Monoidal equivalence of locally compact quantum groups and application to bivariant K-theory

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Introduction

The notion of monoidal equivalence of compact quantum groups has been introduced by Bichon, De Rijdt and Vaes in [5]. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are said to be monoidally equivalent if their categories of representations are equivalent as monoidal C*-categories. They have proved that $\mathbb{G}_1$ and $\mathbb{G}_2$ are monoidally equivalent if and only if there exists a unital C*-algebra equipped with commuting continuous ergodic actions of full multiplicity of $\mathbb{G}_1$ on the left and of $\mathbb{G}_2$ on the right.
Many crucial results of the geometric theory of free discrete quantum groups (random walks and their associated boundaries, Haagerup property, weak amenability, K-amenability and so on) rely on the monoidal equivalence of their dual compact quantum groups. Among the applications of monoidal equivalence to this theory, we will mention the following:

- In [24], Vaes and Vander Vennet have computed the Poisson and Martin boundaries for the dual of the orthogonal quantum groups $\mathrm{A_o}(F)$ by using the monoidal equivalence of $\mathrm{A_o}(F)$ and $\mathrm{SU}_q(2)$ for an appropriate value of q (see [5]) and the results of [15, 25].

- In [12], De Rijdt and Vander Vennet have established a one-to-one correspondence between the continuous actions of two monoidally equivalent compact quantum groups. Moreover, this correspondence exchanges the Poisson (resp. Martin) boundaries of the dual discrete quantum groups. It then follows that the knowledge of the Poisson (resp. Martin) boundary of a discrete quantum group $\widehat{\mathbb{G}}$ provides that of any group whose dual compact quantum group is monoidally equivalent to $\mathbb{G}$. By using this principle, the authors have computed the Poisson boundaries of the duals of the automorphism quantum groups.

- In [11], the authors have established the CCAP property and the Haagerup property for the dual of any orthogonal quantum group by using the same principle. In virtue of the compatibility of the monoidal equivalence with some operations, they have extended these properties to free discrete quantum groups.

- In [28], Voigt has established an equivalence of the categories $\mathsf{KK}^{\mathbb{G}_1}$ and $\mathsf{KK}^{\mathbb{G}_2}$ for two monoidally equivalent compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. It follows that the Baum-Connes conjecture for the duals is invariant by monoidal equvalence. By proving this conjecture for the dual of $\mathrm{SU}_q(2)$ for an appropriate value of q, Voigt has then proved the conjecture for the duals of orthogonal quantum groups $\mathrm{A_o}(F)$ and their K-amenability. In [27], the authors have used this result to establish the conjecture for the duals of the free unitary quantum groups $\mathrm{A_u}(F)$.

In his Ph.D. thesis [9], De Commer has extended the notion of monoidal equivalence to the locally compact case. Two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ (in the sense of Kustermans and Vaes [18]) are said to be monoidally equivalent if there exists a von Neumann algebra equipped with a left Galois action of $\mathbb{G}_1$ and a right Galois action of $\mathbb{G}_2$ that commute. He proved that this notion is completely encoded by a measured quantum groupoid (in the sense of Enock and

Lesieur [14]) on the basis $\mathbb{C}^2$. Such a groupoid is called a colinking measured quantum groupoid. The measured quantum groupoids have been introduced and studied by Lesieur and Enock (see [14, 19]). Roughly speaking, a measured quantum groupoid (in the sense of Enock-Lesieur) is an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$, where N and M are von Neumann algebras (the basis N and M are the algebras of the groupoid corresponding respectively to the space of units and the total space for a classical groupoid), α and β are faithful normal *-homomorphisms from N and N^o (the opposite algebra) to M (corresponding to the source and target maps for a classical groupoid) with commuting ranges, Γ is a coproduct taking its values in a certain fiber product, ν is a normal semi-finite weight on N and T and T' are operator-valued weights satisfying some axioms.
In the case where the basis N is finite-dimensional, the definition has been simplified by De Commer [8, 9] and we will use this point of view in this thesis. Broadly speaking, we can take for ν the non-normalized Markov trace on the C*-algebra $N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$. The relative tensor product of Hilbert spaces (resp. the fiber product of von Neumann algebras) is replaced by the ordinary tensor product of Hilbert spaces (resp. von Neumann algebras). The coproduct takes its values in $M \otimes M$ but is no longer unital.

In this thesis, we introduce a notion of continuous actions on C*-algebras of measured quantum groupoids on a finite basis. We extend the construction of the crossed product, the dual action and we give a version of the Takesaki-Takai duality extending the Baaj-Skandalis duality theorem [2] in this setting.
If a colinking measured quantum groupoid $\mathcal{G}$, associated with a monoidal equivalence of two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, acts continuously on a C*-algebra A, then A splits up as a direct sum $A = A_1 \oplus A_2$ of C*-algebras and the action of $\mathcal{G}$ on A restricts to an action of $\mathbb{G}_1$ (resp. $\mathbb{G}_2$) on A_1 (resp. A_2).
We also extend the induction procedure to the case of monoidally equivalent regular locally compact quantum groups. To any continuous action of $\mathbb{G}_1$ on a C*-algebra A_1, we associate canonically a C*-algebra A_2 endowed with a continuous action of $\mathbb{G}_2$. As important consequences of this construction, we mention:

- A one-to-one functorial correspondence between the continuous actions of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, which generalizes the compact case [12] and the case of deformations by a 2-cocycle [20].
- A Morita equivalence between the crossed product $A_1 \rtimes \mathbb{G}_1$ and $A_2 \rtimes \mathbb{G}_2$.
- A complete description of the continuous actions of colinking measured quantum groupoids.
- The equivalence of the categories $\mathsf{KK}^{\mathbb{G}_1}$ and $\mathsf{KK}^{\mathbb{G}_2}$, which generalizes to the regular locally compact case a result of Voigt [28].

The proofs of the above results rely crucially on the regularity of the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. We prove that the regularity of $\mathbb{G}_1$ and $\mathbb{G}_2$ is equivalent to the regularity in the sense of [13] (see also [21, 22]) of the associated colinking measured quantum groupoid. By passing, this result solves some questions raised in [20] in the case of deformations by a 2-cocycle.

This thesis is organized as follows:

- *Chapter 1.* First, we recall some notations, definitions and results concerning von Neumann algebras, the theory of weights and operator-valued weights. We make a brief review of the

theory of locally compact quantum groups (in the sense of Kustermanns-Vaes [18]). We recall the construction of the Hopf-C*-algebra associated with a locally compact quantum group and the notion of action of locally compact quantum groups in the C*-algebraic setting and also very briefly in the von Neumann algebraic setting (see [23] for more information). We recall the construction of the crossed product, the dual action and we recall the Baaj-Skandalis duality theorem, the version of the Takesaki-Takai duality theorem for actions of regular locally compact quantum groups. Finally, we recall the biduality for crossed products of equivariant Hilbert bimodules.

- *Chapter 2.* We begin with some technical preliminaries concerning the relative tensor product of Hilbert spaces and the fiber product of von Neumann algebras. Then, we make a quick review of the theory of measured quantum groupoid in the sense of Enock-Lesieur (see [14] for more information) and we recall the simplified definition of measured quantum groupoids on a finite basis and the associated C*-algebraic structure provided by De Commer in [8, 9]. In the last paragraph, we make a very brief review of the reflection technique across a Galois object provided by De Commer (see [9]), the construction and the structure of the colinking measured quantum groupoid associated with monoidally equivalent locally compact quantum groups. Finally, we provide a precise description of the C*-algebraic structure of colinking measured quantum groupoids and we obtain some new results.

- *Chapter 3.* We prove that the measured quantum groupoids in general satisfy a condition of irreducibility. We give a more precise result in the case of a measured quantum groupoid on a finite basis and we obtain some useful corollaries. In the second paragraph, we introduce the notion of semi-regularity for a measured quantum groupoid and we also recall the notion of regularity, which already appears in [13] (see also [21]). We prove the regularity (resp. semi-regularity) of the dual measured quantum groupoid in the regular (resp. semi-regular) case and we obtain some useful conditions equivalent to the regularity (resp. semi-regularity). In the case of a colinking measured quantum groupoid associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$, we prove that the regularity (resp. semi-regularity) of the groupoid is equivalent to the regularity (resp. semi-regularity) of both $\mathbb{G}_1$ and $\mathbb{G}_2$.

- *Chapter 4.* We introduce and study the notion of actions of measured quantum groupoids on a finite basis on C*-algebras, we define the crossed product and the dual action. By using the irreducibility property, we obtain a version of the Takesaki-Takai duality and we also provide an important improvement of the result in the regular case.

- *Chapter 5.* We investigate in the minute details the continuous actions of colinking measured quantum groupoids. In the first paragraph, we prove that any C*-algebra A acted upon by a colinking measured quantum groupoid associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ splits up as a direct sum $A = A_1 \oplus A_2$. In particular, we show that the C*-algebras A_1 and A_2 inherit a continuous action of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively. In the second paragraph, we study in detail the crossed product and we obtain a canonical Morita equivalence between the crossed products $A_1 \rtimes \mathbb{G}_1$ and $A_2 \rtimes \mathbb{G}_2$. In the third paragraph, we investigate and provide a thorough description of the double crossed product. We also give a more precise picture in the regular case.

- *Chapter 6.* In the first paragraph, we define an induction procedure which generalizes that of [12] to the regular locally compact case. In particular, we obtain a one-to-one functorial correspondence between the continuous actions of $\mathbb{G}_1$ and $\mathbb{G}_2$. It should be noted that an induction procedure has been developed by De Commer in [9] in the von Neumann algebraic setting. We also complete the description of the continuous actions of a colinking measured quantum groupoid by defining a one-to-one functorial correspondence between the continuous actions of the groupoid and the continuous actions of $\mathbb{G}_1$ (resp. $\mathbb{G}_2$). In the second paragraph, we introduce a notion of equivariant Morita equivalence (the underlying object is called "equivariant linking algebra") for continuous actions of measured quantum groupoids and we prove that the previous functors exchange the equivariant Morita equivalences. In the third paragraph, we use the previous result to extend the induction procedure to equivariant Hilbert modules. In the fourth paragraph, we prove that the double crossed product $A_2 \rtimes \mathbb{G}_2 \rtimes \widehat{\mathbb{G}}_2$ is canonically equivariantly Morita equivalent to the $\mathbb{G}_2$-C*-algebra induced by $A_1 \rtimes \mathbb{G}_1 \rtimes \widehat{\mathbb{G}}_1$. We also investigate the case where A_1 is replaced by an equivariant linking algebra J_1.

- *Chapter 7.* By using the induction procedure developed in the previous chapter, we define in a canonical way an equivalence of the categories $\mathsf{KK}^{\mathbb{G}_1}$ and $\mathsf{KK}^{\mathbb{G}_2}$ which generalizes the correspondence obtained by Voigt in [28].

- *Chapter 8.* We introduce a notion of equivariant Hilbert modules for actions of measured quantum groupoids under three guises as in [3]. We discuss the question of the continuity of the action of the groupoid on the associated linking algebra and we study some examples.

Contents

Basic notations and writing conventions

We give here a short list of the basic notations and symbols that one can find throughout this thesis.

$\mathscr{H}$, $\mathscr{K}$, $\mathcal{H}$, $\mathcal{K}$, $\mathfrak{H}$, $\mathfrak{K}$, ...	:	Hilbert spaces.
M, N, O, P, ...	:	von Neumann algebras.
A, B, C, D, E, S, ...	:	C*-algebras.
$\mathscr{E}$, $\mathscr{F}$, $\mathcal{E}$, $\mathcal{F}$, ...	:	Hilbert C*-modules.
$\mathbb{G}$, $\mathbb{H}$, ...	:	locally compact quantum groups.
$\mathcal{G}$	:	measured quantum groupoids.
φ, ψ, ν, ...	:	weights on von Neumann algebras.
$\omega, \varphi, \psi, ...$	:	normal linear functionals on von Neumann algebras.
$\Sigma, \sigma, \varsigma$	:	(relative) flip maps.
$\Sigma_{\mathscr{H}\otimes\mathscr{K}}$	:	flip map $\Sigma_{\mathscr{H}\otimes\mathscr{K}} : \mathscr{H} \otimes \mathscr{K} \to \mathscr{K} \otimes \mathscr{H}$.
Δ, $\widehat{\Delta}$, $\Delta_{\mathbb{G}}$, Δ_M, ...	:	coproducts of von Neumann algebraic quantum groups.
δ, $\widehat{\delta}$, $\delta_{\mathbb{G}}$	:	coproducts of Hopf-C*-algebras.
α_N, γ_N	:	(left or right) actions of von Neumann algebraic quantum groups on a von Neumann algebra N.
$\langle X \rangle$	:	linear span of a set X in a vector space.
$[X]$	:	closed linear span of a set X in a normed vector space.
$\langle x\,;P(x)\rangle$ (resp. $[x\,;P(x)]$)	:	linear span (resp. closed linear span) of a subset $\{x\,;P(x)\}$ in a vector space (resp. normed vector space).
$X^* \subset \mathcal{A}$	:	the subset $\{x^*\,;\, x \in X\}$, where X is a subset of a *-algebra $\mathcal{A}$.
XY	:	$\{xy\,;\, x \in X,\, y \in Y\}$, where xy is the product/composition of x and y or the evaluation of x at y.
$f\restriction_X$	:	the restriction of a map f to a subset X.
$\langle \cdot, \cdot \rangle$	:	inner product in a Hilbert space always assumed to be anti-linear in the first variable and linear in the second variable.
$\langle \cdot, \cdot \rangle_A$ (resp. ${}_A\langle \cdot, \cdot \rangle$)	:	inner product in a right (resp. left) Hilbert module over a C*-algebra A.
$\overline{\mathscr{H}}$	:	conjugate Hilbert space of $\mathscr{H}$.
$\mathbb{C}^n$	:	canonical Hilbert space of dimension n.
$\mathcal{B}(\mathscr{H}, \mathscr{K})$ (resp. $\mathcal{K}(\mathscr{H}, \mathscr{K})$)	:	bounded (resp. compact) linear operators from $\mathscr{H}$ to $\mathscr{K}$, where $\mathscr{H}$ and $\mathscr{K}$ are two Hilbert spaces. We denote $\mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H}, \mathscr{H})$ (resp. $\mathcal{K}(\mathscr{H}) = \mathcal{K}(\mathscr{H}, \mathscr{H})$).
$\mathsf{M}_n(\mathbb{C})$	:	square matrices of order n with entries in the field of complex numbers.
$\theta_{\xi,\eta} \in \mathcal{B}(\mathscr{H}, \mathscr{K})$	:	rank-one bounded operator given by $\theta_{\xi,\eta}\zeta = \langle \eta, \zeta \rangle \xi$ for all $\zeta \in \mathscr{H}$ where $\eta \in \mathscr{H}$, $\xi \in \mathscr{K}$.

$\mathcal{L}(\mathscr{E},\mathscr{F})$ (resp. $\mathcal{K}(\mathscr{E},\mathscr{F})$)	:	adjointable (resp. "compact") operators from $\mathscr{E}$ to $\mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are Hilbert modules, $\mathcal{L}(\mathscr{E}) = \mathcal{L}(\mathscr{E},\mathscr{E})$ (resp. $\mathcal{K}(\mathscr{E}) = \mathcal{K}(\mathscr{E},\mathscr{E})$).
$\theta_{\xi,\eta} \in \mathcal{L}(\mathscr{E},\mathscr{F})$	:	elementary operator defined by $\theta_{\xi,\eta}\zeta = \xi\langle\eta,\zeta\rangle$ for all $\zeta \in \mathscr{E}$, where $\xi \in \mathscr{F}$, $\eta \in \mathscr{E}$.
$\xi^* \in \mathcal{L}(\mathscr{E}, A)$	:	the operator given by $\xi^*\eta = \langle\xi,\eta\rangle$, for all $\eta \in \mathscr{E}$.
$\mathscr{E}^* = \{\xi^*\,;\,\xi \in \mathscr{E}\}$	:	left Hilbert A-module defined by $a\xi^* := (\xi a^*)^*$, $\langle\xi^*,\eta^*\rangle_{\mathscr{E}^*} := \langle\xi,\eta\rangle$, for all $\xi,\eta \in \mathscr{E}$ and $a \in A$.
$[x,y]$	:	the commutator of x and y in some algebra, that is $[x,y] := xy - yx$.
$\mathcal{M}(A)$	:	multipliers of the C*-algebra A.
$\widetilde{\mathcal{M}}(A \otimes B)$	:	the C*-algebra $\{m \in \mathcal{M}(\widetilde{A}\otimes B)\,;\,(\widetilde{A}\otimes B)m \subset A\otimes B, m(\widetilde{A}\otimes B) \subset A\otimes B\}$ where $\widetilde{A}$ is the C*-algebra obtained from A by adjunction of a unit element (see [3] §1).
$\mathcal{Z}(A)$	:	center of an algebra A, that is $\mathcal{Z}(A) = \{x \in A\,;\,\forall a \in A,\,[x,a] = 0\}$.
A_+	:	cone of positive elements in a C*-algebra A.
M'	:	commutant of $M \subset \mathcal{B}(\mathscr{H})$, that is $M' = \{x \in \mathcal{B}(\mathscr{H})\,;\,\forall y \in M,\,[x,y] = 0\}$.
M^{o}	:	opposite von Neumann algebra of M.
M_* (resp. M_*^+)	:	normal linear forms (resp. positive normal linear forms) on the von Neumann algebra M.
$\odot/\otimes/\otimes_\pi$	:	algebraic tensor product over $\mathbb{C}$ / tensor product of Hilbert spaces, the external tensor product of Hilbert modules, the tensor product of von Neumann algebras or the minimal tensor product of C*-algebras (spatial tensor product) / internal tensor products of Hilbert modules.
resp./*i.e.*/*e.g.*/*cf.*	:	Abbreviation for "respectively"/"*id est*"/"*exampli gratia*"/"*confere*".

Chapter 1

Locally compact quantum groups

1.1 Preliminaries on von Neumann algebras

1.1.1 Normal linear forms

In this paragraph, we very briefly recall the main notations that we will need later on. For more information about von Neumann algebras we refer the reader to classical references.

We recall here some notations that we will use several times. Let M be a von Neumann algebra. Let $\omega \in M_*$ and $a, b \in M$. We define $a\omega \in M_*$ and $\omega b \in M_*$ the normal linear functionals on M given by

$$(a\omega)(x) = \omega(xa), \quad (\omega b)(x) = \omega(bx), \quad x \in M.$$

so that we have $a'(a\omega) = (a'a)\omega$ and $(\omega b)b' = \omega(bb')$, for all $a, a, b, b' \in M$. We also denote

$$a\omega b := a(\omega b) = (a\omega)b \quad ; \quad \omega_a := a^*\omega a.$$

If $\omega \in M_*^+$, then $\omega_a \in M_*^+$. Note that $(\omega_a)_b = \omega_{ab}$ for all $a, b \in M$. If $\omega \in M_*$ we define $\overline{\omega} \in M_*$ by setting:

$$\overline{\omega}(x) = \overline{\omega(x^*)}, \quad x \in M.$$

Let $\mathscr{H}$ be a Hilbert space and let us fix $\xi, \eta \in \mathscr{H}$. We denote $\omega_{\xi,\eta} \in \mathcal{B}(\mathscr{H})_*$, the normal linear form defined by:

$$\omega_{\xi,\eta}(x) = \langle \xi,\, x\eta \rangle, \quad x \in \mathcal{B}(\mathscr{H}).$$

Note that we have $\overline{\omega}_{\xi,\eta} = \omega_{\eta,\xi}$. Furthermore, we also have $a\omega_{\xi,\eta} = \omega_{\xi,a\eta}$ and $\omega_{\xi,\eta}a = \omega_{a^*\xi,\eta}$ for all $a \in \mathcal{B}(\mathscr{H})$.

$\lozenge$ *Tensor product of normal linear forms.* Let M and N be von Neumann algebras, $\phi \in M_*$ and $\psi \in N_*$. Then, there exists a unique $\phi \otimes \psi \in (M \otimes N)_*$ such that $(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y)$ for all $x \in M$ and $y \in N$. Moreover, $\|\phi \otimes \psi\| \leqslant \|\phi\| \cdot \|\psi\|$. Actually, it is known that we have an (completely) isometric identification $M_* \widehat{\otimes}_\pi N_* = (M \otimes N)_*$, where $\widehat{\otimes}_\pi$ denotes the projective tensor product of Banach spaces. In particular, any $\omega \in (M \otimes N)_*$ is the norm limit of finite sums of the form $\sum_i \phi_i \otimes \psi_i$, where $\phi_i \in M_*$ and $\psi_i \in N_*$.

$\lozenge$ *Slicing with normal linear forms.* We will also need to slice maps with normal linear functionals. Let M_1 and M_2 be von Neumann algebras, $\omega_1 \in (M_1)_*$ and $\omega_2 \in (M_2)_*$. Therefore, the maps $\omega_1 \odot \mathrm{id} : M_1 \odot M_2 \to M_1$ and $\mathrm{id} \odot \omega_2 : M_1 \odot M_2 \to M_2$ extend uniquely to norm continuous normal linear maps $\omega_1 \otimes \mathrm{id} : M_1 \otimes M_2 \to M_2$ and $\mathrm{id} \otimes \omega_2 : M_1 \otimes M_2 \to M_1$. We have $\|\omega_1 \otimes \mathrm{id}\| = \|\omega_1\|$

and $\|\mathrm{id}\otimes\omega_2\| = \|\omega_2\|$. If $\omega_1 \in (M_1)_*^+$ (resp. $\omega_2 \in (M_2)_*^+$), then $\omega_1 \otimes \mathrm{id}$ (resp. $\mathrm{id}\otimes\omega_2$) is positive. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, we define

$$\theta_\xi \in \mathcal{B}(\mathscr{K}, \mathscr{H}\otimes\mathscr{K}), \quad \theta'_\eta \in \mathcal{B}(\mathscr{H}, \mathscr{H}\otimes\mathscr{K}), \quad \xi \in \mathscr{H},\, \eta \in \mathscr{K}$$

by setting $\theta_\xi(\zeta) = \xi\otimes\zeta$ for all $\zeta \in \mathscr{K}$ and $\theta'_\eta(\zeta) = \zeta\otimes\eta$ for all $\zeta \in \mathscr{H}$. If $T \in \mathcal{B}(\mathscr{H}\otimes\mathscr{K})$, $\phi \in \mathcal{B}(\mathscr{K})_*$ and $\omega \in \mathcal{B}(\mathscr{H})_*$, then $(\mathrm{id}\otimes\phi)(T) \in \mathcal{B}(\mathscr{H})$ and $(\omega\otimes\mathrm{id})(T) \in \mathcal{B}(\mathscr{K})$ are determined by the formulas:

$$\langle \xi_1,\, (\mathrm{id}\otimes\phi)(T)\xi_2\rangle = \phi(\theta_{\xi_1}^* T \theta_{\xi_2}), \quad \xi_1, \xi_2 \in \mathscr{H},$$

$$\langle \eta_1,\, (\omega\otimes\mathrm{id})(T)\eta_2\rangle = \omega(\theta'^*_{\eta_1} T \theta'_{\eta_2}), \quad \eta_1, \eta_2 \in \mathscr{K}.$$

In particular, we have:

$$(\mathrm{id}\otimes\omega_{\eta_1,\eta_2})(T) = \theta'^*_{\eta_1} T \theta'_{\eta_2}, \quad \eta_1,\eta_2 \in \mathscr{K} \quad ; \quad (\omega_{\xi_1,\xi_2}\otimes\mathrm{id})(T) = \theta^*_{\xi_1} T \theta_{\xi_2}, \quad \xi_1,\xi_2 \in \mathscr{H}.$$

Let us recall some formulas that will be used several times. For all $\phi \in \mathcal{B}(\mathscr{K})_*$, $\omega \in \mathcal{B}(\mathscr{H})_*$ and $T \in \mathcal{B}(\mathscr{H}\otimes\mathscr{K})$, we have:

$$x(\mathrm{id}\otimes\phi)(T)y = (\mathrm{id}\otimes\phi)((x\otimes 1)T(y\otimes 1)) \;;\; (y\omega x\otimes\mathrm{id})(T) = (\omega\otimes\mathrm{id})((x\otimes 1)T(y\otimes 1)),$$

for all $x, y \in \mathcal{B}(\mathscr{H})$ and

$$a(\omega\otimes\mathrm{id})(T)b = (\omega\otimes\mathrm{id})((1\otimes a)T(1\otimes b)) \;;\; (\mathrm{id}\otimes b\phi a)(T) = (\mathrm{id}\otimes\phi)((1\otimes a)T(1\otimes b)),$$

for all $a, b \in \mathcal{B}(\mathscr{K})$. We also have the following formulas:

$$(\mathrm{id}\otimes\phi)(T)^* = (\mathrm{id}\otimes\overline{\phi})(T^*), \quad (\omega\otimes\mathrm{id})(T)^* = (\overline{\omega}\otimes\mathrm{id})(T^*),$$

$$(\phi\otimes\mathrm{id})(\Sigma_{\mathscr{H}\otimes\mathscr{K}} T \Sigma_{\mathscr{K}\otimes\mathscr{H}}) = (\mathrm{id}\otimes\phi)(T), \quad (\mathrm{id}\otimes\omega)(\Sigma_{\mathscr{H}\otimes\mathscr{K}} T \Sigma_{\mathscr{K}\otimes\mathscr{H}}) = (\omega\otimes\mathrm{id})(T),$$

for all $T \in \mathcal{B}(\mathscr{H}\otimes\mathscr{K})$, $\phi \in \mathcal{B}(\mathscr{K})_*$ and $\omega \in \mathcal{B}(\mathscr{H})_*$.

$\Diamond$ *Predual of* $\mathcal{B}(\mathscr{H},\mathscr{K})$. An element of $\mathcal{B}(\mathscr{H},\mathscr{K})_*$ is the restriction to $\mathcal{B}(\mathscr{H},\mathscr{K})$ of some $\omega \in \mathcal{B}(\mathscr{H}\oplus\mathscr{K})_*$. If $\xi \in \mathscr{K}$, $\eta \in \mathscr{H}$, we define $\omega_{\xi,\eta} \in \mathcal{B}(\mathscr{H},\mathscr{K})_*$ as above. We also define $a\omega$, ωb, $a\omega b$ and $\overline{\omega}$ as above. Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{H}'$, $\mathscr{K}'$ be Hilbert spaces, $T \in \mathcal{B}(\mathscr{H}\otimes\mathscr{K}, \mathscr{H}'\otimes\mathscr{K}')$, we define the slice maps:

$$(\mathrm{id}\otimes\phi)(T) \in \mathcal{B}(\mathscr{H},\mathscr{H}'), \quad \phi \in \mathcal{B}(\mathscr{K},\mathscr{K}')_* \quad ; \quad (\omega\otimes\mathrm{id})(T) \in \mathcal{B}(\mathscr{K},\mathscr{K}'), \quad \omega \in \mathcal{B}(\mathscr{H},\mathscr{H}')_*,$$

in a similar way. Moreover, we can easily generalize the above formulas. For instance, we have $(\mathrm{id}\otimes\phi)(T) = (\phi\otimes\mathrm{id})(\Sigma_{\mathscr{H}'\otimes\mathscr{K}'} T \Sigma_{\mathscr{K}\otimes\mathscr{H}})$ for all $\phi \in \mathcal{B}(\mathscr{K},\mathscr{K}')_*$.

1.1.2 Weights on von Neumann algebras

Definition 1.1.1. A weight φ on a von Neumann algebra M is a map $\varphi : M_+ \to [0,\infty]$ such that:

- $\forall x, y \in M_+$, $\varphi(x+y) = \varphi(x) + \varphi(y)$,
- $\forall x \in M_+$, $\forall \lambda \in \mathbb{R}_+$, $\varphi(\lambda x) = \lambda\varphi(x)$,

We denote $\mathfrak{N}_\varphi = \{x \in M\,;\, \varphi(x^*x) < \infty\}$ the left ideal of square φ-integrable elements of M, $\mathfrak{M}_\varphi^+ = \{x \in M_+\,;\, \varphi(x) < \infty\}$ the cone of positive φ-integrable elements of M and $\mathfrak{M}_\varphi = \langle \mathfrak{M}_\varphi^+ \rangle$ the space of φ-integrable elements of M.

