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Yueyuan Gao

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Laboratoire de Mathématiques d'Orsay (UMR 8628)

Mathématiques Appliquées

par

Yueyuan Gao

**Méthodes de volumes finis pour des équations aux
dérivées partielles déterministes et stochastiques**

Thèse présentée et soutenue à Orsay, le 10/12/2015

Composition du jury :

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Méthodes de volumes finis pour des équations aux dérivées partielles déterministes et stochastiques

Résumé:

Le but de cette thèse est de faire l'étude de méthodes de volumes finis pour des équations aux dérivées partielles déterministes et stochastiques; nous effectuons des simulations numériques et démontrons des résultats de convergence d'algorithmes.

Au Chapitre 1, nous appliquons un schéma semi-implicite en temps combiné avec la méthode de volumes finis généralisés SUSHI pour la simulation d'écoulements à densité variable en milieu poreux; il vient à résoudre une équation de convection-diffusion parabolique pour la concentration couplée à une équation elliptique en pression. Nous présentons ensuite une méthode de simulation numérique pour un problème d'écoulements à densité variable couplé à un transfert de chaleur.

Au Chapitre 2, nous effectuons une étude numérique de l'équation de Burgers non visqueuse en dimension un d'espace, avec des conditions aux limites périodiques, un terme source stochastique de moyenne spatiale nulle et une condition initiale déterministe. Nous utilisons un schéma de volumes finis combinant une intégration en temps de type Euler-Maruyama avec le flux numérique de Godunov. Nous effectuons des simulations par la méthode de Monte-Carlo et analysons les résultats pour différentes régularités du terme source. Il apparaît que la moyenne empirique des réalisations converge vers la moyenne en espace de la condition initiale déterministe quand $t \rightarrow \infty$. Par ailleurs, la variance empirique converge elle aussi en temps long, vers une valeur qui dépend de la régularité et de l'amplitude du terme stochastique.

Au Chapitre 3, nous démontrons la convergence d'une méthode de volumes finis pour une loi de conservation du premier ordre avec une fonction de flux monotone et un terme source multiplicatif faisant intervenir un processus Q -Wiener. Le terme de convection est discrétisé à l'aide d'un schéma amont. Nous présentons des estimations a priori pour la solution discrète dont en particulier une estimation de type BV faible. A l'aide d'une interpolation en temps, nous démontrons deux inégalités entropiques vérifiées par la solution discrète, ce qui nous permet de prouver que la solution discrète converge selon une sous-suite vers une solution stochastique faible entropique à valeurs mesurées de la loi de conservation.

Au Chapitre 4, nous obtenons des résultats similaires à ceux du Chapitre 3 dans le cas où la fonction flux n'est pas monotone; le terme de convection est discrétisé à l'aide d'un schéma monotone.

Mots-clés: Écoulements à densité variable - Méthode de volumes finis généralisée SUSHI - Simulations numériques - Loi de conservation stochastique - Convergence au sens des mesures de Young.

AMS subject classifications: 35R60 - 37M05 - 65M08 - 65M12 - 76S05

Finite volume methods for deterministic and stochastic partial differential equations

Abstract :

This thesis bears on numerical methods for deterministic and stochastic partial differential equations; we perform numerical simulations by means of finite volume methods and prove convergence results.

In Chapter 1, we apply a semi-implicit time scheme together with the generalized finite volume method SUSHI for the numerical simulation of density driven flows in porous media; it amounts to solve a nonlinear convection-diffusion parabolic equation for the concentration coupled with an elliptic equation for the pressure. We then propose a numerical scheme to simulate density driven flows in porous media coupled to heat transfer. We use adaptive meshes, based upon square or cubic volume elements.

In Chapter 2, We perform Monte-Carlo simulations in the one-dimensional torus for the first order Burgers equation forced by a stochastic source term with zero spatial integral. We suppose that this source term is a white noise in time, and consider various regularities in space. We apply a finite volume scheme combining the Godunov numerical flux with the Euler-Maruyama integrator in time. It turns out that the empirical mean converges to the space-average of the deterministic initial condition as $t \rightarrow \infty$. The empirical variance also stabilizes for large time, towards a limit which depends on the space regularity and on the intensity of the noise.

In Chapter 3, we study a time explicit finite volume method with an upwind scheme for a first order conservation law with a monotone flux function and a multiplicative source term involving a Q -Wiener process. We present some a priori estimates including a weak BV estimate. After performing a time interpolation, we prove two entropy inequalities for the discrete solution and show that it converges up to a subsequence to a stochastic measure-valued entropy solution of the conservation law in the sense of Young measures.

In Chapter 4, we obtain similar results as in Chapter 3, in the case that the flux function is non-monotone, and that the convection term is discretized by means of a monotone scheme.

Keywords : Groundwater flow in porous media - Generalized finite volume method SUSHI - Numerical simulations - Stochastic conservation law - Convergence in the sense of Young measures.

AMS subject classifications : 35R60 - 37M05 - 65M08 - 65M12 - 76S05

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Introduction

Méthodes de volumes finis pour les équations aux dérivées partielles déterministes et stochastiques

0.1 Introduction en français

Le but de cette thèse est de faire l'étude de méthodes de volumes finis pour des équations aux dérivées partielles déterministes et stochastiques. Nous effectuons tout d'abord des simulations numériques pour un système décrivant un écoulement à densité variable en milieu poreux; nous abordons ensuite la discrétisation d'une loi de conservation pour une équation avec un terme source stochastique, à la fois du point de vue des simulations numériques et de celui de la convergence d'algorithmes.

0.2 Les équations et systèmes étudiés

0.2.1 Ecoulements à densité variable

Nous étudions un problème modélisant des écoulements à densité variable en milieu poreux. Le point de départ est l'étude de l'eau s'écoulant dans une nappe phréatique du sol. Il s'agit d'un problème d'intérêt général, car l'eau potable est extraite des nappes phréatiques. Si une nappe est polluée par un contaminant ou si elle est envahie par de l'eau salée, son eau peut ne plus être buvable, ce qui représente un problème immense pour les villes environnantes. Deux quantités essentielles que l'on doit calculer sont la pression et la vitesse de l'écoulement. Dans ce but, nous appliquons la loi de Darcy, qui exprime le fait que la vitesse est proportionnelle à l'opposé du gradient de la pression. Plus précisément, la loi de Darcy est de la forme

$$\mathbf{q} = -\frac{k}{\mu} \nabla p,$$

où \mathbf{q} est la vitesse, p la pression, k la perméabilité et μ la viscosité dynamique. Dans de nombreux cas, l'effet de la gravité doit également être inclus; la loi de Darcy devient alors

$$\mathbf{q} = -\frac{k}{\mu} (\nabla p - \rho \mathbf{g}),$$

où ρ est la densité du fluide et \mathbf{g} la gravité.

Un exemple typique est le cas d'écoulements à densité variable, où la densité ρ et la viscosité μ dépendent de la concentration c , si bien que $\rho = \rho(c)$ et $\mu = \mu(c)$. Plus précisément, nous étudions le système

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}), \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = Q_\rho, \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = Q, \end{cases}$$

où la constante positive Φ est la porosité, avec des conditions aux limites et des conditions initiales appropriées.

0.2.2 Lois de conservation stochastiques

Nous étudions des lois de conservation du premier ordre avec un terme source stochastique. Dans tout système naturel, il se rajoute toutes sortes d'effets perturbatifs aux équations de base; c'est ce que nous modélisons par des termes de bruit et qui nous conduit à l'étude d'équations aux dérivées partielles stochastiques. Quand un terme stochastique f_ω est introduit dans une équation, la solution $u(x, t)$ devient une variable aléatoire $u_\omega(x, t)$, ce qui conduit à repenser en profondeur les études mathématiques déjà effectuées dans les cas déterministes.

Les motivations pour l'étude de tels modèles sont multiples. Les systèmes spatio-temporels en physique, chimie et biologie sont souvent soumis à des phénomènes aléatoires, qui peuvent intervenir sous la forme de forces fluctuantes, de paramètres incertains, de termes sources ou de conditions aux limites aléatoires. Il s'agit donc de savoir intégrer l'incertitude dans la réalité que le modèle doit représenter, ce qui est souvent fait à l'aide d'équations aux dérivées partielles stochastiques. Pour donner un exemple concret, Debussche-Glatt-Temam-Ziane [21] s'intéressent à la modélisation de l'océan et de l'atmosphère et introduisent un bruit multiplicatif pour représenter les incertitudes dues à la complexité du système climatique terrestre. Un autre exemple est donné par l'équation de Burgers stochastique étudiée dans cette thèse. L'un des problèmes principaux dans le domaine des systèmes complexes est de parvenir à une compréhension suffisamment profonde de la turbulence. Des simulations directes ont beaucoup contribué à la compréhension des écoulements désordonnés qui se produisent quand le nombre de Reynolds est élevé. Toutefois, il manque encore une théorie complète de la turbulence qui permettrait de prédire des phénomènes importants comme les mélanges turbulents, la convection turbulente et la combustion turbulente à partir des des équations de base de la dynamique des fluides. L'équation de Burgers est due au physicien néerlandais J.M. Burgers [13] qui a simplifié les équations de Navier-Stokes en négligeant le terme de pression, obtenant l'équation de Burgers qui peut être étudiée en dimension un d'espace, avec ou sans terme source.

Dans cette thèse, nous étudions l'équation de Burgers en négligeant également le terme diffusif et effectuons une étude numérique en supposant que le terme source a la forme d'un bruit blanc

en temps combiné à un bruit coloré en espace. Il nous faut repenser les méthodes déterministes de façon à parvenir à appréhender l'effet de ce terme source aléatoire. En particulier, nous traçons systématiquement les graphes de deux quantités : celui d'une seule réalisation du bruit, et celui d'une moyenne d'un très grand nombre de réalisations, par exemple 2000 ou 8000. Notre résultat le plus surprenant est le suivant : la moyenne des réalisations semble se comporter comme la solution du problème déterministe *sans* le terme source stochastique. En particulier, elle semble tendre vers une constante, la moyenne de la condition initiale, quand $t \rightarrow \infty$. Par contre, elle diffère de la solution déterministe en ce sens qu'elle semble lisser les sauts de la solution déterministe.

Debussche-Vovelle [23], [25] ont démontré l'existence et l'unicité d'une solution cinétique dans le cas d'un bruit multiplicatif faisant intervenir un processus Q -Wiener tandis que Bauzet-Vallet-Wittbold [6] ont démontré l'existence et l'unicité d'une solution faible entropique stochastique de la solution d'un problème de Cauchy sur \mathbb{R}^d dans le cas d'un bruit multiplicatif faisant intervenir un mouvement brownien unidimensionnel. Nous ne parvenons pas encore à étudier la convergence de la solution approchée vers la solution de l'équation de Burgers, mais seulement la convergence de la solution approchée d'une loi de conservation un peu plus régulière et cela pour deux raisons : nous imposons que le flux est une fonction monotone, ce qui importe peu car il ne s'agit que d'une hypothèse technique dont nous saurions nous affranchir en implémentant un algorithme adapté, comme cela a été fait par Bauzet-Charrier-Gallouët [8]. Par contre, nous supposons également que le flux est une fonction Lipchitzienne, ce qui est une hypothèse encore à améliorer. L'aspect stochastique nous amène à travailler avec des solutions faibles entropiques, définies à l'aide de la formule d'Itô, comme cela a été fait par [6]. Notre résultat étend celui de [9] dans le sens où nous supposons essentiellement que le bruit est un bruit multiplicatif où intervient la dérivée généralisée d'un processus Q -Wiener tandis que [8] et [9] ne considèrent que le cas d'un bruit multiplicatif unidimensionnel dans la variable de temps.

0.3 Méthodes de volumes finis

Pour la résolution numérique, nous nous appuyons sur des méthodes de volumes finis, soit la méthode des volumes finis standard soit la méthode de volumes finis généralisés SUSHI.

Le schéma de volumes finis standard a d'abord été développé par les ingénieurs afin d'étudier des phénomènes physiques couplés complexes où la conservation de quantités extensives (telles que les masses, l'énergie, l'impulsion ...) doit être soigneusement respectée par la solution approchée. Un avantage de cette famille de méthodes est qu'une grande variété de maillages peut être utilisée. L'idée de base est la suivante: on intègre les équations aux dérivées partielles dans chaque élément de volume et on définit des flux approchés à travers les frontières des éléments de volumes. C'est ce schéma que nous utilisons pour la résolution de l'équation de Burgers stochastique en dimension un d'espace.

Pour des équations elliptiques ou paraboliques en dimension supérieure, l'application de la méthode des volumes finis standard présente certains inconvénients. On suppose que le domaine spatial est un sous-ensemble ouvert borné de \mathbb{R}^d de frontière polyédrale, qui est discrétisé

à l'aide d'éléments de volume polyédriques. Une première contrainte est liée au fait qu'une condition, appelée condition d'orthogonalité, doit être satisfaite : pour pouvoir appliquer la méthode des volumes finis standard, il est nécessaire d'associer un point à chaque élément de volume et le segment joignant deux points correspondant à deux éléments de volume voisins doit être orthogonal à l'interface entre les deux éléments de volume. Ensuite, les éléments de volume doivent être compatibles, c'est-à-dire qu'un côté d'élément de volume doit précisément coïncider soit avec un côté d'un élément de volume voisin soit avec un segment de la frontière du domaine. Un autre problème est que l'on ne peut pas appliquer la méthode des volumes finis standard à des équations aux dérivées partielles où la matrice de diffusion est une matrice pleine.

Ceci nous amène à appliquer le schéma de volumes finis SUSHI qui présente notamment les avantages suivants: il n'est pas nécessaire d'imposer la condition d'orthogonalité, les éléments de volume peuvent ne pas être compatibles et de plus, le tenseur de diffusion peut avoir la forme d'une matrice pleine. Alors qu'un degré de liberté est associé à chaque élément de volume dans le cas de la méthode des volumes finis standard, le schéma SUSHI nécessite non seulement un degré de liberté par élément de volume mais également un degré de liberté correspondant à chaque côté. Ce schéma est basé sur la définition d'un gradient discret qui permet de définir une famille de flux numériques associés à une forme bilinéaire continue, coercive, et symétrique. A son tour, le schéma SUSHI peut être considéré comme un cas particulier de la famille plus générale des schémas gradient.

0.4 Plan de la thèse

Cette thèse comporte quatre chapitres. Le Chapitre 1 porte sur l'application de la méthode de volumes finis SUSHI à des problèmes d'écoulements à densité variable en milieux poreux. Le Chapitre 2 porte la simulation numérique de l'équation de Burgers non visqueuse avec un terme source stochastique à l'aide de la méthode des volumes finis standard combinée avec une discrétisation amont du flux. Au Chapitre 3, nous étudions la convergence d'un schéma de volumes finis pour une loi de conservation faisant intervenir un terme source multiplicatif comportant la dérivée généralisée d'un processus Q -Wiener. Au Chapitre 4, nous considérons le même problème qu'au Chapitre 3 mais dans le cas où la fonction flux n'est pas monotone.

0.4.1 Chapitre 1 : Ecoulements à densité variable en milieu poreux

Nous considérons un milieu poreux contenant à la fois de l'eau douce et de l'eau salée; plus précisément nous étudions un système couplé décrivant l'interaction entre l'écoulement et le transport dans un milieu poreux; nous supposons que $\rho(c) = \rho_0(1 + \alpha c)$ où α est une constante positive. Nous résolvons une équation parabolique non linéaire de convection-diffusion pour la concentration de sel couplée à une équation qui peut être perçue comme une équation elliptique en pression que l'on déduit en substituant la loi de Darcy (la première équation du système

ci-dessous) dans la deuxième équation

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}) & \text{dans } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = 0 & \text{dans } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = 0 & \text{dans } \mathbb{D} \times (0, T). \end{cases} \quad (0.4.1)$$

Nous résolvons ces équations avec les conditions aux limites

$$\begin{cases} c = c_D(\mathbf{x}, t) & \text{sur } \partial\mathbb{D}_D^c \times (0, T), \\ \frac{\partial c}{\partial n} = \bar{c}_N(\mathbf{x}, t) & \text{sur } \partial\mathbb{D}_N^c \times (0, T), \\ p = p_D(\mathbf{x}, t) & \text{sur } \partial\mathbb{D}_D^p \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N(\mathbf{x}, t) & \text{sur } \partial\mathbb{D}_N^p \times (0, T), \end{cases}$$

avec $\partial\mathbb{D} = \overline{\partial\mathbb{D}_D^c} \cup \overline{\partial\mathbb{D}_N^c} = \overline{\partial\mathbb{D}_D^p} \cup \overline{\partial\mathbb{D}_N^p}$ où $\partial\mathbb{D}_D^c$ et $\partial\mathbb{D}_D^p$ correspondent à des conditions aux limites de Dirichlet et $\partial\mathbb{D}_N^c$ et $\partial\mathbb{D}_N^p$ correspondent à des conditions aux limites de Neumann pour la concentration et la pression respectivement. La fonction initiale est donnée par

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \quad x \in \mathbb{D}.$$

La porosité Φ est le rapport des vides (espaces vides) sur le volume total. Dans la première équation (A.0.1), la loi de Darcy, \mathbf{q} est la vitesse de lécoulement, p la pression, k la perméabilité, μ la viscosité dynamique, et \mathbf{g} la gravité.

Une caractéristique essentielle du schéma SUSHI est qu'il permet d'utiliser des éléments de volume qui ne se correspondent pas. En particulier, nous pouvons appliquer un maillage adaptatif de carrés ou de cubes de différentes tailles dans les tests numériques. Les fonctions inconnues varient fortement dans certaines parties du domaine alors qu'elles sont presque constantes ailleurs. Nous raffinons le maillage dans les régions de fortes variations des fonctions inconnues alors que nous fusionnons des éléments dans les zones où elles ne subissent que de faibles variations. Ceci permet de réduire le nombre des inconnues (qui dépendent du nombre d'éléments et de côtés d'éléments de volumes) et de diminuer le temps de calcul. Après chaque raffinement ou déraffinement, nous devons nous appuyer sur de nouvelles données pour le maillage et pour les fonctions inconnues.

Nous effectuons des simulations numériques pour le problème d'une interface en rotation, le problème de Henry en dimension deux d'espace et le problème de bassin salé (salt pool) en dimension trois d'espace, en nous inspirant de calculs effectués par Hilhorst-Vu Do-Wang [37] à l'aide méthode de volumes finis standard.

Dans une deuxième partie nous prenons également en compte le transfert de chaleur. La chaleur peut être récupérée à partir de l'eau souterraine. Selon l'application, la chaleur récupérée peut être utilisée pour la production de chaleur ou pour la production d'électricité. Le chauffage

géothermique permet à la fois de remplacer le chauffage conventionnel et la production d'eau chaude. Nous étudions un système couplé décrivant l'interaction entre l'écoulement et le transport dans un milieu poreux, où la densité et la viscosité dépendent de la concentration des espèces à transporter, aussi bien que de la température. Plus précisément, nous résolvons un système de trois équations paraboliques non linéaires pour la concentration de sel C , la température Θ et la charge hydraulique h .

Un logiciel commercial très utilisé pour la simulation du transport d'un soluté multi-espèces et du transport de chaleur est le code SEAWAT [48]. Il combine les logiciels MODFLOW pour le calcul de l'écoulement et MT3DMS pour la résolution numérique des équations de transport de soluté. Le couplage entre l'écoulement et le transport est effectué à l'aide de résolution successives de l'équation pour l'écoulement par MODFLOW et de l'équation de transport par MT3DMS. Les deux logiciels MODFLOW et MT3DMS s'appuient sur une méthode de différences finies et la solution est donnée par les valeurs moyennes assignées au centre de chaque maille. Le logiciel SEAWAT utilise également cette structure.

Nous proposons un code unique qui utilise la méthode de volumes finis généralisée fini SUSHI. Les éléments de volume, en dimension d'espace deux ou trois, peuvent être très généraux et la structure du programme est beaucoup plus simple puisqu'elle s'appuie sur un code unique au lieu de deux codes couplés. Nous montrons les résultats de premiers tests numériques en fin de chapitre. Nos calculs portent sur un modèle proposé dans la documentation de SEAWAT. Le domaine spatial est donné par une section transversale bi-dimensionnelle d'un aquifère captif côtier initialement saturé d'eau de mer relativement froide à la température de $5\text{ }^{\circ}\text{C}$. De l'eau douce plus chaude à la température de $25\text{ }^{\circ}\text{C}$ est injectée dans l'aquifère côtier le long de la frontière de gauche pour représenter le flux d'eau provenant de l'intérieur des terres. L'eau douce chaude s'écoule vers la droite jusqu'à atteindre la frontière avec l'océan. La frontière de l'océan est soumise à des conditions hydrostatiques basées sur la densité de fluide calculée à partir de la salinité de l'eau de mer à $5\text{ }^{\circ}\text{C}$. Des conditions de flux nul sont assignées aux limites supérieure et inférieure du domaine. Ce problème est une représentation simplifiée de ce qui pourrait se produire dans une plate-forme côtière de carbonate. Nos résultats sont très proches de ceux de SEAWAT.

Il s'agit d'un travail en collaboration avec Danielle Hilhorst et Huy Cuong Vu Do.

0.4.2 Chapitre 2 : Simulations numériques de l'équation de Burgers non visqueuse avec des conditions aux limites périodiques et un terme source stochastique

Nous effectuons une étude numérique de l'équation de Burgers non visqueuse avec un terme source stochastique. Nous considérons le problème sur le tore de dimension un, avec des conditions aux limites périodiques et une condition initiale prescrite

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = g, \quad (0.4.2)$$

où le terme source stochastique g est de moyenne spatiale nulle: $\int_{\mathbb{T}^1} g dx = 0$. Nous appliquons un schéma de volumes finis combinant une intégration en temps de type Euler-Maruyama avec le flux numérique de Godunov. Le terme source stochastique discret, qui est de la forme

$$\alpha G_i^n = \alpha \sqrt{\frac{2}{I}} \left(\sum_{k=1}^{\frac{I-1}{2}} \left(\frac{C_k^n}{k^\beta} \cos(2\pi k x_i) - \frac{S_k^n}{k^\beta} \sin(2\pi k x_i) \right) \right) \quad (0.4.3)$$

s'inspire de [27], l'entier impair I est le nombre de volume en espace, α l'amplitude et le paramètre β permet de moduler la régularité en espace. Les coefficients $\{C_k^n\}$ et $\{S_k^n\}$ sont des variables aléatoires indépendantes et identiquement distribuées selon la loi Gaussienne $\mathcal{N}(0, 1)$. Quand $\beta = 0$, les G_i^n sont des variables aléatoires Gaussiennes de moyenne nulle et de covariance:

$$\mathbb{E}[G_i^n G_j^m] = \delta_{m,n} [\delta_{i,j} - \frac{1}{I}], \quad (0.4.4)$$

ce qui correspond à un bruit blanc en temps, et pour $I \rightarrow \infty$, également en espace.

Nous présentons des résultats de simulations numériques effectués à l'aide de la méthode de Monte-Carlo. Comme le terme source G_i^n est stochastique, nous effectuons un certain nombre de réalisations pour chaque α et β fixés. Nous comparons les résultats de réalisations uniques et la moyenne empirique des réalisations avec la solution numérique déterministe ($g = 0$). Les résultats numériques montrent que pour une seule réalisation, quand l'amplitude α est grande, la solution est dispersée autour de la solution déterministe; d'autre part, le terme source stochastique possède un effet de diffusion qui lisse le choc de la solution déterministe, et plus l'amplitude α est grande, plus l'effet de diffusion est fort. Nous obtenons également des résultats numériques en temps long, et observons que pour α et β fixés, la moyenne empirique converge vers la moyenne en espace de la condition initiale. Quand α et β sont plus grands, la moyenne empirique converge vers cette valeur plus vite. La variance empirique se stabilise vers une valeur qui dépend de α et β ; plus α est grand et β est petit, plus cette valeur asymptotique est grande.

Ce chapitre a été écrit en collaboration avec Emmanuel Audusse, Sébastien Boyaval et Danielle Hilhorst [3].

0.4.3 Chapitre 3 : Convergence d'un schéma de volumes finis pour une loi de conservation du premier ordre faisant intervenir un processus Q -Wiener

Un sujet d'intérêt essentiel est la convergence de méthodes numériques pour la discrétisation de lois de conservation stochastiques. Nous étudions le problème

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t), & \text{dans } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \text{pour tout } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (0.4.5)$$

où $W(x, t)$ un processus Q -Wiener. Plus précisément soit Q un opérateur symétrique semi-défini positif sur $L^2(\mathbb{T}^d)$, soit $\{e_m\}_{m \geq 1}$ une base orthonormée de $L^2(\mathbb{T}^d)$ qui diagonalise Q , et

soient $\{\lambda_m\}_{m \geq 1}$ les valeurs propres correspondantes. Le processus W défini par

$$W(x, t) = \sum_{m=1}^{\infty} \beta_m(t) Q^{\frac{1}{2}} e_m(x) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (0.4.6)$$

est un processus Q -Wiener dans $L^2(\mathbb{T}^d)$ si

$$\sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0.$$

Nous supposons que la fonction non linéaire f est Lipschitzienne et monotone et que la fonction $g(u)$ est Lipschitzienne et bornée.

Après avoir présenté la définition d'une solution faible entropique à valeur mesure du problème (A.0.3), nous appliquons une méthode de volumes finis avec un schéma amont pour la discrétisation de ce problème, et nous notons $\{u_{\mathcal{T},k}\}$ la solution discrète. Nous présentons des estimations a priori pour le terme source discret et pour la solution discrète $\{u_{\mathcal{T},k}\}$ ainsi qu'une estimation BV faible. Puis nous prouvons deux inégalités entropiques vérifiées par la solution discrète après l'avoir interpolée en temps. Nous démontrons que $\{u_{\mathcal{T},k}\}$ admet une sous-suite qui converge vers une solution faible entropique à valeur mesure au sens des mesures de Young quand le diamètre maximal des éléments de volume et le pas de temps tendent vers zéro. Ce Chapitre étend des travaux de Bauzet-Charrier-Gallouët [8] dans lequel ils considèrent une loi de conservation avec un terme source multiplicatif faisant intervenir un mouvement Brownien unidimensionnel.

Il s'agit d'un travail en collaboration avec Tadahisa Funaki, Danielle Hilhorst et Hendrik Weber.

0.4.4 Chapitre 4 : Convergence dans le cas où la fonction de flux n'est pas monotone

Nous étudions le problème

$$\begin{cases} du + \operatorname{div}(\mathbf{v}(x, t) f(u)) dt = g(u) dW(x, t), & \text{dans } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \text{pour tout } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (0.4.7)$$

où $W(x, t)$ un processus Q -Wiener. Nous supposons que la fonction non linéaire f est seulement Lipschitzienne.

Nous appliquons une méthode de volumes finis avec un flux numérique monotone pour la discrétisation de ce problème, et nous notons $\{u_{\mathcal{T},k}\}$ la solution discrète. Nous présentons des estimations a priori pour le terme source discret et pour la solution discrète $\{u_{\mathcal{T},k}\}$ ainsi qu'une estimation BV faible. Puis nous prouvons une inégalité entropique vérifiée par la solution discrète après l'avoir interpolée en temps. On peut prouver une inégalité continue entropique et un théorème de convergence par des méthodes similaires à celles utilisées au Chapitre 3. Ce Chapitre étend des travaux de Bauzet-Charrier-Gallouët [9].

Il s'agit d'un travail en collaboration avec Tadahisa Funaki, Danielle Hilhorst et Hendrik Weber.

Finite volume methods for deterministic and stochastic partial differential equations

0.5 Introduction in English

In this thesis we study finite volume methods for deterministic and stochastic partial differential equations. We perform numerical simulations for a system describing a density driven flow in porous media; we then study the discretization of a conservation law for an equation with a stochastic source term, both from the viewpoint of numerical simulations and of the convergence of algorithms.

0.6 The equations and systems which we study

0.6.1 Density driven flows in porous media

We study groundwater flow in an aquifer. This is an essential problem since most drinking water is extracted from aquifers. If an aquifer becomes polluted by some contaminant or if it gets invaded by salt water, its water may not be drinkable any more, which is a huge problem for the surrounding cities. Two essential quantities to compute are the pressure and the velocity of the flow. To that purpose we apply Darcy's law, which states that the velocity is proportional to the opposite of the pressure gradient. More precisely, Darcy's law is of the form

$$\mathbf{q} = -\frac{k}{\mu}\nabla p,$$

where \mathbf{q} is the velocity, p the pressure, k the permeability, and μ the dynamic viscosity. In many cases, the gravity effect should also be included; Darcy's law then becomes

$$\mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho\mathbf{g}),$$

where ρ is the density of the fluid and \mathbf{g} the gravity.

A typical example is the case of density driven flows, where the density ρ and the viscosity μ depend on the concentration c , namely $\rho = \rho(c)$ and $\mu = \mu(c)$. More specifically, we study the system

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}), \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = Q_\rho, \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = Q, \end{cases}$$

where the positive constant Φ is the porosity, together with corresponding boundary and initial conditions.

0.6.2 Stochastic conservation laws

We study first-order conservation laws with a stochastic source term. In every natural system, all kinds of perturbative effects may be added to basic equations; this is what we model by noise. When a stochastic term f_ω is introduced in an equation, the solution $u(x, t)$ becomes a random variable $u_\omega(x, t)$, which leads to revisit the mathematical studies already carried out in the deterministic case.

Spatiotemporal systems in physics, chemistry, and biology are often subject to random phenomena that can occur in the form of fluctuating forces, uncertain parameters, and random source terms or boundary conditions. It is therefore of essential importance to be able to introduce uncertainty in the reality that the model must represent, and this is often done by means of stochastic partial differential equations. As a concrete example, Debussche-Glatt-Temam-Ziane [21], who model the ocean and the atmosphere, introduce a multiplicative noise to represent uncertainties due to the complexity of the earth's climate system. Another example is given by the stochastic Burgers equation studied in this thesis. One of the main problems in the field of complex systems is to achieve a sufficiently deep understanding of turbulence. Direct simulations have greatly contributed to the understanding of disordered flows that occur when the Reynolds number is high. However, one still lacks a complete theory of turbulence that would permit to predict important phenomena such as turbulent mixing, turbulent convection and turbulent combustion starting from the basic equations of fluid dynamics. The Burgers equation is due to the Dutch physicist J.M. Burgers [13] who in 1939 simplified the Navier-Stokes equation by neglecting the pressure term, obtaining the Burgers equation which can be studied in one space dimension, with or without a term source.

In this thesis, we study the Burgers equation also neglecting the diffusive term and carry out a numerical study assuming that the source term has the form of a white noise in time combined with a colored noise in space. We need to revisit the deterministic methods in order to understand the effect of this random source term. In particular, we systematically draw graphs of two quantities: a prescribed single realization of the noise, and an average of a large number of realizations, such as 2000 or 8000. Our most surprising result is the following: the average of the realizations seems to behave as a solution of the problem *without* the stochastic source term. In particular, it seems to tend towards a constant, namely the average of the initial condition, as $t \rightarrow \infty$. However, it differs from the deterministic solution in the sense that it seems to smoothen the discontinuities of the deterministic solution.

As for analytical results, Debussche-Vovelle [23], [25] have shown the existence and uniqueness of a kinetic solution on a torus of \mathbb{R}^d in the case of a multiplicative noise involving a Q -Wiener process, while Bauzet-Vallet-Wittbold [6] have proved the existence and uniqueness of a weak entropy stochastic solution of a related Cauchy problem in \mathbb{R}^d in the case of a multiplicative noise involving a one-dimensional Brownian motion. We still are unable to study the convergence of the approximate solution to the solution of Burgers equation, but only the convergence of the approximate solution of a somewhat more regular conservation law. On the one hand, we impose the nonlinear flux function to be nondecreasing, but we would know, at the cost of writing a more technical proof for a suitable algorithm, how to get rid of

this hypothesis as has been done by Bauzet-ChARRIER-Gallouët [8]. On the other hand, we also assume that the nonlinear flux function is Lipschitz continuous, which is a hypothesis still to be improved. The stochastic force term leads us to work with weak entropy solutions, involving the Itô formula, as has been done by [6]. Our result extends that of [9] in the sense that we assume that the source term has the form of a multiplicative noise involving the derivative of a Q -Wiener process while [8] and [9] only consider the case of a one-dimensional multiplicative noise in the time variable.

0.7 Finite volume methods

For the numerical solution, we apply both the standard finite volume method and the generalized finite volume method SUSHI.

The standard finite volume method has first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, momentum...) must be carefully respected by the approximate solution. Another advantage of such schemes is that a large variety of meshes can be used. The basic idea is the following: one integrates the partial differential equations in each volume element and then approximates the fluxes across the volume boundaries. It is the method which we apply to solve the stochastic Burgers equation in one space dimension.

However, for elliptic or parabolic equations in higher space dimensions, there are some constraints when applying the standard finite volume method. Suppose that the space domain is an open bounded connected polyhedral subset of \mathbb{R}^d , which is discretized by means of polyhedral volume elements. First the so-called orthogonality condition should hold : one associates a point to each volume element and the segment joining two points corresponding to two neighboring volume elements should be orthogonal to the interface between the two volume elements. Then the volume elements should be matching, namely an edge of a polygonal volume element should precisely coincide either with an edge of a neighboring volume element or with an edge of the boundary of the domain. A further problem is that one cannot apply the standard finite volume method to a partial differential equation whose diffusion term involves a full diffusion matrix.

This induces us to apply the so-called SUSHI scheme which has the following advantages : the orthogonality condition is not necessary anymore, the volume elements can be non-matching and moreover, it permits to handle diffusion terms involving a full diffusion matrix. Whereas one degree of freedom is associated to each volume element in the case of the standard finite volume method, the SUSHI scheme involves not only one degree of freedom per volume element but also a degree of freedom associated to each edge. This scheme is based upon the definition of a discrete gradient which, in the case of suitable diffusion terms, permits to define a family of numerical fluxes, which are continuous, coercive, consistent and symmetric. In turn, the SUSHI scheme can be considered as a special case of the more general family of gradient schemes.

0.8 Organization of the thesis

This thesis is composed of four chapters. Chapter 1 concerns the application of the finite volume method SUSHI to density driven flow problems in porous media. We present in Chapter 2 numerical simulations of the inviscid Burgers equation with a stochastic source term. In Chapter 3, we study the convergence of a finite volume scheme for a conservation law with a multiplicative source term involving a Q -Wiener process. In Chapter 4, we consider the same problem as in Chapter 3 in the case that the flux function is non-monotone.

0.8.1 Chapter 1 : Density driven flow in porous media

We consider a porous medium containing both fresh and salt water; more specifically we study a coupled system describing the interaction between flow and transport in porous media and we assume that $\rho(c) = \rho_0(1 + \alpha c)$ where α is a positive constant. In the first part of Chapter 1, we solve a nonlinear parabolic convection-diffusion equation for the salt concentration coupled to an equation which can be considered as an elliptic equation for the pressure when substituting Darcy's law, the first equation of the system below, in the second equation

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}) & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = 0 & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = 0 & \text{in } \mathbb{D} \times (0, T). \end{cases} \quad (0.8.1)$$

We solve these equations together with the boundary conditions

$$\begin{cases} c = c_D(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_D^c \times (0, T), \\ \frac{\partial c}{\partial n} = \bar{c}_N(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_N^c \times (0, T), \\ p = p_D(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_D^p \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_N^p \times (0, T), \end{cases}$$

with $\partial\mathbb{D} = \overline{\partial\mathbb{D}_D^c} \cup \overline{\partial\mathbb{D}_N^c} = \overline{\partial\mathbb{D}_D^p} \cup \overline{\partial\mathbb{D}_N^p}$ where $\partial\mathbb{D}_D^c$ and $\partial\mathbb{D}_D^p$ correspond to the Dirichlet boundary condition and $\partial\mathbb{D}_N^c$ and $\partial\mathbb{D}_N^p$ correspond to the Neumann boundary condition for the concentration and the pressure respectively. The initial condition is given by

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \quad x \in \mathbb{D}.$$

The porosity Φ is the proportion of the empty space over the total volume. In the first equation of (0.8.1), Darcy's law, \mathbf{q} is the velocity, p the pressure, k the permeability, μ the dynamic viscosity, and \mathbf{g} the gravity.

An essential feature of the SUSHI scheme is that it allows to use volume elements which do not match. In particular, we can apply an adaptive mesh composed of squares or cubes of different sizes. Unknown functions vary strongly in certain parts of the field, while they hardly vary elsewhere. We refine the mesh in regions of large variations of the unknown functions while we merge elements in areas where they only undergo small variations. This reduces the number of unknowns and permits to reduce the computation time.

We perform various numerical tests such as the computation of a rotating interface, numerical simulations for Henry's problem in space dimension two and for the saltpool problem in space dimension three, extending a paper by Hilhorst-Vu Do-Wang [37] where they apply the standard finite volume method.

In the second part we also take heat transfer into account. Heat can be recovered from groundwater. Depending on the application, the recovered heat can be used for heat production or for the production of electricity. The geothermal heating allows both to replace the conventional heating and to produce hot water. We study a coupled system describing the interaction between flow and transport in a porous medium, where the density and the viscosity depend on the concentration of the species to be transported, as well as on the temperature. More specifically, we solve a system of three nonlinear parabolic equations for the salt concentration C , the temperature Θ and the hydraulic head h .

A widely used commercial software for the simulation of a multi-species solute and heat transport is the SEAWAT code [48]. It combines the software codes MODFLOW and MT3DMS which permit to solve the flow and the solute-transport equations respectively. The coupling between flow and transport is performed through a synchronous time-stepping approach that cycles between MODFLOW solutions of the flow equation and MT3DMS solutions of the transport equation. Both MODFLOW and MT3DMS use cell-centered grids. In this formulation, the dependent variables obtained in the finite-difference solution represent average values, assumed to be given at the cell center, for the respective cells. Therefore SEAWAT also uses cell-centered grids. We propose a single code using the generalized finite volume method SUSHI. We show results of test cases at the end of the chapter. The volume elements, in space dimensions two or three, can be quite arbitrary and the structure is much simpler since we deal with a single code instead of two heavy codes which are coupled with each other.

We take a model problem proposed in the SEAWAT documentation [48] as a test case for our method. The problem consists in a two-dimensional cross section of a confined coastal aquifer initially saturated with relatively cold seawater. Warmer fresh water is injected into the coastal aquifer along the left boundary to represent the flow from inland areas. The warmer fresh water flows to the right, where it discharges into a vertical ocean boundary. Hydrostatic conditions are imposed on the ocean boundary while no flow conditions are assigned to the top and bottom boundaries. This problem is a simplified representation of what may occur in a coastal carbonate platform. Our results are close to those of SEAWAT.

This Chapter is joint work with Danielle Hilhorst and Huy Cuong Vu Do.

0.8.2 Chapter 2 : Numerical simulations of the inviscid Burgers equation with periodic boundary conditions and a stochastic source term

We perform a numerical study of a non-viscous Burgers equation with a stochastic source term. We study the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = g \quad (0.8.2)$$

on the one-dimensional torus, with periodic boundary conditions and a prescribed initial condition. We suppose that the stochastic source term g satisfies: $\int_{\mathbb{T}^1} g dx = 0$. We apply a finite

volume scheme combining an Euler-Maruyama type scheme for the time discretization with the Godunov scheme for the discretization of the fluxes. The discrete source term, which is of the form

$$\alpha G_i^n = \alpha \sqrt{\frac{2}{I}} \sum_{k=1}^{\frac{I-1}{2}} \left(\frac{C_k^n}{k^\beta} \cos(2\pi k x_i) - \frac{S_k^n}{k^\beta} \sin(2\pi k x_i) \right), \quad (0.8.3)$$

is inspired from [27], the odd integer I is the number of space volumes, α the amplitude, and the parameter β modulates the regularity in space. The coefficients $\{C_k^n\}$ and $\{S_k^n\}$ are independent random variables which are identically distributed according to Gaussian law $\mathcal{N}(0, 1)$. When $\beta = 0$, the G_i^n 's are zero average Gaussian random variables of covariance

$$\mathbb{E}[G_i^n G_j^m] = \delta_{m,n} [\delta_{i,j} - \frac{1}{I}], \quad (0.8.4)$$

which corresponds to Gaussian white noise in time and for $I \rightarrow \infty$, also in space.

We perform a number of realizations for each test case with α and β fixed. We compare the results of the empirical mean with the deterministic numerical solution with $g = 0$. The numerical results show that for a single realization, when the amplitude α is large, the solution is dispersed around the deterministic solution; moreover, the stochastic source term has a diffusive effect which smoothens the shock of the deterministic solution, and this diffusive effect increases with α . We also obtain numerical results in large time, and observe that for α and β fixed, the empirical mean converges to the average of the initial condition. When α and β are larger, the empirical mean converges to this value faster. The empirical variance stabilizes to a value depending on α and β ; as α increases and β decreases, this limit value increases.

This chapter has been written in collaboration with Emmanuel Audusse, Sébastien Boyaval and Danielle Hilhorst [3].

0.8.3 Chapter 3 : Convergence of a finite volume scheme for a first order conservation law involving a Q -Wiener process

A topic of high interest is the convergence of numerical methods for the discretization of stochastic conservation laws. We study the problem

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x) & \text{for all } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (0.8.5)$$

where $W(x, t)$ is a Q -Wiener process. More precisely, let Q be a symmetric semi-definite positive operator on $L^2(\mathbb{T}^d)$, let $\{e_m\}_{m \geq 1}$ be an orthonormal basis of $L^2(\mathbb{T}^d)$ which diagonalizes Q , and let $\{\lambda_m\}_{m \geq 1}$ be the corresponding eigenvalues. The process W defined by

$$W(x, t) = \sum_{m=1}^{\infty} \beta_m(t) Q^{\frac{1}{2}} e_m(x) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (0.8.6)$$

is a Q -Wiener process in $L^2(\mathbb{T}^d)$ if

$$\sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0.$$

We suppose that the function f is Lipschitz continuous and nondecreasing, and that the function $g(u)$ is Lipschitz continuous and bounded.

After having presented the definition of a measure-valued weak entropy solution of Problem (0.8.5), we apply a finite volume method together with an upwind scheme for the space discretization, and we denote by $\{u_{\mathcal{T},k}\}$ its discrete solution. We present a priori estimates for the discrete source term and for the discrete solution $\{u_{\mathcal{T},k}\}$, including a weak BV estimate. We then prove two entropy inequalities satisfied by the discrete solution after interpolation in time. We show that the discrete solution $\{u_{\mathcal{T},k}\}$ converges along a subsequence to a measure-valued entropy solution of Problem (0.8.5) in the sense of Young measures as the maximum diameter of the volume elements and the time step tend to zero. This chapter extends an article by Bauzet-Charrier-Gallouët [8] where they consider a conservation law with a multiplicative noise involving a one-dimensional Brownian motion.

This is joint work with Tadahisa Funaki, Danielle Hilhorst and Hendrik Weber.

0.8.4 Chapter 4 : Convergence of a finite volume scheme in the case that flux function is not monotone

We consider the problem

$$\begin{cases} du + \operatorname{div}(\mathbf{v}(x, t) f(u)) dt = g(u) dW(x, t), & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \text{for all } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (0.8.7)$$

where $W(x, t)$ is a Q -Wiener process. We only suppose that the nonlinear function f is Lipschitz continuous.

We apply a finite volume method together with a monotone scheme for the flux discretization, and we denote by $\{u_{\mathcal{T},k}\}$ its discrete solution. We present a priori estimates for the discrete source term and for the discrete solution $\{u_{\mathcal{T},k}\}$, including a weak BV estimate. We then prove an entropy inequality satisfied by the discrete solution after interpolation in time. The proofs of the continuous entropy inequality and the corresponding convergence theorem are similar to those presented in Chapter 3. This chapter extends an article by Bauzet-Charrier-Gallouët [9].

This is joint work with Tadahisa Funaki, Danielle Hilhorst and Hendrik Weber.

A generalized finite volume method for a density driven flow coupled to heat transport in porous media

Résumé. Nous appliquons un schéma semi-implicite en temps combiné avec la méthode de volumes finis généralisés SUSHI pour la simulation d'écoulements à densité variable en milieu poreux; il vient à résoudre une équation de convection-diffusion parabolique pour la concentration couplée à une équation elliptique en pression. Nous présentons ensuite une méthode de simulation numérique pour un problème d'écoulements à densité variable couplé à un transfert de chaleur. Nous utilisons des maillages adaptatifs, basés sur des éléments de volume carrés ou cubiques.

Abstract. We apply a semi-implicit scheme in time together with the generalized finite volume method SUSHI for the numerical solution of density driven flows in porous media; it amounts to solve a nonlinear convection-diffusion parabolic equation for the concentration coupled to an elliptic equation for the pressure. We compute the solutions of specific problems such as a rotating interface problem, Henry's problem and a saltpool problem. We also consider the case when the density driven flow is coupled to heat transfer in a confined aquifer. We use adaptive meshes, based upon square or cubic volume elements.

1.1 Introduction

We present results which have been obtained in the context of an exploratory project of CNRS (PEPS ECODEVA) on the numerical simulation of flows with variable density for the production of lithium batteries. More precisely, the purpose of this project was related to the exploitation of lithium deposits in salt lakes, also known as “Salars”. In recent years, lithium has become a strategic element for industrial countries because it is the basic element of lithium-ion batteries used for hybrid and electric vehicles. Therefore its production has become of high interest for all major groups involved in car industry as well as suppliers of these groups. Currently the largest deposit in the world is the Salar Uyuni, in the department of Potosí, in South-West Bolivia. This deposit represents a third of the world resources. In March 2008, Bolivia has authorized the exploitation, however reserving this right to its nationals. Chile has the second largest deposit with the Salar Atacama and it has become the world’s largest exporter since 1997, with the German company Chemetall as main operator. Argentina also has a lithium deposit, the Salar Hombre Muerto, which is located in the North-West of the country. Other Salar areas of the Altiplano of Argentina provide mining exploitation concessions to foreign companies, among whom European groups.

Other deposits are exploited, including salt lakes in Tibet as well as mines in Australia, Russia and the United States. They are not accessible to European operators. The largest deposits are either clusters of crystallized salt (solid) or lenses of supersaturated salt water created by evaporation under endorheic conditions. Rational exploitation implies mastering these special aqueous flows whose density depends on the concentration of salt including lithium. An operating technique consists in sweeping the reservoir with fresh water in order to obtain a maximal recovery without earthworks and with a minimal impact on fluid levels, and thus a minimal impact on environment. This explains the need of implementing research methodologies and techniques from hydrogeology. The purpose here is to extract salt water which contains lithium. In a later stage, the lithium will be separated from the salt water.

From a mathematical viewpoint, it amounts to study a coupled system describing the interaction between flow and transport in a porous medium, whose density ρ is a strictly increasing function of the concentration c of the salt water; typically we have that $\rho(c) = \rho_0(1 + \alpha c)$. More specifically, we solve a nonlinear parabolic convection-diffusion equation for the concentration coupled to an elliptic equation for the pressure p , which is derived from Darcy’s law

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}) & \text{in } \mathbb{D} \times (0, T), & (1.1.1a) \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = Q_\rho & \text{in } \mathbb{D} \times (0, T), & (1.1.1b) \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = Q & \text{in } \mathbb{D} \times (0, T), & (1.1.1c) \end{cases}$$

together with suitable boundary conditions. The porosity Φ is the fraction of the voids (empty spaces) over the total volume. In the equation (1.1.1a), namely, Darcy’s law, \mathbf{q} is the velocity, p the pressure, k the permeability, μ the dynamic viscosity, and \mathbf{g} the gravity.

This Chapter is composed of two parts; in the first part, we study a density driven flow problem

by means of generalized finite volume method; in the second part, we study the coupling with heat transport.

In Section 1.2, we present the generalized finite volume method SUSHI, which permits to work with full diffusion matrices and to use non-matching volume elements. In Section 1.3, we present the numerical algorithm, which is based upon the SUSHI method and permits to ensure the mass conservation of both quantities $\rho(c)$ and $\rho(c)c$ in the case of homogeneous Neumann boundary conditions. In Section 1.4, we introduce adaptive meshes and show numerical results for three test cases, a rotating interface problem and Henry’s problem in space dimension two and a three dimensional saltpool problem. The first two test cases are studied by Hilhorst-Vu Do-Wang [37] by means of the standard finite volume method. We also compare the computational time and the number of unknowns (degrees of freedom) of the simulation using uniform meshes and adaptive meshes. These adaptive meshes permit equally accurate numerical computations with a smaller number of volume elements while they require a shorter computing time. In the last test case of Part I, we perform simulations for the saltpool problem, which is a well-known three-dimensional experiment and which was studied by [41] and [56]. We refer to [41] for a finite element discretization of this problem. We compare our numerical results to the experimental results as well as the results of [41] obtained by means of the finite element method. An advantage of the method which we present is that it is rather simple to implement while permitting to perfectly ensure the mass conservation property.

In Part II, we also take heat transfer into account. Heat can be recovered from groundwater. Depending on the application, the recovered heat can be used for heat production or for the production of electricity. The geothermal heating allows both to replace the conventional heating and to produce hot water. We study a coupled system describing the interaction between flow and transport in a porous medium, where the density and the viscosity depend on the concentration of the species to be transported, as well as on the temperature. More specifically, we solve a system of three nonlinear parabolic equations for the salt concentration C , the temperature Θ and the hydraulic head h .

A widely used commercial software for the simulation of a multi-species solute and heat transport is the SEAWAT code [48]. It combines the software codes MODFLOW and MT3DMS which permit to solve the flow and the solute-transport equations respectively. The coupling between flow and transport is performed through a synchronous time-stepping approach that cycles between MODFLOW solutions of the flow equation and MT3DMS solutions of the transport equation. Both MODFLOW and MT3DMS use cell-centered grids. In this formulation, the dependent variables obtained in the finite-difference solution represent average values, assumed to be given at the cell center, for the respective cells. Therefore SEAWAT also uses cell-centered grids. We propose a single code using the generalized finite volume method SUSHI. We show results of test cases at the end of the chapter. The volume elements, in space dimensions two or three, can be quite arbitrary and the structure is much simpler since we deal with a single code instead of two heavy codes which are coupled with each other.

We take a model problem proposed in the SEAWAT documentation [48] as a test case for our method. The problem consists of a two-dimensional cross section of a confined coastal aquifer initially saturated with relatively cold seawater. Warmer fresh water is injected into the coastal aquifer along the left boundary to represent the flow from inland areas. The warmer fresh water flows to the right, where it discharges into a vertical ocean boundary. Hydrostatic

conditions are imposed on the ocean boundary while no flow conditions are assigned to the top and bottom boundaries. This problem is a simplified representation of what may occur in a coastal carbonate platform. Our results are close to those of SEAWAT.

This Chapter is joint work with Danielle Hilhorst and Huy Cuong Vu Do.

PART I:

A generalized finite volume method for density driven flows in porous media

To begin with, we present the generalized finite volume method SUSHI in the case of a linear elliptic problem as it was done by Eymard-Gallouët-Herbin [29].

1.2 Generalized finite volume method SUSHI

We consider the linear elliptic equation

$$-\operatorname{div}(\mathbf{K}(\mathbf{x})\nabla u) = f \quad \text{in } \mathbb{D}, \quad (1.2.1)$$

together with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\mathbb{D},$$

where \mathbb{D} is a polyhedral open bounded connected subset of \mathbb{R}^d , for all $d \in \mathbb{N}^+$ and $\partial\mathbb{D} = \overline{\mathbb{D}} \setminus \mathbb{D}$ is the boundary of the domain \mathbb{D} . We assume that $f \in L^2(\mathbb{D})$ and that \mathbf{K} is a measurable function from \mathbb{D} to $\mathcal{M}_d(\mathbb{R})$, the set of $d \times d$ matrices, such that for a.e. $\mathbf{x} \in \mathbb{D}$, $\mathbf{K}(\mathbf{x})$ is symmetric and its eigenvalues is included in $[\bar{\lambda}, \underline{\lambda}]$ with $0 < \bar{\lambda} < \underline{\lambda}$.

1.2.1 Discretization

Definition 1.1 *Let \mathbb{D} be a polyhedral open bounded connected subset of \mathbb{R}^d and $\partial\mathbb{D} = \overline{\mathbb{D}} \setminus \mathbb{D}$ its boundary.*

- \mathcal{T} is a finite family of nonempty connected open disjoint subsets of \mathbb{D} (“control volumes”) such that $\overline{\mathbb{D}} = \cup_{K \in \mathcal{T}} \overline{K}$. For any $K \in \mathcal{T}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K ; let $|K| > 0$ denote the measure of K and let h_K denote the diameter of K .
- \mathcal{E} is a finite family of disjoint subsets of $\overline{\mathbb{D}}$ (edges of the mesh), such that for all $\sigma \in \mathcal{E}$, σ is a nonempty open subset of a hyperplane of \mathbb{R}^d , whose $d - 1$ -dimensional measure $|\sigma|$ is positive. We also assume that, for all $K \in \mathcal{T}$, there exists a subset \mathcal{E}_K of \mathcal{E} such that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. For all $\sigma \in \mathcal{E}$, we denote by $x\mathcal{T}_\sigma = \{K \in \mathcal{T}, \sigma \in \mathcal{E}_K\}$. We assume that for all $\sigma \in \mathcal{E}$, either \mathcal{T}_σ has exactly one element and then $\sigma \subset \partial\mathbb{D}$ (the set of these interfaces, called boundary interfaces, is denoted by \mathcal{E}_{ext}) or \mathcal{T}_σ has exactly two elements (the set of interfaces, called interior interfaces, is denoted by \mathcal{E}_{int}). For all $\sigma \in \mathcal{E}$, we denote by x_σ the barycenter of σ . For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K .

- \mathcal{P} is a family of points of \mathbb{D} indexed by \mathcal{T} , denoted by $\mathcal{P} = (\mathbf{x}_K)_{K \in \mathcal{T}}$, such that for all $K \in \mathcal{T}$, $\mathbf{x}_K \in K$ and K is assumed to be \mathbf{x}_K -star-shaped, which means that for all $\mathbf{x} \in K$, the inclusion $[\mathbf{x}_K, \mathbf{x}]$ holds. Denoting by $d_{K,\sigma}$ the Euclidean distance between \mathbf{x}_K and the hyperplane including σ , one assumes that $d_{K,\sigma} > 0$. We denote by $D_{K,\sigma}$ the cone with vertex \mathbf{x}_K and basis σ .

We integrate Equation (1.2.1) over the control volume $K \in \mathcal{T}$, which formally gives (assuming sufficient regularity on u and \mathbf{K}) the following balance equation on the control volume K :

$$\sum_{\sigma \in \mathcal{E}_K} \left(- \int_{\sigma} \mathbf{K}(\mathbf{x}) \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x}) \right) = \int_K f(\mathbf{x}) d\mathbf{x}.$$

The flux $-\int_{\sigma} \mathbf{K}(\mathbf{x}) \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x})$ is approximated by a function $F_{K,\sigma}(u)$ of the values $((u_K)_{K \in \mathcal{T}}, (u_{\sigma})_{\sigma \in \mathcal{E}})$ at the centers and at the interfaces of the control volumes. A discrete equation corresponding to (1.2.1) is given by:

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u) = \int_K f(\mathbf{x}) d\mathbf{x} \quad \text{for all } K \in \mathcal{T}. \quad (1.2.2)$$

The values u_{σ} on the interfaces are introduced in such a way that they provide a consistent approximation of the normal fluxes even in the case of an anisotropic operator and on a general, possibly nonconforming mesh. We have $\text{card}(\mathcal{E})$ extra unknowns, and need $\text{card}(\mathcal{E})$ equations to ensure that the problem is well posed. For the boundary faces or edges, these equations are obtained by writting the discrete counterpart of the boundary condition:

$$u_{\sigma} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{ext}.$$

For the interior edges, they are obtained by imposing the continuity of the discrete flux across the edges, namely:

$$F_{K,\sigma}(u) + F_{L,\sigma}(u) = 0, \quad (1.2.3)$$

for $\sigma \in \mathcal{E}_{int}$ such that $\mathcal{T}_{\sigma} = \{K, L\}$. We have then $\text{card}(\mathcal{T}) + \text{card}(\mathcal{E}_{int})$ unknowns and equations.

