha)$, and in order for $m_\alpha(p) = 1$ to be true, p has to be a regular point of a single $\mathrm{Val}(Y_i)$, and then $\mathrm{in}_p(Y_i) \subset (F^*)^n$ must be a subvariety.

The last relevant aspect to be addressed here is the generalized Sturmfels-Tevelev formula for homomorphisms of $\mathbb{K}$-tori, which was first described in [ST07] for the case of trivial valuation. If $\Phi : (\mathbb{K}^*)^n \longrightarrow$

$(\mathbb{K}^*)^m$ is a homomorphism of $\mathbb{K}$-tori, then it induces a homomorphism $\Phi_* : Z_*((\mathbb{K}^*)^n) \longrightarrow Z_*((\mathbb{K}^*)^m)$ as follows: let $[Y]$ be a prime cycle in $(\mathbb{K}^*)^n$ and let Y' be the closure of $\Phi(Y)$ in $(\mathbb{K}^*)^m$. We define

$$\Phi_*([Y]) = \begin{cases} [\mathbb{K}(Y) : \mathbb{K}(Y')][Y'], & \text{if } [\mathbb{K}(Y) : \mathbb{K}(Y')] < +\infty; \\ 0, & \text{if } [\mathbb{K}(Y) : \mathbb{K}(Y')] = +\infty. \end{cases} \tag{1.8}$$

This extends to a homomorphism $\Phi_* : Z_*((\mathbb{K}^*)^n) \longrightarrow Z_*((\mathbb{K}^*)^m)$ by linearity[2].

If $(\mathbb{K}^*)^n$ has coordinates $(x_1, \ldots, x_n)$ and $(\mathbb{K}^*)^m$ has coordinates $(y_1, \ldots, y_m)$, let Φ be induced by the monomial assignment $y_i \mapsto x_1^{a_{i1}} \cdots x_n^{a_{in}}$, $i = 1, \ldots, m$. We denote by $\mathrm{Trop}(\Phi) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the linear function induced by the matrix $(a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ in $\mathbb{Z}_{m \times n}$.

Definition: Let $\Phi : (\mathbb{K}^*)^n \longrightarrow (\mathbb{K}^*)^m$ and $\mathrm{Trop}(\Phi) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be as above. If $\alpha \in Z_*((\mathbb{K}^*)^n)$, then the **tropical push-forward** $(\mathrm{Trop}(\Phi))_*(\mathrm{Trop}(\alpha))$ of $\mathrm{Trop}(\alpha)$ is the tropical cycle $\mathrm{Trop}(\Phi_*(\alpha))$.

The above formula describes the assignment $(\mathrm{Trop}(\Phi))_*(\mathrm{Val}(\alpha), m_\alpha) = (\mathrm{Trop}(\Phi)(\mathrm{Val}(\alpha)), m_{\Phi_*(\alpha)})$. It follows that the function $(\mathrm{Trop}(\Phi))_* : Z_*(\mathbb{R}^n) \longrightarrow Z_*(\mathbb{R}^m)$ is a homomorphism, since the following diagram is commutative:

$$\begin{tikzcd} Z_*((\mathbb{K}^*)^n) \arrow[r, "\Phi_*"] \arrow[d, "\mathrm{Trop}"] & Z_*((\mathbb{K}^*)^m) \arrow[d, "\mathrm{Trop}"'] \\ Z_*(\mathbb{R}^n) \arrow[r, "(\mathrm{Trop}(\Phi))_*"] & Z_*(\mathbb{R}^m) \end{tikzcd}$$

The generalized Sturmfels-Tevelev formula (1.9) describes generically the tropical multiplicity function $m_{\Phi_*(\alpha)}$ associated to the cycle $\Phi_*(\alpha)$ in terms of the function m_α. See Theorem 12.17 in [Gub11].

Theorem 1.3.15 (Sturmfels-Tevelev, Baker-Payne-Rabinoff). *Let $\Phi : (\mathbb{K}^*)^n \longrightarrow (\mathbb{K}^*)^m$, $\mathrm{Trop}(\Phi) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and α as above. Let $p \in \mathrm{Trop}(\Phi)(\mathrm{Val}(\alpha))$ be a regular point, then we have*

$$m_{\Phi_*(\alpha)}(p) = \sum_{q \in \mathrm{Trop}(\Phi)^{-1}(p)} m_\alpha(q)[\Lambda_p : \mathrm{Trop}(\Phi)(\Lambda_q)], \tag{1.9}$$

whenever $\mathrm{Trop}(\Phi)^{-1}(p) \subset \mathrm{Val}(\alpha)$ is finite and consists only of regular points[3].

We can use Equation (1.9) to define the push-forward of tropical cycles defined by a linear function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ induced by a matrix $(a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ in $\mathbb{Z}_{m \times n}$.

Definition: Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a $\mathbb{Z}$-linear function and let $A = (A, w)$ be a tropical k-cycle such that $\phi(A)$ has dimension k in $\mathbb{R}^m$. We define the **tropical push-forward** $\phi_*(A)$ in $\mathbb{R}^m$ by $\phi_*(A) = (\phi(A), m_{\phi_*(A)})$, where

$$m_{\phi_*(A)}(p) = \sum_{q \in \phi^{-1}(p)} m_A(q)[\Lambda_p : \phi(\Lambda_q)].$$

1.3.4 Tropical intersection theory and tropical modifications

We start by reviewing the tropical intersection of two pure dimensional tropical cycles in $\mathbb{R}^n$.

Definition: Let $A = (A, w_A) \in Z_{\ell_1}(\mathbb{R}^n)$ and $B = (B, w_B) \in Z_{\ell_2}(\mathbb{R}^n)$ be tropical cycles in $\mathbb{R}^n$. We denote by $A.B = (A.B, w_{A.B})$ their **stable intersection**, where $A.B$ is the set of all faces of dimension less than or equal to $\ell_1 + \ell_2 - n$ of the polyhedral complex $A \cap B$, and for any facet $F \subset A.B$, we define $w_{A.B}(F)$ by:

1. $w_A(D)w_B(E)[\mathbb{Z}^n : \Lambda_E + \Lambda_D]$, if F is the transverse intersection of the facets $D \subset A$ and $E \subset B$;
2. otherwise, for a generic vector $v \in \mathbb{R}^n$ with non-rational coordinates and $\varepsilon > 0$, in a neighborhood of the facet F, the cycles $A_\varepsilon = A + \varepsilon v$ and B will meet in a finite number of facets $F_1, \ldots, F_s$ parallel to F, such that each F_i is the transverse intersection the facets $D_i \subset A_\varepsilon$ and $E_i \subset B$. We set then $\sum_{i=1}^s w_{A_\varepsilon.B}(F_i)$.

[2] The notation $[F : K]$ in (1.8) denotes the degree of the field extension F/K.
[3] The notation $[G : H]$ in (1.9) denotes the index of a subgroup $H \subset G$ in the group G.

The pair $A.B = (A.B, w_{A.B})$ is a well-defined $(\ell_1 + \ell_2 - n)$-tropical cycle in $\mathbb{R}^n$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial and let $A = (A, w)$ be a tropical k-cycle in $\mathbb{R}^n$. Then $A.\mathrm{div}_{\mathbb{R}^n}(f)$ is a tropical $(k-1)$-cycle in $\mathbb{R}^n$.

Consider the function $\phi = f|_A$. If we denote by $Z_\ell(A)$ the group of tropical ℓ-cycles which are contained in A, then we want to associate to the function ϕ an element $\mathrm{div}_A(\phi) \in Z_{k-1}(A)$.

We will describe the construction of $\mathrm{div}_A(\phi)$ via tropical modifications. This approach can be generalized to functions $\phi : A \longrightarrow \mathbb{R}$ which do not necessarily arise as the restriction to A of a tropical polynomial.

We denote by $\mathbb{T}$ the set $\mathbb{R} \cup \{-\infty\}$, by $H^\circ_\emptyset$ the set $\mathbb{T} \setminus \{-\infty\} = \mathbb{R}$ and by $H^\circ_{[1]}$ the set $\{-\infty\}$. Then for $\emptyset \subseteq J \subset [1]$ we have an inclusion of sets $\mathbb{R}^n \times H^\circ_J \hookrightarrow \mathbb{R}^n \times \mathbb{T}$.

Let A and ϕ be as above. Our aim is to define diagrams of sets:

$$\begin{tikzcd} A_\emptyset(\phi) \arrow[r, hook, "\alpha_\emptyset"] \arrow[d, "\delta_\emptyset"'] & \mathbb{R}^n \times H^\circ_\emptyset \arrow[d, hook] \\ A \arrow[r, hook] & \mathbb{R}^n \times \mathbb{T} \end{tikzcd} \qquad \begin{tikzcd} A_{[1]}(\phi) \arrow[r, hook, "\alpha_{[1]}"] \arrow[d, "\delta_{[1]}"'] & \mathbb{R}^n \times H^\circ_{[1]} \arrow[d, hook] \\ A \arrow[r, hook] & \mathbb{R}^n \times \mathbb{T} \end{tikzcd} \tag{1.10}$$

such that for $\emptyset \subseteq J \subseteq [1]$, $A_J(\phi)$ is a tropical cycle in $\mathbb{R}^n \times H^\circ_J$ and the map α_J is an inclusion. This makes sense since $\mathbb{R}^n \times H^\circ_J$ is isomorphic to $\mathbb{R}^{n+1-\#J}$.

We will start by constructing the cycle $A_\emptyset(\phi)$. The graph of ϕ induces an inclusion of sets $\Gamma_\phi : A \hookrightarrow \mathbb{R}^n \times \mathbb{T}$. Now let $\delta_\emptyset$ be the projection from $\Gamma_\phi(A)$ to A and let $F' \subset A_\emptyset(\phi)$ be a facet projecting onto a facet $F \subset A$, if we set $w_{A_\emptyset(\phi)}(F') = w_A(F)$, then $(\Gamma_\phi(A), w_{A_\emptyset(\phi)})$ is a weighted rational polyhedral complex which is not balanced in codimension one.

The set $A_\emptyset(\phi)$ is constructed by adding to $\Gamma_\phi(A)$ its **undergraph** $U_\phi(A)$ along the set of points $p \in A$ in which ϕ is not locally linear:

$$U_\phi(A) = \{(p, q) \in A \times \mathbb{R} \mid \phi \text{ is not locally linear at } p \text{ and } q \leq \phi(p)\}.$$

If $F' \subset A_\emptyset(\phi)$ is a facet contained in $U_\phi(A)$, then there exists a unique weight $w_{A_\emptyset(\phi)}(F')$ such that $A_\emptyset(\phi)$ is a balanced polyhedral complex in $\mathbb{R}^n \times H^\circ_\emptyset$.

The underlying set of the cycle $A_{[1]}(\phi)$ is the intersection of the closure of the set $A_\emptyset(\phi)$ in $\mathbb{R}^n \times \mathbb{T}$ with the set $\mathbb{R}^n \times H^\circ_{[1]}$. If $F' \subset A_{[1]}(\phi)$ is a facet, then there exists a facet $F \subset A_\emptyset(\phi)$ contained in the undergraph of $\Gamma_\phi(A)$ such that $F' = \overline{F} \cap (\mathbb{R}^n \times H^\circ_\emptyset)$. We set $w_{A_{[1]}(\phi)}(F') = w_{A_\emptyset(\phi)}(F)$.

Consider the projection $\pi : \mathbb{R}^n \times H^\circ_J \longrightarrow \mathbb{R}^n$ for $\emptyset \subseteq J \subset [1]$. Since $\mathbb{R}^n \times H^\circ_J$ is isomorphic to $\mathbb{R}^{n+1-\#J}$ and α_J is an inclusion that satisfies $\pi \circ \alpha_J = \delta_J$, we can define

$$(\delta_J)_*(A_J(\phi)) = \pi_*(A_J(\phi)), \text{ for } \emptyset \subseteq J \subset [1].$$

Definition: Let f be a tropical polynomial on $\mathbb{R}^n$, A an effective tropical k-cycle in $\mathbb{R}^n$ and consider the function $\phi := f|_A$. We call the function $\delta_\emptyset : A_\emptyset(\phi) \longrightarrow A$ the (principal) **tropical modification** of A along ϕ. We call $(\delta_{[1]})_*(A_{[1]}(\phi))$ the **Weil divisor** of ϕ on A, which will be denoted by $\mathrm{div}_A(\phi)$.

Definition: Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be tropical polynomials. We say that the function $h(p) = f(p) - g(p)$ is a **tropical rational function**; it will be denoted by $h =$ " $\frac{f}{g}$ ".

If A is an effective tropical k-cycle in $\mathbb{R}^n$ and $h =$ " $\frac{f_1}{f_2}$ " is a tropical rational function in $\mathbb{R}^n$, we construct a new function $\phi : A \longrightarrow \mathbb{R}$ as follows. Let $\phi_1, \phi_2 : A \longrightarrow \mathbb{R}$ be defined as $\phi_i = f_i|_A$, then we set $\phi(p) = \phi_1(p) - \phi_2(p) = h|_A(p)$.

Definition: Let A, $h =$ " $\frac{f_1}{f_2}$ " and $\phi = h|_A$ be as above. The **Weil divisor** of ϕ on A is $\mathrm{div}_A(\phi) := \mathrm{div}_A(\phi_1) - \mathrm{div}_A(\phi_2)$.

We have the following important result. See [AR09], [Sha13].

Proposition 1.3.16. *Let A be an effective tropical k-cycle in $\mathbb{R}^n$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ a tropical rational function and $\phi = h|_A$.Then we have that $A.\mathrm{div}_{\mathbb{R}^n}(h) = \mathrm{div}_A(\phi)$.*

Remark 1.3.17: *Let A, h and $\phi = h|_A$ be as in Proposition 1.3.16. If the Weil divisor $\mathrm{div}_A(\phi)$ of ϕ on A is effective, then a principal tropical modification $\delta_\emptyset : A_\emptyset(\phi) \longrightarrow A$ of A along ϕ can be constructed as we just did when ϕ was the restriction to A of a tropical polynomial, this is, by balancing the union of the graph $\Gamma_\phi(A)$ and the undergraph $U_\phi(A)$ of ϕ over A. See [Sha13], p.8.*

1.3.5 Local tropical intersection theory

Sources for the following material are [AR09] and [Sha13].

Definition: We say that a tropical k-cycle $A = (A, w)$ in $\mathbb{R}^n$ is a (tropical) **fan k-cycle** if A is a rational polyhedral fan in $\mathbb{R}^n$.

1.3.18 Example (Tropicalization with trivial valuation): Recall that if F is an algebraically closed field of characteristic zero endowed with the trivial absolute value $||\cdot||_0$, then $\mathbb{K} = F((t^{\mathbb{R}}))$ is a non-Archimedean extension of $(F, ||\cdot||_0)$. If $X \subset (F^*)^n$ is a k-dimensional subvariety, then $\mathrm{Trop}(X)$ is a tropical fan k-cycle in $\mathbb{R}^n$ supported on $\mathrm{Val}(X)$. $*$

Definition: Let $A = (A, w)$ be a tropical k-cycle and let $U \subset |A|$ be an open neighborhood of a point $p \in |A|$. We say that U is a **fan neighborhood** of p if there exists a rational polyhedral fan V such that $U - p \subset |V|$ is an open neighborhood of 0 in $|V|$.

In order to define smoothness on tropical cycles, we need to introduce a particular type of tropical fan k-cycles, known as matroidal fans. First we will recall a procedure described in [Sha13] that assigns to a loop-less matroid $M = (\{0, 1, \ldots, n\}, \Lambda(M))$ over the set $\{0, 1, \ldots, n\}$, with lattice of flats $\Lambda(M)$ and rank $k + 1 > 1$, the support of a rational polyhedral fan $\Sigma(M)$ of pure dimension k in $\mathbb{R}^n$.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbb{R}^n$ and set $v_i = -e_i$ for $i = 1, \ldots, n$ and $v_0 = -\sum_{i=1}^n v_i$, so that $\sum_{i=0}^n v_i = 0$. Let $M = (\{0, 1, \ldots, n\}, \Lambda(M))$ be a loop-less matroid over the set $\{0, 1, \ldots, n\}$ with lattice of flats $\Lambda(M)$. For any chain $\emptyset \neq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d \neq [n]$ in $\Lambda(M)$, consider the cone $\mathbb{R}_{\geq 0} v_{F_1} + \cdots + \mathbb{R}_{\geq 0} v_{F_d}$ inside $\mathbb{R}^n$, where $v_{F_j} := \sum_{i \in F_j} v_i$.

Let $\Sigma(M)$ be the union of all such cones in $\mathbb{R}^n$. If the matroid $M = (\{0, 1, \ldots, n\}, \Lambda(M))$ has rank $k+1 > 1$, then $\Sigma(M)$ is the support of a rational polyhedral fan of pure-dimension k in $\mathbb{R}^n$. Furthermore, the set $\Sigma(M)$ can be turned into a tropical k-cycle $(\Sigma(M), w)$ if we endow it with the constant weight function $w \equiv 1$.

Definition: The **matroidal fan** $\Sigma(M)$ associated to $M = (\{0, 1, \ldots, n\}, \Lambda(M))$ is the simple tropical cycle $(\Sigma(M), 1)$.

1.3.19 Example: Let $I \subset F[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be an ideal generated by linear forms and set $X = V(I)$. Then the tropical cycle $\mathrm{Trop}(X)$ is a matroidal fan.

Definition: Let $A = (A, w)$ be a tropical cycle in $\mathbb{R}^n$. We say that A is **smooth** at $p \in |A|$ if for some fan neighborhood $U \subset |A|$ of p, we have that:

1. every regular point $q \in U$ is simple,
2. there exists an element $B \in \mathrm{GL}_n(\mathbb{Z})$ and a matroidal fan $V \subset \mathbb{R}^n$ such that $B(U - p) \subset V$ is an open neighborhood of 0 in V.

If A is smooth at every point, we say that it is **smooth**.

1.3.20 Example (Smooth tropical hypersurfaces in $\mathbb{R}^n$): Let $\phi(p) = \max_{i \in A}\{a_i + \langle i, p\rangle\}$ be a tropical polynomial in $\mathbb{R}^n$ and let $\{\Delta_k\}_k$ be the dual polyhedral subdivision of $\Delta = \mathrm{Conv}(A)$, as introduced in Example 1.7.

If Δ has dimension n, then the tropical cycle $\mathrm{div}_{\mathbb{R}^n}(\phi)$ will be locally matroidal if and only if $\{\Delta_k\}_k$ is unimodular. This assertion rests on the fact that the minimal volume of an n-dimensional convex lattice polytope in $\mathbb{R}^n$ is $1/n!$, and up to an affine translation, such polytopes are convex hulls of $n + 1$ points $\{0, v_1, \ldots, v_n\} \subset \mathbb{Z}^n$, where the coordinates v_{ij} of the points v_i form a matrix $(v_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ in $SL_n(\mathbb{Z})$. $*$

We now discuss the tropical intersection theory of two tropical fan sub-cycles of a tropical fan cycle. The following definition is found in [AR09].

Definition: Let $A = (A, w)$ be a fan k-cycle and let $\phi : A \longrightarrow \mathbb{R}$ be a continuous function. We say that ϕ is a **rational function** if there exists a fan refinement A' of A such that for every $\sigma \in A'$, the restriction $\phi|_\sigma$ is an affine integer function.

We have the following important result, which says that any rational function $\phi : A \longrightarrow \mathbb{R}$ defined on a fan k-cycle A is the restriction to A of some tropical rational function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$. The converse is not true in general.

Lemma 1.3.21 (Shaw). *Let $A \subset \mathbb{R}^n$ be a tropical fan k-cycle and let $\phi : A \longrightarrow \mathbb{R}$ be a continuous piecewise affine integer sloped function with the property that there exists a fan refinement A' of A such that ϕ is linear on each cone of A'. Then ϕ is the restriction of a tropical rational function*

We point out that this result corresponds to Lemma 2.19 on a previous version of [Sha13]. This result does not appear in the latest version of this article.

If $A = (A, w)$ is a fan k-cycle and $\phi : A \longrightarrow \mathbb{R}$ is a rational function, then we can apply the procedure of Section 1.3.4 to construct the tropical modification $\delta_\emptyset : A_\emptyset(\phi) \longrightarrow A$ to define a Weil divisor $\mathrm{div}_A(\phi)$ which will be a fan $(k-1)$-cycle in A. See [AR09].

Let $V = (V, w)$ be an effective fan k-cycle properly contained in $\mathbb{R}^n$, and let A, B be two fan cycles contained in V of dimension ℓ_1 and ℓ_2 respectively. There are two cases in which a tropical intersection product of A and B has been defined, namely:

1. suppose that there exists a rational function $\phi : V \longrightarrow \mathbb{R}$ such that $A = \mathrm{div}_V(\phi)$, then we can apply the procedure of Section 1.3.4 to construct the tropical modification $\delta_\emptyset : B_\emptyset(\phi) \longrightarrow B$ of B along ϕ as well as the Weil divisor $\mathrm{div}_B(\phi)$ of ϕ on B. The tropical intersection of A and B is then by definition the Weil divisor $\mathrm{div}_B(\phi)$, which is denoted $\phi.B$ in [AR09];

2. suppose that the fan V is matroidal, then K. Shaw has given in [Sha13] a method for constructing an intersection product $A.B$ of the fan cycles A, B.

If the fan V is matroidal and there exists a rational function $\phi : V \longrightarrow \mathbb{R}$ such that $A = \mathrm{div}_V(\phi)$, then these two approaches coincide (see [Sha13]), i.e.

$$\mathrm{div}_V(\phi).B = \phi.B.$$

In fact, using the **projection formula** in [AR09] we can say more. Suppose that there is an element $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ such that $V' = \Phi(V)$ is a matroidal fan. Let $\Psi := \Phi^{-1} : V' \longrightarrow V$, then we can consider the cycle $B' = \Phi_*(B)$ in V' and the rational function $\phi' = \Psi^*(\phi) : V' \longrightarrow \mathbb{R}$. The projection formula reads:

$$\phi.B = \phi.(\Psi_*(\Phi_*(B))) = \Psi_*(\Psi^*(\phi).\Phi_*(B)) = \Psi_*(\phi'.B'). \tag{1.11}$$

Since V' is matroidal, we have that $\phi'.B' = \mathrm{div}_{V'}(\phi').B'$, and finally we have $\phi.B = \Psi_*(\mathrm{div}_{V'}(\phi').B')$, where $\Psi_* : Z_*(\mathbb{R}^n) \longrightarrow Z_*(\mathbb{R}^n)$ is the tropical push-forward of tropical cycles induced by $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

1.3.6 Tropical curves in $\mathbb{R}^2$

We now discuss tropical curves. The source for the following material will be [BL12].

By **tropical curve** C in $\mathbb{R}^2$, we mean the tropical 1-cycle $\mathrm{div}_{\mathbb{R}^2}(\phi)$ in $\mathbb{R}^2$ induced by a tropical polynomial $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Let $\{\Delta_k\}_k$ be the subdivision of $\mathrm{New}(\phi)$ induced by the coefficients of ϕ, and let us endow $C = (C, w)$ with the polyhedral structure which is dual to $\{\Delta_k\}_k$, as in Example 1.3.12. Let $\Delta \in \{\Delta_k\}_k$ and let $\Delta^\vee \in C$ be its dual polyhedron (see Example 1.3.12). We say that $\Delta^\vee$ is an **edge** (respectively a **vertex**) of C if the dimension $\dim(\Delta^\vee)$ of $\Delta^\vee$ is one (respectively zero).

If $e \in C$ is a bounded edge, say $\partial e = \{v, w\}$, its length $\ell(e)$ is defined to be $\ell(e) = \frac{||v-w||}{w(e)||v(e,v)||}$, where $||v-w||$ is the Euclidean distance between v, w and $||v(e,v)||$ is the norm of the primitive integer vector orthogonal to v generating e.

Definition: Let C be a tropical curve in $\mathbb{R}^2$. We say that it is **generic** if the lengths of its bounded edges are all distinct.

1.3.22 Example (Stable intersection in $\mathbb{R}^2$): Let $C_i = \mathrm{div}_{\mathbb{R}^2}(\phi_i)$, $i = 1, 2$, be two tropical curves in $\mathbb{R}^2$. Their stable intersection cycle $C_1.C_2 = (C_1.C_2, w_{C_1.C_2})$ is a tropical 0-cycle in $\mathbb{R}^2$ which can be constructed as follows. Consider the tropical curve $C_3 = \mathrm{div}_{\mathbb{R}^2}(\phi_1 + \phi_2)$ and set $C_1.C_2 = \{v \text{ vertex of } C_3 : v \in |C_1| \cap |C_2|\}$.

For $v \in C_1.C_2$ and $i = 1, 2, 3$, there exists a polygon Δ_i in convex polyhedral subdivision of $\mathrm{New}(\phi_i)$ such that v is in the interior of $\Delta_i^\vee$. We set $w_{C_1.C_2}(v) = \mathrm{vol}_{\mathbb{Z}}(\Delta_3^\vee) - \mathrm{vol}_{\mathbb{Z}}(\Delta_1^\vee) - \mathrm{vol}_{\mathbb{Z}}(\Delta_2^\vee)$. $*$

Let $\mathbb{K}$ be the subfield of $\mathbb{C}((t^{\mathbb{R}}))$ consisting of locally convergent generalized Puiseux series. A polynomial $F = \sum_{ij} a_{ij} x_1^i x_2^j$ in $K[x_1^{\pm 1}, x_2^{\pm 1}]$ gives us :

1. an algebraic curve $C^{\mathbb{K}} = V(F)$ in $(\mathbb{K}^*)^2$;

2. a tropical curve $C = \mathrm{div}_{\mathbb{R}^2}(\mathrm{Trop}(F))$ in $\mathbb{R}^2$.

Moreover, if all the coefficients $a_{ij} = \sum_r \alpha_{ijr} t^r$ satisfy $\mathrm{ord}(a_{ij}) \geq 0$ for all i, j and $\alpha_{ijr} \in \mathbb{R}$ for all i, j, r, then we also get

3. a 1-dimensional family $\{C_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of curves $C_\varepsilon \subset (\mathbb{C}^*)^2$ defined over $\mathbb{R}$, where C_ε is the real algebraic curve associated to the real polynomial $F_\varepsilon = \sum_{ij} a_{ij}(\varepsilon) X^i Y^j$.

1.4.– Inflection points of linear series on algebraic curves

Unless otherwise stated, in this part by algebraic curve we will mean a non-singular projective algebraic curve X over a fixed algebraically closed field of characteristic zero $\mathbb{K}$. We will be using the following notation and conventions:

1. if $D \in \mathrm{Div}(X)$, then $\mathcal{L}(D)$ will denote the invertible sheaf induced by D;
2. if L is an invertible sheaf on X, then we will denote $H^0(L)$ its group of global sections $H^0(L) = \Gamma(X, L)$ and $[L]$ its class in $\mathrm{Pic}(X)$;
3. we will use Ω_X and K_X to denote the canonical sheaf and the canonical divisor of X respectively;
4. for a point $x \in X$, we will use ord_x to denote the valuation induced by the local ring $\mathcal{O}_{X,x}$ on the field $\mathbb{K}(X)$.

If $C \subset \mathbb{KP}^n$ is a curve, we denote by $\nu : \widetilde{C} \longrightarrow C$ its normalization morphism. For $p \in C_{\mathrm{Sing}}$, we denote by $\widetilde{\mathcal{O}_{C,p}}$ the integral closure of $\mathcal{O}_{C,p}$ in the field $\mathbb{K}(C)$ and by δ_p the length $\ell_{\mathcal{O}_{C,p}}(\widetilde{\mathcal{O}_{C,p}})$.

Definition: Let $Y \subset \mathbb{KP}^n$ be a subvariety of dimension k.

1. We say that Y is **non-degenerate** if it is not contained in any hyperplane of $\mathbb{KP}^n$.
2. The **multiplicity** $\mathrm{mult}_Y(x)$ of Y at a point x is defined as

$$\mathrm{mult}_Y(x) = \min_W \{\ell(\mathcal{O}_{Y \cap W,x})\}, \tag{1.12}$$

where W runs over the linear spaces of codimension k such that Y and W intersect properly at x.

1.4.1 Linear series on algebraic curves

Let X be an algebraic curve over $\mathbb{K}$. First we will introduce the concepts of linear series (V, L) on X and of inflection point of (V, L), then we discuss some related tools to study them using projective geometry, including the well-known Plücker formulas.

Definition: A **linear series** of degree d and rank r or g^r_d on X is a pair (V, L) consisting of :

1. an invertible sheaf L of degree d on X such that $H^0(L) \neq \{0\}$, and
2. a linear subspace $V \subseteq H^0(L)$ of dimension $r + 1$, with $r \geq 0$.

We say that (V, L) is **complete** if $V = H^0(L)$.

Since L is invertible, for any $x \in X$ we can find an element $h \in \mathbb{K}(X)$ such that $L_x \cong \mathcal{O}_{X,x} \cdot h_x$. Thus for any $s \in H^0(L)$ there exists $f_x \in \mathcal{O}_{X,x}$ such that $s_x = f_x \cdot h_x$. The integer $\mathrm{ord}_x(f_x)$ is independent from this representation and it will be denoted as $\mathrm{ord}_x(s)$.

Definition: Let (V, L) be a g^r_d on X. We say that $x \in X$ is an **inflection point** of (V, L) if there exists $s \in (V \setminus \{0\})$ such that $\mathrm{ord}_x(s) > r$.

1.4.1 Example (The case $r = 0$): Let (V, L) be a g^0_d on X with $L = \mathcal{L}(D)$ for some divisor D of degree $d \geq 0$, then there exists $f \in \mathbb{K}(X)$ such that $V \cong \mathbb{K} \cdot f$. The set of inflection points of (V, L) is just the support of the effective divisor $E := \mathrm{div}_X(f) + D$. It follows that a g^0_d on a curve X has at most $d = \deg(D) \geq 0$ inflection points (in particular, a g^0_0 has none). $*$

For $x \in X$, the function $\mathrm{ord}_x : V \longrightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ has the property that for every $i \in \mathbb{Z}_{\geq 0}$, the set $V_{x,i} := \mathrm{ord}_x^{-1}(\mathbb{Z}_{\geq i} \cup \{+\infty\})$ is a linear subspace of V. An element $s \in V$ is in $V_{x,i}$ if s_x is in $\mathfrak{m}_{X,x}^i \cdot L_x$, and since $\dim\,(\mathfrak{m}_{X,x}^i/\mathfrak{m}_{X,x}^{i+1}) = 1$, then $\dim_{\mathbb{K}}(V_{x,i}/V_{x,i+1}) \leq 1$.

It follows that for $x \in X$, the set $\{\mathrm{ord}_x(s)\}_{s\in V}$ consists of $r+1$ distinct elements $\{a_0(x), \ldots, a_r(x)\}$ with $0 \leq a_0(x) < \ldots < a_r(x)$, and that $V_{x,a_i(x)} = \{s \in V \,:\, \mathrm{ord}_x(s) \geq a_i(x)\}$ is a proper vector subspace of V of dimension $r+1-i$ for $i = 1, \ldots, r$.

Remark 1.4.2:

1. *The sequence $0 \leq a_0(x) < \cdots < a_r(x)$ is known as the* **vanishing sequence** *of (V, L) at $x \in X$, and some authors prefer to use instead the* **gap sequence** *$1 \leq g_1(x) \leq \cdots \leq g_{r+1}(x)$ of (V, L) at $x \in X$, where $g_{i+1}(x) := a_i(x) - 1$ for $i = 0 \ldots, r$.*

2. *If $L \cong L'$, let $V' \subset H^0(L')$ be the subspace isomorphic to $V \subset H^0(L)$ under the isomorphism $H^0(L) \cong H^0(L')$, then the vanishing sequences of (V, L) and (V', L') at $x \in X$ are equal.*

Definition: Let (V, L) be a g_d^r on X. The **ramification sequence** of (V, L) at $x \in X$ is the non-decreasing sequence

$$\lambda(x) := (\alpha_0(x, V) \leq \cdots \leq \alpha_r(x, V)), \text{ where } \alpha_i(x, V) := a_i(x) - i \text{ for } i = 0, \ldots, r. \tag{1.13}$$

Let (V, L) be a g_d^r on a curve X. It turns out that the set of inflection points of (V, L) is the support of the **Wronskian** $\mathrm{Wr}(s_1, \ldots, s_{r+1})$ associated to a basis $\{s_1, \ldots, s_{r+1}\}$ of V, which is a regular section of the sheaf $L^{\otimes(r+1)} \otimes \Omega_X^{\otimes r(r+1)/2}$. The construction of the Wronskian, as well as a sketch of proof for the next result will be presented in the Section (1.4.2).

Proposition 1.4.3. *Let (V, L) be a g_d^r on a curve X of genus g. For $x \in X$, we set $|\lambda(x)| = \sum_{i=0}^r \alpha_i(x, V)$. Then $\mathrm{Wr}(V, L) = \sum_x |\lambda(x)| \cdot x$ is an effective divisor of degree $(r+1)(d + r(g-1))$ supported on the set of inflection points of (V, L).*

We call $|\lambda(x)|$ the **inflection multiplicity** of (V, L) at x and $\mathrm{Wr}(V, L)$ the **inflection divisor** of (V, L).

Remark 1.4.4: *The value $(r+1)(d+r(g-1))$ is zero if and only if $d+r(g-1) = 0$. Since $g, d, r \geq 0$, we conclude that there are two cases in which a g_d^r on a curve X has zero inflection points, namely $r = d = 0$ and $g \geq 0$, or $r = d > 0$ and $g = 0$.*

We close this part with the following definitions.

Definition: Let (V, L) be a g_d^r on X. We say that an inflection point $x \in X$ of (V, L) is

- a **base point**, if $\alpha_0(x, V) = a_0(x) > 0$;
- **honest**, if $\alpha_i(x, V) = 0$ for $i < r$, and
- **simple**, if it is honest and $\alpha_r(x, V) = 1$.

We say that (V, L) is **base point-free** if it has no base-point. If (V, L) is base point-free, then it is **honest** (respectively **simple**) if all its inflection points are honest (respectively simple).

1.4.2 The Wronskian and Gauss maps associated to a linear series

Let X be a curve, (V, L) a g_d^r on X and $\{s_0, \ldots, s_r\}$ a basis for V. For $x \in X$ we choose $h_x \in L_x$ such that $L_x \cong \mathcal{O}_{X,x} \cdot h_x$, then there exists $f_{0,x}, \ldots, f_{r,x} \in \mathcal{O}_{X,x}$ such that $(s_i)_x = f_{i,x} \cdot h_x$ for $i = 0, \ldots, r$.

Let $\tau \in \mathfrak{m}_{X,x}$ be a local parameter of X at x, then $\Omega_{X,x} \cong \mathcal{O}_{X,x} \cdot d\tau$. If $d : \mathcal{O}_{X,x} \longrightarrow \Omega_{X,x}$ is the derivation homomorphism, then for any $f_x \in \mathcal{O}_{X,x}$ there exists a unique function $\frac{df_x}{d\tau} \in \mathcal{O}_{X,x}$ such that $d(f_x) = \frac{df_x}{d\tau} \cdot d\tau$.