Definition 1.1.2. Let φ be a weight on a von Neumann algebra M. The opposite weight of φ is the weight φ^{o} on M^{o} given by $\varphi^{\mathrm{o}}(x^{\mathrm{o}}) = \varphi(x)$ for all $x \in M_+$. Then, we have $\mathfrak{N}_{\varphi^{\mathrm{o}}} = (\mathfrak{N}_\varphi^*)^{\mathrm{o}}$, $\mathfrak{M}_{\varphi^{\mathrm{o}}}^+ = (\mathfrak{M}_\varphi^+)^{\mathrm{o}}$ and $\mathfrak{M}_{\varphi^{\mathrm{o}}} = (\mathfrak{M}_\varphi)^{\mathrm{o}}$.

Definition 1.1.3. A weight φ on a von Neumann algebra M is called:

- semi-finite, if $\mathfrak{N}_\varphi$ is σ-weakly dense in M.
- faithful, if for $x \in M_+$ we have that $\varphi(x) = 0$ implies $x = 0$.
- normal, if $\varphi(\sup_{i\in\mathcal{I}} x_i) = \sup_{i\in\mathcal{I}} \varphi(x_i)$ for all increasing bounded net $(x_i)_{i\in\mathcal{I}}$ of M_+.

Remarks 1.1.4. Note that φ is semi-finite if and only if $\mathfrak{M}_\varphi^+$ is σ-weakly dense in M_+, or again, if and only if $\mathfrak{M}_\varphi$ is σ-weakly dense in M. There are several conditions equivalent to the normality of φ.

From now on, we will mainly use normal semi-finite faithful weights. We will then abbreviate "normal semi-finite faithful weight" to "n.s.f. weight".

Definition 1.1.5. Let M be a von Neumann algebra and φ a n.s.f. weight on M. We define an inner product on $\mathfrak{N}_\varphi$ by setting:

$$\langle x,\, y\rangle_\varphi = \varphi(x^*y), \quad x, y \in \mathfrak{N}_\varphi.$$

We denote $(\mathscr{H}_\varphi, \Lambda_\varphi)$ the Hilbert space completion of $\mathfrak{N}_\varphi$ with respect to this inner product, where $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathscr{H}_\varphi$ is the canonical map. There exists a unique unital normal *-representation $\pi_\varphi : M \to \mathcal{B}(\mathscr{H}_\varphi)$ such that

$$\pi_\varphi(x)\Lambda_\varphi(y) = \Lambda_\varphi(xy), \quad x \in M,\, y \in \mathfrak{N}_\varphi.$$

The triple $(\mathscr{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$ is called the G.N.S. construction for (M, φ).

Remarks 1.1.6. The linear map Λ_φ is called the G.N.S. map. We have that $\Lambda_\varphi(\mathfrak{N}_\varphi)$ is dense in $\mathscr{H}_\varphi$ and $\langle \Lambda_\varphi(x), \Lambda_\varphi(y)\rangle_\varphi = \varphi(x^*y)$ for all $x, y \in \mathfrak{N}_\varphi$. In particular, Λ_φ is injective. Moreover, we also call π_φ the G.N.S. representation.

We recall here the main objects of the Tomita-Takesaki modular theory.

Proposition-Definition 1.1.7. *Let M be a von Neumann algebra and φ a n.s.f. weight on M. The anti-linear map*

$$\begin{array}{ccc} \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi) & \longrightarrow & \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi) \\ \Lambda_\varphi(x) & \longmapsto & \Lambda_\varphi(x^*) \end{array}$$

is closable and its closure is a possibly unbounded anti-linear map $\mathcal{T}_\varphi : D(\mathcal{T}_\varphi) \subset \mathscr{H}_\varphi \to \mathscr{H}_\varphi$ such that $D(\mathcal{T}_\varphi) = \operatorname{Ran} \mathcal{T}_\varphi$ and $\mathcal{T}_\varphi \circ \mathcal{T}_\varphi(x) = x$ for all $x \in D(\mathcal{T}_\varphi)$. Let

$$\mathcal{T}_\varphi = J_\varphi \nabla_\varphi^{1/2}$$

be the polar decomposition of $\mathcal{T}_\varphi$. The anti-unitary $J_\varphi : \mathscr{H}_\varphi \to \mathscr{H}_\varphi$ is called the modular conjugation for φ and the injective positive self-adjoint operator ∇_φ is called the modular operator for φ.

The functional calculus can be applied to ∇_φ so that ∇_φ^{it} is a unitary operator on $\mathscr{H}_\varphi$ for all $t \in \mathbb{R}$ and we have the following commutation relation:

$$J_\varphi \nabla_\varphi^{it} J_\varphi = \nabla_\varphi^{it}, \quad t \in \mathbb{R}.$$

Note that $[J_\varphi, \nabla_\varphi^{it}] = 0$ for all $t \in \mathbb{R}$ since $J_\varphi^2 = 1$.

Let us recall the following definition:

Definition 1.1.8. A one-parameter group of automorphisms on M is a family $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ indexed by $\mathbb{R}$ of *-automorphisms of M such that $\sigma_{s+t} = \sigma_s \circ \sigma_t$ for all $s, t \in \mathbb{R}$ and such that the map $t \mapsto \sigma_t(x)$ is σ-weakly continuous for all $x \in M$.

Proposition-Definition 1.1.9. *There exists a unique one-parameter group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ of automorphisms on M, called the modular automorphism group of φ, such that*

$$\pi_\varphi(\sigma_t^\varphi(x)) = \nabla_\varphi^{it} \pi_\varphi(x) \nabla_\varphi^{-it}, \quad t \in \mathbb{R},\ x \in M.$$

Then, for all $t \in \mathbb{R}$ and $x \in M$ we have $\sigma_t^\varphi(x) \in \mathfrak{N}_\varphi$ and

$$\Lambda_\varphi(\sigma_t^\varphi(x)) = \nabla_\varphi^{it} \Lambda_\varphi(x), \quad t \in \mathbb{R},\ x \in M.$$

The modular conjugation induces a canonical *-isomorphism between M' and M^{o}.

Proposition 1.1.10. *The following map*

$$\begin{array}{ccc} M^{\mathrm{o}} & \longrightarrow & M' \\ x^{\mathrm{o}} & \longmapsto & J_\varphi \pi_\varphi(x)^* J_\varphi. \end{array}$$

*is a unital normal *-isomorphism.*

We are now able to identify all G.N.S. spaces and the corresponding G.N.S. representations given by the n.s.f. weights on M.

Notations 1.1.11. From now on, we will then identify all the G.N.S. spaces $\mathscr{H}_\varphi$ to a fixed Hilbert space denoted $\mathscr{H}_M$. Moreover, we will identify all G.N.S. representations π_φ to a fixed unital normal *-representation π_M of M on $\mathscr{H}$ and we also identify all modular conjugations J_φ to a fixed anti-unitary operator J_M. By abuse of notation, we still denote $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathscr{H}_M$ the G.N.S. map. Then, we call the triple $(\mathscr{H}_M, \pi_M, \Lambda_\varphi)$ the standard G.N.S. construction for (M, φ). We denote $C_M : M \to M'$ the normal unital *-anti-isomomorphism given by

$$C_M(x) = J_M \pi_M(x)^* J_M, \quad x \in M.$$

We also denote $j_M : M^{\mathrm{o}} \to M'$ the normal unital *-isomorphism induced by C_M, that is to say $j_M(x^{\mathrm{o}}) = J_M \pi_M(x)^* J_M$, for all $x \in M$.

Definition 1.1.12. If φ is a n.s.f. weight on M, we denote φ^{c} the n.s.f. weight on M' defined by $\varphi^{\mathrm{c}} = \varphi^{\mathrm{o}} \circ j_M^{-1}$.

There are canonical identifications $\mathscr{H}_{M^{\mathrm{o}}} = \mathscr{H}_M = \mathscr{H}_{M'}$ of the standard G.N.S. spaces. Indeed, let us fix a n.s.f. weight φ on M, then the maps

$$\begin{array}{ccc} \mathscr{H}_{M^{\mathrm{o}}} & \longrightarrow & \mathscr{H}_M \\ \Lambda_{\varphi^{\mathrm{o}}}(x^{\mathrm{o}}) & \longmapsto & J_M \Lambda_\varphi(x^*), \end{array} \qquad \begin{array}{ccc} \mathscr{H}_{M'} & \longrightarrow & \mathscr{H}_M, \\ \Lambda_{\varphi^{\mathrm{c}}}(j_M(x)) & \longmapsto & J_M \Lambda_\varphi(x^*) \end{array} \quad x \in \mathfrak{N}_\varphi^*,$$

are unitaries and are independent of the chosen n.s.f. weight φ.

1.1.3 Operator-valued weights

Definition 1.1.13. Let N be a von Neumann algebra. The extended positive cone of N is the set N_+^{ext} consisting of the maps $m : N_*^+ \to [0,\infty]$ that satisfy:

1. $\forall \omega_1, \omega_2 \in N_*^+$, $m(\omega_1+\omega_2) = m(\omega_1) + m(\omega_2)$.
2. $\forall \omega \in N_*^+$, $\forall \lambda \in \mathbb{R}_+$, $m(\lambda\omega) = \lambda m(\omega)$.
3. m is lower semicontinuous with respect to the norm topology on N_*.

Notations 1.1.14. Let N be a von Neumann algebra.

1. From now on, we will identify N_+ with its part inside N_+^{ext}. Accordingly, if $m \in N_+^{\mathrm{ext}}$ we will denote $\omega(m)$ the evaluation of m at ω, for $\omega \in N_*^+$.
2. Let $a \in N$ and $m \in N_+^{\mathrm{ext}}$, we define $a^*ma \in N_+^{\mathrm{ext}}$ by setting

$$\omega(a^*ma) = a\omega a^*(m), \quad \omega \in N_*^+.$$

   If $m, n \in N_+^{\mathrm{ext}}$ and $\lambda \in \mathbb{R}_+$, we also define $m+n, \lambda m \in N_+^{\mathrm{ext}}$ to be given by

$$\omega(m+n) = \omega(m) + \omega(n), \quad \omega(\lambda m) = \lambda\omega(m), \quad \omega \in N_*^+.$$

Definition 1.1.15. Let $N \subset M$ be a unital normal inclusion of von Neumann algebras. An operator-valued weight from M to N (or N-valued weight on M) is a map $T : M_+ \to N_+^{\mathrm{ext}}$ that satisfies:

1. $\forall x, y \in M_+$, $T(x+y) = T(x) + T(y)$.
2. $\forall x \in M_+$, $\forall \lambda \in \mathbb{R}_+$, $T(\lambda x) = \lambda T(x)$.
3. $\forall x \in M_+$, $\forall a \in N$, $T(a^*xa) = a^*T(x)a$.

By analogy with ordinary weights, if $N \subset M$ is a unital normal inclusion of von Neumann algebras and T is an operator-valued weight from M to N we define:

$$\mathfrak{N}_T = \{x \in M\,;\, T(x^*x) \in N_+\}, \quad \mathfrak{M}_T^+ = \{x \in M_+\,;\, T(x) \in N_+\}, \quad \mathfrak{M}_T = \langle \mathfrak{M}_T^+ \rangle.$$

Then, $\mathfrak{M}_T$ and $\mathfrak{N}_T$ are two-sided modules over N and $\mathfrak{M}_T = \langle \mathfrak{N}_T^* \mathfrak{N}_T \rangle$.

Definition 1.1.16. Let $N \subset M$ be a unital normal inclusion of von Neumann algebras. An operator-valued weight T from M to N is said to be:

1. semi-finite, if $\mathfrak{N}_T$ is σ-weakly dense in M.
2. faithful, if for $x \in M_+$ we have that $T(x) = 0$ implies $x = 0$.
3. normal, if for every increasing bounded net $(x_i)_{i\in\mathcal{I}}$ of elements of M_+, we have

$$\omega(T(\sup_{i\in\mathcal{I}} x_i)) = \lim_{i\in\mathcal{I}} \omega(T(x_i)), \quad \omega \in N_*^+.$$

Note that if $T : M_+ \to N_+^{\mathrm{ext}}$ is an operator-valued weight, it extends uniquely to a semi-linear map $\overline{T} : M_+^{\mathrm{ext}} \to N_+^{\mathrm{ext}}$. This will allow us to compose n.s.f. operator-valued weights. Indeed, let $P \subset N \subset M$ be unital normal inclusions of von Neumann algebras, T and S be respectively a P-valued n.s.f. weight on N and an N-valued n.s.f. weight on M. We define a P-valued n.s.f. weight on M by setting $(S \circ T)(x) := \overline{S}(T(x))$ for $x \in N_+$.

Fundamental example. Let L, O and P be von Neumann algebras, $\pi : L \to O \otimes P$ be a faithful normal unital *-homomorphism and φ a normal faithful weight on P. Then, we define a normal faithful operator-valued weight $T : L_+ \to O_+^{\mathrm{ext}}$ by setting $T(x) = (\mathrm{id} \otimes \varphi)\pi(x)$ for all $x \in L_+$.

1.2 von Neumann algebraic quantum groups

In the following paragraph, we recall the main definitions and results from [18] on locally compact quantum groups.

Definition 1.2.1. A locally compact quantum group is a quadruple

$$\mathbb{G} = (\mathrm{L}^\infty(\mathbb{G}), \Delta_{\mathbb{G}}, \varphi, \psi),$$

where $\mathrm{L}^\infty(\mathbb{G})$ is a von Neumann algebra, $\Delta_{\mathbb{G}} : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \otimes \mathrm{L}^\infty(\mathbb{G})$ is a unital normal *-homomorphism, φ and ψ are n.s.f. weights on $\mathrm{L}^\infty(\mathbb{G})$ satisfying the following conditions:

1. $(\Delta_{\mathbb{G}} \otimes \mathrm{id})\Delta_{\mathbb{G}} = (\mathrm{id} \otimes \Delta_{\mathbb{G}})\Delta_{\mathbb{G}}$ (coassociativity).
2. $\varphi((\omega \otimes \mathrm{id})\Delta_{\mathbb{G}}(x)) = \varphi(x)\omega(1)$, for all $\omega \in \mathrm{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathfrak{M}_\varphi^+$ (left invariance).
3. $\psi((\mathrm{id} \otimes \omega)\Delta_{\mathbb{G}}(x)) = \psi(x)\omega(1)$, for all $\omega \in \mathrm{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathfrak{M}_\psi^+$ (right invariance).

The map $\Delta_{\mathbb{G}}$ is called the comultiplication (or coproduct) of $\mathbb{G}$ and the n.s.f. weight φ (resp. ψ) is called the left (resp. right) Haar weight on $\mathbb{G}$.

The notation $\mathrm{L}^\infty(\mathbb{G})$ suggests the analogy with the fundamental example given by complex-valued essentially bounded measurable functions on a locally compact group. Moreover, by adding commutativity in the previous definition, we get back to the classical case of a locally compact group. However, there is no underlying space $\mathbb{G}$ in the quantum case and $\mathrm{L}^\infty(\mathbb{G})$ is a non necessarily commutative von Neumann algebra.

Proposition 1.2.2. *Let $\mathbb{G}$ be a locally compact quantum group. Let φ and $\widetilde{\varphi}$ be two left (resp. right) invariant n.s.f. weights on $\mathbb{G}$. Then, there exists a real number $r > 0$ such that $\widetilde{\varphi} = r \cdot \varphi$.*

Therefore, since the left (resp. right) invariant n.s.f. weight on a locally compact quantum group $\mathbb{G}$ is unique up to a positive scalar factor, we will always suppose that we have associated a fixed left invariant n.s.f. weight φ on a given locally compact quantum group $\mathbb{G}$.

In the following, we fix a locally compact quantum group $\mathbb{G} = (M, \Delta)$ and a left invariant weight φ on $\mathbb{G}$.

Proposition-Definition 1.2.3. *There exists a unique couple (τ, R), where $\tau = (\tau_t)_{t \in \mathbb{R}}$ is a one-parameter group of *-automorphisms of M and R is an involutive *-anti-automorphism of M satisfying the following statements:*

1. $R \circ \tau_t = \tau_t \circ R$, *for all* $t \in \mathbb{R}$.

2. *Let* $S = R \circ \tau_{-i/2}$. *We have* $(\mathrm{id} \otimes \varphi)(\Delta(y)^*(1 \otimes x)) \in D(S)$ *for all* $x, y \in \mathfrak{N}_\varphi$ *and*

$$S((\mathrm{id} \otimes \varphi)(\Delta(y)^*(1 \otimes x)))^* = (\mathrm{id} \otimes \varphi)(\Delta(x)^*(1 \otimes y)), \text{ for all } x, y \in \mathfrak{N}_\varphi.$$

The one-parameter group τ_t *is called the scaling group of* $\mathbb{G}$. *The automorphism* R *is called the unitary antipode of* $\mathbb{G}$. *The map* S *is called the antipode of* $\mathbb{G}$.

We have that τ_t commutes with σ_s^φ and σ_s^ψ for all $s, t \in \mathbb{R}$, where ψ is a right invariant n.s.f. weight on $\mathbb{G}$. We also have that $R \circ \sigma_t^\varphi = \sigma_{-t}^\psi \circ R$ for all $t \in \mathbb{R}$.

Proposition 1.2.4. *We have:*

1. $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$, *for all* $t \in \mathbb{R}$.

2. $\varsigma \circ (R \otimes R) \circ \Delta = \Delta \circ R$, *where* $\varsigma : M \otimes M \to M \otimes M$ *is the flip *-automorphism.*

In particular, if φ *is a left invariant n.s.f. weight on* $\mathbb{G}$, *we define a right invariant n.s.f. weight* ψ *on* $\mathbb{G}$ *by setting* $\psi(x) = \varphi(R(x))$ *for all* $x \in M_+$.

Henceforth, we will take $\psi = \varphi \circ R$ as the right invariant weight on $\mathbb{G}$.

Proposition-Definition 1.2.5. *There exists a unique number* $\nu > 0$ *such that*

$$\varphi(\tau_t(x)) = \nu^{-t}\varphi(x), \qquad \varphi(\sigma_t^\psi(x)) = \nu^t \varphi(x),$$

$$\psi(\tau_t(x)) = \nu^{-t}\psi(x), \qquad \psi(\sigma_t^\varphi(x)) = \nu^{-t}\psi(x),$$

for all $t \in \mathbb{R}$ *and* $x \in M_+$. *We call* ν *the scaling constant of* $\mathbb{G}$.

Proposition-Definition 1.2.6. *There exists a unique injective positive operator* d *such that*

1. d *is affiliated to* M (*i.e.* $d^{it} \in M$ *for all* $t \in \mathbb{R}$)

2. $(D\psi : D\varphi)_t = \nu^{it^2/2} d^{it}$ *for all* $t \in \mathbb{R}$,

where $(D\psi : D\varphi)$ *is the Radon-Nikodým derivative of* ψ *with respect to* φ (*see* [7]). *We call* d *the modular element of* $\mathbb{G}$.

Multiplicative unitary. The notion of multiplicative unitary has been introduced and studied by Baaj and Skandalis in [2]. Let us recall the leg numbering notation. Let $\mathscr{H}$ be a Hilbert space, for $\xi \in \mathscr{H}$ and $i = 1, 2, 3$ we define $\theta_{i,\xi} \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H}, \mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H})$ by setting:

$$\theta_{1,\xi}(\eta \otimes \zeta) = \xi \otimes \eta \otimes \zeta, \quad \theta_{2,\xi}(\eta \otimes \zeta) = \eta \otimes \xi \otimes \zeta, \quad \theta_{3,\xi}(\eta \otimes \zeta) = \eta \otimes \zeta \otimes \xi, \quad \eta, \zeta \in \mathscr{H}.$$

Let $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$, we define the operators $T_{12}, T_{13}, T_{23} \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H})$ by:

$$T_{12}\theta_{3,\xi} = \theta_{3,\xi}T, \quad T_{13}\theta_{2,\xi} = \theta_{2,\xi}T, \quad T_{23}\theta_{1,\xi} = \theta_{1,\xi}T, \quad \xi \in \mathscr{H}.$$

In other words, we have $T_{12} = T \otimes 1_{\mathscr{H}}$, $T_{23} = 1_{\mathscr{H}} \otimes T$ and $T_{13} = \Sigma_{12} T_{23} \Sigma_{12}$ where $\Sigma \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ is the flip map. We will also use some generalizations of these notations. Let $(\mathscr{H}_j)_{1 \leqslant j \leqslant n}$ be a family of Hilbert spaces. Let $1 \leqslant k \leqslant n$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and $T \in \mathcal{B}(\bigotimes_{1 \leqslant j \leqslant k} \mathscr{H}_{i_l})$. We define similarly $T_{i_1 \cdots i_k} \in \mathcal{B}(\bigotimes_{1 \leqslant j \leqslant n} \mathscr{H}_j)$. Moreover, the k-tuple $(i_1, \cdots, i_k)$ of pairwise distinct indices need not be ordered, *e.g.* if $T \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ we denote $T_{21} = \Sigma T \Sigma$. We can easily extend the leg numbering notation for adjointable operators on an external tensor product of Hilbert C*-modules over possibly different C*-algebras.

Definition 1.2.7. Let $\mathscr{H}$ be a Hilbert space and $V \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ be a unitary operator. We say that V is multiplicative if it satisfies the pentagon equation

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

Let $(\mathscr{H}, \pi, \Lambda_\varphi)$ be the standard G.N.S. construction for (M, φ). Then, the linear map

$$\Lambda_\varphi \odot \Lambda_\varphi : \mathfrak{N}_\varphi \odot \mathfrak{N}_\varphi \to \mathscr{H} \otimes \mathscr{H}$$

is closable with respect to the σ-strong* topology on $M \otimes M$ and the norm topology on $\mathscr{H} \otimes \mathscr{H}$. We denote $\Lambda_\varphi \otimes \Lambda_\varphi$ its closure with respect to these topologies. Therefore, the triple

$$(\mathscr{H} \otimes \mathscr{H}, \pi \otimes \pi, \Lambda_\varphi \otimes \Lambda_\varphi)$$

is a G.N.S. construction for $\varphi \otimes \varphi$. By the left invariance of φ, we have

$$\Delta(y)(x \otimes 1) \in D(\Lambda_\varphi \otimes \Lambda_\varphi), \quad x, y \in \mathfrak{N}_\varphi.$$

Moreover, we have

$$\langle\, (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)(x \otimes 1)),\, (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y')(x' \otimes 1))\,\rangle = \langle\, \Lambda_\varphi(x) \otimes \Lambda_\varphi(y), \Lambda_\varphi(x') \otimes \Lambda_\varphi(y')\,\rangle$$

for all $x, y, x', y' \in \mathfrak{N}_\varphi$.

Proposition-Definition 1.2.8. *There exists a unique $W \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ such that*

$$W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)(x \otimes 1)),$$

for all $x, y \in \mathfrak{N}_\varphi$. Moreover, W is a multiplicative unitary of $\mathscr{H}$. We call W the left regular representation of $\mathbb{G}$.

In a similar way, we consider $(\mathscr{H}, \pi, \Lambda_\psi)$ the standard G.N.S. construction for (M, ψ) and we define $(\mathscr{H} \otimes \mathscr{H}, \pi \otimes \pi, \Lambda_\psi \otimes \Lambda_\psi)$ the G.N.S. construction for $\psi \otimes \psi$. By using the right invariance of ψ, we have:

Proposition-Definition 1.2.9. *There exists a unique $V \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ such that*

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Delta(x)(1 \otimes y)),$$

for all $x, y \in \mathfrak{N}_\psi$. Moreover, V is a multiplicative unitary of $\mathscr{H}$. We call V the right regular representation of $\mathbb{G}$.

From now on, we assume that the underlying von Neumann algebra M is in standard form with respect to a Hilbert space $\mathscr{H}$ (also denoted $\mathrm{L}^2(\mathbb{G})$ by analogy with the classical case). As before, we fix a left invariant weight φ on $\mathbb{G}$ and a G.N.S. construction $(\mathscr{H}, \iota, \Lambda_\varphi)$ where ι denotes the inclusion map $M \subset \mathcal{B}(\mathscr{H})$. We also consider the G.N.S. construction $(\mathscr{H}, \iota, \Lambda_\psi)$ for the right invariant weight $\psi = \varphi \circ R$.

The left (resp. right) regular representation W (resp. V) carries all the relevant data of $\mathbb{G}$. To be more precise, we have the following result:

Proposition 1.2.10. *We have:*

- *M is the σ-strong* closure of the algebra $\{(\mathrm{id} \otimes \omega)(W)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$.*

- $\Delta(x) = W^*(1 \otimes x)W$, *for all* $x \in M$.

Similarly, we have:

- M *is the* σ*-strong* closure of the algebra* $\{(\omega \otimes \mathrm{id})(V)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$.
- $\Delta(x) = V(x \otimes 1)V^*$ *for all* $x \in M$.

Now, we recall the notion of dual of a locally compact quantum group $\mathbb{G} = (M, \Delta)$ as it was originally introduced and studied in [18].

Proposition-Definition 1.2.11. *Let* $\widehat{M}$ *be the* σ*-strong* closure of the algebra*

$$\{(\omega \otimes \mathrm{id})(W)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}.$$

Then, $\widehat{M}$ *is a von Neumann algebra and there exists a unique unital normal *-homomorphism* $\widehat{\Delta}_\lambda : \widehat{M} \to \widehat{M} \otimes \widehat{M}$ *such that*

$$\widehat{\Delta}_\lambda(x) = W(x \otimes 1)W^*, \text{ for all } x \in \widehat{M}.$$

The pair $(\widehat{M}, \widehat{\Delta}_\lambda)$ *is a locally compact quantum group.*

We also recall briefly the construction of the dual weight $\widehat{\varphi}$, where φ is a left invariant weight on $\mathbb{G}$. Let us denote

$$\mathscr{I}_\varphi = \{\omega \in \mathcal{B}(\mathscr{H})_*\,;\, \exists k \in \mathbb{R}_+,\, \forall x \in \mathfrak{N}_\varphi,\, |\omega(x^*)| \leqslant k\|\Lambda_\varphi(x)\|\}.$$

By Riesz' representation theorem, for all $\omega \in \mathscr{I}_\varphi$ there exists a unique $a_\varphi(\omega) \in \mathscr{H}$ such that

$$\forall x \in \mathfrak{N}_\varphi,\, \omega(x^*) = \langle \Lambda_\varphi(x),\, a_\varphi(\omega)\rangle.$$

Then, $\widehat{\varphi}$ is the unique n.s.f. weight on $\widehat{M}$ whose G.N.S. construction is $(\mathscr{H}, \mathrm{id}, \Lambda_{\widehat{\varphi}})$, where $\Lambda_{\widehat{\varphi}}$ is the closure of the operator $(\omega \otimes \mathrm{id})(W) \mapsto a_\varphi(\omega)$. The weight $\widehat{\varphi}$ is then right invariant for $(\widehat{M}, \widehat{\Delta}_\lambda)$. Starting from a right invariant weight for $\mathbb{G}$, we also define a left invariant weight for $(\widehat{M}, \widehat{\Delta}_\lambda)$.

Remark 1.2.12. For the definition of the coproduct, we have used a different convention than in [18]. As a result, we have to keep in mind the fact that the weight $\widehat{\varphi}$ is right (and not left) invariant for $(\widehat{M}, \widehat{\Delta}_\lambda)$. The notation $\widehat{\Delta}_\lambda$ will be clarified later on.

Definition 1.2.13. Let $\mathbb{G} = (M, \Delta)$ be a locally compact quantum group. We can define two new locally compact quantum groups:

1. The opposite locally compact quantum group $\mathbb{G}^{\mathrm{o}} = (M, \Delta^{\mathrm{o}})$ whose coproduct is given by $\Delta^{\mathrm{o}} = \varsigma \circ \Delta$.
2. The commutant locally compact quantum group $\mathbb{G}^{\mathrm{c}} = (M', \Delta^{\mathrm{c}})$ whose coproduct is given by $\Delta^{\mathrm{c}} = (C_M \otimes C_M) \circ \Delta \circ C_M^{-1}$.

If φ and ψ are respectively left and right invariant weights on $\mathbb{G}$, then ψ and φ are respectively left and right invariant on $\mathbb{G}^{\mathrm{o}}$. Moreover, the n.s.f. weights φ^{c} and ψ^{c} are respectively left and right invariant on $\mathbb{G}^{\mathrm{c}}$.

Starting from a locally compact quantum group $\mathbb{G} = (M, \Delta)$, we define the dual quantum group $\widehat{\mathbb{G}}$. This dual will appear better suited to *right* actions of $\mathbb{G}$.

