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Long Hu

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**Docteur de l'Université Pierre et Marie Curie**

Spécialité : Mathématiques Appliquées

par

**Long HU**

sous la direction de

**Jean-Michel CORON et Tatsien LI**

**Contrôle frontière, stabilisation et  
synchronisation pour des systèmes de  
lois de bilan en dimension un d'espace**

après avis des rapporteurs

Mme. Fatiha ALABAU-BOUSSOIRA & M. Günther LEUGERING

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# Introduction

## 1 Classical Topics in Control Theory for Dynamical Systems

What is the controllability and stabilization for a dynamical system? Let us start with the case of Ordinary Differential Equations (ODEs). Consider the following finite dimensional dynamical system

$$\frac{dx}{dt} = f(x, u), \quad (1.1)$$

where  $t \in \mathbb{R}$  is the independent variable (time),  $x \in \mathbb{R}^n$  is the state variable and  $u \in \mathbb{R}^m$  is the control (note that in many cases  $m < n$ ),  $f$  is a function of class  $C^\infty$  with respect to  $x$  and  $u$ . For convenience, we assume that

$$f(0, 0) = 0. \quad (1.2)$$

Let us recall some basic definitions of local exact controllability and stabilization for the system (1.1). One can refer to [19] for an introduction and fruitful results on the controllability and stabilization of linear and nonlinear dynamical systems.

**Definition 1.1** *The control system (1.1) is locally controllable at the equilibrium  $(0, 0)$  on the interval  $[T_0, T_1]$  ( $T_0 < T_1$ ) if, for every real number  $\varepsilon > 0$ , there exists a real number  $\eta > 0$  such that, for every  $x^0 \in B_\eta := \{x \in \mathbb{R}^n; |x| < \eta\}$  and for every  $x^1 \in B_\eta$ , there exists a measurable bounded function  $u : [T_0, T_1] \rightarrow \mathbb{R}^m$  such that the solution  $x = x(t)$  to the Cauchy problem*

$$\begin{cases} \frac{dx}{dt} = f(x, u(t)), \\ x(T_0) = x^0 \end{cases} \quad (1.3)$$

*verifies the final condition*

$$x(T_1) = x^1. \quad (1.4)$$

**Remark 1.1** *The dynamical system  $\dot{x} = f(x, u(t))$  is usually called an open-loop system and  $u$  is called the corresponding open-loop control.*

**Definition 1.2** *The control system (1.1) is locally asymptotically (resp. exponentially) stabilizable at the equilibrium  $(0, 0)$  if, there exists a feedback law, i.e. a continuous map,  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , satisfying*

$$u(0) = 0, \quad (1.5)$$

*such that  $0 \in \mathbb{R}^n$  is a locally asymptotically (resp. exponentially) stable point for the system*

$$\frac{dx}{dt} = f(x, u(x)). \quad (1.6)$$

**Remark 1.2** *The dynamical system  $\dot{x} = f(x, u(x))$  is usually called a closed-loop system and  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called the corresponding closed-loop control.*

It is well-known that if we focus on the linear case, i.e.

$$f(x, u) := Ax + Bu, \quad (1.7)$$

where  $A \in \mathcal{M}_{n,n}(\mathbb{R})$  and  $B \in \mathcal{M}_{n,m}(\mathbb{R})$ , one can find explicit necessary and sufficient conditions for the aforementioned controllability and stabilization properties (see, for example, [19, 34, 62] *et. al.*):

**Theorem 1.1** *System (1.1) with (1.7) is controllable if and only if*

$$\text{rank}(\lambda I - A, B) = n, \quad \forall \lambda \in \mathbb{C}. \quad (1.8)$$

*Here and in what follows,  $\text{rank}(\cdot)$  denotes the rank of the corresponding matrix.*

**Theorem 1.2** *System (1.1) with (1.7) is asymptotically (resp. exponentially) stabilizable if and only if*

$$\text{rank}(\lambda I - A, B) = n, \quad \text{for each } \lambda \in \mathbb{C} \text{ with } \Re \lambda \geq 0. \quad (1.9)$$

**Remark 1.3** *Condition (1.8) is known as the Hautus criterion, which is equivalent to the famous Kalman rank condition:*

$$\text{rank}(B, AB, \dots, A^{n-1}B) = n. \quad (1.10)$$

In the finite dimensional case, from Theorem 1.1 and Theorem 1.2, one can easily see that :

- (1) For linear control system (1.1) with (1.7) which satisfies (1.8) or equivalently (1.10), the open-loop controllability can be realized almost immediately. In other words, a linear control system which is controllable in large time (i.e.  $T_1 - T_0$  is large) is also controllable in small time (i.e.  $T_1 - T_0$  is small);
- (2) Every linear control system which is controllable can be asymptotically stabilized by means of linear feedback laws.

However, (1)–(2) mentioned above usually do not hold for general nonlinear system and for infinite dimensional dynamical systems, which leaves us much research space. In fact, in what follows, we will study three related control problems, i.e. controllability, stabilization and synchronization (a weak kind of controllability) for the linear and nonlinear hyperbolic systems of balance laws, and we will show some new mechanisms for such systems.

## 2 Hyperbolic Systems of Balance Laws and Their Related Control Problems

The concepts in control theory for ODEs in Introduction 1 can be automatically adapted to Partial Differential Equations (PDEs). Before this, let us briefly review our hyperbolic systems of balance laws.

### 2.1 System Description

Consider the following first-order 1-D hyperbolic systems of balance laws

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u), \quad (t, x) \in [0, T] \times [0, L], \quad (2.1)$$

where,  $u = (u_1, \dots, u_n)^T$  is a vector function of  $(t, x)$ ,  $A(u)$  is a given  $n \times n$  matrix with suitably smooth entries  $a_{ij}(u)$  ( $i, j = 1, \dots, n$ ), and  $F(u) = (f_1(u), \dots, f_n(u))^T$  is a given vector function of  $u$  with suitably smooth elements  $f_i(u)$  ( $i = 1, \dots, n$ ).

**Remark 2.1** *In general, we call the following systems*

$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = F(u) \quad (2.2)$$

*to be hyperbolic balance laws, where the flux  $g := (g_1, \dots, g_n)$  is a vector function of  $u$  with suitably smooth elements. Obviously, system (2.2) can be written in the quasilinear form as (2.1) with the Jacobian matrix*

$$A(u) := D(g(u)). \quad (2.3)$$

Therefore, throughout this thesis, unless otherwise indicated, we consider the inhomogeneous quasilinear hyperbolic system (2.1) as the hyperbolic balance laws.

**Remark 2.2** In the special case where there are no source terms (i.e.  $F(u) \equiv 0$ ), the homogeneous quasilinear system (2.1) reduces to

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \quad (2.4)$$

which constitutes a so-called hyperbolic system of conservation laws.

**Definition 2.1** (2.1) is a hyperbolic system, if, for any given  $u$  on the domain under consideration,  $A(u)$  has  $n$  real eigenvalues  $\lambda_i(u)$  ( $i = 1, \dots, n$ ) and a complete set of left (resp. right) eigenvectors  $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$  (resp.  $r_i(u) = (r_{1i}(u), \dots, r_{ni}(u))^T$ , ( $i = 1, \dots, n$ )):

$$l_i(u)A(u) = \lambda_i(u)l_i(u). \quad (2.5)$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u). \quad (2.6)$$

Here, by a complete set we mean a basis for  $\mathbb{R}^n$ .

In what follows, we assume that, maybe after a suitable change of variables,

$$\begin{aligned} \lambda_r(u) < \lambda_p(u) \equiv 0 < \lambda_s(u) \\ r = 1, \dots, l, \quad p = l + 1, \dots, m, \quad s = m + 1, \dots, n, \end{aligned} \quad (2.7)$$

and

$$l_{ij}(0) = \delta_{ij} \quad (i, j = 1, \dots, n), \quad (2.8)$$

where  $\delta_{ij}$  stands for the Kronecker symbol. In particular,

$$A(0) = \begin{pmatrix} \lambda_1(0) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(0) \end{pmatrix}. \quad (2.9)$$

**Definition 2.2** If, for any given  $u$  on the domain under consideration,  $A(u)$  has  $n$  distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u), \quad (2.10)$$

then, the set of left (resp. right) eigenvectors forms a complete set. The corresponding hyperbolic systems (2.1) satisfying (2.10) are called strictly hyperbolic systems.

**Remark 2.3** Here and in what follows, we always suppose that all the real eigenvalues  $\lambda_i(u)$  ( $i = 1, \dots, n$ ) and all left and right eigenvectors  $l_i(u)$  and  $r_i(u)$  ( $i = 1, \dots, n$ ) have the same regularity as  $A(u)$ . However, one can easily check that this is not an issue for strictly hyperbolic systems (see, for example, [72, Appendix A, Pages 195-210]).

To ensure the well-posedness of system (2.1), we consider the following general kind of boundary conditions:

$$x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) \quad (s = m + 1, \dots, n), \quad (2.11)$$

$$x = L : u_r = G_r(t, u_{l+1}, \dots, u_n) + H_r(t) \quad (r = 1, \dots, l), \quad (2.12)$$

where  $G_j$  and  $H_j$  ( $j = 1, \dots, l, m + 1, \dots, n$ ) are functions of  $t$  with suitably smooth components. Here and hereafter, we suppose that

$$G_j(t, 0, \dots, 0) \equiv 0 \quad (j = 1, \dots, l, m + 1, \dots, n). \quad (2.13)$$

**Remark 2.4** Boundary conditions (2.11)-(2.12) are prescribed under the hypothesis (2.8). We could also replace  $u_i$  by  $v_i = l_i(u)u$  in (2.11)-(2.12) in the general situation (see also [69, Remark 1.3 on page 171], [72], [111] and [52]).

Hyperbolic balance laws (2.1), even systems modeled by PDEs, are typically infinite dimensional dynamical system, which not only contains all the cases of conservation laws such as the Saint–Venant equation (including subcritical, supercritical, and critical cases), the isentropic and full Euler equation for one-dimensional gas dynamics (including subsonic, supersonic, and sonic cases), the nonlinear vibrating string equations *et. al.* (see, for example, [18, 23, 40, 41, 44, 48, 69, 102] and the references therein), but also the fundamental dynamics of open conservative systems such as the telegrapher equations for electrical lines, the Saint–Venant–Exner equations *et. al.* (see [12, 32, 101]). It is well-known that the controllability theory of hyperbolic systems (2.1) depends strongly on the matrix  $A(u)$  and how the controls act to the systems (in the equations or to the boundary conditions).

In this thesis, we study the following three control problems, i.e. Problem of Exact Boundary Controllability (EBC), Problem of Boundary Stabilization (BS) and Problem of Weak Boundary Controllability (WBC), for the system (2.1) with boundary conditions (2.11)–(2.12):

- **Problem (EBC).** Let  $\mathfrak{C} \subseteq \{1, \dots, n\}$ . For any given initial data  $\varphi := (\varphi_1, \dots, \varphi_n)^T \in (C^1[0, L])^n$  and final date  $\psi := (\psi_1, \dots, \psi_n)^T \in (C^1[0, L])^n$  with small  $C^1$  norm, can we find a time  $T > 0$  and boundary controls  $H_i \in C^1[0, T]$  ( $i \in \mathfrak{C}$ ) (for  $i \notin \mathfrak{C}$ ,  $H_i(t)$  is given and is assumed to be small) with small  $C^1$  norm, satisfying the  $C^1$  compatibility conditions at the points  $(t, x) = (0, 0)$

and  $(0, L)$  (see Remark 2.5), such that the mixed initial-boundary value problem (2.1), (2.11)-(2.12) with the initial condition

$$t = 0 : u = \varphi(x), 0 \leq x \leq L \quad (2.14)$$

admits a  $C^1$  solution  $u(t, x)$  on the domain

$$R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}, \quad (2.15)$$

which satisfies the final condition

$$t = T : u = \psi(x), 0 \leq x \leq L? \quad (2.16)$$

- **Problem (BS).** Suppose that  $H_i(t) \equiv 0$  ( $i = 1, \dots, n$ ) and that  $\lambda > 0$  is an arbitrary given constant. Suppose furthermore that  $C^1$  compatibility conditions are satisfied at the point  $(t, x) = (0, 0)$  (see Remark 2.5). Does there exist a linear feedback control  $\mathcal{B} : (H^2(0, L))^n \rightarrow \mathbb{R}^l$ , without loss of generality verifying the  $C^1$  compatibility conditions at the point  $(t, x) = (0, 1)$ , such that for some  $\varepsilon > 0$ , every solution  $u$  of the mixed initial-boundary value problem (2.1), (2.11)-(2.12) and (2.14) with  $H(t) = \mathcal{B}(u(t, \cdot))$  satisfies

$$\|u(t, \cdot)\|_{H^2(0, L)} \leq C e^{-\lambda t} \|\varphi(\cdot)\|_{H^2(0, L)}, \quad (2.17)$$

for some  $C > 0$ , provided that  $\|\varphi(\cdot)\|_{H^2(0, L)} \leq \varepsilon$ ?

Moreover, generally speaking, if there are essentially not enough boundary controls, then it is not possible to realize the exact boundary (null) controllability (see [80]). Then, it is nature to ask what happens in the case of lack of boundary controls to the hyperbolic systems (2.1)? Therefore, we are also concerned with

- **Problem (WBC).** Which kind of controllability in a weaker sense can be realized by means of less boundary controls for the general hyperbolic balance laws (2.1) or its special form?

**Remark 2.5** *The  $C^1$  compatibility conditions at the point  $(t, x) = (0, 0)$  are given by*

$$\begin{aligned} \varphi_s(0) &= G_s(0, \varphi_1(0), \dots, \varphi_m(0)) + H_s(0) \quad (s = m + 1, \dots, n), \\ f_s(\varphi(0)) - \sum_{j=1}^n a_{sj}(\varphi(0)) \varphi_{jx}(0) &= \frac{\partial G_s}{\partial t}(0, \varphi_1(0), \dots, \varphi_m(0)) + H_{st}(0) \\ &+ \sum_{r=1}^m \sum_{j=1}^n \frac{\partial G_s}{\partial u_r}(0, \varphi_1(0), \dots, \varphi_m(0)) \left( f_r(\varphi(0)) - a_{rj}(\varphi(0)) \varphi_{jx}(0) \right) \quad (s = m + 1, \dots, n). \end{aligned}$$

The  $C^1$  compatibility conditions at the point  $(t, x) = (0, L)$  are similar.

## 2.2 Concise Review of the Literature and Open Problems

### Exact Boundary Controllability

There are many publications concerning the exact controllability for linear hyperbolic systems (see [88], [100] and the references therein). As a special form of second order hyperbolic equations, the exact boundary controllability for linear wave equations was obtained in a complete manner by using the so-called Hilbert Uniqueness Method (HUM) (see [88]-[89]). In fact, the essence of J.-L. Lions's HUM is to use the duality to get the controllability by a corresponding observability inequality. Using HUM as well as other technical developments, Zuazua [122]-[123], Emanuilov [37] and Lasiecka & Triggiani [66] subsequently established the exact boundary controllability for some semilinear wave equations. As for the general first order hyperbolic systems, one can refer to the prior works [15, 86] and the recent monograph of Li [72] for almost complete results on several kinds of exact boundary controllability for the 1-D quasilinear inhomogeneous hyperbolic systems (2.1) in the context of classical solution (see also Part I). Let us mention [121] for the local boundary controllability of quasilinear wave equations in the multidimensional space. In the framework of BV solutions, the corresponding controllability for hyperbolic conservation laws has been studied in [8, 9, 13, 40, 41, 51, 96].

Since the hyperbolic wave has a finite speed of propagation, the exact boundary control time  $T > 0$  should be suitably large. However, from the point of view of applications,  $T$  should be chosen as small as possible. Thus, one may ask

- (i). Is it possible to get an optimal control time estimate for the hyperbolic balance laws (2.1), homogeneous case or inhomogeneous case?

On the other hand, it is well known that if all the characteristic speeds of the system do not vanish, i.e., all the eigenvalues of  $A(u)$  are nonzero in the domain under consideration, then (2.1) is exactly controllable (at least locally in  $C^1$  class) by boundary controls provided that the control time is sufficiently large. (see Li and Rao [75, 74]; see also Wang [111] for the nonautonomous case). If  $A(u)$  has zero eigenvalue, one can add some internal controls acting on the components corresponding to the zero eigenvalues and boundary controls acting on the other components [82, 111, 76]. It is worth mentioning that Glass recently obtained the controllability (not exact boundary controllability) of 1-D non-isentropic Euler equation, a hyperbolic system of conservation laws with a vanishing characteristic speed, in both Eulerian and Lagrangian coordinates in the context of entropy solutions [41]. Another open question naturally arises:



- (ii). Is it possible to establish the exact controllability for a hyperbolic system of balance laws with vanishing characteristic speeds by only boundary controls?

This situation seems to be complicated. Notice that the solution  $u(t, x)$  of the degenerate case of the equation (essentially an ordinary differential equation)

$$\begin{cases} u_t = 0, & (t, x) \in (0, T) \times (0, 2\pi), \\ u(0, x) = u_0(x), & x \in (0, 2\pi), \end{cases} \quad (2.18)$$

obeys

$$u(t, \cdot) \equiv u_0, \quad \forall t \in [0, T]. \quad (2.19)$$

Obviously, it is impossible to change the value  $u$  by using the boundary controls acting on the end  $x = 0$  and/or  $x = 2\pi$ . In this direction, Gugat and Leugering [48] proved a global boundary controllability result which covers the critical case in which one of the characteristic speed is zero. In spite of the degenerate model (2.18), it is still possible to prove the controllability for some nonlinear systems. The return method of Coron has been applied to many situations, where the linearized system is not controllable while the nonlinearity enables the corresponding nonlinear systems to be controllable (see, in particular, [19] and the references therein). One can apply the return method to realize exact boundary controllability for quasilinear hyperbolic systems with a zero characteristic speed [23], when the possible vanishing characteristic speed can be driven to nonzero after sufficiently long time. However, this approach seems to be no longer valid for the system with identically zero characteristic speeds (see [23, Remark 1.3]).

**Remark 2.6** *Let us also mention [36, 38] for the distributed (internal) controllability of linear/semilinear hyperbolic systems in any space dimension and [90] for the corresponding stochastic problem. The optimal control problems for wave equation are considered with controls in both Dirichlet and Neumann boundary conditions and with pointwise constraints on the control and the state (see [92, 93]).*

## Boundary Stabilization

In the last three decades or so, there has been a resurgence in the analysis of the asymptotical behavior for quasilinear hyperbolic systems (2.1). During this time, three parallel mathematical approaches have emerged. The first one is the so-called “Characteristic method”, i.e. computing corresponding bounds by using explicit evolution of the solution along the characteristic curves. With this method, Problem (BS) has been previously investigated by Greenberg and Li (see [42]) for  $2 \times 2$  (i.e.  $A \in \mathcal{M}_{2,2}(\mathbb{R})$ ) systems and Li and Qin (see [69, 98]) for a generalization to  $n \times n$  homogeneous systems in the framework of  $C^1$  norm. Also, this method was developed

by Li and Rao [75] to study the exact boundary controllability for general inhomogeneous quasilinear hyperbolic systems.

The second method is the “Control Lyapunov Functions method”, which is a useful tool to analyze the asymptotic behavior of dynamical systems. This method was first used by Coron et.al. to enable the design of dissipative boundary conditions for nonlinear homogeneous hyperbolic systems in the context of both  $C^1$  norm and  $H^2$  norm [11, 20, 21, 22, 25]. Although both Characteristic method and Control Lyapunov Functions method have been driven to study exponential stability for hyperbolic conservation laws or even some hyperbolic balance laws, however, these two methods fail when dealing with Problem (BS) for the general inhomogeneous hyperbolic systems (see [10], [11, Section 5.6], [19, Pages 314 and 361–371], [32], [69], [21] and [22]).

The third one is the “Backstepping method”, which has been proven to be effective to stabilize inhomogeneous quasilinear hyperbolic systems (2.1) (see [26] and [31]) and other equations (see [14, 24, 65, 103]). In [26], a full-state feedback control law, with actuation on only one end of the domain, which achieves  $H^2$  exponential stability of the closed-loop  $2 \times 2$  linear and quasilinear hyperbolic system is derived using a backstepping method. Unfortunately, the method presented in [26] can not be directly extended to  $n \times n$  systems, even in linear cases, especially when several states convecting in the same direction are controlled (see also [31]). Therefore, the following question is of great interest and remained open:

- (iii). Can we stabilize the  $3 \times 3$  linear or quasilinear inhomogeneous hyperbolic systems (2.1) by multi-boundary feedback controls? and what about  $n \times n$  cases?

**Remark 2.7** *One can also refer [28], [46], [68] and [97] to the results on stabilization for hyperbolic systems on networks.*

## Weak Boundary Controllability–Exact Boundary Synchronization

As being said before, it is a significant issue from both theoretical and practical points of view that whether we can shorten the control time or number of controls for hyperbolic systems (2.1). For instance, consider the following coupled system of wave equations

$$U_{tt} - U_{xx} + AU = 0, \quad (2.20)$$

where  $U \in \mathbb{R}^N$  and  $A \in \mathcal{M}_{N,N}(\mathbb{R})$ . Based on the results in [55], we can see that one needs  $N$  boundary controls to achieve controllability for  $N$  coupled wave systems (2.20). In view of less controls, although one can not realize the exact controllability in general (see [80] and Remark 2.8), however, the indirect controllability and asymptotical controllability (another kind of controllability which is not as mentioned in

Definition 1.2), both the boundary controllability in a weaker sense, can be realized by means of  $M(< N)$  boundary controls for some special coupled systems of  $N$  wave equations (2.20) (cf. [2], [3], [6], [78]). An interesting open question rises:

- (iv). Can we define another kind of weak boundary controllability without enough controls for hyperbolic systems (2.20), such as the exact boundary synchronization and its related topics? and how to construct boundary controls achieving these synchronizations?

Synchronization is actually a widespread natural phenomenon. Thousands of fireflies may twinkle at the same time; audiences in the theater can applaud with a rhythmic beat; pacemaker cells of the heart function simultaneously; and field crickets give out a unanimous cry—all these are phenomena of synchronization [104], [115].

In principle, synchronization happens when different individuals possess likeness in nature, that is, they conform essentially to the same governing equation, and meanwhile, the individuals should bear a certain coupled relation.

The phenomenon of synchronization was first observed by Huygens in 1665 [60]. The theoretical research on synchronization phenomena dates back to H. Fujisaka and T. Yamada's study of synchronization for coupled equations in 1983 [39]. The previous studies focused on systems described by ODEs, such as

$$\frac{dX_i}{dt} = f(X_i, t) + \sum_{j=1}^N A_{ij}X_j \quad (i = 1, \dots, N), \quad (2.21)$$

and considered the asymptotical synchronization behavior, i.e.,

$$X_i(t) - X_j(t) \rightarrow 0, \text{ as } t \rightarrow +\infty \quad (\forall i \neq j), \quad (2.22)$$

where  $X_i (i = 1, \dots, N)$  are  $n$ -dimensional vectors,  $A_{ij} (i, j = 1, \dots, N)$  are  $n \times n$  matrices, and  $f(X, t)$  is an  $n$ -dimensional vector function independent of  $i$ .

Our goal is to synchronize a hyperbolic system (2.20) or its quasilinear form through boundary controls. Different from the ODE situation, the coupling of systems can then be fulfilled by coupling of the equations or (and) the boundary conditions, which has richer research implications, moreover, boundary controls can be used to realize our goal in a finite time.

Roughly speaking, we want to find a finite time  $T > 0$  and few boundary controls on  $[0, T]$ , such that from the time  $t = T$  on, the system states tend to a state having components which are equal. That is to say, we hope to achieve the synchronization of the system states not only at the moment  $t = T$  under the action of few boundary controls on  $[0, T]$ , but also when  $t \geq T$  and withdrawing all the controls. This is forever, instead of short-lived, synchronization, as is desired in many actual applications.

All the topics in synchronization are almost blank and uncultivated in the field of PDEs, especially for hyperbolic systems of balance laws (2.1) or its special form (2.20).

**Remark 2.8** *One can also find some results on the exact controllability of cascade coupled multi-dimensional hyperbolic systems by a reduced number of controls in [4, 5].*

### 3 Motivation and Outline of the Thesis

This thesis is motivated by the above four questions in Introduction 2.2 and brings precise answers to all of them. It is composed of eight chapters and comprises three main parts numbered I,II and III, where Part I deals with the Questions (i) and (ii), while Parts II and III deal with the Questions (iii) and (iv), respectively. Most of the results presented in the thesis are published or submitted for publication in our papers [56, 52, 59, 58, 53, 54, 55, 57, 81].

#### 3.1 Main Work in Part I

In Part I (Chapters 1–2), we are concerned with the exact boundary controllability problem, i.e. Problem (EBC), for hyperbolic systems of balance laws (2.1).

For this, in Chapter 1, we present our results in [52], which is devoted to give sharp time estimates for local exact boundary controllability of 1-D homogeneous quasilinear hyperbolic systems without zero characteristic speeds. We improve the time for the controllability found by Li [72, Theorem 3.2 and Theorem 3.3 on pages 39-41] and Zhang [120] both for the case of one-sided control and for the case of both-sided controls when the dimension of the controls is the same as the case of one-sided control. We prove that this control time is sharp. Our approach relies on the utilization of constructive methods, the characteristic theory together with a suitable iteration scheme on a suitable time interval.

As being said, despite of the fact that distributed (internal) controls are usually used to obtain controllability for a hyperbolic system (2.1) with vanishing characteristic speeds, Chapter 2 is, however, devoted to study the case where only boundary controls are considered. Following our approach developed in [59], we first prove that a weak coupled hyperbolic system of balance laws (2.1) is not (null) controllable in finite time. Meanwhile, we give a sufficient and necessary condition for the asymptotic stabilization of the system under a natural feedback. This result somehow gives a negative answer to Question (ii), that is even for the coupled hyperbolic system (2.1) with zero characteristic speed, it is also impossible to achieve the corresponding exact boundary controllability. However, one can stabilize these systems in infinite time by means of boundary feedback. This extends the previous works [82, 111, 76] and [23].

### 3.2 Main Work in Part II

Part II (Chapters 3–4) is devoted to study boundary stabilization for inhomogeneous hyperbolic systems. Two chapters are established in order to illustrate this topic.

In Chapter 3, we first introduce the definition of heterodirectional and homodirectional hyperbolic systems, and present our results from [54] that deal with backstepping control design for linear first-order heterodirectional and homodirectional hyperbolic systems featuring arbitrary large (local) coupling coefficients. A particular choice of the target system, featuring a cascade structure similar to [26, Section 3.5], enables the use of a classical Volterra integral transformation. Well-posedness of the system of kernel equations, which is the main technical challenge of this chapter, is proved by a method of successive approximations using a novel recursive bound. Moreover, we design a stabilizing full-state feedback law for heterodirectional systems. Along with a dual observer design, this yields an output feedback controller amenable to implementation. Besides, we solve the problem of boundary motion planning for homodirectional systems. Our results in Chapter 3 not only give a positive answer to Question (iii) and the open question proposed in [26] and [31], but also open the door for a large number of related problems to be solved, e.g. collocated observer design, disturbance rejection, similarly to [1], parameter identification as in [30], stabilization of quasilinear systems as in [26], which directly results in the research activities in Chapter 4.

As an apparent generalization of Chapter 3, in Chapter 4, we present the results of our paper [53], which deal with the problem of boundary stabilization of first-order  $n \times n$  inhomogeneous quasilinear hyperbolic systems (2.1). Based on the results of the linear case [54] and Chapter 3, we use the linearized feedback control to stabilize the nonlinear system as it is mentioned in [26]. Although the target system is a little different from the one in [26] with a linear term involved in the equations, thanks to its special structure, we still find that the natural control Lyapunov functions do lead to arbitrarily large exponential decay rate to the target system (2.1) with more technical developments than that was used in [26]. The main result of this chapter supplements the previous works on how to design multi-boundary feedback controllers to realize (rapid) exponential stability of the original nonlinear system in the spatial  $H^2$  sense. Therefore, Part II completely answers the Question (iii).

### 3.3 Main Work in Part III

In Part III (Chapters 5-6-7-8), we focus on the study of a kind of weak boundary controllability, i.e. exact boundary synchronization, for a coupled system of wave equations (2.20) and its quasilinear form, which, to our best knowledge, is first introduced to the field of PDEs.

In Chapter 5, based on the theory of semi-global classical solutions to quasilinear hyperbolic systems, we first recall our results in [55] that apply a unified constructive method to establish the local exact boundary (null) controllability and the local boundary (weak) observability for the following coupled system of 1-D quasilinear wave equations

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w) \frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^n a_{ij}(w) w_j = 0, \quad (3.1)$$

where  $w = (w_1, \dots, w_n)^T$  is the unknown vector function of  $(t, x)$ ,  $a_i(w)$  and  $a_{ij}(w)$  ( $i, j = 1, \dots, n$ ) are all  $C^1$  functions of  $w$  on the domain under consideration, satisfying

$$a_i(0) > 0, \quad i = 1, \dots, n. \quad (3.2)$$

Four kinds of boundary conditions, i.e. Dirichlet type, Neumann type, Coupled third type and Coupled dissipative type are considered. The conclusions we obtain in Chapter 5 provide significant basis for studying the exact boundary synchronization for a coupled system of wave equations.

In Chapter 6 and Chapter 7, inspired by the natural phenomena of synchronization and combining the null controllability results in Chapter 5, several kinds of exact synchronizations and the generalized exact synchronization are introduced for a coupled system of 1-D wave equations (2.1) with boundary conditions of Dirichlet type, Neumann type, Coupled third type and Coupled dissipative type, respectively. Our results shown in Chapter 6 (see also [81]) indicate that all these synchronizations can be realized by means of less boundary controls with the boundary conditions of Dirichlet type, Neumann type and Coupled third type, while similar results are proven in Chapter 7 (see also [57]) for the hyperbolic system (2.1) with boundary conditions of coupled dissipative type. It should be noted that sufficient and necessary conditions are obtained in Chapter 6 for the cases of exact boundary synchronization and exact boundary null controllability and synchronization by groups for coupled wave systems (2.1) with the first three boundary conditions.

In Chapter 8, we consider the following general kind of coupled system of wave equations

$$\frac{\partial^2 U}{\partial t^2} - a^2(U, U_t, U_x) \frac{\partial^2 U}{\partial x^2} + A(U, U_t, U_x) U = 0, \quad (3.3)$$

where,  $U = (u_1, \dots, u_N)^T$  is an unknown vector function of  $(t, x)$ ,  $A(U, U_t, U_x) = (a_{ij}(U, U_t, U_x))$  is an  $N \times N$  coupling matrix, whose elements are  $C^1$  functions with respect to their arguments.  $a(U, U_t, U_x)$ , the common velocity for the coupled system, is also a  $C^1$  function with respect to its arguments, satisfying

$$a(0, 0, 0) > 0. \quad (3.4)$$

As being said before, it is a significant issue from both theoretical and practical points of view to see if we can shorten the number of controls for the coupled systems of wave

equations (3.3). Based on the theory of semi-global classical  $C^2$  solution for coupled wave equations, using a unified constructive method for one-sided exact boundary controllability (see [55], [74], [75], [72]), by a suitable iteration procedure as well as certain estimates for coupled systems of wave equations with afore mentioned four kinds of boundary conditions, we establish its local exact boundary synchronization around  $U = 0$ . The main work in this chapter is also presented in [56].

## **Part I**

# **Exact Boundary Controllability for Hyperbolic Balance Laws**





# Chapter 1

## Sharp Boundary Control Time Estimates for 1-D Homogeneous Quasilinear Hyperbolic Systems

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### 1.1 Introduction and Main Results

Consider the following 1-D quasilinear system

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0 \quad (t, x) \in [0, T] \times [0, L], \quad (1.1.1)$$

where,  $u = (u_1, \dots, u_n)^T$  is a vector function of  $(t, x)$ ,  $A(u)$  is an  $n \times n$  matrix with  $C^1$  elements  $a_{ij}(u)$  ( $i, j = 1, \dots, n$ ). By the definition of hyperbolicity,  $A(u)$  has  $n$  real eigenvalues  $\lambda_i(u)$  ( $i = 1, \dots, n$ ) and a complete set of left eigenvectors  $l_i(u) =$

$(l_{i1}(u), \dots, l_{in}(u))$  ( $i = 1, \dots, n$ ):

$$l_i(u)A(u) = \lambda_i(u)l_i(u), \quad (1.1.2)$$

and, consequently, a complete set of right eigenvectors  $r_i(u) = (r_{1i}(u), \dots, r_{ni}(u))^T$  ( $i = 1, \dots, n$ ):

$$A(u)r_i(u) = \lambda_i(u)r_i(u). \quad (1.1.3)$$

Without loss of generality, here we assume that

$$l_{ij}(0) = \delta_{ij} \quad (i, j = 1, \dots, n), \quad (1.1.4)$$

where  $\delta_{ij}$  stands for the Kronecker symbol. Moreover, for simplicity of statement, we suppose that the set of eigenvalues  $\Lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$  satisfies the following condition:

$$\lambda_1(0) < \lambda_2(0) < \dots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \dots < \lambda_n(0), \quad (1.1.5)$$

and the number of positive eigenvalues is not bigger than the number of negative eigenvalues, i.e.

$$\overline{m} \stackrel{\text{def.}}{=} n - m \leq m. \quad (1.1.6)$$

The boundary conditions are given as follows:

$$x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) \quad (s = m + 1, \dots, n), \quad (1.1.7)$$

$$x = L : u_r = G_r(t, u_{m+1}, \dots, u_n) + H_r(t) \quad (r = 1, \dots, m), \quad (1.1.8)$$

where  $G_i$  and  $H_i$  ( $i = 1, \dots, n$ ) are  $C^1$  functions, without loss of generality, such that

$$G_i(t, 0, \dots, 0) \equiv 0 \quad (i = 1, \dots, n). \quad (1.1.9)$$

Let us first recall the following well-posedness result due to Li (see [72, Chapter 2 on pages 19-36]) for the system (1.1.1) with (1.1.7)-(1.1.8):

**Lemma 1.1.1** *Suppose that  $\lambda_i, l_i, r_i, G_i$  and  $H_i$  ( $i = 1, \dots, n$ ) are all  $C^1$  functions with respect to their arguments and (1.1.4) holds. Suppose furthermore that the conditions of  $C^1$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$ , respectively. For any given  $T_0 > 0$ , the mixed initial-boundary value problem (1.1.1), (1.1.7)-(1.1.8) with*

$$t = 0 : u = \varphi(x), 0 \leq x \leq L \quad (1.1.10)$$

admits a unique  $C^1$  solution  $u(t, x)$  with sufficiently small  $C^1$  norm on the domain

$$R(T_0) = \{(t, x) | 0 \leq t \leq T_0, 0 \leq x \leq L\}, \quad (1.1.11)$$

provided that  $\|\varphi\|_1$  and  $\|H\|_1$  are small enough (depending on  $T_0$ ).

**Remark 1.1.1** Here and throughout this chapter, for simplicity, we denote  $\|\cdot\|_0$  and  $\|\cdot\|_1$  as the corresponding  $C^0$  and  $C^1$  norms, respectively.

**Remark 1.1.2** Boundary conditions (1.1.7)-(1.1.8) are prescribed under the hypothesis (1.1.4). We could also replace  $u_i$  by the  $v_i = l_i(u)u$  in (1.1.7)-(1.1.8) in the general situation (see also [69, Remark 1.3 on page 171]).

**Remark 1.1.3** The  $C^1$  compatibility conditions at the point  $(t, x) = (0, 0)$  are given by

$$\varphi_s(0) = G_s(0, \varphi_1(0), \dots, \varphi_m(0)) + H_s(0) \quad (s = m + 1, \dots, n), \quad (1.1.12)$$

$$\begin{aligned} \sum_{j=1}^n a_{sj}(\varphi(0))\varphi_{jx}(0) &= \sum_{r=1}^m \sum_{j=1}^n \frac{\partial G_s}{\partial u_r}(0, \varphi_1(0), \dots, \varphi_m(0))a_{rj}(\varphi(0))\varphi_{jx}(0) \\ &\quad - \frac{\partial G_s}{\partial t}(0, \varphi_1(0), \dots, \varphi_m(0)) - H_{st}(0) \quad (s = m + 1, \dots, n). \end{aligned} \quad (1.1.13)$$

The  $C^1$  compatibility conditions at the point  $(t, x) = (0, L)$  are similar.

Based on Lemma 1.1.1, we are concerned with the corresponding exact controllability problem for the system (1.1.1) and (1.1.7)-(1.1.8):

**Problem (EBC).** Let  $\mathfrak{C} \subseteq \{1, \dots, n\}$ . For any given initial data  $\varphi \in (C^1[0, L])^n$  and final data  $\psi \in (C^1[0, L])^n$  with small  $C^1$  norm, can we find a time  $T > 0$  and boundary controls  $H_i \in C^1[0, T]$  ( $i \in \mathfrak{C}$ ) (for  $i \notin \mathfrak{C}$ ,  $H_i(t)$  is given) with small  $C^1$  norm, such that the mixed initial-boundary value problem (1.1.1), (1.1.7)-(1.1.8) and (1.1.10) admits a  $C^1$  solution  $u(t, x)$  on the domain

$$R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}, \quad (1.1.14)$$

which satisfies exactly the final condition

$$t = T : u = \psi(x), 0 \leq x \leq L? \quad (1.1.15)$$

The exact boundary controllability problems for both linear and quasilinear 1-D hyperbolic systems have attracted considerable attention over the past several decades (see, for example, [19, 72, 100] and the numerous references therein). Eventually,

three cases have been distinguished in the literatures, depending on the number of boundary controls, the boundary conditions on non-control side as well as the control time, to study **Problem (EBC)**(see, for instance, [72, Chapter 3 on pages 37-62] and [75, 120]):

- (1) **Both-sided control** ( $\mathfrak{C} = \{1, \dots, n\}$ ). One can choose

$$T > L \max \left( \frac{1}{|\lambda_m(0)|}, \frac{1}{\lambda_{m+1}(0)} \right) \text{ and } H_i (i = 1, \dots, n) \quad (1.1.16)$$

to realize the (local) exact boundary controllability for the general first order 1-D quasilinear hyperbolic system (1.1.1), (1.1.7)-(1.1.8). Since enough boundary controls are acting on both ends, we can drive any initial data  $\varphi$  to any desired data  $\psi$  around equilibrium without any additional assumptions to the boundary conditions (1.1.7)-(1.1.8).

- (2) **One-sided control** ( $\mathfrak{C} = \{1, \dots, m\}$ ). In this situation, in order to handle the **Problem (EBC)**, one should utilize all the boundary functions  $H_r(t)$  ( $r = 1, \dots, m$ ) as boundary controls on  $x = L$ , meanwhile, suppose that the boundary condition (1.1.7) on the non-control side  $x = 0$  in a neighborhood of  $u = 0$  satisfies

$$\begin{aligned} u_s &= G_s(t, u_1, \dots, u_m) + H_s(t) \quad (s = m + 1, \dots, n) \\ \Leftrightarrow u_{\bar{r}} &= \bar{G}_{\bar{r}}(t, u_{\bar{m}+1}, \dots, u_m, u_{m+1}, \dots, u_n) + \bar{H}_{\bar{r}}(t) \quad (\bar{r} = 1, \dots, \bar{m}), \end{aligned} \quad (1.1.17)$$

where  $\bar{G}_{\bar{r}}$  and  $\bar{H}_{\bar{r}}$  are also  $C^1$  functions with respect to their arguments, and

$$\bar{G}_{\bar{r}}(t, 0, \dots, 0) \equiv 0 \quad (\bar{r} = 1, \dots, \bar{m}), \quad (1.1.18)$$

provided that the control time satisfies

$$T > L \left( \frac{1}{|\lambda_m(0)|} + \frac{1}{\lambda_{m+1}(0)} \right). \quad (1.1.19)$$

- (3) **Both-sided with less controls** ( $\mathfrak{C} = \{\bar{m} + 1, \dots, n\}$ ). In this case, under the same assumption (1.1.19) as in the case of “One-sided control”, one needs all (i.e.  $\bar{m}$ ) boundary controls  $H_s$  ( $s = m + 1, \dots, n$ ) acting on  $x = 0$  together with  $m - \bar{m}$  boundary controls  $H_q$  ( $q = \bar{m} + 1, \dots, m$ ) acting on  $x = L$ , provided that in a neighborhood of  $u = 0$  the first  $\bar{m}$  boundary conditions (1.1.8), the non-control components, on  $x = L$  satisfies

$$\begin{aligned} u_{\bar{r}} &= G_{\bar{r}}(t, u_{m+1}, \dots, u_n) + H_{\bar{r}}(t) \quad (\bar{r} = 1, \dots, \bar{m}) \\ \Leftrightarrow u_s &= \bar{G}_s(t, u_1, \dots, u_{\bar{m}}) + \bar{H}_s(t) \quad (s = m + 1, \dots, n), \end{aligned} \quad (1.1.20)$$

where  $\overline{G}_s$  and  $\overline{H}_s$  are also  $C^1$  functions with respect to their arguments, and

$$\overline{G}_s(t, 0, \dots, 0) \equiv 0 \quad (s = m + 1, \dots, n). \quad (1.1.21)$$

Compared with the case “Both-sided control”, the latter two cases reduce the number of boundary controls to  $m$ , however, at the cost of the larger control time. It is easy to show that the control time given by (1.1.16) is sharp for the first case (see [72, Remark 3.11 on page 47]). However, in the cases of “One-sided control” and “Both-sided with less controls”, one verifies that the control time (1.1.19) can be improved for linear hyperbolic systems (see Remark 1.1.4). Taking into account this phenomenon, one may ask the following natural question: is it still valid for the general nonlinear systems? Here, we give a positive answer.

In this chapter, based on the constructive methods, the characteristic theory together with a suitable iteration scheme on a suitable time interval, we shorten the control time for the quasilinear hyperbolic systems (1.1.1) with boundary controls acting on one side or both sides with less controls that was proposed by Li [72, Theorem 3.2 and Theorem 3.3 on pages 39-41] and Zhang [120]. Our main results, which can be applied to control Saint-Venant-Exner system (see [32]), the full Euler equation for 1-D gas dynamics ([23, section 4])etc., are given as follows:

**Theorem 1.1.1 (*One-sided control*)** *Let us assume that the basic hypotheses of Lemma 1.1.1 hold and (1.1.17)-(1.1.18) is valid in a neighborhood of  $u = 0$ . Let  $T > 0$  satisfies*

$$T > L \max \left( \frac{1}{|\lambda_{\overline{m}}(0)|} + \frac{1}{\lambda_{m+1}(0)}, \frac{1}{|\lambda_m(0)|} \right). \quad (1.1.22)$$

*Then there exists  $\varepsilon > 0$  such that for any given initial data  $\varphi \in (C^1[0, L])^n$ , any boundary functions  $H_s \in C^1[0, T]$  ( $s = m + 1, \dots, n$ ),  $\overline{H}_{\overline{r}} \in C^1[0, T]$  ( $\overline{r} = 1, \dots, \overline{m}$ ) and any  $\psi \in (C^1[0, L])^n$  satisfying the  $C^1$  compatibility conditions at the point  $(t, x) = (0, 0)$  and  $(T, 0)$  respectively, and*

$$\|\varphi\|_1 + \|\psi\|_1 + \max_{\overline{r}=1, \dots, \overline{m}} \|\overline{H}_{\overline{r}}\|_1 + \max_{s=m+1, \dots, n} \|H_s\|_1 \leq \varepsilon, \quad (1.1.23)$$

*there exist boundary controls  $H_r \in C^1[0, T]$  ( $r = 1, \dots, m$ ) on  $x = L$ , such that the mixed initial-boundary value problem (1.1.1), (1.1.7)-(1.1.8) and (1.1.10) admits a unique  $C^1$  solution  $u = u(t, x)$  on the domain  $R(T)$ , which satisfies the final condition (1.1.15).*

**Theorem 1.1.2 (*Both-sided with less controls*)** *Let us assume that the basic hypotheses of Lemma 1.1.1 hold and (1.1.20)-(1.1.21) is valid in a neighborhood of  $u = 0$ . Furthermore, let  $T > 0$  satisfies the sharp time estimate (1.1.22). Then there exists  $\varepsilon > 0$  such that for any given initial data  $\varphi \in (C^1[0, L])^n$ , any boundary*

functions  $H_{\bar{r}} \in C^1[0, T]$  ( $\bar{r} = 1, \dots, \bar{m}$ ),  $\bar{H}_s \in C^1[0, T]$  ( $s = m + 1, \dots, n$ ) and any  $\psi \in (C^1[0, L])^n$  satisfying the  $C^1$  compatibility conditions at the point  $(t, x) = (0, L)$  and  $(T, L)$  respectively, and

$$\|\varphi\|_1 + \|\psi\|_1 + \max_{\bar{r}=1, \dots, \bar{m}} \|H_{\bar{r}}\|_1 + \max_{s=m+1, \dots, n} \|\bar{H}_s\|_1 \leq \varepsilon, \quad (1.1.24)$$

there exist boundary controls  $H_p \in C^1[0, T]$  ( $p = \bar{m} + 1, \dots, n$ ), such that the mixed initial-boundary value problem (1.1.1), (1.1.7)-(1.1.8) and (1.1.10) admits a unique  $C^1$  solution  $u = u(t, x)$  on the domain  $R(T)$ , which satisfies the final condition (1.1.15).

In general, when considering our controllability problem for quasilinear hyperbolic systems, the difficulties usually come from two aspects. One is the nonlinearity from the boundary conditions, and the other is that from the governing matrix  $A(u)$ . To overcome the first difficulty, we use the so-called constructive methods (see [72, 75, 111, 120]). It allows us to find a  $C^1$  solution  $u$  to the system (1.1.1) on the domain  $R(T)$ , satisfying simultaneously any given initial condition (1.1.10) and any given final condition (1.1.15), as well as all the boundary conditions without control (i.e. (1.1.17) for the case “One-sided control” or (1.1.20) for the case “Both-sided with less controls”). On the other hand, one easily checks from (1.1.17) and (1.1.20) that the components  $u_q$  ( $q = \bar{m} + 1, \dots, m$ ) are in essence decoupled from  $u_p$  ( $p = 1, \dots, \bar{m}, m + 1, \dots, n$ ) on these boundary conditions. Consequently the main difficulty of this chapter is from the second one, i.e. how to decouple these two parts of components from the governing matrix  $A(u)$  on the whole domain  $R(T)$ . If so, we then respectively control the two different hyperbolic systems, the first are on  $u_q$  ( $q = \bar{m} + 1, \dots, m$ ) and the second are on  $u_p$  ( $p = 1, \dots, \bar{m}, m + 1, \dots, n$ ). Therefore, the control time for the original system (1.1.1) is much smaller than the one in [72, Theorem 3.2 and Theorem 3.3 on pages 39-41] and [120]. To overcome the second difficulty, we will see below that the method used is strongly inspired from a suitable linearized iteration scheme on a suitable time interval. Then based on theory of the linearized hyperbolic systems with nonlinear boundary conditions, not only can we shorten the control time for the linearized system, but also prove the convergence by using constructive methods as well as some appropriate estimates.

**Remark 1.1.4** *In this remark, we first show that the time estimates provided in Theorem 1.1.1 and Theorem 1.1.2 improves the one in [72, Theorem 3.2 and Theorem 3.3 on pages 39-41] and [120]. In fact, noting (1.1.5), one has*

$$L \max \left( \frac{1}{|\lambda_{\bar{m}}(0)|} + \frac{1}{\lambda_{m+1}(0)}, \frac{1}{|\lambda_m(0)|} \right) \leq L \left( \frac{1}{|\lambda_m(0)|} + \frac{1}{\lambda_{m+1}(0)} \right). \quad (1.1.25)$$

Moreover, the two sides are equal if and only if  $\bar{m} = m$ . If  $\bar{m} < m$ , one can use less time given by (1.1.22) to control the nonlinear system (1.1.1). In this sense, our

results indeed improve the one in [72, Theorem 3.2 and Theorem 3.3 on pages 39-41] and [120]. Next, we claim that the exact controllability time  $T > 0$  given by (1.1.22) is sharp for the above two cases. Consider the following linear system

$$\frac{\partial u_i}{\partial t} + \lambda_i \frac{\partial u_i}{\partial x} = 0, \quad (1.1.26)$$

with constant characteristic speeds  $\lambda_i$  ( $i = 1, \dots, n$ ) satisfying

$$\lambda_1 < \dots < \lambda_m < 0 < \lambda_{m+1} < \dots < \lambda_n. \quad (1.1.27)$$

(I) (**One-sided Control**) For the boundary conditions

$$x = 0 : u_s = u_{n+1-s} \quad (s = m + 1, \dots, n), \quad (1.1.28)$$

$$x = L : u_r = H_r(t) \quad (r = 1, \dots, m), \quad (1.1.29)$$

when (1.1.6) holds, the whole system (1.1.26) can be divided into two systems, the one is

$$\begin{cases} \frac{\partial u_p}{\partial t} + \lambda_p \frac{\partial u_p}{\partial x} = 0 & (p = 1, \dots, \bar{m}, m + 1, \dots, n), \\ x = 0 : u_s = u_{n+1-s} & (s = m + 1, \dots, n), \\ x = L : u_{\bar{r}} = H_{\bar{r}}(t) & (\bar{r} = 1, \dots, \bar{m}), \end{cases} \quad (1.1.30)$$

and the other is

$$\begin{cases} \frac{\partial u_q}{\partial t} + \lambda_q \frac{\partial u_q}{\partial x} = 0 & (q = \bar{m} + 1, \dots, m), \\ x = L : u_q = H_q(t) & (q = \bar{m} + 1, \dots, m). \end{cases} \quad (1.1.31)$$

Noting (1.1.27), it is well known that the system (1.1.30) is controllable by boundary controls  $H_{\bar{r}}(\bar{r} = 1, \dots, \bar{m})$  if and only if

$$T > \frac{L}{|\lambda_{\bar{m}}|} + \frac{L}{\lambda_{m+1}}. \quad (1.1.32)$$

While the system (1.1.31) is controllable by boundary controls  $H_q(q = \bar{m} + 1, \dots, m)$  if and only if

$$T > \frac{L}{|\lambda_m|}. \quad (1.1.33)$$

Therefore, the system (1.1.26), (1.1.28)-(1.1.29) is controllable by the boundary con-



trols  $H_r(r = 1, \dots, m)$  if and only if

$$T > L \max \left( \frac{1}{|\lambda_{\bar{m}}|} + \frac{1}{\lambda_{m+1}}, \frac{1}{|\lambda_m|} \right). \quad (1.1.34)$$

(II) (**Both-sided with less controls**) For the boundary conditions

$$x = 0 : u_s = H_s(t) \quad (s = m + 1, \dots, n), \quad (1.1.35)$$

$$x = L : \begin{cases} u_{\bar{r}} = u_{n+1-\bar{r}} \quad (\bar{r} = 1, \dots, \bar{m}), \\ u_q = H_q(t) \quad (q = \bar{m} + 1, \dots, m), \end{cases} \quad (1.1.36)$$

noting (1.1.6), the whole system (1.1.26) can be also divided into two systems, the one is

$$\begin{cases} \frac{\partial u_p}{\partial t} + \lambda_p \frac{\partial u_p}{\partial x} = 0 \quad (p = 1, \dots, \bar{m}, m + 1, \dots, n), \\ x = 0 : u_s = H_s(t) \quad (s = m + 1, \dots, n), \\ x = L : u_{\bar{r}} = u_{n+1-\bar{r}} \quad (\bar{r} = 1, \dots, \bar{m}), \end{cases} \quad (1.1.37)$$

and the other is (1.1.31). An argument similar to the one used in (I) shows that the system (1.1.26), (1.1.35)-(1.1.36) is controllable by the boundary controls  $H_p(p = \bar{m} + 1, \dots, n)$  if and only if (1.1.34) holds.

**Remark 1.1.5** We could also treat the case with source term (see, in particular, Remark 1.3.2 below):

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = F(u), \quad (1.1.38)$$

in which  $F \in C^1$  and

$$\lim_{|u| \rightarrow 0} \frac{|F(u)|}{|u|} = 0. \quad (1.1.39)$$

However, if (1.1.39) is not satisfied, although one can refer [72, Chapter 3 on pages 37-62] and [120] to treat this case by larger control time (i.e. (1.1.19)), the approach of this chapter is not valid anymore. Can we still shorten control time as in (1.1.22) for such systems? To our best knowledge, this question remains open.

**Remark 1.1.6** The condition (1.1.5) implies that (1.1.1) is a strictly hyperbolic system around the equilibrium. For the more general case

$$\lambda_r(0) < 0 < \lambda_s(0) \quad (r = 1, \dots, m, s = m + 1, \dots, n), \quad (1.1.40)$$

*Theorem 1.1.1 and Theorem 1.1.2 still hold if we replace (1.1.22) by*

$$T > L \max \left( \max_{\bar{r}=1, \dots, \bar{m}} \frac{1}{|\lambda_{\bar{r}}(0)|} + \max_{s=m+1, \dots, n} \frac{1}{\lambda_s(0)}, \max_{q=\bar{m}+1, \dots, n} \frac{1}{|\lambda_q(0)|} \right). \quad (1.1.41)$$

The rest of this chapter is organized as follows: In §1.2, we show the  $C^0$ ,  $C^1$  and modulus of continuity estimates for general linear hyperbolic systems with nonlinear boundary conditions, which is proved in §1.5. In §1.3, we use a suitable iteration scheme together with fixed point theory to prove Theorem 1.1.1 and Theorem 1.1.2. In §1.4, we establish a technical result on controllability of the linearized system by using constructive methods.

## 1.2 Some Estimates on Linear Hyperbolic Systems with Nonlinear Boundary Conditions

In order to obtain the desired exact controllability for the nonlinear system (1.1.1), (1.1.7)-(1.1.8), we have to investigate  $C^0$ ,  $C^1$  norms and the modulus of continuity for the corresponding linear problem. Here, we introduce the modulus of continuity for  $f(t, x)$  to be the following non-negative function:

$$\omega(\eta|f) := \sup_{\substack{|t'-t''| \leq \eta, |x'-x''| \leq \eta \\ (t', x'), (t'', x'') \in R(T)}} |f(t', x') - f(t'', x'')|, \quad \eta \geq 0. \quad (1.2.1)$$

Similarly, we can define the modulus of continuity of a vector function  $f = (f_1, \dots, f_n)$  as

$$\omega(\eta|f) := \max_{i=1, \dots, n} \omega(\eta|f_i), \quad (1.2.2)$$

and the modulus of continuity of a matrix function  $A = (a_{ij})_{n \times n}$  can be

$$\omega(\eta|A) := \max_{i,j=1, \dots, n} \omega(\eta|a_{ij}). \quad (1.2.3)$$

Consider the linear hyperbolic system as follows (see also [85, 111]):

$$\frac{\partial u_i}{\partial t} + \lambda_i(t, x) \frac{\partial u_i}{\partial x} = \sum_{k=1}^{\tilde{n}} b_{ik}(t, x) \left( \frac{\partial c_k(t, x)}{\partial t} + \lambda_i(t, x) \frac{\partial c_k(t, x)}{\partial x} \right), \quad (1.2.4)$$

where  $u = (u_1, \dots, u_n)^T$  is an unknown vector function of  $(t, x)$ , we assume that  $\lambda_i, b_{ik}, c_k (i = 1, \dots, n; k = 1, \dots, \tilde{n})$  are all  $C^1$  functions with respect to their arguments. Moreover, without of generality, we assume also that, on the domain under

consideration, the set of eigenvalues  $\Lambda := \{\lambda_1(t, x), \dots, \lambda_n(t, x)\}$  satisfies the following condition:

$$\lambda_1(t, x) < \dots < \lambda_m(t, x) < 0 < \lambda_{m+1}(t, x) < \dots < \lambda_n(t, x), (t, x) \in R(T). \quad (1.2.5)$$

Suppose furthermore that

$$\|\Lambda\|_1 + \|G_u\|_0 \leq K < +\infty, \quad (1.2.6)$$

where  $K$  is a positive constant and

$$\|G_u\|_0 := \max_{\substack{r=1, \dots, m \\ s=m+1, \dots, n}} \left\{ \left\| \frac{\partial G_r}{\partial u_s} \right\|_0, \left\| \frac{\partial G_s}{\partial u_r} \right\|_0 \right\}. \quad (1.2.7)$$

For the mixed initial-boundary value problem (1.2.4) with the initial condition (1.1.10) and the nonlinear boundary conditions (1.1.7)-(1.1.8), where  $G_i (i = 1, \dots, n)$  and  $H_i$  are defined in §1.1, according to the characteristic theory for hyperbolic system (see [56, 84]), we have the following lemma (the details of the proof can be seen in §1.5):

**Lemma 1.2.1** *Suppose  $G_i$  and  $H_i (i = 1, \dots, n)$  are  $C^1$  functions and (1.1.9) together with (1.2.5) hold. Suppose furthermore that the conditions of  $C^1$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$ , respectively. For any given  $T > 0$ , the forward mixed initial-boundary problem (1.2.4), (1.1.7)-(1.1.8) and (1.1.10) admits a unique  $C^1$  solution  $u = u(t, x)$  on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which depends continuously on  $(\varphi, \Lambda, B, c)$ . Moreover, we have*

$$\|u\|_0 \leq C_1(T, K)(\|\varphi\|_0 + \|H\|_0 + \|B\|_0 \|c\|_0), \quad (1.2.8)$$

$$\|u\|_1 \leq C_2(T, K)(\|\varphi\|_1 + \|H\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0), \quad (1.2.9)$$

where  $B := (b_{ik})$ ,  $c := (c_1, \dots, c_{\bar{n}})$ ,  $H := (H_1, \dots, H_n)$  and

$$\|G_t\|_0 := \max_{\substack{r=1, \dots, m \\ s=m+1, \dots, n}} \left\{ \left\| \frac{\partial G_r}{\partial t} \right\|_0, \left\| \frac{\partial G_s}{\partial t} \right\|_0 \right\}. \quad (1.2.10)$$

In addition, if

$$0 < \eta < \min_{i=1, \dots, n} \inf_{(t, x) \in R(T)} \frac{L}{|\lambda_i(t, x)|}, \quad (1.2.11)$$

one gets

$$\omega(\eta|u_t) + \omega(\eta|u_x) \leq \Omega(\eta), \quad (1.2.12)$$

where

$$\begin{aligned} \Omega(\eta) = & C_3(T, K) \left( \omega(\eta|\varphi_x) + \|B\|_1(\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1(\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\ & + (1 + \|\varphi\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|\varphi\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)\omega(\eta|\Gamma)) \\ & \left. + (\|\varphi\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)\omega(\eta|\Lambda_x) + \omega(\eta|H_t) \right), \end{aligned} \quad (1.2.13)$$

in which  $\Gamma$  denotes the following set of functions

$$\Gamma := \left\{ \frac{\partial G_r}{\partial u_s}, \frac{\partial G_r}{\partial t}, \frac{\partial G_s}{\partial u_r}, \frac{\partial G_s}{\partial t}, \quad r = 1, \dots, m; \quad s = m + 1, \dots, n \right\}. \quad (1.2.14)$$

**Remark 1.2.1**  $\omega(\eta|\Gamma)$  denotes the modulus of continuity for  $\Gamma$  on the domain  $\tilde{R} = \{0 \leq t \leq T, 0 \leq \|u\|_0 \leq C_1(T, K)(\|\varphi\|_0 + \|H\|_0 + \|B\|_0\|c\|_0)\}$ .

**Remark 1.2.2** Different from the usual hyperbolic systems, the source term of (1.2.4) is essentially the directional derivative with respect to  $t$  along the  $i$ -th characteristic  $\frac{dx}{dt} = \lambda_i(t, x)$ . One can refer [85, 111] to the well-posedness of this kind of (nonlinear) hyperbolic systems. In this chapter, however, we still show the detail proof of Lemma 1.2.1 in order to (1) emphasize the explicit estimate on the matrix  $B$ , the smallness of which will be utilized when iteration in § 1.4; (2) obtain inequality (1.2.12) for the modulus of continuity under nonlinear boundary conditions (1.1.7)-(1.1.8) on the whole domain  $R(T)$  (not locally), which is not involved in previous references and will be used to prove the existence of fixed point below; (3) show the solution  $u$  depends continuously on  $(\varphi, \Lambda, B, c)$ , which guarantees the continuity of our iteration scheme in § 1.4.

## 1.3 Proof of Main results

### 1.3.1 Proof of Theorem 1.1.1

Since one can recover the boundary controls as the traces of  $u$  afterwards, in order to get Theorem 1.1.1, it suffices to establish the following

**Lemma 1.3.1** Under the assumptions of Theorem 1.1.1, let  $T > 0$  be defined by (1.1.22). For any given initial data  $\varphi \in (C^1[0, L])^n$ , final data  $\psi \in (C^1[0, L])^n$  and the functions  $H_s \in C^1[0, T]$  ( $s = m + 1, \dots, n$ ) satisfying (1.1.23) and the  $C^1$  compatibility conditions at the points  $(t, x) = (0, 0)$  and  $(T, 0)$ , respectively, system (1.1.1) together with the boundary conditions (1.1.7) on  $x = 0$  admits a  $C^1$  solution  $u = u(t, x)$  on the domain  $R(T)$ , which satisfies simultaneously the initial condition (1.1.10) and the final condition (1.1.15).

**Proof of Lemma 1.3.1** Based on the results in §2, we next utilize a suitable iterative scheme, the constructive methods (see, in particular, [72, Proof of Theorem 3.2 on pages 47-52] and [111]) together with the fixed point theory (see, for instance, [19, Chapter 4 on pages 159-180]) to prove Lemma 1.3.1. Firstly, multiplying the  $i$ -th left eigenvector  $l_i(u)$  ( $i = 1, \dots, n$ ) on system (1.1.1) from the left and noting (1.1.2), we get the characteristic form of hyperbolic system (1.1.1):

$$l_i(u) \left( \frac{\partial u}{\partial t} + \lambda_i(u) \frac{\partial u}{\partial x} \right) = 0 \quad (i = 1, \dots, n). \quad (1.3.1)$$

On the domain  $R(T)$ , we introduce some sets of function

$$\Sigma_1 := \left\{ y(t, x) \in (C^1[R(T)])^n \left| \begin{array}{l} y(0, x) = \varphi(x), y(T, x) = \psi(x); \\ y_t(0, 0) = -A(\varphi(0))\varphi_x(0); \\ y_t(0, L) = -A(\varphi(L))\varphi_x(L); \\ y_t(T, 0) = -A(\psi(0))\psi_x(0); \\ y_t(T, L) = -A(\psi(L))\psi_x(L); \end{array} \right. \right\}, \quad (1.3.2)$$

$$\Sigma_2 := \left\{ y(t, x) \in \Sigma_1 \mid \|y\|_0 \leq C_4\varepsilon, \|y\|_1 \leq \kappa(\varepsilon) \right\}, \quad (1.3.3)$$

$$\Sigma_3 := \left\{ y(t, x) \in \Sigma_2 \mid \omega(\eta|y_t) + \omega(\eta|y_x) \leq \Omega_1(\eta) \right\}, \quad (1.3.4)$$

Here and in what follows, for simplicity, when no confusion is possible, we put  $C_i$  ( $i = 4, 5, \dots$ ) as positive constants which are independent of  $y$ , while  $\kappa(\varepsilon)$  and  $\Omega_1(\eta)$  ( $0 \leq \eta \leq \min_{i=1, \dots, n} \inf_{\|u\|_0 \leq C_4\varepsilon} \frac{L}{|\lambda_i(u)|}$ ) denote two continuous nonnegative functions with respect to their arguments, satisfying

$$\kappa(0) = 0, \quad (1.3.5)$$

and

$$\Omega_1(0) = 0. \quad (1.3.6)$$

For any given  $y \in \Sigma_2$ , we consider the following linear system

$$\left\{ \begin{array}{l} \frac{\partial u_i}{\partial t} + \lambda_i(y) \frac{\partial u_i}{\partial x} = \sum_{k=1}^n \Theta_{ik}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_i(y) \frac{\partial y_k}{\partial x} \right) \quad (i = 1, \dots, n), \\ x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) \quad (s = m + 1, \dots, n), \end{array} \right. \quad (1.3.7)$$

$$\left\{ \begin{array}{l} \frac{\partial u_i}{\partial t} + \lambda_i(y) \frac{\partial u_i}{\partial x} = \sum_{k=1}^n \Theta_{ik}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_i(y) \frac{\partial y_k}{\partial x} \right) \quad (i = 1, \dots, n), \\ x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) \quad (s = m + 1, \dots, n), \end{array} \right. \quad (1.3.8)$$

where

$$\Theta_{ik}(y) := \begin{cases} -\frac{l_{ik}(y)}{l_{ii}(y)}, & \text{if } k \neq i, \\ 0, & \text{if } k = i \end{cases}. \quad (1.3.9)$$

It is easy to see that  $\Theta := (\Theta_{ik})_{n \times n}$  is a  $C^1$  matrix function of  $y$ , and

$$\Theta(0) = 0. \quad (1.3.10)$$

In addition, if  $y \in \Sigma_1$ , from (1.3.7), one can check that

$$\begin{aligned} u_t(0, 0) &= y_t(0, 0) = -A(\varphi(0))\varphi_x(0), \\ u_t(0, L) &= y_t(0, L) = -A(\varphi(L))\varphi_x(L), \\ u_t(T, 0) &= y_t(T, 0) = -A(\psi(0))\psi_x(0), \\ u_t(T, L) &= y_t(T, L) = -A(\psi(L))\psi_x(L), \end{aligned} \quad (1.3.11)$$

provided  $u(0, x) = \varphi(x)$  and  $u(T, x) = \psi(x)$ . Therefore, the problem becomes whether there exist a positive constant  $C_4$  and two continuous nonnegative value functions  $\kappa(\varepsilon)$  and  $\Omega_1(\eta)$  satisfying (1.3.5)-(1.3.6), such that

(a) We can find a continuous iterative operator  $\mathfrak{F} : \Sigma_2 \rightarrow \Sigma_2$ :

$$u = \mathfrak{F}(y), \quad (1.3.12)$$

which satisfies

$$u \in \Sigma_3, \quad \text{if } y \in \Sigma_3; \quad (1.3.13)$$

(b) (1.3.7)-(1.3.8) hold under the hypothesis (1.1.22).

In fact, by Ascoli's theorem,  $\Sigma_3$  is a compact subset of  $\Sigma_2$  (see [19, Page 170]). Clearly  $\Sigma_3$  is convex, then (1.3.13) imply that

$$\mathfrak{F}(\Sigma_3) \subset \Sigma_3. \quad (1.3.14)$$

By the Schauder fixed-point theorem, we know  $\mathfrak{F}$  has a fixed point. Thus, when  $T > 0$  satisfies (1.1.22), there exists a  $C^1$  function  $u$  on the domain  $R(T)$  such that

$$\begin{cases} l_i(u)(u_t + \lambda_i(u)u_x) = 0, & (t, x) \in R(T), \\ t = 0 : u = \varphi(x), \\ t = T : u = \psi(x), \\ x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) & (s = m + 1, \dots, n). \end{cases} \quad (1.3.15)$$

This concludes the proof of Lemma 1.3.1, assuming the existence of operator  $\mathfrak{F}$ .  $\blacksquare$

### 1.3.2 Proof of Theorem 1.1.2

Assuming the existence of operator  $\mathfrak{F}$ , it is easy to see that we can find another continuous operator  $\mathfrak{B} : \Sigma_2 \rightarrow \Sigma_2$ , such that

(aa)

$$u = \mathfrak{B}(y), \quad (1.3.16)$$

which satisfy

$$u \in \Sigma_3, \quad \text{if } y \in \Sigma_3; \quad (1.3.17)$$

(bb) under the hypothesis (1.1.22), we have

$$\begin{cases} \frac{\partial u_i}{\partial t} + \lambda_i(y) \frac{\partial u_i}{\partial x} = \sum_{k=1}^n \Theta_{ik}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_i(y) \frac{\partial y_k}{\partial x} \right) & (i = 1, \dots, n), \end{cases} \quad (1.3.18)$$

$$\begin{cases} x = L : u_{\bar{r}} = G_{\bar{r}}(t, u_{m+1}, \dots, u_n) + H_{\bar{r}}(t) & (\bar{r} = 1, \dots, \bar{m}). \end{cases} \quad (1.3.19)$$

An argument similar to the one used in § 1.3.1 shows that  $\mathfrak{B}$  has a fixed point. When  $T > 0$  satisfies (1.1.22), there exists a  $C^1$  function  $u$  on the domain  $R(T)$  such that

$$\begin{cases} l_i(u)(u_t + \lambda_i(u)u_x) = 0, & (t, x) \in R(T), \\ t = 0 : u = \varphi(x), \\ t = T : u = \psi(x), \\ x = L : u_{\bar{r}} = G_{\bar{r}}(t, u_{m+1}, \dots, u_n) + H_{\bar{r}}(t) & (\bar{r} = 1, \dots, \bar{m}). \end{cases} \quad (1.3.20)$$

This concludes the proof of Theorem 1.1.2, assuming the existence of operator  $\mathfrak{F}$ . ■

**Remark 1.3.1** *The purposes of our iteration (1.3.7)-(1.3.8) are (1) to guarantee the boundary conditions without control (1.3.8) are always satisfied. (2) to use the smallness of  $\Theta$  in order to make (a)-(b) work.*

**Remark 1.3.2** *For the system (1.1.38) with (1.1.39), the iteration scheme is*

$$\frac{\partial u_i}{\partial t} + \lambda_i(y) \frac{\partial u_i}{\partial x} = \sum_{j=1}^n \Theta_{ij}(y) \left( \frac{\partial y_j}{\partial t} + \lambda_i(y) \frac{\partial y_j}{\partial x} \right) + \sum_{j=1}^n \frac{l_{ij}(y)}{l_{ii}(y)} F_j(y) \quad (1.3.21)$$

One can check, if (1.1.39) holds, by a similar manner, Theorem 1.1.1 and Theorem 1.1.2 are still valid. However, if (1.1.39) is not satisfied (for example,  $F(y) := Ay$ , where  $A \neq 0$  is a  $n \times n$  matrix with constant elements  $a_{ij}$  ( $i, j = 1, \dots, n$ ) and  $\|A\| := \max_{i,j=1,\dots,n} |a_{ij}|$  is large enough), through our iteration scheme (1.3.21), one can not expect that  $u = \mathfrak{F}(y) \in \Sigma_2$  and (1.3.13) hold.

## 1.4 The Existence of Operator $\mathfrak{F}$

In this section, we establish the existence of mapping  $\mathfrak{F}$  by a modified constructive method. Inspired from the iteration scheme (1.3.7)-(1.3.8), we can divide  $u$  into the following two systems:

$$\frac{\partial u_q}{\partial t} + \lambda_q(y) \frac{\partial u_q}{\partial x} = \sum_{k=1}^n \Theta_{qk}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_q(y) \frac{\partial y_k}{\partial x} \right) \quad (q = \bar{m} + 1, \dots, m), \quad (1.4.1)$$

and

$$\begin{cases} \frac{\partial u_p}{\partial t} + \lambda_p(y) \frac{\partial u_p}{\partial x} = \sum_{k=1}^n \Theta_{pk}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_p(y) \frac{\partial y_k}{\partial x} \right) & (p = 1, \dots, \bar{m}, m + 1, \dots, n), \\ x = 0 : u_s = G_s(t, u_1, \dots, u_m) + H_s(t) & (s = m + 1, \dots, n). \end{cases} \quad (1.4.2)$$

Then the operator  $\mathfrak{F}$  can be established by the following three steps:

### Step 1: Control of the system (1.4.1)

In this step, we will construct a  $C^1$  solution  $u_q = u_q(t, x)$  ( $q = \bar{m} + 1, \dots, m$ ) for the system (1.4.1), which satisfies

$$t = 0 : u_q = \varphi_q(x) \quad (q = \bar{m} + 1, \dots, m), \quad (1.4.3)$$

$$t = T : u_q = \psi_q(x) \quad (q = \bar{m} + 1, \dots, m). \quad (1.4.4)$$

Before proceeding further, let us claim some uniform estimates. Noting (1.1.22), if  $\varepsilon$  is small enough, we have

$$\int_0^T \inf_{0 \leq x \leq L} |\lambda_m(y(t, x))| dt > L. \quad (1.4.5)$$

Then from the above estimate, if  $\varepsilon$  is small enough, there exist  $T_1^{(y)}$  and  $T_2^{(y)}$  satisfying

$$\int_0^{T_1^{(y)}} \inf_{0 \leq x \leq L} |\lambda_m(y(t, x))| dt = \int_{T_2^{(y)}}^T \inf_{0 \leq x \leq L} |\lambda_m(y(t, x))| dt = \frac{L}{2}. \quad (1.4.6)$$

By (1.1.22), if  $\varepsilon$  is small enough, one can easily show that

$$0 < T_1^{(y)} < \frac{T - \delta}{2} < \frac{T}{2} < \frac{T + \delta}{2} < T_2^{(y)} < T, \quad (1.4.7)$$



in which we denote

$$\delta := \frac{1}{2} \left( T - L \max \left( \frac{1}{|\lambda_{\bar{m}}(0)|} + \frac{1}{\lambda_{m+1}(0)}, \frac{1}{|\lambda_m(0)|} \right) \right). \quad (1.4.8)$$

We now turn to the the construction of  $u_q(q = \bar{m} + 1, \dots, m)$ , which is based on the following three steps. The proof is similar to the one in [72], however, they are much more complicate in our situation.