For $i = 0, \ldots, r$ we define $f_{i,x}^{(j)}$ inductively for $j = 0, \ldots, r$ as follows: $f_{i,x}^{(0)} = f_{i,x}$ and $f_{i,x}^{(j)} = \frac{df_{i,x}^{(j-1)}}{d\tau}$ for $j > 0$. Fix $0 \leq \ell \leq r$ and let $J = \{k_0, \ldots, k_\ell\}$ be a subset of $\{0, 1, \ldots, r\}$, we denote $(M_J(s_0, \ldots, s_r))_x$ the element of $(L^{\otimes(\ell+1)} \otimes \Omega_X^{\otimes \ell(\ell+1)/2})_x$ induced by the $(\ell+1) \times (\ell+1)$ minor of the matrix $\left(f_{i,x}^{(j)}\right)_{\substack{0\leq j\leq \ell \\ 0\leq i\leq r}}$ defined by J, i.e.

$$(M_J(s_0, \ldots, s_r))_x = \begin{pmatrix} f_{k_0,x}^{(0)} & \cdots & f_{k_\ell,x}^{(0)} \\ \vdots & \ddots & \vdots \\ f_{k_0,x}^{(\ell)} & \cdots & f_{k_\ell,x}^{(\ell)} \end{pmatrix} \cdot h_x^{\ell+1} (d\tau)^{\ell(\ell+1)/2}, \quad J = \{k_0, \ldots, k_\ell\} \subset \{0, 1, \ldots, r\}.$$

The germs $(M_J(s_0,\dots,s_r))_x$ do not depend on the choice of τ and they are non-zero because the elements $f_{i,x}$ are linearly independent, so they can be glued together to construct an element $M_J(s_0,\dots,s_r)$ of $H^0(L^{\otimes(\ell+1)}\otimes\Omega_X^{\otimes\ell(\ell+1)/2})$.

We now give a sketch of proof for Proposition 1.4.3.

Sketch of the proof. The Wronskian of (V,L) associated to the basis $\{s_0,\dots,s_r\}$ of V is the unique section $M_{\{0,\dots,r\}}(s_0,\dots,s_r)\in H^0(L^{\otimes(r+1)}\otimes\Omega_X^{\otimes r(r+1)/2})$ arising for the case $\ell=r$, and it is denoted by $\mathrm{Wr}(s_0,\dots,s_{r+1})$.

For any $x\in X$, the value $\mathrm{ord}_x(\mathrm{Wr}(s_0,\dots,s_r))$ is independent of the choice of the basis $\{s_0,\dots,s_r\}$ for V, so the divisor of zeroes $\mathrm{Wr}(V,L)$ of $\mathrm{Wr}(s_0,\dots,s_r)$ is a well-defined effective divisor of degree $(r+1)(d+r(g-1))$ associated to (V,L).

Let $0\le a_0(x)<\cdots<a_r(x)$ be the vanishing sequence of (V,L) at $x\in X$. To compute the value $\mathrm{ord}_x(\mathrm{Wr}(s_0,\dots,s_r))$, we use a basis $\{s_0,\dots,s_r\}$ of V that satisfy $\mathrm{ord}_x(s_i)=a_i(x)$ for $i=0,\dots,r$. Then we can write $f_{i,x}=\tau^{a_i(x)}g_i(\tau)$ for some g_i with $g_i(0)\neq 0$, and a computation of the determinant above shows that

$$\mathrm{ord}_x(\mathrm{Wr}(s_0,\dots,s_r))=\sum_{i=0}^{r}a_i(x)-i=\sum_{i=0}^{r}\alpha_i(x,V)=|\lambda(x)|.$$

Finally, note that the points $x\in X$ such that $|\lambda(x)|>0$ are exactly the inflection points of (V,L). ■

It follows from Example 1.4.1 that if $(\mathbb{K}\cdot f,\mathcal{L}(D))$ is a g^0_d on a curve X, then its inflection divisor is just $E:=\mathrm{div}_X(f)+D$. From now on we will suppose that $r>0$.

Consider now the compete flag $V=V_0\supset V_{x,a_1(x)}\supset\cdots\supset V_{x,a_r(x)}\supset\{0\}$ associated to a point $x\in X$. If we denote as $W^\perp$ the dual vector space $\{f\in V^*:f(s)=0 \text{ for all } s\in W\}$ of a linear subspace $W\subseteq V$, then we get a sequence $\mathbb{P}(V^\perp_{x,a_1(x)})\subset\cdots\subset\mathbb{P}(V^\perp_{x,a_r(x)})\subset\mathbb{P}(V^*)$.

Definition: Let (V,L) be a g^r_d on X with $r>0$. For $\ell=0,\dots,r-1$, we define:

1. the ℓ-**th osculating plane** $\phi_\ell(x)$ of (V,L) at x to be the ℓ-dimensional projective subspace $\mathbb{P}(V^\perp_{x,a_{\ell+1}(x)})$;
2. the ℓ-th **Gauss map** $\phi_\ell:X\longrightarrow \mathrm{Gr}(\ell,\mathbb{P}(V^*))$ of (V,L) to be the assignment $x\mapsto\phi_\ell(x)$;
3. the ℓ-th **associated curve** of (V,L) to be $C_\ell:=\phi_\ell(X)$. We will denote d_ℓ the degree of the curve C_ℓ.

The **osculating flag** of ϕ_0 at x is the complete flag $\phi_0(x)\subset\phi_1(x)\subset\cdots\subset\phi_{r-1}(x)\subset\mathbb{P}(V^*)$.

Remark 1.4.5: *If (V,L) is base point-free, then the choice of a basis $\{s_0,\dots,s_r\}$ for V induces an isomorphism $V^*\cong\mathbb{K}^{r+1}$, and the morphism $\phi_0:X\longrightarrow\mathbb{KP}^r$ can be written as $x\mapsto[s_0(x):\cdots:s_r(x)]$. Furthermore, if (V,L) is base point-free and $\phi_0:X\longrightarrow C_0$ is birational (i.e., ϕ_0 is the normalization of C_0), then C_0 does not have a cusp at $p=\phi(x)$ if $\alpha_1(x,V)=a_1(x)-1=0$. See [HM98] p.256.*

1.4.6 Example (The case $r=1$): Let (V,L) be a base point-free g^1_d on X. If $\{s_0,s_1\}$ is a basis for V, then $\phi_0:X\longrightarrow\mathbb{KP}^1$ is a map of degree d given by $x\mapsto[s_0(x):s_1(x)]$. The vanishing sequence of a point $x\in X$ is of the form $0=a_0(x)<a_1(x)$, and if we choose $\{s_0,s_1\}$ such that $\mathrm{ord}_x(s_i)=a_i(x)$, it is easy to see that $|\lambda(x)|=\alpha_1(x):=a_1(x)-1$ is the ramification index of ϕ_0 at x, thus $\mathrm{Wr}(V,L)$ is precisely the ramification divisor of the map ϕ_0. ∗

1.4.7 Example (The case $r\ge 2$ and projective geometry): Let (V,L) be a g^r_d on X with $r\ge 2$, then $C_0\subset\mathbb{P}(V^*)$ is a projective curve of degree d_0 which is non-degenerate. Suppose that (V,L) is also base-point free. If $\alpha_1(x,V)=0$ and if $\phi_0^{-1}(\phi_0(x))=\{x\}$, then ϕ_0 is a closed embedding in an affine neighborhood of x, and in this case the $(r-1)$-osculating plane $H=\phi_{r-1}(x)\subset\mathbb{P}(V^*)$ is a hyperplane satisfying $\ell(\mathcal{O}_{C_0\cap H,\phi_0(x)})=a_r(x)$. It follows that in this case, $x\in X$ is an inflection point of (V,L) and satisfies

$$|\lambda(x)|\ge\ell(\mathcal{O}_{C_0\cap H,\phi_0(x)})-r.$$

We will have equality if x is honest. ∗

Proposition 1.4.8. *The ramification index $\beta_\ell(x)$ of the map $\phi_\ell:X\longrightarrow Gr(\ell,\mathbb{P}(V^*))$ at $x\in X$ is equal to $\beta_\ell(x)=\alpha_{\ell+1}(x,V)-\alpha_\ell(x,V)=a_{\ell+1}(x)-a_\ell(x)-1\ge 0$.*

Sketch of the proof. For the sake of simplicity, we will assume that (V, L) is base point-free. Let $\{s_0, \dots, s_r\}$ be a basis for V. For a fixed $\ell = 0, \dots, r-1$, the family $\{M_A(s_0, \dots, s_r) : A = \{k_0, \dots, k_\ell\} \subset \{0, 1, \dots, r\}\}$ consists of $\binom{r+1}{\ell+1}$ global sections of the sheaf $L^{\otimes(i+1)} \otimes \Omega_X^{\otimes i(i+1)/2}$ that define the Plücker embedding $\mathrm{Pl} \circ \phi_i : X \longrightarrow \mathbb{P}(\Lambda^{\ell+1}\mathbb{K}^{r+1})$.

Let $0 = a_0(x) < \cdots < a_r(x)$ be the vanishing sequence of (V, L) at $x \in X$, and suppose that $\{s_0, \dots, s_r\}$ satisfies $\mathrm{ord}_x(s_i) = a_i(x)$ for $i = 0, \dots, r$. Then a local lifting to $\mathbb{K}^{r+1}$ for the map ϕ_0 near $\phi_0(x)$ is given by $f(\tau) = (1, \tau^{1+\alpha_1} + \cdots, \dots, \tau^{r+\alpha_r} + \cdots)$.

Let $\frac{d^i f}{d\tau^i}$ be the i-th derivative of the function f, then the lift of $\phi_\ell(x)$ to $\mathbb{K}^{r+1}$ is spanned by the first $\ell+1$ linearly independent vectors in the sequence $\frac{d^0 f}{d\tau^0}, \frac{d^1 f}{d\tau^1}, \dots$. If we denote by B the matrix having these vectors as rows, then the Plücker coordinates in $\mathrm{Gr}(\ell+1, \mathbb{K}^{r+1})$ of this lift are the $(\ell+1)$-dimensional minors of B.

Finally, the $(\ell+1)$-dimensional minor Δ_1 which contains the smallest power of τ is formed by the column set $A = \{0, 1, 2, \dots, \ell\}$, and the next smallest power of τ corresponds to the ℓ-dimensional minor Δ_2 formed by the column set $A = \{0, \dots, \ell-1, \ell+1\}$. So, the coordinates of the local lifting of $\phi_\ell(x)$ to $\mathrm{Gr}(\ell+1, \mathbb{K}^{r+1})$ are given by $\tau \mapsto (1, a\tau^{\alpha_{i+1}-\alpha_i+1} + \cdots, \dots)$, where the order of τ in all the remaining coordinates is higher. ■

We denote as $\beta_\ell = \sum_{x \in X} \beta_\ell(x) \cdot x$ the ramification divisor of ϕ_ℓ, and we say that ϕ_ℓ is **unramified** if $\beta_\ell = 0$.

In [GH78] we find a proof for the complex case of the so-called Plücker formulas. This proof can be adapted to an arbitrary algebraically closed field of characteristic zero $\mathbb{K}$.

Theorem 1.4.9 (Plücker formulas). *For $\ell = 0, \dots, r-1$, let $d_\ell = deg(C_\ell)$. Then we have*

$$d_{\ell-1} - 2d_\ell + d_{\ell+1} = 2g - 2 - deg(\beta_\ell). \tag{1.14}$$

Remark 1.4.10: *In general we will be interested in linear series (V, L) on X of degree d and rank $r \geq 2$ with the property that the map $\phi_0 : X \longrightarrow C_0$ is birational. In this case, the map $\phi_0 : X \longrightarrow C_0$ is the normalization of the non-degenerate curve $C_0 \subset \mathbb{P}(V^*)$ of degree $d_0 = d$. Conversely, if $C \subset \mathbb{KP}^r$ is a non-degenerate curve, then we can construct a linear system (V, L) on its normalization $\nu : \tilde{C} \longrightarrow C$ as the linear system associated to the morphism $\tilde{C} \xrightarrow{\nu} C \hookrightarrow \mathbb{KP}^r$. In this case we will refer to the inflection points of (V, L) as the inflection points of the curve C.*

1.4.3 The real case

Let X be a complex algebraic variety. A **real structure** on X is an anti-holomorphic involution $\sigma : X(\mathbb{C}) \longrightarrow X(\mathbb{C})$, and the pair (X, σ) is a real (algebraic) variety. The set $X(\mathbb{C})^\sigma$ of fixed points is the real part of (X, σ) and it is denoted as $X(\mathbb{R})$. A real morphism between two real algebraic varieties $f : (X, \sigma) \longrightarrow (X', \sigma')$ is a morphism of varieties $f : X \longrightarrow X'$ such that $f \circ \sigma = \sigma' \circ f$; it is a real isomorphism if f is an isomorphism. Observe that a real morphism $f : (X, \sigma) \longrightarrow (X', \sigma')$ between real varieties induces a map $f|_{X(\mathbb{R})} : X(\mathbb{R}) \longrightarrow X'(\mathbb{R})$ between the real parts.

Remark 1.4.11: *Let X be a complex variety. The existence of a real structure σ on X is equivalent to the existence of a variety X_0 defined over $\mathbb{R}$ such that $X = (X_0)_\mathbb{C} := X_0 \times_\mathbb{R} Spec(\mathbb{C})$, in this case we will say that (X, σ) has X_0 as real model. Likewise, real morphisms between real algebraic varieties come from morphisms between their real models.*

The main source for the present material is [GH81]. Let (X, σ) be a real curve, its **topological type** is the triple $(g(X), k(X), a(X))$, where $g(X)$ is the genus of X, $k(X)$ is the number of connected components of $X(\mathbb{R})$ and $a(X) \in \{0, 1\}$ is the **type** of (X, σ) : $a = 0$ if and only if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is not connected. The triples $(g, k, a) \in \mathbb{N} \times \mathbb{N} \times \{0, 1\}$ coming from the topological type of real curves are subject to the following conditions :

1. $0 \leq k \leq g + 1$
2. if $k = g + 1$ then $a = 0$; if $k = 0$ then $a = 1$;
3. if $a = 0$ then $k \equiv g + 1 \mod 2$.

The involution σ acts on the group of divisors $\mathrm{Div}(X)$ as $\sum_x d_x \cdot x \mapsto \sum_x d_x \cdot \sigma(x)$ and so it defines a real structure σ on $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ given by $\sigma \cdot [D] = [\sigma(D)]$. Let D be a σ-invariant divisor on (X, σ), then we will express it as

$$D = \sum_{x \notin X(\mathbb{R})} d_x(x + \sigma(x)) + \sum_{x \in X(\mathbb{R})} d_x x. \tag{1.15}$$

Sometimes we will refer to the number $\sum_{x \in X(\mathbb{R})} d_x$ as the **real degree** of D. We will also denote as $L_{\mathbb{R}} = \mathcal{L}_{\mathbb{R}}(D)$ the algebraic real line bundle defined on X by D. In particular, the set of real sections $H^0(L_{\mathbb{R}}) = \Gamma(X, L_{\mathbb{R}})$ of $L_{\mathbb{R}}$ is a real vector space.

The set $\mathrm{Pic}(X)(\mathbb{R})$ of real points of $\mathrm{Pic}(X)$ represents the complex line bundles on X which are isomorphic to their complex conjugate. This group contains the subgroup $\mathrm{Pic}(X)(\mathbb{R})^+$ of those classes represented by a σ-invariant divisor D, which correspond to algebraic line bundles which may be defined over $\mathbb{R}$. It is shown in [GH81], Proposition 2.2 that $\mathrm{Pic}(X)(\mathbb{R})^+ = \mathrm{Pic}(X)(\mathbb{R})$ if $X(\mathbb{R}) \neq \emptyset$, and $[\mathrm{Pic}(X)(\mathbb{R}) : \mathrm{Pic}(X)(\mathbb{R})^+] = 2$ otherwise.

Suppose that (X, σ) is a real curve with $X(\mathbb{R}) \neq \emptyset$, then we write $X(\mathbb{R}) = S_1 \cup \cdots \cup S_{k(X)}$ with $k(X) > 0$. Then there is a well defined **parity homomorphism** $\mathrm{par} : \mathrm{Pic}(X)(\mathbb{R}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{k(X)}$ given by

$$\mathrm{par}([D]) = (\deg(D|_{S_1}) \mod 2, \ldots, \deg(D|_{S_{k(X)}}) \mod 2). \tag{1.16}$$

1.4.12 Example (The parity of the canonical class of a real curve): The canonical class K_X of any real curve (X, σ) is in $\mathrm{Pic}(X)(\mathbb{R})^+$, and if $k(X) > 0$, then it satisfies $\mathrm{par}(K_X) = 0$, i.e., its support has an even number of points on each connected component of $X(\mathbb{R})$ (see [GH81], Proposition 4.2). $*$

1.5.– Integration with respect to the topological Euler characteristic, and projective duality

In this part we retake some concepts from [Vir88]. Let us consider $\mathbb{CP}^n$ endowed with its Zariski topology. If $Y \subset \mathbb{CP}^n$ is a subvariety, let $\mathbf{1}_Y$ be the characteristic function associated to Y, defined as $x \mapsto 1$ if $x \in Y$ and $x \mapsto 0$ otherwise.

Let $\mathcal{F}(\mathbb{CP}^n)$ be the set of constructible functions, since $\mathbb{CP}^n$ is the only n-dimensional closed set, any $f \in F(\mathbb{CP}^n)$ can be written as

$$f = \lambda_{\mathbb{CP}^n} \cdot \mathbf{1}_{\mathbb{CP}^n} + \sum_{Y \subsetneq \mathbb{CP}^n} \lambda_Y \cdot \mathbf{1}_Y, \tag{1.17}$$

where Y runs over the algebraic subvarieties of $\mathbb{CP}^n$, and $\lambda_Y \in \mathbb{Z}$ is equal to zero for almost all Y. We have an inclusion $\mathbb{Z} \hookrightarrow \mathcal{F}(\mathbb{CP}^n)$ given by $\lambda \mapsto \lambda \cdot \mathbf{1}_{\mathbb{CP}^n}$.

We say that $\lambda(f) = \lambda_{\mathbb{CP}^n}$ is the **generic value** of f on $\mathbb{CP}^n$, i.e., the function $\overline{f} = f - \lambda_{\mathbb{CP}^n} \cdot \mathbf{1}_{\mathbb{CP}^n}$ is zero on some non-empty open subset of $\mathbb{CP}^n$. Observe that the closed set $\mathrm{Supp}(\overline{f}) \subset \mathbb{CP}^n$ is empty if and only if f is constant.

Definition: Let $f \in \mathcal{F}(\mathbb{CP}^n)$ be of the form $f = \sum_Y \lambda_Y \cdot \mathbf{1}_Y$ and let $A \subset \mathbb{CP}^n$ be a closed subset. The **integral** with respect to the Euler characteristic $\int_A f(x) d\chi(x)$ of f over A is defined to be $\sum_Y \lambda_Y \chi(Y \cap A)$, where χ denotes the topological Euler characteristic function.

Definition: Let $f \in \mathcal{F}(\mathbb{CP}^n)$, its **dual function** $f^* : \mathbb{CP}^{n*} \longrightarrow \mathbb{Z}$ is defined as $f^*(H) = \int_H f(x) d\chi(x)$, where $H \in \mathbb{CP}^{n*}$ is a hyperplane.

The function f^* belongs to $\mathcal{F}(\mathbb{CP}^{n*})$, and the assignment $f \mapsto f^*$ defines a duality $(\cdot)^* : \mathcal{F}(\mathbb{CP}^n)/\mathbb{Z} \longrightarrow \mathcal{F}(\mathbb{CP}^{n*})/\mathbb{Z}$.

The following theorem due to O. Viro, computes the value $\int_{\mathbb{RP}^{n*}} f^*(x)\, d\chi(x)$ in terms of integral of the function f.

Theorem 1.5.1 (Viro). *If $f \in \mathcal{F}(\mathbb{CP}^n)$, then*

$$\int_{\mathbb{RP}^{n*}} f^*(x)\, d\chi(x) = \begin{cases} \int_{\mathbb{CP}^n \setminus \mathbb{RP}^n} f(x)\, d\chi(x), & \text{if } n \text{ is even,} \\ \int_{\mathbb{RP}^n} f(x)\, d\chi(x), & \text{if } n \text{ is odd.} \end{cases} \tag{1.18}$$

We shall now present some properties of dual varieties of irreducible, non-degenerate algebraic subvarieties of $\mathbb{CP}^n$, then we focus on the structure of dual varieties of curves. The main source for this part is [Tev01].

In this part we will denote by Y a non-degenerate algebraic subvariety of $\mathbb{CP}^n$ of dimension k.

Definition: We say that Y is **ruled** (in projective subspaces of dimension s) if for any $x \in Y$ there exists a projective subspace L of dimension s such that $x \in L \subset Y$.

Remark 1.5.2: *It is sufficient to check this property only for points x in some Zariski open dense subset of Y.*

Definition: A hyperplane $H \subset \mathbb{CP}^n$ is said to be **tangent** to Y if it contains an embedded tangent space $\hat{T}_x Y$ at some smooth point $x \in Y$. The Zariski closure of the set of all tangent hyperplanes to Y is the **dual** Y^* of Y.

Remark 1.5.3: *Some basic facts about the dual Y^* of a subvariety $Y \subset \mathbb{CP}^n$ are:*

1. *Y^* is a subvariety of $\mathbb{CP}^{n*}$;*
2. *the assignment $Y \mapsto Y^*$ is indeed a duality, i.e. $Y^{**} = Y$;*
3. *if Y is smooth, then $Y^* = \{H \subset \mathbb{CP}^{n*} \,:\, Y \cap H \text{ is singular}\}$.*

Definition: The Zariski closure N_Y of the set

$$\{(x, H) \in \mathbb{CP}^n \times \mathbb{CP}^{n*} \,:\, x \in Y_{\text{Smooth}},\ H \text{ tangent to } Y \text{ at } x\}$$

is called the **conormal variety** of Y, and the diagram $Y \xleftarrow{\pi_1} N_Y \xrightarrow{\pi_2} Y^*$ is the **conormal diagram**

The following result can be found in [Tev01] (see Theorem 1.10, p. 6).

Lemma 1.5.4. *Suppose that $Y \subset \mathbb{CP}^n$ is a non-degenerate curve.*

1. *The variety Y^* is a hypersurface ruled in projective subspaces of dimension $n-2$, and the map $\pi_2 : N_Y \longrightarrow Y^*$ is birational.*
2. *If in addition Y is smooth, then $\pi_2 : N_Y \longrightarrow Y^*$ is a resolution of singularities.*

Proof. Let $\nu : \tilde{Y} \longrightarrow Y$ be the normalization of Y, $i : Y \hookrightarrow \mathbb{CP}^n$ the closed embedding of Y in $\mathbb{CP}^n$, and (V, L) the linear system on the smooth curve $\tilde{Y}$ associated to the map $i \circ \nu : \tilde{Y} \longrightarrow \mathbb{CP}^n$.

Let $C_{n-1} = \phi_{n-1}(\tilde{Y})$ be the $(n-1)$-associated curve of (V, L) and let $\phi_{n-1}(x) = p \in (C_{n-1})_{\text{Smooth}}$. Then we have that $\mathbb{P}(V_{x,a_2(x)}) \subset Y^*$, and it follows that Y^* is a hypersurface which at the same time is a fiber bundle with fiber $\mathbb{CP}^{n-2}$ over the smooth part of the curve C_{n-1}. ■

We have the following important result (see [Ern94], Theorem 3.2, p.8).

Theorem 1.5.5 (Ernstrom). *For every non-degenerate subvariety $Y \subset \mathbb{CP}^n$, there exists a unique function $Eu_Y \in \mathcal{F}(\mathbb{CP}^n)$ that satisfies the following properties:*

1. *$Supp(Eu_Y) \subset Y$;*
2. *$Supp(\overline{Eu_Y^*}) \subset Y^*$;*
3. *$Eu_Y(x) = 1$ for $x \notin Y_{Sing}$.*

1.5.6 Example (The local Euler obstruction of a non-degenerate smooth curve): Let $Y \subset \mathbb{CP}^n$ be a non-degenerate smooth curve and let $f \in \mathcal{F}(\mathbb{CP}^n)$ be the function $\mathbf{1}_Y$. If $H \in \mathbb{CP}^{n*}$, then we have

$$(\mathbf{1}_Y)^*(H) = \int_H \mathbf{1}_Y(x)\, d\chi(x) = \chi(Y \cap H) = \#(Y \cap H).$$

If H is generic, then $\#(Y \cap H) = \deg(Y)$, so we have $\overline{\mathbf{1}_Y^*}(H) = \#(Y \cap H) - \deg(Y)$.

Observe that $\overline{\mathbf{1}_Y^*}(H) \neq 0$ if and only if $Y \cap H$ is singular, and since Y is smooth, this says that $\overline{\mathbf{1}_Y^*}(H) \neq 0$ if and only if $H \in C^*$. It follows from Theorem 1.5.5 that $\mathrm{Eu}_Y = \mathbf{1}_Y$. In fact, it is true that $\mathrm{Eu}_Y = \mathbf{1}_Y$ for any non-degenerate smooth subvariety $Y \subset \mathbb{CP}^n$. *

The function Eu_Y was introduced by R. MacPherson and is called the **local Euler obstruction** of Y. In fact, Ernstrom proves that if the dimension of Y is k and the dimension of its dual Y^* is k^*, then

$$(\mathrm{Eu}_Y)^* = (-1)^{k+r-1-k^*}\mathrm{Eu}_{Y^*} + \lambda((\mathrm{Eu}_Y)^*) \cdot \mathbf{1}_{\mathbb{CP}^{n*}}. \tag{1.19}$$

1.5.1 Generalized Viro formulas for non-degenerate smooth curves

Let $C \subset \mathbb{CP}^n$ be a non-degenerate curve. Then the Ernstrom formula (1.19) gives us

$$(\mathrm{Eu}_C)^* = -\mathrm{Eu}_{C^*} + \deg(C) \cdot \mathbf{1}_{\mathbb{CP}^{n*}}, \tag{1.20}$$

where $\mathrm{Eu}_C \in \mathcal{F}(\mathbb{CP}^n)$ and $\mathrm{Eu}_{C^*} \in \mathcal{F}(\mathbb{CP}^{n*})$ are the local Euler obstruction functions associated to C and C^*.

The following Lemma is an easy generalization of Paragraph 6.C in [Vir88].

Lemma 1.5.7. *Let $C \subset \mathbb{CP}^n$ be a non-degenerate curve. Then $\int_{\mathbb{CP}^n} Eu_C(x)\, d\chi(x) = 2deg(C) - deg(C^*)$.*

Proof. Let $\ell \subset \mathbb{CP}^{n*}$ be a line. If ℓ is generic, then $\int_\ell \mathrm{Eu}_{C^*}(x)\, d\chi(x) = \deg(C^*)$, since C^* is a hypersurface and $\mathrm{Eu}_{C^*}(x) = 1$ for $x \in C^*_{\mathrm{Smooth}}$. We now integrate (1.20) over a generic line $\ell \subset \mathbb{CP}^{n*}$ to find:

$$\int_\ell (\mathrm{Eu}_C)^*(x)\, d\chi(x) = \int_\ell [\deg(C)\mathbf{1}_{\mathbb{CP}^n}(x) - \mathrm{Eu}_{C^*}](x)\, d\chi(x) = \chi(\ell)\deg(C) - \deg(C^*).$$

On the other hand

$$\int_\ell (\mathrm{Eu}_C)^*(x)\, d\chi(x) = \int_\ell \left(\int_{y\in\ell} \mathrm{Eu}_C(y)\, d\chi(y) \right) d\chi(x) = \int_{\mathbb{CP}^n} \mathrm{Eu}_C(x)\, d\chi(x),$$

by an application of the Fubini theorem for integration with respect to the Euler characteristic. The result follows. ■

Remark 1.5.8: *If the curve C is non-degenerate and smooth, the previous relation gives the classical formula $\chi(C) = 2deg(C) - deg(C^*)$.*

In [Vir88], O. Viro proves the following result.

Theorem 1.5.9 (Viro). *Let $C \subset \mathbb{CP}^n$ be a non-degenerate curve.*

1. *If $n = 2$, then $Eu_C = mult_C$ and $deg(C) - \int_{\mathbb{RP}^2} mult_C(x) d\chi(x) = deg(C^*) - \int_{\mathbb{RP}^{2*}} mult_{C^*}(x) d\chi(x)$.*
2. *If $n = 3$, then $\int_{\mathbb{RP}^3} Eu_C(x)\, d\chi(x) = -\int_{\mathbb{RP}^{3*}} Eu_{C^*}(x)\, d\chi(x)$*

In fact, the proof of the previous Theorem (given in [Vir88] pp.132-134), can be easily generalized for any $n \geq 2$ by using Ernstrom formula for curves (1.20) and Lemma 1.5.7.

Theorem 1.5.10 (Generalized Viro formulas). *Let $C \subset \mathbb{CP}^n$ be a non-degenerate curve.*

1. *If n is even, then $deg(C) - \int_{\mathbb{RP}^n} Eu_C(x)\, d\chi(x) = deg(C^*) - \int_{\mathbb{RP}^{n*}} Eu_{C^*}(x)\, d\chi(x)$.*
2. *If n is odd, then $\int_{\mathbb{RP}^n} Eu_C(x)\, d\chi(x) = -\int_{\mathbb{RP}^{n*}} Eu_{C^*}(x)\, d\chi(x)$*

Proof. We integrate Equation (1.20) over $\mathbb{RP}^{n*}$ to get:

$$\int_{\mathbb{RP}^{n*}} (\mathrm{Eu}_C)^*(x) d\chi(x) = \int_{\mathbb{RP}^{n*}} [-\mathrm{Eu}_{C^*} + \deg(C)\cdot \mathbf{1}_{\mathbb{CP}^{r*}}](x) d\chi(x) = -\int_{\mathbb{RP}^{n*}} \mathrm{Eu}_{C^*}(x) d\chi(x) + \deg(C)\chi(\mathbb{RP}^{n*}).$$

Since $\chi(\mathbb{RP}^n)$ equals 1 for n even and equals 0 for n odd, we have Then

$$\int_{\mathbb{RP}^{n*}} (\mathrm{Eu}_C)^*(x)\, d\chi(x) = \begin{cases} -\int_{\mathbb{RP}^{n*}} \mathrm{Eu}_{C^*}(x)\, d\chi(x) + \deg(C), & \text{if } n \text{ is even,} \\ -\int_{\mathbb{RP}^{n*}} \mathrm{Eu}_{C^*}(x)\, d\chi(x), & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, we get from (1.18) and Lemma 1.5.7 that

$$\int_{\mathbb{RP}^{n*}} (\mathrm{Eu}_C)^*(x)\, d\chi(x) = \begin{cases} 2\deg(C) - \deg(C^*) - \int_{\mathbb{RP}^n} \mathrm{Eu}_C(x)\, d\chi(x), & \text{if } n \text{ is even,} \\ \int_{\mathbb{RP}^n} \mathrm{Eu}_C(x)\, d\chi(x), & \text{if } n \text{ is odd.} \end{cases}$$

The result follows. ■

We have the following important result for smooth, non-degenerate curves (see [Hol04], p. 15).

Theorem 1.5.11 (Dimca-Nemethi). *If $C \subset \mathbb{CP}^r$ is a smooth, non-degenerate curve, then $Eu_{C^*} = mult_{C^*}$.*

Corolary 1.5.12 (Generalized Viro formulas, smooth case). *Let $C \subset \mathbb{CP}^n$ be a smooth, non-degenerate curve of genus g. Then*

$$\int_{\mathbb{RP}^{n*}} mult_{C^*}(x)\, d\chi(x) = \begin{cases} deg(C) + 2g - 2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.21}$$

Proof. It follows from Example 1.5.6 that $\mathrm{Eu}_C = \mathbf{1}_C$, so $\int_{\mathbb{RP}^n} \mathrm{Eu}_C(x)\, d\chi(x) = \chi(C(\mathbb{R})) = 0$, since $C(\mathbb{R})$ is a disjoint union of circles. By combining Theorem 1.5.10 and Theorem 1.5.11, we get

$$\int_{\mathbb{RP}^{n*}} \mathrm{mult}_{C^*}(x)\, d\chi(x) = \begin{cases} \deg(C^*) - \deg(C), & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.22}$$

The last step is to use that $\deg(C^*) = 2\deg(C) + 2g - 2$ from Remark 1.5.8. ■

1.5.2 Singularities of maps and the incidence variety of a smooth curve

References for the following material are [Ron98] and [GG73].

Definition: Let $U \subset \mathbb{C}^n$ be an open set and let $f : U \longrightarrow \mathbb{C}^n$ be an holomorphic map, or let $U \subset \mathbb{R}^n$ be an open set and let $f : U \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. We define the singular locus $\Sigma^1(f)$ by

$$\Sigma^1(f) = \{x \in U \ : \ \dim \mathrm{Ker}(d_x f) = 1\}. \tag{1.23}$$

If $y \in \Sigma^1(f)$, we can assume (see [Ron98], p. 198) that f can be expressed as

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_n)))$$

with $\frac{\partial g}{\partial x_n}(y) = 0$.

Definition: The function $f : U \longrightarrow \mathbb{C}^n$ is said to be Σ^1-**transversal** at $y \in \Sigma^1(f)$ if there exists $i \in [n] = \{1, \ldots, n\}$ such that $\frac{\partial^2 g}{\partial x_i \partial x_n}(y) \neq 0$.

Definition: Let $f : U \longrightarrow \mathbb{C}^n$ be as above. For $k \geq 1$, the singular locus $\Sigma^k(f)$ is defined by

$$\Sigma^k(f) = \{x \in U \ : \ \tfrac{\partial^\ell g}{\partial x_n^\ell}(x) = 0 \text{ for } \ell = 1, \ldots, k\}. \tag{1.24}$$

The function f is said to be Σ^k-**transversal** at $y \in \Sigma^k(f)$ if the set of equations $\{\frac{\partial^\ell g}{\partial x_n^\ell}(x) = 0\}_{\ell=1,\ldots,k}$ has maximal rank at y.

Let X, Y be smooth manifolds of the same dimension (respectively complex manifolds of the same dimension) and let $f : X \longrightarrow Y$ be a $\mathcal{C}^\infty$ map which is proper[4] (respectively a holomorphic map). Then the singular loci $\Sigma^k(f)$ and the notion of Σ^k-transversality can be introduced by using local coordinates. If $\dim(X) = n$ and $f : X \longrightarrow Y$ is Σ^k-transversal for all $k = 1, \ldots, n$, then we have

1. a flag of smooth submanifolds $\Sigma^n(f) \subset \cdots \subset \Sigma^1(f) \subset X$, where $\Sigma^k(f)$ is a smooth submanifold of codimension k of X.

2. a stratification $X = \bigcup_i \Sigma^{i,\circ}(f)$, where $\Sigma^{i,\circ}(f) := \Sigma^i(f) \setminus \Sigma^{i+1}(f)$ for $i \geq 1$.