Definition 1.2.14. Let $\mathbb{G} = (M, \Delta, \varphi, \psi)$ be a locally compact quantum group. We call the dual quantum group of $\mathbb{G}$, the locally compact quantum group

$$\widehat{\mathbb{G}} = (\widehat{M}', \widehat{\Delta}, \widehat{\psi}^{\mathrm{c}}, \widehat{\varphi}^{\mathrm{c}}),$$

where the coproduct $\widehat{\Delta}$ is given by:

$$\widehat{\Delta}(x) = V^*(1 \otimes x)V, \quad \text{for all } x \in \widehat{M}'.$$

1.3 C*-algebraic quantum groups

In this paragraph, we recall how to associate canonically a Hopf-C*-algebra to any locally compact quantum group.

Definition 1.3.1. A Hopf-C*-algebra is a couple (S, δ) consisting of a C*-algebra S and a faithful non-degenerate *-homomorphism $\delta : S \to \mathcal{M}(S \otimes S)$ such that:

1. δ is coassociative, that is $(\delta \otimes \mathrm{id}_S)\delta = (\mathrm{id}_S \otimes \delta)\delta$,
2. $[\delta(S)(S \otimes 1_S)] = S \otimes S = [\delta(S)(1_S \otimes S)]$.

The *-homomorphism δ is called the comultiplication (or the coproduct).

Remark 1.3.2. Since δ is non-degenerate, $\mathrm{id}_S \otimes \delta$ and $\delta \otimes \mathrm{id}_S$ extend uniquely to strictly continuous unital *-homormorphisms from $\mathcal{M}(S \otimes S)$ to $\mathcal{M}(S \otimes S \otimes S)$. Therefore, the coassociativity condition makes sense.

Let us fix a locally compact quantum group $\mathbb{G} = (M, \Delta)$. Let V be the right regular representation of $\mathbb{G}$. Let us denote J (resp. $\widehat{J}$) the modular conjugation of the left invariant weight on $\mathbb{G}$ (resp. $\widehat{\mathbb{G}}$).

Theorem 1.3.3. *Let S (resp. $\widehat{S}$) be the norm closure of*

$$\{(\omega \otimes \mathrm{id})(V)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H}) \quad (\textit{resp. } \{(\mathrm{id} \otimes \omega)(V)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H})).$$

Then, S (resp. $\widehat{S}$) is a C-algebra and the restriction of Δ (resp. $\widehat{\Delta}$) to S (resp. $\widehat{S}$) defines a non-degenerate *-homomorphism*

$$\delta : S \to \mathcal{M}(S \otimes S) \quad (\textit{resp. } \widehat{\delta} : \widehat{S} \to \mathcal{M}(\widehat{S} \otimes \widehat{S})).$$

Moreover, the pair (S, δ) (resp. $(\widehat{S}, \widehat{\delta})$) is a Hopf-C-algebra.*

Definition 1.3.4. We call the pair (S, δ) (resp. $(\widehat{S}, \widehat{\delta})$) the Hopf-C*-algebra (resp. dual Hopf-C*-algebra) of $\mathbb{G}$.

Note that the pair $(\widehat{S}, \widehat{\delta})$ is the Hopf-C*-algebra of $\widehat{\mathbb{G}}$. If $\mathbb{G} = (\mathrm{L}^\infty(\mathbb{G}), \Delta_{\mathbb{G}})$, we also denote $(\mathrm{C}_0(\mathbb{G}), \delta_{\mathbb{G}})$ the Hopf-C*-algebra associated with $\mathbb{G}$ by analogy with the classical case.

Notations 1.3.5. Let us denote $U := \widehat{J}J \in \mathcal{B}(\mathscr{H})$. Note that U is unitary. Moreover, we have $U^* = J\widehat{J} = \nu^{-i/4}U$, where ν is the scaling constant of $\mathbb{G}$. We can endow the C*-algebras S and $\widehat{S}$ with the following representations:

$$L : S \to \mathcal{B}(\mathscr{H})\,;\, y \mapsto y, \quad R : S \to \mathcal{B}(\mathscr{H})\,;\, y \mapsto UyU^*,$$

$$\rho : \widehat{S} \to \mathcal{B}(\mathscr{H})\,;\, x \mapsto x, \quad \lambda : \widehat{S} \to \mathcal{B}(\mathscr{H})\,;\, x \mapsto UxU^*.$$

It follows from Proposition 2.15 of [18] that

$$W = \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma, \quad [W_{12}, V_{23}] = 0.$$

We denote $\widehat{V} := \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma$ and $\widetilde{V} := \Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma = (U \otimes U)\widehat{V}(U^* \otimes U^*)$ (see Définition 6.2 of [2]). We have $W = \widehat{V}$ and $\widetilde{V}$ is the right regular representation of $\widehat{\mathbb{G}}$.

Remark 1.3.6. With the above notations, we have that the Hopf-C*-algebra associated with $(\widehat{M}, \widehat{\Delta}_\lambda)$ (see Proposition-Definition 1.2.11) is $(\lambda(\widehat{S}), \widehat{\delta}_\lambda)$, where the coproduct $\widehat{\delta}_\lambda$ is given by $\widehat{\delta}_\lambda(x) = W(x \otimes 1)W^*$, for all $x \in \lambda(\widehat{S})$.

Regularity and semi-regularity of locally compact quantum groups

The notion of regularity (resp. semi-regularity) has been introduced and studied in [2] (resp. [1]).

Notation 1.3.7. Let V be a multiplicative unitary on a Hilbert space $\mathscr{H}$. We denote $\mathcal{C}(V)$ the norm closure of the subalgebra $\{(\mathrm{id} \otimes \omega)(\Sigma V)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$ of $\mathcal{B}(\mathscr{H})$.

Proposition 1.3.8. (*see Proposition 3.2 of* [2]) *$\mathcal{C}(V)$ is a norm closed subalgebra of $\mathcal{B}(\mathscr{H})$ and the following statements are equivalent:*

1. $\mathcal{K}(\mathscr{H}) \subset \mathcal{C}(V)$ (*resp.* $\mathcal{K}(\mathscr{H}) = \mathcal{C}(V)$).
2. $\mathcal{K}(\mathscr{H} \otimes \mathscr{H}) \subset [(x \otimes 1)V(1 \otimes y)\,;\, x, y \in \mathcal{K}(\mathscr{H})]$
   (*resp.* $\mathcal{K}(\mathscr{H} \otimes \mathscr{H}) = [(x \otimes 1)V(1 \otimes y)\,;\, x, y \in \mathcal{K}(\mathscr{H})]$).

Definition 1.3.9. Let V be a multiplicative unitary on a Hilbert space $\mathscr{H}$. We say that V is semi-regular (resp. regular) if $\mathcal{K}(\mathscr{H}) \subset \mathcal{C}(V)$ (resp. $\mathcal{K}(\mathscr{H}) = \mathcal{C}(V)$). A locally compact quantum group is said to be semi-regular (resp. regular) if its right regular representation (or equivalently its left regular representation) is semi-regular (resp. regular).

We finish with a result concerning the regularity of a corepresentation of a regular multiplicative unitary. Let us recall a definition:

Definition 1.3.10. (see Définition A.1 in [2]) Let V be a multiplicative unitary on a Hilbert space $\mathscr{H}$. A corepresentation of V on a Hilbert space $\mathscr{K}$ is a unitary $Y \in \mathcal{B}(\mathscr{H} \otimes \mathscr{K})$ such that

$$V_{12}Y_{13}Y_{23} = Y_{23}V_{12}.$$

In the following proposition, if V is a regular multiplicative unitary on a Hilbert space $\mathscr{H}$ we denote $\rho_V(\omega) := (\mathrm{id} \otimes \omega)(V)$ for $\omega \in \mathcal{B}(\mathscr{H})_*$. We also denote $\widehat{S}_V$ the norm closure of $\{\rho_V(\omega)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$. Recall that $\widehat{S}_V$ is a C*-algebra (see Proposition 3.5 in [2]).

Proposition 1.3.11. (*Proposition A.3 and Remarque A.4 of* [2]) *Let V be a regular multiplicative unitary on a Hilbert space $\mathscr{H}$ and Y a corepresentation of V on a Hilbert space $\mathscr{K}$. Then, we have*

$$[\,(\widehat{S}_V \otimes 1_{\mathscr{K}})Y(1_{\mathscr{H}} \otimes \mathcal{K}(\mathscr{K}))\,] = \widehat{S}_V \otimes \mathcal{K}(\mathscr{K}).$$

1.4 Actions of locally compact quantum groups

Definition 1.4.1. Let (S, δ) be a Hopf-C*-algebra. A (right) coaction of (S, δ) on a C*-algebra A is a non-degenerate *-homomorphism $\delta_A : A \to \mathcal{M}(A \otimes S)$ such that

$$(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A.$$

Moreover, the coaction δ_A is said to be (strongly) continuous if it further satisfies

$$[\delta_A(A)(1_A \otimes S)] = A \otimes S.$$

A (S, δ)-C*-algebra is a couple (A, δ_A) consisting of a C*-algebra A and an injective continuous coaction δ_A of (S, δ) on A.

Remarks 1.4.2. 1. Since δ_A and δ are non-degenerate, $\mathrm{id}_A \otimes \delta$ and $\delta_A \otimes \mathrm{id}_S$ extend uniquely to strictly continuous unital *-homomorphisms from $\mathcal{M}(A \otimes S)$ to $\mathcal{M}(A \otimes S \otimes S)$. Then, the condition $(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$ makes sense and we will consider the following *-homomorphism $\delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A : A \to \mathcal{M}(A \otimes S \otimes S)$.

2. We can define left coactions in a similar way. Note that we define a one-to-one correspondence by mapping any left coaction of (S, δ) to a right coaction of the opposite Hopf-C*-algebra $(S, \varsigma \circ \delta)$, where $\varsigma : S \otimes S \to S \otimes S\,;\, a \otimes b \mapsto b \otimes a$ is the flip map. In the following, we will refer to right coactions simply as coactions.

Examples 1.4.3. Let us give two basic examples:

1. Let A be a C*-algebra and (S, δ) a Hopf-C*-algebra. Let us consider the *-homomorphism $\delta_A : A \to \mathcal{M}(A \otimes S)$ given by $\delta_A(a) = a \otimes 1_S$ for all $a \in A$. Then, δ_A is a coaction called the trivial coaction of (S, δ) on A. Besides, the trivial coaction δ_A turns A into a (S, δ)-C*-algebra.

2. If (S, δ) is a Hopf-C*-algebra, the map $\delta : S \to \mathcal{M}(S \otimes S)$ is a right (or left) coaction of (S, δ) called the right (or left) regular coaction.

Definition 1.4.4. Let A and B be two C*-algebras. Let δ_A and δ_B be two coactions of (S, δ) on A and B respectively. A non-degenerate *-homomorphism $f : A \to \mathcal{M}(B)$ is said to be (S, δ)-equivariant if we have $(f \otimes \mathrm{id}_S) \circ \delta_A = \delta_B \circ f$.

Remarks 1.4.5. 1. Note that the condition $(f \otimes \mathrm{id}_S) \circ \delta_A = \delta_B \circ f$ in the above definition does make sense. Indeed, since f is non-degenerate, $f \otimes \mathrm{id}_S$ extends uniquely to a strictly continuous unital *-homomorphism $f \otimes \mathrm{id}_S : \mathcal{M}(A \otimes S) \to \mathcal{M}(B \otimes S)$. Besides, δ_B extends uniquely to a strictly continuous unital *-homomorphism $\delta_B : \mathcal{M}(B) \to \mathcal{M}(B \otimes S)$.

2. It is clear that the class of all (S, δ)-C*-algebras for a given Hopf-C*-algebra (S, δ) is a category with respect to (S, δ)-equivariant non-degenerate *-homomorphisms.

Definition 1.4.6. Let $\mathbb{G}$ be a locally compact quantum group and A a C*-algebra. We will refer to coactions of $(\mathrm{C}_0(\mathbb{G}), \delta_{\mathbb{G}})$ as actions of $\mathbb{G}$. Moreover, we will also refer to $(\mathrm{C}_0(\mathbb{G}), \delta_{\mathbb{G}})$-C*-algebras as $\mathbb{G}$-C*-algebras.

Definition 1.4.7. Let (A, δ_A) and (B, δ_B) be two $\mathbb{G}$-C*-algebras. Let $f : A \to \mathcal{M}(B)$ be a non-degenerate *-homomorphism. We say that f is $\mathbb{G}$-equivariant when it is $(\mathrm{C}_0(\mathbb{G}), \delta_{\mathbb{G}})$-equivariant. We will denote $\mathrm{Mor}_{\mathbb{G}}(A, B)$ the set of all $\mathbb{G}$-equivariant non-degenerate *-homomorphisms from A to $\mathcal{M}(B)$ and $\mathbb{G}$-$\mathsf{C^*\text{-}Alg}$ the category of $\mathbb{G}$-C*-algebras.

1.5 Crossed product and Baaj-Skandalis duality

Let $\mathbb{G}$ be a locally compact quantum group. Let (S, δ) (resp. $(\widehat{S}, \widehat{\delta})$) be the Hopf-C*-algebra associated with $\mathbb{G}$ (resp. $\widehat{\mathbb{G}}$). We denote $L : S \to \mathcal{B}(\mathscr{H})$ and $\rho : \widehat{S} \to \mathcal{B}(\mathscr{H})$ the canonical inclusion maps (see Notations 1.3.5). Since S and $\widehat{S}$ are non-degenerate C*-subalgebras of $\mathcal{B}(\mathscr{H})$, L and ρ extend uniquely to strictly continuous unital *-homomorphisms

$$L : \mathcal{M}(S) \to \mathcal{B}(\mathscr{H}), \quad \rho : \mathcal{M}(\widehat{S}) \to \mathcal{B}(\mathscr{H}).$$

Let A be a C*-algebra. The *-homomorphism $\mathrm{id}_A \otimes L : A \otimes S \to \mathcal{L}(A \otimes \mathscr{H})$ extends uniquely to a strictly continuous unital *-homomrphism $\mathrm{id}_A \otimes L : \mathcal{M}(A \otimes S) \to \mathcal{L}(A \otimes \mathscr{H})$, where we

have used the canonical embedding $A \otimes \mathcal{B}(\mathscr{H}) \hookrightarrow \mathcal{L}(A \otimes \mathscr{H})$. Let δ_A be an action of $\mathbb{G}$ on A. We define the following *-representation of A on the Hilbert A-module $A \otimes \mathscr{H}$:

$$\pi_L := (\mathrm{id}_A \otimes L) \circ \delta_A : A \to \mathcal{L}(A \otimes \mathscr{H}).$$

Proposition-Definition 1.5.1. *Let $A \rtimes_{\delta_A} \mathbb{G}$ (or simply $A \rtimes \mathbb{G}$ when no ambiguity can arise) the norm closure of the linear subspace of $\mathcal{L}(A \otimes \mathscr{H})$ spanned by all products of the form*

$$\pi_L(a)(1_A \otimes \rho(\omega)), \quad a \in A,\, \omega \in \mathcal{B}(\mathscr{H})_*.$$

Therefore, $A \rtimes_{\delta_A} \mathbb{G}$ is a C-algebra called the crossed product of A by the action δ_A of $\mathbb{G}$.*

In particular, π_L induces a non-degenerate *-homomorphism $\pi : A \to \mathcal{M}(A \rtimes \mathbb{G})$. Similarly, ρ induces a *-homomorphism $\widehat{\theta} : \widehat{S} \to \mathcal{M}(A \rtimes \mathbb{G})$ given by $\widehat{\theta}(x) = 1_A \otimes \rho(x)$, for all $x \in \widehat{S}$. Besides, we have

$$A \rtimes \mathbb{G} = [\, \pi(a)\widehat{\theta}(x)\,;\, a \in A,\, x \in \widehat{S}\,].$$

Proposition-Definition 1.5.2. *Let δ_A be an action of $\mathbb{G}$ on a C*-algebra A. There exists a unique *-homomorphism $\delta_{A\rtimes\mathbb{G}} : A \rtimes \mathbb{G} \to \mathcal{M}((A \rtimes \mathbb{G}) \otimes \widehat{S})$ given for all $a \in A$ and $x \in \widehat{S}$ by*

$$\delta_{A\rtimes\mathbb{G}}(\pi(a)\widehat{\theta}(x)) = (\pi(a) \otimes 1_{\widehat{S}})(\widehat{\theta} \otimes \mathrm{id}) \circ \widehat{\delta}(x).$$

Then, $\delta_{A\rtimes\mathbb{G}}$ is an action of $\widehat{\mathbb{G}}$ on $A \rtimes \mathbb{G}$ called the dual action. Moreover, the pair $(A \rtimes \mathbb{G}, \delta_{A\rtimes\mathbb{G}})$ is a $\widehat{\mathbb{G}}$-C-algebra.*

Let us assume that $\mathbb{G}$ is regular. Let (A, δ_A) be a $\mathbb{G}$-C*-algebra. Let δ_0 be the action of $\mathbb{G}$ on $A \otimes \mathcal{K}(\mathscr{H})$ given by

$$\delta_0(a \otimes k) = \delta_A(a)_{13}(1_A \otimes k \otimes 1_S),$$

for all $a \in A$ and $k \in \mathcal{K}(\mathscr{H})$. Let us denote $V_0 \in \mathcal{M}(\widehat{S} \otimes S)$ such that $(\rho \otimes L)(V_0) = V$. Let us then define the unitary $\mathcal{V} = (\rho \otimes \mathrm{id}_S)(V_0) \in \mathcal{L}(\mathscr{H} \otimes S)$. Let $\delta'_A : A \otimes \mathcal{K}(\mathscr{H}) \to \mathcal{M}(A \otimes \mathcal{K}(\mathscr{H}) \otimes S)$ be the *-homomorphism given by

$$\delta'_A(x) = \mathcal{V}_{23}\delta_0(x)\mathcal{V}_{23}^*, \quad x \in A \otimes \mathcal{K}(\mathscr{H}).$$

Therefore, $(A \otimes \mathcal{K}(\mathscr{H}), \delta'_A)$ is a $\mathbb{G}$-C*-algebra.

Theorem 1.5.3. (*Baaj-Skandalis duality theorem*) *Let $\mathbb{G}$ be a regular locally compact quantum group. Let (A, δ_A) be a $\mathbb{G}$-C*-algebra, then the double crossed product $(A \rtimes \mathbb{G}) \rtimes \widehat{\mathbb{G}}$, endowed with the action $\delta_{(A\rtimes\mathbb{G})\rtimes\widehat{\mathbb{G}}}$ of $\mathbb{G}$, is canonically $\mathbb{G}$-equivariantly isomorphic to the $\mathbb{G}$-C*-algebra $(A \otimes \mathcal{K}(\mathscr{H}), \delta'_A)$.*

We will now give some definitions and results concerning actions of locally compact quantum groups in the von Neumann algebraic setting from the seminal paper [23]. Indeed, this will be necessary in §2.5 in order to recall the crucial results of De Commer [9] concerning the so-called reflection technique. Let us fix a locally compact quantum group $\mathbb{G} = (M, \Delta)$ and a left invariant n.s.f. weight φ on $\mathbb{G}$.

Definition 1.5.4. A right (resp. left) action of $\mathbb{G}$ on a von Neumann algebra N is a faithful normal unital *-homomorphism $\alpha : N \to N \otimes M$ (resp. $\alpha : N \to M \otimes N$) such that

$$(\alpha \otimes \mathrm{id}_M)\alpha = (\mathrm{id}_N \otimes \Delta)\alpha \quad (\text{resp. } (\mathrm{id}_M \otimes \alpha)\alpha = (\Delta \otimes \mathrm{id}_N)\alpha).$$

Note that if α is a left (resp. right) action of $\mathbb{G}$ on a von Neumann algebra N, then $\varsigma \circ \alpha$ is a right (resp. left) action of $\mathbb{G}^{\mathrm{o}}$. In what follows, we will refer to actions of $\mathbb{G}$ as right actions of $\mathbb{G}$.

Definition 1.5.5. Let α be an action of $\mathbb{G}$ on a von Neumann algebra N. We call the von Neumann algebra

$$N^{\alpha} := \{x \in N\,;\, \alpha(x) = x \otimes 1\} \subset N,$$

the algebra of α-invariant elements of N. The action α is said to be ergodic if N^{α} is reduced to the scalars, that is $N^{\alpha} = \mathbb{C} \cdot 1_N$.

Proposition-Definition 1.5.6. *Let N be a von Neumann algebra. Let α be an action of $\mathbb{G}$ on N. For $x \in N_+$, we denote*

$$T_{\alpha}(x) = (\mathrm{id} \otimes \varphi)\alpha(x) \in N_+^{\mathrm{ext}}.$$

Then, we have $T_{\alpha}(x) \in (N^{\alpha})_+^{\mathrm{ext}}$ and $T_{\alpha} : N_+ \to (N^{\alpha})_+^{\mathrm{ext}}$ is a normal faithful operator-valued weight from N to N^{α}. We say that the action α is integrable if T_{α} is semi-finite, which means that $\mathfrak{M}_{\mathrm{id}\otimes\varphi} \cap \alpha(N)$ is σ-weakly dense in $\alpha(N)$.

Now, we recall the crossed product construction in this setting.

Proposition-Definition 1.5.7. *Let α be an action of $\mathbb{G}$ on a von Neumann algebra N. Then, the crossed product $N \rtimes_{\alpha} \mathbb{G}$ of N by α is the σ-weak closure in $N \otimes \mathcal{B}(\mathscr{H})$ of the linear span of $\{\alpha(x)(1 \otimes y)\,;\, x \in N,\, y \in \widehat{M}'\}$. Then, $N \rtimes_{\alpha} \mathbb{G}$ is a von Neumann algebra.*

Note that we have $N \rtimes_{\alpha} \mathbb{G} = (\alpha(N) \cup (1 \otimes \widehat{M}'))''$. Now, we give the definition of dual action, which is an action of $\widehat{\mathbb{G}}$ on the crossed product $N \rtimes_{\alpha} \mathbb{G}$.

Proposition-Definition 1.5.8. *There exists a unique action $\widehat{\alpha}$ of $\widehat{\mathbb{G}}$ on $N \rtimes_{\alpha} \mathbb{G}$ such that:*

$$\widehat{\alpha}(\alpha(x)) = \alpha(x) \otimes 1, \quad x \in N \quad ; \quad \widehat{\alpha}(1 \otimes y) = 1 \otimes \widehat{\Delta}(y), \quad y \in \widehat{M}'.$$

We call $\widehat{\alpha}$ the dual action of α.

A Takesaki-Takai duality theorem is given in [23] (see Theorem 2.6).

1.6 Equivariant Hilbert modules and bimodules

In the following, we briefly recall the biduality for crossed products of equivariant Hilbert bimodules (*cf.* [2] and [3]).

Let $\mathbb{G}$ be a locally compact quantum group. Let (S, δ) be the Hopf-C*-algebra associated with $\mathbb{G}$. Let (A, δ_A) and (B, δ_B) be two $\mathbb{G}$-C*-algebras. A $\mathbb{G}$-equivariant Hilbert A-B-bimodule is a Hilbert A-B-bimodule $\mathscr{E}$ such that the C*-algebra $\mathcal{K}(\mathscr{E} \oplus B)$ is endowed with a continuous action

$$\delta_{\mathcal{K}(\mathscr{E}\oplus B)} : \mathcal{K}(\mathscr{E} \oplus B) \to \mathcal{M}(\mathcal{K}(\mathscr{E} \oplus B) \otimes S)$$

of $\mathbb{G}$ compatible with δ_A and δ_B in the following sense:

- The canonical *-homomorphism $\iota_B : B \to \mathcal{K}(\mathscr{E} \oplus B)$ is $\mathbb{G}$-equivariant.
- The left action of A on $\mathscr{E}$, $A \to \mathcal{L}(\mathscr{E})$, is $\mathbb{G}$-equivariant.

We denote $\delta_{\mathscr{E}}$ the restriction of $\delta_{\mathcal{K}(\mathscr{E}\oplus B)}$ to $\mathscr{E}$ (see [3]).

Definition 1.6.1. A linking $\mathbb{G}$-C*-algebra (or a $\mathbb{G}$-equivariant Morita equivalence) is a quintuple (J, δ_J, e_1, e_2) consisting of a $\mathbb{G}$-C*-algebra (J, δ_J) endowed with two nonzero self-adjoint projections $e_1, e_2 \in \mathcal{M}(J)$ such that:

1. $e_1 + e_2 = 1_J$.
2. $[Je_iJ] = J$, $i = 1, 2$.
3. $\delta_J(e_i) = e_i \otimes 1_S$, $i = 1, 2$.

Remark 1.6.2. If (J, δ_J, e_1, e_2) is a linking $\mathbb{G}$-C*-algebra, then by restriction of δ_J, e_1Je_1 and e_2Je_2 are $\mathbb{G}$-C*-algebras and e_1Je_2 is a $\mathbb{G}$-equivariant Hilbert e_1Je_1-e_2Je_2-bimodule (*cf.* [3] page 706). We say that two $\mathbb{G}$-C*-algebras A and B are $\mathbb{G}$-equivariantly Morita equivalent if there exists a linking $\mathbb{G}$-C*-algebra (J, δ_J, e_1, e_2) such that A (resp. B) and e_1Je_1 (resp. e_2Je_2) are isomorphic as $\mathbb{G}$-C*-algebras.

Assume that $\mathbb{G}$ is regular. It follows from §7 [2] that:

1. The $\mathbb{G}$-equivariant Hilbert $A \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}}$-$B \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}}$-bimodule $\mathscr{E} \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}}$ is identified to the $\mathbb{G}$-equivariant Hilbert $A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$-$B \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$-bimodule $\mathscr{E} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$.
2. The $\mathbb{G}$-C*-algebras $A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$ and $B \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$ are endowed with the bidual action of $\mathbb{G}$. The action of $\mathbb{G}$ on the Hilbert $B \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$-module $\mathscr{E} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$ is given by the action $\delta_{\mathscr{E}\otimes\mathcal{K}(\mathrm{L}^2(\mathbb{G}))}$ defined by:

$$\delta_{\mathscr{E}\otimes\mathcal{K}(\mathrm{L}^2(\mathbb{G}))}(\xi) = V_{23}(\mathrm{id} \otimes \sigma)(\delta_{\mathscr{E}} \otimes \mathrm{id}_{\mathcal{K}(\mathrm{L}^2(\mathbb{G}))})(\xi)V_{23}^*, \quad \xi \in \mathscr{E} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})),$$

where $\sigma : S \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))) \to \mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes S$ is the flip map and $V \in \mathcal{M}(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes S)$ is the right regular representation of $\mathbb{G}$.

Chapter 2

Measured quantum groupoids: reminders and supplements

2.1 Preliminaries

In this paragraph, we will recall the definitions, notations and important results concerning the relative tensor product (also known as Connes-Sauvageot tensor product) and the fiber product which are the main technical tools of the theory of measured quantum groupoids. For more information, we refer the reader to [6].

Relative tensor product. In what follows, N is a von Neumann algebra endowed with a n.s.f. weight φ. Let π (resp. γ) be a normal unital *-representation of N (resp. N^{o}) on a Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$). Then, $\mathcal{H}$ (resp. $\mathcal{K}$) may be considered as a left (resp. right) N-module. Moreover, $\mathscr{H}_\varphi$ is an N-bimodule whose actions are given by:

$$x\xi = \pi_\varphi(x)\xi, \quad \xi y = J_\varphi \pi_\varphi(y^*) J_\varphi \xi, \quad \xi \in \mathscr{H}_\varphi,\ x,\ y \in N.$$

Definition 2.1.1. We define the set of right (resp. left) bounded vectors with respect to φ and π (resp. γ) to be:

$$_\varphi(\pi, \mathcal{H}) = \{\xi \in \mathcal{H}\,;\, \exists\, C \in \mathbb{R}_+,\ \forall\, x \in \mathfrak{N}_\varphi,\ \|\pi(x)\xi\| \leqslant C\|\Lambda_\varphi(x)\|\},$$

$$(\text{resp. } (\mathcal{K}, \gamma)_\varphi = \{\xi \in \mathcal{K}\,;\, \exists\, C \in \mathbb{R}_+,\ \forall\, x \in \mathfrak{N}_\varphi^*,\ \|\gamma(x^{\mathrm{o}})\xi\| \leqslant C\|\Lambda_{\varphi^{\mathrm{o}}}(x^{\mathrm{o}})\|\}).$$

If $\xi \in {}_\varphi(\pi, \mathcal{H})$, we then denote $R^{\pi,\varphi}(\xi) \in \mathcal{B}(\mathscr{H}_\varphi, \mathcal{H})$ the unique bounded operator such that

$$R^{\pi,\varphi}(\xi)\Lambda_\varphi(x) = \pi(x)\xi, \text{ for all } x \in \mathfrak{N}_\varphi.$$

Similarly, if $\xi \in (\mathcal{K}, \gamma)_\varphi$ we denote $L^{\gamma,\varphi}(\xi) \in \mathcal{B}(\mathscr{H}_\varphi, \mathcal{K})$ the unique bounded operator such that

$$L^{\gamma,\varphi}(\xi) J_\varphi \Lambda_\varphi(x^*) = \gamma(x^{\mathrm{o}})\xi, \text{ for all } x \in \mathfrak{N}_\varphi^*,$$

where we have used the identification $\mathscr{H}_{\varphi^{\mathrm{o}}} \to \mathscr{H}_\varphi\,;\, \Lambda_{\varphi^{\mathrm{o}}}(x^{\mathrm{o}}) \mapsto J_\varphi \Lambda_\varphi(x^*)$.