(i) On the domain  $R_f = \{(t, x) | 0 \leq t \leq \frac{T+\delta}{2}, 0 \leq x \leq L\}$  (Not  $\hat{R}(y) = \{(t, x) | 0 \leq t \leq T_1^{(y)}, 0 \leq x \leq L\}$  as in [111]), we first consider a forward mixed initial-boundary problem (1.4.1) with the initial condition  $\varphi_q(q = \bar{m} + 1, \dots, m)$  and artificial boundary conditions

$$x = L : u_q = f_q(t) \quad (q = \bar{m} + 1, \dots, m), \quad (1.4.9)$$

where  $f_q$  are any given  $C^1$  functions of  $t$  with

$$\max_{q=\bar{m}+1, \dots, m} \|f_q\|_1 \leq \|\tilde{H}\|_1 \leq \varepsilon, \quad (1.4.10)$$

$$\max_{q=\bar{m}+1, \dots, m} \omega(\eta | \dot{f}_q) \leq \omega(\eta | \dot{\tilde{H}}) \quad (1.4.11)$$

and satisfying the following conditions of  $C^1$  compatibility at the point  $(t, x) = (0, L)$ :

$$\varphi_q(L) = f_q(0) \quad (q = \bar{m} + 1, \dots, m), \quad (1.4.12)$$

$$\sum_{j=1}^n a_{qj}(\varphi(L)) \varphi_{jx}(L) = -f_{qt}(0) \quad (q = \bar{m} + 1, \dots, m). \quad (1.4.13)$$

Hereafter, we denote

$$\tilde{H} := \{H_{m+1}, \dots, H_n, \bar{H}_1, \dots, \bar{H}_{\bar{r}}\}. \quad (1.4.14)$$

We point out here that, in our iteration,  $f_q(q = \bar{m} + 1, \dots, m)$  are considered as chosen: in particular, they do not depend on  $y \in \Sigma_2$ . This is crucial to guarantee the continuity of  $\mathfrak{F}$ . By Lemma 1.2.1, there exists a unique  $C^1$  solution  $u = u_f(t, x)$  on the domain  $R_f$  with

$$\|u_f\|_0 \leq C_5(\|\Theta\|_0 \|y\|_0 + \max_q \|f_q\|_0 + \max_q \|\varphi_q\|_0) \quad (1.4.15)$$

$$\begin{aligned} &\leq C_5(\|\Theta\|_0 C_4 + 2)\varepsilon, \\ \|u_f\|_1 &\leq C_6(\|\Theta\|_1 \|y\|_1 + \max_q \|f_q\|_1 + \max_q \|\varphi_q\|_1) \\ &\leq C_6(\|\Theta\|_1 \kappa(\varepsilon) + 2\varepsilon), \end{aligned} \quad (1.4.16)$$

and

$$\omega(\eta|u_{ft}) + \omega(\eta|u_{fx}) \leq C_7\Omega_2(\eta), \quad (1.4.17)$$

where

$$\begin{aligned} \Omega_2(\eta) = & \omega(\eta|\varphi_x) + \omega(\eta|\psi_x) + \omega(\eta|\tilde{H}_t) + \kappa^2(\varepsilon)(1 + \kappa(\varepsilon))\omega(\eta|\Theta_y) \\ & + \left(1 + \varepsilon + \tilde{G}(\varepsilon) + \|\Theta\|_1\kappa(\varepsilon)\right) \left(\eta + \left(1 + \varepsilon + \tilde{G}(\varepsilon) + \|\Theta\|_1\kappa(\varepsilon)\right)\omega(\eta|\Gamma^*)\right) \\ & + \kappa(\varepsilon)(1 + \kappa(\varepsilon)) \left(\varepsilon + \|\Theta\|_1\kappa(\varepsilon) + \tilde{G}(\varepsilon)\right)\omega(\eta|\Lambda_u) \\ & + \left(\varepsilon + \|\Theta\|_1\kappa(\varepsilon) + \tilde{G}(\varepsilon) + \|\Theta\|_1 + \|\Theta_y\|_0\kappa(\varepsilon)\right)\Omega_1(\eta), \end{aligned} \quad (1.4.18)$$

in which

$$\Lambda_u := \left\{ \frac{\partial \lambda_i}{\partial u_j} \right\}, \quad \Theta_y := \left\{ \frac{\partial \Theta_{ij}}{\partial y_k} \right\}, \quad \|\Theta\|_0 := \|\Theta(y(t, x))\|_{(C^0[R(T)])^n}, \quad \|\Theta\|_1 := \|\Theta(y(t, x))\|_{(C^1[R(T)])^n}, \quad (1.4.19)$$

$$\tilde{G}(\varepsilon) := \max_{\substack{\bar{r}=1, \dots, \bar{m} \\ s=m+1, \dots, n}} \sup_{\substack{0 \leq t \leq T \\ \|u\|_0 \leq C_4\varepsilon}} \left\{ \left| \frac{\partial G_s}{\partial t}(t, u_1, \dots, u_m) \right|, \left| \frac{\partial \bar{G}_{\bar{r}}}{\partial t}(t, u_{\bar{m}+1}, \dots, u_n) \right| \right\}, \quad (1.4.20)$$

$$\Gamma^* := \left\{ \frac{\partial G_s}{\partial u_r}, \frac{\partial \bar{G}_{\bar{r}}}{\partial u_s}, \frac{\partial G_s}{\partial t}, \frac{\partial \bar{G}_{\bar{r}}}{\partial t} \right\} \quad (i = 1, \dots, n; \bar{r} = 1, \dots, \bar{m}; s = m + 1, \dots, n). \quad (1.4.21)$$

Here, we point out that  $\omega(\eta|\Theta_y)$ ,  $\omega(\eta|\Lambda_u)$  and  $\omega(\eta|\Gamma^*)$  denote the modulus of continuity for  $\Theta_y$ ,  $\Lambda_u$  and  $\Gamma^*$  on the domain  $R_1 = \{0 \leq \|y\|_0 \leq C_4\varepsilon\}$ ,  $R_2 = \{0 \leq \|u\|_0 \leq C_4\varepsilon\}$  and  $R_3 = \{0 \leq t \leq T, 0 \leq \|u\|_0 \leq C_4\varepsilon\}$ , respectively. One can easily check that  $\tilde{G}(\varepsilon) \geq 0$  is a  $C^0$  function of  $\varepsilon$  and

$$\tilde{G}(0) = 0. \quad (1.4.22)$$

Then we can determine the value of  $u_f$  at  $x = 0$  as

$$x = 0: \quad u_f = a(t), \quad 0 \leq t \leq \frac{T + \delta}{2}. \quad (1.4.23)$$

and

$$\|a\|_0 \leq C_8(\|\Theta\|_0 C_4 + 2)\varepsilon, \quad (1.4.24)$$

$$\|a\|_1 \leq C_9(\|\Theta\|_1 \kappa(\varepsilon) + 2\varepsilon), \quad (1.4.25)$$

$$\omega(\eta|a_t) \leq C_{10}\Omega_2(\eta). \quad (1.4.26)$$

Here, we denote  $a(t) := (a_{\bar{m}+1}(t), \dots, a_m(t))$ . Evidently, By Lemma 1.2.1, one obtains that  $u_f$  and  $a$  depends continuously on  $y \in \Sigma_2$ .

(ii) Similarly, on the domain  $R_b = \{(t, x) | \frac{T-\delta}{2} \leq t \leq T, 0 \leq x \leq L\}$ , we consider the backward mixed initial-boundary value problem (1.4.1) with the final condition  $\psi_q(q = \bar{m} + 1, \dots, m)$  and artificial boundary conditions

$$x = 0 : u_q = g_q(t), \quad (1.4.27)$$

where  $g_q(q = \bar{m} + 1, \dots, m)$  are any given functions of  $t$  with

$$\max_{q=\bar{m}+1, \dots, m} \|g_q\|_{C^1} \leq \|\tilde{H}\|_1 \leq \varepsilon, \quad (1.4.28)$$

$$\max_{q=\bar{m}+1, \dots, m} \omega(\eta|\dot{g}_q) \leq \omega(\eta|\dot{\tilde{H}}), \quad (1.4.29)$$

such that the conditions of  $C^1$  compatibility are satisfied at the point  $(t, x) = (T, 0)$ . Similarly, in order to have the continuity of  $\mathfrak{F}$ ,  $g_q(q = \bar{m} + 1, \dots, m)$  are considered as chosen. By Lemma 1.2.1, there exists a unique  $C^1$  solution  $u = u_b(t, x)$  on the domain  $R_b$ , which allows us to determine the value of  $u_b$  at  $x = 0$  as

$$x = 0 : u_b = b(t), \quad \frac{T-\delta}{2} \leq t \leq T, \quad (1.4.30)$$

in which

$$\|b\|_0 \leq C_{11}(\|\Theta\|_0 C_4 + 2)\varepsilon, \quad (1.4.31)$$

$$\|b\|_1 \leq C_{12}(\|\Theta\|_1 \kappa(\varepsilon) + 2\varepsilon), \quad (1.4.32)$$

$$\omega(\eta|b_t) \leq C_{13}\Omega_2(\eta). \quad (1.4.33)$$

Here, we denote  $b(t) := (b_{\bar{m}+1}(t), \dots, b_m(t))$ . An argument similar to the one used in (i) shows  $b$  depends continuously on  $y \in \Sigma_2$ .

(iii) Let  $(\alpha, \beta) \in (C^2[\frac{T-\delta}{2}, \frac{T+\delta}{2}])^2$  be such that

$$\alpha\left(\frac{T-\delta}{2}\right) = 1, \alpha\left(\frac{T+\delta}{2}\right) = 0, \alpha_t\left(\frac{T-\delta}{2}\right) = 0, \alpha_t\left(\frac{T+\delta}{2}\right) = 0; \quad (1.4.34)$$

$$\beta\left(\frac{T-\delta}{2}\right) = 0, \beta\left(\frac{T+\delta}{2}\right) = 1, \beta_t\left(\frac{T-\delta}{2}\right) = 0, \beta_t\left(\frac{T+\delta}{2}\right) = 0. \quad (1.4.35)$$

Again, the scalar functions  $\alpha(t)$  and  $\beta(t)$  as the functions  $f_q$  and  $g_q$  ( $q = \bar{m}+1, \dots, m$ ) are considered as chosen. Next, Set

$$c(t) = \begin{cases} a(t), & 0 \leq t \leq \frac{T-\delta}{2}, \\ \alpha(t)a(t) + \beta(t)b(t), & \frac{T-\delta}{2} \leq t \leq \frac{T+\delta}{2}, \\ b(t), & \frac{T+\delta}{2} \leq t \leq T. \end{cases} \quad (1.4.36)$$

Obviously,  $c \in (C^1[0, T])^{m-\bar{m}}$ . On the other hand, it can easily be verified from (1.4.24)-(1.4.26) and (1.4.31)-(1.4.33) that

$$\|c\|_0 \leq C_{14}(\|\Theta\|_0 C_4 + 2)\varepsilon, \quad (1.4.37)$$

$$\|c\|_1 \leq C_{15}(\|\Theta\|_1 \kappa(\varepsilon) + \|\Theta\|_0 C_4 \varepsilon + 2\varepsilon), \quad (1.4.38)$$

$$\omega(\eta|c_t) \leq C_{16}\Omega_2(\eta), \quad (1.4.39)$$

and that  $c$  depends continuously on  $y \in \Sigma_2$ . Here, we denote  $c(t) := (c_{\bar{m}+1}(t), \dots, c_m(t))$ .

Since there are no zero eigenvalues, changing the status of  $t$  and  $x$ , system (1.4.1) can be equivalently rewritten as

$$\frac{\partial u_q}{\partial x} + \frac{1}{\lambda_q(y)} \frac{\partial u_q}{\partial t} = \sum_{k=1}^n \Theta_{qk}(y) \left( \frac{\partial y_k}{\partial x} + \frac{1}{\lambda_q(y)} \frac{\partial y_k}{\partial t} \right). \quad (1.4.40)$$

Now we consider the rightward mixed initial-boundary value problem for the system (1.4.40) with the initial condition

$$x = 0 : u_q = c(t), \quad 0 \leq t \leq T \quad (q = \bar{m} + 1, \dots, m), \quad (1.4.41)$$

and the following boundary conditions reduced from the final data  $\psi_q$ :

$$t = T : u_q = \psi_q(x) \quad (q = \bar{m} + 1, \dots, m). \quad (1.4.42)$$

We easily check that the mixed initial-boundary value problem (1.4.40)-(1.4.42) satisfies the condition of  $C^1$  compatibility at the point  $(t, x) = (T, 0)$ . Therefore, by Lemma 1.2.1, there exists a unique  $C^1$  solution  $u_q = \tilde{u}_q(t, x)$  ( $q = \bar{m} + 1, \dots, m$ ) of (1.4.40)-(1.4.42) with

$$\max_{q=\bar{m}+1, \dots, m} \|\tilde{u}_q\|_0 \leq C_{17}(\|\Theta\|_0 C_4 + 2)\varepsilon, \quad (1.4.43)$$

$$\max_{q=\bar{m}+1, \dots, m} \|\tilde{u}_q\|_1 \leq C_{18}(\|\Theta\|_1 \kappa(\varepsilon) + \|\Theta\|_0 C_4 \varepsilon + 2\varepsilon), \quad (1.4.44)$$

$$\max_{q=\bar{m}+1, \dots, m} \Omega(\eta|\tilde{u}_{qt}) + \Omega(\eta|\tilde{u}_{qx}) \leq C_{19}\Omega_2(\eta), \quad (1.4.45)$$

and  $\tilde{u}_q$  ( $q = \bar{m} + 1, \dots, m$ ) depends continuously on  $y$ . Next, let  $u_q = \tilde{u}_q(t, x)$  ( $q = \bar{m} + 1, \dots, m$ ). By uniqueness (see also [72] and [111]),  $u_q$  ( $q = \bar{m} + 1, \dots, m$ ) satisfies (1.4.3)-(1.4.4) and (1.4.43)-(1.4.45).

## Step 2: Control of the system (1.4.2)

Once the system (1.4.1) is controlled, we can substitute  $u_q(t, 0) = c_q(t)$  ( $q = \bar{m} + 1, \dots, m$ ) to (1.3.8), which leads system (1.4.2) to a new linearized system as

follows:

$$\begin{cases} \frac{\partial u_p}{\partial t} + \lambda_p(y) \frac{\partial u_p}{\partial x} = \sum_{k=1}^n \Theta_{pk}(y) \left( \frac{\partial y_k}{\partial t} + \lambda_p(y) \frac{\partial y_k}{\partial x} \right), (p = 1, \dots, \bar{m}, m+1, \dots, n), \\ x = 0 : u_s = \mathcal{G}_s(t, u_1, \dots, u_{\bar{m}}) + H_s(t), s = m+1, \dots, n, \end{cases} \quad (1.4.46)$$

where

$$\mathcal{G}_s(t, u_1, \dots, u_{\bar{m}}) := G_s(t, u_1, \dots, u_{\bar{m}}, c_{\bar{m}+1}, \dots, c_m), (s = m+1, \dots, n). \quad (1.4.47)$$

Fortunately, noting (1.3.11), one can easily check that the above system (1.4.46) satisfies the same  $C^1$  compatibility conditions at the point  $(t, x) = (0, 0)$  and  $(T, 0)$  as the original one (1.1.1) (see Remark 1.1.3), which guarantees the well-posedness of the system (1.4.46).

Combining the one-side control approach in [72, Chapter 3 on pages 47-52] and an analogous heuristic argument used in **Step 1**, the details of which we omit, one can show that under the hypothesis (1.1.22), there exists a  $C^1$  solution  $u_p = \tilde{u}_p(t, x)$  ( $p = 1, \dots, \bar{m}, m+1, \dots, n$ ) satisfying

$$t = 0 : \tilde{u}_p = \varphi_p(x) \quad (p = 1, \dots, \bar{m}, m, \dots, n), \quad (1.4.48)$$

$$t = T : \tilde{u}_p = \psi_p(x) \quad (p = 1, \dots, \bar{m}, m, \dots, n). \quad (1.4.49)$$

Moreover, we have

$$\max_p \|\tilde{u}_p\|_0 \leq C_{20} (\|\Theta\|_0 C_4 + 2) \varepsilon, \quad (1.4.50)$$

$$\max_p \|\tilde{u}_p\|_1 \leq C_{21} (\|\Theta\|_1 \kappa(\varepsilon) + \|\Theta\|_0 C_4 \varepsilon + 2\varepsilon + \tilde{G}(\varepsilon)), \quad (1.4.51)$$

$$\max_p \omega(\eta | \tilde{u}_{pt}) + \omega(\eta | \tilde{u}_{px}) \leq C_{22} \Omega_2(\eta), \quad (1.4.52)$$

and  $\tilde{u}_p$  ( $p = 1, \dots, \bar{m}, m, \dots, n$ ) depends continuously on  $y \in \Sigma_2$ .

### Step 3: Proof of the existence and continuity of $\mathfrak{F}$

Let

$$C_{23} = \max\{C_{17}, C_{20}\}, \quad (1.4.53)$$

$$C_{24} = \max\{C_{18}, C_{21}\}, \quad (1.4.54)$$

$$C_{25} = \max\{C_{19}, C_{22}\}. \quad (1.4.55)$$

We then choose

$$C_4 = 3C_{23}, \quad (1.4.56)$$

$$\kappa(\varepsilon) = 5C_{24}\varepsilon + 2C_{24}\tilde{G}(\varepsilon), \quad (1.4.57)$$

and

$$\begin{aligned} \Omega_1(\eta) = & 2C_{25} \left( \omega(\eta|\dot{\varphi}) + \omega(\eta|\dot{\psi}) + \omega(\eta|\tilde{H}) + 2\omega(\eta|\Theta_y) \right) \\ & + (2C_{25} + 1) \left( \eta + \left( 1 + \frac{1}{2C_{25}} \right) \omega(\eta|\Gamma^*) \right) + 2\omega(\eta|\Lambda_u), \end{aligned} \quad (1.4.58)$$

by (1.1.4), if  $0 < \varepsilon \ll 1$  is small enough, one gets

$$\|\Theta\|_0 \leq \frac{1}{2C_4}, \quad \|\Theta\|_1 \leq \frac{1}{2C_{24}}, \quad (1.4.59)$$

and

$$\kappa(\varepsilon) < 1, \quad C_{25} \left( \varepsilon + \|\Theta\|_1 \kappa(\varepsilon) + \tilde{G}(\varepsilon) + \|\Theta\|_1 + \|\Theta_y\|_0 \kappa(\varepsilon) \right) \leq \frac{1}{2}. \quad (1.4.60)$$

Let

$$u(t, x) = (\tilde{u}_1, \dots, \tilde{u}_n). \quad (1.4.61)$$

Then noting (1.4.43)-(1.4.45) and (1.4.50)-(1.4.52), under the hypothesis (1.1.22), we have

$$\|u\|_0 \leq C_4\varepsilon, \quad \|u\|_1 \leq \kappa(\varepsilon), \quad (1.4.62)$$

and

$$\omega(\eta|u_t) + \omega(\eta|u_x) \leq \Omega_1(\eta). \quad (1.4.63)$$

Therefore, combining (1.4.48)-(1.4.49) and (1.4.3)-(1.4.4), we show that

$$u \in \Sigma_3. \quad (1.4.64)$$

From the construction of the operator  $\mathfrak{F}$  in **Step 1-Step 2**, one can easily verify it is continuous on  $y$ . The construction of  $\mathfrak{F}$  is now completed.  $\blacksquare$

## 1.5 Appendix: Proof of Lemma 1.2.1

In this section, we give a detail proof of Lemma 1.2.1. For the case of coupled system of wave equations, one can refer to [56].

Set  $w = (w_1, \dots, w_n), p = (p_1, \dots, p_n)$ , where

$$w_i = u_{ix} \quad (i = 1, \dots, n), \quad (1.5.1)$$

$$p_i = u_{it} \quad (i = 1, \dots, n). \quad (1.5.2)$$

Obviously, noting the equations (1.2.4), one can show that

$$p_i + \lambda_i(t, x)w_i = \sum_{k=1}^{\tilde{n}} b_{ik}(t, x) \left( \frac{\partial c_k(t, x)}{\partial t} + \lambda_i(t, x) \frac{\partial c_k(t, x)}{\partial x} \right). \quad (1.5.3)$$

Hence, by straightforward computations, we have

$$\|p\|_0 \leq \tilde{C}_1(K) \left( \|w\|_0 + \|B\|_0 \|c\|_1 \right) \quad (1.5.4)$$

and

$$\omega(\eta|p) \leq \tilde{C}_2(K) \left( (\|w\|_0 + \|B\|_1 \|c\|_1) \eta + \omega(\eta|w) + \|B\|_0 (\omega(\eta|c_t) + \omega(\eta|c_x)) \right). \quad (1.5.5)$$

Here and in what follows, we assume that  $\tilde{C}_n(K) (n \in \mathbb{Z}^+)$  are positive real numbers depending on  $K$ .

Thus, if we want to prove that  $\|u\|_0, \|u\|_1$  and  $\omega(\eta|u_t) + \omega(\eta|u_x)$  satisfy (1.2.8)-(1.2.9) and (1.2.12), respectively, it suffices to show (1.2.8),

$$\|w\|_0 \leq \tilde{C}_3(T, K) \left( \|\varphi\|_1 + \|H\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 \right) \quad (1.5.6)$$

and

$$\begin{aligned} \omega(\eta|w) &\leq \tilde{C}_4(T, K) \left( \omega(\eta|\varphi_x) + \|B\|_1 (\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1 (\omega(\eta|B_t) + \omega(\eta|B_x)) \right) \\ &+ (1 + \|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1) (\eta + (1 + \|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 \\ &+ \|H\|_1) \omega(\eta|\Gamma)) + (\|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1) \omega(\eta|\Lambda_x) + \omega(\eta|H_t) \end{aligned} \quad (1.5.7)$$

hold.

Utilizing the decomposition of waves (see [61] and [111]), one gets

$$\frac{du_i}{d_i t} = \beta_i(t, x, u), \quad (1.5.8)$$

$$\frac{dw_i}{d_i t} = \gamma_i(t, x, u) + \sum_{j=1}^n \gamma_{ij}(t, x) w_j, \quad (1.5.9)$$

where

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(t, x) \frac{\partial}{\partial x} \quad (1.5.10)$$

denotes the directional derivative along the  $i$ -th characteristic,

$$\beta_i(t, x, u) = \sum_{k=1}^{\tilde{n}} b_{ik}(t, x) \frac{dc_k(t, x)}{d_i t}, \quad (1.5.11)$$

and

$$\begin{aligned} \gamma_i(t, x) = \sum_{k=1}^{\tilde{n}} \left[ \frac{d}{d_i t} \left( b_{ik}(t, x) \frac{\partial c_k(t, x)}{\partial x} \right) - \frac{\partial b_{ik}(t, x)}{\partial t} \frac{\partial c_k(t, x)}{\partial x} \right. \\ \left. + \frac{\partial b_{ik}(t, x)}{\partial x} \frac{\partial c_k(t, x)}{\partial t} + b_{ik}(t, x) \frac{\partial \lambda_i(t, x)}{\partial x} \frac{\partial c_k(t, x)}{\partial x} \right], \end{aligned} \quad (1.5.12)$$

$$\gamma_{ij}(t, x) = -\frac{\partial \lambda_i(t, x)}{\partial x} \delta_{ij}. \quad (1.5.13)$$

Let

$$T_1 = \min_{i=1, \dots, n} \inf_{(t, x) \in R(T)} \frac{L}{|\lambda_i(t, x)|}. \quad (1.5.14)$$

Next, we will prove (1.2.8), (1.5.6)-(1.5.7) on the domain

$$R(T_1) = \{(t, x) | 0 \leq t \leq T_1, 0 \leq x \leq L\}. \quad (1.5.15)$$

then claim they still hold on the whole domain  $R(T)$ .

**Step A (estimate of  $\|u\|_0$ ):** For any given point  $(t, x) \in R(T_1)$ , we draw down the  $r$ -th characteristic  $\xi = f_r(\tau; t, x)$  ( $r = 1, \dots, m$ ):

$$\begin{cases} \frac{df_r(\tau; t, x)}{d\tau} = \lambda_r(\tau, f_r(\tau; t, x)), \\ \tau = t : \xi = x. \end{cases} \quad (1.5.16)$$

passing through  $(t, x)$ . Noting (1.2.5) and (1.5.14), there are only two possibilities:

(a) This  $r$ -th characteristic intersects the  $x$ -axis at a point  $(0, \alpha_r)$ . Integrating the  $r$ -th equation in (1.5.8) along this characteristic from  $(0, \alpha_r)$  to  $(t, x)$ , noting (1.5.11),



we get

$$|u_r(t, x)| \leq \|u(0, \cdot)\|_0 + 2\|B\|_0\|c\|_0, \quad (1.5.17)$$

(b) This  $r$ -characteristic intersects  $x = L$  at the point  $(t_r, L)$ , and all  $s$ -th characteristics passing through  $(t_r, L)$  intersect the  $x$ -axis at point  $(0, \beta_s)$  ( $s = m + 1, \dots, n$ ) respectively. Then similarly to (1.5.17), we have

$$|u_r(t, x)| \leq |u_r(t_r, L)| + 2\|B\|_0\|c\|_0. \quad (1.5.18)$$

Moreover, by means of the boundary conditions (1.1.8) and noting (1.1.9), it is easy to get

$$|u(t_r, L)| \leq \tilde{C}_5(K) \max_{m+1 \leq s \leq n} |u_s(t_r, L)| + \|H\|_0, \quad (1.5.19)$$

without loss of generality, we may suppose  $\tilde{C}_5(K) > 1$ .

Similar to (1.5.17), integrating the  $s$ -th equation in (1.5.8) along the  $s$ -th characteristic from  $(0, \beta_s)$  to  $(t_r, L)$  gives

$$|u_s(t_r, L)| \leq \|u(0, \cdot)\|_0 + 2\|B\|_0\|c\|_0. \quad (1.5.20)$$

The combination (1.5.18) to (1.5.20) leads to

$$|u_r(t, x)| \leq \tilde{C}_6(K) \left( \|u(0, \cdot)\|_0 + \|H\|_0 + \|B\|_0\|c\|_0 \right). \quad (1.5.21)$$

Thus, it follows from (1.5.17) and (1.5.21) that

$$|u_r(t, x)| \leq \tilde{C}_7(K) \left( \|u(0, \cdot)\|_0 + \|H\|_0 + \|B\|_0\|c\|_0 \right). \quad (1.5.22)$$

Similar estimates can be obtained for  $u_s(t, x)$  ( $s = m + 1, \dots, n$ ). Hence, we have

$$u(t) \leq \tilde{C}_8(K) (\|u(0, \cdot)\|_0 + \|H\|_0 + \|B\|_0\|c\|_0) \quad (1.5.23)$$

in which we may assume that  $\tilde{C}_8(K) > 1$  and in what follows, we denote

$$u(\tau) := \sup_{0 \leq t \leq \tau} \|u(t, \cdot)\|_0, \quad (1.5.24)$$

$$w(\tau) := \sup_{0 \leq t \leq \tau} \|w(t, \cdot)\|_0. \quad (1.5.25)$$

Taking  $u(T_1, x)$  as the initial data on  $t = T_1$ , and repeating the previous procedure,

we obtain

$$\begin{aligned} u(t) &\leq \tilde{C}_8(K) (\|u(T_1, \cdot)\|_0 + \|H\|_0 + \|B\|_0 \|c\|_0) \\ &\leq \tilde{C}_8^2(K) (\|u(0, \cdot)\|_0 + \|H\|_0 + \|B\|_0 \|c\|_0), \quad T_1 \leq t \leq 2T_1 \end{aligned}$$

Repeating this procedure at most  $M = \left\lceil \frac{T}{T_1} \right\rceil + 1$  times, we get

$$\|u\|_0 \leq \max_{t \in [0, T]} u(t) \leq \tilde{C}_8^M(K) (\|u(0, \cdot)\|_0 + \|H\|_0 + \|B\|_0 \|c\|_0), \quad \forall t \in [0, T].$$

This concludes the proof of (1.2.8).

**Step B (estimate of  $\|w\|_0$ ):** As before, for any given point  $(t, x) \in R(T_1)$ , there are still two possibilities for the  $r$ -th characteristic ( $r = 1, \dots, m$ ).

In case (a), integrating the  $r$ -th equation in (1.5.9) along this  $r$ -th characteristic from  $(0, \alpha_r)$  to  $(t, x)$  yields

$$w_r(t, x) - w_r(0, \alpha_r) = \int_0^t \gamma_r(\tau, f_r(\tau; t, x)) + \sum_{j=1}^n \gamma_{rj}(\tau, f_r(\tau; t, x)) w_j d\tau, \quad (1.5.26)$$

then by (1.5.12)-(1.5.13) and (1.5.25), one can show that

$$|w_r(t, x)| \leq \|w(0, \cdot)\|_0 + \tilde{C}_9(K) \left( \|B\|_1 \|c\|_1 + \int_0^t w(\tau) d\tau \right). \quad (1.5.27)$$

In case (b), similar to (1.5.19), we have

$$|w_r(t, x)| \leq |w_r(t_r, L)| + \tilde{C}_{10}(K) \left( \|B\|_1 \|c\|_1 + \int_{t_r}^t w(\tau) d\tau \right). \quad (1.5.28)$$

In order to estimate  $|w_r(t_r, L)|$ , we seek the boundary conditions satisfied by  $w_r$  on  $x = L$ . Differentiating (1.1.8) with respect to  $t$ , we get

$$x = L : \quad \frac{\partial u_r}{\partial t} = \frac{\partial G_r}{\partial t} + \sum_{s=m+1}^n \frac{\partial G_r}{\partial u_s} \frac{\partial u_s}{\partial t} + H_{rt} \quad (r = 1, \dots, m). \quad (1.5.29)$$

By (1.2.4), together with (1.5.29), we obtain

$$\begin{aligned} w_r(t, L) &= \frac{1}{\lambda_r(t, L)} \left( \sum_{k=1}^{\tilde{n}} b_{rk} \left( \frac{\partial c_k}{\partial t} + \lambda_r \frac{\partial c_k}{\partial x} \right) - \frac{\partial G_r}{\partial t} - H_{rt} \right. \\ &\quad \left. + \sum_{s=m+1}^n \frac{\partial G_r}{\partial u_s} \left( \lambda_s w_s - \sum_{k=1}^{\tilde{n}} b_{sk} \left( \frac{\partial c_k}{\partial t} + \lambda_s \frac{\partial c_k}{\partial x} \right) \right) \right) (t, L). \end{aligned} \quad (1.5.30)$$

Then by noting (1.1.9), one can easily see that

$$|w_r(t_r, L)| \leq \tilde{C}_{11}(K) \left( \max_{m+1 \leq s \leq n} |w_s(t_r, L)| + \|H_t\|_0 + \|B\|_0 \|c\|_1 + \|G_t\|_0 \right), \quad (1.5.31)$$

in which we may assume that  $\tilde{C}_{11}(K) \geq 1$ . Integrating the  $s$ -th equation in (1.5.9) along the corresponding  $s$ -th characteristic from  $(0, \beta_r)$  to  $(t_r, L)$  gives

$$|w_s(t_r, L)| \leq \|w(0, \cdot)\|_0 + \tilde{C}_{12}(K) \left( \|B\|_1 \|c\|_1 + \int_0^{t_r} w(\tau) d\tau \right). \quad (1.5.32)$$

Combining (1.5.28), (1.5.31)-(1.5.32) yields

$$|w_r(t, x)| \leq \tilde{C}_{13}(K) \left( \|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|H\|_1 + \|G_t\|_0 + \int_0^t w(\tau) d\tau \right), \quad (1.5.33)$$

Similar estimates can be obtained for  $w_s(t, x)$  ( $s = m + 1, \dots, n$ ). Hence we get

$$w(t) \leq \tilde{C}_{14}(K) \left( \|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|H\|_1 + \|G_t\|_0 + \int_0^t w(\tau) d\tau \right). \quad (1.5.34)$$

Then, using Gronwall's inequality, for  $0 \leq t \leq T_1$ , one shows

$$w(t) \leq \tilde{C}_{15}(K) (\|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|H\|_1 + \|G_t\|_0), \quad (1.5.35)$$

we assume here that  $\tilde{C}_{15}(K) \geq 1$ .

Repeating above procedures, for  $0 \leq t \leq T$ , we have

$$\|w\|_0 \leq \max_{t \in [0, T]} w(t) \leq \tilde{C}_{16}(T, K) (\|\varphi\|_1 + \|B\|_1 \|c\|_1 + \|H\|_1 + \|G_t\|_0), \quad (1.5.36)$$

which finishes the proof of (1.5.6).

**Step C (estimate of  $\omega(\eta|w)$ ):** By (1.2.5), when  $i = 1, \dots, m$ , let us consider the  $i$ -th characteristic from the point  $(0, L)$  which divides the domain  $R(T_1)$  into  $R_i^+$  and  $R_i^0$ ; Similarly, when  $i = m + 1, \dots, n$ , let us consider the  $i$ -th characteristic from  $(0, 0)$  which divides  $R(T_1)$  into  $R_i^-$  and  $R_i^0$ . For any given  $(t, x)$  in  $R(T_1)$ , let us consider the  $i$ -th characteristic  $\xi_i = f_i(\tau; t, x)$  which intersects the boundary of  $R(T_1)$  (i.e.  $x = 0, x = L$  and  $t = 0$ ) on  $(\tau_k(t, x), \xi_k(t, x))$ , then we have

$$\begin{cases} \tau_i(t, x) = 0, \xi_i(t, x) = f_i(0; t, x), & \text{if } (t, x) \in R_i^0, (i = 1, \dots, n), \\ \xi_i(t, x) = L, f_i(\tau_i(t, x); t, x) = L, & \text{if } (t, x) \in R_i^+, (i = 1, \dots, m), \\ \xi_i(t, x) = 0, f_i(\tau_i(t, x); t, x) = 0, & \text{if } (t, x) \in R_i^-, (i = m + 1, \dots, n). \end{cases} \quad (1.5.37)$$

Before claiming (1.5.7), we need the following estimates, which can be found in many references (see, for instance, [84, Lemma 3.3 on pages 50-51]):

**Proposition 1.5.1** *On the domain  $R(T_1)$ , for any characteristic  $\xi = f_i(\tau; t, x)$ , we have*

$$\left\| \frac{\partial f_i}{\partial x} \right\|_0, \left\| \frac{\partial f_i}{\partial t} \right\|_0, \|\tau_{it}\|_0, \|\tau_{ix}\|_0, \|\xi_{it}\|_0, \|\xi_{ix}\|_0 < \tilde{C}_{17}(K) < +\infty \quad (i = 1, \dots, n). \quad (1.5.38)$$

Integrating the  $r$ -th equation in (1.5.9) along the  $r$ -th characteristic  $\xi_r = f_r(\tau; t, x)$ , we obtain

$$w_r(t, x) = \bar{\psi}_r(\tau_r(t, x), \xi_r(t, x)) + \int_{\tau_r(t, x)}^t (\gamma_r + \sum_{j=1}^n \gamma_{rj} w_j)(\tau, f_r(\tau; t, x)) d\tau, \quad (1.5.39)$$

where

$$\bar{\psi}_r(\tau_r(t, x), \xi_r(t, x)) = \begin{cases} w_r(\tau_r(t, x), L), & \forall (t, x) \in R_r^+, \\ w_r(0, \xi_r(t, x)), & \forall (t, x) \in R_r^0. \end{cases} \quad (1.5.40)$$

Obviously, by the basic proposition of the modulus of continuity (see, for example, [84, Pages 58-59 and 88-90]), one obtains

$$\omega(\eta|w_r(0, \xi_r(t, x))) \leq \tilde{C}_{18}(K)\omega(\eta|w_r(0, \cdot)). \quad (1.5.41)$$

Let

$$\omega(\tau, \eta|w) = \sup_{\substack{(t_1, x_1), (t_2, x_2) \in R(\tau) \\ |t_1 - t_2| \leq \eta, |x_1 - x_2| \leq \eta}} |w(t_1, x_1) - w(t_2, x_2)|. \quad (1.5.42)$$

It is easy to see that

$$\omega(T_1, \eta|w) = \sup_{0 \leq \tau \leq T_1} \omega(\tau, \eta|w). \quad (1.5.43)$$

Utilizing Proposition 1.5.1, for any given  $0 \leq \tau \leq T_1$ , by (1.5.30), one gets

$$\begin{aligned} & \omega(\tau, \eta|w_r(\tau_r(\cdot, \cdot), L)) \leq \tilde{C}_{19}(K)\omega(\tau, \eta|w_r(\cdot, L)) \\ & \leq \tilde{C}_{20}(K) \left( (\|u\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1)\eta + (1 + \|u\|_1)(1 + \|u\|_1 + \|B\|_1 \|c\|_1)\omega(\eta|\Gamma) \right. \\ & \quad \left. + \|B\|_0 (\omega(\eta|c_t) + \omega(\eta|c_x)) + \omega(\eta|H_t) + \max_{s=m+1, \dots, n} \omega(\tau, \eta|w_s(\cdot, L)) \right) \\ & \leq \tilde{C}_{21}(K) \left( (1 + \|u\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|u\|_1 + \|B\|_1 \|c\|_1)\omega(\eta|\Gamma)) \right. \\ & \quad \left. + \|B\|_0 (\omega(\eta|c_t) + \omega(\eta|c_x)) + \omega(\eta|H_t) + \max_{s=m+1, \dots, n} \omega(\tau, \eta|w_s(\cdot, L)) \right). \end{aligned} \quad (1.5.44)$$

For any given point  $(t, L)$  on  $x = L$ , integrating the  $s$ -th equation in (1.5.9) along the  $s$ -th characteristic  $\xi_s = f_s(\tau; t, x)$  from  $(t, L)$  to  $(0, \xi_s)$ , we obtain

$$w_s(t, L) = w_s(0, \xi_s(t, L)) + \int_0^t \gamma_s(\tau, f_s(\tau; t, x)) + \sum_{j=1}^n \gamma_{sj}(\tau, f_s(\tau; t, x)) w_j d\tau. \quad (1.5.45)$$

Next, we will use (1.5.45) in order to estimate  $\omega(\tau, \eta | w_s(\cdot, L))$ , where  $0 \leq \eta \leq T_1$ . Setting

$$\begin{aligned} \mathfrak{L}_{ijk}(t, x) := & -\frac{\partial b_{ik}(t, x)}{\partial t} \frac{\partial c_k(t, x)}{\partial x} + \frac{\partial b_{ik}(t, x)}{\partial x} \frac{\partial c_k(t, x)}{\partial t} \\ & + b_{ik}(t, x) \frac{\partial \lambda_i(t, x)}{\partial x} \frac{\partial c_k(t, x)}{\partial x} - \frac{\partial \lambda_i(t, x)}{\partial x} \delta_{ij} w_j(t, x) \end{aligned} \quad (1.5.46)$$

in which  $i, j = 1, \dots, n$  and  $k = 1, \dots, \tilde{n}$ . For any given  $0 \leq \tau \leq T_1$ , by Proposition 1.5.1, one gets

$$\begin{aligned} & \omega(\tau, \eta | \mathfrak{L}_{ijk}(\tau, f_i(\tau; t, L))) \\ & \leq \tilde{C}_{22}(K) \omega(\tau, \eta | \mathfrak{L}_{ijk}(\tau, \xi)) \\ & \leq \tilde{C}_{23}(K) \left( (\|u\|_1 + \|B\|_1 \|c\|_1) (\eta + \omega(\eta | \Lambda_x)) + \|B\|_1 (\omega(\eta | c_t) + \omega(\eta | c_x)) \right. \\ & \quad \left. + \|c\|_1 (\omega(\eta | B_t) + \omega(\eta | B_x)) + \omega(\tau, \eta | w) \right). \end{aligned} \quad (1.5.47)$$

Thus, noting (1.5.47), for any given  $0 \leq \bar{t} \leq T_1$ , we have

$$\begin{aligned} & \omega(\bar{t}, \eta | \int_0^{\bar{t}} \mathfrak{L}_{ijk}(\tau, f(\tau; t, L)) d\tau) \\ & \leq \tilde{C}_{24}(K) \left( (1 + \|u\|_1 + \|B\|_1 \|c\|_1) \eta + (\|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta | \Lambda_x) \right. \\ & \quad \left. + \|B\|_1 (\omega(\eta | c_t) + \omega(\eta | c_x)) + \|c\|_1 (\omega(\eta | B_t) + \omega(\eta | B_x)) + \int_0^{\bar{t}} \omega(\tau, \eta | w) d\tau \right). \end{aligned} \quad (1.5.48)$$

By (1.5.45), it is easy to show that

$$\begin{aligned} & \omega(\bar{t}, \eta | w_s(\cdot, L)) \\ & \leq \tilde{C}_{25}(K) \left( \omega(\eta | w(0, \cdot)) + (1 + \|u\|_1 + \|B\|_1 \|c\|_1) \eta + (\|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta | \Lambda_x) \right. \\ & \quad \left. + \|B\|_1 (\omega(\eta | c_t) + \omega(\eta | c_x)) + \|c\|_1 (\omega(\eta | B_t) + \omega(\eta | B_x)) + \int_0^{\bar{t}} \omega(\tau, \eta | w) d\tau \right). \end{aligned} \quad (1.5.49)$$

From (1.5.44), we find that

$$\begin{aligned}
 & \omega(\bar{t}, \eta|w_r(\tau_r(\cdot, \cdot), L)) \\
 & \leq \tilde{C}_{26}(K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1(\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1(\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
 & \quad + (1 + \|u\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Gamma)) \\
 & \quad \left. + (\|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Lambda_x) + \omega(\eta|H_t) + \int_0^{\bar{t}} \omega(\tau, \eta|w)d\tau \right).
 \end{aligned} \tag{1.5.50}$$

Combining (1.5.40)-(1.5.41) and (1.5.50), one has

$$\begin{aligned}
 & \omega(\bar{t}, \eta|\bar{\psi}_r(\tau_r(\cdot, \cdot), \xi_r(\cdot, \cdot))) \\
 & \leq \tilde{C}_{27}(K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1(\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1(\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
 & \quad + (1 + \|u\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Gamma)) \\
 & \quad \left. + (\|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Lambda_x) + \omega(\eta|H_t) + \int_0^{\bar{t}} \omega(\tau, \eta|w)d\tau \right).
 \end{aligned} \tag{1.5.51}$$

Next, noting (1.5.39) and (1.5.48) together with Proposition 1.5.1, one can show that

$$\begin{aligned}
 & \omega(\bar{t}, \eta|w_r) \\
 & \leq \tilde{C}_{28}(K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1(\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1(\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
 & \quad + (1 + \|u\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Gamma)) \\
 & \quad \left. + (\|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Lambda_x) + \omega(\eta|H_t) + \int_0^{\bar{t}} \omega(\tau, \eta|w)d\tau \right).
 \end{aligned} \tag{1.5.52}$$

Similar estimates can be obtained for  $w_s(t, x)$  ( $s = m + 1, \dots, n$ ). Thus,

$$\begin{aligned}
 & \omega(\bar{t}, \eta|w) \\
 & \leq \tilde{C}_{29}(K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1(\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1(\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
 & \quad + (1 + \|u\|_1 + \|B\|_1\|c\|_1 + \|G_t\|_0 + \|H\|_1)(\eta + (1 + \|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Gamma)) \\
 & \quad \left. + (\|u\|_1 + \|B\|_1\|c\|_1)\omega(\eta|\Lambda_x) + \omega(\eta|H_t) + \int_0^{\bar{t}} \omega(\tau, \eta|w)d\tau \right),
 \end{aligned} \tag{1.5.53}$$

which, by Gronwall's inequality, leads to

$$\begin{aligned}
& \omega(T_1, \eta|w) \\
& \leq \tilde{C}_{30}(K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1 (\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1 (\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
& \quad + (1 + \|u\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1) (\eta + (1 + \|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta|\Gamma)) \\
& \quad \left. + (\|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta|\Lambda_x) + \omega(\eta|H_t) \right).
\end{aligned} \tag{1.5.54}$$

Let

$$R_i := \{(t, x) | t_{i-1} \leq t \leq t_i, 0 \leq x \leq L\}. \tag{1.5.55}$$

where,  $i = 1, \dots, M$  and

$$t_i - t_{i-1} \leq T_1, i = 1, \dots, M \quad (t_0 = 0, t_M = T). \tag{1.5.56}$$

Obviously, one has

$$R(T) \subseteq \bigcup_{i=1}^M R_i. \tag{1.5.57}$$

Before claiming (1.5.7), we denote that

$$\omega_i(\eta|w) := \sup_{\substack{(t_1, x_1), (t_2, x_2) \in R_i \\ |t_1 - t_2| \leq \eta, |x_1 - x_2| \leq \eta}} |w(t_1, x_1) - w(t_2, x_2)|. \tag{1.5.58}$$

Clearly,  $\omega_1(\eta|w) \leq \omega(T_1, \eta|w)$  and

$$\omega(\eta|w) = \omega(T, \eta|w) \leq \sum_i^M \omega_i(\eta|w). \tag{1.5.59}$$

Then repeating the procedures in **Step C** at most  $M$  times, we can get similar estimates (like (1.5.54)) for  $\omega_i(\eta|w)$  ( $i = 2, \dots, M$ ). Therefore, noting (1.5.59), one obtains

$$\begin{aligned}
\omega(\eta|w) & \leq \tilde{C}_{31}(T, K) \left( \omega(\eta|w(0, \cdot)) + \|B\|_1 (\omega(\eta|c_t) + \omega(\eta|c_x)) + \|c\|_1 (\omega(\eta|B_t) + \omega(\eta|B_x)) \right. \\
& \quad + (1 + \|u\|_1 + \|B\|_1 \|c\|_1 + \|G_t\|_0 + \|H\|_1) (\eta + (1 + \|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta|\Gamma)) \\
& \quad \left. + (\|u\|_1 + \|B\|_1 \|c\|_1) \omega(\eta|\Lambda_x) + \omega(\eta|H_t) \right).
\end{aligned} \tag{1.5.60}$$

Applying (1.5.6) to (1.5.60) concludes the proof of (1.5.7).

**Step D (Continuity on  $(\varphi, \Lambda, B, c)$ ):** For any  $(\varphi^{(1)}, \Lambda^{(1)}, B^{(1)}, c^{(1)}), (\varphi^{(2)}, \Lambda^{(2)}, B^{(2)}, c^{(2)}) \in (C^1[R(T)])^n \times (C^1[R(T)])^n \times (C^1[R(T)])^{n \times \tilde{n}} \times (C^1[R(T)])^n$ , we put  $u^{(1)}(t, x; \varphi^{(1)}, \Lambda^{(1)}, B^{(1)}, c^{(1)})$  and  $u^{(2)}(t, x; \varphi^{(2)}, \Lambda^{(2)}, B^{(2)}, c^{(2)})$  be the  $C^1$  solution for the corresponding mixed initial-boundary problems (1.2.4), (1.1.7)-(1.1.8) and (1.1.10). Taken

$$\begin{aligned} u^*(t, x) &:= u^{(1)}(t, x; \varphi^{(1)}, \Lambda^{(1)}, B^{(1)}, c^{(1)}) - u^{(2)}(t, x; \varphi^{(2)}, \Lambda^{(2)}, B^{(2)}, c^{(2)}), \\ \varphi^*(x) &:= \varphi^{(1)}(x) - \varphi^{(2)}(x), \\ \Lambda^*(t, x) &:= \Lambda^{(1)}(t, x) - \Lambda^{(2)}(t, x), \\ B^*(t, x) &:= B^{(1)}(t, x) - B^{(2)}(t, x), \\ c^*(t, x) &:= c^{(1)}(t, x) - c^{(2)}(t, x). \end{aligned}$$

Next, we will show that

$$\lim_{\|\varphi^*\|_1 + \|\Lambda^*\|_1 + \|B^*\|_1 + \|c^*\|_1 \rightarrow 0} \|u^*\|_1 = 0. \quad (1.5.61)$$

Since  $u^*$  satisfies the following mixed initial-boundary problems

$$\left\{ \begin{aligned} \frac{\partial u_i^*}{\partial t} + \lambda_i^{(1)}(t, x) \frac{\partial u_i^*}{\partial x} &= \lambda_i^*(t, x) \frac{\partial u_i^{(2)}}{\partial x} + \sum_{k=1}^n b_{ik}^*(t, x) \left( \frac{\partial c_k^{(1)}}{\partial t} + \lambda_i^{(1)}(t, x) \frac{\partial c_k^{(1)}}{\partial x} \right) \\ &\quad + \sum_{k=1}^n b_{ik}^{(2)}(t, x) \left( \frac{\partial c_k^*}{\partial t} + \lambda_i^{(1)}(t, x) \frac{\partial c_k^*}{\partial x} + \lambda_i^*(t, x) \frac{\partial c_k^{(2)}}{\partial x} \right) \quad (i = 1, \dots, n), \\ x = 0 : u_s^* &= G_s(t, u_1^{(1)}, \dots, u_m^{(1)}) - G_s(t, u_1^{(2)}, \dots, u_m^{(2)}) \quad (s = m+1, \dots, n), \\ x = L : u_r^* &= G_r(t, u_{m+1}^{(1)}, \dots, u_n^{(1)}) - G_r(t, u_{m+1}^{(2)}, \dots, u_n^{(2)}) \quad (r = 1, \dots, m), \\ t = 0 : u_i^* &= \varphi_i^*(x) \quad (i = 1, \dots, n). \end{aligned} \right.$$

By the result in [111, inequality (2.23)] for the inhomogeneous system, one directly obtains

$$\lim_{\|\varphi^*\|_1 + \|\Lambda^*\|_1 + \|B^*\|_1 + \|c^*\|_1 \rightarrow 0} \|u^*\|_0 = 0. \quad (1.5.62)$$

On the other hand, on the domain  $R(T_1)$ , noting (1.5.39) and (1.5.3), a straightforward computation yields that

$$\lim_{\|\varphi^*\|_1 + \|\Lambda^*\|_1 + \|B^*\|_1 + \|c^*\|_1 \rightarrow 0} \|u_x^*\|_0 + \|u_t^*\|_0 = 0 \quad (\text{on the domain } R(T_1)). \quad (1.5.63)$$

Then repeating the procedures in (1.5.63) at most  $M$  times, one can show that

$$\lim_{\|\varphi^*\|_1 + \|\Lambda^*\|_1 + \|B^*\|_1 + \|c^*\|_1 \rightarrow 0} \|u_x^*\|_0 + \|u_t^*\|_0 = 0 \quad (\text{on the domain } R(T)). \quad (1.5.64)$$

We have thus proved the Lemma 1.2.1. ■





# Chapter 2

## On boundary control of a hyperbolic system with a vanishing characteristic speed

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### 2.1 Introduction and main results

There are many publications concerning the exact boundary controllability for linear and nonlinear hyperbolic systems, including wave equations, Saint-Venant equations, Euler equations, etc. (see [19, 72, 89, 100] and the references therein).

As for the general first order hyperbolic systems, one can refer to the recent monograph of Li [72] for almost complete results on the exact controllability of the following 1-D quasilinear hyperbolic system (including many conservation laws)

$$U_t + A(U)U_x = B(U) \tag{2.1.1}$$

in the context of classical solution. It is well known that if all the characteristic speeds of the system does not vanish, i.e., all the eigenvalues of  $A(U)$  are nonzero in the domain under consideration, then (2.1.1) is exactly controllable (at least locally

in  $C^1$  class) by boundary controls provided that the control time is sufficiently large. (see Li and Rao [75, 74]; see also Wang [111] for the nonautonomous case).

On the other hand, the controllability problems have been studied also in the content of weak entropy solutions for hyperbolic conservation laws since the early result on attainable set by Ancona and Marson [8]. In the scalar case, one can refer to Horsin [51] for boundary controllability result of Burgers equation and to Perrollaz [96] when an additional distributed control appears in the right hand side. In the case of systems, the first result on controllability was due to Bressan and Coclite [13]. For general strictly hyperbolic systems of conservation laws with genuinely nonlinear or linearly degenerate and non-vanishing characteristic fields, it is shown in [13] that a small BV initial state can be driven to a constant state by an open-loop control asymptotically in time (see also [9] for one-sided control). Moreover, a negative result for a class of  $2 \times 2$  hyperbolic systems was also proved in [13] that an initial data with a dense distribution of shocks can not be driven to constant by using boundary control. Different from this negative result, Glass obtained the boundary controllability of 1-D isentropic Euler equation [40]. As a further study of [40], Glass recently obtained the boundary controllability for 1-D non-isentropic Euler equation in both Eulerian and Lagrangian coordinates. It is surprising because the equivalent controllability in  $C^1$  framework is false. Actually, the physical entropy can not be changed by boundary controls since the corresponding equation of the Lagrangian system has identically vanishing characteristic speed (See [41, Remark 1.3]).

A widely open question naturally arises that whether it is possible to establish exact controllability for a hyperbolic system with vanishing characteristic speeds. The situation seems different. Notice that the solution  $u(t, x)$  of the degenerate case of the equation (essentially an ordinary differential equation)

$$\begin{cases} u_t = 0, & (t, x) \in (0, T) \times (0, 2\pi), \\ u(0, x) = u_0(x), & x \in (0, 2\pi), \end{cases} \quad (2.1.2)$$

obeys

$$u(t, \cdot) \equiv u_0, \quad \forall t \in [0, T]. \quad (2.1.3)$$

Obviously, it is impossible to change the value  $u$  by using the boundary controls acting on the end  $x = 0$  and/or  $x = 2\pi$ .

There are several directions that have been attempted to answer the above open question.

One is to add some internal controls. For systems with identically zero characteristic speeds, a general result on exact controllability has been obtained by combining internal controls acted on the components corresponding to the zero eigenvalues and boundary controls acting on the other components [82, 111]. There the internal controls depend on both variables  $(t, x)$  and are globally distributed. Later on, an exact controllability result, with some constraints concerning the compatibility between the

initial and final data, was established by using only the internal controls acted on the components corresponding to nonzero eigenvalues [76]. Again the internal controls are globally distributed. It is worthy of mentioning that exact controllability for a simplified model is also realized in [76] by switching controls where the internal control is locally distributed.

Another is to loose the requirements of exact controllability if one considers only the boundary controls because internal controls are usually not applicable for some physical reasons. It is possible to obtain *partial* exact controllability by boundary controls, if one aims to control only the values of the components corresponding to the nonzero eigenvalues. [113, 114].

While the most interesting problem is whether one could establish exact controllability for a hyperbolic system with zero characteristic speeds by only boundary controls. In this direction, Gugat and Leugering [48] proved a global boundary controllability result which covers the critical case in which one of the characteristic speed is zero. In spite of the degenerate model (2.1.2), it is still possible to prove the controllability for some nonlinear system. The return method of Coron has been applied to the many situations in [19], where the linearized system is not controllable while the nonlinearity enables the corresponding nonlinear system to be controllable. One can apply the return method to realize exact boundary controllability for quasilinear hyperbolic systems with a zero characteristic speed [23], when the possible vanishing characteristic speed can be driven nonzero after sufficiently long time. However, this approach seems to be no longer valid for the system with identically zero characteristic speeds (see [23, Remark 1.3]).

In this chapter, we consider the following linear hyperbolic system with a vanishing characteristic speed:

$$\begin{cases} u_t = \alpha u_x + \beta v, \\ v_t = \gamma u, \end{cases} \quad (2.1.4)$$

where  $\alpha, \beta, \gamma \in \mathbb{R}$  are constants. Without loss of generality, we may assume that  $\alpha < 0$ . The problems that we are interested in are (null) controllability and asymptotic stabilization problems under only boundary controls, and how the control properties are determined by the coefficients  $\alpha, \beta, \gamma$ .

More precisely, we first study the null controllability problem for the system (2.1.4) under the boundary control  $h \in L^2(0, T)$  for some  $T > 0$ :

$$\begin{cases} u_t = \alpha u_x + \beta v, & (t, x) \in (0, T) \times (0, 2\pi), \\ v_t = \gamma u, & (t, x) \in (0, T) \times (0, 2\pi), \\ u(t, 0) - u(t, 2\pi) = h(t), & t \in (0, T). \end{cases} \quad (2.1.5)$$

**Definition 2.1.1** *Let  $T > 0$ , the system (2.1.5) is (null) controllable if: for any  $(u_0, v_0) \in (L^2(0, 2\pi))^2$ , there exists a function  $h \in L^2(0, T)$  such that the solution of*

the mixed initial-boundary problem (2.1.5) with the following initial data

$$(u(0, x), v(0, x)) = (u_0(x), v_0(x)), \quad x \in (0, 2\pi) \quad (2.1.6)$$

satisfies the final condition:

$$(u(T, x), v(T, x)) = (0, 0), \quad x \in (0, 2\pi). \quad (2.1.7)$$

The system (2.1.5) is asymptotically (null) controllable if it is (null) controllable for  $T = +\infty$  in the sense that

$$\lim_{t \rightarrow +\infty} (u(t, \cdot), v(t, \cdot)) = (0, 0) \quad \text{in } L^2(0, 2\pi). \quad (2.1.8)$$

We also consider the asymptotic stabilization problem for system (2.1.4) under a feedback law

$$\begin{cases} u_t = \alpha u_x + \beta v, & (t, x) \in (0, +\infty) \times (0, 2\pi), \\ v_t = \gamma u, & (t, x) \in (0, +\infty) \times (0, 2\pi), \\ u(t, 0) = ku(t, 2\pi), & t \in (0, +\infty). \end{cases} \quad (2.1.9)$$

**Definition 2.1.2** *The system (2.1.5) is asymptotically stabilizable if: there exists  $k \in \mathbb{R}$  such that for any  $(u_0, v_0) \in (L^2(0, 2\pi))^2$ , the solution of the closed-loop system (2.1.9) with the initial data (2.1.6) is asymptotically stable, i.e., (2.1.8) holds.*

The Main results that we obtain in this chapter are the following theorems.

**Theorem 2.1.1** *For any given  $0 < T < +\infty$ , the system (2.1.5) is not (null) controllable.*

**Theorem 2.1.2** *The system (2.1.9) is asymptotically stabilizable if and only if  $\beta\gamma < 0$ .*

**Theorem 2.1.3** *If  $\beta\gamma < 0$ , the system (2.1.5) is asymptotically (null) controllable.*

The rest of this chapter is organized as follows. In section 2.2, we state the well-posedness of both the open-loop system (2.1.5) and the closed-loop system (2.1.9). The proofs of the Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.1.3 are given in Section 2.3, Section 2.4 and Section 2.5, respectively.

## 2.2 Well-posedness of the system (2.1.5) and the system (2.1.9)

In this section, we give the definition and well-posedness of solution to the system (2.1.5) and the system (2.1.9). The well-posedness issue is fundamental to the control problems. Here we only present the results without proof, which can be derived by classical methods, such as characteristic method [84] or by theory of semigroup of linear operators [94].

**Definition 2.2.1** *Let  $0 < T \leq +\infty$ ,  $h \in L^2(0, T)$  and  $(u_0, v_0) \in (L^2(0, 2\pi))^2$  be given. A solution of the initial boundary problem (2.1.5)-(2.1.6) is a function  $(u, v) \in C^0([0, T]; (L^2(0, 2\pi))^2)$ , such that, for every  $\tau \in [0, T]$  (if  $T = +\infty$ ,  $\tau \in [0, +\infty)$ ) and for every  $(\varphi_1, \varphi_2) \in (C^1([0, \tau] \times [0, 2\pi]))^2$  such that  $\varphi_1(t, 0) = \varphi_1(t, 2\pi)$ ,  $\forall t \in [0, \tau]$ , one has*

$$\begin{aligned} & - \int_0^\tau \int_0^{2\pi} [(\varphi_{1t} - \alpha\varphi_{1x})u + \beta v\varphi_1] dxdt + \alpha \int_0^\tau h(t)\varphi_1(t, 2\pi)dt \\ & + \int_0^{2\pi} \varphi_1(\tau, x)u(\tau, x)dx - \int_0^{2\pi} \varphi_1(0, x)u_0(x)dx = 0 \end{aligned} \quad (2.2.1)$$

and

$$- \int_0^\tau \int_0^{2\pi} [\varphi_{2t}v + \gamma u\varphi_2] dxdt + \int_0^{2\pi} \varphi_2(\tau, x)v(\tau, x)dx - \int_0^{2\pi} \varphi_2(0, x)v_0(x)dx = 0. \quad (2.2.2)$$

**Lemma 2.2.1** *Let  $T > 0$ ,  $(u_0, v_0) \in (L^2(0, 2\pi))^2$  and  $u \in L^2(0, T)$  be given. Then the initial boundary problem (2.1.5) with (2.1.6) has a unique solution, which depends continuously on  $(u_0, v_0, h)$ .*

**Definition 2.2.2** *Let  $k \in \mathbb{R}$  and  $(u_0, v_0) \in (L^2(0, 2\pi))^2$  be given. A solution of the initial boundary problem (2.1.9) and (2.1.6) is a function  $(u, v) \in C^0([0, +\infty); (L^2(0, 2\pi))^2)$ , such that, for every  $\tau \in [0, \infty)$  and for every  $(\varphi_1, \varphi_2) \in (C^1([0, \tau] \times [0, 2\pi]))^2$  such that  $k\varphi_1(t, 0) = \varphi_1(t, 2\pi)$ ,  $\forall t \in [0, \tau]$ , one has*

$$- \int_0^\tau \int_0^{2\pi} [(\varphi_{1t} - \alpha\varphi_{1x})u + \beta v\varphi_1] dxdt + \int_0^{2\pi} \varphi_1(\tau, x)u(\tau, x)dx - \int_0^{2\pi} \varphi_1(0, x)u_0(x)dx = 0,$$

and

$$- \int_0^\tau \int_0^{2\pi} [\varphi_{2t}v + \gamma u\varphi_2] dxdt + \int_0^{2\pi} \varphi_2(\tau, x)v(\tau, x)dx - \int_0^{2\pi} \varphi_2(0, x)v_0(x)dx = 0.$$

**Lemma 2.2.2** *Let  $k \in \mathbb{R}$  and  $(u_0, v_0) \in (L^2(0, 2\pi))^2$  be given. Then the initial boundary problem (2.1.9) with (2.1.6) has a unique solution, which depends continuously on  $(u_0, v_0)$ .*

## 2.3 Proof of Theorem 2.1.1

We divide the proof of Theorem 2.1.1 into two cases:  $\beta\gamma = 0$  and  $\beta\gamma \neq 0$ .

**Case 1:**  $\beta\gamma = 0$ . In Case 1, the result is trivial in the sense that (at least) one of the two equations in (2.1.5) is decoupled from the other one.

**Case 1.1:**  $\gamma = 0$ . Since  $\gamma = 0$ , the second equation in (2.1.5) becomes

$$v_t = 0, \quad (t, x) \in (0, T) \times (0, 2\pi), \quad (2.3.1)$$

which implies that  $v(t, \cdot) = v_0(\cdot) \in L^2(0, 2\pi)$  for all  $t \in [0, T]$ . Therefore, it is impossible to drive  $v$  from  $v_0 \neq 0 \in L^2(0, 2\pi)$  to zero at time  $T$  no matter what the control  $h$  is.

**Case 1.2:**  $\beta = 0$  and  $\gamma \neq 0$ . Since  $\beta = 0$ , the first equation in (2.1.5) becomes

$$u_t = \alpha u_x, \quad (2.3.2)$$

which does not depend on  $v$ . Suppose that the system (2.1.5) is (null) controllable. Let

$$(u_0(x), v_0(x)) \equiv (1, 0), \quad x \in (0, 2\pi), \quad (2.3.3)$$

then there exists  $h \in L^2(0, 2\pi)$  driving  $(u, v)$  from  $(u_0, v_0)$  to  $(0, 0)$  at time  $T$ . Multiplying (2.3.2) by  $e^{inx}$  ( $n \in \mathbb{Z}$ ) and integrating on  $(0, T) \times (0, 2\pi)$ , we get from  $\gamma \neq 0$  and (2.1.5), (2.1.7) and (2.3.3) that

$$\begin{aligned} 0 &= \int_0^T \int_0^{2\pi} (u_t - \alpha u_x) e^{inx} dx dt \\ &= \int_0^{2\pi} \left[ u e^{inx} \right]_{t=0}^{t=T} dx - \alpha \int_0^T \left[ u e^{inx} \right]_{x=0}^{x=2\pi} dt + in\alpha \int_0^T \int_0^{2\pi} u e^{inx} dx dt \\ &= - \int_0^{2\pi} u_0(x) e^{inx} + \alpha \int_0^T h(t) dt + \frac{in\alpha}{\gamma} \int_0^{2\pi} \left[ v e^{inx} \right]_{t=0}^{t=T} dx dt \\ &= - \int_0^{2\pi} e^{inx} + \alpha \int_0^T h(t) dt, \quad \forall n \in \mathbb{Z}. \end{aligned} \quad (2.3.4)$$

Therefore  $\int_0^T h(t) dt = 0$  as  $n \neq 0$ , while  $\int_0^T h(t) dt = -\frac{2\pi}{\alpha} \neq 0$  as  $n = 0$ . This contradiction concludes the proof of Theorem 2.1.1 for the **Case 1.2:**  $\beta\gamma = 0$ .

**Case 2:**  $\beta\gamma \neq 0$ . In this case, we will prove Theorem 2.1.1 by contradiction argument. The contradiction is derived by moment theory, with other applications on controllability in [91] and [99].

Let us assume that there exists  $h \in L^2(0, 2\pi)$  such that the solution  $(u, v)$  of (2.1.5) with (2.1.6) satisfies (2.1.7). We introduce the adjoint system of the system (2.1.5):

$$\begin{cases} p_t = \alpha p_x - \gamma q, \\ q_t = -\beta p, \\ p(t, 0) = p(t, 2\pi). \end{cases} \quad (2.3.5)$$

Then for the above solution  $(u, v)$  of the system (2.1.5) satisfying (2.1.7), it holds, by duality, that for all solution  $(p, q)$  of the adjoint system (2.3.5),

$$\begin{aligned} & \int_0^{2\pi} u_0(x)p(0, x) + v_0(x)q(0, x)dx \\ &= - \left[ \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L^2(0, 2\pi)} \right]_{t=0}^{t=T} \\ &= - \int_0^T \frac{d}{dt} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L^2(0, 2\pi)} dt \\ &= - \int_0^T \left\langle \begin{pmatrix} u_t \\ v_t \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle_{L^2(0, 2\pi)} + \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} p_t \\ q_t \end{pmatrix} \right\rangle_{L^2(0, 2\pi)} dt \\ &= - \int_0^T \left[ \alpha u(t, x)p(t, x) \right]_{x=0}^{x=2\pi} dt \\ &= \alpha \int_0^T h(t)p(t, 0)dt. \end{aligned} \quad (2.3.6)$$

Next we look for the solutions of the adjoint system (2.3.5). Clearly, the corresponding eigenvalue  $\lambda$  and the eigenvector  $(\xi, \eta)$  obey

$$\begin{cases} \lambda\xi = \alpha\xi' - \gamma\eta \\ \lambda\eta = -\beta\xi \\ \xi(0) = \xi(2\pi) \end{cases} \quad (2.3.7)$$

By canceling  $\eta$ , (2.3.7) is reduced to

$$\alpha\lambda\xi'(x) = (\lambda^2 - \beta\gamma)\xi(x) \quad \text{with} \quad \xi(0) = \xi(2\pi). \quad (2.3.8)$$

It is easy to see that  $\lambda = 0$  happens only when  $\xi(x) = \eta(x) \equiv 0$  since  $\beta\gamma \neq 0$ . Hence (2.3.7) possesses only nonzero eigenvalues.



Since  $\alpha < 0$ , the solution of (2.3.8) is given by

$$\xi(x) = e^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}x} \xi(0), \quad (2.3.9)$$

where the eigenvalue  $\lambda$  is determined by

$$e^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}2\pi} = 1, \quad (2.3.10)$$

i.e.

$$\frac{\lambda^2 - \beta\gamma}{\lambda\alpha} = in, \quad n \in \mathbb{Z}. \quad (2.3.11)$$

Therefore, we obtain the eigenvalues of the equations (2.3.7):

$$\lambda_n^\pm = \begin{cases} \frac{in\alpha \pm \sqrt{4\beta\gamma - n^2\alpha^2}}{2}, & \text{if } n^2 < \frac{4\beta\gamma}{\alpha^2}, n \in \mathbb{Z} \\ \frac{in\alpha \pm i\sqrt{n^2\alpha^2 - 4\beta\gamma}}{2}, & \text{if } n^2 \geq \frac{4\beta\gamma}{\alpha^2}, n \in \mathbb{Z} \end{cases} \quad (2.3.12)$$

and the corresponding eigenvectors:

$$(\xi_n^\pm(x), \eta_n^\pm(x)) = e^{inx} \left(1, \frac{-\beta}{\lambda_n^\pm}\right), \quad n \in \mathbb{Z}. \quad (2.3.13)$$

Obviously for all  $n \in \mathbb{Z}$ ,  $(p_n^\pm(t, x), q_n^\pm(t, x)) = e^{\lambda_n^\pm t} (\xi_n^\pm(x), \eta_n^\pm(x)) = e^{\lambda_n^\pm t + inx} \left(1, \frac{-\beta}{\lambda_n^\pm}\right)$  satisfies (2.3.5). Substituting  $(p_n^\pm(t, x), q_n^\pm(t, x))$  into (2.3.6) yields that the control  $h \in L^2(0, T)$  driving  $(u_0, v_0)$  to  $(0, 0)$  satisfies that

$$\int_0^T h(t) e^{\lambda_n^\pm t} dt = \frac{1}{\alpha \lambda_n^\pm} \int_0^{2\pi} (\lambda_n^\pm u_0(x) - \beta v_0(x)) e^{inx} dx, \quad \forall n \in \mathbb{Z}. \quad (2.3.14)$$

Let

$$(u_0(x), v_0(x)) \equiv (c_1, c_2), \quad x \in (0, 2\pi), \quad (2.3.15)$$

where  $c_1, c_2 \in \mathbb{R}$  are two constants such that

$$\lambda_0^\pm c_1 - \beta c_2 \neq 0. \quad (2.3.16)$$

Thus

$$\int_0^T h(t) e^{\lambda_n^\pm t} dt = \frac{\lambda_n^\pm c_1 - \beta c_2}{\alpha \lambda_n^\pm} \int_0^{2\pi} e^{inx} dx = 0, \quad \forall n \in \mathbb{Z} \setminus \{0\}, \quad (2.3.17)$$

$$\int_0^T h(t)e^{\lambda_0^\pm t} dt = \frac{2\pi(\lambda_0^\pm c_1 - \beta c_2)}{\alpha \lambda_0^\pm} \neq 0. \quad (2.3.18)$$

Let

$$F(z) = \int_0^T h(t)e^{-izt} dt, \quad z \in \mathbb{C}. \quad (2.3.19)$$

Then it follows that  $F(i\lambda_0^\pm) \neq 0$  and  $F(i\lambda_n^\pm) = 0$  for all  $n \in \mathbb{Z} \setminus \{0\}$ . On the other hand,  $F$  is an entire function on  $\mathbb{C}$  due to Paley-Wiener Theorem [63]. It is easy to get from  $\alpha < 0$  and (2.3.12) that

$$\lim_{n \rightarrow +\infty} \lambda_n^+ = 0. \quad (2.3.20)$$

Therefore  $F$  is an entire function vanishing on the set  $\{z = i\lambda_n^\pm | n \in \mathbb{Z} \setminus \{0\}\}$  which has an accumulation point  $z = 0 \in \mathbb{C}$ . This simply implies  $F \equiv 0$ , however, this contradicts with  $F(i\lambda_0^+) \neq 0$ . Consequently, there exists no function  $h \in L^2(0, T)$  such that (2.3.14) holds for the initial data given by (2.3.15). This concludes the proof of Theorem 2.1.1 for the **Case 2**:  $\beta\gamma \neq 0$ .  $\blacksquare$

## 2.4 Proof of Theorem 2.1.2

In this section, we will prove that  $k \in \mathbb{R}$  does exist such that the closed-loop system (2.1.9) is asymptotically stable if and only if  $\beta\gamma < 0$ . The proof relies on the Riesz basis method and a careful analysis of the spectrum.

**Proposition 2.4.1** *All the the eigenvalues of the closed loop system (2.1.9) have a strictly negative real part if and only if  $\beta\gamma < 0$  and  $0 < |k| < 1$ .*

**Proof of Proposition 2.4.1.** Clearly, the corresponding eigenvalue  $\lambda$  and the eigenvector  $(\phi, \psi)$  satisfy

$$\begin{cases} \lambda(\phi, \psi)^{tr} = \mathcal{A}(\phi, \psi)^{tr} \\ \phi(0) = k\phi(2\pi) \end{cases} \quad (2.4.1)$$

where

$$\mathcal{A} = \begin{pmatrix} \alpha \partial_x & \beta \\ \gamma & 0 \end{pmatrix} \quad (2.4.2)$$

is the infinitesimal generator of the semigroup corresponding to the system (2.1.9). By canceling  $\psi$ , (2.4.1) is reduced to

$$\alpha \lambda \phi'(x) = (\lambda^2 - \beta\gamma)\phi(x) \quad \text{with} \quad \phi(0) = k\phi(2\pi). \quad (2.4.3)$$

It is easy to see that  $\lambda = 0$  happens only when  $\phi(x) = \psi(x) \equiv 0$  since  $\beta\gamma \neq 0$ . Hence (2.4.1) has no zero eigenvalues.

Since  $\alpha < 0$ , the solution of (2.4.3) is given by

$$\phi(x) = e^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}x} \phi(0), \quad (2.4.4)$$

where the eigenvalue  $\lambda$  is determined by

$$e^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}2\pi} = \frac{1}{k}, \quad k \neq 0. \quad (2.4.5)$$

For every  $n \in \mathbb{Z}$ , let

$$\tilde{n} = \begin{cases} n, & \text{if } k > 0, \\ n + \frac{1}{2}, & \text{if } k < 0, \end{cases} \quad (2.4.6)$$

then

$$\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}2\pi = i2\tilde{n}\pi - \ln |k|, \quad (2.4.7)$$

or equivalently,

$$\lambda^2 - \frac{\alpha}{2\pi}(i2\tilde{n}\pi - \ln |k|)\lambda - \beta\gamma = 0. \quad (2.4.8)$$

Therefore, for any  $n \in \mathbb{Z}$  and  $k \neq 0$ , (2.4.8) has two roots  $\lambda_{n,k}^\pm \in \mathbb{C}$  such that

$$\left[ \lambda_{n,k}^\pm - \frac{\alpha}{4\pi}(i2\tilde{n}\pi - \ln |k|) \right]^2 = \beta\gamma + \frac{\alpha^2}{16\pi^2}(i2\tilde{n}\pi - \ln |k|)^2 \quad (2.4.9)$$

Direct computations give us the real and imaginary part of  $\lambda_{n,k}^\pm$ :

$$\Re(\lambda_{n,k}^\pm) = \frac{-\alpha \ln |k|}{4\pi} \pm \sqrt{\frac{c_{n,k} + \sqrt{c_{n,k}^2 + 4d_{n,k}^2}}{2}}, \quad (2.4.10)$$

$$\Im(\lambda_{n,k}^\pm) = \frac{\alpha\tilde{n}}{2} \mp \operatorname{sgn}(\tilde{n} \ln |k|) \sqrt{\frac{-c_{n,k} + \sqrt{c_{n,k}^2 + 4d_{n,k}^2}}{2}}, \quad (2.4.11)$$

where

$$c_{n,k} = \beta\gamma + \frac{\alpha^2}{16\pi^2}(\ln^2 |k| - 4\tilde{n}^2\pi^2), \quad (2.4.12)$$

$$d_{n,k} = \frac{\alpha^2 \ln |k| \tilde{n}}{8\pi}. \quad (2.4.13)$$

Since  $\alpha < 0$ , it is obvious to see that if  $|k| \geq 1$ ,  $\Re(\lambda_{n,k}^+) \geq 0$ . On the other hand, if  $0 < |k| < 1$ , we have

$$\begin{aligned}
 & \Re(\lambda_{n,k}^\pm) < 0, \quad \forall n \in \mathbb{Z} \\
 \Leftrightarrow & \frac{\alpha^2 \ln^2 |k|}{16\pi^2} > \frac{c_{n,k} + \sqrt{c_{n,k}^2 + 4d_{n,k}^2}}{2} \\
 \Leftrightarrow & -\beta\gamma + \frac{\alpha^2(\ln^2 |k| + 4\tilde{n}^2\pi^2)}{16\pi^2} > \sqrt{c_{n,k}^2 + 4d_{n,k}^2} \\
 \Leftrightarrow & -\frac{\beta\gamma\alpha^2 \ln^2 |k|}{4\pi^2} > 0 \\
 \Leftrightarrow & \beta\gamma < 0.
 \end{aligned} \tag{2.4.14}$$

This ends the proof of Proposition 2.4.1. ■

In addition to Proposition 2.4.1, we can obtain, after careful computations to (2.4.10)-(2.4.11), the asymptotic behavior of the eigenvalues  $\lambda_{n,k}^\pm$  as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} \lambda_{n,k}^+ = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} |\lambda_{n,k}^-| = +\infty. \tag{2.4.15}$$

In view of (2.4.4) and (2.4.7), the corresponding eigenvectors can be chosen as

$$E_{n,k}^+ := (\phi_{n,k}^+(x), \psi_{n,k}^+(x))^{tr} = e^{(\frac{-\ln |k|}{2\pi} + i\tilde{n})x} (\lambda_{n,k}^+, \gamma)^{tr}, \quad n \in \mathbb{Z}, \tag{2.4.16}$$

$$E_{n,k}^- := (\phi_{n,k}^-(x), \psi_{n,k}^-(x))^{tr} = e^{(\frac{-\ln |k|}{2\pi} + i\tilde{n})x} (1, \frac{\gamma}{\lambda_{n,k}^-})^{tr}, \quad n \in \mathbb{Z}. \tag{2.4.17}$$

In order to prove Theorem 2.1.2, we first prove that if  $\beta\gamma \geq 0$  and  $k \in \mathbb{R}$ , the system (2.1.9) is not asymptotically stable. According to Proposition 2.4.1, if  $\beta\gamma \geq 0$ , then for any  $k \in \mathbb{R}$ , there exists  $n \in \mathbb{Z}$ , such that  $\Re(\lambda_{n,k}^+) \geq 0$ . Let the initial data be the corresponding eigenvector :  $(u_0, v_0)^{tr} = E_{n,k}^+ \in (L^2(0, 2\pi))^2$ , then the corresponding solution of the system (2.1.9) is given by  $(u(t, \cdot), v(t, \cdot)) = e^{\lambda_{n,k}^+ t} (u_0, v_0)$ , which is obviously not stable.