1.5.13 Example: The following example can be found in [Ron98]. Let $U \subset \mathbb{R}^3$ and let $f : U \longrightarrow \mathbb{R}^3$ be a map which is Σ^k-transversal for $k = 1, 2, 3$. If $p \in \Sigma^{3,\circ}(f)$, then the surface $f(\Sigma^1(f))$ has a swallow-tail singularity at the point $y = f(p)$. See Figure 1.1.

[4] A morphism between topological spaces is proper if the pre-image of a compact set is compact.

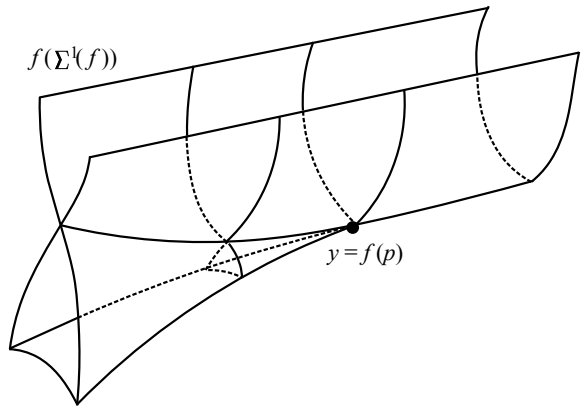


Figure 1.1: The local geometry of the surface $f(\Sigma^1(f))$ at the image $y = f(p)$ of a point $p \in \Sigma^{3,\circ}(f)$.

Definition: Let $f : X \longrightarrow Y$ be a holomorphic map (respectively a proper $\mathcal{C}^\infty$ map) between n-dimensional complex manifolds (respectively smooth manifolds) which is $\Sigma^k(f)$-transversal for all k. We define the singular locus $M_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ}) \subset \Sigma^{i_1,\circ}(f)$

$$\begin{aligned} M_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ}) = \{y_1 \in \Sigma^{i_1,\circ}(f) \mid \exists y_k \in \Sigma^{i_k,\circ}(f) \text{ for } 2 \leq k \leq \ell \text{ such that} \\ y_a \neq y_b \text{ and } f(y_a) = p \text{ for } 1 \leq a \neq b \leq \ell\}. \end{aligned} \tag{1.25}$$

We say that f is $M_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ})$-**transverse** if the vector spaces

$$d_{y_1} f(T_{y_1} \Sigma^{i_1,\circ}(f)), \ldots, d_{y_\ell} f(T_{y_\ell} \Sigma^{i_\ell,\circ}(f))$$

are in general position in $T_p Y$. Finally we set $N_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ}) = f(M_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ}))$.

1.5.14 Example: The following example can be found in [Ron98]. Let $U \subset \mathbb{R}^3$ and let $f : U \longrightarrow \mathbb{R}^3$ be a map which is Σ^k-transversal for $k = 1, 2, 3$. Then the singular loci $M_\ell(f, \Sigma^{i_1,\circ}, \ldots, \Sigma^{i_\ell,\circ})$ has the following components:

1. points $M_2(f, \Sigma^{2,\circ}, \Sigma^{1,\circ})$, which are smooth points of $\Sigma^{2,\circ}$ and of $\overline{M_2(f, \Sigma^{1,\circ}, \Sigma^{1,\circ})}$. See Figure 1.2a);
2. points $M_2(f, \Sigma^{1,\circ}, \Sigma^{2,\circ})$, which are singular points of $\overline{M_2(f, \Sigma^{1,\circ}, \Sigma^{1,\circ})}$. See Figure 1.2b);
3. points $M_3(f, \Sigma^{1,\circ}, \Sigma^{1,\circ}, \Sigma^{1,\circ})$, which are singular points of $\overline{M_2(f, \Sigma^{1,\circ}, \Sigma^{1,\circ})}$. See Figure 1.2c);

The surface $f(\Sigma^1(f))$ is singular along the curve $f(\Sigma^{2,\circ})$ and at the point $p = N_2(\Sigma^{1,\circ}, \Sigma^{2,\circ}) = N_2(\Sigma^{2,\circ}, \Sigma^{1,\circ})$, it has a singularity which is locally the transverse intersection of two real branches of $f(\Sigma^1(f))$ as in Figure 1.2 d). Finally, at the points $N_3(\Sigma^{1,\circ}, \Sigma^{1,\circ}, \Sigma^{1,\circ})$, the surface $f(\Sigma^1(f))$ is locally the transverse intersection of three real branches, as in Figure 1.2 e). $*$

Let X be a non-singular complex curve and (V, L) a g^r_d on X such that $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is a closed embedding.

Definition: The **incidence variety** of the smooth, non-degenerate curve $C_0 \subset \mathbb{P}(V^*)$ is the smooth variety

$$I_{C_0} = \{(x, H) \in \mathbb{P}(V^*) \times \mathbb{P}(V) \ : \ x \in C_0 \cap H\}. \tag{1.26}$$

The second projection $\pi_2 : I_{C_0} \longrightarrow \mathbb{P}(V)$ gives us a holomorphic map between two smooth varieties of the same dimension such that the singular locus $\Sigma^1(\pi_2)$ is the conormal variety N_{C_0} of C_0, which is smooth since C_0 is smooth. It follows that π_2 is Σ^1-transversal and $\pi_2(\Sigma^1(\pi_2)) = C_0^*$.

We begin studying the singular loci $\Sigma^k(\pi_2)$ of the morphism $\pi_2 : I_{C_0} \longrightarrow \mathbb{P}(V)$.

Proposition 1.5.15. *Let X be a non-singular complex curve and (V, L) a g^r_d on X such that $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is a closed embedding. If (V, L) is simple, then the morphism $\pi_2 : I_{C_0} \longrightarrow \mathbb{P}(V)$ is Σ^{1_k}-transversal for all $k = 1, \ldots, r$.*

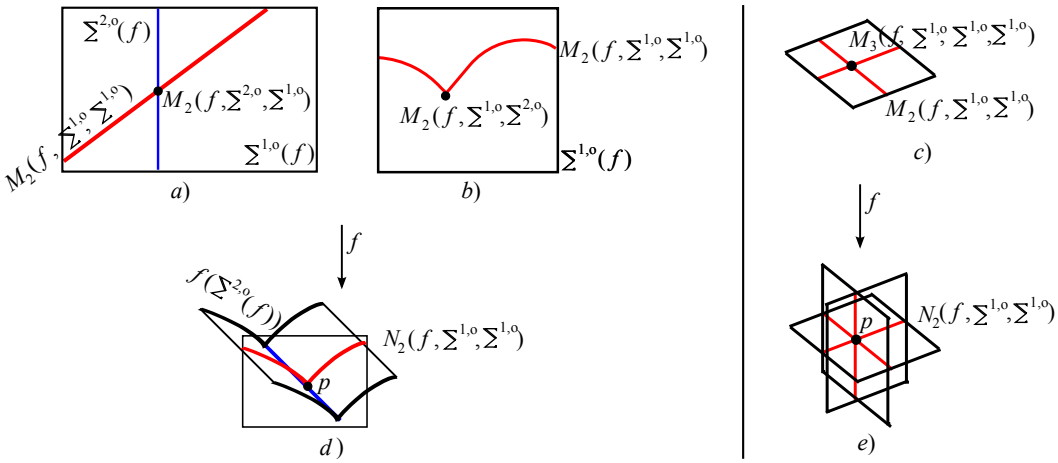


Figure 1.2: The local geometry a) at a point in $N_2(\Sigma^{1,\circ},\Sigma^{2,\circ})$, and b) at a point in $N_3(\Sigma^{1,\circ},\Sigma^{1,\circ},\Sigma^{1,\circ})$.

Proof. For a given point $x \in X$ let τ be a local parameter of X at x and let $\{s_0,\dots,s_r\}$ be a basis for V such that $\mathrm{ord}_x(s_i) = a_i(x)$ for $i = 0,\dots,r$. Then the map $\phi_0 : X \longrightarrow \mathbb{CP}^r$ is given by $x \mapsto [s_0(x) : \cdots : s_r(x)]$. Let $[T_0 : \cdots : T_r]$ be homogeneous coordinates for $\mathbb{CP}^r$, then ϕ_0 is given locally in the affine chart $\{T_0 = 1\}$ by $\tau \mapsto (\tau^{a_1(x)}g_1(\tau),\dots,\tau^{a_r(x)}g_r(\tau))$, where $g_i(\tau)$ are holomorphic near 0 and $g_i(0) \neq 0$.

Let us endow $\mathbb{CP}^{r*}$ with projective coordinates $[U_0 : \cdots : U_r]$, then the set of hyperplanes $H \in \mathbb{CP}^{r*}$ passing through points $p(\tau) = (\tau^{a_1(x)}g_1(\tau),\dots,\tau^{a_r(x)}g_r(\tau))$ is parameterized by $\mathbb{CP}^{r-1}$ as

$$[x_1 : \cdots : x_r] \mapsto \left[-\sum_{i=1}^{r} x_i\tau^{a_i(x)}g_i(\tau) : x_1 : \cdots : x_r\right], \quad [x_1 : \cdots : x_r] \in \mathbb{CP}^{r-1}.$$

Let $\ell = 1,\dots,r$. In the chart $\{U_\ell = 1\}$ of $\mathbb{CP}^{r*}$ we have a local expression of each fiber of the map $\pi_1 : I_{C_0} \longrightarrow \mathbb{CP}^r$, namely $(x_1,\cdots,\hat{x_\ell},\dots,x_r) \mapsto (-\sum_{i\neq\ell} x_i\tau^{a_i(x)}g_i(\tau) - \tau^{a_\ell(x)}g_\ell(\tau), x_1,\dots,\hat{x_\ell},\dots,x_r)$. So we have a local parametrization of I_{C_0}:

$$(x_1,\dots,\hat{x_\ell},\dots,x_r,\tau) \mapsto \big(\tau^{a_1(x)}g_1(\tau),\dots,\tau^{a_r(x)}g_r(\tau)\big), \big(-\sum_{i\neq\ell} x_i\tau^{a_i(x)}g_i(\tau) - \tau^{a_\ell(x)}g_\ell(\tau), x_1,\dots,\hat{x_\ell},\dots,x_r\big)\big).$$

There is a permutation $[S_0 : \cdots : S_r]$ of the projective coordinates $[U_0 : \cdots : U_r]$ for $\mathbb{CP}^{r*}$, such that a local expression for π_2 with respect to the charts $\{T_0 = 1\}$ and $\{U_\ell = 1\}$ of $\mathbb{CP}^r \times \mathbb{CP}^{r*}$, and the chart $\{U_\ell = 1\}$ of $\mathbb{CP}^{r*}$ is

$$(x_1,\dots,\hat{x_\ell},\dots,x_r,\tau) \mapsto \big(x_1,\dots,\hat{x_\ell},\dots,x_r, -\sum_{i\neq\ell} x_i g_i(\tau)\tau^{i+\alpha_i(x)} - g_\ell(\tau)\tau^{\ell+\alpha_\ell(x)}\big), \quad g_i(0) \neq 0. \quad (1.27)$$

Suppose that $\ell = r$, and consider the function $h(x_1,\dots,x_{r-1},\tau) = \sum_{i=1}^{r-1} x_i g_i(\tau)\tau^{i+\alpha_i(x)} + g_r(\tau)\tau^{r+\alpha_r(x)}$. We know that $\frac{\partial h}{\partial \tau} = 0$ gives a local equation for $\Sigma^1(\pi_2)$. If ϕ_0 is unramified at x, then $\alpha_1(x) = 0$ and for $s \in \mathbb{C}$ near 0, we have

$$\frac{\partial h}{\partial \tau}(x_1,\dots,x_{r-1},s) = x_1\frac{\partial}{\partial \tau}(g_1(\tau)\tau)|_{\tau=s} + \sum_{i=2}^{r-1} x_i\frac{\partial}{\partial \tau}(g_i(\tau)\tau^{i+\alpha_i(x)})|_{\tau=s} + \frac{\partial}{\partial \tau}(g_r(\tau)\tau^{r+\alpha_r(x)})|_{\tau=s}.$$

Observe that $\frac{\partial^2 h}{\partial x_1 \partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial h}{\partial x_1}\right) = \frac{\partial}{\partial \tau}(g_1(\tau)\tau)$, so π_2 is Σ^1-transversal at every point $(p_1,\dots,p_{r-1},s) \in \Sigma^1(\pi_2)$ with $g_1(s) + sg_1(s) \neq 0$. In particular, if $s = 0$, we have $\frac{\partial h}{\partial \tau}(x_1,\dots,x_{r-1},0) = x_1 g_1(0) = 0$ if and only if $x_1 = 0$, and since $g_1(0) + 0g_1(0) = g_1(0) \neq 0$, we conclude that π_2 is Σ^1-transversal.

We know that $\frac{\partial^2 h}{\partial \tau^2} = 0$ gives a local equation for $\Sigma^2(\pi_2)$ inside $\Sigma^1(\pi_2)$. If we also have $\alpha_2(x) = 0$, then we have for $s \in \mathbb{C}$ near 0:

$$\frac{\partial^2 h}{\partial \tau^2}(x_1,\dots,x_{r-1},s) = \sum_{i=1}^{2} x_i \frac{\partial^2}{\partial \tau^2}(g_i(\tau)\tau^i)|_{\tau=s} + \sum_{i=3}^{r-1} x_i \frac{\partial^2}{\partial \tau^2}(g_i(\tau)\tau^{i+\alpha_i(x)})|_{\tau=s} + \frac{\partial^2}{\partial \tau^2}(g_r(\tau)\tau^{r+\alpha_r(x)})|_{\tau=s}.$$

Observe that $\frac{\partial^3 h}{\partial x_2 \partial^2 \tau} = \frac{\partial^2}{\partial \tau^2}\left(\frac{\partial h}{\partial x_2}\right) = \frac{\partial^2}{\partial \tau^2}(g_2(\tau)\tau^2)$, so π_2 is Σ^2-transversal at every point $(p_1,\dots,p_{r-1},s) \in \Sigma^2(\pi_2)$ with $g_2(s)s^2 + 2g_1(s)s + 2g_2(s) \neq 0$. In particular, if $s = 0$, we have $\frac{\partial^2 h}{\partial \tau^2}(0,x_2,\dots,x_{r-1},0) = 2x_2 g_2(0) = 0$ if and only if $x_2 = 0$, and since $g_2(0) \neq 0$, π_2 is Σ^2-transversal.

Let us consider $2 < k < r$, then we know that $\frac{\partial^k h}{\partial \tau^k} = 0$ gives a local equation for $\Sigma^k(\pi_2)$ inside $\Sigma^{k-1}(\pi_2)$. Suppose that $\alpha_1(x) = \cdots = \alpha_k(x) = 0$, then $\frac{\partial^{k+1} h}{\partial x_k \partial^k \tau} = \frac{\partial^k}{\partial \tau^k}\left(\frac{\partial h}{\partial x_k}\right) = \frac{\partial^k}{\partial \tau^k}(g_k(\tau)\tau^k)$, so π_2 is Σ^k-transversal at every point $(p_1,\dots,p_{r-1},s) \in \Sigma^k(\pi_2)$ with $\frac{\partial^k}{\partial \tau^k}(g_k(\tau)\tau^k)|_{\tau=s} \neq 0$. In particular, if $s = 0$, we have $\frac{\partial^k h}{\partial \tau^k}(0,\dots,x_k,\dots,x_{r-1},0) = k! x_k g_k(0) = 0$ if and only if $x_k = 0$, and $\frac{\partial^k}{\partial \tau^k}(g_k(\tau)\tau^k)|_{\tau=0} = k! g_k(0) \neq 0$, so π_2 is Σ^k-transversal for $2 < k < r$.

We have just shown that π_2 is Σ^k-transversal for $0 < k < r$ and $s \in \mathbb{C}$ near 0 when $\alpha_1(x) = \cdots = \alpha_{r-1}(x) = 0$. If (V,L) is a simple g^r_d, then either $\alpha_r(x) = 0$ or $\alpha_r(x) = 1$. If $\alpha_r(x) = 0$, then the equation $\frac{\partial^r h}{\partial \tau^r} = 0$ has no solution $(p_1,\dots,p_{r-1},s) \in \Sigma^{r-1}(\pi_2)$. On the other hand, if $\alpha_r(x) = 1$, then $h(x_1,\dots,x_{r-1},\tau) = \sum_{i=1}^{r-1} x_i g_i(\tau)\tau^i + g_r(\tau)\tau^{r+1}$, and the point $(0,0,\dots,0)$ is a solution of the equation $\frac{\partial^r h}{\partial \tau^r} = 0$. Finally, we have that $\frac{\partial^{r+1} h}{\partial \tau^{r+1}}(0,\dots,0) = (r+1)! g_r(0) \neq 0$, so the point $(0,\dots,0)$ is in $\Sigma^r(\pi_2)$ and π_2 is Σ^r-transversal at this point.

It follows that $\pi_2 : I_{C_0} \longrightarrow \mathbb{CP}^{r*}$ is Σ^k-transversal for all $k = 1,\dots,r$ whenever (V,L) is a simple g^r_d. ■

If π_2 is Σ^k-transversal for all $k = 1,\dots,r$, then we can study the topology of the dual hypersurface C_0^* of C_0 by using the tools introduced before. In particular, we can study the singular locus $(C_0^*)_{\mathrm{Sing}}$ of C_0^*, since $\pi_2 : \Sigma^1(\pi_2) \longrightarrow C_0^*$ is a resolution of singularities.

We have the following result (see [Hol04], Proposition 2.1.1 for a proof).

Proposition 1.5.16. *Let $C \subset \mathbb{CP}^r$ be a smooth, non-degenerate curve. Define*

$$C^*_{cusp} = \{H \in C^* \ : \ \exists\, p \in C \ \text{ s. t. } \ \ell(\mathcal{O}_{C\cap H,p}) \geq 3\},$$
$$C^*_{node} = \overline{\{H \in C^* \ : \ \exists p_1 \neq p_2 \in C \text{ s. t. } \ell(\mathcal{O}_{C\cap H,p_1}), \ell(\mathcal{O}_{C\cap H,p_2}) \geq 2\}}$$

Then $C^*_{Sing} = C^*_{cusp} \cup C^*_{node}$.

If C is a smooth curve, since $\mathrm{mult}_{C^*}(H) = \deg(C) - \#(C \cap H)$ we have that $H \in C^*_{\mathrm{Sing}}$ if and only if $\mathrm{mult}_{C^*}(H) \geq 2$. We have that $\mathrm{mult}_{C^*}(H) = 2$ if and only if the 0-cycle $[C \cap H]$ is of the form :

$$[C \cap H] = \begin{cases} 3p + q_1 + \cdots + q_{\deg(C)-3} \text{ and all the points } p, q_i \text{ are different,} \\ 2p_1 + 2p_2 + q_1 + \cdots + q_{\deg(C)-4} \text{ and all the points } p_j, q_i \text{ are different.} \end{cases}$$

In the first case, H represents a generic point of C^*_{cusp}, and in the second case, H represents a general point of C^*_{node}.

If C is non-singular and $\pi_2 : I_C \longrightarrow \mathbb{CP}^{r*}$ is Σ^k-transversal for all $k = 1,\dots,r$, then we have $C^*_{\mathrm{cusp}} = \pi_2(\Sigma^2(\pi_2))$ and $C^*_{\mathrm{node}} = \overline{N_2(\Sigma^{1,\circ}, \Sigma^{1,\circ})}$.

Chapter 2

Algebraic modifications on very affine, generically integral varieties

2.1.– Introduction

Let $\mathbb{K}$ be the non-Archimedean field $F((t^{\mathbb{R}}))$. This chapter is dedicated to the study of the relationship between the algebraic intersection theory in some particular affine $\mathbb{K}$-varieties and the tropical intersection theory in their tropicalization.

In Section 2.2 we present some general properties of very affine varieties, including their intrinsic embedding into an algebraic $\mathbb{K}$-torus. We introduce the concept of a generically integral $\mathbb{K}$-variety as being a very affine $\mathbb{K}$-variety X that admits a closed embedding $g : X \hookrightarrow (\mathbb{K}^*)^n$ such that $g(X)$ has simple tropicalization. We show that the very affine $\mathbb{K}$-varieties which are generically integral are characterized by their intrinsic embedding (see Theorem 2.2.2).

Theorem. *Let X be a very afine $\mathbb{K}$-variety with intrinsic embedding $f : X \longrightarrow (\mathbb{K}^*)^m$. Then X is generically integral if and only if $f(X)$ has simple tropicalization.*

We also introduce the notion of tropical Cartier divisor ϕ defined on a tropical k-cycle A in $\mathbb{R}^n$ as a continuous piecewise integer affine linear function $\phi : A \longrightarrow \mathbb{R}$ with the property that for any regular point $p \in A$ such that the point $q = (p, \phi(p))$ in the graph $\Gamma_\phi(A)$ is regular, then the index $[\Lambda_p : \pi(\Lambda_q)]$ is equal to one (the lattice Λ_p is defined in page 4). If $\phi : A \longrightarrow \mathbb{R}$ is a tropical Cartier divisor and $Y \subset A$ is a tropical ℓ-cycle, we define an intersection product $Y \mapsto Y \cdot \phi$ which generalizes the intersection product introduced by Allermann and Rau in [AR09] to tropical cycles A which are not fans and to functions ϕ which are not necessarily the restriction of a tropical rational function defined on $\mathbb{R}^n$.

Let $X \subset (\mathbb{K}^*)^n$ be a subvariety. Based in previous work [BL12] by E. Brugallé and L. López de Medrano, we develop the concept of (algebraic) $\emptyset$-**modification** of X along a family $\mathbf{f} = (f_1, \ldots, f_b)$ of functions $f_1, \ldots, f_b \in \mathbb{K}[X]$, with $b > 0$. The graph $X_\emptyset(\mathbf{f}) = \{(x, f_1(x), \ldots, f_b(x)) \ : \ x \in X_{f_1 \cdots f_b}\}$ is a closed subscheme of the product $(\mathbb{K}^*)^n \times (\mathbb{K}^*)^b$, and the projection $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^*)^b \longrightarrow (\mathbb{K}^*)^n$ induces an open embedding $\Pi : X_\emptyset(\mathbf{f}) \longrightarrow X$. The tropicalization $\pi : \mathrm{Trop}(X_\emptyset(\mathbf{f})) \longrightarrow \mathrm{Trop}(X)$ of the open embedding $\Pi : X_\emptyset(\mathbf{f}) \longrightarrow X$ is by definition the $\emptyset$-**modification** of X along the family $\mathbf{f} = (f_1, \ldots, f_b)$.

We show that if $X \subset (\mathbb{K}^*)^n$ has simple tropicalization and $b = 1$, then the $\emptyset$-modification $\pi : \mathrm{Trop}(X_\emptyset(f)) \longrightarrow \mathrm{Trop}(X)$ induces a tropical Cartier divisor $\mathcal{T}(f) : \mathrm{Val}(X) \longrightarrow \mathbb{R}$. In particular, if $Y \subset X$ a closed subscheme of pure dimension, then we can define the tropical intersection $\mathrm{Trop}(Y).\mathcal{T}(f)$ of $\mathrm{Trop}(Y)$ with $\mathcal{T}(f)$.

If in addition $X \subset (\mathbb{K}^*)^n$ is non-singular, $Y \subset X$ is a closed subscheme of pure dimension one and the schemes Y and $\mathrm{div}_X(f)$ have proper intersection in X, then the tropical 0-cycles $\mathrm{Trop}(Y \cap \mathrm{div}_X(f))$ and $\mathrm{Trop}(Y).\mathcal{T}(f)$ can be compared in the following sense (See Theorem 2.4.9).

Theorem. *Let $X \subset (\mathbb{K}^*)^n$ be a non-singular variety with simple tropicalization, $C \subset X$ a purely 1-dimensional closed subscheme, and $f \in \mathbb{K}[X]$ such that C and $\mathrm{div}_X(f)$ intersect properly. Let E be a connected component of the set $\mathrm{Val}(C) \cap \mathrm{Val}(\mathrm{div}_X(f))$, then we have*

$$\sum_{\mathrm{Val}(x) \in E} \ell(\mathcal{O}_{C \cap \mathrm{div}_X(f), x}) \leq \sum_{p \in E} w_{\mathrm{Trop}(C).\mathcal{T}(f)}(p),$$

where $\mathrm{Trop}(C).\mathcal{T}(f)$ is the tropical intersection product of $\mathrm{Trop}(C)$ with the tropical Cartier divisor $\mathcal{T}(f) : \mathrm{Val}(X) \longrightarrow \mathbb{R}$. If E is compact, then equality is attained.

This is a generalization of a Theorem in [BL12], which treats the case of two curves intersecting properly in $X = (\mathbb{K}^*)^2$ (see Theorem 2.4.8 in this work).

2.2.– Very affine and generically integral very affine varieties

2.2.1 Very affine varieties

We begin our exposition with the following example. Consider an affine hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ in $\mathbb{K}^d$ and set $X = \mathbb{K}^d \setminus \mathcal{H}$. We have that X is a non-singular affine variety with a morphism $f : X \longrightarrow (\mathbb{K}^*)^m$ induced by the linear equations of the elements of $\mathcal{H}$. We will be interested in the case when $f : X \longrightarrow (\mathbb{K}^*)^m$ is a closed embedding.

Let us endow $\mathbb{K}^d$ with its canonical bilinear form. A hyperplane arrangement $\mathcal{H}$ is said to be essential if and only if the space spanned by the normals to the elements in $\mathcal{H}$ is the whole $\mathbb{K}^d$.

We have that the morphism $f : X \longrightarrow (\mathbb{K}^*)^m$ is a closed embedding if and only if the hyperplane arrangement $\mathcal{H}$ is essential. A natural generalization of this objects is given by the class of very affine $\mathbb{K}$-varieties.

Definition: A $\mathbb{K}$-variety is **very affine** if it admits a closed embedding into some torus $(\mathbb{K}^*)^n$.

If X is such a variety, then it is affine and its ring of regular functions $\mathbb{K}[X]$ is generated by its group of multiplicative units $\mathbb{K}[X]^*$. The fact that $\mathbb{K}[X]^*/\mathbb{K}^*$ is a free abelian group of finite rank is a theorem of P. Samuel that can be deduced also from the Nagata exact sequence.

If we choose a basis $\{[f_1], \ldots, [f_m]\}$ for $\mathbb{K}[X]^*/\mathbb{K}^*$, then the map $p \mapsto (f_1(p), \ldots, f_m(p))$ is a closed embedding $f : X \hookrightarrow (\mathbb{K}^*)^m$ which is well-defined up to the natural multiplicative action of $\mathrm{GL}_m(\mathbb{Z})$ in $(\mathbb{K}^*)^m$. This closed embedding is the **intrinsic embedding** of X, and it controls the image of any morphism $X \longrightarrow (K^*)^n$, as the following result shows.

Lemma 2.2.1. *Let $f : X \longrightarrow (\mathbb{K}^*)^m$ be the intrinsic embedding of a very affine variety X. Then for any morphism $g : X \longrightarrow (\mathbb{K}^*)^n$, there exists a unique homomorphism of tori $\Phi : (\mathbb{K}^*)^m \to (\mathbb{K}^*)^n$ such that the following diagram is commutative*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & (\mathbb{K}^*)^m \arrow[d, "\Phi"] \\ & (\mathbb{K}^*)^n. \end{tikzcd} \tag{2.1}$$

Proof. Suppose that the morphism g is given by $p \mapsto (g_1(p), \ldots, g_n(p))$, then since $\{[f_1], \ldots, [f_m]\}$ is a basis for $\mathbb{K}[X]^\times/\mathbb{K}^*$, we can write $[g_i] = \prod_j [f_j]^{a_{ij}}$ for some $a_{ij} \in \mathbb{Z}$, for $i = 1, \ldots, n$, $j = 1, \ldots, m$. We take Φ to be the homomorphism $\Phi : (\mathbb{K}^*)^m \to (\mathbb{K}^*)^n$ induced by the coefficients of the matrix $(a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}$.
■

Suppose now that $\mathbb{K} = F((t^{\mathbb{R}}))$. Recall that a subvariety $X \subset (\mathbb{K}^*)^n$ has simple tropicalization if $m_X(p) = 1$ for every regular point $p \in \mathrm{Val}(Z)$. We introduce the following concept.

Definition: Let X be a very-affine $\mathbb{K}$-variety. We say that X is **generically integral** if it admits a closed embedding $h : X \hookrightarrow (\mathbb{K}^*)^n$ such that the cycle $[h(X))]$ has simple tropicalization.

The importance of the intrinsic embedding of a very affine variety can be seen also in the following result.

Theorem 2.2.2. *Let X be a very afine variety with intrinsic embedding $f : X \longrightarrow (\mathbb{K}^*)^m$. Then X is generically integral if and only if $[f(X)]$ has simple tropicalization.*

Proof. We will show that if $[f(X)]$ is not a cycle with simple tropicalization, and $g : X \longrightarrow (\mathbb{K}^*)^n$ is any other closed embedding, then $[g(X)]$ is not a cycle with simple tropicalization.

We know by Lemma 2.2.1 that there exists a homomorphism of tori $\Phi : (\mathbb{K}^*)^m \longrightarrow (\mathbb{K}^*)^n$ such that $g(X) = \Phi(f(X))$. It follows that $\mathrm{Trop}(g(X)) = (\mathrm{Trop}(\Phi))_*(\mathrm{Trop}(f(X)))$ and by (1.9) we have

$$m_{g(X)}(p) = \sum_{q\in\phi^{-1}(p)} m_{f(X)}(q)[\Lambda_p : \phi(\Lambda_q)].$$

Since f is not generically integral, there exists a regular point $p \in \mathrm{Val}(g(X))$ such that $\pi^{-1}(p)$ consists of regular points $\{q_1, \ldots, q_s\}$ in $\mathrm{Val}(f(X))$ with $m_{f(X)}(q_i) > 1$ for some $i = 1, \ldots, s$. The result follows. ■

2.3.– Graph embeddings of closed subschemes of very affine varieties

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We will be interested in some stratifications on principal open subsets of non-singular very affine $\mathbb{K}$-varieties induced by finite families of principal Cartier divisors. The strata will consist of locally closed subschemes on these varieties which we will embed as closed subschemes of algebraic $\mathbb{K}$-tori using the graphs of the regular functions that define the family of principal Cartier divisors.

By embedding we mean a locally closed embedding. Let X be a very affine $\mathbb{K}$-variety and let f, g be non-zero elements in $\mathbb{K}[X]$. We denote

1. X_g the open subscheme supported on the principal open subset $\{x \in X \ : \ g(x) \neq 0\} \subset X$. Note that $X_{fg} = X_f \cap X_g$.;
2. $\mathrm{div}_X(g)$ the principal Cartier divisor on X defined by g;
3. if f is non-zero in X_g, then it defines a principal Cartier divisor $X_g \cap \mathrm{div}_X(f)$ of X_g which we will denote $\mathrm{div}_{X_g}(f)$.

Observe that $X_g = X$ if and only if $g \in \mathbb{K}[X]^\times$. We start with the following result.

Lemma 2.3.1. *Let X be a very affine $\mathbb{K}$-variety and let $g, f \in \mathbb{K}[X]$. If $h : X \hookrightarrow (\mathbb{K}^*)^n$ is a closed embedding, then $x \mapsto (h(x), \frac{1}{g(h(x))}, f(h(x)))$ defines a closed embedding $X_g \hookrightarrow (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \mathbb{K})$.*

Proof. Since $h : X \hookrightarrow (\mathbb{K}^*)^n$ is a closed embedding, we have that $x \mapsto (h(x), \frac{1}{g(h(x))})$ defines a closed embedding $\delta : X_g \hookrightarrow (\mathbb{K}^*)^n \times \mathbb{K}^*$. Let $F : (\mathbb{K}^*)^n \times \mathbb{K}^* \longrightarrow \mathbb{K}$ be a regular function such that $F|_{\delta(X_g)} = f$, and let γ be the closed embedding induced by the graph of F. Then the map $x \mapsto (h(x), \frac{1}{g(x)}, f(x))$ equals $\gamma \circ \delta$. ■

Remark 2.3.2: *Let X be a very affine $\mathbb{K}$-variety, $h : X \hookrightarrow (\mathbb{K}^*)^n$ a closed embedding and $g, f_1 \in \mathbb{K}[X]$. We will write $x \mapsto (x, \frac{1}{g(x)}, f_1(x))$ for the embedding $X_g \hookrightarrow (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \mathbb{K})$ of Lemma 2.3.1.*

Consider a closed embedding $X_g \hookrightarrow (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \mathbb{K})$ as in Remark 2.3.2. For $J = \emptyset, [1]$ we denote by:

1. $F_J(X_g) \hookrightarrow X_g$ the closed subscheme defined by the ideal $\langle f_j \ : \ j \in J\rangle \cdot \mathbb{K}[X_g] \subset \mathbb{K}[X_g]$ and $U_J(X_g) \hookrightarrow X_g$ the open subscheme $X_g \cap X_{\prod_{j \notin J} f_j}$;
2. $D^\circ_J(X_g) \hookrightarrow X_g$ the locally closed subscheme $U_J(X_g) \cap F_J(X_g)$ and $D_J(X_g) \hookrightarrow X_g$ the scheme-theoretic closure of $D^\circ_J(X_g)$.

We have the following diagram:

$$\begin{tikzcd} D^\circ_J(X_g) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & F_J(X_g) \arrow[d, hook] \\ U_J(X_g) \arrow[r, hook] & X_g, \end{tikzcd} \tag{2.2}$$

and the open embedding $D^\circ_J(X_g) \hookrightarrow D_J(X_g)$ is $(X_g)_{f_1} \hookrightarrow X_g$ for $J = \emptyset$ and $\mathrm{div}_{X_g}(f_1) = \mathrm{div}_{X_g}(f_1)$ for $J = [1]$.

2.3.3 Example: Let $X = (\mathbb{K}^*)^2 = \mathrm{Spec}(\mathbb{K}[x^{\pm 1}, y^{\pm 1}])$, $g = x - a$, $f_1 = (x-a)(y-b)$, $a, b \neq 0$. Then f_1 defines the regular function $f_1 = y - b$ on X_g. So the open emnedding $D^\circ_\emptyset(X_g) \hookrightarrow D_\emptyset(X_g)$ is just $X_{f_1} \hookrightarrow X_g$. *

We consider the following two Cartesian squares:

$$\begin{tikzcd} X_\emptyset(g^{-1}, f_1) \arrow[r, hook, "\alpha_\emptyset"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & {[(\mathbb{K}^*)^n \times \mathbb{K}^*] \times \mathbb{K}^*} \arrow[d, hook, "\beta_\emptyset"] \\ X_g \arrow[r, hook] & {[(\mathbb{K}^*)^n \times \mathbb{K}^*] \times \mathbb{K}} \end{tikzcd} \qquad \begin{tikzcd} X_{[1]}(g^{-1}, f_1) \arrow[r, hook, "\alpha_{[1]}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & {[(\mathbb{K}^*)^n \times \mathbb{K}^*] \times \{0\}} \arrow[d, hook, "\beta_{[1]}"] \\ X_g \arrow[r, hook] & {[(\mathbb{K}^*)^n \times \mathbb{K}^*] \times \mathbb{K}.} \end{tikzcd} \tag{2.3}$$

Note that the embeddings $\alpha_\emptyset$ and $\alpha_{[1]}$ are closed, but only $\beta_{[1]}$ is a closed embedding ($\beta_\emptyset$ is open). For $\emptyset \subseteq J \subseteq [1]$, the projection $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \mathbb{K}) \longrightarrow (\mathbb{K}^*)^n$ induces a morphism $\Pi_J = \Pi \circ \alpha_J$ from $X_J(g^{-1}, f_1)$ to $(\mathbb{K}^*)^n$ that induces an isomorphism $X_J(g^{-1}, f_1) \cong D^\circ_J(X_g)$. The map Π_J is induced by the torus homomorphism $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \mathbb{K}^*) \longrightarrow (\mathbb{K}^*)^n$ for $J = \emptyset$ and by the torus homomorphism $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^* \times \{0\}) \longrightarrow (\mathbb{K}^*)^n$ for $J = [1]$.