Note that $\xi \in \mathcal{K}$ is left bounded with respect to φ and γ if and only if it is right bounded with respect to the n.s.f. weight φ^{c} on N' and the normal unital *-representation $\gamma^{\mathrm{c}} = \gamma \circ j_N^{-1}$ of N'. It is important to note that $(\mathcal{K}, \gamma)_\varphi$ (resp. ${}_\varphi(\pi, \mathcal{H})$) is dense in $\mathcal{K}$ (resp. $\mathcal{H}$) (see Lemma 2 in [6]).

If $\xi \in {}_\varphi(\pi, \mathcal{H})$ (resp. $\xi \in (\mathcal{K}, \gamma)_\varphi$), we have that $R^{\pi,\varphi}(\xi)$ (resp. $L^{\gamma,\varphi}(\xi)$) is left (resp. right) N-linear. Therefore, for all $\xi, \eta \in {}_\varphi(\pi, \mathcal{H})$ (resp. $(\mathcal{K}, \gamma)_\varphi$) we have

$$R^{\pi,\varphi}(\xi)^* R^{\pi,\varphi}(\eta) \in \pi_\varphi(N)' = j_N(N^{\mathrm{o}}), \quad R^{\pi,\varphi}(\xi) R^{\pi,\varphi}(\eta)^* \in \pi(N)'$$

$$(\text{resp. } L^{\gamma,\varphi}(\xi)^* L^{\gamma,\varphi}(\eta) \in \pi_\varphi(N), \quad L^{\gamma,\varphi}(\xi) L^{\gamma,\varphi}(\eta)^* \in \gamma(N^{\mathrm{o}})').$$

Notations 2.1.2. Let $\xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})$ (resp. $(\mathcal{K}, \gamma)_{\varphi}$), we denote:

$$\langle \xi,\, \eta \rangle_{N^{\mathrm{o}}} = j_N^{-1}(R^{\pi,\varphi}(\xi)^* R^{\pi,\varphi}(\eta)) \in N^{\mathrm{o}} \quad (\text{resp. } \langle \xi,\, \eta \rangle_N = \pi_{\varphi}^{-1}(L^{\gamma,\varphi}(\xi)^* L^{\gamma,\varphi}(\eta)) \in N).$$

Proposition 2.1.3. *For all $\xi, \eta \in {}_{\varphi}(\pi, \mathcal{H})$ (resp. $\xi, \eta \in (\mathcal{K}, \gamma)_{\varphi}$) and $y \in N$ analytic for $(\sigma_t^{\varphi})_{t \in \mathbb{R}}$, we have:*

1. $\langle \xi, \eta \rangle_{N^{\mathrm{o}}}^* = \langle \eta, \xi \rangle_{N^{\mathrm{o}}} \quad (\text{resp. } \langle \xi, \eta \rangle_N^* = \langle \eta, \xi \rangle_N).$
2. $\langle \xi, \eta y^{\mathrm{o}} \rangle_{N^{\mathrm{o}}} = \langle \xi, \eta \rangle_{N^{\mathrm{o}}} \sigma_{i/2}^{\varphi}(y)^{\mathrm{o}} \quad (\text{resp. } \langle \xi, \eta y \rangle_N = \langle \xi, \eta \rangle_N \sigma_{-i/2}^{\varphi}(y)).$

Lemma 2.1.4. *For all $\xi_1,\, \xi_2 \in {}_{\varphi}(\pi, \mathcal{H})$ and $\eta_1,\, \eta_2 \in (\mathcal{K}, \gamma)_{\varphi}$, we have*

$$\langle \eta_1,\, \gamma(\langle \xi_1,\, \xi_2 \rangle_{N^{\mathrm{o}}})\eta_2 \rangle_{\mathcal{K}} = \langle \xi_1, \pi(\langle \eta_1,\, \eta_2 \rangle_N)\xi_2 \rangle_{\mathcal{H}}.$$

Definition 2.1.5. The relative tensor product

$$\mathcal{K} \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \mathcal{H} \quad (\text{or simply denoted } \mathcal{K} \,{}_{\gamma}\otimes_{\pi}\, \mathcal{H})$$

is the Hausdorff completion of the pre-Hilbert space $(\mathcal{K}, \gamma)_{\varphi} \odot {}_{\varphi}(\pi, \mathcal{H})$, whose inner product is given by

$$\langle \xi_1 \otimes \eta_1,\, \xi_2 \otimes \eta_2 \rangle := \langle \eta_1,\, \gamma(\langle \xi_1,\, \xi_2 \rangle_{N^{\mathrm{o}}})\eta_2 \rangle_{\mathcal{K}} = \langle \xi_1,\, \pi(\langle \eta_1,\, \eta_2 \rangle_N)\xi_2 \rangle_{\mathcal{H}},$$

for all $\xi_1,\, \xi_2 \in (\mathcal{K}, \gamma)_{\varphi}$ and $\eta_1, \eta_2 \in {}_{\varphi}(\pi, \mathcal{H})$. If $\xi \in (\mathcal{K}, \gamma)_{\varphi}$ and $\eta \in {}_{\varphi}(\pi, \mathcal{H})$, we will denote

$$\xi \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \eta \quad (\text{or simply denoted } \xi \,{}_{\gamma}\otimes_{\pi}\, \eta)$$

the image of $\xi \otimes \eta$ by the canonical map $(\mathcal{K}, \gamma)_{\varphi} \odot {}_{\varphi}(\pi, \mathcal{H}) \to \mathcal{K} \,{}_{\gamma}\otimes_{\pi}\, \mathcal{H}$ (isometric dense range).

Remarks 2.1.6.

1. By applying this construction to $(N^{\mathrm{o}}, \varphi^{\mathrm{o}})$ instead of (N, φ) we obtain the relative tensor product $\mathcal{H} \underset{\varphi^{\mathrm{o}}}{{}_{\pi}\otimes_{\gamma}} \mathcal{K}$.
2. The relative tensor product $\mathcal{K} \,{}_{\gamma}\otimes_{\pi}\, \mathcal{H}$ is also the Hausdorff completion of the pre-Hilbert space $\mathcal{K} \odot {}_{\varphi}(\pi, \mathcal{H})$ (resp. $(\mathcal{K}, \gamma)_{\varphi} \odot \mathcal{H}$), whose inner product is given by:

$$\langle \xi_1 \otimes \eta_1,\, \xi_2 \otimes \eta_2 \rangle := \langle \xi_1,\, \pi(\langle \eta_1,\, \eta_2 \rangle_N)\xi_2 \rangle_{\mathcal{H}}$$

$$(\text{resp. } \langle \xi_1 \otimes \eta_1,\, \xi_2 \otimes \eta_2 \rangle := \langle \eta_1,\, \gamma(\langle \xi_1,\, \xi_2 \rangle_{N^{\mathrm{o}}})\eta_2 \rangle_{\mathcal{K}}).$$

3. Moreover, for all $\xi \in \mathcal{K}$, $\eta \in {}_{\varphi}(\pi, \mathcal{H})$ and $y \in N$ analytic for $(\sigma_t^{\varphi})_{t \in \mathbb{R}}$, we have

$$\gamma(y^{\mathrm{o}})\xi \,{}_{\gamma}\otimes_{\pi}\, \eta = \xi \,{}_{\gamma}\otimes_{\pi}\, \pi(\sigma_{-i/2}^{\varphi}(y))\eta.$$

The relative flip map is the isomorphism $\sigma_{\varphi}^{\gamma,\pi}$ from $\mathcal{K} \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \mathcal{H}$ onto $\mathcal{H} \underset{\varphi^{\mathrm{o}}}{{}_{\pi}\otimes_{\gamma}} \mathcal{K}$ given by:

$$\sigma_{\varphi}^{\gamma,\pi}(\xi \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \eta) := \eta \underset{\varphi^{\mathrm{o}}}{{}_{\pi}\otimes_{\gamma}} \xi, \quad \xi \in (\mathcal{K}, \gamma)_{\varphi},\, \eta \in {}_{\varphi}(\pi, \mathcal{H}) \quad (\text{or simply } \sigma^{\gamma,\pi}).$$

Note that $\sigma_{\varphi}^{\gamma,\pi}$ is unitary and $(\sigma_{\varphi}^{\gamma,\pi})^* = \sigma_{\varphi^{\mathrm{o}}}^{\pi,\gamma}$. Then, we can define a relative flip *-homomorphism

$$\varsigma_{\varphi}^{\gamma,\pi} : \mathcal{B}(\mathcal{K} \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \mathcal{H}) \to \mathcal{B}(\mathcal{H} \underset{\varphi^{\mathrm{o}}}{{}_{\pi}\otimes_{\gamma}} \mathcal{K})$$

(or simply denoted $\varsigma^{\gamma,\pi}$) by setting:

$$\varsigma_{\varphi}^{\gamma,\pi}(X) := \sigma_{\varphi}^{\gamma,\pi} X (\sigma_{\varphi}^{\gamma,\pi})^*, \quad X \in \mathcal{B}(\mathcal{K} \underset{\varphi}{{}_{\gamma}\otimes_{\pi}} \mathcal{H}).$$

Fiber product of von Neumann algebras. We continue to use the notations of the previous paragraph.

Proposition-Definition 2.1.7. *Let $\mathcal{K}_i$ and $\mathcal{H}_i$ be Hilbert spaces, and $\gamma_i : N^{\rm o} \to \mathcal{B}(\mathcal{K}_i)$ and $\pi_i : N \to \mathcal{B}(\mathcal{H}_i)$ be unital normal *-homomorphisms for $i = 1, 2$. Let $T \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ and $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that*

$$T \circ \gamma_1(n^{\rm o}) = \gamma_2(n^{\rm o}) \circ T, \quad S \circ \pi_1(n) = \pi_2(n) \circ S, \quad n \in N.$$

Then, the linear map

$$\begin{array}{rcl} (\mathcal{K}_1, \gamma_1)_\varphi \odot {}_\varphi(\pi_1, \mathcal{H}_1) & \longrightarrow & \mathcal{K}_2 \,{}_{\gamma_2}\!\otimes_{\pi_2} \mathcal{H}_2 \\ \xi \odot \eta & \longmapsto & T\xi \,{}_{\gamma_2}\!\otimes_{\pi_2} S\eta \end{array}$$

extends uniquely to a bounded operator

$${}_{\gamma_2}T \,{}_{\gamma_1}\!\otimes_{\pi_2} S_{\pi_1} \in \mathcal{B}(\mathcal{K}_1 \,{}_{\gamma_1}\!\otimes_{\pi_1} \mathcal{H}_1, \mathcal{K}_2 \,{}_{\gamma_2}\!\otimes_{\pi_2} \mathcal{H}_2) \quad (\textit{or simply denoted } T \,{}_{\gamma_1}\!\otimes_{\pi_2} S),$$

whose adjoint operator is ${}_{\gamma_1}T^* \,{}_{\gamma_2}\!\otimes_{\pi_1} S^*{}_{\pi_2}$ (or simply $T^* \,{}_{\gamma_2}\!\otimes_{\pi_1} S^*$). In particular, let $x \in \gamma(N^{\rm o})'$ and $y \in \pi(N)'$, then the linear map

$$\begin{array}{rcl} (\mathcal{K}, \gamma)_\varphi \odot {}_\varphi(\pi, \mathcal{H}) & \longrightarrow & \mathcal{K} \,{}_{\gamma}\!\otimes_{\pi} \mathcal{H} \\ \xi \odot \eta & \longmapsto & x\xi \,{}_{\gamma}\!\otimes_{\pi} y\eta \end{array}$$

extends uniquely to a bounded operator on $\mathcal{K} \,{}_{\gamma}\!\otimes_{\pi} \mathcal{H}$, denoted $x \,{}_{\gamma}\!\otimes_{\pi} y \in \mathcal{B}(\mathcal{K} \,{}_{\gamma}\!\otimes_{\pi} \mathcal{H})$.

Remark 2.1.8. With the notations of Proposition-Definition 2.1.7, let $T : \mathcal{K}_1 \to \mathcal{H}_2$ and $S : \mathcal{H}_1 \to \mathcal{K}_2$ be bounded antilinear maps such that

$$T \circ \gamma_1(n^{\rm o})^* = \pi_2(n) \circ T, \quad S \circ \pi_1(n) = \gamma_2(n^{\rm o})^* \circ S, \quad n \in N.$$

In a similar way, we define the operator

$${}_{\pi_2}T \,{}_{\gamma_1}\!\otimes_{\gamma_2} S_{\pi_1} \in \mathcal{B}(\mathcal{K}_1 \,{}_{\gamma_1}\!\otimes_{\pi_1} \mathcal{H}_1, \mathcal{H}_2 \,{}_{\pi_2}\!\otimes_{\gamma_2} \mathcal{K}_2), \quad (\text{or simply denoted } T \,{}_{\gamma_1}\!\otimes_{\gamma_2} S),$$

whose adjoint operator is ${}_{\gamma_1}T^* \,{}_{\pi_2}\!\otimes_{\pi_1} S^*{}_{\gamma_2}$ (or simply $T^* \,{}_{\pi_2}\!\otimes_{\pi_1} S^*$). Note that these notations are different from those used in [14, 19].

Let $M \subset \mathcal{B}(\mathcal{K})$ and $P \subset \mathcal{B}(\mathcal{H})$ be two von Neumann algebras. Let us assume that $\pi(N) \subset P$ and $\gamma(N^{\rm o}) \subset M$.

Definition 2.1.9. The fiber product

$$M \underset{N}{{}_{\gamma}\!\star_{\pi}} P \quad (\text{or simply denoted } M \,{}_{\gamma}\!\star_{\pi} P)$$

of M and P over N is the commutant of

$$\{x \,{}_{\gamma}\!\otimes_{\pi} y \,;\, x \in M',\, y \in P'\} \subset \mathcal{B}(\mathcal{K} \,{}_{\gamma}\!\otimes_{\pi} \mathcal{H}).$$

Then, $M \,{}_{\gamma}\!\star_{\pi} P$ is a von Neumann algebra. We also denote $M' \,{}_{\gamma}\!\otimes_{\pi} P'$ the von Neumann algebra generated by the set $\{x \,{}_{\gamma}\!\otimes_{\pi} y \,|\, x \in M',\, y \in P'\}$, called the relative tensor product of M' and P'. We have

$$M \,{}_{\gamma}\!\star_{\pi} P = (M' \,{}_{\gamma}\!\otimes_{\pi} P')'.$$

Note that we have $\varsigma^{\gamma,\pi}(M\,{}_\gamma\!\star_\pi P) = P\,{}_\pi\!\star_\gamma M$. We still denote $\varsigma^{\gamma,\pi} : M\,{}_\gamma\!\star_\pi P \to P\,{}_\pi\!\star_\gamma M$ the restriction of $\varsigma^{\gamma,\pi}$ to $M\,{}_\gamma\!\star_\pi P$.

$\lozenge$ *Slicing with normal linear forms.* Now, let us recall how to slice with normal linear forms. For $\xi \in (\mathcal{K},\gamma)_\varphi, \eta \in {}_\varphi(\pi,\mathcal{H})$, we consider the following bounded linear maps

$$\begin{array}{rccl} \lambda_\xi^{\gamma,\pi} : & \mathcal{H} & \to & \mathcal{K}\,{}_\gamma\!\otimes_\pi \mathcal{H}, \\ & \zeta & \mapsto & \xi\,{}_\gamma\!\otimes_\pi \zeta \end{array} \qquad \begin{array}{rccl} \rho_\eta^{\gamma,\pi} : & \mathcal{K} & \to & \mathcal{K}\,{}_\gamma\!\otimes_\pi \mathcal{H}. \\ & \zeta & \mapsto & \zeta\,{}_\gamma\!\otimes_\pi \eta \end{array}$$

Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert spaces, and $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{K})$ and $\pi : N \to \mathcal{B}(\mathcal{H})$ be unital normal *-homomorphisms, $T \in \mathcal{B}(\mathcal{K}\,{}_\gamma\!\otimes_\pi \mathcal{H})$ and $\omega \in \mathcal{B}(\mathcal{K})_*$ (resp. $\omega \in \mathcal{B}(\mathcal{H})_*$). By using the fact that ${}_\varphi(\pi,\mathcal{H})$ (resp. $(\mathcal{K},\gamma)_\varphi$) is dense in $\mathcal{H}$ (resp. $\mathcal{K}$), there exists a unique

$$(\omega\,{}_\gamma\!\star_\pi \mathrm{id})(T) \in \mathcal{B}(\mathcal{H}) \quad (\text{resp. } (\mathrm{id}\,{}_\gamma\!\star_\pi \omega)(T) \in \mathcal{B}(\mathcal{K})),$$

which satisfies:

$$\langle \eta_2,\, (\omega\,{}_\gamma\!\star_\pi \mathrm{id})(T)\eta_1\rangle = \omega((\rho_{\eta_2}^{\gamma,\pi})^* T \rho_{\eta_1}^{\gamma,\pi}), \quad \eta_1,\eta_2 \in {}_\varphi(\pi,\mathcal{H})$$

$$(\text{resp. } \langle \xi_2,\, (\mathrm{id}\,{}_\gamma\!\star_\pi \omega)(T)\xi_1\rangle = \omega((\lambda_{\xi_2}^{\gamma,\pi})^* T \lambda_{\xi_1}^{\gamma,\pi}), \quad \xi_1,\xi_2 \in (\mathcal{K},\gamma)_\varphi).$$

In particular, we have the formulas:

$$(\omega_{\xi_2,\xi_1}\,{}_\gamma\!\star_\pi \mathrm{id})(T) = (\lambda_{\xi_2}^{\gamma,\pi})^* T \lambda_{\xi_1}^{\gamma,\pi} \in \mathcal{B}(\mathcal{H}), \quad \xi_1,\xi_2 \in (\mathcal{K},\gamma)_\varphi,$$

$$(\mathrm{id}\,{}_\gamma\!\star_\pi \omega_{\eta_2,\eta_1})(T) = (\rho_{\eta_2}^{\gamma,\pi})^* T \rho_{\eta_1}^{\gamma,\pi} \in \mathcal{B}(\mathcal{K}), \quad \eta_1,\eta_2 \in {}_\varphi(\pi,\mathcal{H}).$$

If $x \in M\,{}_\gamma\!\star_\pi P$, then for all $\omega \in \mathcal{B}(\mathcal{H})_*$ (resp. $\omega \in \mathcal{B}(\mathcal{K})_*$) we have

$$(\mathrm{id}\,{}_\gamma\!\star_\pi \omega)(x) \in M \quad (\text{resp. } (\omega\,{}_\gamma\!\star_\pi \mathrm{id})(x) \in P).$$

We refrain from writing the details but we can easily define the slice maps if T takes its values in a different relative tensor product. Note that we can extend the notion of slice maps for normal linear forms to normal semi-finite weights.

Fiber product over a finite dimensional von Neumann algebra. Now, let us assume that

$$N = \bigoplus_{1\leqslant l\leqslant k} \mathsf{M}_{n_l}(\mathbb{C}), \quad \varphi = \bigoplus_{1\leqslant l\leqslant k} \mathrm{Tr}_l(F_l -),$$

where F_l is a positive invertible matrix of $\mathsf{M}_{n_l}(\mathbb{C})$ and Tr_l is the non-normalized trace on $\mathsf{M}_{n_l}(\mathbb{C})$. Let us denote $(F_{l,i})_{1\leqslant i\leqslant n_l}$ the eigenvalues of F_l. We have:

Proposition 2.1.10. (*see* §7 *of* [8]) *The bounded linear map*

$$\begin{array}{rccl} v_\varphi^{\gamma,\pi} : & \mathcal{K}\otimes\mathcal{H} & \longrightarrow & \mathcal{K}\,{}_\gamma\!\underset{\varphi}{\otimes}_\pi \mathcal{H} \\ & \xi\otimes\eta & \longmapsto & \xi\,{}_\gamma\!\underset{\varphi}{\otimes}_\pi \eta \end{array} \quad (\textit{or simply denoted } v^{\gamma,\pi})$$

is a coisometry if and only if $\sum_{1\leqslant i\leqslant n_l} F_{l,i}^{-1} = 1$ *for all* $1 \leqslant l \leqslant k$. *In the following, we assume that this condition is satisfied. Let us denote*

$$q_\varphi^{\gamma,\pi} = (v_\varphi^{\gamma,\pi})^* v_\varphi^{\gamma,\pi} \quad (\textit{or simply } q^{\gamma,\pi}).$$

Then, $q_\varphi^{\gamma,\pi}$ is a self-adjoint projection of $\mathcal{B}(\mathcal{K}\otimes\mathcal{H})$ such that

$$q_\varphi^{\gamma,\pi} = \sum_{1\leqslant l\leqslant k}\sum_{1\leqslant i,j\leqslant n_l} F_{l,i}^{-1/2}F_{l,j}^{-1/2}\gamma(e_{ij}^{(l)\,\mathrm{o}})\otimes\pi(e_{ji}^{(l)}),$$

where, for all $1\leqslant l\leqslant k$, $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ is a system of matrix units such that $F_l=\sum_{1\leqslant i\leqslant n_l}F_{l,i}e_{ii}^{(l)}$. Furthermore, the map

$$\begin{aligned} M\underset{N}{{}_\gamma\star_\pi}P &\longrightarrow q_\varphi^{\gamma,\pi}(M\otimes P)q_\varphi^{\gamma,\pi}\\ x &\longmapsto (v_\varphi^{\gamma,\pi})^*xv_\varphi^{\gamma,\pi}\end{aligned}$$

*is a unital normal *-isomorphism.*

Now, let us take for φ the non-normalized Markov trace on $N=\bigoplus_{1\leqslant l\leqslant k}\mathsf{M}_{n_l}(\mathbb{C})$, that is

$$\epsilon=\bigoplus_{1\leqslant l\leqslant k} n_l\cdot\mathrm{Tr}_l.$$

As a corollary of Proposition 2.1.10, we have:

Proposition 2.1.11. *For all system of matrix units $(e_{ij}^{(l)})$, $1\leqslant l\leqslant k$, $1\leqslant i,j\leqslant n_l$, of N we have:*

$$q_\epsilon^{\gamma,\pi}=\sum_{1\leqslant l\leqslant k} n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\gamma(e_{ij}^{(l)\mathrm{o}})\otimes\pi(e_{ji}^{(l)}).$$

The following result is a slight generalization of the previous proposition to the setting of C*-algebras. This result will be used several times in the subsequent chapters.

Proposition-Definition 2.1.12. *Let A, B be two C*-algebras. We consider two non-degenerate *-homomorphisms $\gamma_A: N^{\mathrm{o}}\to\mathcal{M}(A)$ and $\pi_B: N\to\mathcal{M}(B)$. There exists a unique self-adjoint projection $q^{\gamma_A,\pi_B}\in\mathcal{M}(A\otimes B)$ such that*

$$q^{\gamma_A,\pi_B}=\sum_{1\leqslant l\leqslant k} n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\gamma_A(e_{ij}^{(l)\mathrm{o}})\otimes\pi_B(e_{ji}^{(l)}),$$

for all system of matrix units $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ for $\mathsf{M}_{n_l}(\mathbb{C})$, $1\leqslant l\leqslant k$.

Proof. The uniqueness of such a self-adjoint projection is straightforward. In virtue of Gelfand-Naimark theorem, we can consider faithful non-degenerate *-homomorphisms $\theta_A: A\to\mathcal{B}(\mathcal{K})$ and $\theta_B: B\to\mathcal{B}(\mathcal{H})$. Let us denote $\gamma:=\theta_A\circ\gamma_A$ and $\pi:=\theta_B\circ\pi_B$. Then, $\gamma: N^{\mathrm{o}}\to\mathcal{B}(\mathcal{K})$ and $\pi: N\to\mathcal{B}(\mathcal{H})$ are normal unital *-representations. Let us fix an arbitrary system of matrix units $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ for $\mathsf{M}_{n_l}(\mathbb{C})$ for each $1\leqslant l\leqslant k$. We define a self-adjoint projection $q^{\gamma_A,\pi_B}\in\mathcal{M}(A\otimes B)$ by setting:

$$q^{\gamma_A,\pi_B}:=\sum_{1\leqslant l\leqslant k} n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\gamma_A(e_{ij}^{(l)\mathrm{o}})\otimes\pi_B(e_{ji}^{(l)}).$$

By Proposition 2.1.11, we have $q_\epsilon^{\gamma,\pi}=(\theta_A\otimes\theta_B)(q^{\gamma_A,\pi_B})$. By using again Proposition 2.1.11 and the fact that $\theta_A\otimes\theta_B$ is faithful, we obtain that q^{γ_A,π_B} is independent of the chosen systems of matrix units. Moreover, the definition of q^{γ_A,π_B} shows that q^{γ_A,π_B} is also independent of the chosen faithful non-degenerate *-homomorphisms θ_A and θ_B. □

Remark 2.1.13. In a similar way, there also exists a unique self-adjoint projection q^{π_B,γ_A} such that

$$q^{\pi_B,\gamma_A}=\sum_{1\leqslant l\leqslant k} n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\pi_B(e_{ij}^{(l)})\otimes\gamma_A(e_{ji}^{(l)\mathrm{o}}),$$

for all system of matrix units $(e_{ij}^{(l)})_{1\leqslant i,j\leqslant n_l}$ for $\mathsf{M}_{n_l}(\mathbb{C})$, $1\leqslant l\leqslant k$.

2.2 Enock-Lesieur's measured quantum groupoids

Definition 2.2.1. We call measured quantum groupoid an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$, where:

- M and N are von Neumann algebras,
- $\Gamma : M \to M \,{}_\beta\!\star_\alpha\, M$ is a faithful normal unital *-homomorphism, called the coproduct,
- $\alpha : N \to M$ and $\beta : N^{\mathrm{o}} \to M$ are faithful normal unital *-homormorphisms, called the range and source maps of $\mathcal{G}$,
- $T : M_+ \to \alpha(N)^{\mathrm{ext}}_+$ and $T' : M_+ \to \beta(N^{\mathrm{o}})^{\mathrm{ext}}_+$ are n.s.f. operator-valued weights,
- ν is a n.s.f. weight on N,

such that the following conditions are satisfied:

1. $\alpha(n')$ and $\beta(n^{\mathrm{o}})$ commute for all $n, n' \in N$.
2. $\forall n \in N$, $\Gamma(\alpha(n)) = \alpha(n) \,{}_\beta\!\otimes_\alpha\, 1$ and $\Gamma(\beta(n^{\mathrm{o}})) = 1 \,{}_\beta\!\otimes_\alpha\, \beta(n^{\mathrm{o}})$.
3. Γ is coassociative, that is $(\Gamma \,{}_\beta\!\star_\alpha\, \mathrm{id})\Gamma = (\mathrm{id} \,{}_\beta\!\star_\alpha\, \Gamma)\Gamma$.
4. The n.s.f. weights φ and ψ on M given by $\varphi = \nu \circ \alpha^{-1} \circ T$ and $\psi = \nu \circ \beta^{-1} \circ T'$ satisfy:
   - $\forall x \in \mathfrak{M}_T^+$, $T(x) = (\mathrm{id} \,{}_\beta\!\star_\alpha\, \varphi)\Gamma(x) \quad ; \quad \forall x \in \mathfrak{M}_{T'}^+$, $T'(x) = (\psi \,{}_\beta\!\star_\alpha\, \mathrm{id})\Gamma(x)$.
   - σ_t^φ and σ_s^ψ commute for all $s, t \in \mathbb{R}$.

Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid. We denote $(\mathscr{H}, \pi, \Lambda)$ the G.N.S. construction for (M, φ) where $\varphi = \nu \circ \alpha^{-1} \circ T$. Let $(\sigma_t)_{t\in\mathbb{R}}$, ∇ and J be respectively the modular automorphism group, the modular operator and the modular conjugation for φ. In the following, we identify M with its image by π in $\mathcal{B}(\mathscr{H})$. We have:

- $R_\mathcal{G} : M \to M$ a *-antihomomorphism satisfying $R_\mathcal{G}^2 = \mathrm{id}$ and $\varsigma^{\alpha,\beta} \circ (R_\mathcal{G} \,{}_\beta\!\star_\alpha\, R_\mathcal{G}) \circ \Gamma = \Gamma \circ R_\mathcal{G}$.