Now it remains to prove that if  $\beta\gamma < 0$ , there exists suitable  $k \in \mathbb{R}$  such that the system (2.1.9) is asymptotically stable. More precisely, we apply the Riesz basis approach to prove that the solution of the system (2.1.9) satisfies (2.1.8) if  $\beta\gamma < 0$  and  $0 < |k| < 1$ .

Let us emphasize that it is possible that some eigenvalues may coincide. Actually, the occurrence of an eigenvalue with multiplicity greater than one happens if and only if

$$\beta\gamma = -\frac{\alpha^2 \ln^2 k}{16\pi^2} < 0 \quad \text{and} \quad 0 < k < 1. \tag{2.4.18}$$

Consequently, the only multiple eigenvalue is given by

$$\lambda_{0,k}^+ = \lambda_{0,k}^- = \lambda_0 := -\frac{\alpha \ln k}{4\pi} < 0. \quad (2.4.19)$$

In this case, the dimension of the eigenspace of  $\lambda_0$  is one and the corresponding eigenvector can be chosen as

$$E_{0,k}^+ := e^{\frac{-\ln k}{2\pi}x}(\lambda_0, \gamma)^{tr}. \quad (2.4.20)$$

Let  $E_{0,k}^-$  be the root vector corresponding to  $\lambda_0$ :

$$\mathcal{A}E_{0,k}^- = \lambda_0 E_{0,k}^- + E_{0,k}^+ \quad (2.4.21)$$

where the operator  $\mathcal{A}$  is given by (2.4.2). Then by (2.4.18), (2.4.19), it is easy to find a typical root vector

$$E_{0,k}^- := e^{\frac{-\ln k}{2\pi}x}(\lambda_0 + 1, \gamma)^{tr} \quad (2.4.22)$$

which satisfies (2.4.21) and is linearly independent of  $E_{0,k}^+$ .

Now we claim the following proposition:

**Proposition 2.4.2** *For any fixed  $\beta\gamma < 0$  and  $0 < |k| < 1$ ,  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$  forms a Riesz basis of the complex Hilbert space  $(L^2(0, 2\pi))^2$ .*

**Proof of Proposition 2.4.2.** We introduce the classical orthogonal basis  $\{(e_{n,k}^+, e_{n,k}^-), n \in \mathbb{Z}\}$  of  $(L^2(0, 2\pi))^2$ , where

$$e_{n,k}^+ = e^{inx}(1, 0)^{tr}, \quad e_{n,k}^- = e^{inx}(0, 1)^{tr}, \quad n \in \mathbb{Z}. \quad (2.4.23)$$

Clearly, we have

$$\langle e_{n,k}^+, e_{m,k}^- \rangle_{(L^2(0,\pi))^2} = 0, \quad \forall n, m \in \mathbb{Z}, \quad (2.4.24)$$

$$\langle e_{n,k}^+, e_{m,k}^+ \rangle_{(L^2(0,\pi))^2} = \langle e_{n,k}^-, e_{m,k}^- \rangle_{(L^2(0,\pi))^2} = 2\pi\delta_{nm}, \quad \forall n, m \in \mathbb{Z}, \quad (2.4.25)$$

where  $\delta_{nm}$  stands for the Kronecker's delta. The proof will be divided into two cases due to the possibility of the occurrence of multiple eigenvalues.

**Case 1. (2.4.18) is not true.** In this case, all the eigenvalues are distinct. Since  $\lambda_{n,k}^+ \neq \lambda_{n,k}^-$  for all  $n \in \mathbb{Z}$ , the matrix  $\begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \lambda_{n,k}^- \end{pmatrix}$  is invertible. Then it is easy to get from

$$(E_{n,k}^+, E_{n,k}^-) = e^{\frac{-\ln |k|}{2\pi}x}(e_{n,k}^+, e_{n,k}^-) \begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \lambda_{n,k}^- \end{pmatrix}. \quad (2.4.26)$$

that  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$  forms a basis of the complex Hilbert space  $(L^2(0, 2\pi))^2$ .

For any  $(f, g) \in (L^2(0, 2\pi))^2$ , there is a series of complex pairs  $\{(\alpha_n^+, \alpha_n^-) \in \mathbb{C}^2\}_{n \in \mathbb{Z}}$  such that

$$(f, g)^{tr} = \sum_{n \in \mathbb{Z}} (\alpha_n^+ E_{n,k}^+ + \alpha_n^- E_{n,k}^-) = \sum_{n \in \mathbb{Z}} (E_{n,k}^+, E_{n,k}^-) (\alpha_n^+, \alpha_n^-)^{tr}. \quad (2.4.27)$$

By (2.4.26), it follows that

$$(f, g)^{tr} = \sum_{n \in \mathbb{Z}} (\beta_n^+ e_{n,k}^+ + \beta_n^- e_{n,k}^-) e^{-\frac{\ln|k|}{2\pi}x}, \quad (2.4.28)$$

where

$$\begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} = \begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \frac{\gamma}{\lambda_{n,k}^-} \end{pmatrix} \begin{pmatrix} \alpha_n^+ \\ \alpha_n^- \end{pmatrix}. \quad (2.4.29)$$

Notting that  $0 < |k| < 1$  and  $e^{\frac{\ln|k|}{2\pi}x} \leq 1, \forall x \in [0, 2\pi]$ . Then we get, from (2.4.23) and (2.4.28) that

$$2\pi \sum_{n \in \mathbb{Z}} (|\beta_n^+|^2 + |\beta_n^-|^2) \leq \|(f, g)\|_{(L^2(0,2\pi))^2}^2 \leq e^{-2\ln|k|} 2\pi \sum_{n \in \mathbb{Z}} (|\beta_n^+|^2 + |\beta_n^-|^2). \quad (2.4.30)$$

Moreover, it is easy to see from (2.4.29) and (2.4.15) that there exists a constant  $C_1 > 0$  such that

$$|\beta_n^+|^2 + |\beta_n^-|^2 \leq C_1 (|\alpha_n^+|^2 + |\alpha_n^-|^2), \quad \forall n \in \mathbb{Z}. \quad (2.4.31)$$

On the other hand, we can also derive from (2.4.29) that

$$\begin{aligned} \begin{pmatrix} \alpha_n^+ \\ \alpha_n^- \end{pmatrix} &= \begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \frac{\gamma}{\lambda_{n,k}^-} \end{pmatrix}^{-1} \begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} \\ &= \begin{pmatrix} a_{n,k}^+ & b_{n,k}^+ \\ a_{n,k}^- & b_{n,k}^- \end{pmatrix} \begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} \end{aligned} \quad (2.4.32)$$

where

$$a_{n,k}^+ = \frac{1}{\lambda_{n,k}^+ - \lambda_{n,k}^-}, \quad a_{n,k}^- = \frac{-\lambda_{n,k}^-}{\lambda_{n,k}^+ - \lambda_{n,k}^-}, \quad (2.4.33)$$

$$b_{n,k}^+ = \frac{-\lambda_{n,k}^-}{\gamma(\lambda_{n,k}^+ - \lambda_{n,k}^-)}, \quad b_{n,k}^- = \frac{\lambda_{n,k}^+ \lambda_{n,k}^-}{\gamma(\lambda_{n,k}^+ - \lambda_{n,k}^-)}. \quad (2.4.34)$$

Thanks to (2.4.15), we get additionally that

$$\lim_{n \rightarrow \infty} a_{n,k}^+ = 0, \quad \lim_{n \rightarrow \infty} a_{n,k}^- = 1, \quad \lim_{n \rightarrow \infty} b_{n,k}^+ = \frac{1}{\gamma}, \quad \lim_{n \rightarrow \infty} b_{n,k}^- = 0. \quad (2.4.35)$$

Therefore we can see that there exists a constant  $C_2 > 0$  such that

$$|\alpha_n^+|^2 + |\alpha_n^-|^2 \leq C_2(|\beta_n^+|^2 + |\beta_n^-|^2), \quad \forall n \in \mathbb{Z}. \quad (2.4.36)$$

Obviously, (2.4.30)-(2.4.31) and (2.4.36) show that  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$  forms a Riesz basis of  $(L^2(0, 2\pi))^2$  in **Case 1**.

**Case 2.** (2.4.18) **is true.** In this case, we still have  $\lambda_{n,k}^+ \neq \lambda_{n,k}^-$  and (2.4.26) for all  $n \in \mathbb{Z} \setminus \{0\}$ . While for  $n = 0$ , we get from (2.4.20), (2.4.22) and (2.4.23) that

$$(E_{0,k}^+, E_{0,k}^-) = e^{\frac{-\ln k}{2\pi}x} (e_{0,k}^+, e_{0,k}^-) \begin{pmatrix} \lambda_0 & \lambda_0 + 1 \\ \gamma & \gamma \end{pmatrix}. \quad (2.4.37)$$

Obviously, the matrix  $\begin{pmatrix} \lambda_0 & \lambda_0 + 1 \\ \gamma & \gamma \end{pmatrix}$  is reversible since  $\gamma \neq 0$ . Then it is similar to prove that  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$  still forms a Riesz basis of  $(L^2(0, 2\pi))^2$  as in **Case 1**. This ends the proof of Proposition 2.4.2.  $\blacksquare$

Thanks to Proposition 2.4.2, the solution  $(u, v)$  of the initial-boundary problem (2.1.9) can be decomposed with respect to the Riesz basis  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$ . Actually, for any given initial data  $(u_0, v_0)^{tr} \in (L^2(0, 2\pi))^2$ , there exists a series  $\{(c_{n,k}^+, c_{n,k}^-) \in \mathbb{C}^2\}_{n \in \mathbb{Z}}$  such that

$$(u_0, v_0)^{tr} = \sum_{n \in \mathbb{Z}} (c_{n,k}^+ E_{n,k}^+ + c_{n,k}^- E_{n,k}^-). \quad (2.4.38)$$

In order to decompose the solution in terms of the Riesz basis  $\{(E_{n,k}^+, E_{n,k}^-), n \in \mathbb{Z}\}$ , we have to discuss the various cases whether the eigenvalues are distinct or not.

**Case 1.** (2.4.18) **is not true.** In this case, all the eigenvalues are distinct. Since  $\{E_{n,k}^\pm\}$  is the eigenvector corresponding to the eigenvalue  $\lambda_{n,k}^\pm$ , it follows from (2.4.38) that

$$(u(t, \cdot), v(t, \cdot))^{tr} = \sum_{n \in \mathbb{Z}} \left( c_{n,k}^+ e^{\lambda_{n,k}^+ t} E_{n,k}^+ + c_{n,k}^- e^{\lambda_{n,k}^- t} E_{n,k}^- \right). \quad (2.4.39)$$

Then by Proposition 2.4.1, we get that in the case of  $\beta\gamma < 0$ ,

$$|e^{\lambda_{n,k}^\pm t}| < 1, \quad \forall n \in \mathbb{Z} \quad (2.4.40)$$

and

$$\lim_{t \rightarrow +\infty} |e^{\lambda_{n,k}^{\pm} t}| \rightarrow 0, \quad \forall n \in \mathbb{Z}. \quad (2.4.41)$$

Therefore, there exists  $C_3 > 0$  independent of  $(u_0, v_0)$  such that

$$\begin{aligned} & \| (u(t, \cdot), v(t, \cdot)) \|_{(L^2(0, 2\pi))^2}^2 \\ & \leq \sum_{n \in \mathbb{Z}} \left\| c_{n,k}^+ E_{n,k}^+ + c_{n,k}^- E_{n,k}^- \right\|_{(L^2(0, 2\pi))^2}^2 \\ & \leq C_3 \| (u_0, v_0) \|_{(L^2(0, 2\pi))^2}^2, \end{aligned} \quad (2.4.42)$$

which implies that the series on the right hand side of (2.4.39) converges uniformly and strongly in  $(L^2(0, 2\pi))^2$ . Then taking the  $(L^2(0, 2\pi))^2$  norm of (2.4.39) and letting  $t \rightarrow +\infty$ , we easily conclude by (2.4.41) that

$$\lim_{t \rightarrow +\infty} (u(t, \cdot), v(t, \cdot))^{tr} = \sum_{n \in \mathbb{Z}} \lim_{t \rightarrow +\infty} \left( c_{n,k}^+ e^{\lambda_{n,k}^+ t} E_{n,k}^+ + c_{n,k}^- e^{\lambda_{n,k}^- t} E_{n,k}^- \right) = 0. \quad (2.4.43)$$

This ends the proof that if  $\beta\gamma < 0$ , then the system (2.1.9) can be stabilized by taking the tuning parameter  $k$  such that  $0 < |k| < 1$ .

**Case 2.** (2.4.18) **is true.** In this case, the only eigenvalue, whose multiplicity is more than one, is  $\lambda_0$  given by (2.4.19). We get from (2.4.20), (2.4.22), (2.4.23) and (2.4.38) that

$$\begin{aligned} (u(t, \cdot), v(t, \cdot))^{tr} &= e^{\lambda_0 t} [(c_{0,k}^- t + c_{0,k}^+) E_{0,k}^+ + c_{0,k}^- E_{0,k}^-] \\ &+ \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( c_{n,k}^+ e^{\lambda_{n,k}^+ t} E_{n,k}^+ + c_{n,k}^- e^{\lambda_{n,k}^- t} E_{n,k}^- \right). \end{aligned} \quad (2.4.44)$$

Again by Proposition 2.4.1, we have (2.4.41) and

$$\lim_{t \rightarrow +\infty} |e^{\lambda_0 t} t| \rightarrow 0. \quad (2.4.45)$$

Then we easily conclude the asymptotic stability of the closed-loop system (2.1.9), similarly as in **Case 1**. This completes the proof of Theorem 2.1.2.  $\blacksquare$

## 2.5 Proof of Theorem 2.1.3

Thanks to Theorem 2.1.2, one has that if  $\beta\gamma < 0$ , there exists  $k \in \mathbb{R}$  such that the system (2.1.9) is asymptotically stable, i.e., (2.1.8) is true for all initial data  $(u_0, v_0) \in (L^2(0, 2\pi))^2$ . Let  $(u(t, x), v(t, x))$  be the solution of the closed-loop system (2.1.9) with (2.1.6). Then according to the uniqueness of solution, it is clear that the



system (2.1.5) is asymptotically (null) controllable by taking the control as:

$$h(t) = u(t, 0) - u(t, 2\pi). \quad (2.5.1)$$

This concludes the proof of Theorem 2.1.3. ■

## Part II

# Boundary Stabilization for Inhomogeneous Hyperbolic Systems



# Chapter 3

## Feedback Control Design of Homodirectional and General Heterodirectional Linear Coupled Hyperbolic PDEs

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## 3.1 Introduction

**a) Background:** Coupled first-order linear hyperbolic systems, typically formulated on a 1-D spatial domain normalized to the interval  $(0, 1)$ , are common in modeling of traffic flow [7], heat exchangers [116], open channel flow [16, 28] or multiphase flow [29, 33, 35].

Research on stabilization of such PDEs has been dominated by the focus on pairs of counter-convecting transport PDEs with distributed local coupling. In [26] a first solution allowing actuation on only one boundary and permitting coupling coefficients of arbitrary size was presented. A recent extension [31] allows stabilization using only one control for a system containing an arbitrary number of coupled transport PDEs that convect at different speeds against the direction of the PDE whose boundary is actuated.

In this paper we present a solution to the fully general case of coupled hyperbolic PDEs. We divide such PDE systems into two categories:

- *homodirectional* systems of  $m$  transport PDEs, for which all the  $m$  transport velocities have the same signs, i.e., all of the PDEs convect in the same direction. Because of the finite length of the spatial domain, these are inherently stable but the coupling between states can cause undesirable transient behaviors and the trajectory planning problem is non-trivial.
- *heterodirectional* systems of  $n + m$  transport PDEs, for which there exist at least two transport velocities with opposite signs, i.e., where  $m$  PDEs convect in one direction and  $n$  PDEs convect in the opposite direction. The coupling between states traveling in opposite directions may cause instability.

In this chapter, we present control designs for the fully general case of coupled heterodirectional hyperbolic PDEs, allowing the numbers  $m$  and  $n$  of PDEs in either direction to be arbitrary, and with actuation applied on only one boundary (to all the  $m$  PDEs that convect downstream from that boundary). To solve this general problem, we solve, as a special case, the heretofore unsolved problem of control of coupled

homodirectional hyperbolic linear PDEs, where multiple transport PDEs convect in the same direction, have possibly distinct speeds, and arbitrary local coupling.

Our approach is based on PDE backstepping and yields solutions to stabilization, by both full-state and observer-based output feedback, trajectory planning, and trajectory tracking problems.

**b) Literature:** Controllability of hyperbolic systems has first been investigated using explicit computation of the solution along the characteristic curves in the framework of  $C^1$  norm [42, 69, 98]. Later, so-called Control Lyapunov Functions methods emerged, enabling the design of dissipative boundary conditions for nonlinear hyperbolic systems in the context of both  $C^1$  norm and  $H^2$  norm [19, 21, 20]. Further, using Lyapunov functions method, sufficient boundary conditions for the exponential stability of linear [32] or nonlinear [46, 47] hyperbolic systems of balance laws have been derived. All of these results impose restrictions on the magnitude of the coupling coefficients, which are responsible for potential instabilities.

In [26], a full-state feedback control law, with actuation only on one end of the domain, which achieves  $H^2$  exponential stability of closed-loop 2–state heterodirectional linear and quasilinear hyperbolic systems is derived using a backstepping method. With a similar backstepping transformation, an output-feedback controller is designed in [31] for heterodirectional systems with  $m = 1$  (controlled) negative velocity and  $n$  (arbitrary) positive ones. These results hold regardless of the (bounded) magnitude of the coupling coefficients. Unfortunately, the method presented in [26, 31] can not be extended to the case  $m > 1$ .

**c) Contribution:** The first step towards this chapter’s general solution for  $m > 1$  was presented (but not published as a paper) in [26] for  $m = 2$  and  $n = 0$ . In conference paper [53], an extension to  $m = 2$  and  $n = 1$  is achieved.

The contribution of this chapter is two-fold. For  $(n + m)$ –state heterodirectional systems, we derive a stabilizing boundary feedback law that ensures finite-time convergence of all the states to zero. For homodirectional systems (for which stability is not an issue), we design a boundary control law ensuring tracking of a given reference trajectory at the uncontrolled boundary.

Both designs rely on the backstepping approach. A particular choice of the target system, featuring a cascade structure similar to [26, Section 3.5], enables the use of a classical Volterra integral transformation. Well-posedness of the system of kernel equations, which is the main technical challenge of this chapter, is proved by a method of successive approximations using a novel recursive bound.

In the case of heterodirectional systems, the approach yields a full-state feedback law that would necessitate full distributed measurements to be implemented, which is not realistic in practice. For this reason, we derive an observer relying on measurements of the states at a single boundary (the anti-controlled one). Along with the full-state feedback law, this yields an output feedback controller amenable to

implementation.

**d) Organization:** In Section 3.2 we introduce the model equations. In Section 3.3 we present the stabilization result for heterodirectional systems: the target system is presented in Section 3.3.1 while the backstepping transformation is derived in Section 3.3.2. The design is summarized in Section 3.3.3. In Section 3.4 we present the boundary observer design. In Section 3.5 we present the motion planning result for homodirectional systems. Section 3.6 contains the main technical difficulty of the paper, i.e. the proof of well-posedness of the backstepping transformation. We conclude in Section 3.7 by discussing open problems.

## 3.2 System description

We consider the following general linear hyperbolic system

$$u_t(t, x) + \Lambda^+ u_x(t, x) = \Sigma^{++} u(t, x) + \Sigma^{+-} v(t, x) \quad (3.2.1)$$

$$v_t(t, x) - \Lambda^- v_x(t, x) = \Sigma^{-+} u(t, x) + \Sigma^{--} v(t, x) \quad (3.2.2)$$

with the following boundary conditions

$$u(t, 0) = Q_0 v(t, 0), \quad v(t, 1) = R_1 u(t, 1) + U(t) \quad (3.2.3)$$

where

$$u = (u_1 \ \cdots \ u_n)^T, \quad v = (v_1 \ \cdots \ v_m)^T \quad (3.2.4)$$

$$\Lambda^+ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix} \quad (3.2.5)$$

with

$$-\mu_1 < \cdots < -\mu_m < 0 < \lambda_1 \leq \cdots \leq \lambda_n \quad (3.2.6)$$

and

$$\begin{aligned} \Sigma^{++} &= \{\sigma_{ij}^{++}\}_{1 \leq i \leq n, 1 \leq j \leq n}, & \Sigma^{+-} &= \{\sigma_{ij}^{+-}\}_{1 \leq i \leq n, 1 \leq j \leq m}, \\ \Sigma^{-+} &= \{\sigma_{ij}^{-+}\}_{1 \leq i \leq m, 1 \leq j \leq n}, & \Sigma^{--} &= \{\sigma_{ij}^{--}\}_{1 \leq i \leq m, 1 \leq j \leq m} \end{aligned} \quad (3.2.7)$$

$$Q_0 = \{q_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m}, \quad R_1 = \{\rho_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n}, \quad U(t) = (U_1(t) \ \cdots \ U_m(t))^T \quad (3.2.8)$$

**Remark 3.2.1** *We consider here constant coupling coefficients and transport velocities for the sake of readability. The method straightforwardly extends to spatially varying coefficients, with more involved technical developments.*

Besides, we also make the following assumption without loss of generality

$$\forall j = 1, \dots, m \quad \sigma_{jj}^- = 0, \quad (3.2.9)$$

i.e. there are no (internal) diagonal coupling terms for  $v$ -system. Such coupling terms can be removed using a change of coordinates as presented in, e.g., [26] and [53]. This yields spatially-varying coupling terms, which is not an issue in the light of Remark 3.2.1.

**Remark 3.2.2** *If two or more states have the same transport speeds (i.e.  $\mu_i = \mu_j$  for some  $i \neq j$ ) we refer to those states as isotachic. This case was intentionally avoided in (3.2.6). To deal with isotachic states, we consider the change of coordinates  $\bar{v}(t, x) = A(x)v(t, x)$ . The matrix  $A(x)$  is a block-diagonal matrix, with  $A_{ii} = 1$  if  $\mu_i \neq \mu_j$  for  $j \neq i$ . If there is a set of  $n_i$  isotachic states (i.e. there is  $i$  such that  $\mu_j = \mu_i$  for  $j = i + 1, \dots, i + n_i - 1$ , then there is in  $A(x)$  a corresponding block  $B(x)$  of dimension  $n_i \times n_i$  in  $A(x)$ . Each of these  $B(x)$  is computed independently for each isotachic set of states. If we call  $\Sigma_{iso}$  the matrix of coupling coefficients among these isotachic states (i.e. with coefficients  $\sigma_{jk}^-$  for  $j, k = i, i + 1, \dots, i + n_i - 1$ ), then  $B(x)$  is computed from the initial value problem  $B(x) = 1/\mu_i B(x)\Sigma_{iso}$ ,  $B(0) = I_{n_i \times n_i}$ . It is easy to see that this transformation is invertible, since one can define a matrix  $C(x)$  from  $C(x) = 1/\mu_i \Sigma_{iso} B(x)$ ,  $C(0) = I_{n_i \times n_i}$ . One has that  $C(x)$  is the inverse of  $B(x)$  as  $B(0)C(0) = I_{n_i \times n_i}$  and  $\frac{d}{dx} B(x)C(x) = 0$ . Applying this invertible transformation eliminates the coupling coefficients between isotachic states, but results in some spatially-varying coupling terms, which is not an issue as explained in Remark 3.2.1.*

### 3.3 Finite-time stabilization of heterodirectional systems

In this section, we derive a stabilizing feedback law for the general  $(n + m)$ -state system. Notice that this is interesting only in the case  $n \neq 0$ , since instability arises from coupling between states traveling in opposite directions. Following the backstepping approach, we seek to map system (3.2.1)–(3.2.3) to a target system with desirable stability properties using an invertible Volterra transformation.



### 3.3.1 Target system

#### Target system design

The strategy we used here is so-called backstepping method, which transforms the original system into a “good” (for example, exponentially stable) target system by using invertible backstepping transformation (see, for example, [19, Chapter 12.5] and [65]). Here, we want to map system (3.2.1)–(3.2.3) to the following target system

$$\alpha_t(t, x) + \Lambda^+ \alpha_x(t, x) = \Sigma^{++} \alpha(t, x) + \Sigma^{+-} \beta(t, x) + \int_0^x C^+(x, \xi) \alpha(\xi) d\xi + \int_0^x C^-(x, \xi) \beta(\xi) d\xi \quad (3.3.1)$$

$$\beta_t(t, x) - \Lambda^- \beta_x(t, x) = G(x) \beta(0) \quad (3.3.2)$$

with the following boundary conditions

$$\alpha(t, 0) = Q_0 \beta(t, 0), \quad \beta(t, 1) = 0 \quad (3.3.3)$$

where  $C^+$  and  $C^-$  are  $L^\infty$  matrix functions on the domain

$$\mathcal{T} = \{0 \leq \xi \leq x \leq 1\}, \quad (3.3.4)$$

while  $G \in L^\infty(0, 1)$  is a lower triangular matrix with the following structures

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}. \quad (3.3.5)$$

The coefficients of  $C^+$ ,  $C^-$  and  $G$  will be determined in section 3.3.2.

#### Stability of the target system

The following lemma assesses the finite-time stability of the target system.

**Lemma 3.3.1** *Consider system (3.3.1),(3.3.2) with boundary conditions (3.3.3). Its zero equilibrium is reached in finite time  $t = t_F$ , where*

$$t_F := \frac{1}{\lambda_1} + \sum_{j=1}^m \frac{1}{\mu_j}. \quad (3.3.6)$$

Noting (3.3.2)–(3.3.3) and (3.3.5), we find that the  $\beta$ -system is in fact a cascade system, which allows us to explicitly solve it by recursion as follows. The explicit

solution of  $\beta_1$  is given by

$$\beta_1(t, x) = \begin{cases} \beta_1(0, x + \mu_1 t) & \text{if } t < \frac{1-x}{\mu_1}, \\ 0 & \text{if } t \geq \frac{1-x}{\mu_1}. \end{cases} \quad (3.3.7)$$

Notice in particular that  $\beta_1$  is identically zero for  $t \geq \mu_1^{-1}$ . From the time  $t \geq \mu_1^{-1}$  on, we have that  $\beta_2(t, x)$  satisfies the following equation

$$\beta_{2t}(t, x) - \mu_2 \beta_{2x}(t, x) = 0. \quad (3.3.8)$$

Similarly, by expressing the solution along the characteristic lines, one obtains that

$$\beta_2(t, x) \equiv 0 \quad \forall t \geq \mu_1^{-1} + \mu_2^{-1}. \quad (3.3.9)$$

Thus, by mathematical induction, one can easily get that  $\beta_j (j = 1, \dots, m)$  vanishes after

$$t = \sum_{k=1}^j \frac{1}{\mu_k}. \quad (3.3.10)$$

This yields that

$$\beta(t, x) \equiv 0, \quad t > \sum_{j=1}^m \frac{1}{\mu_j}. \quad (3.3.11)$$

When  $t > \sum_{j=1}^m \frac{1}{\mu_j}$ , the  $\alpha$ -system becomes

$$\alpha_t(t, x) + \Lambda^+ \alpha_x(t, x) = \Sigma^{++} \alpha(t, x) + \int_0^x C^+(x, \xi) \alpha(\xi) d\xi \quad (3.3.12)$$

with the boundary conditions

$$\alpha(t, 0) = 0. \quad (3.3.13)$$

Since there are no zero eigenvalues for the  $\alpha$ -system (see (3.2.6)), we may change the status of  $t$  and  $x$ , and Equations (3.3.12) can be rewritten as

$$\alpha_x(t, x) + (\Lambda^+)^{-1} \alpha_t(t, x) = (\Lambda^+)^{-1} \Sigma^{++} \alpha(t, x) + \int_0^x (\Lambda^+)^{-1} C^+(x, \xi) \alpha(\xi) d\xi \quad (3.3.14)$$

with the initial condition (3.3.13). Then by the uniqueness of the system (3.3.13), (3.3.14), and noting the order of the transport speeds of the  $\alpha$ -system (see (3.2.6)), this yields

(see [64]) that  $\alpha$  identically vanishes for

$$t \geq \frac{1}{\lambda_1} + \sum_{j=1}^m \frac{1}{\mu_j} \quad (3.3.15)$$

This concludes the proof.

### 3.3.2 Backstepping transformation

To map system (3.2.1)–(3.2.3) to the target system (3.3.1)–(3.3.3), we consider the following backstepping (Volterra) transformation

$$\alpha(t, x) = u(t, x) \quad (3.3.16)$$

$$\beta(t, x) = v(t, x) - \int_0^x [K(x, \xi)u(\xi) + L(x, \xi)v(\xi)] d\xi \quad (3.3.17)$$

where the kernels to be determined  $K$  and  $L$  are defined on the triangular domain  $\mathcal{T}$ . Deriving (3.3.17) with respect to space and time yields, respectively (omitting the time argument)

$$\begin{aligned} \beta_t(x) &= \Lambda^- v_x(x) + \Sigma^{-+} u(x) + \Sigma^{--} v(x) + K(x, x)\Lambda^+ u(x) - K(x, 0)\Lambda^+ u(0) \\ &\quad - \int_0^x [K_\xi(x, \xi)\Lambda^+ + K(x, \xi)\Sigma^{++}] u(\xi) d\xi - \int_0^x [K(x, \xi)\Sigma^{+-}] v(\xi) d\xi \\ &\quad - L(x, x)\Lambda^- v(x) + L(x, 0)\Lambda^- v(0) \\ &\quad + \int_0^x [L_\xi(x, \xi)\Lambda^- - L(x, \xi)\Sigma^{--}] v(\xi) d\xi - \int_0^x [L(x, \xi)\Sigma^{-+}] u(\xi) d\xi \end{aligned}$$

and

$$\beta_x(x) = v_x(x) - K(x, x)u(x) - L(x, x)v(x) - \int_0^x [K_x(x, \xi)u(\xi) + L_x(x, \xi)v(\xi)] d\xi$$

Plugging into the target system equations and noting  $\beta(0) \equiv v(0)$  yields the following system of kernel equations

$$0 = K(x, x)\Lambda^+ + \Lambda^- K(x, x) + \Sigma^{-+} \quad (3.3.18)$$

$$0 = \Lambda^- L(x, x) - L(x, x)\Lambda^- + \Sigma^{--} \quad (3.3.19)$$

$$L(x, 0)\Lambda^- = K(x, 0)\Lambda^+ Q_0 + G(x) \quad (3.3.20)$$

$$0 = \Lambda^- K_x(x, \xi) - K_\xi(x, \xi)\Lambda^+ - K(x, \xi)\Sigma^{++} - L(x, \xi)\Sigma^{-+} \quad (3.3.21)$$

$$0 = \Lambda^- L_x(x, \xi) + L_\xi(x, \xi)\Lambda^- - L(x, \xi)\Sigma^{--} - K(x, \xi)\Sigma^{-+} \quad (3.3.22)$$

and yields the following equations for  $C^-(x, \xi)$  and  $C^+(x, \xi)$

$$C^-(x, \xi) = \Sigma^{+-} L(x, \xi) + \int_{\xi}^x C^-(x, s) L(s, \xi) d\xi \quad (3.3.23)$$

$$C^+(x, \xi) = \Sigma^{+-} K(x, \xi) + \int_{\xi}^x C^-(x, s) K(s, \xi) d\xi \quad (3.3.24)$$

**Remark 3.3.1** For each  $x \in [0, 1]$ , Equation (3.3.23) is a Volterra equation of the second kind on  $[0, x]$  with  $C^-(x, \cdot)$  as the unknown. Besides, Equation (3.3.24) explicitly gives  $C^+(x, \xi)$  as a function of  $C^-(x, \xi)$  and  $K(x, \xi)$ . Therefore, provided the kernels  $K$  and  $L$  are well-defined and bounded, so are  $C^+$  and  $C^-$ .

Developing equations (3.3.18)–(3.3.22) leads to the following set of kernel PDEs

for  $1 \leq i \leq m, 1 \leq j \leq n$

$$\mu_i \partial_x K_{ij}(x, \xi) - \lambda_j \partial_{\xi} K_{ij}(x, \xi) = \sum_{k=1}^n \sigma_{kj}^{++} K_{ik}(x, \xi) + \sum_{p=1}^m \sigma_{pj}^{-+} L_{ip}(x, \xi) \quad (3.3.25)$$

for  $1 \leq i \leq m, 1 \leq j \leq m$

$$\mu_i \partial_x L_{ij}(x, \xi) + \mu_j \partial_{\xi} L_{ij}(x, \xi) = \sum_{p=1}^m \sigma_{pj}^{--} L_{ip}(x, \xi) + \sum_{k=1}^n \sigma_{kj}^{+-} K_{ik}(x, \xi) \quad (3.3.26)$$

along with the following set of boundary conditions

$$K_{ij}(x, x) = -\frac{\sigma_{ij}^{-+}}{\mu_i + \lambda_j} \triangleq k_{ij} \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (3.3.27)$$

$$L_{ij}(x, x) = -\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \triangleq l_{ij} \quad \text{for } 1 \leq i, j \leq m, \quad i \neq j \quad (3.3.28)$$

$$\mu_j L_{ij}(x, 0) = \sum_{k=1}^n \lambda_k K_{ik}(x, 0) q_{k,j} \quad \text{for } 1 \leq i \leq j \leq m. \quad (3.3.29)$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for  $L_{ij}(i > j)$

$$L_{ij}(1, \xi) = l_{ij}, \quad \text{for } 1 \leq j < i \leq m \quad (3.3.30)$$

While the  $g_{ij}$ , for  $1 \leq j < i \leq n$ , are given by

$$g_{ij}(x) = \mu_j L_{ij}(x, 0) - \sum_{p=1}^n \lambda_p q_{pj} K_{ip}(x, 0) \quad (3.3.31)$$

provided the  $K$  and  $L$  kernels are properly defined by (3.3.25)–(3.3.30), which we prove in the next section.

**Remark 3.3.2** *The choice of imposing (3.3.30) as the boundary condition for  $L_{ij}(1 \leq j < i \leq m)$ , on the boundary  $x = 1$  is arbitrary and was designed to ensure continuity of some of the kernels. This degree of freedom in the control design had never appeared in previous backstepping designs for hyperbolic system [26, 31]. The impact of the boundary values of  $L_{ij}$ ,  $1 \leq j < i \leq m$  on the transient behavior of the closed-loop system remains an open question, out of the scope of this chapter.*

**Remark 3.3.3** *If there are isotachic states, and the transformation explained in Remark 3.2.2 is applied, then the  $L_{ij}$  kernels for  $i, j$  corresponding to isotachic states ( $\mu_i = \mu_j$ ) have all boundary conditions of the type (3.3.29) instead of (3.3.28)—which would become singular—or (3.3.30). The results that follow do not change, but we have omitted the case for the sake of brevity.*

The well-posedness of the target system equations is assessed in the following Theorem.

**Theorem 3.3.1** *Consider system (3.3.25)–(3.3.30). There exists a unique solution  $K$  and  $L$  in  $L^\infty(\mathcal{T})$ . Moreover, all the boundary traces for the  $K$ -kernel and  $L$ -kernel are functions of  $L^\infty(0, 1)$ .*

The proof of this Theorem is the main technical difficulty of the chapter and is presented in Section 3.6.

### 3.3.3 Control law and main stabilization result

We are now ready to state the main stabilization result as follows.

**Theorem 3.3.2** *Consider system (3.2.1)–(3.2.2) with boundary conditions (3.2.3) and the following feedback control law (3.3.32)*

$$U(t) = \int_0^1 [K(1, \xi)u(\xi) + L(1, \xi)v(\xi)] d\xi - R_1 u(t, 1), \quad (3.3.32)$$

where  $K$  and  $L$  are the solutions of system (3.3.25)–(3.3.30), the existence of which are guaranteed by the Theorem 3.3.1. For any initial condition  $(u_0, v_0) \in (L^\infty(0, 1))^{(n+m) \times (n+m)}$ , the zero equilibrium is reached in finite time  $t = t_F$ , where  $t_F$  is given by (3.3.6).

First, notice that evaluating transformation (3.3.17) at  $x = 1$  yields (3.3.32). Besides, rewriting transformation (3.3.17) as follows

$$\begin{pmatrix} \alpha(t, x) \\ \beta(t, x) \end{pmatrix} = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & 0 \\ K(x, \xi) & L(x, \xi) \end{pmatrix} \begin{pmatrix} u(t, \xi) \\ v(t, \xi) \end{pmatrix} d\xi. \quad (3.3.33)$$

one notices that it is a classical Volterra equation of the second kind. One can check from, e.g., [50] and [108] that there exists a unique matrix function  $\mathcal{R} \in (L^\infty(\mathcal{T}))^{(n+m) \times (n+m)}$  such that

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} \alpha(t, x) \\ \beta(t, x) \end{pmatrix} - \int_0^x \mathcal{R}(x, \xi) \begin{pmatrix} \alpha(t, \xi) \\ \beta(t, \xi) \end{pmatrix} d\xi. \quad (3.3.34)$$

Applying Lemma 3.3.1 implies that  $(\alpha, \beta)$  go to zero in finite time  $t = t_F$ , therefore, by (3.3.34),  $(u, v)$  also converge to zero in finite time.

### 3.4 Uncollocated observer design and output feedback controller

In this section, we derive an observer that relies on the measurement of the  $v$  states at the left boundary, i.e.

$$y(t) = v(t, 0) \quad (3.4.1)$$

Then, using the estimates from the observer along with the control law (3.3.32), we derive an output feedback controller.

#### 3.4.1 Observer design

The observer equations read as follows

$$\hat{u}_t(t, x) + \Lambda^+ \hat{u}_x(t, x) = \Sigma^{++} \hat{u}(t, x) + \Sigma^{+-} \hat{v}(t, x) - P^+(x)(\hat{v}(t, 0) - v(t, 0)) \quad (3.4.2)$$

$$\hat{v}_t(t, x) - \Lambda^- \hat{v}_x(t, x) = \Sigma^{-+} \hat{u}(t, x) + \Sigma^{--} \hat{v}(t, x) - P^-(x)(\hat{v}(t, 0) - v(t, 0)) \quad (3.4.3)$$

with the following boundary conditions

$$\hat{u}(t, 0) = Q_0 v(t, 0), \quad \hat{v}(t, 1) = R_1 \hat{u}(t, 1) + \hat{u}(t) \quad (3.4.4)$$

where  $P^+(\cdot)$  and  $P^-(\cdot)$  have yet to be designed. This yields the following error system

$$\tilde{u}_t(t, x) + \Lambda^+ \tilde{u}_x(t, x) = \Sigma^{++} \tilde{u}(t, x) + \Sigma^{+-} \tilde{v}(t, x) - P^+(x) \tilde{v}(t, 0) \quad (3.4.5)$$

$$\tilde{v}_t(t, x) - \Lambda^- \tilde{v}_x(t, x) = \Sigma^{-+} \tilde{u}(t, x) + \Sigma^{--} \tilde{v}(t, x) - P^-(x) \tilde{v}(t, 0) \quad (3.4.6)$$

with the following boundary conditions

$$\tilde{u}(t, 0) = 0, \quad \tilde{v}(t, 1) = R_1 \tilde{u}(t, 1) \quad (3.4.7)$$

**Remark 3.4.1** *One should notice that the output is directly injected at the left boundary, which means potential sensor noise is only filtered throughout the spatial domain. Combining the approach of [31] and the cascade structure of (3.3.1)–(3.3.3), we now derive a target system and backstepping transformation to design observer gains  $P^+(\cdot)$  and  $P^-(\cdot)$  that yield finite-time stability of the error system (3.4.5)–(3.4.7).*

### 3.4.2 Target system and backstepping transformation

We map system (3.4.5)–(3.4.7) to the following target system

$$\tilde{\alpha}_t(t, x) + \Lambda^+ \tilde{\alpha}_x(t, x) = \Sigma^{++} \tilde{\alpha}(t, x) + \int_0^x D^+(x, \xi) \tilde{\alpha}(\xi) d\xi \quad (3.4.8)$$

$$\tilde{\beta}_t(t, x) - \Lambda^- \tilde{\beta}_x(t, x) = \Sigma^{-+} \tilde{\alpha}(t, x) + \int_0^x D^-(x, \xi) \tilde{\alpha}(\xi) d\xi \quad (3.4.9)$$

with the following boundary conditions

$$\tilde{\alpha}(t, 0) = 0, \quad \tilde{\beta}(t, 1) = R_1 \tilde{\alpha}(t, 1) - \int_0^1 H(\xi) \tilde{\beta}(\xi) d\xi \quad (3.4.10)$$

where  $D^+$  and  $D^-$  are  $L^\infty$  matrix functions on the domain  $\mathcal{T}$  and  $H \in L^\infty(0, 1)$  is an upper triangular matrix with the following structure

$$H(x) = \begin{pmatrix} 0 & h_{1,2}(x) & \cdots & h_{1,m}(x) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{m-1,m}(x) \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad (3.4.11)$$

all of which have yet to be determined.

**Proposition 3.4.1 (Stability of the observer target system)** *The solutions of system (3.4.8)–(3.3.3) converge to zero in finite time. More precisely, one has*

$$\forall t \geq t_F, \quad \tilde{\alpha} \equiv \tilde{\beta} \equiv 0 \quad (3.4.12)$$

where  $t_F$  is defined by (3.3.6).

The system consists in a cascade of the  $\tilde{\alpha}$ -system (that has zero input at the left boundary) into the  $\tilde{\beta}$ -system. Further, the  $\tilde{\beta}$  is a cascade of its slow states into its fast states. The rigorous proof follows the same steps that the proof of Lemma 3.3.1 and is therefore omitted here.

To map system (3.4.5)–(3.4.7) to the target system (3.4.8)–(3.4.10), we consider the following backstepping (Volterra) transformation

$$\tilde{u}(t, x) = \tilde{\alpha}(t, x) + \int_0^x M(x, \xi) \tilde{\beta}(\xi) d\xi \quad (3.4.13)$$

$$\tilde{v}(t, x) = \tilde{\beta}(t, x) + \int_0^x N(x, \xi) \tilde{\beta}(\xi) d\xi \quad (3.4.14)$$

where the kernels to be determined  $M$  and  $N$  are defined on the triangular domain  $\mathcal{T}$ . Deriving (3.4.13), (3.4.14) with respect to space and time yields (omitting the time argument)

$$\begin{aligned} 0 &= \tilde{u}_t(x) + \Lambda^+ \tilde{u}_x(x) - \Sigma^{++} \tilde{u}(x) - \Sigma^{+-} \tilde{v}(x) + P^+(x) \tilde{v}(0) \\ &= [-P^+(x) - M(x, 0) \Lambda^-] \tilde{\beta}(0) + [\Lambda^+ M(x, x) + M(x, x) \Lambda^- - \Sigma^{+-}] \tilde{\beta}(x) \\ &\quad + \int_0^x \left[ D^+(x, \xi) + M(x, \xi) \Sigma^{-+} + \int_\xi^x M(x, s) D^-(s, \xi) ds \right] \tilde{\alpha}(\xi) d\xi \\ &\quad + \int_0^x [\Lambda^+ M_x(x, \xi) - M_\xi(x, \xi) \Lambda^- - \Sigma^{+-} N(x, \xi) - \Sigma^{++} M(x, \xi)] \tilde{\beta}(\xi) d\xi \end{aligned}$$

and

$$\begin{aligned} 0 &= \tilde{v}_t(x) + \Lambda^+ \tilde{v}_x(x) - \Sigma^{-+} \tilde{u}(x) - \Sigma^{--} \tilde{v}(x) + P^-(x) \tilde{v}(0) \\ &= [-P^-(x) + N(x, 0) \Lambda^-] \tilde{\beta}(0) + [\Lambda^- N(x, x) - N(x, x) \Lambda^-] \tilde{\beta}(t, x) \\ &\quad + \int_0^x \left[ D^-(x, \xi) + N(x, \xi) \Sigma^{-+} + \int_\xi^x N(x, s) D^-(s, \xi) ds \right] \tilde{\alpha}(\xi) d\xi \\ &\quad - \int_0^x [\Lambda^- N_x(x, \xi) + N_\xi(x, \xi) \Lambda^- + \Sigma^{-+} M(x, \xi) + \Sigma^{--} N(x, \xi)] \tilde{\beta}(\xi) d\xi \end{aligned}$$

which yields

$$\begin{aligned} &\underline{\text{for } 1 \leq i \leq n, 1 \leq j \leq m} \\ \lambda_i \partial_x M_{ij}(x, \xi) - \mu_j \partial_\xi M_{ij}(x, \xi) &= \sum_{k=1}^n \sigma_{ik}^{++} M_{kj}(x, \xi) + \sum_{p=1}^m \sigma_{ip}^{+-} N_{pj}(x, \xi) \quad (3.4.15) \end{aligned}$$

$$\begin{aligned} &\underline{\text{for } 1 \leq i \leq m, 1 \leq j \leq m} \\ \mu_i \partial_x N_{ij}(x, \xi) + \mu_j \partial_\xi N_{ij}(x, \xi) &= \sum_{k=1}^n \sigma_{ik}^{-+} M_{kj}(x, \xi) + \sum_{p=1}^m \sigma_{ip}^{--} N_{pj}(x, \xi) \quad (3.4.16) \end{aligned}$$

along with the following set of boundary conditions

$$M_{ij}(x, x) = \frac{\sigma_{ij}^{+-}}{\mu_i + \lambda_j} \triangleq m_{ij} \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (3.4.17)$$



$$N_{ij}(x, x) = 0 \quad \text{for } 1 \leq i, j \leq m, \quad i \neq j \quad (3.4.18)$$

besides, evaluating (3.4.13),(3.4.14) at  $x = 1$  yields

$$N_{ij}(1, x) = \sum_{k=1}^n \rho_{ik} M_{kj}(1, x) \quad \text{for } 1 \leq j \leq i \leq m. \quad (3.4.19)$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for  $N_{ij}(i < j)$

$$N_{ij}(x, 0) = 0, \quad \text{for } 1 \leq i < j \leq m \quad (3.4.20)$$

while the  $d_{ij}^+$ ,  $d_{ij}^-$  and  $h_{ij}$  are given by

$$h_{ij}(x) = N_{ij}(1, x) - \sum_{k=1}^n \rho_{ik} M_{kj}(1, x) \quad (3.4.21)$$

$$d_{ij}^+(x, \xi) = - \sum_{k=1}^m M_{ik}(x, \xi) \sigma_{kj}^{-+} + \int_{\xi}^x \sum_{k=1}^m M_{ik}(x, s) d_{kj}^-(s, \xi) ds \quad (3.4.22)$$

$$d_{ij}^-(x, \xi) = - \sum_{k=1}^m N_{ik}(x, \xi) \sigma_{kj}^{-+} + \int_{\xi}^x \sum_{k=1}^m N_{ik}(x, s) d_{kj}^-(s, \xi) ds \quad (3.4.23)$$

provided the  $M$  and  $N$  kernels are properly defined. Interestingly, the well-posedness of the system of kernel equations of the observer (3.4.15)–(3.4.20) is equivalent to that of the controller kernels (3.3.25)–(3.3.30). Indeed, considering the following alternate variables

$$\bar{M}_{ij}(\chi, y) = M_{ij}(1 - y, 1 - \chi) = M_{ij}(x, \xi), \quad \bar{N}_{ij}(\chi, y) = N_{ij}(1 - y, 1 - \chi) = N_{ij}(x, \xi) \quad (3.4.24)$$

yields

$$\underline{\text{for } 1 \leq i \leq n, 1 \leq j \leq m}$$

$$\mu_j \partial_{\chi} \bar{M}_{ij}(\chi, y) - \lambda_i \partial_y \bar{M}_{ij}(\chi, y) = - \sum_{k=1}^n \sigma_{ik}^{++} \bar{M}_{kj}(\chi, y) - \sum_{p=1}^m \sigma_{ip}^{+-} \bar{N}_{pj}(\chi, y) \quad (3.4.25)$$

$$\underline{\text{for } 1 \leq i \leq m, 1 \leq j \leq m}$$

$$\mu_j \partial_{\chi} \bar{N}_{ij}(\chi, y) + \mu_i \partial_y \bar{N}_{ij}(\chi, y) = - \sum_{k=1}^n \sigma_{ik}^{-+} \bar{M}_{kj}(\chi, y) - \sum_{p=1}^m \sigma_{ip}^{--} \bar{N}_{pj}(\chi, y) \quad (3.4.26)$$

along with the following set of boundary conditions

$$\bar{M}_{ij}(\chi, \chi) = \frac{\sigma_{ij}^{+-}}{\mu_i + \lambda_j} \triangleq m_{ij} \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n \quad (3.4.27)$$

$$\bar{N}_{ij}(\chi, \chi) = 0 \quad \text{for } 1 \leq i, j \leq m, \quad i \neq j \quad (3.4.28)$$

$$\bar{N}_{ij}(\chi, 0) = \sum_{k=1}^n \rho_{ik} \bar{M}_{kj}(\chi, 0) \quad \text{for } 1 \leq j \leq i \leq m. \quad (3.4.29)$$

$$\bar{N}_{ij}(1, y) = 0, \quad \text{for } 1 \leq i < j \leq m \quad (3.4.30)$$

which has the exact same structure as the controller kernel system, the well-posedness of which is assessed in Theorem 3.3.1.

### 3.4.3 Output feedback controller

The estimates can be used in an observer-controller scheme to derive an output feedback law yielding finite-time stability of the zero equilibrium. More precisely, we have the following Lemma.

**Lemma 3.4.1** *Consider the system composed of the original (3.2.1)–(3.2.3) and target systems (3.4.2)–(3.4.4) with the following control law*

$$U(t) = \int_0^1 [K(1, \xi)\hat{u}(\xi) + L(1, \xi)\hat{v}(\xi)] d\xi - R_1\hat{u}(t, 1) \quad (3.4.31)$$

where  $K$  and  $L$  are defined by (3.3.25)–(3.3.30). Its solutions  $(u, v, \hat{u}, \hat{v})$  converge in finite time to zero.

Proposition 3.4.1 along with the existence of the observer backstepping transformation (3.4.13),(3.4.14) yields convergence of the observer error states  $\tilde{u}, \tilde{v}$  defined by (3.4.5)–(3.4.7) to zero for  $t \geq t_F^*$ . Therefore, for  $t \geq t_F$ , one has  $v(t, 0) = \hat{v}(t, 0)$  and Theorem 3.3.2 applies to the observer system (3.4.2)–(3.4.4). Therefore, for  $t \geq 2t_F$ , one has  $(\tilde{u}, \tilde{v}, \hat{u}, \hat{v}) \equiv 0$  which also yields  $(u, v) \equiv 0$ .

## 3.5 Motion planning for homodirectional systems

### 3.5.1 Definition of the motion planning problem

Consider now the case  $n = 0$ . Then system (3.2.1)–(3.2.3) reduces to

$$v_t(t, x) - \Lambda^- v_x(t, x) = \Sigma^{--} v(t, x), \quad (3.5.1)$$

---

\*the proof of this claim follows the exact same steps as in the controller case, see Section 3.3.3

where coefficients  $\Lambda^-$  and  $\Sigma^{--}$  defined as in (3.2.5) and (3.2.7), with boundary conditions

$$v(t, 1) = U(t). \quad (3.5.2)$$

For simplicity in this section we drop the super-indices in the coefficients.

Equation (3.5.1) represents a system of  $m$  states moving in the same direction (in this case, from right to left). We call such a system *homodirectional* (in opposition with *heterodirectional* systems, whose states move in different directions, such as (3.2.1)–(3.2.3) with  $n, m \neq 0$ ). Homodirectional systems are inherently finite-time stable. Physically, this is due to the fact that they are transport equations with information flowing only in one direction; thus, setting  $U(t)$  to zero in (3.5.2) and solving the equations with the method of characteristics, we obtain  $u(t, x) \equiv 0$  for  $t \geq \frac{1}{\mu_m}$  (the slowest transport time in (3.5.1)).

For (3.5.1)–(3.5.2) we consider the following *motion planning* problem. Given  $\Phi(t)$ , a known function defined as

$$\Phi(t) = (\Phi_1(t) \ \cdots \ \Phi_n(t))^T, \quad (3.5.3)$$

find the value of  $U(t)$  so that  $v(t, 0) \equiv \Phi(t)$  for  $t \geq t_M$ , for some  $t_M > 0$ .

**Remark 3.5.1** *Even though the plant (3.5.1) is finite-time stable, and a formula for the states can be written by using the method of characteristics, the motion planning problem is not trivial to solve. The entanglement of different states moving with different speeds severely complicates finding a solution. This design difficulty will be explicitly shown with an example in Section 3.5.3.*

### 3.5.2 Tracking control design

The following result solves the motion planning problem.

**Theorem 3.5.1** *Consider system (3.5.1) with boundary conditions (3.5.2), initial condition  $v_0 \in (L^2(0, 1))^m$ , and feedback control law*

$$U_i(t) = \Phi_i \left( t + \frac{1}{\mu_i} \right) - \sum_{j=1}^{i-1} \int_0^1 \frac{\mu_j}{\mu_i} L_{ij}(\xi, 0) \Phi_j \left( t + \frac{1-\xi}{\mu_i} \right) d\xi + \sum_{j=1}^{i=m} \int_0^1 L_{ij}(1, \xi) v_j(\xi) d\xi. \quad (3.5.4)$$

*Then,  $v(t, 0) \equiv \Phi(t)$  if  $t \geq t_M$ , for  $t_M = \sum_{j=1}^m \frac{1}{\mu_j}$ .*

**Remark 3.5.2** *The motion planning problem has been solved for the homodirectional case for the sake of clarity. However, it can be formulated for the full heterodirectional system (3.2.1)–(3.2.3) with only minor modifications. Noting  $u(t, 0) = Q_0 v(t, 0)$ , the values of some  $u_i$ 's could be chosen as part of the output instead of some of the  $v_i$ 's, for a total of  $m$  states. The only condition would be that all the rows of the output vector (written in terms of the  $v_i$ 's) are linearly independent.*

We start by using the backstepping transformation (3.3.17)—where the kernels  $K$  are zero due to  $n$  being zero—to map (3.5.1)–(3.5.2) into the target system

$$\beta_t(t, x) - \Lambda \beta_x(t, x) = G(x)\beta(0), \quad (3.5.5)$$

where  $G(x)$  was defined in Section 3.3.2 as a function of the kernels, with the following boundary conditions

$$\beta(t, 1) = B(t), \quad (3.5.6)$$

where  $B(t)$  in (3.5.6) is a function defined as

$$B(t) = (B_1(t) \ \cdots \ B_n(t))^T, \quad (3.5.7)$$

with components to be determined.  $B$  represents an extra degree of freedom that did not appear in the target system for the homodirectional control problem (Equation 3.3.3). It will be used to solve the motion planning problem. The presence of  $B(t)$  in the boundary conditions does not change the backstepping transformation; however it modifies the feedback control law to

$$U(t) = B(t) + \int_0^1 L(1, \xi)v(\xi)d\xi. \quad (3.5.8)$$

Now, noticing that if one sets  $x = 0$  in the transformation (3.3.17) one obtains  $v_i(t, 0) = \beta_i(t, 0)$ , it is clear that we only need to solve the motion planning problem for the target  $\beta$  system by using  $B(t)$ . The next steps of the proof are devoted to finding the value of  $B(t)$ .

Using the method of characteristics, the explicit solution for each state  $\beta_i(t, x)$  of (3.5.5) with boundary condition (3.5.6) at time  $t \geq \frac{1-x}{\mu_i}$  is

$$\beta_i(t, x) = B_i \left( t + \frac{x-1}{\mu_i} \right) + \sum_{j=1}^m \frac{1}{\mu_i} \int_x^1 G_{ij}(\xi) \beta_j \left( t + \frac{x-\xi}{\mu_i}, 0 \right) d\xi, \quad (3.5.9)$$

Using (3.3.5) and (3.3.31) in (3.5.9), we obtain

$$\beta_i(t, x) = B_i \left( t + \frac{x-1}{\mu_i} \right) + \sum_{j=1}^{i-1} \int_x^1 \frac{\mu_j}{\mu_i} L_{ij}(\xi, 0) \beta_j \left( t + \frac{x-\xi}{\mu_i}, 0 \right) d\xi. \quad (3.5.10)$$

To solve now the motion planning problem, consider first (3.5.10) for  $i = 1$  and  $x = 0$ , for  $t \geq \frac{1}{\mu_1}$ . Imposing  $\beta_1(t, 0) = \Phi_1(t)$ , we obtain:

$$\Phi_1(t) = B_1 \left( t - \frac{1}{\mu_1} \right), \quad (3.5.11)$$

thus, setting  $B_1(t) = \Phi_1 \left( t + \frac{1}{\mu_1} \right)$  for  $t \geq 0$ , we obtain the desired behavior for  $\beta_1(t, 0)$  for  $t \geq \frac{1}{\mu_1}$ . Now consider (3.5.10) for  $i = 2$  and  $x = 0$ , for  $t \geq \frac{1}{\mu_2}$ . Imposing  $\beta_2(t, 0) = \Phi_2(t)$ , we obtain:

$$\Phi_2(t) = B_2 \left( t - \frac{1}{\mu_2} \right) + \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi, 0) \beta_1 \left( t - \frac{\xi}{\mu_2}, 0 \right) d\xi. \quad (3.5.12)$$

Solving for  $B_2$  as before

$$B_2(t) = \Phi_2 \left( t + \frac{1}{\mu_2} \right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi, 0) \beta_1 \left( t + \frac{1-\xi}{\mu_2}, 0 \right) d\xi. \quad (3.5.13)$$

To be able to substitute  $\beta_1(t, 0)$  for  $\Phi_1(t)$  in the whole domain of the integral in (3.5.13) we need to wait until  $t = \frac{1}{\mu_1}$ . Thus choosing

$$B_2(t) = \Phi_2 \left( t + \frac{1}{\mu_2} \right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi, 0) \Phi_1 \left( t + \frac{1-\xi}{\mu_2}, 0 \right) d\xi, \quad (3.5.14)$$

we get that  $\beta_2(t, 0) = \Phi_2(t)$  for  $t \geq \frac{1}{\mu_1} + \frac{1}{\mu_2}$  (as we have to wait an extra  $\frac{1}{\mu_2}$  time for (3.5.14) to propagate). It is clear that this procedure can be continued for  $i = 3, \dots, m$ . Thus we obtain that

$$B_i(t) = \Phi_i \left( t + \frac{1}{\mu_i} \right) - \sum_{j=1}^{i-1} \int_0^1 \frac{\mu_j}{\mu_i} L_{ij}(\xi, 0) \Phi_j \left( t + \frac{1-\xi}{\mu_i} \right) d\xi \quad (3.5.15)$$

solves the motion problem for  $\beta_i$  for  $t \geq \sum_{j=1}^i \frac{1}{\mu_j}$ . Applying (3.5.15) for  $i = 1, \dots, m$  and substituting in (3.5.8) produces the feedback law (3.5.4), thus solving the motion planning problem in time  $t_M = \sum_{j=1}^m \frac{1}{\mu_j}$ .

**Remark 3.5.3** *Theorem 3.5.1 gives in fact tracking (in finite-time) of the desired output signal, a result stronger than pure motion planning. To obtain a pure motion planning result, one should take (3.5.10)—the explicit solutions of the target system obtained in the proof of the theorem—and substitute the values of  $B_i$  found in (3.5.15), so that the  $\beta_i$ 's are explicit functions of the  $\Phi_i$ 's. Then, using the inverse backstepping transformation (3.3.34), find the  $v_i$ 's as explicit functions of the  $\Phi_i$ 's and substitute them in the control law (3.5.4), which would then be an exclusive function of the outputs. We omit this result for the sake of brevity purposes.*

### 3.5.3 An explicit motion planning example

Next we present an specific example of a motion planning problem for  $m = 2$ . Consider the plant

$$v_{1t}(t, x) - \mu_1 v_{1x}(t, x) = \sigma_{12} v_2(t, x), \quad (3.5.16)$$

$$v_{2t}(t, x) - \mu_2 v_{2x}(t, x) = \sigma_{21} v_1(t, x), \quad (3.5.17)$$

with boundary conditions

$$v_1(t, 1) = U_1(t), \quad v_2(t, 1) = U_2(t). \quad (3.5.18)$$

The objective is to design  $U_1(t)$  and  $U_2(t)$  so that  $v_1(t, 0) = \Phi_1(t)$  and  $v_2(t, 0) = \Phi_2(t)$  for some functions  $\Phi_1, \Phi_2$  for  $t \geq t_M$ . Notice that since (3.5.16)–(3.5.18) is explicitly solvable, one might think that the inputs can be directly designed. Using the method of characteristics to explicitly write a solution of the system, one gets, after time  $t = \frac{1}{\mu_2}$ ,

$$v_1(t, 0) = U_1 \left( t - \frac{1}{\mu_1} \right) + \frac{1}{\mu_1} \int_0^1 \sigma_{12} v_2 \left( t - \frac{\xi}{\mu_1}, \xi \right) d\xi, \quad (3.5.19)$$

$$v_2(t, 0) = U_2 \left( t - \frac{1}{\mu_2} \right) + \frac{1}{\mu_2} \int_0^1 \sigma_{21} v_1 \left( t - \frac{\xi}{\mu_2}, \xi \right) d\xi. \quad (3.5.20)$$

However, if one tries to proceed as in the proof of Theorem 3.5.1, by plugging in  $\Phi_1(t)$  in (3.5.19) and  $\Phi_2(t)$  in (3.5.20), and then solve for  $U_1(t)$  and  $U_2(t)$ , one ends up with a feedback law that requires knowing *future* values of  $v_1$  and  $v_2$ , i.e., a *non-causal* (and therefore not implementable) feedback law. Thus, a direct approach does not work even for the  $m = 2$  case. To solve the motion planning problem, we resort to Theorem 3.5.1; in this particular case, the motion planning problem is solved by the inputs

$$U_1(t) = \Phi_1 \left( t + \frac{1}{\mu_1} \right) + \int_0^1 L_{11}(1, \xi) v_1(\xi) d\xi + \int_0^1 L_{12}(1, \xi) v_2(\xi) d\xi, \quad (3.5.21)$$

$$U_2(t) = \Phi_2 \left( t + \frac{1}{\mu_2} \right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi, 0) \Phi_1 \left( t + \frac{1 - \xi}{\mu_2} \right) d\xi \\ + \int_0^1 L_{21}(1, \xi) v_1(\xi) d\xi + \int_0^1 L_{22}(1, \xi) v_2(\xi) d\xi, \quad (3.5.22)$$

where the kernels  $L_{11}, L_{12}, L_{21}$  and  $L_{22}$  verify

$$\mu_1 \partial_x L_{11}(x, \xi) + \mu_1 \partial_\xi L_{11}(x, \xi) = \sigma_{21} L_{12}(x, \xi) \quad (3.5.23)$$

$$\mu_1 \partial_x L_{12}(x, \xi) + \mu_2 \partial_\xi L_{12}(x, \xi) = \sigma_{12} L_{11}(x, \xi), \quad (3.5.24)$$

$$\mu_2 \partial_x L_{21}(x, \xi) + \mu_1 \partial_\xi L_{21}(x, \xi) = \sigma_{21} L_{22}(x, \xi) \quad (3.5.25)$$

$$\mu_2 \partial_x L_{22}(x, \xi) + \mu_2 \partial_\xi L_{22}(x, \xi) = \sigma_{12} L_{21}(x, \xi), \quad (3.5.26)$$

with boundary conditions

$$\begin{aligned} L_{11}(x, 0) &= L_{12}(x, 0) = L_{22}(x, 0) = 0, \\ L_{12}(x, x) &= \frac{\sigma_{12}}{\mu_2 - \mu_1}, L_{21}(x, x) = \frac{\sigma_{21}}{\mu_1 - \mu_2}, \end{aligned} \quad (3.5.27)$$

plus the artificial boundary condition  $L_{21}(1, \xi) = l_{21}(\xi)$ , where the function  $l_{21}$  is arbitrary. These kernel PDEs can be explicitly solved using techniques akin to those used in [107]. The resulting kernels (whose validity can be verified by substitution in the kernel equations) are

$$\begin{aligned} L_{11}(x, \xi) &= \begin{cases} \frac{\sqrt{\sigma_{12}\sigma_{21}}}{\mu_2 - \mu_1} \sqrt{\frac{\mu_1\xi - \mu_2x}{\mu_1(x - \xi)}} I_1 \left( \frac{2}{\mu_1 - \mu_2} \sqrt{\frac{\sigma_{12}\sigma_{21}(x - \xi)(\mu_1\xi - \mu_2x)}{\mu_1}} \right), & \xi \geq \frac{\mu_2}{\mu_1}x \\ 0, & \xi < \frac{\mu_2}{\mu_1}x \end{cases} \\ L_{12}(x, \xi) &= \begin{cases} \frac{\sigma_{21}}{\mu_2 - \mu_1} I_0 \left( \frac{2}{\mu_1 - \mu_2} \sqrt{\frac{\sigma_{12}\sigma_{21}(x - \xi)(\mu_1\xi - \mu_2x)}{\mu_1}} \right), & \xi \geq \frac{\mu_2}{\mu_1}x \\ 0, & \xi < \frac{\mu_2}{\mu_1}x \end{cases} \\ L_{21}(x, \xi) &= \frac{\sigma_{21}\xi}{\mu_1x - \mu_2\xi} J_0 \left( \frac{2}{\mu_1 - \mu_2} \sqrt{\frac{\sigma_{12}\sigma_{21}(x - \xi)(\mu_1x - \mu_2\xi)}{\mu_2}} \right) \\ &\quad + \mu_1 \sqrt{\frac{\sigma_{21}\mu_2(x - \xi)}{\sigma_{12}(\mu_1x - \mu_2\xi)^3}} J_1 \left( \frac{2}{\mu_1 - \mu_2} \sqrt{\frac{\sigma_{12}\sigma_{21}(x - \xi)(\mu_1x - \mu_2\xi)}{\mu_2}} \right), \\ L_{22}(x, \xi) &= \xi \sqrt{\frac{\sigma_{12}\sigma_{21}}{\mu_2(x - \xi)(\mu_1x - \mu_2\xi)}} J_1 \left( \frac{2}{\mu_1 - \mu_2} \sqrt{\frac{\sigma_{12}\sigma_{21}(x - \xi)(\mu_1x - \mu_2\xi)}{\mu_2}} \right) \end{aligned}$$

where  $I_0$  and  $I_1$  are the modified Bessel functions of order 0 and 1, and  $J_0$  and  $J_1$  are the (regular) Bessel functions of order 0 and 1, respectively.

The kernels appearing in (3.5.19)–(3.5.20) are depicted in Fig 1 for the case  $\mu_1 = 1$ ,  $\mu_2 = 0.2$  and  $\sigma_{12} = 2$ ,  $\sigma_{21} = 5$ . It can be seen that  $L_{11}(1, \xi)$  and  $L_{12}(1, \xi)$  have a monotone behaviour (they are always negative or zero), whereas  $L_{21}(1, \xi)$ ,  $L_{21}(\xi, 0)$ , and  $L_{22}(1, \xi)$  are oscillatory. Fig. 2 shows  $L_{11}$  and  $L_{12}$  in the whole domain  $\mathcal{T}$ ; notice that  $L_{12}(x, \xi)$  is discontinuous along the line  $\xi = \frac{\mu_2}{\mu_1}x$  (which is the lower domain on Figure 4c), whereas  $L_{11}(x, \xi)$  is not discontinuous. On the other hand, it is evident that  $l_{21}(\xi) = L_{21}(1, \xi)$  is rather non-trivial. In fact, the procedure that was followed to find these explicit solutions was not setting a value of  $l_{21}$  a priori, but rather *extending* the domain shown in Figure 4a up to  $x = \frac{\mu_1}{\mu_1 - \mu_2}$ , so that boundary condition (3.5.27) can be used to actually find the value of  $l_{21}$ .

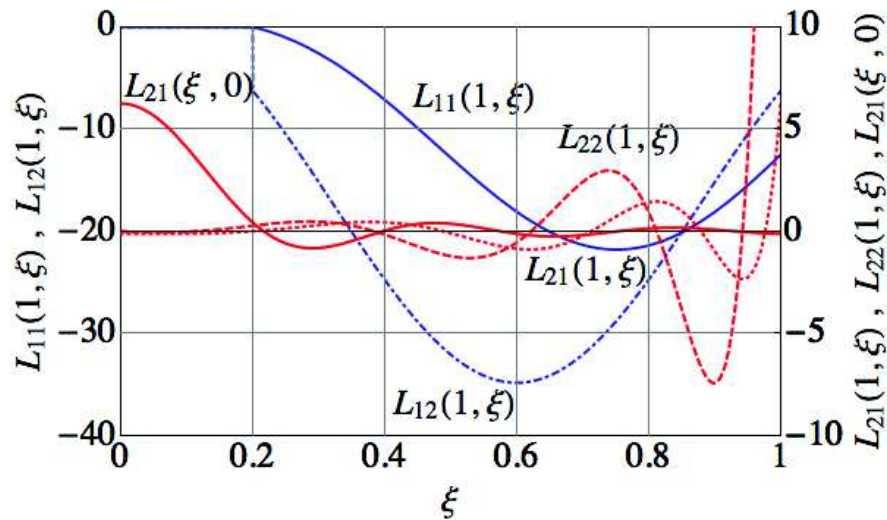


Figure 1: Motion planning kernels ( $n = 0, m = 2$ ). Solid:  $L_{11}(1, \xi)$  and  $L_{21}(\xi, 0)$ . Dash-dotted:  $L_{12}(1, \xi)$ . Dotted:  $L_{21}(1, \xi)$ . Dashed:  $L_{22}(1, \xi)$ .

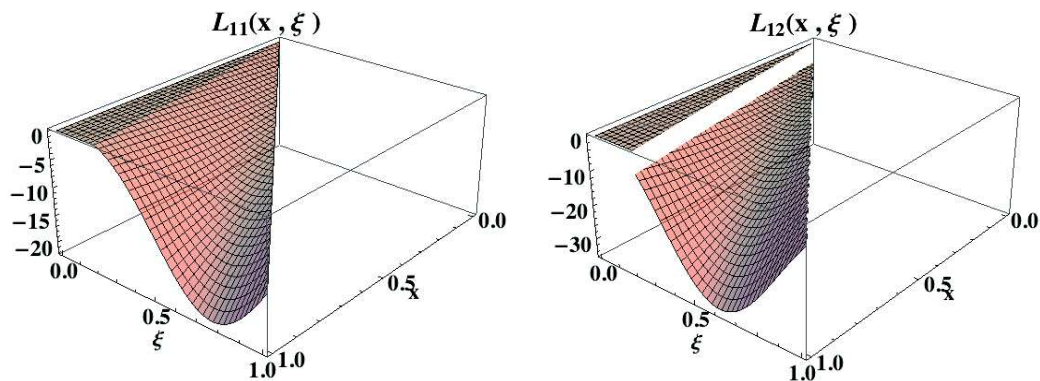


Figure 2: Motion planning kernels  $L_{11}(x, \xi)$  and  $L_{12}(x, \xi)$  ( $n = 0, m = 2$ ).

### 3.6 Proof of Theorem 3.3.1: well-posedness of the kernel equations

To prove well-posedness of the kernel equations, we classically transform them into integral equations and use the method of successive approximations.