So the map $\Pi_\emptyset : X_\emptyset(g^{-1}, f) \longrightarrow D_\emptyset(X_g)$ is just the open embedding $(X_g)_f \hookrightarrow X_g$, which is birational since $D_\emptyset(X) = X_g$ is a variety, and the map $\Pi_{[1]} : X_{[1]}(g^{-1}, f) \longrightarrow D_{[1]}(X_g)$ is an isomorphism. So, for any subset $\emptyset \subseteq J \subseteq [1]$, we have defined a diagram

$$\begin{tikzcd} X_J(g^{-1}, f) \arrow[r, hook, "\alpha_J"] \arrow[d, hook, "\Pi_J"] & {[(\mathbb{K}^*)^n \times \mathbb{K}^*] \times (\mathbb{K}^*)^{\#J}} \\ D_J(X_g) \arrow[r, hook] & X_g \end{tikzcd}$$

where Π_J is an open embedding and the horizontal arrows are closed embedding of schemes.

Our next task is to extend the preceding construction for the case when the functions g, f are expressed as $g = g_1 \cdots g_a$ and $f = f_1 \cdots f_b$ with $a \geq 0, b \geq 1$. We construct a closed embedding $X_g \hookrightarrow (\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times \mathbb{K}^b$ as

$$x \mapsto \left(x, \tfrac{1}{g_1(x)}, \ldots, \tfrac{1}{g_a(x)}, f_1(x), \ldots, f_b(x)\right).$$

If $b > 0$ then we will use the stratification $\mathbb{K}^b = \coprod_{\emptyset \subseteq J \subseteq [b]} H^\circ_J$ by locally closed subschemes of the affine space $\mathbb{K}^b$ defined by $H^\circ_J := V(x_j \,:\, j \in J) \cap (\mathbb{K}^b)_{\prod_{j \notin J} x_j}$ for $\emptyset \subseteq J \subseteq [b] := \{1, 2, \ldots, b\}$, so we have that $\mathrm{Supp}(H^\circ_J) = \{(p_1, \ldots, p_b) \in \mathbb{K}^b \mid p_i = 0 \text{ if and only if } i \in J\}$.

For $\emptyset \subseteq J \subseteq [b]$ we construct the Cartesian square :

$$\begin{tikzcd} X_J(\mathbf{g}^{-1}, \mathbf{f}) \arrow[r, hook, "\alpha_J"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & {[(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times H^\circ_J} \arrow[d, hook, "\beta_J"] \\ X_g \arrow[r, hook] & {[(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times \mathbb{K}^b.} \end{tikzcd} \tag{2.4}$$

Note that α_J is a closed embedding, while β_J is open (respectively closed, locally closed) for $J = \emptyset$ (respectively $J = [b]$, $J \neq \emptyset, [b]$).

Since $H^\circ_J \cong (\mathbb{K}^*)^{b-\#J}$, the projection $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times \mathbb{K}^b \longrightarrow (\mathbb{K}^*)^n$ induces a torus homomorphism $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times H^\circ_J \longrightarrow (\mathbb{K}^*)^n$, and the composition $\Pi_J := \Pi \circ \alpha_J$ induces an isomorphism $X_J(\mathbf{g}^{-1}, \mathbf{f}) \cong D^\circ_J(X_g)$. It follows that the map $\Pi_J : X_J(g^{-1}, f) \hookrightarrow D_J(X_g)$ is the open embedding $D^\circ_J(X_g) \hookrightarrow D_J(X_g)$.

2.3.4 Example: Let $X \subset (\mathbb{K}^*)^n$ be a subvariety, $\mathbf{g} = (g_1, \ldots, g_a)$, $\mathbf{f} = (f_1, \ldots, f_b)$, $g = g_1 \cdots g_a$ and $\Pi_J : X_J(g^{-1}, f) \hookrightarrow D_J(X_g)$ be as above.

1. Let $J = \emptyset$. If we denote the product $f_1 \cdots f_b$ by f, and f is not the zero function in X_g, then $X_\emptyset(\mathbf{g}^{-1}, \mathbf{f}) \xrightarrow{\Pi_\emptyset} D_\emptyset(X_g)$ represents the diagram
$$(X_g)_f \hookrightarrow X_g.$$
2. If $J = [b]$, then $D^\circ_{[b]}(X_g)$ is the closed subscheme of X_g defined by the ideal $\langle f_1, \ldots, f_b \rangle \cdot \mathbb{K}[X_g] \subset \mathbb{K}[X_g]$, so $D_{[b]}(X_g) = D^\circ_{[b]}(X_g)$ and $X_{[b]}(\mathbf{g}^{-1}, \mathbf{f}) \xrightarrow{\Pi_{[b]}} D_{[b]}(X_g)$ represents:
$$D^\circ_{[b]}(X_g) = D^\circ_{[b]}(X_g)\ .$$

Our next task is to extend the previous construction for a closed subscheme of a very affine variety. Let X be a very affine variety and let $Y \subset X$ be a closed subscheme defined by the ideal $I(Y) \subset \mathbb{K}[X]$. If $g \in \mathbb{K}[X]$ is a non-zero function we will denote also by g the image of g in $\mathbb{K}[Y]$ under the isomorphism $\mathbb{K}[Y] \cong \mathbb{K}[X]/I(Y)$.

Consider the families of regular functions $\mathbf{g} = (g_1, \ldots, g_a)$ and $\mathbf{f} = (f_1, \ldots, f_b)$ on X and let $g = g_1 \cdots g_a$, then we have a closed embedding $Y_g \hookrightarrow X_g$ (since localization is an exact functor) and for any $\emptyset \subseteq J \subseteq [b]$ we extend the diagram (2.4) to get:

$$\begin{tikzcd} Y_J(\mathbf{g}^{-1},\mathbf{f}) \arrow[r, hook, "\gamma_J"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & X_J(\mathbf{g}^{-1},\mathbf{f}) \arrow[r, hook, "\alpha_J"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & [(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times H_J^\circ \arrow[d, hook, "\beta_J"] \\ Y_g \arrow[r, hook] & X_g \arrow[r, hook] & [(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times \mathbb{K}^b, \end{tikzcd} \tag{2.5}$$

where γ_J is a closed embedding, and since α_J is also a closed embedding, it follows that $\alpha_J \circ \gamma_J : Y_J(\mathbf{g}^{-1},\mathbf{f}) \hookrightarrow [(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times H_J^\circ$ is a closed embedding.

Remark 2.3.5: *The morphism $\Pi_J := \Pi \circ \alpha_J \circ \gamma_J$ from $Y_J(\mathbf{g}^{-1},\mathbf{f})$ to $(\mathbb{K}^*)^n$ induces an isomorphism between $Y_J(\mathbf{g}^{-1},\mathbf{f})$ and the intersection scheme $Y_g \cap D_J^\circ(X_g)$ in X_g. The construction (2.5) gives us then a closed embedding $\alpha_J \circ \gamma_J$ of the intersection scheme $Y_g \cap D_J^\circ(X_g)$ in X_g to the torus $[(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a] \times H_J^\circ$.*

We denote by $D_J(Y_g) \hookrightarrow Y_g$ the scheme-theoretic closure of $Y_g \cap D_J^\circ(X_g)$. Observe that $Y_g \cap D_J(X_g)$ is a closed subscheme of Y_g containing $Y_g \cap D_J^\circ(X_g)$, so we get a closed embedding $D_J(Y_g) \hookrightarrow Y_g \cap D_J(X_g)$. So we have the diagram

$$\begin{tikzcd} & Y_J(\mathbf{g}^{-1},\mathbf{f}) \arrow[dl, "\cong"'] \arrow[d, "\Pi_J"] \\ Y_g \cap D_J^\circ(X_g) \arrow[r, hook] & D_J(Y_g), \end{tikzcd}$$

with $\Pi_J : Y_J(\mathbf{g}^{-1},\mathbf{f}) \longrightarrow D_J(Y_g)$ an open embedding.

2.3.6 Example: Let $Y \subset X \subset (\mathbb{K}^*)^n$, $\mathbf{g} = (g_1, \ldots, g_a)$, $\mathbf{f} = (f_1, \ldots, f_b)$, $g = g_1 \cdots g_a$ and $Y_J(\mathbf{g}^{-1},\mathbf{f}) \xrightarrow{\Pi_J} D_J(Y_g)$ be as above. We recall that any non-empty subset of an irreducible topological space is irreducible and dense (see [Har77], p.3).

1. Let $J = \emptyset$ and let us denote the product $f_1 \cdots f_b$ by f. Suppose that $\mathrm{div}_{X_g}(f)$ has proper intersection with every irreducible component of Y_g, then $Y_\emptyset(\mathbf{g}^{-1},\mathbf{f}) \xrightarrow{\Pi_\emptyset} D_\emptyset(Y_g)$ represents the diagram
$$(Y_g)_f \hookrightarrow Y_g.$$
2. Let $J = [b]$. Recall from Example 2.3.4 that $D_{[b]}^\circ(X_g)$ is the closed subscheme of X_g defined by the ideal $\langle f_1, \ldots, f_b \rangle \cdot \mathbb{K}[X_g] \subset \mathbb{K}[X_g]$, so $D_{[b]}^\circ(Y_g) = Y_g \cap D_{[b]}^\circ(X_g)$ is the intersection of two closed subschemes of X_g. The diagram $Y_\emptyset(\mathbf{g}^{-1},\mathbf{f}) \xrightarrow{\Pi_\emptyset} D_\emptyset(Y_g)$ represents:
$$Y_g \cap D_{[b]}^\circ(X_g) = Y_g \cap D_{[b]}^\circ(X_g)\ .$$

Remark 2.3.7: *The definition and notation of the embeddings $X_J(\mathbf{g}^{-1},\mathbf{f})$ is inspired by the so-called* Laurent domains *of the spectrum of affinoid algebras in the theory of Berkovich analytic spaces (if $a = 0$, then these domains are called Weierstrass domains). See [Ber90] for further information.*

2.4.– Algebraic modifications of closed subschemes on very affine varieties

Let us recall our previous notation and concepts: $X \subset (\mathbb{K}^*)^n$ is a subvariety, $\mathbf{g} = (g_1, \ldots, g_a)$, $\mathbf{f} = (f_1, \ldots, f_b)$ are families of regular functions on X, $g = g_1 \cdots g_a$ and J a set $\emptyset \subseteq J \subseteq [b]$. Then $X_J(\mathbf{g}^{-1},\mathbf{f})$ is a closed subscheme of the torus $(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times H_J^\circ$ which is isomorphic to the locally closed subscheme $D_J^\circ(X_g)$ of X_g, which is supported in the set $\{x \in X_g \ : \ f_j(x) = 0 \text{ for } j \in J \text{ and } f_j(x) \neq 0 \text{ for } j \notin J\}$.

Let $\mathbb{K}$ be the Mal'cev-Neumann field $F((t^{\mathbb{R}}))$. Let $[X_\emptyset(\mathbf{g}^{-1},\mathbf{f})]$ be the fundamental cycle associated to the subvariety $X_\emptyset(\mathbf{g}^{-1},\mathbf{f})$ of the torus $(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times H_\emptyset^\circ$, and let $\Pi : (\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times H_\emptyset^\circ \longrightarrow (\mathbb{K}^*)^n$, then we have that

$$\Pi_*([X_\emptyset(\mathbf{g}^{-1},\mathbf{f})]) = [X].$$

Let us denote by $\pi = \mathrm{Trop}(\Pi)$ the projection $\mathbb{R}^n \times \mathbb{R}^a \times \mathrm{Val}(H_\emptyset^\circ) \longrightarrow \mathbb{R}^n$ and by $\pi_* : Z_*(\mathbb{R}^n \times \mathbb{R}^a \times \mathrm{Val}(H_\emptyset^\circ)) \longrightarrow Z_*(\mathbb{R}^n)$ its induced homomorphism on tropical cycles. Then we know that by the definition of tropical push-forward that

$$\pi_*(\mathrm{Trop}([X_\emptyset(\mathbf{g}^{-1},\mathbf{f})])) = \mathrm{Trop}(\Pi_*([X_\emptyset(\mathbf{g}^{-1},\mathbf{f})])) = \mathrm{Trop}(X). \tag{2.6}$$

Finally, since $X_\emptyset(\mathbf{g}^{-1},\mathbf{f})$ is a variety, we have $\mathrm{Trop}([X_\emptyset(\mathbf{g}^{-1},\mathbf{f})]) = \mathrm{Trop}(X_\emptyset(\mathbf{g}^{-1},\mathbf{f}))$.

Definition: We call $\pi_* : \mathrm{Trop}(X_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \longrightarrow \mathrm{Trop}(X)$ the (algebraic) $\emptyset$-**modification** of X induced by the families $\mathbf{g}$ and $\mathbf{f}$.

It follows from (1.9) that an $\emptyset$-modification $\pi_* : \mathrm{Trop}(X_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \longrightarrow \mathrm{Trop}(X)$ consists of a surjective function of sets $\pi : \mathrm{Val}(X_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \longrightarrow \mathrm{Val}(X)$ and an expression for the tropical multiplicity $m_X(p)$ of a regular point $p \in \mathrm{Val}(X)$ in terms of the tropical multiplicities $m_{X_\emptyset(\mathbf{g}^{-1},\mathbf{f})}(q_i)$ of the points $\{q_1, \ldots, q_m\} = \pi^{-1}(p)$, namely

$$m_X(p) = \sum_{i=1}^{m} m_{X_\emptyset(\mathbf{g}^{-1},\mathbf{f})}(q_i)[\Lambda_p : \pi(\Lambda_{q_i})].$$

Recall that the points q_i must be regular points in $\mathrm{Val}(X_\emptyset(\mathbf{g}^{-1}, \mathbf{f}))$.

Likewise, if $Y \subseteq X$ is a closed subscheme, then $Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})$ is a closed subscheme of $(\mathbb{K}^*)^n \times (\mathbb{K}^*)^a \times H_\emptyset^\circ$ which is isomorphic to the intersection scheme $Y_g \cap D_\emptyset^\circ(X_g)$ of Y_g and $D_\emptyset^\circ(X_g)$ in X_g. Let $[Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})]$ be its fundamental cycle, then we have that $\Pi_*([Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})]) = [D_\emptyset(Y_g)]$.

Definition: We call $\pi_* : \mathrm{Trop}(Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \longrightarrow \mathrm{Trop}(D_\emptyset(Y_g))$ the $\emptyset$-**modification** of the closed subscheme $Y_g \subseteq X_g$ induced by $\mathbf{g}$ and $\mathbf{f}$.

2.4.1 Example: Let $\pi_* : \mathrm{Trop}(Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \longrightarrow \mathrm{Trop}(D_\emptyset(Y_g))$ be the $\emptyset$-**modification** of $Y_g \subseteq X_g$ induced by $\mathbf{g}$ and $\mathbf{f}$.

1. The case $Y = X = (\mathbb{K}^*)^n$, $a = 0$, $b = 1$ is the principal contraction introduced by G. Mikhalkin (see [Mik06]).

2. Let $X = (\mathbb{K}^*)^2 = \mathrm{Spec}(\mathbb{K}[x^{\pm 1}, y^{\pm 1}])$, $a = 0$, and let $f, h \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ such that $Y = V(h)$ and $Z = V(f)$ intersect properly in X. Then $Y_\emptyset(f) \cong Y \setminus Z$ and thus $\Pi_*([Y_\emptyset(f)]) = [Y]$. The $\emptyset$-modification $\pi_* : \mathrm{Trop}([Y_\emptyset(f)]) \longrightarrow \mathrm{Trop}(Y)$ of $Y \subset X$ was introduced in [BL12].

*

We have a diagram:

$$\begin{tikzcd}
\mathrm{Val}(Y_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \arrow[r, hook] \arrow[d, "\pi_*"'] & \mathrm{Val}(X_\emptyset(\mathbf{g}^{-1}, \mathbf{f})) \arrow[d, "\pi_*"] \\
\mathrm{Val}(D_\emptyset(Y_g)) \arrow[r, hook] & \mathrm{Val}(X)
\end{tikzcd}$$

Suppose that $a = 0$ and let $f = f_1 \cdots f_b$. The set $\mathrm{Supp}(\mathrm{div}_X(f)) = \mathrm{Supp}(D_\emptyset(X)) \setminus \mathrm{Supp}(D_\emptyset^\circ(X))$ is a closed algebraic subset of X with the property that for any $p \in \mathrm{Val}(\mathrm{div}_X(f))$, the fiber $\pi^{-1}(p)$ is not finite. Furthermore, if $\mathrm{div}_X(f)$ has proper intersection with every irreducible component of Y, then $D_\emptyset(Y) = Y$, and we have the following diagram:

$$\begin{tikzcd}
\mathrm{Trop}(Y_\emptyset(\mathbf{f})) \arrow[r, hook] \arrow[d, "\pi_*"'] & \mathrm{Trop}(X_\emptyset(\mathbf{f})) \arrow[d, "\pi_*"] & \\
\mathrm{Trop}(Y) \arrow[r, hook] & \mathrm{Trop}(X) & \mathrm{Trop}(\mathrm{div}_X(f)) \arrow[l, hook']
\end{tikzcd}$$

Definition: Let $\pi_* : \mathrm{Trop}(X_\emptyset(\mathbf{f})) \longrightarrow \mathrm{Trop}(X)$ be the $\emptyset$-modification of X induced by $\mathbf{f} = (f_1, \ldots, f_b)$. We call $\mathrm{Trop}(\mathrm{div}_X(f))$ the **divisor** of the modification.

Consider an $\emptyset$-modification $\pi_* : \mathrm{Trop}(X_\emptyset(\mathbf{f})) \longrightarrow \mathrm{Trop}(X)$. We can consider the partial compactification $\overline{\mathrm{Val}(X_\emptyset(\mathbf{f}))}$ of the set $\mathrm{Val}(X_\emptyset(\mathbf{f})) \subset \mathbb{R}^n \times \mathbb{R}^b$ inside $\mathbb{R}^n \times \mathbb{T}^b$. Observe that for any $\emptyset \subset J \subset [b]$, we have

$$\overline{\mathrm{Val}(X_\emptyset(\mathbf{f}))} \cap (\mathbb{R}^n \times \mathrm{Val}(H_J^\circ)) = \mathrm{Val}(X_J(\mathbf{f})).$$

Since $X_J(\mathbf{f}) \cong D_J^\circ(X)$, we see that the set $\mathrm{Val}(X_\emptyset(\mathbf{f}))$ separates the elements of the stratification $X = \coprod_J D_J^\circ(X)$ induced by the family $\mathbf{f} = (f_1, \ldots, f_b)$.

2.4.1 Algebraic modifications on generically integral algebraic cycles

In the rest of this part, we will suppose that X is a generically integral $\mathbb{K}$-variety which is already embedded as a subvariety of $(\mathbb{K}^*)^n$ with simple tropicalization.

Lemma 2.4.2. *Let $X \subset (\mathbb{K}^*)^n$ be a variety with simple tropicalization and let $f \in \mathbb{K}[X]$ be a non-zero regular function. Then there exists a continuous piecewise integer affine linear function $\mathcal{T}(f) : Val(X) \longrightarrow \mathbb{R}$ with the property that for any regular point $p \in Val(X)$ such that $q = (p, \mathcal{T}(f)(p)) \in \Gamma_{\mathcal{T}(f)}(Val(X))$ is regular, we have that $[\Lambda_p : \pi(\Lambda_q)] = 1$, where $\pi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the projection onto the first factor.*

Proof. Let $\pi_\emptyset : \mathrm{Trop}(X_\emptyset(f)) \longrightarrow \mathrm{Trop}(X)$ be the $\emptyset$-**modification** induced by f on X, so that $\pi_\emptyset = \pi|_{\mathrm{Val}(X_\emptyset(f))}$. Let $U \subset \mathrm{Val}(X)$ be the set of regular points $p \in \mathrm{Val}(X)$ such that $(\pi_\emptyset)^{-1}(p)$ is finite. According to (1.9), for every $p \in U$ we have

$$m_X(p) = 1 = \sum_{q \in (\pi_\emptyset)^{-1}(p)} m_{X_\emptyset(\mathbf{f})}(q)[\Lambda_p : \pi(\Lambda_q)],$$

which says at once that $(\pi_\emptyset)^{-1}(p) = \{q\}$ is a singleton and that $[\Lambda_p : \pi(\Lambda_q)] = 1$. The assignment $p \mapsto (\pi_\emptyset)^{-1}(p)$ gives us then a function $F : U \longrightarrow \mathbb{R}$, and since U is open in $\mathrm{Val}(X)$, we conclude the existence of the continuous function $\mathcal{T}(f)$ whose graph is precisely $\Gamma_{\mathcal{T}(f)}(\mathrm{Val}(X))$ the closure of the graph $\Gamma_F(U) \subset \mathbb{R}^n \times \mathbb{R}$.

Finally, the fact that the graph $\Gamma_{\mathcal{T}(f)}(\mathrm{Val}(X))$ has structure of a rational polyhedral complex comes from the expression $\mathrm{Val}(X_\emptyset(f)) = \Gamma_{\mathcal{T}(f)}(\mathrm{Val}(X)) \cup \pi_\emptyset^{-1}(\mathrm{Val}(\mathrm{div}_X(f)))$ and the fact that $\mathrm{Val}(X_\emptyset(f))$ has structure of a rational polyhedral complex. ■

We now generalize this definition to any effective tropical k-cycle in $\mathbb{R}^n$.

Definition: Let A be an effective tropical k-cycle in $\mathbb{R}^n$ and let $\phi : A \longrightarrow \mathbb{R}$ be a continuous piecewise integer affine linear function. Let $\Gamma_\phi(A) \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of ϕ and let $\pi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be the projection onto the first factor. We say that ϕ is a **tropical Cartier divisor** on A if for any regular point $p \in A$ such that $q = (p, \phi(p)) \in \Gamma_\phi(A)$ is regular, we have that $[\Lambda_p : \pi(\Lambda_q)] = 1$.

2.4.3 Example: Let $X \subset (\mathbb{K}^*)^n$ be a variety and $f \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the restriction of the tropical polynomial $\mathrm{Trop}(f) : \mathbb{R}^n \longrightarrow \mathbb{R}$ to $\mathrm{Val}(X) \subset \mathbb{R}^n$ gives us a tropical Cartier divisor $\phi : \mathrm{Trop}(X) \longrightarrow \mathbb{R}$.

When X has simple tropicalization and $f|_X \neq 0$, then according to Lemma 2.4.2 we can construct another tropical Cartier divisor on X, namely $\mathcal{T}(f|_X)$. In this latter case, it may not be possible to find a tropical polynomial (or a tropical rational function) $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\phi|_{\mathrm{Val}(X)} = \mathcal{T}(f|_X)$. ∗

We will show now that a tropical Cartier divisor on an effective tropical k-cycle is locally a tropical rational function in the following sense.

Lemma 2.4.4. *Let A be a tropical k-cycle in $\mathbb{R}^n$ and let $\phi : A \longrightarrow \mathbb{R}$ be a tropical Cartier divisor. Then for any $p \in A$ there exists a fan neighborhood $U \subset A$ of p and a tropical rational function h such that $\phi|_U = h|_U$.*

Proof. Let $p \in A$ be a regular point such that $q = (p, \phi(p)) \in \Gamma_\phi(A)$ is regular. We will show that there exists $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ such that $\phi(p) = \langle i, p \rangle$ in a small neighborhood of p.

Let $\{v_1, \ldots, v_k, w_1, \ldots, w_{n-k}\} \subset \mathbb{Z}^n$ be a basis for $\mathbb{Z}^n$ such that $\{v_1, \ldots, v_k\}$ is a basis for Λ_p. We shall construct a set $\{v'_1, \ldots, v'_k, w'_1, \ldots, w'_{n+1-k}\} \subset \mathbb{Z}^{n+1}$ such that it is a basis for $\mathbb{Z}^{n+1}$ and $\{v'_1, \ldots, v'_k\}$ is a basis for Λ_q. Let $v'_j = (v_j, \phi(v_j))$ for $j = 1, \ldots, k$, $w'_j = (w_j, 0)$, for $j = 1, \ldots, n-k$ and $w'_{n+1-k} = e_{n+1}$.

Let B be the matrix whose j-th row is the vector v_j, if $1 \leq j \leq k$, or the vector w_j, if $1 \leq j-k \leq n-k$, and let C be the n-dimensional column vector $[\phi(v_1), \ldots, \phi(v_k), 0, \ldots, 0]$, then it is easy to see that in a small neighborhood U of p in A consisting of regular points, the vector $i = (i_1, \ldots, i_n)$ that we are looking for is $B^{-1} \cdot C$.

Let $p \in A$ and consider a fan neighborhood $U \subset A$. We have shown that the restriction $\phi|_U$ is a continuous piecewise-linear function with integer slopes whose graph is a finite polyhedral complex. Let A' be a polyhedral subdivision on A such that $\phi|_\sigma$ is $\mathbb{Z}$-affine linear for each polygon $\sigma \in A'$. By shrinking U if necessary, we can suppose that the subdivision induced by A'on U is that of a fan. Then it follows from Lemma 1.3.21 that there exists a tropical rational function h such that $h|_U = \phi_U$. ■

We shall now give a concrete description of the set $\mathrm{Val}(X_\emptyset(\mathbf{f}))$ for the case when $X \subset (\mathbb{K}^*)^n$ is a subvariety with simple tropicalization, which is a generalization of Lemma 2.4.2 for $b > 1$.

Proposition 2.4.5. *Let $X \subset (\mathbb{K}^*)^n$ be a subvariety of dimension k with simple tropicalization and let $\pi_\emptyset : Trop(X_\emptyset(\mathbf{f})) \longrightarrow Trop(X)$ be the $\emptyset$-**modification** induced by $\mathbf{f} = (f_1, \dots, f_b)$. Then there exists a continuous, piecewise affine $\mathbb{Z}$-linear function $\mathcal{T}(\mathbf{f}) : Val(X) \longrightarrow \mathbb{R}^b$ such that*

$$Val(X_\emptyset(\mathbf{f})) = \Gamma_{\mathcal{T}(\mathbf{f})}(Val(X)) \cup (\pi_\emptyset)^{-1}(Val(div_X(f)),$$

where $f = f_1 \cdots f_b$.

Proof. According to the Sturmfels-Tevelev formula (1.9), if $p \in (\mathrm{Val}(X) \setminus \mathrm{Val}(\mathrm{div}_X(f))$ is a regular point, then we have $m_X(p) = 1 = \sum_{q \in (\pi_\emptyset)^{-1}(p)} m_{X_\emptyset(\mathbf{f})}(q)[\Lambda_p : \pi_\emptyset(\Lambda_q)]$, which says in particular that $(\pi_\emptyset)^{-1}(p)$ is a singleton. Since the set of such points is open in $\mathrm{Val}(X)$, we conclude the existence of $\mathcal{T}(\mathbf{f})$.

Let $p \in \mathrm{Val}(X)$ be a regular point such that $q = (p, \mathcal{T}(\mathbf{f})(p)) \in \Gamma_\phi(Z)$ is regular. We will show that there exists $i_1, \dots, i_b \in \mathbb{Z}^n$ such that $\mathcal{T}(\mathbf{f})(p) = (\langle i_1, p\rangle, \dots, \langle i_b, p\rangle)$ in a small neighborhood of p.

Let $\{v_1, \dots, v_k, w_1, \dots, w_{n-k}\} \subset \mathbb{Z}^n$ be a basis for $\mathbb{Z}^n$ such that $\{v_1, \dots, v_k\}$ is a basis for Λ_p. We shall construct a set $\{v'_1, \dots, v'_k, w'_1, \dots, w'_{n+b-k}\} \subset \mathbb{Z}^{n+b}$ such that it is a basis for $\mathbb{Z}^{n+b}$ and $\{v'_1, \dots, v'_k\}$ is a basis for Λ_q. Let $v'_j = (v_j, \mathcal{T}(\mathbf{f})(v_j))$ for $j = 1, \dots, k$, $w'_j = (w_j, 0)$, for $j = 1, \dots, n-k$ and $w'_j = e_{n+i}$ for $j = n-k+i$.

Let A be the matrix whose j-th row is the vector v_j, if $1 \le j \le k$, or the vector w_j, if $1 \le j-k \le n-k$ and for every $j = 1, \dots, k$, let $\mathcal{T}(\mathbf{f})(v_j) = (v_{j1}, \dots, v_{jb})$. For $\ell = 1, \dots, b$, let B_ℓ be the n-dimensional column vector $[v_{1\ell}, \dots, v_{k\ell}, 0, \dots, 0]$, then it is easy to see that the vector $i_\ell = (i_1, \dots, i_n)$ is $A^{-1} \cdot B_\ell$.

Since $\mathrm{Val}(W_\emptyset)$ has codimension at least one in $\mathrm{Val}(X)$, we conclude that $(\pi_\emptyset)^{-1}(\mathrm{Val}(W_\emptyset))$ has dimension at least one in $\mathrm{Val}(X_\emptyset(\mathbf{f}))$ as well, because this last object is a purely-dimensional polyhedral set. This also implies that $\mathcal{T}(\mathbf{f})$ is locally affine piecewise linear. ■

Definition: The function $\mathcal{T}(\mathbf{f}) : \mathrm{Val}(X) \longrightarrow \mathbb{R}^b$ is the **tropicalization** of the family $\mathbf{f}$, and $\mathcal{W}(\mathbf{f}) := \pi_\emptyset)^{-1}(\mathrm{Val}(\mathrm{div}_X(f_1 \cdots f_b)$ is the **wall set** of the $\emptyset$-modification $\pi_\emptyset : \mathrm{Trop}(X_\emptyset(\mathbf{f})) \longrightarrow \mathrm{Trop}(X)$.

2.4.6 Example: If $b = 1$, then the $\emptyset$-modification $\pi_\emptyset : \mathrm{Trop}(X_\emptyset(f) \longrightarrow \mathrm{Trop}(X)$ is completely described by the tropicalization $\mathcal{T}(f)$ of the function f. This is no longer true for $b > 1$, for example, take $X = \mathbb{K}^*$ and let $f_1(x) = \alpha_{11}x + \alpha_{12}$, $f_2(x) = \alpha_{21}x + \alpha_{22}$, with $\mathrm{val}(\alpha_{ij}) = a_{ij}$ for $1 \le i, j \le 2$. Then the tropicalization $\mathcal{T}(\mathbf{f}) : \mathbb{R} \longrightarrow \mathbb{R}^2$ of the family $\mathbf{f} = (f_1, f_2)$ is given by $p \mapsto (\max(a_{11} + p, a_{12}), \max(a_{21} + p, a_{22}))$.

Suppose that there exists $p_0 \in \mathbb{R}$ such that $a_{11} + p_0 = a_{12}$ and $a_{21} + p_0 = a_{22}$, then it follows that $\mathcal{T}(\mathbf{f})$ is described by

$$\mathcal{T}(\mathbf{f})(p) = \begin{cases} (a_{12}, a_{22}), & \text{if } p \le p_0, \\ (a_{11} + p, a_{21} + p), & \text{if } p \ge p_0. \end{cases}$$

This happens if and only if $a_{12} - a_{11} = a_{22} - a_{21}$. We have $V(f_1) = -\frac{\alpha_{12}}{\alpha_{11}}$ and $V(f_2) = -\frac{\alpha_{22}}{\alpha_{21}}$, and there are two different cases, either $V(f_1) = V(f_2)$, in which case the wall set $\mathrm{Val}(X_\emptyset(\mathbf{f}))$ consists of an infinite ray in direction $(0, -1, -1)$ starting at the point (p_0, a_{12}, a_{22}), or $V(f_1) \neq V(f_2)$ in which case the wall set $\mathrm{Val}(X_\emptyset(\mathbf{f}))$ consists of two infinite rays, one in direction $(0, -1, 0)$ and the other in direction $(0, 0, -1)$ starting at the point (p_0, a_{12}, a_{22}). *

Let A be an effective tropical k-cycle in $\mathbb{R}^n$, $Y \subset A$ a tropical ℓ-cycle and $\phi : A \longrightarrow \mathbb{R}$ be a tropical Cartier divisor. We construct the modification $\delta_\emptyset : A_\emptyset(\phi) \longrightarrow A$ as well as the intersection product $Y \cdot \phi$ using the process described in Section 1.3.4.

Definition: For A and ϕ as above, we say that ϕ is an **effective tropical Cartier divisor** if $\mathrm{div}_A(\phi)$ is an effective tropical cycle.

Definition: Let A be an effective tropical k-cycle in $\mathbb{R}^n$ and let $\phi : A \longrightarrow \mathbb{R}$ be an effective tropical Cartier divisor. If $Y \subset A$ is a tropical ℓ-cycle, we define the intersection cycle $Y.\phi$ as $\mathrm{div}_Y(\phi)$.

Remark 2.4.7: *Let A be an effective tropical k-cycle in $\mathbb{R}^n$, $\phi : A \longrightarrow \mathbb{R}$ an effective tropical Cartier divisor, $Y \subset A$ a tropical ℓ-cycle and $Y.\phi$ their tropical intersection cycle.*

1. *By Lemma 2.4.4, if A is a fan cycle, ϕ is a rational function on A and Y is a fan cycle, then $Y.\phi$ coincides with the one defined in [AR09].*
2. *If A and Y are not necessarily fan cycles, but $\phi = h|_A$ for some tropical rational function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ with effective divisor $A.\mathrm{div}_{\mathbb{R}^n}(h)$, then $Y.\phi$ coincides with the intersection product defined for generalized principal modifications, as in Section 1.3.4.*

3. *If A is smooth, then by Lemma 2.4.4, the intersection product $Y.\mathrm{div}_A(\phi)$ defined in [Sha13] can be computed locally, using fan neighborhoods. In this case we have that $Y.\phi = Y.\mathrm{div}_A(\phi)$.*

We have seen that if X is a generically integral variety and $f \in \mathbb{K}[X]$ is a non-zero regular function, then we can construct an effective tropical Cartier divisor $\mathcal{T}(\mathbf{f}) : \mathrm{Val}(X) \longrightarrow \mathbb{R}$. The next section will discuss this operation.