From now on, we will assume that $T' = R_\mathcal{G} T R_\mathcal{G}$ and then also $\psi = \varphi \circ R_\mathcal{G}$.

- There exist self-adjoint positive non-singular operators λ and d respectively affiliated to $\mathcal{Z}(M)$ and M such that
$$(D\psi : D\varphi)_t = \lambda^{it^2/2} d^{it}, \quad t \in \mathbb{R}.$$
The operators λ and d are respectively called the scaling operator and the modular operator of $\mathcal{G}$.
- The G.N.S. construction for (M, ψ) is given by $(\mathscr{H}, \pi_\psi, \Lambda_\psi)$, where Λ_ψ is the closure of the operator which sends any element $x \in M$ such that $xd^{1/2}$ is closable and its closure $\overline{xd^{1/2}} \in \mathfrak{N}_\varphi$ to $\Lambda_\varphi(\overline{xd^{1/2}})$, $\pi_\psi : M \to \mathcal{B}(\mathscr{H})$ is given by the formula $\pi_\psi(a)\Lambda_\psi(x) = \Lambda_\psi(ax)$.
- The modular conjugation J_ψ for ψ is given by $J_\psi = \lambda^{i/4} J$.

- We will denote
$$\mathscr{W}_{\mathcal{G}} : \mathscr{H}\ {}_{\beta}\otimes_{\alpha}\mathscr{H} \to \mathscr{H}\ {}_{\alpha}\otimes_{\widehat{\beta}}\mathscr{H}$$
the left regular pseudo-multiplicative unitary of $\mathcal{G}$ (see [26]) : $\mathscr{W}_{\mathcal{G}}$ is a unitary map, which satisfies a pentagon relation and some commutation relations with respect to α, $\widehat{\beta}$ and β, that is for all $n \in N$ we have:
$$\begin{aligned} \mathscr{W}_{\mathcal{G}}(\alpha(n)\ {}_{\beta}\otimes_{\alpha} 1) &= (1\ {}_{\alpha}\otimes_{\widehat{\beta}}\alpha(n))\mathscr{W}_{\mathcal{G}}, \quad \mathscr{W}_{\mathcal{G}}(1\ {}_{\beta}\otimes_{\alpha}\beta(n^{\mathrm{o}})) = (1\ {}_{\alpha}\otimes_{\widehat{\beta}}\beta(n^{\mathrm{o}}))\mathscr{W}_{\mathcal{G}}, \\ \mathscr{W}_{\mathcal{G}}(\widehat{\beta}(n^{\mathrm{o}})\ {}_{\beta}\otimes_{\alpha} 1) &= (\widehat{\beta}(n^{\mathrm{o}})\ {}_{\alpha}\otimes_{\widehat{\beta}} 1)\mathscr{W}_{\mathcal{G}}, \quad \mathscr{W}_{\mathcal{G}}(1\ {}_{\beta}\otimes_{\alpha}\widehat{\beta}(n^{\mathrm{o}})) = (\beta(n^{\mathrm{o}})\ {}_{\alpha}\otimes_{\widehat{\beta}} 1)\mathscr{W}_{\mathcal{G}}. \end{aligned} \tag{2.2.1}$$
Moreover, M is the weak closure of $\{(\mathrm{id}\star\omega)(\mathscr{W}_{\mathcal{G}})\,;\,\omega\in\mathcal{B}(\mathscr{H})_*\}$ and we have:
$$\Gamma(x) = \mathscr{W}_{\mathcal{G}}^*(1\ {}_{\alpha}\otimes_{\widehat{\beta}} x)\mathscr{W}_{\mathcal{G}}, \quad x \in M.$$

Starting from a measured quantum groupoid $\mathcal{G}$, we are able to build new ones: the dual, opposite and commutant measured quantum groupoids respectively denoted $\widehat{\mathcal{G}}$, $\mathcal{G}^{\mathrm{o}}$ and $\mathcal{G}^{\mathrm{c}}$.

Proposition-Definition 2.2.2. *Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid. Let $\mathscr{W}_{\mathcal{G}}$ be the regular pseudo-multiplicative unitary for $\mathcal{G}$. We define the dual measured quantum groupoid of $\mathcal{G}$ to be the octuple $\widehat{\mathcal{G}} = (N, \widehat{M}, \alpha, \widehat{\beta}, \widehat{\Gamma}, \widehat{T}, \widehat{R}\circ\widehat{T}\circ\widehat{R}, \nu)$, where:*

- *$\widehat{M} \subset \mathcal{B}(\mathscr{H})$ is the von Neumann algebra generated by*
$$\{(\omega\star\mathrm{id})(\mathscr{W}_{\mathcal{G}})\,;\,\omega\in\mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H}).$$
- *$\widehat{\beta} : N^{\mathrm{o}} \to \widehat{M}$ is given by $\widehat{\beta}(n^{\mathrm{o}}) = J\alpha(n^*)J$ for all $n \in N$.*
- *$\widehat{\Gamma} : \widehat{M} \to \widehat{M}\ {}_{\widehat{\beta}}\star_{\alpha}\widehat{M}$ is given for all $x \in \widehat{M}$ by*
$$\widehat{\Gamma}(x) = \sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}(x\ {}_{\beta}\otimes_{\alpha} 1)\mathscr{W}_{\mathcal{G}}^*\sigma^{\widehat{\beta},\alpha}.$$
- *There exists a unique n.s.f. weight $\widehat{\varphi}$ on $\widehat{M}$ whose G.N.S. construction is $(\mathscr{H}, \mathrm{id}, \Lambda_{\widehat{\varphi}})$, where the G.N.S. map $\Lambda_{\widehat{\varphi}}$ is the closure of the operator*
$$(\omega\star\mathrm{id})(\mathscr{W}_{\mathcal{G}}) \mapsto a_{\varphi}(\omega),$$
defined for normal linear forms ω in a dense subspace of
$$\mathscr{I}_{\varphi} = \{\omega\in\mathcal{B}(\mathscr{H})_*\,;\,\exists k\in\mathbb{R}_+,\ \forall x\in\mathfrak{N}_{\varphi},\ |\omega(x^*)|^2 \leqslant k\varphi(x^*x)\}$$
and $a_{\varphi}(\omega) \in \mathscr{H}$ satisfies
$$\forall x\in\mathfrak{N}_{\varphi},\ \omega(x^*) = \langle\Lambda_{\varphi}(x),\, a_{\varphi}(\omega)\rangle.$$
- *$\widehat{T}$ is the unique n.s.f. operator-valued weight from $\widehat{M}$ to $\alpha(N)$ such that $\widehat{\varphi} = \nu\circ\alpha^{-1}\circ\widehat{T}$ and $\widehat{T}' = R_{\widehat{\mathcal{G}}}\widehat{T}R_{\widehat{\mathcal{G}}}$, where $R_{\widehat{\mathcal{G}}} : \widehat{M} \to \widehat{M}$ is given by $R_{\widehat{\mathcal{G}}}(x) = Jx^*J$ for all $x \in \widehat{M}$.*

We will also consider the n.s.f. weight $\widehat{\psi} = \widehat{\varphi}\circ R_{\widehat{\mathcal{G}}}$. We will denote $\widehat{\nabla}$ and $\widehat{J}$ the modular operator and the modular conjugation for $\widehat{\varphi}$.

Note that the scaling operator of $\widehat{\mathcal{G}}$ is λ^{-1}. In particular, we have $\lambda^{it} \in \mathcal{Z}(M)\cap\mathcal{Z}(\widehat{M})$ for all $t \in \mathbb{R}$. The regular pseudo-multiplicative unitary $\mathscr{W}_{\widehat{\mathcal{G}}}$ of $\widehat{\mathcal{G}}$ is given by
$$\mathscr{W}_{\widehat{\mathcal{G}}} = \sigma^{\beta,\alpha}\mathscr{W}_{\mathcal{G}}^*\sigma^{\widehat{\beta},\alpha}.$$

Theorem 2.2.3. *Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid. We have:*

1. $\mathscr{W}_{\mathcal{G}}({}_{\beta}\widehat{J}_{\alpha}\otimes_{\alpha} J_{\widehat{\beta}}) = ({}_{\alpha}\widehat{J}_{\beta}\otimes_{\widehat{\beta}} J_{\alpha})\mathscr{W}_{\mathcal{G}}^{*}$.
2. $\widehat{J}J = \lambda^{i/4}J\widehat{J}$.
3. *Let us define* $\widehat{\alpha}(n) := J\beta(n^{\mathrm{o}})^{*}J = \widehat{J}\,\widehat{\beta}(n^{\mathrm{o}})^{*}\widehat{J}$ *for* $n \in N$. *We have the Heisenberg type relations:*

$$M \cap \widehat{M} = \alpha(N), \quad M \cap \widehat{M}' = \beta(N^{\mathrm{o}}), \quad M' \cap \widehat{M} = \widehat{\beta}(N^{\mathrm{o}}), \quad M' \cap \widehat{M}' = \widehat{\alpha}(N).$$

Proposition-Definition 2.2.4. *Let $\mathcal{G}$ be a measured quantum groupoid. We have*

$$\lambda^{it}J = J\lambda^{-it}, \quad \lambda^{it}\widehat{J} = \widehat{J}\lambda^{-it}, \quad \textit{for all } t \in \mathbb{R}.$$

Let us denote $U = \widehat{J}J \in \mathcal{B}(\mathscr{H})$. Then, $U^{*} = \lambda^{-i/4}U$ and $U^{2} = \lambda^{i/4}$. In particular, U is unitary. We also have:

$$\widehat{\alpha}(n) = U^{*}\alpha(n)U, \quad \widehat{\beta}(n^{\mathrm{o}}) = U\beta(n^{\mathrm{o}})U^{*}, \quad n \in N.$$

Remark 2.2.5. We have $\lambda^{-i/4} \in \mathcal{Z}(M)$ and $U^{*} = \lambda^{-i/4}U$. Hence,

$$\widehat{\alpha}(n) = U\alpha(n)U^{*}, \quad \widehat{\beta}(n^{\mathrm{o}}) = U^{*}\beta(n^{\mathrm{o}})U, \quad n \in N.$$

Proposition-Definition 2.2.6. *Let us fix $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ a measured quantum groupoid.*

1. *The octuple $(N^{\mathrm{o}}, M, \beta, \alpha, \varsigma^{\beta,\alpha} \circ \Gamma, R_{\mathcal{G}}TR_{\mathcal{G}}, T, \nu^{\mathrm{o}})$ is a measured quantum groupoid denoted $\mathcal{G}^{\mathrm{o}}$ and called the opposite measured quantum groupoid (of $\mathcal{G}$). The regular pseudo-multiplicative unitary $\mathscr{W}_{\mathcal{G}^{\mathrm{o}}}$ of $\mathcal{G}^{\mathrm{o}}$ is given by:*

$$\mathscr{W}_{\mathcal{G}^{\mathrm{o}}} = ({}_{\beta}\widehat{J}_{\alpha}\otimes_{\widehat{\alpha}} \widehat{J}_{\widehat{\beta}})\mathscr{W}_{\mathcal{G}}({}_{\beta}\widehat{J}_{\alpha}\otimes_{\alpha} \widehat{J}_{\beta}).$$

2. *Let $C_M : M \to M'$ be the canonical *-antihomomorphism given by $C_M(x) = Jx^{*}J$, $x \in M$. Let us define:*

$$\Gamma^{\mathrm{c}} = (C_M \,{}_{\beta}\star_{\alpha}\, C_M) \circ \Gamma \circ C_M^{-1}, \quad R_{\mathcal{G}}^{\mathrm{c}} = C_M \circ R_{\mathcal{G}} \circ C_M^{-1}, \quad T^{\mathrm{c}} = C_M \circ T \circ C_M^{-1}.$$

Then, the octuple $(N^{\mathrm{o}}, M', \widehat{\beta}, \widehat{\alpha}, \Gamma^{\mathrm{c}}, T^{\mathrm{c}}, R^{\mathrm{c}}T^{\mathrm{c}}R^{\mathrm{c}}, \nu^{\mathrm{o}})$ is a measured quantum groupoid denoted $\mathcal{G}^{\mathrm{c}}$ and called the commutant measured quantum groupoid (of $\mathcal{G}$). The regular pseudo-multiplicative unitary $\mathscr{W}_{\mathcal{G}^{\mathrm{c}}}$ of $\mathcal{G}^{\mathrm{c}}$ is given by:

$$\mathscr{W}_{\mathcal{G}^{\mathrm{c}}} = ({}_{\widehat{\beta}}J_{\alpha}\otimes_{\alpha} J_{\widehat{\beta}})\mathscr{W}_{\mathcal{G}}({}_{\beta}J_{\widehat{\alpha}}\otimes_{\alpha} J_{\widehat{\beta}}).$$

Proposition 2.2.7. *Let $\mathcal{G}$ be a measured quantum groupoid. We have the following formulas:*

$$(\mathcal{G}^{\mathrm{o}})^{\mathrm{o}} = \mathcal{G}, \quad (\mathcal{G}^{\mathrm{c}})^{\mathrm{c}} = \mathcal{G}, \quad (\mathcal{G}^{\mathrm{o}})^{\mathrm{c}} = (\mathcal{G}^{\mathrm{c}})^{\mathrm{o}}.$$

$$\widehat{\mathcal{G}^{\mathrm{o}}} = (\widehat{\mathcal{G}})^{\mathrm{c}}, \quad \widehat{\mathcal{G}^{\mathrm{c}}} = (\widehat{\mathcal{G}})^{\mathrm{o}}.$$

Remark 2.2.8. From §3.2 on, we will refer to $(\widehat{\mathcal{G}})^{\mathrm{c}}$ instead of $\widehat{\mathcal{G}}$ as the dual of $\mathcal{G}$. Indeed, this dual will appear better suited to right actions of $\mathcal{G}$. Note that:

$$(\widehat{\mathcal{G}})^{\mathrm{c}} = (N^{\mathrm{o}}, \widehat{M}', \beta, \widehat{\alpha}, \widehat{\Gamma}^{\mathrm{c}}, \widehat{T}^{\mathrm{c}}, \widehat{T}^{\mathrm{c}\prime}, \nu^{\mathrm{o}}),$$

where the coproduct and the operator valued weights are given by:

- $\widehat{\Gamma}^{\mathrm{c}}(x) = (\mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}})^*(1\,{}_\beta\otimes_\alpha x)\mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}}$, for all $x \in \widehat{M}'$.
- $\widehat{T}^{\mathrm{c}} = C_{\widehat{M}} \circ \widehat{T} \circ C_{\widehat{M}}^{-1}$, where $C_{\widehat{M}} : \widehat{M} \to \widehat{M}'$; $x \mapsto \hat{J}x^*\hat{J}$.
- $\widehat{T}^{\mathrm{c}\prime} = R_{(\widehat{\mathcal{G}})^{\mathrm{c}}} \circ \widehat{T}^{\mathrm{c}} \circ R_{(\widehat{\mathcal{G}})^{\mathrm{c}}}$.
- Note also that the commutant weight $\widehat{\varphi}\,' = \nu^{\mathrm{o}} \circ \beta^{-1} \circ \widehat{T}^{\mathrm{c}}$ derived from the weight $\widehat{\varphi}$ is left invariant for the coproduct $\widehat{\Gamma}^{\mathrm{c}}$.

Notations 2.2.9. For a given measured quantum groupoid $\mathcal{G}$, we will need the following pseudo-multiplicative unitaries:

$$\widehat{\mathscr{V}} := \mathscr{W}_{\mathcal{G}}, \quad \mathscr{V} := \mathscr{W}_{\widehat{(\mathcal{G}^{\mathrm{o}})}} = \mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}}, \quad \widetilde{\mathscr{V}} := \mathscr{W}_{(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}}.$$

2.3 De Commer's weak Hopf-von Neumann algebras with finite basis

In [9], De Commer provides an equivalent definition of a measured quantum groupoid on a finite basis. This definition is far more tractable since it avoids the use of relative tensor products and fiber products. Let us assume that:

$$N = \bigoplus_{1\leqslant l\leqslant k} \mathsf{M}_{n_l}(\mathbb{C}), \quad \epsilon = \bigoplus_{1\leqslant l\leqslant k} n_l \cdot \mathrm{Tr}_l.$$

Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \epsilon)$ be a measured quantum groupoid. Let us recall that we have a unital normal *-isomorphism:

$$\begin{aligned} M\,{}_\beta\star_\alpha\, M &\longrightarrow q^{\beta,\alpha}(M \otimes M)q^{\beta,\alpha} \\ x &\longmapsto (v^{\beta,\alpha})^* x v^{\beta,\alpha} \end{aligned}$$

Let us then denote $\Delta : M \to M \otimes M$ the (non necessarily unital) faithful normal *-homomorphism given by

$$\Delta(x) = (v^{\beta,\alpha})^*\Gamma(x)v^{\beta,\alpha}, \quad x \in M.$$

We have $\Delta(1) = q^{\beta,\alpha}$. We verify easily that for all $T \in \mathcal{B}(\mathscr{H}\,{}_\beta\otimes_\alpha\,\mathscr{H})$ and $\omega \in \mathcal{B}(\mathscr{H})_*$ we have:

$$(\omega\,{}_\beta\star_\alpha\,\mathrm{id})(T) = (\omega \otimes \mathrm{id})((v^{\beta,\alpha})^*Tv^{\beta,\alpha}), \quad (\mathrm{id}\,{}_\beta\star_\alpha\,\omega)(T) = (\mathrm{id} \otimes \omega)((v^{\beta,\alpha})^*Tv^{\beta,\alpha}).$$

In particular, we obtain:

$$(\omega\,{}_\beta\star_\alpha\,\mathrm{id})\Gamma(x) = (\omega \otimes \mathrm{id})\Delta(x), \quad (\mathrm{id}\,{}_\beta\star_\alpha\,\omega)\Gamma(x) = (\mathrm{id} \otimes \omega)\Delta(x),$$

for all $x \in M$, $\omega \in \mathcal{B}(\mathscr{H})_*$. The coassociativity of Δ is derived from that of Γ and for all $n \in N$ we have:

$$\Delta(\alpha(n)) = (\alpha(n) \otimes 1)\Delta(1) = \Delta(1)(\alpha(n) \otimes 1),$$

$$\Delta(\beta(n^{\mathrm{o}})) = (1 \otimes \beta(n^{\mathrm{o}}))\Delta(1) = \Delta(1)(1 \otimes \beta(n^{\mathrm{o}})).$$

This leads De Commer to build the following equivalent definition of a measured quantum groupoid on a finite basis (called "weak Hopf-von Neumann algebra with finite basis and integrals" in Definition 11.1.2 of [9]):

Definition 2.3.1. A measured quantum groupoid on the finite-dimensional basis

$$N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$$

is an octuple $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$, where:

- M is a von Neumann algebra, $\alpha : N \to M$ and $\beta : N^{\mathrm{o}} \to M$ are unital faithful normal *-homomorphisms,
- $\Delta : M \to M \otimes M$ is a faithful normal *-homomorphism,
- $T : M_+ \to \alpha(N)_+^{\mathrm{ext}}$ and $T' : M_+ \to \beta(N^{\mathrm{o}})_+^{\mathrm{ext}}$ are n.s.f. operator-valued weights,
- $\epsilon = \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l$ is the non-normalized Markov trace on N,

such that the following conditions are satisfied:

1. $\alpha(n')$ and $\beta(n^{\mathrm{o}})$ commute for all $n, n' \in N$.
2. $\Delta(1) = q^{\beta,\alpha}$.
3. Δ is coassociative, that is $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
4. $\forall n \in N$, $\Delta(\alpha(n)) = \Delta(1)(\alpha(n) \otimes 1)$, $\Delta(\beta(n^{\mathrm{o}})) = \Delta(1)(1 \otimes \beta(n^{\mathrm{o}}))$.
5. The n.s.f. weights φ and ψ on M given by $\varphi = \epsilon \circ \alpha^{-1} \circ T$ and $\psi = \epsilon \circ \beta^{-1} \circ T'$ satisfy:
$$T(x) = (\mathrm{id} \otimes \varphi)\Delta(x), \quad x \in \mathfrak{M}_T^+ \quad ; \quad T'(x) = (\psi \otimes \mathrm{id})\Delta(x), \quad x \in \mathfrak{M}_{T'}^+.$$
6. $\forall t \in \mathbb{R}$, $\sigma_t^T \circ \alpha = \alpha$, $\sigma_t^{T'} \circ \beta = \beta$.

Let us fix a measured quantum groupoid $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$.

Notations 2.3.2. Let us consider the injective bounded linear map

$$\iota_{\widehat{\alpha},\alpha}^{\beta} : \mathcal{B}(\mathscr{H}\, {}_{\widehat{\alpha}}\otimes_{\beta} \mathscr{H}, \mathscr{H}\, {}_{\beta}\otimes_{\alpha} \mathscr{H}) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H}) \quad ; \quad X \mapsto (v^{\beta,\alpha})^* X v^{\widehat{\alpha},\beta}.$$

Similarly, we also define $\iota_{\beta,\widehat{\beta}}^{\alpha}$ and $\iota_{\widehat{\beta},\beta}^{\widehat{\alpha}}$. Then, we define:

$$V := \iota_{\widehat{\alpha},\alpha}^{\beta}(\mathscr{V}), \quad W := \iota_{\beta,\widehat{\beta}}^{\alpha}(\widehat{\mathscr{V}}), \quad \widetilde{V} := \iota_{\widehat{\beta},\beta}^{\widehat{\alpha}}(\widetilde{\mathscr{V}}),$$

where we recall that $\mathscr{V} = \mathscr{W}_{(\widehat{\mathcal{G}})^{\mathrm{c}}} = \mathscr{W}_{\widehat{(\mathcal{G}^{\mathrm{o}})}}$, $\widehat{\mathscr{V}} = \mathscr{W}_{\mathcal{G}}$ and $\widetilde{\mathscr{V}} = \mathscr{W}_{(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}}$.

Proposition 2.3.3. *The operators V, W and $\widetilde{V}$ are multiplicative partial isometries acting on $\mathscr{H} \otimes \mathscr{H}$ whose initial and final supports are given by*

$$V^*V = q^{\widehat{\alpha},\beta} = \widetilde{V}\widetilde{V}^*, \quad W^*W = q^{\beta,\alpha} = VV^*, \quad WW^* = q^{\alpha,\widehat{\beta}}, \quad \widetilde{V}^*\widetilde{V} = q^{\widehat{\beta},\widehat{\alpha}}.$$

Moreover, we have:

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Delta(x)(1 \otimes y)), \quad x, y \in \mathfrak{N}_\psi,$$

$$W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)(x \otimes 1)), \quad x, y \in \mathfrak{N}_\varphi.$$

It is also easy to see that

$$(\omega \star \mathrm{id})(\mathscr{W}_{\mathcal{G}}) = (\omega \otimes \mathrm{id})(W), \quad (\mathrm{id} \star \omega)(\mathscr{W}_{\mathcal{G}}) = (\mathrm{id} \otimes \omega)(W), \quad \omega \in \mathcal{B}(\mathscr{H})_*.$$

In particular, M (resp. $\widehat{M}$) is the weak closure of

$$\{(\mathrm{id} \otimes \omega)(W)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H}) \quad (\text{resp. } \{(\omega \otimes \mathrm{id})(W)\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H})).$$

Proposition 2.3.4. *The multiplicative partial isometries $V, W, \widetilde{V}$ are related to each other by the following formulas:*

$$W = \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma, \quad \widetilde{V} = \Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma, \quad \textit{where} \quad U = \widehat{J}J.$$

Furthermore, we have

$$V^* = (J \otimes \widehat{J})V(J \otimes \widehat{J}), \quad W^* = (\widehat{J} \otimes J)W(\widehat{J} \otimes J).$$

Proof. This is a consequence of the definition of V, W and $\widetilde{V}$, Theorem 2.2.3 1 and Lemma 3.1.1 (proved independently in §3.1). □

Following the notations of [2], we denote $\widehat{V} = \Sigma(U \otimes 1)V(U^* \otimes 1)\Sigma$. The previous proposition says that $W = \widehat{V}$. Note that we have $\widetilde{V} = (U \otimes U)W(U^* \otimes U^*)$. Note also that we have:

$$W \in M \otimes \widehat{M}, \quad V \in \widehat{M}' \otimes M, \quad \widetilde{V} \in M' \otimes \widehat{M}'. \tag{2.3.1}$$

Proposition 2.3.5. *We have the following commutation relations:*

1. $W_{12}V_{23} = V_{23}W_{12} \quad , \quad V_{12}\widetilde{V}_{23} = \widetilde{V}_{23}V_{12}$.

For all $n \in N$, we also have:

2. $[V, \alpha(n) \otimes 1] = 0, \quad [V, \widehat{\beta}(n^{\mathrm{o}}) \otimes 1] = 0, \quad [V, 1 \otimes \widehat{\alpha}(n)] = 0, \quad [V, 1 \otimes \widehat{\beta}(n^{\mathrm{o}})] = 0.$
3. $V(1 \otimes \alpha(n)) = (\widehat{\alpha}(n) \otimes 1)V, \quad V(\beta(n^{\mathrm{o}}) \otimes 1) = (1 \otimes \beta(n^{\mathrm{o}}))V.$
4. $[W, \widehat{\beta}(n^{\mathrm{o}}) \otimes 1] = 0, \quad [W, \widehat{\alpha}(n) \otimes 1] = 0, \quad [W, 1 \otimes \beta(n^{\mathrm{o}})] = 0, \quad [W, 1 \otimes \widehat{\alpha}(n)] = 0.$
5. $W(1 \otimes \widehat{\beta}(n^{\mathrm{o}})) = (\beta(n^{\mathrm{o}}) \otimes 1)W, \quad W(\alpha(n) \otimes 1) = (1 \otimes \alpha(n))W.$
6. $[\widetilde{V}, \alpha(n) \otimes 1] = 0, \quad [\widetilde{V}, \beta(n^{\mathrm{o}}) \otimes 1] = 0, \quad [\widetilde{V}, 1 \otimes \alpha(n)] = 0, \quad [\widetilde{V}, 1 \otimes \widehat{\beta}(n^{\mathrm{o}})] = 0.$
7. $\widetilde{V}(1 \otimes \beta(n^{\mathrm{o}})) = (\widehat{\beta}(n^{\mathrm{o}}) \otimes 1)\widetilde{V}, \quad \widetilde{V}(\widehat{\alpha}(n) \otimes 1) = (1 \otimes \widehat{\alpha}(n))\widetilde{V}.$

Proof. The first statement is a consequence of (2.3.1). For the statements 4 and 5, see Lemma 11.1.2 of [9] which is derived from the formulas in Definition 3.2. (i) and Theorem 3.6. (ii) of [14] (see the formulas (2.2.1) above). The formula $[W, 1 \otimes \widehat{\alpha}(n)] = 0$ is a consequence of (2.3.1) and Theorem 2.2.3 3. The remaining statements follow from Proposition 2.3.4. □

Furthermore, we also have some formulas of a different kind (these are not commutation relations).