We first define the sets $X_{\mathcal{D}}$ and $X_{\mathcal{D},0}$

$$X_{\mathcal{D}} = \{v = ((v_K)_{K \in \mathcal{T}}, (v_{\sigma})_{\sigma \in \mathcal{E}}), v_K \in \mathbb{R}, v_{\sigma} \in \mathbb{R}\}, \quad (1.2.4)$$

$$X_{\mathcal{D},0} = \{v \in X_{\mathcal{D}} \text{ such that } v_{\sigma} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{ext}\}. \quad (1.2.5)$$

For all $v \in X_{\mathcal{D},0}$, multiplying Equation (1.2.2) by the value v_K of v on the control volume K and summing over $K \in \mathcal{T}$ leads to:

$$\sum_{K \in \mathcal{T}} \left(v_K \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u) \right) = \sum_{K \in \mathcal{T}} \left(v_K \int_K f(\mathbf{x}) d\mathbf{x} \right).$$

Also using (1.2.3), we deduce the following discrete weak formulation,

$$\begin{cases} \text{Find } u \in X_{\mathcal{D},0} \text{ such that :} \\ \langle u, v \rangle_F = \sum_{K \in \mathcal{T}} v_K \int_K f(\mathbf{x}) d\mathbf{x}, \quad \text{for all } v \in X_{\mathcal{D},0}, \end{cases}$$

with

$$\langle u, v \rangle_F = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma). \quad (1.2.6)$$

1.2.2 The stable discrete gradient

The discrete flux $F_{K,\sigma}$ is expressed in terms of the discrete unknowns. The idea is based upon the identification of the numerical fluxes $F_{K,\sigma}(u)$ through the mesh dependent bilinear form $\langle \cdot, \cdot \rangle_F$, using the expression of the discrete gradient. Assuming that, for all $u \in X_{\mathcal{D}}$, a discrete gradient $\nabla_{\mathcal{D}}u$ has been constructed, we seek a family $(F_{K,\sigma}(u))_{K \in \mathcal{T}, \sigma \in \mathcal{E}_K}$ such that

$$\langle u, v \rangle_F = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u)(v_K - v_\sigma) = \int_{\mathbb{D}} \nabla_{\mathcal{D}}u(\mathbf{x}) \cdot \mathbf{K}(\mathbf{x}) \nabla_{\mathcal{D}}v(\mathbf{x}) \, d\mathbf{x} \quad \forall u, v \in X_{\mathcal{D},0}. \quad (1.2.7)$$

To this purpose, we first define

$$\nabla_K u = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (u_\sigma - u_K) \mathbf{n}_{K,\sigma} \quad \forall K \in \mathcal{T}, \forall u \in X_{\mathcal{D}}. \quad (1.2.8)$$

The consistency of formula (1.2.8) follows from the geometrical relation:

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_{K,\sigma} (\mathbf{x}_\sigma - \mathbf{x}_K)^T = |K| \mathbf{Id} \quad \forall K \in \mathcal{T}, \quad (1.2.9)$$

where $(\mathbf{x}_\sigma - \mathbf{x}_K)^T$ is the transpose of $\mathbf{x}_\sigma - \mathbf{x}_K \in \mathbb{R}^d$ and \mathbf{Id} is the $d \times d$ identity matrix.

The approximation formula (1.2.8) is exact for linear functions. Indeed, for any linear function defined on \mathbb{D} by $\varphi(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x}$ with $\mathbf{G} \in \mathbb{R}^d$, assuming that $u_\sigma = \varphi(\mathbf{x}_\sigma)$ and $u_K = \varphi(\mathbf{x}_K)$, we obtain $u_\sigma - u_K = (\mathbf{x}_\sigma - \mathbf{x}_K)^T \mathbf{G} = (\mathbf{x}_\sigma - \mathbf{x}_K)^T \nabla \varphi$; hence (1.2.8) leads to $\nabla_K u = \nabla \varphi$. We also remark that

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_{K,\sigma} = \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \mathbf{n}_{K,\sigma} \, d\gamma = \int_K (\nabla 1) \, d\mathbf{x} = 0,$$

which means that the coefficient of u_K in (1.2.8) is equal to zero. Thus, a reconstruction of the discrete gradient based only on (1.2.8) cannot lead to a definite discrete bilinear form in the general case. Therefore we introduce the stabilized gradient

$$\nabla_{K,\sigma} u = \nabla_K u + R_{K,\sigma} u \mathbf{n}_{K,\sigma}, \quad (1.2.10)$$

where

$$R_{K,\sigma} u = \frac{\sqrt{d}}{d_{K,\sigma}} \left(u_\sigma - u_K - \nabla_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) \right). \quad (1.2.11)$$

We may then define $\nabla_{\mathcal{D}}u$ as the piecewise constant function equal to $\nabla_{K,\sigma} u$ a.e. in the cone $D_{K,\sigma}$

$$\nabla_{\mathcal{D}}u(\mathbf{x}) = \nabla_{K,\sigma} u \quad \text{for a.e. } \mathbf{x} \in D_{K,\sigma}. \quad (1.2.12)$$

Note that, from the definition (1.2.11), in view of (1.2.9) and (1.2.8), we deduce that

$$\sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma| d_{K,\sigma}}{d} R_{K,\sigma} u \mathbf{n}_{K,\sigma} = 0 \quad \text{for all } K \in \mathcal{T}. \quad (1.2.13)$$

In order to identify the numerical fluxes $F_{K,\sigma}(u)$ through relation (1.2.7), we put the discrete gradient in the form

$$\nabla_{\mathcal{D}}u = \sum_{\sigma' \in \mathcal{E}_K} (u_{\sigma'} - u_K) \mathbf{y}_K^{\sigma\sigma'}, \quad (1.2.14)$$

with

$$\mathbf{y}_K^{\sigma\sigma'} = \begin{cases} \frac{|\sigma|}{|K|} \mathbf{n}_{K,\sigma} + \frac{\sqrt{d}}{d_{K,\sigma}} \left(1 - \frac{|\sigma|}{|K|} \mathbf{n}_{K,\sigma} \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) \right) \mathbf{n}_{K,\sigma} & \text{if } \sigma = \sigma', \\ \frac{|\sigma'|}{|K|} \mathbf{n}_{K,\sigma'} - \frac{\sqrt{d}}{d_{K,\sigma}} \frac{|\sigma'|}{|K|} \mathbf{n}_{K,\sigma'} \cdot (\mathbf{x}_\sigma - \mathbf{x}_K) \mathbf{n}_{K,\sigma} & \text{otherwise.} \end{cases}$$

Thus

$$\int_K \nabla_{\mathcal{D}}u(\mathbf{x}) \cdot \mathbf{K}(x) \nabla_{\mathcal{D}}v(\mathbf{x}) d\mathbf{x} = \sum_{\sigma \in \mathcal{E}_K} \sum_{\sigma' \in \mathcal{E}_K} A_K^{\sigma\sigma'} (u_\sigma - u_K) (v_{\sigma'} - v_K^n),$$

with $\sigma, \sigma' \in \mathcal{E}_K$ and

$$A_K^{\sigma\sigma'} = \sum_{\sigma'' \in \mathcal{E}_K} \mathbf{y}_K^{\sigma''\sigma} \cdot \mathbf{K}_K^{\sigma''} \mathbf{y}_K^{\sigma''\sigma'}, \quad \mathbf{K}_K^{\sigma''} = \int_{D_{K,\sigma''}} \mathbf{K}(x) d\mathbf{x}.$$

The local matrices $A_K^{\sigma\sigma'}$ are symmetric and positive, and the identification of the numerical fluxes using relation (1.2.7) leads to the expression:

$$F_{K,\sigma}(u) = \sum_{\sigma' \in \mathcal{E}_K} A_K^{\sigma\sigma'} (u_K - u_{\sigma'}). \quad (1.2.15)$$

1.3 Application of the SUSHI method to the density driven flow problem

1.3.1 Approximate fluxes

Mathematically, we propose to solve the system of partial differential equations (1.1.1), together with the boundary conditions

$$\begin{cases} c = c_D(\mathbf{x}, t) & \text{on } \partial\mathcal{D}_D^c \times (0, T), \\ \frac{\partial c}{\partial n} = \bar{c}_N(\mathbf{x}, t) & \text{on } \partial\mathcal{D}_N^c \times (0, T), \\ p = p_D(\mathbf{x}, t) & \text{on } \partial\mathcal{D}_D^p \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N(\mathbf{x}, t) & \text{on } \partial\mathcal{D}_N^p \times (0, T), \end{cases} \quad (1.3.1)$$

with $\partial\mathcal{D} = \overline{\partial\mathcal{D}_D^c} \cup \overline{\partial\mathcal{D}_N^c} = \overline{\partial\mathcal{D}_D^p} \cup \overline{\partial\mathcal{D}_N^p}$ where $\partial\mathcal{D}_D^c$ and $\partial\mathcal{D}_D^p$ correspond to Dirichlet boundary conditions and $\partial\mathcal{D}_N^c$ and $\partial\mathcal{D}_N^p$ correspond to Neumann boundary condition for the concentration and the pressure respectively. The initial condition is given by

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \quad \text{in } \mathcal{D}. \quad (1.3.2)$$

Our purpose is to compute the pressure p and the concentration c . Rather than discretizing the term $\nabla \cdot (\rho(c) \mathbf{D} \nabla c)$, we propose to discretize the equivalent form $\nabla \cdot (\mathbf{D} \nabla \varphi(u))$, where

$u = \rho(c)c$ and φ is defined as follows : since ρ is strictly increasing in c , the function $u \rightarrow c$ is invertible, so that we can rewrite the concentration as a function of u , namely $c = r(u)$ and $\rho(c) = \rho(r(u)) = R(u)$.

We set $\theta(c) = \int_0^c \rho(s) ds$, which implies that $\nabla \cdot (\mathbf{D}\nabla\theta(c)) = \nabla \cdot (\mathbf{D}\rho(c)\nabla c)$, and we define

$$\varphi(u) = \theta(c) = \theta(r(u)). \quad (1.3.3)$$

Then the system takes the form:

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - R(u)\mathbf{g}) & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial R(u)}{\partial t} + \nabla \cdot (\mathbf{q}R(u)) = Q_\rho & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{q}u) - \nabla \cdot (\mathbf{D}\nabla\varphi(u)) = Q & \text{in } \mathbb{D} \times (0, T), \end{cases} \quad (1.3.4)$$

where the unknown functions are p and u . This form has been studied by [37] who applied the standard finite volume method for its numerical solution. In order to apply the SUSHI scheme, we rewrite the system (1.3.4) in the form

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \varrho(w)\mathbf{g}) & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial \varrho(w)}{\partial t} + \nabla \cdot (\mathbf{q}\varrho(w)) = Q_\rho & \text{in } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial m(w)}{\partial t} + \nabla \cdot (\mathbf{q}m(w)) - \nabla \cdot (\mathbf{D}\nabla w) = Q & \text{in } \mathbb{D} \times (0, T), \end{cases} \quad (1.3.5)$$

where $w = \varphi(u)$, $m = \varphi^{-1}$ and $\varrho = R(m)$. The main reason for this transformation is to apply the SUSHI scheme to a linear laplacian operator and then be able to handle a standard bilinear form. We remark that $m(w)$ and $\varrho(w)$ coincide with the mass function $c\rho(c)$ and the density function $\rho(c)$ respectively. In what follows, we discretize system (1.3.5) together with the following boundary conditions

$$\begin{cases} w = \theta(c_D(\mathbf{x}, t)) & \text{on } \partial\mathbb{D}_D^c \times (0, T), \\ \frac{\partial w}{\partial n} = \theta'(w(\mathbf{x}, t))\bar{c}_N(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_N^c \times (0, T), \\ p = p_D(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_D^p \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N(\mathbf{x}, t) & \text{on } \partial\mathbb{D}_N^p \times (0, T), \end{cases} \quad (1.3.6)$$

and the initial condition

$$w(\mathbf{x}, 0) = \theta(c_0(\mathbf{x})) \quad \text{on } \mathbb{D}. \quad (1.3.7)$$

In this Chapter, we suppose that Φ is a positive constant. We use the space discretization described in Definition 1.1 and, for the time discretization, we fix a positive integer N and define the time step as $\delta t = \frac{T}{N}$ and $t^n = n \cdot \delta t$ for all $n \in \{0, 1, \dots, N\}$.

After formally integrating the two last equations of (1.3.5) on the domain $K \times (t_{n-1}, t_n)$ for all $K \in \mathcal{T}$ and $n = 1, \dots, N$, we obtain

$$\begin{aligned} & \Phi \int_K \left(\varrho(w(\mathbf{x}, t_n)) - \varrho(w(\mathbf{x}, t_{n-1})) \right) d\mathbf{x} \\ & \quad + \sum_{\sigma \in \mathcal{E}_K} \int_{t_{n-1}}^{t_n} \int_{\sigma} \mathbf{q} \varrho(w) \cdot \mathbf{n}_{K,\sigma} d\gamma dt = \int_{t_{n-1}}^{t_n} \int_K Q_\rho d\mathbf{x} dt, \\ & \Phi \int_K \{m(w(\mathbf{x}, t_n)) - m(w(\mathbf{x}, t_{n-1}))\} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_K} \int_{t_{n-1}}^{t_n} \int_{\sigma} \mathbf{q} m(w) \cdot \mathbf{n}_{K,\sigma} d\gamma dt \\ & \quad - \sum_{\sigma \in \mathcal{E}_K} \int_{t_{n-1}}^{t_n} \int_{\sigma} \mathbf{D} \nabla w \cdot \mathbf{n}_{K,\sigma} d\gamma dt = \int_{t_{n-1}}^{t_n} \int_K Q d\mathbf{x} dt. \end{aligned} \tag{1.3.8}$$

In what follows we define numerical fluxes $F_{K,\sigma}^D(w)$ and $F_{K,\sigma}^C(p, w)$ in order to approximate the diffusion flux $\int_{\sigma} -\mathbf{D} \nabla w \cdot \mathbf{n}_{K,\sigma} d\gamma$ and the convection flux $\int_{\sigma} \mathbf{q} \cdot \mathbf{n}_{K,\sigma} d\gamma$ respectively. We express the discrete fluxes $F_{K,\sigma}^D(w)$ and $F_{K,\sigma}^C(p, w)$ in terms of the discrete unknowns.

We recall below formulas which have been derived by [29]. We define

$$F_{K,\sigma}^D(w) = \sum_{\sigma' \in \mathcal{E}_K} A_K^{\sigma\sigma'} (w_K - w_{\sigma'}), \tag{1.3.9}$$

for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$. Next we approximate the convection flux in (1.3.8). To that purpose we apply an idea which has been used in [37]. We set

$$F_{K,\sigma}^C(p, w) = \frac{k}{\mu} \left(\sum_{\sigma' \in \mathcal{E}_K} \overline{A_K^{\sigma\sigma'}} (p_K - p_{\sigma'}) + |\sigma| \mathbf{g} \cdot \mathbf{n}_{K,\sigma} \cdot \varrho(w_\sigma) \right), \tag{1.3.10}$$

where

$$\overline{A_K^{\sigma\sigma'}} = \sum_{\sigma'' \in \mathcal{E}_K} \mathbf{y}^{\sigma''\sigma} \cdot \left(\int_{D_{K,\sigma''}} \mathbf{I} d\mathbf{x} \right) \mathbf{y}^{\sigma''\sigma'}, \tag{1.3.11}$$

with \mathbf{I} denotes the unit tensor.

1.3.2 Numerical scheme

We present below a semi-implicit finite volume scheme corresponding to the system (1.3.5):

The initial condition for the scheme is given by

$$w_K^0 = \frac{1}{|K|} \int_K \theta(c_0(\mathbf{x})) d\mathbf{x}, \quad w_\sigma^0 = \frac{1}{|\sigma|} \int_{\sigma} \theta(c_0(\mathbf{x})) d\gamma. \tag{1.3.12}$$

We define $\tilde{w}_{K,\sigma}$ according to the upwind scheme

$$\tilde{w}_{K,\sigma} = \begin{cases} w_K & \text{if } F_{K,\sigma}^C(p, w) > 0. \\ w_\sigma & \text{otherwise.} \end{cases}$$

For all $n \in \{1, \dots, N\}$, we propose the following scheme:

Knowing w^{n-1} and w^{n-2} , find $p^n \in X_D$ such that

$$\left\{ \begin{array}{ll} \Phi|K| \left(\varrho(w_K^{n-1}) - \varrho(w_K^{n-2}) \right) \\ \quad + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^C(p^n, w^{n-1}) \cdot \varrho(w_\sigma^{n-1}) = (Q_\rho)_K^n & \text{for all } K \in \mathcal{T}, \quad (1.3.13a) \\ \sum_{K \in \mathcal{T}_\sigma} F_{K,\sigma}^C(p^n, w^{n-1}) \cdot \varrho(w_\sigma^{n-1}) = 0 & \text{for all } \sigma \in \mathcal{E}_{int}, \quad (1.3.13b) \\ p_\sigma^n = p_D(\mathbf{x}_\sigma, t_n) & \text{for all } \mathbf{x}_\sigma \in \partial\mathbb{D}_D^p, \quad (1.3.13c) \\ F_{K,\sigma}^C(p^n, w^{n-1}) = |\sigma| \bar{q}_N(\mathbf{x}_\sigma, t_n) & \text{for all } \mathbf{x}_\sigma \in \partial\mathbb{D}_N^p. \quad (1.3.13d) \end{array} \right.$$

Knowing p^n and w^{n-1} , find $w^n \in X_D$ such that

$$\left\{ \begin{array}{ll} \Phi|K| \left(m(w_K^n) - m(w_K^{n-1}) \right) + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^D(w^n) \\ \quad + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^C(p^n, w^{n-1}) \cdot m(\tilde{w}_{K,\sigma}^n) = Q_K^n & \text{for all } K \in \mathcal{T}, \quad (1.3.14a) \\ \sum_{K \in \mathcal{T}_\sigma} \left\{ F_{K,\sigma}^D(w^n) + F_{K,\sigma}^C(p^n, w^{n-1}) \cdot m(\tilde{w}_{K,\sigma}^n) \right\} = 0 & \text{for all } \sigma \in \mathcal{E}_{int}, \quad (1.3.14b) \\ w_\sigma = \theta \left(c_D(\mathbf{x}_\sigma, t_n) \right) & \text{for all } \mathbf{x}_\sigma \in \partial\mathbb{D}_D^c, \quad (1.3.14c) \\ F_{K,\sigma}^D(w^n) = -D|\sigma| \theta' \left(\theta^{-1}(w_\sigma^{n-1}) \right) \bar{c}_N(\mathbf{x}_\sigma, t_n) & \text{for all } \mathbf{x}_\sigma \in \partial\mathbb{D}_N^c. \quad (1.3.14d) \end{array} \right.$$

where $(Q_\rho)_K^n = \int_{t^{n-1}}^{t^n} \int_K Q_\rho d\mathbf{x} dt$ and $Q_K^n = \int_{t^{n-1}}^{t^n} \int_K Q d\mathbf{x} dt$. The approximate fluxes $F_{K,\sigma}^D$ and $F_{K,\sigma}^C$ are defined by (1.3.9) and (1.3.10), respectively.

1.3.3 Mass conservation

In this section we discuss the conservation of mass property in the case that the concentration c and flux \mathbf{q} satisfy the boundary conditions:

$$\left\{ \begin{array}{ll} \frac{\partial c}{\partial n} = \bar{c}_N = 0 & \text{on } \partial\mathbb{D} \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N = 0 & \text{on } \partial\mathbb{D} \times (0, T), \end{array} \right.$$

and that the source terms Q and Q_ρ are equal to zero, first from the analytical viewpoint and then from the numerical algorithm's viewpoint. For all $t \in (0, T)$, we integrate the transport equation (1.1.1c) on $\mathbb{D} \times (0, t)$:

$$\begin{aligned} & \int_{\mathbb{D}} \Phi \left(\rho(c(\mathbf{x}, t)) c(\mathbf{x}, t) - \rho(c_0(\mathbf{x})) c_0(\mathbf{x}) \right) d\mathbf{x} \\ & + \int_0^t \int_{\mathbb{D}} \nabla \cdot (\mathbf{q} \rho(c(\mathbf{x}, \tau)) c(\mathbf{x}, \tau)) d\mathbf{x} d\tau - \int_0^t \int_{\mathbb{D}} \nabla \cdot (\mathbf{D} \rho(c) \nabla(c(\mathbf{x}, \tau))) d\mathbf{x} d\tau = 0, \end{aligned}$$

which yields

$$\begin{aligned} & \int_{\mathbb{D}} \Phi \rho(c(\mathbf{x}, t)) c(\mathbf{x}, t) \, d\mathbf{x} - \int_{\mathbb{D}} \Phi \rho(c_0(\mathbf{x})) c_0(\mathbf{x}) \, d\mathbf{x} \\ &= - \int_0^t \int_{\partial \mathbb{D}} \rho(c(\mathbf{x}, \tau)) c(\mathbf{x}, \tau) \mathbf{q} \cdot \mathbf{n} \, d\gamma d\tau + \int_0^t \int_{\partial \mathbb{D}} \mathbf{D} \rho(c) \nabla c(\mathbf{x}, \tau) \cdot \mathbf{n} \, d\gamma d\tau. \end{aligned} \quad (1.3.15)$$

In this case we deduce from (1.3.15) that

$$\int_{\mathbb{D}} \rho(c(\mathbf{x}, t)) c(\mathbf{x}, t) \, d\mathbf{x} = \int_{\mathbb{D}} \rho(c_0(\mathbf{x})) c_0(\mathbf{x}) \, d\mathbf{x} \quad \text{for all } t > 0,$$

or else

$$\int_{\mathbb{D}} m(w(\mathbf{x}, t)) \, d\mathbf{x} = \int_{\mathbb{D}} m(w_0(\mathbf{x})) \, d\mathbf{x} \quad \text{for all } t > 0.$$

It follows in a similar way from the equation (1.1.1b) that

$$\int_{\mathbb{D}} \rho(c(\mathbf{x}, t)) \, d\mathbf{x} = \int_{\mathbb{D}} \rho(c_0(\mathbf{x})) \, d\mathbf{x} \quad \text{for all } t > 0,$$

or else

$$\int_{\mathbb{D}} \varrho(w(\mathbf{x}, t)) \, d\mathbf{x} = \int_{\mathbb{D}} \varrho(w_0(\mathbf{x})) \, d\mathbf{x} \quad \text{for all } t > 0.$$

Next we consider the numerical algorithm. We add up the semi-implicit discrete equations (1.3.14b) on all control volumes, and set $Q = 0$ to obtain:

$$\sum_{K \in \mathcal{T}} \Phi |K| (m(w_K^n) - m(w_K^{n-1})) + \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \left(F_{K,\sigma}^D(w^n) + F_{K,\sigma}^C(p^n, w^{n-1}) m(\tilde{w}_{K,\sigma}^n) \right) = 0.$$

We define the total mass at the time $t_n = n\delta t$ by $\sum_{K \in \mathcal{T}} |K| m(w_K^n)$. It is obvious that

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \left(F_{K,\sigma}^D(w^n) + F_{K,\sigma}^C(p^n, w^{n-1}) m(\tilde{w}_{K,\sigma}^n) \right) \\ &= \sum_{\sigma \in \mathcal{E}_{int}} \sum_{K \in \mathcal{T}_\sigma} \left(F_{K,\sigma}^D(w^n) + F_{K,\sigma}^C(p^n, w^{n-1}) m(\tilde{w}_{K,\sigma}^n) \right) \\ &+ \sum_{\sigma \in \mathcal{E}_{ext}} \left(F_{K,\sigma}^D(w^n) + F_{K,\sigma}^C(p^n, w^{n-1}) m(\tilde{w}_{K,\sigma}^n) \right). \end{aligned} \quad (1.3.16)$$

Using the local conservation of the discrete fluxes on the interior edges, the first term of the right-hand side in (1.3.16) vanishes. Because of the homogeneous Neumann boundary condition and by the definition of the approximate fluxes $F_{K,\sigma}^C(p^n, w^{n-1})$ and $F_{K,\sigma}^D(w^n)$ at the Neumann boundary (1.3.13d) and (1.3.14d) respectively, we obtain

$$\begin{cases} F_{K,\sigma}^C(p^n, w^{n-1}) = |\sigma| \bar{q}_N(\mathbf{x}_\sigma, t_n) = 0 & \text{on } \partial \mathbb{D} \times (0, T), \\ F_{K,\sigma}^D(w^n) = -D |\sigma| \theta'(\theta^{-1}(w_\sigma^{n-1})) \bar{c}_N(\mathbf{x}_\sigma, t_n) = 0 & \text{on } \partial \mathbb{D} \times (0, T). \end{cases}$$

As a result, the second term on the right-hand side of (1.3.16) vanishes. Therefore we deduce that:

$$\sum_{K \in \mathcal{T}} |K| m(w_K^n) = \sum_{K \in \mathcal{T}} |K| m(w_K^0) \quad \text{for all } n \in \{1, \dots, N\}, \quad (1.3.17)$$

which expresses the fact that our algorithm conserves the mass. Similarly one can show that

$$\sum_{K \in \mathcal{T}} |K| \varrho(w_K^n) = \sum_{K \in \mathcal{T}} |K| \varrho(w_K^0) \quad \text{for all } n \in \{1, \dots, N\}. \quad (1.3.18)$$

We present the mass conservation properties in the numerical results of rotating interface problem.

1.4 Numerical tests

An essential feature of the SUSHI scheme is that it allows to use non-matching volume elements. Therefore, we can apply an adaptive mesh with squares or cubes of various sizes in the numerical tests. The unknown functions strongly vary in part of the domain whereas they are nearly constant elsewhere. We refine the mesh in regions of strong variations of the unknown functions while we merge elements in areas where they only undergo small variations. This permits to reduce the number of unknowns (which depends on the number of elements and edges) and helps the process run faster. After each refinement or derefinement, we have to deal with new data for the mesh and the unknown functions.

To this purpose, we define the maximum of the discrete difference of unknowns at the cell center and at its edges: $U_K = \max_{\sigma \in \mathcal{E}_K} |u_K - u_\sigma|$ where u is an arbitrary unknown. Then we sort all the elements in order of increasing U_K . The element L is added to a list called “refined-list” if

$$\frac{U_L - \min_{K \in \mathcal{T}} U_K}{\max_{K \in \mathcal{T}} U_K - \min_{K \in \mathcal{T}} U_K} > \alpha, \quad (1.4.1)$$

where $0 < \alpha < 1$. It means that the “refined list” includes the elements whose unknown undergoes the highest of discrete difference. The scalar α controls the amount of refined elements (the length of the refined-list). If $\alpha = 0.75$, around one fourth of the elements of the current mesh will be refined. If α is close to 1, the refined-list is very short. If the element L satisfies

$$\frac{U_L - \min_{K \in \mathcal{T}} U_K}{\max_{K \in \mathcal{T}} U_K - \min_{K \in \mathcal{T}} U_K} < \beta, \quad (1.4.2)$$

where $0 < \beta < \alpha$, it is added to the “coarse-list”. In a similar way, the scalar β controls the length of the coarse-list. If β tends to 0, the coarse-list becomes empty. Figure 1.2 demonstrates our statements. Moreover we wish to make the difference of diameter of neighboring volume elements not too high. This leads us to also refine some elements closed to the ones in the refined-list and derefine elements neighboring ones of the coarse-list.

After each time step, we compute a new coarse-list and a new refined-list. If an element belongs to the refined-list, it is divided into smaller ones (4 elements in the two-dimensional square mesh or 8 elements in the three-dimensional cubic mesh). If some neighbor elements belong to the coarse-list, they are merged together.

We use the constant interpolation and linear interpolation to assign values to the new unknowns after each refinement. In general, the linear interpolation is used very often because it gives a more accurate result. However the solution maybe does not satisfy the mass conservation law. In that case, we use the constant interpolation or we have to modify the linear interpolation.

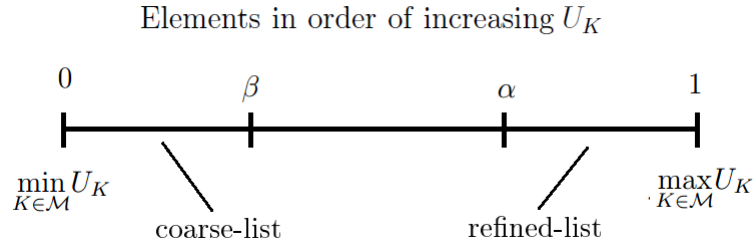


Figure 1.1: The refined-list and the coarse-list.

1.4.1 The rotating interface problem

We now consider the case of a rotating interface between fresh and salt water. The salt water, which is heavier than the fresh water, has a tendency to flow to the bottom, while the fresh water flows to the top. At the initial time, the space domain is divided into two rectangles as shown in Figure 1.3. The left rectangle is filled with salt water while the right one is filled with fresh water. The interface between these two liquids rotates counterclockwise until it becomes horizontal. The simulations described below permit to follow the rotation of the interface.

The rotating interface problem is defined with $Q_\rho = Q = 0$ and c and \mathbf{q} satisfying the boundary conditions:

$$\begin{cases} \frac{\partial c}{\partial n} = 0 & \text{on } \partial\mathbb{D} \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{D} \times (0, T). \end{cases} \quad (1.4.3)$$

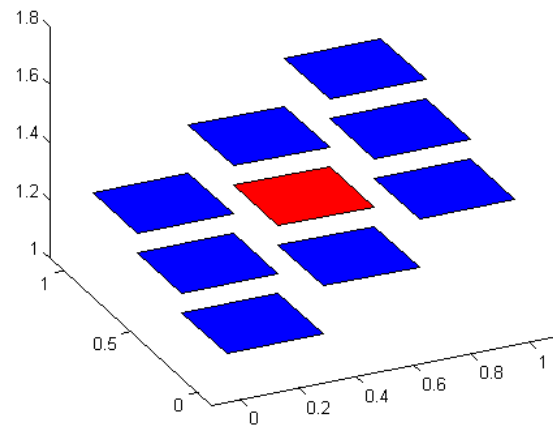
A technical problem is that if p is a solution, also $p + \bar{p}$ is a solution, with \bar{p} an arbitrary

Symbol	Value	Unit
k	3.10×10^{-12}	m^2
D	3.3×10^{-6}	$m^2.s^{-1}$
\mathbf{g}	9.81	$m.s^{-2}$
Φ	0.5	—
μ	10^{-3}	$Pa.s$
ρ_0	10^3	$kg.m^{-3}$
α	0.3	—
ε	10^{-5}	—

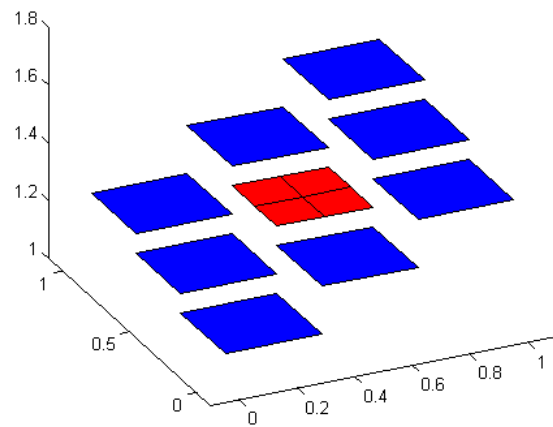
Table 1.1: Parameter values for the rotating interface problem.

constant. This leads us to transform the elliptic equation for p into a parabolic equation by adding the extra term $\varepsilon \frac{\partial p(\mathbf{x}, t)}{\partial t}$ to the pressure equation, which makes the model slightly compressible. In literature, this is referred to as a pseudo-compressibility approximation (cf. [16]). The second equation in system (1.3.5) becomes

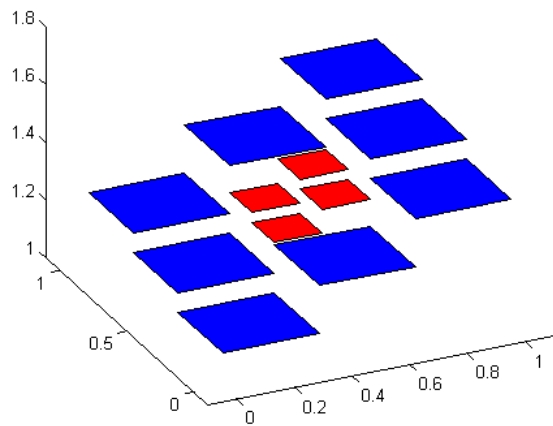
$$\varepsilon \frac{\partial p}{\partial t} + \Phi \frac{\partial \varrho(w)}{\partial t} + \nabla \cdot (\mathbf{q}\varrho(w)) = 0 \quad \text{in } \Omega \times (0, T). \quad (1.4.4)$$



(a) The original mesh



(b) The constant interpolation



(c) The linear interpolation

Figure 1.2: The constant and linear interpolations

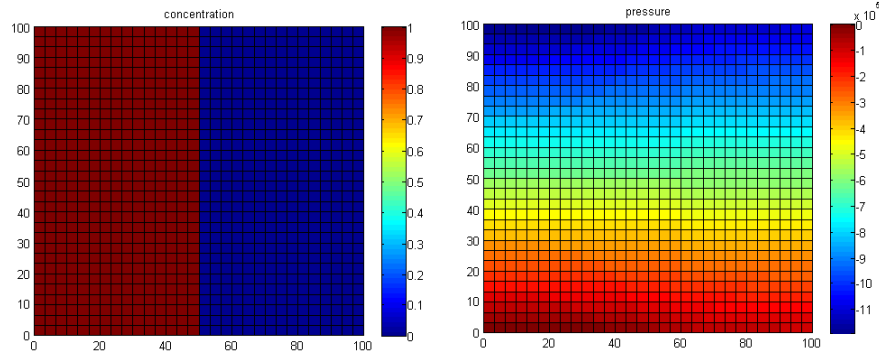


Figure 1.3: Initial conditions for concentration and pressure.

Therefore, the semi-implicit finite volume scheme (1.3.13a) for this new system is given by:

$$\begin{aligned} \varepsilon|K|(p^n - p^{n-1}) + \Phi|K|(\varrho(w_K^{n-1}) - \varrho(w_K^{n-2})) \\ + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^C(p^n, w^{n-1}) \cdot \varrho(w_\sigma^{n-1}) = 0 \quad \text{for all } K \in \mathcal{T}. \end{aligned}$$

We take as space domain the domain $\mathbb{D} = (0, 100) \times (0, 100)$ and consider the time interval $(0, T)$ with $T = 500$ (days). We use 21×21 square control volumes. The values of the parameters are given in Table 1.1 and the time evolution of the densities is plotted in Figure 1.4. In this test case, the dispersion-diffusion tensor is defined by $\mathbf{D} = D\mathbf{I}$.

The initial concentration is given by $c(\mathbf{x}, 0) = 0$ if \mathbf{x} belongs the right half domain and $c(\mathbf{x}, 0) = 1$ otherwise. The pressure p is such that $p(\mathbf{x}, 0) = 0$ at $\mathbf{x}_0 = (0, 0)$ and satisfies $\nabla \cdot (\mathbf{q}\rho(c)) = 0$ for all $\mathbf{x} \in \mathbb{D} \setminus \mathbf{x}_0$. Figure 1.3 presents the functions $c(\mathbf{x}, 0)$ and $p(\mathbf{x}, 0)$.

The salt water diffuses from left to right. Under the influence of gravity, the heavier fluid (the salt water) goes down while the light fluid (fresh water or slightly salted water) goes up. Therefore, the interface slowly moves from a vertical to a horizontal line. After a long time (about 500 days), the fluid is in balance and the motion stops.

Because of the influence of gravity, the salt water, which is heavier, diffuses towards the bottom; the pure water, which is lighter, diffuses towards the top. Therefore, the interface slowly evolves from a vertical to a horizontal line. After about 300 days, the fluid is in equilibrium and the motion stops. One can observe the velocity field in Figure 1.5 and the time evolution of the concentration and of the rotating interface in Figure 1.4. The refined elements are located in the neighborhood of the interface, which they follow at time evolves.

Note that because of the homogeneous Neumann boundary condition (1.4.3), the conservation of mass property holds, as it has been discussed in Section 1.3.3. We can check in Table 1.2 that both the standard finite volume scheme and the SUSHI scheme permit to conserve mass [61].

In the simulation, we start with a very coarse mesh of 5×5 square elements and take $\delta t = 0.5$ day as the time step. A new mesh is generated after each time step. At the beginning, because

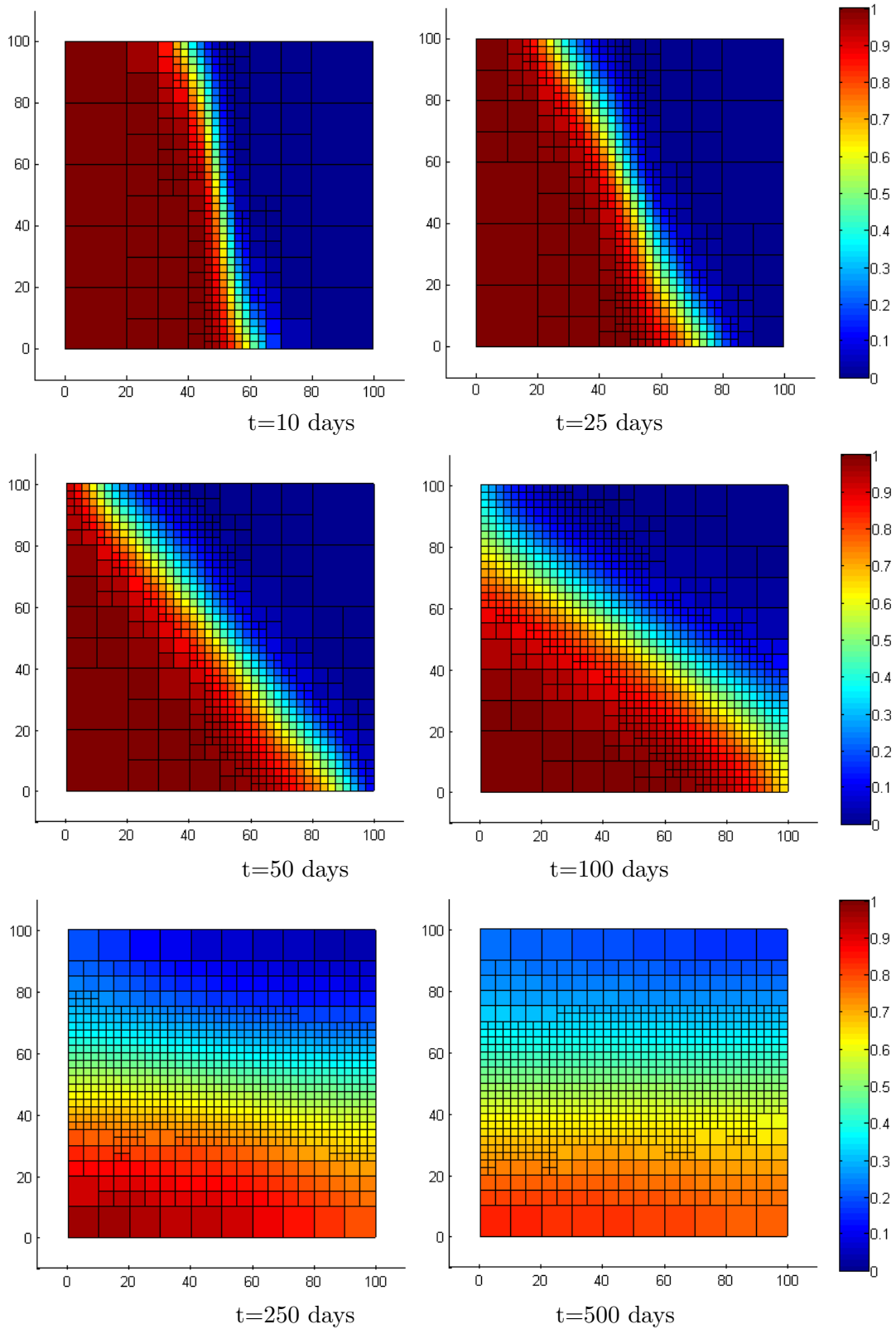


Figure 1.4: Time evolution of the rotating interface problem using an adaptive mesh.

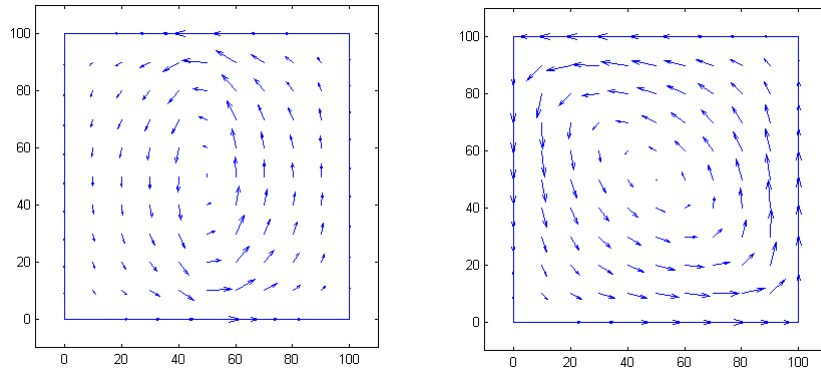


Figure 1.5: The velocity field in the interface rotating problem at $t = 0$ and $t = 200$ days.

time	SUSHI	SFVM
t=0	6.5×10^6	6.5×10^6
t=10	$6.500000000000002 \times 10^6$	$6.499999999999998 \times 10^6$
t=20	$6.500000000000004 \times 10^6$	$6.499999999999996 \times 10^6$
t=30	$6.499999999999998 \times 10^6$	$6.500000000000002 \times 10^6$
t=50	$6.500000000000006 \times 10^6$	$6.499999999999998 \times 10^6$
t=100	$6.499999999999990 \times 10^6$	$6.499999999999986 \times 10^6$
t=200	$6.499999999999996 \times 10^6$	$6.499999999999991 \times 10^6$
t=300	$6.499999999999975 \times 10^6$	$6.499999999999992 \times 10^6$
t=500	$6.499999999999961 \times 10^6$	$6.499999999999963 \times 10^6$

Table 1.2: Total mass with the standard finite volume method and with the generalized finite volume method SUSHI

the mesh is coarse, we choose $\alpha = 0.75$ to create more refined elements. Then we choose $\beta = 0.2$ to make a balance between refinement and derefinement. At $t = 50$ days, the number of elements is large enough and we choose $\alpha = 0.85$ in order to reduce the rate of the refinement. We also choose $\beta = 0.4$ to increase the rate of the derefinement. From $t = 250$ to the end, the evolution is slow, so that we choose $\alpha = 0.95$. Table 1.3 shows the number of elements of the adaptive mesh at various times. At the beginning, the adaptive mesh has a small number of elements, which increases in time. In Figure 1.6 we compare the CPU times with different meshes, namely three uniform meshes with respectively 100, 400 and 1600 volume elements and an adaptive mesh.

Time	NoE	NoU	Time	NoE	NoU
t=0 days	120	420			
t=10 days	355	1123	t=100 days	934	2882
t=20 days	499	1567	t=200 days	988	3038
t=30 days	583	1825	t=300 days	964	2966
t=50 days	597	2171	t=500 days	1024	3146

Table 1.3: The number of volume elements (NoE) and number of unknowns (NoU) in the rotating interface problem using an adaptive mesh.

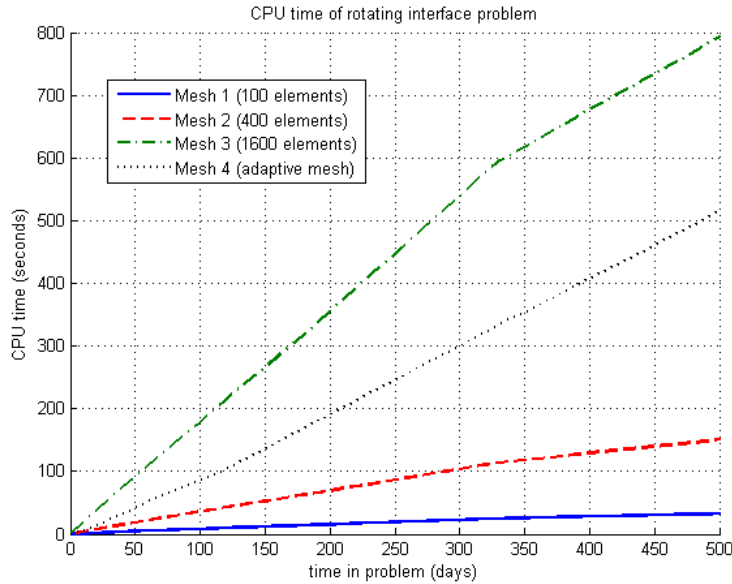


Figure 1.6: CPU time for the rotating interface using an adaptive mesh.

1.4.2 Henry's problem

Henry's problem describes the advance of a salt water front in a confined aquifer which is initially charged with fresh water. Henry developed a solution method to compute the steady-state distribution of the solute. To this end, he applied the Boussinesq approximation, which involves a stream function. Henry derived analytical expressions for the stream function and the concentration in the form of a Fourier series, the resulting algebraic equations for the determination of the Fourier coefficients must be obtained numerically. The 'mystery' of Henry's problem is that no numerical model so far has been able to closely reproduce his semi-analytical solution. Nevertheless, because of the absence of other non numerical solution for this kind of nonlinear problems, Henry's solution has become one of the standard tests for variable density groundwater models.

Mathematically, Henry's problem is defined as System (1.1.1) in the rectangle \mathbb{D} together with

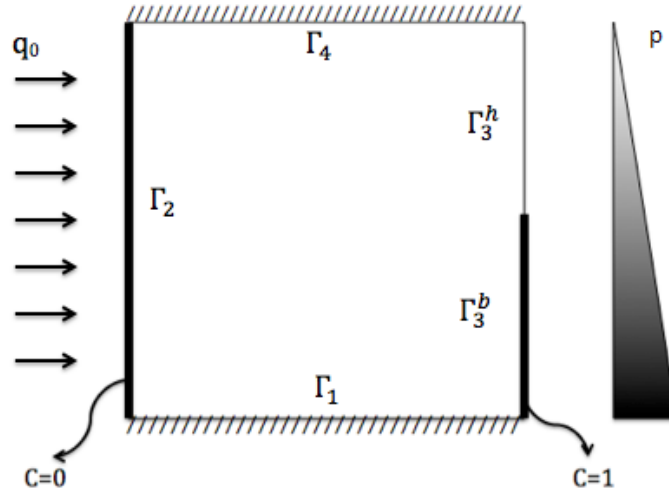


Figure 1.7: The various parts of the boundary for Henry's problem.

$Q_\rho = Q = 0$ and the boundary conditions

$$\begin{cases} c = 0 & \text{over } \Gamma_2 \times (0, T), \\ c = 1 & \text{over } \Gamma_3^b \times (0, T), \\ \frac{\partial c}{\partial n} = 0 & \text{on } (\Gamma_1 \cup \Gamma_4 \cup \Gamma_3^h) \times (0, T), \\ p = \rho_0 |\mathbf{g}| (\alpha(1-y) - y) & \text{on } \Gamma_3 \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = 0 & \text{on } (\Gamma_1 \cup \Gamma_4) \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = q_0 & \text{on } \Gamma_2 \times (0, T), \end{cases} \quad (1.4.5)$$

where $\Gamma_3 = \Gamma_3^h \cup \Gamma_3^b$ and y is y -coordinate. Figure 1.7 shows the configuration of the boundary conditions. The initial conditions are such that $c(\mathbf{x}, 0) = 0$ and that the pressure $p(\mathbf{x}, 0)$ satisfies $\nabla \cdot (\mathbf{q} \rho(c)) = 0$ for all $\mathbf{x} \in \mathcal{D}$.

We perform the numerical tests on the space domain $\mathcal{D} = (0, 1) \times (0, 1)$, with $T = 0.05$ days. The parameters are given in Table 1.4. In this test case, the dispersion-diffusion tensor is defined by $\mathbf{D} = D\mathbf{I}$.

We use 20×20 square control volumes in \mathcal{D} . The simulations describe the intrusion of salt water in a confined aquifer. The salt water enters from the right-hand-side, while the fresh water, of density ρ_0 , flows in from the left-hand-side at a constant rate. Therefore, the concentration in the area near the coastal side increases in time. At first the interface between the fresh and the salt water coming from the right-bottom corner has a large slope, which slowly decreases in time while the salt concentration enters deeper in the domain. Figure 1.8 presents the velocity fields at the initial and at the end times and Figure 1.9 presents the time evolution of the salt concentration in Henry's problem.

Symbol	Value	Unit
k	1.02×10^{-9}	m^2
D	6.6×10^{-6}	$m^2 \cdot s^{-1}$
\mathbf{g}	9.81	$m \cdot s^{-2}$
Φ	0.3	–
μ	10^{-3}	$Pa \cdot s$
ρ_0	10^3	$kg \cdot m^{-3}$
α	0.025	–
q_0	-6.6×10^{-5}	$m \cdot day^{-1}$

Table 1.4: Parameter values for Henry's problem.

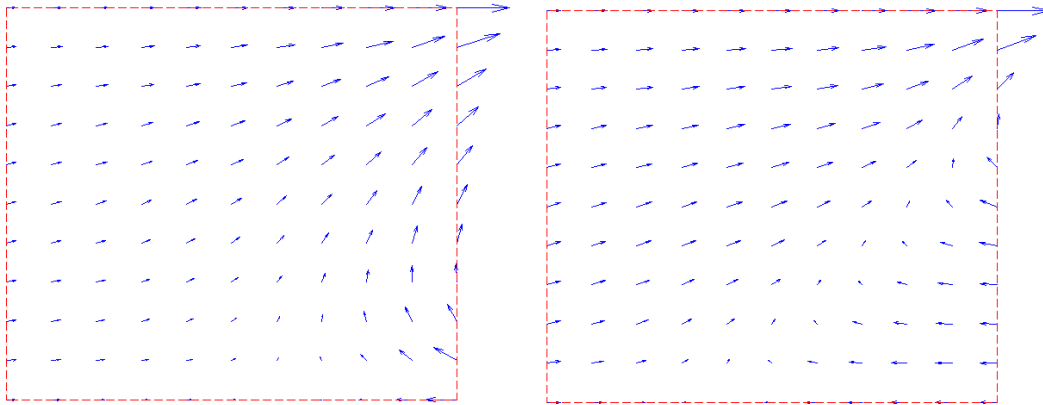


Figure 1.8: The velocity field at the initial and the end times for Henry's problem.

In the numerical simulation, we choose $\alpha = 0.8$ in order to make the refine-list long enough. We set $\beta = 0.15$ in order to keep the process stable enough. Table 1.5 gives the number of elements of the adaptive mesh at various times. The number is much smaller than that of the refined uniform mesh. The number of elements explains why the computational time is small in comparison with the computational time with refined uniform meshes at it is showed in Figure 1.10. As a conclusion, we save much time by using an adaptive mesh.

Time	NoE	NoU	Time	NoE	NoU
t=0 days	120	420			
t=0.001 days	262	844	t=0.01 days	469	1477
t=0.002 days	307	983	t=0.02 days	586	1834
t=0.003 days	328	1046	t=0.03 days	649	2025
t=0.005 days	331	1053	t=0.05 days	673	2099

Table 1.5: The number of elements (NoE) and the number of unknowns (NoU) in Henry's problem using an adaptive mesh.

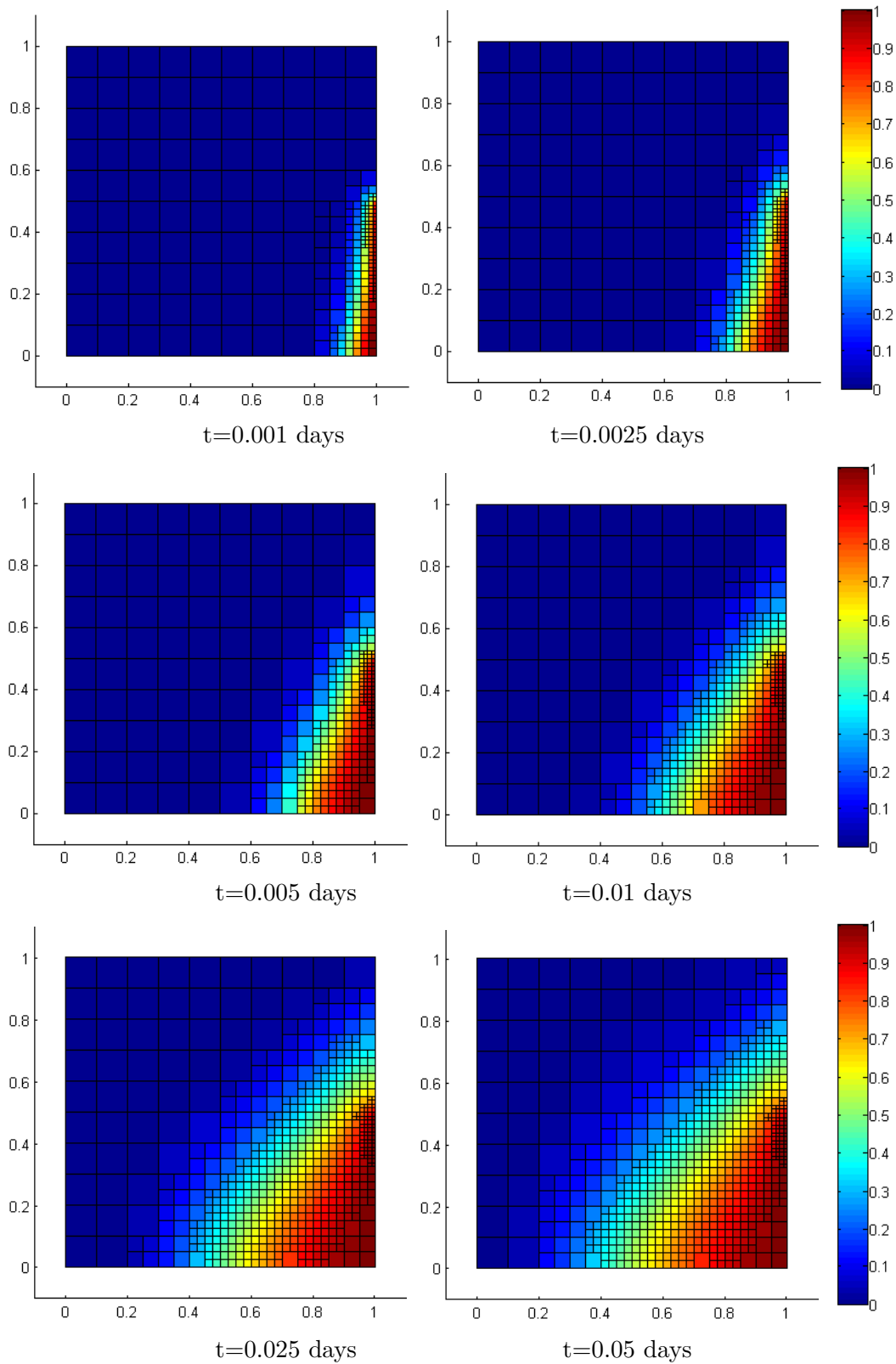


Figure 1.9: Time evolution of the salt concentration in Henry's problem using an adaptive mesh.