**Remark 3.6.1** *Similar proofs have been derived for less general systems, e.g. in [26] or [31]. The proof is more involved here due to the existence of homodirectional controlled states, which lead to the homodirectional kernel PDEs (3.3.26).*



### 3.6.1 Method of characteristics

#### Characteristics of the $K$ kernels

For each  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $(x, \xi) \in \mathcal{T}$ , we define the following characteristic lines  $(x_{ij}(x, \xi; \cdot), \xi_{ij}(x, \xi; \cdot))$  corresponding to Equations (3.3.25)

$$\begin{cases} \frac{dx_{ij}}{ds}(x, \xi; s) = -\mu_i, & s \in [0, s_{ij}^F(x, \xi)] \\ x_{ij}(x, \xi; 0) = x, & x_{ij}(x, \xi; s_{ij}^F(x, \xi)) = x_{ij}^F(x, \xi) \end{cases}, \quad (3.6.1)$$

$$\begin{cases} \frac{d\xi_{ij}}{ds}(x, \xi; s) = \lambda_j, & s \in [0, s_{ij}^F(x, \xi)] \\ \xi_{ij}(x, \xi; 0) = \xi, & \xi_{ij}(x, \xi; s_{ij}^F(x, \xi)) = \xi_{ij}^F(x, \xi) \end{cases}$$

These lines, depicted on Figure 3, originate at the point  $(x, \xi)$  and terminate on the hypotenuse at the point  $(x_{ij}^F(x, \xi), \xi_{ij}^F(x, \xi))$ . The expressions of  $x_{ij}(x, \xi; s)$ ,  $\xi_{ij}(x, \xi; s)$ ,  $s_{ij}^F(x, \xi)$  and  $x_{ij}^F(x, \xi)$  are given in Appendix 3.8.1. Integrating (3.3.25)

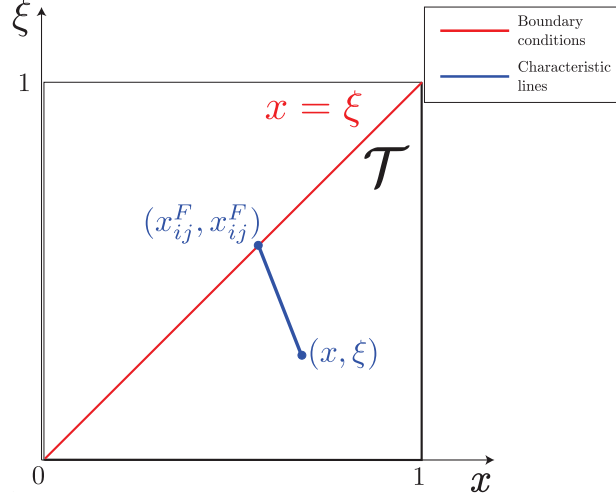


Figure 3: Characteristic lines of the  $K$  kernels

along these characteristic lines and plugging in the boundary condition (3.3.27) yields

$$K_{ij}(x, \xi) = k_{ij} + \int_0^{s_{ij}^F(x, \xi)} \left[ \sum_{k=1}^n \sigma_{kj}^{++} K_{ik}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) + \sum_{p=1}^m \sigma_{pj}^{--} L_{ip}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) \right] ds \quad (3.6.2)$$

### Characteristics of the $L$ kernels

For each  $1 \leq i \leq m$ ,  $1 \leq j \leq m$ , and  $(x, \xi) \in \mathcal{T}$ , we define the following characteristic lines  $(\chi_{ij}(x, \xi; \cdot), \zeta_{ij}(x, \xi; \cdot))$  corresponding to Equations (3.3.26)

$$\begin{cases} \frac{d\chi_{ij}}{d\nu}(x, \xi; \nu) = \varepsilon_{ij}\mu_i, & \nu \in [0, \nu_{ij}^F(x, \xi)] \\ \chi_{ij}(x, \xi; 0) = x, & \chi_{ij}(x, \xi; \nu_{ij}^F(x, \xi)) = \chi_{ij}^F(x, \xi) \end{cases}, \quad (3.6.3)$$

$$\begin{cases} \frac{d\zeta_{ij}}{d\nu}(x, \xi; \nu) = \varepsilon_{ij}\mu_j, & \nu \in [0, \nu_{ij}^F(x, \xi)] \\ \zeta_{ij}(x, \xi; 0) = \xi, & \zeta_{ij}(x, \xi; \nu_{ij}^F(x, \xi)) = \zeta_{ij}^F(x, \xi) \end{cases}$$

where  $\varepsilon_{ij}$  is defined by

$$\varepsilon_{ij}(x, \xi) = \begin{cases} 1 & \text{if } i > j \\ -1 & \text{otherwise} \end{cases} \quad (3.6.4)$$

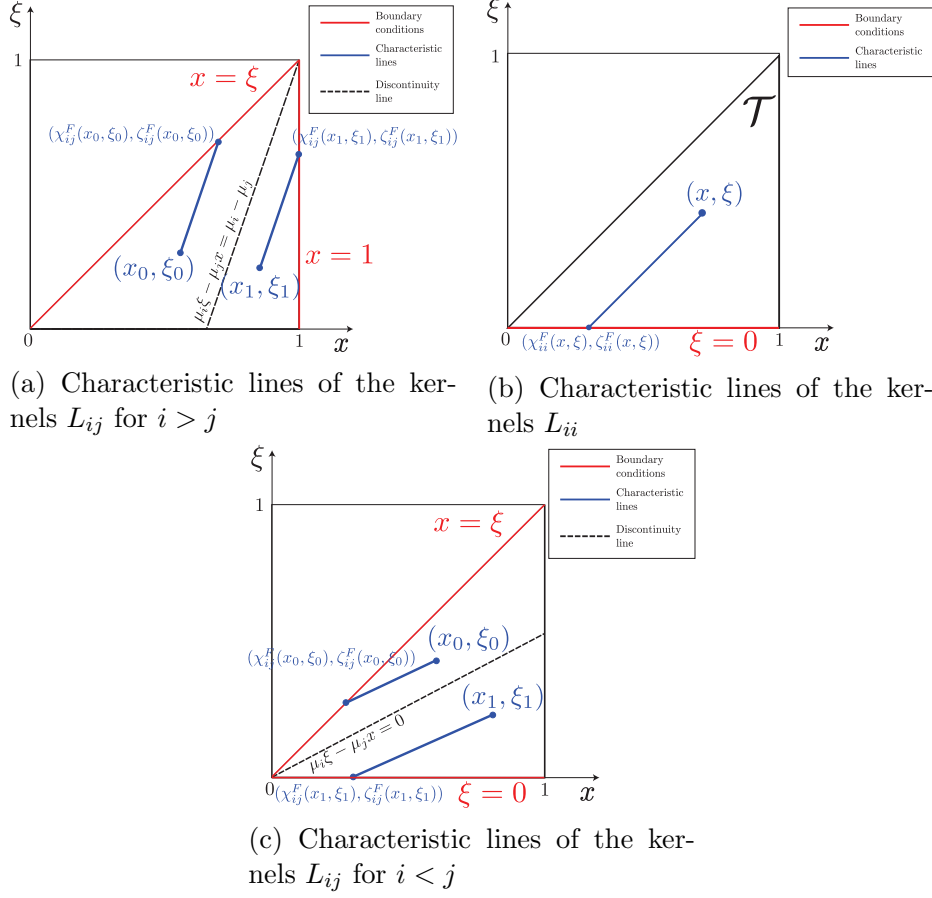
These lines all originate at  $(x, \xi)$  and terminate on  $\partial\mathcal{T}$  at the point  $(\chi_{ij}^F(x, \xi), \zeta_{ij}^F(x, \xi))$ . They are depicted on Figure 4 in the three distinct cases  $i < j$ ,  $i = j$  and  $i > j$ . The detailed expressions of  $\chi_{ij}(x, \xi; s)$ ,  $\zeta_{ij}(x, \xi; s)$ ,  $\nu_{ij}^F(x, \xi)$ ,  $\chi_{ij}^F(x, \xi)$  and  $\zeta_{ij}^F(x, \xi)$  are given in Appendix 3.8.2. Integrating (3.3.26) along these characteristics and plugging in the boundary conditions (3.3.28), (3.3.29) and (3.3.30) yields

$$\begin{aligned} L_{ij}(x, \xi) &= \delta_{ij}(x, \xi)l_{ij} + (1 - \delta_{ij}(x, \xi)) \frac{1}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} K_{ir}(\chi_{ij}^F(x, \xi), 0) \\ &- \varepsilon_{ij} \int_0^{\nu_{ij}^F(x, \xi)} \left[ \sum_{p=1}^m \sigma_{pj}^- L_{ip}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) + \sum_{k=1}^n \sigma_{kj}^+ K_{ik}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) \right] d\nu \end{aligned} \quad (3.6.5)$$

where the coefficient  $\delta_{ij}(x, \xi)$ , defined by

$$\delta_{ij}(x, \xi) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i < j \text{ and } \mu_i \xi - \mu_j x \leq 0, \\ 1 & \text{otherwise} \end{cases} \quad (3.6.6)$$

reflects the fact that some characteristics terminate on the  $\xi = 0$  boundary of  $\mathcal{T}$ , while others terminate on the hypotenuse or on the  $x = 1$  boundary of  $\mathcal{T}$ . Plugging in (3.6.2) evaluated at  $(\chi_{ij}^F(x, \xi), 0)$  yields

Figure 4: Characteristic lines of the  $L$  kernels

$$\begin{aligned}
L_{ij}(x, \xi) &= \delta_{ij}(x, \xi) l_{ij} + (1 - \delta_{ij}(x, \xi)) \frac{1}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} k_{ir} \\
&+ \frac{1 - \delta_{ij}(x, \xi)}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} \int_0^{s_{ir}^F(\chi_{ij}^F(x, \xi), 0)} \left[ \sum_{k=1}^n \sigma_{kr}^{++} K_{ik}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \right. \\
&\quad \left. + \sum_{p=1}^m \sigma_{pr}^{-+} L_{ip}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \right] ds \\
&- \varepsilon_{ij} \int_0^{\nu_{ij}^F(x, \xi)} \left[ \sum_{p=1}^m \sigma_{pj}^{--} L_{ip}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) \right. \\
&\quad \left. + \sum_{k=1}^n \sigma_{kj}^{+-} K_{ik}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) \right] d\nu
\end{aligned} \tag{3.6.7}$$

### 3.6.2 Method of successive approximations

We now use the method of successive approximations to solve equations (3.6.2),(3.6.7). Define first

$$\varphi_{ij}(x, \xi) = k_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n \quad (3.6.8)$$

$$\psi_{ij}(x, \xi) = \delta_{ij}(x, \xi)l_{ij} + (1 - \delta_{ij}(x, \xi)) \frac{1}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} k_{ir}, \quad 1 \leq i \leq m, 1 \leq j \leq m. \quad (3.6.9)$$

Besides, we define  $\mathbf{H}$  as the vector containing all the kernels, reordered line by line and stacked up as follows

$$\mathbf{H} = \begin{pmatrix} H_1 \\ \vdots \\ H_n \\ H_{n+1} \\ \vdots \\ H_{nm} \\ H_{nm+1} \\ \vdots \\ H_{nm+m^2} \end{pmatrix} = \begin{pmatrix} K_{11} \\ \vdots \\ K_{1n} \\ K_{21} \\ \vdots \\ K_{mn} \\ L_{11} \\ \vdots \\ L_{mm} \end{pmatrix}, \quad \text{and similarly} \quad \phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \\ \phi_{n+1} \\ \vdots \\ \phi_{nm} \\ \phi_{nm+1} \\ \vdots \\ \phi_{nm+m^2} \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{1n} \\ \varphi_{21} \\ \vdots \\ \varphi_{mn} \\ \psi_{11} \\ \vdots \\ \psi_{mm} \end{pmatrix} \quad (3.6.10)$$

we consider the following linear operators acting on  $\mathbf{H}$ , for  $1 \leq i \leq m, 1 \leq j \leq n$

$$\Phi_{ij}[\mathbf{H}](x, \xi) = \int_0^{s_{ij}^F(x, \xi)} \left[ \sum_{k=1}^n \sigma_{kj}^{++} K_{ik}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) + \sum_{p=1}^m \sigma_{pj}^{-+} L_{ip}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) \right] ds \quad (3.6.11)$$

and for  $1 \leq i \leq m, 1 \leq j \leq m$

$$\Psi_{ij}[\mathbf{H}](x, \xi) = (1 - \delta_{ij}(x, \xi)) \frac{1}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} \int_0^{s_{ir}^F(\chi_{ij}^F(x, \xi), 0)} \left[ \sum_{k=1}^n \sigma_{kr}^{++} K_{ik}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \right]$$

$$\begin{aligned}
& + \sum_{p=1}^m \sigma_{pr}^{-+} L_{ip}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \Big] ds \\
& - \varepsilon_{ij} \int_0^{\nu_{ij}^F(x, \xi)} \left[ \sum_{p=1}^m \sigma_{pj}^{-} L_{ip}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) + \right. \\
& \qquad \qquad \qquad \left. \sum_{k=1}^n \sigma_{kj}^{+-} K_{ik}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) \right] d\nu. \quad (3.6.12)
\end{aligned}$$

Define then the following sequence

$$\mathbf{H}^0(x, \xi) = 0, \quad (3.6.13)$$

$$\mathbf{H}^q(x, \xi) = \phi(x, \xi) + \Phi[\mathbf{H}^{q-1}](x, \xi) \quad (3.6.14)$$

$$= \begin{pmatrix} \varphi_{11}(x, \xi) + \Phi_{11}[\mathbf{H}^{q-1}](x, \xi) \\ \vdots \\ \varphi_{1n}(x, \xi) + \Phi_{1n}[\mathbf{H}^{q-1}](x, \xi) \\ \varphi_{21}(x, \xi) + \Phi_{21}[\mathbf{H}^{q-1}](x, \xi) \\ \vdots \\ \varphi_{mn}(x, \xi) + \Phi_{mn}[\mathbf{H}^{q-1}](x, \xi) \\ \psi_{11}(x, \xi) + \Psi_{11}[\mathbf{H}^{q-1}](x, \xi) \\ \vdots \\ \psi_{mm}(x, \xi) + \Psi_{mm}[\mathbf{H}^{q-1}](x, \xi) \end{pmatrix} \quad (3.6.15)$$

One should notice that if the limit exists, then  $\mathbf{H} = \lim_{q \rightarrow +\infty} \mathbf{H}^q(x, \xi)$  is a solution of the integral equations, and thus solves the original hyperbolic system. Besides, define for  $q \geq 1$  the increment  $\Delta \mathbf{H}^q = \mathbf{H}^q - \mathbf{H}^{q-1}$ , with  $\Delta \mathbf{H}^0 = \phi$  by definition. Since the functional  $\Phi$  is linear, the following equation  $\Delta \mathbf{H}^q(x, \xi) = \Phi[\mathbf{H}^{q-1}](x, \xi)$  holds. Using the definition of  $\Delta \mathbf{H}^q$ , it follows that if the sum  $\sum_{q=0}^{+\infty} \Delta \mathbf{H}^q(x, \xi)$  is finite, then

$$\mathbf{H}(x, \xi) = \sum_{q=0}^{+\infty} \Delta \mathbf{H}^q(x, \xi) \quad (3.6.16)$$

In the next section, we prove convergence of the series in  $L^\infty$ .

### 3.6.3 Convergence of the successive approximation series

To prove convergence of the series, we look for a recursive upper bound, similarly to, e.g. [64]. More precisely, let  $\varepsilon$  be such that

$$0 < \varepsilon < 1 - \max_{1 \leq j < i \leq m} \left\{ \frac{\mu_i}{\mu_j} \right\}. \quad (3.6.17)$$

Then, the following result holds

**Proposition 3.6.1** *For  $q \geq 1$ , assume that*

$$\forall (x, \xi) \in \mathcal{T}, \forall i = 1, \dots, nm + m^2 \quad |\Delta H_i^q(x, \xi)| \leq \bar{\phi} \frac{M^q (x - (1 - \varepsilon)\xi)^q}{q!} \quad (3.6.18)$$

where  $\Delta H_i^q(x, \xi)$  denotes the  $i$ -th ( $i = 1, \dots, mn + m^2$ ) component of  $\Delta H^q(x, \xi)$ , then it follows that

$$\begin{aligned} \forall (x, \xi) \in \mathcal{T}, \forall i = 1, \dots, m \forall j = 1, \dots, n \\ |\Phi_{ij}[\Delta \mathbf{H}^q](x, \xi)| \leq \bar{\phi} \frac{M^{q+1} (x - (1 - \varepsilon)\xi)^{q+1}}{(q + 1)!} \end{aligned} \quad (3.6.19)$$

and

$$\begin{aligned} \forall (x, \xi) \in \mathcal{T}, \forall i = 1, \dots, m \forall j = 1, \dots, m \\ |\Psi_{ij}[\Delta \mathbf{H}^q](x, \xi)| \leq \bar{\phi} \frac{M^{q+1} (x - (1 - \varepsilon)\xi)^{q+1}}{(q + 1)!} \end{aligned} \quad (3.6.20)$$

The proof of this proposition relies on the following Lemma, which is crucial and different with previous works.

**Lemma 3.6.1** *For  $q \in \mathbb{N}$ ,  $(x, \xi) \in \mathcal{T}$ , and  $s_{ij}^F(x, \xi)$ ,  $\nu_{ij}^F(x, \xi)$ ,  $\chi_{ij}(x, \xi, \cdot)$ ,  $\eta_{ij}(x, \xi, \cdot)$  defined as in (3.8.2), (3.8.4), (3.6.1), (3.6.3), respectively, the following inequalities holds*

$$\int_0^{s_{ij}^F(x, \xi)} (x_{ij}(x, \xi; s) - (1 - \varepsilon)\xi_{ij}(x, \xi; s))^q ds \leq M_\lambda \frac{(x - (1 - \varepsilon)\xi)^{q+1}}{q + 1}, \quad 1 \leq i \leq m, 1 \leq j \leq n \quad (3.6.21)$$

$$\int_0^{\nu_{ij}^F(x, \xi)} (\chi_{ij}(x, \xi; \nu) - (1 - \varepsilon)\zeta_{ij}(x, \xi; \nu))^q d\nu \leq M_\lambda \frac{(x - (1 - \varepsilon)\xi)^{q+1}}{q + 1}, \quad 1 \leq i, j \leq m \quad (3.6.22)$$

where

$$M_\lambda = \max_{i, s=1, \dots, m, j=1, \dots, n} \left\{ \frac{1}{\mu_i + (1 - \varepsilon)\lambda_j}, \frac{1}{-\varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_s)} \right\} \quad (3.6.23)$$

Consider the following change of variables, noting (3.6.1),

$$\tau = x_{ij}(x, \xi; s) - (1 - \varepsilon)\xi_{ij}(x, \xi; s), \quad d\tau = \left[ \frac{dx_{ij}}{ds}(x, \xi; s) - (1 - \varepsilon)\frac{d\xi_{ij}}{ds}(x, \xi; s) \right] ds \quad (3.6.24)$$

$$= (-\mu_i - (1 - \varepsilon)\lambda_j) ds \quad (3.6.25)$$

The left-hand-side of (3.6.21) becomes

$$\begin{aligned} & \int_0^{s_{ij}^F(x, \xi)} (x_{ij}(x, \xi; s) - (1 - \varepsilon)\xi_{ij}(x, \xi; s))^q ds \\ &= \int_{x - (1 - \varepsilon)\xi}^{x_{ij}^F(x, \xi) - (1 - \varepsilon)\xi_{ij}^F(x, \xi)} \frac{-\tau^q}{\mu_i + (1 - \varepsilon)\lambda_j} d\tau \\ &= \frac{(x - (1 - \varepsilon)\xi)^{q+1} - (x_{ij}^F(x, \xi) - (1 - \varepsilon)\xi_{ij}^F(x, \xi))^{q+1}}{(\mu_i + (1 - \varepsilon)\lambda_j)(q + 1)} \\ &\leq M_\lambda \frac{(x - (1 - \varepsilon)\xi)^{q+1}}{q + 1} \end{aligned} \quad (3.6.26)$$

where we have used the fact that for all  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , one has

$$x_{ij}^F(x, \xi) - (1 - \varepsilon)\xi_{ij}^F(x, \xi) \geq 0 \quad (3.6.27)$$

which is trivially satisfied since  $(x_{ij}^F(x, \xi), \xi_{ij}^F(x, \xi)) \in \partial\mathcal{T}$  and  $\varepsilon > 0$ . Consider now the following change of variables

$$\tau = \chi_{ij}(x, \xi; s) - (1 - \varepsilon)\zeta_{ij}(x, \xi; s), \quad d\tau = \left[ \frac{d\chi_{ij}}{ds}(x, \xi; s) - (1 - \varepsilon)\frac{d\zeta_{ij}}{ds}(x, \xi; s) \right] ds \quad (3.6.28)$$

$$= \varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_j) ds \quad (3.6.29)$$

Thus, left-hand-side of (3.6.22) becomes

$$\begin{aligned} & \int_0^{\nu_{ij}^F(x, \xi)} (\chi_{ij}(x, \xi; \nu) - (1 - \varepsilon)\zeta_{ij}(x, \xi; \nu))^q d\nu \\ &= \int_{x - (1 - \varepsilon)\xi}^{\chi_{ij}^F(x, \xi) - (1 - \varepsilon)\zeta_{ij}^F(x, \xi)} \frac{\tau^q}{\varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_j)} d\tau \\ &= \frac{(x - (1 - \varepsilon)\xi)^{q+1} - (\chi_{ij}^F(x, \xi) - (1 - \varepsilon)\zeta_{ij}^F(x, \xi))^{q+1}}{-\varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_j)(q + 1)} \end{aligned} \quad (3.6.30)$$

Given the definition of  $\varepsilon_{ij}$  given by (3.6.4), one has

$$-\varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_j) = \begin{cases} \mu_i - (1 - \varepsilon)\mu_j & \text{if } i \leq j \\ (1 - \varepsilon)\mu_j - \mu_i & \text{if } i > j \end{cases} \quad (3.6.31)$$

Therefore, given the definition of  $\varepsilon$  (Equation (3.6.17)) in the case  $i > j$  and the ordering of the  $\mu_i$  in the case  $i \leq j$ , one has

$$-\varepsilon_{ij}(\mu_i - (1 - \varepsilon)\mu_j) > 0 \quad (3.6.32)$$

Besides, since  $(\chi_{ij}^F(x, \xi), \zeta_{ij}^F(x, \xi)) \in \mathcal{T}$ , one has  $(\chi_{ij}^F(x, \xi) - (1 - \varepsilon)\zeta_{ij}^F(x, \xi)) > 0$  and (3.6.30) becomes

$$\int_0^{\nu_{ij}^F(x, \xi)} (\chi_{ij}(x, \xi; \nu) - (1 - \varepsilon)\zeta_{ij}(x, \xi; \nu))^q d\nu \leq M_\lambda \frac{(x - (1 - \varepsilon)\xi)^{q+1}}{q+1} \quad (3.6.33)$$

which concludes the proof.

**Remark 3.6.2** Notice that (3.6.32) also implies that, for any  $(x, \xi) \in \mathcal{T}$  and  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  the function

$$\nu \in [0, \nu_{ij}^F(x, \xi)] \mapsto \chi_{ij}(x, \xi; \nu) - (1 - \varepsilon)\zeta_{ij}(x, \xi; \nu) \quad (3.6.34)$$

is strictly decreasing, in particular the following inequality holds

$$0 \leq \chi_{ij}^F(x, \xi) - (1 - \varepsilon)\zeta_{ij}^F(x, \xi) \leq x - (1 - \varepsilon)\xi \quad (3.6.35)$$

which will be useful in the proof of Proposition 3.6.1.

**Proof of Proposition 3.6.1** Define

$$\bar{\lambda} = \max\{\lambda_n, \mu_1\}, \quad \underline{\lambda} = \max\left\{\frac{1}{\lambda_1}, \frac{1}{\mu_n}\right\}, \quad (3.6.36)$$

$$\bar{\sigma} = \max_{i,j} \{\sigma^{++}, \sigma^{-+}, \sigma^{+-}, \sigma^{--}\}, \quad \bar{q} = \max_{i,j} \{q_{ij}\} \quad (3.6.37)$$

$$M = (n\bar{\lambda}\underline{\lambda}\bar{q} + 1)(n + m)\bar{\sigma}M_\lambda, \quad \bar{\phi} = \max_{i,j} \max_{(x, \xi) \in \mathcal{T}} \{|\varphi_{i,j}(x, \xi)|, |\psi_{i,j}(x, \xi)|\} \quad (3.6.38)$$

Let now  $q \in \mathbb{N}$  and assume that

$$\forall (x, \xi) \in \mathcal{T}, \forall i = 1, \dots, nm + m^2 \quad |\Delta H_i^q(x, \xi)| \leq \bar{\phi} \frac{M^q (x - (1 - \xi))^q}{q!} \quad (3.6.39)$$



Then, for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ ,  $(x, \xi) \in \mathcal{T}$  one has

$$\begin{aligned} |\Phi_{ij}[\Delta \mathbf{H}^q](x, \xi)| &\leq \int_0^{s_{ij}^F(x, \xi)} \left| \sum_{k=1}^n \sigma_{kj}^{++} K_{ik}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) \right. \\ &\quad \left. + \sum_{p=1}^m \sigma_{pj}^{-+} L_{ip}(x_{ij}(x, \xi; s), \xi_{ij}(x, \xi; s)) \right| ds \end{aligned} \quad (3.6.40)$$

using (3.6.21) and (3.6.39), this yields

$$\begin{aligned} |\Phi_{ij}[\Delta \mathbf{H}^q](x, \xi)| &\leq (n+m)\bar{\sigma} \int_0^{s_{ij}^F(x, \xi)} \bar{\phi} \frac{M^q (x_{ij}(x, \xi; s) - (1-\varepsilon)\xi_{ij}(x, \xi; s))^q}{q!} ds \\ &\leq (n+m)\bar{\sigma} \frac{\bar{\phi} M^q}{q!} M_\lambda \frac{(x - (1-\xi))^{q+1}}{q+1} \\ &\leq \bar{\phi} \frac{M^{q+1} (x - (1-\varepsilon)\xi)^{q+1}}{(q+1)!} \end{aligned} \quad (3.6.41)$$

Similarly, for  $1 \leq i, j \leq m$ , one gets that

$$\begin{aligned} |\Psi_{ij}[\Delta \mathbf{H}^q](x, \xi)| &\leq \\ (1 - \delta_{ij}(x, \xi)) \frac{1}{\mu_j} \sum_{r=1}^n \lambda_r q_{rj} \int_0^{s_{ir}^F(\chi_{ij}^F(x, \xi), 0)} &\left| \sum_{k=1}^n \sigma_{kr}^{++} K_{ik}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \right. \\ &\quad \left. + \sum_{p=1}^m \sigma_{pr}^{-+} L_{ip}(x_{ir}(\chi_{ij}^F(x, \xi), 0; s), \xi_{ir}(\chi_{ij}^F(x, \xi), 0; s)) \right| ds \\ + \int_0^{\nu_{ij}^F(x, \xi)} &\left| \sum_{p=1}^m \sigma_{pj}^{-+} L_{ip}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) + \sum_{k=1}^n \sigma_{kj}^{+-} K_{ik}(\chi_{ij}(x, \xi; \nu), \zeta_{ij}(x, \xi; \nu)) \right| d\nu \end{aligned} \quad (3.6.42)$$

Using (3.6.39) yields

$$\begin{aligned} |\Psi_{ij}[\Delta \mathbf{H}^q](x, \xi)| &\leq \\ \bar{\lambda} \bar{\lambda} \bar{q} (n+m)\bar{\sigma} \sum_{r=1}^n \int_0^{s_{ir}^F(\chi_{ij}^F(x, \xi), 0)} &\bar{\phi} \frac{M^q (x_{ir}(\chi_{ij}^F(x, \xi), 0; s) - (1-\varepsilon)\xi_{ir}(\chi_{ij}^F(x, \xi), 0; s))^q}{q!} ds \\ + (n+m)\bar{\sigma} \int_0^{\nu_{ij}^F(x, \xi)} &\bar{\phi} \frac{M^q (\chi_{ij}(x, \xi; \nu) - (1-\varepsilon)\zeta_{ij}(x, \xi; \nu))^q}{q!} d\nu \end{aligned} \quad (3.6.43)$$

Then, using (3.6.21) at  $(x, \xi) = (\chi_{ij}^F(x, \xi), 0)$  and (3.6.22) yields

$$\begin{aligned}
 |\Psi_{ij}[\Delta\mathbf{H}^q](x, \xi)| &\leq \bar{\lambda}\lambda\bar{q}(n+m)\bar{\sigma}n\bar{\phi}M_\lambda M^q \frac{(\chi_{ij}^F(x, \xi) - (1-\varepsilon)\zeta_{ij}^F(x, \xi))^{q+1}}{(q+1)!} \\
 &\quad + (n+m)\bar{\sigma}\bar{\phi} \frac{M^q M_\lambda (x - (1-\varepsilon)\xi)^{q+1}}{(q+1)!} \quad (3.6.44)
 \end{aligned}$$

Inequality (3.6.35) yields

$$|\Psi_{ij}[\Delta\mathbf{H}^q](x, \xi)| \leq (n\bar{\lambda}\lambda\bar{q} + 1) (n+m)\bar{\sigma}\bar{\phi}M_\lambda \frac{M^q (x - (1-\varepsilon)\xi)^{q+1}}{(q+1)!} \quad (3.6.45)$$

$$\leq \bar{\phi} \frac{M^{q+1} (x - (1-\varepsilon)\xi)^{q+1}}{(q+1)!} \quad (3.6.46)$$

which concludes the proof.

Proposition 3.6.1 directly leads to Theorem 3.3.1, since by the same procedures presented in [26] and [31], one has that (3.6.16) converges and

$$|\mathbf{H}(x, \xi)| = \left| \sum_{q=0}^{+\infty} \Delta\mathbf{H}^q(x, \xi) \right| \leq \bar{\phi} e^{M(x-(1-\varepsilon)\xi)}. \quad (3.6.47)$$

### 3.7 Concluding remarks

We have presented boundary control designs for a general class of linear first-order hyperbolic systems: an output-feedback law for stabilization of heterodirectional systems and a tracking controller for motion planning for homodirectional systems.

These results bridge the gap with the results of, e.g. [77], where the null (or weak) controllability of  $(n+m)$ -state heterodirectional states is proved but no explicit design is given.

Our results open the door for a large number of related problems to be solved, e.g. collocated observer design, disturbance rejection, similarly to [1], parameter identification as in [30], stabilization of quasilinear systems as in [26].

Another important question concerns the degree of freedom given by Equation (3.3.30) in the control design. The effect of the boundary value of the kernels on the transient performances of the closed-loop system is non-trivial, yet crucial for applications.

## 3.8 Appendix: Explicit definition of the characteristic lines

In this appendix, we give the detailed expressions of the characteristic lines of the kernel equations (3.3.25),(3.3.26).

### 3.8.1 Characteristics of the $K$ kernels

The characteristic lines corresponding to Equations (3.3.25), for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  are given by

$$x_{ij}(x, \xi; s) = x - \mu_i s, \quad \xi_{ij}(x, \xi; s) = \xi + \lambda_j s, \quad s \in [0, s_{ij}^F(x, \xi)] \quad (3.8.1)$$

These lines, depicted on Figure 3 originate at  $(x, \xi)$  and arrive on the hypotenuse at  $(x_{ij}^F, \xi_{ij}^F)$ , with

$$s_{ij}^F(x, \xi) = \frac{x - \xi}{\mu_i + \lambda_j}, \quad x_{ij}^F(x, \xi) = \frac{\lambda_j x + \mu_i \xi}{\lambda_j + \mu_i} \quad (3.8.2)$$

### 3.8.2 Characteristics of the $L$ kernels

The characteristic lines corresponding to Equations (3.3.26), for  $1 \leq i \leq m$ ,  $1 \leq j \leq m$  are given by

$$\chi_{ij}(x, \xi; \nu) = x + \varepsilon_{ij} \mu_i \nu, \quad \zeta_{ij}(x, \xi; \nu) = \xi + \varepsilon_{ij} \mu_j \nu, \quad \nu \in [0, \nu_{ij}^F(x, \xi)] \quad (3.8.3)$$

where  $\varepsilon_{ij}(x, \xi)$  are given by (3.6.4) and

$$\nu_{ij}^F(x, \xi) = \begin{cases} \frac{\xi}{\mu_j} & \text{if } i = j \\ \frac{\xi}{\mu_j} & \text{if } i < j \text{ and } \mu_i \xi - \mu_j x \leq 0 \\ \frac{\mu_j}{x - \xi} & \text{if } (i < j \text{ and } \mu_i \xi - \mu_j x \geq 0) \text{ or } (i > j \text{ and } \mu_i \xi - \mu_j x \geq \mu_i - \mu_j) \\ \frac{\mu_i - \mu_j}{1 - x} & \text{if } i > j \text{ and } \mu_i \xi - \mu_j x \leq \mu_i - \mu_j. \end{cases} \quad (3.8.4)$$

Therefore the terminating points  $(\chi_{ij}^F(x, \xi), \zeta_{ij}^F(x, \xi)) \in \partial\mathcal{T}$  of the characteristics are given by

$$\chi_{ij}^F(x, \xi) = \begin{cases} x - \xi & \text{if } i = j \\ x - \frac{\mu_i}{\mu_j}\xi & \text{if } i < j \text{ and } \mu_i\xi - \mu_jx \leq 0 \\ \frac{\mu_i\xi - \mu_jx}{\mu_i - \mu_j} & \text{if } i < j \text{ and } \mu_i\xi - \mu_jx \geq 0 \\ \frac{\mu_jx - \mu_i\xi}{\mu_i - \mu_j} & \text{if } i > j \text{ and } \mu_i\xi - \mu_jx \geq \mu_i - \mu_j \\ 1 & \text{if } i > j \text{ and } \mu_i\xi - \mu_jx \leq \mu_i - \mu_j \end{cases} \quad (3.8.5)$$

and

$$\zeta_{ij}^F(x, \xi) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i < j \text{ and } \mu_i\xi - \mu_jx \leq 0 \\ \frac{\mu_i\xi - \mu_jx}{\mu_i - \mu_j} & \text{if } i < j \text{ and } \mu_i\xi - \mu_jx \geq 0 \\ \frac{\mu_jx - \mu_i\xi}{\mu_i - \mu_j} & \text{if } i > j \text{ and } \mu_i\xi - \mu_jx \geq \mu_i - \mu_j \\ \xi + \mu_j \frac{1-x}{\mu_i} & \text{if } i > j \text{ and } \mu_i\xi - \mu_jx \leq \mu_i - \mu_j. \end{cases} \quad (3.8.6)$$



# Chapter 4

## Boundary Exponential Stabilization of 1-D Inhomogeneous Quasilinear Hyperbolic Systems

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## 4.1 Introduction and Main Result

Consider the following 1-D  $n \times n$  inhomogeneous quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(x, u) \frac{\partial u}{\partial x} = F(x, u), \quad x \in [0, 1], \quad t \in [0, +\infty), \quad (4.1.1)$$

where,  $u = (u_1, \dots, u_n)^T$  is an unknown vector function of  $(t, x)$ ,  $A(x, u)$  is an  $n \times n$  matrix with  $C^2$  entries  $a_{ij}(x, u)$  ( $i, j = 1, \dots, n$ ),  $F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a vector valued function with  $C^2$  components  $f_i(x, u)$  ( $i = 1, \dots, n$ ) with respect to  $u$  and

$$F(x, 0) \equiv 0. \quad (4.1.2)$$

Denote

$$\frac{\partial F}{\partial u}(x, 0) := (f_{ij}(x))_{n \times n}, \quad (4.1.3)$$

we assume that  $f_{ij} \in C^2([0, 1])$

By the definition of hyperbolicity, we assume that  $A(x, 0)$  is a diagonal matrix with distinct and nonzero eigenvalues  $A(x, 0) = \text{diag}(\Lambda_1(x), \dots, \Lambda_n(x))$ , which are, without loss of generality, ordered as follows:

$$\Lambda_1(x) < \Lambda_2(x) < \dots < \Lambda_m(x) < 0 < \Lambda_{m+1}(x) < \dots < \Lambda_n(x), \quad \forall x \in [0, 1]. \quad (4.1.4)$$

Here and in what follows,  $\text{diag}(\Lambda_1(x), \dots, \Lambda_n(x))$  denotes the diagonal matrix whose  $i$ -th element on the diagonal is  $\Lambda_i(x)$ .

Under the assumption (4.1.4), a general kind of boundary conditions which guarantee the well-posedness of the forward problem on the domain  $\{(t, x) | t \geq 0, 0 \leq x \leq 1\}$  can be written as (see [69]):

$$x = 0 : u_s = G_s(u_1, \dots, u_m), \quad s = m + 1, \dots, n, \quad (4.1.5)$$

$$x = 1 : u_r = h_r(t), \quad r = 1, \dots, m, \quad (4.1.6)$$

where  $G_s$  are  $C^2$  functions, and we assume that they vanish at the origin, i.e.

$$G_s(0, \dots, 0) \equiv 0, \quad s = m + 1, \dots, n. \quad (4.1.7)$$

while,  $H = (h_1, \dots, h_m)^T$  are boundary controls. Our concern, in this paper, is to design a feedback control law for  $H(t)$  in order to ensure that the closed-loop system is locally exponentially stable in the  $H^2$  norm.

In other words, we are interested in the following stabilization problem for the system (4.1.1) and (4.1.5)-(4.1.6):

**Problem (ES).** For any given  $\lambda > 0$ . Suppose that  $C^1$  compatibility conditions

are satisfied at the point  $(t, x) = (0, 0)$ . Does there exist a linear feedback control  $\mathcal{B} : (H^2(0, 1))^n \rightarrow \mathbb{R}^m$ , verifying the  $C^1$  compatibility conditions at the point  $(t, x) = (0, 1)$ , such that for some  $\varepsilon > 0$ , the mixed initial-boundary value problem (4.1.1), (4.1.5)-(4.1.6) and the initial conditions

$$t = 0 : u(0, x) := \phi(x) = (\phi_1(x), \dots, \phi_n(x)), \quad (4.1.8)$$

with  $H(t) = \mathcal{B}(u(t, \cdot))$  admits a unique  $C^0([0, \infty); (H^2(0, 1))^n)$  solution  $u = u(t, x)$ , which satisfies

$$\|u(t, \cdot)\|_{H^2(0,L)} \leq C e^{-\lambda t} \|\phi(\cdot)\|_{H^2(0,L)}, \quad (4.1.9)$$

for some  $C > 0$ , provided that  $\|\phi(\cdot)\|_{H^2(0,L)} \leq \varepsilon$ ?

The boundary stabilization problem for linear and nonlinear hyperbolic system has been widely studied in the last three decades or so. During this time, three parallel mathematical approaches have emerged. The first one is the so-called ‘‘Characteristic method’’, i.e. computing corresponding bounds by using explicit evolution of the solution along the characteristic curves. With this method, Problem (ES) has been previously investigated by Greenberg and Li (see [42]) for  $2 \times 2$  systems and Li and Qin (see [69, 98]) for a generalization to  $n \times n$  homogeneous systems in the framework of  $C^1$  norm. Also, this method was developed by Li and Rao [75] to study the exact boundary controllability for general inhomogeneous quasilinear hyperbolic systems.

The second method is the ‘‘Control Lyapunov Functions method’’, which is a useful tool to analyze the asymptotic behavior of dynamical systems. This method was first used by Coron et.al. to design dissipative boundary conditions for nonlinear homogeneous hyperbolic systems in the context of both  $C^1$  and  $H^2$  norm [20, 21, 22]. More recently, it has been shown in [25] that the exponential stability strongly depends on the considered norm, i.e. a previously known sufficient condition for exponential stability with respect to the  $H^2$  norm is not sufficient in the framework of  $C^1$  norm. Although the Control Lyapunov Functions method has been introduced to study exponential stability for hyperbolic systems of balance laws, however, finding a ‘‘good’’ Lyapunov Function is the main difficulty, especially when the ‘‘natural’’ control Lyapunov functions do not lead to arbitrarily large exponential decay rate to the original system (see [10], [19, Pages 314 and 361–371]). This phenomenon indeed happens when we deal with Problem (ES) for the inhomogeneous hyperbolic systems (see [21] and [22]).

The third one is the ‘‘Backstepping method’’, which is now a popular mathematical tool to stabilize the finite dimensional and infinite dimensional dynamic systems (see [64, 65, 103, 105, 106]). In [26], a full-state feedback control law, with actuation on only one end of the domain, which achieves  $H^2$  exponential stability of the closed-loop  $2 \times 2$  linear and quasilinear hyperbolic system is derived using a backstepping method. Moreover, this method ensures that the linear hyperbolic system vanishes in finite



time. Unfortunately, the method presented in [26] can not be directly extended to  $n \times n$  cases, especially when several states convecting in the same direction are controlled (see also [31]). In [53], a first step towards generalization to  $3 \times 3$  linear hyperbolic systems is addressed, in the case where two controlled states are considered. With a similar Volterra transformation, designing an appropriate form of the target system, Hu et.al.[54] adopt a classical backstepping controller to handle the Problem (ES) for general  $n \times n$  linear hyperbolic systems. Well-posedness of the system of kernel equations, which is the main technical challenge, is shown there by an improved successive approximation method.

In this paper, based on the results for the linear case [54], we will use the linearized feedback control to stabilize the nonlinear system as it is mentioned in [26]. Although the target system is a little different from the one in [26] with a linear term involved in the equations, thanks to its special structure, we show that all the procedures to handle nonlinearities in [26] can be also adapted in this paper with more technical developments. Let us recall some definitions and statements [26]. Define the norms

$$\begin{aligned}\|u(t, \cdot)\|_{H^1} &= \|u(t, \cdot)\|_{L^2} + \|u_x(t, \cdot)\|_{L^2}, \\ \|u(t, \cdot)\|_{H^2} &= \|u(t, \cdot)\|_{H^1} + \|u_{xx}(t, \cdot)\|_{L^2}.\end{aligned}$$

Our main result is given by

**Theorem 4.1.1** *Under the assumptions in §4.1, suppose furthermore that  $C^1$  compatibility conditions are satisfied at the point  $(t, x) = (0, 0)$ , there exists a continuous linear feedback control laws  $\mathcal{B} : (H^2(0, 1))^n \rightarrow \mathbb{R}^m$ , satisfying the  $C^1$  compatibility conditions at the points  $(t, x) = (0, 1)$ , then for every  $\lambda > 0$ , there exist  $\delta > 0$  and  $c > 0$ , such that the mixed initial-boundary value problem (4.1.1), (4.1.5), (4.1.6) and (4.1.8) with  $H(t) = \mathcal{B}(u(t, \cdot))$  admits a unique  $C^0([0, \infty), (H^2(0, 1))^n)$  solution  $u = u(t, x)$ , which verifies*

$$\|u(t, \cdot)\|_{H^2} \leq ce^{-\lambda t} \|\phi\|_{H^2}, \quad (4.1.10)$$

provided that  $\|\phi\|_{H^2} \leq \delta$ .

**Remark 4.1.1** *The  $C^1$  compatibility conditions at the point  $(t, x) = (0, 0)$  are given by*

$$\phi_s(0) = G_s(\phi_1(0), \dots, \phi_m(0)) \quad s = m + 1, \dots, n, \quad (4.1.11)$$

$$\begin{aligned}f_s(0, \phi(0)) - \sum_{j=1}^n a_{sj}(0, \phi(0))\phi'_j(0) = \\ \sum_{r=1}^m \frac{\partial G_s}{\partial u_r}(\phi_1(0), \dots, \phi_m(0)) \cdot \left( f_r(0, \phi(0)) - \sum_{j=1}^n a_{rj}(0, \phi(0))\phi'_j(0) \right) \quad s = m + 1, \dots, n.\end{aligned} \quad (4.1.12)$$

The  $C^1$  compatibility conditions at the point  $(t, x) = (0, 1)$  are similar.

**Remark 4.1.2** For convenience, we always assume that the feedback controls  $H(t) = \mathcal{B}(u(t, \cdot))$  satisfy the  $C^1$  compatibility conditions at the point  $(t, x) = (0, 1)$ . However, if this property fails, one can add some dynamic terms to the controllers (see also Remark 4.3.1 and [26, Section 4]).

The rest of this paper is organized as follows. In §4.2, we review a former result on the boundary backstepping controls for  $n \times n$  linear hyperbolic system. Besides, we design a Lyapunov function to stabilize the linear system in the  $L^2$  norm. In §4.3, we input the corresponding linearized closed-loop control to the original nonlinear system and give the feedback control design. In §4.4, we prove exponential stability of zero equilibrium for the quasilinear system by using the Control Lyapunov Function method. We finally include two appendices with some technical details.

## 4.2 Preliminaries—Linear Case

In this section, we review the results on stabilization of  $n \times n$  hyperbolic linear system by using the backstepping method. Similar to the situation in [26], this procedure can be applied to locally stabilize the original nonlinear system. Consider the following  $n \times n$  hyperbolic systems

$$w_t(t, x) + \Lambda(x)w_x(t, x) = \Sigma(x)w(t, x) \quad (4.2.1)$$

where,  $w = (w_1, \dots, w_n)^T$  is a vector function of  $(t, x)$ ,  $\Lambda: [0, 1] \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$  is an  $n \times n$   $C^2$  diagonal matrix, i.e.

$$\Lambda(x) = \begin{pmatrix} \Lambda_-(x) & 0 \\ 0 & \Lambda_+(x) \end{pmatrix} \quad (4.2.2)$$

in which  $\Lambda_-(x) := \text{diag}(\lambda_1(x), \dots, \lambda_m(x))$  and  $\Lambda_+(x) := \text{diag}(\lambda_{m+1}(x), \dots, \lambda_n(x))$  are diagonal submatrices, without loss of generality, satisfying

$$\lambda_1(x) < \dots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \dots < \lambda_n(x), \forall x \in [0, 1]. \quad (4.2.3)$$

On the other hand,  $\Sigma: [0, 1] \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$  is a  $n \times n$  matrix with

$$\Sigma(x) = \begin{pmatrix} \Sigma^{--}(x) & \Sigma^{-+}(x) \\ \Sigma^{+-}(x) & \Sigma^{++}(x) \end{pmatrix} \quad (4.2.4)$$

in which  $\Sigma^{--} \in \mathcal{M}_{m,m}(\mathbb{R})$ ,  $\Sigma^{-+} \in \mathcal{M}_{m,n-m}(\mathbb{R})$ ,  $\Sigma^{+-} \in \mathcal{M}_{n-m,m}(\mathbb{R})$  and  $\Sigma^{++} \in \mathcal{M}_{n-m,n-m}(\mathbb{R})$  are all  $C^2$  submatrices with respect to  $x$ . Moreover, for any  $i =$

$1, \dots, n$ , we assume that

$$\Sigma_{ii}(x) \equiv 0, \quad \forall x \in [0, 1]. \quad (4.2.5)$$

The boundary conditions for the linear hyperbolic system (4.2.1) are given by

$$x = 0 : w_+(t, 0) = Qw_-(t, 0), \quad (4.2.6)$$

and

$$x = 1 : w_-(t, 1) = U(t). \quad (4.2.7)$$

where  $w_- \in \mathbb{R}^m, w_+ \in \mathbb{R}^{n-m}$  are defined by requiring that  $w := (w_-, w_+)^T$ ,  $U = (U_1, \dots, U_m)^T$  are boundary feedback controls,  $Q \in \mathcal{M}_{n-m, m}$  is a constant matrix. Our purpose in this section is to find a full-state feedback control law for  $U(t)$  to ensure that the closed-loop system (4.2.1), (4.2.6)-(4.2.7) is globally asymptotically stable in the  $L^2$  norm, which is defined by  $\|w(t, \cdot)\|_{L^2} = \sqrt{\sum_{i=1}^n \int_0^1 w_i^2(t, x) dx}$ .

### 4.2.1 Target System

In Section 4.2.2, it will be shown that we can transform the system (4.2.1), (4.2.6)-(4.2.7) into the following cascade system

$$\gamma_t(t, x) + \Lambda(x)\gamma_x(t, x) = G(x)\gamma(t, 0) \quad (4.2.8)$$

with the boundary conditions

$$x = 0 : \gamma_+(t, 0) = Q\gamma_-(t, 0) \quad (4.2.9)$$

and

$$x = 1 : \gamma_-(t, 1) = 0, \quad (4.2.10)$$

where  $\gamma_- \in \mathbb{R}^m, \gamma_+ \in \mathbb{R}^{n-m}$  are defined by requiring that  $\gamma := (\gamma_-, \gamma_+)^T$ ,  $G$  is a lower triangular matrix with following structure

$$G(x) = \begin{pmatrix} \mathcal{G}_1(x) & 0 \\ \mathcal{G}_2(x) & 0 \end{pmatrix} \quad (4.2.11)$$

in which  $\mathcal{G}_1 \in \mathcal{M}_{m, m}(\mathbb{R})$  is a lower triangular matrix, i.e.

$$\mathcal{G}_1(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}, \quad (4.2.12)$$

and  $\mathcal{G}_2(x) \in \mathcal{M}_{n-m,m}(\mathbb{R})$ . The coefficients of both  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are to be determined in §2.2. Next, we prove that the cascade system (4.2.8)-(4.2.10) verifies the following proposition.

**Proposition 4.2.1** *For any given matrix function  $G(\cdot) \in C^1(0,1)$ , the mixed initial-boundary value problem (4.2.8)-(4.2.10) with initial condition*

$$t = 0 : \gamma(0, x) = \gamma_0(x), \quad (4.2.13)$$

where  $\gamma_0 \in (L^2(0,1))^n$  admits a  $C^0([0, \infty); (L^2(0,1))^n)$  solution  $\gamma = \gamma(t, x)$ , which is globally exponentially stable in the  $L^2$  norm, i.e. for every  $\lambda > 0$ , there exists  $c > 0$  such that

$$\|\gamma(t, \cdot)\|_{L^2} \leq ce^{-\lambda t} \|\gamma_0\|_{L^2}. \quad (4.2.14)$$

In fact, this solution vanishes in finite time  $t > t_F$ , where  $t_F$  is given by

$$t_F = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^m \frac{1}{|\lambda_r(s)|} ds. \quad (4.2.15)$$

**Proof.** Equations (4.2.8) can be rewritten as

$$\begin{aligned} \partial_t \gamma_-(t, x) + \Lambda_-(x) \partial_x \gamma_-(t, x) &= \mathcal{G}_1(x) \gamma_-(t, 0), \\ \partial_t \gamma_+(t, x) + \Lambda_+(x) \partial_x \gamma_+(t, x) &= \mathcal{G}_2(x) \gamma_-(t, 0), \end{aligned} \quad (4.2.16)$$

then consider the following Lyapunov functional

$$V_0(t) = \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \gamma_+(t, x) dx - \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx \quad (4.2.17)$$

where  $\delta > 0$  is a parameter,  $B = \text{diag}(b_1, \dots, b_m)$  (with  $b_r > 0, r = 1, \dots, m$ ) whose coefficients are to be determined. Obviously,  $\sqrt{V_0}$  is a norm equivalent to  $\|\gamma(t, \cdot)\|_{L^2}$ . Differentiating  $V_0$  with respect to  $t$  and integrating by parts yields

$$\dot{V}_0(t) = I + II + III + IV$$

with

$$\begin{aligned} I &= \left[ -e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) + e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) \right]_0^1 \\ II &= - \int_0^1 \delta e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx - \int_0^1 \delta e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx \\ III &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \mathcal{G}_2(x) \gamma_-(t, 0) dx \end{aligned}$$

$$IV = -2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \mathcal{G}_1(x) \gamma_-(t, 0) dx$$

Noting the boundary conditions (4.2.9)-(4.2.10), we have that

$$I = -e^{\delta} \gamma_+(t, 1)^T \gamma_+(t, 1) - \gamma_-(t, 0)^T (B - Q^T Q) \gamma_-(t, 0) \quad (4.2.18)$$

$$\begin{aligned} III &\leq \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx + \gamma_-(t, 0)^T \int_0^1 e^{-\delta x} \mathcal{G}_2^T(x) (\Lambda_+(x))^{-2} \mathcal{G}_2(x) dx \gamma_-(t, 0) \\ &\leq \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx + \gamma_-(t, 0)^T \int_0^1 \mathcal{G}_2^T(x) (\Lambda_+(x))^{-2} \mathcal{G}_2(x) dx \gamma_-(t, 0) \end{aligned} \quad (4.2.19)$$

$$\begin{aligned} IV &= -2 \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \gamma_i(t, x) \frac{b_i}{\Lambda_i(x)} g_{ij}(x) \gamma_j(t, 0) dx \\ &\leq -M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx - M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_j^2(t, 0) dx \\ &\leq -M \int_0^1 e^{\delta x} \sum_{m \geq i > j \geq 1} \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx + M \mu e^{\delta} \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0) \\ &\leq -mM \int_0^1 e^{\delta x} \sum_{i=2}^m \frac{b_i}{\Lambda_i(x)} \gamma_i^2(t, x) dx + M \mu e^{\delta} \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0) \\ &\leq -mM \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx + M \mu e^{\delta} \gamma_-(t, 0)^T \mathcal{C} \gamma_-(t, 0) \end{aligned} \quad (4.2.20)$$

in which

$$M := \|G\|_{L^\infty}, \quad \mathcal{C} := \text{diag}(\mathcal{C}_1, \dots, \mathcal{C}_m) \quad (4.2.21)$$

with

$$\mathcal{C}_r := \begin{cases} \sum_{j=r+1}^m b_j, & 1 \leq r \leq m-1 \\ 0, & r = m. \end{cases} \quad (4.2.22)$$

and

$$\mu := \max_i \left\{ \frac{1}{\|\Lambda_i\|_{C^0}} \right\}. \quad (4.2.23)$$

Let

$$P = Q^T Q + \int_0^1 \mathcal{G}_2^T(x) (\Lambda_-(x))^{-2} \mathcal{G}_2(x) dx. \quad (4.2.24)$$

There exists a diagonal matrix  $S = \text{diag}(s_1, \dots, s_m)$  with  $s_r > 0$  ( $r = 1, \dots, m$ ) being large enough, such that

$$P \prec S. \quad (4.2.25)$$

where  $P \prec S$  denotes that  $S - P$  is a positive-definite matrix. This yields

$$\begin{aligned} \dot{V}_0(t) \leq & -\gamma_-(t, 0)^T \left( B - S - M\mu e^{\delta} \mathcal{C} \right) \gamma_-(t, 0) - (\delta - 1) \int_0^1 e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx \\ & - (\delta - mM\mu) \int_0^1 e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx. \end{aligned}$$

Thus, for any given  $\lambda > 0$ , picking

$$\delta > \max \{ \lambda\mu + mM\mu, \lambda\mu + 1 \} \quad (4.2.26)$$

$$b_r > \begin{cases} M\mu e^{\delta} \sum_{j=r+1}^m b_j + s_r, & 1 \leq r \leq m-1 \\ s_m, & r = m. \end{cases} \quad (4.2.27)$$

we have

$$\dot{V}_0 \leq -\lambda V_0 \quad (4.2.28)$$

where  $\lambda$  can be chosen as large as desired. It is easy to see that Parameter matrix  $B$  does exist, since one can easily check (4.2.27) by induction. This shows exponential stability of  $\gamma$  system.

To show finite-time convergence to the origin, one can find the explicit solution of (4.2.8)-(4.2.10) as follows. Define

$$\phi_i(x) = \int_0^x \frac{1}{|\lambda_i(\xi)|} d\xi, \quad 1 \leq i \leq n. \quad (4.2.29)$$

Notice that every  $\phi_i$  ( $1 \leq i \leq n$ ) is monotonically increasing  $C^2$  functions of  $x$ , and thus invertible. With the same statement in [26] and noting (4.2.8)-(4.2.12), one can express the explicit solution of  $\gamma_1$  by

$$\gamma_1(t, x) = \begin{cases} \gamma_1(0, \phi_1^{-1}(\phi_1(x) + t)) & \text{if } t < \phi_1(1) - \phi_1(x), \\ 0 & \text{if } t \geq \phi_1(1) - \phi_1(x) \end{cases} \quad (4.2.30)$$

Notice in particular that  $\gamma_1$  is identically zero for  $t \geq \phi_1(1)$ . From (4.2.8) and (4.2.12), we obtain that  $\gamma_2(t, x)$  satisfies the following equation for  $t \geq \phi_1(1)$

$$\partial_t \gamma_2(t, x) + \lambda_2(x) \partial_x \gamma_2(t, x) = 0, \quad (4.2.31)$$

with

$$\gamma_2(t, 1) = 0, \quad (4.2.32)$$

which ensures the explicit expression of  $\gamma_2(t, x)$  to be

$$\gamma_2(t, x) = \begin{cases} \gamma_2(\phi_1(1), \phi_2^{-1}(\phi_2(x) + t)) & \text{if } \phi_1(1) < t < \phi_1(1) + \phi_2(1) - \phi_2(x), \\ 0 & \text{if } t \geq \phi_1(1) + \phi_2(1) - \phi_2(x). \end{cases} \quad (4.2.33)$$

Therefore, by induction, one has that  $\gamma_r(t, x)$  ( $2 \leq r \leq m$ ) satisfies the following equations, for  $t > \sum_{k=1}^{r-1} \phi_k(1)$ ,

$$\partial_t \gamma_r(t, x) + \lambda_r(x) \partial_x \gamma_r(t, x) = 0, \quad (4.2.34)$$

with the boundary condition

$$\gamma_r(t, 1) = 0. \quad (4.2.35)$$

Thus, when  $t > \sum_{k=1}^{r-1} \phi_k(1)$ , we have

$$\gamma_r(t, x) = \begin{cases} \gamma_r\left(\sum_{k=1}^{r-1} \phi_k(1), \phi_r^{-1}(\phi_r(x) + t)\right) & \text{if } \sum_{k=1}^{r-1} \phi_k(1) < t < \sum_{k=1}^r \phi_k(1) - \phi_r(x), \\ 0 & \text{if } t \geq \sum_{k=1}^r \phi_k(1) - \phi_r(x). \end{cases} \quad (4.2.36)$$

This yields that  $\gamma_-(t, x) \equiv 0$  ( $t > \sum_{k=1}^m \phi_k(1)$ ). From the time  $t = \sum_{k=1}^m \phi_k(1)$  on, we find  $\gamma_+$  becomes the solution of the following system

$$\partial_t \gamma_+(t, x) + \Lambda_+(x) \partial_x \gamma_+(t, x) = 0 \quad (4.2.37)$$

with

$$x = 0 : \gamma_+(t, 0) \equiv 0. \quad (4.2.38)$$

Since (4.2.37)-(4.2.38) is a completely decoupled system, by the characteristic method, after  $t = t_F$ , where

$$t_F = \phi_{m+1}(1) + \sum_{r=1}^m \phi_r(1) = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^m \frac{1}{|\lambda_r(s)|} ds, \quad (4.2.39)$$

one can see that  $\gamma_+(t, x) \equiv 0 (t \geq t_F)$ , which concludes the Proof of Proposition 4.2.1.  $\blacksquare$

## 4.2.2 Backstepping transformation and Kernel Equations

To map the original system (4.2.1) into the target system (4.2.8), we use the following Volterra transformation of the second kind, which is similar to the one in [26] and [31]:

$$\gamma(t, x) = w(t, x) - \int_0^x K(x, \xi) w(t, \xi) d\xi. \quad (4.2.40)$$

We point out here that this transformation yields that  $w(t, 0) \equiv \gamma(t, 0) (\forall t > 0)$ , which is crucial to design our feedback law.

Utilizing (4.2.1) and straightforward computations, one can show that

$$\begin{aligned} \gamma_t + \Lambda(x)\gamma_x = & - \int_0^x (K_\xi(x, \xi)\Lambda(\xi) + \Lambda(x)K_x(x, \xi) + K(x, \xi)\Sigma(\xi) + K(x, \xi)\Lambda_\xi(\xi))w(t, \xi)d\xi \\ & + (\Sigma(x) + K(x, x)\Lambda(x) - \Lambda(x)K(x, x))w(t, x) - K(x, 0)\Lambda(0) \begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix} w(t, 0) \end{aligned} \quad (4.2.41)$$

The original system (4.2.1) is mapped into the target system (4.2.8) if one has the following kernel equations:

$$\Lambda(x)K_x(x, \xi) + K_\xi(x, \xi)\Lambda(\xi) + K(x, \xi)\Sigma(\xi) + K(x, \xi)\Lambda_\xi(\xi) = 0 \quad (4.2.42)$$

$$\Sigma(x) + K(x, x)\Lambda(x) - \Lambda(x)K(x, x) = 0 \quad (4.2.43)$$

$$G(x) = -K(x, 0)\Lambda(0) \begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix} \quad (4.2.44)$$

Developing equations (4.2.42)–(4.2.44) leads to the following set of kernel PDEs

$$\lambda_i(x)\partial_x K_{ij}(x, \xi) + \lambda_j(\xi)\partial_\xi K_{ij}(x, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj}\lambda'_j(\xi))K_{ik}(x, \xi) \quad (4.2.45)$$



along with the following set of boundary conditions

$$K_{ij}(x, x) = \frac{\sigma_{ij}(x)}{\lambda_i(x) - \lambda_j(x)} \triangleq k_{ij}(x) \quad \text{for } 1 \leq i, j \leq n (i \neq j), \quad (4.2.46)$$

$$K_{ij}(x, 0) = -\frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0) K_{i, m+k}(x, 0) q_{k,j} \quad \text{for } 1 \leq i \leq j \leq m. \quad (4.2.47)$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for  $K_{ij}(m \geq i > j \geq 1, n \geq j > i \geq m+1)$  on  $x = 1$ :

$$K_{ij}(1, \xi) = k_{ij}^{(1)}(\xi), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (4.2.48)$$

and the boundary conditions for  $K_{ij}(n \geq i \geq j \geq m+1)$  on  $\xi = 0$ :

$$K_{ij}(x, 0) = k_{ij}^{(2)}(x), \quad \text{for } m+1 \leq j \leq i \leq n. \quad (4.2.49)$$

where  $k_{ij}^{(1)}$  and  $k_{ij}^{(2)}$  are chosen as functions of  $C^\infty[0, 1]$  satisfying the  $C^1$  compatibility conditions at the point  $(x, \xi) = (1, 1)$  (see Remark 4.2.1). The equations evolve in the triangular domain  $\mathcal{T} = \{(x, \xi) : 0 \leq \xi \leq x \leq 1\}$ . By Theorem 4.5.1, one finds that there exists a unique piecewise  $C^2(\mathcal{T})$  solution  $K(x, \xi)$  to (4.2.45)-(4.2.49) with  $K(x, x), K(x, 0) \in C^1(0, 1)$ , provided that  $\sigma_{ij}(x)$  are  $C^2[0, 1]$ ,  $\lambda_i(x)$  are  $C^2[0, 1]$ . While  $G(x) \in C^1$  (with bounded  $C^1$  norm) is given by (4.2.44) under the well-posedness of  $K(x, 0)$ , which is proved in Theorem 4.5.1.

**Remark 4.2.1** *The  $C^1$  compatibility conditions at the point  $(x, \xi) = (1, 1)$  are given by*

$$k_{ij}(1) = k_{ij}^{(1)}(1), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n, \quad (4.2.50)$$

$$\dot{k}_{ij}^{(1)}(1) = \frac{\lambda_i(1)k'_{ij}(1) + \sum_{k=1}^n (\sigma_{kj}(1) + \delta_{kj}\lambda'_j(1))k_{ik}(1)}{\lambda_i(1) - \lambda_j(1)}, \quad (4.2.51)$$

$$\text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n.$$

### 4.2.3 The inverse transformation and stabilization for linear system

Transformation (4.2.40) is a classical Volterra equation of the second kind, one can check from Theorem 4.5.2 that there exists a unique piecewise  $C^2(\mathcal{T})$  matrix function  $L(x, \xi)$  such that

$$w(t, x) = \gamma(t, x) + \int_0^x L(x, \xi)\gamma(t, \xi)d\xi. \quad (4.2.52)$$

From the transformation (4.2.40) evaluated at  $x = 1$ , one gets the following feedback control laws

$$U_i(t) = \int_0^1 \sum_{j=1}^n K_{ij}(1, \xi) w_j(t, \xi) d\xi, \quad (i = 1, \dots, m), \quad (4.2.53)$$

which immediately leads to our feedback stabilization result for the linear system as follows:

**Theorem 4.2.1** *The mixed initial-boundary value problem (4.2.1) with the boundary conditions (4.2.6), the feedback control law (4.2.53) and initial condition*

$$t = 0 : w(0, x) = w_0(x), \quad (4.2.54)$$

*in which  $w_0 \in (L^2(0, 1))^n$ , admits a  $(L^2(0, 1))^n$  solution  $w = w(t, x)$ . Moreover, for every  $\eta > 0$ , there exists  $c > 0$  such that*

$$\|w(\cdot, t)\|_{L^2} \leq ce^{-\eta t} \|w_0\|_{L^2}. \quad (4.2.55)$$

*In fact,  $w$  vanishes in finite time  $t > t_F$ , where  $t_F$  is given by (4.2.15).*

**Remark 4.2.2** *If we focus on the linear problem,  $\Lambda$  and  $\Sigma$  can be assumed to be  $C^1([0, 1])$  and  $C^0([0, 1])$  functions. The corresponding kernels  $K$  and  $L$  are then both functions of  $L^\infty(\mathcal{T})$ .*

### 4.3 Backstepping boundary control design for non-linear system

As mentioned in [26], we wish the linear controller (4.2.53) designed by backstepping method to work locally for the corresponding nonlinear system. Let us show that this is indeed the case. Introduce

$$\varphi_i(x) := \exp\left(-\int_0^x \frac{f_{ii}(s)}{\Lambda_i(s)} ds\right) \quad i = 1, \dots, n. \quad (4.3.1)$$

One can make the following coordinates transformation

$$w(t, x) = \begin{pmatrix} \varphi_1(x) & & \\ & \ddots & \\ & & \varphi_n(x) \end{pmatrix} u(t, x) = \Phi(x)u(t, x). \quad (4.3.2)$$

Then the original control system  $u$  is transformed into the following system expressed in the new coordinates:

$$w_t(t, x) + \bar{A}(x, w)w_x(t, x) = \tilde{F}(x, w), \quad (4.3.3)$$

in which

$$\bar{A}(x, w) = \Phi(x)A(x, \Phi^{-1}(x)w)\Phi^{-1}(x), \quad (4.3.4)$$

$$\tilde{F}(x, w) = \Phi(x)F(x, \Phi^{-1}(x)w) - \bar{A}(x, w) \begin{pmatrix} \frac{f_{11}(x)}{\Lambda_1(x)} & & \\ & \ddots & \\ & & \frac{f_{nn}(x)}{\Lambda_n(x)} \end{pmatrix} w. \quad (4.3.5)$$

Obviously, one can check that

$$\tilde{F}(x, 0) = 0, \quad (4.3.6)$$

$$\bar{A}(x, 0) = \Phi(x)A(x, 0)\Phi^{-1}(x) = A(x, 0). \quad (4.3.7)$$

Moreover, define

$$\Sigma(x) = \left. \frac{\partial \tilde{F}(x, w)}{\partial w} \right|_{w=0}, \quad (4.3.8)$$

we have that

$$\Sigma_{ij}(x) = \begin{cases} \frac{\varphi_i(x)}{\varphi_j(x)} f_{ij}(x), & i \neq j, \\ 0, & i = j. \end{cases} \quad (4.3.9)$$

Therefore, we may rewrite (4.3.3) as a linear system with the same structure as (4.2.1) plus nonlinear terms:

$$w_t(t, x) + \Lambda(x)w_x(t, x) = \Sigma(x)w(t, x) + \Lambda_{NL}(x, w)w_x(t, x) + f_{NL}(x, w), \quad (4.3.10)$$

where

$$\Lambda(x) = A(x, 0), \quad (4.3.11)$$

and

$$\Lambda_{NL}(x, w) = \Lambda(x) - \bar{A}(x, w), \quad f_{NL}(x, w) = \tilde{F}(x, w) - \Sigma(x)w(t, x). \quad (4.3.12)$$

For the boundary conditions of the system (4.3.10), defining

$$Q = \left( \frac{\partial G_s}{\partial u_r} \right)_{(n-m) \times m} \Big|_{u=0} \quad \text{and} \quad G_{NL}(w_-(t, 0)) = G(w_-(t, 0)) - Qw_-(t, 0), \quad (4.3.13)$$

one obtains that

$$x = 0 : w_+(t, 0) = Qw_-(t, 0) + G_{NL}(w_-(t, 0)) \quad (4.3.14)$$

and

$$x = 1 : w_-(t, 1) = U(t). \quad (4.3.15)$$

where

$$U(t) = \begin{pmatrix} \varphi_1(1) & & \\ & \ddots & \\ & & \varphi_m(1) \end{pmatrix} H(t) = \tilde{\Phi}(1)H(t). \quad (4.3.16)$$

It is easily verified that

$$\Lambda(x, 0) = 0, \quad f_{NL}(x, 0) = \frac{\partial f_{NL}}{\partial w}(x, 0) = 0 \quad (4.3.17)$$

and

$$G_{NL}(0) = \frac{\partial G_{NL}}{\partial w}(0) = 0. \quad (4.3.18)$$

Thus, the feedback control law can be chosen as

$$h_r(t) = \tilde{\Phi}_{rr}^{-1}(1)U_r(t) = \tilde{\Phi}_{rr}^{-1}(1) \int_0^1 \sum_{j=1}^n K_{rj}(1, \xi) \tilde{\Phi}_{jj}(\xi) u_j(t, \xi) d\xi, \quad r = 1, \dots, m \quad (4.3.19)$$

where the kernels are computed from (4.2.45)–(4.2.49) with the coefficients  $\Sigma(x)$  and  $\Lambda(x)$  obtained from (4.3.9) and (4.3.11). One easily verifies that under the assumptions of §4.1, both  $\Sigma$  and  $\Lambda$  are functions of  $C^2$ .

**Remark 4.3.1** *The  $C^1$  compatibility conditions at the point  $(t, x) = (0, 1)$  for system (4.1.1) with boundary conditions (4.3.15) should be*

$$\phi_r(1) = \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi \quad r = 1, \dots, m, \quad (4.3.20)$$

$$f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi'_j(1) = \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left( f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi'_j(1) \right) \quad r = 1, \dots, m, \quad (4.3.21)$$

where  $\tilde{k}_{rk}(\xi)$  are the elements of the matrix  $\tilde{K}(\xi)$  with

$$\tilde{K}(\xi) = \tilde{\Phi}^{-1}(1)K(1, \xi)\tilde{\Phi}(\xi). \quad (4.3.22)$$

Notice that (4.3.20)-(4.3.21) depend on the feedback control design, however, there are no physical reasons that the initial data should satisfy them. In order to guarantee the initial conditions independent of these artificial conditions, we, following [26], modify the boundary controls on  $x = 1$  as

$$x = 1: \quad u_r = h_r(t) + a_r(t) + b_r(t) \quad r = 1, \dots, m, \quad (4.3.23)$$

where  $a_r$  and  $b_r$  are the state of the following dynamic systems

$$\dot{a}_r(t) = -d_r a_r(t), \quad \dot{b}_r(t) = -\tilde{d}_r b_r(t) \quad r = 1, \dots, m \quad (4.3.24)$$

with  $d_r > 0, \tilde{d}_r > 0$  and  $d_r \neq \tilde{d}_r, r = 1, \dots, m$ . By the modified control designs (4.3.23), the compatibility conditions on  $x = 1$  are rewritten by

$$\phi_r(1) = \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi + a_r(0) + b_r(0) \quad r = 1, \dots, m, \quad (4.3.25)$$

$$\begin{aligned} f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi_j'(1) = \\ \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left( f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi_j'(1) \right) - d_r a_r(0) - \tilde{d}_r b_r(0) \quad r = 1, \dots, m. \end{aligned} \quad (4.3.26)$$

For any  $1 \leq r \leq m$ , call

$$\mathcal{P}_r(\phi) = \phi_r(1) - \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi) \phi_j(\xi) d\xi \quad (4.3.27)$$

$$\begin{aligned} \mathcal{M}_r(\phi) = f_r(1, \phi(1)) - \sum_{j=1}^n a_{rj}(1, \phi(1)) \phi_j'(1) \\ - \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi) \left( f_k(1, \phi(1)) - \sum_{j=1}^n a_{kj}(1, \phi(1)) \phi_j'(1) \right) \end{aligned} \quad (4.3.28)$$

Picking

$$a_r(0) = -\frac{\mathcal{M}_r(\phi) + \tilde{d}_r \mathcal{P}_r(\phi)}{d_r - \tilde{d}_r}, \quad b_r(0) = \frac{d_r \mathcal{P}_r(\phi) + \mathcal{M}_r(\phi)}{d_r - \tilde{d}_r} \quad (4.3.29)$$

*the compatibility conditions are automatically verified. Similar stabilization results as Theorem 4.1.1 are still valid for the closed-loop system (4.1.1), (4.1.5) and (4.3.23) (see [26, Theorem 4.1]). In fact, this dynamic extension is designed to avoid restriction for artificial boundary conditions due to the compatibility conditions at the points  $(t, x) = (0, 1)$ , and it has been introduced in [17] to deal with the stabilization of the Euler equations of incompressible fluids (see also [108]).*

## 4.4 Proof of Theorem 4.1.1

In this section, we will prove the exponential stability for the system (4.1.1), (4.1.5) and (4.1.6) under the boundary feedback controls (4.3.19) by Control Lyapunov Function method. The whole proof is divided into the following steps.

### 4.4.1 Definitions

We first define some notations (omitting the time argument):

$$\|\gamma\|_\infty := \operatorname{ess\,sup}_{x \in [0,1]} |\gamma(x)|, \quad \|\gamma\|_{L^p} := \left( \int_0^1 |\gamma(\xi)|^p d\xi \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty. \quad (4.4.1)$$

For a  $n \times n$  matrix, denote

$$|M| := \max\{\|M\gamma\|_{L^\infty} : \gamma \in \mathbb{R}^n, |\gamma| = 1\}. \quad (4.4.2)$$

For a piecewise kernel matrix  $K(x, \xi)$ , which is a continuous function on each domain  $D_i (i = 1, \dots, \mathcal{S})$ , respectively, with

$$\mathcal{T} = \bigcup_{i=1}^{\mathcal{S}} D_i, \quad (4.4.3)$$

$$D_i \cap D_j = \emptyset, (i \neq j). \quad (4.4.4)$$

Let

$$\|K\|_\infty := \max_i \sup_{(x,\xi) \in D_i} |K(x, \xi)|. \quad (4.4.5)$$

As before, we recall the following symbols of [26] for simplicity:

$$\mathcal{K}[\gamma](t, x) = \gamma(t, x) - \int_0^x K(x, \xi) \gamma(t, \xi) d\xi, \quad (4.4.6)$$

$$\mathcal{L}[\gamma](t, x) = \gamma(t, x) + \int_0^x L(x, \xi) \gamma(t, \xi) d\xi, \quad (4.4.7)$$

$$\mathcal{K}_1[\gamma](t, x) = -K(x, x)\gamma(t, x) + \int_0^x K_\xi(x, \xi)\gamma(t, \xi)d\xi, \quad (4.4.8)$$

$$\mathcal{K}_2[\gamma](t, x) = -K(x, x)\gamma(t, x) - \int_0^x K_x(x, \xi)\gamma(t, \xi)d\xi, \quad (4.4.9)$$

$$\mathcal{L}_1[\gamma](t, x) = L(x, x)\gamma(t, x) + \int_0^x L_x(x, \xi)\gamma(t, \xi)d\xi. \quad (4.4.10)$$

Define  $F_1[\gamma]$  and  $F_2[\gamma]$  as

$$F_1[\gamma] := \Lambda_{NL}(x, \mathcal{L}[\gamma]), \quad F_2[\gamma] := f_{NL}(x, \mathcal{L}[\gamma]). \quad (4.4.11)$$

To prove our result, we notice that if we apply the (inverse) backstepping transformation (4.2.40) to the nonlinear system (4.3.10), we obtain the following transformed system

$$\begin{aligned} & \gamma_t(t, x) + \Lambda(x)\gamma_x(t, x) - G(x)\gamma(t, 0) \\ &= \mathcal{K}[\Lambda_{NL}(x, w)w_x] + \mathcal{K}[f_{NL}(x, w)] \\ &= \mathcal{K}[\Lambda_{NL}(x, w)\gamma_x] + \mathcal{K}[\Lambda_{NL}(x, w)\mathcal{L}_1[\gamma]] + \mathcal{K}[f_{NL}(x, w)] \\ &= F_3[\gamma, \gamma_x] + F_4[\gamma] \end{aligned} \quad (4.4.12)$$

where

$$\begin{aligned} F_3 &= \mathcal{K}[F_1[\gamma]\gamma_x], \\ F_4 &= \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\gamma] + F_2[\gamma]]. \end{aligned}$$

The boundary conditions are

$$x = 0 : \gamma_+(t, 0) = Q\gamma_-(t, 0) + G_{NL}(\gamma_-(t, 0)) \quad (4.4.13)$$

and

$$x = 1 : \gamma_-(t, 1) = 0. \quad (4.4.14)$$

Notice that here we may lose the regularity on the point  $(0, 0)$  for the kernels  $K$  and  $L$ , which leads both of them to be discontinuous (see [54]). However, by the assumptions on the coefficients and applying Theorem 4.5.1 and Theorem 4.5.2, the direct and inverse transformations (4.2.40) and (4.2.52) have  $C^2$  piecewise kernels functions. Fortunately, differentiating twice with respect to  $x$  in these transformations, by the similar argument in [26] and [108, Proposition 3.1] as well as the additive property of the integral, it can be shown that the  $H^2$  norm of  $\gamma$  is equivalent to the  $H^2$  norm of  $w$ . Thus, if we show  $H^2$  local stability of the origin for (4.4.12)-(4.4.14), the same holds for  $w$  i.e.  $u$ .

In order to get the desired  $H^2$  estimation for  $\gamma$ , the things left are just estimating

the growth of  $\|\gamma\|_{L^2}$ ,  $\|\gamma_t\|_{L^2}$  and  $\|\gamma_{tt}\|_{L^2}$ , respectively.

#### 4.4.2 Analyzing the growth of $\|\gamma\|_{L^2}$

Let

$$F_3[\gamma, \gamma_x] = (F_3^-[\gamma, \gamma_x], F_3^+[\gamma, \gamma_x])^T, \quad F_4[\gamma] = (F_4^-[\gamma], F_4^+[\gamma])^T. \quad (4.4.15)$$

where  $F_3^-$  and  $F_4^- \in \mathbb{R}^m$ ,  $F_3^+$  and  $F_4^+ \in \mathbb{R}^{n-m}$ .