2.4.2 Intersecting with a tropical Cartier divisor in generically integral tropical cycles

Let $X = (\mathbb{K}^*)^2$ and let $D_i = V(f_i)$, $i = 1, 2$, be two effective divisors with proper intersection. Algebraic modifications were introduced in [BL12] to link the algebraic intersection numbers of D_1 and D_2 with the tropical intersection numbers of their tropicalization in $\mathbb{R}^2$ along a 1-dimensional connected component of $\mathrm{Val}(D_1) \cap \mathrm{Val}(D_2)$. Let $\mathrm{Trop}(D_i)$ be the correspondent tropical cycles in $\mathbb{R}^2$ and let $\mathrm{Trop}(D_1).\mathrm{Trop}(D_2)$ be their stable intersection.

Theorem 2.4.8 (Brugallé-López de Medrano). *Let E be a connected component of $\mathrm{Val}(D_1) \cap \mathrm{Val}(D_2)$, then we have*

$$\sum_{\mathrm{Val}(x)\in E} i(x, D_1.D_2; X) \leq \sum_{p\in E} w_{\mathrm{Trop}(D_1).\mathrm{Trop}(D_2)}(p). \tag{2.7}$$

Equality is attained if E is compact.

We want to find a similar statement for the following situation: let $X \subset (\mathbb{K}^*)^n$ be a non-singular variety with simple tropicalization, $C \subset X$ a closed subscheme of pure dimension one and $\mathrm{div}_X(f)$ a principal Cartier divisor such that C and $\mathrm{div}_X(f)$ have proper intersection in X. In this case, the cycle $[C \cap \mathrm{div}_X(f)]$ associated to the intersection scheme of C and $[\mathrm{div}_X(f)]$ in X is a 0-dimensional cycle in X, so we have that its tropicalization $\mathrm{Trop}(\mathrm{div}_X(f)) = (\mathrm{Val}(C \cap \mathrm{div}_X(f)), m_{C\cap \mathrm{div}_X(f)})$ satisfies:

$$m_{C\cap \mathrm{div}_X(f)}(p) = \sum_{\mathrm{Val}(x)=p} \ell(\mathcal{O}_{C\cap \mathrm{div}_X(f)),x}), \quad p \in \mathrm{Val}(C \cap \mathrm{div}_X(f)).$$

Let us consider the $\emptyset$-modification $\pi_\emptyset : \mathrm{Trop}(C_\emptyset(f)) \longrightarrow \mathrm{Trop}(C)$ of C along f; the divisor of the modification is $\mathrm{Trop}(C \cap \mathrm{div}_X(f))$. If $p \in \mathrm{Val}(C \cap \mathrm{div}_X(f))$ and if $q << 0$ then (p, q) is a regular point of $\mathrm{Val}(C_\emptyset(f))$ and we have that $m_{C_\emptyset(f)}(p, q) = m_{C\cap \mathrm{div}_X(f)}(p)$ (see Definition 3.1, p.7 in [Sturmfels-Tevelev]).

We can now state the generalization that we have mentioned before.

Theorem 2.4.9. *Let $X \subset (\mathbb{K}^*)^n$ be a non-singular variety with simple tropicalization, $C \subset X$ a purely 1-dimensional closed subscheme and $f \in \mathbb{K}[X]$ such that C and $\mathrm{div}_X(f)$ intersect properly. Let E be a connected component of the set $\mathrm{Val}(C) \cap \mathrm{Val}(\mathrm{div}_X(f))$, then we have*

$$\sum_{\mathrm{Val}(x)\in E} \ell(\mathcal{O}_{C\cap \mathrm{div}_X(f)),x}) \leq \sum_{p\in E} w_{\mathrm{Trop}(C).\mathcal{T}(f)}(p), \tag{2.8}$$

where $\mathrm{Trop}(C).\mathcal{T}(f)$ is the tropical intersection product of $\mathrm{Trop}(C)$ with the tropical Cartier divisor $\mathcal{T}(f) : \mathrm{Val}(X) \longrightarrow \mathbb{R}$. If E is compact, then equality is attained.

Proof. First endow the tropical 1-cycle $A := \mathrm{Trop}(C_\emptyset(f))$ with a structure of finite polyhedral complex. Let $\mathcal{A} = \{(v, e) \text{ flag in } A : v \in \partial e, \pi_\emptyset(v) \in E\}$ and we endow each edge $e \subset \mathcal{A}$ with the same weight that it possesses in A (i.e., the multiplicity of any regular point in $\mathrm{relint}(e)$). This is a balanced polyhedral complex in $\mathrm{Trop}(X_\emptyset(f))$, so it is true that

$$\sum_{(v,e)\in\mathcal{A}} w_A(e) s_{n+1}(v, e) = 0,$$

where $s(v, e) = (s_1(v, e), \ldots, s_{n+1}(v, e))$ is the primitive integer vector pointing outwards v in direction e. Let us define :

1. $\mathcal{A}_1 = \{(v, e) \in \mathcal{A} : \pi_\emptyset(e) \subseteq E \text{ is bounded}\}$;
2. $\mathcal{A}_2 = \{(v, e) \in \mathcal{A} : \pi_\emptyset(e) \nsubseteq E\}$;

3. $\mathcal{A}_3 = \{(v,e) \in \mathcal{A} \,:\, s(v,e) = (0,\dots,0,-1)\}$;

4. $\mathcal{A}_4 = \{(v,e) \in \mathcal{A} \,:\, \pi_\emptyset(e) \subseteq E \text{ is unbounded}\}$;

Note that $\mathcal{A}$ is the disjoint union of the sets $\mathcal{A}_i$. For every $i = 1,\dots,4$ let us denote by $S_i(\mathcal{A})$ the sum $\sum_{(v,e)\in\mathcal{A}_i} w_A(e)s_{n+1}(v,e) = 0$, then we have that $\sum_{i=1}^4 S_i(\mathcal{A}) = 0$. In particular we have that $S_1(\mathcal{A}) = 0$, $S_3(\mathcal{A}) = \sum_{\mathrm{Trop}(x)\in E}(C \cap \mathrm{div}_X(f))_x$, and $S_4(\mathcal{A}) = 0$ when E is compact.

Let $\delta_\emptyset : (\mathrm{Trop}(C))_\emptyset(\mathcal{T}(f)) \longrightarrow \mathrm{Trop}(C)$ be the tropical modification of $\mathrm{Trop}(C)$ along $\mathcal{T}(f)$. We now repeat the preceding process with the tropical 1-cycle $B := (\mathrm{Trop}(C))_\emptyset(\mathcal{T}(f))$ to get the set of flags $\mathcal{B} = \{(v,e) \text{ flag in } B \;:\; v \in \partial e,\ \pi_\emptyset(v) \in E\}$ satisfying $\sum_{(v,e)\in\mathcal{B}} w_B(e)s_{n+1}(v,e) = 0$, as well as its decomposition into the sets $\mathcal{B}_1,\dots,\mathcal{B}_4$. For every $i = 1,\dots,4$ let us denote by $S_i(\mathcal{B})$ the sum $\sum_{(v,e)\in\mathcal{B}_i} w_B(e)s_{n+1}(v,e) = 0$, then we have that $\sum_{i=1}^4 S_i(\mathcal{B}) = 0$. In particular we have that $S_1(\mathcal{B}) = 0$, $S_3(\mathcal{B}) = \sum_{p\in E}(\mathrm{Trop}(C).\mathcal{T}(f))_p$, and $S_4(\mathcal{B}) = 0$ when E is compact.

We have

$$\sum_{i=1}^4 S_i(\mathcal{A}) = \sum_{(v,e)\in\mathcal{A}} w_A(e)s_{n+1}(v,e) = 0 = \sum_{(v,e)\in\mathcal{B}} w_B(e)s_{n+1}(v,e) = \sum_{i=1}^4 S_i(\mathcal{B}),$$

and since $S_2(\mathcal{A}) = S_2(\mathcal{B})$, then we have $S_4(\mathcal{A})+\sum_{\mathrm{Trop}(x)\in E}(C\cap\mathrm{div}_X(f))_x = S_4(\mathcal{B})+\sum_{p\in E}(\mathrm{Trop}(C).\mathcal{T}(f))_p$, so we just need to see what happens with $S_4(\mathcal{A})$ and $S_4(\mathcal{B})$.

Let $o \subset E$ be an unbounded edge such that there exists a flag (v,e) of type 4 in $\mathcal{A}$ with $\pi_\emptyset(e) = o$. Since $\mathcal{A}$ project onto a subset of $\mathrm{Val}(C)$ we have

$$m_C(o) = \sum_{i=1}^k m_{C_\emptyset(f)}(e_i)[\Lambda_o : \pi_\emptyset(\Lambda_{e_i})],$$

where $e_1,\dots,e_k \in \mathcal{A}$ are the edges which project onto o.

The complex $\mathcal{B}$ has only one edge a such that $\pi_\emptyset(a) = o$; its weight is precisely $m_C(o)$. Since the edge a lies on the uppergraph of $\mathrm{Trop}(X_\emptyset(f))$, we have that $s_{n+1}(v,e_i) \le s_{n+1}(v,a)$ for every $i = 1,\dots,k$. Thus

$$\sum_i m_{C_\emptyset(f)}(e_i)s_{n+1}(v,e_i) \le (\sum_i m_{C_\emptyset(f)}(e_i))s_{n+1}(v,a) \le m_C(o)s_{n+1}(v,a).$$

This gives us the relation $S_4(\mathcal{A}) \le S_4(\mathcal{B})$, which finishes the proof. ■

Remark 2.4.10: *Let $X \subset (\mathbb{K}^*)^n$, $C \subset X$ and $f \in \mathbb{K}[X]$ be as in Theorem 2.4.9.*

1. *If $\mathrm{Trop}(X)$ is a smooth tropical cycle, then we can replace in (2.8) the intersection product $\mathrm{Trop}(C).\mathcal{T}(f)$ of the tropical cycle $\mathrm{Trop}(C)$ with the tropical Cartier divisor $\mathcal{T}(f)$ with Shaw's tropical intersection product of tropical cycles in matroidal fans $\mathrm{Trop}(C).\mathrm{Trop}(\mathrm{div}_X(f))$.*

2. *If X is a non-singular surface and both C and $\mathrm{div}_X(f))$ are reduced, then it follows from Example 1.1.6 that we can replace in (2.8) the length $\ell(\mathcal{O}_{C\cap\mathrm{div}_X(f)),x})$ with the refined intersection multiplicity $i(x, C\cdot \mathrm{div}_X(f)); X)$.*

By combining both conditions of Remark 2.4.10, we get the following result.

Corolary 2.4.11. *Let $X \subset (\mathbb{K}^*)^n$ be a non-singular surface, $C \subset X$ a purely 1-dimensional closed subscheme and $f \in K[X]$ such that*

1. *the 2-tropical cycle $\mathrm{Trop}(X)$ is smooth, and*

2. *C and $\mathrm{div}_X(f)$ are both reduced and intersect properly in X.*

Let E be a connected component of the set $\mathrm{Val}(C) \cap \mathrm{Val}(\mathrm{div}_X(f))$, then we have

$$\sum_{\mathrm{Val}(x)\in E} i(x, C\cdot \mathrm{div}_X(f)); X) \le \sum_{p\in E} w_{\mathrm{Trop}(C).\mathrm{Trop}(\mathrm{div}_X(f))}(p). \tag{2.9}$$

If E is compact, then equality is attained.

Chapter 3

Real inflection points of real linear series on real curves

3.1.–Introduction

In this chapter we study the possible distributions of real inflection points of a real linear series defined on a real algebraic curve.

In Section 3.2 we introduce the concepts of real linear series and real inflection point of a real linear series on a real algebraic curve, and make some general remarks. In Theorem 3.2.5 we classify all possible distributions of real inflection points of a real complete linear series of degree $d \geq 2$ on a real elliptic curve (X, σ).

Theorem. *Let $X = (X, \sigma)$ be a real algebraic curve of genus 1 with $X(\mathbb{R}) \neq \emptyset$, and let Q be a real complete linear series of degree $d \geq 2$. Then Q has exactly d^2 complex inflection points. Moreover Q has exactly either 0, d, or $2d$ real inflection points according to the following cases:*

- *if $X(\mathbb{R})$ is connected, then Q has d real inflection points;*
- *if $X(\mathbb{R})$ has two connected components and d is odd, then Q has d real inflection points; these points are located on the connected component of $X(\mathbb{R})$ on which Q has odd degree;*
- *if $X(\mathbb{R})$ has two connected components and d is even, then*
  - *if Q has even degree in both connected components, then Q has exactly d real inflection points on each connected component (hence Q has $2d$ real inflection points);*
  - *if Q has odd degree in both connected components, then Q has no real inflection point.*

In particular, this result shows that the number of such real inflection points is at most twice the square root of the total number of inflection points. Theorem 3.2.5 follows from the study of torsion points on $\mathrm{Pic}_0(X)$, study that we first recall.

Next, we show that if $C \subset \mathbb{CP}^r$ is a smooth real curve of degree $d < 2r+2$ having just simple inflection points, then the number of real inflection points $w_{\mathbb{R}}(C)$ of C can be read from the real part of the dual variety $C^*(\mathbb{R})$ of C. In Proposition 3.2.9 we show that:

$$w_{\mathbb{R}}(C) = \#\{H \in C^*(\mathbb{R}) \,:\, \exists\, p \in C_0 \text{ such that } \ell(\mathcal{O}_{C\cap H,p}) = r+1\}. \tag{3.1}$$

In Section 3.3 we use the Proposition 3.2.9 in the case of a smooth, simple real curve $C \subset \mathbb{CP}^3$ of genus four and degree six. First we find a decomposition $\mathrm{mult}_{C^*} = \sum_{i=1}^{s} \lambda_i \mathbf{1}_{Y_i}$, for some subvarieties $Y_1, \ldots, Y_s \subset C^*$, and apply Corollary 1.5.12 for $r = 3$ to get

$$\sum_{i=1}^{s} \lambda_i \chi(Y_i(\mathbb{R})) = 0.$$

Then we compute $\chi(Y_i(\mathbb{R}))$ for $i = 1, \ldots, s$, and using the projection $\pi_2 : I_C \longrightarrow \mathbb{CP}^{3*}$ from the incidence variety of C to $\mathbb{CP}^{3*}$, we obtain the following result (see Theorem 3.3.6).

Theorem. *Let $C \subset \mathbb{CP}^3$ be a smooth, simple real curve of genus four and degree six. Then*

$$w_{\mathbb{R}}(C) = -\chi(\pi_2^{-1}(C^*(\mathbb{R}))). \tag{3.2}$$

Let X be a curve and let (V, L) be a linear series on X such that the map $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ induced by (V, L) is birational. If $C_0 = \phi_0(X)$ is smooth, the local Euler obstruction Eu_{C_0} is equal to the multiplicity function of the curve $\mathrm{mult}_{C_0} = \mathbf{1}_{C_0}$. In Section 3.4, we extend the equalitiy $\mathrm{Eu}_{C_0} = \mathrm{mult}_{C_0}$ for non-degenerate curves such that all of its singularities consists of ordinary m-fold points (see Proposition 3.4.3). We say that a singular point p in a curve $C \subset \mathbb{CP}^n$ with $\mathrm{mult}_C(p) = m$ is an ordinary m-fold point if $\#\nu^{-1}(p) = m$, where $\nu : \tilde{C} \longrightarrow C$ is the normalization map.

Finally, by applying Viro's theorem to Proposition 3.4.3, we get the following result (see Theorem 3.4.5):

Theorem (Generalized Viro formula, controlled singularities case). *Let $C \subset \mathbb{CP}^n$ be a non-degenerate curve whose only singularities are ordinary $r(p)$-fold points. For any $p \in C_{Sing}(\mathbb{R})$, we denote by $r''(p)$ the number of complex branches passing through p. Then*

$$\int_{\mathbb{RP}^*} Eu_{C^*}(x)\, d\chi(x) = \begin{cases} deg(C^*) - deg(C) + \sum_{p \in C_{Sing}(\mathbb{R})} r''(p), & \text{if } n \text{ is even,} \\ -\sum_{p \in C_{Sing}(\mathbb{R})} r''(p), & \text{if } n \text{ is odd.} \end{cases} \tag{3.3}$$

3.2.– Real linear series on real algebraic curves

Let $X = (X, \sigma)$ be a real curve, $D \in \mathrm{Div}(X)$ a σ-invariant divisor and $L_{\mathbb{R}} = \mathcal{L}_{\mathbb{R}}(D)$ the algebraic real line bundle on X induced by D.

Definition: A **real linear series** (of degree d and rank r) or real g^r_d on (X, σ) is a pair $(V_{\mathbb{R}}, L_{\mathbb{R}})$ consisting of

1. an algebraic real line bundle $L_{\mathbb{R}}$ of degree d on X such that $H^0(L_{\mathbb{R}}) \neq \{0\}$, and
2. a real linear subspace $\{0\} \neq V_{\mathbb{R}} \subseteq H^0(L_{\mathbb{R}})$ of dimension $r + 1$, with $r \geq 0$.

A real linear series $(V_{\mathbb{R}}, L_{\mathbb{R}})$ on a real curve (X, σ) induces a linear series (V, L) on the complex curve X, with $L = L_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Definition: Let $(V_{\mathbb{R}}, L_{\mathbb{R}})$ be a real g^r_d on a real curve (X, σ) and let (V, L) the g^r_d defined on the complex curve X. We define the inflection divisor $\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}})$ of $(V_{\mathbb{R}}, L_{\mathbb{R}})$ to be the inflection divisor of $\mathrm{Wr}(V, L)$.

We have the following result.

Proposition 3.2.1. *The inflection divisor $Wr(V_{\mathbb{R}}, L_{\mathbb{R}})$ of a real linear series $(V_{\mathbb{R}}, L_{\mathbb{R}})$ on a real curve (X, σ) is σ-invariant.*

Proof. Since a product of the form $L^{\otimes a} \otimes \Omega_X^{\otimes b}$ is the line bundle associated to the divisor $aD + bK_X$, we see that it is σ-invariant whenever D and K_X are σ-invariant, which is the case.

We know that $\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}})$ is the divisor of zeroes of an element $\mathrm{Wr}(s_1, \ldots, s_{r+1})$ in $H^0(L^{\otimes(r+1)} \otimes \Omega_X^{\otimes r(r+1)/2})$ associated to a basis $\{s_1, \ldots, s_{r+1}\}$ of V. Hence we can write $\mathrm{Wr}(V, L) = (r + 1)D + \frac{r(r+1)}{2} K_X + \mathrm{div}(f)$ for some $f \in \mathbb{C}(X)$.

If we choose $\{s_1, \ldots, s_{r+1}\}$ as a basis for $V_{\mathbb{R}}$, then we see that $f \in \mathbb{R}(X)$, which finishes the proof. ■

It follows from Proposition 1.4.3, Proposition 3.2.1 and from (1.15) that the inflection divisor $\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}})$ of a real linear series $(V_{\mathbb{R}}, L_{\mathbb{R}})$ on a real curve (X, σ) can be expressed as:

$$\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}}) = \sum_{x \notin X(\mathbb{R})} |\lambda(x)|(x + \sigma(x)) + \sum_{x \in X(\mathbb{R})} |\lambda(x)|x.$$

Definition: The set of **real inflection points** of the real linear series $(V_{\mathbb{R}}, L_{\mathbb{R}})$ is $\mathrm{Supp}(\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}})) \cap X(\mathbb{R})$. The number of real inflection points $w_{\mathbb{R}}((V_{\mathbb{R}}, L_{\mathbb{R}}))$ of $(V_{\mathbb{R}}, L_{\mathbb{R}})$ is $w_{\mathbb{R}}(V_{\mathbb{R}}, L_{\mathbb{R}}) = \sum_{x \in X(\mathbb{R})} |\lambda(x)|$.

Remark 3.2.2: *A real linear series $(V_{\mathbb{R}}, L_{\mathbb{R}})$ of degree d and rank r on a real curve (X, σ) of genus g has always $(r + 1)(d + r(g - 1))$ inflection points. However, the number of real inflection points $w_{\mathbb{R}}(V_{\mathbb{R}}, L_{\mathbb{R}})$ of $(V_{\mathbb{R}}, L_{\mathbb{R}})$ depends on the curve (X, σ), its topological type $(g, k(X), a(X))$, the class $[L_{\mathbb{R}}]$ of $L_{\mathbb{R}}$ in $Pic^d(X)(\mathbb{R})$ and $V_{\mathbb{R}} \in Gr(r + 1, H^0(L_{\mathbb{R}}))$.*

Let (X, σ) be a real curve with $X(\mathbb{R}) = S_1 \cup \cdots \cup S_{k(X)}$ and $k(X) > 0$. Let $\mathrm{par} : \mathrm{Pic}(X)(\mathbb{R}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{k(X)}$ be the parity homomorphism (1.16). We have the following result.

Proposition 3.2.3. *Let (X,σ) be a real curve with $X(\mathbb{R}) = S_1 \cup \cdots \cup S_{k(X)}$ and $k(X) > 0$. Let $(V_\mathbb{R}, L_\mathbb{R})$ be a real g^r_d on (X,σ) with $L_\mathbb{R} = L_\mathbb{R}(D)$ for some σ-invariant divisor D. Then*

$$par(Wr(V_\mathbb{R}, L_\mathbb{R})) = \begin{cases} par(D), & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases} \tag{3.4}$$

In particular, if r is odd, then each component $S_i \subset X(\mathbb{R})$ will contain an even number of real inflection points of $(V_\mathbb{R}, L_\mathbb{R})$, counted with multiplicity.

Proof. In Proposition 3.2.1 we showed that $\mathrm{Wr}(V_\mathbb{R}, L_\mathbb{R})$ is linearly equivalent to $(r+1)D + \frac{r(r+1)}{2}K_X$, so we have

$$\mathrm{par}(\mathrm{Wr}(V_\mathbb{R}, L_\mathbb{R})) = \mathrm{par}((r+1)D + \frac{r(r+1)}{2}K_X) = (r+1)\mathrm{par}(D),$$

since $\mathrm{par}(K_X) = 0$ by Example 1.4.12. ■

3.2.4 Example (Real complete linear series on real elliptic curves): Let $X = (X,\sigma)$ be a real curve of genus 1 with $X(\mathbb{R}) = S_1 \cup \cdots \cup S_{k(X)}$ and $k(X) > 0$. It follows that the topological type $(g, k(X), a(X))$ of (X,σ) is either $(1,1,1)$ or $(1,2,0)$. Let D be a σ-invariant divisor on X of degree d, and let $Q = (V_\mathbb{R}, L_\mathbb{R})$ be the real complete g^r_d on X induced by D, i.e., $L_\mathbb{R} = \mathcal{L}_\mathbb{R}(D)$ and $V_\mathbb{R} = H^0(L_\mathbb{R})$. By the Riemann-Roch Theorem, we have that $\dim H^0(L_\mathbb{R}) = d$, so $r = d-1$.

If $(g, k(X), a(X)) = (1,1,1)$, then by Proposition 3.2.3 we have

$$\mathrm{par}(\mathrm{Wr}(V_\mathbb{R}, L_\mathbb{R})) = \mathrm{par}(D) = \begin{cases} (0), & \text{if } d \text{ is even,} \\ (1), & \text{if } d \text{ is odd.} \end{cases}$$

If $(g, k(X), a(X)) = (1,2,1)$, then $\mathrm{par}(\mathrm{Wr}(V_\mathbb{R}, L_\mathbb{R})) = \mathrm{par}(D)$ unless $\mathrm{par}(D) = (1,1)$, in which case we have $\mathrm{par}(\mathrm{Wr}(V_\mathbb{R}, L_\mathbb{R})) = (0,0)$. ∗

Let (X,σ) be a real curve and suppose that $(V_\mathbb{R}, L_\mathbb{R})$ is a real base point-free g^r_d on it. In this case we know that the 0-th Gauss map $\phi_0 : X \longrightarrow \mathbb{CP}^r$ can be written as $\phi_0(x) = [s_1(x) : \ldots : s_{r+1}(x)]$, where $\{s_1, \ldots, s_{r+1}\}$ is a basis for V. If we choose $\{s_1, \ldots, s_{r+1}\}$ to be a basis of $V_\mathbb{R}$ and if we endow $\mathbb{CP}^r$ with the usual real structure $\sigma'([z_0 : \cdots : z_r]) \mapsto [\overline{z_0} : \cdots : \overline{z_r}]$, then ϕ_0 becomes a real morphism between real varieties $\phi_0 : (X,\sigma) \longrightarrow (\mathbb{CP}^r, \sigma')$.

If the real morphism $\phi_0 : (X,\sigma) \longrightarrow (\mathbb{CP}^r, \sigma')$ described above is a birational map, then it is just the normalization of the non-degenerate real curve $C_0 \subset \mathbb{CP}^r$. Once again, instead of starting out with a real linear series $(V_\mathbb{R}, L_\mathbb{R})$ on a real curve (X,σ), we can start out with a non-degenerated real curve $C_0 \subset (\mathbb{CP}^r, \sigma')$.

3.2.1 Inflection points of complete linear series on real elliptic curves

In this part we classify all possible distributions of real inflection points of a real complete linear series induced by a σ-invariant divisor $D \in \mathrm{Div}(X)$ of degree $d \geq 2$ on a real elliptic curve (X,σ). Since the situation when $X(\mathbb{R}) = \emptyset$ is trivial regarding real inflection points of linear series on X, we assume from now on that $X(\mathbb{R}) \neq \emptyset$.

Let X be a complex elliptic curve. A choice of $p_0 \in X$ induces an isomorphism

$$\begin{array}{rccc} \psi : & X & \longrightarrow & \mathrm{Pic}_0(X) \\ & p & \longmapsto & p - p_0 \end{array}$$

which induces in its turn a group structure on X. Geometrically, writing X as the quotient of $\mathbb{C}$ by a full rank lattice Λ for which p_0 is the orbit of 0, the group structure induced by ψ on X is simply the group structure inherited from $(\mathbb{C}, +)$ by the quotient map. This description allows to easily describe torsion points of order d on X. Indeed, if $\Lambda = u\mathbb{Z} + v\mathbb{Z}$ with u and v two complex numbers which are linearly independent over $\mathbb{R}$, then the solutions of

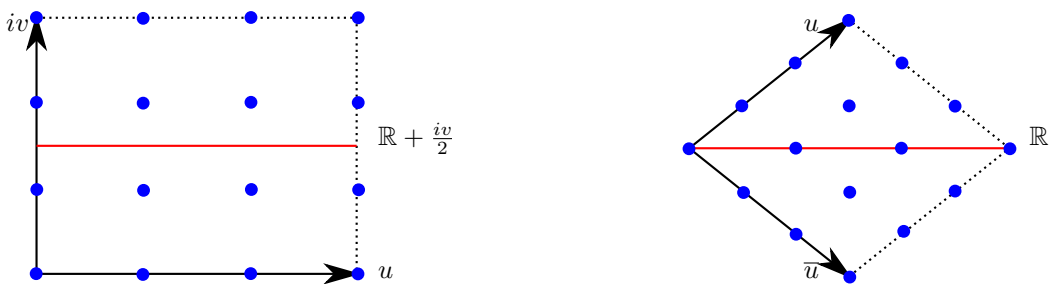
$$dp = 0 \tag{3.5}$$

are precisely the elements of X of the form $\frac{k}{d}u + \frac{l}{d}v$ with $k, l \in \{0, \ldots, d-1\}$. In particular, Equation (3.5) has d^2 solutions.

If (X,σ) is real and $p_0 \in X(\mathbb{R})$, then the map ψ induces a real structure on $\mathrm{Pic}^0(X)$. Recall (see [Nat90]) that X can be expressed as $\mathbb{C}/\Lambda$ with the real structure inherited by the complex conjugation on $\mathbb{C}$, where Λ has one of the following forms

- $\Lambda = u\mathbb{Z} + iv\mathbb{Z}$ with u and v two real numbers; in this case $X(\mathbb{R})$ has two connected components: $\mathbb{R}/u\mathbb{Z}$ and $(\mathbb{R} + \frac{iv}{2})/u\mathbb{Z}$ (see Figure 3.1a); when d is even, both connected components of $X(\mathbb{R})$ contain exactly d solutions of Equation (3.5); when d is odd, Equation (3.5) has exactly d real solutions, all located on $\mathbb{R}/u\mathbb{Z}$;

- $\Lambda = u\mathbb{Z}+\overline{u}\mathbb{Z}$ with u a complex number with non-zero imaginary part; in this case $X(\mathbb{R}) = \mathbb{R}/(u+\overline{u})\mathbb{Z}$ is connected (see Figure 3.1b); Equation (3.5) has exaclty d real solutions for any d.



a) A maximal real elliptic curve

b) A real elliptic curve with a connected real part

Figure 3.1: Uniformization of real elliptic curves with a non-empty real part. The blue points represent the solutions of $3p = 0$.

Theorem 3.2.5. *Let (X, σ) be a real algebraic curve of genus 1 with $X(\mathbb{R}) \neq \emptyset$, and let Q be a complete linear series of degree $d \geq 2$. Then Q has exactly d^2 complex inflection points. Moreover $w_{\mathbb{R}}(Q) \in \{0, d, 2d\}$ according to the following cases:*

- *if $X(\mathbb{R})$ is connected, then $w_{\mathbb{R}}(Q) = d$;*
- *if $X(\mathbb{R})$ has two connected components and d is odd, then Q has d inflection points located on the connected component of $X(\mathbb{R})$ on which Q has odd degree;*
- *if $X(\mathbb{R})$ has two connected components and d is even, then*
  - *if $\mathrm{par}(Q) = (0, 0)$, then Q has exactly d real inflection points on each of them (hence $w_{\mathbb{R}}(Q) = 2d$);*
  - *if $\mathrm{par}(Q) = (1, 1)$, then $w_{\mathbb{R}}(Q) = 0$.*

Proof. Let X be a complex curve of genus 1 and let D be a divisor on X of degree d such that $L_{\mathbb{R}} = \mathcal{L}_{\mathbb{R}}(D)$. A point $p \in X$ is an inflection point of $Q = (H^0(L_{\mathbb{R}}), L_{\mathbb{R}})$ if and only if dp is linearly equivalent to D. In particular, two inflection points of Q differ by a solution of Equation (3.5). The map ψ induces the following series of bijection Ψ_d with $d \in \mathbb{Z}$:

$$\begin{array}{rccc} \Psi_d : & \mathrm{Pic}_d(X) & \longrightarrow & X \\ & \sum_{i=1}^d p_i & \longmapsto & \psi^{-1}\left(\sum_{i=1}^d p_i - dp_0\right) \end{array}$$

satisfying

$$\Psi_d(D) + \Psi_{d'}(D') = \Psi_{d+d'}(D + D').$$

Hence $\frac{1}{d}\Psi_d(D)$ is a solution in $\mathrm{Pic}^1(X)$ of the equation $dp = D$, and we deduce that Q has d^2 inflection points.

Consider now $X = (X, \sigma)$ a real curve of genus 1 with $X(\mathbb{R}) \neq \emptyset$, a σ-invariant divisor $D \in \mathrm{Div}(X)$ and suppose that the point p_0 in the definition of the maps ψ and Ψ_d lies on $X(\mathbb{R})$. In particular, the map Ψ_d induces a real structure on $\mathrm{Pic}_d(X)$. Analogously to the complex case, the distribution of real inflection points of $Q = (H^0(L_{\mathbb{R}}), L_{\mathbb{R}})$ depends on:

1. wether the equation $dp = D$ has a real solution in $\mathrm{Pic}_1(X)$ or not;
2. the distribution of real solutions of $dp = 0$ in $\mathrm{Pic}_0(X)$.

If $X(\mathbb{R})$ is connected, then $\frac{1}{d}\Psi_d(D) \in X(\mathbb{R})$ (see Figure 3.2a), and Q has d real inflection points. Let us assume from now on that $X(\mathbb{R})$ has two connected components.

If d is odd, then $\Psi_d(D)$ is on the connected component $\mathcal{O}$ of $X(\mathbb{R})$ containing an odd number of points in the support of D. In particular $\frac{1}{d}\Psi_d(D) \in \mathcal{O}$ (see Figures 3.2b and c), and V has exactly d real inflection points, all contained in $\mathcal{O}$.

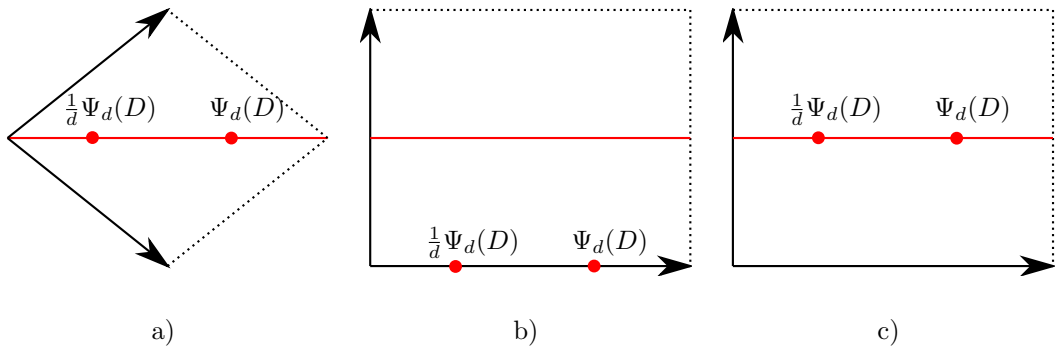


Figure 3.2: Real inflection points of real elliptic curves

If d is even and $\Psi(D) \in \mathbb{R}/u\mathbb{Z}$, then $\frac{1}{d}\Psi_d(D) \in \mathbb{R}/u\mathbb{Z}$. Hence V has exactly $2d$ real inflection points, and each connected component of $X(\mathbb{R})$ contains d of them.

If d is even and $\Psi(D) \in (\mathbb{R} + \frac{iv}{2})/u\mathbb{Z}$, then the equation $dp = D$ has no real solution. Indeed since d is even, we have $dp \in \mathbb{R}/u\mathbb{Z}$ for any $p \in X(\mathbb{R})$. ■

3.2.6 Example (Real plane elliptic curves): Applying Theorem 3.2.5 with $d = 3$, we find again that a non-singular real algebraic cubic curve X in $\mathbb{CP}^2$ has exactly 3 real inflection points, which are located on the connected component of $X(\mathbb{R})$ realizing the non-trivial class in $H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$. $*$

3.2.2 The case of dimension two ($r = 2$)

Let $C \subset \mathbb{CP}^2$ be a non-degenerate curve of degree d and let $\nu : X \longrightarrow C$ be its normalization. If (V, L) is the g^r_d on X associated to the morphism $X \longrightarrow C \hookrightarrow \mathbb{CP}^2$, then it is a base-point free linear series of degree d and rank 2 on X, and thus it has $3(d + 2g - 2)$ inflection points, where g is the genus of X.

Recall that C is said to have **traditional singularities** (see [GH78]) if it only has

1. nodes and cusps as singularities,
2. inflection points of multiplicity one, and
3. bi-tangents as multitangents.

Suppose that C has traditional singularities and for any point $x \in X$, let $(0, \alpha_1(x, V), \alpha_2(x, V))$ be its ramification sequence and let $p = \nu(x)$ be its image in C. Then we have that p is a cusp of C if and only if the ramification sequence of x is $(0, 1, 1)$, and p is an inflection point (of multiplicity one) of C if and only if the ramification sequence of x is $(0, 0, 1)$.