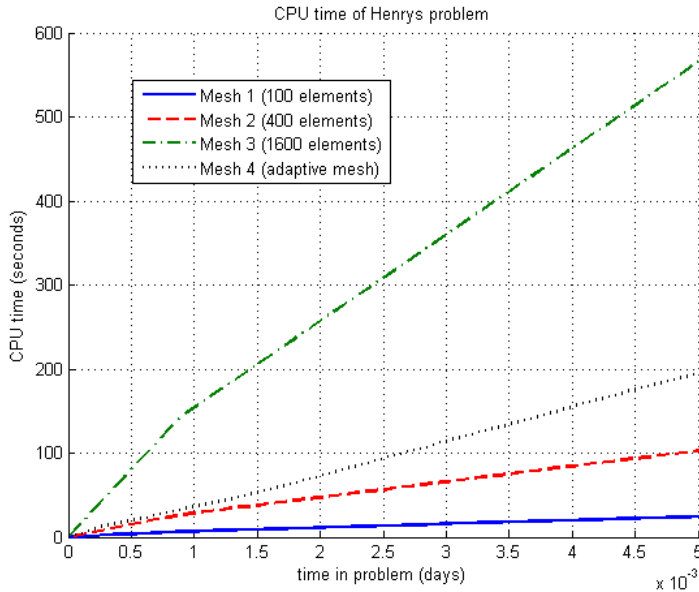


Figure 1.10: The computational time for Henry's problem using an adaptive mesh.

1.4.3 The saltpool problem

We present below a three dimensional numerical test which is studied in [41] and [56]. We consider a $0.2 \times 0.2 \times 0.2 \text{ m}^3$ cube where fresh water is injected into the top of the domain while the bottom is full of salt water.

Initial conditions are such that freshwater lies above saltwater with an 8 millimeter wide transition zone centered at $z = 0.06 \text{ m}$. In the transition zone, the mass fraction varies linearly from $c = 0$ at the top boundary to $c = 1$ at the bottom boundary.

Symbol	Value	Unit
k	1.19×10^{-9}	m^2
D	1.06×10^{-9}	$\text{m}^2 \text{s}^{-1}$
g	9.81	ms^{-2}
Φ	0.357	-
μ	10^{-3}	$\text{kgm}^{-1} \text{s}^{-1}$
ρ_0	998.2	kgm^{-3}
α	0.009	-
α_l	4.3×10^{-5}	m
α_t	1.2×10^{-3}	m
Q	1.89×10^{-6}	$\text{m}^2 \text{day}^{-1}$

Table 1.6: Parameters for the saltpool problem.

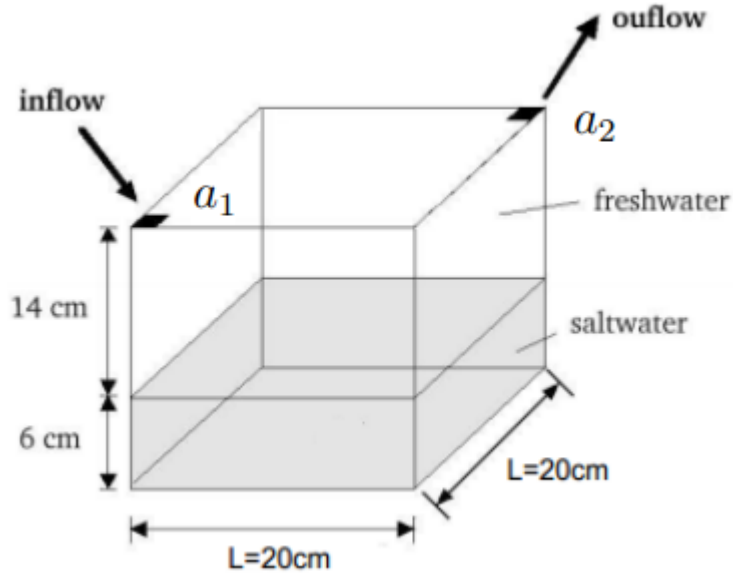


Figure 1.11: The geometry and parameters of the saltpool problem.

Boundary conditions for the flow equation are modeled using an inlet normal flux Neumann condition defined as $Q\rho_0$ in the inlet area, where Q is the inflow flux and ρ_0 the fresh water density. A Dirichlet freshwater piezometric head condition corresponding to a pressure value zero was used at the outlet. On all other parts of the domain, no flow boundary conditions were prescribed. For the transport equation, a zero normalized mass fraction Dirichlet condition corresponding to freshwater inflow is specified at the inlet. On the remaining part of the boundary, a zero advective and diffusion normal flux boundary condition is set. In summary, the boundary conditions for the saltpool are given by

$$\begin{cases} c = 0 & \text{on } a_1, \\ \frac{\partial c}{\partial n} = 0 & \text{on } \partial\mathcal{D} \setminus a_1, \\ \frac{\partial p}{\partial n} = 0 & \text{on } \partial\mathcal{D} \setminus \{a_1 \cup a_2\}, \\ q = -Q & \text{on } a_1, \\ p = 0 & \text{on } a_2. \end{cases} \quad (1.4.6)$$

In this test, we perform a simulation corresponding to test case 1 (1% salt mass fraction) of the article by K. Johannesen et. al. [41]. We start with a very coarse mesh with $5 \times 5 \times 5$ elements. The regions around the inlet and outlet as well as at the interface are refined to capture the boundary effect and prepare for the simulation. The model is expressed as Figure 1.11 with fresh water of density $\rho_0 = 998.2$ at the top and salt water of density $\rho = 1007.6$ (corresponding to $c = 1$) at the bottom. We use a fixed time step $\delta t = 100$ and the values of the parameters which are showed on Table 1.6 in our simulation. For the dispersion-diffusion

tensor, the Scheidegger's dispersion model is used

$$\mathbf{D}(\mathbf{q}) = D_m \mathbf{I} + \alpha_t |\mathbf{q}| \mathbf{I} + (\alpha_l - \alpha_t) \frac{\mathbf{q}^T \mathbf{q}}{|\mathbf{q}|}, \quad (1.4.7)$$

where α_l and α_t are the longitudinal and transverse dispersivity of the isotropic porous medium, respectively. Here, the numerical flux in each element is computed as follows:

$$\mathbf{q}_K^n = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_K} \frac{1}{|\sigma|} F_{K,\sigma}^C(p^n, w^{n-1}) \mathbf{n}_{K,\sigma} \quad \text{for all } K \in \mathcal{T}.$$

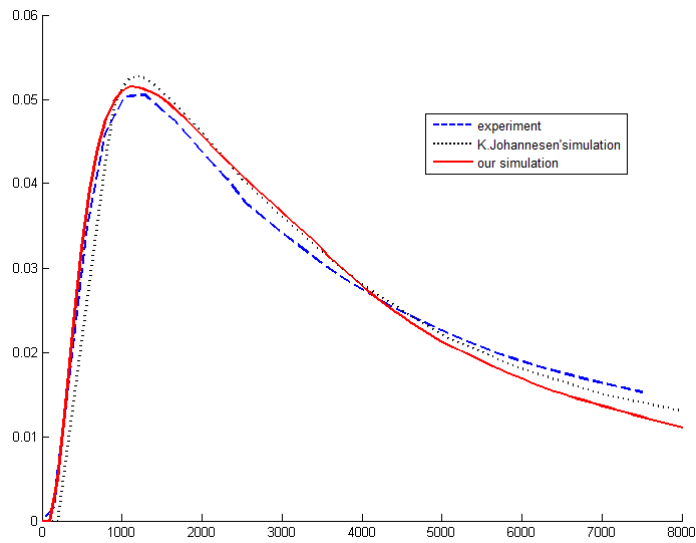


Figure 1.12: Outlet concentration of the saltpool problem.

Figure 1.13 shows the time evolution of the process while Figure 1.12 shows the outlet concentration obtained by means of (i) our simulation; (ii) the simulation of [41]; (iii) the real experiment. In a first period, our result is very close to the experimental result. In the time interval [1000, 4000] seconds, it is a little further from the experimental curve but it is slightly better than the result from [41]. In the last period, the result from [41] is closer to the experimental result than ours.

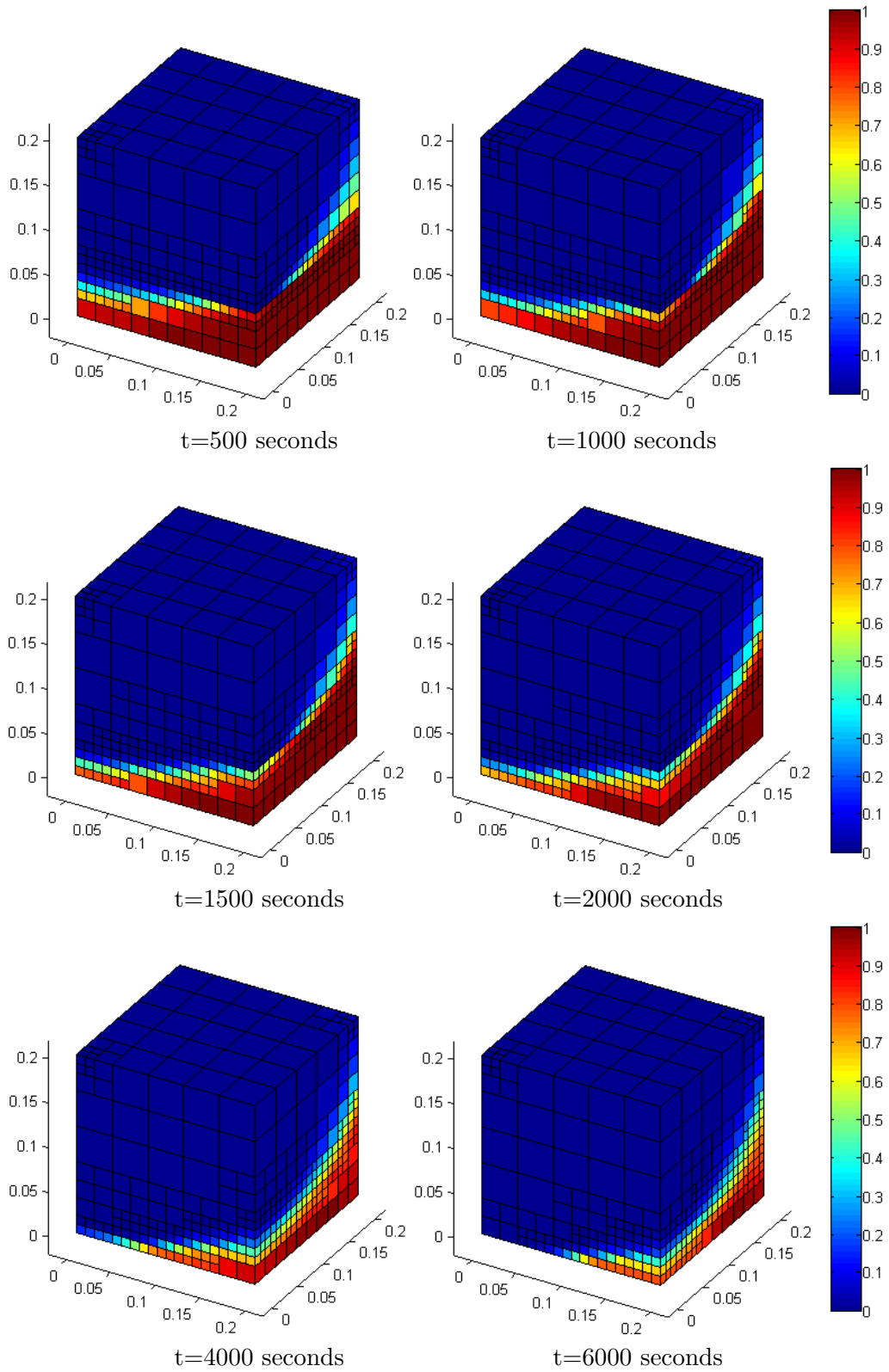


Figure 1.13: Time evolution of the salt concentration in the saltpool problem using an adaptive mesh.

PART II:

Numerical simulation of density a driven flow coupled to heat transport in porous media

1.5 A density driven flow coupled to heat transport

1.5.1 Variable density groundwater equation

We study a system of equations presented in SEAWAT and present below the case of a single species. We consider the following form of the variable density groundwater flow equation

$$S_s \rho \frac{\partial h}{\partial t} + \theta \frac{\partial \rho}{\partial C} \frac{\partial C}{\partial t} + \nabla \cdot (\mathbf{q}\rho) = q_s \rho_s \quad \text{in } \mathbb{D} \times (0, T), \quad (1.5.1)$$

where $\rho = \rho(h, C, \Theta)$ is the fluid density, θ the porosity, q_s a source or sink of the fluid with density ρ_s , and S_s the specific storage, defined as the volume of the water released from the storage per unit decline of h . Here the velocity \mathbf{q} is given by Darcy's law:

$$\mathbf{q}(h, \mu, \rho) = -\frac{\mu_0}{\mu} \mathbf{K}_0 \left(\nabla h + \frac{\rho - \rho_0}{\rho_0} \nabla z \right), \quad (1.5.2)$$

where μ is the dynamic viscosity and μ_0 is a reference viscosity, ρ_0 is the fluid density at the reference concentration C_0 and reference temperature Θ_0 . \mathbf{K}_0 is the hydraulic conductivity tensor of material saturated with the reference fluid.

1.5.2 The fluid density in equation (1.5.1)

For the transport of the solute species, where the concentration of the species can affect the fluid density, Hughes and Sanford [40] have implemented the following equation of state

$$\rho(h, C, \Theta) = \rho_0 + \frac{\partial \rho}{\partial C} (C - C_0) + \frac{\partial \rho}{\partial \Theta} (\Theta - \Theta_0) + \frac{\partial \rho}{\partial P} (P - P_0). \quad (1.5.3)$$

The third term on the right-hand side of (1.5.3) is reformulated here using the height of a water column l of density ρ_0 . This variable, which can be thought of as the pressure head in terms of the reference density, is related to the pressure by

$$P = l\rho_0 g,$$

where $l = h - z$. After some simple rearrangements, the final form of the equation of state is given by

$$\rho(h, C, \Theta) = \rho_0 + \frac{\partial \rho}{\partial C} (C - C_0) + \frac{\partial \rho}{\partial \Theta} (\Theta - \Theta_0) + \frac{\partial \rho}{\partial l} (l - l_0).$$

We can thus rewrite the formula for the fluid density as a function of the concentration, the temperature and the hydraulic head, namely

$$\rho(h, C, \Theta) = \rho_0 + \frac{\partial \rho}{\partial C}(C - C_0) + \frac{\partial \rho}{\partial \Theta}(\Theta - \Theta_0) + \frac{\partial \rho}{\partial l}(h - h_0). \quad (1.5.4)$$

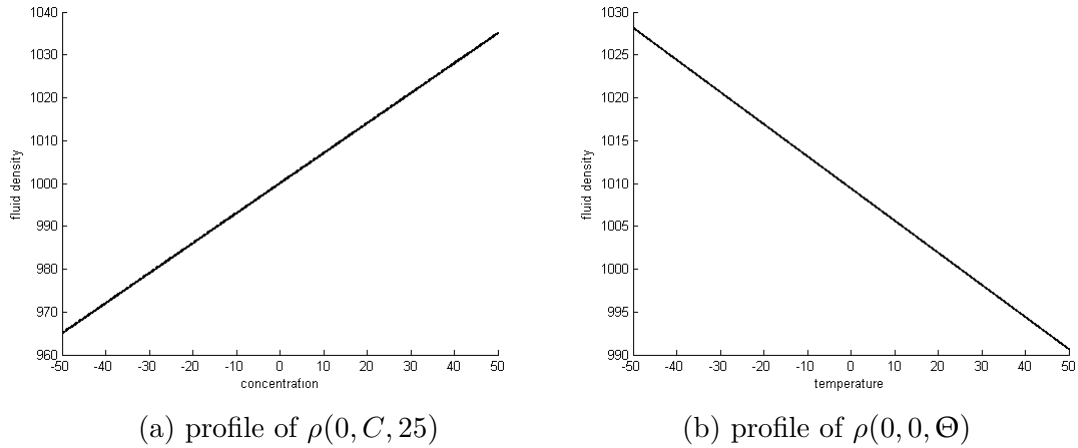


Figure 1.14: The fluid density $\rho(C, \Theta, h)$. The parameters are given in Table 1.7

1.5.3 The fluid viscosity in equation (1.5.1)

The dynamic viscosity is considered to be a function of temperature and solute concentration, which is a typical approach in [20]; we neglect the dependence of the viscosity on the fluid pressure. A general equation for representing the fluid viscosity as a function of concentration and temperature is given by

$$\mu(C, \Theta) = \mu_0 + \frac{\partial \mu}{\partial C}(C - C_0) + \frac{\partial \mu}{\partial \Theta}(\Theta - \Theta_0).$$

On a number of temperature ranges, the linear approximation does not adequately represent the effect of temperature on the dynamic viscosity. For this reason, an alternative equation for the dynamic viscosity has been implemented, namely

$$\mu(C, \Theta) = \mu_{\Theta}(\Theta) + \frac{\partial \mu}{\partial C}(C - C_0).$$

There are many options for representing $\mu_{\Theta}(\Theta)$ as a function of temperature. Here we use the following viscosity and temperature relation

$$\mu_{\Theta}(\Theta) = A_1 A_2 \left(\frac{A_3}{\Theta + A_4} \right),$$

where the values A_1, A_2, A_3 and A_4 are positive constants (cf. C.I. Voss [60]). As a result we obtain the following formula for the dynamic viscosity

$$\mu(C, \Theta) = A_1 A_2 \left(\frac{A_3}{\Theta + A_4} \right) + \frac{\partial \mu}{\partial C}(C - C_0). \quad (1.5.5)$$

1.5.4 The solute transport and heat transport equations

The equation for solute transport in groundwater is an advection-dispersion equation. A general form is given by

$$\left(1 + \frac{\rho_b K_d^C}{\theta}\right) \frac{\partial(\theta C)}{\partial t} + \nabla \cdot (\mathbf{q}C) - \nabla \cdot \left((\theta D_m^C + \mathbf{a} \cdot \mathbf{q}) \nabla C \right) = q_s C_s \quad \text{in } \mathbf{D} \times (0, T), \quad (1.5.6)$$

where ρ_b is the bulk density, K_d^C is the distribution coefficient for salinity, D_m^C is the diffusion coefficient and \mathbf{a} is the dispersivity tensor. Next we present a possible form for the heat transport equation, which was proposed by Thorne et al [59] to highlight the similarity with the solute transport equation

$$\left(1 + \frac{\rho_b K_d^\Theta}{\theta}\right) \frac{\partial(\theta \Theta)}{\partial t} + \nabla \cdot (\mathbf{q}\Theta) - \nabla \cdot \left((\theta D_m^\Theta + \mathbf{a} \cdot \mathbf{q}) \nabla \Theta \right) = q_s \Theta_s \quad \text{in } \mathbf{D} \times (0, T), \quad (1.5.7)$$

where K_d^Θ is the distribution coefficient for the temperature and D_m^Θ is the bulk thermal diffusivity. Mathematically, we propose to solve the system of partial differential equations (1.5.1), (1.5.2) (1.5.6) and (1.5.7) together with the boundary conditions

$$\begin{cases} h = h_D(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_D^h \times (0, T), \\ \mathbf{q} \cdot \mathbf{n} = \bar{q}_N(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_N^h \times (0, T), \\ C = C_D(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_D^C \times (0, T), \\ \frac{\partial C}{\partial n} = \bar{C}_N(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_N^C \times (0, T), \\ \Theta = \Theta_D(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_D^\Theta \times (0, T), \\ \frac{\partial \Theta}{\partial n} = \bar{\Theta}_N(\mathbf{x}, t) & \text{on } \partial \mathbf{D}_N^\Theta \times (0, T), \end{cases} \quad (1.5.8)$$

with $\partial \mathbf{D} = \overline{\partial \mathbf{D}_D^h} \cup \overline{\partial \mathbf{D}_N^h} = \overline{\partial \mathbf{D}_D^C} \cup \overline{\partial \mathbf{D}_N^C} = \overline{\partial \mathbf{D}_D^\Theta} \cup \overline{\partial \mathbf{D}_N^\Theta}$ where $\partial \mathbf{D}_D^h$, $\partial \mathbf{D}_D^C$ and $\partial \mathbf{D}_D^\Theta$ correspond to Dirichlet boundary conditions and $\partial \mathbf{D}_N^h$, $\partial \mathbf{D}_N^C$ and $\partial \mathbf{D}_N^\Theta$ correspond to Neumann boundary conditions for the hydraulic head, the concentration, and the temperature respectively. The initial conditions are given by

$$\begin{cases} h(\mathbf{x}, 0) = h_{ini}(\mathbf{x}) & \text{in } \mathbf{D}, \\ C(\mathbf{x}, 0) = C_{ini}(\mathbf{x}) & \text{in } \mathbf{D}, \\ \Theta(\mathbf{x}, 0) = \Theta_{ini}(\mathbf{x}) & \text{in } \mathbf{D}. \end{cases} \quad (1.5.9)$$

1.6 The generalized finite volume method SUSHI for fluxes approximation

1.6.1 Approximation of fluxes

After formally integrating equation (1.5.1) on the volume element K for each $K \in \mathcal{T}$ we obtain

$$\begin{aligned} S_s \int_K \rho \frac{\partial h}{\partial t} d\mathbf{x} + \int_K \theta \frac{\partial \rho}{\partial C} \frac{\partial C}{\partial t} d\mathbf{x} - \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \frac{\mu_0}{\mu} \mathbf{K}_0 \nabla h \rho \cdot \mathbf{n}_{K,\sigma} d\gamma \\ - \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \frac{\mu_0}{\mu} \mathbf{K}_0 \frac{\rho - \rho_0}{\rho_0} \nabla z \rho \cdot \mathbf{n}_{K,\sigma} d\gamma = \int_K q_s \rho_s d\mathbf{x}, \end{aligned} \quad (1.6.1)$$

The diffusion flux $-\int_{\sigma} \frac{\mu_0}{\mu} \mathbf{K}_0 \nabla h \rho \cdot \mathbf{n}_{K,\sigma} d\gamma$ is approximated by the numerical flux $F_{K,\sigma}^h(h) \rho_{\sigma}$ which have been derived in SUSHI scheme framework [29]. More precisely we define

$$F_{K,\sigma}^h(v) = \frac{\mu_0}{\mu_{\sigma}} \sum_{\sigma' \in \mathcal{E}_K} A_K^{\sigma\sigma'} (v_K - v_{\sigma'}) \quad \text{for } v \in X_{\mathcal{D}}, \quad (1.6.2)$$

where the matrices $A_K^{\sigma\sigma'}$, which are symmetric and positive-definite, are given by

$$A_K^{\sigma\sigma'} = \sum_{\sigma'' \in \mathcal{E}_K} \mathbf{y}_K^{\sigma''\sigma} \cdot \left(\int_{V_{K,\sigma''}} \mathbf{K}_0 d\mathbf{x} \right) \mathbf{y}_K^{\sigma''\sigma'}, \quad (1.6.3)$$

where $\mathbf{y}_K^{\sigma\sigma'}$ is defined by (1.2.15).

We then integrate equations (1.5.6) and (1.5.7) on the volume element K for each $K \in \mathcal{T}$. This gives

$$\begin{aligned} \int_K \left(1 + \frac{\rho_b K_d^C}{\theta}\right) \frac{\partial(\theta C)}{\partial t} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \mathbf{q} C \cdot \mathbf{n}_{K,\sigma} d\gamma \\ - \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} (\theta D_m^C + \mathbf{a} \cdot \mathbf{q}) \nabla C \cdot \mathbf{n}_{K,\sigma} d\gamma = \int_K q_s C_s d\mathbf{x}, \\ \int_K \left(1 + \frac{\rho_b K_d^{\Theta}}{\theta}\right) \frac{\partial(\theta \Theta)}{\partial t} d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \mathbf{q} \Theta \cdot \mathbf{n}_{K,\sigma} d\gamma \\ - \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} (\theta D_m^{\Theta} + \mathbf{a} \cdot \mathbf{q}) \nabla \Theta \cdot \mathbf{n}_{K,\sigma} d\gamma = \int_K q_s \Theta_s d\mathbf{x}. \end{aligned} \quad (1.6.4)$$

In what follows we define the numerical fluxes $F_{K,\sigma}^C(v)$ and $F_{K,\sigma}^{\Theta}(v)$ in order to approximate the diffusion fluxes $-\int_{\sigma} (\theta D_m^C + \mathbf{a} \cdot \mathbf{q}) \nabla v \cdot \mathbf{n}_{K,\sigma} d\gamma$ and $-\int_{\sigma} (\theta D_m^{\Theta} + \mathbf{a} \cdot \mathbf{q}) \nabla v \cdot \mathbf{n}_{K,\sigma} d\gamma$ respectively. Following the SUSHI scheme, we express the discrete fluxes $F_{K,\sigma}^C(v)$ and $F_{K,\sigma}^{\Theta}(v)$ in terms of the discrete unknowns as:

$$F_{K,\sigma}^C(v) = \sum_{\sigma' \in \mathcal{E}_K} \bar{A}_K^{\sigma\sigma'} (v_K - v_{\sigma'}), \quad \bar{A}_K^{\sigma\sigma'} = \sum_{\sigma'' \in \mathcal{E}_K} \mathbf{y}_K^{\sigma''\sigma} \cdot \left(\int_{V_{K,\sigma''}} (\theta D_m^C + \mathbf{a} \cdot \mathbf{q}) d\mathbf{x} \right) \mathbf{y}_K^{\sigma''\sigma'}, \quad (1.6.5)$$

and

$$F_{K,\sigma}^{\Theta}(v) = \sum_{\sigma' \in \mathcal{E}_K} \hat{A}_K^{\sigma\sigma'} (v_K - v_{\sigma'}), \quad \hat{A}_K^{\sigma\sigma'} = \sum_{\sigma'' \in \mathcal{E}_K} \mathbf{y}_K^{\sigma''\sigma} \cdot \left(\int_{V_{K,\sigma''}} (\theta D_m^{\Theta} + \mathbf{a} \cdot \mathbf{q}) d\mathbf{x} \right) \mathbf{y}_K^{\sigma''\sigma'}. \quad (1.6.6)$$

1.6.2 Numerical scheme

In the numerical scheme, almost all the parameters are constant. Moreover the functions $\partial\rho/\partial h$, $\partial\rho/\partial C$, $\partial\rho/\partial\Theta$ in the equation (1.5.4) and $\partial\mu/\partial C$ in the equation (1.5.5) are also constant. For the sake of simplicity, we set $\Phi_C = \theta + \rho_b K_d^C$ and $\Phi_{\Theta} = \theta + \rho_b K_d^{\Theta}$. We present

below a semi-implicit finite volume scheme corresponding to the system (1.5.1) - (1.5.7) and the initial conditions (1.5.9):

The initial conditions for the scheme are given by

$$\begin{aligned} h_K^0 &= \frac{1}{|K|} \int_K h_{ini}(\mathbf{x}) \, d\mathbf{x}, \\ C_K^0 &= \frac{1}{|K|} \int_K C_{ini}(\mathbf{x}) \, d\mathbf{x}, \\ \Theta_K^0 &= \frac{1}{|K|} \int_K \Theta_{ini}(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (1.6.7)$$

For each $n \in \{1, \dots, N\}$:

Knowing $h^{n-1}, C^{n-1}, C^{n-2}$ and Θ^{n-1} , find $h^n \in X_D$ such that

$$\left\{ \begin{aligned} & S_s |K| \rho_K^{n-1} (h_K^n - h_K^{n-1}) + \theta |K| \frac{\partial \rho}{\partial C} (C_K^{n-1} - C_K^{n-2}) + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^h(h^n) \rho_\sigma^{n-1} \\ & \quad + \delta t \sum_{\sigma \in \mathcal{E}_K} \frac{\mu_0}{\mu_\sigma^{n-1}} \frac{\rho_\sigma^{n-1} - \rho_0}{\rho_0} \rho_\sigma^{n-1} \mathbf{K}_0 \nabla z \cdot \mathbf{n}_{K\sigma} |\sigma| = \rho_s Q_K^n \text{ for all } K \in \mathcal{T}, \quad (1.6.8a) \\ & \sum_{K \in \mathcal{T}_\sigma} F_{K,\sigma}^h(h^n) \rho_\sigma^{n-1} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{int}, \quad (1.6.8b) \\ & h_\sigma^n = h_D(\mathbf{x}_\sigma, t_n) \quad \text{for all } \mathbf{x}_\sigma \in \partial \mathcal{D}_D^h, \quad (1.6.8c) \\ & \mathbf{q}_\sigma^n = |\sigma| \bar{q}_N(\mathbf{x}_\sigma, t_n) \quad \text{for all } \mathbf{x}_\sigma \in \partial \mathcal{D}_N^h. \quad (1.6.8d) \end{aligned} \right.$$

where $Q_K^n = \int_{t^{n-1}}^{t^n} \int_K q_s \, d\mathbf{x} dt$, $F_{K,\sigma}^h(v)$, ρ_σ^{n-1} , and μ_σ^{n-1} are given by (1.6.2), (1.6.12) and (1.6.13) below, respectively. The approximate velocity \mathbf{q}_σ^n is computed as (1.6.11) below.

Knowing \mathbf{q}^n, C^{n-1} , find $C^n \in X_D$ such that

$$\left\{ \begin{aligned} & \Phi_C |K| (C_K^n - C_K^{n-1}) + \delta t \sum_{\sigma \in \mathcal{E}_K} \mathbf{q}_\sigma^n \tilde{C}_{K\sigma}^n \\ & \quad + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^C(C^n) = C_s Q_K^n \quad \text{for all } K \in \mathcal{T}, \quad (1.6.9a) \\ & \sum_{K \in \mathcal{T}_\sigma} \left\{ \mathbf{q}_\sigma^n \tilde{C}_{K\sigma}^n + F_{K,\sigma}^C(C^n) \right\} = 0 \quad \text{for all } \sigma \in \mathcal{E}_{int}, \quad (1.6.9b) \\ & C_\sigma^n = C_D(\mathbf{x}_\sigma, t_n) \quad \text{for all } \mathbf{x}_\sigma \in \partial \mathcal{D}_D^c, \quad (1.6.9c) \\ & F_{K,\sigma}^C(C^n) = -(\theta D_m^C + \mathbf{a} \cdot \mathbf{q}) |\sigma| \bar{C}_N(\mathbf{x}_\sigma, t_n) \quad \text{for all } \mathbf{x}_\sigma \in \partial \mathcal{D}_N^c. \quad (1.6.9d) \end{aligned} \right.$$

Knowing \mathbf{q}^n , Θ^{n-1} , find $\Theta^n \in X_D$ such that

$$\left\{ \begin{array}{l} \Phi_{\Theta}|K|(\Theta_K^n - \Theta_K^{n-1}) + \delta t \sum_{\sigma \in \mathcal{E}_K} \mathbf{q}_{\sigma}^n \tilde{\Theta}_{K\sigma}^n \\ \quad + \delta t \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{\Theta}(\Theta^n) = \Theta_s Q_K^n \end{array} \right. \quad \text{for all } K \in \mathcal{T}, \quad (1.6.10a)$$

$$\left\{ \begin{array}{l} \sum_{K \in \mathcal{T}_{\sigma}} \left\{ \mathbf{q}_{\sigma}^n \tilde{\Theta}_{K\sigma}^n + F_{K,\sigma}^{\Theta}(\Theta^n) \right\} = 0 \end{array} \right. \quad \text{for all } \sigma \in \mathcal{E}_{int}, \quad (1.6.10b)$$

$$\left\{ \begin{array}{l} \Theta_{\sigma}^n = \Theta_D(\mathbf{x}_{\sigma}, t_n) \\ F_{K,\sigma}^{\Theta}(\Theta^n) = -(\theta D_m^{\Theta} + \mathbf{a} \cdot \mathbf{q})|\sigma| \bar{\Theta}_N(\mathbf{x}_{\sigma}, t_n) \end{array} \right. \quad \begin{array}{l} \text{for all } \mathbf{x}_{\sigma} \in \partial \mathcal{D}_D^{\Theta}, \\ \text{for all } \mathbf{x}_{\sigma} \in \partial \mathcal{D}_N^{\Theta}. \end{array} \quad (1.6.10c) \quad (1.6.10d)$$

The approximate fluxes $F_{K,\sigma}^C(v)$ and $F_{K,\sigma}^{\Theta}(v)$ are defined by (1.6.5) and (1.6.6), respectively. The approximate velocity \mathbf{q}_{σ}^n is given by

$$\mathbf{q}_{\sigma}^n = F_{K,\sigma}^h(h^n) + \frac{\mu_0}{\mu_{\sigma}^n} \frac{\rho_{\sigma}^n - \rho_0}{\rho_0} \mathbf{K}_0 \nabla z \cdot \mathbf{n}_{K\sigma} |\sigma|, \quad (1.6.11)$$

where

$$\rho_{\sigma}^n = \rho_0 + \frac{\partial \rho}{\partial C}(C_{\sigma}^n - C_0) + \frac{\partial \rho}{\partial \Theta}(\Theta_{\sigma}^n - \Theta_0) + \frac{\partial \rho}{\partial l}(h_{\sigma}^n - h_0), \quad (1.6.12)$$

and

$$\mu_{\sigma}^n = A_1 A_2 \left(\frac{A3}{\Theta_{\sigma}^n + A4} \right) + \frac{\partial \mu}{\partial C}(C_{\sigma}^n - C_0), \quad (1.6.13)$$

We remark that when $n = 1$, the second term on the left-hand side of the equation (1.6.8a) is omitted. The three equations (1.6.8b), (1.6.9b) and (1.6.10b) come from the local conservation of the discrete fluxes on the interior edges. Note that we use an upwind scheme for the convection term in (1.6.9a) and (1.6.10a) with

$$\tilde{v}_{K,\sigma} = \begin{cases} v_K & \text{if } \mathbf{q}_{\sigma} > 0, \\ v_{\sigma} & \text{otherwise.} \end{cases}$$

1.7 Numerical tests

We propose an alternative simulation for a model problem developed in the SEAWAT documentation [48]. The problem consists in a two-dimensional cross section of a confined coastal aquifer initially saturated with relatively cold seawater at the temperature of $5^{\circ}C$. Warmer fresh water with the temperature of $25^{\circ}C$ is injected into the coastal aquifer along the left boundary to represent the flow from inland areas. The warmer fresh water flows to the right, where it discharges into a vertical ocean boundary. The ocean boundary is represented with hydrostatic conditions based on the fluid density calculated from seawater salinity at $5^{\circ}C$. No flow conditions are assigned to the top and bottom boundaries. This problem is a simplified representation of what may occur in a coastal carbonate platform.

In this section, we demonstrate the process of numerical simulation of this problem in three cases. The simulations describe the transfer of fresh water in a confined aquifer. The fresh water

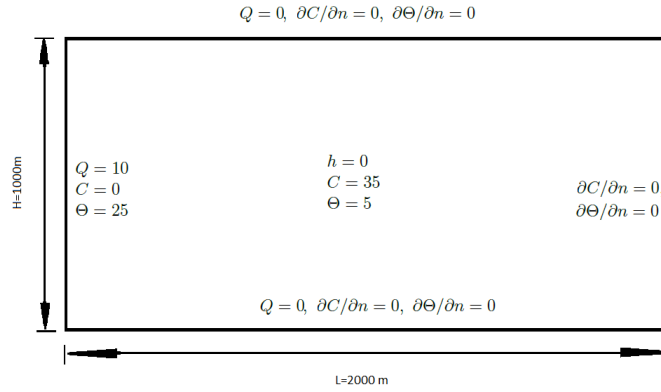


Figure 1.15: The model problem.

($C = 0, \Theta = 25$) enters from the left hand side and replaces the salt water ($C = 35, \Theta = 5$) little after little. Because of the gravity and no flux at the top and bottom boundaries, the velocity is different at the top, bottom and the middle of the aquifer. At the bottom, the flow transfers very slowly so that the values of the concentration and of the temperature do not change much. At the top, the flux is very strong, the values of the concentration and of the temperature change fast.

At first we use a uniform 20×40 squares mesh to discretize the domain. However, the variation only takes place in a small area of the domain, this is why we replace the uniform mesh by an adaptive mesh. We choose α close to 0.8 in order to make the refined-list long enough, and we choose β around 0.2 in order to keep the process stable.

1.7.1 Test case 1 and test case 2

Test case 1: We consider the simplest case that the fluid density is only affected by salinity while the fluid viscosity is constant. Then the fluid density can be rewritten as follows:

$$\rho = \rho_0 + \frac{\partial \rho}{\partial C}(C - C_0). \quad (1.7.1)$$

In this case, the heat effect is neglected, so that we use the scheme (1.6.8) and (1.6.9) for the numerical simulations. The following figures 1.16 and 1.17 show the time evolution of the concentration in this test case.

Test case 2: In this test case, the density ρ is a function of salinity and temperature. A linear expression is used to relate the fluid density to the concentration and the temperature:

$$\rho = \rho_0 + \frac{\partial \rho}{\partial C}(C - C_0) + \frac{\partial \rho}{\partial \Theta}(\Theta - \Theta_0), \quad (1.7.2)$$

The slope of the linear relation $\frac{\partial \rho}{\partial \Theta}$ is about -0.375 . The negative sign indicates that the fluid density increases as the temperature decreases (as in Figure 1.14(b)).

1. A GENERALIZED FINITE VOLUME METHOD FOR A DENSITY DRIVEN FLOW COUPLED TO HEAT TRANSPORT IN POROUS MEDIA

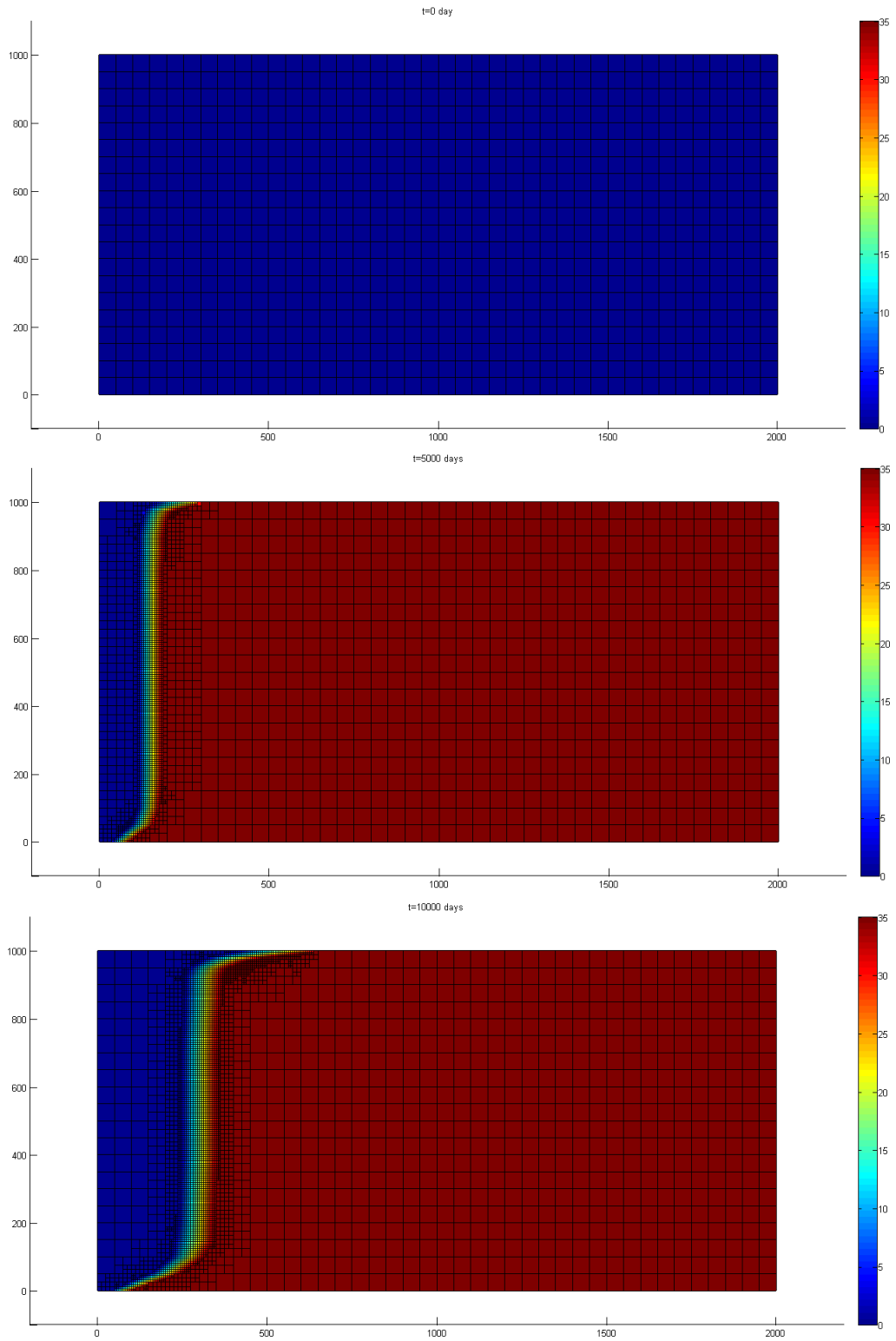


Figure 1.16: Concentration profile at the initial time and at the times $t = 5000$ and $t = 10000$ days for the test case 1.

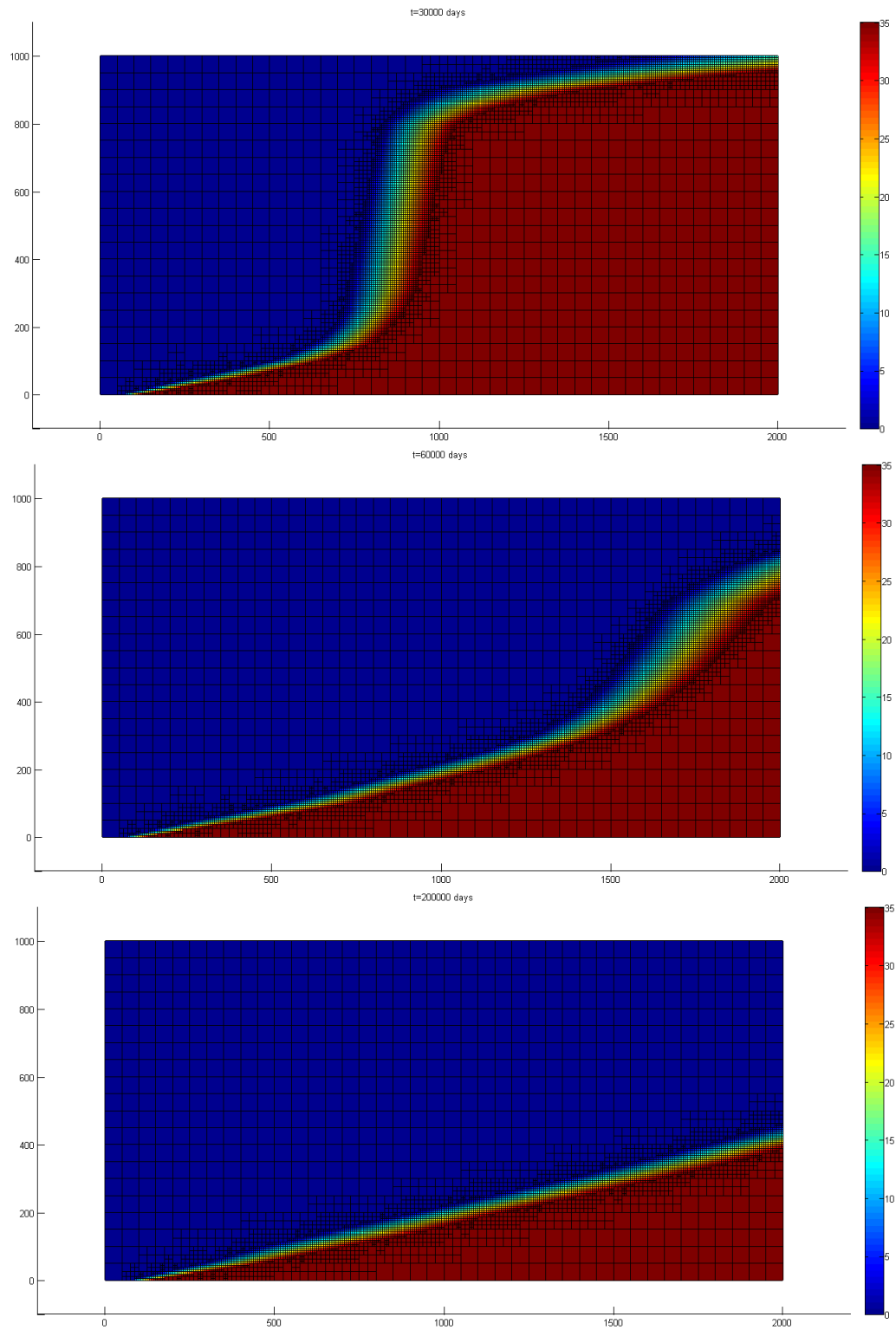


Figure 1.17: Concentration profile at the times $t = 30000$, $t = 60000$ and $t = 200000$ days for the test case 1.

Here the temperature is a simple linear function of the concentration:

$$T = \left(\frac{T_{inflow} - T_{ocean}}{C_{inflow} - C_{ocean}} \right) C + (T_{inflow} - T_{ocean}) = -\frac{4}{7}C + 20, \quad (1.7.3)$$

where $T_{inflow} = 25^\circ C$, $T_{ocean} = 5^\circ C$, $C_{inflow} = 0$ and $C_{ocean} = 35$. The figures 1.18 and 1.19 show the evolution of the concentration.

Comparison between the test cases 1 and 2: Next we compare the results which are obtained in the two cases above. It turns out that the velocity of displacement of the front is lower in the test case 2 than in the test case 1; moreover we also observe that the transition front width is thinner in the test case 2. We also compare our results to those in SEAWAT [48]. The transient movement of the freshwater-saltwater transition zone in the simulations performed in SEAWAT corresponding test case 1 and test case 2 are shown in Figure 1.20. Our results look very similar to those of SEAWAT.

1.7.2 Test case 3

In this test case, we consider the situation of the test case 2 coupled with the heat transfer equation. Then the fluid density is given by (1.7.2) while the temperature is obtained from the scheme (1.6.10). An adaptive mesh corresponding to the concentration variation is used for the simulations shown in the figures 1.21 and 1.22. For the numerical solution of temperature, we perform the simulation using an adaptive mesh corresponding to the temperature variation. The time evolution of the temperature is shown in Figure 1.23. Our results are compared to those in SEAWAT to show that both are similar. The transient motion of the freshwater-saltwater transition zone and that of temperature transition zone in SEAWAT corresponding to the test case 3 are shown in Figure 1.24.

We remark that the width of the transition fronts is thicker in the test case 3 where there is a real coupling with the temperature equation.

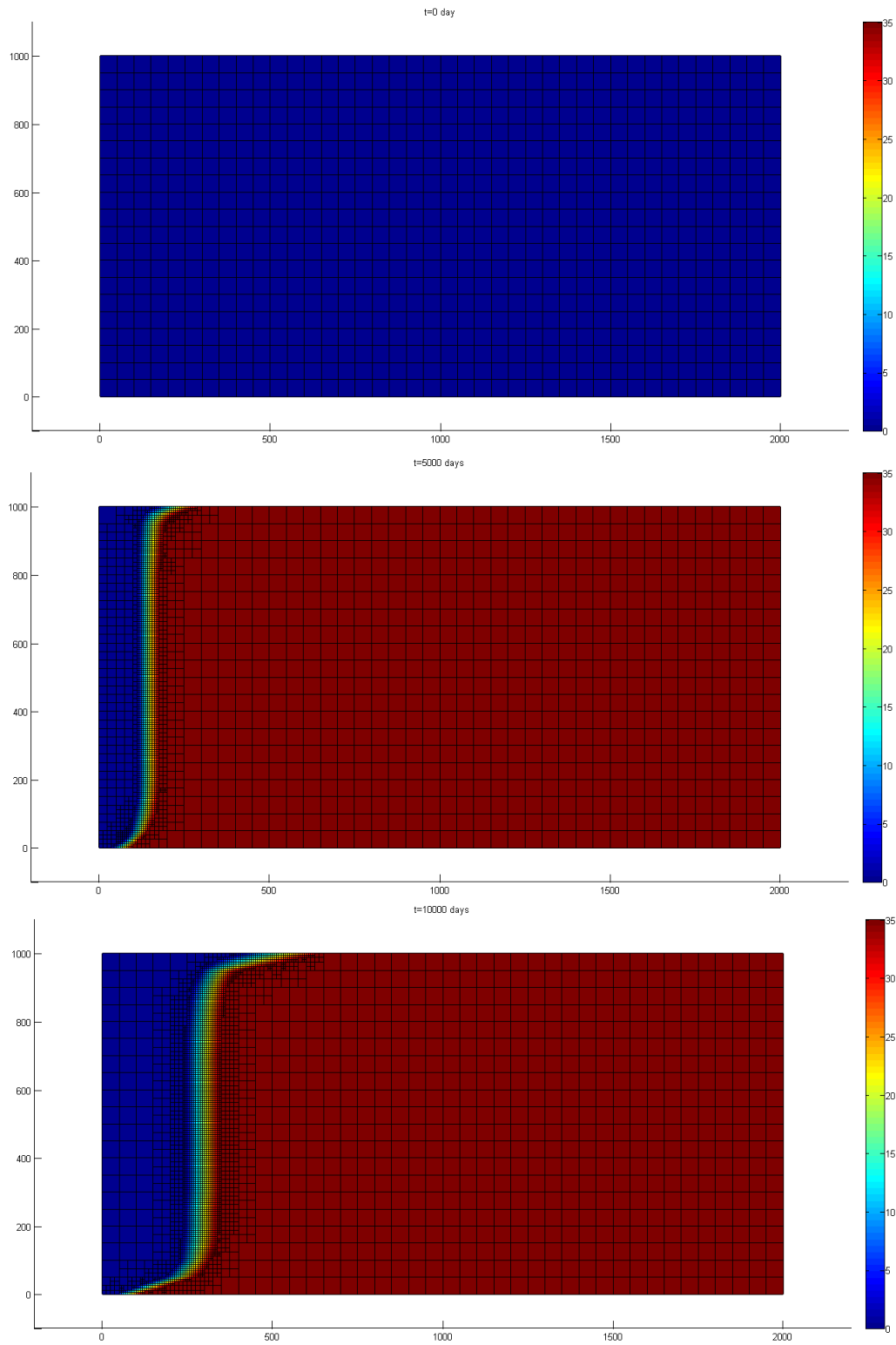


Figure 1.18: Concentration profile at the initial time and at the times $t = 5000$ and $t = 10000$ days for the test case 2.

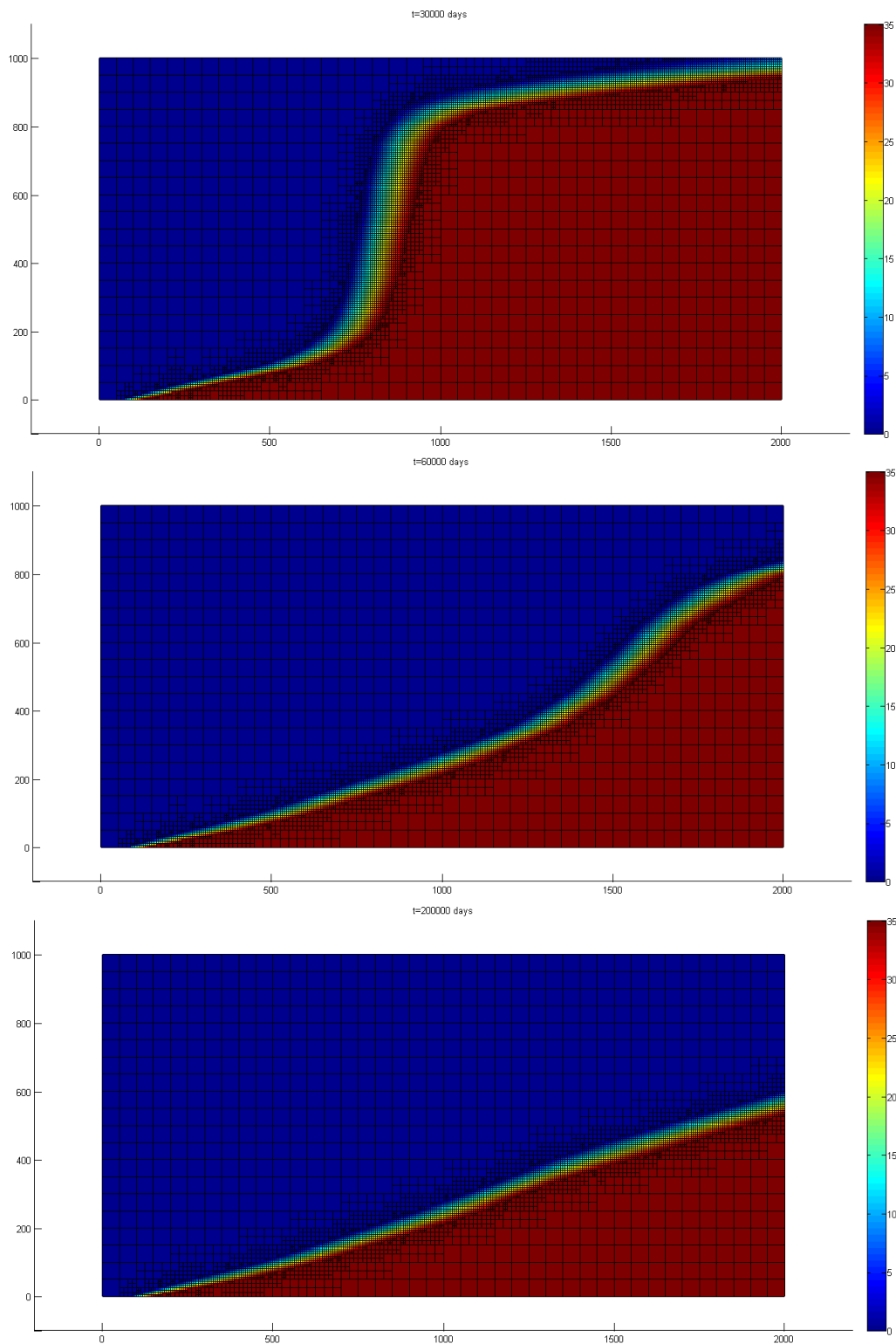
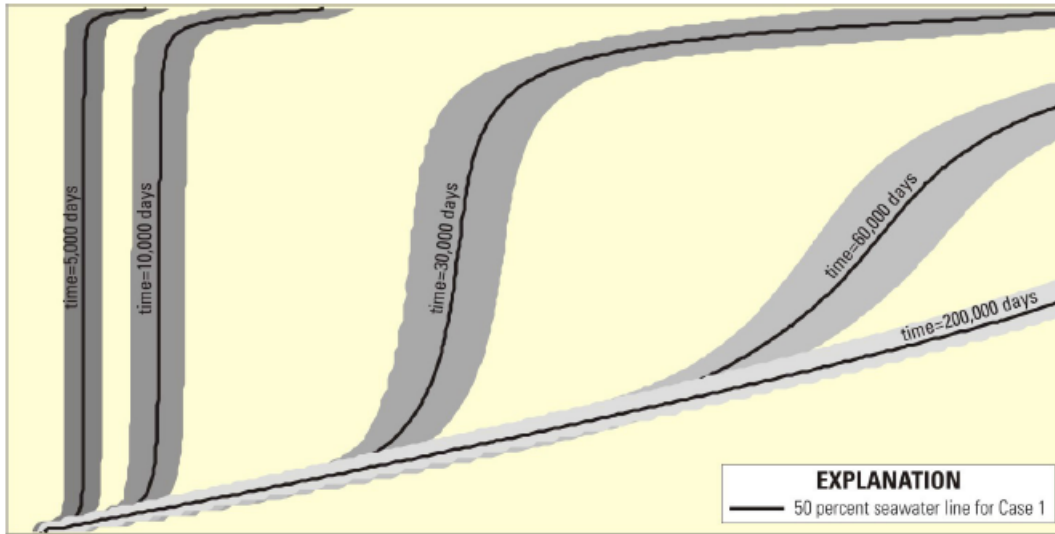
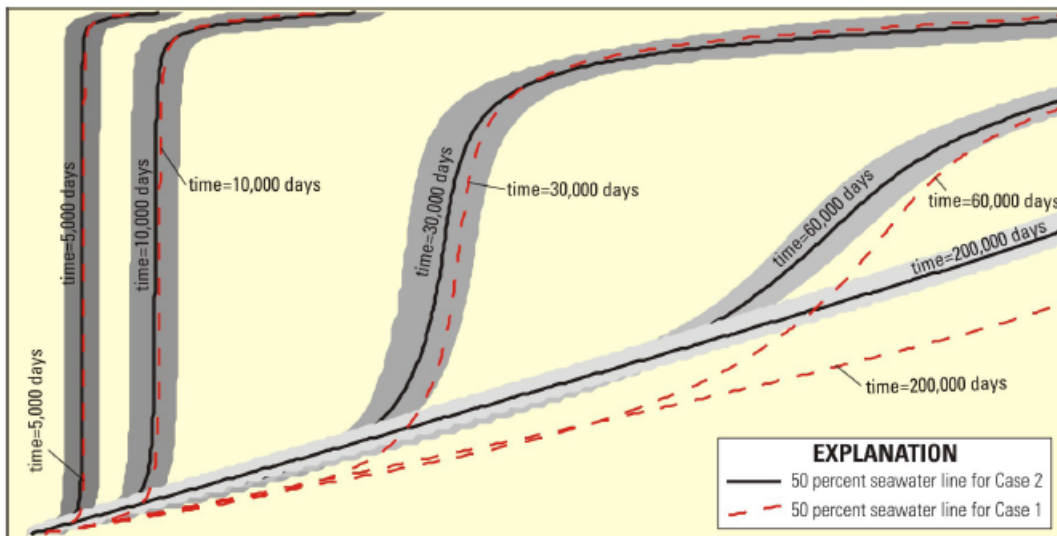


Figure 1.19: Concentration profile at the times $t = 30000$, $t = 60000$ and $t = 200000$ days for the test case 2.