Proposition 2.3.6. *For all $n \in N$, we have:*

1. $W(\beta(n^{\mathrm{o}}) \otimes 1) = W(1 \otimes \alpha(n)), \quad (1 \otimes \widehat{\beta}(n^{\mathrm{o}}))W = (\alpha(n) \otimes 1)W.$
2. $V(1 \otimes \beta(n^{\mathrm{o}})) = V(\widehat{\alpha}(n) \otimes 1), \quad (1 \otimes \alpha(n))V = (\beta(n^{\mathrm{o}}) \otimes 1)V.$

3. $\widetilde{V}(\widehat{\beta}(n^{\mathrm{o}})\otimes 1)=\widetilde{V}(1\otimes\widehat{\alpha}(n)),\quad (1\otimes\beta(n^{\mathrm{o}}))\widetilde{V}=(\widehat{\alpha}(n)\otimes 1)\widetilde{V}.$

Proof. See Lemma 11.1.4 of [9] and Proposition 2.3.4 above. □

It should be mentioned that these formulas rely heavily on the fact that ϵ is a faithful positive tracial functional. Indeed, if ν is some faithful positive functional we only have

$$\beta(n^{\mathrm{o}})\xi\underset{\nu}{{}_{\beta}\otimes_{\alpha}}\eta=\xi\underset{\nu}{{}_{\beta}\otimes_{\alpha}}\sigma^{\nu}_{-i}(\alpha(n))\eta,\quad n\in N,\ \xi,\eta\in\mathscr{H}.$$

Corollary 2.3.7. *For all $n\in N$, we have:*

1. $q^{\beta,\alpha}(\beta(n^{\mathrm{o}})\otimes 1)=q^{\beta,\alpha}(1\otimes\alpha(n)),\quad (1\otimes\widehat{\beta}(n^{\mathrm{o}}))q^{\alpha,\widehat{\beta}}=(\alpha(n)\otimes 1)q^{\alpha,\widehat{\beta}}.$
2. $q^{\widehat{\alpha},\beta}(1\otimes\beta(n^{\mathrm{o}}))=q^{\widehat{\alpha},\beta}(\widehat{\alpha}(n)\otimes 1),\quad (1\otimes\alpha(n))q^{\beta,\alpha}=(\beta(n^{\mathrm{o}})\otimes 1)q^{\beta,\alpha}.$
3. $q^{\widehat{\beta},\widehat{\alpha}}(\widehat{\beta}(n^{\mathrm{o}})\otimes 1)=q^{\widehat{\beta},\widehat{\alpha}}(1\otimes\widehat{\alpha}(n)),\quad (1\otimes\beta(n^{\mathrm{o}}))q^{\widehat{\alpha},\beta}=(\widehat{\alpha}(n)\otimes 1)q^{\widehat{\alpha},\beta}.$

Proof. The corollary is a straightforward consequence of Propositions 2.3.3 and 2.3.6. □

The coproduct $\Delta : M\to M\otimes M$ is given by:

$$\Delta(x)=V(x\otimes 1)V^{*}=W^{*}(1\otimes x)W,\quad x\in M.$$

We denote $\widehat{\Delta}:\widehat{M'}\to\widehat{M'}\otimes\widehat{M'}$ the coproduct of $(\widehat{\mathcal{G}})^{\mathrm{c}}$. We have

$$\widehat{\Delta}(x)=V^{*}(1\otimes x)V=\widetilde{V}(x\otimes 1)\widetilde{V}^{*},\quad x\in\widehat{M'}.$$

Finally, we will use a coproduct $\widehat{\Delta}_{\lambda}$ of $\widehat{\mathcal{G}}$ different from the one given in [9]. The coproduct $\widehat{\Delta}_{\lambda}:\widehat{M}\to\widehat{M}\otimes\widehat{M}$ is given by:

$$\widehat{\Delta}_{\lambda}(x)=W(x\otimes 1)W^{*},\quad x\in\widehat{M}.$$

2.4 Weak Hopf-C*-algebra associated with a measured quantum groupoid on a finite basis

We recall - with different notations and conventions - the construction provided by De Commer in [9] of the weak Hopf-C*-algebra associated with a measured quantum groupoid on a finite basis. Let us fix a measured quantum groupoid $\mathcal{G}=(N,M,\alpha,\beta,\Delta,T,T',\epsilon)$ on the finite-dimensional basis $N=\bigoplus_{1\leqslant l\leqslant k}\mathsf{M}_{n_l}(\mathbb{C})$. Following [2], we denote:

$$L(\omega)=(\omega\otimes\mathrm{id})(V),\quad \rho(\omega)=(\mathrm{id}\otimes\omega)(V),\quad \omega\in\mathcal{B}(\mathscr{H})_{*}.$$

We define S (resp. $\widehat{S}$) to be the norm closure of the subalgebra

$$\{L(\omega)\,;\,\omega\in\mathcal{B}(\mathscr{H})_{*}\}\subset\mathcal{B}(\mathscr{H})\quad(\text{resp. }\{\rho(\omega)\,;\,\omega\in\mathcal{B}(\mathscr{H})_{*}\}\subset\mathcal{B}(\mathscr{H})).$$

In [9], one denotes D (resp. $\widehat{D}$) the norm closure of the subalgebra

$$\{(\mathrm{id}\otimes\omega)(W)\,;\,\omega\in\mathcal{B}(\mathscr{H})_{*}\}\subset\mathcal{B}(\mathscr{H})\quad(\text{resp. }\{(\omega\otimes\mathrm{id})(W)\,;\,\omega\in\mathcal{B}(\mathscr{H})_{*}\}\subset\mathcal{B}(\mathscr{H})).$$

According to Proposition 11.2.1 of [9], D and $\widehat{D}$ are C*-subalgebras of $\mathcal{B}(\mathscr{H})$. Note that we have:

Lemma 2.4.1. $S = D, \quad \widehat{S} = U\widehat{D}U^*$.

Proof. Indeed, since $W = \widehat{V}$ we have $(\mathrm{id}\otimes\omega)(W) = L(\omega_U)$ for all $\omega \in \mathcal{B}(\mathscr{H})_*$ (see the notations of §1.1.1). This proves that $D \subset S$. Conversely, if $\omega \in \mathcal{B}(\mathscr{H})_*$ we have $L(\omega) = (\mathrm{id} \otimes \omega_{U^*})(W)$ since $\omega = (\omega_{U^*})_U$. Hence, $S = D$. Since $(\omega \otimes \mathrm{id})(W) = U^*\rho(\omega)U$ for all $\omega \in \mathcal{B}(\mathscr{H})_*$, we obtain $\widehat{S} = U\widehat{D}U^*$. □

Proposition 2.4.2. *S and $\widehat{S}$ are non-degenerate* C*-*subalgebras of $\mathcal{B}(\mathscr{H})$ weakly dense in respectively M and $\widehat{M}'$.*

Proof. The fact that S and $\widehat{S}$ are C*-subalgebras of $\mathcal{B}(\mathscr{H})$ is a straightforward consequence of Proposition 11.2.1 of [9] and Lemma 2.4.1. The fact that S (resp. $\widehat{S}$) is weakly dense in M (resp. $\widehat{M}'$) follows from the fact that M (resp. $\widehat{M}'$) is the weak closure of $\{L(\omega)\,;\,\omega \in \mathcal{B}(\mathscr{H})_*\}$ (resp. $\{\rho(\omega)\,;\,\omega \in \mathcal{B}(\mathscr{H})_*\}$) in $\mathcal{B}(\mathscr{H})$ (*cf.* Lemma 11.1.5 [9]). The non-degeneracy of S (resp. $\widehat{S}$) follows directly from the fact that S (resp. $\widehat{S}$) is strongly dense in M (resp. $\widehat{M}'$). □

Notations 2.4.3. We endow S and $\widehat{S}$ with two faithful non-degenerate *-representations:

$$L : S \to \mathcal{B}(\mathscr{H})\,;\, y \mapsto y, \quad R : S \to \mathcal{B}(\mathscr{H})\,;\, y \mapsto UyU^*,$$

$$\rho : \widehat{S} \to \mathcal{B}(\mathscr{H})\,;\, x \mapsto x, \quad \lambda : \widehat{S} \to \mathcal{B}(\mathscr{H})\,;\, x \mapsto UxU^*.$$

We also have $R(y) = U^*yU$ and $\lambda(x) = U^*xU$ for all $y \in S$ and $x \in \widehat{S}$.

Proposition 2.4.4. (*see Proposition 11.2.2 of* [9])

1. $\alpha(N) \subset \mathcal{M}(S), \quad \beta(N^{\mathrm{o}}) \subset \mathcal{M}(S), \quad \beta(N^{\mathrm{o}}) \subset \mathcal{M}(\widehat{S}), \quad \widehat{\alpha}(N) \subset \mathcal{M}(\widehat{S})$.
2. $V \in \mathcal{M}(\widehat{S} \otimes S), \quad W \in \mathcal{M}(S \otimes \lambda(\widehat{S})), \quad \widetilde{V} \in \mathcal{M}(R(S) \otimes \widehat{S})$.
3. Δ (*resp.* $\widehat{\Delta}$) *restricts to a* *-*homomorphism* $\delta : S \to \mathcal{M}(S \otimes S)$ (*resp.* $\widehat{\delta} : \widehat{S} \to \mathcal{M}(\widehat{S} \otimes \widehat{S})$).
4. δ (*resp.* $\widehat{\delta}$) *extends uniquely to a strictly continuous* *-*homomorphism*

$$\delta : \mathcal{M}(S) \to \mathcal{M}(S \otimes S) \quad (\textit{resp. } \widehat{\delta} : \mathcal{M}(\widehat{S}) \to \mathcal{M}(\widehat{S} \otimes \widehat{S})),$$

   which satisfies $\delta(1_S) = q^{\beta,\alpha}$ (*resp.* $\widehat{\delta}(1_{\widehat{S}}) = q^{\widehat{\alpha},\beta}$).

5. δ *and* $\widehat{\delta}$ *are coassociative, that is* $(\delta \otimes \mathrm{id}_S)\delta = (\mathrm{id}_S \otimes \delta)\delta$ *and* $(\widehat{\delta} \otimes \mathrm{id}_{\widehat{S}})\widehat{\delta} = (\mathrm{id}_{\widehat{S}} \otimes \widehat{\delta})\widehat{\delta}$.
6. *We have:*

$$[\delta(S)(1_S \otimes S)] = \delta(1_S)(S \otimes S) = [\delta(S)(S \otimes 1_S)],$$

$$[\delta(\widehat{S})(1_{\widehat{S}} \otimes \widehat{S})] = \widehat{\delta}(1_{\widehat{S}})(\widehat{S} \otimes \widehat{S}) = [\widehat{\delta}(\widehat{S})(\widehat{S} \otimes 1_{\widehat{S}})].$$

7. *The unital faithful* *-*homomorphisms* $\alpha : N \to \mathcal{M}(S)$ *and* $\beta : N^{\mathrm{o}} \to \mathcal{M}(S)$ *satisfy:*

$$\delta(\alpha(n)) = \delta(1_S)(\alpha(n) \otimes 1_S), \quad \delta(\beta(n^{\mathrm{o}})) = \delta(1_S)(1_S \otimes \beta(n^{\mathrm{o}})), \quad n \in N.$$

8. *The unital faithful* *-*homomorphisms* $\beta : N^{\mathrm{o}} \to \mathcal{M}(\widehat{S})$ *and* $\widehat{\alpha} : N \to \mathcal{M}(\widehat{S})$ *satisfy:*

$$\widehat{\delta}(\beta(n^{\mathrm{o}})) = \widehat{\delta}(1_{\widehat{S}})(\beta(n^{\mathrm{o}}) \otimes 1_{\widehat{S}}), \quad \widehat{\delta}(\widehat{\alpha}(n)) = \widehat{\delta}(1_{\widehat{S}})(1_{\widehat{S}} \otimes \widehat{\alpha}(n)), \quad n \in N.$$

Note that we have:

$$\delta(y) = W^*(1 \otimes y)W = V(y \otimes 1)V^*, \quad y \in S \quad ; \quad \widehat{\delta}(x) = V^*(1 \otimes x)V = \widetilde{V}(x \otimes 1)\widetilde{V}^*, \quad x \in \widehat{S}.$$

Definition 2.4.5. With the above notations, we call the pair (S, δ) (resp. $(\widehat{S}, \widehat{\delta})$) the weak Hopf-C*-algebra (resp. dual weak Hopf-C*-algebra) associated with the measured quantum groupoid $\mathcal{G}$.

Remarks 2.4.6. With the notations of the above definition, we have:

1. $(\widehat{S}, \widehat{\delta})$ is the weak Hopf-C*-algebra of $\widehat{\mathcal{G}}^{\mathrm{c}}$ while its dual weak Hopf-C*-algebra is $(R(S), \delta_R)$, where $R(S) = USU^*$ and the coproduct δ_R is given by $\delta_R(y) = \widetilde{V}^*(1 \otimes y)\widetilde{V}$ for all $y \in R(S)$.
2. The weak Hopf-C*-algebra of $\widehat{\mathcal{G}}$ is $(\lambda(\widehat{S}), \widehat{\delta}_\lambda)$, where $\widehat{\delta}_\lambda$ is given by $\widehat{\delta}_\lambda(x) = W(x \otimes 1)W^*$, for all $x \in \lambda(\widehat{S})$.

2.5 Measured quantum groupoids and monoidal equivalence of locally compact quantum groups

We will recall the construction of the measured quantum groupoid associated with a monoidal equivalence between two locally compact quantum groups provided by De Commer in his thesis [9]. First of all, we will need to recall the definitions and the crucial results of De Commer.

Definition 2.5.1. Let $\mathbb{G}$ be a locally compact quantum group. A right (resp. left) Galois action of $\mathbb{G}$ on a von Neumann algebra N is an ergodic integrable coaction $\alpha_N : N \to N \otimes \mathrm{L}^\infty(\mathbb{G})$ (resp. $\gamma_N : N \to \mathrm{L}^\infty(\mathbb{G}) \otimes N$) such that the crossed product $N \rtimes_{\alpha_N} \mathbb{G}$ (resp. $\mathbb{G} \ltimes_{\gamma_N} N$) is a type I factor. Then, the pair (N, α_N) (resp. (N, γ_N)) is called a right (resp. left) Galois object for $\mathbb{G}$.

Let $\mathbb{G}$ be a locally compact quantum group and let us fix a right Galois object (N, α_N) for $\mathbb{G}$. In his thesis, De Commer was able to build a locally compact quantum group $\mathbb{H}$ equipped with a left Galois action γ_N on N commuting with α_N, that is $(\mathrm{id} \otimes \alpha_N)\gamma_N = (\gamma_N \otimes \mathrm{id})\alpha_N$. This construction is called the *reflection technique* and $\mathbb{H}$ is called the *reflected locally compact quantum group across* (N, α_N).
In a canonical way, he was also able to associate a right Galois object (O, α_O) for $\mathbb{H}$ and a left Galois action $\gamma_O : O \to \mathrm{L}^\infty(\mathbb{G}) \otimes O$ of $\mathbb{G}$ on O commuting with α_O. Finally, De Commer has built a measured quantum groupoid

$$\mathcal{G}_{\mathbb{H},\mathbb{G}} = (\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon),$$

where $M = \mathrm{L}^\infty(\mathbb{H}) \oplus N \oplus O \oplus \mathrm{L}^\infty(\mathbb{G})$, $\Delta : M \to M \otimes M$ is made up of the coactions and coproducts of the constituents of M, the operator-valued weights T and T' are given by the invariants weights and the non-normalized Markov trace ϵ on $\mathbb{C}^2$ is simply given by $\epsilon(a, b) = a + b$, $(a, b) \in \mathbb{C}^2$. Moreover, the source and target maps α and β have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$.
Conversely, if $\mathcal{G} = (\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ is a measured quantum groupoid whose source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$, then $\mathcal{G}$ is of the form $\mathcal{G}_{\mathbb{H},\mathbb{G}}$ in a unique way, where $\mathbb{H}$ and $\mathbb{G}$ are locally compact quantum groups canonically associated with $\mathcal{G}$.

In what follows, we fix a measured quantum groupoid $\mathcal{G} = (\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ whose source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$. It is worth noticing that for such a groupoid we have:

Lemma 2.5.2. $\widehat{\alpha} = \beta, \quad \widehat{\beta} = \alpha.$

Proof. We recall that we have $\widehat{\beta}(n) = J\alpha(n)^*J$, for all $n \in \mathbb{C}^2$. For $n \in \mathbb{C}^2$, $\alpha(n) \in \mathcal{Z}(M)$, hence $J\alpha(n)^*J = \alpha(n)$. Thus, we have $\widehat{\beta}(n) = \alpha(n)$, for all $n \in \mathbb{C}^2$. Since $\widehat{\alpha}(n) = U^*\alpha(n)U$ and $\widehat{\beta}(n) = U\beta(n)U^*$ for all $n \in \mathbb{C}^2$, we also have $\beta = \widehat{\alpha}$. □

Following the notations introduced in [9], we want to investigate more precisely the left and right regular representations W and V of $\mathcal{G}$ introduced in the previous section. We identify M with its image by π in $\mathcal{B}(\mathscr{H})$, where $(\mathscr{H}, \pi, \Lambda)$ is the G.N.S. construction for M endowed with the n.s.f. weight $\varphi = \epsilon \circ \alpha^{-1} \circ T$. We also consider the n.s.f. weight $\psi = \epsilon \circ \beta^{-1} \circ T'$. Let us denote $(\varepsilon_1, \varepsilon_2)$ the canonical basis of the vector space $\mathbb{C}^2$.

Notations 2.5.3. Let us introduce some useful notations and make some remarks concerning them:

- For $i, j = 1, 2$, we define the following nonzero central self-adjoint projection of M:

  $$p_{ij} = \alpha(\varepsilon_i)\beta(\varepsilon_j).$$

  It follows from $\beta(\varepsilon_1) + \beta(\varepsilon_2) = 1_M$ and $\alpha(\varepsilon_1) + \alpha(\varepsilon_2) = 1_M$ that:

  $$\alpha(\varepsilon_i) = p_{i1} + p_{i2}, \quad \beta(\varepsilon_j) = p_{1j} + p_{2j}, \quad i, j = 1, 2.$$

- We have $\Delta(1) = \alpha(\varepsilon_1) \otimes \beta(\varepsilon_1) + \alpha(\varepsilon_2) \otimes \beta(\varepsilon_2)$ and $\widehat{\Delta}(1) = \beta(\varepsilon_1) \otimes \beta(\varepsilon_1) + \beta(\varepsilon_2) \otimes \beta(\varepsilon_2)$ since $\widehat{\alpha} = \beta$.

- Let us denote $M_{ij} = p_{ij}M$, for $i, j = 1, 2$. Then, M_{ij} is a von Neumann subalgebra of M.

- Let us also denote $\mathscr{H}_{ij} = p_{ij}\mathscr{H}$, for $i, j = 1, 2$. Then, $\mathscr{H}_{ij}$ is a nonzero Hilbert subspace of $\mathscr{H}$ for all $i, j = 1, 2$.

- $\varphi_{ij} = \varphi\restriction_{(M_{ij})_+}$, $\psi_{ij} = \psi\restriction_{(M_{ij})_+}$, for $i, j = 1, 2$. Then, φ_{ij} and ψ_{ij} are n.s.f. weights on M_{ij}.

- For all $i, j, k = 1, 2$, we denote $\Delta_{ij}^k : M_{ij} \to M_{ik} \otimes M_{kj}$ the unital normal *-homomorphism given by

  $$\Delta_{ij}^k(x_{ij}) = (p_{ik} \otimes p_{kj})\Delta(x_{ij}), \quad x_{ij} \in M_{ij}.$$

- We have $Jp_{kl} = p_{kl}J$, $\widehat{J}p_{kl} = p_{lk}\widehat{J}$ and $Up_{kl} = p_{lk}U$ for $k, l = 1, 2$. We define the anti-unitaries $J_{kl} : \mathscr{H}_{kl} \to \mathscr{H}_{kl}$, $\widehat{J}_{kl} : \mathscr{H}_{kl} \to \mathscr{H}_{lk}$ and the unitary $U_{kl} : \mathscr{H}_{kl} \to \mathscr{H}_{lk}$ by setting $J_{kl} = p_{kl}Jp_{kl}$, $\widehat{J}_{kl} = p_{lk}\widehat{J}p_{kl}$ and $U_{kl} = p_{lk}Up_{kl} = \widehat{J}_{kl}J_{kl}$.

- For all $i, j, k, l = 1, 2$, we denote $\Sigma_{ij\otimes kl} := \Sigma_{\mathscr{H}_{ij}\otimes\mathscr{H}_{kl}} : \mathscr{H}_{ij} \otimes \mathscr{H}_{kl} \to \mathscr{H}_{kl} \otimes \mathscr{H}_{ij}$ the flip map.

We readily obtain:

$$M = \bigoplus_{i,j=1,2} M_{ij}, \quad \mathscr{H} = \bigoplus_{i,j=1,2} \mathscr{H}_{ij}, \quad \Delta(p_{ij}) = p_{i1} \otimes p_{1j} + p_{i2} \otimes p_{2j}, \text{ for all } i, j = 1, 2.$$

Note that in terms of the parts Δ_{ij}^k of Δ, the coassociativity condition reads as follows:

$$(\Delta_{ik}^l \otimes \mathrm{id}_{M_{kj}})\Delta_{ij}^k(x_{ij}) = (\mathrm{id}_{M_{il}} \otimes \Delta_{lj}^k)\Delta_{ij}^l(x_{ij}), \quad x_{ij} \in M_{ij}, \quad i, j, k, l = 1, 2.$$

The G.N.S. representation for (M_{ij}, φ_{ij}) is obtained by restriction of the G.N.S. representation of (M, φ) to M_{ij}. In particular, the G.N.S. space $\mathscr{H}_{\varphi_{ij}}$ is identified with $\mathscr{H}_{ij}$.

Proposition 2.5.4. *For all $i,j,k,l=1,2$, we have:*

$$(p_{ij}\otimes 1_{\mathscr{H}})V(p_{kl}\otimes 1_{\mathscr{H}})=\delta^i_k\cdot(p_{ij}\otimes p_{jl})V(p_{il}\otimes p_{jl}),$$
$$(1_{\mathscr{H}}\otimes p_{ij})W(1_{\mathscr{H}}\otimes p_{kl})=\delta^l_j\cdot(p_{ik}\otimes p_{ij})W(p_{ik}\otimes p_{kj}),$$
$$(1_{\mathscr{H}}\otimes p_{ji})\widetilde{V}(1_{\mathscr{H}}\otimes p_{lk})=\delta^l_j\cdot(p_{ki}\otimes p_{ji})\widetilde{V}(p_{ki}\otimes p_{jk}).$$

Notations 2.5.5. Therefore, V, W and $\widetilde{V}$ each splits up into eight unitaries

$$V^i_{jl}:\mathscr{H}_{il}\otimes\mathscr{H}_{jl}\to\mathscr{H}_{ij}\otimes\mathscr{H}_{jl},\quad W^j_{ik}:\mathscr{H}_{ik}\otimes\mathscr{H}_{kj}\to\mathscr{H}_{ik}\otimes\mathscr{H}_{ij},\quad \widetilde{V}^j_{ki}:\mathscr{H}_{ki}\otimes\mathscr{H}_{jk}\to\mathscr{H}_{ki}\otimes\mathscr{H}_{ji},$$

for $i,j,k,l=1,2$ given by

$$V^i_{jl}=(p_{ij}\otimes p_{jl})V(p_{il}\otimes p_{jl}),\quad W^j_{ik}=(p_{ik}\otimes p_{ij})W(p_{ik}\otimes p_{kj}),\quad \widetilde{V}^j_{ki}=(p_{ki}\otimes p_{ji})\widetilde{V}(p_{ki}\otimes p_{jk}).$$

It follows from Proposition 2.3.4 that these unitaries are related to each other by the following relations:

$$W^j_{ik}=\Sigma_{ij\otimes ik}(U_{ji}\otimes 1_{\mathscr{H}_{ik}})V^j_{ik}(U^*_{jk}\otimes 1_{\mathscr{H}_{ik}})\Sigma_{ik\otimes kj},\quad \widetilde{V}^j_{ki}=\Sigma_{ji\otimes ki}(1_{\mathscr{H}_{ji}}\otimes U_{ik})V^j_{ik}(1_{\mathscr{H}_{jk}}\otimes U^*_{ik})\Sigma_{ki\otimes jk},$$

$$\widetilde{V}^j_{ki}=(U_{ik}\otimes U_{ij})W^j_{ik}(U^*_{ik}\otimes U^*_{kj}).$$

Furthermore, we also have:

$$(V^i_{jl})^*=(J_{il}\otimes\widehat{J}_{lj})V^i_{lj}(J_{ij}\otimes\widehat{J}_{jl}),\quad (W^l_{ik})^*=(\widehat{J}_{ki}\otimes J_{kj})W^j_{ki}(\widehat{J}_{ik}\otimes J_{ij}).$$

Moreover, these unitaries satisfy the following pentagon equations

$$\begin{gathered}(V^i_{jk})_{12}(V^i_{kl})_{13}(V^j_{kl})_{23}=(V^j_{kl})_{23}(V^i_{jl})_{12},\quad (W^k_{ij})_{12}(W^l_{ij})_{13}(W^l_{jk})_{23}=(W^l_{ik})_{23}(W^k_{ij})_{12},\\ (\widetilde{V}^k_{ji})_{12}(\widetilde{V}^l_{ji})_{13}(\widetilde{V}^l_{kj})_{23}=(\widetilde{V}^l_{ki})_{23}(\widetilde{V}^k_{ji})_{12},\end{gathered}\tag{2.5.1}$$

and the following commutation relations

$$(V^l_{kj})_{23}(W^j_{ll'})_{12}=(W^k_{ll'})_{12}(V^{l'}_{kj})_{23},\quad (V^l_{ki})_{12}(\widetilde{V}^j_{ki})_{23}=(\widetilde{V}^j_{ki})_{23}(V^l_{ki})_{12}.\tag{2.5.2}$$

We derive easily the relations (2.5.1) and (2.5.2) from the corresponding relations satisfied by V and W. Furthermore, we have the formulas

$$\Delta^k_{ij}(x_{ij})=(W^j_{ik})^*(1_{\mathscr{H}_{ik}}\otimes x_{ij})W^j_{ik}=V^i_{kj}(x_{ij}\otimes 1_{\mathscr{H}_{kj}})(V^i_{kj})^*,\quad x_{ij}\in M_{ij},\tag{2.5.3}$$

which follow from $\Delta(x)=W^*(1_{\mathscr{H}}\otimes x)W=V(x\otimes 1_{\mathscr{H}})V^*$, $x\in M$. Note that for all $\omega\in\mathcal{B}(\mathscr{H})_*$ we have:

$$\begin{gathered}(\mathrm{id}\otimes p_{jl}\omega p_{jl})(V^i_{jl})=p_{ij}(\mathrm{id}\otimes\omega)(V)p_{il},\quad (p_{ik}\omega p_{ik}\otimes\mathrm{id})(W^j_{ik})=p_{ij}(\omega\otimes\mathrm{id})(W)p_{kj},\\ (p_{ki}\omega p_{ki}\otimes\mathrm{id})(\widetilde{V}^j_{ki})=p_{ji}(\mathrm{id}\otimes\omega)(\widetilde{V})p_{jk}.\end{gathered}\tag{2.5.4}$$

Proposition 2.5.6. *Let $i,j=1,2$, $i\neq j$, we have:*

1. *$\mathbb{G}_i:=(M_{ii},\Delta^i_{ii},\varphi_{ii},\psi_{ii})$ is a locally compact quantum group whose left (resp. right) regular representation is W^i_{ii} (resp. V^i_{ii}).*
2. *(M_{ij},Δ^j_{ij}) is a right Galois object for $\mathbb{G}_j$ and V^i_{jj} is its canonical implementation.*
3. *(M_{ij},Δ^i_{ij}) is a left Galois object for $\mathbb{G}_i$ and W^j_{ii} is its canonical implementation.*

4. *The actions Δ_{ij}^{j} and Δ_{ij}^{i} on M_{ij} commute.*

5. *$W_{12}^{2} = \Sigma\widetilde{G}^{*}\Sigma$ and $W_{21}^{1} = \Sigma\widetilde{H}^{*}\Sigma$, where $\widetilde{G}$ and $\widetilde{H}$ are the Galois isometries associated with the right Galois object $(M_{12}, \Delta_{12}^{2})$ for $\mathbb{G}_2$ and the right Galois object $(M_{21}, \Delta_{21}^{1})$ for $\mathbb{G}_1$ respectively (see Lemma 6.4.1 and Definition 6.4.2 of* [9]).

Let (S, δ) be the weak Hopf-C*-algebra associated with $\mathcal{G}$. Note that

$$p_{ij} = \alpha(\varepsilon_i)\beta(\varepsilon_j) \in \mathcal{Z}(\mathcal{M}(S)), \quad i, j = 1, 2.$$

Let us introduce some notations:

1. Let us denote $S_{ij} = p_{ij}S$, for $i, j = 1, 2$. Then, S_{ij} is a C*-algebra (actually a closed two-sided ideal) of S weakly dense in M_{ij}.