Define

$$V_1(t) = \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \gamma_+(t, x) dx - \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \gamma_-(t, x) dx \quad (4.4.16)$$

Differentiating  $V_1$  with respect to time and integrating by parts yields

$$\dot{V}_1(t) = V + VI + VII + VIII + IX + X$$

with

$$\begin{aligned} V &= [-e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) + e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x)]_0^1 \\ VI &= - \int_0^1 \delta e^{-\delta x} \gamma_+(t, x)^T \gamma_+(t, x) dx - \int_0^1 \delta e^{\delta x} \gamma_-(t, x)^T B \gamma_-(t, x) dx \\ VII &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} \mathcal{G}_2(x) \gamma_-(t, 0) dx \\ VIII &= - 2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} \mathcal{G}_1(x) \gamma_-(t, 0) dx \\ IX &= 2 \int_0^1 e^{-\delta x} \gamma_+(t, x)^T (\Lambda_+(x))^{-1} (F_3^+[\gamma, \gamma_x] + F_4^+[\gamma]) dx \\ X &= - 2 \int_0^1 e^{\delta x} \gamma_-(t, x)^T B (\Lambda_-(x))^{-1} (F_3^-[\gamma, \gamma_x] + F_4^-[\gamma]) dx \end{aligned}$$

By the same argument in [26] and noting Lemma 4.6.2, we have

$$\begin{aligned} IX + X &\leq K_1 \int_0^1 |\gamma| (|F_3[\gamma, \gamma_x]| + |F_4[\gamma]|) dx \\ &\leq K_2 (\|\gamma_x\|_\infty V_1 + V_1^{\frac{3}{2}}). \end{aligned} \quad (4.4.17)$$



Moreover, for  $\|\gamma\|_\infty \leq \delta$ ,  $|G_{NL}(\gamma_-(t, 0))| \leq K_3|\gamma_-(t, 0)|$ , then

$$\begin{aligned} V &= -e^{-\delta}\gamma_+(t, 1)^T\gamma_+(t, 1) + e^\delta\gamma_-(t, 1)^TB\gamma_-(t, 1) + \gamma_+(t, 0)^T\gamma_+(t, 0) - \gamma_-(t, 0)^TB\gamma_-(t, 0) \\ &\leq -\gamma_-(t, 0)^T\left(B - Q^TQ - K_3^2I_m\right)\gamma_-(t, 0) \end{aligned} \quad (4.4.18)$$

By (4.2.19) and (4.2.20), one immediately obtains

$$\begin{aligned} \dot{V}_1(t) &\leq -\gamma_-(t, 0)^T\left(B - \tilde{S} - M\mu e^\delta\mathcal{C}\right)\gamma_-(t, 0) - (\delta - 1)\int_0^1 e^{-\delta x}\gamma_+(t, x)^T\gamma_+(t, x)dx \\ &\quad - (\delta - mM\mu)\int_0^1 e^{\delta x}\gamma_-(t, x)^TB\gamma_-(t, x)dx + K_2(V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1) \end{aligned}$$

where  $M$ ,  $\mathcal{C}$ ,  $\mu$  are given by (4.2.21) and (4.2.23),  $\tilde{S} := S + K_3^2I_m$  with  $S$  stated in (4.2.25). Thus, for any given  $\lambda_1 > 0$ , picking

$$\delta > \max\{\lambda_1\mu + mM\mu, \lambda_1\mu + 1\} \quad (4.4.19)$$

$$b_r := \begin{cases} M\mu e^\delta \sum_{j=r+1}^m b_j + \tilde{s}_r, & 1 \leq r \leq m-1 \\ \tilde{s}_m, & r = m. \end{cases} \quad (4.4.20)$$

we have the following

**Proposition 4.4.1** *For any given  $\lambda_1 > 0$ , there exists  $\delta_1 > 0$  and  $K_2 > 0$ , such that*

$$\dot{V}_1 \leq -\lambda_1 V_1 + K_2(V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1), \quad (4.4.21)$$

provided  $\|\gamma\|_\infty \leq \delta_1$ .

### 4.4.3 Analyzing the growth of $\|\gamma_t\|_{L^2}$

Let  $\zeta = \gamma_t$ . Taking the partial derivative with  $t$  in (4.4.12) yields:

$$\zeta_t(t, x) + (\Lambda(x) - F_1[\gamma])\zeta_x(t, x) - G(x)\zeta(t, 0) = F_5[\gamma, \gamma_x, \zeta] + F_6[\gamma, \zeta], \quad (4.4.22)$$

where

$$F_5 = \mathcal{K}_1[F_1[\gamma]\zeta] + \int_0^x K(x, \xi)F_{12}[\gamma, \gamma_x]\zeta(\xi)d\xi + K(x, 0)\Lambda_{NL}(0, \gamma(0))\zeta(0) + \mathcal{K}[F_{11}[\gamma, \zeta]\gamma_x], \quad (4.4.23)$$

$$F_6 = \mathcal{K}[F_{11}[\gamma, \zeta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_{21}[\gamma, \zeta]], \quad (4.4.24)$$

with

$$\begin{aligned} F_{11} &= \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta], \\ F_{12} &= \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])(\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial \Lambda_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma]), \end{aligned} \quad (4.4.25)$$

and

$$F_{21} = \frac{\partial f_{NL}}{\partial \gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]. \quad (4.4.26)$$

The boundary conditions are given by

$$x = 0 : \zeta_+(t, 0) = Q\zeta_-(t, 0) + \frac{\partial G_{NL}}{\partial \gamma_-}(\gamma_-(t, 0))\zeta_-(t, 0) \quad (4.4.27)$$

and

$$x = 1 : \zeta_-(t, 1) = 0, \quad (4.4.28)$$

in which  $\zeta_- \in \mathbb{R}^m, \zeta_+ \in \mathbb{R}^{n-m}$  are defined by requiring that  $\zeta := (\zeta_-, \zeta_+)^T$ .

Similarly as in [26], we need the following lemma in order to find a Lyapunov function for  $\zeta(t, x)$ :

**Lemma 4.4.1** *There exists  $\delta > 0$  such that, for any  $\|\gamma\|_\infty \leq \delta$ , there exists a symmetric matrix  $R[\gamma]$  satisfying the identity*

$$R[\gamma](\Lambda(x) - F_1[\gamma]) - (\Lambda(x) - F_1[\gamma])^T R[\gamma] = 0. \quad (4.4.29)$$

Moreover, we have that

$$|R[\gamma](x)| \leq c_1 + c_2\|\gamma\|_\infty, \quad (4.4.30)$$

$$\left| ((R[\gamma] - D(x))\Lambda(x))_x \right| \leq c_2(\|\gamma\|_\infty + \|\gamma_x\|_\infty) \quad (4.4.31)$$

$$|(R[\gamma])_t| \leq c_3(|\zeta| + \|\zeta\|_{L^1}), \quad (4.4.32)$$

where  $c_1, c_2$  and  $c_3$  are positive constants, and

$$D(x) = \begin{pmatrix} -e^{\delta x} B(\Lambda_-(x))^{-1} & 0 \\ 0 & e^{-\delta x} (\Lambda_+(x))^{-1} \end{pmatrix}. \quad (4.4.33)$$

**Proof:** Denote  $\mathcal{D}_n(x)$  as the set of  $n \times n$  diagonal matrices with  $C^1$  elements. Let  $\Lambda(x) := \text{diag}(\Lambda_1(x), \dots, \Lambda_n(x)) \in \mathcal{D}_n(x)$  be such that  $\Lambda_i(x) \neq \Lambda_j(x) (i \neq j \forall x \in [0, 1])$  holds. Notice that  $D \in \mathcal{D}_n(x)$ . Based on the proof in [21, Lemma 4.1], one can easily see that there exist a positive real number  $\eta$  and a map  $\mathcal{N} : \{M \in$

$\mathcal{M}_{n,n}(\mathbb{R}; x; \|M(x) - \Lambda(x)\|_{C^1} < \eta\} \rightarrow \mathcal{S}_n$  of class  $C^\infty$  such that

$$\mathcal{N}(\Lambda(x)) = D(x), \quad (4.4.34)$$

and

$$\mathcal{N}(M)M - M^T\mathcal{N}(M) = 0 \quad \forall M \in \mathcal{M}_{n,n}(\mathbb{R}; x), \quad \|M(x) - \Lambda(x)\|_{C^1} < \eta. \quad (4.4.35)$$

It then suffices to define  $R[\gamma]$  by

$$R[\gamma] = \mathcal{N}(\Lambda(x) - F_1[\gamma]). \quad (4.4.36)$$

Moreover, by the regularity of  $\mathcal{N}$  and Lemma 4.6.2–4.6.3, one can show that

$$\begin{aligned} |R[\gamma]| &\leq |D(x)| + |R[\gamma] - D(x)| \\ &\leq c_4 + c_5|F_1[\gamma]| \\ &\leq c_4 + c_6\|\gamma\|_\infty, \end{aligned} \quad (4.4.37)$$

$$\begin{aligned} \left| ((R[\gamma] - D(x))\Lambda(x))_x \right| &\leq |(R[\gamma] - D(x))_x\Lambda(x)| + |(R[\gamma] - D(x))\Lambda_x(x)| \\ &\leq c_7|F_{12}| + c_8|F_1| \\ &\leq c_9(\|\gamma\|_\infty + \|\gamma_x\|_\infty) \end{aligned} \quad (4.4.38)$$

and

$$|R[\gamma]_t| \leq c_{10} \left| \frac{\partial F_1[\gamma]}{\partial t} \right| \quad (4.4.39)$$

$$\leq c_{10}|F_{11}[\gamma, \zeta]| \quad (4.4.40)$$

$$\leq c_{11}(|\zeta| + \|\zeta\|_{L^1}). \quad (4.4.41)$$

This concludes the proof of Lemma 4.1. ■

Define

$$V_2(t) = \int_0^1 \zeta^T(t, x)R[\gamma]\zeta(t, x)dx. \quad (4.4.42)$$

Using (4.4.29) and straightforward computations, one can show that

$$\dot{V}_2(t) = XI + XII + XIII + XIV + XV$$

with

$$XI = \int_0^1 \zeta^T(t, x)(R[\gamma](\Lambda(x) - F_1[\gamma]))_x\zeta(t, x)dx$$

$$\begin{aligned}
 XII &= -[\zeta^T(t, x)R[\gamma](\Lambda(x) - F_1[\gamma])\zeta(t, x)]_{x=0}^{x=1} \\
 XIII &= \int_0^1 \zeta(t, x)(R[\gamma])_t \zeta(t, x) dx \\
 XIV &= 2 \int_0^1 \zeta^T(t, x)R[\gamma]F_5[\gamma, \gamma_x, \zeta, \zeta_x] dx + 2 \int_0^1 \zeta^T(t, x)R[\gamma]F_6[\gamma, \zeta] dx \\
 XV &= 2 \int_0^1 \zeta^T(t, x)R[\gamma]G(x)\zeta(t, 0) dx.
 \end{aligned}$$

For  $XII$  and  $XV$ , by the boundary conditions (4.4.27)–(4.4.28), we have

$$\begin{aligned}
 XII + XV &= -[\zeta^T(t, x)(D(x) + \Theta[\gamma])(\Lambda(x) - F_1[\gamma])\zeta(t, x)]_{x=0}^{x=1} \\
 &\quad + 2 \int_0^1 \zeta^T(t, x)(D(x) + \Theta[\gamma])G(x)\zeta(t, 0) dx \\
 &= -[\zeta^T(t, x)(D(x)\Lambda(x) + \Theta[\gamma]\Lambda(x) - D(x)F_1[\gamma] - \Theta[\gamma]F_1[\gamma])\zeta(t, x)]_{x=0}^{x=1} \\
 &\quad + 2 \int_0^1 \zeta^T(t, x)D(x)G(x)\zeta(t, 0) dx + 2 \int_0^1 \zeta^T(t, x)\Theta[\gamma]G(x)\zeta(t, 0) dx \\
 &\leq -\zeta_-(t, 0)^T \left( B - \tilde{S} - M\mu e^{\delta} \mathcal{C} - K_3 \|\gamma\|_{\infty} I_m \right) \zeta_-(t, 0) \\
 &\quad + \int_0^1 e^{-\delta x} \zeta_+(t, x)^T \zeta_+(t, x) dx + mM\mu \int_0^1 e^{\delta x} \zeta_-(t, x)^T B \zeta_-(t, x) dx \\
 &\quad + K_4 \|\gamma\|_{\infty} V_2
 \end{aligned}$$

As stated in [26], we obtain

$$XI \leq -\lambda_2 V_2 + K_4 \|\zeta\|_{L^2}^2 (\|\gamma\|_{\infty} + \|\gamma_x\|_{\infty}), \quad (4.4.43)$$

$$XIII \leq K_5 \|\zeta\|_{L^2}^2 \|\zeta\|_{\infty}, \quad (4.4.44)$$

$$XIV \leq K_6 \left( \|\zeta\|_{L^2}^2 (\|\gamma\|_{\infty} + \|\gamma_x\|_{\infty}) + \|\zeta\|_{L^2} |\zeta(t, 0)| |\gamma(t, 0)| \right). \quad (4.4.45)$$

Following Lemma 4.6.5, we are in the position to conclude that

**Proposition 4.4.2** *For any given  $\lambda_2 > 0$ , there exists  $\delta_2 > 0$  and  $K_7 > 0$ , such that*

$$\dot{V}_2 \leq -\lambda_2 V_2 + K_7 (\|\zeta\|_{\infty} + \|\gamma\|_{\infty}) V_2, \quad (4.4.46)$$

*provided that  $\|\gamma\|_{\infty} \leq \delta_2$ .*

#### 4.4.4 Analyzing the growth of $\|\gamma_{tt}\|_{L^2}$

We next deal with  $\|\gamma_{tt}\|_{L^2}$ . Define  $\theta = \gamma_{tt}$ . Taking a partial derivative with respect to  $t$  for (4.4.22), one obtains an equation of  $\theta$ :

$$\theta_t + [\Lambda(x) - F_1[\gamma]]\theta_x = G(x)\theta(t, 0) + F_7[\gamma, \gamma_x, \zeta, \zeta_x, \theta] + F_8[\gamma, \zeta, \theta], \quad (4.4.47)$$

where

$$\begin{aligned} F_7 &= \mathcal{K}_1[F_{11}[\gamma, \zeta]\zeta] + \int_0^x K(x, \xi)F_{12}[\gamma, \gamma_x]\theta(\xi)d\xi + \mathcal{K}_1[F_1[\gamma]\theta] \\ &\quad + \int_0^x K(x, \xi)F_{14}[\gamma, \gamma_x, \zeta, \zeta_x]\zeta(\xi)d\xi + K(x, 0)\frac{\partial\Lambda_{NL}}{\partial\gamma}(0, \gamma(0))\zeta(0)\zeta(0) \\ &\quad + K(x, 0)\Lambda_{NL}(0, \gamma(0))\theta(0) + \mathcal{K}[F_{11}[\gamma, \zeta]\zeta_x] + \mathcal{K}[F_{13}[\gamma, \zeta, \theta]\gamma_x], \\ F_8 &= 2\mathcal{K}[F_{11}[\gamma, \zeta]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\theta]] + \mathcal{K}[F_{13}[\gamma, \zeta, \theta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_{22}[\gamma, \zeta, \theta]] \end{aligned} \quad (4.4.48)$$

$$(4.4.49)$$

with

$$F_{13} = \frac{\partial\Lambda_{NL}^2}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial\Lambda_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\theta], \quad (4.4.50)$$

$$\begin{aligned} F_{14} &= \frac{\partial\Lambda_{NL}^2}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta](\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial\Lambda_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])(\zeta_x + \mathcal{L}_1[\zeta]) \\ &\quad + \frac{\partial^2\Lambda_{NL}}{\partial x\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta], \end{aligned} \quad (4.4.51)$$

$$F_{22} = \frac{\partial^2 f_{NL}}{\partial\gamma^2}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial f_{NL}}{\partial\gamma}(x, \mathcal{L}[\gamma])\mathcal{L}[\theta]. \quad (4.4.52)$$

The boundary conditions of  $\theta$  are given by

$$\begin{aligned} x = 0 : \theta_+(t, 0) &= Q\theta_-(t, 0) + \frac{\partial G_{NL}}{\partial\gamma_-}(\gamma_-(t, 0))\theta_-(t, 0) \\ &\quad + \frac{\partial^2 G_{NL}}{\partial\gamma_-^2}(\gamma_-(t, 0))\zeta_-(t, 0)\zeta_-(t, 0) \end{aligned} \quad (4.4.53)$$

and

$$x = 1 : \theta_-(t, 1) = 0. \quad (4.4.54)$$

where  $\theta_- \in \mathbb{R}^m, \theta_+ \in \mathbb{R}^{n-m}$  are defined by requiring that  $\theta := (\theta_-, \theta_+)^T$ .

In order to control  $\|\theta\|_{L^2}$ , we introduce

$$V_3(t) = \int_0^1 \theta^T(t, x)R[\gamma]\theta(t, x)dx, \quad (4.4.55)$$

then it is easy to see that

$$\dot{V}_3(t) = XVI + XVII + XVIII + XIX + XX \quad (4.4.56)$$

with

$$\begin{aligned} XVI &= \int_0^1 \theta^T(t, x)(R[\gamma](\Lambda(x) - F_1[\gamma]))_x \theta(t, x) dx \\ XVII &= - [\theta^T(t, x)R[\gamma](x)(\Lambda(x) - F_1[\gamma](x))\theta(t, x)]_{x=0}^{x=1} \\ XVIII &= + \int_0^1 \theta^T(t, x)(R[\gamma])_t \theta(t, x) dx \\ XIX &= + 2 \int_0^1 \theta^T(t, x)R[\gamma]F_7[\gamma, \gamma_x, \zeta, \zeta_x, \theta] dx + 2 \int_0^1 \theta^T(t, x)R[\gamma]F_8[\gamma, \zeta, \theta] dx \\ XX &= + 2 \int_0^1 \theta^T(t, x)R[\gamma]G(x)\theta(t, 0) dx. \end{aligned}$$

Let us first look at the second and the last term of (4.4.56)(i.e. XVII and XX), by some straight computations, one gets

$$\begin{aligned} XVII + XX &\leq -\theta_-(t, 0)^T \left( B - \tilde{S} - M\mu e^\delta \mathcal{C} - K_8 \|\gamma\|_\infty I_m \right) \theta_-(t, 0) \\ &\quad + \int_0^1 e^{-\delta x} \theta_+(t, x)^T \theta_+(t, x) dx + mM\mu \int_0^1 e^{\delta x} \theta_-(t, x)^T B \theta_-(t, x) dx \\ &\quad + K_9 \|\gamma\|_\infty V_3 \end{aligned} \quad (4.4.57)$$

Then by the same procedures in [26], we have the following

**Proposition 4.4.3** *For any given  $\lambda_3 > 0$ , there exists  $\delta_3 > 0$  and positive constants  $K_{10}$ ,  $K_{11}$ ,  $K_{12}$ ,  $K_{13}$  and  $K_{14}$ , such that*

$$\dot{V}_3 \leq -\lambda_3 V_3 + K_{10} \|\gamma\|_\infty V_3 + K_{11} V_3 V_2^{\frac{1}{2}} + K_{12} V_2 V_3^{\frac{1}{2}} + K_{13} V_3^{\frac{3}{2}} + K_{14} \|\zeta\|_\infty^3, \quad (4.4.58)$$

provided that  $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta_3$ .

#### 4.4.5 Proof of the $H^2$ stability for $\gamma$

Denote  $W = V_1 + V_2 + V_3$ , by Proposition 4.4.1, 4.4.2 and 4.4.3 as well as Lemma 4.6.7, one can show that for any given  $\lambda > 0$ , there exists  $\delta > 0$  and  $K_{15} > 0$ , such that

$$\dot{W} \leq -\lambda W + K_{15} W^{\frac{3}{2}}, \quad (4.4.59)$$

provided that  $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta$ . This concludes the whole proof of Theorem 4.1.1. ■

## 4.5 Appendix A

In this section, we will show the well-posedness and piecewise smoothness of the Kernel  $K$  and  $L$  which are given by the following Theorems.

**Theorem 4.5.1** *Let  $N \in \mathbb{N}^+$ . Under the assumption that  $\sigma_{ij} \in C^N[0, 1]$ ,  $\lambda_i \in C^N[0, 1]$  ( $i, j = 1, \dots, n$ ), there exists a unique piecewise  $C^N(\mathcal{T})$  solution  $K$  to the hyperbolic system (4.2.45)-(4.2.49). Moreover, if the  $C^{N-1}$  compatibility conditions at the point  $(x, \xi) = (1, 1)$  are satisfied, then  $K(\cdot, \cdot) \in C^{N-1}(0, 1)$ ,  $K(\cdot, 0) \in C^{N-1}(0, 1)$  with bounded  $C^{N-1}$  norm.*

**Proof.** We divided the proof into two parts. For the first part, we prove the regularity of the kernels. For this, we only prove the case  $N = 1$ . For  $N \geq 1$ , the results can be obtained by induction. In the case  $N = 1$ , one can, in fact, refer [54] and Remark 4.5.1 to find there exists a piecewise  $C^0$  kernel  $K$  for the boundary problem (4.2.45)-(4.2.49), where though only constant coupling coefficients and transport velocities are considered. However, the method in [54] straightforwardly extends to spatially varying coefficients with more involved technical developments. Next, we will improve the regularity of  $K$ . Let  $\mathcal{H}_{ij} = \partial_x K_{ij}(x, \xi)$  and  $\mathcal{Y}_{ij} = \partial_\xi K_{ij}(x, \xi)$ . By differentiating with respect to  $x$  in (4.2.45), one can show that

$$\lambda_i(x) \partial_x \mathcal{H}_{ij}(x, \xi) + \lambda_j(\xi) \partial_\xi \mathcal{H}_{ij}(x, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj} \lambda_j'(\xi)) \mathcal{H}_{ik}(x, \xi) - \lambda_i'(x) \mathcal{H}_{ij}(x, \xi). \quad (4.5.1)$$

Differentiating the boundary conditions in (4.2.46) and (4.2.47), we have

$$\mathcal{H}_{ij}(x, x) + \mathcal{Y}_{ij}(x, x) = k_{ij}'(x) \quad \text{for } 1 \leq i, j \leq n (i \neq j), \quad (4.5.2)$$

$$\mathcal{H}_{ij}(x, 0) = - \frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0) \mathcal{H}_{i, m+k}(x, 0) q_{k,j} \quad \text{for } 1 \leq i \leq j \leq m. \quad (4.5.3)$$

Next, differentiating the boundary conditions in (??)-(3.3.30), we have

$$\mathcal{Y}_{ij}(1, \xi) = \dot{k}_{ij}^{(1)}(\xi), \quad \text{for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (4.5.4)$$

and the boundary conditions for  $\mathcal{H}_{ij}$  ( $n \geq i \geq j \geq m+1$ ) on  $\xi = 0$ :

$$\mathcal{H}_{ij}(x, 0) = \dot{k}_{ij}^{(2)}(x), \quad \text{for } m+1 \leq j \leq i \leq n. \quad (4.5.5)$$

In view of the equations (4.2.45), it is easy to see that

$$\lambda_i(x)\mathcal{H}_{ij}(x, x) + \lambda_j(x)\mathcal{Y}_{ij}(x, x) = - \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x))K_{ik}(x, x) \quad (4.5.6)$$

$$\lambda_i(1)\mathcal{H}_{ij}(1, \xi) + \lambda_j(\xi)\mathcal{Y}_{ij}(1, \xi) = - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi))K_{ik}(1, \xi) \quad (4.5.7)$$

Combining (4.5.2) and (4.5.6), we have

$$\mathcal{H}_{ij}(x, x) = \frac{\lambda_j(x)k'_{ij}(x) + \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x))K_{ik}(x, x)}{\lambda_j(x) - \lambda_i(x)} \quad \text{for } 1 \leq i, j \leq n (i \neq j). \quad (4.5.8)$$

Similarly, plugging (4.5.7) into (4.5.4), one immediately obtains, for  $1 \leq j < i \leq m \cup m+1 \leq i < j \leq n$ , we have

$$\mathcal{H}_{ij}(1, \xi) = -\frac{1}{\lambda_i(1)} \left( \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi))K_{ik}(1, \xi) + \lambda_j(\xi)k_{ij}^{(1)}(\xi) \right), \quad (4.5.9)$$

which are piecewise  $C^0(0, 1)$  function. By the theory in [54], we can prove that there exists a unique piecewise  $\mathcal{H} \in C^0(\mathcal{T})$  for the boundary value problem (4.5.1), (4.5.3), (4.5.5) and (4.5.8)–(4.5.9). Noting the equations (4.2.45), we know that  $\mathcal{Y}$  shares the same regularity as  $\mathcal{H}$ .

Next, we prove the regularity of  $K(\cdot, 0)$ . Obviously, for  $N = 1$ , by the theory in [54] and Remark 4.5.1, one can prove that both  $K(\cdot, \cdot)$  and  $K(\cdot, 0) \in C^0(0, 1)$  with bounded  $C^0$  norm, provided that the  $C^0$  compatibility conditions (4.2.50) are satisfied at the the point  $(x, \xi) = (1, 1)$ . Next, we prove the case  $N = 2$ . Taking an  $\xi$ -derivative in (4.2.45) yields

$$\begin{aligned} \lambda_i(x)\partial_x\mathcal{Y}_{ij}(x, \xi) + \lambda_j(\xi)\partial_\xi\mathcal{Y}_{ij}(x, \xi) &= - \sum_{k=1}^n (\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi))\mathcal{Y}_{ik}(x, \xi) - \lambda_j'(\xi)\mathcal{Y}_{ij}(x, \xi) \\ &\quad - \sum_{k=1}^n (\sigma'_{kj}(\xi) + \delta_{kj}\lambda_j''(\xi))\mathcal{K}_{ik}(x, \xi) \end{aligned} \quad (4.5.10)$$

Combining (4.5.2) and (4.5.6), we have

$$\mathcal{Y}_{ij}(x, x) = \frac{\lambda_i(x)k'_{ij}(x) + \sum_{k=1}^n (\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x))K_{ik}(x, x)}{\lambda_i(x) - \lambda_j(x)} \quad \text{for } 1 \leq i, j \leq n (i \neq j). \quad (4.5.11)$$



Since

$$\lambda_i(x)\mathcal{H}_{ij}(x, 0) + \lambda_j(0)\mathcal{Y}_{ij}(x, 0) = - \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) \quad (4.5.12)$$

Plugging (4.5.3) and (4.5.5), respectively, one obtains

$$\mathcal{Y}_{ij}(x, 0) = - \frac{1}{\lambda_j(0)} \left( \lambda_i(x)\dot{k}_{ij}^{(2)}(x) + \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) \right), \quad (4.5.13)$$

for  $m+1 \leq j \leq i \leq n$

and

$$\begin{aligned} \mathcal{Y}_{ij}(x, 0) = & - \frac{1}{\lambda_j(0)} \sum_{k=1}^n (\sigma_{kj}(0) + \delta_{kj}\lambda'_j(0))K_{ik}(x, 0) + \frac{1}{\lambda_j^2(0)} \sum_{k=1}^{n-m} \lambda_{m+k}^2(0)q_{k,j}\mathcal{Y}_{i,m+k}(x, 0) \\ & + \frac{1}{\lambda_j^2(0)} \sum_{k=1}^{n-m} \sum_{s=1}^n \lambda_{m+k}(0)q_{k,j}(\sigma_{s,m+k}(0) + \delta_{s,m+k}\lambda'_{m+k}(0))K_{is}(x, 0) \end{aligned}$$

for  $1 \leq i \leq j \leq m$ .

(4.5.14)

Noting (4.5.4), (4.5.11), (4.5.13) and  $K(\cdot, 0) \in C^0$ , we know that  $\mathcal{Y}_{ij}(\cdot, \cdot) \in C^0(0, 1)$  ( $i \neq j$ ).  $\mathcal{Y}_{ij}(1, \cdot) \in C^0(0, 1)$  (for  $1 \leq j < i \leq m \cup m+1 \leq i < j \leq n$ ) and  $\mathcal{Y}_{ij}(\cdot, 0) \in C^0(0, 1)$  (for  $m+1 \leq j \leq i \leq n$ ). By the  $C^1$  compatibility conditions (4.2.51) at the point  $(x, \xi) = (1, 1)$  and using the theory in [54] and Remark 4.5.1, we can prove that there exists a unique piecewise  $C^0$  function  $\mathcal{Y} = \mathcal{Y}(x, \xi)$  for the boundary value problem (4.5.10), (4.5.11), (4.5.13), (4.5.4) and (4.5.14), which satisfies  $\mathcal{Y}(\cdot, \cdot), \mathcal{Y}(\cdot, 0) \in C^0(0, 1)$ . Noting (4.5.12) and (4.5.6), we know that  $\mathcal{H}(\cdot, \cdot), \mathcal{H}(\cdot, 0) \in C^0(0, 1)$ . This finishes the proof.  $\blacksquare$

**Remark 4.5.1** *It is worthy of mentioning that in [54], we only prove  $K \in L^\infty(\mathcal{T})$  and do not clarify the regularity of the kernel because of brevity purposes. However, with the same procedure in [26, Section A.3] and [31], one can prove that  $K$  is a piecewise  $C^0$  function with  $K(\cdot, \cdot), K(\cdot, 0) \in C^0(0, 1)$  and  $K(1, \cdot)$  being a function of piecewise  $C^0(0, 1)$  for the boundary problem (4.2.45)-(4.2.49), provided  $\sigma_{ij} \in C^0[0, 1], \lambda_i \in C^1[0, 1]$  ( $i, j = 1, \dots, n$ ) and the  $C^0$  compatibility conditions (4.2.50) are satisfied at the the point  $(x, \xi) = (1, 1)$ .*

**Theorem 4.5.2** *Under the assumptions of Theorem 4.5.1, For any  $N \in \mathbb{N}$ , there exists a unique piecewise  $C^N(\mathcal{T})$  kernel  $L$  to the inverse transformation (4.2.52). Moreover,  $L(x, x), L(x, 0) \in C^{N-1}(0, 1)$ .*

**Proof.** Substituting (4.2.40) for (4.2.52), it is easy to see that  $L$  is the solution of the following Volterra equations

$$L(x, \xi) = K(x, \xi) + \int_{\xi}^x K(x, s)L(s, \xi)ds \quad (4.5.15)$$

which yields that

$$L(x, x) = K(x, x) \in C^{N-1}(0, 1). \quad (4.5.16)$$

Noting (4.2.43), we have

$$\Sigma(x) + L(x, x)\Lambda(x) - \Lambda(x)L(x, x) = 0. \quad (4.5.17)$$

Next, Taking a partial derivative in  $x$  and  $\xi$  in (4.5.15), respectively, one obtains

$$L_x(x, \xi) = K_x(x, \xi) + K(x, x)L(x, \xi) + \int_{\xi}^x K_x(x, \xi)L(s, \xi)ds, \quad (4.5.18)$$

$$L_{\xi}(x, \xi) = K_{\xi}(x, \xi) - K(x, \xi)L(\xi, \xi) + \int_{\xi}^x K(x, s)L_{\xi}(s, \xi)ds. \quad (4.5.19)$$

Substituting (4.5.18) and (4.5.19) for (4.2.42) and using integration by parts, one has

$$\Lambda(x)L_x(x, \xi) + L_{\xi}(x, \xi)\Lambda(\xi) = (\Sigma(x) - \Lambda_{\xi}(\xi))L(x, \xi) \quad (4.5.20)$$

Again by (4.5.15), we have

$$L(x, 0) = K(x, 0) + \int_0^x K(x, s)L(s, 0)ds \quad (4.5.21)$$

since both  $K(x, 0)$  and  $K(x, x)$  are  $C^{N-1}$  continuous functions, by a suitable iteration procedure (see [87, Theorem 3.2, Pages 32–34]), it easy to see that there exists  $L(x, 0) = l(x) \in C^{N-1}(0, 1)$  for the Volterra equation of the second kind (4.5.21).

On the other hand, substituting (4.2.52) for (4.2.40), one gets

$$L(x, \xi) = K(x, \xi) + \int_{\xi}^x L(x, s)K(s, \xi)ds \quad (4.5.22)$$

then

$$L(1, \xi) = K(1, \xi) + \int_{\xi}^1 L(1, s)K(s, \xi)ds \quad (4.5.23)$$

With the same argument above, we can see that  $L_{ij}(1, \xi) = \tilde{l}_{ij}(\xi)$  ( $m \geq i > j \geq 1, n \geq j > i \geq m + 1$ ) on  $x = 1$  are functions of piecewise  $C^N(0, 1)$ . Then, For the boundary

problem (4.5.20) with the boundary conditions (4.5.17), and

$$L_{ij}(1, \xi) = \tilde{l}_{ij}(\xi), \text{ for } 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (4.5.24)$$

$$L_{ij}(x, 0) = l_{ij}(x), \text{ for } 1 \leq i \leq j \leq m \cup m+1 \leq j \leq i \leq n. \quad (4.5.25)$$

by Theorem 4.5.1, one immediately gets Theorem 4.5.2.  $\blacksquare$

## 4.6 Appendix B

In this appendix, we first sketch out four useful lemmas (the details can be found in [26]).

**Lemma 4.6.1** *There exists a positive real number  $c_1$ , such that*

$$|\mathcal{K}[\gamma]| + |\mathcal{L}[\gamma]| + |\mathcal{K}_1[\gamma]| + |\mathcal{K}_2[\gamma]| + |\mathcal{L}_1[\gamma]| \leq c_1(|\gamma| + \|\gamma\|_{L^1}). \quad (4.6.1)$$

**Lemma 4.6.2** *Suppose  $\|\gamma\|_\infty$  is suitable small, one can see that*

$$|F_1| \leq c_2(|\gamma| + \|\gamma\|_{L^1}), \quad (4.6.2)$$

$$|F_2| \leq c_3(|\gamma|^2 + \|\gamma\|_{L^1}^2), \quad (4.6.3)$$

$$|F_3| \leq c_4(|\gamma| + \|\gamma\|_{L^1})(\|\gamma_x\|_{L^2} + |\gamma_x|), \quad (4.6.4)$$

$$|F_4| \leq c_5(|\gamma|^2 + \|\gamma\|_{L^1}^2). \quad (4.6.5)$$

**Lemma 4.6.3**

$$|F_{11}| \leq c_6(|\zeta| + \|\zeta\|_{L^1}), \quad (4.6.6)$$

$$|F_{12}| \leq c_7(|\gamma_x| + |\gamma| + \|\gamma\|_{L^1}), \quad (4.6.7)$$

$$|F_{21}| \leq c_8(|\gamma| + \|\gamma\|_{L^1})(|\zeta| + \|\zeta\|_{L^1}), \quad (4.6.8)$$

$$|F_5| \leq c_9(|\zeta| + \|\zeta\|_{L^2})(|\gamma| + \|\gamma\|_{L^2}) + c_{10}(|\zeta| + \|\zeta\|_{L^2})(|\gamma_x| + \|\gamma_x\|_{L^2}) + c_{11}|\gamma(0)||\zeta(0)|, \quad (4.6.9)$$

$$|F_6| \leq c_{12}(|\gamma| + \|\gamma\|_{L^2})(|\zeta| + \|\zeta\|_{L^2}). \quad (4.6.10)$$

**Lemma 4.6.4**

$$|F_{13}| \leq c_{13}(|\zeta|^2 + \|\zeta\|_{L^2}^2) + c_{14}(|\theta| + \|\theta\|_{L^1}), \quad (4.6.11)$$

$$|F_{14}| \leq c_{14}(|\zeta| + \|\zeta\|_{L^1})(1 + |\gamma_x| + |\gamma| + \|\gamma\|_{L^1}) + c_{15}(|\zeta| + |\zeta_x| + \|\zeta\|_{L^1}), \quad (4.6.12)$$

$$|F_{22}| \leq c_{16}(|\gamma| + \|\gamma\|_{L^1})(|\theta| + \|\theta\|_{L^1}) + c_{17}(|\zeta|^2 + \|\zeta\|_{L^2}^2), \quad (4.6.13)$$

$$|F_7| \leq c_{18}(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + |\gamma| + \|\gamma_x\|) + c_{19}(|\zeta| + \|\zeta\|_{L^2})(|\zeta_x| + \|\zeta\|_{L^2}) \quad (4.6.14)$$

$$\begin{aligned}
 & + c_{20}(|\gamma| + \|\gamma\|_{L^2} + |\gamma_x|)(|\theta| + \|\theta\|_{L^2}) \\
 & + c_{21}(|\zeta(0)|^2 + |\gamma(0)|\|\theta(0)|), \\
 |F_8| \leq & c_{22}(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + \|\gamma\|_\infty) + c_{23}(|\gamma| + \|\gamma\|_{L^2})(|\theta| + \|\theta\|_{L^2}). \tag{4.6.15}
 \end{aligned}$$

Next, we show the following proposition which is also mentioned in [26], however here more technical developments are involved.

**Proposition 4.6.1** *There exists  $\delta > 0$  such that for any  $|\gamma| + |\zeta| \leq \delta$ , one has*

$$\|\theta\|_\infty \leq C_1(\|\gamma_{xx}\|_\infty + \|\gamma_x\|_\infty + \|\gamma\|_\infty), \tag{4.6.16}$$

$$\|\theta\|_{L^2} \leq C_2(\|\gamma_{xx}\|_{L^2} + \|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{4.6.17}$$

$$\|\gamma_{xx}\|_\infty \leq C_3(\|\theta\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \tag{4.6.18}$$

$$\|\gamma_{xx}\|_{L^2} \leq C_4(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \tag{4.6.19}$$

where  $C_1, C_2, C_3$  and  $C_4$  are positive constants.

**Proof.** We prove the next three lemmas to get Proposition 4.6.1.

**Lemma 4.6.5** *There exists  $\delta$  such that, if  $|\gamma| \leq \delta$ , then the following inequalities hold:*

$$\|\zeta\|_\infty \leq c_1(\|\gamma_x\|_\infty + \|\gamma\|_\infty) \tag{4.6.20}$$

$$\|\zeta\|_{L^2} \leq c_2(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{4.6.21}$$

$$\|\gamma_x\|_\infty \leq c_3(\|\zeta\|_\infty + \|\gamma\|_\infty), \tag{4.6.22}$$

$$\|\gamma_x\|_{L^2} \leq c_4(\|\zeta\|_{L^2} + \|\gamma\|_{L^2}) \tag{4.6.23}$$

**Proof.** Noting (4.4.12), one can easily see that

$$\zeta(t, x) + \Lambda(x)\gamma_x(t, x) - G(x)\gamma(t, 0) = F_3[\gamma, \gamma_x] + F_4[\gamma] \tag{4.6.24}$$

The difference between our proof and the proof in [26, Lemma B.6] is the appearance of the term  $G(x)\gamma(t, 0)$  in (4.6.24). Noting (4.2.44) and Theorem 4.5.1, we have  $G(\cdot) \in C^1(0, 1)$  with bounded  $C^1$  norm. Then since one can show that

$$\|G(\cdot)\gamma(t, 0)\|_{L^2} \leq c_5\|G(\cdot)\gamma(t, 0)\|_\infty \leq c_6\|\gamma\|_\infty \leq c_7(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{4.6.25}$$

which yields, by the same argument in [26, Lemma B.6], (4.6.20)-(4.6.22).

On the other hand, by the special structure of  $G(x)$ , we have

$$\|\partial_x \gamma_1\|_{L^2} \leq c_8(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \tag{4.6.26}$$

$$\|\partial_x \gamma_2\|_{L^2} \leq c_9(\|\zeta\|_{L^2} + \|\gamma_1\|_\infty + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty), \tag{4.6.27}$$

$$\begin{array}{cccc}
 \vdots & & \vdots & \\
 \vdots & & \vdots & \\
 \vdots & & \vdots & \\
 \vdots & & \vdots & 
 \end{array}$$

$$\|\partial_x \gamma_m\|_{L^2} \leq c_{m+7} (\|\zeta\|_{L^2} + \sum_{r=1}^{m-1} \|\gamma_r\|_\infty + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty), \quad (4.6.28)$$

$$\|\partial_x \gamma_s\|_{L^2} \leq c_{s+7} (\|\zeta\|_{L^2} + \sum_{r=1}^m \|\gamma_r\|_\infty + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty) \quad (4.6.29)$$

in which  $s = m + 1, \dots, n$ . Noting the classical Sobolev's inequality

$$\|\gamma\|_{L^\infty} \leq \tilde{C} (\|\gamma\|_{L^2} + \|\gamma_x\|_{L^2}) \leq \tilde{C} \|\gamma\|_{H^1}, \quad (4.6.30)$$

one gets that

$$\|\partial_x \gamma_1\|_{L^2} \leq C_1 (\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty), \quad (4.6.31)$$

$$\|\partial_x \gamma_2\|_{L^2} \leq C_2 (\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \|\partial_x \gamma_1\|_{L^2} + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty), \quad (4.6.32)$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$\|\partial_x \gamma_m\|_{L^2} \leq C_m (\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^{m-1} \|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty), \quad (4.6.33)$$

$$\|\partial_x \gamma_s\|_{L^2} \leq C_s (\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^m \|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2} \|\gamma\|_{L^\infty} + \|\gamma\|_{L^2} \|\gamma\|_\infty) \quad (4.6.34)$$

where  $s = m + 1, \dots, n$ . Then, we can easily obtain by induction that

$$\|\gamma_x\|_{L^2} \leq \tilde{c}_1 (\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2} \|\gamma\|_\infty + \|\gamma\|_{L^2}), \quad (4.6.35)$$

which concludes (4.6.23), under the assumption that  $\|\gamma\|_\infty$  is small enough.  $\blacksquare$

Combining the same technical approach as in [26, Lemma B.7 and Lemma B.8] and an analogous argument used in the proof of Lemma 4.6.5 and noting  $G \in C^1$ , the details of which we omit, one can show the next two lemmas.

**Lemma 4.6.6** *There exists  $\delta$  such that, if  $\|\gamma\|_\infty \leq \delta$ , then the following inequalities hold:*

$$\|\gamma_{xx}\|_\infty \leq c_1 (\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty) \quad (4.6.36)$$

$$\|\gamma_{xx}\|_{L^2} \leq c_2 (\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}) \quad (4.6.37)$$

$$\|\zeta_x\|_\infty \leq c_3 (\|\gamma_{xx}\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty) \quad (4.6.38)$$

$$\|\zeta_x\|_{L^2} \leq c_4 (\|\gamma_{xx}\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (4.6.39)$$

where  $c_1, c_2, c_3$  and  $c_4$  are positive constants.

and

**Lemma 4.6.7** *There exists  $\delta$  such that, if  $\|\gamma\|_\infty + \|\zeta\|_\infty \leq \delta$ , then the following inequalities hold:*

$$\|\theta\|_\infty \leq c_1(\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty) \quad (4.6.40)$$

$$\|\theta\|_{L^2} \leq c_2(\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}) \quad (4.6.41)$$

$$\|\zeta_x\|_\infty \leq c_3(\|\theta\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty) \quad (4.6.42)$$

$$\|\zeta_x\|_{L^2} \leq c_4(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \quad (4.6.43)$$

where  $c_1$ ,  $c_2$ ,  $c_3$  and  $c_4$  are positive constants.

The above three Lemma 4.6.5–4.6.7 immediately yield Proposition 4.6.1. ■



## Part III

# Exact Boundary Synchronization for a Coupled System of Wave Equations





# Chapter 5

## Exact Boundary Controllability and Observability for a Coupled System of Quasilinear Wave Equations

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### 5.1 Introduction

There are many publications concerning the exact controllability for linear hyperbolic systems (see [88]-[100] and the references therein). As a special form of second order hyperbolic equations, the exact boundary controllability for linear wave equation was obtained in a complete manner (see [88]-[89]). Using the Hilbert Uniqueness Method (HUM) suggested in [88]-[89], *Zuazua* [122]-[123], *Emanuilov* [37] and *Lasiecka & Triggiani* [66] subsequently established the exact boundary controllability for some semilinear wave equations. In the quasilinear case, based on the result on the semi-global  $C^2$  solution, by a direct constructive method, Li and Yu established the local exact boundary controllability for single 1-D quasilinear wave equation with boundary conditions of various types (see [70]-[83]). Later, this result has been applied to get the exact boundary controllability of nodal profile and on a

tree-like network for quasilinear wave equations respectively (see [43], [110]). For the following second order quasilinear hyperbolic system

$$u_{tt} + A(u, u_x, u_t)u_{tx} + B(u, u_x, u_t)u_{xx} = C(u, u_x, u_t) \quad (5.1.1)$$

under some different hypothesis on matrices  $A$ ,  $B$  and  $C$ , the corresponding local exact boundary controllability was obtained by Yu [118] and Wang [109] respectively.

On the other hand, as a dual problem of controllability, the exact boundary observability for wave equations has been widely studied (see [27], [67], [88]-[89]). In fact, The essence of J.-L. Lions's HUM is to use the duality to get the controllability by a corresponding observability inequality. Based on the result on semi-global classical solutions for quasilinear hyperbolic systems, by a constructive method, the exact boundary observability for single quasilinear wave equation was established by Li (see [72],[71]) and Guo(see [49]) respectively, and some implicit dualities have been shown between the exact boundary controllability and the exact boundary observability in the quasilinear case. As for the second order quasilinear hyperbolic system (5.1.1), the corresponding local exact boundary observability was obtained by Yu [119] under some hypothesis on matrices  $A$ ,  $B$  and  $C$ .

In this chapter, we consider a kind of coupled system of 1-D quasilinear wave equations, which can be rewritten in the form of second order quasilinear hyperbolic systems discussed in [109] and [119], but the systems in [109] and [119] were too general, which is not convenient to get the results we want. Therefore, for the coupled system of quasilinear wave equations under consideration, we restudy its controllability and observability and get brief results. In addition, the corresponding discussion of exact boundary null controllability and weak observability are added. Based on the existence and uniqueness of semi-global  $C^1$  solution and the local exact boundary controllability and observability for first order quasilinear hyperbolic systems, by a constructive method developed by Li (see [74] and [71] et.al.), we can obtain the local exact boundary (null) controllability and (weak) observability for a coupled system of quasilinear wave equations with boundary conditions of different types. The conclusions we get provide basis for studying the exact boundary synchronization for the coupled system of wave equations (see [81]).

Consider the following the coupled system of 1-D quasilinear wave equations

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w) \frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^n a_{ij}(w) w_j = 0, \quad (5.1.2)$$

where  $w = (w_1, \dots, w_n)^T$  is the unknown vector function of  $(t, x)$ ,  $a_i(w)$  and  $a_{ij}(w)$  ( $i, j = 1, \dots, n$ ) are all  $C^1$  functions of  $w$  on the domain under consideration, satisfying

$$a_i(0) > 0, i = 1, \dots, n. \quad (5.1.3)$$

On one end  $x = 0$ , we prescribe any one of the following Dirichlet type, Neumann type, Coupled third type and Coupled dissipative type boundary conditions:

$$x = 0 : w_i = h_i(t) \quad (i = 1, \dots, n), \quad (5.1.4a)$$

$$x = 0 : \frac{\partial w_i}{\partial x} = h_i(t) \quad (i = 1, \dots, n), \quad (5.1.4b)$$

$$x = 0 : \frac{\partial w_i}{\partial x} - \sum_{j=1}^n b_{ij}(w)w_j = h_i(t) \quad (i = 1, \dots, n), \quad (5.1.4c)$$

$$x = 0 : \frac{\partial w_i}{\partial x} - \sum_{j=1}^n c_{ij}(w)\frac{\partial w_j}{\partial t} = h_i(t) \quad (i = 1, \dots, n), \quad (5.1.4d)$$

where  $b_{ij} = b_{ij}(w)$  and  $c_{ij} = c_{ij}(w)$  are  $C^1$  functions of  $w$ ,  $h_i(t)$  are  $C^2$  functions (in case(5.1.4a)) or  $C^1$  functions (in case(5.1.4b)-(5.1.4d)).

Similarly, on another end  $x = L$ , the boundary conditions are

$$x = L : w_i = \bar{h}_i(t) \quad (i = 1, \dots, n), \quad (5.1.5a)$$

$$x = L : \frac{\partial w_i}{\partial x} = \bar{h}_i(t) \quad (i = 1, \dots, n), \quad (5.1.5b)$$

$$x = L : \frac{\partial w_i}{\partial x} + \sum_{j=1}^n \bar{b}_{ij}(w)w_j = \bar{h}_i(t) \quad (i = 1, \dots, n), \quad (5.1.5c)$$

$$x = L : \frac{\partial w_i}{\partial x} + \sum_{j=1}^n \bar{c}_{ij}(w)\frac{\partial w_j}{\partial t} = \bar{h}_i(t) \quad (i = 1, \dots, n), \quad (5.1.5d)$$

where  $\bar{b}_{ij} = \bar{b}_{ij}(w)$  and  $\bar{c}_{ij} = \bar{c}_{ij}(w)$  are  $C^1$  functions of  $w$ ,  $\bar{h}_i(t)$  are  $C^2$  functions (in case(5.1.5a)) or  $C^1$  functions (in case(5.1.5b)-(5.1.5d)).

The initial conditions are given by

$$t = 0 : (w, w_t) = (\varphi(x), \psi(x)), \quad 0 \leq x \leq L, \quad (5.1.6)$$

where  $\varphi = (\varphi_1, \dots, \varphi_n)^T$  is a  $C^2$  vector function on  $[0, L]$ ,  $\psi = (\psi_1, \dots, \psi_n)^T$  is a  $C^1$  vector function on  $[0, L]$ , they are all with small  $C^2$  norm or  $C^1$  norm, such that the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$  are satisfied respectively.

Obviously,  $w = 0$  is an equilibrium of system (5.1.2). Based on the theory of semi-global  $C^2$  solution, by a constructive method (see [74] and [71]), we will establish local exact boundary controllability and observability around  $w = 0$ .

This chapter is organized as follows. The existence and uniqueness of semi-global  $C^2$  solution to the coupled system of quasilinear wave equations (5.1.2) with boundary conditions (5.1.4) and (5.1.5) of different types will be presented in § 5.2. Based on

this, in § 5.3, we obtain the corresponding local exact boundary (null) controllability with boundary controls on one end or on two ends and in § 5.4, we obtain the corresponding local exact boundary (weak) observability with observed values on one end or on two ends.

## 5.2 Existence and Uniqueness of Semi-global $C^2$ Solution

For the purpose of getting the local exact boundary controllability and observability for system (5.1.2) with boundary conditions (5.1.4)-(5.1.5), we should first prove the existence and uniqueness of semi-global  $C^2$  solution for mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6). In order to get it in a unified way, the best way is to reduce the equations to a first order quasilinear hyperbolic system and use the corresponding results of semi-global  $C^1$  solution.

Setting

$$u_i = \frac{\partial w_i}{\partial x}, v_i = \frac{\partial w_i}{\partial t} (i = 1, \dots, n), \quad (5.2.1)$$

$$u = (u_1, \dots, u_n)^T, v = (v_1, \dots, v_n)^T, \quad (5.2.2)$$

equations (5.1.2) can be reduced to the following first order quasilinear system

$$\begin{cases} \frac{\partial w}{\partial t} = v, \\ \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \\ \frac{\partial v}{\partial t} - \Lambda^2(w) \frac{\partial u}{\partial x} = -A(w)w, \end{cases} \quad (5.2.3)$$

where  $\Lambda(w) = \text{diag}\{a_1(w), \dots, a_n(w)\}$  and  $A(w) = (a_{ij}(w))_{n \times n}$ . Its equivalent matrix form can be written as

$$\frac{\partial}{\partial t} \begin{pmatrix} w \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & -\Lambda^2(w) & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w \\ u \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ -A(w)w \end{pmatrix}. \quad (5.2.4)$$

The characteristic equation of (5.2.3) or (5.2.4) is

$$\det \begin{pmatrix} \lambda I_n & 0 & 0 \\ 0 & \lambda I_n & I_n \\ 0 & \Lambda^2(w) & \lambda I_n \end{pmatrix} = \lambda^n |\lambda^2 I_n - \Lambda^2(w)| = 0, \quad (5.2.5)$$

whose solutions, the real eigenvalues of system (5.2.3) or (5.2.4), are

$$\lambda_i^- = -a_i(w), \quad \lambda_i^0 = 0, \quad \lambda_i^+ = a_i(w) \quad (i = 1, \dots, n), \quad (5.2.6)$$

the corresponding left eigenvectors, which constitute a complete set, can be chosen as

$$l_i^- = (\mathbf{0}, a_i(w)\mathbf{e}_i, \mathbf{e}_i), l_i^0 = (\mathbf{e}_i, \mathbf{0}, \mathbf{0}), l_i^+ = (\mathbf{0}, -a_i(w)\mathbf{e}_i, \mathbf{e}_i), \quad (i = 1, \dots, n), \quad (5.2.7)$$

in which  $\mathbf{0} = (0, \dots, 0)$  is the  $n - D$  zero vector,  $\mathbf{e}_i = (0, \dots, \overset{(i)}{1}, \dots, 0)$  is a  $n - D$  row vector. Thus, (5.2.3) or (5.2.4) reduced from the system (5.1.2) is a first order quasilinear hyperbolic system.

Let

$$U = (w, u, v)^T. \quad (5.2.8)$$

Setting diagonal variables as

$$\begin{cases} V_i^- = l_i^- U = a_i(w)u_i + v_i, \\ V_i^0 = l_i^0 U = w_i, \\ V_i^+ = l_i^+ U = -a_i(w)u_i + v_i, \end{cases} \quad (i = 1, \dots, n), \quad (5.2.9)$$

namely,

$$\begin{cases} V^- = \Lambda(w)u + v, \\ V^0 = w, \\ V^+ = -\Lambda(w)u + v, \end{cases} \quad (5.2.10)$$

we have

$$\begin{cases} w = V^0, \\ u = \frac{1}{2}\Lambda^{-1}(V^0)(V^- - V^+), \\ v = \frac{1}{2}(V^- + V^+). \end{cases} \quad (5.2.11)$$

When equations (5.1.2) are reduced to system (5.2.3) or (5.2.4), the boundary conditions (5.1.4) will be correspondingly replaced by

$$x = 0 : v = \dot{H}(t), \quad (5.2.12a)$$

$$x = 0 : u = H(t), \quad (5.2.12b)$$

$$x = 0 : u - B(w)w = H(t), \quad (5.2.12c)$$

$$x = 0 : u - C(w)v = H(t), \quad (5.2.12d)$$

in which  $H(t) = (h_1(t), \dots, h_n(t))^T$ ,  $B(w) = (b_{ij}(w))_{n \times n}$ ,  $C(w) = (c_{ij}(w))_{n \times n}$ .

Noting (5.2.11), if

$$\det(\Lambda^{-1}(0) + C(0)) \neq 0, \quad (5.2.13)$$

it is easy to see that, at least in a neighborhood of  $U = 0$ , the boundary conditions

(5.2.12) (namely, (5.1.4)) on  $x = 0$  can be rewritten as

$$x = 0 : \quad V^+ = -V^- + 2\dot{H}(t), \quad (5.2.14a)$$

$$x = 0 : \quad V^+ = V^- - 2\Lambda(V^0)H(t), \quad (5.2.14b)$$

$$x = 0 : \quad V^+ = V^- - 2\Lambda(V^0)B(V^0)V^0 - 2\Lambda(V^0)H(t), \quad (5.2.14c)$$

$$x = 0 : \quad V^+ = \left(\Lambda^{-1}(V^0) + C(V^0)\right)^{-1} \left(\Lambda^{-1}(V^0) - C(V^0)\right) V^- \\ - 2\left(\Lambda^{-1}(V^0) + C(V^0)\right)^{-1} H(t), \quad (5.2.14d)$$

which can be uniformly presented as

$$x = 0 : \quad V^+ = G(t, V^-, V^0) + \tilde{H}(t), \quad (5.2.15)$$

where  $G$  and  $\tilde{H}$  are  $C^1$  functions with respect to their arguments, and without loss of generality, we may assume that

$$G(t, 0, 0) \equiv 0. \quad (5.2.16)$$

Similarly, if

$$\det(\Lambda^{-1}(0) + \bar{C}(0)) \neq 0, \quad (5.2.17)$$

then at least in a neighborhood of  $U = 0$ , the boundary conditions (5.1.5) on  $x = L$  are all of the form

$$x = L : \quad V^- = \bar{G}(t, V^0, V^+) + \tilde{\bar{H}}(t), \quad (5.2.18)$$

in which  $\bar{G}$  and  $\tilde{\bar{H}}$  are  $C^1$  functions with respect to their arguments, and without loss of generality, we may assume that

$$\bar{G}(t, 0, 0) \equiv 0. \quad (5.2.19)$$

Meanwhile, the corresponding initial condition (5.1.6) can be reduced as

$$t = 0 : \quad U = (\varphi(x), \varphi'(x), \psi(x))^T, \quad 0 \leq x \leq L. \quad (5.2.20)$$

Together with the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  and  $(0, L)$  for the coupled system of wave equations (5.1.2) with the boundary conditions (5.1.4)-(5.1.5) on  $x = 0$  and  $x = L$  respectively and the initial condition(5.1.6), it is easy to see that the conditions of  $C^1$  compatibility at these two points for the mixed initial-boundary value problem (5.2.3), (5.2.15), (5.2.18) and (5.2.20) are also satisfied.

For the convenience of statement, we denote in the whole chapter that

$$d = \begin{cases} 2 & \text{for(5.1.4a),} \\ 1 & \text{for(5.1.4b) - (5.1.4d)} \end{cases} \quad (5.2.21)$$

and

$$\bar{d} = \begin{cases} 2 & \text{for(5.1.5a),} \\ 1 & \text{for(5.1.5b) - (5.1.5d),} \end{cases} \quad (5.2.22)$$

Based on the theory of semi-global  $C^1$  solution for first order quasilinear hyperbolic system with zero eigenvalues (see [71], [73], [70] and [117]), it is easy to get the following

**Lemma 5.2.1** *Under the hypotheses given in §5.1, suppose that the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$ , respectively. Suppose furthermore that (5.2.13) and (5.2.17) hold. For any given and possibly quite large  $T > 0$ , if  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$ ,  $\|h_i\|_{C^d[0,T]}$  and  $\|\bar{h}_i\|_{C^{\bar{d}}[0,T]}$  ( $i = 1, \dots, n$ ) are suitably small, then the forward mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6) admits a unique semi-global  $C^2$  solution  $w = w(t, x)$  on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , and*

$$\|w\|_{C^2[R(T)]} \leq C \left( \sum_{i=1}^n \|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]} + \sum_{i=1}^n \|(h_i, \bar{h}_i)\|_{C^d[0,L] \times C^{\bar{d}}[0,L]} \right), \quad (5.2.23)$$

where  $C$  is a positive constant possibly depending on  $T$ .

**Corollary 5.2.1** *Under the hypotheses given in §5.1, if  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  ( $i = 1, \dots, n$ ) are suitably small, then Cauchy problem (5.1.2) and (5.1.6) admits a unique global  $C^2$  solution  $w = w(t, x)$  on the whole maximum determinate domain (see, for example, [84, Remark 4.2 on Chapter 1 pages 72-73]) and*

$$\|w\|_{C^2} \leq C \sum_{i=1}^n \|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}, \quad (5.2.24)$$

where  $C$  is a positive constant.

As for the backward mixed initial-boundary value problem (5.1.2), (5.1.4)-(5.1.5) with final condition

$$t = T : (w, w_t) = (\Phi(x), \Psi(x)), \quad 0 \leq x \leq L. \quad (5.2.25)$$



in which  $\Phi = (\Phi_1, \dots, \Phi_n)^T$  is a  $C^2$  vector function on  $[0, L]$ ,  $\Psi = (\Psi_1, \dots, \Psi_n)^T$  is a  $C^1$  vector function on  $[0, L]$ , they are all with small  $C^2$  norm or  $C^1$  norm, such that the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$  are satisfied respectively, similar results can be obtained as follows

**Lemma 5.2.2** *Under the hypotheses given in §5.1, suppose that the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively. Suppose furthermore that*

$$\det(\Lambda^{-1}(0) - C(0)) \neq 0, \quad (5.2.26)$$

and

$$\det(\Lambda^{-1}(0) - \bar{C}(0)) \neq 0, \quad (5.2.27)$$

hold. For any given and possibly quite large  $T > 0$ , if  $\|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]}$ ,  $\|h_i\|_{C^d[0,T]}$  and  $\|\bar{h}_i\|_{C^{\bar{d}}[0,T]}$  ( $i = 1, \dots, n$ ) are suitably small, then the backward mixed initial-boundary value problem (5.1.2), (5.2.25) and (5.1.4)-(5.1.5) admits a unique semi-global  $C^2$  solution  $w = w(t, x)$  on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$  and

$$\|w\|_{C^2[R(T)]} \leq C \left( \sum_{i=1}^n \|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]} + \sum_{i=1}^n \|(h_i, \bar{h}_i)\|_{C^d[0,L] \times C^{\bar{d}}[0,L]} \right), \quad (5.2.28)$$

where  $C$  is a positive constant possibly depending on  $T$ .

**Remark 5.2.1** *If  $a_i, a_{ij}, b_{ij}, \bar{b}_{ij}, c_{ij}, \bar{c}_{ij}$  are all independent of  $w$ , then the problem is linear and it is not necessary to assume the small data for initial (resp. final) value as well as boundary functions in Lemma 5.2.1 (resp. Lemma 5.2.2). Moreover, the  $C^2$  solution obtained is global.*

**Remark 5.2.2** *Suppose  $a_i(w) \equiv a > 0$  ( $i = 1, \dots, n$ ) are the same constant, the conditions (5.2.13) and (5.2.17) in Lemma 5.2.1 mean  $-a$  is not the eigenvalue of both matrices  $C(0)$  and  $\bar{C}(0)$ , while the conditions (5.2.26) and (5.2.27) in Lemma 5.2.2 mean  $a$  is not the eigenvalue of both matrices  $C(0)$  and  $\bar{C}(0)$ .*

**Remark 5.2.3** *By Lemma 5.2.1, conditions (5.2.13) and (5.2.17) for matrices  $C$  and  $\bar{C}$  in the coupled dissipative type boundary conditions (5.1.4d) and (5.1.5d) are imposed for guaranteeing the well-posedness of the corresponding forward mixed initial-boundary value problem, however, by Lemma 5.2.2, conditions (5.2.26) and (5.2.27) for matrices  $C$  and  $\bar{C}$  in the coupled dissipative type boundary conditions (5.1.4d) and (5.1.5d) are imposed for guaranteeing the well-posedness of the corresponding backward*

*mixed initial-boundary value problem. Other coupled matrices  $A$ ,  $B$  and  $\bar{B}$ , however, can be completely arbitrary.*

**Remark 5.2.4** *Lemma 5.2.1, Corollary 5.2.1 and Lemma 5.2.2 are still valid for the following equations*

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w) \frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^n \bar{a}_{ij}(w) \frac{\partial w_j}{\partial t} = 0, \quad (5.2.29)$$

or

$$\frac{\partial^2 w_i}{\partial t^2} - a_i^2(w) \frac{\partial^2 w_i}{\partial x^2} + \sum_{j=1}^n a_{ij}(w) w_j + \sum_{j=1}^n \bar{a}_{ij}(w) \frac{\partial w_j}{\partial t} = 0, \quad (5.2.30)$$

*in which  $a_i(w)$  and  $a_{ij}(w)$ ,  $\bar{a}_{ij}(w)$  are all  $C^1$  functions with respect to their arguments on the domain under consideration, satisfying (5.1.3).*

### 5.3 Local Exact Boundary Controllability

**Theorem 5.3.1 (Two-sided control)** *Under the hypotheses given in §5.1, suppose furthermore that (5.2.13) and (5.2.17) hold. Let*

$$T > L \max_{i=1, \dots, n} \left( \frac{1}{a_i(0)} \right). \quad (5.3.1)$$

*For any given initial data  $(\varphi, \psi)$  and final data  $(\Phi, \Psi)$  with small norms  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$ ,  $\|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  ( $i = 1, \dots, n$ ), there exist boundary controls  $H = (h_1, \dots, h_n)$  and  $\bar{H} = (\bar{h}_1, \dots, \bar{h}_n)$  with small norms  $\|h_i\|_{C^q[0,T]}$  and  $\|\bar{h}_i\|_{C^{\bar{q}}[0,T]}$  ( $i = 1, \dots, n$ ), such that the mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6) admits a unique semi-global  $C^2$  solution  $w = w(t, x)$  with small  $C^2$  norm on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which exactly satisfies the final condition (5.2.25).*

In order to prove Theorem 5.3.1, it suffices to use the constructive method suggested in [74] and [83] to prove the following lemma, the details of whose proofs are skipped here.

**Lemma 5.3.1** *Under the assumptions of Theorem 5.3.1, for any given initial data  $(\varphi, \psi)$  and final data  $(\Phi, \Psi)$  with small norms  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  and  $\|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  ( $i = 1, \dots, n$ ), the coupled system of quasilinear wave equations (5.1.2) admits a  $C^2$  solution  $w = w(t, x)$  with small  $C^2$  norm on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which satisfies simultaneously the initial condition (5.1.6) and the final condition (5.2.25).*

**Remark 5.3.1** If  $a_i, a_{ij}, b_{ij}, \bar{b}_{ij}, c_{ij}, \bar{c}_{ij}$  are all independent of  $w$ , then the problem is linear, Theorem 5.3.1 implies the corresponding global exact boundary controllability.

**Remark 5.3.2** The exact controllability time (5.3.1) given in Theorem 5.3.1 is sharp.

**Theorem 5.3.2 (One-sided control)** Under the hypotheses given in §5.1, suppose furthermore that (5.2.13), (5.2.17) and (5.2.26) hold. Let

$$T > 2L \max_{i=1, \dots, n} \left( \frac{1}{a_i(0)} \right). \quad (5.3.2)$$

For any given initial data  $(\varphi, \psi)$  and final data  $(\Phi, \Psi)$  with small norms  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}, \|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  ( $i = 1, \dots, n$ ) and for any given boundary functions  $H = (h_1, \dots, h_n)$  on  $x = 0$  with small norms  $\|h_i\|_{C^d[0,T]}$  ( $i = 1, \dots, n$ ), such that the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(T, 0)$  respectively, there exist boundary controls  $\bar{H} = (\bar{h}_1, \dots, \bar{h}_n)$  on  $x = L$  with small norms  $\|\bar{h}_i\|_{C^{\bar{d}}[0,T]}$  ( $i = 1, \dots, n$ ), such that the mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6) admits a unique semi-global  $C^2$  solution  $w = w(t, x)$  with small  $C^2$  norm on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which exactly satisfies the final condition (5.2.25).

In order to get Theorem 5.3.2, similarly, it suffices to prove the following

**Lemma 5.3.2** Under the assumptions of Theorem 5.3.2, for any given initial data  $(\varphi, \psi)$  and final data  $(\Phi, \Psi)$  with small norms  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  and  $\|(\Phi_i, \Psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  ( $i = 1, \dots, n$ ), for any given boundary functions  $H = (h_1, \dots, h_n)$  on  $x = 0$  with small norm  $\|h_i\|_{C^d[0,T]}$  ( $i = 1, \dots, n$ ), such that the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(T, 0)$  respectively, the coupled system of quasilinear wave equations (5.1.2) with the boundary condition (5.1.4) on  $x = 0$  admits a  $C^2$  solution  $w = w(t, x)$  with small  $C^2$  norm on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which satisfies simultaneously the initial condition (5.1.6) and the final condition (5.2.25).

**Remark 5.3.3** Similar results hold if we take the boundary controls  $H(t)$  ( $0 \leq t \leq T$ ) at  $x = 0$  and hypothesis (5.2.26) is replaced by (5.2.27).

**Remark 5.3.4** If  $a_i, a_{ij}, b_{ij}, \bar{b}_{ij}, c_{ij}, \bar{c}_{ij}$  are all independent of  $w$ , then the problem is linear, Theorem 5.3.2 implies the corresponding global exact boundary controllability.

**Remark 5.3.5** *The exact controllability time (5.3.2) given in Theorem 5.3.2 is sharp.*

If we only consider the corresponding null controllability (see [77]), for which the final data (5.2.25) are specially taken as

$$t = T : w = 0, w_t = 0, \quad 0 \leq x \leq L, \tag{5.3.3}$$

then for the coupled dissipative type boundary conditions (5.1.4d) (resp. (5.1.5d)), the condition (5.2.26) (resp. (5.2.27)) is not necessary. In fact, in this situation, similar to [77], we have

**Theorem 5.3.3 (One-sided null control)** *Let  $T > 0$  satisfies (5.3.2). Suppose that (5.2.13) and (5.2.17) hold. Suppose furthermore that*

$$H(t) \equiv 0. \tag{5.3.4}$$

*at  $x = 0$ . For any given initial data  $(\varphi, \psi)$ , where  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]} (i = 1, \dots, n)$  are small enough, such that the conditions of  $C^2$  compatibility are satisfied at the point  $(t, x) = (0, 0)$ , there exists boundary controls  $\bar{H} = (\bar{h}_1, \dots, \bar{h}_n)$  on  $x = L$  with small norms  $\|\bar{h}_i\|_{C^{\bar{a}}[0,T]} (i = 1, \dots, n)$ , such that the mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6) admits a unique semi-global  $C^2$  solution  $w = w(t, x)$  with small  $C^2$  norm on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , which exactly satisfies the final condition (5.3.3).*

**Remark 5.3.6** *Similar results hold if we take the boundary controls  $H(t) (0 \leq t \leq T)$  on  $x = 0$ , and hypothesis (5.3.4) is replaced by*

$$\bar{H}(t) \equiv 0. \tag{5.3.5}$$

**Remark 5.3.7** *Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.3.3 are still valid for the coupled system (5.2.29) or (5.2.30), which satisfies (5.1.3).*

## 5.4 Local Exact Boundary Observability

We now consider the exact boundary observability for the mixed initial-boundary value problem (5.1.2) and (5.1.4)-(5.1.6), in which the boundary functions  $h_i(t)$  and  $\bar{h}_i(t) (i = 1, \dots, n)$  are given.

The principle of choosing the observed values on the boundary is that the observed values together with the boundary condition can uniquely determine the values  $(w, w_x)$  on the boundary.

Hence, the observed values at  $x = 0$  can be taken as

1.  $w_{ix} = k_i(t) (i = 1, \dots, n)$ , for the *Dirichlet* type boundary conditions (5.1.4a), then

$$x = 0 : (w_i, w_{ix}) = (h_i(t), k_i(t)); \quad (5.4.1a)$$

2.  $w_i = k_i(t) (i = 1, \dots, n)$ , for the *Neumann* type boundary conditions (5.1.4b), then

$$x = 0 : (w_i, w_{ix}) = (k_i(t), h_i(t)); \quad (5.4.1b)$$

3.  $w_i = k_i(t) (i = 1, \dots, n)$ , for the Coupled third type boundary conditions (5.1.4c), then

$$x = 0 : (w_i, w_{ix}) = (k_i(t), \sum_{j=1}^n b_{ij} k_j(t) + h_i(t)); \quad (5.4.1c)$$

4.  $w_i = k_i(t) (i = 1, \dots, n)$ , for the Coupled dissipative type boundary conditions (5.1.4d), then

$$x = 0 : (w_i, w_{ix}) = (k_i(t), \sum_{j=1}^n c_{ij} k_j'(t) + h_i(t)). \quad (5.4.1d)$$

where  $k_i(t) (i = 1, \dots, n) \in C^1$  for (5.1.4a) or  $\in C^2$  for (5.1.4b)-(5.1.4d). Thus, by means of the observed values at  $x = 0$ , we have

$$x = 0 : (w, w_x) = (a(t), b(t)), \quad (5.4.2)$$

and for any given  $T > 0$ ,

$$\|(a, b)\|_{C^2[0, T] \times C^1[0, T]} \leq C \left( \sum_{i=1}^n \|k_i\|_{C^l[0, T]} + \sum_{i=1}^n \|h_i\|_{C^d[0, T]} \right), \quad (5.4.3)$$

where  $C$  is a positive constant,  $d$  is given by (5.2.21) and

$$l = \begin{cases} 1 & \text{for (5.1.4a),} \\ 2 & \text{for (5.1.4b) - (5.1.4d),} \end{cases} \quad (5.4.4)$$

The corresponding observed values  $\bar{k}_i(t) (i = 1, \dots, n)$  at  $x = L$  can be similarly taken, then we have

$$x = 0 : (w, w_x) = (\bar{a}(t), \bar{b}(t)), \quad (5.4.5)$$

and for any given  $T > 0$ ,

$$\|(\bar{a}, \bar{b})\|_{C^2[0,T] \times C^1[0,T]} \leq C \left( \sum_{i=1}^n \|\bar{k}_i\|_{C^{\bar{l}}[0,T]} + \sum_{i=1}^n \|\bar{h}_i\|_{C^{\bar{d}}[0,T]} \right), \quad (5.4.6)$$

where  $C$  is a positive constant,  $\bar{d}$  is given by (5.2.22) and

$$\bar{l} = \begin{cases} 1 & \text{for (5.1.5a),} \\ 2 & \text{for (5.1.5b) - (5.1.5d).} \end{cases} \quad (5.4.7)$$

By the constructive method suggested in [49],[77],[72] and [71], we can prove following theorems.

**Theorem 5.4.1 (Two-sided observation)** *Under the hypotheses given in §5.1, suppose furthermore that (5.2.13) and (5.2.17) hold. Let  $T > 0$  satisfies (5.3.1), for any given initial condition  $(\varphi, \psi)$ , such that  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  are suitably small and the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$  respectively. If we have the observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$  and  $\bar{k}_i(t)$  ( $i = 1, \dots, n$ ) at  $x = L$  on the interval  $[0, T]$ , then the initial data  $(\varphi, \psi)$  can be uniquely determined and the following observability inequality holds:*

$$\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} \leq C \left( \sum_{i=1}^n \|(k_i, \bar{k}_i)\|_{C^{\bar{l}}[0,T] \times C^{\bar{l}}[0,T]} + \sum_{i=1}^n \|(h_i, \bar{h}_i)\|_{C^{\bar{d}}[0,T] \times C^{\bar{d}}[0,T]} \right), \quad (5.4.8)$$

where  $C$  is a positive constant.

**Remark 5.4.1** *If  $a_i, a_{ij}, b_{ij}, \bar{b}_{ij}, c_{ij}, \bar{c}_{ij}$  are all independent of  $w$ , then the problem is linear, Theorem 5.4.1 implies the corresponding global exact boundary observability.*

**Remark 5.4.2** *The exact observability time (5.3.1) given in Theorem 5.4.1 is sharp.*

**Theorem 5.4.2 (One-sided observability)** *Under the hypotheses given in §5.1, suppose furthermore that (5.2.13), (5.2.17) and (5.2.27) hold. Let  $T > 0$  satisfies (5.3.2), for any given initial condition  $(\varphi, \psi)$ , such that  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  are suitably small and the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$  respectively. If we have the observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$  on the interval  $[0, T]$ , then the initial data  $(\varphi, \psi)$  can be uniquely determined and the following observability inequality holds:*

$$\|(\varphi, \psi)\|_{C^2[0,L] \times C^1[0,L]} \leq C \left( \sum_{i=1}^n \|k_i\|_{C^{\bar{l}}[0,T]} + \sum_{i=1}^n \|(h_i, \bar{h}_i)\|_{C^{\bar{d}}[0,T] \times C^{\bar{d}}[0,T]} \right), \quad (5.4.9)$$

where  $C$  is a positive constant.

**Remark 5.4.3** *Similar results hold if we take observed values  $\bar{k}_i(t)$  ( $i = 1, \dots, n$ ) at  $x = L$  and hypothesis (5.2.27) is replaced by (5.2.26).*

**Remark 5.4.4** *If  $a_i, a_{ij}, b_{ij}, \bar{b}_{ij}, c_{ij}, \bar{c}_{ij}$  are all independent of  $w$ , then the problem is linear, Theorem 5.4.2 implies the corresponding global exact boundary observability.*

**Remark 5.4.5** *The exact observability time (5.3.2) given in Theorem 5.4.2 is sharp.*

If we only consider the corresponding weak observability (see [77]), for which the final condition (5.2.25) can be uniquely determined by the observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$ , then for the coupled dissipative type boundary conditions (5.1.4d) (resp. (5.1.5d)), the condition (5.2.26) (resp. (5.2.27)) is not necessary. In fact, in this situation, we have

**Theorem 5.4.3 (One-sided weak observability)** *Under the hypotheses given in §5.1, suppose furthermore that (5.2.13), (5.2.17) hold. Let  $T > 0$  satisfies (5.3.2), for any given initial condition  $(\varphi, \psi)$ , such that  $\|(\varphi_i, \psi_i)\|_{C^2[0,L] \times C^1[0,L]}$  are suitably small and the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$  respectively. If we have the observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$  on the interval  $[0, T]$ , then the final data  $(\Phi, \Psi)$  can be uniquely determined by the observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$  and the boundary conditions  $h_i(t)$  and  $\bar{h}_i(t)$  on the interval  $[0, T]$ , in addition, the following weak observability inequality holds:*

$$\|(\Phi, \Psi)\|_{C^2[0,L] \times C^1[0,L]} \leq C \left( \sum_{i=1}^n \|k_i\|_{C^1[0,T]} + \sum_{i=1}^n \|(h_i, \bar{h}_i)\|_{C^d[0,T] \times C^{\bar{d}}[0,T]} \right), \quad (5.4.10)$$

where  $C$  is a positive constant.