The transient motion of the freshwater-saltwater transition zone for the test case 1 in SEAWAT.



The transient motion of the freshwater-saltwater transition zone for the test case 2 in SEAWAT.

Figure 1.20: Simulation for the test case 1 and the test case 2 using SEAWAT.

1. A GENERALIZED FINITE VOLUME METHOD FOR A DENSITY DRIVEN FLOW COUPLED TO HEAT TRANSPORT IN POROUS MEDIA

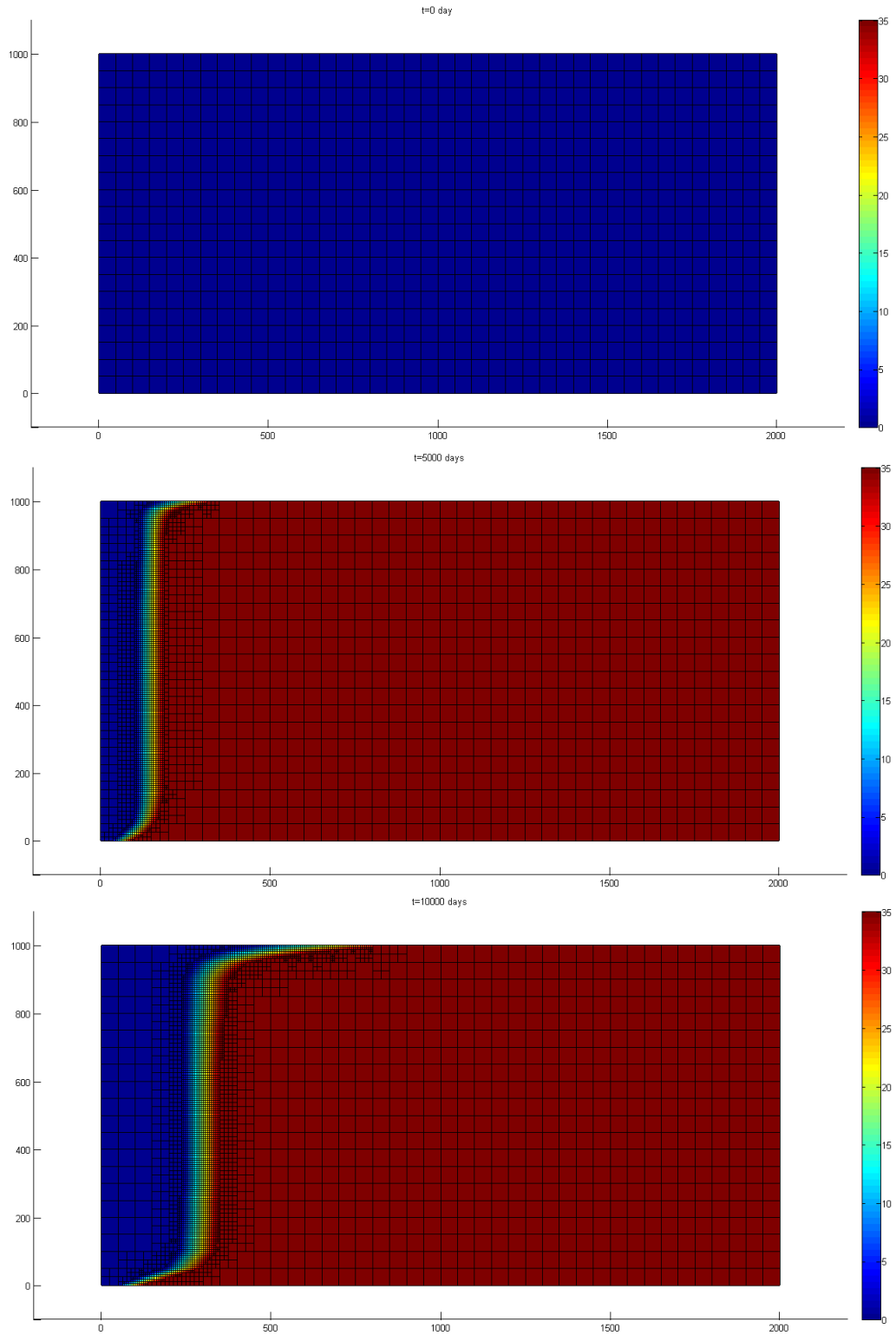


Figure 1.21: Concentration profile at the initial time and at the times $t = 5000$ and $t = 10000$ days for the test case 3.

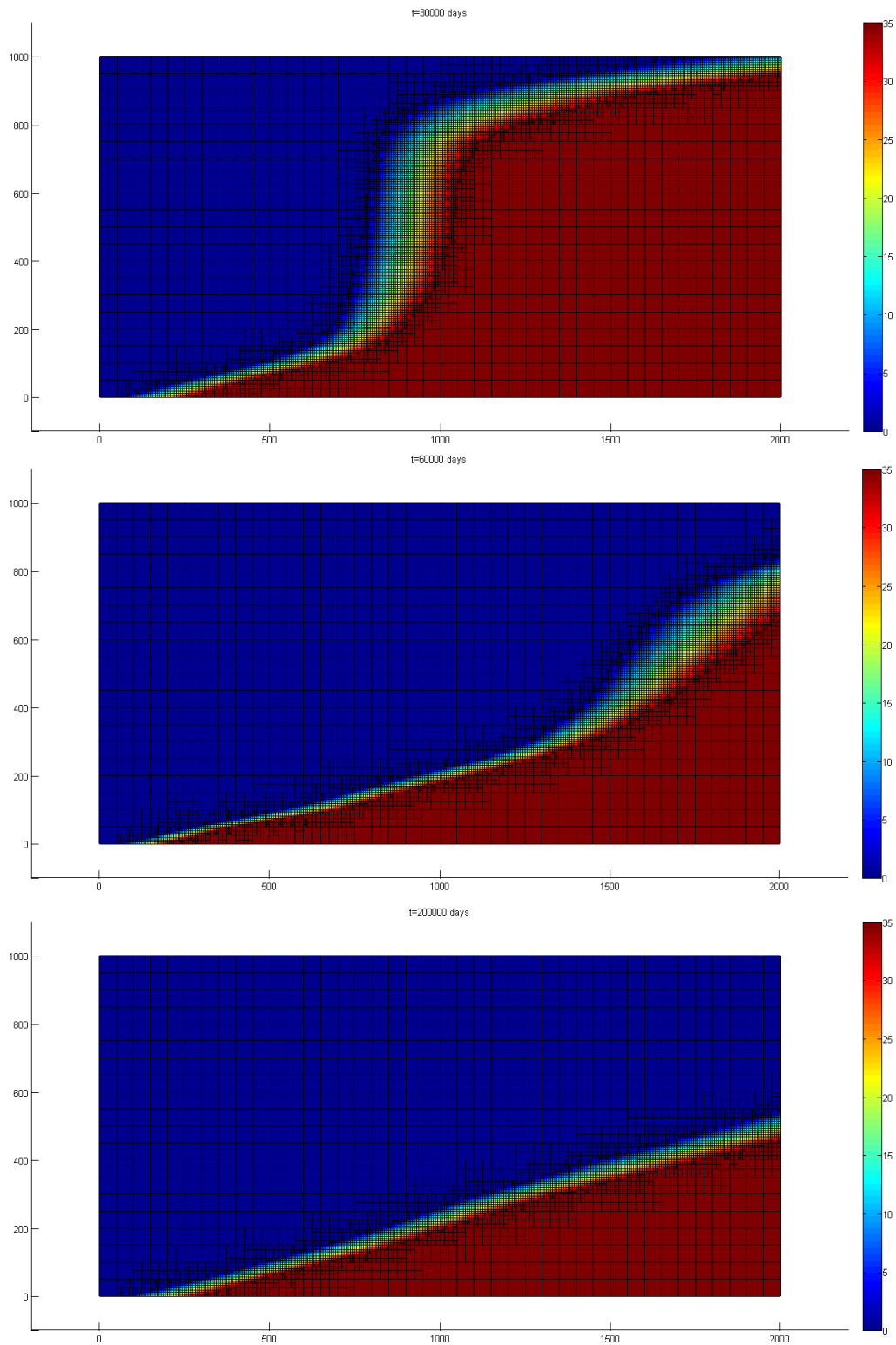


Figure 1.22: Concentration profile at the times $t = 30000$, $t = 60000$ and $t = 200000$ days for the test case 3.

1. A GENERALIZED FINITE VOLUME METHOD FOR A DENSITY DRIVEN FLOW COUPLED TO HEAT TRANSPORT IN POROUS MEDIA

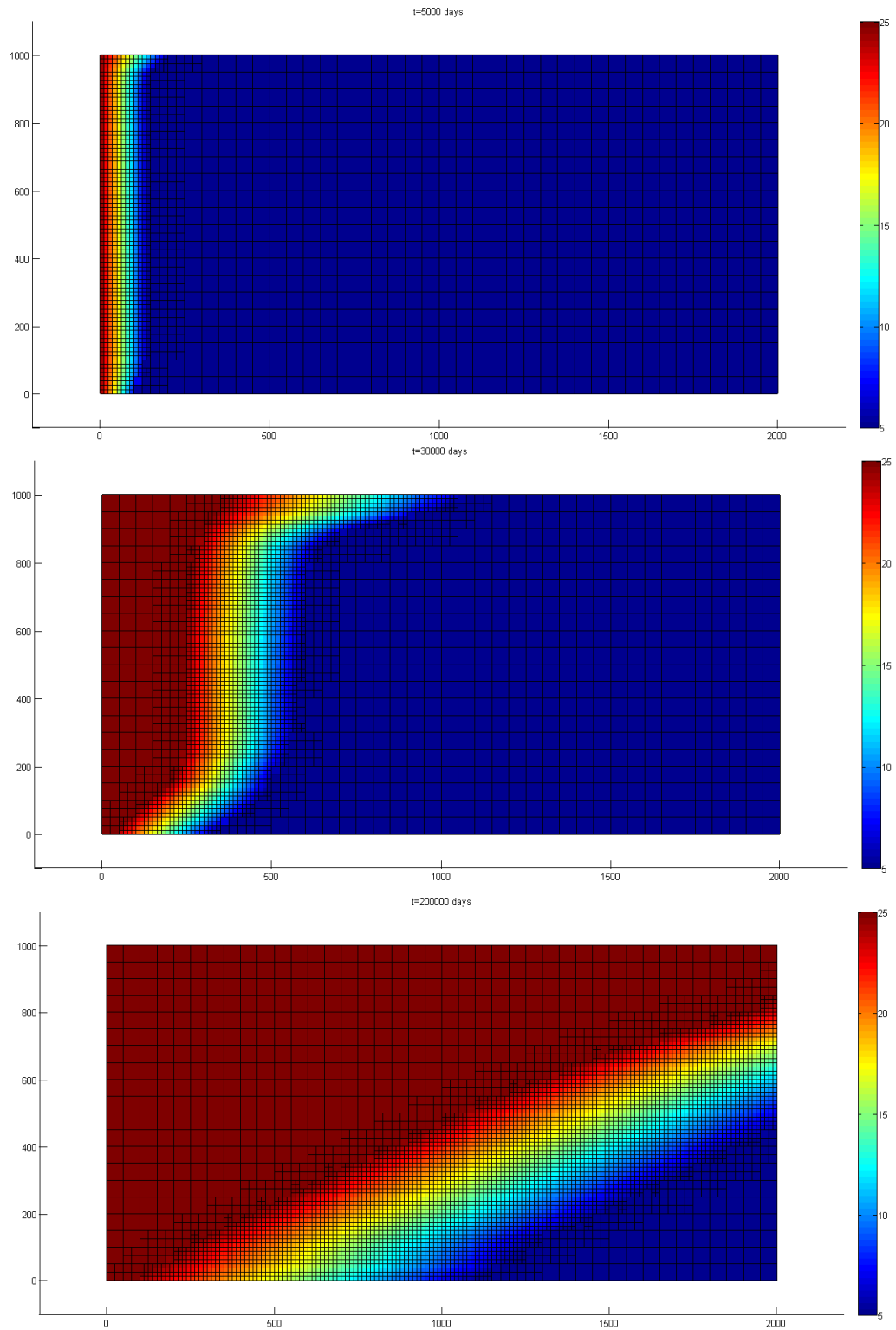
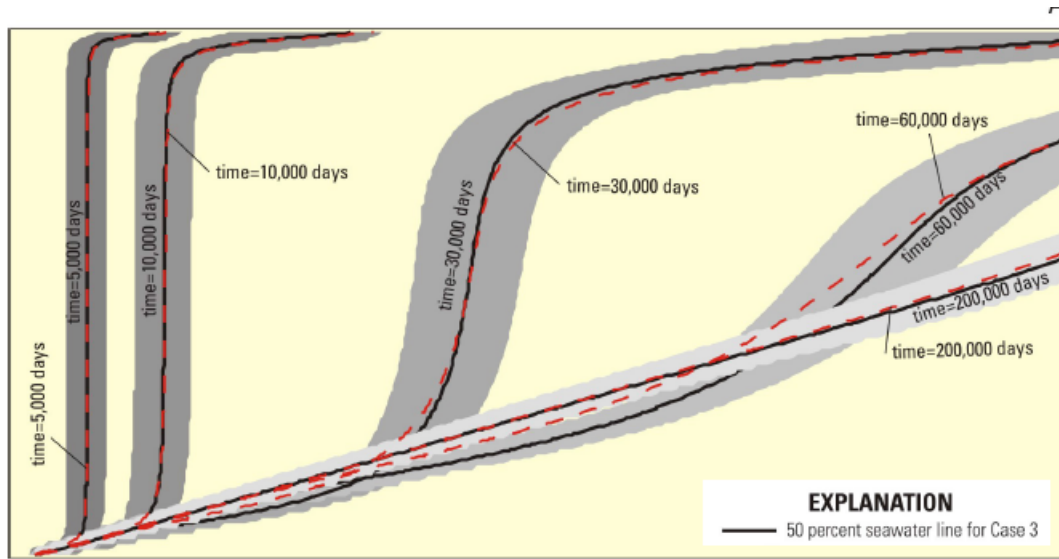
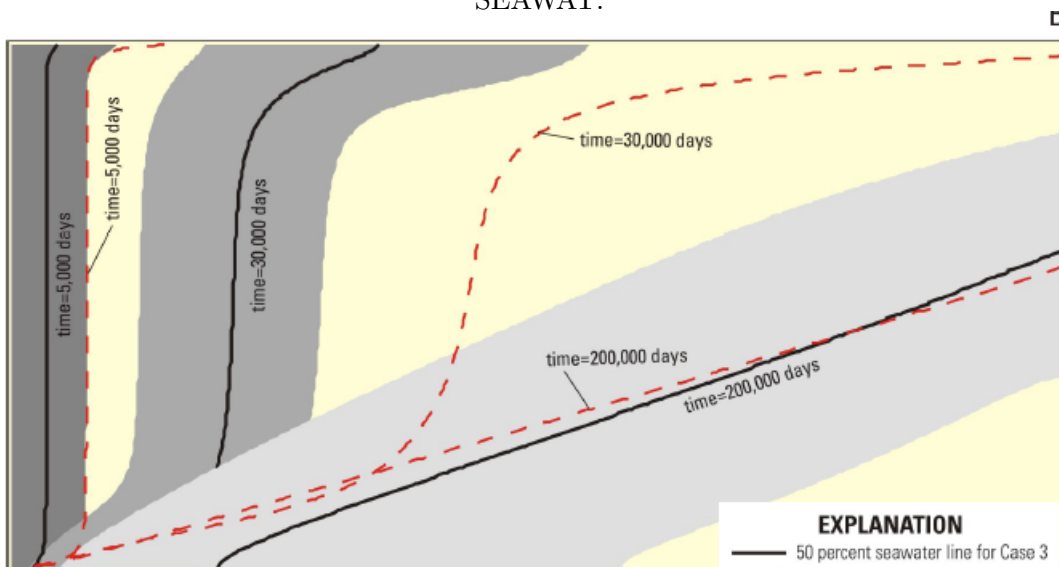


Figure 1.23: Temperature profile at the time $t = 5000$, $t = 30000$ and $t = 200000$ days.



The transient motion of the freshwater-saltwater transition zone for the test case 3 in SEAWAT.



The transient motion of the temperature transition zone for the test case 3 in SEAWAT.

Figure 1.24: Simulation for the test case 3 using SEAWAT.

Parameter	Value	Unit
Specific storage S_s	1.00×10^{-5}	m
Porosity θ	0.35	–
Reference viscosity μ_0	0.001	kg/m^3
Reference density ρ_0	1000	kg/m^3
Bulk density ρ_b	1761.5	kg/m^3
Reference concentration C_0	0	kg/m^3
Reference temperature Θ_0	25	C
Horizontal hydraulic conductivity \mathbf{K}_{h0}	10	m/d
Vertical hydraulic conductivity \mathbf{K}_{v0}	0.1	m/d
Longitudinal dispersivity \mathbf{a}_L	1	m
Transverse dispersivity \mathbf{a}_T	0.1	m
Diffusion coefficient D_m^C	1.00×10^{-10}	m/d
Bulk thermal diffusivity D_m^Θ	0.150309621	m/d
Distribution coefficient for concentration K_d^C	0	m^3/kg
Distribution coefficient for temperature K_d^Θ	2.00×10^{-4}	m^3/kg
A_1	2.394×10^{-5}	–
A_2	10	–
A_3	248.37	–
A_4	133.15	–
$\delta\mu/\delta C$	1.92×10^{-6}	m^2/d
$\delta\rho/\delta h$	4.46×10^{-3}	kg/m^4
$\delta\rho/\delta C$	0.7	–
$\delta\rho/\delta\Theta$	-0.375	$kg/(m^3C)$

Table 1.7: Parameter for the example problem

Numerical simulations of the inviscid Burgers equation with periodic boundary conditions and stochastic forcing

Résumé. Nous effectuons une étude numérique de l'équation de Burgers non visqueuse en dimension un d'espace, avec des conditions aux limites périodiques et un terme source stochastique de moyenne spatiale nulle. Ce terme source possède la régularité d'un bruit blanc en temps, tandis que nous considérons différentes régularités en espace. Pour les tests numériques, nous utilisons un schéma de volumes finis combinant une intégration en temps de type Euler-Maruyama avec le flux numérique de Godunov. Nous effectuons des simulations avec la méthode de Monte-Carlo et analysons les résultats pour différentes régularités en accordant une attention particulière au comportement en temps long. Il apparaît que la moyenne empirique des réalisations converge toujours vers la moyenne en espace de la condition initiale (déterministe) quand $t \rightarrow \infty$, comme c'est le cas pour la solution du problème sans terme source, même dans le cas où le terme stochastique est peu régulier en espace. Par ailleurs, la variance empirique converge elle aussi en temps long, vers une valeur qui dépend de la régularité et de l'amplitude du terme stochastique.

Abstract. We perform numerical simulations in the one-dimensional torus for the first order Burgers equation forced by a stochastic source term with zero spatial integral. We suppose that this source term is a white noise in time, and consider various regularities in space. For the numerical tests, we apply a finite volume scheme combining the Godunov numerical flux with the Euler-Maruyama integrator in time. Our Monte-Carlo simulations are analyzed in bounded time intervals as well as in the large time limit, for various regularities in space. The empirical mean always converges to the space-average of the (deterministic) initial condition as $t \rightarrow \infty$, just as the solution of the deterministic problem without source term, even if the stochastic source term is very rough. The empirical variance also stabilizes for large time, towards a limit which depends on the space regularity and on the intensity of the noise.

2.1 Introduction

Our aim is to numerically approximate solutions of the stochastically forced inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = g \quad (2.1.1)$$

in a bounded domain of unit length with periodic boundary conditions or, equivalently, on the torus $x \in \mathbb{S}^1$.

We suppose that the stochastic source term in (2.1.1) has zero space average

$$\int_{\mathbb{S}^1} g = 0 \quad (2.1.2)$$

in order to preserve the conservative character of the inviscid Burgers equation on the torus. Indeed, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the Cauchy problem for (2.1.1) together with a deterministic initial condition $u(\cdot, t = 0) = u_0$. Then, (2.1.2) implies that solutions satisfy the conservation property

$$\int_{\mathbb{S}^1} u = \int_{\mathbb{S}^1} u_0 \quad \forall t > 0.$$

Moreover, the stochastic source term g is assumed to behave as a white noise with respect to time $t \in [0, T)$. Then, the exact interpretation of (2.1.1) is not obvious. We recall that in the deterministic case where g is a given *smooth* function of time and space (continuously differentiable for instance, see e.g. [46]), only *discontinuous* solutions to the Cauchy problem for (2.1.1) can be rigorously defined for large time (discontinuities appear even if the initial condition is smooth), and a notion of *entropic* solution is necessary for uniqueness. In the case that g is stochastic, further notions of solutions are necessary [25, 26].

In this note, we consider space-time discrete versions of (2.1.1). Considering the inherent difficulties of (2.1.1), our numerical scheme combines standard discretization techniques for scalar first order conservation laws such as the inviscid Burgers equation with periodic boundary conditions [50, 12] together with standard discretization techniques for stochastic differential equations.

More precisely, we consider a finite volume discretization. Given $I \in \mathbb{N}^+$, we split the one-dimensional torus \mathbb{S}^1 into cells of uniform volume $\Delta x := 1/I$. Then, a discretization of (2.1.1) between two times $0 < t^n < t^{n+1}$ is typically obtained from the integral formula

$$\begin{aligned} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u(x, t^{n+1}) dx &= \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u(x, t^n) dx \\ &+ \int_{t^n}^{t^{n+1}} \left((u^2/2)((i-1/2)\Delta x, t) - (u^2/2)((i+1/2)\Delta x, t) \right) dt + \int_{t^n}^{t^{n+1}} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} g dx dt \end{aligned} \quad (2.1.3)$$

by defining a numerical solution u^n with a fixed time step $\Delta t = \Delta t^n := t^{n+1} - t^n > 0$ for all $n \in \{0, 1, \dots, N_T - 1\}$ with $N_T = T/\Delta t$ through the relation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{\Delta x} (F_{i+1/2}^n - F_{i-1/2}^n) + \alpha \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n \quad (2.1.4)$$

for all $n \in \{0, 1, \dots, N_T - 1\}$ and all $i \in \{1, \dots, I\}$, where the positive constant α represents the intensity of the noise and the space derivative is approximated in a conservative way through the definition of the fluxes $F_{i+1/2}^n$ [50]; the time process is discretized by an explicit Euler-Maruyama time integrator [43, 52], recalling that g is a white noise in time. We have thereby defined a Markov process $(u^n)_{n \in \mathbb{N}}$ with values in \mathbb{R}^I to discretize the cell-averages of the solution of (2.1.1), starting with the finite volume approximation of the deterministic initial condition u^0

$$u_i^0 = \frac{1}{\Delta x} \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} u_0(x).$$

We make precise definitions of the flux $F_{i+1/2}^n$ and of the stochastic source term G_i^n in the next section.

A first important question is whether, and how, the Markov process defined by (2.1.4) allows to approximate solutions of (2.1.1). In the deterministic case with $G^n = 0$, our discrete sequences $(u^n)_{n \in \mathbb{N}}$ are already known to converge for all $T > 0$, as the space and time steps tend to zero under a CFL condition [54, 31], toward the unique entropic solution of the inviscid Burgers equation with periodic boundary conditions

$$\frac{\partial u}{\partial t}(x, t) + \frac{\partial}{\partial x} \left(\frac{u^2}{2}(x, t) \right) = 0 \quad \text{for all } (x, t) \in \mathbb{S}^1 \times [0, T]. \quad (2.1.5)$$

But in order to be able to consider stochastic cases where $G^n \neq 0$, one still needs to define a notion of solution to (2.1.1). For the “space-time white noise” case in the sense of zero correlation-length in space and time, which is among the cases numerically approximated in this work, we are not aware of any existing mathematical notion of solution. A *stochastic entropic solution* has however already been rigorously defined in the case of a *colored noise* where g has sufficiently regular space variation.

In our one-dimensional setting with additive stochastic forcing, solutions of the *viscous* stochastically forced Burgers equation, that is with an additional diffusion term $-\nu \partial_{xx}^2 u$ on the left-hand-side of (2.1.1) ($\nu > 0$), have been shown to trajectoryally (i.e. for each realization) converge to the entropic solutions when $\nu \rightarrow 0$, as in the deterministic case. Note that the viscous stochastically forced Burgers equation has been much more studied than the inviscid one, from the pure theoretical as well as numerical [1] viewpoints. Additionally, let us mention [32] where *stochastic entropic solutions* to one-dimensional scalar conservation laws have been univoquely defined. In [25], *kinetic solutions* have been recently defined in any dimension (they coincide with the former in the case of space dimension one). Moreover we refer to [26] for a Hamilton-Jacobi reformulation of Burgers equation.

However, it is not clear whether, and how, a conservative numerical scheme as ours does indeed approximate solutions. In particular, it has been shown in [36] that the discretization of the nonlinear term has a lot of impact on the continuous limit of the viscous stochastic case. Although we use here a more usual discretization of the nonlinear term (flux) than in [36], which actually allows one to define the entropic solutions of Burgers equation in the deterministic case, we are not able to pass to the limit $\Delta t \rightarrow 0$ yet. During the completion of the present work, we remarked that approximations by a flux-splitting scheme (i.e. a special choice of the numerical flux) have been recently shown to converge to the stochastic entropic

solution of a scalar conservation law under a fixed CFL condition on the time step [8], in the case of a multiplicative noise which vanishes when $u = 0$.

Another important question with respect to the stochastic dynamics is the behaviour of u^n as $n \rightarrow \infty$, with the discretization of (2.1.1) being fixed. Note that, in deterministic cases without noise [22] or with a time-independent forcing with zero space average [2], the solutions of (2.1.5) are indeed already known to converge for large times to the constant $\int_{\mathbb{S}^1} u_0(x) dx$. Moreover, the existence of an invariant measure has been proved for sufficiently regular noise in space, using the equivalence with a Hamilton-Jacobi equation in [26], and with kinetic formalism in [25]. In this work, we numerically study the large time behaviour when the noise G^n is defined with various regularities in space and various intensities.

The stochastic Burgers equation with periodic boundary conditions has already been studied numerically by numerous authors, including its long-time statistics (usually from a physical more than mathematical point of view). Let us mention [27, 28], where some interesting physical motivations are explained. The numerical scheme which we propose here does not seem to have been used yet, in particular for large time simulations.

2.2 Numerical method

We define the two-point numerical flux $(u^2/2)((i-1/2)\Delta x, t)$ by the Godunov scheme [50]

$$F_{i-1/2} = \begin{cases} \frac{u_i^2}{2} & \text{if } u_{i-1} \leq u_i \leq 0 \\ \frac{u_{i-1}^2}{2} & \text{if } 0 \leq u_{i-1} \leq u_i \\ 0 & \text{if } u_{i-1} \leq 0 \leq u_i \\ \frac{u_{i-1}^2}{2} & \text{if } u_i \leq u_{i-1} \text{ and } \frac{u_i + u_{i-1}}{2} \geq 0 \\ \frac{u_i^2}{2} & \text{if } u_i \leq u_{i-1} \text{ and } \frac{u_i + u_{i-1}}{2} \leq 0. \end{cases} \quad (2.2.1)$$

This flux is known to be consistent in a weak sense, see e.g. [31, 12]. Furthermore, in the deterministic case with a smooth source term G_i^n , the numerical scheme is stable (i.e. entropy-satisfying) under the classical CFL condition

$$\Delta t^n \leq \frac{\Delta x}{\max_{i \in \{1, 2, \dots, I\}} \{|u_i^n|\}}, \quad (2.2.2)$$

so that the numerical solution converges to the unique entropy solution of (2.1.1) on $[0, T)$ for all $T > 0$ as $\max_{n=0 \dots N-1} \Delta t^n \rightarrow 0$, see e.g. [14].

We now turn to the stochastic source term g . On the one hand, we would like the function $t \rightarrow g(x, t)$ to have the regularity of a white noise for all x , because this is a natural source term from the physical viewpoint. On the other hand, a notion of stochastic entropic solution has only been defined when the function $x \rightarrow g(x, t)$ is smooth enough. In this note, we would like to explore numerically our model with various source terms of different space regularities.

Restricting to odd integers I , we define the discrete noise as the finite volume approximation

$$G_i^n = \sqrt{\frac{2}{I}} \left(\sum_{k=1}^{\frac{I-1}{2}} \left(\frac{C_k^n}{k^\beta} \cos(2\pi k \Delta x i) - \frac{S_k^n}{k^\beta} \sin(2\pi k \Delta x i) \right) \right) \quad (2.2.3)$$

of a Fourier series where C_k^n and S_k^n , in the Fourier coefficients C_k^n/k^β , S_k^n/k^β , are mutually independent identically distributed (i.i.d.) random variables with normal distribution $\mathcal{N}(0, 1)$ for all k and n , where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2 , and where the parameter β controls the regularity of the variations with respect to space: $\beta = 0$ corresponds to white noise in space whereas $\beta > 0$ corresponds to colored noise. The well-posedness theory for the corresponding continuous problem is only valid for $\beta > \frac{1}{2}$ [24]. We will check in the appendix that if $\beta = 0$,

$$\mathbb{E}(G_i^n G_j^m) = \delta_{m,n} [\delta_{i,j} - \frac{1}{I}]. \quad (2.2.4)$$

We will also check that for all $\beta \geq 0$, $\sum_{i=1}^I G_i^n = 0$ for all n , which is the discrete counterpart of the conservativity requirement $\int_{\mathcal{S}^1} g = 0$ on the source term. Note that the discrete noise (2.2.3) has the same scaling in Δt^n and Δx as in [32, 36].

We compute the empirical mean and variance estimators of the Markov chain entries (2.1.4) by the Monte-Carlo method

$$\mathbb{E}_M(u_i^n) := \frac{1}{M} \sum_{m=1}^M u_i^n(\omega_m), \quad \text{Var}_M(u_i^n) := \frac{1}{M} \sum_{m=1}^M |u_i^n(\omega_m) - \mathbb{E}_M(u_i^n)|^2,$$

invoking M i.i.d. realizations $u_i^n(\omega_m)$, $m = 1 \dots M$, computed with INM Gaussian numbers $\mathcal{N}(0, 1)$ (cf. (2.2.3)). Here N denotes the number of time iterations that one needs for the average of the realizations to reach the stationary solution zero of the deterministic problem with a good precision, see next section.

The number of volume elements is first fixed equal to $I = 101$, so that the number of Fourier modes in (2.2.3) is equal to $I - 1 = 100$ (recall that the constant mode is eliminated in order to ensure a zero space average).

The CFL condition (2.2.2) is naturally stochastic, i.e. dependent on the realization. To maintain a fixed number of time steps for each realization, we assume that $|u_i^n| \leq \bar{u}$ with $\bar{u} := 10$ set arbitrarily, which allows to fix the CFL condition as $\Delta t = 0.1 \Delta x$. Of course, we are aware that this may introduce a bias in the probability law which is simulated. However, we reject the realization in the case that $\max_{i \in \{1, 2, \dots, I\}} \{|u_i^n|\} > 10$ for some $n \in \{1, \dots, N\}$. Our assumption $|u_i^n| \leq \bar{u}$ was hardly violated. When $\alpha = 1$ and $\beta = 0$, which corresponds to the case of high amplitude and low space regularity on the noise term, the frequency of explosion is around 1 time per 4000 realisations. Meanwhile for small amplitude ($\alpha = 0.1$) or higher space regularity ($\beta = 1$ or $\beta = 2$), there is no explosion in the simulations.

We have performed numerical tests with a variety of values of α and β . In each case, we chose the number of realizations M large enough so that the confidence intervals for $\mathbb{E}_M(u_i^n)$ and $\text{Var}_M(u_i^n)$ are small enough.

The influence of the number of realizations on the evaluations of $\|\text{Var}_U(x)\|_{L^1(0,1)}(t)$, in the cases that $\alpha = 1, \beta = 0$ and $\alpha = 1, \beta = 1$ with initial condition $u_0(x) = \sin(2\pi x)$ is presented in Figure 2.2.

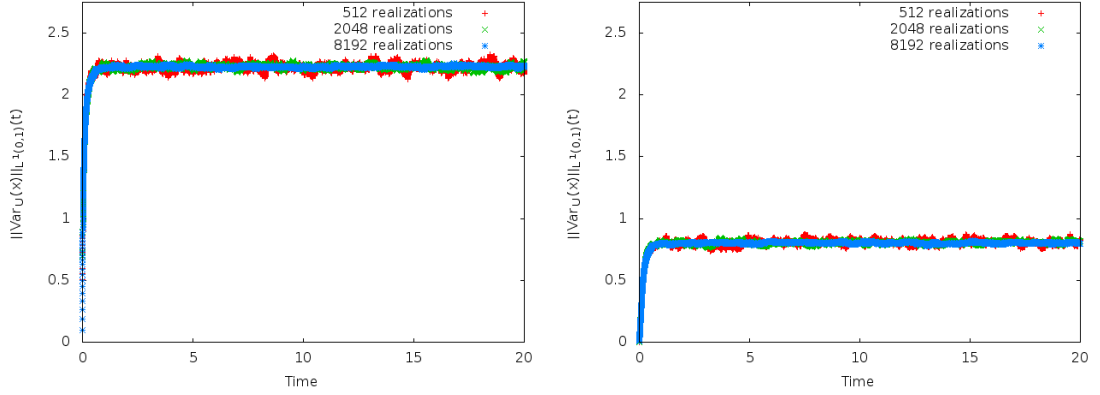


Figure 2.1: The confidence interval of $\bar{\mu}$ at level 0.95

It is remarkable that large time convergence seems to occur even when $\beta < \frac{1}{2}$. However, to conclude about the continuous limit, one should still discuss the space and time discretization in those cases. Moreover, a complete understanding of ergodic properties would require the study of higher-order moments as well.

In order to discuss the number of realizations, we define the random variable X_m for the m -th realization by

$$X_m = \int_0^1 |u(\omega_m)(x, T)| dx. \quad (2.2.5)$$

For each realization, we use the same scheme. Thus if we perform M realizations, the series $\{X_m\}_{m=1,2,\dots,M}$ is independently and identically distributed. We denote the corresponding expectation by $\bar{\mu}$ and its variance by $\bar{\sigma}^2$.

According to the central limit theorem, there holds

$$\mathbb{P}\left(-z \leq \frac{\frac{1}{M} \sum_{m=1}^M X_m - \bar{\mu}}{\bar{\sigma}^2/M} \leq z\right) \xrightarrow{M \rightarrow +\infty} \int_{-z}^z \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \quad \forall z \in \mathbb{R}. \quad (2.2.6)$$

The confidence interval $[I_1(M), I_2(M)]$ of $\bar{\mu}$ at the confidence level $1 - p$ is given by

$$[I_1(M), I_2(M)] = \left[\frac{1}{M} \sum_{m=1}^M X_m - z_p \cdot \sqrt{\bar{\sigma}^2/M}, \frac{1}{M} \sum_{m=1}^M X_m + z_p \cdot \sqrt{\bar{\sigma}^2/M}\right], \quad (2.2.7)$$

where the quantile z_p is such that $\int_{-z_p}^{z_p} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt = 1 - p$. Since the variance $\bar{\sigma}^2$ is unknown, we need to approximate it by:

$$\bar{\sigma}^2 \approx \text{Var}^M(X) := \frac{1}{M} \sum_{m=1}^M X_m^2 - \left(\frac{1}{M} \sum_{m=1}^M X_m\right)^2 \quad (2.2.8)$$

for a large enough value of M . Thus

$$[I_1(M), I_2(M)] \approx \left[\frac{1}{M} \sum_{m=1}^M X_m - z_p \cdot \sqrt{\frac{\text{Var}^M(X)}{M}}, \frac{1}{M} \sum_{m=1}^M X_m + z_p \cdot \sqrt{\frac{\text{Var}^M(X)}{M}} \right] \quad (2.2.9)$$

We fix $\alpha = 1$, $\beta = 0$, $T = 20$. We choose 95% as the confidence level, which corresponds to $z_{0.05} \approx 1.96$.

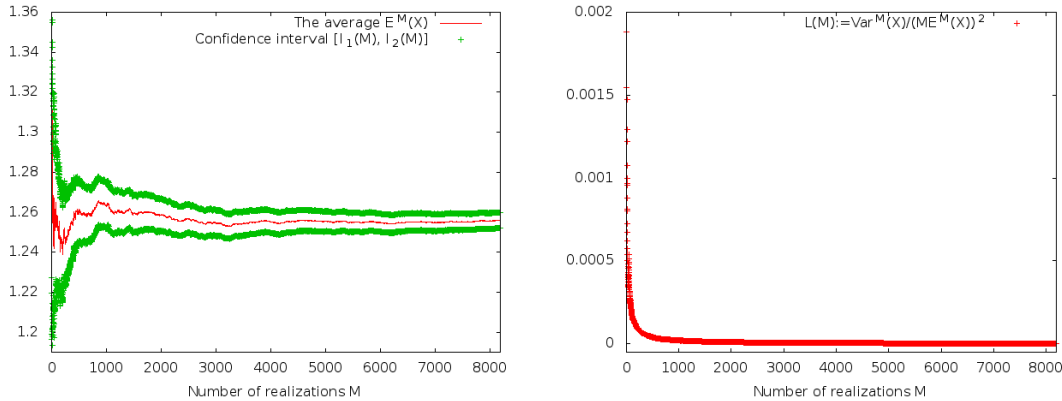


Figure 2.2: Confidence interval of $\bar{\mu}$ at level 0.95

On the left-hand side, the average $E^M(X) := \frac{1}{M} \sum_{m=1}^M X_m$ and the confidence interval $[I_1(M), I_2(M)]$ are plotted as functions of M , while the function $L(M) := \frac{\text{Var}^M(X)}{M(E^M(X))^2}$ is plotted on the right-hand figure. It turns out that $\frac{\text{Var}^M(X)}{M(E^M(X))^2}$ is a decreasing function of M and that $\frac{\text{Var}^{8192}(X)}{8192(E^{8192}(X))^2} \approx 2.5 \times 10^{-6}$.

Next, we study the second moment of $\{X_m\}_{m=1,2,\dots,M}$. We recall that $\{X_m\}_{m=1,2,\dots,M}$ is independently and identically distributed, so that we can use the notations $E(X_m^2)$ for the second moment and $\text{Var}(X_m^2)$ for the corresponding variance for all $m = 1, 2, \dots, M$. According to (2.2.6) and (2.2.7), the confidence interval of $E(X_m^2)$, namely $[I_3(M), I_4(M)]$, is given by

$$[I_3(M), I_4(M)] = \left[\frac{1}{M} \sum_{m=1}^M X_m^2 - z_p \cdot \sqrt{\text{Var}(X_m^2)/M}, \frac{1}{M} \sum_{m=1}^M X_m^2 + z_p \cdot \sqrt{\text{Var}(X_m^2)/M} \right], \quad (2.2.10)$$

where $\text{Var}(X_m^2) = E((X_m^2)^2) - (E(X_m^2))^2$, which we approximate by

$$\text{Var}(X_m^2) \approx \text{Var}^M(X^2) := \frac{1}{M} \sum_{m=1}^M X_m^4 - \left(\frac{1}{M} \sum_{m=1}^M X_m^2 \right)^2. \quad (2.2.11)$$

We fix $\alpha = 1$, $\beta = 0$, $T = 20$. We choose 95% as the confidence level, which corresponds to $z_{0.05} \approx 1.96$.

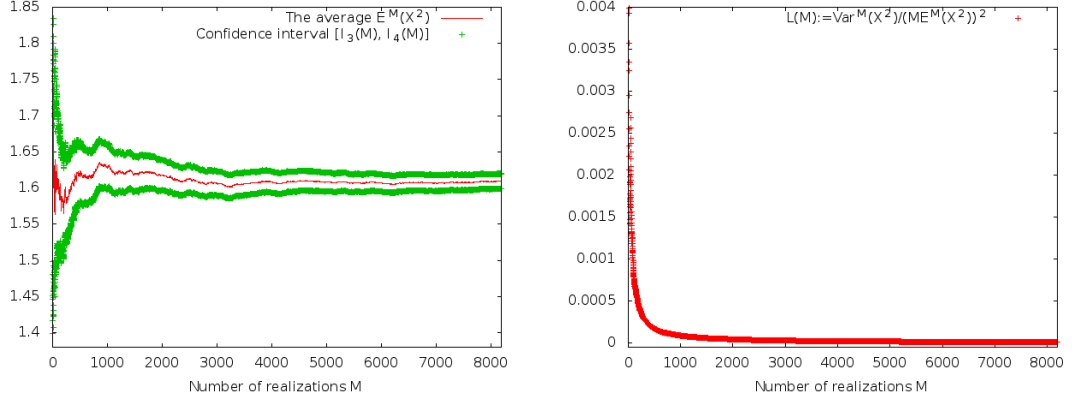


Figure 2.3: Confidence interval of $E(X_m^2)$ at level 0.95

On the left-hand side, the average $E^M(X^2) := \frac{1}{M} \sum_{m=1}^M X_m^2$ and the confidence interval $[I_3(M), I_4(M)]$ are plotted as functions of M , while the function $L(M) := \frac{\text{Var}^M(X^2)}{M(E^M(X^2))^2}$ is plotted on the right-hand figure. It turns out that $\frac{\text{Var}^M(X^2)}{M \cdot (E^M(X^2))^2}$ is a decreasing function of M and that $\frac{\text{Var}^{8192}(X^2)}{8192(E^{8192}(X^2))^2} \approx 1.04 \times 10^{-5}$.

2.3 Results and discussion

The numerical tests are all performed with the deterministic initial condition $u_0(x) = \sin(2\pi x)$, $x \in \mathbb{S}^1$. We performed 8192 realizations for each computation. In the figures 2.4, 2.5 and 2.6, we present on the left-hand side comparisons between $E_M(u_i^n)$, and \bar{u}_i^n in the case without noise, for different values of the discrete time $t^n = n\Delta t$. On the right-hand side, we show comparisons between u_i^n for some realization and \bar{u}_i^n in the case without noise, at the same values of the discrete time $t^n = n\Delta t$.

We first fix three cases $\alpha = 0.1, \beta = 0$, $\alpha = 0.1, \beta = 1$ and $\alpha = 0.1, \beta = 2$.

In Figure 2.9 we rescale the comparisons for the empirical average at the times $t = 1$ and $t = 20$.

We then fix $\alpha = 1, \beta = 0$, $\alpha = 1, \beta = 1$ and $\alpha = 1, \beta = 2$. We present the corresponding results in the figures 2.10, 2.11 and 2.12.

In Figure 2.13 we rescale again the comparisons at the times $t = 1$ and $t = 20$.

Numerical solutions of the periodic inviscid Burgers equation (2.1.1) look quite different according to the space regularity of the stochastic source term.

Whereas single realizations look like discontinuous functions in the case that $\beta = 1$, they do not even look like functions in the case that $\beta = 0$.

We remark that the average of the realizations, also on a large time interval, is very close to the deterministic solution, namely the solution of the partial differential equation where the source term g is such that $g = 0$. However, this average appears to smoothen the discontinuities, so

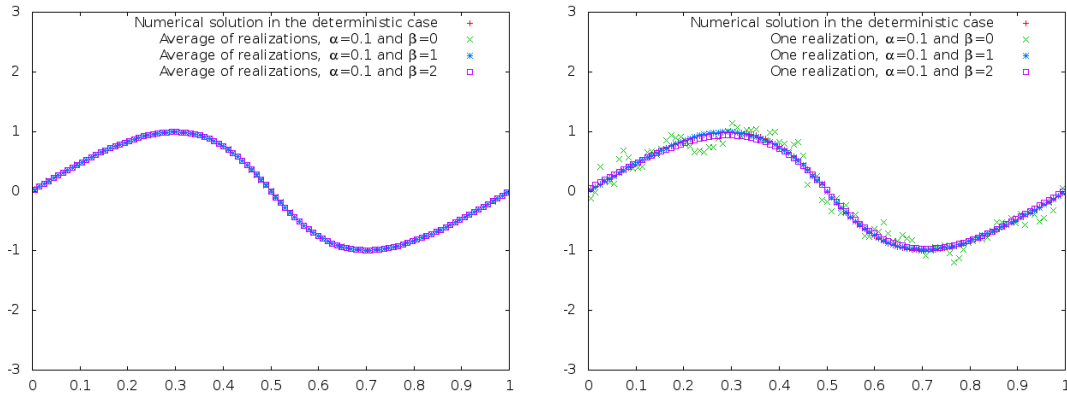


Figure 2.4: Empirical average (left) and one realization (right) at $t = 0.05$

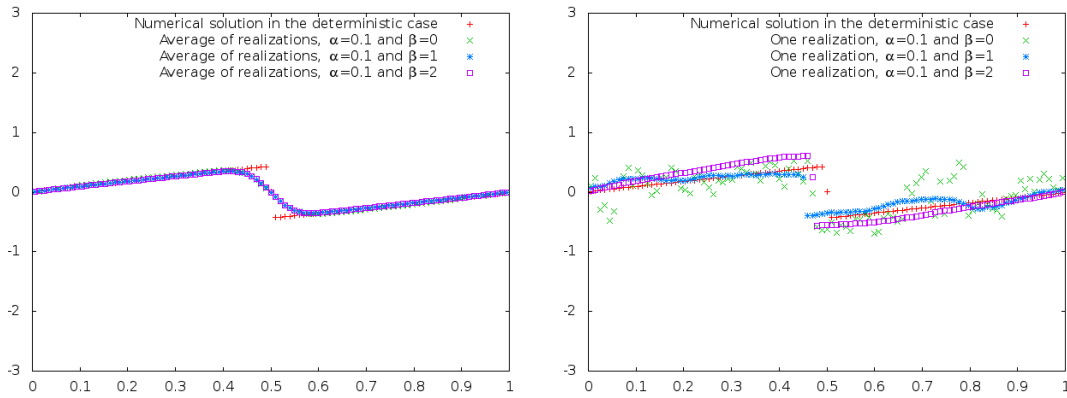


Figure 2.5: Empirical average (left) and one realization (right) at $t = 1$

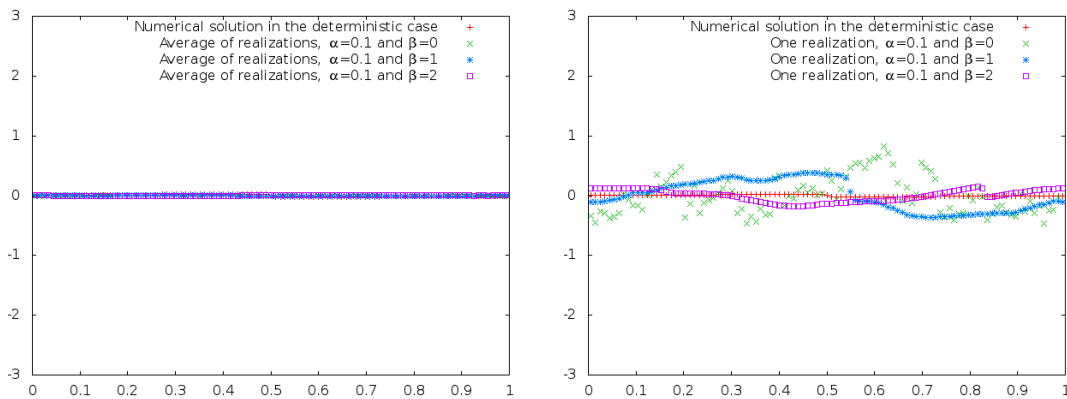


Figure 2.6: Empirical average (left) and one realization (right) at $t = 20$

2. NUMERICAL SIMULATIONS OF THE INVISCID BURGERS EQUATION WITH PERIODIC BOUNDARY CONDITIONS AND STOCHASTIC FORCING

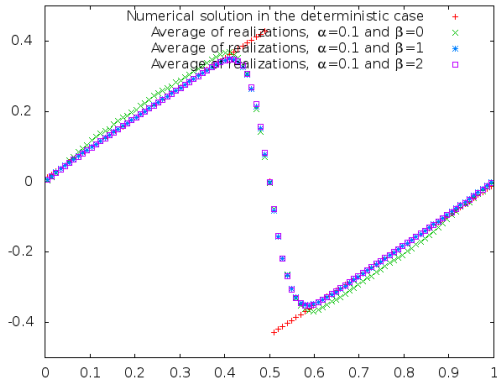


Figure 2.7: $t = 1$

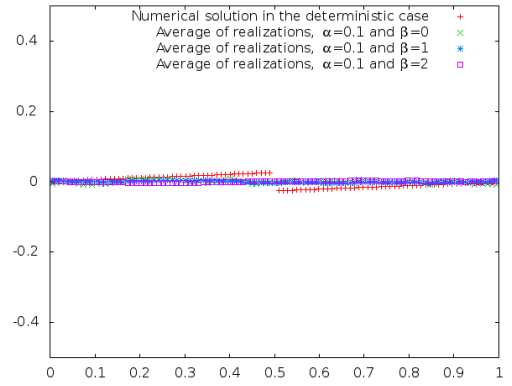


Figure 2.8: $t = 20$

Figure 2.9: Rescaled graphs for $t = 1$ and $t = 20$

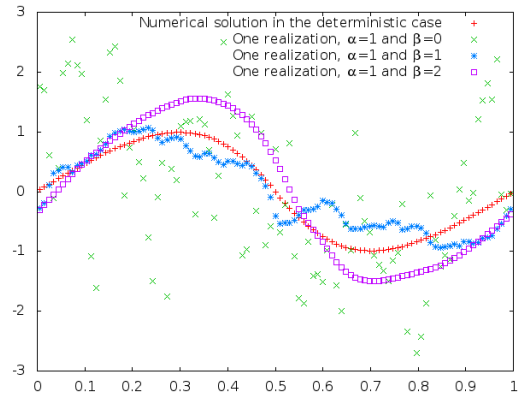
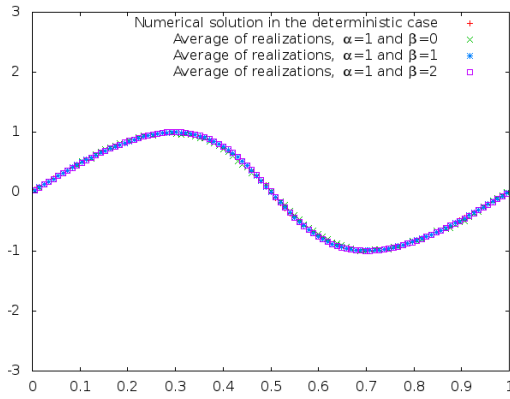


Figure 2.10: Empirical average (left) and one realization (right) at $t = 0.05$

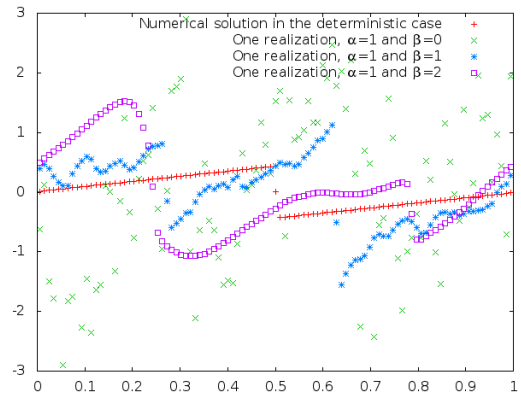
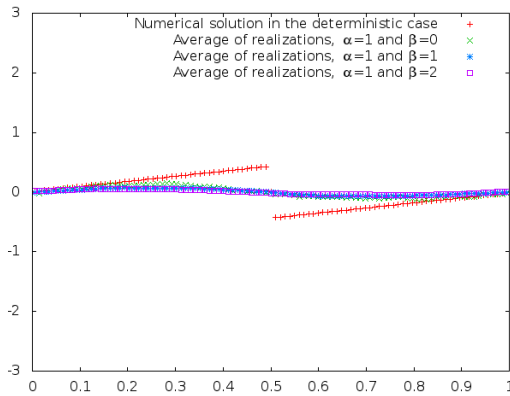


Figure 2.11: Empirical average (left) and one realization (right) at $t = 1$

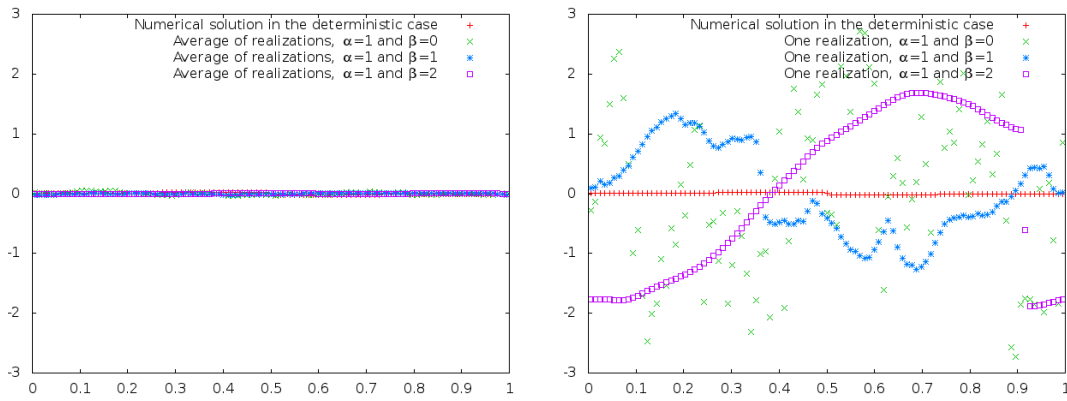


Figure 2.12: Empirical average (left) and one realization (right) at $t = 20$

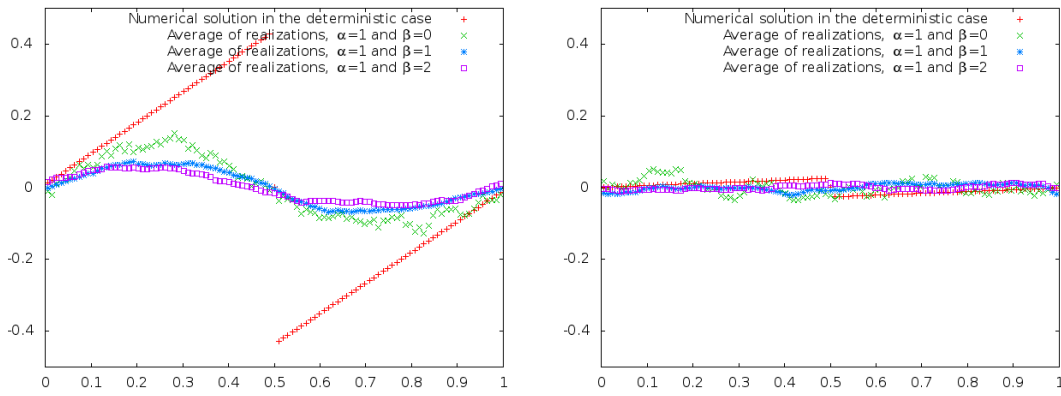


Figure 2.13: Empirical average (left) and one realization (right) at $t = 20$

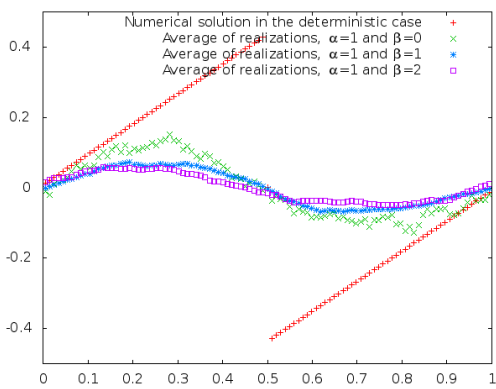


Figure 2.14: $t = 1$

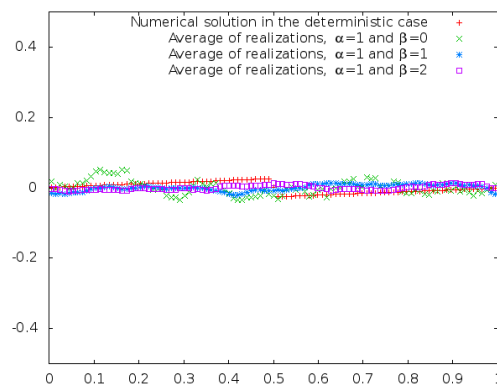


Figure 2.15: $t = 20$

that the addition of a stochastic right-hand-side seems to have a similar effect as the addition of a small diffusion term.