Suppose now that C is real, and let $\delta''(C)$, $\kappa_{\mathbb{R}}(C)$, $w_{\mathbb{R}}(C)$ and $t''(C)$ be the total number of real solitary nodes[1], real cusps, real regular inflection points and real bi-tangents at a pair of complex conjugated points, then we have the following result.

Theorem 3.2.7 (Klein). *Let $C \subset \mathbb{CP}^2$ be a non-degenerate real algebraic curve with traditional singularities. Then*

$$deg(C) + w_{\mathbb{R}}(C) + 2t''(C) = deg(C^*) + \kappa_{\mathbb{R}}(C) + 2\delta''(C). \tag{3.6}$$

Consider now $C \subset \mathbb{CP}^2$ a non-degenerate plane curve and recall Viro's formula from Theorem 1.5.9

$$\deg(C) - \int_{\mathbb{RP}^2} \mathrm{mult}_C(x)\, d\chi(x) = \deg(C^*) - \int_{\mathbb{RP}^{2*}} \mathrm{mult}_{C^*}(x)\, d\chi(x).$$

[1] A real node of a real curve is solitary if its branches are complex.

In this context, Viro's formula is a generalization of (3.6) and a reformulation of a result obtained in [Sch03]. Suppose that $C \subset \mathbb{CP}^2$ is smooth, then we have $\int_{\mathbb{RP}^2} \mathrm{mult}_C(x) d\chi(x) = 0$ and $\deg(C^*) - \deg(C) = d(d-2)$, so Viro's formula takes the form

$$\int_{\mathbb{RP}^{2*}} \mathrm{mult}_{C^*}(x)\, d\chi(x) = d(d-2). \tag{3.7}$$

If in addition C has traditional singularities, then (3.7) takes the form

$$w_{\mathbb{R}}(C) + 2t''(C) = d(d-2). \tag{3.8}$$

It follows in particular that a smooth real plane curve can have at most $d(d-2)$ distinct real inflection points, and we recall that F. Klein showed that this bound is sharp by constructing examples of real curves C with $t''(C) = 0$.

Finally, if $C \subset \mathbb{CP}^2$ is smooth, then the restriction of $\mathcal{O}_{\mathbb{CP}^2}(1)$ to C gives a divisor class $[D] \in \mathrm{Pic}(C)(\mathbb{R})^+$ with $h^0(D) = 3$ (see [GH81]). Since in this case we have that $X = C$, it follows that the linear series (V, L) defined on X is complete and has $3d(d-2)$ distinct inflection points. We can reformulate Theorem 3.2.7 as follows.

Theorem 3.2.8. *Let (X, σ) be a real curve and let $Q = (V_{\mathbb{R}}, L_{\mathbb{R}})$ be a real g^2_d such that $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is a closed embedding. Then Q is complete and has $3d(d-2)$ inflection points. Moreover Q has at most $d(d-2)$ real inflection points and this bound is sharp.*

We have the following result.

Proposition 3.2.9. *Let (X, σ) be a real curve and let $(V_{\mathbb{R}}, L_{\mathbb{R}})$ be a real, simple g^r_d such that $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is a closed embedding. If $2r + 2 > d$, then we have a bijection*

$$w_{\mathbb{R}}(V_{\mathbb{R}}, L_{\mathbb{R}}) = \#\{H \in C_0^*(\mathbb{R}) \,:\, \exists\, p \in C_0 \text{ such that } \ell(\mathcal{O}_{C_0 \cap H, p}) = r+1\}.$$

Proof. Suppose that $L_{\mathbb{R}} = L_{\mathbb{R}}(D)$ for some σ-invariant divisor $D \in \mathrm{Div}(X)$. Let $x \in \mathrm{Supp}(\mathrm{Wr}(V_{\mathbb{R}}, L_{\mathbb{R}})) \cap X(\mathbb{R})$ and let $\phi_0(x) = p \in C_0$. Since $|\lambda(x)| = 1$, the osculating hyperplane $H = \phi_{r-1}(x) \in \mathbb{P}(V)$ satisfies $\ell(\mathcal{O}_{C_0 \cap H, p}) = r + 1$.

The intersection scheme $H \cap C_0$ can be written as $D + \mathrm{div}(f)$ for some $f \in V$. We can write $D + \mathrm{div}(f) = E_{\mathbb{R}} + E_{\mathbb{C}\setminus\mathbb{R}}$, where the σ-invariant part $E_{\mathbb{R}}$ contains the term $(r+1)p$. If $f \notin V_{\mathbb{R}}$, then the effective divisor $D + \mathrm{div}(\overline{f})$ contains also $(r+1)p$, but this contradicts the uniqueness of the $(r-1)$-osculating hyperplane of x. It follows that $H \in C_0^*(\mathbb{R})$.

Conversely, let $H \in (C_0)^*(\mathbb{R})$ such that there exists $p \in C_0$ with $\ell(\mathcal{O}_{C_0 \cap H, p}) = r + 1$. We will show that $p \in C_0(\mathbb{R})$. Suppose that $p \notin C_0(\mathbb{R})$, then since $H \in (C_0)^*(\mathbb{R})$ vanishes with multiplicity $(r+1)$ at p, it follows that it also vanishes with multiplicity $(r+1)$ at $\sigma(p)$. Then $\deg(C_0) = d \geq 2r + 2$, which is a contradiction. We conclude that $p \in C_0(\mathbb{R})$. ■

It follows that if $C \subset \mathbb{CP}^r$ is a smooth real curve of degree $d < 2r + 2$ and genus g having just simple inflection points, then the number of real inflection points $w_{\mathbb{R}}(C)$ of C is precisely $\#\{H \in C^*(\mathbb{R}) \,:\, \exists\, p \in C$ such that $\ell(\mathcal{O}_{C \cap H, p}) = r + 1\}$.

Suppose now that we can express the function of mult_{C^*} as $\mathrm{mult}_{C^*} = \sum_{i=1}^s \lambda_i \mathbf{1}_{Y_i}$, for some subvarieties $Y_1, \ldots, Y_s \subset C^*$. Then Corollary 1.5.12 gives us

$$\sum_{i=1}^s \lambda_i \chi(Y_i(\mathbb{R})) = \begin{cases} d + 2g - 2, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd,} \end{cases}$$

which is a generalization of (3.7) for any $r \geq 2$. We wil find the expression $\mathrm{mult}_{C^*} = \sum_{i=1}^s \lambda_i \mathbf{1}_{Y_i}$ for mult_{C^*} and compute $\chi(Y_i(\mathbb{R}))$, $i = 1, \ldots, s$ for the case of a real, generic non-singular curve of degree 4 and genus 6 in $\mathbb{CP}^3$.

3.2.3 Codifying real hyperplane sections on a smooth curve

Let $C \subset \mathbb{CP}^r$ be a non-degenerate smooth curve and let $H \in C^*$. If $[C \cap H] = \sum_x n_x \cdot x$, we know that $\mathrm{mult}_{C^*}(H) = \sum_x (n_x - 1)$, so a convenient way to store these summands is in a decreasing sequence $\lambda(H) = (n_{x_1} - 1 \geq n_{x_2} - 1 \geq \cdots \geq n_{x_k} - 1)$.

Definition: A **partition** is a decreasing sequence $\mathbf{a} = (a_1 \geq a_2 \geq \cdots \geq a_k)$ of $k > 0$ positive integers. We define its **depth** $|\mathbf{a}|$ as $|\mathbf{a}| = \sum_{i=1}^k a_i$, and we say that it is of **degree** $d > 1$ if $|\mathbf{a}| + k \leq d$. If $|\mathbf{a}| \leq r - 1$, then we say that $\mathbf{a}$ is r-**simple**.

In particular, if $\mathbf{a} = (a_1 \geq a_2 \geq \ldots \geq a_k)$ has degree d, then $1 \leq a_i \leq d-1$ for all $i = 1, \ldots, k$.

3.2.10 Example: The partitions of degree $d = 6$ are

$$(1), (2), (1 \geq 1), (3), (2 \geq 1), (1 \geq 1 \geq 1), (4), (3 \geq 1), (2 \geq 2), \text{ and } (5).$$

The partitions of degree $d = 6$ which are 4-simple are

depth			
1	(1)		
2	(2)	$(1 \geq 1)$	
3	(3)	$(2 \geq 1)$	$(1 \geq 1 \geq 1)$

*

Suppose now that $C \subset \mathbb{CP}^r$ is a real smooth curve, and let $H \in C^*(\mathbb{R})$ be a real hyperplane. Then $[C \cap H]$ is a σ-invariant divisor, and thus it can be expressed as

$$[C \cap H] = \sum_{x \in X(\mathbb{R})} n_x \cdot x + \sum_{x \notin X(\mathbb{R})} m_x \cdot (x + \sigma(x)).$$

We can now form two decreasing sequences $(n_{x_1} - 1 \geq n_{x_2} - 1 \geq \cdots \geq n_{x_k} - 1)$, $(m_{x_1} - 1 \geq m_{x_2} - 1 \geq \cdots \geq m_{x_\ell} - 1)$ and say that the pair

$$\lambda^{\mathbb{R}}(H) = ((n_{x_1} - 1 \geq n_{x_2} - 1 \geq \cdots \geq n_{x_k} - 1), (m_{x_1} - 1 \geq m_{x_2} - 1 \geq \cdots \geq m_{x_\ell} - 1))$$

is the real partition associated to H.

Definition: A **real partition** is a pair of decreasing sequences

$$(\mathbf{a})(\mathbf{b}) = (a_1 \geq \cdots \geq a_k)_{\mathbb{R}}(b_1 \geq \ldots \geq b_\ell)_{\mathbb{C}\setminus\mathbb{R}}, \quad k, \ell \geq 0.$$

We define its **depth** $|(\mathbf{a})(\mathbf{b})|$ as $|(\mathbf{a})(\mathbf{b})| = |\mathbf{a}| + 2|\mathbf{b}|$ and we say that it is of **degree** $d > 1$ if $|\mathbf{a}| + k + 2(|\mathbf{b}| + \ell) \leq d$. If $|\mathbf{a}| + 2|\mathbf{b}| \leq r - 1$, then we say that the real divisor partition $(\mathbf{a})(\mathbf{b})$ is r-**simple**.

If $\ell = 0$, we write $(\mathbf{a}) = (a_1, \ldots, a_k)_{\mathbb{R}}$, and if $k = 0$, we write $(\mathbf{b}) = (b_1, \ldots, b_\ell)_{\mathbb{C}\setminus\mathbb{R}}$. In particular, if $(\mathbf{a})(\mathbf{b})$ is a real divisor partition of degree d, then $1 \leq a_i \leq d-1$ and $1 \leq b_j \leq \frac{d-2}{2}$.

3.2.11 Example: The real divisor partitions of degree $d = 6$ which are 4-simple arranged by their depth are

depth				
1	$(1)_{\mathbb{R}}$			
2	$(2)_{\mathbb{R}}$	$(1 \geq 1)_{\mathbb{R}}$	$(1)_{\mathbb{C}\setminus\mathbb{R}}$	
3	$(3)_{\mathbb{R}}$	$(1 \geq 2)_{\mathbb{R}}$	$(1 \geq 1 \geq 1)_{\mathbb{R}}$	$(1)_{\mathbb{R}}(1)_{\mathbb{C}\setminus\mathbb{R}}$

*

3.3.– The case of the canonical embedding of a non-hyperelliptic genus four curve

Let X be a non-singular and non-hyperelliptic projective algebraic curve of genus 4 over $\mathbb{C}$. The canonical map φ_X of X embeds this curve as a non-degenerate smooth curve of degree six in $\mathbb{CP}^3$. Moreover, the family of such curves can be identified with the family of smooth curves of genus four and degree six in $\mathbb{CP}^3$. See Proposition 4.2.1 in page 48.

In this Section we will denote by $C \subset \mathbb{CP}^3$ a smooth, 4-simple real curve of genus four and degree six, and by $\pi_2 : I_C \longrightarrow \mathbb{CP}^{3*}$ the projection from the incidence variety I_C of C to $\mathbb{CP}^{3*}$. The main result of this part is the following (see Theorem 3.3.6).

Theorem. *Let $C \subset \mathbb{CP}^3$ be a smooth, 4-simple real curve of genus four and degree six and consider the projection $\pi_2 : I_C \longrightarrow \mathbb{CP}^{3*}$. Then*

$$w_{\mathbb{R}}(C) = -\chi(\pi_2^{-1}(C^*(\mathbb{R}))).$$

3.3.1 Computations

Let $C \subset \mathbb{CP}^3$ be a curve as above, and let $C^*_{\text{Sing}} = C^*_{\text{cusp}} \cup C^*_{\text{node}}$ be the decomposition of C^*_{Sing} from Proposition 1.5.16.

Lemma 3.3.2. *The multiplicity function $mult_{C^*}$ of the dual surface $C^* \subset \mathbb{CP}^{3*}$ of C can be expressed as*

$$mult_{C^*} = \mathbf{1}_{C^*} + \mathbf{1}_{C^*_{cusp}} + \mathbf{1}_{C^*_{node}} - \mathbf{1}_{C^*_{cusp} \cap C^*_{node}} + \mathbf{1}_F, \tag{3.9}$$

where $F = mult^{-1}_{C^*}(3)$.

Proof. Since the curve C is 4-simple, we have that $\text{mult}_{C^*}(C^*) \subset \{1, 2, 3\}$, with $\text{mult}^{-1}_{C^*}(\{2, 3\}) = C^*_{\text{Sing}}$ and $\text{mult}^{-1}_{C^*}(3) = F$ a finite set. Then we can write

$$\text{mult}_{C^*} = \mathbf{1}_{C^*} + \mathbf{1}_{C^*_{\text{Sing}}} + \mathbf{1}_F.$$

Since $C^*_{\text{Sing}} = C^*_{\text{cusp}} \cup C^*_{\text{node}}$, we have that $\mathbf{1}_{C^*_{\text{Sing}}} = \mathbf{1}_{C^*_{\text{cusp}}} + \mathbf{1}_{C^*_{\text{node}}} - \mathbf{1}_{C^*_{\text{cusp}} \cap C^*_{\text{node}}}$ and the result follows. ■

Let $\mathbf{a}_1 = (3)_{\mathbb{R}}$, $\mathbf{a}_2 = (2 \geq 1)_{\mathbb{R}}$, $\mathbf{a}_3 = (1 \geq 1 \geq 1)_{\mathbb{R}}$ and $\mathbf{a}_4 = (1)_{\mathbb{R}}(1)_{\mathbb{C}\setminus\mathbb{R}}$ be all the real partitions of degree 6 and depth 3 which are 4-simple (see Example 3.2.11). We set

$$V(\mathbf{a}_i) = \{H \in C^*(\mathbb{R}) \,:\, \lambda^{\mathbb{R}}(H) = \mathbf{a}_i\}, \quad i = 1, \ldots, 4.$$

Since $\text{mult}^{-1}_{C^*}(\{3\}) = F$, we have that $F \cap \mathbb{RP}^{3*} = \bigcup_{i=1}^4 V(\mathbf{a}_i)$, and it follows from Proposition 3.2.9 that $\#V(\mathbf{a}_1) = w_{\mathbb{R}}(C)$.

Lemma 3.3.3. *Let $C \subset \mathbb{CP}^3$ be a smooth, 4-simple real curve of genus four and degree six. Then $C^*_{cusp} \cap C^*_{node} \cap \mathbb{RP}^{3*} = V(\mathbf{a}_1) \cup V(\mathbf{a}_2)$.*

Proof. If C is non-singular and 4-simple, then $\pi_2 : I_C \longrightarrow \mathbb{CP}^{3*}$ is Σ^k-transversal for all $k = 1, 2, 3$, and we have $C^*_{\text{cusp}} = \pi_2(\Sigma^2(\pi_2))$ and $C^*_{\text{node}} = \overline{N_2(\Sigma^{1,\circ}, \Sigma^{1,\circ})} = \overline{\pi_2(M_2(\Sigma^{1,\circ}, \Sigma^{1,\circ}))}$. We shall now analyze the subvarieties $\overline{M_2(\Sigma^{1,\circ}, \Sigma^{1,\circ})} \subset I_C$ and $\Sigma^2(\pi_2) \subset I_C$.

It follows from Proposition 3.3 in [Ron98], p.206 that

$$\overline{M_2(\Sigma^{1,\circ}, \Sigma^{1,\circ})} = M_2^\circ(\pi_2, \Sigma^{1,\circ}, \Sigma^{1,\circ}) \cup M_2^\circ(\pi_2, \Sigma^{2,\circ}, \Sigma^{1,\circ}) \cup M_2^\circ(\pi_2, \Sigma^{1,\circ}, \Sigma^{2,\circ}) \cup M_3^\circ(\pi_2, \Sigma^{1,\circ}, \Sigma^{1,\circ}, \Sigma^{1,\circ}) \cup \Sigma^3(\pi_2).$$

And we have that $M_2^\circ(\pi_2, \Sigma^{2,\circ}, \Sigma^{1,\circ}), \Sigma^3(\pi_2) \subset \Sigma^2(\pi_2)$. Finally we have that

$$\begin{aligned} V(\mathbf{a}_1) &= \pi_2(\Sigma^3(\pi_2))(\mathbb{R}), \\ V(\mathbf{a}_2) &= \pi_2(M_2^\circ(\pi_2, \Sigma^{2,\circ}, \Sigma^{1,\circ}))(\mathbb{R}) = \pi_2(M_2^\circ(\pi_2, \Sigma^{1,\circ}, \Sigma^{2,\circ}))(\mathbb{R}), \\ V(\mathbf{a}_3) \cup V(\mathbf{a}_4) &= \pi_2(M_3^\circ(\pi_2, \Sigma^{1,\circ}, \Sigma^{1,\circ}, \Sigma^{1,\circ}))(\mathbb{R}). \end{aligned}$$

In particular, the curve $C^*_{\text{node}}(\mathbb{R})$ contains all the elements of the finite set $F \cap \mathbb{RP}^{3*}$. The result follows. ■

Lemma 3.3.4. *Let $C \subset \mathbb{CP}^3$ be a smooth, 4-simple real curve of genus four and degree six. Then we have*

$$\chi(C^*(\mathbb{R})) + \chi(C^*_{node}(\mathbb{R})) + \#V(\mathbf{a}_3) + \#V(\mathbf{a}_4) = 0. \tag{3.10}$$

Proof. By Theorem 1.5.12 we know that $\int_{\mathbb{RP}^{3*}} \text{mult}_{C^*}(x)\, d\chi(x) = 0$, so we apply this to (3.9) and find

$$0 = \chi(C^*(\mathbb{R})) + \chi(C^*_{\text{cusp}}(\mathbb{R})) + \chi(C^*_{\text{node}}(\mathbb{R})) + \#(F \cap \mathbb{RP}^{3*}) - \#(C^*_{\text{cusp}} \cap C^*_{\text{node}} \cap \mathbb{RP}^{3*}).$$

Since the curve C is 4-simple, the curve $C^*_{\text{cusp}}(\mathbb{R})$ is a real curve with the real cusps $V(\mathbf{a}_1)$ as only singularities, so $\chi(C^*_{\text{cusp}}(\mathbb{R})) = 0$. Finally, from Lemma 3.3.3, we have that $\#(F \cap \mathbb{RP}^{3*}) - \#(C^*_{\text{cusp}} \cap C^*_{\text{node}} \cap \mathbb{RP}^{3*}) = \#V(\mathbf{a}_3) + \#V(\mathbf{a}_4)$. ■

We want to compute the terms $\chi(C^*_{\text{node}}(\mathbb{R}))$ and $\chi(C^*(\mathbb{R}))$ from Equation 3.10. See Section 1.5.2 for more details.

If $H \in C^*_{\text{node}}(\mathbb{R}) \setminus F$, then we have that its real partition $\lambda^{\mathbb{R}}(H)$ is either $(1 \geq 1)_{\mathbb{R}}$ or $(1)_{\mathbb{C}\setminus\mathbb{R}}$. Let $E_{\mathbb{R}}$ and $E_{\mathbb{C}\setminus\mathbb{R}}$ denote the set of connected components of the sets $\{H \in (C^*_{\text{node}}(\mathbb{R}) \setminus F) \,:\, \lambda^{\mathbb{R}}(H) = (1 \geq 1)_{\mathbb{R}}\}$ and $\{H \in (C^*_{\text{node}}(\mathbb{R}) \setminus F) \,:\, \lambda^{\mathbb{R}}(H) = (1)_{\mathbb{C}\setminus\mathbb{R}}\}$, respectively. The elements of $E_{\mathbb{R}}$ are the parts of $C^*_{\text{node}}(\mathbb{R})$ where two real sheets of C^* intersect, and the elements of $E_{\mathbb{C}\setminus\mathbb{R}}$ are the parts of $C^*_{\text{node}}(\mathbb{R})$ where two complex conjugated sheets of C^* intersect. Consider $p \in F \cap \mathbb{RP}^{3*}$.

1. If $p \in V(\mathbf{a}_1)$, then it is a swallow-tail singularity (see Figure 1.1 in page 22) of $C^*(\mathbb{R})$ and
   (a) p is a cusp of the curve $C^*_{\text{cusp}}(\mathbb{R})$;
   (b) p separates an element in $E_{\mathbb{R}}$ and an element in $E_{\mathbb{C}\setminus\mathbb{R}}$ of the curve $C^*_{\text{node}}(\mathbb{R})$. See Figure 3.3 a).
2. if $p \in V(\mathbf{a}_2)$, then it is a cusp of $C^*_{\text{node}}(\mathbb{R})$ and a smooth point of $C^*_{\text{cusp}}(\mathbb{R})$. See Figure 3.3 b).
3. if $p \in V(\mathbf{a}_3)$, then it the intersection of three real sheets of C^* and it is a triple point of $C^*_{\text{node}}(\mathbb{R})$. See Figure 3.3 c).
4. if $p \in V(\mathbf{a}_4)$, then it is the intersection of three sheets of C^*, two of them imaginary. See Figure 3.3 d).

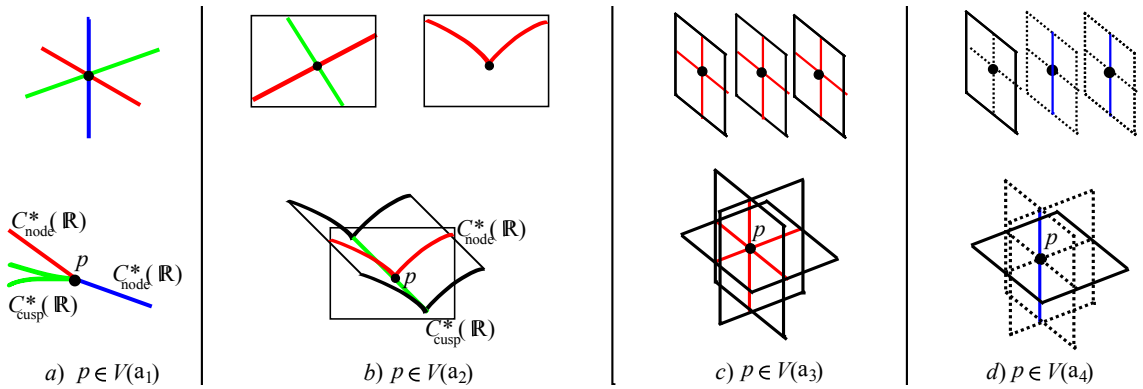


Figure 3.3: Local geometry of the map $\pi_2 : I_C \longrightarrow \mathbb{RP}^{3*}$ at the points $F \cap \mathbb{RP}^{3*} = \bigcup_{i=1}^4 V(\mathbf{a}_i)$.

Lemma 3.3.5. *With the above notation, we have* $\chi(C^*_{node}(\mathbb{R})) = -2\#V(\mathbf{a}_3)$.

Proof. Let $G = (V, E)$ be the graph having $F \cap \mathbb{RP}^{3*}$ as vertex set and $E_{\mathbb{R}} \cup E_{\mathbb{C}\setminus\mathbb{R}}$ as edge set. This is a triangulation for the real curve $C^*_{\text{node}}(\mathbb{R})$, and the incidence conditions among its elements are as follows:

1. $v \in V(\mathbf{a}_1)$ has degree 2 and they connect an edge in $E_{\mathbb{R}}$ and one edge in $E_{\mathbb{C}\setminus\mathbb{R}}$;
2. $v \in V(\mathbf{a}_2)$ has degree 2, and it connects only edges in $E_{\mathbb{R}}$;
3. $v \in V(\mathbf{a}_3)$ has degree 6, and it connects only edges in $E_{\mathbb{R}}$;
4. $v \in V(\mathbf{a}_4)$ has degree 2, and it connects only edges in $E_{\mathbb{C}\setminus\mathbb{R}}$.

The equation $\sum_{v\in V} \deg(v) = 2\#E$ gives us the relation $\#V(\mathbf{a}_1) + \#V(\mathbf{a}_2) + 3\#V(\mathbf{a}_3) + \#V(\mathbf{a}_4) = \#E_{\mathbb{R}} + \#E_{\mathbb{C}\setminus\mathbb{R}}$, which yields $\chi(G) = \chi(C^*_{\text{node}}(\mathbb{R})) = -2\#V(\mathbf{a}_3)$. ■

Theorem 3.3.6. *With the above notation, we have* $\chi(\pi_2^{-1}(C^*(\mathbb{R}))) = -w_{\mathbb{R}}(C)$.

Proof. Let $G = (V, E)$ be the triangulation for C^*_{node} introduced in Lemma 3.3.5, with $V = F \cap \mathbb{RP}^{3*}$ and $E = E_{\mathbb{R}} \cup E_{\mathbb{C}\setminus\mathbb{R}}$. We construct a triangulation $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{T}$ of $C^*(\mathbb{R})$ such that

1. the set $\mathcal{V} \cup \mathcal{E}$ is a refinement of G, so that $\mathcal{V} = V \cup V_5 \cup V_6$ where $V_5 \subset C^*_{\text{node}}(\mathbb{R})$ and $V_6 \subset C^*(\mathbb{R}) \setminus C^*_{\text{node}}$, and $\mathcal{E} = E \cup F'$, where each $f = \langle v, w\rangle \in F$ has at least one of its vertices v, w in V_6, so it lifts to a single edge in $\pi_2^{-1}(C^*(\mathbb{R}))$;
2. the preimages $\mathcal{V}'$, $\mathcal{E}'$ and $\mathcal{T}'$ of $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{T}$ lift to a triangulation of $\pi_2^{-1}(C^*(\mathbb{R}))$

Since the ramification locus is of codimension one, we have $\#\mathcal{T}' = \#\mathcal{T} = t$. We also have $\#\mathcal{V}' = \#V(\mathbf{a}_1) + 2\#V(\mathbf{a}_2) + 3\#V(\mathbf{a}_3) + 3\#V(\mathbf{a}_4) + 2\#V_5 + \#V_6$ and $\#\mathcal{E}' = \#2E_{\mathbb{R}} + 2\#E_{\mathbb{C}\setminus\mathbb{R}} + \#F'$, so

$$\begin{aligned}\chi(\pi_2^{-1}(C^*(\mathbb{R}))) &= \chi(C^*(\mathbb{R})) + \#V(\mathbf{a}_2) + 2\#V(\mathbf{a}_3) + 2\#V(\mathbf{a}_4) + \#V_5 - (E_{\mathbb{R}} + E_{\mathbb{C}\setminus\mathbb{R}})\\ &= \chi(C^*(\mathbb{R})) - \#V(\mathbf{a}_1) - \#V(\mathbf{a}_3) + \#V(\mathbf{a}_4),\end{aligned}$$

since $\#V(\mathbf{a}_1) + \#V(\mathbf{a}_2) + 3\#V(\mathbf{a}_3) + \#V(\mathbf{a}_4) + \#V_5 = \#E_{\mathbb{R}} + \#E_{\mathbb{C}\setminus\mathbb{R}}$. Since $\chi(C^*_{\text{node}}(\mathbb{R})) = -2\#V(\mathbf{a}_3)$, we deduce from (3.10) that

$$\chi(C^*(\mathbb{R})) = \#V(\mathbf{a}_3) - \#V(\mathbf{a}_4).$$

The result follows. ■

3.4.– Generalized Viro formulas for non-degenerated projective curves with unramified normalization

Let X be a nonsingular complex curve and (V, L) a g^r_d on it. If ϕ_0 is a closed embedding, we have defined the incidence variety I_{C_0} of C_0 as the set of pairs $(x, H) \in \mathbb{P}(V^*) \times \mathbb{P}(V)$ such that $x \in C_0 \cap H$. In this part we extend the definition of the incidence variety for linear series (V, L) having the property that $\phi_0 : X \longrightarrow C_0$ is birational.

Let X be a nonsingular complex curve and (V, L) a g^r_d on it. Let $s \in V$ and let $Z(s)$ be the zero set of s, then if $x \in Z(s)$, the **Milnor number** $\mu_x(Z(s))$ of $Z(s)$ at x satisfies

$$\mu_x(Z(s)) = \mathrm{ord}_x(s) - 1.$$

See [NBT08] for more details.

We say that $Z(s)$ is singular at $x \in X$ if $\mu_x(Z(s)) > 0$, and we set $\mu(Z(s)) := \sum_{x \in Z(s)} \mu_x(s)$. Note that $\mu(Z(s)) = d - \#\{x : s(x) = 0\}$ is independent of the class $[s]$ of s in $\mathbb{P}(V)$, so we get a well-defined function $\mu : \mathbb{P}(V) \longrightarrow \mathbb{N}$ given by $\mu([s]) = \mu(Z(s))$.

Note that if there exists $x \in X$ such that $\alpha_1(x, v) = a_1(x) - 1 > 0$, then all the points $[s] \in \mathbb{P}(V)$ of the projective hyperplane $\mathbb{P}(V_{x,a_1(x)}) \subset \mathbb{P}(V)$ will satisfy $\mu([s]) > 0$.

Lemma 3.4.1. *If ϕ_0 is base-point free, then the set $I_X = \{(x, [s]) \in X \times \mathbb{P}(V) : \mu_x([s]) \geq 0\}$ is a non-singular projective variety of dimension r called the* **incidence variety** *of X.*

Proof. For every $x \in X$, we have that $\{[s] \in \mathbb{P}(V) : \mu_x([s]) \geq 0\} = \mathbb{P}(V_{x,a_1(x)})$, so we can see I_X as a projective bundle over X with fiber $\mathbb{P}^{r-1}$, thus it is a non-singular projective variety of dimension r. ■

Let $\pi_2 : I_X \longrightarrow \mathbb{P}(V)$ be the morphism induced by projecting onto the second factor. This is a morphism between two non-singular projective varieties of the same dimension.

Proposition 3.4.2. *If $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is unramified, then $N_X = \{(x, [s]) \in I_X : \mu_x([s]) > 0\}$ is a a non-singular projective subvariety of codimension one in I_X such that $\pi_2 : N_X \longrightarrow C_0^*$ is a resolution of singularities.*

Proof. For $x \in X$ and $s \in V$, note that $\mu_x([s]) > 0$ if and only if $\mathrm{ord}_x(s) > 1$. Since ϕ_0 is unramified, we have that $a_1(x) = 1$, so we see that $[s] \in \mathbb{P}(V_{x,a_2(x)})$ and thus N_X is a projective bundle over X with fiber $\mathbb{P}(V_{x,a_2(x)})$. It follows from proof of Lemma 1.5.4 that $\pi_2 : N_X \longrightarrow C_0^*$ is a resolution of singularities. ■

We conclude that if $\phi_0 : X \longrightarrow \mathbb{P}(V^*)$ is unramified, then $\pi_2(N_X) = (C_0)^* = \{[s] \in \mathbb{P}(V) : [s] \text{ is singular}\}$, and we get a diagram

$$\begin{tikzcd} N_X \arrow[r, hook] \arrow[d, "\pi_2"'] & I_X \arrow[d, "\pi_2"] & \\ C_0^* \arrow[r, hook] & \mathbb{P}(V) \arrow[r, "\mu"] & \mathbb{Z}_{\geq 0}, \end{tikzcd} \tag{3.11}$$

where $\mathrm{Supp}(\mu) = C_0^*$.

Definition: Let $C \subset \mathbb{CP}^r$ be a curve and let $\nu : \tilde{C} \longrightarrow C$ be its normalization map. We say that $p \in C_{\mathrm{Sing}}$ is an **ordinary m-fold point** if $\mathrm{mult}_C(p) = m$ and $\#\nu^{-1}(p) = m$.

Proposition 3.4.3. *Let $C \subset \mathbb{CP}^r$ be a non-degenerate curve such that if $p \in C_{Sing}$, then p is an ordinary $r(p)$-fold point. We have $Eu_C = mult_C$.*

Proof. If C is as above, then the multiplicity $\mathrm{mult}_C(p)$ of a point $p \in C_{\mathrm{Sing}}$ coincides with the number of branches $r(p)$ of C passing through p. We have to show that if $f = \mathrm{mult}_C = \mathbf{1}_C + \sum_{x \in C_{\mathrm{Sing}}} [r(x) - 1]\mathbf{1}_{\{x\}}$ and $H \in \mathbb{CP}^{r*}$ satisfies $\overline{f^*}(H) = \#(C \cap H) + \sum_{x \in C_{\mathrm{Sing}} \cap H} [r(x) - 1] - d \neq 0$, then $H \in C^*$.

We begin by showing that if $\nu : X \longrightarrow C$ is the normalization map of C, then it is unramified. Any $H \in \mathbb{CP}^{r*}$ induces a divisor on X as follows: let f be the linear polynomial such that $H = V(f)$, then $\nu^*(f)$ is a section on X and if $f(p) = 0$, then we have that $\ell(\mathcal{O}_{C \cap H,p}) = \mathrm{ord}_p(f) = \sum_{y \in \nu^{-1}(p)} \mathrm{ord}_y(\nu^*(f))$, and since $\mu_y(\nu^*(f)) = \mathrm{ord}_y(\nu^*(f)) - 1$, we have

$$\sum_{y \in \nu^{-1}(p)} \mu_y(\nu^*(f)) = \ell(\mathcal{O}_{C \cap H,p}) - r(p) = \ell(\mathcal{O}_{C \cap H,p}) - \mathrm{mult}_C(p).$$

In particular we have that if $p \in C$ is a smooth point such that $\nu^{-1}(p) = y$, then $\mu_y(\nu^*(f)) = \ell(\mathcal{O}_{C\cap H,p}) - 1$, and if $p \in C_{\mathrm{Sing}}$ and the hyperplane H is such that $\ell(\mathcal{O}_{C\cap H,p}) = r$, then $\sum_{y\in\nu^{-1}(p)} \mu_y(\nu^*(f)) = 0$, so $\mathrm{ord}_y(\nu^*(f)) = 1$ for all $y \in \nu^{-1}(p)$ and this implies that ν is unramified.