**Remark 5.4.6** *Similar results hold if we take observed values  $k_i(t)$  ( $i = 1, \dots, n$ ) at  $x = 0$ .*

**Remark 5.4.7** *Theorem 5.4.1, Theorem 5.4.2 and Theorem 5.4.3 are still valid for the coupled system (5.2.29) or (5.2.30), which satisfies (5.1.3).*

# Chapter 6

## Exact Boundary Synchronization for a Coupled System of Linear Wave Equations

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### 6.1 Introduction

Synchronization is a widespread natural phenomenon. Thousands of fireflies may twinkle at the same time; audiences in the theater can applaud with a rhythmic beat; pacemaker cells of the heart function simultaneously; and field crickets give out a unanimous cry—all these are phenomena of synchronization [104], [115].

In principle, synchronization happens when different individuals possess likeness in nature, that is, they conform essentially to the same governing equation, and meanwhile, the individuals should bear a certain coupled relation.

The phenomenon of synchronization was first observed by Huygens in 1665 [60]. The theoretical research on synchronization phenomena dates back to H. Fujisaka



and T. Yamada's study of synchronization for coupled equations in 1983 [39]. The previous studies focused on systems described by ODEs, such as

$$\frac{dX_i}{dt} = f(X_i, t) + \sum_{j=1}^N A_{ij}X_j \quad (i = 1, \dots, N), \quad (6.1.1)$$

where  $X_i (i = 1, \dots, N)$  are  $n$ -dimensional vectors,  $A_{ij} (i, j = 1, \dots, N)$  are  $n \times n$  matrices, and  $f(X, t)$  is an  $n$ -dimensional vector function independent of  $i$ . The right-hand side of (6.1.1) shows that every  $X_i (i = 1, \dots, N)$  possesses two basic features: satisfying a fundamental governing equation and bearing a coupled relation among one another.

Our goal is to synchronize a hyperbolic system through boundary controls. Different from the ODE situation, the coupling of systems can then be fulfilled by coupling of the equations or (and) the boundary conditions, which has richer research implications, moreover, boundary controls can be used to realize our goal in a finite time.

Roughly speaking, we want to find a  $T > 0$  and some boundary controls on  $[0, T]$ , such that from the time  $t = T$  on, the system states tend to the same. That is to say, we hope to achieve the synchronization of the system states not only at the moment  $t = T$  under the action of boundary controls on  $[0, T]$ , but also when  $t \geq T$  and withdrawing all the controls. This is forever, instead of short-lived, synchronization, as is desired in many actual applications.

Obviously, if the system has the exact boundary null controllability ([77], [88], [89], [100]), it must have the exact boundary synchronization, but this is a trivial situation that should be excluded beforehand. The exact boundary null controllability here also means that the system states remain null not only at the moment  $t = T$  under the action of boundary controls on  $[0, T]$ , but also when  $t \geq T$  and withdrawing all the controls.

In this chapter, we consider the exact boundary synchronization for a coupled system of 1-D wave equations with various boundary condition in the framework of classical solutions. The same problem for a coupled system of  $n$ -dimensional wave equations with Dirichlet boundary controls in the framework of weak solutions can be found in [79]. We point out that in 1-D case, thanks to the theory on semi-global classical solutions (see [72] and [73]), there is a unified theory on the exact boundary (null) controllability for a coupled system of wave equations with various kinds of coupled boundary conditions (see [55], [72], [77], [109]), which enables us to get a complete result on the exact boundary synchronization. However, in the framework of weak solutions in  $n$ -dimensional case, for getting the corresponding exact boundary synchronization, we have to study the exact boundary (null) controllability for a coupled system of wave equations with every prescribed kind of boundary conditions separately, and, up to now, only the Dirichlet type of boundary conditions

was discussed in [79]. In addition, the generalized exact boundary synchronization is introduced and discussed in this chapter.

Precisely speaking, we will consider the following coupled system of wave equations

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + AU = 0, \quad (6.1.2)$$

where

$$U = (u_1, \dots, u_N)^T \quad (6.1.3)$$

is an unknown vector function of  $(t, x)$ ,  $A = (a_{ij})$  is a  $N \times N$  coupling matrix with constant elements. (6.1.2) can be written as

$$\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial^2 u_i}{\partial x^2} + \sum_{j=1}^N a_{ij} u_j = 0 \quad (i = 1, \dots, N). \quad (6.1.4)$$

At the end  $x = 0$ , we prescribe anyone of the following boundary conditions:

$$x = 0 : U = H(t) \quad (\text{Dirichlet type}), \quad (6.1.5a)$$

$$x = 0 : U_x = H(t) \quad (\text{Neumann type}), \quad (6.1.5b)$$

$$x = 0 : U_x - BU = H(t) \quad (\text{Coupled third type}), \quad (6.1.5c)$$

where  $B = (b_{ij})$  is an  $N \times N$  coupling matrix with constant elements, and

$$H(t) = (h_1(t), \dots, h_N(t))^T, \quad (6.1.6)$$

the components of which will be totally or partially taken as boundary controls.

Similarly, at the end  $x = L$ , since no boundary controls are concerned for one-sided control, we prescribe anyone of the following homogeneous boundary conditions:

$$x = L : U = 0, \quad (6.1.7a)$$

$$x = L : U_x = 0, \quad (6.1.7b)$$

$$x = L : U_x + \bar{B}U = 0, \quad (6.1.7c)$$

where  $\bar{B} = (\bar{b}_{ij})$  is an  $N \times N$  coupling matrix with constant elements.

The initial condition is given by

$$t = 0 : U = \Phi(x), \quad U_t = \Psi(x), \quad (6.1.8)$$

where  $\Phi(x) = (\varphi_1(x), \dots, \varphi_N(x))^T$  and  $\Psi(x) = (\psi_1(x), \dots, \psi_N(x))^T$ .

For the forward mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) of the coupled system of wave equations, by means of its reduction to the mixed initial-boundary value problem of a corresponding first order hyperbolic system and using the theory of global  $C^1$  solutions to first order linear hyperbolic systems, we have (see [55], [72])

**Lemma 6.1.1** *For any given  $T > 0$ , assume that  $\Phi \in (C^2[0, L])^N$ ,  $\Psi \in (C^1[0, L])^N$  and  $H \in (C^2[0, T])^N$  (case (6.1.5a)) or  $(C^1[0, T])^N$  (cases (6.1.5b)-(6.1.5c)) satisfy the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$ , respectively (see Remark 6.1.2). For any given coupling matrices  $A$ ,  $B$  and  $\bar{B}$ , the forward mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) admits a unique  $C^2$  solution  $U = U(t, x)$  on the domain  $R(T) = \{(t, x) \mid 0 \leq t \leq T, 0 \leq x \leq L\}$ .*

**Remark 6.1.1** *In what follows, (6.1.5) always means anyone of (6.1.5a), (6.1.5b) and (6.1.5c), and it is similar for (6.1.7).*

**Remark 6.1.2** *The conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$  can be precisely written as*

$$\begin{cases} \Phi(L) = 0, \\ \Psi(L) = 0, \\ \Phi''(L) = 0, \end{cases} \quad \text{for (6.1.7a);} \quad (6.1.9a)$$

$$\begin{cases} \Phi'(L) = 0, \\ \Psi'(L) = 0, \end{cases} \quad \text{for (6.1.7b);} \quad (6.1.9b)$$

or

$$\begin{cases} \Phi'(L) + \bar{B}\Phi(L) = 0, \\ \Psi'(L) + \bar{B}\Psi(L) = 0, \end{cases} \quad \text{for (6.1.7c).} \quad (6.1.9c)$$

The conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  can be similarly obtained.

**Remark 6.1.3** *By Lemma 6.1.1, in order to guarantee the well-posedness of the forward mixed problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8), no assumptions on the coupling matrices  $A$ ,  $B$  and  $\bar{B}$  are needed.*

**Remark 6.1.4** *For the backward mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) with the final condition*

$$t = T : U = \bar{\Phi}(x), \quad U_t = \bar{\Psi}(x). \quad (6.1.10)$$

where  $\bar{\Phi}(x) = (\bar{\varphi}_1(x), \dots, \bar{\varphi}_N(x))^T$  and  $\bar{\Psi}(x) = (\bar{\psi}_1(x), \dots, \bar{\psi}_N(x))^T$ , the conclusion of Lemma 6.1.1 is still valid.

By Lemma 6.1.1 and using a constructive method, we have the following result on the exact boundary null controllability (also see [55], [109], [118]).

**Lemma 6.1.2** *Let*

$$T > 2L. \tag{6.1.11}$$

For any given initial state  $(\Phi, \Psi) \in (C^2[0, L])^N \times (C^1[0, L])^N$ , satisfying the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , there exist boundary controls  $H \in (C^2[0, T])^N$  (case (6.1.5a)) or  $(C^1[0, T])^N$  (cases (6.1.5b)–(6.1.5c)) with support on  $[0, T]$  at the end  $x = 0$ , such that the corresponding mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) admits a unique  $C^2$  solution  $U = U(t, x)$  on  $t \geq 0$ , and as  $t \geq T$  we have

$$U(t, x) \equiv 0, \quad 0 \leq x \leq L. \tag{6.1.12}$$

In what follows, we will use Lemma 6.1.1 and Lemma 6.1.2 to consider the exact boundary synchronization, the exact boundary synchronization by groups, and the exact boundary null controllability and synchronization by groups in §6.2, §6.3 and §6.4, respectively, and the generalized exact boundary synchronization in §6.5, for the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7). Moreover, several remarks are given in §6.6.

## 6.2 Exact boundary synchronization

**Definition 6.2.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi(x), \Psi(x))$ , we can find some boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}(t, x), \quad 0 \leq x \leq L, \tag{6.2.1}$$

then the coupled system (6.1.2), (6.1.5) and (6.1.7) is said to possess the **exact boundary synchronization** or is said to be **exactly synchronizable**, and  $\tilde{u} = \tilde{u}(t, x)$  is called the corresponding **synchronizable state**.

Obviously, if the system possesses the exact boundary null controllability, then it must be exactly synchronizable. However, this trivial situation should be excluded in the discussion.

**Lemma 6.2.1** *If the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7) is exactly synchronizable, but not exactly null controllable, then the coupling matrices  $A$ ,  $B$  and  $\bar{B}$  should satisfy the following necessary conditions: The values of the sum of every row of these matrices,  $\sum_{j=1}^N a_{ij}$ ,  $\sum_{j=1}^N b_{ij}$  and  $\sum_{j=1}^N \bar{b}_{ij}$  are independent of  $i = 1, \dots, N$ , namely,*

$$\sum_{j=1}^N a_{ij} \stackrel{\text{def.}}{=} \tilde{a} \quad (i = 1, \dots, N), \quad (6.2.2)$$

$$\sum_{j=1}^N b_{ij} \stackrel{\text{def.}}{=} \tilde{b} \quad (i = 1, \dots, N), \quad (6.2.3)$$

$$\sum_{j=1}^N \bar{b}_{ij} \stackrel{\text{def.}}{=} \tilde{\bar{b}} \quad (i = 1, \dots, N), \quad (6.2.4)$$

where  $\tilde{a}$ ,  $\tilde{b}$  and  $\tilde{\bar{b}}$  are constants independent of  $i = 1, \dots, N$ .

**Proof.** By synchronization, there exists  $T > 0$  and a synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  such that (6.2.1) holds. Then, it follows from (6.1.4) that for  $t \geq T$  we have

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \left( \sum_{j=1}^N a_{ij} \right) \tilde{u} = 0, \quad 0 \leq x \leq L, \quad i = 1, \dots, N. \quad (6.2.5)$$

In particular, for  $t \geq T$  we get

$$\left( \sum_{j=1}^N a_{ij} \right) \tilde{u} = \left( \sum_{j=1}^N a_{kj} \right) \tilde{u}, \quad 0 \leq x \leq L, \quad i, k = 1, \dots, N.$$

By the non exact null controllability, there exists at least an initial data  $(\Phi, \Psi)$  for which the corresponding solution  $U$ , or equivalently  $\tilde{u}$ , does not identically vanish for  $t \geq T$  whatever boundary controls  $H$  are chosen. This leads to (6.2.2). In the meantime, noting (6.2.5), the synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  satisfies the following wave equation

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{a} \tilde{u} = 0, \quad (6.2.6)$$

where  $\tilde{a}$  is given by (6.2.2).

We now prove (6.2.3). (6.2.4) can be similarly obtained.

Similarly to (6.2.5), it follows from (6.1.5c) that for  $t \geq T$  we have

$$x = 0 : \frac{\partial \tilde{u}}{\partial x} - \left( \sum_{j=1}^N b_{ij} \right) \tilde{u} = 0, \quad i = 1, \dots, N, \quad (6.2.7)$$

then for  $t \geq T$  we get

$$x = 0 : \left( \sum_{j=1}^N b_{ij} \right) \tilde{u} = \left( \sum_{j=1}^N b_{kj} \right) \tilde{u}, \quad i, k = 1, \dots, N.$$

Since the system is not exactly null controllable, we claim that at least for an initial data  $(\Phi, \Psi)$ ,  $\tilde{u}$  does not identically vanish at  $x = 0$  for  $t \geq T$ , then (6.2.3) holds. Otherwise, noting (6.2.7) we have

$$x = 0 : \tilde{u} \equiv \frac{\partial \tilde{u}}{\partial x} \equiv 0, \quad t \geq T,$$

then, by the exact boundary observability ([71], [72]) for the wave equation (6.2.6),  $\tilde{u}$  should identically vanish for  $t \geq T$  and  $0 \leq x \leq L$ , which gives a contradiction to the non exact null controllability.

This completes the proof of Lemma 6.2.1.  $\square$

**Remark 6.2.1** *Conditions (6.2.2)–(6.2.4) mean that all the matrices  $A$ ,  $B$  and  $\bar{B}$  have a common right eigenvector  $(1, 1, \dots, 1)^T$  with the corresponding eigenvalues  $\tilde{a}$ ,  $\tilde{b}$  and  $\tilde{b}$ , respectively.*

**Theorem 6.2.1** *Suppose that (6.2.2)–(6.2.4) hold and*

$$T > 2L. \quad (6.2.8)$$

*For any given initial state  $(\Phi, \Psi) \in (C^2[0, L])^N \times (C^1[0, L])^N$ , satisfying the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , there exist  $(N - 1)$   $C^2$  (case (6.1.5a)) or  $C^1$  (cases (6.1.5b)–(6.1.5c)) boundary controls with support on  $[0, T]$  in  $H(t)$  (for example, take  $h_2(t), \dots, h_N(t)$  with  $h_1(t) \equiv 0$ ), such that the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the exact boundary synchronization.*

**Proof** Let

$$w_i = u_i - u_{i+1} \quad (i = 1, \dots, N - 1). \quad (6.2.9)$$

It is easily shown (see [79]) that by (6.2.2), the original system (6.1.2) for the variable  $U$  can be reduced to a self-closing system of the same kind for the variable  $W =$

$(w_1, \dots, w_{N-1})^T$  as follows:

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} + \tilde{A}W = 0, \quad (6.2.10)$$

where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N-1) \times (N-1)$  matrix with the entries

$$\tilde{a}_{ij} = - \sum_{p=j+1}^N (a_{ip} - a_{i+1,p}) = - \sum_{p=1}^j (a_{i+1,p} - a_{ip}), \quad i, j = 1, \dots, N-1. \quad (6.2.11)$$

Similarly, by (6.2.3)–(6.2.4), the original boundary conditions (6.1.5) and (6.1.7) for the variable  $U$  can be reduced, respectively, to self-closing boundary conditions of the same kind for the variable  $W$  as follows:

$$x = 0 : W = \tilde{H}(t), \quad (6.2.12a)$$

$$x = 0 : W_x = \tilde{H}(t), \quad (6.2.12b)$$

$$x = 0 : W_x - \tilde{B}W = \tilde{H}(t) \quad (6.2.12c)$$

and

$$x = L : W = 0, \quad (6.2.13a)$$

$$x = L : W_x = 0, \quad (6.2.13b)$$

$$x = L : W_x + \tilde{B}W = 0, \quad (6.2.13c)$$

where  $\tilde{B} = (\tilde{b}_{ij})$  and  $\tilde{\bar{B}} = (\tilde{\bar{b}}_{ij})$  are  $(N-1) \times (N-1)$  coupling matrices reduced by  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  in the following way:

$$\tilde{b}_{ij} = - \sum_{p=j+1}^N (b_{ip} - b_{i+1,p}) = - \sum_{p=1}^j (b_{i+1,p} - b_{ip}), \quad i, j = 1, \dots, N-1 \quad (6.2.14)$$

etc. Moreover,

$$\tilde{H}(t) = (\tilde{h}_1(t), \dots, \tilde{h}_{N-1}(t))^T \quad (6.2.15)$$

with

$$\tilde{h}_i(t) = h_i(t) - h_{i+1}(t), \quad i = 1, \dots, N-1. \quad (6.2.16)$$

The initial condition of  $W$  is given by

$$t = 0 : W = \tilde{\Phi}(x), \quad W_t = \tilde{\Psi}(x) \quad (6.2.17)$$

with

$$\begin{cases} \tilde{\Phi}(x) = (\varphi_1(x) - \varphi_2(x), \varphi_2(x) - \varphi_3(x), \dots, \varphi_{N-1}(x) - \varphi_N(x))^T, \\ \tilde{\Psi}(x) = (\psi_1(x) - \psi_2(x), \psi_2(x) - \psi_3(x), \dots, \psi_{N-1}(x) - \psi_N(x))^T. \end{cases} \quad (6.2.18)$$

Thus, according to Lemma 6.1.2, by means of  $(N - 1)$   $C^2$  (case(6.2.12a)) or  $C^1$  (cases (6.2.12b)-(6.2.12c)) boundary controls  $\tilde{H}(t)$  (for instance, take  $h_2(t), \dots, h_n(t)$  with  $h_1(t) \equiv 0$ ) at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly synchronizable. This finished the proof.

**Remark 6.2.2** *Noting Remark 6.1.3, by Lemma 6.1.1 and Lemma 6.1.2, the reduced mixed initial-boundary value problem (6.2.10), (6.2.12)–(6.2.13) and (6.2.17) for the variable  $W$  is always well-posed and exactly null controllable.*

**Remark 6.2.3** *As  $t \geq T$ , the synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  defined by (6.2.1) satisfies the wave equation (6.2.6) with the following boundary conditions:*

$$x = 0 : \tilde{u} = 0, \quad (6.2.19a)$$

$$x = 0 : \tilde{u}_x = 0, \quad (6.2.19b)$$

$$x = 0 : \tilde{u}_x - \tilde{b}\tilde{u} = 0 \quad (6.2.19c)$$

and

$$x = L : \tilde{u} = 0, \quad (6.2.20a)$$

$$x = L : \tilde{u}_x = 0, \quad (6.2.20b)$$

$$x = L : \tilde{u}_x + \tilde{b}\tilde{u} = 0, \quad (6.2.20c)$$

where  $\tilde{a}$ ,  $\tilde{b}$  and  $\tilde{b}$  are given by (6.2.2)–(6.2.4), respectively.

*Noting Remark 6.1.3, the forward mixed initial-boundary value problem for (6.2.6) and (6.2.19)–(6.2.20) is always well-posed. Hence, if we know the initial state of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ :*

$$t = T : \tilde{u} = \varphi(x), \quad \tilde{u}_t = \psi(x), \quad 0 \leq x \leq L, \quad (6.2.21)$$

*we can completely determine the evolution of  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$ .*

**Remark 6.2.4** *Under assumptions (6.2.2)–(6.2.4), in order to realize the exact boundary synchronization, the  $(N - 1)$  boundary controls in  $H(t)$  can be chosen in infinitely many ways, therefore, the initial state  $(\varphi, \psi)$  in (6.2.21) will be not unique.*



Moreover, it can be shown that the attainable set of all possible initial data of synchronizable state is the whole space  $C^2[0, L] \times C^1[0, L]$  associated with the conditions of  $C^2$  compatibility at  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (6.2.19) and (6.2.20).

To illustrate this, for any given  $(\varphi, \psi) \in C^2[0, L] \times [C^1[0, L]$  satisfying the conditions of  $C^2$  compatibility with the boundary conditions (6.2.19)–(6.2.20) at  $(t, x) = (T, 0)$  and  $(T, L)$  respectively, we can solve the backward mixed initial-boundary value problem (6.2.6) and (6.2.19)–(6.2.21) to get a unique  $C^2$  solution  $u = u(t, x)$  and its initial value at  $t = 0$

$$t = 0 : u = \widehat{\varphi}(x), \quad u_t = \widehat{\psi}(x). \quad (6.2.22)$$

Then, by assumptions (6.2.2)–(6.2.4), it is easy to see that

$$U(t, x) = (u, u, \dots, u)^T(t, x) \quad (6.2.23)$$

is the  $C^2$  solution to the original problem (6.1.2), (6.1.5) and (6.1.7) with the null control  $H \equiv 0$  and the initial condition

$$\begin{aligned} t = 0 : U &= \widehat{\Phi}(x) = (\widehat{\varphi}(x), \widehat{\varphi}(x), \dots, \widehat{\varphi}(x))^T, \\ U_t &= \widehat{\Psi}(x) = (\widehat{\psi}(x), \widehat{\psi}(x), \dots, \widehat{\psi}(x))^T. \end{aligned} \quad (6.2.24)$$

Then, by solving the mixed problem (6.1.2), (6.1.5), (6.1.7) and (6.2.24) with null boundary controls, we can reach any given synchronizable state  $(\varphi, \psi)$  at the moment  $t = T$ .

### 6.3 Exact boundary synchronization by groups

The concept of exact boundary synchronization can be generalized to the exact boundary synchronization by groups. Without loss of generality, we consider here the exact boundary synchronization by 2-groups. For this purpose, setting  $U = \begin{pmatrix} U^{(1)} \\ U^{(2)} \end{pmatrix}$  with  $U^{(1)} = (u_1, \dots, u_m)^T$  and  $U^{(2)} = (u_{m+1}, \dots, u_N)^T$ , we look for some boundary controls in  $H(t)$ , such that  $U^{(1)}$  and  $U^{(2)}$  are independently synchronizable.

**Definition 6.3.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi(x), \Psi(x))$ , we can find some boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and for  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_m(t, x) \stackrel{\text{def.}}{=} \widetilde{u}_1(t, x), \quad 0 \leq x \leq L, \quad (6.3.1)$$

$$u_{m+1}(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \widetilde{u}_2(t, x), \quad 0 \leq x \leq L, \quad (6.3.2)$$

then the coupled system (6.1.2), (6.1.5) and (6.1.7) is said to possess the **exact boundary synchronization by 2-groups** or is said to be **exactly synchronizable by 2-groups**, and  $\tilde{U} = \tilde{U}(t, x) = \begin{pmatrix} \tilde{u}_1(t, x) \\ \tilde{u}_2(t, x) \end{pmatrix}$  is called the corresponding **synchronizable state by 2-groups**.

Obviously, if the coupled system possesses the exact boundary synchronization, then it must be exactly synchronizable by 2-groups. However, this trivial situation should be excluded in advance. For this purpose we need to impose suitable assumptions on the synchronizable states by 2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$ . We have

**Lemma 6.3.1** *Suppose that the coupled system (6.1.2), (6.1.5) and (6.1.7) is exactly synchronizable by 2-groups. Suppose furthermore that at least for an initial state  $(\Phi(x), \Psi(x))$  the synchronizable states by 2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent. Then the coupling matrix  $A = (a_{ij})$  in system (6.1.2) of wave equations should satisfy the following necessary conditions:*

$$\sum_{j=1}^m a_{ij} \stackrel{\text{def.}}{=} \begin{cases} \tilde{a}_{11} & (i = 1, \dots, m), \\ \tilde{a}_{21} & (i = m + 1, \dots, N), \end{cases} \quad (6.3.3)$$

$$\sum_{j=m+1}^N a_{ij} \stackrel{\text{def.}}{=} \begin{cases} \tilde{a}_{12} & (i = 1, \dots, m), \\ \tilde{a}_{22} & (i = m + 1, \dots, N), \end{cases} \quad (6.3.4)$$

where  $\tilde{a}_{11}$  and  $\tilde{a}_{12}$  are independent of  $i = 1, \dots, m$ , while,  $\tilde{a}_{21}$  and  $\tilde{a}_{22}$  are independent of  $i = m + 1, \dots, N$ .

**Proof.** By Definition 6.3.1, for  $t \geq T$  we have

$$\frac{\partial^2 \tilde{u}_1}{\partial t^2} - \frac{\partial^2 \tilde{u}_1}{\partial x^2} + \left( \sum_{p=1}^m a_{ip} \right) \tilde{u}_1 + \left( \sum_{p=m+1}^N a_{ip} \right) \tilde{u}_2 = 0, \quad 0 \leq x \leq L, \quad i = 1, \dots, m, \quad (6.3.5)$$

$$\frac{\partial^2 \tilde{u}_2}{\partial t^2} - \frac{\partial^2 \tilde{u}_2}{\partial x^2} + \left( \sum_{p=1}^m a_{ip} \right) \tilde{u}_1 + \left( \sum_{p=m+1}^N a_{ip} \right) \tilde{u}_2 = 0, \quad 0 \leq x \leq L, \quad i = m + 1, \dots, N. \quad (6.3.6)$$

Hence we get

$$t \geq T : \left( \sum_{p=1}^m a_{ip} - \sum_{p=1}^m a_{kp} \right) \tilde{u}_1 + \left( \sum_{p=m+1}^N a_{ip} - \sum_{p=m+1}^N a_{kp} \right) \tilde{u}_2 = 0, \quad 0 \leq x \leq L$$

for  $i, k = 1, \dots, m$  and  $i, k = m + 1, \dots, N$ , respectively. Since at least for an initial data  $(\Phi(x), \Psi(x))$ , the synchronizable states by 2-groups,  $\tilde{u}_1$  and  $\tilde{u}_2$  are linearly

independent, we get (6.3.3)–(6.3.4) immediately.

For the coupling matrices  $B$  and  $\bar{B}$ , we can not get the corresponding necessary conditions under the assumptions of Lemma 6.3.1 (unless we assume that at least for an initial data  $(\Phi(x), \Psi(x))$ ,  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent at  $x = 0$  and at  $x = L$ , respectively). However, based on Lemma 6.3.1, in what follows we impose the following similar assumptions:

$$\sum_{j=1}^m b_{ij} (\text{resp. } \bar{b}_{ij}) \stackrel{\text{def.}}{=} \begin{cases} \tilde{\tilde{b}}_{11} (\text{resp. } \tilde{\tilde{b}}_{11}) & (i = 1, \dots, m), \\ \tilde{\tilde{b}}_{21} (\text{resp. } \tilde{\tilde{b}}_{21}) & (i = m + 1, \dots, N), \end{cases} \quad (6.3.7)$$

$$\sum_{j=m+1}^N b_{ij} (\text{resp. } \bar{b}_{ij}) \stackrel{\text{def.}}{=} \begin{cases} \tilde{\tilde{b}}_{12} (\text{resp. } \tilde{\tilde{b}}_{12}) & (i = 1, \dots, m), \\ \tilde{\tilde{b}}_{22} (\text{resp. } \tilde{\tilde{b}}_{22}) & (i = m + 1, \dots, N), \end{cases} \quad (6.3.8)$$

where  $\tilde{\tilde{b}}_{11}$  (resp.  $\tilde{\tilde{b}}_{11}$ ) and  $\tilde{\tilde{b}}_{12}$  (resp.  $\tilde{\tilde{b}}_{12}$ ) are independent of  $i = 1, \dots, m$ , while,  $\tilde{\tilde{b}}_{21}$  (resp.  $\tilde{\tilde{b}}_{21}$ ) and  $\tilde{\tilde{b}}_{22}$  (resp.  $\tilde{\tilde{b}}_{22}$ ) are independent of  $i = m + 1, \dots, N$ .

**Theorem 6.3.1** *Suppose that (6.3.3)–(6.3.4) and (6.3.7)–(6.3.8) hold and  $T > 0$  satisfies (6.2.8). For any given initial state  $(\Phi(x), \Psi(x))$ , there exist  $(N - 2)$   $C^2$  (case (6.1.5a)) or  $C^1$  (cases (6.1.5b)–(6.1.5c)) boundary controls with support on  $[0, T]$  in  $H(t)$  (for instance,  $h_2(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_1(t) \equiv h_{m+1}(t) \equiv 0$ ), such that the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the exact boundary synchronization by 2-groups.*

**Proof** Let

$$\begin{cases} w_i = u_i - u_{i+1} & (i = 1, \dots, m - 1), \\ w_i = u_{i+1} - u_{i+2} & (i = m, \dots, N - 2). \end{cases} \quad (6.3.9)$$

Similarly to the proof of Theorem 6.2.1 (cf. [79]), under assumptions (6.3.3)–(6.3.4), the original system (6.1.2) for the variable  $U$  can be reduced to a self-closing system (6.2.10) of the same kind for the variable  $W = (w_1, \dots, w_{N-2})^T$ , where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N - 2) \times (N - 2)$  matrix with the entries

$$\tilde{a}_{ij} = \begin{cases} - \sum_{p=j+1}^m (a_{ip} - a_{i+1,p}) = - \sum_{p=1}^j (a_{i+1,p} - a_{ip}), & i = 1, \dots, N - 2; j = 1, \dots, m - 1, \\ - \sum_{p=j+2}^N (a_{ip} - a_{i+1,p}) = - \sum_{p=m+1}^{j+1} (a_{i+1,p} - a_{ip}), & i = 1, \dots, N - 2; j = m, \dots, N - 2. \end{cases} \quad (6.3.10)$$

Similarly, under assumptions (6.3.7)–(6.3.8), the original boundary conditions (6.1.5) and (6.1.7) for the variable  $U$  can be reduced to self-closing boundary condi-

tions (6.2.12) and (6.2.13) of the same kind for the variable  $W$ , in which the coupling matrices  $\tilde{B} = (\tilde{b}_{ij})$  and  $\tilde{\bar{B}} = (\tilde{\bar{b}}_{ij})$  are  $(N-2) \times (N-2)$  matrices reduced by  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  in the following way:

$$\tilde{b}_{ij} = \begin{cases} -\sum_{p=j+1}^m (b_{ip} - b_{i+1,p}) = -\sum_{p=1}^j (b_{i+1,p} - b_{ip}), & i = 1, \dots, N-2; j = 1, \dots, m-1, \\ -\sum_{p=j+2}^N (b_{ip} - b_{i+1,p}) = -\sum_{p=m+1}^{j+1} (b_{i+1,p} - b_{ip}), & i = 1, \dots, N-2; j = m, \dots, N-2 \end{cases} \quad (6.3.11)$$

etc. Moreover,

$$\tilde{H}(t) = (\tilde{h}_1(t), \dots, \tilde{h}_{N-2}(t)) \quad (6.3.12)$$

with

$$\tilde{h}_i(t) = \begin{cases} h_i(t) - h_{i+1}(t), & i = 1, \dots, m-1, \\ h_{i+1}(t) - h_{i+2}(t), & i = m, \dots, N-2. \end{cases} \quad (6.3.13)$$

The initial condition (6.2.17) of  $W$  is given by

$$\tilde{\Phi}(x) = (\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_{N-2}(x))^T, \quad \tilde{\Psi}(x) = (\tilde{\psi}_1(x), \dots, \tilde{\psi}_{N-2}(x))^T \quad (6.3.14)$$

with

$$\begin{aligned} \tilde{\varphi}_i(x) &= \begin{cases} \varphi_i(x) - \varphi_{i+1}(x), & i = 1, \dots, m-1, \\ \varphi_{i+1}(x) - \varphi_{i+2}(x), & i = m, \dots, N-2, \end{cases} \\ \tilde{\Psi}_i(x) &= \begin{cases} \psi_i(x) - \psi_{i+1}(x), & i = 1, \dots, m-1, \\ \psi_{i+1}(x) - \psi_{i+2}(x), & i = m, \dots, N-2. \end{cases} \end{aligned} \quad (6.3.15)$$

Thus, according to Lemma 6.1.2, by means of  $(N-2)$   $C^2$  (case(6.2.12a)) or  $C^1$  (cases (6.2.12b)-(6.2.12c)) boundary controls  $\tilde{H}(t)$  (for instance, take  $h_2(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_1(t) \equiv h_{m+1}(t) \equiv 0$ ) at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly synchronizable by 2-groups. This completes the proof.

**Remark 6.3.1** *Lemma 6.1.1 and Lemma 6.1.2 are always available for the reduced mixed initial-boundary value problem (6.2.10), (6.2.12)–(6.2.13) and (6.3.14) for the variable  $W$ .*

**Remark 6.3.2** *As  $t \geq T$ , the synchronizable state by 2-groups  $\tilde{\tilde{U}} = \tilde{\tilde{U}}(t, x) = \begin{pmatrix} \tilde{\tilde{u}}_1(t, x) \\ \tilde{\tilde{u}}_2(t, x) \end{pmatrix}$  satisfies the following coupled system of wave equations*

$$\frac{\partial^2 \tilde{\tilde{U}}}{\partial t^2} - \frac{\partial^2 \tilde{\tilde{U}}}{\partial x^2} + \tilde{\tilde{A}} \tilde{\tilde{U}} = 0, \quad (6.3.16)$$

where

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}, \quad (6.3.17)$$

and  $\tilde{a}_{11}$ ,  $\tilde{a}_{12}$ ,  $\tilde{a}_{21}$  and  $\tilde{a}_{22}$  are given by (6.3.3)–(6.3.4). Moreover,  $\tilde{U} = \tilde{U}(t, x)$  satisfies the following boundary conditions:

$$x = 0 : \tilde{U} = 0, \quad (6.3.18a)$$

$$x = 0 : \tilde{U}_x = 0, \quad (6.3.18b)$$

$$x = 0 : \tilde{U}_x - \tilde{B}\tilde{U} = 0 \quad (6.3.18c)$$

and

$$x = 0 : \tilde{U} = 0, \quad (6.3.19a)$$

$$x = 0 : \tilde{U}_x = 0, \quad (6.3.19b)$$

$$x = 0 : \tilde{U}_x + \tilde{B}\tilde{U} = 0, \quad (6.3.19c)$$

where

$$\tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix}, \quad (6.3.20)$$

the elements of which are provided by (6.3.7)–(6.3.8).

Hence, by Lemma 6.1.1, if we know the initial state of  $\tilde{U} = \tilde{U}(t, x)$  at the moment  $t = T$ :

$$t = T : \tilde{U} = (\varphi^{(1)}(x), \varphi^{(2)}(x))^T, \quad \tilde{U}_t = (\psi^{(1)}(x), \psi^{(2)}(x))^T, \quad 0 \leq x \leq L, \quad (6.3.21)$$

then the evolution of  $\tilde{U} = \tilde{U}(t, x)$  with respect to  $t$  can be completely determined.

**Remark 6.3.3** Under assumptions (6.3.3)–(6.3.4) and (6.3.7)–(6.3.8), in order to realize the exact boundary synchronization by 2-groups, one has infinitively many ways to choose  $(N - 2)$  boundary controls at the end  $x = 0$ . Moreover, as in Remark 6.2.4, any given state  $(\varphi^{(1)}, \varphi^{(2)})^T$  and  $(\psi^{(1)}, \psi^{(2)})^T$  in  $(C^2[0, L])^2 \times (C^1[0, L])^2$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (6.3.18) and (6.3.19), belongs to the attainable set of the initial data of synchronizable state by 2-groups.

Thus, at least for some initial states  $(\Phi(x), \Psi(x))$ , the synchronizable states by

2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent on  $t = T$  then for  $t \geq T$ . It shows that the corresponding requirement on  $\tilde{u}_1$  and  $\tilde{u}_2$ , given in Lemma 6.3.1, is reasonable.

**Remark 6.3.4** In the special case that  $m = N - 1$ ,  $U^{(2)}(t, x)$  is composed of only one component  $u_N(t, x)$ , then assumptions (6.3.4) and (6.3.8) reduce to

$$a_{iN} = \tilde{a}_{12} \quad (i = 1, \dots, N - 1), \quad (6.3.22)$$

and

$$b_{iN} \text{ (resp. } \bar{b}_{iN}) = \tilde{b}_{12} \text{ (resp. } \bar{\tilde{b}}_{12}) \quad (i = 1, \dots, N - 1), \quad (6.3.23)$$

where  $\tilde{a}_{12}$  and  $\tilde{b}_{12}$  (resp.  $\bar{\tilde{b}}_{12}$ ) are independent of  $i = 1, \dots, N - 1$ , namely, the first  $(N - 1)$  components of the last column of the coupling matrices  $A$  and  $B$  (resp.  $\bar{B}$ ) are the same, respectively.

**Remark 6.3.5** The exact boundary synchronization by  $k$ -groups can be treated in a similar way.

In order to realize the exact boundary synchronization by 2-groups, the number of boundary controls is equal to  $(N - 2)$ , while, in order to realize the exact boundary synchronization by  $k$ -groups, the number of boundary controls is equal to  $(N - k)$ .

## 6.4 Exact boundary null controllability and synchronization by groups

For the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7), by Lemma 6.1.2, we can use  $N$  boundary controls to realize the exact boundary null controllability. When the number of boundary controls is less than  $N$ , however, generically speaking, it is impossible to realize the same requirement. A natural question is whether we can get the exact boundary null controllability for a part of state variables when the number of boundary controls is less than  $N$ . For instance, is it possible or not to realize the exact boundary null controllability for  $(N - 1)$  state variables by means of only  $(N - 1)$  boundary controls? Since all the state variables are coupled each other, generally speaking, this requirement is impossible to be realized. However, following the idea given in §6.3, under certain assumptions on the coupled system, if we divide the state variables into two groups  $U^{(1)} = (u_1, \dots, u_m)^T$  and  $U^{(2)} = (u_{m+1}, \dots, u_N)^T$  such that  $U^{(1)}$  is exactly null controllable, while  $U^{(2)}$  is exactly synchronizable, then we can realize the previous requirement in some sense. It leads to the following

**Definition 6.4.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi(x), \Psi(x))$ , we can find some boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_m(t, x) \equiv 0, \quad 0 \leq x \leq L, \quad (6.4.1)$$

$$u_{m+1}(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}(t, x), \quad 0 \leq x \leq L, \quad (6.4.2)$$

then the coupled system (6.1.2), (6.1.5) and (6.1.7) is said to possess **the exact boundary null controllability and synchronization by 2-groups** or is said to be **exactly null controllable and synchronizable by 2-groups**, and  $\tilde{u} = \tilde{u}(t, x)$  is called to be the **partially synchronizable state**.

Obviously, if the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the exact null controllability, then it must satisfy the requirement of Definition 6.4.1. This situation should be excluded in advance.

**Lemma 6.4.1** *Suppose that the coupled system (6.1.2), (6.1.5) and (6.1.7) is exactly null controllable and synchronizable by 2-groups, but not exactly null controllable, then the coupling matrix  $A = (a_{ij})$  in system (6.1.2) must satisfy the following necessary conditions:*

$$\sum_{j=m+1}^N a_{ij} = 0 \quad (i = 1, \dots, m), \quad (6.4.3)$$

$$\sum_{j=m+1}^N a_{ij} \stackrel{\text{def.}}{=} \tilde{a} \quad (i = m+1, \dots, N), \quad (6.4.4)$$

where  $\tilde{a}$  is a constant independent of  $i = m+1, \dots, N$ .

For the coupling matrices  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  in the boundary conditions (6.1.5) and (6.1.7), similar necessary conditions hold, namely,

$$\sum_{j=m+1}^N b_{ij} \text{ (resp. } \bar{b}_{ij}) = 0 \quad (i = 1, \dots, m), \quad (6.4.5)$$

$$\sum_{j=m+1}^N b_{ij} \text{ (resp. } \bar{b}_{ij}) \stackrel{\text{def.}}{=} \tilde{b} \text{ (resp. } \tilde{\bar{b}}) \quad (i = m+1, \dots, N), \quad (6.4.6)$$

where  $\tilde{b}$  and  $\tilde{\bar{b}}$  are constants independent of  $i = m+1, \dots, N$ .

**Proof** By Definition 6.4.1, it follows from (6.1.4) that for  $t \geq T$  we have

$$\left( \sum_{j=m+1}^N a_{ij} \right) \tilde{u} = 0, \quad 0 \leq x \leq L, \quad i = 1, \dots, m, \quad (6.4.7)$$

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \left( \sum_{j=m+1}^N a_{ij} \right) \tilde{u} = 0, \quad 0 \leq x \leq L, \quad i = m+1, \dots, N. \quad (6.4.8)$$

In particular, for  $t \geq T$  we get from (6.4.8) that

$$\left( \sum_{j=m+1}^N a_{ij} \right) \tilde{u} = \left( \sum_{j=m+1}^N a_{kj} \right) \tilde{u}, \quad 0 \leq x \leq L, \quad i, k = m+1, \dots, N. \quad (6.4.9)$$

Then, (6.4.3)–(6.4.4) follows directly from the non exact null controllability.

Similarly to the proof of Lemma 6.2.1, we get (6.4.5)–(6.4.6).

**Theorem 6.4.1** *Suppose that (6.4.3)–(6.4.6) hold and  $T > 0$  satisfies (6.2.8). For any given initial state  $(\Phi(x), \Psi(x))$ , we can find  $(N-1)$   $C^2$  (case (6.1.5a)) or  $C^1$  (cases (6.1.5b)–(6.1.5c)) boundary controls with support on  $[0, T]$  in  $H(t)$  (for instance, take  $h_1(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_{m+1}(t) \equiv 0$ ), such that the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the exact boundary null controllability and synchronization by 2-groups.*

**Proof** Let

$$\begin{cases} w_i = u_i & (i = 1, \dots, m), \\ w_i = u_i - u_{i+1} & (i = m+1, \dots, N-1). \end{cases} \quad (6.4.10)$$

Similarly to the proof of Theorem 6.2.1 (cf.[79]), under assumptions (6.4.3)–(6.4.4), the original system (6.1.2) for the variable  $U$  can be reduced to a self-closing system (6.2.10) of the same kind for the variable  $W = (w_1, \dots, w_{N-1})^T$ , where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N-1) \times (N-1)$  matrix with

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & j = 1, \dots, m, \\ -\sum_{p=j+1}^N a_{ip}, & j = m+1, \dots, N-1, \end{cases} \quad \text{for } i = 1, \dots, m, \quad (6.4.11)$$

$$\tilde{a}_{ij} = \begin{cases} a_{ij} - a_{i+1,j}, & j = 1, \dots, m, \\ -\sum_{p=j+1}^N (a_{ip} - a_{i+1,p}) = -\sum_{p=m+1}^j (a_{i+1,p} - a_{ip}), & j = m+1, \dots, N-1, \end{cases} \quad \text{for } i = m+1, \dots, N-1, \quad (6.4.12)$$



Similarly, by (6.4.5)–(6.4.6), the original boundary conditions (6.1.5) and (6.1.7) for the variable  $U$  can be reduced, respectively, to a self-closing boundary conditions (6.2.12) and (6.2.13) of the same kind for the variable  $W$ , in which the coupling matrices  $\tilde{B} = (\tilde{b}_{ij})$  and  $\tilde{\bar{B}} = (\tilde{\bar{b}}_{ij})$  are  $(N-1) \times (N-1)$  matrices reduced by  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  in the following way:

$$\tilde{b}_{ij} = \begin{cases} b_{ij}, & j = 1, \dots, m, \\ -\sum_{p=j+1}^N b_{ip}, & j = m+1, \dots, N-1, \end{cases} \quad \text{for } i = 1, \dots, m, \quad (6.4.13)$$

$$\tilde{\bar{b}}_{ij} = \begin{cases} b_{ij} - b_{i+1,j}, & j = 1, \dots, m, \\ -\sum_{p=j+1}^N (b_{ip} - b_{i+1,p}) = -\sum_{p=1}^j (b_{i+1,p} - b_{ip}), & j = m+1, \dots, N-1, \end{cases} \quad \text{for } i = m+1, \dots, N-1, \quad (6.4.14)$$

etc. Moreover,  $\tilde{H}(t)$  is given by (6.2.15) with

$$\tilde{h}_i(t) = \begin{cases} h_i(t), & i = 1, \dots, m, \\ h_i(t) - h_{i+1}(t), & i = m+1, \dots, N-1. \end{cases} \quad (6.4.15)$$

The initial condition (6.2.17) of  $W$  is given by

$$\begin{aligned} \tilde{\varphi}_i(x) &= \begin{cases} \varphi_i(x), & i = 1, \dots, m, \\ \varphi_i(x) - \varphi_{i+1}(x), & i = m+1, \dots, N-1, \end{cases} \\ \tilde{\psi}_i(x) &= \begin{cases} \psi_i(x), & i = 1, \dots, m, \\ \psi_i(x) - \psi_{i+1}(x), & i = m+1, \dots, N-1. \end{cases} \end{aligned} \quad (6.4.16)$$

Thus, according to Lemma 6.1.2, by means of  $(N-1)$   $C^2$  (case (6.2.12a)) or  $C^1$  (cases (6.2.12b)–(6.2.12c)) boundary controls  $\tilde{H}(t)$  (for instance, take  $h_1(t), \dots, h_m(t), h_{m+1}(t), \dots, h_N(t)$  with  $h_{m+1}(t) \equiv 0$ ) at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly null controllable and synchronizable by 2-groups. This proves Theorem 6.4.1.

**Remark 6.4.1** *Lemma 6.1.1 and Lemma 6.1.2 are always available for the reduced initial-boundary value problem for the variable  $W$ .*

**Remark 6.4.2** *As  $t \geq T$ , the partially synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  satisfies wave equation (6.2.6) and boundary conditions (6.2.19) and (6.2.20), where  $\tilde{a}$ ,  $\tilde{\bar{b}}$  and  $\tilde{\bar{b}}$  are given by (6.4.4) and (6.4.6). Lemma 6.1.1 is always available for this system (6.2.6) and (6.2.19)–(6.2.20).*

Hence, if we know this initial condition (6.2.21) of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ , then the evolution of the partially synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$  can be completely determined.

Moreover, any given state  $(\varphi, \psi)$  in  $C^2[0, L] \times C^1[0, L]$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (6.2.19) and (6.2.20), belongs to the attainable set of the initial data of partially synchronizable state.

**Remark 6.4.3** In the special case that  $m = N - 1$ ,  $U^{(2)}(t, x)$  is composed of only one component  $u_N(t, x)$ , then assumptions (6.4.3) and (6.4.5) become

$$a_{iN} = 0 \quad (i = 1, \dots, N - 1), \quad (6.4.17)$$

$$b_{iN} \text{ (resp. } \bar{b}_{iN}) = 0 \quad (i = 1, \dots, N - 1), \quad (6.4.18)$$

namely, the first  $(N - 1)$  components of the last column of the coupling matrices  $A$  and  $B$  (resp.  $\bar{B}$ ) are all zero. In this case, by Theorem 6.4.1,  $(N - 1)$  boundary controls can be used to realize the exact boundary null controllability for the first  $(N - 1)$  state variables of  $U$ , however, since the first  $(N - 1)$  wave equations and boundary conditions constitute a self-closing system, this result is obvious.

Now, taking  $m = N - 2$  in Theorem 6.4.1, assumptions (6.4.3) and (6.4.6) become

$$\sum_{j=N-1}^N a_{ij} = 0 \quad (i = 1, \dots, N - 2), \quad (6.4.19)$$

$$\sum_{j=N-1}^N a_{ij} = \tilde{a} \quad (i = N - 1, N) \quad (6.4.20)$$

and

$$\sum_{j=N-1}^N b_{ij} \text{ (resp. } \bar{b}_{ij}) = 0, \quad (i = 1, \dots, N - 2), \quad (6.4.21)$$

$$\sum_{j=N-1}^N b_{ij} \text{ (resp. } \bar{b}_{ij}) = \tilde{b} \text{ (resp. } \tilde{\tilde{b}}) \quad (i = N - 1, N). \quad (6.4.22)$$

Thus, we can use  $(N - 1)$  (instead of  $(N - 2)!$ ) boundary controls to realize the exact boundary null controllability for  $(N - 2)$  state variables in  $U$ .

## 6.5 Generalized exact boundary synchronization

We now consider the problem of synchronization from a more general mathematical point of view.

For the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7), based on the previous discussions, we can define the corresponding generalized exact boundary synchronization.

Let

$$w_i = \sum_{j=1}^N \theta_{ij} u_j \quad (i = 1, \dots, M) \quad (6.5.1)$$

or

$$W = \textcircled{\mathbf{H}} U, \quad (6.5.2)$$

where  $W = (w_1, \dots, w_M)^T$ ,  $0 < M < N$  and  $\textcircled{\mathbf{H}} = (\theta_{ij})_{M \times N}$  is of full row-rank.

Suppose that for any given initial state  $(\Phi(x), \Psi(x))$ , the mixed initial-boundary value problem (6.1.2), (6.1.5), (6.1.7) and (6.1.8) for the variable  $U$  can be reduced to a self-closing mixed initial-boundary values problem of the same kind for the variable  $W$ , and we can use Lemma 6.1.2 to find  $M$   $C^2$  (case (6.1.5a)) or  $C^1$  (cases (6.1.5b)-(6.1.5c)) boundary controls in  $H(t)$  with support on  $[0, T]$  with  $T > 2L$ , such that as  $t \geq T$  we have

$$W(t, x) \equiv 0, \quad 0 \leq x \leq L, \quad (6.5.3)$$

i.e.,

$$\textcircled{\mathbf{H}} U(t, x) \equiv 0, \quad 0 \leq x \leq L. \quad (6.5.4)$$

Thus, the coupled system (6.1.2), (6.1.5) and (6.1.7) is said to have the **generalized exact boundary synchronization associated with  $\textcircled{\mathbf{H}}$**  and  $U = U(t, x)$  is the corresponding generalized synchronizable state.

It is easily seen that this definition covers all the cases discussed in the previous sections.

(1) If  $M = N - 1$  and we take

$$w_i = u_i - u_{i+1} \quad (i = 1, \dots, N - 1), \quad (6.5.5)$$

i.e.,

$$\textcircled{\text{H}} = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(N-1) \times N}, \quad (6.5.6)$$

then the generalized exact synchronization leads to the exact synchronization given in §6.2.

(2) If  $M = N - 1$  and we take

$$\begin{cases} w_i = u_i & (i = 1, \dots, m), \\ w_i = u_i - u_{i+1} & (i = m + 1, \dots, N - 1), \end{cases} \quad (6.5.7)$$

i.e.,

$$\textcircled{\text{H}} = \begin{pmatrix} \text{I}_m & & \\ & \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(N-m-1) \times (N-m)} & \\ & & & \end{pmatrix}_{(N-1) \times N}, \quad (6.5.8)$$

where  $\text{I}_m$  is the unit matrix of order  $m$ , then the generalized exact synchronization leads to the exact null controllability and synchronization by 2-groups discussed in §6.4.

(3) If  $M = N - 2$  and we take

$$\begin{cases} w_i = u_i - u_{i+1} & (i = 1, \dots, m - 1), \\ w_i = u_{i+1} - u_{i+2} & (i = m, \dots, N - 2), \end{cases} \quad (6.5.9)$$

i.e.,

$$\textcircled{\text{H}} = \begin{pmatrix} & \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(m-1) \times m} & & \\ & & \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix}_{(N-m-1) \times (N-m)} & \\ & & & \end{pmatrix}_{(N-2) \times N}, \quad (6.5.10)$$

then the generalized exact synchronization leads to the exact synchronization by 2-groups discussed in §6.3.

A natural question is under what conditions the original coupled system for the variable  $U$  can be reduced to a self-closing coupled system of the same type for the variable  $W$ .

Multiplying the matrix  $\textcircled{\mathbb{H}}$  to the system (6.1.2), (6.1.5) and (6.1.7) from the left, it is easy to get

**Theorem 6.5.1** *For any given coupling  $N \times N$  matrices  $A$ ,  $B$  and  $\bar{B}$ , if there exist a full row-rank matrix  $\textcircled{\mathbb{H}} = (\theta_{ij})_{M \times N}$ , and  $M \times M$  matrices  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{\bar{B}}$  such that*

$$\textcircled{\mathbb{H}}A = \tilde{A}\textcircled{\mathbb{H}}, \quad (6.5.11)$$

$$\textcircled{\mathbb{H}}B = \tilde{B}\textcircled{\mathbb{H}}, \quad (6.5.12)$$

$$\textcircled{\mathbb{H}}\bar{B} = \tilde{\bar{B}}\textcircled{\mathbb{H}}, \quad (6.5.13)$$

then we can get a self-closing coupled system (6.2.10) and (6.2.12)–(6.2.13) with  $\tilde{H}(t) = \textcircled{\mathbb{H}}H(t)$  for the variable  $W$ , so that the generalized exact boundary synchronization associated with  $\textcircled{\mathbb{H}}$  for the variable  $U$  can be realized by means of  $M$  boundary control functions.

**Remark 6.5.1** *Lemma 6.1.1 and Lemma 6.1.2 are always available for the reduced coupled system for the variable  $W$ .*

The next question is for any given coupling  $N \times N$  matrices  $A$ ,  $B$  and  $\bar{B}$ , how to guarantee the existence of such  $M \times N$  matrix  $\textcircled{\mathbb{H}}$  and  $M \times M$  matrices  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{\bar{B}}$  in Theorem 6.5.1.

We first consider the case  $M = N - 1$ .

**Theorem 6.5.2** *Suppose that  $M = N - 1$ . For the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7), if the coupling matrices  $A$ ,  $B$  and  $\bar{B}$  possess a common real right eigenvector  $X \in \mathbb{R}^N \setminus \{0\}$ :*

$$AX = \tilde{\tilde{a}}X, \quad (6.5.14)$$

$$BX = \tilde{\tilde{b}}X, \quad (6.5.15)$$

$$\bar{B}X = \tilde{\tilde{b}}X, \quad (6.5.16)$$

where  $\tilde{\tilde{a}}$ ,  $\tilde{\tilde{b}}$  and  $\tilde{\tilde{b}}$  are corresponding eigenvalues, respectively, then there exist an  $(N - 1) \times N$  full row-rank matrix  $\textcircled{\mathbb{H}} = (\theta_{ij})$  and  $(N - 1) \times (N - 1)$  matrices  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{\bar{B}}$

such that (6.5.11)–(6.5.13) hold. Moreover,  $X$  constitutes a system of basic solutions to the linear algebraic system

$$\textcircled{\text{H}}Y = 0. \tag{6.5.17}$$

**Proof** There exist  $(N - 1)$  linearly independent column vectors  $Y_1, \dots, Y_{N-1}$  perpendicular to  $X$ ;

$$\langle X, Y_j \rangle = 0, \quad j = 1, \dots, N - 1. \tag{6.5.18}$$

Thus, noting (6.5.14) we have

$$\langle X, A^T Y_j \rangle = \langle AX, Y_j \rangle = 0, \quad j = 1, \dots, N - 1. \tag{6.5.19}$$

Then,

$$A^T Y_j = \sum_{k=1}^{N-1} \hat{a}_{jk} Y_k, \quad j = 1, \dots, N - 1, \tag{6.5.20}$$

where  $\hat{a}_{jk}$  are constants. Let

$$\tilde{A} = (\hat{a}_{jk})^T = (\tilde{a}_{jk})_{(N-1) \times (N-1)}. \tag{6.5.21}$$

It follows from (6.5.20) that

$$A^T(Y_1, \dots, Y_{N-1}) = (Y_1, \dots, Y_{N-1})\tilde{A}^T, \tag{6.5.22}$$

then, taking

$$\textcircled{\text{H}} = (Y_1, \dots, Y_{N-1})^T, \tag{6.5.23}$$

we get (6.5.11). Moreover, it is easy to see that  $X$  constitutes a system of basic solutions to the linear algebraic system (6.5.17).

The proof of (6.5.12)–(6.5.13) is completely similar.

**Remark 6.5.2** In case (1),  $M = N - 1$ , noting (6.5.6),  $X = (1, \dots, 1)^T$  is a system of basic solutions to (6.5.17). The requirement that the coupling matrices  $A$ ,  $B$  and  $\bar{B}$  possess a common right eigenvector  $X$  implies (6.2.2)–(6.2.4). Thus, Theorem 6.5.2 leads to the conclusion given in §6.2.

**Remark 6.5.3** In case (2),  $M = N - 1$ , noting (6.5.8),  $X = (\overbrace{0, \dots, 0}^m, \overbrace{1, \dots, 1}^{N-m})^T$  is a system of basic solutions to (6.5.17). The requirement that the coupled matrices  $A$ ,  $B$  and  $\bar{B}$  possess a common right eigenvector  $X$  implies (6.4.3)–(6.4.6). Thus, the conclusion given in §6.4 follows from Theorem 6.5.2.

On the other hand, we have

**Theorem 6.5.3** *Suppose that  $M = N - 1$ . If the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the generalized exact boundary synchronization associated with  $\textcircled{\mathbf{H}}$ , but is not exactly boundary null controllable, then the coupling matrices  $A$ ,  $B$  and  $\bar{B}$  must have a common real right eigenvector  $X$ , and  $X$  constitutes a system of basic solutions to the linear algebraic system (6.5.17).*

**Proof** Since the rank of  $\textcircled{\mathbf{H}}$  is  $(N - 1)$ , the linear algebraic system (6.5.17) possesses a basic solution  $X = (\alpha_1, \dots, \alpha_N)^T \in \mathbb{R}^N \setminus \{0\}$ . Thus, when  $t \geq T$ , it follows from (6.5.4) that the generalized synchronizable state  $U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  can be written as

$$u_i(t, x) = \alpha_i \tilde{u}(t, x), \quad 0 \leq x \leq L, \quad (6.5.24)$$

where  $\tilde{u} = \tilde{u}(t, x)$  is a  $C^2$  function.

Substituting (6.5.24) into (6.1.4), when  $t \geq T$  we have

$$\alpha_i \left( \frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} \right) + \left( \sum_{j=1}^N a_{ij} \alpha_j \right) \tilde{u} = 0, \quad i = 1, \dots, N. \quad (6.5.25)$$

Let

$$\aleph = \{i \mid \alpha_i \neq 0, i = 1, \dots, N\}. \quad (6.5.26)$$

When the number of  $\aleph$  is bigger than or equal to 2, it is easy to see from (6.5.25) that for any given  $i, k \in \aleph$ ,

$$\frac{1}{\alpha_i} \left( \sum_{j=1}^N a_{ij} \alpha_j \right) \tilde{u} = \frac{1}{\alpha_k} \left( \sum_{j=1}^N a_{kj} \alpha_j \right) \tilde{u},$$

then by the non exact null controllability, we get

$$\frac{1}{\alpha_i} \left( \sum_{j=1}^N a_{ij} \alpha_j \right) = \frac{1}{\alpha_k} \left( \sum_{j=1}^N a_{kj} \alpha_j \right) \stackrel{\text{def.}}{=} \tilde{a}, \quad \forall i, k \in \aleph,$$

namely, for  $i \in \aleph$  we have

$$\sum_{j=1}^N a_{ij} \alpha_j = \tilde{a} \alpha_i, \quad (6.5.27)$$

where  $\tilde{a}$  is a constant independent of  $i \in \aleph$ . On the other hand, for  $i \notin \aleph$ , since  $\alpha_i = 0$ , by (6.5.25) and the non exact null controllability, (6.5.27) is still valid.

When the number of  $\aleph$  is equal to 1, without loss of generality, we may suppose that  $\aleph = \{1\}$ , namely,  $\alpha_1 \neq 0$  and  $\alpha_2 = \dots = \alpha_N = 0$ . Taking  $\tilde{a} = a_{11}$ , we still have (6.5.27) for  $i = 1$ . Moreover, as in the previous situation, (6.5.27) is still valid for  $i \notin \aleph$ .

Thus, for  $i = 1 \cdots, N$  we always have (6.5.27), where  $\tilde{a}$  is independent of  $i$ . This proves (6.5.14). Moreover, it follows from (6.5.25) and (6.5.14) that  $\tilde{u} = \tilde{u}(t, x)$  satisfies the wave equation (6.2.6) for  $t \geq T$ .

We next prove (6.5.15). The proof of (6.5.16) is similar.

Substituting (6.5.24) into the boundary condition (6.1.5c), for  $t \geq T$  we have

$$x = 0 : \alpha_i \frac{\partial \tilde{u}}{\partial x} - \left( \sum_{j=1}^N b_{ij} \alpha_j \right) \tilde{u} = 0, \quad i = 1, \cdots, N. \quad (6.5.28)$$

As in the proof of Lemma 6.2.1, we may suppose that  $\tilde{u}(t, 0) \neq 0$  for  $t \geq T$ , then similarly we can get

$$\sum_{j=1}^N b_{ij} \alpha_j = \tilde{b} \alpha_i, \quad i = 1, \cdots, N,$$

where  $\tilde{b}$  is a constant independent of  $i$ . This proves (6.5.15).

**Remark 6.5.4** *As  $t \geq T$ , for the generalized synchronizable state  $U(t, x) = \tilde{u}(t, x)X$ , where  $X = (\alpha_1, \cdots, \alpha_N)^T$  is a basic solution of (6.5.17),  $\tilde{u} = \tilde{u}(t, x)$  satisfies the wave equation (6.2.6) and the boundary conditions (6.2.19) and (6.2.20). Then, if we know the initial state (6.2.21) of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ , we can completely determine the evolution of  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$ .*

*Moreover, as in Remark 6.2.4, the set of all possible initial data  $(\varphi, \psi)$  of  $\tilde{u} = \tilde{u}(t, x)$  is the whole space  $C^2[0, L] \times C^1[0, L]$  associated with the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (6.2.19) and (6.2.20).*

For the case  $M = N - 2$ , we only consider the special situation that

$$\textcircled{\text{H}} = \begin{pmatrix} \textcircled{\text{H}}_1 & \\ & \textcircled{\text{H}}_2 \end{pmatrix}_{(N-2) \times N}, \quad (6.5.29)$$

where  $\textcircled{\text{H}}_1$  is a full row-rank  $(N_1 - 1) \times N_1$  matrix, while  $\textcircled{\text{H}}_2$  is a full row-rank  $(N_2 - 1) \times N_2$  matrix with  $N_1 + N_2 = N$ .

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \text{ etc.}, \quad (6.5.30)$$

where  $A_{11}$  (resp.  $\tilde{A}_{11}$ ) is an  $N_1 \times N_1$  (resp.  $(N_1 - 1) \times (N_1 - 1)$ ) matrix,  $A_{21}$  (resp.  $\tilde{A}_{21}$ ) is an  $N_2 \times N_1$  (resp.  $(N_2 - 1) \times (N_1 - 1)$ ) matrix,  $A_{12}$  (resp.  $\tilde{A}_{12}$ ) is an  $N_1 \times N_2$  (resp.



$(N_1 - 1) \times (N_2 - 1)$  matrix,  $A_{22}$  (resp.  $\tilde{A}_{22}$ ) is an  $N_2 \times N_2$  (resp.  $(N_2 - 1) \times (N_2 - 1)$ ) matrix etc. Thus, (6.5.11)–(6.5.13) in Theorem 6.5.1 can be equivalently written as

$$\textcircled{\mathbb{H}}_i A_{ij} = \tilde{A}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2, \quad (6.5.31)$$

$$\textcircled{\mathbb{H}}_i B_{ij} = \tilde{B}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2, \quad (6.5.32)$$

$$\textcircled{\mathbb{H}}_i \bar{B}_{ij} = \bar{B}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2. \quad (6.5.33)$$

**Theorem 6.5.4** *Suppose that  $M = N - 2$ . For the coupled system of wave equations (6.1.2), (6.1.5) and (6.1.7), if there exist real vectors  $X_1 \in \mathbb{R}^{N_1} \setminus \{0\}$  and  $X_2 \in \mathbb{R}^{N_2} \setminus \{0\}$  and real numbers  $\tilde{a}_{ij}$ ,  $\tilde{b}_{ij}$  and  $\bar{b}_{ij}$  such that*

$$A_{ij} X_j = \tilde{a}_{ij} X_i, \quad i, j = 1, 2, \quad (6.5.34)$$

$$B_{ij} X_j = \tilde{b}_{ij} X_i, \quad i, j = 1, 2, \quad (6.5.35)$$

$$\bar{B}_{ij} X_j = \bar{b}_{ij} X_i, \quad i, j = 1, 2, \quad (6.5.36)$$

then we can find a  $(N - 2) \times N$  full row-rank matrix  $\textcircled{\mathbb{H}} = \begin{pmatrix} \textcircled{\mathbb{H}}_1 & \textcircled{\mathbb{H}}_2 \end{pmatrix}$  and  $(N - 2) \times (N - 2)$  matrices  $\tilde{A}$ ,  $\tilde{B}$  and  $\bar{B}$  such that (6.5.31)–(6.5.33) hold, moreover,  $X_1$  (resp.  $X_2$ ) constitutes a system of basic solutions to the linear algebraic system

$$\textcircled{\mathbb{H}}_1 Y_1 = 0 \quad (\text{resp.} \quad \textcircled{\mathbb{H}}_2 Y_2 = 0). \quad (6.5.37)$$

**Proof** Specially taking  $i = j$  in (6.5.34)–(6.5.36), we get

$$A_{ii} X_i = \tilde{a}_{ii} X_i, \quad i = 1, 2, \quad (6.5.38)$$

$$B_{ii} X_i = \tilde{b}_{ii} X_i, \quad i = 1, 2, \quad (6.5.39)$$

$$\bar{B}_{ii} X_i = \bar{b}_{ii} X_i, \quad i = 1, 2. \quad (6.5.40)$$

Then, by the proof of Theorem 6.5.2, there exist  $N_1 - 1$  (resp.  $N_2 - 1$ ) linearly independent column vectors  $Y_1, \dots, Y_{N_1-1} \in \mathbb{R}^{N_1} \setminus \{0\}$  (resp.  $Y_{N_1}, \dots, Y_{N-2} \in \mathbb{R}^{N_2} \setminus \{0\}$ ) perpendicular to  $X_1$  (resp.  $X_2$ ) such that (6.5.31)–(6.5.33) hold for  $i = j$  with some matrices  $\tilde{A}_{ii}$ ,  $\tilde{B}_{ii}$  and  $\bar{B}_{ii}$  and

$$\textcircled{\mathbb{H}}_1 = (Y_1, \dots, Y_{N_1-1})^T, \quad \textcircled{\mathbb{H}}_2 = (Y_{N_1}, \dots, Y_{N-2})^T. \quad (6.5.41)$$

We now prove that there exist matrices  $\tilde{A}_{ij}$ ,  $\tilde{B}_{ij}$  and  $\bar{B}_{ij}$  for  $i \neq j$  such that (6.5.31)–(6.5.33) are still valid for  $i \neq j$ , in which  $\textcircled{\mathbb{H}}_1$  and  $\textcircled{\mathbb{H}}_2$  are given by (6.5.41). For fixing the data, in what follows we only prove (6.5.31) for  $i = 1$  and  $j = 2$ .

Noting (6.5.34), we have

$$\langle X_2, A_{12}^T Y_j \rangle = \langle A_{12} X_2, Y_j \rangle = \langle \tilde{a}_{12} X_1, Y_j \rangle = 0, \quad j = 1, \dots, N_1 - 1. \quad (6.5.42)$$

Then

$$A_{12}^T Y_j = \sum_{k=N_1}^{N-2} \hat{a}_{jk} Y_k, \quad j = 1, \dots, N_1 - 1. \quad (6.5.43)$$

where  $\hat{a}_{jk}$  are constants. Let

$$\tilde{A}_{12} = (\hat{a}_{jk})^T = (\tilde{a}_{jk}). \quad (6.5.44)$$

By (6.5.43), we have

$$A_{12}^T (Y_1, \dots, Y_{N_1-1}) = (Y_1, \dots, Y_{N_1-1}) \tilde{A}_{12}^T, \quad (6.5.45)$$

which together with (6.5.41) leads to (6.5.31) for  $i = 1$  and  $j = 2$ .

**Remark 6.5.5** *In case (3),  $M = N - 2$ , noting (6.5.10), we have  $X_1 = (\overbrace{1, \dots, 1}^m)^T$  and  $X_2 = (\overbrace{1, \dots, 1}^{N-m})^T$ . The requirements (6.5.34)–(6.5.36) imply (6.3.3)–(6.3.4) and (6.3.7)–(6.3.8), then Theorem 6.5.4 leads to the conclusion given in §6.3.*

By (6.5.29) and noting that the rank of both  $\textcircled{\mathbb{H}}_1$  and  $\textcircled{\mathbb{H}}_2$  is  $(N - 1)$ , linear algebraic system (6.5.37) possesses a basic solution  $X_1 = (\alpha_1, \dots, \alpha_{N_1})^T \in \mathbb{R}^{N_1} \setminus \{0\}$  (resp.  $X_2 = (\alpha_{N_1+1}, \dots, \alpha_N)^T \in \mathbb{R}^{N_2} \setminus \{0\}$ ). Thus, noting (6.5.4), as  $t \geq T$  the generalized synchronizable state  $U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  can be written as

$$u_p(t, x) = \alpha_p \tilde{u}_1(t, x), \quad p = 1, \dots, N_1, \quad 0 \leq x \leq L, \quad (6.5.46)$$

$$u_q(t, x) = \alpha_q \tilde{u}_2(t, x), \quad q = N_1 + 1, \dots, N, \quad 0 \leq x \leq L. \quad (6.5.47)$$

Similarly to Theorem 6.5.3, we have

**Theorem 6.5.5** *Suppose that  $M = N - 2$ . If the coupled system (6.1.2), (6.1.5) and (6.1.7) possesses the generalized exact boundary synchronization associated with (6.5.29), and at least for an initial state  $(\Phi(x), \Psi(x))$ ,  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  given by (6.5.46)–(6.5.47) for the generalized synchronizable state are linearly independent. Then there exist real vectors  $X_1 \in \mathbb{R}^{N_1} \setminus \{0\}$  and  $X_2 \in \mathbb{R}^{N_2} \setminus \{0\}$  and real numbers  $\tilde{a}_{ij}$  ( $i, j = 1, 2$ ) such that (6.5.34) holds.*

**Proof** Substituting (6.5.46)–(6.5.47) into (6.1.4), we have

$$\alpha_i \left( \frac{\partial^2 \tilde{u}_1}{\partial t^2} - \frac{\partial^2 \tilde{u}_1}{\partial x^2} \right) + \left( \sum_{p=1}^{N_1} a_{ip} \alpha_p \right) \tilde{u}_1 + \left( \sum_{q=N_1+1}^N a_{iq} \alpha_q \right) \tilde{u}_2 = 0, \quad i = 1, \dots, N_1, \quad (6.5.48)$$

$$\alpha_i \left( \frac{\partial^2 \tilde{u}_2}{\partial t^2} - \frac{\partial^2 \tilde{u}_2}{\partial x^2} \right) + \left( \sum_{p=1}^{N_1} a_{ip} \alpha_p \right) \tilde{u}_1 + \left( \sum_{q=N_1+1}^N a_{iq} \alpha_q \right) \tilde{u}_2 = 0, \quad i = N_1 + 1, \dots, N. \quad (6.5.49)$$

Let

$$\aleph_1 = \{p \mid \alpha_p \neq 0, p = 1, \dots, N_1\}. \quad (6.5.50)$$

When the number  $\aleph_1$  is bigger than or equal to 2, it is easy to see from (6.5.48) that for any given  $i, k \in \aleph_1$

$$\begin{aligned} & \frac{1}{\alpha_i} \left[ \left( \sum_{p=1}^{N_1} a_{ip} \alpha_p \right) \tilde{u}_1 + \left( \sum_{q=N_1+1}^N a_{iq} \alpha_q \right) \tilde{u}_2 \right] \\ &= \frac{1}{\alpha_k} \left[ \left( \sum_{p=1}^{N_1} a_{kp} \alpha_p \right) \tilde{u}_1 + \left( \sum_{q=N_1+1}^N a_{kq} \alpha_q \right) \tilde{u}_2 \right], \end{aligned}$$

then, since we may assume that  $\tilde{u}_1$  and  $\tilde{u}_2$  are linearly independent, we get

$$\begin{aligned} \frac{1}{\alpha_i} \left( \sum_{p=1}^{N_1} a_{ip} \alpha_p \right) &= \frac{1}{\alpha_k} \left( \sum_{p=1}^{N_1} a_{kp} \alpha_p \right) \stackrel{\text{def.}}{=} \tilde{a}_{11}, \quad \forall i, k \in \aleph_1, \\ \frac{1}{\alpha_i} \left( \sum_{q=N_1+1}^N a_{iq} \alpha_q \right) &= \frac{1}{\alpha_k} \left( \sum_{q=N_1+1}^N a_{kq} \alpha_q \right) \stackrel{\text{def.}}{=} \tilde{a}_{12}, \quad \forall i, k \in \aleph_1, \end{aligned}$$

namely, for  $i \in \aleph_1$ , we have

$$\sum_{p=1}^{N_1} a_{ip} \alpha_p = \tilde{a}_{11} \alpha_i, \quad (6.5.51)$$

$$\sum_{q=N_1+1}^N a_{iq} \alpha_q = \tilde{a}_{12} \alpha_i, \quad (6.5.52)$$

where  $\tilde{a}_{11}$  and  $\tilde{a}_{12}$  are constants in dependent of  $i \in \aleph_1$ . On the other hand, for  $i \notin \aleph_1$ , since  $\alpha_i = 0$ , by (6.5.48) and noting that  $\tilde{u}_1$  and  $\tilde{u}_2$  are linearly independent, (6.5.51)–(6.5.52) are still valid.

When the number of  $\aleph_1$  is equal to 1, without loss of generality, we may suppose that  $\aleph_1 = \{1\}$ , namely,  $\alpha_1 \neq 0$  and  $\alpha_2 = \dots = \alpha_{N_1} = 0$ . Taking  $\tilde{\tilde{a}}_{11} = a_{11}$  and  $\tilde{\tilde{a}}_{12} = \frac{1}{\alpha_1} \left( \sum_{q=N_1+1}^N a_{1q} \alpha_q \right)$ , we still have (6.5.51)–(6.5.52) for  $i = 1$ . Moreover, as in the previous situation, (6.5.51)–(6.5.52) are still valid for  $i \notin \aleph_1$ .

Thus, for  $i = 1, \dots, N$ , we always have (6.5.51)–(6.5.52), where  $\tilde{\tilde{a}}_{11}$  and  $\tilde{\tilde{a}}_{12}$  are independent of  $i$ . This proves (6.5.34) for  $i = j = 1$  and  $i = 1, j = 2$ .

Similarly, by (6.5.49), we get (6.5.34) for  $i = 2, j = 1$  and  $i = j = 2$ .  $\square$

**Remark 6.5.6** *As  $t \geq T$ , for the generalized synchronizable state  $U(t, x) = \begin{pmatrix} \tilde{\tilde{u}}_1(t, x) X_1 \\ \tilde{\tilde{u}}_2(t, x) X_2 \end{pmatrix}$  given by (6.5.46)–(6.5.47), where  $X_1 = (\alpha_1, \dots, \alpha_{N_1})^T$  (resp.  $X_2 = (\alpha_{N_1+1}, \dots, \alpha_N)^T$ ) is a basic solution to (6.5.37),  $\tilde{\tilde{U}}(t, x) = \begin{pmatrix} \tilde{\tilde{u}}_1(t, x) \\ \tilde{\tilde{u}}_2(t, x) \end{pmatrix}$  satisfies the coupled system (6.3.16) of wave equations with (6.3.17), in which  $\tilde{\tilde{a}}_{ij}$  ( $i, j = 1, 2$ ) are given by (6.5.34), and the boundary conditions (6.3.18) and (6.3.19) with (6.3.20), in which  $\tilde{\tilde{b}}_{ij}$  and  $\tilde{\tilde{c}}_{ij}$  ( $i, j = 1, 2$ ) are given by (6.5.35)–(6.5.36). Hence, if we know the initial state (6.3.21) of  $\tilde{\tilde{U}} = \tilde{\tilde{U}}(t, x)$  at the moment  $t = T$ , then the evolution of  $\tilde{\tilde{U}} = \tilde{\tilde{U}}(t, x)$  with respect to  $t$  can be completely determined.*

Moreover, any given state  $(\varphi^{(1)}(x), \varphi^{(2)}(x))^T$  and  $(\psi^{(1)}(x), \psi^{(2)}(x))^T$  in  $(C^2[0, L])^2 \times (C^1[0, L])^2$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (6.3.18) and (6.3.19), can be the initial data of  $\tilde{\tilde{U}} = \tilde{\tilde{U}}(t, x)$ .