2. In order to provide a description of the coproduct δ, for $i, j, k = 1, 2$ we will consider

$$\iota_{ij}^{k} : \mathcal{M}(S_{ik} \otimes S_{kj}) \to \mathcal{M}(S \otimes S)$$

the unique strictly continuous extension of the inclusion $S_{ik} \otimes S_{kj} \subset S \otimes S$ satisfying $\iota_{ij}^{k}(1_{S_{ik}\otimes S_{kj}}) = p_{ik} \otimes p_{kj}$. Now, let $\delta_{ij}^{k} : S_{ij} \to \mathcal{M}(S_{ik} \otimes S_{kj})$ be the unique *-homomorphism such that

$$\iota_{ij}^{k} \circ \delta_{ij}^{k}(s_{ij}) = (p_{ik} \otimes p_{kj})\delta(s_{ij}), \text{ for all } s_{ij} \in S_{ij}.$$

With these notations, we have:

Proposition 2.5.7. (*see Lemmas 7.4.13 and 7.4.14 of* [9]*, Proposition 2.4.4*)
For all $i, j, k, l = 1, 2$, we have:

1. $(\delta_{ik}^{l} \otimes \mathrm{id}_{S_{kj}})\delta_{ij}^{k} = (\mathrm{id}_{S_{il}} \otimes \delta_{lj}^{k})\delta_{ij}^{l}$.

2. $\delta_{ij}^{k}(s) = (W_{ik}^{j})^{*}(1_{\mathscr{H}_{ik}} \otimes s)W_{ik}^{j} = V_{kj}^{i}(s \otimes 1_{\mathscr{H}_{kj}})(V_{kj}^{i})^{*}$, *for all $s \in S_{ij}$.*

3. $[\delta_{ij}^{k}(S_{ij})(1_{S_{ik}} \otimes S_{kj})] = S_{ik} \otimes S_{kj} = [\delta_{ij}^{k}(S_{ij})(S_{ik} \otimes 1_{S_{kj}})]$. *In particular, we have*

$$S_{kj} = [(\mathrm{id}_{S_{ik}} \otimes \omega)\delta_{ij}^{k}(s)\,;\, s \in S_{ij},\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_{*}].$$

The next result will play a crucial role in the following.

Proposition 2.5.8. *The self-adjoint projections $\beta(\varepsilon_1)$ and $\beta(\varepsilon_2)$ are also multipliers of $\widehat{S}$. They satisfy the following facts:*

1. $\beta(\varepsilon_1) + \beta(\varepsilon_2) = 1_{\widehat{S}} \quad ; \quad \widehat{\delta}(\beta(\varepsilon_j)) = \beta(\varepsilon_j) \otimes \beta(\varepsilon_j)$, $j = 1, 2$.

2. $[\widehat{S}\beta(\varepsilon_j)\widehat{S}] = \widehat{S}$, $j = 1, 2$.

In order to prove the proposition, we will need the following lemma, which is valid for any measured quantum groupoid on a finite basis.

Lemma 2.5.9. *Let us assume that for some self-adjoint projection $n \in N$ the linear span of $\widehat{M}\alpha(n)\widehat{M}$ is weakly dense in $\widehat{M}$. Then, we have $[\widehat{S}\widehat{\alpha}(n)\widehat{S}] = \widehat{S}$.*

Proof. Let us prove that $[(1\otimes\lambda(\widehat{S}))V(1\otimes\lambda(\widehat{S}))V^*] = (\rho(\widehat{S})\otimes\lambda(\widehat{S}))q^{\beta,\alpha}$. By Proposition 2.4.4 6, we have

$$[(\rho(\widehat{S})\otimes 1)\widehat{\delta}(\rho(\widehat{S}))] = (\rho(\widehat{S})\otimes\rho(\widehat{S}))q^{\widehat{\alpha},\beta}. \tag{2.5.5}$$

The following formula

$$V(1\otimes\lambda(x))V^* = (1\otimes U)\Sigma V^*(1\otimes\rho(x))V\Sigma(1\otimes U^*), \quad x\in\widehat{S},$$

is a consequence of Corollary 3.1.4 2 that will be established independently in the next chapter for a general measured quantum groupoid. Since $\widehat{\delta}(\rho(x)) = V^*(1\otimes\rho(x))V$ for all $x\in\widehat{S}$, we obtain

$$(1\otimes\lambda(x'))V(1\otimes\lambda(x))V^* = (1\otimes U)\Sigma(\rho(x')\otimes 1)\widehat{\delta}(\rho(x))\Sigma(1\otimes U^*), \quad x,x'\in\widehat{S}. \tag{2.5.6}$$

It follows from (2.5.5) and (2.5.6) that

$$[(1\otimes\lambda(\widehat{S}))V(1\otimes\lambda(\widehat{S}))V^*] = (1\otimes U)\Sigma(\rho(\widehat{S})\otimes\rho(\widehat{S}))q^{\widehat{\alpha},\beta}\Sigma(1\otimes U^*) = (\rho(\widehat{S})\otimes\lambda(\widehat{S}))q^{\beta,\alpha}. \tag{2.5.7}$$

Now, since $V = Vq^{\widehat{\alpha},\beta}$ and $\lambda(\widehat{S})\subset\widehat{M}\subset\beta(N^{\mathrm{o}})'$ we have

$$(1\otimes\lambda(x'))V(1\otimes\lambda(x)) = (1\otimes\lambda(x'))V(1\otimes\lambda(x))q^{\widehat{\alpha},\beta} = (1\otimes\lambda(x'))V(1\otimes\lambda(x))V^*V. \tag{2.5.8}$$

Since $q^{\beta,\alpha}V = V$, it follows from (2.5.7) and (2.5.8) that

$$[(1\otimes\lambda(\widehat{S}))V(1\otimes\lambda(\widehat{S}))] = (\rho(\widehat{S})\otimes\lambda(\widehat{S}))q^{\beta,\alpha}V = (\rho(\widehat{S})\otimes\lambda(\widehat{S}))V. \tag{2.5.9}$$

Therefore, we have

$$[(\mathrm{id}\otimes\omega)(1\otimes\lambda(x'))V(1\otimes\lambda(x))\,;\, x,x'\in\widehat{S},\,\omega\in\mathcal{B}(\mathscr{H})_*] = \rho(\widehat{S}). \tag{2.5.10}$$

Note that (2.5.10) follows more directly from the non-degeneracy of λ. By combining (2.5.10) with the assumption, we obtain

$$[(\mathrm{id}\otimes\omega_{\xi,\eta})(1\otimes\lambda(x'))V(1\otimes\lambda(x)\alpha(n))\,;\, x,x'\in\widehat{S},\,\xi,\eta\in\mathscr{H}] = \widehat{S}.$$

In virtue of (2.5.9), we have

$$(1\otimes\lambda(x'))V(1\otimes\lambda(x)\alpha(n))\in(\widehat{S}\otimes\lambda(\widehat{S}))V(1\otimes\alpha(n)), \quad x,x'\in\widehat{S}.$$

However, we have $V(1\otimes\alpha(n)) = (\widehat{\alpha}(n)\otimes 1)V$ (see Proposition 2.3.5 3). If $u,u'\in\widehat{S}$ and $\omega\in\mathcal{B}(\mathscr{H})_*$, we have

$$(\mathrm{id}\otimes\omega)((u\otimes\lambda(u'))V(1\otimes\alpha(n))) = u\widehat{\alpha}(n)(\mathrm{id}\otimes\omega\lambda(u'))(V)\in\widehat{S}\widehat{\alpha}(n)\widehat{S}.$$

As a result, we finally obtain $\widehat{S}\subset\widehat{S}\widehat{\alpha}(n)\widehat{S}$ and the converse inclusion is obvious since we have $\widehat{\alpha}(N)\subset\mathcal{M}(\widehat{S})$ (see Proposition 2.4.4 1). □

Proof of Proposition 2.5.8. Firstly, we have $\beta(\varepsilon_1)+\beta(\varepsilon_2) = 1_{\widehat{S}}$ as β is unital. Secondly, we have $\widehat{\delta}(\beta(\varepsilon_j)) = \widehat{\delta}(1_{\widehat{S}})(1_{\widehat{S}}\otimes\beta(\varepsilon_j))$ and $\widehat{\delta}(1_{\widehat{S}}) = \beta(\varepsilon_1)\otimes\beta(\varepsilon_1)+\beta(\varepsilon_2)\otimes\beta(\varepsilon_2)$ as $\widehat{\alpha}=\beta$. Hence, $\widehat{\delta}(\beta(\varepsilon_j)) = \beta(\varepsilon_j)\otimes\beta(\varepsilon_j)$ since $\beta(\varepsilon_k)\beta(\varepsilon_j) = \delta_k^j\beta(\varepsilon_j)$, $k=1,2$. In virtue of Proposition 7.4.3 of [9], the linear span of $\widehat{M}\alpha(\varepsilon_j)\widehat{M}$ is weakly dense in $\widehat{M}$. By the previous lemma, it then follows that $[\widehat{S}\widehat{\alpha}(\varepsilon_j)\widehat{S}] = \widehat{S}$. The Proposition is then proved since in our case we have $\widehat{\alpha}=\beta$. □

Notations 2.5.10. Let $i, j, k = 1, 2$. Let $x \in \widehat{S}$ and $y \in \lambda(\widehat{S})$. We denote:

$$x_{jk} = \beta(\varepsilon_j)x\beta(\varepsilon_k) \in \widehat{S}, \quad \widehat{\pi}_i(x_{jk}) = p_{ij}xp_{ik} \quad ; \quad y_{jk} = \alpha(\varepsilon_j)y\alpha(\varepsilon_k) \in \lambda(\widehat{S}), \quad \widehat{\pi}^i(y_{jk}) = p_{ji}yp_{ki}.$$

Note that we have $\lambda(x)_{jk} = \lambda(x_{jk})$ since $\widehat{\alpha} = \beta$. We also denote E^i_{jk} (resp. $E^i_{jk,\lambda}$) the norm closed linear subspace

$$\{\widehat{\pi}_i(x_{jk})\,;\, x \in \widehat{S}\} \subset \mathcal{B}(\mathscr{H}_{ik}, \mathscr{H}_{ij}) \quad (\text{resp. } \{\widehat{\pi}^i(y_{jk})\,;\, y \in \lambda(\widehat{S})\} \subset \mathcal{B}(\mathscr{H}_{ki}, \mathscr{H}_{ji})).$$

Note that we have $E^i_{jk,\lambda} = U_{ij}E^i_{jk}U^*_{ik}$.

Proposition 2.5.11. *Let $i, j, k, l = 1, 2$, we have:*

1. *Let $\omega \in \mathcal{B}(\mathscr{H})_*$, if $x = (\mathrm{id} \otimes \omega)(V)$ we have:*

$$\widehat{\pi}_i(x_{jl}) = (\mathrm{id} \otimes p_{jl}\omega p_{jl})(V^i_{jl}), \quad \widehat{\pi}^j(\lambda(x_{ik})) = (p_{ik}\omega p_{ik} \otimes \mathrm{id})(W^j_{ik}).$$

2. $[E^i_{jl}\mathscr{H}_{il}] = \mathscr{H}_{ij}$.

3. $(E^i_{jl})^* = E^i_{lj}$, $[E^i_{jk}E^i_{kl}] = E^i_{jl}$.

In particular, E^i_{jj} is a non-degenerate C**-subalgebra of $\mathcal{B}(\mathscr{H}_{ij})$. Moreover, E^i_{12} is a Morita equivalence between E^i_{11} and E^i_{22}.*

Proof. The first assertion is just a restatement of (2.5.4). 2) follows from the first statement and the fact that V^i_{jl} is unitary (the proof is very similar to that of Proposition 1.4 in [2]).
3) The equality $(E^i_{jl})^* = E^i_{lj}$ and the inclusion $[E^i_{jk}E^i_{kl}] \subset E^i_{jl}$ are straightforward. Now, let $\omega, \omega' \in \mathcal{B}(\mathscr{H})_*$ and let us set $x = \rho(\omega)$ and $x' = \rho(\omega')$. Let $\phi \in \mathcal{B}(\mathscr{H})_*$ be the normal linear form given for all $z \in \mathcal{B}(\mathscr{H})$ by:

$$\phi(z) := (p_{jk}\omega p_{jk} \otimes p_{kl}\omega' p_{kl})(V(z \otimes 1)V^*) = (p_{jk}\omega p_{jk} \otimes p_{kl}\omega' p_{kl})(V^j_{kl}(p_{jl}zp_{jl} \otimes 1)(V^j_{kl})^*).$$

Let us set $y = \rho(\phi)$. By using the first statement, the fact that $p_{jl}\phi p_{jl} = \phi$ and the pentagon equation, we obtain

$$\begin{aligned}
\widehat{\pi}_i(y_{jl}) &= (\mathrm{id} \otimes \phi)(V^i_{jl}) \\
&= (\mathrm{id} \otimes p_{jk}\omega p_{jk} \otimes p_{kl}\omega' p_{kl})((V^j_{kl})_{23}(V^i_{jl})_{12}(V^j_{kl})^*_{23}) \\
&= (\mathrm{id} \otimes p_{jk}\omega p_{jk} \otimes p_{kl}\omega' p_{kl})((V^i_{jk})_{12}(V^i_{kl})_{13}) \\
&= \widehat{\pi}_i(x_{jk})\widehat{\pi}_i(x'_{kl}).
\end{aligned}$$

Since each operator V^j_{kl} is unitary, it follows that the linear subspace of $\mathcal{B}(\mathscr{H}_{jl})_*$ spanned by the normal linear forms ϕ, where ω and ω' run through $\mathcal{B}(\mathscr{H})_*$, is norm dense in $\mathcal{B}(\mathscr{H}_{jl})_*$. Hence, $E^i_{jl} \subset [E^i_{jk}E^i_{kl}]$. □

Remark 2.5.12. Since $E^i_{jk,\lambda} = U_{ij}E^i_{jk}U^*_{ik}$, we also have:

$$[E^i_{jl,\lambda}\mathscr{H}_{li}] = \mathscr{H}_{ji}, \quad (E^i_{jl,\lambda})^* = E^i_{lj,\lambda}, \quad [E^i_{jk,\lambda}E^i_{kl,\lambda}] = E^i_{jl,\lambda}.$$

Corollary 2.5.13. *For all $i = 1, 2$, we have the following faithful non-degenerate *-representations:*

$$\widehat{\pi}_i : \widehat{S} \to \mathcal{B}(\mathscr{H}_{i1} \oplus \mathscr{H}_{i2}), \quad \widehat{\pi}^i : \lambda(\widehat{S}) \to \mathcal{B}(\mathscr{H}_{1i} \oplus \mathscr{H}_{2i})$$

given for all $x \in \widehat{S}$ and $y \in \lambda(\widehat{S})$ by:

$$\widehat{\pi}_i(x) = \begin{pmatrix} p_{i1}xp_{i1} & p_{i1}xp_{i2} \\ p_{i2}xp_{i1} & p_{i2}xp_{i2} \end{pmatrix}, \quad \widehat{\pi}^i(y) = \begin{pmatrix} p_{1i}yp_{1i} & p_{1i}yp_{2i} \\ p_{2i}yp_{1i} & p_{2i}yp_{2i} \end{pmatrix}.$$

Note that we have $\widehat{\pi}_i(x) = \alpha(\varepsilon_i)x\alpha(\varepsilon_i)$ and $\widehat{\pi}^i(y) = \beta(\varepsilon_i)y\beta(\varepsilon_i)$ for all $x \in \widehat{S}$ and $y \in \lambda(\widehat{S})$. For all $i, j, k, l = 1, 2$, we have:

$$(\widehat{\pi}_k \otimes \widehat{\pi}_l)\,\widehat{\delta}(x_{ij}) = (V^k_{li})^*(1_{\mathscr{H}_{kl}} \otimes \widehat{\pi}_l(x_{ij}))V^k_{lj} = \widetilde{V}^l_{ki}(\widehat{\pi}_k(x_{ij}) \otimes 1_{\mathscr{H}_{lk}})(\widetilde{V}^l_{kj})^*, \quad x \in \widehat{S},$$

$$(\widehat{\pi}^k \otimes \widehat{\pi}^l)\widehat{\delta}_\lambda(y_{ij}) = W^l_{ik}(\widehat{\pi}^k(y_{ij}) \otimes 1_{\mathscr{H}_{kl}})(W^l_{jk})^*, \quad y \in \lambda(\widehat{S}).$$

Proof. The non-degeneracy of the *-representation $\widehat{\pi}_i$ follows from the second statement of Proposition 2.5.11. Let $x \in \widehat{S}$ such that $\widehat{\pi}_i(x) = 0$, which means that $\alpha(\varepsilon_i)x\alpha(\varepsilon_i) = 0$. However, $\alpha(\varepsilon_j)y\alpha(\varepsilon_i) = \delta^i_j\alpha(\varepsilon_i)y\alpha(\varepsilon_i)$ for all $y \in \widehat{S}$ and $j = 1, 2$. Hence, $x\alpha(\varepsilon_i) = 0$.
Furthermore, the projection $1 - \alpha(\varepsilon_i)$ belongs to $\widehat{M}$, then $1 - \alpha(\varepsilon_i)$ is the weak limit of finite sums of elements of the form $y\alpha(\varepsilon_i)y'$, where $y, y' \in \widehat{M}$. Therefore, $x = x(1 - \alpha(\varepsilon_i))$ is the weak limit of finite sums of elements of the form $xy\alpha(\varepsilon_i)y'$, where $y, y' \in \widehat{M}$. However, since $\widehat{S} \subset \widehat{M}'$ we have $xy\alpha(\varepsilon_i)y' = yx\alpha(\varepsilon_i)y' = 0$ for all $y, y' \in \widehat{M}$. Hence, $x = 0$.
Since $\widehat{\pi}_i$ is non-degenerate, $\widehat{\pi}_k \otimes \widehat{\pi}_l$ extends uniquely to a unital strictly continuous *-representation of $\mathcal{M}(\widehat{S} \otimes \widehat{S})$. Therefore, for all $i, j = 1, 2$ we have

$$(\widehat{\pi}_k \otimes \widehat{\pi}_l)((\beta(\varepsilon_i) \otimes \beta(\varepsilon_i))z(\beta(\varepsilon_j) \otimes \beta(\varepsilon_j))) = (p_{ki} \otimes p_{li})z(p_{kj} \otimes p_{lj}), \quad z \in \mathcal{M}(\widehat{S} \otimes \widehat{S}).$$

In particular, if $x \in \widehat{S}$ we have

$$\begin{aligned}(\widehat{\pi}_k \otimes \widehat{\pi}_l)\widehat{\delta}(x_{ij}) &= (p_{ki} \otimes p_{li})\widehat{\delta}(x)(p_{kj} \otimes p_{lj}) \\ &= (p_{ki} \otimes p_{li})V^*(1 \otimes x)V(p_{kj} \otimes p_{lj}) \\ &= (V^k_{li})^*(p_{kl} \otimes p_{li}xp_{lj})V^k_{lj} \\ &= (V^k_{li})^*(\mathrm{id}_{\mathscr{H}_{kl}} \otimes \widehat{\pi}_l(x_{ij}))V^k_{lj},\end{aligned}$$

where we have used $\widehat{\delta}(\beta(\varepsilon_r)) = \beta(\varepsilon_r) \otimes \beta(\varepsilon_r)$ for $r = 1, 2$ and the fact that the coproduct $\widehat{\delta} : \widehat{S} \to \mathcal{M}(\widehat{S} \otimes \widehat{S})$ satisfies $\widehat{\delta}(x) = V^*(1 \otimes x)V$. Since we also have $\widehat{\delta}(x) = \widetilde{V}(x \otimes 1)\widetilde{V}^*$ we obtain $(\widehat{\pi}_k \otimes \widehat{\pi}_l)\,\widehat{\delta}(x_{ij}) = \widetilde{V}^l_{ki}(\widehat{\pi}_k(x_{ij}) \otimes 1_{\mathscr{H}_{lk}})(\widetilde{V}^l_{kj})^*$. The proof of the corresponding statements for the *-representation $\widehat{\pi}^i$ is similar. □

Corollary 2.5.14. *Let us recall that we denote*

$$E^i_{kl} = \widehat{\pi}_i(\beta(\varepsilon_k)\widehat{S}\beta(\varepsilon_l)), \quad E^i_{kl,\lambda} = \widehat{\pi}^i(\alpha(\varepsilon_k)\lambda(\widehat{S})\alpha(\varepsilon_l)), \quad i, k, l = 1, 2.$$

For all $i, j, k, l = 1, 2$, $E^i_{lk} \otimes S_{jl}$ and $E^i_{jk} \otimes S_{jl}$ (resp. $S_{ik} \otimes E^j_{kl,\lambda}$ and $S_{ik} \otimes E^j_{il,\lambda}$) are Hilbert C^*-modules over $E^i_{kk} \otimes S_{jl}$ (resp. $S_{ik} \otimes E^j_{ll,\lambda}$) and we have:

$$V^i_{jl} \in \mathcal{L}(E^i_{lk} \otimes S_{jl}, E^i_{jk} \otimes S_{jl}), \quad W^j_{ik} \in \mathcal{L}(S_{ik} \otimes E^j_{kl,\lambda}, S_{ik} \otimes E^j_{il,\lambda}).$$

In particular, we have $V^i_{jj} \in \mathcal{M}(E^i_{jj} \otimes S_{jj})$ and $W^j_{ii} \in \mathcal{M}(S_{ii} \otimes E^j_{ii,\lambda})$.

Proof. This follows immediately from $V \in \mathcal{M}(\widehat{S} \otimes S)$ and $W \in \mathcal{M}(S \otimes \lambda(\widehat{S}))$. □

We conclude this paragraph with the following definitions:

Definition 2.5.15. A measured quantum groupoid $(\mathbb{C}^2, M, \alpha, \beta, \Delta, T, T', \epsilon)$ such that the source and target maps have range in $\mathcal{Z}(M)$ and generate a copy of $\mathbb{C}^4$ will be denoted $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$, where $\mathbb{G}_i = (M_{ii}, \Delta^i_{ii}, \varphi_{ii}, \psi_{ii})$ (see Notations 2.5.3) and will be called a colinking measured quantum groupoid.

Definition 2.5.16. Let $\mathbb{G}$ and $\mathbb{H}$ be two locally compact quantum groups. We say that $\mathbb{G}$ and $\mathbb{H}$ are monoidally equivalent if there exists a colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between two locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ such that $\mathbb{H}$ (resp. $\mathbb{G}$) is isomorphic to $\mathbb{G}_1$ (resp. $\mathbb{G}_2$).

Chapter 3

Irreducibility and regularity for measured quantum groupoids

3.1 Irreducibility for measured quantum groupoids

Let us fix some notations. If $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ is a measured quantum groupoid, we denote $\mathscr{W}_{\mathcal{G}}$ its left regular pseudo-multiplicative unitary. Let $\mathcal{G}$ be a measured quantum groupoid, we recall the following notations:

$$\widehat{\mathscr{V}} := \mathscr{W}_{\mathcal{G}}, \quad \mathscr{V} := \mathscr{W}_{\widehat{(\mathcal{G}^{o})}} = \mathscr{W}_{(\widehat{\mathcal{G}})^{c}}, \quad \widetilde{\mathscr{V}} := \mathscr{W}_{(\mathcal{G}^{o})^{c}}, \quad U := \widehat{J}J.$$

Lemma 3.1.1. *We have the formulas*

$$\widehat{\mathscr{V}} = \sigma^{\widehat{\beta},\alpha}(U\,{}_{\beta}\otimes_{\alpha} 1)\mathscr{V}(U^{*}\,{}_{\alpha}\otimes_{\beta} 1)\sigma^{\beta,\alpha}, \quad \widetilde{\mathscr{V}} = \sigma^{\beta,\widehat{\alpha}}(1\,{}_{\beta}\otimes_{\widehat{\alpha}} U)\mathscr{V}(1\,{}_{\widehat{\alpha}}\otimes_{\beta} U^{*})\sigma^{\widehat{\beta},\widehat{\alpha}}.$$

Proof. In virtue of Theorem 3.11 (iii) and Theorem 3.12 (v) of [14], we have

$$\mathscr{W}_{\mathcal{G}^{o}}({}_{\alpha}\widehat{J}\,{}_{\beta}\otimes_{\beta} J_{\widehat{\alpha}}) = ({}_{\beta}\widehat{J}\,{}_{\alpha}\otimes_{\widehat{\alpha}} J_{\beta})\mathscr{W}^{*}_{\mathcal{G}^{o}}, \quad \widehat{\mathscr{V}} = ({}_{\alpha}\widehat{J}\,{}_{\beta}\otimes_{\widehat{\beta}} \widehat{J}_{\widehat{\alpha}})\mathscr{W}_{\mathcal{G}^{o}}({}_{\alpha}\widehat{J}\,{}_{\beta}\otimes_{\beta} \widehat{J}_{\alpha}).$$

Therefore, we obtain

$$\widehat{\mathscr{V}} = ({}_{\alpha}\widehat{J}\,{}_{\beta}\otimes_{\widehat{\beta}} \widehat{J}_{\widehat{\alpha}})\mathscr{W}_{\mathcal{G}^{o}}({}_{\alpha}\widehat{J}\,{}_{\beta}\otimes_{\beta} J_{\widehat{\alpha}})(1\,{}_{\beta}\otimes_{\widehat{\alpha}} U^{*}) = (1\,{}_{\alpha}\otimes_{\widehat{\beta}} U)\mathscr{W}^{*}_{\mathcal{G}^{o}}(1\,{}_{\beta}\otimes_{\widehat{\alpha}} U^{*}).$$

Hence, $\widehat{\mathscr{V}} = \sigma^{\widehat{\beta},\alpha}(U\,{}_{\beta}\otimes_{\alpha} 1)\mathscr{V}(U^{*}\,{}_{\alpha}\otimes_{\beta} 1)\sigma^{\beta,\alpha}$ as $\mathscr{V} = \mathscr{W}_{\widehat{\mathcal{G}^{o}}} = \sigma^{\alpha,\beta}\mathscr{W}^{*}_{\mathcal{G}^{o}}\sigma^{\alpha,\beta}$.

By using Theorem 3.12 (vi) of [14], we have

$$\widetilde{\mathscr{V}} = (U^{*}\,{}_{\alpha}\otimes_{\beta} U^{*})\widehat{\mathscr{V}}(U\,{}_{\widehat{\beta}}\otimes_{\alpha} U)$$

and the second formula follows. □

Remarks 3.1.2. It follows from this lemma that we can express each unitary $\mathscr{V}$, $\widehat{\mathscr{V}}$ and $\widetilde{\mathscr{V}}$ in terms of one of the others. Indeed, we have:

i. $\mathscr{V} = (U^{*}\,{}_{\widehat{\beta}}\otimes_{\alpha} 1)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\sigma^{\alpha,\beta}(U\,{}_{\widehat{\alpha}}\otimes_{\beta} 1) = \sigma^{\alpha,\beta}(1\,{}_{\alpha}\otimes_{\beta} U^{*})\widehat{\mathscr{V}}(1\,{}_{\beta}\otimes_{\alpha} U)\sigma^{\widehat{\alpha},\beta}$,

ii. $\mathscr{V} = (1\,{}_{\beta}\otimes_{\alpha} U^{*})\sigma^{\widehat{\alpha},\beta}\widetilde{\mathscr{V}}\sigma^{\widehat{\alpha},\widehat{\beta}}(1\,{}_{\widehat{\alpha}}\otimes_{\widehat{\beta}} U) = \sigma^{\alpha,\beta}(U^{*}\,{}_{\widehat{\alpha}}\otimes_{\beta} 1)\widetilde{\mathscr{V}}(U\,{}_{\beta}\otimes_{\widehat{\alpha}} 1)\sigma^{\widehat{\alpha},\beta}$,

iii. $\widetilde{\mathscr{V}} = (U^{*}\,{}_{\alpha}\otimes_{\beta} U^{*})\widehat{\mathscr{V}}(U\,{}_{\widehat{\beta}}\otimes_{\alpha} U)$ and $\widehat{\mathscr{V}} = (U\,{}_{\widehat{\alpha}}\otimes_{\widehat{\beta}} U)\widetilde{\mathscr{V}}(U^{*}\,{}_{\beta}\otimes_{\widehat{\alpha}} U^{*})$.