We now compare the ways in which the expectation and the variance converge towards their stationary values when the parameters α and β vary. In particular we observe in Figure 2.16 faster convergence in time as α and β increase, and larger variance in the large time limit as α increases and β decreases; of course, this is precisely as it could be expected.

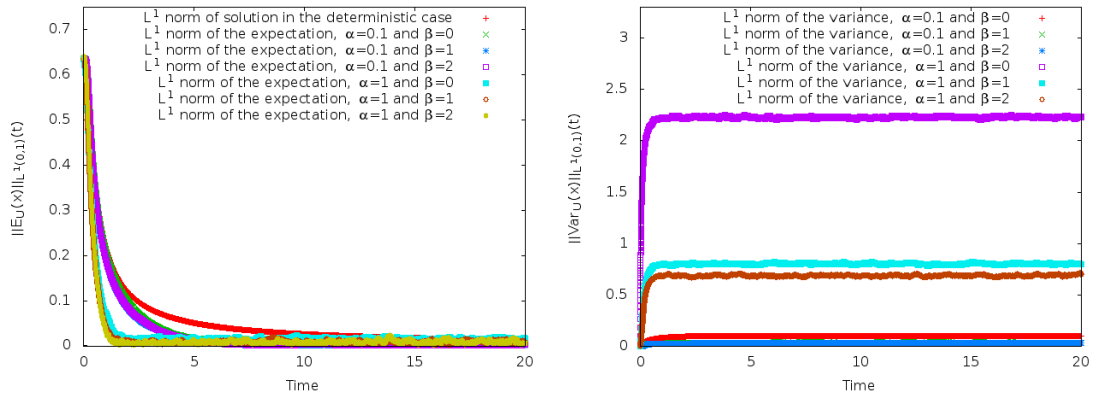


Figure 2.16: L^1 -norm of the expectation (left) and of the variance (right) of the solution

2.4 Some conclusions

We have considered cases of a white noise in space and time ($\alpha = 0.1, \beta = 0$) and that of a smoother in space version ($\alpha = 0.1, \beta = 1$ and $\alpha = 0.1, \beta = 2$). In those cases, the average of the realizations is a good approximation of the deterministic solution and as time tends to infinity it converges to the space-average of the initial function [22]. However, while the deterministic solution is discontinuous, the average of the realizations smoothens it out.

We have also considered corresponding cases with a larger amplitude, namely ($\alpha = 1, \beta = 0$, $\alpha = 1, \beta = 1$ and $\alpha = 1, \beta = 2$); then our numerical results for one realization are very dispersed. The average smoothens the deterministic solution and goes faster to equilibrium than in the cases where $\alpha = 0.1$. However, when $\beta = 0$, a single realization at $t = 20$ does not seem to be a function anymore, which is consistent with the fact that the existence proof of an invariant measure for the solution of the continuous problem does not hold in this case.

Our results are still far from complete. A forthcoming work will also involve a numerical study of the limit $I \rightarrow \infty$, as well as a more detailed study of single realizations. We also propose to compute various distribution functions. Moreover we will be performing other numerical simulations with an initial condition which is not symmetric. In this case the shock travels with a certain speed and we will try to analyse the influence of the noise on the shock.

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2.5 APPENDIX: the numerical approximation of the noise

We recall the numerical scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) + \alpha \Delta t \sqrt{\frac{1}{\Delta t \Delta x}} G_i^n \quad \forall n \in \mathbb{N} \quad \forall i \in \{1, \dots, I\}, \quad (2.5.1)$$

and propose to study the the formula

$$G_i^n = \sqrt{\frac{2}{I}} \left(\sum_{k=1}^{\frac{I-1}{2}} \left(\frac{C_k^n}{k^\beta} \cos(2\pi k \Delta x i) - \frac{S_k^n}{k^\beta} \sin(2\pi k \Delta x i) \right) \right), \quad (2.5.2)$$

where I is an odd number, and C_k^n and S_k^n follow the Gaussian law $\mathcal{N}(0, 1)$ for fixed k and n . In the following, we note $x_i := \Delta x i$ and $x_j := \Delta x j$.

We first check the conservation property, namely that $\sum_{i=1}^I G_i^n = 0$ for all $n \in \mathbb{N}$. Indeed

$$\begin{aligned} \sum_{i=1}^I G_i^n &= \sum_{i=1}^I \sqrt{\frac{2}{I}} \left(\sum_{k=1}^{\frac{I-1}{2}} \left(\frac{C_k^n}{k^\beta} \cos(2\pi k x_i) - \frac{S_k^n}{k^\beta} \sin(2\pi k x_i) \right) \right) \\ &= \sqrt{\frac{2}{I}} \sum_{k=1}^{\frac{I-1}{2}} \frac{1}{k^\beta} C_k^n \sum_{i=1}^I \cos(2\pi k x_i) - \sqrt{\frac{2}{I}} \sum_{k=1}^{\frac{I-1}{2}} \frac{1}{k^\beta} S_k^n \sum_{i=1}^I \sin(2\pi k x_i). \end{aligned} \quad (2.5.3)$$

In order to compute $\sum_{i=1}^I \cos(2\pi k x_i)$ and $\sum_{i=1}^I \sin(2\pi k x_i)$, we recall the Dirichlet kernel:

$$D_n(2\pi x) = \sum_{p=-n}^n e^{2\pi i p x} = 1 + 2 \sum_{p=1}^n \cos(2\pi p x) = \frac{\sin((n + \frac{1}{2})2\pi x)}{\sin(\frac{2\pi x}{2})} \quad (2.5.4)$$

where \mathbf{i} is the imaginary unit. We deduce from (2.5.4) the equality:

$$\sum_{p=1}^n \cos(2\pi p x) = \left(\frac{\sin((n + \frac{1}{2})2\pi x)}{\sin(\frac{2\pi x}{2})} - 1 \right) \cdot \frac{1}{2}, \quad (2.5.5)$$

so that

$$\sum_{i=1}^I \cos(2\pi k x_i) = \sum_{i=1}^I \cos(2\pi i \frac{k}{I}) = \left(\frac{\sin((I + \frac{1}{2})\frac{2\pi k}{I})}{\sin(\frac{2\pi k}{2I})} - 1 \right) \cdot \frac{1}{2}. \quad (2.5.6)$$

Since $\sin((I + \frac{1}{2})\frac{2\pi k}{I}) = \sin(\frac{2\pi k}{2I} + 2\pi k) = \sin(\frac{2\pi k}{2I})$, we deduce from (2.5.6) that

$$\sum_{i=1}^I \cos(2\pi k x_i) = (1 - 1) \cdot \frac{1}{2} = 0.$$

Next we check that $\sum_{i=1}^I \sin(2\pi k x_i) = 0$

$$\begin{aligned}
 \sum_{i=1}^I \sin(2\pi k x_i) &= \sum_{i=1}^I \sin(2\pi k \frac{i}{I}) \\
 &= \sum_{i=1}^{\frac{I-1}{2}} [\sin(2\pi k \frac{i}{I}) + \sin(2\pi k \frac{I-i}{I})] + \sin(2\pi k \frac{I}{I}) \\
 &= \sum_{i=1}^{\frac{I-1}{2}} [\sin(2\pi k \frac{i}{I}) + \sin(2\pi k \frac{I}{I} - 2\pi k \frac{i}{I})] \\
 &= \sum_{i=1}^{\frac{I-1}{2}} [\sin(2\pi k \frac{i}{I}) - \sin(2\pi k \frac{i}{I})] \\
 &= 0
 \end{aligned} \tag{2.5.7}$$

We conclude that $\sum_{i=1}^I G_i^n = 0$ for all $n \in \mathbb{N}$ and $\beta \geq 0$.

Next, we compute $E(G_i^n G_j^m)$ in the case of the white noise in space, namely when $\beta = 0$. We have that:

$$\begin{aligned}
 G_i^n G_j^m &= \frac{2}{I} \left(\sum_{k=1}^{\frac{I-1}{2}} \{C_k^n \cos(2\pi k x_i) - S_k^n \sin(2\pi k x_i)\} \right) \left(\sum_{l=1}^{\frac{I-1}{2}} \{C_l^m \cos(2\pi l x_j) - S_l^m \sin(2\pi l x_j)\} \right) \\
 &= \frac{2}{I} \sum_{k=1}^{\frac{I-1}{2}} \sum_{l=1}^{\frac{I-1}{2}} (C_k^n C_l^m \cos(2\pi k x_i) \cos(2\pi l x_j) + S_k^n S_l^m \sin(2\pi k x_i) \sin(2\pi l x_j) \\
 &\quad - C_k^n S_l^m \cos(2\pi k x_i) \sin(2\pi l x_j) - S_k^n C_l^m \sin(2\pi k x_i) \cos(2\pi l x_j)).
 \end{aligned} \tag{2.5.8}$$

We recall that S_k^n and C_l^m are independent random variables for all k and l , which implies that

$$E(S_k^n C_l^m) = E(S_k^n) E(C_l^m); \tag{2.5.9}$$

moreover also remembering that both

$$S_k^n \sim \mathcal{N}(0, 1) \text{ and } C_l^m \sim \mathcal{N}(0, 1), \tag{2.5.10}$$

we deduce from (2.5.9)

$$E(S_k^n C_l^m) = 0, \tag{2.5.11}$$

and that

$$E(C_k^n S_l^m) = 0 \tag{2.5.12}$$

as well. Similarly, if $k \neq l$ or $n \neq m$, C_k^n and C_l^m are i.i.d. random variables, so that

$$E(C_k^n C_l^m) = 0 \quad \text{if } k \neq l \text{ or } n \neq m \tag{2.5.13}$$

and

$$\mathbb{E}(S_k^n S_l^m) = 0 \quad \text{if } k \neq l \text{ or } n \neq m. \quad (2.5.14)$$

If $m = n$ and $k = l$, then in view of (2.5.10)

$$\mathbb{E}(C_k^n C_l^m) = \mathbb{E}(C_k^n \cdot C_k^n) = \text{Var}(C_k^n) - (\mathbb{E}(C_k^n))^2 = 1 \quad (2.5.15)$$

and we have a similar result for $\mathbb{E}(S_k^n S_l^m)$.

We conclude that $\mathbb{E}(C_k^n C_l^m) = \mathbb{E}(S_k^n S_l^m) = \delta_{m,n} \delta_{k,l}$. We deduce from (2.5.8) that:

$$\begin{aligned} \mathbb{E}(G_i^n G_j^m) &= \delta_{m,n} \frac{2}{I} \sum_{k=1}^{\frac{I-1}{2}} \sum_{l=1}^{\frac{I-1}{2}} \delta_{k,l} (\cos(2\pi k x_i) \cos(2\pi l x_j) + \sin(2\pi k x_i) \sin(2\pi l x_j)) \\ &= \delta_{m,n} \frac{2}{I} \sum_{k=1}^{\frac{I-1}{2}} (\cos(2\pi k x_i) \cos(2\pi k x_j) + \sin(2\pi k x_i) \sin(2\pi k x_j)) \\ &= \delta_{m,n} \frac{2}{I} \sum_{k=1}^{\frac{I-1}{2}} \cos(2\pi k(x_i - x_j)) \\ &= \delta_{m,n} \frac{2}{I} \sum_{k=1}^{\frac{I-1}{2}} \frac{e^{2\pi i k(x_i - x_j)} + e^{-2\pi i k(x_i - x_j)}}{2} \\ &= \delta_{m,n} \frac{1}{I} \sum_{k=1}^{\frac{I-1}{2}} (e^{2\pi i k(x_i - x_j)} + e^{-2\pi i k(x_i - x_j)}) \\ &= \delta_{m,n} \frac{1}{I} \sum_{k=-\frac{I-1}{2}, k \neq 0}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} \\ &= \frac{\delta_{m,n}}{I} \left[\sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} - 1 \right] \end{aligned} \quad (2.5.16)$$

Next we consider the term $\sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)}$:

if $i = j$, $x_i - x_j = 0$, then

$$\sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} = I \cdot 1 = I,$$

but if $i \neq j$, $x_i - x_j \neq 0$, then we use again the Dirichlet kernel (2.5.4) to obtain:

$$\begin{aligned} \sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} &= \frac{\sin((\frac{I-1}{2} + \frac{1}{2}) \cdot 2\pi(x_i - x_j))}{\sin(\pi(x_i - x_j))} \\ &= \frac{\sin(\pi I(i - j))}{\sin(\pi(x_i - x_j))} \\ &= 0. \end{aligned} \tag{2.5.17}$$

Thus

$$\sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} = I\delta_{i,j}.$$

Finally we deduce that

$$\begin{aligned} \mathbb{E}(G_i^n G_j^m) &= \frac{\delta_{m,n}}{I} \left[\sum_{k=-\frac{I-1}{2}}^{\frac{I-1}{2}} e^{2\pi i k(x_i - x_j)} - 1 \right] \\ &= \frac{\delta_{m,n}}{I} [I\delta_{i,j} - 1] \\ &= \delta_{m,n} \left[\delta_{i,j} - \frac{1}{I} \right], \end{aligned} \tag{2.5.18}$$

so that G_i^n corresponds to a discrete white noise.

Convergence of a finite volume scheme for a first order conservation law involving a Q -Wiener process

Résumé. Nous démontrons la convergence d'une méthode de volumes finis pour une loi de conservation avec un terme source multiplicatif faisant intervenir un processus Q -Wiener. Nous présentons des estimations a priori pour la solution discrète dont en particulier une estimation de type BV faible. A l'aide d'une interpolation en temps, nous démontrons deux inégalités entropiques vérifiées par la solution discrète, ce qui nous permet de prouver que la solution discrète converge selon une sous-suite vers une solution stochastique faible entropique à valeurs mesures de la loi de conservation.

Abstract. We study a time explicit finite volume method with an upwind scheme for a conservation law driven by a multiplicative source term involving a Q -Wiener process. We present some a priori estimates including a weak BV estimate. After performing a time interpolation, we prove two entropy inequalities for the discrete solution and show that it converges up to a subsequence to a stochastic measure-valued entropy solution of the conservation law in the sense of Young measures.

3.1 Introduction

In this Chapter, we study the convergence of a finite volume scheme for the discretization of the stochastic scalar conservation law

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x) & \omega \in \Omega, \quad x \in \mathbb{T}^d, \end{cases} \quad (3.1.1)$$

where \mathbb{T}^d is the d -dimensional torus and that $W(x, t)$ is a Q -Wiener process [18]. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(\mathbb{T}^d)$ with $\operatorname{Tr} Q < +\infty$, $\{e_m\}_{m \geq 1}$ be an orthonormal basis in $L^2(\mathbb{T}^d)$ diagonalizing Q , and $\{\lambda_m\}_{m \geq 1}$ be the corresponding eigenvalues, so that

$$Qe_m = \lambda_m e_m$$

for all $m \geq 1$. Since Q is of trace-class,

$$\operatorname{Tr} Q = \sum_{m=1}^{\infty} \langle Qe_m, e_m \rangle_{L^2(\mathbb{T}^d)} = \sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0. \quad (3.1.2)$$

for some positive constant Λ_0 . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration (\mathcal{F}_t) [42] and $\{\beta_m(t)\}_{m \geq 1}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$; the process W defined by

$$W(x, t) = \sum_{m=1}^{\infty} \beta_m(t) Q^{\frac{1}{2}} e_m(x) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (3.1.3)$$

is a Q -Wiener process in $L^2(\mathbb{T}^d)$ [cf. [35], Definition 2.6, p. 20], and the series defined by (3.1.3) is convergent in $C([0, T], L^2(\Omega \times \mathbb{T}^d))$ [cf. [35], p. 20]. We recall that a Brownian motion $\beta(t)$ is called an (\mathcal{F}_t) Brownian motion if it is (\mathcal{F}_t) -adapted and the increment $\beta(t) - \beta(s)$ is independent of \mathcal{F}_s for every $0 \leq s < t$. We suppose furthermore that $e_m \in L^\infty(\mathbb{T}^d)$ for $m = 1, 2, \dots$ and that there exist a positive constant Λ_1 such that

$$\sum_{m=1}^{\infty} \lambda_m \|e_m\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (3.1.4)$$

Moreover we assume the following hypotheses which we denote by (H) :

- $u_0 \in L^2(\mathbb{T}^d)$
- $f : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing Lipschitz continuous function with Lipschitz constant C_f such that $f(0) = 0$,
- $g : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Lipschitz continuous function with Lipschitz constant C_g such that $|g(u)| \leq M_g$ for some positive constant M_g ,
- The vector $\mathbf{v} \in \mathbb{R}^d$ is constant and its Euclidian norm is given by V .

Next we introduce some further notations,

- Let $E[\cdot]$ denote the expectation, namely the integral over Ω with respect to the probability measure P . In general for a random variable Y from the probability space (Ω, \mathcal{F}, P) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, if Y is integrable for the measure P , then $E[Y] = \int_{\Omega} Y(\omega) P(d\omega)$.
- We denote by $\mathcal{N}_{\omega}^2(0, T; L^2(\mathbb{T}^d))$ the subclass of $L^2(\Omega \times \mathbb{T}^d \times (0, T))$ consisting of predictable $L^2(\mathbb{T}^d)$ -valued processes [cf. [19], p. 98].

3.1.1 Previous results

A number of articles have been devoted to the study of scalar conservation laws with a multiplicative stochastic forcing term involving a white noise in time.

Let us mention the one-dimensional study of Feng-Nualart [32], where the authors introduced a notion of entropy solution in order to prove the existence and uniqueness of an entropy solution. Chen-Ding-Karlsen [15] extended the work of Feng-Nualart to the multi-dimensional case. They proved a uniform spatial BV bound by means of vanishing viscosity approximations. Moreover they proved the temporal equicontinuity of approximations in $L^1(\Omega \times D \times (0, T))$, uniformly in the viscosity coefficient.

Debussche-Vovelle [23] proved the existence and uniqueness of a kinetic solution for multi-dimensional scalar conservation laws in a d -dimensional torus driven by a general multiplicative space-time noise. Hofmanová [38] then presented a Bhatnagar-Gross-Krook-like approximation of this problem. Applying the stochastic characteristics method, the author established the existence of an approximate solution and proved its convergence to the kinetic solution introduced by [23].

Under the assumption (H) and by the definition of Young measure-valued solutions, Bauzet-Vallet-Wittbold [6] proved the existence and uniqueness of the solution to the multi-dimensional Cauchy problem in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$ in the case of a multiplicative one-dimensional white noise in time. Since the method consists in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in this case), the authors required rather mild assumptions on the data. In Bauzet-Vallet-Wittbold [7] the authors investigated a corresponding Dirichlet problem in a bounded domain of \mathbb{R}^d .

Bauzet-Charrier-Gallouët [8] studied explicit flux-splitting finite volume discretizations of multi-dimensional nonlinear scalar conservation laws with monotonic flux perturbed by a multiplicative one-dimensional white noise in time with a given initial function in $L^2(\mathbb{R}^d)$. Under a stability condition on the time step, they proved the convergence of the finite volume approximation towards the unique stochastic entropy solution of the corresponding initial value problem. Then Bauzet-Charrier-Gallouët [9] studied the case of a more general flux. Let us also mention results of Holden-Risebro [39] and Bauzet [5] on a bounded domain of \mathbb{R}^d , as well as an article of Kröker-Rohde [45] in a one-dimensional context.

In a recent study, Audusse-Boyaval-Gao-Hilhorst [3] performed numerical simulations in the one-dimensional torus for the first order Burgers equation forced by a stochastic source term. The source term is supposed to be a white noise in time and various regularities in space are considered. The authors applied the Monte-Carlo method, and observed that the empirical

mean converges to the space average of the initial condition as the time t tends to infinity and that the empirical variance stabilizes for large time.

The organisation of this Chapter is as follows: In section 3.1.2 we define a weak stochastic entropy solution and a measure-valued entropy solution of Problem (3.1.1) and recall basic results from probability theory. In section 3.2, we apply a finite volume method to the Problem (3.1.1) and define the discrete solution $\{u_{\mathcal{T},k}\}$. In section 3.3, we present an estimate of the discrete noise term as well as an a priori estimate on the discrete solution $\{u_{\mathcal{T},k}\}$. We then give a weak BV estimate which is essential in the sequel. In section 3.4, we prove two inequalities, a discrete entropy inequality and a continuous entropy inequality on the discrete solution which are fundamental for the convergence proof. Then, in section 3.5, we prove that the discrete solution $\{u_{\mathcal{T},k}\}$ converges up to a subsequence to a measure-valued entropy solution \mathbf{u} in the sense of Young measures. In the appendix, we present an example of a space time noise which was previously used in numerical computations [3] and we discuss the conditions (3.1.2) and (3.1.4) in that special case.

Funaki-Gao-Hilhorst-Weber [34] will show that the measured-value entropy solution \mathbf{u} is unique and coincides with the unique weak stochastic entropy solution; this will ensure that the whole approximate sequence $\{u_{\mathcal{T},k}\}$ converges to u .

3.1.2 Continuous problem

Next we define a weak stochastic entropy solution of Problem (3.1.1) as it has been done in [8]:

Definition 3.1 (*Entropy solution of Problem (3.1.1)*)

A function $u \in \mathcal{N}_{\omega}^2(0, T, L^2(\mathbb{T}^d)) \cap L^{\infty}(0, T; L^2(\Omega \times \mathbb{T}^d))$ is a weak entropy solution of the stochastic scalar conservation law (3.1.1) with the initial condition $u_0 \in L^2(\mathbb{T}^d)$, if P-a.s in Ω ,

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \{ \eta(u) \partial_t \varphi(x, t) + F^{\eta}(u) \mathbf{v} \cdot \nabla_x \varphi(x, t) \} dx dt \\ & + \int_{\mathbb{T}^d} \int_0^T \eta'(u) g(u) \varphi(x, t) dW(x, t) dx \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \eta''(u) g^2(u) \varphi(x, t) e_m^2(x) dx dt \geq 0 \end{aligned} \quad (3.1.5)$$

with

$$F^{\eta}(\tau) = \int_0^{\tau} \eta'(\sigma) f'(\sigma) d\sigma \quad (3.1.6)$$

for all $\varphi \in \mathcal{C} := \{ \varphi \in C_0^{\infty}(\mathbb{T}^d \times [0, T]), \varphi \geq 0 \}$ and for all $\eta \in \mathcal{A}$ where \mathcal{A} is the set of $C^3(\mathbb{R})$ convex functions η such that the support of η'' is compact.

Definition 3.2 (*Measure-valued entropy solution of Problem (3.1.1)*)

A function \mathbf{u} of $\mathcal{N}_{\omega}^2(0, T, L^2(\mathbb{T}^d \times (0, 1))) \cap L^{\infty}(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ is a measure-valued entropy solution of the stochastic scalar conservation law (3.1.1) with the initial condition

$u_0 \in L^2(\mathbb{T}^d)$, if P-a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \{ \eta(\mathbf{u}(\cdot, \alpha)) \partial_t \varphi(x, t) + F^\eta(\mathbf{u}(\cdot, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) \} d\alpha dx dt \\ & + \int_{\mathbb{T}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha)) g(\mathbf{u}(\cdot, \alpha)) \varphi(x, t) d\alpha dW(x, t) dx \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha)) g^2(\mathbf{u}(\cdot, \alpha)) \varphi(x, t) e_m^2(x) d\alpha dx dt \geq 0 \end{aligned} \quad (3.1.7)$$

We introduce Itô formula (cf. [42], p. 153) which is a basic tool in this article.

Lemma 3.3 (*Itô formula*)

Assume that the stochastic process $W(\tau)$ is an (\mathcal{F}_t) -Brownian motion multiplied by a positive constant $q^{\frac{1}{2}}$ and $X(\tau)$ is an \mathbb{R} -valued stochastic process given by

$$X(\tau) = X(0) + \int_0^\tau \psi(t) dt + \int_0^\tau \theta(t) dW(t), \quad 0 \leq t \leq T,$$

where $X(0)$ is \mathcal{F}_0 -measurable, ψ is (\mathcal{F}_t) -adapted and measurable in (ω, t) such that

$$\int_0^T \|\psi(t)\|_{\mathbb{R}} dt < \infty \quad \text{P-a.s.}$$

and θ is an (\mathcal{F}_t) -adapted and measurable process such that

$$\text{P} \left(\int_0^T \theta^2(t) dt < \infty \right) = 1.$$

Suppose that the function $\mathcal{G}(x_1, x_2) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is such that its partial derivatives $\frac{\partial \mathcal{G}}{\partial x_2}$, $\frac{\partial \mathcal{G}}{\partial x_1}$ and $\frac{\partial^2 \mathcal{G}}{\partial x_1^2}$ are continuous on $\mathbb{R} \times [0, T]$. Then P-a.s. for all $\tau \in [0, T]$,

$$\begin{aligned} \mathcal{G}(X(\tau), \tau) &= \mathcal{G}(X(0), 0) + \int_0^\tau \left\{ \frac{\partial \mathcal{G}}{\partial x_2}(X(t), t) + \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \psi(t) \right\} dt \\ &+ \int_0^\tau \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \cdot \theta(t) dW(t) \\ &+ \frac{q}{2} \int_0^\tau \frac{\partial^2 \mathcal{G}}{\partial x_1^2}(X(t), t) \cdot \theta^2(t) dt. \end{aligned} \quad (3.1.8)$$

□

Lemma 3.4 (*Itô isometry*) (cf. [42], p. 144)

Assume that $\{\beta_i(t)\}_{i=1,2}$ are independent (\mathcal{F}_t) -Brownian motions, and let $X = (X_t(\omega))_{t \in (0, T)}$ and $Y = (Y_t(\omega))_{t \in (0, T)}$ be (\mathcal{F}_t) -adapted and measurable stochastic processes such that $X, Y \in L^2(\Omega \times (0, T))$. Then, it holds

$$\text{E} \left[\left(\int_0^T X_t d\beta_i(t) \right) \left(\int_0^T Y_t d\beta_j(t) \right) \right] = \text{E} \left[\int_0^T X_t Y_t dt \right] \cdot \delta_{i,j}, \quad (3.1.9)$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, which in turn implies that

$$\mathbb{E} \left[\left(\int_0^T X_t d\beta(t) \right)^2 \right] = \mathbb{E} \left[\int_0^T X_t^2 dt \right], \quad (3.1.10)$$

where $\beta(t)$ is an (\mathcal{F}_t) -Brownian motion. □

3.2 The numerical scheme

Definition 3.5 (*Admissible mesh*) An admissible mesh \mathcal{T} of \mathbb{T}^d for the discretization is given by a family of disjoint polygonal connected subset of \mathbb{T}^d such that \mathbb{T}^d is the union of the closure of the elements of \mathcal{T} and the common interface of any two control volumes is included in a hyperplane of \mathbb{T}^d . It is assumed that

$$h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty,$$

and that, for some $\bar{\alpha} \in \mathbb{R}^+$, we have

$$\bar{\alpha}h^d \leq |K| \text{ and } |\partial K| \leq \frac{1}{\bar{\alpha}}h^{d-1}, \quad \text{for all } K \in \mathcal{T}, \quad (3.2.1)$$

so that

$$\frac{|\partial K|}{|K|} \leq \frac{1}{\bar{\alpha}^2 h}, \quad (3.2.2)$$

where we note

- ∂K the boundary of the control volume K .
- x_K a point in the control volume K .
- $|K|$ the d -dimensional Lebesgue measure of K .
- $|\partial K|$ the $(d-1)$ -dimensional Lebesgue measure of ∂K .
- $\mathcal{N}(K)$ the set of control volumes neighbors of the control volume K .
- $\sigma_{K,L}$ the common interface between K and L for all $L \in \mathcal{N}(K)$.
- \mathcal{E}_K the set of interfaces of the control volume K .
- \mathcal{E} the set of all the interfaces of the mesh \mathcal{T} .
- $\tilde{\mathcal{E}}$ the set of pairs of adjacent control volumes, which is given by $\tilde{\mathcal{E}} = \{(K, L) \in \mathcal{T}^2, K \neq L, |\sigma_{K,L}| \neq 0\}$.
- $\mathbf{n}_{K,L}$ the unit normal vector which is perpendicular to the interface $\sigma_{K,L}$, outward to the control volume K , for all $L \in \mathcal{N}(K)$.

Consider an admissible mesh \mathcal{T} in the sense of Definition 3.5. In order to compute an approximation of u on $(0, T)$, we take $N \in \mathbb{N}^+$ and define the time step $k = \frac{T}{N}$. In this way

$(0, T) = \bigcup_{n=0}^{N-1} (nk, (n+1)k)$. We set $t^n = nk$ for all $n = 0, 1, 2, \dots, N$ and assume that k and h satisfy a Courant-Friedrichs-Lewy (CFL) condition: $k \leq \frac{\bar{\alpha}^2}{VC_f} h$.

The discrete unknowns are u_K^n , for all $n \in \{1, 2, \dots, N\}$ and for all $K \in \mathcal{T}$. The discrete initial condition $\{u_K^0, K \in \mathcal{T}\}$ is given by:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx \quad \text{for all } K \in \mathcal{T}, \quad (3.2.3)$$

and $\{u_K^n\}$ satisfies the following upwind scheme:

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} f(u_{K|L}^n) = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n), \quad (3.2.4)$$

for all $n \in \{0, 1, \dots, N-1\}$ and for all $K \in \mathcal{T}$, where the upwind value is defined by

$$u_{K|L}^n = \begin{cases} u_K^n & \text{if } \mathbf{v} \cdot \mathbf{n}_{K,L} \geq 0 \\ u_L^n & \text{if } \mathbf{v} \cdot \mathbf{n}_{K,L} < 0. \end{cases} \quad (3.2.5)$$

And the discrete noise terms are given by

$$\begin{aligned} W_{M,K}(t) &= \sum_{m=1}^M \sqrt{\lambda_m} \beta_m(t) e_K^m, \\ W_K(t) &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_K^m, \end{aligned} \quad (3.2.6)$$

where $e_K^m = \frac{1}{|K|} \int_K e_m(x) dx$ for all $K \in \mathcal{T}$. Moreover we denote by $W_{M,K}^n$ and W_K^n the values of $W_{M,K}$ and W_K at time $t = nk$ respectively.

Next we define the approximate finite volume solution $u_{\mathcal{T},k}$ on $\mathbb{T}^d \times (0, T)$ from the discrete unknowns u_K^n , for all $n \in \{0, 1, \dots, N-1\}$ and for all $K \in \mathcal{T}$, that:

$$u_{\mathcal{T},k}(x, t) = u_K^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k), \quad (3.2.7)$$

where the set $\{u_K^0, K \in \mathcal{T}\}$ is defined by (3.2.3).

The main result of this Chapter is the following Theorem:

Theorem 3.6 (Convergence to a measure-valued entropy solution of Problem (3.1.1)) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfying that*

$$\frac{k}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Then there exist a function $\mathbf{u} \in \mathcal{N}_{\omega}^2(0, T, L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $\{u_{\mathcal{T},k}\}$ converges to \mathbf{u} in the sense of Young measures. Moreover \mathbf{u} is measure-valued entropy solution of Problem (3.1.1) in the sense of Definition 3.2.

In a forthcoming article, we propose to prove that there exists a unique measure-valued entropy solution \mathbf{u} of Problem (3.1.1), which coincides with the unique stochastic entropy solution u in the sense of Definition 3.1. This property implies that the whole sequence $\{u_{\mathcal{T},k}\}$ converges weakly to u .

Next we discuss the noise term. We define for later use that:

$$\begin{aligned} W_{M,\mathcal{T},k}(x,t) &= W_{M,K}^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k), \\ W_{\mathcal{T},k}(x,t) &= W_K^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k). \end{aligned} \quad (3.2.8)$$

$$W_{\mathcal{T}}(x,t) = W_K(t) \quad \text{for } x \in K \quad \text{and } t \in (0, T). \quad (3.2.9)$$

Lemma 3.7 *Given by the definitions of $W_{M,K}(t)$ and $W_K(t)$ as in (3.2.6), we have that*

$$\begin{aligned} \mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] &= 0 \\ \text{and } \mathbb{E}[W_K^{n+1} - W_K^n] &= 0 \end{aligned}$$

Proof. Remark that for each $m \in \mathbb{N}^+$ and each $n \in \{0, 1, \dots, N-1\}$

$$\beta_m(t^{n+1}) - \beta_m(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n),$$

where $\mathcal{N}(0, t^{n+1} - t^n)$ denotes the Gaussian law $\mathcal{N}(\mu, \sigma^2)$ with density $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$ in the case $\mu = 0$ and $\sigma^2 = t^{n+1} - t^n$, so that, given M and K

$$\mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] = 0.$$

and that

$$\mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] = \sum_{m=1}^M \sqrt{\lambda_m} \mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] e_K^m = 0. \quad (3.2.10)$$

Since

$$(e_K^m)^2 = \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \leq \frac{1}{|K|} \int_K e_m^2(x) dx \leq \frac{1}{|K|},$$

we have that

$$\mathbb{E}[(W_K^n)^2] = \sum_{m=1}^{\infty} \lambda_m t^n (e_K^m)^2 \leq \frac{t^n}{|K|} \sum_{m=1}^{\infty} \lambda_m < \infty,$$

where we have used that $\text{Var}[\beta_m(t_n)] = \text{Var}[\beta_m(t_n) - \beta(0)] = t_n$. Thus, $W_K^n \in L^2(\Omega)$ and in particular $W_K^n \in L^1(\Omega)$, i.e. $\mathbb{E}[|W_K^n|] < \infty$. Therefore

$$\begin{aligned} \mathbb{E}[W_K^{n+1} - W_K^n] &= \mathbb{E}\left[\sum_{m=1}^{\infty} \sqrt{\lambda_m} (\beta_m(t^{n+1}) - \beta_m(t^n)) e_K^m \right] \\ &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] e_K^m \\ &= 0. \end{aligned} \quad (3.2.11)$$

Lemma 3.8 *Suppose that the coefficients $\{\lambda_m\}_{m \geq 1}$ satisfy (3.1.2), then*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0, \quad (3.2.12)$$

holds for all $M \in \mathbb{N}^+$, $n \in \{0, 1, \dots, N-1\}$ and $K \in \mathcal{T}$.

Proof: We have that for fixed M , n and K ,

$$W_{M,K}^{n+1} - W_{M,K}^n = \sum_{m=1}^M \sqrt{\lambda_m} (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{1}{|K|} \int_K e_m(x) dx, \quad (3.2.13)$$

therefore

$$\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 = \left(\sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2.$$

We take the expectation of both sides to obtain:

$$\begin{aligned} & \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{m_1=1}^M (\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) \sqrt{\lambda_{m_1}} e_K^{m_1} \right) \right. \\ & \quad \left. \times \left(\sum_{m_2=1}^M (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)) \sqrt{\lambda_{m_2}} e_K^{m_2} \right) \right] \\ &= \mathbb{E} \left[\sum_{m=1}^M (\beta_k(t^{n+1}) - \beta_k(t^n))^2 \lambda_m \right] \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ & \quad + \mathbb{E} \left[2 \sum_{m_1 \neq m_2} (\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)) \sqrt{\lambda_{m_1}} e_K^{m_1} \sqrt{\lambda_{m_2}} e_K^{m_2} \right] \\ &= \sum_{m=1}^M \mathbb{E} \left[(\beta_k(t^{n+1}) - \beta_k(t^n))^2 \right] \lambda_m \\ & \quad + 2 \sum_{m_1 \neq m_2} \mathbb{E} \left[(\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)) \right] \sqrt{\lambda_{m_1}} e_K^{m_1} \sqrt{\lambda_{m_2}} e_K^{m_2}. \end{aligned}$$

Since $\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)$ and $\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)$ are independent and

$$\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n)$$

$$\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n)$$

we have

$$\begin{aligned} & \mathbb{E}[(\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n))(\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n))] \\ &= \mathbb{E}[\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)] \mathbb{E}[\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)] \\ &= 0. \end{aligned} \quad (3.2.14)$$

Therefore

$$\mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] = \sum_{m=1}^M \mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))^2] \lambda_m \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2. \quad (3.2.15)$$

Next we use that

$$\text{Var}[\beta_m(t^{n+1}) - \beta_m(t^n)] = \mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))^2] - \left(\mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))] \right)^2, \quad (3.2.16)$$

and

$$\text{Var}[\beta_m(t^{n+1}) - \beta_m(t^n)] = t^{n+1} - t^n, \quad (3.2.17)$$

$$\mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))] = 0, \quad (3.2.18)$$

to deduce that

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] &= \sum_{m=1}^M (t^{n+1} - t^n) \lambda_m \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &= (t^{n+1} - t^n) \sum_{m=1}^M \lambda_m \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &\leq (t^{n+1} - t^n) \sum_{m=1}^M \lambda_m \sum_{K \in \mathcal{T}} \int_K e_m^2(x) dx \\ &\leq (t^{n+1} - t^n) \sum_{m=1}^{\infty} \lambda_m \\ &\leq (t^{n+1} - t^n) \Lambda_0. \end{aligned} \quad (3.2.19)$$

Corollary 3.9 *There holds:*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0. \quad (3.2.20)$$

Proof: We first show the limiting property

$$\|W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t)\|_{L^2(\Omega \times \mathbb{T}^d)} \rightarrow 0 \text{ as } M \rightarrow \infty, \quad (3.2.21)$$

for all $t \in (0, T)$.

$$\begin{aligned}
 & \|W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t)\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\
 &= \mathbb{E} \left[\int_{\mathbb{T}^d} (W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t))^2 dx \right] \\
 &= \mathbb{E} \left[\sum_{K \in \mathcal{T}} |K| \left(\frac{1}{|K|} \int_K (W_M(x, t) - W(x, t)) dx \right)^2 \right] \\
 &\leq \mathbb{E} \left[\sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \int_K 1 dx \int_K (W_M(x, t) - W(x, t))^2 dx \right] \\
 &\leq \mathbb{E} \left[\int_{\mathbb{T}^d} dx (W_M(x, t) - W(x, t))^2 \right] \\
 &\rightarrow 0 \text{ as } M \rightarrow \infty,
 \end{aligned}$$

since the series defined by (3.1.3) is convergent in $C([0, T], L^2(\Omega \times \mathbb{T}^d))$. We have that,

$$\begin{aligned}
 & \left| \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} - \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \right| \\
 &\leq \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)) - (W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\
 &= \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})) - (W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\
 &\leq \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})\|_{L^2(\Omega \times \mathbb{T}^d)} + \|W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \\
 &\rightarrow 0 \text{ as } M \rightarrow \infty,
 \end{aligned}$$

that is to say

$$\lim_{M \rightarrow \infty} \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} = \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)}.$$

In view of (3.2.12), we obtain that

$$\|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} = \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0,$$

where we take the limit $M \rightarrow \infty$ to obtain:

$$\lim_{M \rightarrow \infty} \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] = \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0,$$

as the upper bound $(t^{n+1} - t^n) \Lambda_0$ does not depend on M .

3.3 A priori estimates

Lemma 3.10 *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfy the CFL condition*

$$k \leq \frac{\bar{\alpha}^2}{VC_f} h. \tag{3.3.1}$$

Then there hold

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2 \quad (3.3.2)$$

and

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2\Lambda_0 M_g^2. \quad (3.3.3)$$

where $Q_T = \mathbb{T}^d \times (0, T)$ and M_g is the upper bound of $|g|$.

Proof. Since $\operatorname{div} \mathbf{v} = 0$,

$$\int_K \operatorname{div} \mathbf{v} dx = \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} = 0$$

for all $K \in \mathcal{T}$, so that the finite volume scheme (3.2.4) can be rewritten as:

$$\frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} (f(u_{K|L}^n) - f(u_K^n)) = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n).$$

We apply the definition of $u_{K|L}^n$ to obtain:

$$\frac{|K|}{k}(u_K^{n+1} - u_K^n) - \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n), \quad (3.3.4)$$

where $s^- = -\min(s, 0)$. We multiply both sides of (3.3.4) by ku_K^n :

$$\begin{aligned} |K|(u_K^{n+1} - u_K^n)u_K^n &= k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n, \end{aligned}$$

and by applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$, we obtain that

$$\begin{aligned} \frac{|K|}{2}[(u_K^{n+1})^2 - (u_K^n)^2 - (u_K^{n+1} - u_K^n)^2] &= k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|K|}{2}[(u_K^{n+1})^2 - (u_K^n)^2] &= \frac{|K|}{2}(u_K^{n+1} - u_K^n)^2 \\ &\quad + k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned} \quad (3.3.5)$$

We substitute (3.3.4) into (3.3.5) to deduce that:

$$\begin{aligned}
 & \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2] \\
 &= \frac{|K|}{2} \left(\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) + g(u_K^n) (W_K^{n+1} - W_K^n) \right)^2 \\
 & \quad + k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) u_K^n \\
 & \quad + |K| g(u_K^n) (W_K^{n+1} - W_K^n) u_K^n.
 \end{aligned}$$

We take the expectation of both sides to obtain:

$$\begin{aligned}
 & \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] \\
 &= \frac{|K|}{2} \mathbb{E} \left[\left(\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 + g^2(u_K^n) (W_K^{n+1} - W_K^n)^2 \right] \\
 & \quad + 2 \frac{|K|}{2} \mathbb{E} \left[\frac{k g(u_K^n)}{|K|} (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right] \\
 & \quad + \mathbb{E} \left[k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) u_K^n \right] \\
 & \quad + \mathbb{E} [g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n].
 \end{aligned}$$

Next we show that some of the terms in the above equality vanish. Indeed since $W_K^{n+1} - W_K^n$ and u_K^n are independent variables, we have that, in view of (3.2.11)

$$\mathbb{E} \left[g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n \right] = \mathbb{E} [W_K^{n+1} - W_K^n] \mathbb{E} [g(u_K^n) u_K^n |K|] = 0,$$

and similarly

$$\begin{aligned}
 & \mathbb{E} \left[g(u_K^n) (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right] \\
 &= \mathbb{E} [W_K^{n+1} - W_K^n] \mathbb{E} \left[g(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right] \\
 &= 0.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{|K|}{2} \mathbb{E} \left[(u_K^{n+1})^2 - (u_K^n)^2 \right] \\
&= \frac{k^2}{2|K|} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right]^2 \\
&+ \frac{|K|}{2} \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \mathbb{E} \left[g^2(u_K^n) \right] \\
&+ k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) u_K^n \right].
\end{aligned} \tag{3.3.6}$$

Next we remark that by (3.2.2)

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \leq V |\partial K| \leq \frac{V|K|}{\bar{\alpha}^2 h},$$

which implies that

$$\begin{aligned}
& \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \\
&\leq \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))^2 \right) \\
&\leq V |\partial K| \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))^2 \right) \\
&\leq \frac{V|K|}{\bar{\alpha}^2 h} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))^2 \right) \\
&= \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(\frac{V|K|}{\bar{\alpha}^2 h} (f(u_{K|L}^n) - f(u_K^n))^2 \right).
\end{aligned} \tag{3.3.7}$$

Substituting (3.3.7) in (3.3.6), we deduce that

$$\frac{|K|}{2} \mathbb{E} \left[(u_K^{n+1})^2 - (u_K^n)^2 \right] \tag{3.3.8}$$

$$\leq k \mathbb{E} \left[\frac{k}{2|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))^2 \right] \tag{3.3.9}$$

$$- k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_K^n) - f(u_{K|L}^n)) u_K^n \right] \tag{3.3.10}$$

$$+ \frac{|K|}{2} \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \mathbb{E} \left[g^2(u_K^n) \right] \tag{3.3.11}$$

$$\leq k \mathbf{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(\frac{kV}{2\bar{\alpha}^2 h} (f(u_K^n) - f(u_{K|L}^n))^2 - (f(u_K^n) - f(u_{K|L}^n)) u_K^n \right) \right] \quad (3.3.12)$$

$$+ \frac{|K|}{2} \mathbf{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \mathbf{E} \left[g^2(u_K^n) \right]. \quad (3.3.13)$$

Before continuing with the proof of Lemma 3.10, we show the following auxiliary result:

Lemma 3.11 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing Lipschitz continuous function with Lipschitz constant $C_f > 0$. Then there holds*

$$\left| \int_a^b (f(s) - f(a)) ds \right| \geq \frac{1}{2C_f} (f(b) - f(a))^2, \quad (3.3.14)$$

for all $a, b \in \mathbb{R}$.

Proof. We assume that $a < b$ and set $l = \frac{f(b) - f(a)}{C_f}$ and

$$h(s) = \begin{cases} f(a) & \text{if } s \in [a, b-l], \\ f(a) + (s-b+l)C_f & \text{if } s \in [b-l, b]. \end{cases} \quad (3.3.15)$$

Then, one has $f(s) \geq h(s)$, for all $s \in [a, b]$, and therefore:

$$\begin{aligned} \int_a^b (f(s) - f(a)) ds &\geq \int_a^b (h(s) - f(a)) ds \\ &= C_f \left(\int_{b-l}^b s ds - \int_{b-l}^b (b-l) ds \right) \\ &= \frac{C_f}{2} (b^2 - (b-l)^2 - 2bl + 2l^2) \\ &= \frac{C_f}{2} l^2 \\ &= \frac{1}{2C_f} (f(b) - f(a))^2. \end{aligned}$$

□

Returning to the proof of Lemma 3.10, we define the function

$$\Phi(a) = \int_0^a s f'(s) ds \quad \text{for all } a \in \mathbb{R}. \quad (3.3.16)$$

It follows from (3.3.14) that

$$\begin{aligned}
 b(f(b) - f(a)) - (\Phi(b) - \Phi(a)) &= b(f(b) - f(a)) - \int_a^b s f'(s) ds \\
 &= bf(b) - bf(a) - \left(bf(b) - af(a) - \int_a^b f(s) ds \right) \\
 &= \int_a^b (f(s) - f(a)) ds \\
 &\geq \frac{1}{2C_f} (f(b) - f(a))^2.
 \end{aligned} \tag{3.3.17}$$

Let $a = u_{K|L}^n$ and $b = u_K^n$; we deduce from (3.3.17) that

$$(f(u_K^n) - f(u_{K|L}^n))u_K^n + \Phi(u_{K|L}^n) - \Phi(u_K^n) - \frac{1}{2C_f} (f(u_K^n) - f(u_{K|L}^n))^2 \geq 0 \tag{3.3.18}$$

in which we substitute the CFL condition (3.3.1) to obtain

$$(f(u_K^n) - f(u_{K|L}^n))u_K^n + \Phi(u_{K|L}^n) - \Phi(u_K^n) - \frac{kV}{2\bar{\alpha}^2 h} (f(u_K^n) - f(u_{K|L}^n))^2 \geq 0$$

so that

$$\frac{kV}{2\bar{\alpha}^2 h} (f(u_K^n) - f(u_{K|L}^n))^2 - (f(u_K^n) - f(u_{K|L}^n))u_K^n \leq \Phi(u_{K|L}^n) - \Phi(u_K^n).$$

In turn, we substitute this inequality into (3.3.8); which yields

$$\begin{aligned}
 \frac{|K|}{2} \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] &\leq k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[\Phi(u_{K|L}^n) - \Phi(u_K^n)] \\
 &\quad + \frac{|K|}{2} \mathbb{E}[(W_K^{n+1} - W_K^n)^2] \mathbb{E}[g^2(u_K^n)].
 \end{aligned} \tag{3.3.19}$$

Next, we consider the first term on the right hand side of (3.3.19). Summing on each control volume K and using the fact that $\operatorname{div} \mathbf{v} = 0$, we obtain

$$\begin{aligned}
 &\sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[\Phi(u_{K|L}^n) - \Phi(u_K^n)] \\
 &= \sum_{K \in \mathcal{T}} \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} \left\{ -|\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L}) \mathbb{E}[\Phi(u_{K|L}^n)] + |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L}) \mathbb{E}[\Phi(u_K^n)] \right\} \\
 &= \sum_{K \in \mathcal{T}} \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} \left\{ |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{L,K}) \mathbb{E}[\Phi(u_L^n)] + |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L}) \mathbb{E}[\Phi(u_K^n)] \right\} \\
 &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L}) \mathbb{E}[\Phi(u_K^n)] \\
 &= \sum_{K \in \mathcal{T}} \mathbb{E}[\Phi(u_K^n)] \int_K \operatorname{div} \mathbf{v} dx = 0.
 \end{aligned} \tag{3.3.20}$$

Next using (3.3.20) and Corollary 3.9, we sum the inequality (3.3.19) on each control volume K to obtain:

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)^2] \mathbb{E}[g^2(u_K^n)] \\ &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + M_g^2 \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)^2] \\ &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + k\Lambda_0 M_g^2. \end{aligned}$$

We deduce from the inequalities :

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-1})^2] + k\Lambda_0 M_g^2 \\ \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-2})^2] + k\Lambda_0 M_g^2 \\ \dots \\ \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^1)^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2] + k\Lambda_0 M_g^2 \end{aligned}$$

that

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] \leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2] + nk\Lambda_0 M_g^2 \quad \text{for all } n \in \{1, 2, \dots, N\},$$

which implies that

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2.$$

As a consequence, there holds

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2\Lambda_0 M_g^2.$$

3.3.1 Weak BV estimate

Lemma 3.12 *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfy the CFL condition*

$$k \leq \frac{(1-\xi)\bar{\alpha}^2 h}{VC_f} \quad (3.3.21)$$

for some $\xi \in (0, 1)$. Then there exist a positive constant C_1 , depending on u_0 , M_g , Λ_0 , T , ξ and C_f that

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] \leq C_1, \quad (3.3.22)$$

and a positive constant C_2 , depending on u_0 , M_g , Λ_0 , T , \mathbb{T}^d , ξ , C_f and $\bar{\alpha}$ such that

$$\sum_{n=0}^{N-1} k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} [|f(u_K^n) - f(u_{K|L}^n)|] \leq C_2 h^{-\frac{1}{2}} \quad (3.3.23)$$

for all $K \in \mathcal{T}$.

3. CONVERGENCE OF A FINITE VOLUME SCHEME FOR A FIRST ORDER CONSERVATION LAW INVOLVING A Q -WIENER PROCESS

Proof: Multiplying the equation (3.3.4) by ku_K^n , taking the expectation and summing over $K \in \mathcal{T}$ and $n = 0, 1, \dots, N-1$ yields

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) u_K^n \right] \\ &\quad + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)], \end{aligned}$$

which we rewrite as $A + B = D$ with

$$\begin{aligned} A &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] \\ B &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[(f(u_{K|L}^n) - f(u_K^n))u_K^n] \\ D &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)]. \end{aligned}$$

Note that the term D is equal to 0 since u_K^n is independent of the increment $(W_K^{n+1} - W_K^n)$ and since by (3.2.11), $\mathbb{E}[W_K^{n+1} - W_K^n] = 0$.

Applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$ we obtain:

$$\begin{aligned} A &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2], \end{aligned} \tag{3.3.24}$$

where we set

$$A_1 = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2],$$

and

$$A_2 = \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2].$$

We deduce from (3.3.4) and (3.3.7) that

$$\begin{aligned} A_1 &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{k^2}{|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_K^n) - f(u_{K|L}^n)) \right)^2 \right] \Big\} \\
 & + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\mathbb{E}[g(u_K^n)(W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_K^n) - f(u_{K|L}^n))] \right) \\
 \geq & - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right. \\
 & \left. + \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[(f(u_K^n) - f(u_{K|L}^n))^2] \right) \right\} \\
 \geq & - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right. \\
 & \left. + \frac{k^2}{|K|} |\partial K| V \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[(f(u_K^n) - f(u_{K|L}^n))^2 \right] \right\} \\
 \geq & - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right. \\
 & \left. + (1 - \xi) \frac{k}{C_f} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[(f(u_K^n) - f(u_{K|L}^n))^2 \right] \right\},
 \end{aligned}$$

so that

$$\begin{aligned}
 A_1 \geq & - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right. \\
 & \left. + (1 - \xi) \frac{k}{C_f} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[(f(u_K^n) - f(u_{K|L}^n))^2 \right] \right\}, \tag{3.3.25}
 \end{aligned}$$

where we have used (3.2.1) and (3.3.21). Next we remark that

$$\begin{aligned}
 A_2 = & \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2] \geq - \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2] \\
 & \geq - \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^d)}. \tag{3.3.26}
 \end{aligned}$$

We deduce from (3.3.25), (3.3.26) and Corollary 3.9 that there exists a positive constant \tilde{C}_1 depending on $\|u_0\|_{L^2(\mathbb{T}^d)}$, M_g , Λ_0 and T such that:

$$\begin{aligned}
A &\geq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left\{ |K| \mathbb{E}[g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \right. \\
&\quad \left. + (1 - \xi) \frac{k}{C_f} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] \right\} - \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^d)} \\
&\geq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} (1 - \xi) \frac{k}{C_f} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] - \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^d)} \\
&\quad - \frac{1}{2} M_g^2 \Lambda_0 T \\
&\geq -\frac{1 - \xi}{2C_f} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] - \tilde{C}_1.
\end{aligned} \tag{3.3.27}$$

Next we estimate the term $B = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[(f(u_K^n) - f(u_{K|L}^n))u_K^n]$.

We recall the function Φ defined by (3.3.16), by using Lemma 3.11, the inequality (3.3.18) and the equality (3.3.20), we obtain:

$$\begin{aligned}
B &\geq \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[-(\Phi(u_{K|L}^n) - \Phi(u_K^n)) + \frac{1}{2C_f} \left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] \\
&\geq -\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[\Phi(u_{K|L}^n) - \Phi(u_K^n)] \\
&\quad + \frac{1}{2C_f} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] \\
&\geq \frac{1}{2C_f} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right].
\end{aligned}$$

Therefore, since $A + B = D$ and $D = 0$, we deduce that

$$0 = A + B \geq \frac{\xi}{2C_f} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] - \tilde{C}_1$$

Thus there exists a positive constant C_1 which depends on u_0 , M_g , Λ_0 , T , ξ and C_f such that

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_K^n) - f(u_{K|L}^n) \right)^2 \right] \leq C_1.$$

Applying Cauchy-Schwarz inequality, in view of (3.2.1) and (3.3.22), we deduce that

$$\begin{aligned}
 & \left(\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_K^n) - f(u_{K|L}^n)|] \right)^2 \\
 & \leq \left(\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_K^n) - f(u_{K|L}^n)|^2] \right) \\
 & \quad \times \left(\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \\
 & \leq C_1 VT \sum_{K \in \mathcal{T}} |\partial K| \\
 & \leq C_1 VT \frac{1}{\bar{\alpha}^2 h} \sum_{K \in \mathcal{T}} |K| \\
 & \leq C_1 VT \frac{|\mathbb{T}^d|}{\bar{\alpha}^2 h}.
 \end{aligned}$$

Setting $C_2 := (C_1 VT \frac{|\mathbb{T}^d|}{\bar{\alpha}^2})^{\frac{1}{2}}$, we deduce that

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_K^n) - f(u_{K|L}^n)|] \leq C_2 h^{-\frac{1}{2}}. \quad (3.3.28)$$

Since for each $K \in \mathcal{T}$, the term $\sum_{n=0}^{N-1} k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_K^n) - f(u_{K|L}^n)|]$ is nonnegative, we deduce that

$$\sum_{n=0}^{N-1} k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_K^n) - f(u_{K|L}^n)|] \leq C_2 h^{-\frac{1}{2}} \quad \text{for each } K \in \mathcal{T} \quad (3.3.29)$$

which concludes the proof.

3.4 Convergence of the scheme

3.4.1 A time-continuous approximation

We recall the finite volume scheme (3.3.4),

For all $n \in \{0, 1, 2, \dots, N-1\}$ and for all $K \in \mathcal{T}$

$$\begin{cases} \frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_K^n) - f(u_{K|L}^n)) = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n) \\ u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \end{cases}$$

where $W_K^n = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t^n) \frac{1}{|K|} \int_K e_m(x) dx$. For all $n \in \{0, 1, 2, \dots, N-1\}$ and for all $K \in \mathcal{T}$ we define v_K as the continuous in time stochastic process

$$\begin{aligned} v_K(t) &= u_K^n + \frac{t - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma|(\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(f(u_{K|L}^n) - f(u_K^n) \right) + g(u_K^n)(W_K(t) - W_K^n) \\ &= u_K^n + \int_{nk}^t \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(\mathbf{v} \cdot \mathbf{n}_{K,L})^- \frac{f(u_{K|L}^n) - f(u_K^n)}{|K|} ds + \int_{nk}^t g(u_K^n) dW_K(t). \end{aligned} \quad (3.4.1)$$

on the domain $\Omega \times (nk, (n+1)k)$. In this way, $v_K(nk) = u_K^n$ and $v_K((n+1)k) = u_K^{n+1}$. We define the time-continuous approximate solution $v_{\mathcal{T},k}$ on $\Omega \times \mathbb{T}^d \times (0, T)$ that:

$$v_{\mathcal{T},k}(x, t) = v_K(t^n) \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k). \quad (3.4.2)$$

Next we estimate the difference between the time-continuous approximation $v_{\mathcal{T},k}$ and the finite volume solution $u_{\mathcal{T},k}$ which is defined in (3.2.7).

Lemma 3.13 *Let $u_0 \in L^2(\mathbb{T}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 3.5. Let $k = \frac{T}{N}$ satisfy the CFL condition (3.3.21). Then there exist $\bar{C}_1, \bar{C}_2 \in \mathbb{R}^+$, independent of k and h such that*

$$\|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq \bar{C}_1 k + \bar{C}_2 h \quad (3.4.3)$$

Proof: There holds

$$\begin{aligned} & \|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \\ & \quad \times \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(f(u_{K|L}^n) - f(u_K^n) \right) + g(u_K^n)(W_K(s) - W_K^n) \right)^2 \right] dx ds \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n)(W_K(s) - W_K^n)^2 \right] dx ds \\ & \quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(f(u_{K|L}^n) - f(u_K^n) \right) \right)^2 \right] dx ds \\ & \quad + 2 \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \\ & \quad \times \mathbb{E} \left[\frac{s - nk}{|K|} g(u_K^n)(W_K(s) - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(\mathbf{v} \cdot \mathbf{n}_{K,L})^- \left(f(u_{K|L}^n) - f(u_K^n) \right) \right] dx ds \end{aligned}$$

therefore

$$\begin{aligned} \|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n) (W_K(s) - W_K^n)^2 \right] dx ds \\ &+ \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] dx ds \end{aligned} \quad (3.4.4)$$

where we used the fact that

$$\mathbb{E} [W_K(s) - W_K^n] = 0. \quad (3.4.5)$$

We remark that (3.4.5) can be proved in the same way as (3.2.11). Applying (3.2.20)

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0,$$

we deduce that

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n) (W_K(s) - W_K^n)^2 \right] dx ds \\ &\leq \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} (s - t^n) \Lambda_0 M_g^2 ds \\ &\leq \frac{1}{2} T \Lambda_0 M_g^2 k. \end{aligned} \quad (3.4.6)$$

Moreover, using the inequality:

$$\begin{aligned} &\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \\ &\leq \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n))^2 \right), \end{aligned}$$

we deduce in view of (3.3.7) that

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] dx ds \\ &= \sum_{K \in \mathcal{T}} \frac{1}{|K|^2} \sum_{n=0}^{N-1} \left(\int_{nk}^{(n+1)k} (s - nk)^2 ds \right) \int_K \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] dx \\ &\leq \sum_{K \in \mathcal{T}} \frac{1}{3|K|^2} \sum_{n=0}^{N-1} \left\{ k^3 |K| \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \right\} \end{aligned}$$

$$\begin{aligned} & \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_{K|L}^n) - f(u_K^n) \right)^2 \right] \Big\} \\ & \leq \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \left\{ \frac{k^3}{3|K|} |\partial K| V \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_{K|L}^n) - f(u_K^n) \right)^2 \right] \right\}. \end{aligned}$$

Using (3.2.1), (3.3.21) and (3.3.22), we obtain

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} \left\{ \frac{k^3}{3|K|} |\partial K| V \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_{K|L}^n) - f(u_K^n) \right)^2 \right] \right\} \\ & \leq \left\{ (1-\xi)^2 \bar{\alpha}^4 \frac{h^2}{3V^2 C_f^2} \frac{1}{\bar{\alpha} h^d} \frac{h^{d-1}}{\bar{\alpha}} V \right\} \sum_{K \in \mathcal{T}} k \sum_{n=0}^{N-1} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left(f(u_{K|L}^n) - f(u_K^n) \right)^2 \right] \\ & \leq (1-\xi)^2 \frac{\bar{\alpha}^2}{3V C_f^2} C_2 h, \end{aligned} \tag{3.4.7}$$

which together with (3.4.4) imply that

$$\|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq \frac{1}{2} T \Lambda_0 M_g^2 \cdot k + (1-\xi)^2 \frac{\bar{\alpha}^2}{3V C_f^2} C_2 \cdot h.$$

This yields the result of Lemma 3.13.

Lemma 3.14 (*Discrete entropy inequality*) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3.5, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \tag{3.4.8}$$

Then \mathbb{P} -a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$:

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m^2(x) dx \right)^2 dx dt \\ & \geq R^{k,h} \end{aligned} \tag{3.4.9}$$

where for all P -measurable set A , $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$.

Before proving Lemma 3.14, we show a relation based upon Itô formula:

Lemma 3.15 *For all $\eta \in \mathcal{A}$, there holds:*

$$\begin{aligned}
 & \eta(v_K((n+1)k)) - \eta(v_K(nk)) \\
 &= \int_{nk}^{(n+1)k} \eta'(v_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \frac{f(u_{K|L}^n) - f(u_K^n)}{|K|} dt \\
 & \quad + \int_{nk}^{(n+1)k} \eta'(v_K(t)) g(u_K^n) dW_K(t) \\
 & \quad + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 \int_{nk}^{(n+1)k} \eta''(v_K(t)) g^2(u_K^n) dt
 \end{aligned} \tag{3.4.10}$$

P-a.s in Ω .

Proof:

We apply the Itô formula stated in Lemma 3.3 to the case that :

- $\mathcal{G}(x_1, x_2) = \eta(x_1)$,
- $X(t) = v_K(t)$ which is defined in (3.4.1) on the time interval $(nk, (n+1)k)$,
- $\psi = \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \frac{f(u_{K|L}^n) - f(u_K^n)}{|K|}$,
- $\theta = g(u_K^n)$
- The stochastic process $W(t)$ is given by $W_K(t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \beta_m(t)$,

Thus:

$$\frac{\partial \mathcal{G}}{\partial x_2} = 0, \quad \frac{\partial \mathcal{G}}{\partial x_1} = \eta'(x_1) \quad \text{and} \quad \frac{\partial^2 \mathcal{G}}{\partial x_1^2} = \eta''(x_1)$$

with $x_1 = X(t) = v_K(t)$, $t \in (nk, (n+1)k)$. According to the assumption of Lemma 3.3, the stochastic process $W_K(t) = q^{\frac{1}{2}} \beta(t)$ where β is a standard Brownian motion. Therefore

$$W_K(t^{n+1}) - W_K(t^n) = q^{\frac{1}{2}} (\beta(t^{n+1}) - \beta(t^n)) \sim \mathcal{N}(0, q(t^{n+1} - t^n)). \tag{3.4.11}$$

On the other hand since $W_K(t) = \sum_{m=1}^{\infty} \left(\sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \right) \beta_m(t)$, we have that

$$W_K(t^{n+1}) - W_K(t^n) = \sum_{m=1}^{\infty} \left(\sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \right) (\beta_m(t^{n+1}) - \beta_m(t^n))$$

where

$$\beta_m(t^{n+1}) - \beta_m(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n)$$

so that

$$\begin{aligned} & \sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx (\beta_m(t^{n+1}) - \beta_m(t^n)) \\ & \sim \mathcal{N} \left(0, \left(\sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right) \end{aligned}$$

for all $m \in \{1, 2, \dots\}$. Thus since the Brownian motions $\{\beta_m(t)\}_{m \geq 1}$ are independent,

$$W_K(t^{n+1}) - W_K(t^n) \sim \mathcal{N} \left(0, \sum_{m=1}^{\infty} \left(\sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right) \quad (3.4.12)$$

So that in view of (3.4.11)

$$q = \sum_{m=1}^{\infty} \left(\sqrt{\lambda_m} \frac{1}{|K|} \int_K e_m(x) dx \right)^2.$$

We finally deduce that

$$\begin{aligned} \eta(v_K((n+1)k)) &= \eta(v_K(nk)) \\ &+ \int_{nk}^{(n+1)k} \eta'(v_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \frac{f(u_{K|L}^n) - f(u_K^n)}{|K|} dt \\ &+ \int_{nk}^{(n+1)k} \eta'(v_K(t)) g(u_K^n) dW_K(t) \\ &+ \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 \int_{nk}^{(n+1)k} \eta''(v_K(t)) g^2(u_K^n) dt \end{aligned} \quad (3.4.13)$$

\mathbb{P} -a.s in Ω .

Proof of Lemma 3.14. Since by (3.4.40) $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$, we may suppose that the CFL condition

$$k \leq \frac{(1-\xi)\bar{\alpha}^2 h}{VC_f} \quad (3.4.14)$$

holds for some $\xi \in (0, 1)$.

Step 1: We multiply equation (3.4.10) by $|K|\varphi_K^n$, with $\varphi_K^n = \frac{1}{|K|} \int_K \varphi(x, nk) dx$, and sum for all $n \in \{0, 1, \dots, N-1\}$ and for all $K \in \mathcal{T}$. Recalling that $v_K(nk) = u_K^n$ and $v_K((n+1)k) =$

u_K^{n+1} , one obtains that P $-a.s.$ in Ω ,

$$\begin{aligned}
 & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
 = & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \int_{nk}^{(n+1)k} \eta'(v_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- [f(u_{K|L}^n) - f(u_K^n)] dt \\
 & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \int_{nk}^{(n+1)k} \eta'(v_K(t)) g(u_K^n) dW_K(t) \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left(\int_K e_m(x) dx \right)^2 \varphi_K^n \int_{nk}^{(n+1)k} \eta''(v_K(t)) g^2(u_K^n) dt
 \end{aligned}$$

The equation above can be written in the form $A^{k,h} = B^{k,h} + C^{k,h} + D^{k,h}$, where

$$\begin{aligned}
 A^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K \varphi(x, nk) dx [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
 B^{k,h} &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \varphi_K^n \eta'(v_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- [f(u_{K|L}^n) - f(u_K^n)] dt \\
 C^{k,h} &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \varphi(x, nk) \eta'(v_K(t)) g(u_K^n) dW_K(t) \\
 D^{k,h} &= \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \varphi(x, nk) \eta''(v_K(t)) g^2(u_K^n) \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt
 \end{aligned} \tag{3.4.15}$$

We analyze these terms separately.

Study of $B^{k,h}$: we decompose $B^{k,h}$ as

$$B^{k,h} = B^{k,h} - \tilde{B}^{k,h} + (\tilde{B}^{k,h} - B_1^{k,h}) + B_1^{k,h} \tag{3.4.16}$$

where

$$\begin{aligned}
 \tilde{B}^{k,h} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \eta'(u_K^n) [f(u_{K|L}^n) - f(u_K^n)], \\
 B_1^{k,h} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- [F^\eta(u_{K|L}^n) - F^\eta(u_K^n)].
 \end{aligned} \tag{3.4.17}$$

Note that

$$\begin{aligned}
 \tilde{B}^{k,h} - B_1^{k,h} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \\
 & \quad \times [\eta'(u_K^n) (f(u_{K|L}^n) - f(u_K^n)) - (F^\eta(u_{K|L}^n) - F^\eta(u_K^n))].
 \end{aligned} \tag{3.4.18}$$

Since f and η' are nondecreasing functions, one deduces that:

$$\begin{aligned}
 & \eta'(u_K^n)(f(u_{K|L}^n) - f(u_K^n)) - (F^\eta(u_{K|L}^n) - F^\eta(u_K^n)) \\
 &= \eta'(u_K^n) \int_{u_K^n}^{u_{K|L}^n} f'(s) ds - \int_{u_K^n}^{u_{K|L}^n} \eta'(s) f'(s) ds \\
 &= \int_{u_K^n}^{u_{K|L}^n} (\eta'(u_K^n) - \eta'(s)) f'(s) ds, \\
 &\leq 0,
 \end{aligned} \tag{3.4.19}$$

so that $\tilde{B}_1^{k,h} - B_1^{k,h} \leq 0$.

Since $\operatorname{div} \mathbf{v} = 0$,

$$\begin{aligned}
 0 &= \int_K \operatorname{div} \mathbf{v} dx = \sum_{L \in \mathcal{N}} \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} d\gamma \\
 &= \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} + \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \geq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L}.
 \end{aligned} \tag{3.4.20}$$

Thus

$$\begin{aligned}
 B_1^{k,h} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- [F^\eta(u_{K|L}^n) - F^\eta(u_K^n)] \\
 &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \left(- \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} [F^\eta(u_{K|L}^n) - F^\eta(u_K^n)] \right) \\
 &= - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_{K|L}^n) \\
 &\quad + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_K^n)
 \end{aligned} \tag{3.4.21}$$

Next we apply (3.4.20) and the definition of $u_{K|L}^n$ to deduce that

$$\begin{aligned}
 B_1^{k,h} &= - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \leq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_{K|L}^n) \\
 &\quad - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \varphi_K^n \sum_{\substack{L \in \mathcal{N}(K) \\ \mathbf{v} \cdot \mathbf{n}_{K,L} \geq 0}} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_K^n) \\
 &= - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_{K|L}^n) \varphi_K^n.
 \end{aligned} \tag{3.4.22}$$

Denoting $x_{\sigma_{K,L}}$ the center of the edge σ and $\varphi_{\sigma_{K,L}}^n$ the value $\varphi(x_{\sigma_{K,L}}, nk)$, we deduce the identity:

$$\begin{aligned}
 & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_{K|L}^n) \varphi_{\sigma_{K,L}}^n \right) \\
 &= \sum_{n=0}^{N-1} k \sum_{(K,L) \in \tilde{\mathcal{E}}} \left(|\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_{K|L}^n) \varphi_{\sigma_{K,L}}^n \right. \\
 & \quad \left. + |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{L,K} F^\eta(u_{L|K}^n) \varphi_{\sigma_{K,L}}^n \right) \\
 &= 0.
 \end{aligned} \tag{3.4.23}$$

We combine (3.4.21) and (3.4.23) to obtain that

$$B_1^{k,h} = - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L}) F^\eta(u_{K|L}^n) [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right),$$

which can be rewritten as

$$\begin{aligned}
 B_1^{k,h} &= - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_K^n) [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right) \\
 & \quad + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} [F^\eta(u_K^n) - F^\eta(u_{K|L}^n)] [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right),
 \end{aligned}$$

where we set

$$R_1^{k,h} = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} [F^\eta(u_K^n) - F^\eta(u_{K|L}^n)] [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right). \tag{3.4.24}$$

Using the fact that $\operatorname{div} \mathbf{v} = 0$, we obtain

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} F^\eta(u_K^n) \varphi_K^n = 0$$

so that

$$B_1^{k,h} = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi_{\sigma_{K,L}}^n \right) F^\eta(u_K^n) + R_1^{k,h},$$

with

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi_{\sigma_{K,L}}^n \right) F^\eta(u_K^n) \\ &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\sum_{L \in \mathcal{N}(K)} \left[|\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi_{\sigma_{K,L}}^n - \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right] \right) \\ &+ \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\sum_{L \in \mathcal{N}(K)} \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right), \end{aligned}$$

and we set

$$R_2^{k,h} = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\sum_{L \in \mathcal{N}(K)} \left[|\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi_{\sigma_{K,L}}^n - \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right] \right) \quad (3.4.25)$$

thus

$$B_1^{k,h} = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\sum_{L \in \mathcal{N}(K)} \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right) + R_2^{k,h} + R_1^{k,h}$$

Moreover

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\sum_{L \in \mathcal{N}(K)} \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right) \\ &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\int_{\partial K} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right) \\ &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \left(\int_K \operatorname{div}(\mathbf{v} \varphi(x, nk)) dx \right) \\ &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx, \end{aligned} \quad (3.4.26)$$

thus

$$B_1^{k,h} = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx + R_2^{k,h} + R_1^{k,h}.$$

Finally in view of (3.4.16) and (3.4.19), we deduce that

$$\begin{aligned} B^{k,h} &\leq B^{k,h} - \tilde{B}^{k,h} + B_1^{k,h} \\ &= B^{k,h} - \tilde{B}^{k,h} + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx + R_2^{k,h} + R_1^{k,h}. \end{aligned}$$

Study of $C^{k,h}$: we decompose $C^{k,h}$ as

$$C^{k,h} = C^{k,h} - \tilde{C}^{k,h} + \tilde{C}^{k,h}$$

with

$$\tilde{C}^{k,h} = \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx. \quad (3.4.27)$$

Study of $D^{k,h}$: we decompose $D^{k,h}$ as

$$D^{k,h} = D^{k,h} - \tilde{D}^{k,h} + \tilde{D}^{k,h},$$

with

$$\tilde{D}^{k,h} = \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left[\int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt \right]. \quad (3.4.28)$$

Since P-a.s. in Ω , $A^{k,h} = B^{k,h} + C^{k,h} + D^{k,h}$, we obtain

$$\begin{aligned} B^{k,h} &= A^{k,h} - C^{k,h} - D^{k,h} \\ &\leq \tilde{B}^{k,h} - \tilde{B}^{k,h} + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx + R_2^{k,h} + R_1^{k,h}, \end{aligned}$$

so that

$$\begin{aligned} &- A^{k,h} + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx \\ &\geq \tilde{B}^{k,h} - B^{k,h} - C^{k,h} - D^{k,h} - R_1^{k,h} - R_2^{k,h}, \end{aligned}$$

and that

$$\begin{aligned} &- A^{k,h} + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx + \tilde{C}^{k,h} + \tilde{D}^{k,h} \\ &\geq \tilde{B}^{k,h} - B^{k,h} + \tilde{C}^{k,h} - C^{k,h} + \tilde{D}^{k,h} - D^{k,h} - R_1^{k,h} - R_2^{k,h}. \end{aligned} \quad (3.4.29)$$

Substituting the definition of $A^{k,h}$ in (3.4.15) into (3.4.29) yields

$$\begin{aligned} &- \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ &+ \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\ &+ \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ &+ \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt \\ &\geq (\tilde{B}^{k,h} - B^{k,h}) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}) - R_1^{k,h} - R_2^{k,h} \end{aligned}$$

We define

$$R^{k,h} = (\tilde{B}^{k,h} - B^{k,h}) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}) - R_1^{k,h} - R_2^{k,h} \quad (3.4.30)$$

to deduce the inequality (3.4.9).

Step 2: Next we prove that for all P-mesurable sets A , $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. Let A be a P-measurable set and we analyze separately the convergence of $\mathbb{E}[\mathbf{1}_A(\tilde{B}^{k,h} - B^{k,h})]$, $\mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})]$, $\mathbb{E}[\mathbf{1}_A(\tilde{D}^{k,h} - D^{k,h})]$, $\mathbb{E}[\mathbf{1}_A(R_1^{k,h})]$ and $\mathbb{E}[\mathbf{1}_A(R_2^{k,h})]$. Note that the assumption that

$\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$ is crucial.

Convergence of $\mathbb{E}[\mathbf{1}_A(\tilde{B}^{k,h} - B^{k,h})]$: In view of (3.4.1), the definitions of $B^{k,h}$ in (3.4.15) and of $\tilde{B}^{k,h}$ in (3.4.17) one obtains,

$$\begin{aligned} & \left| \mathbb{E}[\mathbf{1}_A(\tilde{B}^{k,h} - B^{k,h})] \right| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} (\eta'(v_K(t)) - \eta'(u_K^n)) dt \right) \right. \right. \\ & \quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right) \right] \Big| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) (v_K(t) - u_K^n) dt \right) \right. \right. \\ & \quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right) \right] \Big| \\ &\leq \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) \frac{t - nk}{|K|} dt \right) \right. \right. \\ & \quad \left. \left. \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] \right| \\ &+ \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) g(u_K^n) (W_K(t) - W_K(nk)) dt \right) \right. \right. \\ & \quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right) \right] \Big|, \end{aligned}$$

where $\xi_K^n \in (\min(v_K(t), u_K^n), \max(v_K(t), u_K^n))$ for $t \in (nk, (n+1)k)$. We set

$$T_1^{k,h} = \left| \mathbf{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) \frac{t-nk}{|K|} dt \right) \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] \right|$$

and

$$T_2^{k,h} = \left| \mathbf{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) g(u_K^n) (W_K(t) - W_K(nk)) dt \right) \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right] \right|,$$

and analyze separately $T_1^{k,h}$ and $T_2^{k,h}$. Therefore

$$\left| \mathbf{E} \left[\mathbf{1}_A (\tilde{B}^{k,h} - B^{k,h}) \right] \right| \leq T_1^{k,h} + T_2^{k,h} \quad (3.4.31)$$

In view of (3.2.2), there holds that

$$\begin{aligned} & |T_1^{k,h}| \\ &= \left| \mathbf{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \left(\int_{nk}^{(n+1)k} \eta''(\xi_K^n) \frac{t-nk}{|K|} dt \right) \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right)^2 \right] \right| \\ &\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left\{ \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \right) \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbf{E}[(f(u_{K|L}^n) - f(u_K^n))^2] \right) \right\} \\ &\leq \|\eta''\|_\infty \|\varphi\|_\infty V \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \left(\frac{k^2 |\partial K|}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbf{E}[(f(u_{K|L}^n) - f(u_K^n))^2] \right) \\ &\leq \|\eta''\|_\infty \|\varphi\|_\infty V \frac{k}{\bar{\alpha}^2 h} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbf{E}[(f(u_{K|L}^n) - f(u_K^n))^2]. \end{aligned}$$

We deduce from (3.3.22) that $|T_1^{k,h}| \leq \|\eta''\|_\infty \|\varphi\|_\infty V \frac{k}{\bar{\alpha}^2 h} C_2 \rightarrow 0$ as $h \rightarrow 0$.

And for $T_2^{k,h}$, since

$$|\mathbf{E}[Y]|^2 \leq \mathbf{E}[|Y|^2]$$

holds for all random variables Y , we have that

$$\begin{aligned} |T_2^{k,h}|^2 &= \left| \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \varphi_K^n \mathbf{1}_A \eta''(\xi_K^n) g(u_K^n) (W_K(t) - W_K(nk)) dt \right. \right. \\ &\quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right] \right|^2 \\ &\leq \mathbb{E} \left[\left| \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \varphi_K^n \mathbf{1}_A \eta''(\xi_K^n) g(u_K^n) (W_K(t) - W_K(nk)) dt \right. \right. \\ &\quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right| \right]^2 \end{aligned}$$

We set

$$\begin{aligned} Y_1^{n,K} &= \sqrt{|K|} \mathbf{1}_A \varphi_K^n \eta''(\xi_K^n) g(u_K^n) \\ Y_2^{n,K} &= \frac{1}{\sqrt{|K|}} (W_K(t) - W_K(nk)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \end{aligned}$$

and apply the Cauchy-Schwarz inequality to the inequality above to obtain:

$$\begin{aligned} |T_2^{k,h}|^2 &= \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} Y_1^{n,K} Y_2^{n,K} dt \right)^2 \right] \\ &\leq \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} 1^2 dt \right] \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \left(\sum_{K \in \mathcal{T}} Y_1^{n,K} Y_2^{n,K} \right)^2 dt \right] \\ &\leq T \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_1^{n,K})^2 dt \right) \right] \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_2^{n,K})^2 dt \right) \right] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_1^{n,K})^2 dt \right) \right] \\ &= \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} |K| |\mathbf{1}_A \varphi_K^n g(u_K^n) \eta''(\xi_K^n)|^2 dt \right) \right] \\ &\leq T |\mathbf{T}^d| \|\varphi\|_\infty^2 \|\eta''\|_\infty^2 M_g^2. \end{aligned}$$

In view of (3.3.7) and using

$$\mathbb{E} \left[(W_K(s) - W_K(nk))^2 \right] \leq (s - nk) \Lambda_0$$

which can be proved as in (3.2.19), we have that

$$\begin{aligned}
 & \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_2^{n,K})^2 dt \right) \right] \\
 &= \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left| \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- (f(u_{K|L}^n) - f(u_K^n)) \right|^2 \left| W_K(t) - W_K(nk) \right|^2 dt \right) \right] \\
 &\leq \frac{k}{2h} \frac{V}{\bar{\alpha}^2} \Lambda_0 \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[(f(u_{K|L}^n) - f(u_K^n))^2] \\
 &\leq \frac{k}{2h} \frac{C_1 V}{\bar{\alpha}^2} \Lambda_0.
 \end{aligned}$$

Thus

$$\begin{aligned}
 |T_2^{k,h}|^2 &\leq T \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_1^{n,K})^2 dt \right) \right] \mathbb{E} \left[\left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} (Y_2^{n,K})^2 dt \right) \right] \\
 &\leq \frac{k}{2h} \frac{C_1 V}{\bar{\alpha}^2} \Lambda_0 T^2 |\mathbb{T}^d| \|\varphi\|_\infty^2 \|\eta''\|_\infty^2 M_g^2 \\
 &\rightarrow 0 \text{ as } h \rightarrow 0.
 \end{aligned}$$

Therefore, it follows from (3.4.31) that

$$\mathbb{E} \left[\mathbf{1}_A (\tilde{B}^{k,h} - B^{k,h}) \right] \rightarrow 0 \text{ as } h \rightarrow 0.$$

Convergence of $\mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})]$

Form (3.4.15) and (3.4.27), we have

$$\begin{aligned}
 & \left| \mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})] \right| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_K(t)) - \eta'(u_K^n)] g(u_K^n) \varphi(x, nk) dW_K(t) dx \right] \right| \\
 &\leq \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_K(t)) - \eta'(u_K^n)] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) dx \right] \right| \\
 &\quad + \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_K(t)) - \eta'(u_K^n)] g(u_K^n) \varphi(x, t) dW_K(t) dx \right] \right| \\
 &= S_1^{k,h} + S_2^{k,h}
 \end{aligned} \tag{3.4.32}$$

where we set

$$S_1^{k,h} = \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_K(t)) - \eta'(u_K^n)] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) dx \right] \right|$$

and

$$S_2^{k,h} = \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_K(t)) - \eta'(u_K^n)] g(u_K^n) \varphi(x, t) dW_K(t) dx \right] \right|.$$

Applying Cauchy-Schwarz inequality on $\Omega \times \mathbb{T}^d$ yields,

$$\begin{aligned} S_1^{k,h} &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) dx \right] \right| \\ &\leq \sum_{n=0}^{N-1} \left| \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\mathbf{1}_A \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) dx \right] \right| \\ &\leq \sum_{n=0}^{N-1} \left\{ \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E} [1^2] dx \right)^{\frac{1}{2}} \right. \\ &\quad \left. \times \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) \right)^2 dx \right]^{\frac{1}{2}} \right\} \end{aligned}$$

so that

$$S_1^{k,h} \leq |\mathbb{T}^d|^{\frac{1}{2}} \sum_{n=0}^{N-1} \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) \right)^2 dx \right]^{\frac{1}{2}} \right). \quad (3.4.33)$$

In order to study $S_1^{k,h}$, we first consider the term

$$\mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_{M,K}(t) \right)^2 \right]$$

We denote $\sqrt{\lambda_m} e_K^m \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_m(t)$ by \tilde{G}_m , so that,

$$\begin{aligned} &\mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_{M,K}(t) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{m=1}^M \sqrt{\lambda_m} e_K^m \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_m(t) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{m=1}^M \tilde{G}_m \right)^2 \right] = \mathbb{E} \left[\left(\sum_{m=1}^M \tilde{G}_m^2 + 2 \sum_{m_1 \neq m_2} \tilde{G}_{m_1} \tilde{G}_{m_2} \right) \right] \\ &= \mathbb{E} \left[\sum_{m=1}^M \tilde{G}_m^2 \right] + 2 \sum_{m_1 \neq m_2} \mathbb{E} [\tilde{G}_{m_1} \tilde{G}_{m_2}]. \end{aligned}$$

We have, by the Itô isometry (3.1.10) that that

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{m=1}^M \tilde{G}_m^2 \right] \\
 &= \mathbb{E} \left[\sum_{m=1}^M \left(\sqrt{\lambda_m} e_K^m \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_m(t) \right)^2 \right] \\
 &= \sum_{m=1}^M \lambda_m (e_K^m)^2 \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_m(t) \right)^2 \right] \\
 &= \sum_{m=1}^M \lambda_m (e_K^m)^2 \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 dt \right].
 \end{aligned} \tag{3.4.34}$$

On the other hand if $m_1 \neq m_2$; we deduce from the Itô isometry (3.1.9) that

$$\begin{aligned}
 & \mathbb{E} [\tilde{G}_{m_1} \tilde{G}_{m_2}] \\
 &= \sqrt{\lambda_{m_1} \lambda_{m_2}} e_K^{m_1} e_K^{m_2} \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_{m_1}(t) \right. \\
 & \quad \times \left. \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_{m_2}(t) \right] \\
 &= \sqrt{\lambda_{m_1} \lambda_{m_2}} e_K^{m_1} e_K^{m_2} \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 dt \right] \delta_{m_1, m_2} \\
 &= 0.
 \end{aligned}$$

We also have that

$$\begin{aligned}
 & \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) \right)^2 \right] \\
 &= \mathbb{E} \left[\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m} e_K^m \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] d\beta_m(t) \right)^2 \right] \\
 &= \mathbb{E} \left[\left(\sum_{m=1}^{\infty} \tilde{G}_m \right)^2 \right] = \mathbb{E} \left[\sum_{m=1}^{\infty} \tilde{G}_m^2 \right] + 2 \sum_{m_1 \neq m_2}^{\infty} \mathbb{E} [\tilde{G}_{m_1} \tilde{G}_{m_2}] \\
 &= \mathbb{E} \left[\sum_{m=1}^{\infty} \tilde{G}_m^2 \right].
 \end{aligned} \tag{3.4.35}$$

Since $\tilde{G}_m^2 \geq 0$, the sum $\sum_{m=1}^M \tilde{G}_m^2$ is increasing as M tends to infinity, towards the infinite sum

$\sum_{m=1}^{\infty} \tilde{G}_m^2$, thus it follows from Lebesgue's monotone convergence theorem that

$$\mathbb{E} \left[\sum_{m=0}^{\infty} \tilde{G}_m^2 \right] = \mathbb{E} \left[\lim_{M \rightarrow \infty} \sum_{m=0}^M \tilde{G}_m^2 \right] = \lim_{M \rightarrow \infty} \mathbb{E} \left[\sum_{m=0}^M \tilde{G}_m^2 \right]$$

Therefore it follows from (3.4.34) that

$$\mathbb{E} \left[\sum_{m=1}^{\infty} \tilde{G}_m^2 \right] = \sum_{m=1}^{\infty} \lambda_m (e_K^m)^2 \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 dt \right]. \quad (3.4.36)$$

Combining (3.4.35) and (3.4.36), yields

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) [\varphi(x, nk) - \varphi(x, t)] dW_K(t) \right)^2 \right] \\ &= \sum_{m=1}^{\infty} \lambda_m (e_K^m)^2 \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 dt \right]. \end{aligned} \quad (3.4.37)$$

Substituting (3.4.37) into (3.4.33), we deduce that

$$\begin{aligned} S_1^{k,h} &\leq |\mathbb{T}^d|^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \lambda_m (e_K^m)^2 \right)^{\frac{1}{2}} \\ &\quad \times \sum_{n=0}^{N-1} \left[\sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 dt \right] dx \right]^{\frac{1}{2}} \\ &\leq (|\mathbb{T}^d| \Lambda_0)^{\frac{1}{2}} \\ &\quad \times \sum_{n=0}^{N-1} \left[\sum_{K \in \mathcal{T}} \int_K \int_{nk}^{(n+1)k} \mathbb{E} \left[[\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)]^2 \right] dt dx \right]^{\frac{1}{2}} \\ &\leq (|\mathbb{T}^d| \Lambda_0)^{\frac{1}{2}} \sum_{n=0}^{N-1} \left[\sum_{K \in \mathcal{T}} \int_K \int_{nk}^{(n+1)k} \mathbb{E} \left[\|2\eta'\|_{\infty}^2 \|g\|_{\infty}^2 [\partial_t \varphi(x, \xi_n)(nk - t)]^2 \right] dt dx \right]^{\frac{1}{2}} \\ &\leq \sqrt{\frac{4}{3}} |\mathbb{T}^d| T (\Lambda_0)^{\frac{1}{2}} \|\eta'\|_{\infty} \|\partial_t \varphi\|_{\infty} M_g k^{\frac{1}{2}} \\ &\rightarrow 0 \quad \text{as } h \rightarrow 0, \end{aligned}$$

where $\xi_n \in (nk, (n+1)k)$. Applying Cauchy-Schwarz inequality on the integral over $\Omega \times \mathbb{T}^d$, we deduce that

$$\begin{aligned} (S_2^{k,h})^2 &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_K^n) \varphi(x, t) dW_K(t) dx \right] \right|^2 \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K 1 \times \int_0^T [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_{\mathcal{T},k}) \varphi(x, t) dW_K(t) dx \right] \right|^2 \\ &\leq \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E}[1^2] dx \right) \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_0^T [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_{\mathcal{T},k}) \varphi(x, t) dW_K(t) \right)^2 \right] dx \right) \\ &= |\mathbb{T}^d| \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_0^T [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_{\mathcal{T},k}) \varphi(x, t) dW_K(t) \right)^2 \right] dx. \end{aligned}$$

Using a similar idea as in (3.4.35) and (3.4.36), we deduce from Lemma 3.13 that

$$\begin{aligned}
 & |\mathbb{T}^d| \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_0^T [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})] g(u_{\mathcal{T},k}) \varphi(x, t) dW_K(t) \right)^2 \right] dx \\
 &= |\mathbb{T}^d| \sum_{K \in \mathcal{T}} \int_K \sum_{m=1}^{\infty} \lambda_m (e_K^m)^2 \mathbb{E} \left[\int_0^T [\eta'(v_{\mathcal{T},k}) - \eta'(u_{\mathcal{T},k})]^2 g^2(u_{\mathcal{T},k}) \varphi^2(x, t) dt \right] dx \\
 &\leq \Lambda_0 |\mathbb{T}^d| \|\varphi\|_{\infty}^2 \|\eta''\|_{\infty}^2 \|v_{\mathcal{T},k} - u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0, T))}^2 \|g\|_{\infty}^2 \\
 &\rightarrow 0 \quad \text{as } h \rightarrow 0.
 \end{aligned}$$

Thus

$$S_1^{k,h} \rightarrow 0 \text{ and } S_2^{k,h} \rightarrow 0 \text{ as } h \rightarrow 0,$$

which implies that

$$\mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})] \rightarrow 0 \text{ as } h \rightarrow 0.$$

Convergence of $\mathbb{E}[\mathbf{1}_A(\tilde{D}^{k,h} - D^{k,h})]$

From the definition of $D^{k,h}$ in (3.4.15) and $\tilde{D}^{k,h}$ in (3.4.28), we have

$$\begin{aligned}
 & \mathbb{E}[\mathbf{1}_A(\tilde{D}^{k,h} - D^{k,h})] \\
 &= \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m a_m^2 \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K \tilde{e}_m(x) dx \right)^2 \right. \\
 & \quad \left. \times \int_K [\eta''(u_K^n) - \eta''(v_K(t))] g^2(u_K^n) \varphi(x, nk) dx dt \right] \\
 &\leq \frac{1}{2} \Lambda_0 \|\varphi\|_{\infty} \|\eta''(u_{\mathcal{T},k}) - \eta''(v_{\mathcal{T},k})\|_{L^1(\Omega \times \mathbb{T}^d \times (0, T))} M_g^2 \\
 &\leq \frac{1}{2} \Lambda_0 \sqrt{T} |\mathbb{T}^d| \|\varphi\|_{\infty} M_g^2 \|\eta''\|_{\infty} \|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0, T))} \\
 &\rightarrow 0 \quad \text{as } h \rightarrow 0 \text{ by using Lemma 3.13 and the fact that } \eta \in C^3(\mathbb{R}).
 \end{aligned} \tag{3.4.38}$$

Convergence of $\mathbb{E}[\mathbf{1}_A R_1^{k,h}]$

Recall the definition of $R_1^{k,h}$ in (3.4.24) and using (3.3.28) stated in the proof of Lemma 3.12, one obtains that $\mathbb{E}[\mathbf{1}_A R_1^{k,h}] \rightarrow 0$ as $h \rightarrow 0$.

$$\begin{aligned}
 & |\mathbb{E}[\mathbf{1}_A R_1^{k,h}]| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} [F^\eta(u_K^n) - F^\eta(u_{K|L}^n)] [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right] \right| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- [F^\eta(u_K^n) - F^\eta(u_{K|L}^n)] [\varphi_K^n - \varphi_{\sigma_{K,L}}^n] \right] \right| \\
 &\leq \|\varphi_x\|_{\infty} h \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E} \left[\left| \int_{u_{K|L}^n}^{u_K^n} \eta'(s) f'(s) ds \right| \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \|\varphi_x\|_\infty \|\eta'\|_\infty h \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (\mathbf{v} \cdot \mathbf{n}_{K,L})^- \mathbb{E}[|f(u_{K|L}^n) - f(u_K^n)|] \\
&\leq h^{\frac{1}{2}} C \|\varphi_x\|_\infty \|\eta'\|_\infty \\
&\rightarrow 0 \text{ as } h \rightarrow 0.
\end{aligned}$$

Convergence of $\mathbb{E}[\mathbf{1}_A(R_2^{k,h})]$

We recall that in (3.4.25),

$$\begin{aligned}
R_2^{k,h} &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \sum_{L \in \mathcal{N}(K)} \left[|\sigma_{K,L}| \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(x_{\sigma_{K,L}}, nk) - \int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma \right] \\
&= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \sum_{L \in \mathcal{N}(K)} \left[\int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} (\varphi(x_{\sigma_{K,L}}, nk) - \varphi(\gamma, nk)) d\gamma \right]
\end{aligned}$$

Using the regularity of φ we have for all $x \in \sigma_{K,L}$

$$\varphi(x, nk) = \varphi(x_{\sigma_{K,L}}, nk) + \varphi'(x_{\sigma_{K,L}}, nk)(x - x_{\sigma_{K,L}}) + (x - x_{\sigma_{K,L}}) \cdot r(x - x_{\sigma_{K,L}})$$

where $r(x - x_{\sigma_{K,L}}) \rightarrow 0$ when $x - x_{\sigma_{K,L}} \rightarrow 0$, so that

$$\begin{aligned}
R_2^{k,h} &= - \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} F^\eta(u_K^n) \sum_{L \in \mathcal{N}(K)} \left[\int_{\sigma_{K,L}} \mathbf{v} \cdot \mathbf{n}_{K,L} (\varphi'(x_{\sigma_{K,L}}, nk)(x - x_{\sigma_{K,L}}) \right. \\
&\quad \left. + (x - x_{\sigma_{K,L}}) \cdot r(x - x_{\sigma_{K,L}})) d\gamma \right]
\end{aligned}$$

Since $x_{\sigma_{K,L}}$ denotes the center of mass of the edge $\sigma_{K,L}$,

$$\int_{\sigma_{K,L}} \varphi'(x_{\sigma_{K,L}}, nk)(x - x_{\sigma_{K,L}}) d\gamma = 0,$$

and

$$\left| \int_{\sigma_{K,L}} (x - x_{\sigma_{K,L}}) \cdot \varepsilon(x - x_{\sigma_{K,L}}) d\gamma \right| \leq |\sigma_{K,L}| h \cdot r(h).$$

Next we use the equality

$$\begin{aligned}
\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| \mathbb{E}[|u_K^n|] &= \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} 1 \times |u_{\mathcal{T},k}| dx dt \right] \\
&\leq \left(\mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} |u_{\mathcal{T},k}|^2 dx dt \right] \right)^{\frac{1}{2}} \sqrt{T |\mathbb{T}^d|} \\
&= \sqrt{T |\mathbb{T}^d|} \|u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0,T))},
\end{aligned} \tag{3.4.39}$$

the property of the admissible mesh (3.2.2), and the a priori estimate (3.3.3), we deduce that:

$$\begin{aligned}
 |\mathbb{E}[\mathbf{1}_A R_2^{k,h}]| &\leq V \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \mathbb{E}[|F^\eta(u_K^n)|] \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| h \cdot r(h) \\
 &\leq r(h) V C_f \|\eta'\|_\infty \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \frac{h^d}{\bar{\alpha}} \mathbb{E}[|u_K^n|] \\
 &\leq r(h) \frac{V}{\bar{\alpha}^2} C_f \|\eta'\|_\infty \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| \mathbb{E}[|u_K^n|] \\
 &\leq r(h) \frac{V}{\bar{\alpha}^2} C_f \|\eta'\|_\infty \sqrt{T |\mathbb{T}^d|} \|u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0,T))} \\
 &\rightarrow 0 \text{ as } h \rightarrow 0
 \end{aligned}$$

We have thus proved that $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$, which concludes the proof of Lemma 3.14.

Lemma 3.16 (*Continuous entropy inequality on the discrete solution*) Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3.5, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \tag{3.4.40}$$

Then P -a.s. in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$, there holds

$$\begin{aligned}
 &\int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
 &+ \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \\
 &+ \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt \\
 &\geq \tilde{R}^{k,h},
 \end{aligned} \tag{3.4.41}$$

where for all P -measurable set A , $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$ as $h \rightarrow 0$.

Proof. Step 1: We first show that the inequality (3.4.41) holds for a suitable $\tilde{R}^{k,h}$. Note that the first term of inequality (3.4.9) in Lemma 3.14. can be rewritten as:

$$\begin{aligned}
& - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} [\eta(u_K^{n+1}) - \eta(u_K^n)] \int_K \varphi(x, nk) dx \\
&= - \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} [\eta(u_K^{n+1}) - \eta(u_K^n)] \varphi(x, nk) dx \\
&= - \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} [\eta(u_K^{n+1}) - \eta(u_K^n)] \varphi(x, nk) dx + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^N) \varphi(x, Nk) dx \\
&= \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^N \eta(u_K^n) \varphi(x, nk) dx - \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \eta(u_K^{n+1}) \varphi(x, nk) dx \\
&= \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \eta(u_K^n) \varphi(x, nk) dx + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^N) \varphi(x, Nk) dx \\
&\quad - \sum_{K \in \mathcal{T}} \int_K \sum_{n=1}^{N-1} \eta(u_K^n) \varphi(x, (n-1)k) dx \\
&= \sum_{K \in \mathcal{T}} \int_K \sum_{n=1}^{N-1} \eta(u_K^n) [\varphi(x, nk) - \varphi(x, (n-1)k)] dx + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx \\
&= \sum_{K \in \mathcal{T}} \int_K \sum_{n=1}^{N-1} \int_{nk}^{(n+1)k} \eta(u_K^n) \partial_t \varphi(x, t-k) dt dx + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx \\
&= \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t-k) dx dt + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx
\end{aligned} \tag{3.4.42}$$

where we have used the fact that for all x in K and k small enough, $\varphi(x, Nk) = \varphi(x, (N-1)k) = 0$. Defining

$$\begin{aligned}
C_1^{k,h} &= \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) dW_{\mathcal{T}}(x, t) dx \\
D_1^{k,h} &= \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt,
\end{aligned}$$

and recalling that in (3.4.27) and (3.4.28)

$$\begin{aligned}
\tilde{C}^{k,h} &= \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx. \\
\tilde{D}^{k,h} &= \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left[\int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt \right].
\end{aligned}$$

Substituting the equality (3.4.42) into the discrete entropy inequality (3.4.9) yields

$$\begin{aligned}
 & \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t-k) dx dt + \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx \\
 & + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\
 & + \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt \\
 & \geq R^{k,h}
 \end{aligned}$$

so that

$$\begin{aligned}
 & \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
 & + \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \\
 & + \frac{1}{2} \left(\sum_{m=1}^{\infty} \lambda_m \right) \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) dx dt \\
 & \geq \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
 & + C_1^{k,h} + D_1^{k,h} \\
 & - \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx - \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t-k) dx dt \\
 & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt - \tilde{C}^{k,h} - \tilde{D}_1^{k,h} + R^{k,h} \\
 & = \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx - \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0) \varphi(x, 0) dx \\
 & + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt - \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t-k) dx dt \\
 & + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
 & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\
 & + C_1^{k,h} - \tilde{C}^{k,h} + D_1^{k,h} - \tilde{D}^{k,h} + R^{k,h}
 \end{aligned}$$

where $R^{k,h}$ is given by (3.4.30) in the proof of the Lemma 3.14. We define

$$\begin{aligned} \tilde{R}^{k,h} &= \int_{\mathbb{T}^d} \eta(u_0)\varphi(x,0)dx - \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0)\varphi(x,0)dx \\ &+ \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k})\partial_t\varphi(x,t)dxdt - \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k})\partial_t\varphi(x,t-k)dxdt \\ &+ \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k})\mathbf{v} \cdot \nabla_x\varphi(x,t)dxdt \\ &- \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n)\mathbf{v} \cdot \nabla_x\varphi(x,nk)dxdt \\ &+ C_1^{k,h} - \tilde{C}^{k,h} + D_1^{k,h} - \tilde{D}^{k,h} + R^{k,h} \end{aligned}$$

to obtain the inequality (3.4.41).

Step 2: We will show in this step that for all P -measurable set A , $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. In the proof of Lemma 3.14, we proved that for all P -measurable set A , $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. It remains to prove the convergence to zero of the following quantities:

$$\begin{aligned} I_1 &:= \mathbb{E} \left[\mathbf{1}_A \left(\int_{\mathbb{T}^d} \eta(u_0)\varphi(x,0)dx - \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0)\varphi(x,0)dx \right) \right], \\ I_2 &:= \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k})\partial_t\varphi(x,t)dxdt - \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k})\partial_t\varphi(x,t-k)dxdt \right) \right], \\ I_3 &:= \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k})\mathbf{v} \cdot \nabla_x\varphi(x,t)dxdt \right. \right. \\ &\quad \left. \left. - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_{\mathcal{T},k})\mathbf{v} \cdot \nabla_x\varphi(x,nk)dxdt \right) \right], \\ &\mathbb{E}[\mathbf{1}_A(C_1^{k,h} - \tilde{C}^{k,h})] \text{ and } \mathbb{E}[\mathbf{1}_A(D_1^{k,h} - \tilde{D}^{k,h})]. \end{aligned}$$

Convergence of I_1

We recall that $u_0(x) \in L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$, so that

$$\begin{aligned} I_1 &= \mathbb{E} \left[\mathbf{1}_A \left(\int_{\mathbb{T}^d} \eta(u_0)\varphi(x,0)dx - \sum_{K \in \mathcal{T}} \int_K \eta(u_K^0)\varphi(x,0)dx \right) \right] \\ &= \mathbb{E} \left[\mathbf{1}_A \left(\sum_{K \in \mathcal{T}} \int_K (\eta(u_0) - \eta(u_K^0))\varphi(x,0)dx \right) \right] \\ &\leq \|\eta'\|_\infty \|\varphi\|_\infty \sum_{K \in \mathcal{T}} \int_K |u_0 - u_K^0| dx \\ &\leq \|\eta'\|_\infty \|\varphi\|_\infty \int_{\mathbb{T}^d} |u_0 - u_{\mathcal{T},k}(x,0)| dx \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

Convergence of I_2

$$\begin{aligned}
 I_2 &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x,t) dx dt - \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x,t-k) dx dt \right) \right] \right| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\int_0^k \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x,t) dx dt + \int_k^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) (\partial_t \varphi(x,t) - \partial_t \varphi(x,t-k)) dx dt \right) \right] \right| \\
 &\leq k |\mathbb{T}^d| \cdot \|\eta\|_\infty \|\partial_t \varphi\|_\infty + k(T-k) |\mathbb{T}^d| \cdot \|\eta\|_\infty \|\varphi_{tt}\|_\infty \\
 &\rightarrow 0 \text{ as } h \rightarrow 0.
 \end{aligned}$$

Convergence of I_3

$$\begin{aligned}
 I_3 &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x,t) dx dt \right. \right. \right. \\
 &\quad \left. \left. - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x,nk) dx dt \right) \right] \right| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot (\nabla_x \varphi(x,t) - \nabla_x \varphi(x,nk)) dx dt \right) \right] \right| \\
 &\leq V \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[F^\eta(u_K^n)]| \|\nabla_x \varphi_t\|_\infty k dt \\
 &\leq V \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[f(u_K^n)]| \\
 &\leq VC_f \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[|u_K^n|]| \\
 &\leq VC_f \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \cdot \sqrt{T |\mathbb{T}^d|} \|u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0,T))} \\
 &\rightarrow 0 \text{ as } h \rightarrow 0
 \end{aligned}$$

where we have used (3.4.39) and (3.3.3).

Convergence of $\mathbb{E}[\mathbf{1}_A(C_1^{k,h} - \tilde{C}^{k,h})]$

Using Cauchy-Schwarz inequality on $\Omega \times \mathbb{T}^d$ and a similar idea as for proving (3.4.37), one obtains

$$\begin{aligned}
 & \left| \mathbb{E}[\mathbf{1}_A(C_1^{k,h} - \tilde{C}^{k,h})] \right| \\
 &= \left| \mathbb{E}[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) (\varphi(x,nk) - \varphi(x,t)) dW_K(t) dx] \right| \\
 &\leq \sum_{n=0}^{N-1} \left| \mathbb{E}[\mathbf{1}_A \sum_{K \in \mathcal{T}} \int_K \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) (\varphi(x,nk) - \varphi(x,t)) dW_K(t) dx] \right| \\
 &\leq \sum_{n=0}^{N-1} \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E}[1^2] dx \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) (\varphi(x, nk) - \varphi(x, t)) dW_K(t) \right)^2 \right] dx \right)^{\frac{1}{2}} \\
& = \sqrt{|\mathbb{T}^d|} \sum_{n=0}^{N-1} \left(\left(\sum_{m=1}^{\infty} \lambda_m (e_K^m)^2 \right) \sum_{K \in \mathcal{T}} \int_K \int_{nk}^{(n+1)k} \mathbb{E} \left[(\eta'(u_K^n) g(u_K^n) (\varphi(x, nk) - \varphi(x, t)))^2 \right] dt dx \right)^{\frac{1}{2}} \\
& \leq \frac{1}{\sqrt{3}} T \sqrt{\Lambda_0} |\mathbb{T}^d| \|\partial_t \varphi\|_{\infty} \|\eta'\|_{\infty} \|g\|_{\infty} k^{\frac{1}{2}} \\
& \rightarrow 0 \text{ as } h \rightarrow 0.
\end{aligned}$$

Convergence of $\mathbb{E}[\mathbf{1}_A(D_1^{k,h} - \tilde{D}^{k,h})]$:

$$\begin{aligned}
& \left| \mathbb{E}[\mathbf{1}_A(D_1^{k,h} - \tilde{D}^{k,h})] \right| \\
& \leq \left| \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K \mathbf{1}_A \eta''(u_K^n) g^2(u_K^n) [\varphi(x, nk) - \varphi(x, t)] \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 dx dt \right] \right| \\
& \quad + \left| \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K \mathbf{1}_A \eta''(u_K^n) g^2(u_K^n) \varphi(x, t) \left(e_m^2(x) - \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 \right) dx dt \right] \right| \\
& \leq \frac{1}{4} k \Lambda_0 \|\eta''\|_{\infty} \|\partial_t \varphi\|_{\infty} T |\mathbb{T}^d| \|g\|_{\infty}^2 + \frac{1}{2} \Lambda_0 \|\eta''\|_{\infty} \|\varphi\|_{\infty} \|g\|_{\infty}^2 T |\mathbb{T}^d| \max_{K \in \mathcal{T}} \left(e_m^2(x) - \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 \right) \\
& \rightarrow 0 \text{ as } h \rightarrow 0,
\end{aligned}$$

where we have used that as $\frac{1}{|K|} \int_K e_m(x) dx \rightarrow e_m(x)$ as $h \rightarrow 0$ where x is a point in K , for all $K \in \mathcal{T}$.