We conclude that

$$\mu([\nu^*(f)]) = \sum_p \sum_{y\in\nu^{-1}(p)} \mu_y([\nu^*(f)]) = \sum_{p\in C\cap H} [\ell(\mathcal{O}_{C\cap H,p}) - \mathrm{mult}_C(p)] = -\overline{f^*}(H),$$

so the condition $\overline{f^*}(H) \neq 0$ is equivalent to $\mu([\nu^*(f)]) > 0$ for $H = V(f)$, and this implies that $H \in C^*$ since $\pi_2(N_X) = C^*$ by Proposition 3.4.2 ■

Remark 3.4.4: *We point out that contrary to the smooth case, the function $Eu_{C^*} : C^* \longrightarrow \mathbb{Z}_{>0}$ is not necessarily the multiplicity function m_{C^*} associated to the dual variety when $C \subset \mathbb{CP}^r$ is a non-degenerated curve such that all of its singularities consists of ordinary r fold points.*

Theorem 3.4.5 (Generalized Viro formula, controlled singularities case). *Let $C \subset \mathbb{CP}^n$ be a non-degenerated curve whose only singularities are ordinary r-tuple points. For any $p \in C_{Sing}(\mathbb{R})$, we denote by $r''(p)$ the number of complex branches passing through p. Then*

$$\int_{\mathbb{RP}^*} Eu_{C^*}(x)\, d\chi(x) = \begin{cases} deg(C^*) - deg(C) + \sum_{p\in C_{Sing}(\mathbb{R})} r''(p), & \text{if } n \text{ is even,} \\ -\sum_{p\in C_{Sing}(\mathbb{R})} r''(p), & \text{if } n \text{ is odd.} \end{cases} \tag{3.12}$$

Proof. For any $p \in C_{\mathrm{Sing}}(R)$, let $r'(p)$ be the number of real branches passing through p. Then we have $\chi(C(\mathbb{R})) = -\sum_{p\in C_{\mathrm{Sing}}(\mathbb{R})}[r'(p) - 1]$, and it follows that

$$\int_{\mathbb{RP}^r} \mathrm{mult}_C(x)\, d\chi(x) = \chi(C(\mathbb{R})) + \sum_{p\in C_{\mathrm{Sing}}(\mathbb{R})} [r(p) - 1] = \sum_{p\in C_{\mathrm{Sing}}(\mathbb{R})} r''(p),$$

since $r(p) = r'(p) + r''(p)$. The result now follows by applying Theorem 1.5.10. ■

Chapter 4

A lower bound for real Weierstrass points on a genus 4 real curve

4.1.–Introduction

The main purpose of this chapter is to construct examples of non-singular, non-hyperelliptic real curves of degree six and genus four in $\mathbb{RP}^3$ having 30 real inflection points. Our result (see Theorem 4.5.1) can be stated as follows.

Theorem. *There exist non-singular, non-hyperelliptic real algebraic curves of genus four having 30 real Weierstrass points.*

Our work uses Viro's patchworking theorem to construct real curves in the (normal) toric surface $\mathbb{RP}(2,1,1)$ associated to the convex lattice polygon $\Delta := \mathrm{Conv}\{(0,0),(2,0),(0,1)\}$.

In Section 4.2 we use the fact that over an algebraically closed field of characteristic zero $\mathbb{K}$, the smooth curves of degree six and genus four on $\mathbb{KP}^3$ are the smooth complete intersections of a quadric S_2 and a cubic S_3 in $\mathbb{KP}^3$. We fix the quadric S_2 to be the rational normal cone in $\mathbb{KP}^3$, which can be seen as the image of the canonical embedding φ of the (normal) toric surface $\mathbb{KP}(2,1,1)$. We use the morphism $\varphi^{-1} : S_2 \longrightarrow \mathbb{KP}(2,1,1)$ to study in $\mathbb{KP}(2,1,1)$ the inflection points of curves in $\mathbb{KP}^3$ which arise as complete intersections of S_2 with a surface $S_k \subset \mathbb{KP}^3$ of degree $k = 1, 2$ or 3. We introduce the concept of **admissible polygon** of degree d and genus g, which are precisely the 2-dimensional convex lattice polygons in $\mathbb{R}^2$ which arise as Newton polygons of non-singular algebraic curves $C \subset (\mathbb{K}^*)^2$ such that $\overline{\varphi(C)} \subset \mathbb{KP}^3$ is a complete intersection $S_2 \cap S_k$. In Proposition 4.2.6 we characterize in terms of the geometry of the curve C the points $p \in C$ which are images of inflection points of the spatial curve $\overline{\varphi(C)}$ under the map φ^{-1}.

Proposition 4.1.2. *Let $\Theta \subset 3\Delta$ be an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$, let $C \subset (\mathbb{K}^*)^2$ be a non-singular curve having Newton polygon Θ. Then $p \in C$ is the image of an inflection point of multiplicity one of the linear series $\mathcal{O}_{\mathbb{KP}^3}(1)|_{\overline{\varphi(C)}}$ on $\overline{\varphi(C)}$ if and only if either:*

1. *there is a curve $H = V(a_3 + a_1X + a_3X^2 + a_2Y) \subset (\mathbb{K}^*)^2$ with $a_2 \neq 0$ satisfying $\ell(\mathcal{O}_{C\cap H,p}) = 4$, or*
2. *the tangent line to C at p is vertical.*

Points $p \in C$ satisfying the first condition (respectively the second condition) are called inflection points of type I (respectively of type II). Using this toric reformulation, in Section 4.3 we give a method to compute the number of inflection points of type I and type II of smooth curves $C \subset (\mathbb{K}^*)^2$ with admissible Newton polygon.

We denote by $\mathcal{O}_{S_2}(1)$ the restriction of $\mathcal{O}_{\mathbb{KP}^3}(1)$ on S_2. In Proposition 4.3.1 we prove the following result.

Proposition 4.1.3. *Let Θ be an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$ and let $C = V(f)$ be a non-singular curve in $(\mathbb{K}^*)^2$ with Newton polygon Θ. Then the inflection points of type I of C satisfy the system $\mathcal{E}_1 = \{f, \Phi(f)\}$, where $\Phi(f)$ is given by:*

$$\Phi(f) := \begin{aligned}(f_{XXX}f_Y - 3f_{XX}f_{XY} - 3f_Xf_{XXY})f_Y^3 + 3(f_{XX}f_{YY} + 2f_{XY}^2 + f_Xf_{XYY})f_Xf_Y^2 - \\ -(9f_{XY}f_{YY} + f_{YYY}f_X)f_X^2f_Y + 3f_{YY}^2f_X^3\end{aligned}$$

Here f_X represents the partial derivative $\frac{\partial f}{\partial X}$ of f with respect to X.

Conversely, if $(x,y) \in (\mathbb{K}^*)^2$ is a solution for the system $\mathcal{E}_1 = \{f, \Phi(f)\}$, then $f_Y(x,y) \neq 0$ and

$$z = \left(\frac{-f_X^2 f_{YY} + 2f_X f_Y f_{XY} - f_Y^2 f_{XX}}{2f_Y^3}\right)(x,y), \quad w = -\left(\frac{f_X}{f_Y} + 2zX\right)(x,y), \quad v = y - zx^2 - wx$$

defines a polynomial $g(X,Y) = Y - zX^2 + wX + v$ such that $\ell(\mathcal{O}_{C\cap V(g),(x,y)}) \geq 4$.

In Section 4.4, we study real inflection points of real elliptic curves with a particular admissible Newton polygon. In particular, in Proposition 4.4.3 we construct explicit examples of real elliptic curves having 8 real inflection points.

Proposition 4.1.4. *Let $f \in \mathbb{R}[X^{\pm 1}, Y^{\pm 1}]$ be of the form $f(X,Y) = u_{02}Y^2 + P_1(X)Y + u_{20}X^2$, with $P_1(X) = u_{01} + u_{11}X + u_{21}X^2$, satisfying $u_{11} \neq 0$, $u_{21}u_{01} < 0$ and $u_{20}u_{02} < 0$. Then the restriction of $\mathcal{O}_{S_2}(1)$ to $\overline{\varphi(V(f))}$ is a real linear series which has eight real inflection points. Furthermore, these are all of type I.*

4.2.– Curves in $\mathbb{KP}(2,1,1)$

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We start with the following fact.

Proposition 4.2.1. *The non-hyperelliptic, smooth complete curves of genus four over $\mathbb{K}$ are the smooth complete intersections of a quadric and a cubic in $\mathbb{KP}^3$.*

Proof. We give here the general idea of this result. The details can be consulted in [GH78], p. 258. Let X be a non-singular and non-hyper-elliptic complete algebraic curve of genus 4 over $\mathbb{K}$ with canonical divisor K_X. The canonical map $\varphi_{K_X} : X \longrightarrow |K_X|$ embeds X as a non-degenerate, smooth curve of degree six $C = \phi_K(X)$.

We consider the pull-back $\varphi^*_{K_X} : H^0(\mathcal{O}_{|K_X|}(2)) \longrightarrow H(2K_X)$, and since the first has dimension ten and the second has dimension nine, we have that C lies on a quadric S_2. This quadric is irreducible since C is non-degenerate.

We also have that $\varphi^*_{K_X} : H^0(\mathcal{O}_{|K_X|}(3)) \longrightarrow H(3K_X)$ is a morphism from a space of dimension twenty to a space of dimension fifteen, and it follows that the space of cubics S_3 containing C has dimension four. The reducible cubics containing S_2 are the form $S_3 = S_2 \cup H$ for H an hyperplane; the space of these cubics has dimension three, and since C is irreducible, then there exists an irreducible cubic S_3 containing it. Finally, since C is a sextic, we have $C = S_2 \cap S_3$ by Bézout Theorem.

Conversely, let $C \subset K\mathbb{P}^3$ be a smooth complete intersection of a quadric S_2 and a cubic S_3. Then we have that C has genus four (by the adjunction formula) and degree six. Finally, if $L = \mathcal{O}(1)_C = \mathcal{O}_{|K_X|}(1)|_C$, then $h^0(L) = 3 + h^0(\Omega_C \otimes L^*) \geq 4$ and $\deg(\Omega_C \otimes L^*) = 0$ implies $\Omega_C \cong L$. ■

Let us endow $\mathbb{KP}^3$ with projective coordinates $[X_0 : X_1 : X_2 : X_3]$ and let $S_2 \subset \mathbb{KP}^3$ be the quadric defined by the polynomial $F_2(X_0, \ldots, X_3) = X_0X_3 - X_1^2$. Let's consider the following family of curves:

$$\begin{aligned}\mathscr{C} = \{C \subset S_2 \ : \ & C \text{ complete intersection in } \mathbb{KP}^3 \text{ of } S_2 \text{ with a surface} \\ & S_k \subset \mathbb{KP}^3 \text{ of degree } k \leq 3 \text{ and } C \cap (\mathbb{K}^*)^3 \text{ smooth}\}.\end{aligned} \tag{4.1}$$

In particular, $\mathscr{C}$ contains a subset of the non-hyperelliptic, smooth complete curves of genus four over $\mathbb{K}$, by Proposition 4.2.1.

Let us endow $(\mathbb{K}^*)^2$ with affine coordinates (X,Y) and let $\mathbb{KP}(2,1,1)$ be the normal toric surface associated to the convex lattice polygon $\Delta = \mathrm{Conv}\{(0,0),(2,0),(0,1)\}$. Let φ be the tautological linear system of $\mathbb{KP}(2,1,1)$ given in coordinates by

$$\begin{array}{rccc} \varphi : & (\mathbb{K}^*)^2 & \longrightarrow & \mathbb{KP}^3 \\ & (x,y) & \longmapsto & [x^2, x, y, 1] \end{array}$$

Then we have that $S_2 = \overline{\varphi((\mathbb{K}^*)^2)} \cong \mathbb{KP}(2,1,1)$. Any curve $C \in \mathscr{C}$ will define a smooth curve $C' = \varphi^{-1}(C) \cap (\mathbb{K}^*)^2$ in $(\mathbb{K}^*)^2$ whose Newton polygon $\mathrm{New}(C')$ is contained in the polygon 3Δ. Conversely, a smooth curve $D \subset (\mathbb{K}^*)^2$ with Newton polygon $\mathrm{New}(D)$ contained in 3Δ will define a curve $D' = \overline{\varphi(D)}$ in S_2. Let us define

$$\mathscr{D} = \{D \subset (\mathbb{K}^*)^2 \ : \ \overline{\varphi(D)} \in \mathscr{C}\}. \tag{4.2}$$

Observe that any $D \in \mathscr{D}$ is smooth in $(\mathbb{K}^*)^2$, but $\overline{D} \subset \mathbb{KP}(2,1,1)$ may have singularities in $\mathbb{KP}(2,1,1) \setminus (\mathbb{K}^*)^2$.

We want to study the relationship between the sets $\mathscr{C}$ and $\mathscr{D}$. To start with, let $U_3 \subset \mathbb{KP}^3$ be the affine chart defined by $\{X_3 = 1\}$ with affine coordinates $x_i = X_i/X_3$, $i = 0,1,2$ and let $\Delta' := \mathrm{Conv}\{(0,0,0),(3,0,0),(0,0,3),(0,1,0),(2,1,0),(0,1,2)\}$.

1. To a polynomial $f = \sum a_{ijk} x_0^i x_1^j x_2^k$ in $\mathbb{K}[x_0,x_1,x_2]$ with $\mathrm{New}(f) \subset \Delta'$ we associate the polynomial $\pi(f) \in \mathbb{K}[X^{\pm 1}, Y^{\pm 1}]$ with $\mathrm{New}(\pi(f)) \subset 3\Delta$ given by
$$\pi(f)(X,Y) = \sum a_{ijk} X^{2i+j} Y^k.$$
2. To a polynomial $g = \sum a_{ij} X^i Y^j$ in $\mathbb{K}[X^{\pm 1}, Y^{\pm 1}]$ with with $\mathrm{New}(g) \subset 3\Delta$ we associate the polynomial $\rho(g) \in \mathbb{K}[x_0,x_1,x_2]$ with $\mathrm{New}(\rho(g)) \subset \Delta'$ given by
$$\rho(g)(x_0,x_1,x_2) = \sum a_{ij} x_0^k x_1^\varepsilon x_2^j, \qquad \text{where } i = 2k+\varepsilon,\ \varepsilon \in \{0,1\}.$$

We have the following result.

Proposition 4.2.2.

1. *For $C \in \mathscr{C}$, there exists $f \in \mathbb{K}[x_0,x_1,x_2]$ with New$(f) \subset \Delta'$ such that $C = S_2 \cap \overline{V(f)}$ and $C' = \varphi^{-1}(C) \cap (\mathbb{K}^*)^2 = V(\pi(f))$. In particular, New$(\pi(f)) \subset 3\Delta$.*
2. *For $D \in \mathscr{D}$, there exists $g \in \mathbb{K}[X^{\pm 1}, Y^{\pm 1}]$ with New$(g) \subset 3\Delta$ such that $D = V(g)$ and $\overline{\varphi(D)} = S_2 \cap V(\rho(g))$. In particular, New$(\rho(g)) \subset \Delta'$.*

Proof. The space of curves $C \subset S_2$ induced by homogeneous polynomials $F \in \mathbb{K}[X_0 : X_1 : X_2 : X_3]$ of degree at most 3 is a projective space of dimension 15. This is because in S_2 we have the relation $X_0X_3 = X_1^2$, hence some identifications happen among the monomials of $F(X_0,\dots,X_3)$, namely $X_0^2X_3 = X_1^2X_0$, $X_0X_1X_3 = X_1^3$, $X_0X_2X_3 = X_1^2X_2$ and $X_0X_3^2 = X_1^2X_3$.

In the affine chart $\{X_3 = 1\}$, the equation $X_0X_3 = X_1^2$ becomes $x_0 = x_1^2$ and the above identifications of monomials become $x_0^2 = x_1^2x_0$, $x_0x_1 = x_1^3$, $x_0x_2 = x_1^2x_2$ and $x_0 = x_1^2$. So the vertices of the convex lattice polytope $\mathrm{Conv}\{(0,2,0),(1,2,0),(0,2,1),(0,3,0)\}$ inside $\mathrm{Conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,3)\}$ are identified with the vertices of the polytope $\mathrm{Conv}\{(1,0,0),(2,0,0),(1,1,0),(1,0,1)\}$ inside Δ', as in Figure 4.1 a).

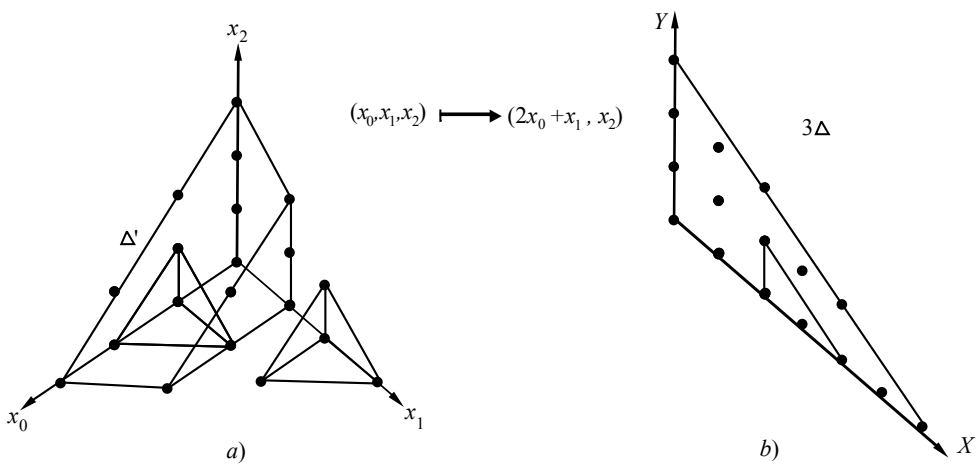


Figure 4.1: Projecting monomials.

The canonical embedding $\varphi : \mathbb{KP}(2,1,1) \longrightarrow \mathbb{KP}^3$ induces a morphism of rings $\mathbb{K}[x_0,x_1,x_2] \longrightarrow [X^{\pm 1}, Y^{\pm 1}]$ given by $x_0 \mapsto X^2$, $x_1 \mapsto X$, $x_2 \mapsto Y$. This morphism induces the bijection between $\Delta' \cap \mathbb{Z}^3 \longrightarrow (3\Delta) \cap \mathbb{Z}^2$ which gives the application $f \mapsto \pi(f)$, as in Figure 4.1 b). The inverse bijection $(3\Delta) \cap \mathbb{Z}^2 \longrightarrow \Delta' \cap \mathbb{Z}^3$ gives us the application $g \mapsto \rho(g)$. The result follows. ■

We want to consider the curves $C \in \mathscr{C}$ as closures of images of smooth curves $D \subset (\mathbb{K}^*)^2$ under the map $\varphi : \mathbb{KP}(2,1,1) \longrightarrow \mathbb{KP}^3$. For $g \in \mathbb{K}[X^{\pm 1}, Y^{\pm 1}]$ let $G \in \mathbb{K}[X_0,\dots,X_3]$ be the homogeneization of $\rho(g) \in \mathbb{K}[x_0,x_1,x_2]$ with respect to the variable X_3. If the curve $D = V(g)$ is smooth, then it will belong to $\mathscr{D}$ if and only if the ideal $\langle G, X_0X_3 - X_1^2 \rangle \subset \mathbb{K}[X_0,\dots,X_3]$ is a complete intersection ideal.

Definition: A convex lattice polygon $\Theta \subset 3\Delta$ is **admissible** if there exists $D \in \mathscr{D}$ with Newton polygon Θ.

With this definition, we can describe the set $\mathscr{D}$ as the set of smooth curves $D \subset (\mathbb{K}^*)^2$ with admissible Newton polygon $\mathrm{New}(D)$. The polygons $k\Delta$ for $k = 1, 2, 3$ are clearly admissible. We now give an example of a polygon which is not admissible.

4.2.3 Example (A non-admissible polygon): Let $\Theta = \mathrm{Conv}\{(0,2),(1,2),(2,1),(2,0)\}$; this polygon is not admissible. Indeed, let $g(X,Y) = \alpha XY^2 + \beta Y^2 + \gamma XY + \delta X^2Y + \epsilon X^2$ be a polynomial with Newton polygon Θ, then the homogeneization with respect to X_3 of the polynomial $\rho(g)$ is the polynomial

$$G = \alpha X_1X_2^2 + X_3(\beta X_2^2 + \gamma X_1X_2 + \delta X_0X_2 + \epsilon X_0X_3).$$

We see that $V(G, X_0X_3 - X_1^2)$ is not irreducible, since it consists of a curve and the line $\{X_1 = X_3 = 0\}$. $*$

Remark 4.2.4: *Let $\Theta \subset 3\Delta$ be an admissible polygon, $D \in \mathscr{D}$ a curve with Newton polygon Θ and $\nu : X \longrightarrow \overline{D}$ the normalization of $\overline{D} \subset \mathbb{KP}(2,1,1)$.*

1. *The degree d of the projective curve $\overline{\varphi(D)} \subset \mathbb{KP}^3$ is $d = 2k$, where $\overline{\varphi(D)} = S_2 \cap S_k$.*
2. *The genus $g(\overline{D})$ of the proper curve $\overline{D} \subset \mathbb{KP}(2,1,1)$ satisfies $g(\overline{D}) = g + \sum_{x \in \overline{D}_{Sing}} \delta_x$, where $g = g(X)$ is the number of inner lattice points of Θ.*

Since the numbers g and d do not depend on the particular choice of D, we will say that the admissible convex lattice polygon Θ has genus g and degree $d = 2k$. The possible pairs (d,g) for an admissible polygon Θ are $(2,0)$, $(4,0),(4,1)$ and $(6,0),(6,1),(6,2),(6,3),(6,4)$. The possible pairs (d,g) for an admissible polygon Θ such that $\overline{\varphi(D)} \subset \mathbb{KP}^3$ is smooth are $(2,0),(4,1),(6,4)$.

Proposition 4.2.5. *Let $\Theta \subset 3\Delta$ be an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$, and let $D \in \mathscr{D}$ be a curve with Newton polygon Θ. Then the linear series (V,L) induced on the normalization $\nu : X \longrightarrow \overline{D}$ of $\overline{D}$ has $\mu(\Theta) = 4d + 12(g-1)$ inflection points.*

Proof. If $(d,g) = (2,0)$, then the curve $\overline{\varphi(D)} \subset \mathbb{KP}^3$ is degenerate. If $(d,g) \neq (2,0)$, then the morphism $X \xrightarrow{\nu} \overline{D} \hookrightarrow \mathbb{KP}^3$ gives us a linear series (V,L) of degree d and rank $r = 3$ on the curve X of genus g. This g^3_d has thus $(r+1)(d + r(g-1)) = 4d + 12(g-1)$ inflection points. ■

Note that if Θ has degree d and genus g, with $(d,g) = (4,1)$ or $(d,g) = (6,4)$, then $\overline{D}$ is smooth and the restriction of $\mathcal{O}_{\mathbb{KP}^3}(1)$ to $\overline{\varphi(D)}$ gives a divisor class $[E]$ with $h^0(E) = 4$, so the closed embedding $\overline{\varphi(D)} \hookrightarrow \mathbb{KP}^3$ is the 0-th Gauss map of a complete g^3_d on $\overline{\varphi(D)}$. If $(d,g) = (4,0)$, then $\overline{D}$ has a singular point $x \in (\overline{D} \setminus D)$ with $\delta_x = 1$, and it follows that x is either a cuspidal point or a nodal point of $\overline{D}$.

From now on, we will only consider admissible polygons $\Theta \subset 3\Delta$ of degree d and genus g with $(d,g) \neq (2,0)$.

Definition: Let $\Theta \subset 3\Delta$ be an admissible polygon of degree d and genus g, and let $D \in \mathscr{D}$ be a curve with Newton polygon Θ.

1. We say that $\mu(\Theta) = 4d + 12(g-1)$ is the inflection multiplicity of Θ.
2. If $q \in \overline{\varphi(D)}$ is an inflection point of multiplicity m, then we say that $p = \varphi^{-1}(q)$ is an inflection point of multiplicity m of $\overline{D} \subset \mathbb{KP}(2,1,1)$.

Let $D \in \mathscr{D}$. Our next task is to find the inflection points of the curve $\overline{\varphi(D)} \subset \mathbb{KP}^3$ that lie in D using the geometry in $(\mathbb{K}^*)^2$ induced by the map $\varphi : (\mathbb{K}^*)^2 \longrightarrow \mathbb{KP}^3$.

Proposition 4.2.6. *Let $\Theta \subset 3\Delta$ be an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$, let $D \in \mathscr{D}$ be a curve with Newton polygon Θ. Then $p \in D$ is an inflection point of multiplicity one if and only if :*

1. *there is a curve $H' = V(a_3 + a_1X + a_3X^2 + a_2Y) \subset (\mathbb{K}^*)^2$ with $a_2 \neq 0$ satisfying $\ell(\mathcal{O}_{D \cap H',p}) = 4$, or*
2. *the tangent line to D at p is vertical.*

Proof. Let $\Theta \subset 3\Delta$ and $D \in \mathscr{D}$ be as in the statement of the Proposition. A regular point $q \in \overline{\varphi(D)}$ will be an inflection point of multiplicity one if there exists a hyperplane $H \subset \mathbb{KP}^3$ such that $\ell(\mathcal{O}_{\overline{\varphi(D)}\cap H,q}) = 4$.

Let $H \subset \mathbb{KP}^3$ be a hyperplane defined by the polynomial $F = a_0X_0 + \cdots + a_3X_3$, then using $X_0X_3 = X_1^2$ we get

$$F \cdot X_3 = (a_0X_1^2 + a_1X_1X_3 + a_3X_3^2) + a_2X_2X_3,$$

and we conclude that $[H \cap S_2] = [L_1] + [L_2]$ if and only if $a_2 = 0$, i.e., the hyperplane H pass through the singular point of S_2, which is $[0:0:1:0]$.

The surface S_2 is ruled: it contains the line in $\mathbb{KP}^3$ joining the points $[q_0 : q_1 : 0 : q_3], [0:0:1:0]$ satisfying $q_0q_3 = q_1^2$. If $q = [q_0 : q_1 : q_2 : q_3] \in \overline{\varphi(D)}$ is a point such that its embedded tangent line L is one of the lines of the ruling of S_2, then it is an inflection point of $\overline{\varphi(D)}$, since there exists a hyperplane H such that $[H \cap S_2] = 2L$, and thus $\ell(\mathcal{O}_{\overline{\varphi(D)}\cap H,p}) = 4$. If $p \in D$, then in the affine chart $\{X_3 = 1\}$ the line L is sent to a vertical line in $(\mathbb{K}^*)^2$.

If $\varphi(p) = q \in \overline{\varphi(D)}$ is an inflection point of $\overline{\varphi(D)}$ with osculating hyperplane $H = V(a_0X_0+\cdots+a_3X_3)$ satisfying $a_2 \neq 0$ and $p \in D$, then in the affine chart $\{X_3 = 1\}$, the map φ^{-1} sends H to $H' = V(a_3 + a_1X + a_3X^2 + a_2Y)$. The result follows ■

Definition: Let Θ be an admissible polygon and let $f \in \mathbb{K}[X^{\pm1}, Y^{\pm1}]$ such that $\mathrm{New}(f) = \Theta$ and $C = V(f) \subset (\mathbb{K}^*)^2$ is non-singular. We say that a point $p \in C$ is:

1. an **inflection point of type I** if there exists a polynomial $g \in \mathbb{K}[X^{\pm1}, Y^{\pm1}]$ of the form $g = zX^2 + wX + v - Y$ such that $\ell(\mathcal{O}_{C\cap V(g),p}) \geq 4$;
2. an **inflection point of type II** if $\frac{\partial f}{\partial Y}(p) = 0$.

We say that $p \in C$ is an inflection point if it is either an inflection point of type I or II.

4.3.– Computational tools

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. If Θ is an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$, then we know that the curve $C = V(f)$ has at most $\mu(\Theta) = 4d + 12(g-1)$ inflection points in $(\mathbb{K}^*)^2$ for a generic polynomial $f \in \mathbb{K}[X^{\pm1}, Y^{\pm1}]$ with Newton polygon Θ.

We will develop a method to compute the number of inflection points in $(\mathbb{K}^*)^2$ of such a curve $C = V(f)$.

Proposition 4.3.1. *Let Θ be an admissible polygon of degree d and genus g, with $(d,g) \neq (2,0)$ and let $C = V(f)$ be a non-singular curve in $(\mathbb{K}^*)^2$ with Newton polygon Θ. Then the inflection points of type I of C satisfy the system $\mathcal{E}_1 = \{f, \Phi(f)\}$, where $\Phi(f)$ is given by:*

$$\Phi(f) := \begin{array}{r} (f_{XXX}f_Y - 3f_{XX}f_{XY} - 3f_Xf_{XXY})f_Y^3 + 3(f_{XX}f_{YY} + 2f_{XY}^2 + f_Xf_{XYY})f_Xf_Y^2 - \\ -(9f_{XY}f_{YY} + f_{YYY}f_X)f_X^2f_Y + 3f_{YY}^2f_X^3 \end{array} \tag{4.3}$$

Conversely, if $(x,y) \in (\mathbb{K}^*)^2$ is a solution for the system $\mathcal{E}_1 = \{f, \Phi(f)\}$, then $f_Y(x,y) \neq 0$ and

$$z = \left(\frac{-f_X^2f_{YY} + 2f_Xf_Yf_{XY} - f_Y^2f_{XX}}{2f_Y^3}\right)(x,y), \quad w = -\left(\frac{f_X}{f_Y} + 2zX\right)(x,y), \quad v = y - zx^2 - wx$$

defines a polynomial $g(X,Y) = Y - zX^2 + wX + v$ such that $\ell(\mathcal{O}_{C\cap V(g),(x,y)}) \geq 4$.

Proof. Let $f(X,Y) \in \mathbb{K}[X^{\pm1}, Y^{\pm1}]$ be a polynomial with Newton polygon Θ expressed as $f(X,Y) = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij}X^iY^j$, $u_{ij} \in \mathbb{K}$. Let $C = V(f)$ and let D be the curve defined by a polynomial of the form $Y - Q(X)$, where $Q(X) = zX^2 + wX + v$ and $zv \neq 0$.

Since D is a rational curve, for any point $(x,y) \in D$ we can take a rational parameterization $\varphi : \mathbb{A}^1_{\mathbb{K}} \longrightarrow D$ such that $\varphi(0) = (x,y)$, namely $T \mapsto (T + x, T^2 + Q'(x)T + y)$.

Let $(x,y) \in C \cap D$, $x, y \neq 0$, and consider the above rational parametrization of D. Then we get an univariate polynomial $f(X(T), Y(T)) = \sum_{k=0}^N a_k(u_{ij}, x, y, z, w)T^k$ given by

$$f(X(T), Y(T)) = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} \sum_{k=0}^{i} \binom{i}{k} T^k x^{i-k} \sum_{\ell=0}^{j} \binom{j}{\ell} T^\ell y^{j-\ell} \sum_{m=0} \binom{\ell}{m} T^m z^m (2zx + w)^{\ell-m}.$$

If the point $(x,y) \in C \cap D$ satisfies $\ell(\mathcal{O}_{C\cap D,(x,y)}) \geq 4$, then we have that $a_k(u_{ij},x,y,z,w) = 0$ for $k = 0,\dots,3$, i.e., the point (x,y) is a solution of the following system of equations:

$$\begin{aligned}
0 &= a_0 = f \\
0 &= a_1 = f_X + Q'(x) f_Y \\
0 &= a_2 = z f_Y + \frac{Q'(x)^2}{2} f_{YY} + Q'(x) f_{XY} + \frac{1}{2} f_{XX} \\
0 &= a_3 = z f_{XY} + z Q'(x) f_{YY} + Q'(x)^3 + \frac{Q'(x)^2}{2} f_{XYY} + \frac{Q'(x)}{2} f_{XXY} + \frac{1}{6} f_{XXX}
\end{aligned}$$

If $a_1(x,y) = 0$ and $f_Y(x,y) = 0$, then $f_X(x,y) = 0$, which is a contradiction since (x,y) is in the smooth curve C. Hence $f_Y(x,y) \neq 0$ and the equation $a_1(x,y) = 0$ gives us the condition $Q'(x) = -\frac{f_X(x,y)}{f_Y(x,y)}$, i.e. $w = -\frac{f_X(x,y)}{f_Y(x,y)} - 2zx$, and together with the equation $a_2 = 0$ gives us

$$z = \frac{-1}{f_Y(x,y)} \left(\frac{Q'(x)^2}{2} f_{YY}(x,y) + Q'(x) f_{XY}(x,y) + \frac{1}{2} f_{XX}(x,y) \right) = \left(\frac{-f_X^2 f_{YY} + 2 f_X f_Y f_{XY} - f_Y^2 f_{XX}}{2 f_Y^3} \right)(x,y).$$

We get the equation $\Phi(f)$ by plugging these two formulas into the equation $a_3 = 0$. ■

Let Θ be an admissible polygon. Following [LR04], we denote by $U = \mathbb{C}^{\#(\Theta\cap\mathbb{Z}^2)}$ the space of parameters of polynomials with Newton polygon contained in Θ and equip it with coordinates $\{u_{ij} \mid (i,j) \in \Theta \cap \mathbb{Z}^2\}$, and by $\mathbb{C}^2$ the space of unknowns with coordinates $\{(x,y)\}$. The total space in which we will be working is $\mathbb{C}^{\#(\Theta\cap\mathbb{Z}^2)} \times \mathbb{C}^2$.

For each $(u_{ij})_{(i,j)} \in U$, we have a polynomial $f(X,Y) = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} X^i Y^j$ with $\mathrm{New}(f) \subseteq \Theta$. The **discriminant hypersurface** $\mathrm{Disc}(\Theta)$ of Θ is the set of $(u_{ij})_{(i,j)} \in U$ such that the curve $V(f(X,Y)) \subset (\mathbb{C}^*)^2$ is singular. This is an algebraic hypersurface in U, and we denote by $\mathcal{D}(\Theta)$ the polynomial that defines it.

Definition: Let Θ be an admissible polygon with set of vertices $\mathrm{Vert}(\Theta)$, $f(X,Y) = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} X^i Y^j$, $\Phi(f)$ the Equation (4.3) and $\mathcal{D}(\Theta)$ the polynomial defining the discriminant hypersurface.

We define $\mathcal{C}_1(\Theta) \subset \mathbb{C}^{\#(\Theta\cap\mathbb{Z}^2)} \times \mathbb{C}^2$ to be the locally closed set defined by

$$\mathcal{E}_1(\Theta) = \{f = 0, \Phi(f) = 0, \mathcal{D}(\Theta) \neq 0, u_{ij} \neq 0 \mid (i,j) \in \mathrm{Vert}(\Theta)\}.$$

Let $\pi : \mathcal{C}_1(\Theta) \longrightarrow \mathbb{C}^{\#(\Theta\cap\mathbb{Z}^2)}$ be the projection $\pi((u_{ij})_{i,j},(x,y)) = (u_{ij})_{i,j}$, let $\Pi \subset \mathbb{C}^{\#(\Theta'\cap\mathbb{Z}^2)}$ be the Zariski closure of $\pi(\mathcal{C}_1(\Theta'))$ and let n be the dimension of Π. We have the following important result.