Thus, at least for some initial states  $(\Phi(x), \Psi(x))$ ,  $\tilde{\tilde{u}}_1(t, x)$  and  $\tilde{\tilde{u}}_2(t, x)$  are linearly independent on  $t = T$  then for  $t \geq T$ . This shows that the corresponding requirement given in Theorem 6.5.5 is reasonable.

**Remark 6.5.7** *The situation  $M = N - k$  ( $k > 2$ ) can be similarly discussed.*

## 6.6 Remarks

**Remark 6.6.1** *The whole previous discussion is still valid for the case of two-sided controls, provided that (6.2.8) is replaced by*

$$T > L \tag{6.6.1}$$

*and there is an inhomogeneous term  $\overline{H}(t)$  on the right-hand side of (6.1.7).*

**Remark 6.6.2** *It is possible that, different from Lemma 6.1.2, in the case of one-sided controls, the exact boundary (null) controllability may be realized (perhaps*

in a weaker sense) by means of  $M(< N)$  boundary controls for some special coupled systems of  $N$  wave equations (cf. [2], [3], [6], [78]). Thus, the exact boundary synchronization (perhaps in a weaker sense) could be realized by only  $P(< M)$  boundary controls.

**Remark 6.6.3** *The problem of synchronization might be also considered as an optimal control problem, where an objective function of the type*

$$\sum_{i=1}^{N-1} \|u_i - u_{i+1}\|^2$$

*is minimized. This kind of optimal control is similar to the optimal control of partial differential equations on networks (cf. [45] for an application in flood management) and should be studied in the framework of weak solutions.*

# Chapter 7

## Exact Boundary Synchronization for a Coupled System of Linear Wave Equations with Coupled Boundary Conditions of Dissipative Type

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### 7.1 Introduction

Several kinds of exact synchronizations for a coupled system of 1-D wave equations with boundary conditions of Dirichlet type, Neumann type and coupled third type are discussed in [81] in the framework of classical solutions. A similar result for a coupled system of n-dimensional wave equations with Dirichlet boundary controls in the framework of weak solutions can be found in [79]. In this chapter, we will consider the corresponding exact synchronizations for the coupled boundary conditions of dissipative type.

More precisely, we consider the following coupled system of 1-D wave equations with coupled boundary conditions of dissipative type:

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + AU = 0, & (7.1.1) \\ x = 0 : U_x - CU_t = H(t), & (7.1.2) \\ x = L : U_x + \bar{C}U_t = 0, & (7.1.3) \end{cases}$$

where  $U = (u_1, \dots, u_N)^T$  is the unknown vector function of  $(t, x)$ ,  $A = (a_{ij})$ ,  $C = (c_{ij})$  and  $\bar{C} = (\bar{c}_{ij})$  are  $N \times N$  coupling matrices with constant elements,  $H(t) = (h_1(t), \dots, h_N(t))^T$  is a  $C^1$  vector function, the components of which will be totally or partially taken as boundary controls. Our goal is to find a control time  $T > 0$  and some boundary controls in  $H(t)$  supported on  $[0, T]$ , such that from the time  $t = T$  on, the system states tend to the same. That is to say, we hope to achieve the synchronization of system states not only at the moment  $t = T$  under the action of boundary controls on  $[0, T]$ , but also when  $t \geq T$  and withdrawing all the controls (see also [79]-[81]).

Obviously, if the system has the exact boundary null controllability ([55], [77], [88], [89] and [100]), it must have the exact boundary synchronization, but this is a trivial situation that should be excluded beforehand. The exact boundary null controllability here also means that the system states remain null not only at the moment  $t = T$  under the action of boundary controls on  $[0, T]$ , but also when  $t \geq T$  and withdrawing all the controls.

For the forward mixed initial-boundary value problem (7.1.1)-(7.1.3) with the following initial condition

$$t = 0 : U = \Phi(x), U_t = \Psi(x), \quad (7.1.4)$$

we have (see [55])

**Lemma 7.1.1** *For any given  $T > 0$ , assume that  $\Phi \in (C^2[0, L])^N$  and  $\Psi \in (C^1[0, L])^N$  satisfy the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$ , respectively (see Remark 7.1.1). For any given coupling matrices  $A$ ,  $C$  and  $\bar{C}$ , if*

$$\det(C + I_N) \neq 0 \quad (7.1.5)$$

and

$$\det(\bar{C} + I_N) \neq 0, \quad (7.1.6)$$

where  $I_N$  denotes the unit matrix of order  $N$ , namely,  $-1$  is not an eigenvalue of both  $C$  and  $\bar{C}$ , then the forward mixed initial-boundary value problem (7.1.1)-(7.1.4) admits a unique  $C^2$  solution  $U = U(t, x)$  on the domain  $R(T) = \{(t, x) | 0 \leq t \leq$

$T, 0 \leq x \leq L\}$ .

**Remark 7.1.1** *The conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$  can be written as*

$$\begin{cases} \Phi'(L) + \bar{C}\Psi(L) = 0, \\ \Psi'(L) + \bar{C}(\Phi''(L) - A\Phi(L)) = 0. \end{cases} \quad (7.1.7)$$

*The conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  can be similarly obtained.*

**Remark 7.1.2** *For the backward mixed initial-boundary value problem (7.1.1)-(7.1.3) with the final condition*

$$t = T : U = \bar{\Phi}(x), \quad U_t = \bar{\Psi}(x), \quad (7.1.8)$$

*where  $\bar{\Phi} \in (C^2[0, L])^N$ ,  $\bar{\Psi} \in (C^1[0, L])^N$  and the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively. The conclusion of Lemma 7.1.1 is still valid, provided that assumptions (7.1.5)-(7.1.6) are replaced by*

$$\det(C - I_N) \neq 0 \quad (7.1.9)$$

*and*

$$\det(\bar{C} - I_N) \neq 0, \quad (7.1.10)$$

*namely, 1 is not an eigenvalue of both  $C$  and  $\bar{C}$ .*

By Lemma 7.1.1 and using a constructive method, we have the following result on the exact boundary null controllability(see [55], [72], [83], [109], [118]).

**Lemma 7.1.2** *Let*

$$T > 2L. \quad (7.1.11)$$

*Under assumptions (7.1.5)-(7.1.6), for any given initial data  $(\Phi, \Psi) \in (C^2[0, L])^N \times (C^1[0, L])^N$ , satisfying the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , there exist  $N$   $C^1$  boundary controls  $H(t) = (h_1(t), \dots, h_N(t))^T$  with support on  $[0, T]$  at the end  $x = 0$ , such that the corresponding mixed initial-boundary value problem (7.1.1)-(7.1.4) admits a unique  $C^2$  solution  $U = U(t, x)$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$U(t, x) \equiv 0, \quad 0 \leq x \leq L. \quad (7.1.12)$$



**Remark 7.1.3** *Different from the situation discussed in [81], in the case of coupled boundary conditions of dissipative type, in order to guarantee the well-posedness and the exact boundary null controllability for the forward mixed initial-boundary value problem, we should impose assumptions (7.1.5)-(7.1.6) for matrices  $C$  and  $\bar{C}$ .*

In what follows, we will use Lemma 7.1.1 and Lemma 7.1.2 to consider the exact boundary synchronization, the exact boundary synchronization by groups, and the exact boundary null controllability and synchronization by groups for the coupled system of wave equations (7.1.1)-(7.1.3) in §7.2, §7.3 and §7.4, respectively, and then the generalized exact boundary synchronization in §7.5. The whole forthcoming discussion is based on the situation of one-sided controls, similar results are still valid for the case of two-sided controls (see [81]).

## 7.2 Exact boundary synchronization

**Definition 7.2.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi, \Psi)$ , we can find some boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (7.1.1)-(7.1.4) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}(t, x), \quad 0 \leq x \leq L, \quad (7.2.1)$$

*then the coupled system (7.1.1)-(7.1.3) is said to possess the **exact boundary synchronization** or is said to be **exactly synchronizable**, and  $\tilde{u} = \tilde{u}(t, x)$  is called the corresponding **synchronizable state**.*

Obviously, if the system possesses the exact boundary null controllability, then it must be exactly synchronizable. However, this trivial situation should be excluded in the discussion. Similarly to [81], we have

**Lemma 7.2.1** *Under assumptions (7.1.5) and (7.1.6), if the coupled system of wave equations (7.1.1)-(7.1.3) is exactly synchronizable, but not exactly null controllable, then the coupling matrix  $A$  should satisfy the following necessary conditions: The values of the sum of every row of  $A$  are independent of  $i = 1, \dots, N$ , namely,*

$$\sum_{j=1}^N a_{ij} \stackrel{\text{def.}}{=} \tilde{a} \quad (i = 1, \dots, N), \quad (7.2.2)$$

*where  $\tilde{a}$  is a constant independent of  $i = 1, \dots, N$ .*

Different from the situation discussed in [81], Lemma 7.2.1 is no longer valid in general for the coupling matrices  $C$  and  $\bar{C}$  in the coupled dissipative boundary

conditions (7.1.2)-(7.1.3). However, based on Lemma 7.2.1, for matrices  $C$  and  $\bar{C}$  we impose the following similar assumptions:

$$\sum_{j=1}^N c_{ij} \stackrel{\text{def.}}{=} \tilde{c} \quad (i = 1, \dots, N), \tag{7.2.3}$$

$$\sum_{j=1}^N \bar{c}_{ij} \stackrel{\text{def.}}{=} \tilde{\bar{c}} \quad (i = 1, \dots, N), \tag{7.2.4}$$

where  $\tilde{c}$  and  $\tilde{\bar{c}}$  are constants independent of  $i = 1, \dots, N$ .

**Remark 7.2.1** *Suppose furthermore that the constant  $\tilde{a}$  given by (7.2.2) is positive, it is easy to get the necessity of (7.2.3)-(7.2.4). In fact, for a coupled system of  $n$ -dimensional wave equations with Neumann boundary conditions, in order to guarantee the uniqueness of solution, one often assumes that  $A$  is a positive definite matrix, then the hypothesis  $\tilde{a} > 0$  is reasonable since  $\tilde{a}$  is an eigenvalue of matrix  $A$ . However, in more general cases, we do not know the necessity of (7.2.3)-(7.2.4). It is a very interesting problem if we can realize the boundary synchronization for the coupled system (7.1.1)-(7.1.3) without additional assumptions (7.2.3)-(7.2.4).*

**Remark 7.2.2** *Conditions (7.2.2)-(7.2.4) mean that all the matrices  $A$ ,  $C$  and  $\bar{C}$  have a common right eigenvector  $(1, 1, \dots, 1)^T$  with the corresponding eigenvalues  $\tilde{a}$ ,  $\tilde{c}$  and  $\tilde{\bar{c}}$ , respectively.*

**Theorem 7.2.1** *Under assumptions (7.1.5)-(7.1.6), suppose furthermore that (7.2.2) – (7.2.4) hold. Let*

$$T > 2L. \tag{7.2.5}$$

*For any given initial state  $(\Phi, \Psi) \in (C^2[0, L])^N \times (C^1[0, L])^N$ , satisfying the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , there exist  $(N - 1)$   $C^1$  boundary controls with support on  $[0, T]$  in  $H(t)$  (for example, take  $h_2(t), \dots, h_N(t)$  with  $h_1(t) \equiv 0$ ), such that the coupled system (7.1.1)-(7.1.3) possesses the exact boundary synchronization.*

**Proof** Let

$$w_i = u_{i+1} - u_i \quad (i = 1, \dots, N - 1). \tag{7.2.6}$$

It is easily shown (see [79]-[81]) that by (7.2.2), the original system (7.1.1) for the variable  $U$  can be reduced to a self-closing system of the same kind for the variable

$W = (w_1, \dots, w_{N-1})^T$  as follows:

$$\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} + \tilde{A}W = 0, \quad (7.2.7)$$

where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N-1) \times (N-1)$  matrix with the entries

$$\tilde{a}_{ij} = \sum_{p=j+1}^N (a_{i+1,p} - a_{ip}) = \sum_{p=1}^j (a_{ip} - a_{i+1,p}), \quad i, j = 1, \dots, N-1. \quad (7.2.8)$$

Similarly, by (7.2.3)-(7.2.4), the original boundary conditions (7.1.2)-(7.1.3) for the variable  $U$  can be reduced, respectively, to self-closing boundary conditions of the same kind for the variable  $W$  as follows:

$$x = 0 : W_x - \tilde{C}W_t = \tilde{H}(t), \quad (7.2.9)$$

$$x = L : W_x + \tilde{C}W_t = 0, \quad (7.2.10)$$

where  $\tilde{C} = (\tilde{c}_{ij})$  and  $\tilde{\bar{C}} = (\tilde{\bar{c}}_{ij})$  are  $(N-1) \times (N-1)$  coupling matrices reduced from  $C = (c_{ij})$  and  $\bar{C} = (\bar{c}_{ij})$  in the following way:

$$\tilde{c}_{ij} = \sum_{p=j+1}^N (c_{i+1,p} - c_{ip}) = \sum_{p=1}^j (c_{ip} - c_{i+1,p}), \quad i, j = 1, \dots, N-1, \quad (7.2.11)$$

$$\tilde{\bar{c}}_{ij} = \sum_{p=j+1}^N (\bar{c}_{i+1,p} - \bar{c}_{ip}) = \sum_{p=1}^j (\bar{c}_{ip} - \bar{c}_{i+1,p}), \quad i, j = 1, \dots, N-1. \quad (7.2.12)$$

Moreover,

$$\tilde{H}(t) = (\tilde{h}_1(t), \dots, \tilde{h}_{N-1}(t))^T \quad (7.2.13)$$

with

$$\tilde{h}_i(t) = h_{i+1}(t) - h_i(t), \quad i = 1, \dots, N-1. \quad (7.2.14)$$

Meanwhile, the initial condition of  $W$  is given by

$$t = 0 : W = \tilde{\Phi}(x), \quad W_t = \tilde{\Psi}(x) \quad (7.2.15)$$

with

$$\begin{cases} \tilde{\Phi}(x) = (\varphi_2(x) - \varphi_1(x), \varphi_3(x) - \varphi_2(x), \dots, \varphi_N(x) - \varphi_{N-1}(x))^T, \\ \tilde{\Psi}(x) = (\psi_2(x) - \psi_1(x), \psi_3(x) - \psi_2(x), \dots, \psi_N(x) - \psi_{N-1}(x))^T. \end{cases} \quad (7.2.16)$$

Different from the discussion in [79]-[81], in the case of coupled dissipative boundary conditions, we should verify the corresponding conditions (7.1.5)-(7.1.6) in Lemma 7.1.1 and Lemma 7.1.2 for the reduced system (7.2.7) and (7.2.9)-(7.2.10) in order to guarantee the well-posedness and the exact boundary null controllability of the reduced mixed initial-boundary value problem. For this purpose, under assumption (7.2.3) we will prove

$$\det(C + I_N) = (\tilde{c} + 1)\det(\tilde{C} + I_{N-1}). \quad (7.2.17)$$

In fact, let  $e_i = (0, \dots, \overset{(i)}{1}, \dots, 0)^T$ ,  $E_j = \sum_{i=1}^j e_i$ , ( $i, j = 1, \dots, N$ ). Noting (7.2.11), it is easy to see that

$$\tilde{c}_{ij} = (e_i - e_j)^T C E_j, \quad 1 \leq i, j \leq N - 1. \quad (7.2.18)$$

Moreover, by (7.2.3), we have

$$(e_i - e_{i+1})^T C E_N = 0, \quad 1 \leq i \leq N - 1 \quad (7.2.19)$$

and

$$e_N^T C E_j = \sum_{i=1}^j C_{Ni} \stackrel{\text{def.}}{=} x_j, \quad 1 \leq j \leq N \quad (7.2.20)$$

with  $x_N = \tilde{c}$ . Therefore, by elementary transformation we have

$$X_N C Y_N = \begin{pmatrix} \tilde{C} & \mathbf{0} \\ x & \tilde{c} \end{pmatrix}, \quad (7.2.21)$$

where

$$X_N = \begin{pmatrix} (e_1 - e_2)^T \\ (e_2 - e_3)^T \\ \vdots \\ (e_{N-1} - e_N)^T \\ e_N^T \end{pmatrix} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}_{N \times N}, \quad (7.2.22)$$

$$Y_N = (E_1, \dots, E_N) = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ & 1 & \cdots & \cdots & 1 \\ & & \ddots & \cdots & \vdots \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}_{N \times N}, \quad (7.2.23)$$

$x = (x_1, \dots, x_{N-1})$  is a row vector of order  $N - 1$ , and  $\mathbf{0}$  is the zero vector of order  $N - 1$ . Obviously, both  $X_N$  and  $Y_N$  are invertible matrices, and

$$X_N Y_N = I_N. \quad (7.2.24)$$

Then

$$X_N(C + I_N)Y_N = X_N C Y_N + I_N = \begin{pmatrix} \tilde{C} + I_{N-1} & \mathbf{0} \\ x & \tilde{c} + 1 \end{pmatrix}, \quad (7.2.25)$$

from which we get immediately (7.2.17).

Thus, assumption (7.1.5) implies that

$$\det(\tilde{C} + I_{N-1}) \neq 0 \quad (7.2.26)$$

and

$$\tilde{c} \neq -1. \quad (7.2.27)$$

Similarly, by (7.2.4) and (7.1.6) we have

$$\det(\tilde{\tilde{C}} + I_{N-1}) \neq 0 \quad (7.2.28)$$

and

$$\tilde{\tilde{c}} \neq -1. \quad (7.2.29)$$

Therefore, under assumptions (7.1.5)-(7.1.6), by Lemma 7.1.1, the reduced mixed initial-boundary value problem (7.2.7),(7.2.9)-(7.2.10) and (7.2.15) for the variable  $W$  is always well-posed, moreover, according to Lemma 7.1.2, by means of  $(N - 1)$   $C^1$  boundary controls  $\tilde{H}(t)$  (for instance, take  $h_2(t), \dots, h_n(t)$  with  $h_1(t) \equiv 0$ ) at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly synchronizable. This finishes the proof.

**Remark 7.2.3** As  $t \geq T$ , the synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  defined by (7.2.1) satisfies the following wave equation and boundary conditions:

$$\begin{cases} \frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{a} \tilde{u} = 0, & (7.2.30) \\ x = 0 : \tilde{u}_x - \tilde{c} \tilde{u}_t = 0, & (7.2.31) \\ x = L : \tilde{u}_x + \tilde{\tilde{c}} \tilde{u}_t = 0, & (7.2.32) \end{cases}$$

where  $\tilde{a}$ ,  $\tilde{c}$  and  $\tilde{\tilde{c}}$  are given by (7.2.2)-(7.2.4), respectively. Noting (7.2.27) and (7.2.29), by Lemma 7.1.1, this forward mixed system (7.2.30)-(7.2.32) is also well-

posed. Hence, if we know the initial state of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ :

$$t = T : \tilde{u} = \varphi(x), \tilde{u}_t = \psi(x), \quad 0 \leq x \leq L, \quad (7.2.33)$$

we can completely determine the evolution of  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$ .

**Remark 7.2.4** Under assumptions (7.2.2)-(7.2.4), in order to realize the exact boundary synchronization, the  $(N - 1)$   $C^1$  boundary controls in  $H(t)$  can be chosen in infinitely many ways, therefore, the initial state  $(\varphi, \psi)$  in (7.2.33) will be not unique. Moreover, it can be shown that under additional assumptions  $\tilde{c} \neq 1$  and  $\tilde{\tilde{c}} \neq 1$ , the attainable set of all possible initial data of synchronizable state is the whole space  $C^2[0, L] \times C^1[0, L]$  associated with the conditions of  $C^2$  compatibility at  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (7.2.31)-(7.2.32).

To illustrate this, for any given  $(\varphi, \psi) \in C^2[0, L] \times [C^1[0, L]$  satisfying the conditions of  $C^2$  compatibility with the boundary conditions (7.2.31)-(7.2.32) at  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, noting  $\tilde{c} \neq 1$  and  $\tilde{\tilde{c}} \neq 1$ , by Remark 1.2, we can solve the backward mixed initial-boundary value problem (7.2.30)-(7.2.33) to get a unique  $C^2$  solution  $u = u(t, x)$  and then its initial value at  $t = 0$ :

$$t = 0 : u = \hat{\varphi}(x), u_t = \hat{\psi}(x). \quad (7.2.34)$$

Then, by assumptions (7.2.2)-(7.2.4), it is easy to see that

$$U(t, x) = (u, u, \dots, u)^T(t, x) \quad (7.2.35)$$

is the  $C^2$  solution on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$  to the original system (7.1.1)-(7.1.3) with the null control  $H \equiv 0$  and the initial condition

$$\begin{aligned} t = 0 : U &= \hat{\Phi}(x) = (\hat{\varphi}(x), \hat{\varphi}(x), \dots, \hat{\varphi}(x))^T, \\ U_t &= \hat{\Psi}(x) = (\hat{\psi}(x), \hat{\psi}(x), \dots, \hat{\psi}(x))^T. \end{aligned} \quad (7.2.36)$$

Then, by solving the mixed problem (7.1.1)-(7.1.3) and (7.2.36) with null boundary controls, we can reach any given synchronizable state  $(\varphi, \psi)$  at the moment  $t = T$  (see also [79], [81]).

### 7.3 Exact boundary synchronization by groups

Similarly to [81], the concept of exact boundary synchronization can be generalized to the exact boundary synchronization by groups. Without loss of generality, we consider here the exact boundary synchronization by 2-groups. For this purpose, setting  $U = \begin{pmatrix} U^{(1)} \\ U^{(2)} \end{pmatrix}$  with  $U^{(1)} = (u_1, \dots, u_m)^T$  and  $U^{(2)} = (u_{m+1}, \dots, u_N)^T$ , we

look for some boundary controls in  $H(t)$ , such that  $U^{(1)}$  and  $U^{(2)}$  are independently synchronizable.

**Definition 7.3.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi, \Psi)$ , we can find some  $C^1$  boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (7.1.1)-(7.1.4) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_m(t, x) \stackrel{\text{def.}}{=} \tilde{u}_1(t, x), \quad 0 \leq x \leq L, \quad (7.3.1)$$

$$u_{m+1}(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}_2(t, x), \quad 0 \leq x \leq L, \quad (7.3.2)$$

then the coupled system (7.1.1)-(7.1.3) is said to possess the **exact boundary synchronization by 2-groups** or is said to be **exactly synchronizable by 2-groups**, and  $\tilde{U} = \tilde{U}(t, x) = \begin{pmatrix} \tilde{u}_1(t, x) \\ \tilde{u}_2(t, x) \end{pmatrix}$  is called the corresponding **synchronizable state by 2-groups**.

Obviously, if the coupled system possesses the exact boundary synchronization, then it must be exactly synchronizable by 2-groups. However, this trivial situation should be excluded in advance. For this purpose we need to impose suitable assumptions on the synchronizable states by 2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  (see also [79]-[81]). We have

**Lemma 7.3.1** *Suppose that the coupled system (7.1.1)-(7.1.3) is exactly synchronizable by 2-groups. Suppose furthermore that at least for an initial state  $(\Phi, \Psi)$  the synchronizable states by 2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent. Then the coupling matrix  $A = (a_{ij})$  in system (7.1.1) of wave equations should satisfy the following necessary conditions:*

$$\sum_{j=1}^m a_{ij} \stackrel{\text{def.}}{=} \begin{cases} \tilde{a}_{11} & (i = 1, \dots, m), \\ \tilde{a}_{21} & (i = m + 1, \dots, N), \end{cases} \quad (7.3.3)$$

$$\sum_{j=m+1}^N a_{ij} \stackrel{\text{def.}}{=} \begin{cases} \tilde{a}_{12} & (i = 1, \dots, m), \\ \tilde{a}_{22} & (i = m + 1, \dots, N), \end{cases} \quad (7.3.4)$$

where  $\tilde{a}_{11}$  and  $\tilde{a}_{12}$  are constants independent of  $i = 1, \dots, m$ , while,  $\tilde{a}_{21}$  and  $\tilde{a}_{22}$  are constants independent of  $i = m + 1, \dots, N$ .

For the coupling matrices  $C$  and  $\bar{C}$  in boundary conditions (7.1.2)-(7.1.3), we can not get the corresponding necessary conditions under the assumptions of Lemma 7.3.1 (unless we assume that at least for an initial state  $(\Phi, \Psi)$ ,  $\tilde{u}_{1t}(t, x)$  and  $\tilde{u}_{2t}(t, x)$

are linearly independent at  $x = 0$  and at  $x = L$ , respectively). However, based on Lemma 7.3.1, similarly to [81], we impose the following assumptions:

$$\sum_{j=1}^m c_{ij} \text{ (resp. } \bar{c}_{ij}) \stackrel{\text{def.}}{=} \begin{cases} \tilde{\tilde{c}}_{11} \text{ (resp. } \tilde{\tilde{c}}_{11}) & (i = 1, \dots, m), \\ \tilde{\tilde{c}}_{21} \text{ (resp. } \tilde{\tilde{c}}_{21}) & (i = m + 1, \dots, N), \end{cases} \quad (7.3.5)$$

$$\sum_{j=m+1}^N c_{ij} \text{ (resp. } \bar{c}_{ij}) \stackrel{\text{def.}}{=} \begin{cases} \tilde{\tilde{c}}_{12} \text{ (resp. } \tilde{\tilde{c}}_{12}) & (i = 1, \dots, m), \\ \tilde{\tilde{c}}_{22} \text{ (resp. } \tilde{\tilde{c}}_{22}) & (i = m + 1, \dots, N), \end{cases} \quad (7.3.6)$$

where  $\tilde{\tilde{c}}_{11}$  (resp.  $\tilde{\tilde{c}}_{11}$ ) and  $\tilde{\tilde{c}}_{12}$  (resp.  $\tilde{\tilde{c}}_{12}$ ) are constants independent of  $i = 1, \dots, m$ , while,  $\tilde{\tilde{c}}_{21}$  (resp.  $\tilde{\tilde{c}}_{21}$ ) and  $\tilde{\tilde{c}}_{22}$  (resp.  $\tilde{\tilde{c}}_{22}$ ) are constants independent of  $i = m+1, \dots, N$ .

**Theorem 7.3.1** *Under assumptions (7.1.5)-(7.1.6), suppose furthermore that (7.3.3)-(7.3.6) hold and  $T > 0$  satisfies (7.2.5). For any given initial state  $(\Phi, \Psi)$ , there exist  $(N - 2)$   $C^1$  boundary controls with support on  $[0, T]$  in  $H(t)$  (for instance, take  $h_2(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_1(t) \equiv h_{m+1}(t) \equiv 0$ ), such that the coupled system (7.1.1)-(7.1.3) possesses the exact boundary synchronization by 2-groups.*

**Proof** Let

$$\begin{cases} w_i = u_{i+1} - u_i & (i = 1, \dots, m - 1), \\ w_i = u_{i+2} - u_{i+1} & (i = m, \dots, N - 2). \end{cases} \quad (7.3.7)$$

Similarly to the proof of Theorem 7.2.1 (see also [79],[81]), under assumptions (7.3.3)–(7.3.4), the original system (7.1.1) for the variable  $U$  can be reduced to a self-closing system (7.2.7) of the same kind for the variable  $W = (w_1, \dots, w_{N-2})^T$ , where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N - 2) \times (N - 2)$  matrix with the entries

$$\tilde{a}_{ij} = \begin{cases} \sum_{p=j+1}^m (a_{i+1,p} - a_{ip}) = \sum_{p=1}^j (a_{ip} - a_{i+1,p}), & i = 1, \dots, N - 2; j = 1, \dots, m - 1, \\ \sum_{p=j+2}^N (a_{i+1,p} - a_{ip}) = \sum_{p=m+1}^{j+1} (a_{ip} - a_{i+1,p}), & i = 1, \dots, N - 2; j = m, \dots, N - 2. \end{cases} \quad (7.3.8)$$

Similarly, under assumptions (7.3.5)–(7.3.6), the original boundary conditions (7.1.2)-(7.1.3) for the variable  $U$  can be reduced to self-closing boundary conditions (7.2.9)-(7.2.10) of the same kind for the variable  $W$ , in which the coupling matrices  $\tilde{C} = (\tilde{c}_{ij})$  and  $\tilde{\tilde{C}} = (\tilde{\tilde{c}}_{ij})$  are  $(N - 2) \times (N - 2)$  matrices reduced from  $C = (c_{ij})$  and



$\bar{C} = (\bar{c}_{ij})$  in the following way:

$$\tilde{c}_{ij} = \begin{cases} \sum_{p=j+1}^m (c_{i+1,p} - c_{ip}) = \sum_{p=1}^j (c_{ip} - c_{i+1,p}), & i = 1, \dots, N-2; j = 1, \dots, m-1, \\ \sum_{p=j+2}^N (c_{i+1,p} - c_{ip}) = \sum_{p=m+1}^{j+1} (c_{ip} - c_{i+1,p}), & i = 1, \dots, N-2; j = m, \dots, N-2, \end{cases} \quad (7.3.9)$$

$$\tilde{\bar{c}}_{ij} = \begin{cases} \sum_{p=j+1}^m (\bar{c}_{i+1,p} - \bar{c}_{ip}) = \sum_{p=1}^j (\bar{c}_{ip} - \bar{c}_{i+1,p}), & i = 1, \dots, N-2; j = 1, \dots, m-1, \\ \sum_{p=j+2}^N (\bar{c}_{i+1,p} - \bar{c}_{ip}) = \sum_{p=m+1}^{j+1} (\bar{c}_{ip} - \bar{c}_{i+1,p}), & i = 1, \dots, N-2; j = m, \dots, N-2. \end{cases} \quad (7.3.10)$$

Moreover,

$$\tilde{H}(t) = (\tilde{h}_1(t), \dots, \tilde{h}_{N-2}(t)) \quad (7.3.11)$$

with

$$\tilde{h}_i(t) = \begin{cases} h_{i+1}(t) - h_i(t), & i = 1, \dots, m-1, \\ h_{i+2}(t) - h_{i+1}(t), & i = m, \dots, N-2. \end{cases} \quad (7.3.12)$$

The initial condition (7.2.15) of  $W$  is given by

$$\tilde{\Phi}(x) = (\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_{N-2}(x))^T, \quad \tilde{\Psi}(x) = (\tilde{\psi}_1(x), \dots, \tilde{\psi}_{N-2}(x))^T \quad (7.3.13)$$

with

$$\tilde{\varphi}_i(x) = \begin{cases} \varphi_{i+1}(x) - \varphi_i(x), & i = 1, \dots, m-1, \\ \varphi_{i+2}(x) - \varphi_{i+1}(x), & i = m, \dots, N-2, \end{cases} \quad (7.3.14)$$

$$\tilde{\psi}_i(x) = \begin{cases} \psi_{i+1}(x) - \psi_i(x), & i = 1, \dots, m-1, \\ \psi_{i+2}(x) - \psi_{i+1}(x), & i = m, \dots, N-2. \end{cases}$$

In this situation, in order to guarantee the well-posedness and the exact boundary null controllability for the reduced system of  $W$ , we should also verify the corresponding conditions (7.1.5)–(7.1.6) in Lemma 7.1.1 and Lemma 7.1.2. Similarly to § 7.2, under assumptions (7.3.5)–(7.3.6) we will prove

$$\det(C + I_N) = \det(\tilde{\tilde{C}} + I_2) \det(\tilde{C} + I_{N-2}), \quad (7.3.15)$$

where  $\tilde{\tilde{C}} = (\tilde{\tilde{c}}_{ij})$  given by (7.3.5)–(7.3.6) is a  $2 \times 2$  matrix.

In order to get (7.3.15), let

$$C = \left( \begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right), \quad (7.3.16)$$

where, by(7.3.5)-(7.3.6), the values of the sum of every row of partitioned matrices  $C_{ij}(1 \leq i, j \leq 2)$  are  $\tilde{c}_{ij}(1 \leq i, j \leq 2)$ , respectively.

By elementary transformation, similarly to (7.2.21), it is easy to get

$$\begin{aligned} \left( \begin{array}{c|c} X_m & \\ \hline & X_{N-m} \end{array} \right) C \left( \begin{array}{c} Y_m \\ \hline Y_{N-m} \end{array} \right) &= \left( \begin{array}{cc|cc} X_m C_{11} Y_m & X_m C_{12} Y_{N-m} & & \\ X_{N-m} C_{21} Y_m & X_{N-m} C_{22} Y_{N-m} & & \end{array} \right) \\ &= \left( \begin{array}{cc|cc} \tilde{C}_{11} & \mathbf{0} & \tilde{C}_{12} & \mathbf{0} \\ x_{11} & \tilde{c}_{11} & x_{12} & \tilde{c}_{12} \\ \hline \tilde{C}_{21} & \mathbf{0} & \tilde{C}_{22} & \mathbf{0} \\ x_{21} & \tilde{c}_{21} & x_{22} & \tilde{c}_{22} \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \end{array}} \right\} m \\ &\qquad\qquad\qquad \left. \vphantom{\begin{array}{c} \\ \\ \\ \end{array}} \right\} N - m, \end{aligned} \quad (7.3.17)$$

where  $x_{11}$  and  $x_{21}$  are some row vectors of order  $(m - 1)$ ,  $x_{12}$  and  $x_{22}$  are some row vectors of order  $(N - m - 1)$ . Deleting the m-th and N-th rows and the m-th and N-th columns of the matrix on the right-hand side of (7.3.17), we get

$$\tilde{C} = \left( \begin{array}{c|c} \tilde{C}_{11} & \tilde{C}_{12} \\ \hline \tilde{C}_{21} & \tilde{C}_{22} \end{array} \right). \quad (7.3.18)$$

Noting (7.2.24), we have

$$\begin{aligned} &\left( \begin{array}{c|c} X_m & \\ \hline & X_{N-m} \end{array} \right) (C + I_N) \left( \begin{array}{c} Y_m \\ \hline Y_{N-m} \end{array} \right) \\ &= \left( \begin{array}{c|c} X_m & \\ \hline & X_{N-m} \end{array} \right) C \left( \begin{array}{c} Y_m \\ \hline Y_{N-m} \end{array} \right) + I_N \\ &= \left( \begin{array}{cc|cc} \tilde{C}_{11} + I_{m-1} & \mathbf{0} & \tilde{C}_{12} & \mathbf{0} \\ x_{11} & \tilde{c}_{11} + 1 & x_{12} & \tilde{c}_{12} \\ \hline \tilde{C}_{21} & \mathbf{0} & \tilde{C}_{22} + I_{N-m-1} & \mathbf{0} \\ x_{21} & \tilde{c}_{21} & x_{22} & \tilde{c}_{22} + 1 \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \end{array}} \right\} m \\ &\qquad\qquad\qquad \left. \vphantom{\begin{array}{c} \\ \\ \\ \end{array}} \right\} N - m. \end{aligned} \quad (7.3.19)$$

Taking the determinants of both sides of (7.3.19), by Laplace expansion theorem, we immediately get (7.3.15).

Thus, under assumption (7.1.5) we have

$$\det(\tilde{C} + I_{N-2}) \neq 0 \quad (7.3.20)$$

and

$$\det(\tilde{\tilde{C}} + I_2) \neq 0. \quad (7.3.21)$$

Similarly, it follows from (7.3.5)-(7.3.6) and (7.1.6) that

$$\det(\tilde{\tilde{C}} + I_{N-2}) \neq 0 \quad (7.3.22)$$

and

$$\det(\tilde{\tilde{\tilde{C}}} + I_2) \neq 0, \quad (7.3.23)$$

where  $\tilde{\tilde{C}} = (\tilde{\tilde{c}}_{ij})$  given by (7.3.5)-(7.3.6) is a  $2 \times 2$  matrix.

Therefore, under assumptions (7.1.5)-(7.1.6), according to Lemma 7.1.1, the reduced mixed initial-boundary problem (7.2.7),(7.2.9)-(7.2.10) and (7.2.15) for the variable  $W$  is always well-posed, then using Lemma 7.1.2, by means of  $(N-2)$   $C^1$  boundary controls  $\tilde{H}(t)$  (for instance, take  $h_2(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_1(t) \equiv h_{m+1}(t) \equiv 0$ ) on  $[0, T]$  at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly synchronizable by 2-groups. This completes the proof.

**Remark 7.3.1** *As  $t \geq T$ , the synchronizable state by 2-groups  $\tilde{U} = \tilde{U}(t, x) = \begin{pmatrix} \tilde{u}_1(t, x) \\ \tilde{u}_2(t, x) \end{pmatrix}$  satisfies the following coupled system of wave equations*

$$\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial x^2} + \tilde{\tilde{A}} \tilde{U} = 0, \quad (7.3.24)$$

$$x = 0 : \tilde{U}_x - \tilde{\tilde{C}} \tilde{U}_t = 0, \quad (7.3.25)$$

$$x = L : \tilde{U}_x + \tilde{\tilde{C}} \tilde{U}_t = 0, \quad (7.3.26)$$

where

$$\tilde{\tilde{A}} = \begin{pmatrix} \tilde{\tilde{a}}_{11} & \tilde{\tilde{a}}_{12} \\ \tilde{\tilde{a}}_{21} & \tilde{\tilde{a}}_{22} \end{pmatrix}, \quad (7.3.27)$$

and  $\tilde{\tilde{a}}_{11}$ ,  $\tilde{\tilde{a}}_{12}$ ,  $\tilde{\tilde{a}}_{21}$  and  $\tilde{\tilde{a}}_{22}$  are given by (7.3.3)-(7.3.4). Noting (7.3.21) and (7.3.23), by Lemma 1.1 the forward mixed system (7.3.24)-(7.3.26) is well-posed. Hence, if we

know the initial state of  $\tilde{U} = \tilde{U}(t, x)$  at the moment  $t = T$ :

$$t = T : \tilde{U} = (\varphi^{(1)}(x), \varphi^{(2)}(x))^T, \quad \tilde{U}_t = (\psi^{(1)}(x), \psi^{(2)}(x))^T, \quad 0 \leq x \leq L, \quad (7.3.28)$$

then the evolution of  $\tilde{U} = \tilde{U}(t, x)$  with respect to  $t$  can be completely determined.

**Remark 7.3.2** Under assumptions (7.3.3)-(7.3.6), in order to realize the exact boundary synchronization by 2-groups, one has infinitively many ways to choose  $(N - 2)$   $C^1$  boundary controls at the end  $x = 0$ , then the initial synchronizable state in (7.3.28) is not unique. Moreover, similarly to Remark 2.2, suppose furthermore that

$$\det(\tilde{C} - I_2) \neq 0 \quad (7.3.29)$$

and

$$\det(\tilde{C} - I_2) \neq 0, \quad (7.3.30)$$

namely, 1 is not an eigenvalue of both  $\tilde{C}$  and  $\tilde{C}$ , any given state  $(\varphi^{(1)}, \varphi^{(2)})^T$  and  $(\psi^{(1)}, \psi^{(2)})^T$  in  $(C^2[0, L])^2 \times (C^1[0, L])^2$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (7.3.25)-(7.3.26), belongs to the attainable set of the initial data of synchronizable state by 2-groups.

Thus, at least for some initial states  $(\Phi, \Psi)$ , the synchronizable states by 2-groups  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent on  $t = T$  then for  $t \geq T$ . It shows that the corresponding requirement in Lemma 3.1 is reasonable.

**Remark 7.3.3** The exact boundary synchronization by  $k(k \geq 2)$ -groups can be treated in a similar way.

In order to realize the exact boundary synchronization by 2-groups, the number of boundary controls is equal to  $(N - 2)$ , while, in order to realize the exact boundary synchronization by  $k$ -groups, the number of boundary controls is equal to  $(N - k)$ .

## 7.4 Exact boundary null controllability and synchronization by groups

Similarly to [79],[81], under some hypotheses on the coupled system, we can divide the state variables into two groups  $U^{(1)} = (u_1, \dots, u_m)^T$  and  $U^{(2)} = (u_{m+1}, \dots, u_N)^T$  such that  $U^{(1)}$  is exactly null controllable, while  $U^{(2)}$  is exactly synchronizable.

**Definition 7.4.1** *If there exists  $T > 0$  such that for any given initial state  $(\Phi, \Psi)$ , we can find some boundary controls with support on  $[0, T]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (7.1.1)-(7.1.4) admits a unique  $C^2$  solution  $U = U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  on  $t \geq 0$ , and as  $t \geq T$  we have*

$$u_1(t, x) \equiv \dots \equiv u_m(t, x) \equiv 0, \quad 0 \leq x \leq L, \quad (7.4.1)$$

$$u_{m+1}(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}(t, x), \quad 0 \leq x \leq L, \quad (7.4.2)$$

then the coupled system (7.1.1)-(7.1.3) is said to possess **the exact boundary null controllability and synchronization by 2-groups** or is said to be **exactly null controllable and synchronizable by 2- groups**, and  $\tilde{u} = \tilde{u}(t, x)$  is called to be the **partially synchronizable state**.

Obviously, when the coupled system (7.1.1)-(7.1.3) possesses the exact null controllability, it must satisfy the requirement of Definition 7.4.1. This situation should be excluded in advance. Similarly to [81], we have

**Lemma 7.4.1** *Under assumptions (7.1.5)-(7.1.6), suppose furthermore that the coupled system (7.1.1)-(7.1.3) is exactly null controllable and synchronizable by 2-groups, but not exactly null controllable, then the coupling matrix  $A$  must satisfy the following necessary conditions:*

$$\sum_{j=m+1}^N a_{ij} = 0 \quad (i = 1, \dots, m), \quad (7.4.3)$$

$$\sum_{j=m+1}^N a_{ij} \stackrel{\text{def.}}{=} \tilde{a} \quad (i = m + 1, \dots, N), \quad (7.4.4)$$

where  $\tilde{a}$  is a constant independent of  $i = m + 1, \dots, N$ .

Different from the situation in [81], the conclusion of Lemma 7.4.1 is no longer valid in general for the coupling matrices  $C$  and  $\bar{C}$  in the coupled dissipative boundary conditions (7.1.2)-(7.1.3). However, based on Lemma 7.4.1, we impose the following similar assumptions:

$$\sum_{j=m+1}^N c_{ij} (\text{resp. } \bar{c}_{ij}) = 0 \quad (i = 1, \dots, m), \quad (7.4.5)$$

$$\sum_{j=m+1}^N c_{ij} (\text{resp. } \bar{c}_{ij}) \stackrel{\text{def.}}{=} \tilde{c} (\text{resp. } \bar{\tilde{c}}) \quad (i = m + 1, \dots, N), \quad (7.4.6)$$

where  $\tilde{c}$  and  $\bar{\tilde{c}}$  are constants independent of  $i = m + 1, \dots, N$ .

**Remark 7.4.1** *Similarly to Remark 7.2.1, if we suppose furthermore that the constant  $\tilde{a}$  given by (7.4.4) is positive, it is easy to get the necessity of (7.4.5)-(7.4.6) for the exact boundary null controllability and synchronization by 2-groups.*

**Theorem 7.4.1** *Under assumptions (7.1.5)-(7.1.6), suppose furthermore that (7.4.3)-(7.4.6) hold and  $T > 0$  satisfies (7.2.5). For any given initial state  $(\Phi, \Psi)$ , we can find  $(N - 1)$   $C^1$  boundary controls with support on  $[0, T]$  in  $H(t)$  (for instance, take  $h_1(t), \dots, h_m(t), h_{m+2}(t), \dots, h_N(t)$  with  $h_{m+1}(t) \equiv 0$ ), such that the coupled system (7.1.1)-(7.1.3) possesses the exact boundary null controllability and synchronization by 2-groups.*

**Proof** Let

$$\begin{cases} w_i = u_i & (i = 1, \dots, m), \\ w_i = u_{i+1} - u_i & (i = m + 1, \dots, N - 1). \end{cases} \quad (7.4.7)$$

Similarly to the proof of Theorem 7.2.1 and Theorem 7.3.1 (see [79]-[81]), under assumptions (7.4.3)-(7.4.4), the original system (7.1.1) for the variable  $U$  can be reduced to a self-closing system (7.2.7) of the same kind for the variable  $W = (w_1, \dots, w_{N-1})^T$ , where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N - 1) \times (N - 1)$  matrix with

$$\tilde{a}_{ij} = \begin{cases} a_{ij}, & j = 1, \dots, m, \\ \sum_{p=j+1}^N a_{ip} = - \sum_{p=m+1}^j a_{ip}, & j = m + 1, \dots, N - 1, \end{cases} \quad \text{for } i = 1, \dots, m, \quad (7.4.8)$$

$$\tilde{a}_{ij} = \begin{cases} a_{i+1,j} - a_{ij}, & j = 1, \dots, m, \\ \sum_{p=j+1}^N (a_{i+1,p} - a_{ip}) = \sum_{p=m+1}^j (a_{ip} - a_{i+1,p}), & j = m + 1, \dots, N - 1, \end{cases} \quad \text{for } i = m + 1, \dots, N - 1. \quad (7.4.9)$$

Similarly, by (7.4.5)-(7.4.6), the original boundary conditions (7.1.2)-(7.1.3) for the variable  $U$  can be reduced, respectively, to a self-closing boundary conditions (7.2.9)-(7.2.10) of the same kind for the variable  $W$ , in which the  $(N - 1) \times (N - 1)$  coupling matrices  $\tilde{C} = (\tilde{c}_{ij})$  and  $\tilde{\bar{C}} = (\tilde{\bar{c}}_{ij})$  are reduced from  $C = (c_{ij})$  and  $\bar{C} = (\bar{c}_{ij})$  in the following way:

$$\tilde{c}_{ij} = \begin{cases} c_{ij}, & j = 1, \dots, m, \\ \sum_{p=j+1}^N c_{ip} = - \sum_{p=m+1}^j c_{ip}, & j = m + 1, \dots, N - 1, \end{cases} \quad \text{for } i = 1, \dots, m, \quad (7.4.10)$$

$$\tilde{c}_{ij} = \begin{cases} c_{i+1,j} - c_{ij}, & j = 1, \dots, m, \\ \sum_{p=j+1}^N (c_{i+1,p} - b_{ip}) = \sum_{p=m+1}^j (c_{ip} - c_{i+1,p}), & j = m+1, \dots, N-1, \end{cases} \quad \text{for } i = m+1, \dots, N-1, \quad (7.4.11)$$

and similar expressions for  $\tilde{c}_{ij}$  ( $i, j = 1, \dots, N-1$ ). Moreover,  $\tilde{H}(t)$  is given by (7.2.13) with

$$\tilde{h}_i(t) = \begin{cases} h_i(t), & i = 1, \dots, m, \\ h_{i+1}(t) - h_i(t), & i = m+1, \dots, N-1. \end{cases} \quad (7.4.12)$$

The initial condition (7.2.15) of  $W$  is given by

$$\begin{aligned} \tilde{\varphi}_i(x) &= \begin{cases} \varphi_i(x), & i = 1, \dots, m, \\ \varphi_{i+1}(x) - \varphi_i(x), & i = m+1, \dots, N-1, \end{cases} \\ \tilde{\psi}_i(x) &= \begin{cases} \psi_i(x), & i = 1, \dots, m, \\ \psi_{i+1}(x) - \psi_i(x), & i = m+1, \dots, N-1. \end{cases} \end{aligned} \quad (7.4.13)$$

By (7.4.5)-(7.4.6), we can prove

$$\det(C + I_N) = (\tilde{c} + 1)\det(\tilde{C} + I_{N-1}). \quad (7.4.14)$$

In fact, let

$$C = \left( \begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right), \quad (7.4.15)$$

in which the sums of every row of partitioned matrices  $C_{12}$  and  $C_{22}$  are 0 and  $\tilde{c}$ , respectively.

By elementary transformation, using (7.2.21) and noting (7.4.10)-(7.4.11), we have

$$\begin{aligned} & \begin{pmatrix} -I_m & \\ & X_{N-m} \end{pmatrix} C \begin{pmatrix} -I_m & \\ & Y_{N-m} \end{pmatrix} \\ &= \begin{pmatrix} C_{11} & -C_{12}Y_{N-m} \\ -X_{N-m}C_{21} & X_{N-m}C_{22}Y_{N-m} \end{pmatrix} = \begin{pmatrix} \tilde{C} & \mathbf{0} \\ x & \tilde{c} \end{pmatrix}, \end{aligned} \quad (7.4.16)$$

where  $x$  is a row vector of order  $(N - 1)$ . Thus,

$$\begin{aligned} & \begin{pmatrix} -I_m & \\ & X_{N-m} \end{pmatrix} (C + I_N) \begin{pmatrix} -I_m & \\ & Y_{N-m} \end{pmatrix} \\ &= \begin{pmatrix} -I_m & \\ & X_{N-m} \end{pmatrix} C \begin{pmatrix} -I_m & \\ & Y_{N-m} \end{pmatrix} + I_N \\ &= \begin{pmatrix} \tilde{C} + I_{N-1} & \mathbf{0} \\ x & \tilde{c} + 1 \end{pmatrix}. \end{aligned} \tag{7.4.17}$$

Taking the determinants of both sides of (7.4.17), we immediately get (7.4.14).

Thus, by (7.1.5) we have

$$\det(\tilde{C} + I_{N-1}) \neq 0 \tag{7.4.18}$$

and

$$\tilde{c} \neq -1. \tag{7.4.19}$$

Similarly, by (7.1.6) we can get

$$\det(\tilde{\tilde{C}} + I_{N-1}) \neq 0 \tag{7.4.20}$$

and

$$\tilde{\tilde{c}} \neq -1. \tag{7.4.21}$$

Therefore, under assumptions (7.1.5)-(7.1.6), by Lemma 7.1.1 the mixed initial-boundary problem (7.2.7),(7.2.9)-(7.2.10) and (7.2.15) for the variable  $W$  is always well-posed, then using Lemma 7.1.2, by means of  $(N - 1)$   $C^1$  boundary controls  $\tilde{H}(t)$  (for instance, take  $h_1(t), \dots, h_m(t), h_{m+1}(t), \dots, h_N(t)$  with  $h_{m+1}(t) \equiv 0$ ) at the end  $x = 0$ , the variable  $W$  is exactly null controllable, then the variable  $U$  is exactly null controllable and synchronizable by 2-groups. This proves Theorem 7.4.1.

**Remark 7.4.2** *As  $t \geq T$ , the partially synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  satisfies the wave system (7.2.30)-(7.2.32), where  $\tilde{a}$ ,  $\tilde{c}$  and  $\tilde{\tilde{c}}$  are given by (7.4.4) and (7.4.6). Noting (7.4.19) and (7.4.21), this forward mixed system is also well-posed. Hence, if we know the initial condition (7.2.33) of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ , then the evolution of the partially synchronizable state  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$  can be completely determined.*

*Moreover, suppose furthermore that  $\tilde{c} \neq 1$  and  $\tilde{\tilde{c}} \neq 1$ , any given state  $(\varphi, \psi)$  in  $C^2[0, L] \times C^1[0, L]$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) =$*



$(T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (7.2.31)-(7.2.32), belongs to the attainable set of the initial data of partially synchronizable state.

## 7.5 Generalized exact boundary synchronization

We now consider the problem of synchronization from a more general mathematical point of view.

Let

$$w_i = \sum_{j=1}^N \theta_{ij} u_j \quad (i = 1, \dots, M) \quad (7.5.1)$$

or

$$W = \textcircled{\mathbf{H}}U, \quad (7.5.2)$$

where  $W = (w_1, \dots, w_M)^T$ ,  $M \leq N - 1$ , and  $\textcircled{\mathbf{H}} = (\theta_{ij})_{M \times N}$  is full-rank.

Suppose that for any given initial state  $(\Phi, \Psi)$ , the mixed initial-boundary value problem (7.1.1)-(7.1.4) for the variable  $U$  can be reduced to a self-closing mixed initial-boundary values problem of the same kind for the variable  $W$ , then, if  $T > 2L$ , we can use Lemma 7.1.2 to find  $M$   $C^1$  boundary controls in  $H(t)$  with support on  $[0, T]$ , such that as  $t \geq T$  we have

$$W(t, x) \equiv 0, \quad 0 \leq x \leq L, \quad (7.5.3)$$

i.e.,

$$\textcircled{\mathbf{H}}U(t, x) \equiv 0, \quad 0 \leq x \leq L. \quad (7.5.4)$$

Thus, the coupled system (7.1.1)-(7.1.3) is said to have the **generalized exact boundary synchronization associated with  $\textcircled{\mathbf{H}}$**  and  $U = U(t, x)$  is the corresponding **generalized synchronizable state**.

As in [81], under hypotheses (7.1.5)-(7.1.6), this definition covers all the cases discussed in the previous sections.

A natural question is under what conditions the original coupled system for the variable  $U$  can be reduced to a self-closing coupled system of the same type for the variable  $W$  by means of (7.5.2).

**Theorem 7.5.1** *For any given coupling  $N \times N$  matrices  $A$ ,  $C$  and  $\bar{C}$ , if there*

exist a full-rank matrix  $\textcircled{\mathbb{H}} = (\theta_{ij})_{M \times N}$  and  $M \times M$  matrices  $\tilde{A}$ ,  $\tilde{C}$  and  $\tilde{\bar{C}}$  such that

$$\textcircled{\mathbb{H}}A = \tilde{A}\textcircled{\mathbb{H}}, \quad (7.5.5)$$

$$\textcircled{\mathbb{H}}C = \tilde{C}\textcircled{\mathbb{H}}, \quad (7.5.6)$$

$$\textcircled{\mathbb{H}}\bar{C} = \tilde{\bar{C}}\textcircled{\mathbb{H}}, \quad (7.5.7)$$

then, by (7.5.2) we can get a self-closing coupled system (7.2.7) and (7.2.9)-(7.2.10) (in which the numbers of both wave equations and the boundary conditions on two sides are all equal to  $M$ ) with  $\tilde{H}(t) = \textcircled{\mathbb{H}}H(t)$  for the variable  $W$ . Thus, we can realize the generalized exact boundary synchronization associated with  $\textcircled{\mathbb{H}}$  for the variable  $U$  by  $M C^1$  boundary controls on  $x = 0$ . Moreover,

$$\text{Spectrum}(\tilde{C}) \subseteq \text{Spectrum}(C), \quad (7.5.8)$$

$$\text{Spectrum}(\tilde{\bar{C}}) \subseteq \text{Spectrum}(\bar{C}), \quad (7.5.9)$$

where  $\text{Spectrum}(C)$  denotes the set of all the eigenvalues of matrix  $C$ , etc.

**Proof** Similarly to [81], here, we need only to prove (7.5.8), and (7.5.9) can be obtained in a similar way. In fact, for the full-rank matrix  $\textcircled{\mathbb{H}}$ , by Gram-Schmidt process on its column vectors, there exists a nonsingular matrix  $P \in \mathbb{R}^{N \times N}$  such that

$$\textcircled{\mathbb{H}}P = (I_M, \mathbf{0}). \quad (7.5.10)$$

Let

$$P^{-1}CP \stackrel{\text{def.}}{=} \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{pmatrix}, \quad (7.5.11)$$

where  $\hat{C}_{11}$  and  $\hat{C}_{22}$  are matrices of order  $M$  and order  $(N-M)$ , respectively. It is easy to see that

$$\textcircled{\mathbb{H}}CP = (\textcircled{\mathbb{H}}P)(P^{-1}CP) = (\hat{C}_{11}, \hat{C}_{12}), \quad (7.5.12)$$

$$\tilde{C}\textcircled{\mathbb{H}}P = (\tilde{C}, \mathbf{0}). \quad (7.5.13)$$

Hence, by(7.5.6) we have

$$\hat{C}_{11} = \tilde{C}, \quad \hat{C}_{12} = \mathbf{0}, \quad (7.5.14)$$

then, noting (7.5.11), we get (7.5.8). This ends the proof.

The next question is for any given coupling  $N \times N$  matrices  $A$ ,  $C$  and  $\bar{C}$ , how to guarantee the existence of such  $M \times N$  matrix  $\textcircled{\mathbb{H}}$  and  $M \times M$  matrices  $\tilde{A}$ ,  $\tilde{C}$  and  $\tilde{\bar{C}}$  in Theorem 7.5.1.

We first consider the case  $M = N - 1$ . Similarly to [81], we have

**Theorem 7.5.2** *Suppose that  $M = N - 1$ . For the coupled system (7.1.1)-(7.1.3) of wave equations, if the coupling matrices  $A$ ,  $C$  and  $\bar{C}$  possess a common real right eigenvector  $X \in \mathbb{R}^N \setminus \{0\}$ :*

$$AX = \tilde{a}X, \quad (7.5.15)$$

$$CX = \tilde{c}X, \quad (7.5.16)$$

$$\bar{C}X = \tilde{\bar{c}}X, \quad (7.5.17)$$

where  $\tilde{a}$ ,  $\tilde{c}$  and  $\tilde{\bar{c}}$  are corresponding eigenvalues, respectively, then there exist a  $(N - 1) \times N$  full-rank matrix  $\textcircled{\text{H}} = (\theta_{ij})$  and  $(N - 1) \times (N - 1)$  matrices  $\tilde{A}$ ,  $\tilde{C}$  and  $\tilde{\bar{C}}$  such that (7.5.5)–(7.5.7) hold. Moreover,  $X$  constitutes a system of basic solutions to the linear algebraic system

$$\textcircled{\text{H}}Y = 0, \quad (7.5.18)$$

and

$$\text{Spectrum}(C) = \{\tilde{c}\} + \text{Spectrum}(\tilde{C}), \quad (7.5.19)$$

$$\text{Spectrum}(\bar{C}) = \{\tilde{\bar{c}}\} + \text{Spectrum}(\tilde{\bar{C}}). \quad (7.5.20)$$

**Proof** Similarly, we need only to prove (7.5.19). In fact, for the full-rank matrix  $\textcircled{\text{H}} \in \mathbb{R}^{(N-1) \times N}$ , by Gram-Schmidt process on its row vectors, it is easy to see that there exists a matrix  $D = (d_1, \dots, d_{N-1})$ , in which  $d_k (1 \leq k \leq N - 1)$  are column vectors of order  $N$ , such that

$$\textcircled{\text{H}}D = I_{N-1}. \quad (7.5.21)$$

Noting  $\textcircled{\text{H}}X = 0$ , we can take  $Z \in \mathbb{R}^{N \times 1}$  such that

$$Z \in (\text{Span}\{d_1, \dots, d_{N-1}\})^\perp \quad (7.5.22)$$

and

$$Z^T \cdot X = 1. \quad (7.5.23)$$

Then, for any given  $\lambda \in \mathbb{C}$ , noting (7.5.6), (7.5.16) and (7.5.18), we have

$$\begin{pmatrix} \textcircled{\text{H}} \\ Z^T \end{pmatrix} (C - \lambda I_N)(D, X) = \begin{pmatrix} \textcircled{\text{H}}CD & \textcircled{\text{H}}CX \\ Z^TCD & Z^TCX \end{pmatrix} - \lambda I_N = \begin{pmatrix} \tilde{C} - \lambda I_{N-1} & 0 \\ Z^TCD & \tilde{c} - \lambda \end{pmatrix}. \quad (7.5.24)$$

Taking the determinants of both sides of (7.5.24), we immediately get (7.5.19). The proof of (7.5.20) is completely similar. This finishes the proof.

**Remark 7.5.1** Under assumptions (7.1.5)-(7.1.6), it follows from (7.5.19)-(7.5.20) that the reduced coupled system (7.2.7) and (7.2.9)-(7.2.10) for the variable  $W$  is always well-posed.

**Remark 7.5.2** As  $t \geq T$ , by (7.5.4) and (7.5.18), the generalized synchronizable state can be written as  $U(t, x) = \tilde{u}(t, x)X$ , in which  $X = (\alpha_1, \dots, \alpha_N)^T$  is a system of basic solutions to (7.5.18), and  $\tilde{u} = \tilde{u}(t, x)$  satisfies the wave system (7.2.30)-(7.2.32). Under assumptions (7.1.5)-(7.1.6), noting that by (7.5.19)-(7.5.20) we have  $\tilde{c} \neq -1$  and  $\tilde{\bar{c}} \neq -1$ , according to Lemma 7.1.1, if we know the initial condition (7.2.33) of  $\tilde{u} = \tilde{u}(t, x)$  at the moment  $t = T$ , then the evolution of  $\tilde{u} = \tilde{u}(t, x)$  with respect to  $t$  can be completely determined.

Moreover, suppose furthermore that  $\tilde{c} \neq 1$  and  $\tilde{\bar{c}} \neq 1$ , then any given state  $(\varphi, \psi)$  in  $C^2[0, L] \times C^1[0, L]$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (7.2.31)-(7.2.32), can be the initial data of a generalized synchronizable state.

**Remark 7.5.3** In the case of exact boundary synchronization,  $M = N - 1$  and  $X = (1, \dots, 1)^T$  is a system of basic solutions to (7.5.18). The requirement that the coupling matrices  $A$ ,  $C$  and  $\bar{C}$  possess a common right eigenvector  $X$  implies (7.2.2)-(7.2.4). Thus, Theorem 7.5.2 leads to the conclusion given in §7.2.

**Remark 7.5.4** In the case of exact boundary null controllability and synchronization by 2-groups,  $M = N - 1$  and  $X = (\overbrace{0, \dots, 0}^m, \overbrace{1, \dots, 1}^{N-m})^T$  is a system of basic solutions to (7.5.18). The requirement that the coupling matrices  $A$ ,  $C$  and  $\bar{C}$  possess a common right eigenvector  $X$  implies (7.4.3)-(7.4.6). Thus, the conclusion given in §7.4 follows from Theorem 7.5.2.

On the other hand, similarly to [81], we have

**Theorem 7.5.3** Suppose that  $M = N - 1$ , if the coupled system (7.1.1)-(7.1.3) possesses the generalized exact boundary synchronization associated with  $\textcircled{\text{H}}$ , but is not exactly boundary null controllable, then there exist a real right eigenvector  $X \in \mathbb{R}^N \setminus \{0\}$  and a real number  $\tilde{a}$  such that (7.5.15) holds.

For the case  $M = N - 2$ , we only consider the special situation that

$$\textcircled{\text{H}} = \begin{pmatrix} \textcircled{\text{H}}_1 & \\ & \textcircled{\text{H}}_2 \end{pmatrix}_{(N-2) \times N}, \tag{7.5.25}$$

where  $\textcircled{\text{H}}_1$  is a full-rank  $(N_1 - 1) \times N_1$  matrix, while  $\textcircled{\text{H}}_2$  is a full-rank  $(N_2 - 1) \times N_2$  matrix with  $N_1 + N_2 = N$ .

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \text{ etc.}, \quad (7.5.26)$$

where  $A_{11}$  (resp.  $\tilde{A}_{11}$ ) is an  $N_1 \times N_1$  (resp.  $(N_1 - 1) \times (N_1 - 1)$ ) matrix,  $A_{21}$  (resp.  $\tilde{A}_{21}$ ) is an  $N_2 \times N_1$  (resp.  $(N_2 - 1) \times (N_1 - 1)$ ) matrix,  $A_{12}$  (resp.  $\tilde{A}_{12}$ ) is an  $N_1 \times N_2$  (resp.  $(N_1 - 1) \times (N_2 - 1)$ ) matrix,  $A_{22}$  (resp.  $\tilde{A}_{22}$ ) is an  $N_2 \times N_2$  (resp.  $(N_2 - 1) \times (N_2 - 1)$ ) matrix etc. Thus, (7.5.5)–(7.5.7) in Theorem 7.5.1 can be equivalently written as

$$\textcircled{\mathbb{H}}_i A_{ij} = \tilde{A}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2, \quad (7.5.27)$$

$$\textcircled{\mathbb{H}}_i C_{ij} = \tilde{C}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2, \quad (7.5.28)$$

$$\textcircled{\mathbb{H}}_i \bar{C}_{ij} = \tilde{\bar{C}}_{ij} \textcircled{\mathbb{H}}_j, \quad i, j = 1, 2. \quad (7.5.29)$$

Similarly to [81], we have

**Theorem 7.5.4** *Suppose that  $M = N - 2$ . For the coupled system of wave equations (7.1.1)–(7.1.3), if there exist real vectors  $X_1 \in R^{N_1} \setminus \{0\}$  and  $X_2 \in R^{N_2} \setminus \{0\}$  and real numbers  $\tilde{a}_{ij}$ ,  $\tilde{c}_{ij}$  and  $\tilde{\bar{c}}_{ij}$  ( $i, j = 1, 2$ ) such that*

$$A_{ij} X_j = \tilde{a}_{ij} X_i, \quad i, j = 1, 2, \quad (7.5.30)$$

$$C_{ij} X_j = \tilde{c}_{ij} X_i, \quad i, j = 1, 2, \quad (7.5.31)$$

$$\bar{C}_{ij} X_j = \tilde{\bar{c}}_{ij} X_i, \quad i, j = 1, 2, \quad (7.5.32)$$

then we can find an  $(N - 2) \times N$  full rank matrix  $\textcircled{\mathbb{H}} = \begin{pmatrix} \textcircled{\mathbb{H}}_1 & \\ & \textcircled{\mathbb{H}}_2 \end{pmatrix}$  and  $(N - 2) \times (N - 2)$  matrices  $\tilde{A}$ ,  $\tilde{C}$  and  $\tilde{\bar{C}}$  such that (7.5.27)–(7.5.29) hold, moreover,  $X_1$  (resp.  $X_2$ ) constitutes a system of basic solutions to the linear algebraic system

$$\textcircled{\mathbb{H}}_1 Y_1 = 0 \text{ (resp. } \textcircled{\mathbb{H}}_2 Y_2 = 0). \quad (7.5.33)$$

In addition, let

$$\tilde{\bar{C}} = \begin{pmatrix} \tilde{\bar{c}}_{11} & \tilde{\bar{c}}_{12} \\ \tilde{\bar{c}}_{21} & \tilde{\bar{c}}_{22} \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{c}_{11} & \bar{c}_{12} \\ \bar{c}_{21} & \bar{c}_{22} \end{pmatrix}. \quad (7.5.34)$$

We have

$$\text{Spectrum}(C) = \text{Spectrum}(\bar{C}) + \text{Spectrum}(\tilde{\bar{C}}), \quad (7.5.35)$$

$$\text{Spectrum}(\bar{C}) = \text{Spectrum}(\tilde{\bar{C}}) + \text{Spectrum}(\tilde{\bar{C}}). \quad (7.5.36)$$

**Proof** We need only to prove (7.5.35). Similarly to the proof of Theorem 7.5.2, by Gram-Schmidt process on the row vectors of matrices  $\textcircled{H}_1$  and  $\textcircled{H}_2$ , respectively, it is easy to see that there exist matrices  $D_1 = (d_1, \dots, d_{N_1-1})$  and  $D_2 = (d_{N_1}, \dots, d_{N-2})$ , in which  $d_k (1 \leq k \leq N-2)$  are column vectors, such that

$$\textcircled{H}_1 D_1 = I_{N_1-1}, \tag{7.5.37}$$

$$\textcircled{H}_2 D_2 = I_{N_2-1}. \tag{7.5.38}$$

Take  $Z_1 \in \mathbb{R}^{N_1}$  and  $Z_2 \in \mathbb{R}^{N_2}$  such that

$$Z_1 \in (\text{Span}\{d_1, \dots, d_{N_1-1}\})^\perp, \quad Z_2 \in (\text{Span}\{d_{N_1}, \dots, d_{N-2}\})^\perp, \tag{7.5.39}$$

$$Z_1^T \cdot X_1 = 1, \quad Z_2^T \cdot X_2 = 1. \tag{7.5.40}$$

For any given  $\lambda \in \mathbb{C}$ , noting (7.5.28) and (7.5.31) we have

$$\begin{aligned} & \begin{pmatrix} \begin{pmatrix} \textcircled{H}_1 \\ Z_1^T \end{pmatrix} \\ \begin{pmatrix} \textcircled{H}_2 \\ Z_2^T \end{pmatrix} \end{pmatrix} \begin{pmatrix} C_{11} - \lambda I_{N_1} & C_{12} \\ C_{21} & C_{22} - \lambda I_{N_2} \end{pmatrix} \begin{pmatrix} (D_1, X_1) \\ (D_2, X_2) \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} \textcircled{H}_1 \\ Z_1^T \end{pmatrix} C_{11}(D_1, X_1) & \begin{pmatrix} \textcircled{H}_1 \\ Z_1^T \end{pmatrix} C_{12}(D_2, X_2) \\ \begin{pmatrix} \textcircled{H}_2 \\ Z_2^T \end{pmatrix} C_{21}(D_1, X_1) & \begin{pmatrix} \textcircled{H}_2 \\ Z_2^T \end{pmatrix} C_{22}(D_2, X_2) \end{pmatrix} - \lambda I_N \\ &= \left( \begin{array}{cc|cc} \tilde{C}_{11} & \mathbf{0} & \tilde{C}_{12} & \mathbf{0} \\ * & \tilde{c}_{11} & * & \tilde{c}_{12} \\ \tilde{C}_{21} & \mathbf{0} & \tilde{C}_{22} & \mathbf{0} \\ * & \tilde{c}_{21} & * & \tilde{c}_{22} \end{array} \right) - \lambda I_N \\ &= \left( \begin{array}{cc|cc} \tilde{C}_{11} - \lambda I_{m-1} & \mathbf{0} & \tilde{C}_{12} & \mathbf{0} \\ * & \tilde{c}_{11} - \lambda & * & \tilde{c}_{12} \\ \tilde{C}_{21} & \mathbf{0} & \tilde{C}_{22} - \lambda I_{N-m-1} & \mathbf{0} \\ * & \tilde{c}_{21} & * & \tilde{c}_{22} - \lambda \end{array} \right), \end{aligned} \tag{7.5.41}$$

where ”\*” denotes some unmarked values. Taking the determinants of both sides of (7.5.41), by Laplace expansion theorem, we immediately get (7.5.35). (7.5.36) can be obtained in a similar way. This ends the proof.

**Remark 7.5.5** Under assumptions (7.1.5)-(7.1.6), by (7.5.35)-(7.5.36) we can guarantee the well-posedness of the reduced system (7.2.7) and (7.2.9)-(7.2.10) for the variable  $W$ .

By (7.5.25) and noting that the rank of  $\textcircled{H}_1$  (resp.  $\textcircled{H}_2$ ) is  $N_1 - 1$  (resp.  $N_2 - 1$ ), the linear algebraic system (7.5.33) possesses a system of basic solutions  $X_1 =$

$(\alpha_1, \dots, \alpha_{N_1})^T \in \mathbb{R}^{N_1} \setminus \{0\}$  (resp.  $X_2 = (\alpha_{N_1+1}, \dots, \alpha_N)^T \in \mathbb{R}^{N_2} \setminus \{0\}$ ). Thus, noting (7.5.4), as  $t \geq T$  the generalized synchronizable state  $U(t, x) = (u_1(t, x), \dots, u_N(t, x))^T$  can be written as

$$u_p(t, x) = \alpha_p \tilde{u}_1(t, x), \quad p = 1, \dots, N_1, \quad 0 \leq x \leq L, \quad (7.5.42)$$

$$u_q(t, x) = \alpha_q \tilde{u}_2(t, x), \quad q = N_1 + 1, \dots, N, \quad 0 \leq x \leq L. \quad (7.5.43)$$

Completely similarly to [81], we have

**Theorem 7.5.5** *Suppose that  $M = N - 2$ . If the coupled system (7.1.1)-(7.1.3) possesses the generalized exact boundary synchronization associated with (7.5.25), and at least for an initial state  $(\Phi, \Psi)$ ,  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  given by (7.5.42)-(7.5.43) for the generalized synchronizable state are linearly independent, then there exist real vectors  $X_1 \in \mathbb{R}^{N_1} \setminus \{0\}$  and  $X_2 \in \mathbb{R}^{N_2} \setminus \{0\}$  and real numbers  $\tilde{a}_{ij}$  ( $i, j = 1, 2$ ) such that (7.5.30) hold.*

**Remark 7.5.6** *As  $t \geq T$ , for the generalized synchronizable state  $U(t, x) = \begin{pmatrix} \tilde{u}_1(t, x) X_1 \\ \tilde{u}_2(t, x) X_2 \end{pmatrix}$  given by (7.5.42)-(7.5.43), in which  $X_1 = (\alpha_1, \dots, \alpha_{N_1})^T$  (resp.  $X_2 = (\alpha_{N_1+1}, \dots, \alpha_N)^T$ ) is a system of basic solutions to (7.5.33),  $\tilde{U}(t, x) = \begin{pmatrix} \tilde{u}_1(t, x) \\ \tilde{u}_2(t, x) \end{pmatrix}$  satisfies the coupled system (7.3.24)-(7.3.26) of wave equations, in which  $\tilde{a}_{ij}, \tilde{c}_{ij}$  are  $\tilde{c}_{ij}$  ( $i, j = 1, 2$ ) given by (7.5.30)-(7.5.32) respectively. By (7.1.5)-(7.1.6), and noting (7.5.35)-(7.5.36), according to Lemma 7.1.1, if we know the initial state (7.3.28) of  $\tilde{U} = \tilde{U}(t, x)$  at the moment  $t = T$ , then the evolution of  $\tilde{U} = \tilde{U}(t, x)$  with respect to  $t$  can be completely determined.*

Moreover, under the additional assumption that 1 is not an eigenvalue of both  $\tilde{C}$  and  $\tilde{\bar{C}}$ , any given state  $(\varphi^{(1)}, \varphi^{(2)})^T$  and  $(\psi^{(1)}, \psi^{(2)})^T$  in  $C^2[0, L] \times C^1[0, L]$ , satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, with boundary conditions (7.3.25)-(7.3.26), can be the initial data of a generalized synchronizable state  $\tilde{U} = \tilde{U}(t, x)$ . Thus, at least for some initial states  $(\Phi, \Psi)$ , the generalized synchronizable states  $\tilde{u}_1(t, x)$  and  $\tilde{u}_2(t, x)$  are linearly independent on  $t \geq T$ , which means the corresponding requirement in Theorem 7.5.5 is reasonable.

**Remark 7.5.7** *In the case of exact boundary synchronization by 2-groups,  $M = N - 2$  and  $X_1 = (\overbrace{1, \dots, 1}^m)^T$ ,  $X_2 = (\overbrace{1, \dots, 1}^{N-m})^T$ . The requirements (7.5.30)-(7.5.32) imply (7.3.3)-(7.3.6), then, Theorem 7.5.4 leads to the conclusion given in §7.3.*

**Remark 7.5.8** *The situation  $M = N - k$  ( $k \geq 2$ ) can be similarly discussed.*

# Chapter 8

## Exact Boundary Synchronization for a Coupled System of Quasilinear Wave Equations

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### 8.1 Introduction and Main Result

Synchronization is a widespread natural phenomenon. How to describe the phenomenon of synchronization for a coupled system has been widely studied by biologists, engineers and sociologists (see [60], [104]). The previous research activities, however, only focused on the systems governed by ordinary differential equations or complex networks (see [39], [95] and [115]). Recently, several kinds of exact synchronizations are introduced by Li et.al. (see [57], [81]) for a coupled system of 1-D wave



equations with boundary conditions of Dirichlet type, Neumann type, coupled third type and coupled dissipative type in the framework of  $C^2$  solutions, and it is shown that these synchronizations can be realized by means of less boundary controls. For the multi-dimensional case, by indirectly using the HUM method (see [88]–[89]), a similar result for a coupled system of wave equations with Dirichlet boundary controls can be also established in the framework of weak solutions (see [79]). In this chapter, based on the theory of semi-global classical solutions for quasilinear hyperbolic systems, under suitable hypotheses, we will present an iteration procedure given by a unified constructive method to establish the local exact boundary synchronization for a coupled system of 1-D quasilinear wave equations with afore mentioned boundary conditions.