To summarize, we have proved that $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$ as $h \rightarrow 0$ which concludes the proof of Lemma 3.16.

3.5 Convergence proof

Theorem 3.17 (Convergence to a measure-valued entropy solution of Problem (3.1.1)) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfying that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then there exist a function $\mathbf{u} \in \mathcal{N}_{\omega}^2(0, T, L^2(\mathbb{T}^d \times (0, 1))) \cap L^{\infty}(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $\{u_{\mathcal{T},k}\}$ converges to \mathbf{u} in the sense of Young measures. Moreover \mathbf{u} is measure-valued entropy solution of Problem (3.1.1) in the sense of Definition 3.2.

Proof. We multiply the inequality (3.4.41) by $\mathbf{1}_A$, namely the characteristic function of the P -measurable set A . We take the expectation, which yields:

$$\begin{aligned}
 & \mathbb{E} \left[\mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx \right] \\
 & + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \\
 & + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right] \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt \right] \\
 & \geq \mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}].
 \end{aligned} \tag{3.5.1}$$

It follows from Lemma 3.10 that there exist an entropy process [4, 30] \mathbf{u} of $L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $u_{\mathcal{T},k}$ converges to \mathbf{u} in the sense of Young measures. Moreover it follows from [4] and [8] that $\mathbf{u} \in \mathcal{N}_\omega^2(0, T, L^2(\mathbb{T}^d \times (0, 1)))$. More precisely the convergence in the sense of Young measures means that given a Carathéodory function $\Psi : \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is uniformly integrable, one has:

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} \Psi(\cdot, u_{\mathcal{T},k}) dx dt \right] \rightarrow \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha dx dt \right].$$

Recall that a function $\Psi : \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function if for almost all $(\omega, x, t) \in \Omega \times \mathbb{T}^d \times (0, T)$ the function $\nu \mapsto \Psi(\omega, x, t, \nu)$ is continuous and for all $\nu \in \mathbb{R}$, the function $(\omega, x, t) \mapsto \Psi(\omega, x, t, \nu)$ is measurable. In order to prove the convergence of $\{u_{\mathcal{T},k}\}$ to a measure-valued stochastic entropy solution of Problem (3.1.1), we aim to pass to the limit in the inequality (3.5.1) as $h \rightarrow 0$. We have proved that $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. It remains to study the convergence of the terms on the left-hand side of (3.5.1).

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right]$

Note that

$$\Psi : (\omega, x, t, \nu) \in \Omega \times L^2(\mathbb{T}^d) \times [0, T] \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \partial_t \varphi(x, t) \eta(\nu) \in \mathbb{R}$$

is a Carathéodory function such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q_T)$. Thus

$$\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] \rightarrow \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha)) d\alpha \partial_t \varphi(x, t) dx dt \right]$$

as $h \rightarrow 0$.

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right]$

Since F^η is bounded in $L^2(\Omega \times \mathbb{T}^d \times (0, T))$, using the same arguments as previously, we deduce that

$$\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \rightarrow \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt \right]$$

as $h \rightarrow 0$.

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right]$

We denote by Ψ the mapping $\Psi : (\omega, x, t, \nu) \mapsto \mathbf{1}_A(\omega) \eta'(\nu) g(\nu) \varphi(x, t) \in \mathbb{R}$; $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^\infty(\Omega \times Q_T)$, so that up to a subsequence $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ converges weakly in $L^2(\Omega \times Q_T)$ to a function $\chi \in L^2(\Omega \times Q_T)$. Thus

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\int_{Q_T} \Psi \varphi dx dt \right] = \mathbb{E} \left[\int_{Q_T} \chi \varphi dx dt \right].$$

For any $\varphi \in L^2(\Omega \times Q_T)$, $(\omega, x, t, \nu) \mapsto \varphi(\omega, x, t) \Psi(\omega, x, t, \nu)$ is a Carathéodory function such that $\varphi \Psi(\cdot, u_{\mathcal{T},k})$ is uniformly integrable. It is based on the fact that there exists a positive constant C_3 such that

$$\mathbb{E} \left[\int_{Q_T} |\varphi \Psi(\cdot, u_{\mathcal{T},k})| dx dt \right] \leq C_3 \|\Psi(\cdot, u_{\mathcal{T},k})\|_{L^2(\Omega \times Q_T)} \|\varphi\|_{L^2(\Omega \times Q_T)} \quad (3.5.2)$$

Thus

$$\lim_{h \rightarrow 0} \int_{\Omega \times Q_T} \Psi(\cdot, u_{\mathcal{T},k}) \varphi dx dt dP = \int_{\Omega \times Q_T} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \varphi dx dt dP.$$

By identification,

$$\Psi(\cdot, u_{\mathcal{T},k}) \rightharpoonup \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \quad (3.5.3)$$

weakly in $L^2(\Omega \times Q_T)$. We have that

$$\begin{aligned} & \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW_{\mathcal{T}}(x, t) \\ &= \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW_{\mathcal{T}}(x, t) - \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \\ & \quad + \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \\ &= \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi d(W_{\mathcal{T}}(x, t) - W(x, t)) + \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \end{aligned}$$

We first study the term $\int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t)$; for $\mathcal{U} \in L^2(\Omega \times Q_T)$, we define the operator $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}}(\mathcal{U}) = \int_0^T \mathcal{U} dW(x, t) \quad (3.5.4)$$

where $W(x, t)$ is the Q -Wiener process defined by (3.1.3).

In view of the Ito isometry (Lemma 3.4), and the hypothesis (3.1.4), we have that:

$$\begin{aligned}
 & \mathbb{E} \left[\int_{\mathbb{T}^d} (\tilde{\mathcal{T}}(\mathcal{U}))^2 dx \right] = \mathbb{E} \left[\int_{\mathbb{T}^d} \left(\int_0^T \mathcal{U} dW(x, t) \right)^2 dx \right] \\
 & = \mathbb{E} \left[\int_{\mathbb{T}^d} \sum_{m=1}^{\infty} \lambda_m \left(\int_0^T \mathcal{U}(x, t) d\beta_m(t) \right)^2 e_m^2(x) dx \right. \\
 & \quad \left. + \int_{\mathbb{T}^d} 2dx \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} e_{m_1}(x) e_{m_2}(x) \int_0^T \mathcal{U}(x, t) d\beta_{m_1}(t) \int_0^T \mathcal{U}(x, t) d\beta_{m_2}(t) \right] \\
 & = \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \left(\sum_{m=1}^{\infty} \lambda_m e_m^2(x) \right) \mathcal{U}^2(x, t) dt dx \right] + 0 \\
 & \leq \Lambda_1 \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \mathcal{U}^2(x, t) dt dx \right] \\
 & \leq \Lambda_1 \|\mathcal{U}\|_{L^2(\Omega \times Q_T)}^2.
 \end{aligned}$$

This proves that $\tilde{\mathcal{T}}$ is a bounded operator from $L^2(\Omega \times Q_T)$ to $L^2(\Omega \times \mathbb{T}^d)$. Let $\tilde{\mathcal{T}}^* : L^2(\Omega \times \mathbb{T}^d) \rightarrow L^2(\Omega \times Q_T)$ be the adjoint operator of $\tilde{\mathcal{T}}$; then

$$(\psi, \tilde{\mathcal{T}}\mathcal{U})_{L^2(\Omega \times \mathbb{T}^d)} = (\tilde{\mathcal{T}}^*\psi, \mathcal{U})_{L^2(\Omega \times Q_T)}$$

for all $\psi \in L^2(\Omega \times \mathbb{T}^d)$. Next we set $\mathcal{U} = \eta'(u_{\mathcal{T}, k})g(u_{\mathcal{T}, k})\varphi$. We recall that by (3.5.3),

$$\eta'(u_{\mathcal{T}, k})g(u_{\mathcal{T}, k})\varphi \rightharpoonup \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi$$

weakly in $L^2(\Omega \times Q_T)$ along a subsequence as h and k tend to zero. Thus as h and $k \rightarrow 0$,

$$\begin{aligned}
 & (\psi, \tilde{\mathcal{T}}(\eta'(u_{\mathcal{T}, k})g(u_{\mathcal{T}, k})\varphi))_{L^2(\Omega \times \mathbb{T}^d)} \\
 & = (\tilde{\mathcal{T}}^*\psi, \eta'(u_{\mathcal{T}, k})g(u_{\mathcal{T}, k})\varphi)_{L^2(\Omega \times Q_T)} \\
 & \rightarrow (\tilde{\mathcal{T}}^*\psi, \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi)_{L^2(\Omega \times Q_T)} \\
 & = (\psi, \tilde{\mathcal{T}}(\left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi))_{L^2(\Omega \times \mathbb{T}^d)}.
 \end{aligned} \tag{3.5.5}$$

We conclude that along a subsequence

$$\tilde{\mathcal{T}}(\eta'(u_{\mathcal{T}, k})g(u_{\mathcal{T}, k})\varphi) \rightharpoonup \tilde{\mathcal{T}} \left(\left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right) \quad \text{weakly in } L^2(\Omega \times \mathbb{T}^d), \tag{3.5.6}$$

as h and $k \rightarrow 0$, or in other words, in view of the definition (3.5.4) of $\tilde{\mathcal{T}}$,

$$\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) \rightarrow \int_0^T \left(\int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x,t) \quad (3.5.7)$$

Therefore, one obtains

$$\mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) dx \right] \rightarrow \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \left(\int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x,t) dx \right] \quad (3.5.8)$$

as h and $k \rightarrow 0$.

Next, we consider the term

$$I_h := I_h(\omega, x) = \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi d(W_{\mathcal{T}}(x,t) - W(x,t)). \quad (3.5.9)$$

To begin with, we prove below the following result:

Lemma 3.18 *Suppose that $A_h \in L^2(\Omega \times \mathbb{T}^d \times (0, T))$ is (\mathcal{F}_t) -adapted (i.e., $A_h(\cdot, t, x)$ is \mathcal{F}_t -measurable in ω for every $x \in \mathbb{T}^d$ and for every $t > 0$), such that*

$$\|A_h\|_{L^\infty(\Omega \times \mathbb{T}^d \times (0, T))} \leq C, \quad (3.5.10)$$

for some positive constant C which does not depend on h . Then

$$\mathcal{I}_h := \mathcal{I}_h(\omega, x) = \int_0^T A_h d(W_{\mathcal{T}}(x,t) - W(x,t)) \rightarrow 0 \quad (3.5.11)$$

strongly in $L^2(\Omega \times \mathbb{T}^d)$ as $h \rightarrow 0$.

Proof: We first rewrite

$$\begin{aligned} W_{\mathcal{T}}(x,t) &= \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K W(y,t) dy \cdot 1_K(x) \\ &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \sum_{K \in \mathcal{T}} e_K^m 1_K(x) \cdot \beta_m(t), \end{aligned}$$

where

$$e_K^m = \frac{1}{|K|} \int_K e_m(y) dy.$$

Then, the difference of stochastic integrals in (3.5.11) can be rewritten as

$$\mathcal{I}_h(\omega, x) := \sum_{m=1}^{\infty} \sqrt{\lambda_m} \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right) \int_0^T A_h(x,t) d\beta_m(t).$$

We have that

$$\begin{aligned}
\|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 &= \int_{\mathbb{T}^d} dx \|\mathcal{I}_h(\cdot, x)\|_{L^2(\Omega)}^2 \\
&= \int_{\mathbb{T}^d} dx \mathbb{E} \left[\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m} \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right) \int_0^T A_h(x, t) d\beta_m(t) \right)^2 \right] \\
&= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \mathbb{E} \left[\left(\int_0^T A_h(x, t) d\beta_m(t) \right)^2 \right] \\
&\quad + \int_{\mathbb{T}^d} 2dx \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} \left(\sum_{K \in \mathcal{T}} e_K^{m_1} 1_K(x) - e_{m_1}(x) \right) \left(\sum_{K \in \mathcal{T}} e_K^{m_2} 1_K(x) - e_{m_2}(x) \right) \\
&\quad \times \mathbb{E} \left[\left(\int_0^T A_h(x, t) d\beta_{m_1}(t) \int_0^T A_h(x, t) d\beta_{m_2}(t) \right) \right]
\end{aligned} \tag{3.5.12}$$

Therefore, by the independence of the Brownian motions $\{\beta_m(t)\}$ and by the Itô isometry for the stochastic integrals (3.1.9) and (3.1.10), in view of (3.5.10)

$$\begin{aligned}
&\|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\
&= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \mathbb{E} \left[\left(\int_0^T A_h^2(x, t) dt \right) \right] \\
&\quad + \int_{\mathbb{T}^d} dx \left[2 \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} \left(\sum_{K \in \mathcal{T}} e_K^{m_1} 1_K(x) - e_{m_1}(x) \right) \left(\sum_{K \in \mathcal{T}} e_K^{m_2} 1_K(x) - e_{m_2}(x) \right) \right. \\
&\quad \left. \times \mathbb{E} \left[\int_0^T A_h^2(x, t) dt \right] \delta_{m_1, m_2} \right] \\
&= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \int_0^T \mathbb{E}[A_h(x, t)^2] dt \\
&\leq C \sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2.
\end{aligned} \tag{3.5.13}$$

By Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}
\left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 &= \sum_{K \in \mathcal{T}} (e_K^m)^2 |K| \\
&= \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left(\int_K e_m(y) dy \right)^2 \leq \sum_{K \in \mathcal{T}} \int_K e_m^2(y) dy = \|e_m\|_{L^2(\mathbb{T}^d)}^2 = 1,
\end{aligned} \tag{3.5.14}$$

for all $m \in \{1, 2, \dots\}$, which implies that

$$\left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \quad \text{for each } m \in \{1, 2, \dots\}.$$

Next we prove that

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$$

tends to zero as $\text{size}(\mathcal{T})$ tends to zero.

Fix $\varepsilon > 0$, since $\sum_{m=1}^{\infty} \lambda_m$ is a converging series, there exists a $M \in \mathbb{N}$, such that

$$\sum_{m=M+1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \sum_{m=M+1}^{\infty} \lambda_m \leq \frac{\varepsilon}{2}. \quad (3.5.15)$$

Next we consider the term $\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$. Let $m \in \{1, 2, \dots\}$ be arbitrary; there holds

$$\begin{aligned} & \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= \lambda_m \left\| \sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right) 1_K(x) \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= \lambda_m \int_{\mathbb{T}^d} dx \left(\sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right) 1_K(x) \right)^2 \\ &= \lambda_m \int_{\mathbb{T}^d} dx \left(\sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 1_K(x) \right) \\ & \quad + \lambda_m \int_{\mathbb{T}^d} dx \sum_{K_1 \neq K_2} \left(\frac{1}{|K_1|} \int_{K_1} e_m(y) dy - e_m(x) \right) \left(\frac{1}{|K_2|} \int_{K_2} e_m(y) dy - e_m(x) \right) 1_{K_1}(x) 1_{K_2}(x) \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 dx \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{|K|} \int_K e_m(y) dy - \frac{1}{|K|} \int_K e_m(x) dy \right)^2 dx \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \frac{1}{|K|^2} \left(\int_K (e_m(y) - e_m(x)) dy \right)^2 dx \\ &\leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy \end{aligned}$$

We denote by $B(h)$ the ball with center 0 and radius h with $h = \text{size}(\mathcal{T})$. Then

$$\begin{aligned}
 & \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy \\
 & \leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K dx \int_{B(h)} dz (e_m(x+z) - e_m(x))^2 \\
 & = \lambda_m \int_{B(h)} dz \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
 & \leq \lambda_m |B(h)| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx.
 \end{aligned}$$

Since the condition (3.2.1) holds, it follows that

$$|B(h)| = C_d h^d \leq \frac{C_d}{\bar{\alpha}} |K|,$$

for some positive constant C_d . Let $\mathbb{T}^d \subset \subset \mathcal{O}$ where \mathcal{O} is an open set. We suppose that e_m is prolonged by periodicity on \mathbb{R}^d and that $\text{dist}(\mathbb{T}^d, \partial\mathcal{O}) \leq \text{size}(\mathcal{T})$. Therefore

$$\begin{aligned}
 & \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\
 & \leq \lambda_m \frac{C_d}{\bar{\alpha}} |K| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
 & = \lambda_m \frac{C_d}{\bar{\alpha}} \sup_{z \in B(h)} \|e_m(\cdot+z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)}^2.
 \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. It follows from the density of $C(\mathcal{O})$ into $L^2(\mathcal{O})$ that there exists a function $g \in C(\mathcal{O})$ such that

$$\|e_m(\cdot+z) - g(\cdot+z)\|_{L^2(\mathbb{T}^d)} + \|e_m(x) - g(x)\|_{L^2(\mathbb{T}^d)} \leq \frac{2\varepsilon}{3} \cdot \frac{\bar{\alpha}}{2M\lambda_m C_d},$$

for all $z \in B(h)$. Thus there exists a positive constant δ such that for all $z \in B(h)$

$$\begin{aligned}
 & \|e_m(\cdot+z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
 & \leq \|e_m(\cdot+z) - g(\cdot+z)\|_{L^2(\mathbb{T}^d)} + \|g(\cdot+z) - g(\cdot)\|_{L^2(\mathbb{T}^d)} + \|g(\cdot) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
 & \leq \frac{\bar{\alpha}}{2M\lambda_m C_d} \varepsilon
 \end{aligned}$$

for all $h < \delta$. Using the fact that the sum is finite, we have

$$\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \frac{\varepsilon}{2}. \quad (3.5.16)$$

Combining (3.5.15) and (3.5.16), we deduce that if $h < \delta$

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \varepsilon.$$

which in view of (3.5.13) completes the proof of (3.5.11).

It follows from Lemma 3.18 that the integral I_h defined by (3.5.9) converges to zero as h and k tend to zero.

Study of the term $\frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt]$:

Since $\Psi : (\omega, x, t, \nu) \in \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \eta''(\nu) g(\nu) \varphi(x, t) e_m^2(x) \in \mathbb{R}$ is a Carathéodory function such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times \mathbb{T}^d \times (0, T))$, we deduce that:

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt] \\ & \rightarrow \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x, t, \alpha)) g^2(\mathbf{u}(x, t, \alpha)) \varphi(x, t) e_m^2(x) \alpha dx dt], \end{aligned}$$

as $h \rightarrow 0$, for all $m \geq 1$.

We deduce that for all P -measurable set A , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx] + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha)) d\alpha \partial_t \varphi(x, t) dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x, t, \alpha)) \varphi(x, t) d\alpha dW(x, t) dx] \tag{3.5.17} \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x, t, \alpha)) g^2(\mathbf{u}(x, t, \alpha)) \varphi(x, t) e_m^2(x) d\alpha dx dt] \\ & \geq 0. \end{aligned}$$

In turn (3.5.17) implies that for all $\chi \in L^2(\Omega)$

$$\begin{aligned} & \mathbb{E}[\chi \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx] + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha)) d\alpha \partial_t \varphi(x, t) dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x, t, \alpha)) \varphi(x, t) d\alpha dW(x, t) dx] \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x, t, \alpha)) g^2(\mathbf{u}(x, t, \alpha)) \varphi(x, t) e_m^2(x) d\alpha dx dt] \\ & \geq 0. \end{aligned}$$

Hence \mathbf{u} is a measure-valued entropy solution of Problem (3.1.1) in the sense of Definition 3.2.

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3.6 Appendix

An example of a space-time stochastic forcing term in one space dimension

In [3], the authors have proposed the following numerical scheme for the stochastic Burgers equation in one space dimension:

$$\frac{u_K^{n+1} - u_K^n}{\Delta t} + \frac{1}{\Delta x} \left(F_{K+\frac{1}{2}}^n - F_{K-\frac{1}{2}}^n \right) = \alpha \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n$$

with

$$G_i^n = \sqrt{\frac{2}{I}} \left(\sum_{m=1}^{\frac{I-1}{2}} \left(\frac{C_m^n}{m^\gamma} \cos(2\pi m x_i) - \frac{S_m^n}{m^\gamma} \sin(2\pi m x_i) \right) \right)$$

where I is the number of the volume elements, C_m^n and S_m^n are independent and identically distributed random variables which follow the Gaussian distribution $\mathcal{N}(0, 1)$. We set

$$C_m^n = \frac{\beta_m(t^{n+1}) - \beta_m(t^n)}{\sqrt{\Delta t}} \quad \text{and} \quad S_m^n = \frac{\tilde{\beta}_m(t^{n+1}) - \tilde{\beta}_m(t^n)}{\sqrt{\Delta t}}$$

where β_m and $\tilde{\beta}_m$ are standard Brownian motions. We recall that for a standard Brownian motion $\beta_m(t)$, $\beta_m(t^{n+1}) - \beta_m(t^n)$ follows the Gaussian distribution $\mathcal{N}(0, \Delta t)$ (cf. [47], Definition 2.1.1, p. 7) and $(\beta_m(t^{n+1}) - \beta_m(t^n))/\sqrt{\Delta t}$ follows the Gaussian distribution $\mathcal{N}(0, 1)$. Moreover note that

$$\sqrt{2} \cos(2\pi m x) \approx \frac{1}{|\Delta x|} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sqrt{2} \cos(2\pi m x) dx,$$

and

$$\sqrt{2} \sin(2\pi m x) \approx \frac{1}{|\Delta x|} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sqrt{2} \sin(2\pi m x) dx.$$

Therefore

$$\begin{aligned} \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n \approx & \sqrt{\frac{1}{\Delta t}} \left(\sum_{m=1}^{\frac{I-1}{2}} \left(\frac{(\beta_m(t^{n+1}) - \beta_m(t^n))/\sqrt{\Delta t}}{m^\gamma} \frac{1}{|\Delta x|} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sqrt{2} \cos(2\pi m x) dx \right. \right. \\ & \left. \left. - \frac{(\tilde{\beta}_m(t^{n+1}) - \tilde{\beta}_m(t^n))/\sqrt{\Delta t}}{m^\gamma} \frac{1}{|\Delta x|} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sqrt{2} \sin(2\pi m x) dx \right) \right), \end{aligned} \quad (3.6.1)$$

with $\gamma > 0$. If we set

$$\begin{cases} e_{2m}(x) = -\sqrt{2} \sin(2\pi m x), & \text{for } m = 1, 2, \dots \\ e_{2m+1}(x) = \sqrt{2} \cos(2\pi m x), & \text{for } m = 0, 1, 2, \dots \end{cases} \quad (3.6.2)$$

and $\sqrt{\lambda_m} = \frac{1}{m^\gamma}$, the discrete noise in (3.2.4) (see also (3.2.6)) is essentially given by

$$\frac{1}{\Delta t} (W_K^{n+1} - W_K^n) = \frac{1}{\Delta t} \sum_{m=1}^{\infty} \sqrt{\lambda_m} (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{1}{|K|} \int_K e_m(x) dx. \quad (3.6.3)$$

We remark that the condition (3.1.2) amounts to

$$\sum_{m=1}^{\infty} \lambda_m = \sum_{m=1}^{\infty} \frac{1}{m^{2\gamma}} \leq \Lambda_0, \quad (3.6.4)$$

for some positive constant Λ_0 so that $\gamma > \frac{1}{2}$.

We recall the condition (3.1.4)

$$\sum_{m=1}^{\infty} \lambda_m \|e_m\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1,$$

so that in the corresponding one space dimension case (3.6.2), $\|e_m\|_{L^\infty(\mathbb{T}^d)} \leq \sqrt{2}$ for all $m \in \{1, 2, \dots\}$. Therefore if the condition (3.1.2) is satisfied, the condition (3.1.4) is satisfied as well.

A finite volume scheme for a first order conservation law with a non-monotone flux function involving a Q -Wiener process

Résumé. Nous démontrons la convergence d'une méthode de volumes finis pour une loi de conservation avec une fonction flux non monotone et un terme source multiplicatif faisant intervenir un processus Q -Wiener. Nous nous appuyons sur un flux numérique monotone pour la discrétisation de ce problème, et nous notons $\{u_{\mathcal{T},k}\}$ la solution discrète. Nous présentons des estimations a priori pour le terme source discret et pour la solution discrète $\{u_{\mathcal{T},k}\}$ ainsi qu'une estimation BV faible. Puis nous prouvons une inégalité entropique vérifiée par la solution discrète après l'avoir interpolée en temps. On peut prouver une inégalité continue entropique et un théorème de convergence par des méthodes similaires à celles utilisées au Chapitre 3.

Abstract. We study a time explicit finite volume method for a conservation law with a non-monotone flux function and a multiplicative source term involving a Q -Wiener process. We apply a monotone scheme for the flux discretization, and we denote by $\{u_{\mathcal{T},k}\}$ its discrete solution. We present a priori estimates for the discrete source term and for the discrete solution $\{u_{\mathcal{T},k}\}$, including a weak BV estimate. We then prove an entropy inequality satisfied by the discrete solution after interpolation in time. The proofs of the continuous entropy inequality and the corresponding convergence theorem are similar to those presented in Chapter 3.

4.1 Introduction

In this Chapter, we study exactly the same problem as in Chapter 3, but without supposing the monotonicity of the nonlinear flux function f

$$\begin{cases} du + \operatorname{div}(\mathbf{v}(x, t)f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x) & \omega \in \Omega, \quad x \in \mathbb{T}^d, \end{cases} \quad (4.1.1)$$

where \mathbb{T}^d is the d -dimensional torus and that

$$W(x, t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (4.1.2)$$

is a Q -Wiener process in $L^2(\mathbb{T}^d)$ [cf. [35], Definition 2.6, p. 20]. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(\mathbb{T}^d)$ with $\operatorname{Tr} Q < +\infty$, $\{e_m\}_{m \geq 1}$ be an orthonormal basis in $L^2(\mathbb{T}^d)$ diagonalizing Q , and $\{\lambda_m\}_{m \geq 1}$ be the corresponding eigenvalues, so that

$$Qe_m = \lambda_m e_m$$

for all $m \geq 1$. Since Q is of trace-class,

$$\operatorname{Tr} Q = \sum_{m=1}^{\infty} \langle Qe_m, e_m \rangle_{L^2(\mathbb{T}^d)} = \sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0. \quad (4.1.3)$$

for some positive constant Λ_0 . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration (\mathcal{F}_t) [42] and $\{\beta_m(t)\}_{\{m \geq 1\}}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose furthermore that $e_m \in L^\infty(\mathbb{T}^d)$ for $m = 1, 2, \dots$ and that there exist positive constant Λ_1 such that

$$\sum_{m=1}^{\infty} \lambda_m \|e_m\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (4.1.4)$$

Moreover we assume the following hypotheses which we denote by (\overline{H}) :

- $u_0 \in L^2(\mathbb{T}^d)$,
- $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant C_f such that $f(0) = 0$,
- $g : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Lipschitz continuous function with Lipschitz constant C_g such that $|g(u)| \leq M_g$ for some positive constant M_g ,
- $\mathbf{v}(x, t) \in C^1(\mathbb{T}^d \times [0, T])$ and $\operatorname{div} \mathbf{v} = 0$ on $\mathbb{T}^d \times [0, T]$, and $|\mathbf{v}| \leq V$ on $\mathbb{T}^d \times [0, T]$ for some positive constant V .

Next we introduce some general notations,

- Let $E[\cdot]$ denote the expectation, namely the integral over Ω with respect to the probability measure \mathbb{P} . In general for a random variable Y from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, if Y is integrable for the measure \mathbb{P} , then $E[Y] = \int_{\Omega} Y(\omega) \mathbb{P}(d\omega)$.

- We denote by $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d))$ the subclass of $L^2(\Omega \times \mathbb{T}^d \times (0, T))$ consisting of predictable $L^2(\mathbb{T}^d)$ -valued processes [cf. [19], p. 98].

This chapter extends the article by Bauzet, Charrier, and Gallouët [9] where they consider a conservation law with a non-monotone flux function and a multiplicative noise involving one-dimensional Brownian motion.

The organisation of the Chapter is as follows: In section 4.1.1 we define a measure-valued stochastic entropy solution of Problem (4.1.1) which coincides with the definition given in Chapter 3. In section 4.2, we apply a monotone finite volume scheme to the Problem (4.1.1) and define the discrete solution $\{u_{\mathcal{T},k}\}$. In section 4.3, we present an estimate of the discrete noise term as well as an a priori estimate on the discrete solution $\{u_{\mathcal{T},k}\}$. We then give a weak BV estimate which is essential in the sequel. In section 4.4, we prove a discrete entropy inequality. Using similar arguments as in Chapter 3 and as in [9], one can prove a continuous entropy inequality for the discrete solution; one can also show that the discrete solution $\{u_{\mathcal{T},k}\}$ converges up to a subsequence to a measure-valued entropy solution \mathbf{u} in the sense of Young measures.

Funaki-Gao-Hilhorst-Weber [34] will show that the measured-value entropy solution \mathbf{u} is unique and coincides with the unique weak stochastic entropy solution; this will ensure that the whole approximate sequence $\{u_{\mathcal{T},k}\}$ converges to u as the maximum diameters of the volume elements and the time step tend to zero.

4.1.1 The continuous problem

We define a measure-valued weak stochastic entropy solution of Problem (4.1.1) as it has been done in Chapter 3, namely

Definition 4.1 (*Measure-valued entropy solution of Problem (4.1.1)*)

A function \mathbf{u} of $\mathcal{N}_\omega^2(0, T, L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ is a measure-valued entropy solution of the stochastic scalar conservation law (4.1.1) with the initial condition $u_0 \in L^2(\mathbb{T}^d)$, if *P*-a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \{ \eta(\mathbf{u}(\cdot, \alpha)) \partial_t \varphi(x, t) + F^\eta(\mathbf{u}(\cdot, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) \} d\alpha dx dt \\ & + \int_{\mathbb{T}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha)) g(\mathbf{u}(\cdot, \alpha)) \varphi(x, t) d\alpha dW(x, t) dx \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha)) g^2(\mathbf{u}(\cdot, \alpha)) \varphi(x, t) e_m^2(x) d\alpha dx dt \geq 0. \end{aligned} \tag{4.1.5}$$

with

$$F^\eta(\tau) = \int_0^\tau \eta'(\sigma) f'(\sigma) d\sigma \tag{4.1.6}$$

for all $\varphi \in \mathcal{C} := \{\varphi \in C_0^\infty(\mathbb{T}^d \times [0, T]), \varphi \geq 0\}$ and for all $\eta \in \mathcal{A}$ where \mathcal{A} is the set of $C^3(\mathbb{R})$ convex functions η such that the support of η'' is compact.

4.2 The numerical scheme

Definition 4.2 (*Admissible mesh*) An admissible mesh \mathcal{T} of \mathbb{T}^d for the discretization is given by a family of disjoint polygonal connected subset of \mathbb{T}^d such that \mathbb{T}^d is the union of the closure of the elements of \mathcal{T} and the common interface of any two control volumes is included in a hyperplane of \mathbb{T}^d . It is assumed that

$$h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty,$$

and that, for some $\bar{\alpha} \in \mathbb{R}^+$, we have

$$\bar{\alpha}h^d \leq |K| \text{ and } |\partial K| \leq \frac{1}{\bar{\alpha}}h^{d-1}, \quad \forall K \in \mathcal{T}, \quad (4.2.1)$$

so that

$$\frac{|\partial K|}{|K|} \leq \frac{1}{\bar{\alpha}^2 h}, \quad (4.2.2)$$

where we note

- ∂K the boundary of the control volume K .
- x_K a point in the control volume K .
- $|K|$ the d -dimensional Lebesgue measure of K .
- $|\partial K|$ the $(d-1)$ -dimensional Lebesgue measure of ∂K .
- \mathcal{E}_K the set of interfaces of the control volume K .
- $\mathcal{N}(K)$ the set of control volumes neighbors of the control volume K .
- $\sigma_{K,L}$ the common interface between K and L for all $L \in \mathcal{N}(K)$.
- $\tilde{\mathcal{E}}$ the set of pairs of adjacent control volumes, which is given by $\tilde{\mathcal{E}} = \{(K, L) \in \mathcal{T}^2, K \neq L, |\sigma_{K,L}| \neq 0\}$.
- \mathcal{E} the set of all the interfaces of the mesh \mathcal{T} .
- $\mathbf{n}_{K,L}$ the unit normal vector which is perpendicular to the interface $\sigma_{K,L}$, outward to the control volume K , for all $L \in \mathcal{N}(K)$.

Consider an admissible mesh \mathcal{T} in the sense of Definition 4.2. In order to compute an approximation of u on $(0, T)$, we take $N \in \mathbb{N}^+$ and define the time step $k = \frac{T}{N}$. In this way

$$(0, T) = \bigcup_{n=0}^{N-1} (nk, (n+1)k). \text{ We set } t^n = nk \text{ for all } n = 0, 1, 2, \dots, N \text{ and assume that } k \leq \frac{\bar{\alpha}^2 h}{VC_f}.$$

The discrete unknowns are u_K^n , for all $n \in \{1, 2, \dots, N\}$ and for all $K \in \mathcal{T}$.

We now define the monotone finite volume scheme. For discretization of such a problem, we need to define the numerical flux.

Definition 4.3 [9] (*monotone numerical flux*) We say that a function $F \in C^2(\mathbb{R}^2, \mathbb{R})$ is a monotone numerical flux if it satisfies the following properties

1. $F(a, b)$ is nondecreasing with respect to a and nonincreasing with respect to b .
2. There exist $F_1, F_2 > 0$ such that for all $a, b \in \mathbb{R}$ there holds $|F(b, a) - F(a, a)| \leq F_1|a - b|$ and $|F(a, b) - F(a, a)| \leq F_2|a - b|$
3. $F(a, a) = f(a)$ for all $a \in \mathbb{R}$

The discrete initial condition $\{u_K^0, K \in \mathcal{T}\}$ is given by:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx \quad \text{for all } K \in \mathcal{T}, \quad (4.2.3)$$

and $\{u_K^n\}$ satisfies the following explicit scheme: for all $K \in \mathcal{T}$ and all $n \in \{0, 1, \dots, N-1\}$

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \right\} = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n), \quad (4.2.4)$$

where denoting by $d\gamma$ the $(d-1)$ -dimensional Lebesgue measure,

$$v_{K,L}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^+ d\gamma(x) dt, \quad (4.2.5)$$

$$\begin{aligned} v_{L,K}^n &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{L,K})^+ d\gamma(x) dt \\ &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^- d\gamma(x) dt. \end{aligned}$$

Note that

$$v_{K,L}^n - v_{L,K}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \quad (4.2.6)$$

and

$$v_{K,L}^n + v_{L,K}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} |\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L}| d\gamma(x) dt \quad (4.2.7)$$

Moreover, since $\operatorname{div}[\mathbf{v}(x, t)] = 0$ for all $(x, t) \in \mathbb{T} \times (0, T)$, we have

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) = 0. \quad (4.2.8)$$

Indeed

$$\begin{aligned} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) &= \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left(\frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right) \\ &= \frac{1}{k} \int_{nk}^{(n+1)k} \int_K \operatorname{div}[\mathbf{v}(x, t)] dx dt = 0 \end{aligned}$$

As in Chapter 3, we define the discrete noise terms as

$$\begin{aligned} W_{M,K}(t) &= \sum_{m=1}^M \sqrt{\lambda_m} \beta_m(t) e_K^m, \\ W_K(t) &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_K^m, \end{aligned} \tag{4.2.9}$$

where $e_K^m = \frac{1}{|K|} \int_K e_m(x) dx$ for all $K \in \mathcal{T}$. Moreover we denote by $W_{M,K}^n$ and W_K^n the values of $W_{M,K}$ and W_K at time $t = nk$ respectively.

The results below follow as in the lemmas 3.7 , 3.8 and 3.9 in Chapter 3.

Lemma 4.4 *Given by the definitions of $W_{M,K}(t)$ and $W_K(t)$ as in (4.2.9), we have that*

$$\mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] = 0 \tag{4.2.10}$$

and

$$\mathbb{E}[W_K^{n+1} - W_K^n] = 0 \tag{4.2.11}$$

Lemma 4.5 *Suppose that the coefficients $\{\lambda_m\}_{m \geq 1}$ satisfy (4.1.3), then*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0, \tag{4.2.12}$$

holds for all $M \in \mathbb{N}^+$, $n \in \{0, 1, \dots, N-1\}$ and $K \in \mathcal{T}$.

Corollary 4.6 *There holds:*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0, \tag{4.2.13}$$

for all $n \in \{0, 1, \dots, N-1\}$ and $K \in \mathcal{T}$.

We repeat below definitions and notations from Chapter 3

$$\begin{aligned} W_{\mathcal{T},k}(x, t) &= W_K^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k), \\ W_{M,\mathcal{T},k}(x, t) &= W_{M,K}^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k). \end{aligned} \tag{4.2.14}$$

and

$$W_{\mathcal{T}}(x, t) = W_K(t) \quad \text{for } x \in K \quad \text{and } t \in (0, T). \tag{4.2.15}$$

Next we define the approximate finite volume solution $u_{\mathcal{T},k}$ on $\mathbb{T}^d \times (0, T)$ from the discrete unknowns u_K^n , for all $n \in \{0, 1, \dots, N-1\}$ and for all $K \in \mathcal{T}$, that:

$$u_{\mathcal{T},k}(x, t) = u_K^n \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k), \tag{4.2.16}$$

where the set $\{u_K^0, K \in \mathcal{T}\}$ is defined by (4.2.3).

4.3 A priori estimates

Lemma 4.7 *Assume that hypotheses (\bar{H}) hold. Let $T > 0$, \mathcal{T} be an admissible mesh in the sense of Definition 4.2 and h and k satisfy the Courant-Friedrichs-Levy (CFL) condition:*

$$k \leq \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)}. \quad (4.3.1)$$

There hold

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2 \quad (4.3.2)$$

and

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2\Lambda_0 M_g^2. \quad (4.3.3)$$

where $Q_T = \mathbb{T}^d \times (0, T)$.

Proof. We recall the numerical scheme

$$\frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} = \frac{|K|}{k} g(u_K^n)(W_K^{n+1} - W_K^n). \quad (4.3.4)$$

We multiply both sides of (4.3.4) by ku_K^n :

$$\begin{aligned} |K|(u_K^{n+1} - u_K^n)u_K^n &= -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

Applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$, we obtain that

$$\begin{aligned} \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2 - (u_K^{n+1} - u_K^n)^2] &= -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2] &= \frac{|K|}{2} (u_K^{n+1} - u_K^n)^2 \\ &\quad - k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned} \quad (4.3.5)$$

We substitute (4.3.4) into (4.3.5) to deduce that:

$$\begin{aligned} &\frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2] \\ &= \frac{|K|}{2} \left(-\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} + g(u_K^n)(W_K^{n+1} - W_K^n) \right)^2 \\ &\quad - k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

We take the expectation of both sides to obtain:

$$\begin{aligned}
& \frac{|K|}{2} \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\
&= \frac{|K|}{2} \mathbb{E} \left[\left(\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \right)^2 + g^2(u_K^n) (W_K^{n+1} - W_K^n)^2 \right] \\
&\quad - 2 \frac{|K|}{2} \mathbb{E} \left[\frac{kg(u_K^n)}{|K|} (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \right] \\
&\quad - \mathbb{E} \left[k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \right] \\
&\quad + \mathbb{E} [g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n].
\end{aligned}$$

Next we show that some of the terms in the equality above vanish. Indeed since $W_K^{n+1} - W_K^n$ and u_K^n are independent variables, we have that, in view of (4.2.11)

$$\begin{aligned}
\mathbb{E} \left[g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n \right] &= \mathbb{E} \left[W_K^{n+1} - W_K^n \right] \mathbb{E} \left[g(u_K^n) u_K^n |K| \right] \\
&= 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \mathbb{E} \left[g(u_K^n) (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \right] \\
&= \mathbb{E} \left[W_K^{n+1} - W_K^n \right] \mathbb{E} \left[g(u_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \right] \\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] \\
&= \frac{k^2}{2|K|} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} \right]^2 \\
&\quad + \frac{|K|}{2} \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \mathbb{E} [g^2(u_K^n)] \\
&\quad - k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)\} u_K^n \right].
\end{aligned} \tag{4.3.6}$$

We recall (4.2.8), which states that $\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(v_{K,L}^n - v_{L,K}^n) = 0$, so that

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}|(v_{K,L}^n - v_{L,K}^n)f(u_K^n) = 0, \quad (4.3.7)$$

then the equality (4.3.6) can be rewritten as

$$\frac{|K|}{2} \mathbb{E} \left[(u_K^{n+1})^2 - (u_K^n)^2 \right] = B_1 - B_2 + D, \quad (4.3.8)$$

where

$$\begin{aligned} B_1 &= \frac{k^2}{2|K|} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right]^2 \\ B_2 &= k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ D &= \frac{|K|}{2} \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 \right] \mathbb{E} \left[g^2(u_K^n) \right] \end{aligned} \quad (4.3.9)$$

We define that

$$\mathcal{I}_n = \left\{ (K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n \right\},$$

and $B_3 = \sum_{K \in \mathcal{T}} B_2$, so that

$$\begin{aligned} B_3 &= k \sum_{u_K^n > u_L^n} \mathbb{E} \left[|\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ &\quad + k \sum_{u_L^n > u_K^n} \mathbb{E} \left[|\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ &= k \sum_{u_K^n > u_L^n} \mathbb{E} \left[|\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ &\quad + k \sum_{u_K^n > u_L^n} \mathbb{E} \left[|\sigma_{K,L}| \{v_{L,K}^n (F(u_L^n, u_K^n) - f(u_L^n)) - v_{K,L}^n (F(u_K^n, u_L^n) - f(u_L^n))\} u_L^n \right] \\ &= k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{u_K^n (F(u_K^n, u_L^n) - f(u_K^n)) - u_L^n (F(u_K^n, u_L^n) - f(u_L^n))\} \right. \\ &\quad \left. - v_{L,K}^n \{u_K^n (F(u_L^n, u_K^n) - f(u_K^n)) - u_L^n (F(u_L^n, u_K^n) - f(u_L^n))\} \right] \end{aligned} \quad (4.3.10)$$

We recall the function Φ defined in (3.3.16) by $\Phi(a) = \int_0^a s f'(s) ds$; an integration by parts yields, for all $(a, b) \in \mathbb{R}^2$

$$\Phi(b) - \Phi(a) = \int_a^b s f'(s) ds = b(f(b) - F(a, b)) - a(f(a) - F(a, b)) - \int_a^b (f(s) - F(a, b)) ds. \quad (4.3.11)$$

Next we rewrite B_3 in the form

$$B_3 = B_4 - B_5$$

where

$$B_4 = \mathbb{E} \left[\sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \left\{ v_{K,L}^n \left(\int_{u_K^n}^{u_L^n} (f(s) - F(u_K^n, u_L^n)) ds \right) + v_{L,K}^n \left(\int_{u_L^n}^{u_K^n} (f(s) - F(u_L^n, u_K^n)) ds \right) \right\} \right] \quad (4.3.12)$$

$$B_5 = \mathbb{E} \left[\sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) \{ \Phi(u_K^n) - \Phi(u_L^n) \} \right] \quad (4.3.13)$$

so that $B_3 = B_4 - B_5$. We show below that $B_5 = 0$. Indeed, we deduce from (4.2.6)

$$B_5 = \mathbb{E} \left[\sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \left(\frac{1}{k |\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right) \{ \Phi(u_K^n) - \Phi(u_L^n) \} \right]$$

$$\begin{aligned} B_5 &= \mathbb{E} \left[\sum_{(K,L) \in \mathcal{I}_n} \left\{ \left(\int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right) \Phi(u_K^n) + \left(\int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{L,K} d\gamma(x) dt \right) \Phi(u_L^n) \right\} \right] \\ &= \mathbb{E} \left[\sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \Phi(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right] \\ &= \mathbb{E} \left[\sum_{K \in \mathcal{T}} \Phi(u_K^n) \int_{nk}^{(n+1)k} \int_K \operatorname{div}[\mathbf{v}(x, t)] dx dt \right] \\ &= 0. \end{aligned}$$

Before continuing with the proof of Lemma 4.7, we recall the following auxiliary result:

Lemma 4.8 *Let $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone Lipschitz continuous function with Lipschitz constant $C_{\mathcal{G}} > 0$. Then there holds*

$$\left| \int_a^b (\mathcal{G}(s) - \mathcal{G}(a)) ds \right| \geq \frac{1}{2C_{\mathcal{G}}} (\mathcal{G}(b) - \mathcal{G}(a))^2, \quad (4.3.14)$$

for all $a, b \in \mathbb{R}$.

Lemma 4.8 coincides with Lemma 3.11 in Chapter 3. Next we search for a lower bound for $B_3 = B_4$. We deduce from Definition 4.3 and Lemma 4.8 that for all $a, b \in \mathbb{R}$, we have

$$\int_a^b (f(t) - F(a, b)) dt \geq \int_a^b (F(a, t) - F(a, a)) dt \geq \frac{1}{2F_2} (f(a) - F(a, b))^2 \quad (4.3.15)$$

and

$$\int_a^b (f(t) - F(a, b)) dt \geq \int_a^b (F(t, b) - F(a, b)) dt \geq \frac{1}{2F_1} (f(b) - F(a, b))^2 \quad (4.3.16)$$

Multiplying (4.3.15) by $\frac{F_2}{F_1 + F_2}$ and (4.3.16) by $\frac{F_1}{F_1 + F_2}$ and adding the two inequalities yields:

$$\int_a^b (f(t) - F(a, b)) dt \geq \frac{1}{2(F_1 + F_2)} \left[(f(a) - F(a, b))^2 + (f(b) - F(a, b))^2 \right]. \quad (4.3.17)$$

We can deduce from this inequality that

$$\begin{aligned} B_3 = B_4 \geq k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} & \left[\frac{v_{K,L}^n}{2(F_1 + F_2)} \left\{ (F(u_K^n, u_L^n) - f(u_K^n))^2 + (F(u_K^n, u_L^n) - f(u_L^n))^2 \right\} \right. \\ & \left. + \frac{v_{L,K}^n}{2(F_1 + F_2)} \left\{ (F(u_L^n, u_K^n) - f(u_K^n))^2 + (F(u_L^n, u_K^n) - f(u_L^n))^2 \right\} \right] \end{aligned} \quad (4.3.18)$$

We have thus obtained a lower bound for B_3 .

Next we study B_1 .

$$\begin{aligned} \sum_{K \in \mathcal{T}} B_1 = \sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \mathbb{E} & \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{ v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\ & \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n)) \} \right)^2 \right]. \end{aligned} \quad (4.3.19)$$

We define $A = F(u_K^n, u_L^n) - f(u_K^n)$, $B = F(u_L^n, u_K^n) - f(u_K^n)$, $\zeta = \frac{v_{K,L}^n}{v_{K,L}^n + v_{L,K}^n}$, and deduce from Cauchy-Schwarz inequality that

$$\begin{aligned} & \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{ v_{K,L}^n A - v_{L,K}^n B \} \right)^2 \\ &= \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \{ \zeta A - (1 - \zeta) B \} \right)^2 \\ &\leq \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \{ \zeta A + (1 - \zeta)(-B) \}^2 \\ &\leq \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \{ \zeta A^2 + (1 - \zeta) B^2 \} \end{aligned}$$

Since $(v_{K,L}^n + v_{L,K}^n)\zeta = v_{K,L}^n$ and $(v_{K,L}^n + v_{L,K}^n)(1 - \zeta) = v_{L,K}^n$, we obtain the following estimates

$$\begin{aligned} \sum_{K \in \mathcal{T}} B_1 &\leq \sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\ &\quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \end{aligned} \quad (4.3.20)$$

Using the fact that, in view of (4.2.7),

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \leq V |\partial K| \quad (4.3.21)$$

which together with the mesh properties (4.2.2) and the CFL condition (4.3.1) implies

$$\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \leq kV \frac{|\partial K|}{|K|} \leq \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)} V \frac{1}{\bar{\alpha}^2 h} = \frac{1}{F_1 + F_2}, \quad (4.3.22)$$

so that

$$\begin{aligned} \sum_{K \in \mathcal{T}} B_1 &\leq \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \mathbb{E} \left[\frac{v_{K,L}^n}{2(F_1 + F_2)} \{ (F(u_K^n, u_L^n) - f(u_K^n))^2 + (F(u_K^n, u_L^n) - f(u_L^n))^2 \} \right. \\ &\quad \left. + \frac{v_{L,K}^n}{2(F_1 + F_2)} \{ (F(u_L^n, u_K^n) - f(u_K^n))^2 + (F(u_L^n, u_K^n) - f(u_L^n))^2 \} \right] \end{aligned} \quad (4.3.23)$$

Using (4.3.18) and (4.3.23), we deduce

$$\sum_{K \in \mathcal{T}} (B_1 - B_2) = \left(\sum_{K \in \mathcal{T}} B_1 \right) - B_3 \leq 0,$$

which we substitute in (4.3.8) and also use the Definition of D in (4.3.9) and the inequality (4.6) to obtain:

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)^2] \mathbb{E}[g^2(u_K^n)] \\ &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + M_g^2 \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)^2] \\ &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] + k\Lambda_0 M_g^2 \end{aligned}$$

We deduce from the inequalities :

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-1})^2] + k\Lambda_0 M_g^2, \\ \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n-2})^2] + k\Lambda_0 M_g^2 \end{aligned}$$

...

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^1)^2] \leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2] + k\Lambda_0 M_g^2$$

that

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^n)^2] \leq \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2] + nk\Lambda_0 M_g^2 \quad \text{for all } n \in \{1, 2, \dots, N\},$$

which implies that

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2.$$

As a consequence, there holds

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2\Lambda_0 M_g^2.$$

4.3.1 Weak BV estimate

Lemma 4.9 *Assume that hypotheses (\overline{H}) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 4.2, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfy the CFL condition*

$$k \leq \frac{(1-\xi)\bar{\alpha}^2 h}{V(F_1 + F_2)} \quad (4.3.24)$$

for some $\xi \in (0, 1)$. Then we have the following estimates

1. There exists a positive constant C_1 , depending on Λ_0 , T , α , u_0 , ξ , C_f and M_g such that

$$\begin{aligned} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F(u_K^n, u_L^n) - f(u_K^n)\}^2 \right. \\ \left. + v_{L,K}^n \{F(u_L^n, u_K^n) - f(u_K^n)\}^2 \right] \leq C_1 \end{aligned} \quad (4.3.25)$$

2. There exists a positive constant $C_2 \in \mathbb{R}$ depending on, T , $\bar{\alpha}$, u_0 , ξ , F_1 , F_2 and C_g such that

$$\begin{aligned} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c)) \right\} \right. \\ \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(d) - F(c,d)) + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(c) - F(c,d)) \right\} \right] \leq C_2 h^{-\frac{1}{2}} \end{aligned} \quad (4.3.26)$$

Proof: 1, Multiplying the equation (4.3.4) by ku_K^n , also using (4.3.7) taking the expectation and summing over $K \in \mathcal{T}$ and $n = 0, 1, \dots, N-1$ yields

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] \\ & + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ & = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)], \end{aligned}$$

which we rewrite as $A + B = D$ with

$$\begin{aligned} A &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] \\ B &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}[\{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} u_K^n] \\ D &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)]. \end{aligned}$$

Note that the term $D = 0$ since u_K^n is independent of the increment $(W_K^{n+1} - W_K^n)$ and since by (4.2.11), $\mathbb{E}[W_K^{n+1} - W_K^n] = 0$. Applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$ we obtain:

$$\begin{aligned} A &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2], \end{aligned} \quad (4.3.27)$$

where we set

$$A_1 = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2],$$

and

$$A_2 = \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2] \geq -\frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2]. \quad (4.3.28)$$

Next, we substitute into A_1 . Also using (4.3.7) and the fact that $W_K^{n+1} - W_K^n$ and any function for u_K^n , u_L^n , $v_{K,L}^n$ and $v_{L,K}^n$ are independent, we deduce that

$$\begin{aligned} A_1 &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \left\{ \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 g^2(u_K^n) \right] \right. \\ & \quad \left. + \frac{k^2}{|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \right\} \end{aligned} \quad (4.3.29)$$

It follows from the CFL condition (4.3.24), the mesh properties (4.2.2) and (4.2.7) that

$$\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \leq \frac{1-\xi}{F_1 + F_2}. \quad (4.3.30)$$

Using a similar idea of (4.3.19) and (4.3.20), we deduce that

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \\ & \leq \sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\ & \quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\ & \leq \sum_{K \in \mathcal{T}} \frac{k}{2} \left(\frac{1-\xi}{F_1 + F_2} \right) \\ & \quad \times \left[\mathbb{E} \left[\sum_{u_K^n > u_L^n} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\sum_{u_L^n > u_K^n} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \right] \\ & \leq \sum_{K \in \mathcal{T}} \frac{k}{2} \left(\frac{1-\xi}{F_1 + F_2} \right) \\ & \quad \times \left[\mathbb{E} \left[\sum_{u_K^n > u_L^n} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \right. \\ & \quad \left. + \mathbb{E} \left[\sum_{u_K^n > u_L^n} |\sigma_{K,L}| \{v_{L,K}^n (F(u_L^n, u_K^n) - f(u_L^n))^2 + v_{K,L}^n (F(u_K^n, u_L^n) - f(u_L^n))^2\} \right] \right] \\ & \leq \frac{1-\xi}{2} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{F_1 + F_2} \\ & \quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \right. \\ & \quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\} \right] \end{aligned}$$

where we define

$$\mathcal{C}(a, b) = \{(c, d) \in [\min(a, b), \max(a, b)]^2 : (d - c)(b - a) \geq 0\}.$$

Therefore

$$\begin{aligned}
 A_1 &\geq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{|K| \in \mathcal{T}} \mathbb{E} \left[(W_K^{n+1} - W_K^n)^2 g^2(u_K^n)^2 \right] \\
 &\quad - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{F_1 + F_2} \\
 &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \right. \\
 &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\} \right]
 \end{aligned}$$

where we recall that $\mathcal{I}_n = \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}$. Next we substitute (4.6) and (4.3.28) to deduce that

$$\begin{aligned}
 A &= A_1 + A_2 \\
 &\geq -\frac{1}{2} (\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T \Lambda_0 M_g^2 |\mathbb{T}^d|) \\
 &\quad - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{F_1 + F_2} \\
 &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \right. \\
 &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\} \right]
 \end{aligned}$$

We now consider the term B . We remark, in view of (4.3.9) and (4.3.11) that

$$B = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} B_2 = \sum_{n=0}^{N-1} B_3 = \sum_{n=0}^{N-1} B_4$$

so that by (4.3.12),

$$B = \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[\left\{ v_{K,L}^n \int_{u_K^n}^{u_L^n} (f(s) - F(u_K^n, u_L^n)) ds + v_{L,K}^n \int_{u_L^n}^{u_K^n} (f(s) - F(u_L^n, u_K^n)) ds \right\} \right] \quad (4.3.31)$$

Next we search for a lower bound for B . Let $a, b \in \mathbb{R}$. In view of the monotonicity properties of F , the following inequality holds for all $(c, d) \in \mathcal{C}(a, b)$:

$$\int_a^b (f(s) - F(a, b)) ds \geq \int_c^d (f(s) - F(a, b)) ds \geq \int_c^d (f(s) - F(c, d)) ds.$$

We now use again the monotonicity of F and Lemma 4.8 to deduce that for all $(c, d) \in \mathcal{C}(a, b)$:

$$\int_a^b (f(s) - F(a, b)) ds \geq \int_c^d (f(s) - F(c, d)) ds \geq \int_c^d (F(c, s) - F(c, d)) ds \geq \frac{1}{2F_2} (f(c) - F(c, d))^2 \quad (4.3.32)$$

and

$$\int_a^b (f(s) - F(a, b)) ds \geq \int_c^d (f(s) - F(c, d)) ds \geq \int_c^d (F(s, d) - F(c, d)) ds \geq \frac{1}{2F_1} (f(d) - F(c, d))^2 \quad (4.3.33)$$

Multiplying (4.3.32) by $\frac{F_2}{F_1 + F_2}$ and (4.3.33) by $\frac{F_1}{F_1 + F_2}$ respectively, taking the maximum for $(c, d) \in \mathcal{C}(a, b)$ and adding the two inequalities yields:

$$\int_a^b (f(s) - F(a, b)) ds \geq \frac{1}{2(F_1 + F_2)} \left[\max_{(c,d) \in \mathcal{C}(a,b)} (F(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(a,b)} (F(c, d) - f(d))^2 \right].$$

Taking the sum over n , we deduce from substituting the inequality above into (4.3.31) that

$$\begin{aligned} B &\geq \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2(F_1 + F_2)} \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c, d) - f(d))^2 \right\} \right] \quad (4.3.34) \end{aligned}$$

Therefore, since $A + B = 0$,

$$\begin{aligned} &\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2 \\ &\geq \frac{\xi}{(F_1 + F_2)} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c, d) - f(d))^2 \right\} \right] \end{aligned}$$

which, in turn, implies the bound

$$\begin{aligned}
 & \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\
 & \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \right. \\
 & \quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\} \right] \\
 & \leq C_1,
 \end{aligned} \tag{4.3.35}$$

where the positive constant C_1 depends on $\Lambda_0, T, C_f, M_g, \xi, F_1, F_2$ and $\|u_0\|_{L^2(\mathbb{T}^d)}$. We then use a reordering of the summation to deduce that

$$\begin{aligned}
 & \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ (F(u_K^n, u_L^n) - f(u_K^n))^2 + (F(u_K^n, u_L^n) - f(u_L^n))^2 \right\} \right. \\
 & \quad \left. + v_{L,K}^n \left\{ (F(u_L^n, u_K^n) - f(u_K^n))^2 + (F(u_L^n, u_K^n) - f(u_L^n))^2 \right\} \right] \\
 & = \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F(u_K^n, u_L^n) - f(u_K^n)\}^2 + v_{L,K}^n \{F(u_L^n, u_K^n) - f(u_K^n)\}^2 \right]
 \end{aligned}$$

which together with (4.3.35) implies

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F(u_K^n, u_L^n) - f(u_K^n)\}^2 + v_{L,K}^n \{F(u_L^n, u_K^n) - f(u_K^n)\}^2 \right] \leq C_1.$$

This completes the proof of Lemma 4.9, 1.