Theorem 4.3.2 (Lazard-Rouillier). *If the system $\mathcal{E}_1(\Theta)$ is defined over $\mathbb{Q}$, then there exists an algebraic variety $W_D(\mathbb{C}) \subseteq \Pi$ defined over $\mathbb{Q}$ such that*

1. *each connected component $\mathcal{U} \subseteq (\Pi \setminus W_D(\mathbb{C}))$ is an analytic sub-manifold of dimension n and $\pi|_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is an analytic covering;*
2. *$W_D(\mathbb{C})$ is included in any other variety that satisfies the precedent property.*

The variety W_D is called the **minimal discriminant variety** with respect to π. Under these conditions, every fiber of $\pi|_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is finite and share the same cardinal.

We conclude that for an admissible polygon Θ, the generic number of complex solutions (x,y) of the system $\mathcal{E}_1(\Theta)$ has a unique value $\mu_I^{\mathbb{C}}(\Theta)$, which is the number of inflection points of type I of a smooth curve $C = V(f)$ for a generic polynomial f with $\mathrm{New}(f) = \Theta$.

A similar discussion applies for the locally closed $\mathcal{C}_2(\Theta) \subset \mathbb{C}^{\#(\Theta\cap\mathbb{Z}^2)} \times \mathbb{C}^2$ defined by

$$\mathcal{E}_2(\Theta) = \{f = 0, f_Y = 0, \mathcal{D}(\Theta) \neq 0, u_{ij} \neq 0 \mid (i,j) \in \mathrm{Vert}(\Theta)\}.$$

We conclude that for an admissible polygon Θ, the number of complex solutions (x,y) of the system $\mathcal{E}_2(\Theta)$ has a unique value $\mu_{II}^{\mathbb{C}}(\Theta)$, which is the number of inflection points of type II of a smooth curve $C = V(f)$ for a generic polynomial f with $\mathrm{New}(f) = \Theta$.

Definition: Let Θ be an admissible polygon. We define its complex inflection multiplicity $\mu^{\mathbb{C}}(\Theta)$ as the sum $\mu_I^{\mathbb{C}}(\Theta) + \mu_{II}^{\mathbb{C}}(\Theta)$.

Note that $\mu^{\mathbb{C}}(\Theta) \leq \mu(\Theta)$, and we have equality if the closure $\overline{C} \subset \mathbb{CP}(2,1,1)$ of the generic smooth curve $C \subset (\mathbb{C}^*)^2$ with Newton polygon Θ has no inflection points in $\mathbb{CP}(2,1,1) \setminus (\mathbb{C}^*)^2$.

Remark 4.3.3: *The toric involution $(\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^2$ given by $(x,y) \mapsto (x^{-1}, yx^{-2})$ transforms the curve $D = V(zX^2 + wX + v - Y)$ into the curve $D' = V(vX^2 + wX + z - Y)$. So if $D = V(zX^2 + wX + v - Y)$ satisfies $\ell(\mathcal{O}_{C\cap D,(x,y)}) \geq 4$ for a curve $C = V(f)$, then $\ell(\mathcal{O}_{C'\cap D',(x^{-1},yx^{-2})}) \geq 4$, where C' is the curve associated to the polynomial $f' = f(X^{-1}, YX^{-2})$, and we conclude that $\mu^{\mathbb{C}}(\Theta) = \mu^{\mathbb{C}}(\Theta')$, where Θ' is the image of Θ under the linear transformation $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ induced by the matrix $\left(\begin{smallmatrix} -1 & -2 \\ 0 & 1 \end{smallmatrix}\right)$.*

In the real case the restriction to $\mathcal{C}_1(\Theta)(\mathbb{R}) = \mathcal{C}_1(\Theta) \cap \mathbb{R}^{\#(\Theta\cap\mathbb{Z}^2)+2}$ of the covering induced by the discriminant variety is a real analytic covering. Thus the the conditions $\{u_{ij} \neq 0 \mid (i,j) \in \mathrm{Vert}(\Theta)\} \cup \{\mathcal{D}(\Theta) \neq 0\}$ have constant sign and do not vanish on the connected components of this covering.

This is, for any connected component $\mathcal{U} \subseteq \Pi \setminus W_D(\mathbb{C})$ and any family of inequalities $\mathcal{F}(\sigma) = \{\sigma_{ij}u_{ij} > 0, \sigma_{\mathcal{D}}\mathcal{D}(\Theta) > 0\}$ induced by a **distribution of signs** $\Sigma = ((\sigma_{ij} \mid (i,j) \in \mathrm{Vert}(\Theta)), \sigma_{\mathcal{D}}) \in \{\pm 1\}^{\mathrm{Vert}(\Theta)+1}$, every real fiber is finite and share the same cardinal. A similar discussion applies for the case of the set $\mathcal{C}_2(\Theta)(\mathbb{R}) = \mathcal{C}_2(\Theta) \cap \mathbb{R}^{\#(\Theta\cap\mathbb{Z}^2)+2}$.

Given Θ an admissible polygon, we want to find the distributions of signs Σ and conditions on the coefficients $(u_{ij})_{(i,j)} \in (\mathbb{R}_{>0})^{\#(\Theta\cap\mathbb{Z}^2)}$ such that the non-singular real curve obtained from the polynomial $\sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} \sigma_{i,j}u_{ij}X^iY^j$ has the maximal number of real inflection points.

We will now show how we can apply the preceding ideas to prove that if $C = V(u_{10}X + u_{01}Y + u_{02}Y^2 + u_{11}XY) \subset (\mathbb{C}^*)^2$ is a smooth real curve, then C has four inflection points and two of them are real.

4.3.4 Example: Consider polygon $\Theta = \mathrm{Conv}\{(1,0),(0,1),(1,1),(0,2)\} \subset 2\Delta$. A polynomial $f(X,Y) = u_{02}Y^2 + u_{11}XY + u_{01}X + u_{10}Y$ defines a smooth algebraic curve $C = V(f) \subset (\mathbb{K}^*)^2$ if and only if $\mathcal{D}(\Theta) = u_{01}u_{11} - u_{02}u_{10} \neq 0$.

Let $F(X_0,X_1,X_2,X_3) = u_{02}X_2^2 + u_{11}X_1X_2 + X_3(u_{01}X_2 + u_{10}X_1)$ with $\mathcal{D}(\Theta) \neq 0$, then $V(F, X_0X_3 - X_1^2) \subset \mathbb{KP}^3$ defines a curve of arithmetic genus 1 with a singular point (a node) at $[1:0:0:0]$. So Θ is an admissible polygon of degree $d = 4$ and genus $g = 0$, hence a generic curve C will have $\mu(\Theta) = 4(4) + 12(0-1) = 4$ inflection points, by Proposition 4.2.5.

According to Theorem 4.3.1, to find the inflection points of type I of C (respectively the inflection points of type II), we must find the solutions $(x,y) \in (\mathbb{K}^*)^2$ of the system $\mathcal{E}_1 = \{f, \Phi(f)\}$ (respectively the system $\mathcal{E}_2 = \{f, f_Y\}$). Let us define $\mathcal{R} = u_{02}u_{10}\mathcal{D}(\Theta)$, then each system $\mathcal{E}_i$ has two solutions $(x,y) \in (\mathbb{K}^*)^2$, namely:

$$\left(\frac{2u_{02}u_{10} - u_{01}u_{11}}{u_{11}^2}, \frac{-u_{02}u_{10} \pm \sqrt{\mathcal{R}}}{u_{02}u_{11}}\right), \quad \text{and} \quad \left(\frac{2u_{02}u_{10} - u_{01}u_{11} \pm \sqrt{-\mathcal{R}}}{u_{11}^2}, \frac{-u_{02}u_{10} \mp \sqrt{-\mathcal{R}}}{u_{02}u_{11}}\right).$$

Suppose that $\mathbb{K} = \mathbb{C}$, then we can consider the case in which $f(X,Y) \in \mathbb{R}[X^{\pm 1}, Y^{\pm 1}]$. Since $\mathcal{R} \in \mathbb{R}$, we can analyze the reality of the 4 inflection points of $C = V(f)$ by choosing a distribution of signs for the coefficients u_{ij} and for $\mathcal{D}(\Theta)$. See Table 4.1.

We observe that no matter the choice of the coefficients $(u_{10}, u_{01}u_{11}, u_{02}) \in ((\mathbb{R}^*)^4 \setminus V(\mathcal{D}(\Theta)))$, half of the inflection points of the curve $V(u_{10}X + u_{01}Y + u_{02}Y^2 + u_{11}XY)$ will be real.

*

4.4.– Some real curves of genus 1 in $\mathbb{CP}(2,1,1)$

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $\Theta \subset 2\Delta$ be any 2-dimensional convex lattice polygon containing one inner lattice point. Since the complete smooth intersection of two quadrics in $\mathbb{KP}^3$ is an elliptic curve, we see that the polygon Θ is admissible of degree 4 and genus 1. If $C \subset (\mathbb{K}^*)^2$ is a generic curve with Newton polygon Θ, we know that the restriction of $\mathcal{O}_{\mathbb{KP}^3}(1)$ to $\overline{\varphi(C)}$ gives us a divisor class $[E]$ with $h^0(E) = 4$ so that the closed embedding $\overline{\varphi(C)}$ is the 0-th Gauss map of a complete g_4^3 on $\overline{\varphi(C)}$. It follows that $\overline{C} \subset \mathbb{KP}(2,1,1)$ has 16 inflection points.

Consider the case $\mathbb{K} = \mathbb{C}$. Let $C \subset (\mathbb{C}^*)^2$ be a smooth, real curve with Newton polygon Θ and $C(\mathbb{R}) \neq \emptyset$. Then by Theorem 3.2.5 we know that the amount of real inflection points of the real curve $\overline{C} \subset \mathbb{CP}(2,1,1)$ depends on the number of connected components of $\overline{C}(\mathbb{R})$ and of the parity of the divisor class $\mathrm{par}([E])$ which is just the parity of a divisor $[\overline{C} \cap \overline{H}]$, where $\overline{H} \subset \mathbb{CP}(2,1,1)$ is the closure of a generic real curve $H \subset (\mathbb{C}^*)^2$ with Newton polygon Δ. Let $w_{\mathbb{R}}(\overline{C})$ be the number of real inflection points

$\sigma = (u_{10}, u_{01}, u_{11}, u_{02}, \mathcal{D}(\Theta))$	(Type I real, Type II real)
$(-,+,+,+,+)$	$(0,2)$
$(+,-,+,+,-)$	$(0,2)$
$(+,+,-,+,-)$	$(0,2)$
$(+,+,+,-,+)$	$(0,2)$
$(+,+,+,+,+)$	$(2,0)$
$(+,+,+,+,-)$	$(0,2)$
$(+,-,-,+,+)$	$(2,0)$
$(+,-,-,+,-)$	$(0,2)$
$(+,+,-,-,-)$	$(2,0)$
$(+,+,-,-,+)$	$(0,2)$
$(+,-,+,-,-)$	$(2,0)$
$(+,-,+,-,+)$	$(0,2)$

Table 4.1: Number of real inflection points (of type I and II) for a distribution of signs of u_{ij} and $\mathcal{D}(\Theta)$ of a smooth real curve $C = V(u_{10}X + u_{01}Y + u_{02}Y^2 + u_{11}XY)$, with $\mathcal{D}(\Theta) = u_{01}u_{11} - u_{02}u_{10}$.

of the curve $\overline{C}$, then we have:

$$w_{\mathbb{R}}(\overline{C}) = \begin{cases} 0, & \text{if } C(\mathbb{R}) \text{ is not connected and } \mathrm{par}([E]) = (1,1), \\ 4, & \text{if } C(\mathbb{R}) \text{ is connected,} \\ 8, & \text{if } C(\mathbb{R}) \text{ is not connected and } \mathrm{par}([E]) = (0,0). \end{cases}$$

In this section we want to use Proposition 4.3.1 to study the real inflection points of smooth real elliptic curves in $(\mathbb{C}^*)^2$.

Let $f \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ be a polynomial with Newton polygon 2Δ, then we can write $f(X,Y) = u_{02}Y^2 + P_1(X)Y + P_2(X)$, for $P_1(X) = u_{01} + u_{11}X + u_{21}X^2$ and $P_2(X) = u_{00} + u_{10}X + u_{20}X^2 + u_{30}X^3 + u_{40}X^4$, $u_{ij} \in \mathbb{C}$, $u_{00}, u_{40}, u_{02} \neq 0$.

Lemma 4.4.1. *Let $f(X,Y) = u_{02}Y^2 + P_1(X)Y + P_2(X)$ be as above. A generic non-singular curve $C = V(f)$ has four points of type II*

$$\left\{ \left(x_0, \tfrac{-P_1(x_0)}{2u_{02}} \right) \,:\, 4u_{02}P_2(x_0) = P_1(x_0)^2 \right\}.$$

Proof. The inflection points of type II of the curve $C = V(f)$ can be found by solving the system $\mathcal{E}_2 = \{f, f_Y\}$. We have that $f_Y(X,Y) = 2u_{02}Y + P_1(X)$, so the equation $f_Y = 0$ gives us $Y = \frac{-P_1(X)}{2u_{02}}$. We now plug this expression in $f(X,Y)$ to find

$$f(X,Y) = P_2(X) - \frac{P_1(X)^2}{4u_{02}}.$$

For any root x_0 of $P_2(X) - \frac{P_1^2(X)}{4u_{02}}$, we get the quadratic equation $f(x_0,Y) = u_{02}Y^2 + P_1(x_0)Y + \frac{P_1(x_0)^2}{4u_{02}}$ and an unique solution $y = \frac{-P_1(x_0)}{2u_{02}}$. ■

4.4.1 Generic curves of degree 4 with polygon Conv$\{(0,1),(0,2),(2,0),(2,1)\}$

Consider the admissible polygon $\Theta = \mathrm{Conv}\{(0,1),(0,2),(2,0),(2,1)\}$ of degree 4 and genus 1 and let $f(X,Y)$ be a generic polynomial of the form $u_{02}Y^2 + P_1(X)Y + P_2(X)$ with $P_1(X) = u_{01} + u_{11}X + u_{21}X^2$, $u_{11} \neq 0$ and $P_2(X) = u_{20}X^2$. Let us define $\mathcal{D}_+(\Theta) = (u_{11} + 2\sqrt{u_{20}u_{02}})^2 - 4u_{21}u_{01}$ and $\mathcal{D}_-(\Theta) = (u_{11} - 2\sqrt{u_{20}u_{02}})^2 - 4u_{21}u_{01}$. Then the polynomial defining the discriminant hypersurface $\mathrm{Disc}(\Theta)$ of Θ is $\mathcal{D}(\Theta) = \mathcal{D}_+(\Theta)\mathcal{D}_-(\Theta)$ if $u_{11} \neq 0$.

To find the remaining twelve inflection points of type I of generic real elliptic curves with Newton polygon $\mathrm{Conv}\{(0,1),(0,2),(2,0),(2,1)\}$, we first compute a Groebner basis $\{P(X), f(X,Y)\}$ for the system $\mathcal{E}_1 = \{f, \Phi(f)\}$ with respect to the monomial order $Y \leq X$, and then we compute the roots of the polynomial $P(X)$. Let us define

$$m = \frac{u_{01}}{u_{21}}, \quad n = \frac{u_{11}^2 - 4(u_{02}u_{20} - u_{21}u_{01})}{u_{11}u_{21}}.$$

Then we have that $P(X) = (X^2 - m)R(X)$, where $R(X) := X^4 + nX^3 + 6mX^2 + mnX + m^2$. We introduce an auxiliary polynomial $Q(X) = X^2 + \frac{n}{2}X + m$ and we will denote as $\mathcal{D}(Q)$ its discriminant $\frac{n^2-16m}{4}$. We have that:

$$Q(X)^2 = X^4 + nX^3 + 2mX^2 + \frac{n^2}{4}X^2 + mnX + m^2 = R(X) + \left(\frac{n^2 - 16m}{4}\right)X^2 = R(X) + (\sqrt{\mathcal{D}(Q)}X)^2.$$

This gives us the following factorization:

$$P(X) = (X - \sqrt{m})(X + \sqrt{m})(Q(X) + \sqrt{\mathcal{D}(Q)}X)(Q(X) - \sqrt{\mathcal{D}(Q)}X). \tag{4.4}$$

If x_0 is a root for $P(X)$, we get the corresponding values for y_0 from the solutions of the quadratic equation $f(x_0, Y) = u_{02}Y^2 + P_1(x_0)Y + u_{20}x_0^2$, and they will be real if and only if

$$\left(\frac{P_1(x_0)}{2x_0}\right)^2 > u_{20}u_{02}. \tag{4.5}$$

Let us define $\mathcal{G}_+(\Theta) = (u_{11} + 2\sqrt{u_{21}u_{01}})^2 - 4u_{02}u_{20}$ and $\mathcal{G}_-(\Theta) = (u_{11} - 2\sqrt{u_{21}u_{01}})^2 - 4u_{02}u_{20}$. We have the following result.

Corolary 4.4.2. *Let $C \subset (\mathbb{R}^*)^2$ be the real curve defined by the polynomial $f(X, Y) = u_{02}Y^2 + P_1(X)Y + u_{20}X^2$ with $\mathcal{D}(\Theta) \neq 0$. Let $\mathcal{F}(\sigma) = (\sigma_1\mathcal{G}_+(\Theta) > 0, \sigma_2\mathcal{G}_-(\Theta) > 0)$ for $\sigma \in \{\pm 1\}^2$ be a distribution of signs.*

1. *If $u_{21}u_{01} > 0$, then $\overline{C} \subset \mathbb{RP}(2, 1, 1)$ is a non-singular real elliptic curve having $2k$ real inflection points of type I corresponding to the factor $(X^2 - m)$ of $P(X)$, where k is the number of $+1$ in σ.*
2. *If $u_{21}u_{01} < 0$, then $\overline{C} \subset \mathbb{RP}(2, 1, 1)$ is a non-singular real elliptic curve with 0 real inflection points of type I corresponding to the factor $(X^2 - m)$ of $P(X)$.*

Proof. From the Equation 4.4, the polynomial $P(X)$ has two real roots $\pm\sqrt{m}$ if and only if $m = \frac{u_{01}}{u_{21}} > 0$. We have $\left(\frac{P_1(\pm\sqrt{m})}{2\sqrt{m}}\right)^2 = \left(\frac{u_{11}}{2} \pm \sqrt{u_{21}u_{01}}\right)^2$, and the assertion follows from the Equation (4.5). ■

We now analyze the real roots of the factor $R(X)$ of the polynomial $P(X)$.

Proposition 4.4.3. *A non-singular real curve defined by a polynomial $f(X, Y) = u_{02}Y^2 + P_1(X)Y + u_{20}X^2$ satisfying $u_{11} \neq 0$, $u_{21}u_{01} < 0$ and $u_{20}u_{02} < 0$ will have eight real inflection points of type I.*

Proof. The roots of the polynomial $R(X)$ are

$$\frac{-\left(\frac{n}{2} + \sqrt{\mathcal{D}(Q)}\right) \pm \sqrt{D_+}}{2}, \text{ and } \quad \frac{-\left(\frac{n}{2} - \sqrt{\mathcal{D}(Q)}\right) \pm \sqrt{D_-}}{2},$$

for $D_\pm(Q) = \left(\frac{n}{2} \pm \sqrt{\mathcal{D}(Q)}\right)^2 - 4m$.

The polynomials $Q(X) \pm \sqrt{\mathcal{D}(Q)}X$ are both real if and only if $\mathcal{D}(Q) > 0$, i.e., if and only if $16m < n^2$, and each one will have two distinct real roots if and only if $D_\pm > 0$. These two conditions are true if $m < 0$, i.e., $u_{01}u_{21} < 0$ and $u_{20}u_{02} < 0$. ■

4.5.– Construction of a real curve of genus four with 30 real inflection points

The purpose of this section is to construct a smooth real curve $C' \subset \mathbb{CP}^3$ of genus four and degree six with 30 real inflection points. Observe that since the polygon $\Theta = \mathrm{Conv}\{(0, 1), (0, 3), (2, 0), (4, 0), (4, 1)\}$ is admissible of degree six and genus four, then it will be enough to construct a smooth real curve $C \subset (\mathbb{C}^*)^2$ with Newton polygon Θ having 30 real inflection points of type I.

Since the condition for $p \in C$ to be a real inflection point is local and stable under small deformations of the curve, we can use Viro's Patchworking Theorem to construct a real patchworking polynomial $F_t = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} \sigma_{ij}u_{ij}t^{-\nu(i,j)}X^iY^j \in \mathbb{R}[X^{\pm 1}, Y^{\pm 1}]$, where for any $(i, j) \in \Theta \cap \mathbb{Z}^2$

1. $u_{ij} \in \mathbb{R}_{>0}$;

2. $\sigma_{ij} \in \{\pm 1\}$;

3. $\nu(i,j) \in \mathbb{Z}_{>0}$.

Let Θ' be the parallelogram $\mathrm{Conv}\{(0,1),(0,2),(2,0),(2,1)\}$. As we have seen in Proposition 4.4.3, a generic polynomial $f(X,Y) = u_{02}Y^2 + (u_{01} + u_{11}X + u_{21}X^2)Y + u_{20}X^2$ will define a non-singular real curve with Newton polygon Θ' with eight real inflection points of type I as long as the following two conditions are met:

1. the products $u_{21}u_{01}$ and $u_{20}u_{02}$ of the coefficients of the two diagonals of Θ' are negative;

2. the coefficient u_{11} is non-zero

We want to choose a function $\nu : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{\geq 0}$ which induces a regular subdivision $\mathcal{S}$ on the polygon Θ having three copies of Θ', as depicted in Figure 4.2 a).

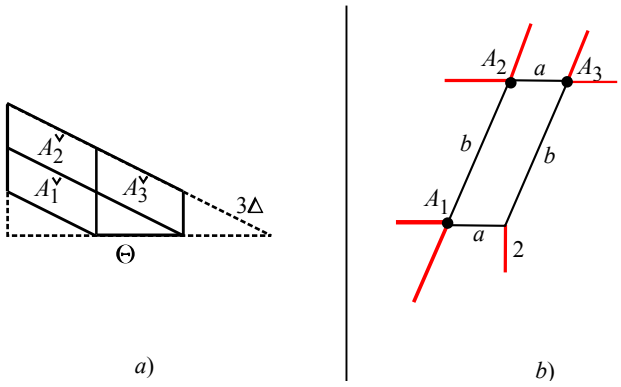


Figure 4.2: a) The polygon $\Theta = \mathrm{Conv} = \{(0,1),(0,3),(2,0),(4,0),(4,1)\}$ together with the regular subdivision $\mathcal{S}$ that we will use. b) A tropical curve having Newton polygon Θ and combinatorial type $\mathcal{S}$.

For each polygon $A_k^\vee \in \mathcal{S}$, $k = 1,2,3$, the real inflection points of type I the restriction $F_t^{A_k^\vee} = \sum_{(i,j)\in A_k^\vee \cap \mathbb{Z}^2} \sigma_{ij} u_{ij} t^{-\nu(i,j)} X^i Y^j$ of the patchworking polynomial F_t will depend only on the collection of signs $\{\sigma_{ij} \,|\, (i,j) \in A_k^\vee \cap \mathbb{Z}^2\}$. There exists two distribution of signs $\{\sigma_{ij} \in \{\pm 1\} : (i,j) \in \Theta \cap \mathbb{Z}^2, i \neq 1,3\}$ with the property that the product of the signs of the diagonals of each polygon $A_k^\vee$, $k = 1,2,3$, are negative. See Figure 4.3.

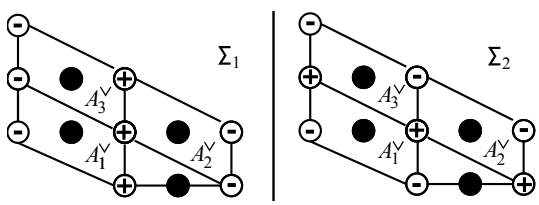


Figure 4.3: Possible distributions of signs such that the product of the signs of the diagonals of each polygon $A_k^\vee$, $k = 1,2,3$, are negative.

The values of a function $\nu_{a,b} : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{\geq 0}$ such that the tropical curve defined by the tropical polynomial $\phi(p_1,p_2) = \max_{(i,j)\in\Theta\cap\mathbb{Z}^2}\{-\nu_{a,b}(i,j) + \langle (p_1,p_2),(i,j)\rangle\}$ are shown in Figure 4.4.

We want to find values $a, b \in \mathbb{Z}_{>0}$ and a distribution of signs Σ of $\Theta \cap \mathbb{Z}^2$ extending those depicted in Figure 4.3 such that the curve associated to the polynomial

$$F_t = \sum_{\substack{(i,j)\in\Theta\cap\mathbb{Z}^2 \\ \sigma\in\Sigma}} \sigma_{ij} t^{-\nu_{a,b}(i,j)} X^i Y^j,$$

has many real inflection points of type I for small values of $t \in \mathbb{R}_{>0}$.

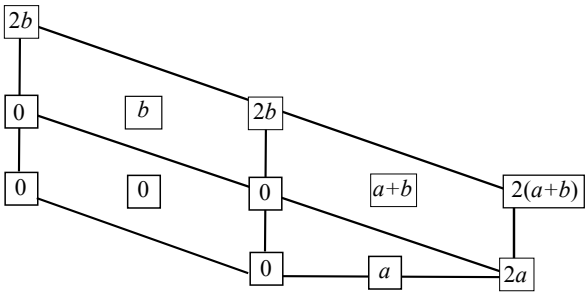


Figure 4.4: Values of the function $\nu_{a,b} : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{\geq 0}$ that induces the tropical curve of Figure 4.2b).

4.5.1 Patchworking of a real curve with 30 real inflection points

The condition for $p \in C$ to be a real inflection point is local and stable under small deformations of C, thus we can use the Viro's patchworking method to construct real curves, which we now describe briefly.

Let $\Theta \subset \mathbb{R}^2$ be a two-dimensional convex lattice polygon and let $\{\Theta_k\}_k$ be the convex polyhedral subdivision of Θ induced by the function $\nu : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{Q}_{\geq 0}$.

Consider a family of polynomials $\{f_k\}_k$ such that

1. $\mathrm{New}(f_k) = \Theta_k$ and f_k is completely nondegenerate for any k;
2. $f_k^{\Theta_{k\ell}} = f_\ell^{\Theta_{k\ell}}$ for any k, ℓ, where $\Theta_{k\ell} = \Theta_k \cap \Theta_\ell$.

There exists a unique polynomial F such that $F^{\Theta_k} = f_k$ for all k; furthermore, since $\cup_k \Theta_k = \Theta$, the polynomial F has Θ as Newton polygon and we can write $F = \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} X^i Y^j$. Viro's patchworking theorem asserts that if $\nu : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{N} \cup \{0\}$ induces $\{\Theta_k\}_k$ and we set $F_t := \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} t^{\nu(i,j)} X^i Y^j$, then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we have that the polynomial F_ε is completely nondegenerate. We say that $\{f_k\}_k$ is a **patchworking family** for $\{\Theta_k\}_k$ and that $F_t := \sum_{(i,j)\in\Theta\cap\mathbb{Z}^2} u_{ij} t^{\nu(i,j)} X^i Y^j$ the **patchworking polynomial** of the family.

Let $\{f_k\}_k$ be a patchworking family for the subdivision $\{\Theta_k\}_k$. It is enough to choose f_k for the two-dimensional elements of $\{\Theta_k\}_k$ that satisfy the compatibility conditions along the one-dimensional elements of the subdivision.

We now consider the following distributions of signs on $\Theta \cap \mathbb{Z}^2$:

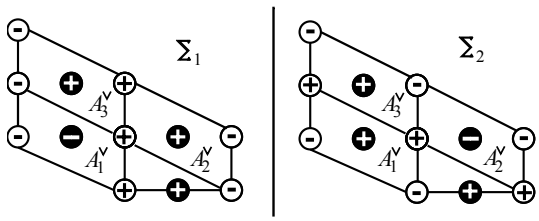


Figure 4.5: Distributions of signs Σ_1 and Σ_2.

We have the following result.

Theorem 4.5.1. *Let $\nu_{a,b} : \Theta \cap \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{\geq 0}$ be the function of Figure 4.4 and let Σ be one of the distribution of signs of Figure 4.5. Then the real curve defined by the polynomial*

$$F_t = \sum_{\substack{(i,j)\in\Theta\cap\mathbb{Z}^2 \\ \sigma\in\Sigma}} \sigma_{ij} t^{-\nu_{1,1}(i,j)} X^i Y^j \tag{4.6}$$

is a real smooth curve of degree 6 and genus 4 having 30 real inflection points for $t < \frac{1}{1000}$.

Proof. The proof reduces to a computation of the real roots of the system $\mathcal{E}_1 = \{F_t, \Phi(F_t)\}$ for $t < \frac{1}{1000}$. ■

4.5.2 Code for the curve with the 30 real Weierstrass points

The first paragraph of the following program defines the polynomial f as F_t, and the second paragraph defines the polynomial g as $\Phi(F_t)$. The third paragraph computes the real solutions to the system $\mathcal{E}_1 = \{f, \Phi(f)\}$, which are the real inflection points of type I of the curve $V(f)$, and the fourth paragraph gives the number of real solutions to the system $\mathcal{E}_1$.

In the next example, we use the distribution of signs Σ_1, the values $a = b = 1$ for the function $\nu_{a,b}$ and $t = \frac{1}{1000}$.

```
f[x,y] := -(1/1000)^{2} y^3 - y^2 - y + x^2 ((1/1000)^{2} y^2 + y + 1) -
x^4 ((1/1000)^{4} y + (1/1000)^{2}) + x ((1/1000)^{1} y^2 - y) +
x^3 ((1/1000)^{2} y + (1/1000))

g[x,y]:=(D[f[x,y],{x,3}]*D[f[x,y],{y,1}]-3D[f[x,y],{x,2}]*D[f[x,y],{x,1},{y,1}]-
-3D[f[x,y],{x,1}]*D[f[x,y],{x,2},{y,1}])*D[f[x,y],{y,1}]^3+
+3(D[f[x,y],{x,2}]*D[f[x,y],{y,2}]+2D[f[x,y],{x,1},{y,1}]^2+
+D[f[x,y],{x,1}]*D[f[x,y],{x,1},{y,2}])*D[f[x,y],{x,1}]*D[f[x,y],{y,1}]^2-
-(9D[f[x,y],{x,1},{y,1}]*D[f[x,y],{y,2}]+D[f[x,y],{x,1}]*D[f[x,y],{y,3}])*
*D[f[x,y],{x,1}]^2*D[f[x,y],{y,1}]+3D[f[x,y],{y,2}]^2*D[f[x,y],{x}]^3

Z = NSolve[{f[x,y] == 0, g[x,y] == 0}, {x,y}, Reals]

Length[Z]
```

And the answer that we get is the following:

```
{{x -> -2.05275*10^6, y -> -705921.}, {x -> -2.04607*10^6, y -> 5.94046*10^6},
{x -> -343938., y -> 168171.}, {x -> -336227., y -> -712077.},
{x -> -165273., y -> -845054.}, {x -> -2904.59, y -> -1.41499*10^6},
{x -> -2904.59, y -> 5.94705*10^6}, {x -> -998.004, y -> 0.994015},
{x -> -487.783, y -> -1.41792*10^6}, {x -> -487.783, y -> 168066.},
{x -> -4.04581, y -> 20.1421}, {x -> -2.20426, y -> 6.76474},
{x -> -2.0478, y -> -0.703758}, {x -> -0.344045, y -> -0.705149},
{x -> -0.343947, y -> 0.167636}, {x -> 0.487151, y -> 0.167527},
{x -> 0.488742, y -> -1.41899}, {x -> 2.9075, y -> -1.42164},
{x -> 2.97418, y -> 6.29885}, {x -> 6.05059, y -> 30.9371},
{x -> 344.282, y -> 167719.}, {x -> 344.282, y -> -704906.},
{x -> 1002., y -> -0.997995}, {x -> 2050.09, y -> 5.95933*10^6},
{x -> 2050.09, y -> -706360.}, {x -> 247169., y -> -1.23053*10^6},
{x -> 453667., y -> -1.39228*10^6}, {x -> 488329., y -> 167822.},
{x -> 2.9066*10^6, y -> 5.95731*10^6}, {x -> 2.90743*10^6, y -> -1.41705*10^6}}

30
```

In the next example, we use the distribution of signs Σ_2, the values $a = b = 1$ for the function $\nu_{a,b}$ and $t = \frac{1}{1000}$.

```
f[x,y] := -(1/1000)^{2} y^3 + y^2 - y + x^2 (-(1/1000)^{2} y^2 + y - 1) +
x^4 (-(1/1000)^{4} y + (1/1000)^{2}) + x ((1/1000)^{1} y^2 + y) +
x^3 (-(1/1000)^{2} y + (1/1000))

g[x,y]:=(D[f[x,y],{x,3}]*D[f[x,y],{y,1}]-3D[f[x,y],{x,2}]*D[f[x,y],{x,1},{y,1}]-
-3D[f[x,y],{x,1}]*D[f[x,y],{x,2},{y,1}])*D[f[x,y],{y,1}]^3+
+3(D[f[x,y],{x,2}]*D[f[x,y],{y,2}]+2D[f[x,y],{x,1},{y,1}]^2+
+D[f[x,y],{x,1}]*D[f[x,y],{x,1},{y,2}])*D[f[x,y],{x,1}]*D[f[x,y],{y,1}]^2-
-(9D[f[x,y],{x,1},{y,1}]*D[f[x,y],{y,2}]+D[f[x,y],{x,1}]*D[f[x,y],{y,3}])*
*D[f[x,y],{x,1}]^2*D[f[x,y],{y,1}]+3D[f[x,y],{y,2}]^2*D[f[x,y],{x}]^3

Z = NSolve[{f[x,y] == 0, g[x,y] == 0}, {x,y}, Reals]

Length[Z]
```

And the answer that we get is the following:

```
{{x -> -2.90743*10^6, y -> 1.41705*10^6}, {x -> -2.9066*10^6, y -> -5.95731*10^6},
{x -> -488329., y -> -167822.}, {x -> -453667., y -> 1.39228*10^6},
{x -> -247169., y -> 1.23053*10^6}, {x -> -2050.09, y -> 706360.},
{x -> -2050.09, y -> -5.95933*10^6}, {x -> -1002., y -> 0.997995},
{x -> -344.282, y -> 704906.}, {x -> -344.282, y -> -167719.},
{x -> -6.05059, y -> -30.9371}, {x -> -2.97418, y -> -6.29885},
{x -> -2.9075,  y -> 1.42164}, {x -> -0.488742, y -> 1.41899},
{x -> -0.487151, y -> -0.167527}, {x -> 0.343947, y -> -0.167636},
{x -> 0.344045, y -> 0.705149}, {x -> 2.0478, y -> 0.703758},
{x -> 2.20426, y -> -6.76474}, {x -> 4.04581, y -> -20.1421},
{x -> 487.783, y -> -168066.}, {x -> 487.783, y -> 1.41792*10^6},
{x -> 998.004, y -> -0.994015}, {x -> 2904.59, y -> -5.94705*10^6},
{x -> 2904.59, y -> 1.41499*10^6}, {x -> 165273., y -> 845054.},
{x -> 336227., y -> 712077.}, {x -> 343938., y -> -168171.},
{x -> 2.04607*10^6, y -> -5.94046*10^6}, {x -> 2.05275*10^6, y -> 705921.}}
```

```
30
```

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