Precisely speaking, we consider the following coupled system of quasilinear wave equations:

$$\frac{\partial^2 U}{\partial t^2} - a^2(U, U_t, U_x) \frac{\partial^2 U}{\partial x^2} + A(U, U_t, U_x)U = 0, \quad (8.1.1)$$

where  $U = (u_1, \dots, u_N)^T$  is an unknown vector function of  $(t, x)$ ,  $A(U, U_t, U_x) = (a_{ij}(U, U_t, U_x))$  is an  $N \times N$  coupling matrix, whose elements are  $C^1$  functions with respect to their arguments,  $a(U, U_t, U_x)$ , the common velocity of propagation for the coupled system, is also a  $C^1$  function with respect to its arguments, and

$$a(0, 0, 0) > 0. \quad (8.1.2)$$

(8.1.1) can be written as

$$\frac{\partial^2 u_i}{\partial t^2} - a^2(U, U_t, U_x) \frac{\partial^2 u_i}{\partial x^2} + \sum_{j=1}^N a_{ij}(U, U_t, U_x) u_j = 0 \quad (i = 1, \dots, N). \quad (8.1.3)$$

For fixing the idea, we study only the case of one-sided controls, the case of two-sided controls can be similarly treated. At the end  $x = 0$ , we prescribe any one of the following boundary conditions:

$$x = 0 : U = H(t) \text{ (Dirichlet type),} \quad (8.1.4a)$$

$$x = 0 : U_x = H(t) \text{ (Neumann type),} \quad (8.1.4b)$$

$$x = 0 : U_x - B(U)U = H(t) \text{ (Coupled third type),} \quad (8.1.4c)$$

where  $B(U) = (b_{ij}(U))$  is an  $N \times N$  boundary coupling matrix with  $C^1$  elements, and

$$H(t) = (h_1(t), \dots, h_N(t))^T \quad (8.1.5)$$

are  $C^2$  (case (8.1.4a)) or  $C^1$  (cases (8.1.4b)–(8.1.4c)) functions of  $t$ , the components of which will be totally or partially taken as boundary controls.

At the end  $x = L$ , since no boundary controls are concerned, we prescribe any

one of the following homogeneous boundary conditions:

$$x = L : U = 0 \text{ (Dirichlet type),} \quad (8.1.6a)$$

$$x = L : U_x = 0 \text{ (Neumann type),} \quad (8.1.6b)$$

$$x = L : U_x + \bar{B}(U)U = 0 \text{ (Coupled third type),} \quad (8.1.6c)$$

where  $\bar{B}(U) = (\bar{b}_{ij}(U))$  is an  $N \times N$  boundary coupling matrix with  $C^1$  elements.

The initial condition is given by

$$t = 0 : (U, U_t) = (\Phi(x), \Psi(x)), \quad 0 \leq x \leq L, \quad (8.1.7)$$

in which both  $\Phi = (\varphi_1, \dots, \varphi_N)^T$  and  $\Psi = (\psi_1, \dots, \psi_N)^T$  are  $C^2$  and  $C^1$  vector functions of  $x$  on  $[0, L]$ , respectively, such that the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$  are satisfied, respectively. Moreover, we assume that

$$\|(\Phi, \Psi)\|_{(C^2[0,L])^N \times (C^1[0,L])^N} \leq \varepsilon. \quad (8.1.8)$$

where  $\varepsilon > 0$  is suitably small.

**Definition 8.1.1** *The coupled system (8.1.1), (8.1.4) and (8.1.6) is said to possess the exact boundary synchronization if there exist two positive constants  $T_0$  and  $T$  with  $T > T_0$ , such that for any given initial state  $(\Phi(x), \Psi(x))$  with small norm  $\|(\Phi, \Psi)\|_{(C^2[0,L])^N \times (C^1[0,L])^N}$  and satisfying the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , we can find some boundary controls with support on  $[0, T_0]$  in  $H(t)$ , such that the corresponding mixed initial-boundary value problem (8.1.1), (8.1.4) and (8.1.6)–(8.1.7) admits a unique  $C^2$  solution  $U(x, t) = (u_1(t, x), \dots, u_N(t, x))^T$  on the domain*

$$R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}, \quad (8.1.9)$$

and as  $T_0 \leq t \leq T$  we have

$$u_1(t, x) \equiv \dots \equiv u_N(t, x) \stackrel{\text{def.}}{=} \tilde{u}(t, x), \quad 0 \leq x \leq L. \quad (8.1.10)$$

$\tilde{u} = \tilde{u}(t, x)$ , being a priori unknown, is called the corresponding synchronizable state. ■

For the exact boundary synchronization for a coupled system of 1-D linear wave equations, it is shown (cf. [81]) that there is a necessary condition that the sum of the elements in every row of the coupling matrices  $A = (a_{ij})$ ,  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  should be independent of  $i = 1, \dots, N$ . In this chapter, for the corresponding

quasilinear case, we impose the similar assumption that

$$\sum_{j=1}^n a_{ij}(U, U_t, U_x) \stackrel{\text{def.}}{=} \tilde{\tilde{a}}(U, U_t, U_x), \quad (8.1.11)$$

$$\sum_{j=1}^n b_{ij}(U) \stackrel{\text{def.}}{=} \tilde{\tilde{b}}(U), \quad (8.1.12)$$

$$\sum_{j=1}^n \bar{b}_{ij}(U) \stackrel{\text{def.}}{=} \tilde{\tilde{b}}(U), \quad (8.1.13)$$

where  $\tilde{\tilde{a}}$ ,  $\tilde{\tilde{b}}$  and  $\tilde{\tilde{b}}$  are all independent of  $i = 1, \dots, N$ .

Obviously,  $U = 0$  is an equilibrium of system (8.1.1) with boundary conditions (8.1.4) (in which  $H \equiv 0$ ) and (8.1.6). Based on the theory of semi-global  $C^2$  solutions, using a unified constructive method for one-sided exact boundary controllability (see [55], [74], [75], [72]), by a suitable iteration procedure as well as certain estimates for coupled system of wave equations, we will establish the local exact boundary synchronization around  $U = 0$ . The main result in this chapter is

**Theorem 8.1.1** *Suppose that  $a$ ,  $a_{ij}$ ,  $b_{ij}$  and  $\bar{b}_{ij}$  are all  $C^1$  functions with respect to their arguments on the domain under consideration. suppose furthermore that (8.1.2) and (8.1.11)–(8.1.13) hold. Let*

$$T > T_0 > 2 \frac{L}{a(0, 0, 0)}. \quad (8.1.14)$$

*For any given initial state  $(\Phi(x), \Psi(x)) \in (C^2[0, L])^N \times (C^1[0, L])^N$ , satisfying simultaneously (8.1.8), the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$ , as well as the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  for the boundary conditions without control in (8.1.4), there exist  $(N - 1)$  boundary controls with support on  $[0, T_0]$  in  $H(t)$  (for example, we take  $h_2(t), \dots, h_N(t)$  as controls with  $h_1(t) \equiv 0$ ) with small  $(C^2[0, T])^N$  norm (case (8.1.4a)) or  $(C^1[0, T])^N$  norm (cases (8.1.4b)–(8.1.4c)), such that the mixed initial-boundary value problem (8.1.1), (8.1.4) and (8.1.6)–(8.1.7) admits a unique  $C^2$  solution  $U = U(t, x)$  on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , and as  $T_0 \leq t \leq T$ ,  $U = U(t, x)$  possesses the exact boundary synchronization shown by (8.1.10). ■*

The chapter is organized as follows: In § 8.2 we present the existence and uniqueness of semi-global  $C^2$  solution to the coupled system of quasilinear wave equations with boundary conditions of various types, meanwhile, for the sake in what follows, we establish the estimates on the  $C^1$ ,  $C^2$  and continuity modulus of the second order partial derivatives of the  $C^2$  solutions for a coupled system of nonautonomous linear

wave equations. In § 8.3, we use a suitable iteration scheme to establish local exact boundary synchronization for a coupled system of quasilinear wave equations. Some remarks are given in § 8.4. Moreover, an appendix (see §8.5) is attached at the end for the precise proof of a technical lemma in the main text.

**Remark 8.1.1** *Similar results are still valid for the coupled boundary conditions of dissipative type, see Remark 8.4.2.*

**Remark 8.1.2** *Precisely speaking, the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, L)$  are given respectively for the boundary conditions (8.1.6a)–(8.1.6c) as follows:*

$$\begin{cases} \Phi(L) = 0, \\ \Psi(L) = 0, \\ \Phi''(L) = 0; \end{cases} \quad (8.1.15a)$$

$$\begin{cases} \Phi'(L) = 0, \\ \Psi'(L) = 0; \end{cases} \quad (8.1.15b)$$

$$\begin{cases} \Phi'(L) + \bar{B}(\Phi(L))\Phi(L) = 0, \\ \psi'_i(L) + \sum_{k,j=1}^N \frac{\partial \bar{b}_{ij}}{\partial u_k}(\Phi(L))\psi_k(L)\varphi_j(L) + \sum_{j=1}^N \bar{b}_{ij}(\Phi(L))\psi_j(L) = 0, \end{cases} \quad (8.1.15c)$$

where  $i = 1, \dots, N$ .

Similarly, for any given  $H(t)$ , the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  are given by

$$\begin{cases} \Phi(0) = H(0), \\ \Psi(0) = H'(0), \\ a^2(\Phi(0), \Psi(0), \Phi'(0))\Phi''(0) - A(\Phi(0), \Psi(0), \Phi'(0))\Phi(0) = H''(0); \end{cases} \quad (8.1.16a)$$

$$\begin{cases} \Phi'(0) = H(0), \\ \Psi'(0) = H'(0); \end{cases} \quad (8.1.16b)$$

$$\begin{cases} \Phi'(0) - B(\Phi(0))\Phi(0) = H(0), \\ \psi'_i(0) - \sum_{k,j=1}^N \frac{\partial b_{ij}}{\partial u_k}(\Phi(0))\psi_k(0)\varphi_j(0) - \sum_{j=1}^N b_{ij}(\Phi(0))\psi_j(0) = h'_i(0), \end{cases} \quad (8.1.16c)$$

where  $i = 1, \dots, N$ , respectively. We point out that if some components in  $H(t)$  are known in advance, then the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  involves only those formulas in (8.1.16) where the corresponding components of  $H(t)$  are given. For example, suppose that  $h_1(t) \equiv 0$ , and the other components of  $H(t)$  are taken as boundary controls, the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  are just given by the first formulas of (8.1.16), the right-hand side of which is zero.

**Remark 8.1.3** For the boundary conditions of Neumann type (8.1.4b) and (8.1.6b), by noting (8.1.15b) and (8.1.16b), the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$  and  $(0, L)$  are only related to the initial data and do not depend on the coupled system of wave equations. This fact will play an important role in establishing a unified constructive method independent of iteration below.

## 8.2 Preliminaries

### 8.2.1 Semi-global classical solutions for a coupled system of quasilinear wave equations

In order to get the well-posedness of the mixed initial-boundary value problem (8.1.1), (8.1.4), (8.1.6) and (8.1.7) in the framework of  $C^2$  solutions, we first reduce the system to a first order quasilinear hyperbolic system, then use the corresponding results of semi-global  $C^1$  solutions.

Let

$$V = (v_1, \dots, v_N)^T = \frac{\partial U}{\partial x}, W = (w_1, \dots, w_N)^T = \frac{\partial U}{\partial t}. \quad (8.2.1)$$

System (8.1.1) can be reduced to the following first order quasilinear system:

$$\begin{cases} \frac{\partial U}{\partial t} = W, \\ \frac{\partial V}{\partial t} - \frac{\partial W}{\partial x} = 0, \\ \frac{\partial W}{\partial t} - a^2(U, W, V)\frac{\partial V}{\partial x} = -A(U, W, V)U. \end{cases} \quad (8.2.2)$$

Let  $I_N$  be the identity matrix of order  $N$ . (8.2.2) can be written in the following

matrix form:

$$\frac{\partial}{\partial t} \begin{pmatrix} U \\ V \\ W \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_N \\ 0 & -a^2(U, W, V)I_N & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} W \\ 0 \\ -A(U, W, V)U \end{pmatrix}. \quad (8.2.3)$$

The characteristic equation of (8.2.2) is

$$\det \begin{pmatrix} \lambda I_N & 0 & 0 \\ 0 & \lambda I_N & I_N \\ 0 & a^2(U, W, V)I_N & \lambda I_N \end{pmatrix} = \lambda^N |\lambda^2 I_N - a^2 I_N| = 0, \quad (8.2.4)$$

whose solutions, the eigenvalues of system (8.2.2), are all real:

$$\lambda_i^- = -a, \quad \lambda_i^0 = 0, \quad \lambda_i^+ = a \quad (i = 1, \dots, N), \quad (8.2.5)$$

and the corresponding left eigenvectors, which constitute a complete set, can be chosen as

$$l_i^- = (\mathbf{0}, a(U, W, V)\mathbf{e}_i, \mathbf{e}_i), \quad l_i^0 = (\mathbf{e}_i, \mathbf{0}, \mathbf{0}), \quad l_i^+ = (\mathbf{0}, -a(U, W, V)\mathbf{e}_i, \mathbf{e}_i) \quad (i = 1, \dots, N), \quad (8.2.6)$$

where  $\mathbf{0} = (0, \dots, 0)$  is the zero vector of order  $N$ , and  $\mathbf{e}_i = (0, \dots, \overset{(i)}{1}, \dots, 0)$  is a unit row vector of order  $N$ . Thus, (8.2.2) is a first order quasilinear hyperbolic system.

Let

$$\tilde{U} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \quad (8.2.7)$$

and

$$\begin{cases} V_i^- = l_i^- \tilde{U} = av_i + w_i, \\ V_i^0 = l_i^0 \tilde{U} = u_i, \\ V_i^+ = l_i^+ \tilde{U} = -av_i + w_i \end{cases} \quad (i = 1, \dots, N). \quad (8.2.8)$$

We have

$$\begin{cases} V^- = aV + W, \\ V^0 = U, \\ V^+ = -aV + W, \end{cases} \quad (8.2.9)$$

where  $V^- = (V_1^-, \dots, V_N^-)^T$ ,  $V^0 = (V_1^0, \dots, V_N^0)^T$  and  $V^+ = (V_1^+, \dots, V_N^+)^T$ . Then

$$\begin{cases} U = V^0, \\ V = \frac{1}{2a}(V^- - V^+), \\ W = \frac{1}{2}(V^- + V^+). \end{cases} \quad (8.2.10)$$

Under the above transformation, the boundary condition (8.1.4) can be correspondingly replaced by

$$x = 0 : W = \dot{H}(t), \quad (8.2.11a)$$

$$x = 0 : V = H(t), \quad (8.2.11b)$$

$$x = 0 : V - B(U)U = H(t). \quad (8.2.11c)$$

Notice (8.2.10), in a neighborhood of  $\tilde{U} = 0$ , i.e.,  $\tilde{V} \stackrel{\text{def.}}{=} \begin{pmatrix} V^- \\ V^0 \\ V^+ \end{pmatrix} = 0$ , the boundary conditions (8.2.11) on  $x = 0$  can be rewritten as

$$x = 0 : V^+ = -V^- + 2\dot{H}(t), \quad (8.2.12a)$$

$$x = 0 : V^+ = V^- - 2a(V^0, \frac{V^- + V^+}{2}, H(t))H(t), \quad (8.2.12b)$$

$$x = 0 : V^+ = V^- - 2a(V^0, \frac{V^- + V^+}{2}, B(V^0)V^0 + H(t))(B(V^0)V^0 + H(t)), \quad (8.2.12c)$$

which can be uniformly expressed as

$$x = 0 : P(t, V^-, V^0, V^+) = 0, \quad (8.2.13)$$

where

$$P(t, V^-, V^0, V^+) = V^+ + V^- - 2\dot{H}(t) \text{ for (8.2.12a)}, \quad (8.2.14a)$$

$$P(t, V^-, V^0, V^+) = V^+ - V^- + 2a(V^0, \frac{V^- + V^+}{2}, H(t))H(t) \text{ for (8.2.12b)}, \quad (8.2.14b)$$

$$P(t, V^-, V^0, V^+) = V^+ - V^- + 2a(V^0, \frac{V^- + V^+}{2}, B(V^0)V^0 + H(t))(B(V^0)V^0 + H(t)) \text{ for (8.2.12c)}. \quad (8.2.14c)$$

For the boundary conditions (8.2.13) on  $x = 0$ , regarding  $(t, V^-, V^0, V^+)$  as variables, noting (8.2.9), under the hypothesis that  $\|H\|_{C^2}$  (case (8.1.4a)) or  $\|H\|_{C^1}$  (cases

(8.1.4b)–(8.1.4c)) and  $\|\tilde{U}\|_{C^0}$  are suitably small (the validity of this hypothesis will be shown by Lemma 8.2.1), it is easy to see that

$$\det \left( \frac{\partial P_i}{\partial V_j^+} \right) \neq 0, \quad i, j = 1, \dots, N. \quad (8.2.15)$$

Therefore, using the implicit function theorem, for any given  $T > 0$ , on the interval  $[0, T]$  the boundary condition on  $x = 0$  can be locally rewritten around  $\tilde{U} = 0$  to the following unified form:

$$x = 0 : \quad V^+ = G(t, V^-, V^0) + \tilde{H}(t), \quad (8.2.16)$$

where  $G$  and  $\tilde{H}$  are  $C^1$  functions with respect to their arguments, and without loss of generality, we assume that

$$G(t, 0, 0) \equiv 0. \quad (8.2.17)$$

Obviously, the  $C^1$  norm of  $\tilde{H}(t)$  is small enough.

Similarly, the boundary conditions (8.1.6a)–(8.1.6c) on  $x = L$  can be also locally rewritten around  $\tilde{U} = 0$  to the following unified form:

$$x = L : \quad V^- = \bar{G}(V^0, V^+), \quad (8.2.18)$$

in which  $\bar{G}$  is a  $C^1$  function with respect to its arguments and

$$\bar{G}(0, 0) \equiv 0. \quad (8.2.19)$$

Meanwhile, the initial condition (8.1.7) can be correspondingly written as

$$t = 0 : \quad \tilde{U}(0, x) = \begin{pmatrix} \Phi(x) \\ \Phi'(x) \\ \Psi(x) \end{pmatrix} \stackrel{\text{def.}}{=} \tilde{U}_0(x), \quad 0 \leq x \leq L. \quad (8.2.20)$$

By the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$  for the mixed initial-boundary value problem (8.1.1), (8.1.4) and (8.1.6)–(8.1.7), it is easy to see that the conditions of  $C^1$  compatibility at these two points are also satisfied for the mixed initial-boundary value problem (8.2.2), (8.2.20), (8.2.16) and (8.2.18). Based on the theory of semi-global  $C^1$  solutions to the first order quasilinear hyperbolic system with zero eigenvalues (see [73], [70], [81], [111]–[112]), we get immediately the following lemma:

**Lemma 8.2.1** *Suppose that on the domain under consideration,  $a$ ,  $a_{ij}$ ,  $b_{ij}$  and  $\bar{b}_{ij}$  are  $C^1$  functions with respect to their arguments. Suppose furthermore that the conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, 0)$  and  $(0, L)$ ,*



respectively. For any given and possibly quite large  $T > 0$ , the forward mixed initial-boundary value problem (8.1.1), (8.1.4) and (8.1.6)–(8.1.7) admits a unique  $C^2$  solution  $U = U(t, x)$  with small  $C^2$  norm on the domain

$$R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}, \quad (8.2.21)$$

provided that  $\|(\Phi, \Psi)\|_{(C^2[0,L])^N \times (C^1[0,L])^N}$  and  $\|H\|_{(C^2[0,T])^N}$ , (case (8.1.4a)) or  $\|H\|_{(C^1[0,T])^N}$  (cases (8.1.4b)–(8.1.4c)) are suitably small (possibly depending on  $T$ ). ■

### 8.2.2 Estimates on the $C^1$ , $C^2$ norms of solutions and the continuity modulus of the second order partial derivatives of solutions for a coupled system of nonautonomous linear wave equations

In order to obtain the exact boundary synchronization for the quasilinear system (8.1.1), (8.1.4) and (8.1.6), we have to investigate the  $C^1$ ,  $C^2$  norms of solutions and the continuity modulus of the second order partial derivatives of solutions for a coupled system of nonautonomous linear wave equations. Here, the continuity modulus of a function  $f(t, x)$  on  $R(T)$  is the following non-negative function:

$$\omega(\eta) = \omega(\eta|f) \stackrel{\text{def.}}{=} \sup_{\substack{|t'-t''| \leq \eta, |x'-x''| \leq \eta \\ (t', x'), (t'', x'') \in R(T)}} |f(t', x') - f(t'', x'')|, \quad \eta \geq 0. \quad (8.2.22)$$

Similarly, the continuity modulus of a vector function  $f$  can be defined by

$$\omega(\eta|f) \stackrel{\text{def.}}{=} \max_{i=1, \dots, n} \omega(\eta|f_i), \quad (8.2.23)$$

and the continuity modulus of a matrix function  $A = (a_{ij})_{n \times n}$  can be defined by

$$\omega(\eta|A) \stackrel{\text{def.}}{=} \max_{i,j=1, \dots, n} \omega(\eta|a_{ij}). \quad (8.2.24)$$

In this subsection, we still denote  $U = U(t, x)$  as the unknown vector function,  $a = a(t, x)$  as the common velocity of propagation and  $A = A(t, x)$ ,  $B = B(t)$  and  $\bar{B} = \bar{B}(t)$  as coupling matrices. Consider the following coupled system of nonautonomous wave equations:

$$\frac{\partial^2 U}{\partial t^2} - a^2(t, x) \frac{\partial^2 U}{\partial x^2} + A(t, x)U = F(t, x), \quad (8.2.25)$$

where  $U = (u_1, \dots, u_N)^T$  is the unknown vector function of  $(t, x)$ ,  $A(t, x) = (a_{ij}(t, x))$  is an  $N \times N$  matrix with  $C^1$  elements,  $a(t, x) > 0$ , the common velocity of propagation

for the coupled system, is a  $C^1$  function of  $(t, x)$ , while  $F(t, x) = (f_1(t, x), \dots, f_N(t, x))^T$  is also a  $C^1$  vector function of  $(t, x)$ .

At the end  $x = 0$ , corresponding to (8.1.4a)–(8.1.4c), we prescribe any one of the following boundary conditions:

$$x = 0 : U = H(t) \text{ (Dirichlet type),} \quad (8.2.26a)$$

$$x = 0 : U_x = H(t) \text{ (Neumann type),} \quad (8.2.26b)$$

$$x = 0 : U_x - B(t)U = H(t) \text{ (Coupled third type),} \quad (8.2.26c)$$

in which  $B(t) = (b_{ij}(t))$  is an  $N \times N$  matrix with  $C^1$  elements, and  $H(t) = (h_1(t), \dots, h_N(t))^T$  are  $C^2$  (case (8.2.26a)) or  $C^1$  (cases (8.2.26b)–(8.2.26c)) functions of  $t$ .

At the end  $x = L$ , corresponding to (8.1.6a)–(8.1.6c), we prescribe any one of the following boundary conditions:

$$x = L : U = \bar{H}(t) \text{ (Dirichlet type),} \quad (8.2.27a)$$

$$x = L : U_x = \bar{H}(t) \text{ (Neumann type),} \quad (8.2.27b)$$

$$x = L : U_x + \bar{B}(t)U = \bar{H}(t) \text{ (Coupled third type),} \quad (8.2.27c)$$

in which  $\bar{B}(t) = (\bar{b}_{ij}(t))$  is an  $N \times N$  matrix with  $C^1$  elements, and  $\bar{H}(t) = (\bar{h}_1(t), \dots, \bar{h}_N(t))^T$  are  $C^2$  (case (8.2.27a)) or  $C^1$  (cases (8.2.27b)–(8.2.27c)) functions of  $t$ .

Denote

$$l = \begin{cases} 1 & \text{case (8.2.26a),} \\ 0 & \text{cases (8.2.26b) – (8.2.26c)} \end{cases}, \quad \bar{l} = \begin{cases} 1 & \text{case (8.2.27a),} \\ 0 & \text{cases (8.2.27b) – (8.2.27c)} \end{cases}, \quad (8.2.28)$$

$$s = l + 1, \quad \bar{s} = \bar{l} + 1. \quad (8.2.29)$$

Suppose that on the domain under consideration, we have

$$\|a\|_1 + \|A\|_1 + \|B\|_1 + \|\bar{B}\|_1 \leq D < +\infty, \quad (8.2.30)$$

where  $D$  is a positive constant. Here and hereafter, for simplicity we denote  $\|\cdot\|_0$ ,  $\|\cdot\|_1$  and  $\|\cdot\|_2$  as the corresponding  $C^0$ ,  $C^1$  and  $C^2$  norms, respectively.

For the mixed initial-boundary value problem (8.2.25), (8.2.26)–(8.2.27) and (8.1.7), by the estimates established in [73] and [72] for the  $C^1$  and  $C^2$  norms of solutions, as well as the continuity modulus of the second order partial derivatives of solutions, obtained by a similar manner, we have

**Lemma 8.2.2** *For any given  $T > 0$ , suppose that  $a$ ,  $a_{ij}$ ,  $b_{ij}$ ,  $\bar{b}_{ij}$  and  $f_i$  ( $i, j = 1, \dots, N$ ) are all  $C^1$  functions on the domain  $R(T) = \{(t, x) | 0 \leq t \leq T, 0 \leq x \leq L\}$ , suppose furthermore that the conditions of  $C^2$  compatibility are satisfied at the points*

$(t, x) = (0, 0)$  and  $(0, L)$ , respectively. Then the forward mixed initial-boundary value problem (8.2.25)–(8.2.27) and (8.1.7) admits a unique  $C^2$  solution  $U = U(t, x)$  on the domain  $R(T)$ , and we have

$$\begin{aligned} \|U\|_{(C^1[R(T)])^N} \leq & C_1 \left( \|(\Phi, \Psi)\|_{(C^1[0,L])^N \times (C^0[0,L])^N} \right. \\ & \left. + \|(H, \overline{H})\|_{(C^l[0,T])^N \times (C^{\overline{l}}[0,T])^N} + \|F\|_{(C^0[R(T)])^N} \right) \end{aligned} \quad (8.2.31)$$

and

$$\begin{aligned} \|U\|_{(C^2[R(T)])^N} \leq & C_2 \left( \|(\Phi, \Psi)\|_{(C^2[0,L])^N \times (C^1[0,L])^N} \right. \\ & \left. + \|(H, \overline{H})\|_{(C^s[0,T])^N \times (C^{\overline{s}}[0,T])^N} + \|F\|_{(C^1[R(T)])^N} \right), \end{aligned} \quad (8.2.32)$$

where  $C_1 = C_1(T, D)$  and  $C_2 = C_2(T, D)$  are positive constants depending only on  $T$  and  $D$ . Moreover, if

$$F(t, x) \equiv 0, \quad (8.2.33)$$

then for any given  $\eta$  with

$$0 < \eta \leq \min_{(t,x) \in R(T)} \frac{L}{a(t, x)}, \quad (8.2.34)$$

we have

$$\omega\left(\eta \left| \frac{\partial^2 U}{\partial t^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U}{\partial x^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U}{\partial t \partial x} \right.\right) \leq \Omega(\eta), \quad (8.2.35)$$

where

$$\begin{aligned} \Omega(\eta) = & C_3 \left( \omega(\eta |\Phi_{xx}) + \omega(\eta |\Psi_x) + \omega\left(\eta \left| \frac{d^s H}{dt^s} \right.\right) + \omega\left(\eta \left| \frac{d^{\overline{s}} \overline{H}}{dt^{\overline{s}}} \right.\right) + (\|\Phi\|_2 + \|\Psi\|_1 + \|H\|_s + \|\overline{H}\|_{\overline{s}}) \right. \\ & \left. \cdot (\eta + \omega(\eta |a_x) + \omega(\eta |a_t) + \omega(\eta |A_x) + \omega(\eta |B_t) + \omega(\eta |\overline{B}_t)) \right) \end{aligned} \quad (8.2.36)$$

is a non-negative function of  $\eta$ , and  $\Omega(\eta) \rightarrow 0$ , when  $\eta \rightarrow 0$ ,  $C_3 = C_3(T, D)$  being a positive constant depending on  $T$  and  $D$ . ■

**Remark 8.2.1** For the backward mixed initial-boundary value problem (8.2.25)–(8.2.27) with final condition

$$t = T : (U, U_t) = (\overline{\Phi}(x), \overline{\Psi}(x)), \quad 0 \leq x \leq L, \quad (8.2.37)$$

satisfying the conditions of  $C^2$  compatibility at the points  $(t, x) = (T, 0)$  and  $(T, L)$ , respectively, similar results can be obtained.

### 8.3 Proof of Theorem 8.1.1

Arbitrarily choose  $H^{(0)}(t)$  (where  $h_1^{(0)}(t) \equiv 0$ ) to be a  $C^2$  (case (8.1.4a)) or  $C^1$  (cases (8.1.4b)–(8.1.4c)) functions of  $t$ , satisfying

$$\|H^{(0)}\|_{(C^s[0,T])^N} \leq C_4\varepsilon, \quad (8.3.1)$$

where  $C_4$  is a positive constant,

$$s = \begin{cases} 2, & \text{case (8.1.4a),} \\ 1, & \text{cases (8.1.4b) – (8.1.4c),} \end{cases} \quad (8.3.2)$$

and  $\varepsilon > 0$ , given by (8.1.8), is a suitably small positive constant, such that the conditions of  $C^2$  compatibility are satisfied at the point  $(t, x) = (0, 0)$ , and

$$H^{(0)}(t) \equiv 0, \quad T_0 \leq t \leq T. \quad (8.3.3)$$

**Remark 8.3.1** *Since we have assumed  $h_1^{(0)}(t) \equiv 0$ , by the the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$ , the initial condition (8.1.7) can not be arbitrarily given near  $x = 0$ . Although this is a special case in Theorem 8.1.1, it is not an issue for the generality of the whole proof.*

By Lemma 8.2.1, the initial-boundary value problem (8.1.1), (8.1.4) (in which  $H(t) = H^{(0)}(t)$ ) and (8.1.6)–(8.1.7) admits a unique  $C^2$  solution  $U = U^{(0)}(t, x)$  on the domain  $R(T)$ , satisfying

$$\|U^{(0)}\|_{(C^2[R(T)])^N} \leq G(\varepsilon), \quad (8.3.4)$$

where  $G(\varepsilon)$  is a  $C^0$  function of  $\varepsilon \geq 0$ , satisfying

$$G(0) = 0, \quad (8.3.5)$$

and, without loss of generality, we may assume that

$$1 \gg G(\varepsilon) \geq C_5\varepsilon, \quad (8.3.6)$$

where  $C_5$  is a positive constant. Since  $U^{(0)} \in C^2$ , there exists a non negative function  $\Omega_0(\eta)$  of  $\eta$  (with  $\Omega_0(\eta) \rightarrow 0$ , as  $\eta \rightarrow 0$ ), such that

$$\omega\left(\eta \left| \frac{\partial^2 U^{(0)}}{\partial t^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(0)}}{\partial x^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(0)}}{\partial t \partial x} \right.\right) \leq \Omega_0(\eta). \quad (8.3.7)$$

Here we point out that, being the first step of the iteration,  $U^{(0)}$  is not asked to be synchronized on  $T_0 \leq t \leq T$ .

Let

$$U^{(-1)}(t, x) \stackrel{\text{def.}}{=} U^{(0)}(t, x). \quad (8.3.8)$$

For any given  $m \geq 0$ , we use an iteration procedure to get  $U^{(m)}(t, x)$  from  $U^{(m-1)}(t, x)$  on  $R(T)$ :

$$\frac{\partial^2 U^{(m)}}{\partial t^2} - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \frac{\partial^2 U^{(m)}}{\partial x^2} + A(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)})U^{(m)} = 0, \quad (8.3.9)$$

the boundary condition on  $x = 0$  is given by any one of :

$$x = 0 : U^{(m)} = H^{(m)}(t), \quad (8.3.10a)$$

$$x = 0 : U_x^{(m)} = H^{(m)}(t), \quad (8.3.10b)$$

$$x = 0 : U_x^{(m)} - B(U^{(m-1)})U^{(m)} = H^{(m)}(t), \quad (8.3.10c)$$

the boundary condition on  $x = L$  is given by any one of

$$x = L : U^{(m)} = 0, \quad (8.3.11a)$$

$$x = L : U_x^{(m)} = 0, \quad (8.3.11b)$$

$$x = L : U_x^{(m)} - \bar{B}(U^{(m-1)})U^{(m)} = 0, \quad (8.3.11c)$$

and the initial condition is

$$t = 0 : (U^{(m)}, U_t^{(m)}) = (\Phi(x), \Psi(x)), \quad 0 \leq x \leq L. \quad (8.3.12)$$

In this iteration scheme, for  $m = 0$ , since  $H^{(0)}(t)$  has been chosen, the corresponding solution is just  $U^{(0)}(t, x)$ ; while, for each  $m \geq 1$ ,  $H^{(m)}(t)$  is a  $C^2$  (case (8.3.10a)) or  $C^1$  (cases (8.3.10b)–(8.3.10c)) functions of  $t$  to be determined. Our goal is to find a suitable  $H^{(m)}(t)$  (where  $h_1^{(m)}(t) \equiv 0$ ), for each  $m \geq 1$ , such that the corresponding mixed initial-boundary value problem (8.3.9)–(8.3.12) admits a  $C^2$  solution  $U^{(m)} = U^{(m)}(t, x)$  on the domain  $R(T)$ , satisfying the exact boundary synchronization (8.1.10). Notice that in the whole iteration procedure, the form of the conditions of  $C^2$  compatibility (8.1.15)–(8.1.16) at the points  $(t, x) = (0, 0)$  and  $(0, L)$  never changes (just replacing  $H(t)$  by  $H^{(m)}(t)$ ).

If the synchronization sequences  $\{U^{(m)}(t, x)\} (m \geq 1)$  are obtained by means of suitable boundary controls  $H^{(m)}(t)$  (in which  $h_1^{(m)}(t) \equiv 0, 0 \leq t \leq T$ ), and if one can prove that

$$U^{(m)}(t, x) \rightarrow U(t, x) \quad \text{in } (C^2[R(T)])^N \quad \text{as } m \rightarrow +\infty, \quad (8.3.13)$$

then  $U(t, x)$  should satisfy (8.1.1) and (8.1.6)–(8.1.7) and possesses the synchronization (8.1.10), and the boundary controls correspondingly to (8.1.4) are given by

$$H(t) = \lim_{m \rightarrow +\infty} H^{(m)}(t), \quad (8.3.14)$$

with  $h_1(t) \equiv 0$  ( $0 \leq t \leq T$ ). Therefore, in order to get Theorem 8.1.1, it suffices to establish the following

**Lemma 8.3.1** *Under the assumptions of Theorem 8.1.1, for any given integer  $m \geq 0$ , there exist a boundary control  $H^{(m)}(t)$  (in which  $h_1^{(m)}(t) \equiv 0$  ( $0 \leq t \leq T$ )) and positive constants  $C_6 = C_6(T_0, T)$ ,  $C_7 = C_7(T_0, T)$  and  $C_8 = C_8(T_0, T)$ , such that the mixed initial-boundary value problem (8.3.9)–(8.3.12) admits a unique  $C^2$  solution  $U^{(m)} = U^{(m)}(t, x)$  on the domain  $R(T)$ , which satisfies*

$$\|U^{(m)}\|_{(C^2[R(T)])^N} \leq C_6 G(\varepsilon), \quad (8.3.15)$$

$$\|U^{(m)} - U^{(m-1)}\|_{(C^1[R(T)])^N} \leq C_7^m G^m(\varepsilon), \quad (8.3.16)$$

$$\omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial t^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial x^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial t \partial x} \right.\right) \leq \Omega_1(\eta), \quad (8.3.17)$$

where

$$\Omega_1(\eta) = \frac{C_8 \left( \omega(\eta |\Phi_{xx}) + \omega(\eta |\Psi_x) + \Omega_0(\eta) + G(\varepsilon) \left( \eta + \omega(\eta |\Gamma^*) \right) \right)}{1 - C_8 G(\varepsilon)} \quad (8.3.18)$$

is a non-negative function of  $\eta$  with  $\Omega_1(\eta) \rightarrow 0$  as  $\eta \rightarrow 0$ , while  $G(\varepsilon)$  and  $\Omega_0(\eta)$  are given by (8.3.4) and (8.3.7),

$$\Gamma^* = \left\{ \frac{\partial a}{\partial u_k}, \frac{\partial a}{\partial u_{kt}}, \frac{\partial a}{\partial u_{kx}}, \frac{\partial a_{ij}}{\partial u_k}, \frac{\partial a_{ij}}{\partial u_{kt}}, \frac{\partial a_{ij}}{\partial u_{kx}}, \frac{\partial b_{ij}}{\partial u_k}, \frac{\partial \bar{b}_{ij}}{\partial u_k}; i, j, k = 1, \dots, N \right\}. \quad (8.3.19)$$

Moreover, for any given  $m \geq 1$ ,  $U^{(m)}(t, x)$  possesses the synchronization (8.1.10) for  $T_0 \leq t \leq T$ . ■

**Proof:** For  $m = 0$ , since  $H^{(0)}(t)$  has been determined, taking  $C_6 \geq 1$ , (8.3.4) immediately yields (8.3.15). Moreover, we have (8.3.16) because of (8.3.8). Combining (8.3.7) and (8.3.18), one gets (8.3.17), provided that  $\varepsilon > 0$  is suitably small and  $C_8 \geq 1$ .

By induction, suppose that for  $m \geq 1$  we have

$$\|U^{(m-1)}\|_{(C^2[R(T)])^N} \leq C_6 G(\varepsilon), \quad (8.3.20)$$

$$\|U^{(m-1)} - U^{(m-2)}\|_{(C^1[R(T)])^N} \leq C_7^{m-1} (G(\varepsilon))^{m-1}, \quad (8.3.21)$$

$$\omega\left(\eta\left|\frac{\partial^2 U^{(m-1)}}{\partial t^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 U^{(m-1)}}{\partial x^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 U^{(m-1)}}{\partial t \partial x}\right.\right) \leq \Omega_1(\eta). \quad (8.3.22)$$

Then for any given  $m \geq 1$ , let

$$\begin{cases} w_i^{(m)} = u_{i+1}^{(m)} - u_i^{(m)}, \\ \tilde{h}_i^{(m)} = h_{i+1}^{(m)} - h_i^{(m)}, \end{cases} \quad i = 1, \dots, N-1. \quad (8.3.23)$$

Under assumption (8.1.11), it is easy to see that from the original system (8.3.9) for the variable  $U^{(m)} = (u_1^{(m)}, \dots, u_N^{(m)})$ , we can get the following self-closing system for the variable  $W^{(m)} = (w_1^{(m)}, \dots, w_{N-1}^{(m)})$  (see also [81]):

$$\frac{\partial^2 W^{(m)}}{\partial t^2} - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \frac{\partial^2 W^{(m)}}{\partial x^2} + \tilde{A}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) W^{(m)} = 0, \quad (8.3.24)$$

where  $\tilde{A} = (\tilde{a}_{ij})$  is an  $(N-1) \times (N-1)$  reduced matrix with

$$\begin{aligned} & \tilde{a}_{ij}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \\ &= \sum_{p=j+1}^N \left( a_{i+1,p}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - a_{ip}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \right) \\ &= \sum_{p=1}^j \left( a_{i,p}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - a_{i+1,p}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \right), \quad i, j = 1, \dots, N-1. \end{aligned} \quad (8.3.25)$$

Similarly, by assumptions (8.1.12)–(8.1.13), from the original boundary conditions (8.3.10)–(8.3.11) for the variable  $U^{(m)}$ , we can get, respectively, the self-closing boundary conditions for the variable  $W^{(m)}$ :

$$x = 0 : W^{(m)} = \tilde{H}^{(m)}(t), \quad (8.3.26a)$$

$$x = 0 : W_x^{(m)} = \tilde{H}^{(m)}(t), \quad (8.3.26b)$$

$$x = 0 : W_x^{(m)} - \tilde{B}(U^{(m-1)}) W^{(m)} = \tilde{H}^{(m)}(t) \quad (8.3.26c)$$

and

$$x = L : W^{(m)} = 0, \quad (8.3.27a)$$

$$x = L : W_x^{(m)} = 0, \quad (8.3.27b)$$

$$x = L : W_x^{(m)} + \tilde{B}(U^{(m-1)}) W^{(m)} = 0, \quad (8.3.27c)$$

in which  $\tilde{H}^{(m)}(t) = (\tilde{h}_1^{(m)}(t), \dots, \tilde{h}_{N-1}^{(m)}(t))^T$ , and  $\tilde{B} = (\tilde{b}_{ij})$  and  $\tilde{\bar{B}} = (\tilde{\bar{b}}_{ij})$  are  $(N - 1) \times (N - 1)$  matrices reduced by  $B = (b_{ij})$  and  $\bar{B} = (\bar{b}_{ij})$  in the following way:

$$\begin{aligned} \tilde{b}_{ij}(U^{(m-1)}) &= \sum_{p=j+1}^n \left( b_{i+1,p}(U^{(m-1)}) - b_{ip}(U^{(m-1)}) \right) \\ &= \sum_{p=1}^j \left( b_{i,p}(U^{(m-1)}) - b_{i+1,p}(U^{(m-1)}) \right), \quad i, j = 1, \dots, N - 1 \end{aligned} \quad (8.3.28)$$

etc.

The initial condition of  $W^{(m)}$  is given by

$$t = 0 : (W^{(m)}, W_t^{(m)}) = (\tilde{\Phi}(x), \tilde{\Psi}(x)), \quad 0 \leq x \leq L, \quad (8.3.29)$$

where

$$\begin{cases} \tilde{\Phi}(x) = (\varphi_2(x) - \varphi_1(x), \dots, \varphi_N(x) - \varphi_{N-1}(x))^T, \\ \tilde{\Psi}(x) = (\psi_2(x) - \psi_1(x), \dots, \psi_N(x) - \psi_{N-1}(x))^T. \end{cases} \quad (8.3.30)$$

Moreover, it easily follows from the condition of  $C^2$  compatibility at the point  $(t, x) = (0, L)$  for  $U^{(m)}$  that the condition of  $C^2$  compatibility at the point  $(t, x) = (0, L)$  for  $W^{(m)}$  is still satisfied.

Noting (8.3.23), once we achieve the exact null controllability for  $W^{(m)}$ , we certainly have the exact synchronization for  $U^{(m)}$ . Similarly to the method in [55] and [77], our next step is to construct a  $C^2$  solution  $W^{(m)} = W^{(m)}(t, x)$  on  $R(T)$  for each  $m \geq 1$ , satisfying simultaneously the system (8.3.24), the initial condition (8.3.29), the null final condition

$$W^{(m)}(t, x) \equiv 0, \quad T_0 \leq t \leq T \quad (8.3.31)$$

and the boundary condition (8.3.27) on  $x = L$ . To this end, we need a suitable treatment on the constructive method for realizing the exact boundary null controllability of  $W^{(m)}$ . In order that, regarding the conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$ , the artificial boundary conditions on  $x = 0$  do not depend on the iteration (8.3.24), we take *Neumann* type artificial boundary conditions on  $x = 0$ , which is independent of  $m$ , instead of the commonly used *Dirichlet* type boundary conditions (see Remark 8.1.2–8.1.3). Thus, for the sequence  $\{W^{(m)}\}$ , we can prove the following Lemma 8.3.2 (for the details of its proof, see Appendix 8.5):

**Lemma 8.3.2** *For any given  $m \geq 1$ , under the assumptions of Theorem 1.1, there exist a boundary control  $\tilde{H}^{(m)}(t)$  on  $x = 0$ , composed of  $C^2$  (case (8.3.26a)) or  $C^1$  (cases (8.3.26b)–(8.3.26c)) functions of  $t$  with support on  $[0, T_0]$ , and the positive constants  $C_9(T_0, T)$ ,  $C_{10}(T_0, T)$  and  $C_{11}(T_0, T)$ , such that the mixed initial-boundary*



value problem (8.3.24), (8.3.26)–(8.3.27) and (8.3.29) admits a unique  $C^2$  solution  $W^{(m)} = W^{(m)}(t, x)$  on the domain  $R(T)$ , satisfying (8.3.31) and

$$\|W^{(m)}\|_{(C^2[R(T)])^N} \leq C_9 G(\varepsilon), \quad (8.3.32)$$

$$\|W^{(m)} - W^{(m-1)}\|_{(C^1[R(T)])^N} \leq C_{10} C_7^{m-1} (G(\varepsilon))^m, \quad (8.3.33)$$

and

$$\omega\left(\eta \left| \frac{\partial^2 W^{(m)}}{\partial t^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 W^{(m)}}{\partial x^2} \right.\right) + \omega\left(\eta \left| \frac{\partial^2 W^{(m)}}{\partial t \partial x} \right.\right) \leq \Omega_2(\eta), \quad (8.3.34)$$

where

$$\Omega_2(\eta) = C_{11} \left( \omega(\eta | \Phi_{xx}) + \omega(\eta | \Psi_x) + \Omega_0(\eta) + G(\varepsilon) \left( \eta + \Omega_1(\eta) + \Omega(\eta | \Gamma^*) \right) \right), \quad (8.3.35)$$

and  $G(\varepsilon)$ ,  $\Omega_0(\eta)$  and  $\Omega_1(\eta)$  are given by (8.3.4), (8.3.7) and (8.3.18), respectively. ■

Noting (8.3.32)–(8.3.34), by the way to obtain the corresponding boundary controls in the constructive method (see [77]), we have

$$\|\tilde{H}^{(m)}\|_{(C^\varepsilon[0, T])^N} \leq C_{12} \|W^{(m)}\|_{(C^2[R(T)])^N} \leq C_{13} G(\varepsilon), \quad (8.3.36)$$

$$\|\tilde{H}^{(m)} - \tilde{H}^{(m-1)}\|_{(C^\varepsilon[0, T])^N} \leq C_{14} \|W^{(m)} - W^{(m-1)}\|_{(C^1[R(T)])^N} \leq C_{15} C_7^{m-1} (G(\varepsilon))^m \quad (8.3.37)$$

and

$$\omega\left(\eta \left| \frac{d^e \tilde{H}^{(m)}}{dt^e} \right.\right) \leq C_{16} \Omega_2(\eta), \quad (8.3.38)$$

where

$$e = \begin{cases} 2, & \text{case (8.3.26a),} \\ 1, & \text{cases (8.3.26b) – (8.3.26c),} \end{cases}, \quad \bar{e} = \begin{cases} 1, & \text{case (8.3.26a),} \\ 0, & \text{cases (8.3.26b) – (8.3.26c),} \end{cases}. \quad (8.3.39)$$

and  $C_{12}$ ,  $C_{13}$ ,  $C_{14}$ ,  $C_{15}$  and  $C_{16}$  are all positive constants depending only on  $T_0$  and  $T$ .

Thus, by (8.3.23) and noting  $h_1^{(m)} \equiv 0$ , we get

$$\|H^{(m)}\|_{(C^r[R(T)])^N} \leq C_{17} G(\varepsilon) \quad (8.3.40)$$

$$\|H^{(m)} - H^{(m-1)}\|_{(C^r[R(T)])^N} \leq C_{18} C_7^{m-1} (G(\varepsilon))^m \quad (8.3.41)$$

and

$$\omega\left(\eta\left|\frac{d^r H^{(m)}}{dt^r}\right.\right) \leq C_{19}\Omega_2(\eta), \quad (8.3.42)$$

where

$$r = \begin{cases} 2, & \text{case (8.3.10a),} \\ 1, & \text{cases (8.3.10b) – (8.3.10c),} \end{cases}, \quad \bar{r} = \begin{cases} 1, & \text{case (8.3.10a),} \\ 0, & \text{cases (8.3.10b) – (8.3.10c),} \end{cases}. \quad (8.3.43)$$

and  $C_{17}$ ,  $C_{18}$ ,  $C_{19}$  are positive constants depending only on  $T_0$  and  $T$ .

Obviously, for  $m \geq 1$ , the previously obtained boundary control  $H^{(m)}$  satisfies the conditions of  $C^2$  compatibility at the points  $(t, x) = (0, 0)$  and  $(0, L)$  for the mixed initial-boundary problem (8.3.9)–(8.3.12), respectively. Notice (8.3.22), (8.3.40) and (8.3.42), it is easy to see from Lemma 8.2.2 that

$$\|U^{(m)}\|_{(C^2[R(T)])^N} \leq C_{20}G(\varepsilon) \quad (8.3.44)$$

and

$$\begin{aligned} & \omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial t^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial x^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial t \partial x}\right.\right) \\ & \leq C_{21}\left(\omega(\eta|\Phi_{xx}) + \omega(\eta|\Psi_x) + \Omega_0(\eta) + G(\varepsilon)\left(\eta + \Omega_1(\eta) + \Omega(\eta|\Gamma^*)\right)\right), \end{aligned} \quad (8.3.45)$$

where  $C_{20}$  and  $C_{21}$  are positive constants depending only on  $T$  and  $T_0$ . Therefore, taking  $C_6 \geq C_{20}$ ,  $C_8 \geq C_{21}$ , we get (8.3.15) and (8.3.17).

On the other hand, let

$$V^{(m)} = U^{(m)} - U^{(m-1)}. \quad (8.3.46)$$

$V^{(m)}$  satisfies the following mixed initial-boundary value problem:

$$\begin{aligned} & \frac{\partial^2 V^{(m)}}{\partial t^2} - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)})\frac{\partial^2 V^{(m)}}{\partial x^2} + A(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)})V^{(m)} \\ & + (a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - a^2(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)}))\frac{\partial^2 U^{(m-1)}}{\partial x^2} \\ & + \left(A(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - A(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)})\right)U^{(m-1)} = 0 \end{aligned} \quad (8.3.47)$$

with the initial condition

$$t = 0 : (V^{(m)}, V_t^{(m)}) = (0, 0), \quad 0 \leq x \leq L, \quad (8.3.48)$$

and the boundary conditions

$$x = 0 : V^{(m)} = H^{(m)} - H^{(m-1)}, \quad (8.3.49a)$$

$$x = 0 : V_x^{(m)} = H^{(m)} - H^{(m-1)}, \quad (8.3.49b)$$

$$x = 0 : V_x^{(m)} - B(U^{(m-1)})V^{(m)} = \left( B(U^{(m-1)}) - B(U^{(m-2)}) \right) U^{(m-1)} + H^{(m)} - H^{(m-1)} \quad (8.3.49c)$$

and

$$x = L : V^{(m)} = 0, \quad (8.3.50a)$$

$$x = L : V_x^{(m)} = 0, \quad (8.3.50b)$$

$$x = L : V_x^{(m)} + \bar{B}(U^{(m-1)})V^{(m)} = \left( \bar{B}(U^{(m-2)}) - \bar{B}(U^{(m-1)}) \right) U^{(m-1)} \quad (8.3.50c)$$

Noting (8.3.20)–(8.3.21), we have

$$\begin{aligned} & \left\| \left( A(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - A(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)}) \right) U^{(m-1)} \right\|_{(C^0[R(T)])^N} \\ & \leq C_{22} C_7^{m-1} (G(\varepsilon))^m, \end{aligned} \quad (8.3.51)$$

$$\begin{aligned} & \left\| \left( a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - a^2(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)}) \right) \frac{\partial^2 U^{(m-1)}}{\partial x^2} \right\|_{(C^0[R(T)])^N} \\ & \leq C_{23} C_7^{m-1} (G(\varepsilon))^m, \end{aligned} \quad (8.3.52)$$

$$\left\| \left( B(U^{(m-1)}) - B(U^{(m-2)}) \right) U^{(m-1)} \right\|_{(C^0[0,T])^N} \leq C_{24} C_7^{m-1} (G(\varepsilon))^m, \quad (8.3.53)$$

$$\left\| \left( \bar{B}(U^{(m-1)}) - \bar{B}(U^{(m-2)}) \right) U^{(m-1)} \right\|_{(C^0[0,T])^N} \leq C_{25} C_7^{m-1} (G(\varepsilon))^m, \quad (8.3.54)$$

where  $C_{22}$ ,  $C_{23}$ ,  $C_{24}$  and  $C_{25}$  are positive constants depending only on  $T$  and  $T_0$ .

Noting (8.3.41), by Lemma 8.2.2 there exists a positive constant  $C_{26} = C_{26}(T, T_0)$  such that

$$\|V^{(m)}\|_{C^1[R(T)]} = \|U^{(m)} - U^{(m-1)}\|_{C^1[R(T)]} \leq C_{26} C_7^{m-1} (G(\varepsilon))^m. \quad (8.3.55)$$

Therefore, choosing  $C_7 \geq C_{26}$ , we get (8.3.16). The proof of Theorem 8.1.1 is complete.  $\blacksquare$

## 8.4 Remarks

**Remark 8.4.1** *Theorem 8.1.1 is still valid in the case of two-sided controls, provided that (8.1.14) is replaced by*

$$T > T_0 > \frac{L}{a(0, 0, 0)}, \quad (8.4.1)$$

and there is an inhomogeneous control  $\bar{H}(t)$  on the right-hand side of (8.1.6).

**Remark 8.4.2** Similar results hold for the following boundary conditions of coupled dissipative type:

$$x = 0 : U_x - C(U)U_t = H(t), \quad (8.4.2)$$

$$x = L : U_x + \bar{C}(U)U_t = 0. \quad (8.4.3)$$

where  $C(U) = (c_{ij}(U))_{N \times N}$  and  $\bar{C}(U) = (\bar{c}_{ij}(U))_{N \times N}$  are  $C^1$  matrix functions with respect to their arguments. However, different from boundary conditions (8.1.4) and (8.1.6), in the case of boundary conditions (8.4.2)–(8.4.3), in order to guarantee the well-posedness and the exact boundary null controllability for the forward mixed initial-boundary value problem, we should impose the following assumptions:

$$\det \left( C(0) + \frac{1}{a(0,0,0)} I_N \right) \neq 0 \quad (8.4.4)$$

and

$$\det \left( \bar{C}(0) + \frac{1}{a(0,0,0)} I_N \right) \neq 0. \quad (8.4.5)$$

for the boundary coupling matrices  $C(U)$  and  $\bar{C}(U)$ . Then, combining the proof of Theorem 8.1.1 and the method presented in [57], one can realize the corresponding local exact boundary synchronization for the coupled system of wave equations (8.1.1) and (8.4.2)–(8.4.3) by  $(N - 1)$  boundary controls on  $x = 0$ , provided that

$$\sum_{j=1}^n c_{ij}(U) \stackrel{\text{def.}}{=} \tilde{c}(U) \quad (8.4.6)$$

and

$$\sum_{j=1}^n \bar{c}_{ij}(U) \stackrel{\text{def.}}{=} \tilde{\bar{c}}(U), \quad (8.4.7)$$

where  $\tilde{c}(U)$  and  $\tilde{\bar{c}}(U)$  are independent of  $i = 1, \dots, N$ .

**Remark 8.4.3** Theorem 8.1.1 and Remark 8.4.2 are also valid for the corresponding nonautonomous quasilinear system, in which

$$\begin{aligned} a &= a(t, x, U, U_t, U_x), \quad A = A(t, x, U, U_t, U_x) \\ B &= B(t, U), \quad \bar{B} = \bar{B}(t, U) \quad (\text{or } C = C(t, U), \quad \bar{C} = \bar{C}(t, U)), \end{aligned} \quad (8.4.8)$$

with suitable regularity with respect to  $t$  and  $x$ .

**Remark 8.4.4** For the following more general coupled system of quasilinear wave equations:

$$\frac{\partial^2 U}{\partial t^2} - a^2(U, U_t, U_x) \frac{\partial^2 U}{\partial x^2} + F(U, U_t, U_x) = 0, \quad (8.4.9)$$

where  $F$  is a  $C^2$  vector function with respect to its arguments and

$$F(0, 0, 0) = 0. \quad (8.4.10)$$

Thus, one can find  $C^1$  matrices  $A(U, U_t, U_x)$ ,  $\check{A}(U, U_t, U_x)$  and  $\hat{A}(U, U_t, U_x)$  such that

$$F(U, U_t, U_x) = A(U, U_t, U_x)U + \check{A}(U, U_t, U_x)U_t + \hat{A}(U, U_t, U_x)U_x, \quad (8.4.11)$$

and the corresponding local exact boundary synchronization can be realized, provided that  $A$ ,  $\check{A}$  and  $\hat{A}$  satisfy similar assumptions as in (8.1.11), respectively.

## 8.5 Appendix: Proof of Lemma 8.3.2

In order to get Lemma 8.3.2, it suffices to prove that for any given integer  $m \geq 1$ , there exists a unique  $C^2$  solution  $W^{(m)} = W^{(m)}(t, x)$  to the coupled system (8.3.24) of wave equations, which satisfies simultaneously the initial condition (8.3.29), the final condition (8.3.31) and the boundary condition (8.3.27) on  $x = L$  as well as the estimates (8.3.32)–(8.3.34).

Let

$$T^{(m)} = \int_0^{T_0} \inf_{0 \leq x \leq L} a(U^{(m-1)}(t, x), U_t^{(m-1)}(t, x), U_x^{(m-1)}(t, x)) dt. \quad (8.5.1)$$

Due to (8.1.14) and (8.3.20), for suitably small  $\varepsilon > 0$  we have

$$T^{(m)} > 2L. \quad (8.5.2)$$

Choose  $T_1^{(m)}$  and  $T_2^{(m)}$  such that

$$\begin{aligned} & \int_0^{T_1^{(m)}} \inf_{0 \leq x \leq L} a(U^{(m-1)}(t, x), U_t^{(m-1)}(t, x), U_x^{(m-1)}(t, x)) dt \\ &= \int_{T_2^{(m)}}^{T_0} \inf_{0 \leq x \leq L} a(U^{(m-1)}(t, x), U_t^{(m-1)}(t, x), U_x^{(m-1)}(t, x)) dt \\ &= L. \end{aligned} \quad (8.5.3)$$

By (8.1.14) and (8.3.20), for  $\varepsilon > 0$  small enough, we have

$$0 < T_1^{(m)} < \frac{T_0 - \delta}{2} < \frac{T_0}{2} < \frac{T_0 + \delta}{2} < T_2^{(m)} < T_0, \quad (8.5.4)$$

in which

$$\delta = \frac{1}{2} \left( T_0 - 2 \frac{L}{a(0, 0, 0)} \right). \quad (8.5.5)$$

(i) We first consider the forward mixed initial-boundary value problem for system (8.3.24) with the initial condition (8.3.29), the boundary condition (8.3.27) on  $x = L$  and the following artificial *Neumann* boundary condition

$$x = 0 : W_x^{(m)} = \mathfrak{F}(t), \quad 0 \leq t \leq \frac{T_0}{2}, \quad (8.5.6)$$

where  $\mathfrak{F}(\cdot)$  is an arbitrarily given  $C^1$  function of  $t$ , satisfying the following conditions of  $C^2$  compatibility at the point  $(t, x) = (0, 0)$ :

$$\begin{cases} \mathfrak{F}(0) = \tilde{\Phi}'(0), \\ \mathfrak{F}'(0) = \tilde{\Psi}'(0) \end{cases} \quad (8.5.7)$$

(see Remark 8.1.2) and verifying

$$\|\mathfrak{F}\|_{C^1[0, \frac{T_0}{2}]} \leq \varepsilon, \quad (8.5.8)$$

$$\Omega(\eta|\mathfrak{F}) \leq \Omega_0(\eta). \quad (8.5.9)$$

Due to (8.5.7)-(8.5.9), this artificial boundary condition (8.5.6) can be chosen to be independent of the iteration (i.e., independent of  $m$ ), namely, we can take the same artificial boundary condition (8.5.6) in every iteration step. Meanwhile, in order that the constants, obtained from all the estimates of the solutions in the iteration procedure, to be also independent of  $m$ , each iteration should be discussed on the same domain, i.e., on

$$R_f = \left\{ (t, x) \mid 0 \leq t \leq \frac{T_0}{2}, 0 \leq x \leq L \right\} \quad (8.5.10)$$

or

$$R_b = \left\{ (t, x) \mid \frac{T_0}{2} \leq t \leq T, 0 \leq x \leq L \right\}. \quad (8.5.11)$$

All these are different from the previous works on the constructive method.

Let

$$\tilde{\Gamma}^* = \left\{ \frac{\partial a}{\partial u_k}, \frac{\partial a}{\partial u_{kt}}, \frac{\partial a}{\partial u_{kx}}, \frac{\partial \tilde{a}_{ij}}{\partial u_k}, \frac{\partial \tilde{a}_{ij}}{\partial u_{kt}}, \frac{\partial \tilde{a}_{ij}}{\partial u_{kx}}, \frac{\partial \tilde{b}_{ij}}{\partial u_k}, \frac{\partial \tilde{b}_{ij}}{\partial u_k} \right\}, \quad (8.5.12)$$

$$i, j, k = 1, \dots, N.$$

Obviously, there exists a positive constant  $\tilde{C}_1$  such that

$$\omega(\eta|\tilde{\Gamma}^*) \leq \tilde{C}_1 \omega(\eta|\Gamma^*). \quad (8.5.13)$$

Here and hereafter,  $\tilde{C}_i (i = 1, 2, \dots)$  denote positive constants,  $\tilde{C}_i(T_0)$  denote positive constants depending only on  $T_0$ , and  $\tilde{C}_i(T, T_0)$  denote positive constants depending only on  $T$  and  $T_0$ .

By Lemma 8.2.2, there exists a unique  $C^2$  solution  $W^{(m)} = W_f^{(m)}(t, x)$  on the domain  $R_f$  (see (8.5.10)). In particular, noting (8.3.20), (8.3.22) and the properties of continuity modulus (see [84]), we have

$$\|W_f^{(m)}(t, x)\|_2 \leq \tilde{C}_2(T_0)\varepsilon \leq \tilde{C}_3(T_0)G(\varepsilon), \quad \forall (t, x) \in R_f, \quad (8.5.14)$$

$$\begin{aligned} & \omega\left(\eta\left|\frac{\partial^2 W_f^{(m)}}{\partial t^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 W_f^{(m)}}{\partial x^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 W_f^{(m)}}{\partial t \partial x}\right.\right) \\ & \leq \tilde{C}_4(T_0) \left( \omega(\eta|\tilde{\Phi}_{xx}) + \omega(\eta|\tilde{\Psi}_x) + \Omega_0(\eta) + \varepsilon \left( \eta + \Omega_1(\eta) + (1 + G(\varepsilon))\Omega(\eta|\tilde{\Gamma}^*) \right) \right) \\ & \leq \tilde{C}_5(T_0) \left( \omega(\eta|\Phi_{xx}) + \omega(\eta|\Psi_x) + \Omega_0(\eta) + G(\varepsilon) \left( \eta + \Omega_1(\eta) + \Omega(\eta|\Gamma^*) \right) \right). \end{aligned} \quad (8.5.15)$$

Thus we can uniquely determine the value of  $(W_f^{(m)}, W_{fx}^{(m)})$  on  $x = L$ :

$$x = L : (W_f^{(m)}, W_{fx}^{(m)}) = (a^{(m)}(t), \bar{a}^{(m)}(t)), \quad 0 \leq t \leq \frac{T_0}{2}, \quad (8.5.16)$$

and we have

$$\|(a^{(m)}(t), \bar{a}^{(m)}(t))\|_{(C^2[0, \frac{T_0}{2}])^N \times (C^1[0, \frac{T_0}{2}])^N} \leq \tilde{C}_6(T_0)G(\varepsilon), \quad (8.5.17)$$

$$\begin{aligned} & \omega(\eta|\ddot{a}^{(m)}) + \omega(\eta|\dot{\bar{a}}^{(m)}) \\ & \leq \tilde{C}_7(T_0) \left( \omega(\eta|\Phi_{xx}) + \omega(\eta|\Psi_x) + \Omega_0(\eta) + G(\varepsilon) \left( \eta + \Omega_1(\eta) + \Omega(\eta|\Gamma^*) \right) \right). \end{aligned} \quad (8.5.18)$$

On the other hand, denoting

$$Z_f^{(m)} = W_f^{(m)} - W_f^{(m-1)}, \quad m = 0, 1, 2, \dots \quad (8.5.19)$$

it is easy to see that  $Z_f^{(m)}$  satisfies the following mixed initial-boundary value problem:

$$\begin{aligned} & \frac{\partial^2 Z_f^{(m)}}{\partial t^2} - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \frac{\partial^2 Z_f^{(m)}}{\partial x^2} + \tilde{A}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) Z^{(m)} \\ & + (\tilde{A}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - \tilde{A}(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)})) W_f^{(m-1)} \\ & - (a^2(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)}) - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)})) \frac{\partial^2 W_f^{(m-1)}}{\partial x^2} = 0 \end{aligned} \quad (8.5.20)$$

with the initial condition

$$t = 0 : (Z_f^{(m)}, Z_{ft}^{(m)}) = (0, 0), \quad 0 \leq x \leq L, \quad (8.5.21)$$

the boundary condition on  $x = 0$

$$x = 0 : Z_{fx}^{(m)} = 0, \quad (8.5.22)$$

and the boundary condition on  $x = L$ :

$$x = L : Z_f^{(m)} = 0, \quad (8.5.23a)$$

$$x = L : Z_{fx}^{(m)} = 0, \quad (8.5.23b)$$

$$x = L : Z_{fx}^{(m)} + \tilde{B}(U^{(m-1)}) Z_f^{(m)} = (\tilde{B}(U^{(m-2)}) - \tilde{B}(U^{(m-1)})) W_f^{(m-1)}. \quad (8.5.23c)$$

Noting (8.3.20)–(8.3.21), (8.3.25) and (8.5.14), we get

$$\begin{aligned} I_0 & \stackrel{\text{def.}}{=} \left\| \left( a^2(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)}) - a^2(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) \right) \frac{\partial^2 W_f^{(m-1)}}{\partial x^2} \right\|_{(C^0[R_f])^N} \\ & \leq \tilde{C}_8(T_0) C_7^{m-1} (G(\varepsilon))^m, \end{aligned} \quad (8.5.24)$$

$$\begin{aligned} I_1 & \stackrel{\text{def.}}{=} \left\| (\tilde{A}(U^{(m-1)}, U_t^{(m-1)}, U_x^{(m-1)}) - \tilde{A}(U^{(m-2)}, U_t^{(m-2)}, U_x^{(m-2)})) W_f^{(m-1)} \right\|_{(C^0[R_f])^N} \\ & \leq \tilde{C}_9(T_0) C_7^{m-1} (G(\varepsilon))^m, \end{aligned} \quad (8.5.25)$$

$$I_2 \stackrel{\text{def.}}{=} \left\| (\tilde{B}(U^{(m-1)}) - \tilde{B}(U^{(m-2)})) W_f^{(m-1)} \right\|_{(C^0[R_f])^N} \leq \tilde{C}_{10}(T_0) C_7^{m-1} (G(\varepsilon))^m. \quad (8.5.26)$$

Therefore, by Lemma 8.2.2, there exists a positive constant  $\tilde{C}_{11}(T_0)$  independent of  $m$ , such that

$$\|W^{(m)} - W^{(m-1)}\|_{(C^1[R_f])^N} \leq \tilde{C}_{11}(T_0) C_7^{m-1} (G(\varepsilon))^m. \quad (8.5.27)$$



In particular, we have

$$\|(a^{(m)}(t) - a^{(m-1)}(t), \bar{a}^{(m)}(t) - \bar{a}^{(m-1)}(t))\|_{(C^1[0, \frac{T_0}{2}])^N \times (C^0[0, \frac{T_0}{2}])^N} \leq \tilde{C}_{12}(T_0)C_7^{m-1}(G(\varepsilon))^m. \quad (8.5.28)$$

(ii) On the domain  $R_b$  (see (8.5.11)), let  $W_b^{(m)}(t, x) \equiv 0$ . Obviously,  $W = W_b^{(m)}(t, x)$  verifies system (8.3.24), final condition (8.3.31) and boundary conditions (8.3.27).

(iii) Thus, we can find the vector function  $(c^{(m)}(t), \bar{c}^{(m)}(t)) \in (C^2[0, T])^N \times (C^1[0, T])^N$  (for example, by polynomial interpolation), such that

$$(c^{(m)}(t), \bar{c}^{(m)}(t)) = \begin{cases} (a^{(m)}(t), \bar{a}^{(m)}(t)), & 0 \leq t \leq \frac{T_0 - \delta}{2}, \\ (0, 0), & \frac{T_0 + \delta}{2} \leq t \leq T, \end{cases} \quad (8.5.29)$$

and  $(c^{(m)}(t), \bar{c}^{(m)}(t))$  verifies the boundary condition (8.3.27) at  $x = L$  on the whole interval  $0 \leq t \leq T$ , moreover, we have

$$\|(c^{(m)}(t), \bar{c}^{(m)}(t))\|_{C^2[0, T] \times C^1[0, T]} \leq \tilde{C}_{13}(T, T_0)G(\varepsilon), \quad (8.5.30)$$

$$\|(c^{(m)}(t) - c^{(m-1)}(t), \bar{c}^{(m)}(t) - \bar{c}^{(m-1)}(t))\|_{C^2[0, T] \times C^1[0, T]} \leq \tilde{C}_{14}(T, T_0)C_6^{m-1}(G(\varepsilon))^m, \quad (8.5.31)$$

$$\begin{aligned} & \omega(\eta)|\ddot{c}^{(m)} + \omega(\eta)|\dot{\bar{c}}^{(m)} \\ & \leq \tilde{C}_{15}(T, T_0) \left( \omega(\eta)|\Phi_{xx} + \omega(\eta)|\Psi_x + \Omega_0(\eta) + G(\varepsilon) \left( \eta + \Omega_1(\eta) + \Omega(\eta)|\Gamma^* \right) \right). \end{aligned} \quad (8.5.32)$$

We now change the status of  $t$  and  $x$ , and consider the following leftward mixed initial-boundary value problem on the domain  $R(T)$  for system (8.3.24) with the initial condition

$$x = L : W_l^{(m)} = c^{(m)}(t), W_{lx}^{(m)} = \bar{c}^{(m)}(t), \quad 0 \leq t \leq T \quad (8.5.33)$$

and the boundary conditions

$$t = 0 : W_{tt}^{(m)} = \tilde{\Psi}(x), \quad 0 \leq x \leq L, \quad (8.5.34)$$

$$t = T : W_{tt}^{(m)} = 0, \quad 0 \leq x \leq L \quad (8.5.35)$$

where  $\tilde{\Psi}(x)$  is given by (8.3.30).

Obviously, the corresponding conditions of  $C^2$  compatibility are satisfied at the points  $(t, x) = (0, L)$  and  $(T, L)$ , respectively. Hence, by Lemma 8.2.2, there exists a

unique  $C^2$  solution  $W^{(m)} = W_l^{(m)}(t, x)$  on the domain  $R(T)$ , such that

$$\|W_l^{(m)}(t, x)\|_{C^2[R(T)]} \leq \tilde{C}_{16}(T, T_0)G(\varepsilon), \quad \forall (t, x) \in R(T), \quad (8.5.36)$$

$$\begin{aligned} & \omega\left(\eta\left|\frac{\partial^2 W_l^{(m)}}{\partial t^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 W_l^{(m)}}{\partial x^2}\right.\right) + \omega\left(\eta\left|\frac{\partial^2 W_l^{(m)}}{\partial t \partial x}\right.\right) \\ & \leq \tilde{C}_{17}(T, T_0)\left(\omega(\eta|\Phi_{xx}) + \omega(\eta|\Psi_x) + \Omega_0(\eta) + G(\varepsilon)\left(\eta + \Omega_1(\eta) + \Omega(\eta|\Gamma^*)\right)\right). \end{aligned} \quad (8.5.37)$$

Thus,  $Z_l^{(m)} = W_l^{(m)} - W_l^{(m-1)}$  satisfies the mixed initial-boundary value problem of system (8.5.20) with the homogeneous boundary condition

$$t = 0 : Z_x^{(m)} = 0, \quad (8.5.38)$$

$$t = T : Z_x^{(m)} = 0 \quad (8.5.39)$$

and the initial condition

$$x = L : (Z^{(m)}, Z_x^{(m)}) = (c^{(m)}(t) - c^{(m-1)}(t), \bar{c}^{(m)}(t) - \bar{c}^{(m-1)}(t)). \quad (8.5.40)$$

Combining (8.5.31), (8.5.36) and Lemma 8.2.2, we get

$$\|W_l^{(m)}(t, x) - W_l^{(m-1)}(t, x)\|_{(C^2[R(T)])^N} \leq \tilde{C}_{18}(T, T_0)C_7^{m-1}(G(\varepsilon))^m. \quad (8.5.41)$$

(iv) By the uniqueness of the  $C^2$  solution for one-sided mixed initial-boundary value problem (see [72], [84], [111] and [112]), the  $C^2$  function  $W^{(m)} = W_l^{(m)}(t, x)$  on  $R(T)$  satisfies the system (8.3.24), the initial condition (8.3.30), the final condition (8.3.31) and the boundary condition (8.3.27) on  $x = L$ . Moreover, noting (8.5.36)–(8.5.37) and (8.5.41), we get (8.3.32)–(8.3.34). This concludes the proof of Lemma 8.3.2. ■



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## Boundary controllability, stabilization and synchronization for 1-D hyperbolic balance laws

### Abstract :

This thesis is devoted to three topics in the control field, namely, exact boundary controllability, boundary stabilization and exact boundary synchronization, for hyperbolic systems of balance laws. For the exact boundary controllability part, we first improve the boundary control time for hyperbolic systems of conservation laws with general boundary conditions and show that this control time is sharp. Then for a coupled hyperbolic system with zero characteristic speed, we prove that it is impossible to achieve the corresponding exact boundary controllability even with inner couplings in the equation. However, one can stabilize the system in infinite time by means of boundary feedback laws. For the boundary stabilization part, we show how to stabilize both the  $n \times n$  linear and quasilinear hyperbolic systems by means of one-sided closed-loop boundary controls. For that a backstepping method is developed. For the exact boundary synchronization part, we first recall both the exact boundary controllability and observability results for coupled systems of quasilinear wave equations. Then several kinds of exact synchronizations are introduced for a coupled system of 1-D linear and quasilinear wave equations with boundary conditions of Dirichlet type, Neumann type, coupled third type and coupled dissipative type in the framework of  $C^2$  solutions. We show that all these synchronizations can be realized by means of few boundary controls.

**Keywords :** Exact boundary controllability, boundary stabilization, boundary synchronization, 1-D hyperbolic balance laws, coupled wave systems

## Contrôle frontière, stabilisation et synchronisation pour des systèmes de lois de bilan en dimension un d'espace

### Résumé :

Cette thèse est consacrée à trois sujets dans le domaine du contrôle, qui sont la contrôlabilité exacte frontière, la stabilisation frontière et la synchronisation exacte frontière, des systèmes hyperboliques de lois de bilan. Pour la partie sur la contrôlabilité exacte frontière, on améliore le temps de contrôlabilité exacte pour les systèmes hyperboliques de lois de conservation pour des conditions aux limites générales. On montre aussi que ce temps est optimal. En ce qui concerne les systèmes hyperboliques couplés avec une vitesse caractéristique nulle, nous prouvons que l'on n'a pas la contrôlabilité exacte, même avec des couplages internes dans les équations. Cependant, on montre que l'on peut stabiliser les systèmes par les lois de rétroaction à la frontière du domaine. Dans la deuxième partie, nous nous intéressons à la stabilisation frontière des systèmes hyperboliques de lois de bilan. En utilisant une approche "backstepping", on montre comment stabiliser des systèmes d'abord dans les cas linéaires puis dans les cas quasi-linéaires. La troisième partie concerne la synchronisation exacte frontière. Nous rappelons d'abord les résultats de contrôlabilité et d'observabilité exacte frontière pour les systèmes couplés d'équations des ondes quasi-linéaires. Puis nous introduisons plusieurs types de synchronisations pour un système d'équations des ondes linéaires, puis quasi-linéaires, couplées avec des conditions aux limites de type Dirichlet, de type Neumann, de type Robin et de type dissipatif dans le cadre de solutions de classe  $C^2$ . Nous montrons que toutes ces synchronisations peuvent être réalisées en imposant peu de contrôles aux frontières.

**Mot clés :** Contrôlabilité exacte frontière, stabilisation frontière, synchronisation frontière, systèmes hyperboliques de lois de bilan 1-D, systèmes d'équations des ondes couplées