Proof of 2: Now we estimate the term

$$\begin{aligned}
 \bar{T}^2 = & \left\{ \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d)) \right\} \right. \right. \\
 & \quad \left. \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(c) - F(c,d)) + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(d) - F(c,d)) \right\} \right] \right\}^2
 \end{aligned}$$

We denote by

$$T_1 = \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))$$

and

$$T_2 = \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(c) - F(c,d)) + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(d) - F(c,d))$$

We deduce from the Cauchy-Schwarz inequality that:

$$\begin{aligned}
\bar{T}^2 &= \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbf{E} \left[v_{K,L}^n T_1 + v_{L,K}^n T_2 \right] \right)^2 \\
&\leq \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbf{E} \left[\frac{(v_{K,L}^n T_1 + v_{L,K}^n T_2)^2}{v_{K,L}^n + v_{L,K}^n} \right] \right) \\
&\leq \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbf{E} \left[v_{K,L}^n T_1^2 + v_{L,K}^n T_2^2 \right] \right),
\end{aligned} \tag{4.3.36}$$

where we have applied the Jensen inequality

$$\left(\frac{v_{K,L}^n T_1 + v_{L,K}^n T_2}{v_{K,L}^n + v_{L,K}^n} \right)^2 \leq \frac{v_{K,L}^n}{v_{K,L}^n + v_{L,K}^n} T_1^2 + \frac{v_{L,K}^n}{v_{K,L}^n + v_{L,K}^n} T_2^2.$$

It follows from (4.2.2), namely $|\partial K| \leq \frac{|K|}{\bar{\alpha}^2 h}$, that

$$\begin{aligned}
&\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \\
&= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| v_{K,L}^n \\
&\leq T \sum_{K \in \mathcal{T}} |\partial K| V \\
&\leq \frac{TV|\mathbb{T}^d|}{\bar{\alpha}^2 h}.
\end{aligned} \tag{4.3.37}$$

Moreover, we remark that

$$\begin{aligned}
T_1^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \\
T_2^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\}
\end{aligned}$$

which we substitute into (4.3.36) to deduce that

$$\begin{aligned}
\bar{T}^2 &\leq \frac{2TV|\mathbb{T}^d|}{\bar{\alpha}^2 h} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\
&\times \mathbf{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d))^2 \right\} \right. \\
&\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (F(c,d) - f(d))^2 \right\} \right].
\end{aligned} \tag{4.3.38}$$

Substituting (4.3.35) into (4.3.38), we deduce that

$$\bar{T}^2 \leq \frac{2TV|\mathbb{T}^d|}{\bar{\alpha}^2 h} C_1$$

which combined with the definition of \bar{T}^2 implies

$$\begin{aligned} & \left\{ \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F(c,d) - f(d)) \right\} \right. \right. \\ & \quad \left. \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(c) - F(c,d)) + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} (f(d) - F(c,d)) \right\} \right] \right\}^2 \\ & \leq \frac{2TC_1|\mathbb{T}^d|V}{\bar{\alpha}^2 h}. \end{aligned}$$

This concludes the proof of Lemma 4.9, 2.

4.4 Convergence of the scheme

4.4.1 A time-continuous approximation

We define \bar{u}_K as the continuous in time stochastic process

$$\begin{aligned} \bar{u}_K(t) &= u_K^n - \frac{t - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left(v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \right) \\ & \quad + g(u_K^n)(W_K(t) - W_K^n) \\ &= u_K^n - \int_{nk}^t \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n)}{|K|} ds \\ & \quad + \int_{nk}^t g(u_K^n) dW_K(t). \end{aligned} \tag{4.4.1}$$

on the domain $\Omega \times (nk, (n+1)k)$. In this way, $\bar{u}_K(nk) = u_K^n$ and $\bar{u}_K((n+1)k) = u_K^{n+1}$. We define the time-continuous approximate solution $\bar{u}_{\mathcal{T},k}$ on $\Omega \times \mathbb{T}^d \times (0, T)$ that:

$$\bar{u}_{\mathcal{T},k}(x, t) = \bar{u}_K(t^n) \quad \text{for } x \in K \quad \text{and } t \in (nk, (n+1)k). \tag{4.4.2}$$

Next we estimate the difference between the time-continuous approximation $\bar{u}_{\mathcal{T},k}$ and the finite volume solution $u_{\mathcal{T},k}$ which is defined in (4.2.16).

Lemma 4.10 *Let $u_0 \in L^2(\mathbb{T}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 4.2. Let $k = \frac{T}{N}$ satisfy the CFL condition (4.3.24). Then there exists $C \in \mathbb{R}^+$ depending on $T, C_g, F_1, F_2, \bar{\alpha}, V$ and u_0 such that*

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq C(h+k). \tag{4.4.3}$$

Proof: There holds

$$\begin{aligned}
& \|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \\
&= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\
&\quad \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} + g(u_K^n)(W_K(s) - W_K^n) \right)^2 \right] dx ds \\
&= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n)(W_K(s) - W_K^n)^2 \right] dx ds \\
&\quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\
&\quad \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dx ds \\
&\quad + 2 \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\frac{s-nk}{|K|} g(u_K^n)(W_K(s) - W_K^n) \right. \\
&\quad \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right],
\end{aligned}$$

so that

$$\begin{aligned}
\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n)(W_K(s) - W_K^n)^2 \right] dx ds \\
&\quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\
&\quad \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dx ds
\end{aligned} \tag{4.4.4}$$

where we used the fact that

$$\mathbb{E} [W_K(s) - W_K^n] = 0. \tag{4.4.5}$$

We remark that (4.4.5) can be proved in the same way as (4.2.11). Applying the counter part of (4.2.13)

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[(W_K(s) - W_K^n)^2 \right] \leq (s - t^n) \Lambda_0,$$

for all $s > nk$, we deduce that

$$\begin{aligned}
 & \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g^2(u_K^n) (W_K(s) - W_K^n)^2 \right] dx ds \\
 & \leq \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} (s - t^n) \Lambda_0 M_g^2 ds \\
 & \leq T \Lambda_0 M_g^2 k.
 \end{aligned} \tag{4.4.6}$$

Moreover

$$\begin{aligned}
 & \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\
 & \quad \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n)) \} \right)^2 \right] dx ds \\
 & \leq \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \frac{k^2 |K|}{|K|^2} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\
 & \quad \left. \left. - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n)) \} \right)^2 \right].
 \end{aligned}$$

As it has been done in (4.3.19) and (4.3.20), also using the inequality (4.3.22), we deduce that

$$\begin{aligned}
 & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \frac{k^2 |K|}{|K|^2} \times \\
 & \quad \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \\
 & \leq \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \\
 & \quad \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
 & \leq \frac{k}{F_1 + F_2} \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \times \\
 & \quad \mathbb{E} \left[\{v_{K,L}^n (F(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
 & \leq C_1 \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)^2},
 \end{aligned}$$

where we have also used (4.3.24) and (4.3.25). Therefore

$$\|u_{\mathcal{T},k} - v_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \Lambda_0 M_g^2 k + C_1 \frac{\bar{\alpha}^2 h}{V(F_1 + F_2)^2}. \tag{4.4.7}$$

Finally we set $C = \max(T \Lambda_0 M_g^2, \frac{C_1 \bar{\alpha}^2}{V(F_1 + F_2)^2})$ to deduce the result of Lemma 4.10.

4.4.2 Entropy inequalities for the approximate solution

In this section, we show entropy inequalities satisfied by the approximate solution and use them in the proof of convergence of the numerical scheme. In order to obtain these entropy inequalities, we first derive some discrete entropy inequalities satisfied by the approximate solution. We first assume that F is a Godunov flux, namely:

$$F^G(a, b) = \begin{cases} \min_{s \in [a, b]} f(s) & \text{if } a \leq b \\ \max_{s \in [b, a]} f(s) & \text{if } a \geq b. \end{cases} \quad (4.4.8)$$

For all $(a, b) \in \mathbb{R}^2$ we denote by $s(a, b) \in [\min(a, b), \max(a, b)]$ a real number such that $F^G(a, b) = f(s(a, b))$. We recall that $F^\eta(a) = \int_0^a \eta'(s) f'(s) ds$ and define the associated entropy numerical flux G by $G(a, b) = F^\eta(s(a, b))$ for all $a, b \in \mathbb{R}$. Note that for all $a \in \mathbb{R}$, $G(a, a) = F^\eta(a)$.

Lemma 4.11 (*Discrete entropy inequality*) *Assume that hypotheses (\overline{H}) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 4.2, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (4.4.9)$$

Then P -a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$:

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) \left(\frac{1}{|K|} \int_K e_m^2(x) dx \right)^2 dx dt \\ & \geq R^{k, h} \end{aligned} \quad (4.4.10)$$

where for all P -measurable set A , $E[\mathbf{1}_A R^{k, h}] \rightarrow 0$ as $h \rightarrow 0$.

As it has been done in Chapter 3, one can prove a relation based upon the Itô formula, which is applied to the function $\bar{u}_K(t)$.

Lemma 4.12 *For all $\eta \in \mathcal{A}$, there holds:*

$$\begin{aligned}
 & \eta(\bar{u}_K((n+1)k)) - \eta(\bar{u}_K(nk)) \\
 = & - \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)}{|K|} dt \\
 & + \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt
 \end{aligned} \tag{4.4.11}$$

P -a.s in Ω .

Remembering that $\bar{u}_K(nk) = u_K^n$, $\bar{u}_K((n+1)k) = u_K^{n+1}$, and using (4.2.8), after multiplying (4.4.11) by $|K|\varphi_K^n$, one deduces that P -a.s. in Ω ,

$$\begin{aligned}
 & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
 = & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \\
 & \times \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} dt \\
 & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
 & + \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left(\int_K e_m(x) dx \right)^2 \varphi_K^n \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt
 \end{aligned} \tag{4.4.12}$$

Next we recall the formula

$$\varphi_K^n = \frac{1}{|K|} \int_K \varphi(x, nk) dx.$$

The equation (4.4.12) can be written in the form $A^{k,h} = -B^{k,h} + C^{k,h} + D^{k,h}$, with

$$\begin{aligned}
 A^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
 B^{k,h} &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \varphi_K^n \eta'(\bar{u}_{\mathcal{T},k}(t)) \\
 & \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} dt \\
 C^{k,h} &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
 D^{k,h} &= \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(x) dx \right)^2 |K| \varphi_K^n \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt
 \end{aligned} \tag{4.4.13}$$

Since the terms $A^{k,h}$, $C^{k,h}$ and $D^{k,h}$ are identical to those in Chapter 3, we only study the term $B^{k,h}$

Study of $B^{k,h}$: we decompose $B^{k,h}$ as

$$B^{k,h} = B^{k,h} - \tilde{B}^{k,h} + (\tilde{B}^{k,h} - \bar{B}^{k,h}) + (\bar{B}^{k,h} - B_1^{k,h}) + B_1^{k,h} \quad (4.4.14)$$

where

$$\begin{aligned} \tilde{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta'(u_K^n) \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ \bar{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n G(u_K^n, u_L^n) - v_{L,K}^n G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ B_1^{k,h} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \end{aligned}$$

We first prove that for any measurable set A , $\mathbb{E} [\mathbf{1}_A(B^{k,h} - \tilde{B}^{k,h})] \rightarrow 0$ as $h \rightarrow 0$. For almost all $\omega \in \Omega$, all $K \in \mathcal{T}$ and all $n \in \{0, 1, \dots, N-1\}$, there exists $\zeta_K^n(\omega) \in \mathbb{R}$ such that

$$\begin{aligned} B^{k,h} - \tilde{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta'(\bar{u}_{\mathcal{T},k}(t)) - \eta'(\bar{u}_K^n)] \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n)(\bar{u}_{\mathcal{T},k}(t) - \bar{u}_K^n)] \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ &= T_1^{h,k} + T_2^{h,k} \end{aligned}$$

where we set

$$\begin{aligned} T_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n) \frac{t - nk}{|K|} \varphi(x, nk)] dx dt \\ &\quad \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \end{aligned} \quad (4.4.15)$$

$$\begin{aligned} T_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n) g(u_K^n) (W_K(t) - W_K(nk))] \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \end{aligned} \quad (4.4.16)$$

Let A be a measurable set, we first study $\mathbb{E} \left[\mathbf{1}_A T_1^{h,k} \right]$. Note that here the assumption $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$ is crucial. Using Cauchy-Schwarz inequality, the assumption (4.2.2) on the mesh and the estimate (4.3.25), we deduce that

$$\begin{aligned}
 & \left| \mathbb{E} \left[\mathbf{1}_A T_1^{h,k} \right] \right| \\
 &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(\zeta_K^n) \frac{t-nk}{|K|} \varphi(x, nk) dx dt \right. \right. \\
 & \quad \left. \left. \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \right| \\
 &\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\
 & \quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
 &\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} V |\partial K| \\
 & \quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
 &\leq C_1 \|\eta''\|_\infty \|\varphi\|_\infty \frac{k}{\bar{\alpha}^2 h} V,
 \end{aligned}$$

which tends to 0 as h tends to 0; the constant C_1 is the same as that in (4.3.25).

Next we estimate the term $\mathbb{E} \left[\mathbf{1}_A T_2^{h,k} \right]$. Using Cauchy-Schwarz inequality, the assumption that $\operatorname{div} \mathbf{v} = 0$ and the weak BV estimate (4.3.25),

$$\begin{aligned}
 & \left(\mathbb{E} \left[\mathbf{1}_A T_2^{h,k} \right] \right)^2 \\
 &= \left(\mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \{\eta''(\zeta_K^n) g(u_K^n) (W_K(t) - W_K(nk))\} \right. \right. \\
 & \quad \left. \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \right] \right)^2 \\
 &\leq \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K |\mathbf{1}_A \eta''(\zeta_K^n) g(u_K^n) \varphi(x, nk)|^2 dx dt \right] \\
 & \quad \times \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \right. \\
 & \quad \left. \times \left((W_K(t) - W_K(nk)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right)^2 dt \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \|\varphi\|_\infty^2 \|\eta''\|_\infty^2 C_g^2 \|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q)}^2 \times \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \right. \\
&\quad \left. \times \left((W_K(t) - W_K(nk)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right)^2 dt \right] \\
&\leq C \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{k}{|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right)^2 \right] dt \\
&\leq C \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dt \\
&\leq C \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\
&\quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
&\leq CV \frac{k}{\bar{\alpha}^2 h}
\end{aligned}$$

which tends to 0 as h tends to 0; C is a constant which depends on φ , η , C_g , T , u_0 , ξ , F_1 , F_2 and Λ_0 .

We show below that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$ almost surely. First we remark that

$$\sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) = \sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) f(u_K^n) = 0.$$

we recall that for all $K \in \mathcal{T}$, $F^G(u_K^n, u_K^n) = f(u_K^n)$ and $G(u_K^n, u_K^n) = F^\eta(u_K^n)$. This permits us to rewrite $\tilde{B}^{h,k} - \bar{B}^{h,k}$ as:

$$\begin{aligned}
\tilde{B}^{h,k} - \bar{B}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} \left\{ v_{K,L}^n \left[\eta'(u_K^n) (F^G(u_K^n, u_L^n) - f(u_K^n)) - (G(u_K^n, u_L^n) - F^\eta(u_K^n)) \right] \right. \\
&\quad \left. - v_{L,K}^n \left[\eta'(u_K^n) (F^G(u_L^n, u_K^n) - f(u_K^n)) - (G(u_L^n, u_K^n) - F^\eta(u_K^n)) \right] \right\} \int_K \varphi(x, nk) dx.
\end{aligned} \tag{4.4.17}$$

Let $K, L \in \mathcal{T}$ and suppose that $u_K^n < u_L^n$. We first study the sign of $\eta'(u_K^n) (F^G(u_K^n, u_L^n) - f(u_K^n)) - (G(u_K^n, u_L^n) - F^\eta(u_K^n))$. Using the fact that F^G is a Godunov numerical flux, we know that there exists $s(u_K^n, u_L^n) \in [u_K^n, u_L^n]$ such that $F^G(u_K^n, u_L^n) = f(s(u_K^n, u_L^n)) = \min_{t \in [u_K^n, u_L^n]} f(t)$. Thus

$$\begin{aligned}
&\eta'(u_K^n) (F^G(u_K^n, u_L^n) - f(u_K^n)) - (G(u_K^n, u_L^n) - F^\eta(u_K^n)) \\
&= \eta'(u_K^n) (f(s(u_K^n, u_L^n)) - f(u_K^n)) - (F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n)) \\
&= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(t) \eta'(u_K^n) dt - \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(t) \eta'(t) dt
\end{aligned}$$

$$\begin{aligned}
&= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(t)(\eta'(u_K^n) - \eta'(t))dt \\
&= \int_{u_K^n}^{s(u_K^n, u_L^n)} f(t)\eta''(t)dt + f(s(u_K^n, u_L^n))\{\eta'(u_K^n) - \eta'(s(u_K^n, u_L^n))\} \\
&\geq \int_{u_K^n}^{s(u_K^n, u_L^n)} f(s(u_K^n, u_L^n))\eta''(t)dt + f(s(u_K^n, u_L^n))\{\eta'(u_K^n) - \eta'(s(u_K^n, u_L^n))\} \\
&\geq 0
\end{aligned}$$

Similarly one can show that $\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G(u_L^n, u_K^n) - F^\eta(u_K^n)) \leq 0$. Indeed, since $u_L^n > u_K^n$, there exists $s(u_L^n, u_K^n) \in [u_K^n, u_L^n]$ such that $F^G(u_L^n, u_K^n) = f(s(u_L^n, u_K^n)) = \max_{t \in [u_K^n, u_L^n]} f(t)$ and we have

$$\begin{aligned}
&\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G(u_L^n, u_K^n) - F^\eta(u_K^n)) \\
&= \eta'(u_K^n)(f(s(u_L^n, u_K^n)) - f(u_K^n)) - (F^\eta(s(u_L^n, u_K^n)) - F^\eta(u_K^n)) \\
&= \int_{u_K^n}^{s(u_L^n, u_K^n)} f'(t)(\eta'(u_K^n) - \eta'(t))dt \\
&= \int_{u_K^n}^{s(u_L^n, u_K^n)} f(t)\eta''(t)dt + f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} \\
&\leq \int_{u_K^n}^{s(u_L^n, u_K^n)} f(s(u_L^n, u_K^n))\eta''(t)dt + f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} \\
&\leq 0
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&v_{K,L}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
&- v_{L,K}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
&\geq 0,
\end{aligned} \tag{4.4.18}$$

from which we deduce that for almost all $\omega \in \Omega$, $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$.

Next we prove that for all measurable sets A , we have that $\mathbb{E} \left[\mathbf{1}_A(\bar{B}^{h,k} - B_1^{h,k}) \right] \rightarrow 0$ as $h \rightarrow 0$.

Using the fact that

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) = 0,$$

we deduce that

$$\begin{aligned}
\bar{B}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_K^n)) \\
&\quad - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \int_K \varphi(x, nk) dx
\end{aligned} \tag{4.4.19}$$

so that, $\bar{B}^{h,k} = \bar{T}_1^{h,k} - \bar{T}_2^{h,k}$, where

$$\begin{aligned} \bar{T}_1^{h,k} &= \sum_{n=0}^N \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|K|} |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_K^n)) \\ &\quad - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \int_K \varphi(x, nk) dx \end{aligned} \quad (4.4.20)$$

and

$$\begin{aligned} \bar{T}_2^{h,k} &= \sum_{n=0}^N \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|L|} |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_L^n)) \\ &\quad - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_L^n))\} \int_L \varphi(x, nk) dx. \end{aligned} \quad (4.4.21)$$

Similarly,

$$\begin{aligned} B_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\ &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \operatorname{div}[\mathbf{v}(x, t) \varphi(x, nk)] dx dt \\ &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} \varphi(x, nk) d\gamma(x) dt \\ &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(x, nk) F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &\quad - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(x, nk) F^\eta(u_L^n) \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &= T_1^{h,k} - T_2^{h,k} \end{aligned}$$

where

$$\begin{aligned} T_1^{h,k} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{v_{K,L}^n G(u_K^n, u_L^n) - v_{L,K}^n G(u_L^n, u_K^n) \\ &\quad - F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L}\} \varphi(x, nk) d\gamma(x) dt \end{aligned} \quad (4.4.22)$$

$$\begin{aligned} T_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{v_{K,L}^n G(u_K^n, u_L^n) - v_{L,K}^n G(u_L^n, u_K^n) \\ &\quad - F^\eta(u_L^n) \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L}\} \varphi(x, nk) d\gamma(x) dt \end{aligned} \quad (4.4.23)$$

We need to estimate the terms $|\bar{T}_1^{h,k} - T_1^{h,k}|$ and $|\bar{T}_2^{h,k} - T_2^{h,k}|$. To that purpose, we rewrite them as follows:

$$\bar{T}_1^{h,k} - T_1^{h,k}$$

$$\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \\
&\quad \times \left\{ \frac{1}{|K|} \int_K \varphi(y, nk) dy - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) \right\} \\
&+ \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k \int_{\sigma_{K,L}} \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \varphi(x, nk) d\gamma(x) \\
&- \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{v_{K,L}^n G(u_K^n, u_L^n) - v_{L,K}^n G(u_L^n, u_K^n) \\
&\quad - F^\eta(u_K^n) \mathbf{v}(x, s) \cdot \mathbf{n}_{K,L}\} \varphi(x, nk) d\gamma(x) ds \\
&= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \\
&\quad \times \left\{ \frac{1}{|K|} \int_K \varphi(y, nk) dy - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) \right\} \\
&+ \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left[\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n) \right] \varphi(x, nk) d\gamma(x) ds.
\end{aligned}$$

We also have that

$$\begin{aligned}
&\bar{T}_2^{h,k} - T_2^{h,k} \\
&= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} k |\sigma_{K,L}| \{v_{K,L}^n (G(u_K^n, u_L^n) - F^\eta(u_L^n)) - v_{L,K}^n (G(u_L^n, u_K^n) - F^\eta(u_L^n))\} \\
&\quad \times \left\{ \frac{1}{|L|} \int_K \varphi(y, nk) dy - \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) \right\} \\
&+ \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_L^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left[\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n) \right] \varphi(x, nk) d\gamma(x) ds.
\end{aligned}$$

In order to estimate $|\bar{T}_1^{h,k} - T_1^{h,k}|$ and $|\bar{T}_2^{h,k} - T_2^{h,k}|$, we need to bound the following quantities:

$$\begin{aligned}
&G(u_K^n, u_L^n) - F^\eta(u_K^n), G(u_K^n, u_L^n) - F^\eta(u_L^n) \\
&G(u_L^n, u_K^n) - F^\eta(u_K^n), G(u_L^n, u_K^n) - F^\eta(u_L^n) \\
&\frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) - \frac{1}{|K|} \int_K \varphi(y, nk) dy,
\end{aligned}$$

and

$$\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left[\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n) \right] \varphi(x, nk) d\gamma(x) ds$$

Let us start with $G(u_K^n, u_L^n) - F^\eta(u_K^n)$. Set $(K, L) \in \mathcal{I}_n$; then $u_K^n > u_L^n$ which implies that $f(s(u_K^n, u_L^n)) = \max_{t \in [u_L^n, u_K^n]} f(t)$ and hence

$$\begin{aligned} G(u_K^n, u_L^n) - F^\eta(u_K^n) &= F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n) \\ &= \int_{u_K^n}^{s(u_K^n, u_L^n)} \eta'(s) f'(s) ds. \end{aligned} \quad (4.4.24)$$

First case: $\int_{u_K^n}^{s(u_K^n, u_L^n)} \eta'(t) f'(t) dt > 0$. Integrating by parts, we obtain with d such that $f(d) = \min_{t \in [s(u_K^n, u_L^n), u_K^n]} f(t)$:

$$\begin{aligned} &\int_{u_K^n}^{s(u_K^n, u_L^n)} \eta'(t) f'(t) dt \\ &= - \int_{s(u_K^n, u_L^n)}^{u_K^n} \eta''(t) [f(t) - f(s(u_K^n, u_L^n))] dt + [\eta'(t) (f(t) - f(s(u_K^n, u_L^n)))]_{s(u_K^n, u_L^n)}^{u_K^n} \\ &\leq \eta'(u_K^n) [f(u_K^n) - f(d)] - \eta'(s(u_K^n, u_L^n)) [f(s(u_K^n, u_L^n)) - f(d)] \\ &\leq \|\eta'\|_\infty |f(u_K^n) - f(d)| + \|\eta'\|_\infty |f(s(u_K^n, u_L^n)) - f(d)| \\ &\leq 2\|\eta'\|_\infty |f(s(u_K^n, u_L^n)) - f(d)| \\ &\leq 2\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - F^G(d, d)|, \end{aligned}$$

since $f(s(u_K^n, u_L^n)) = \max_{t \in [s(u_K^n, u_L^n), d]} f(t) = F^G(d, s(u_K^n, u_L^n))$.

Second case: $\int_{u_K^n}^{s(u_K^n, u_L^n)} \eta'(t) f'(t) dt < 0$. Similarly we have:

$$\begin{aligned} &\int_{u_K^n}^{s(u_K^n, u_L^n)} \eta'(t) f'(t) dt \\ &= - \int_{s(u_K^n, u_L^n)}^{u_K^n} \eta''(t) [f(t) - f(s(u_K^n, u_L^n))] dt + [\eta'(t) (f(t) - f(s(u_K^n, u_L^n)))]_{s(u_K^n, u_L^n)}^{u_K^n} \\ &\geq \eta'(u_K^n) [f(u_K^n) - f(s(u_K^n, u_L^n))] \\ &\geq -\|\eta'\|_\infty |F^G(u_K^n, u_K^n) - F^G(u_K^n, s(u_K^n, u_L^n))| \\ &\geq -\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(d)|, \end{aligned}$$

since $f(s(u_K^n, u_L^n)) = \max_{t \in [s(u_K^n, u_L^n), u_K^n]} f(t) = F^G(u_K^n, s(u_K^n, u_L^n))$.

We deduce in both cases that we have:

$$|G(u_K^n, u_L^n) - F^\eta(u_K^n)| \leq 2\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(d)|. \quad (4.4.25)$$

Similarly, one can show that for all $(K, L) \in \mathcal{I}_n$

$$|G(u_L^n, u_K^n) - F^\eta(u_K^n)| \leq 2\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(d)| \quad (4.4.26)$$

$$|G(u_K^n, u_L^n) - F^\eta(u_L^n)| \leq 2\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(c)| \quad (4.4.27)$$

$$|G(u_L^n, u_K^n) - F^\eta(u_L^n)| \leq 2\|\eta'\|_\infty \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(c)| \quad (4.4.28)$$

We now estimate the term $\frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) - \frac{1}{|K|} \int_K \varphi(y, nk) dy$. Using the regularity of φ , we deduce the bound

$$\left| \frac{1}{|\sigma_{K,L}|} \int_{\sigma_{K,L}} \varphi(x, nk) d\gamma(x) - \frac{1}{|K|} \int_K \varphi(y, nk) dy \right| \leq 2h \|\nabla_x \varphi\|_\infty \quad (4.4.29)$$

Next we estimate the term

$$\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} [\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n)] \varphi(x, nk) d\gamma(x) ds$$

which is equal to

$$\sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left[\frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{\mathbf{v}(\bar{x}, s) - \mathbf{v}(\bar{x}, t)\} \cdot \mathbf{n}_{K,L} d\gamma(\bar{x}) dt \right] \varphi(x, nk) d\gamma(x) ds.$$

Using the regularity of \mathbf{v} , we deduce that there exists a constant $c(\mathbf{v})$ only depending on \mathbf{v} , such that

$$\frac{1}{k|\sigma_{K,L}|} \left| \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{\mathbf{v}(\bar{x}, s) - \mathbf{v}(\bar{x}, t)\} \cdot \mathbf{n}_{K,L} d\gamma(\bar{x}) dt \right| \leq c(\mathbf{v})(k+h). \quad (4.4.30)$$

Using the fact that $\operatorname{div} \mathbf{v} = 0$, we deduce that

$$\begin{aligned} & \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n)\} \varphi(x, nk) d\gamma(x) ds \\ &= \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \left(\frac{1}{k|\sigma_{K,L}|} \int_{\sigma_{K,L}} \int_{nk}^{(n+1)k} \{\mathbf{v}(x, s) - \mathbf{v}(x, t)\} \cdot \mathbf{n}_{K,L} d\gamma(x) \right) \\ & \quad \times [\varphi(x, nk) - \varphi(x_\sigma, nk)] d\gamma(x) ds, \end{aligned}$$

from which we deduce

$$\begin{aligned} & \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} \{\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n)\} \varphi(x, nk) d\gamma(x) ds \\ & \leq c(\mathbf{v}) \|\varphi\|_\infty h^2 k |\sigma_{K,L}|. \end{aligned}$$

Therefore, there exists a constant C which only depends on \mathbf{v} , φ , u_0 , T , C_g , α , η and f , such that

$$\begin{aligned}
 & \left| \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_K^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} [\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n)] \varphi(x, nk) d\gamma(x) ds \right| \\
 & \leq Ch^2 \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| |F^\eta(u_K^n)| \\
 & \leq Ch^2 \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} k |\partial K| |u_K^n| \\
 & \leq \frac{C}{\bar{\alpha}^2} h \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} k |K| |u_K^n| \\
 & \leq Ch \|u_{\mathcal{T},k}\|_{\mathbb{T}^d \times (0,T)} \\
 & \leq Ch.
 \end{aligned} \tag{4.4.31}$$

Similarly, we deduce that for some constant C which only depends on \mathbf{v} , φ , u_0 , T , C_g , α , η and f , there holds

$$\begin{aligned}
 & \left| \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} F^\eta(u_L^n) \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} [\mathbf{v}(x, s) \cdot \mathbf{n}_{K,L} - (v_{K,L}^n - v_{L,K}^n)] \varphi(x, nk) d\gamma(x) ds \right| \\
 & \leq Ch
 \end{aligned} \tag{4.4.32}$$

Next we compare $\bar{B}^{h,k}$ to $B_1^{h,k}$. We recall that $\bar{B}^{h,k} - B_1^{h,k} = (\bar{T}_1^{h,k} - T_1^{h,k}) - (\bar{T}_2^{h,k} - T_2^{h,k})$. Using (4.4.25), (4.4.26), (4.4.29) and (4.4.31), we deduce that for some constant C only depending on \mathbf{v} , φ , u_0 , T , C_g , $\bar{\alpha}$, η , f that

$$\begin{aligned}
 |\bar{T}_1^{h,k} - T_1^{h,k}| & \leq C \sum_{n=0}^{N-1} \sum_{\mathcal{I}_n} k |\sigma_{K,L}| h \left(v_{K,L}^n \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(d)| \right. \\
 & \quad \left. + v_{K,L}^n \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(d)| \right) + Ch.
 \end{aligned} \tag{4.4.33}$$

It follows from (4.4.27), (4.4.28), (4.4.29) and (4.4.32), that there exists a positive constant C only depending on \mathbf{v} , φ , u_0 , T , C_g , $\bar{\alpha}$, η , f such that

$$\begin{aligned}
 |\bar{T}_2^{h,k} - T_2^{h,k}| & \leq C \sum_{n=0}^{N-1} \sum_{\mathcal{I}_n^R} k |\sigma_{K,L}| h \left(v_{K,L}^n \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(c)| \right. \\
 & \quad \left. + v_{K,L}^n \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(c)| \right) + Ch.
 \end{aligned} \tag{4.4.34}$$

Combining these two inequalities and using the weak BV estimates (4.3.25) and (4.3.26), we deduce that for some positive constant C only depending on \mathbf{v} , φ , u_0 , T , C_g , $\bar{\alpha}$, η , f for almost

all ω

$$\begin{aligned}
 & |\bar{B}^{h,k} - B_1^{h,k}| \\
 & \leq C \sum_{n=0}^{N-1} \sum_{\mathcal{I}_n^R} k |\sigma_{K,L}| h \\
 & \quad \times \left(v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(c)| + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(d, c) - f(d)| \right\} \right. \\
 & \quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(d)| + \max_{(c,d) \in \mathcal{C}(u_L^n, u_K^n)} |F^G(c, d) - f(c)| \right\} \right) + Ch \\
 & \leq Ch^{\frac{1}{2}}.
 \end{aligned} \tag{4.4.35}$$

We conclude that for all measurable sets A , $E \left[\mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k}) \right] \rightarrow 0$ as $h \rightarrow 0$.

In order to show the convergence of the finite volume scheme, we apply the following result, which states that any general monotone numerical flux can be split into the sum of a Godunov flux and a modified Lax-Friedrichs flux (also known as Rusanov flux).

Lemma 4.13 [9] *Any numerical monotone flux F (i.e. any numerical flux satisfying Definition 4.3) can be written as a convex combination of a Godunov flux and a modified Lax-Friedrichs flux as follows: For any $a, b \in \mathbb{R}$, there exists $\theta(a, b) \in [0, 1]$ such that*

$$F(a, b) = \theta(a, b) F^G(a, b) + (1 - \theta(a, b)) F_D^{LF}(a, b), \tag{4.4.36}$$

where F^G is a Godunov flux (cf. (4.4.8)) and F_D^{LF} is a modified Lax-Friedrichs flux with parameter $D = \max(F_1, F_2)$ satisfying:

$$F_D^{LF}(a, b) = \frac{f(a) + f(b)}{2} - D(b - a). \tag{4.4.37}$$

Note that since D is fixed independently from a and b , F_D^{LF} is a flux-splitting type scheme.

Proof. We give the proof in the case that $a < b$, the case that $b < a$ can be proved similarly. If $a < b$, one has $F^G(a, b) = f(c)$ where $c \in [a, b]$ is such that $f(c) = \min\{f(d), d \in [a, b]\}$. Then using the fact that $F(c, c) = f(c)$ and the monotonicity properties of F , one deduces that

$$F(a, b) \leq F(c, b) \leq F(c, c) = f(c) = F^G(a, b).$$

On the other hand, the property 2 of Definition 4.3 implies

$$F(a, b) \geq F(a, a) - F_2 \cdot (b - a) \geq f(a) - D \cdot (b - a)$$

and

$$F(a, b) \geq F(b, b) - F_1 \cdot (b - a) \geq f(b) - D \cdot (b - a).$$

Adding this two inequalities leads to

$$F(a, b) \geq \frac{1}{2}(f(a) + f(b)) - D \cdot (b - a) = F_D^{LF}(a, b).$$

Therefore $F(a, b)$ is a convex combination of $F^G(a, b)$ and $F_D^{LF}(a, b)$, which concludes the proof of Lemma 4.13.

Lemma 4.14 *Lemma 4.11 holds for a general monotone flux F , with the same assumptions.*

Proof. In order to adapt the proof of Lemma 4.11 where we have used the fact that F was Godunov numerical flux, we use the decomposition

$$F(a, b) = \theta(a, b)F^G(a, b) + (1 - \theta(a, b))F_D^{LF}(a, b).$$

First we give a definition of the entropy numerical flux G , which uses the above decomposition: for all $a, b \in \mathbb{R}$

$$G(a, b) = \theta(a, b)G^G(a, b) + (1 - \theta(a, b))G_D^{LF}(a, b),$$

where $G^G(a, b) = F^\eta(s(a, b))$ and $G_D^{LF}(a, b) = \frac{F^\eta(a) + F^\eta(b)}{2} - D(\eta(b) - \eta(a))$. In order to prove that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$ almost surely, we split the sum into two terms:

$$\begin{aligned} & \tilde{B}^{h,k} - \bar{B}^{h,k} \\ = & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} \theta(u_K^n, u_L^n) \left\{ v_{K,L}^n [\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \right. \\ & \left. - v_{L,K}^n [\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G^G(u_L^n, u_K^n) - F^\eta(u_K^n))] \right\} \int_K \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} (1 - \theta(u_K^n, u_L^n)) \\ & \times \left\{ v_{K,L}^n [\eta'(u_K^n)(F_D^{LF}(u_K^n, u_L^n) - f(u_K^n)) - (G_D^{LF}(u_K^n, u_L^n) - F^\eta(u_K^n))] \right. \\ & \left. - v_{L,K}^n [\eta'(u_K^n)(F_D^{LF}(u_L^n, u_K^n) - f(u_K^n)) - (G_D^{LF}(u_L^n, u_K^n) - F^\eta(u_K^n))] \right\} \int_K \varphi(x, nk) dx. \end{aligned}$$

In order to treat the first sum, we apply the same argument as in the proof that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$, whereas the second term can be dealt by using similar argument to the one used in the proof of Proposition 4 of [8], since we recall that the modified Lax-Friedrichs scheme belongs to the family of flux-splitting schemes. For example, considering $\eta'(u_K^n)(F_D^{LF}(u_K^n, u_L^n) - f(u_K^n)) - (G_D^{LF}(u_K^n, u_L^n) - F^\eta(u_K^n))$,

$$\begin{aligned} & \eta'(u_K^n)(F_D^{LF}(u_K^n, u_L^n) - f(u_K^n)) - (G_D^{LF}(u_K^n, u_L^n) - F^\eta(u_K^n)) \\ = & \eta'(u_K^n) \left(\frac{1}{2}(f(u_K^n) + f(u_L^n)) - D(u_L^n - u_K^n) - f(u_K^n) \right) \\ & - \left(\frac{1}{2}(F^\eta(u_K^n) + F^\eta(u_L^n)) - D(\eta(u_L^n) - \eta(u_K^n)) - F^\eta(u_K^n) \right) \\ = & \eta'(u_K^n) \left(\frac{1}{2}(f(u_L^n) - f(u_K^n)) - D(u_L^n - u_K^n) \right) - \left(\frac{1}{2}(F^\eta(u_L^n) - F^\eta(u_K^n)) - D(\eta(u_L^n) - \eta(u_K^n)) \right) \\ = & \int_{u_K^n}^{u_L^n} \left(\eta'(u_K^n - \eta(s)) \left(\frac{1}{2}f'(s) - D \right) \right) ds \\ \geq & 0 \end{aligned}$$

In order to prove that for all measurable sets A , $\mathbb{E} \left[\mathbf{1}_A(\bar{B}^{h,k} - B_1^{h,k}) \right]$ tends to 0 as h tends to 0, we split again the sum into two parts:

$$\bar{B}^{h,k} - B_1^{h,k}$$

$$\begin{aligned}
&= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} \theta(u_K^n, u_L^n) |\sigma_{K,L}| \\
&\quad \times \left\{ v_{K,L}^n G^G(u_K^n, u_L^n) - v_{L,K}^n G^G(u_L^n, u_K^n) \right\} \varphi(x, nk) dx dt \\
&+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{\text{Card}(\mathcal{N}(K))} \int_K \sum_{L \in \mathcal{N}(K)} \theta(u_K^n, u_L^n) F^\eta(u_k^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\
&+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} (1 - \theta(u_K^n, u_L^n)) |\sigma_{K,L}| \\
&\quad \times \left\{ v_{K,L}^n G_D^{LF}(u_K^n, u_L^n) - v_{L,K}^n G_D^{LF}(u_L^n, u_K^n) \right\} \varphi(x, nk) dx dt \\
&+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{\text{Card}(\mathcal{N}(K))} \int_K \sum_{L \in \mathcal{N}(K)} (1 - \theta(u_K^n, u_L^n)) F^\eta(u_k^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt
\end{aligned}$$

To conclude, in order to treat the sum of the first two terms, we apply a similar argument as in the proof that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$, whereas the sum of the third and the fourth term can be dealt by using similar arguments as the one used in the proof of Proposition 4 of [8], since the modified Lax-Friedrichs scheme belongs to the family of flux-splitting schemes.

We can then deduce from the discrete entropy inequality 4.11 the following Continuous entropy inequality

Lemma 4.15 (*Continuous entropy inequality on the discrete solution*) Assume that hypotheses (\bar{H}) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 4.2, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (4.4.38)$$

Then P -a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$:

$$\begin{aligned}
&\int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
&+ \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \\
&+ \frac{1}{2} \sum_{m=1}^{\infty} \lambda_m \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) e_m^2(x) dx dt \\
&\geq \tilde{R}^{k,h},
\end{aligned} \quad (4.4.39)$$

where for all P -measurable set A , $\mathbb{E} \left[\mathbf{1}_A \tilde{R}^{k,h} \right] \rightarrow 0$ as $h \rightarrow 0$.

Proof. The main difference with the proof of Lemma 3.14 is the definition of the term F^η in the definition of the term I_3 ,

$$I_3 := \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right. \right. \\ \left. \left. - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \right) \right],$$

The main idea is to use the fact that f is Lipschitz continuous with Lipschitz constant C_f and that $f(0) = 0$. Indeed

$$\begin{aligned} I_3 &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right. \right. \right. \\ &\quad \left. \left. - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \right) \right] \Big| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \left(\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot (\nabla_x \varphi(x, t) - \nabla_x \varphi(x, nk)) dx dt \right) \right] \right| \\ &\leq V \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[|F^\eta(u_K^n)|]| \|\nabla_x \varphi_t\|_\infty k dt \\ &\leq V \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[|f(u_K^n)|]| \\ &\leq V C_f \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} |K| |\mathbb{E}[|u_K^n|]| \\ &\leq V C_f \|\eta'\|_\infty \|\nabla_x \varphi_t\|_\infty k \cdot \sqrt{T |\mathbb{T}^d|} \|u_{\mathcal{T},k}\|_{L^2(\Omega \times \mathbb{T}^d \times (0, T))} \\ &\rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

As a conclusion, we have proved the following result

Theorem 4.16 (Convergence to a measure-valued entropy solution of Problem (4.1.1)) *Assume that hypotheses (\bar{H}) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 4.2, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (4.4.40)$$

There exist a function $\mathbf{u} \in \mathcal{N}_\omega^2(0, T, L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $\{u_{\mathcal{T},k}\}$ converges to \mathbf{u} in the sense of Young measures. Moreover \mathbf{u} is measure-valued entropy solution of Problem (4.1.1) in the sense of Definition 4.1.

Synthèse



Titre: Méthodes de volumes finis pour des équations aux dérivées partielles déterministes et stochastiques

Écoulements à densité variable en milieu poreux

Nous considérons un milieu poreux contenant à la fois de l'eau douce et de l'eau salée; plus précisément nous étudions un système couplé décrivant l'interaction entre l'écoulement et le transport dans un milieu poreux. Nous résolvons le système suivant

$$\begin{cases} \mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho(c)\mathbf{g}) & \text{dans } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial \rho(c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)) = 0 & \text{dans } \mathbb{D} \times (0, T), \\ \Phi \frac{\partial (\rho(c)c)}{\partial t} + \nabla \cdot (\mathbf{q}\rho(c)c - \rho(c)\mathbf{D}\nabla c) = 0 & \text{dans } \mathbb{D} \times (0, T). \end{cases} \quad (\text{A.0.1})$$

avec les conditions initiales et les conditions aux limites correspondantes. La porosité Φ est le rapport des vides (espaces vides) sur le volume total. Dans la première équation (A.0.1), la loi de Darcy, \mathbf{q} est la vitesse de l'écoulement, p la pression, k la perméabilité, μ la viscosité dynamique, et \mathbf{g} la gravité.

Une caractéristique essentielle du schéma SUSHI est qu'il permet d'utiliser des éléments de volume qui ne se correspondent pas. En particulier, nous pouvons appliquer un maillage adaptatif de carrés ou de cubes de différentes tailles dans les tests numériques. Les fonctions inconnues varient fortement dans certaines parties du domaine alors qu'elles sont presque constantes ailleurs. Nous raffinons le maillage dans les régions de fortes variations des fonctions inconnues alors que nous fusionnons des éléments dans les zones où elles ne subissent que de faibles variations. Ceci permet de réduire le nombre des inconnues (qui dépendent du nombre d'éléments

et de côtés d'éléments de volumes) et de diminuer le temps de calcul. Après chaque raffinement ou déraffinement, nous devons nous appuyer sur de nouvelles données pour le maillage et pour les fonctions inconnues.

Nous effectuons des simulations numériques pour le problème d'une interface en rotation, le problème de Henry en dimension deux d'espace et le problème de bassin salé (saltpool) en dimension trois d'espace, en nous inspirant de calculs effectués par Hilhorst-Vu Do-Wang [37] à l'aide méthode de volumes finis standard.

Dans une deuxième partie nous prenons également en compte le transfert de chaleur. La chaleur peut être récupérée à partir de l'eau souterraine. Selon l'application, la chaleur récupérée peut être utilisée pour la production de chaleur ou pour la production d'électricité. Le chauffage géothermique permet à la fois de remplacer le chauffage conventionnel et la production d'eau chaude. Nous étudions un système couplé décrivant l'interaction entre l'écoulement et le transport dans un milieu poreux, où la densité et la viscosité dépendent de la concentration des espèces à transporter, aussi bien que de la température. Plus précisément, nous résolvons un système de trois équations paraboliques non linéaires pour la concentration de sel C , la température Θ et la charge hydraulique h .

Un logiciel commercial très utilisé pour la simulation du transport d'un soluté multi-espèces et du transport de chaleur est le code SEAWAT [48]. Il combine les logiciels MODFLOW pour le calcul de l'écoulement et MT3DMS pour la résolution numérique des équations de transport de soluté. Le couplage entre l'écoulement et le transport est effectué à l'aide de résolution successives de l'équation pour l'écoulement par MODFLOW et de l'équation de transport par MT3DMS. Les deux logiciels MODFLOW et MT3DMS s'appuient sur une méthode de différences finies et la solution est donnée par les valeurs moyennes assignées au centre de chaque maille. Le logiciel SEAWAT utilise également cette structure.

Nous proposons un code unique qui utilise la méthode de volumes finis généralisée fini SUSHI. Les éléments de volume, en dimension d'espace deux ou trois, peuvent être très généraux et la structure du programme est beaucoup plus simple puisqu'elle s'appuie sur un code unique au lieu de deux codes couplés. Nous montrons les résultats de premiers tests numériques en fin de chapitre. Nos calculs portent sur un modèle proposé dans la documentation de SEAWAT. Le domaine spatial est donné par une section transversale bi-dimensionnelle d'un aquifère captif côtier initialement saturé d'eau de mer relativement froide à la température de $5\text{ }^{\circ}\text{C}$. De l'eau douce plus chaude à la température de $25\text{ }^{\circ}\text{C}$ est injectée dans l'aquifère côtier le long de la frontière de gauche pour représenter le flux d'eau provenant de l'intérieur des terres. L'eau douce chaude s'écoule vers la droite jusqu'à atteindre la frontière avec l'océan. La frontière de l'océan est soumise à des conditions hydrostatiques basées sur la densité de fluide calculée à partir de la salinité de l'eau de mer à $5\text{ }^{\circ}\text{C}$. Des conditions de flux nul sont assignées aux limites supérieure et inférieure du domaine. Ce problème est une représentation simplifiée de ce qui pourrait se produire dans une plate-forme côtière de carbonate. Nos résultats sont très proches de ceux de SEAWAT.

Il s'agit d'un travail en collaboration avec Danielle Hilhorst et Huy Cuong Vu Do.

Simulations numériques de l'équation de Burgers stochastique

Nous effectuons une étude numérique de l'équation de Burgers non visqueuse avec un terme source stochastique. Nous considérons le problème sur le tore de dimension un, avec des conditions aux limites périodiques et une condition initiale prescrite

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = g, \quad (\text{A.0.2})$$

où le terme source stochastique g est de moyenne spatiale nulle: $\int_{\mathbb{T}^1} g dx = 0$. Nous appliquons un schéma de volumes finis combinant une intégration en temps de type Euler-Maruyama avec le flux numérique de Godunov.

Nous présentons des résultats de simulations numériques effectués à l'aide de la méthode de Monte-Carlo. Comme le terme source G_i^n est stochastique, nous effectuons un certain nombre de réalisations pour chaque α et β fixés. Nous comparons les résultats de réalisations uniques et la moyenne empirique des réalisations avec la solution numérique déterministe ($g = 0$). Les résultats numériques montrent que pour une seule réalisation, quand l'amplitude α est grande, la solution est dispersée autour de la solution déterministe; d'autre part, le terme source stochastique possède un effet de diffusion qui lisse le choc de la solution déterministe, et plus l'amplitude α est grande, plus l'effet de diffusion est fort. Nous obtenons également des résultats numériques en temps long, et observons que pour α et β fixés, la moyenne empirique converge vers la moyenne en espace de la condition initiale. Quand α et β sont plus grands, la moyenne empirique converge vers cette valeur plus vite. La variance empirique se stabilise vers une valeur qui dépend de α et β ; plus α est grand et β est petit, plus cette valeur asymptotique est grande.

Ce travail a été écrit en collaboration avec Emmanuel Audusse, Sébastien Boyaval et Danielle Hilhorst [3].

Convergence d'un schéma de volumes finis

Un sujet d'intérêt essentiel est la convergence de méthodes numériques pour la discrétisation de lois de conservation stochastiques. Nous étudions le problème

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t), & \text{dans } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \text{pour tout } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (\text{A.0.3})$$

Après avoir présenté la définition d'une solution faible entropique à valeur mesure du problème (A.0.3), nous appliquons une méthode de volumes finis avec un schéma amont pour la discrétisation de ce problème, et nous notons $\{u_{\mathcal{T},k}\}$ la solution discrète. Nous présentons des estimations a priori pour le terme source discret et pour la solution discrète $\{u_{\mathcal{T},k}\}$ ainsi qu'une estimation BV faible. Puis nous prouvons deux inégalités entropiques vérifiées par la solution discrète après l'avoir interpolée en temps. Nous démontrons que $\{u_{\mathcal{T},k}\}$ admet une sous-suite qui

converge vers une solution faible entropique à valeur mesure au sens des mesures de Young quand le diamètre maximal des éléments de volume et le pas de temps tendent vers zéro. Ce travail étend des travaux de Bauzet-Charrier-Gallouët [8] dans lequel ils considèrent une loi de conservation avec un terme source multiplicatif faisant intervenir un mouvement Brownien unidimensionnel.

Il s'agit d'un travail en collaboration avec Tadahisa Funaki, Danielle Hilhorst et Hendrik Weber.

Appendix B

Glossaries

- \mathbb{T}^d : d -dimensional torus
- \mathbb{D} : Space domain in the numerical tests
- $(\mathcal{T}, \mathcal{E}, \mathcal{P})$: A space discretisation with the set of control volumes \mathcal{T} , the set of the edges \mathcal{E} and the set of points \mathcal{P}
- $(\Omega, \mathcal{F}, \mathbb{P})$: Probability space which consists of the sample space Ω , the set of events \mathcal{F} , and the probability measure \mathbb{P}
- $E[Y]$: Expectation of random variable Y
- $\text{Var}[Y]$: Variance of random variable Y
- $\mathcal{N}(\mu, \sigma^2)$: Gaussian distribution with expectation μ and variance σ , the corresponding probability density function is given by $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- i.i.d: A sequence of random variables is independent and identically distributed (i.i.d) if each random variable has the same probability distribution as the others and all are mutually independent.
- $\text{Card}(A)$: Cardinality of the set A , i.e. the number of elements of set A .
- $\mathbf{1}_A$: Characteristic function of the set A , i.e. $\mathbf{1}_A(x) = 1$, if $x \in A$ and $\mathbf{1}_A(x) = 0$, otherwise.
- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$: natural, integer and real numbers
- $\mathbb{N}^+, \mathbb{Z}^+, \mathbb{R}^+$: positive natural, positive integer and positive real numbers

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