We will also need the following lemma:

Lemma 3.1.3. *Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid. Let us denote $\widehat{\mathcal{G}} = (N, \widehat{M}, \alpha, \widehat{\beta}, \widehat{\Gamma}, \widehat{T}, \widehat{R} \circ \widehat{T} \circ \widehat{R}, \nu)$ the dual measured quantum groupoid of $\mathcal{G}$. Then, we have*

1. *a)* $\widehat{\mathcal{V}}(x\,{}_\beta\otimes_\alpha 1) = (x\,{}_\alpha\otimes_{\widehat{\beta}} 1)\widehat{\mathcal{V}}$, *b)* $\mathcal{V}(1\,{}_{\widehat{\alpha}}\otimes_\beta x) = (1\,{}_\beta\otimes_\alpha x)\mathcal{V}$, $x \in M'$.
2. *a)* $\mathcal{V}(x\,{}_{\widehat{\alpha}}\otimes_\beta 1) = (x\,{}_\beta\otimes_\alpha 1)\mathcal{V}$, *b)* $\widetilde{\mathcal{V}}(1\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} x) = (1\,{}_{\widehat{\alpha}}\otimes_\beta x)\widetilde{\mathcal{V}}$, $x \in \widehat{M}$.
3. *a)* $\mathcal{V}(x\,{}_{\widehat{\alpha}}\otimes_\beta 1)\mathcal{V}^* = \widehat{\mathcal{V}}^*(1\,{}_\alpha\otimes_{\widehat{\beta}} x)\widehat{\mathcal{V}}$, *b)* $\widetilde{\mathcal{V}}(x\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} 1) = (x\,{}_{\widehat{\alpha}}\otimes_\beta 1)\widetilde{\mathcal{V}}$, $x \in M$.
4. *a)* $\mathcal{V}^*(1\,{}_\beta\otimes_\alpha x)\mathcal{V} = \widetilde{\mathcal{V}}(x\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} 1)\widetilde{\mathcal{V}}^*$, *b)* $\widehat{\mathcal{V}}(1\,{}_\beta\otimes_\alpha x) = (1\,{}_\alpha\otimes_{\widehat{\beta}} x)\widehat{\mathcal{V}}$, $x \in \widehat{M}'$.

Proof. All operators appearing in this statement are well defined thanks to the Heisenberg type relations:

$$M \cap \widehat{M} = \alpha(N), \quad M \cap \widehat{M}' = \beta(N^{\rm o}), \quad M' \cap \widehat{M} = \widehat{\beta}(N^{\rm o}), \quad M' \cap \widehat{M}' = \widehat{\alpha}(N).$$

The proof relies only on Theorems 3.8, 3.11 and 3.12 of [14]. The formula 1 a) follows from the fact that M is the weak closure of the subalgebra $\{(\mathrm{id} \star \omega)(\widehat{\mathcal{V}})\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\} \subset \mathcal{B}(\mathscr{H})$. The formula 1 b) follows by using the formula $\widehat{\mathcal{V}} = \sigma^{\widehat{\beta},\alpha}(U\,{}_\beta\otimes_\alpha 1)\mathcal{V}(U^*\,{}_\alpha\otimes_\beta 1)\sigma^{\beta,\alpha}$. Similarly, 2 follows from the fact that $\widetilde{\mathcal{V}} = \sigma^{\beta,\widehat{\alpha}}(1\,{}_\beta\otimes_{\widehat{\alpha}} U)\mathcal{V}(1\,{}_{\widehat{\alpha}}\otimes_\beta U^*)\sigma^{\widehat{\beta},\widehat{\alpha}}$ and the fact that $\widehat{M}$ is the weak closure of the subalgebra $\{(\omega \star \mathrm{id})(\widehat{\mathcal{V}})\,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$ of $\mathcal{B}(\mathscr{H})$. The formula 3 a) (resp. 4 a)) follows from the first (resp. last) formula of Theorem 3.12 (v) (resp. Theorem 3.12 (vi)). □

Corollary 3.1.4.

1. $\mathcal{V}^*(UxU^*\,{}_\beta\otimes_\alpha 1)\mathcal{V} = (U\,{}_\alpha\otimes_\beta 1)\sigma^{\beta,\alpha}\mathcal{V}(x\,{}_{\widehat{\alpha}}\otimes_\beta 1)\mathcal{V}^*\sigma^{\alpha,\beta}(U^*\,{}_{\widehat{\alpha}}\otimes_\beta 1)$, $x \in M$.
2. $\mathcal{V}(1\,{}_{\widehat{\alpha}}\otimes_\beta UyU^*)\mathcal{V}^* = (1\,{}_\beta\otimes_\alpha U)\sigma^{\widehat{\alpha},\beta}\mathcal{V}^*(1\,{}_\beta\otimes_\alpha y)\mathcal{V}\sigma^{\beta,\widehat{\alpha}}(1\,{}_\beta\otimes_{\widehat{\alpha}} U^*)$, $y \in \widehat{M}'$.
3. $\widetilde{\mathcal{V}}(1\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} UyU^*)\widetilde{\mathcal{V}}^* = (1\,{}_{\widehat{\alpha}}\otimes_\beta U)\sigma^{\widehat{\beta},\widehat{\alpha}}\widetilde{\mathcal{V}}^*(1\,{}_{\widehat{\alpha}}\otimes_\beta y)\widetilde{\mathcal{V}}\sigma^{\widehat{\alpha},\widehat{\beta}}(1\,{}_{\widehat{\alpha}}\otimes_{\widehat{\beta}} U^*)$, $y \in M'$.
4. $[\widehat{\mathcal{V}}_{12}, \mathcal{V}_{23}] = 0$, $\quad [\mathcal{V}_{12}, \widetilde{\mathcal{V}}_{23}] = 0$.

Proof. The first (resp. the second) assertion follows from the formula 3 a) (resp. 4 a)) of Lemma 3.1.3. The third one is obtained by replacing $\mathcal{G}$ by the measured quantum groupoid $(\widehat{\mathcal{G}})^{\rm c}$ in 2. The first (resp. the second) formula of 4 is equivalent to 2 a) (resp. 3 b)). □

We are now able to provide an irreducibility result for measured quantum groupoid.

Theorem 3.1.5. *We have* $(1\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathcal{V}}\mathcal{V}\widetilde{\mathcal{V}} \in \widehat{\beta}(N^{\rm o})\,{}_{\widehat{\beta}}\star_{\widehat{\alpha}}\,\widehat{\alpha}(N)$.

Note that $(1\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathcal{V}}\mathcal{V}\widetilde{\mathcal{V}}$ is a unitary.

Proof. Let us denote $T = (1\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathcal{V}}\mathcal{V}\widetilde{\mathcal{V}}$ for short. By definition, we have

$$\widehat{\beta}(N^{\rm o})\,{}_{\widehat{\beta}}\star_{\widehat{\alpha}}\,\widehat{\alpha}(N) = (\widehat{\beta}(N^{\rm o})'\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}}\,\widehat{\alpha}(N)')',$$

where $\widehat{\beta}(N^{\rm o})'\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}}\,\widehat{\alpha}(N)'$ is the von Neumann algebra generated by elements of the form $x\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} y$, $x \in \widehat{\beta}(N^{\rm o})'$ and $y \in \widehat{\alpha}(N)'$. Then, it amounts to proving that

$$[T, x\,{}_{\widehat{\beta}}\otimes_{\widehat{\alpha}} y] = 0, \quad \text{for all } x \in \widehat{\beta}(N^{\rm o})' \text{ and } y \in \widehat{\alpha}(N)'.$$

First, let us prove that $[T, x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1] = 0$, for all $x \in \widehat{\beta}(N^{\mathrm{o}})'$.

- Let $x \in M$, we have

$$\begin{aligned} T(x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1) &= (1\,\widehat{\otimes}_{\widehat{\beta}\alpha}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}(x\,\widehat{\otimes}_{\widehat{\alpha}\beta}\,1)\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\alpha}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}(x\,\widehat{\otimes}_{\widehat{\alpha}\beta}\,1)\mathscr{V}^*\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\alpha}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\widehat{\mathscr{V}}^*(1\,\widehat{\otimes}_{\alpha\widehat{\beta}}\,x)\widehat{\mathscr{V}}\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\alpha}\,U)\sigma^{\alpha,\widehat{\beta}}(1\,\widehat{\otimes}_{\alpha\widehat{\beta}}\,x)\widehat{\mathscr{V}}\mathscr{V}\widetilde{\mathscr{V}} \\ &= (x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1)T, \end{aligned}$$

where we have successively used Lemma 3.1.3 3 b), the fact that $\mathscr{V}$ is unitary, Lemma 3.1.3 3 a) and the fact that $\widehat{\mathscr{V}}$ is unitary.

- Let $x \in \widehat{M}'$, we have

$$\begin{aligned} T(x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1) &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}\widetilde{\mathscr{V}}(x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1)\widetilde{\mathscr{V}}^*\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}\mathscr{V}^*(1\,\widehat{\otimes}_{\beta\alpha}\,x)\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}(1\,\widehat{\otimes}_{\beta\alpha}\,x)\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}(1\,\widehat{\otimes}_{\alpha\widehat{\beta}}\,x)\widehat{\mathscr{V}}\mathscr{V}\widetilde{\mathscr{V}} \\ &= (x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1)T, \end{aligned}$$

where we have successively used the fact that $\widetilde{\mathscr{V}}$ is unitary, Lemma 3.1.3 4 a), the fact that $\mathscr{V}$ is unitary and Lemma 3.1.3 4 b). Hence, $[T, x\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,1] = 0$ for all $x \in \widehat{\beta}(N^{\mathrm{o}})'$ since $\widehat{\beta}(N^{\mathrm{o}}) = M' \cap \widehat{M}$.

Similarly, let us prove now that $[T, 1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,y] = 0$ for all $y \in \alpha(N)'$.

- Let $y \in \widehat{M}$, we have

$$\begin{aligned} T(1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,y) &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}(1\,\widehat{\otimes}_{\widehat{\alpha}\beta}\,y)\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}\mathscr{V}(1\,\widehat{\otimes}_{\widehat{\alpha}\beta}\,y)\mathscr{V}^*\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}(1\,\widehat{\otimes}_{\beta\alpha}\,U)\sigma^{\widehat{\alpha},\beta}\mathscr{V}^*(1\,\widehat{\otimes}_{\beta\alpha}\,U^*yU)\mathscr{V}\sigma^{\beta,\widehat{\alpha}}(1\,\widehat{\otimes}_{\beta\widehat{\alpha}}\,U^*)\mathscr{V}\widetilde{\mathscr{V}}, \end{aligned}$$

where we have used Lemma 3.1.3 2. b), the fact that $\mathscr{V}$ is unitary and Corollary 3.1.4 2. applied to $U^*yU \in \widehat{M}'$. Now, by using the first formula of Lemma 3.1.1, we obtain

$$(1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)\sigma^{\alpha,\widehat{\beta}}\widehat{\mathscr{V}}(1\,\widehat{\otimes}_{\beta\alpha}\,U)\sigma^{\widehat{\alpha},\beta} = (U\,\widehat{\otimes}_{\beta\widehat{\alpha}}\,U)\mathscr{V}.$$

Therefore, since $\mathscr{V}\mathscr{V}^* = 1$ we have

$$\begin{aligned} T(1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,y) &= (U\,\widehat{\otimes}_{\beta\widehat{\alpha}}\,U)(1\,\widehat{\otimes}_{\beta\alpha}\,U^*yU)\mathscr{V}\sigma^{\beta,\widehat{\alpha}}(1\,\widehat{\otimes}_{\beta\widehat{\alpha}}\,U^*)\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,y)(1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,U)(U\,\widehat{\otimes}_{\beta\alpha}\,1)\mathscr{V}(U^*\,\widehat{\otimes}_{\alpha\beta}\,1)\sigma^{\beta,\alpha}\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1\,\widehat{\otimes}_{\widehat{\beta}\widehat{\alpha}}\,y)T, \end{aligned}$$

by using again the first formula of Lemma 3.1.1.

- Let $y \in M$, we have

$$\begin{aligned} T(1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} y) &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}(\widehat{\mathscr{V}})\mathscr{V}(\widetilde{\mathscr{V}})(1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} y)(\widetilde{\mathscr{V}})^*\widetilde{\mathscr{V}} \\ &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}(\widehat{\mathscr{V}})\mathscr{V}(1_{\widehat{\alpha}} \otimes_{\beta} U)\sigma^{\widehat{\beta},\widehat{\alpha}}(\widetilde{\mathscr{V}})^*(1_{\widehat{\alpha}} \otimes_{\beta} U^*yU)\widetilde{\mathscr{V}}\sigma^{\widehat{\alpha},\widehat{\beta}}(1_{\widehat{\alpha}} \otimes_{\widehat{\beta}} U^*)\widetilde{\mathscr{V}}, \end{aligned}$$

where we have used $(\widetilde{\mathscr{V}})^*\widetilde{\mathscr{V}} = 1$ and Corollary 3.1.4 3 applied to $U^*yU \in M'$. Note that $U^* = \lambda^{-i/4}U = U\lambda^{-i/4}$ and $\lambda^{-i/4} \in \mathcal{Z}(M) \cap \mathcal{Z}(\widehat{M})$. Then, the second formula of Lemma 3.1.1 can be written as follows:

$$\widetilde{\mathscr{V}} = \sigma^{\beta,\widehat{\alpha}}(1_{\beta} \otimes_{\widehat{\alpha}} U^*)\mathscr{V}(1_{\widehat{\alpha}} \otimes_{\beta} U)\sigma^{\widehat{\beta},\widehat{\alpha}}.$$

Hence, $(1_{\beta} \otimes_{\alpha} U)\sigma^{\widehat{\alpha},\beta} = \mathscr{V}(1_{\widehat{\alpha}} \otimes_{\beta} U)\sigma^{\widehat{\beta},\widehat{\alpha}}(\widetilde{\mathscr{V}})^*$. Therefore, we have

$$\begin{aligned} T(1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} y) &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}(\widehat{\mathscr{V}})(1_{\beta} \otimes_{\alpha} U)\sigma^{\widehat{\alpha},\beta}(1_{\widehat{\alpha}} \otimes_{\beta} U^*yU)\widetilde{\mathscr{V}}\sigma^{\widehat{\alpha},\widehat{\beta}}(1_{\widehat{\alpha}} \otimes_{\widehat{\beta}} U^*)\widetilde{\mathscr{V}} \\ &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}(\widehat{\mathscr{V}})(U^*yU_{\beta} \otimes_{\alpha} 1)(1_{\beta} \otimes_{\alpha} U)\sigma^{\widehat{\alpha},\beta}\widetilde{\mathscr{V}}\sigma^{\widehat{\alpha},\widehat{\beta}}(1_{\widehat{\alpha}} \otimes_{\widehat{\beta}} U^*)\widetilde{\mathscr{V}} \\ &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} U)\sigma^{\alpha,\widehat{\beta}}(U^*yU_{\alpha} \otimes_{\widehat{\beta}} 1)\widehat{\mathscr{V}}\mathscr{V}\widetilde{\mathscr{V}} \\ &= (1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} y)T, \end{aligned}$$

by using Remarks 3.1.2 ii and Lemma 3.1.3 1 a). Therefore, since $\widehat{\alpha}(N) = M' \cap \widehat{M}'$ we have $[T, 1_{\widehat{\beta}} \otimes_{\widehat{\alpha}} y] = 0$ for all $y \in \widehat{\alpha}(N)'$ and the theorem is proved. □

In the case of a measured quantum groupoid on a finite-dimensional basis, we obtain a more precise result. In the following, we assume that $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ is a measured quantum groupoid on the finite-dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$ equipped with the non-normalized Markov trace $\epsilon = \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l$. We denote $(e^{(l)})_{1 \leqslant l \leqslant k}$ the minimal central self-adjoint projections of N.

Theorem 3.1.6. *There exist unimodular complex numbers ν_l, $1 \leqslant l \leqslant k$, unique, such that*

$$(1 \otimes U)\Sigma WV\widetilde{V} = q^{\widehat{\beta},\widehat{\alpha}} \sum_{1 \leqslant l \leqslant k} \nu_l\, \widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)}) = \sum_{1 \leqslant l \leqslant k} \nu_l\, \widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)})\, q^{\widehat{\beta},\widehat{\alpha}}.$$

For the proof, we will need the following result:

Lemma 3.1.7. *Let $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{K})$ and $\pi : N \to \mathcal{B}(\mathcal{H})$ be unital *-homomorphisms. Then, we have*

$$q^{\gamma,\pi}(\gamma(N^{\mathrm{o}}) \otimes \pi(N))q^{\gamma,\pi} = \bigoplus_{1 \leqslant l \leqslant k} \mathbb{C} \cdot (\gamma(e^{(l)\mathrm{o}}) \otimes \pi(e^{(l)}))q^{\gamma,\pi}.$$

Proof. More precisely, we have

$$q^{\gamma,\pi}(\gamma(x^{\mathrm{o}}) \otimes \pi(y))q^{\pi,\gamma} = \sum_{1 \leqslant l \leqslant k} n_l^{-2}\epsilon(yxe^{(l)})(\gamma(e^{(l)\,\mathrm{o}}) \otimes \pi(e^{(l)}))q^{\gamma,\pi}, \tag{3.1.1}$$

for all $x, y \in N$. By bilinearity, it suffices to check it out for $x = e^{(l)}_{rs}$ and $y = e^{(l')}_{pq}$, where $1 \leqslant l, l' \leqslant k$, $1 \leqslant r, s \leqslant n_l$ and $1 \leqslant p, q \leqslant n_{l'}$.

First, if $l' \neq l$ we have $q^{\gamma,\pi}(\gamma(e_{rs}^{(l)\,\mathrm{o}}) \otimes \pi(e_{pq}^{(l')})) = 0$ and $\epsilon(e_{pq}^{(l')} e_{rs}^{(l)} e^{(l'')}) = 0$ for all $1 \leqslant l'' \leqslant k$, then (3.1.1) is trivially true if $l' \neq l$. Now let $1 \leqslant l \leqslant k$ and $1 \leqslant r, s, p, q \leqslant n_l$, we have

$$\begin{aligned} q^{\gamma,\pi}(\gamma(e_{rs}^{(l)\,\mathrm{o}}) \otimes \pi(e_{pq}^{(l)})) &= \sum_{1 \leqslant l' \leqslant k} n_{l'}^{-1} \sum_{1 \leqslant i,j \leqslant n_{l'}} \gamma(e_{ij}^{(l')\,\mathrm{o}} e_{rs}^{(l)\,\mathrm{o}}) \otimes \pi(e_{ji}^{(l')} e_{pq}^{(l)}) \\ &= \delta_p^s n_l^{-1} \sum_{1 \leqslant j \leqslant n_l} \gamma(e_{rj}^{(l)\,\mathrm{o}}) \otimes \pi(e_{jq}^{(l)}). \end{aligned}$$

Moreover, we also have

$$\begin{aligned} (\gamma(e_{rj}^{(l)\,\mathrm{o}}) \otimes \pi(e_{jq}^{(l)})) q^{\gamma,\pi} &= \sum_{1 \leqslant l' \leqslant k} n_{l'}^{-1} \sum_{1 \leqslant u,v \leqslant n_{l'}} \gamma(e_{rj}^{(l)\,\mathrm{o}} e_{uv}^{(l')\,\mathrm{o}}) \otimes \pi(e_{jq}^{(l)} e_{vu}^{(l')}) \\ &= \delta_q^r n_l^{-1} \sum_{1 \leqslant u \leqslant n_l} \gamma(e_{uj}^{(l)\,\mathrm{o}}) \otimes \pi(e_{ju}^{(l)}), \end{aligned}$$

for all $1 \leqslant j \leqslant n_l$. Therefore, it follows that

$$q^{\gamma,\pi}(\gamma(e_{rs}^{(l)\,\mathrm{o}}) \otimes \pi(e_{pq}^{(l)})) q^{\gamma,\pi} = \delta_p^s \delta_q^r n_l^{-2} \sum_{1 \leqslant j,u \leqslant n_l} \gamma(e_{uj}^{(l)\,\mathrm{o}}) \otimes \pi(e_{ju}^{(l)}). \tag{3.1.2}$$

Furthermore, since $e^{(l_1)} e_{ij}^{(l_2)} = \delta_{l_1}^{l_2} e_{ij}^{(l_2)}$ for $1 \leqslant l_1, l_2 \leqslant k$ and $1 \leqslant i, j \leqslant n_{l_2}$ we have

$$(\gamma(e^{(l)\,\mathrm{o}}) \otimes \pi(e^{(l)})) q^{\gamma,\pi} = n_l^{-1} \sum_{1 \leqslant j,u \leqslant n_l} \gamma(e_{uj}^{(l)\,\mathrm{o}}) \otimes \pi(e_{ju}^{(l)}), \tag{3.1.3}$$

for all $1 \leqslant l \leqslant k$. Therefore, since $\epsilon(e_{pq}^{(l)} e_{rs}^{(l)} e^{(l')}) = \delta_l^{l'} \delta_q^r \epsilon(e_{ps}^{(l)}) = \delta_l^{l'} \delta_q^r \delta_p^s n_l$ we have

$$\begin{aligned} \sum_{1 \leqslant l' \leqslant k} n_{l'}^{-2} \epsilon(e_{pq}^{(l)} e_{rs}^{(l)} e^{(l')}) (\gamma(e^{(l')\,\mathrm{o}}) \otimes \pi(e^{(l')})) q^{\gamma,\pi} &= \delta_q^r \delta_p^s n_l^{-1} (\gamma(e^{(l)\,\mathrm{o}}) \otimes \pi(e^{(l)})) q^{\gamma,\pi} \\ &= q^{\gamma,\pi}(\gamma(e_{rs}^{(l)\,\mathrm{o}}) \otimes \pi(e_{pq}^{(l)})) q^{\gamma,\pi} \end{aligned}$$

in virtue of (3.1.3) and (3.1.2). □

Proof of theorem 3.1.6. We have $W = (v^{\alpha,\widehat{\beta}})^* \widehat{\mathscr{V}} v^{\beta,\alpha}$, $V = (v^{\beta,\alpha})^* \mathscr{V} v^{\widehat{\alpha},\beta}$ and $\widetilde{V} = (v^{\widehat{\alpha},\beta})^* \widetilde{\mathscr{V}} v^{\widehat{\beta},\widehat{\alpha}}$. Let us denote $X = (1 \otimes U)\Sigma W V \widetilde{V}$ and $T = (1 \,{}_{\widehat{\beta}}\!\otimes_{\widehat{\alpha}}\, U)\sigma^{\alpha,\widehat{\beta}} \widehat{\mathscr{V}} \mathscr{V} \widetilde{\mathscr{V}}$. Since the operators $v^{\widehat{\alpha},\beta}$ and $v^{\beta,\alpha}$ are coisometries, we have

$$(v^{\widehat{\beta},\widehat{\alpha}})^* T v^{\widehat{\beta},\widehat{\alpha}} = X.$$

Since T is unitary, X is a partial isometry such that

$$X^* X = q^{\widehat{\beta},\widehat{\alpha}} = X X^*.$$

Now, according to Proposition 2.1.10 we have

$$\widehat{\beta}(N^{\mathrm{o}}) \,{}_{\widehat{\beta}}\!\star_{\widehat{\alpha}}\, \widehat{\alpha}(N) = v^{\widehat{\beta},\widehat{\alpha}} (\widehat{\beta}(N^{\mathrm{o}}) \otimes \widehat{\alpha}(N)) (v^{\widehat{\beta},\widehat{\alpha}})^*.$$

In virtue of Theorem 3.1.5 and Lemma 3.1.7 (with $\mathcal{K} := \mathscr{H} =: \mathcal{H}$, $\gamma := \widehat{\beta}$, $\pi := \widehat{\alpha}$), we have

$$X \in q^{\widehat{\beta},\widehat{\alpha}} (\widehat{\beta}(N^{\mathrm{o}}) \otimes \widehat{\alpha}(N)) q^{\widehat{\beta},\widehat{\alpha}} = \bigoplus_{1 \leqslant l \leqslant k} \mathbb{C} \cdot (\widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)})) q^{\widehat{\beta},\widehat{\alpha}}.$$

Then, there exists $\nu_l \in \mathbb{C}$ for $1 \leqslant l \leqslant k$ such that

$$X = \sum_{1 \leqslant l \leqslant k} \nu_l \, \widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)}) \, q^{\widehat{\beta},\widehat{\alpha}} = q^{\widehat{\beta},\widehat{\alpha}} \sum_{1 \leqslant l \leqslant k} \nu_l \, \widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)}).$$

It then follows that

$$q^{\widehat{\beta},\widehat{\alpha}} = X^*X = q^{\widehat{\beta},\widehat{\alpha}} \sum_{1 \leqslant l \leqslant k} |\nu_l|^2 \widehat{\beta}(e^{(l)\mathrm{o}}) \otimes \widehat{\alpha}(e^{(l)}).$$

Hence, $|\nu_l| = 1$ for all $1 \leqslant l \leqslant k$ by injectivity of $\widehat{\beta}$ and $\widehat{\alpha}$. □

Notation 3.1.8. Let $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{K})$ and $\pi : N \to \mathcal{B}(\mathcal{H})$ be *-homomorphisms. We consider the following operator

$$\mathsf{T}_{\gamma,\pi} := \sum_{1 \leqslant l \leqslant n_l} \nu_l \, \gamma(e^{(l)\,\mathrm{o}}) \otimes \pi(e^{(l)}) \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H}),$$

where $\nu_l \in \mathbb{C}$ are the unimodular complex numbers defined in Theorem 3.1.6. We have

$$(1 \otimes U)\Sigma W V \widetilde{V} = q^{\widehat{\beta},\widehat{\alpha}} \, \mathsf{T}_{\widehat{\beta},\widehat{\alpha}} \quad (\text{with } [q^{\widehat{\beta},\widehat{\alpha}}, \mathsf{T}_{\widehat{\beta},\widehat{\alpha}}] = 0).$$

We will also denote $\mathsf{T}_{\pi,\gamma} := \Sigma_{\mathcal{K}\otimes\mathcal{H}} \mathsf{T}_{\gamma,\pi} \Sigma_{\mathcal{H}\otimes\mathcal{K}} = \sum_{1 \leqslant l \leqslant k} \nu_l \, \pi(e^{(l)}) \otimes \gamma(e^{(l)\,\mathrm{o}}) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

Corollary 3.1.9. *We have the following formulas:*

1. $V\widetilde{V} = W^*\Sigma(1 \otimes U^*)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}$.
2. $WV = \Sigma(1 \otimes U^*)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}\widetilde{V}^*$.
3. $V = W^*\Sigma(1 \otimes U^*)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}\widetilde{V}^*$.

Proof. We have

$$V\widetilde{V} = (VV^*)V\widetilde{V} = (W^*W)V\widetilde{V} = W^*\Sigma(1 \otimes U^*)q^{\widehat{\beta},\widehat{\alpha}}\mathsf{T}_{\widehat{\beta},\widehat{\alpha}} = W^*\Sigma(1 \otimes U^*)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}},$$

since V is a partial isometry, $VV^* = W^*W$, $\Sigma(1 \otimes U^*)q^{\widehat{\beta},\widehat{\alpha}} = q^{\alpha,\widehat{\beta}}\Sigma(1 \otimes U^*)$ and $W^*q^{\alpha,\widehat{\beta}} = W^*$. The second formula follows similarly from the facts that V is a partial isometry and $\widetilde{V}\widetilde{V}^* = V^*V$. The third formula is derived from the first one by the same argument. □

Remarks 3.1.10. We also have the following formulas:

1. $(\Sigma(1 \otimes U)V)^3 = q^{\widehat{\alpha},\beta}\mathsf{T}_{\widehat{\alpha},\beta}(1 \otimes \lambda^{i/4})$.
2. $(\Sigma(1 \otimes U)W)^3 = q^{\beta,\alpha}\mathsf{T}_{\beta,\alpha}(\lambda^{i/4} \otimes 1)$.
3. $(\Sigma(1 \otimes U)\widetilde{V})^3 = (\lambda^{i/4} \otimes 1)q^{\widehat{\beta},\widehat{\alpha}}\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}$.

Let us prove the first formula. We recall that we denote $X = (1 \otimes U)\Sigma W V \widetilde{V}$. We have

$$\begin{aligned} X &= \Sigma(U \otimes 1)\Sigma(U^* \otimes 1)V(U \otimes 1)\Sigma V\Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma \\ &= (U^* \otimes U)V\Sigma(1 \otimes U)V\Sigma(1 \otimes U)V(1 \otimes U^*)\Sigma. \end{aligned}$$

By multiplying on the left by $\Sigma(U\otimes 1)$ and on the right by $(U\otimes 1)\Sigma$, we then obtain

$$\Sigma(U\otimes 1)X(U\otimes 1)\Sigma = (\Sigma(1\otimes U)V)^3.$$

However, since we have $X = q^{\widehat{\beta},\widehat{\alpha}}\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}$ (Theorem 3.1.6) we also have

$$\begin{aligned}\Sigma(U\otimes 1)X(U\otimes 1)\Sigma &= \Sigma(U\otimes 1)q^{\widehat{\beta},\widehat{\alpha}}\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}(U\otimes 1)\Sigma\\ &= q^{\widehat{\alpha},\beta}\Sigma(U\otimes 1)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}(U\otimes 1)\Sigma\\ &= q^{\widehat{\alpha},\beta}\mathsf{T}_{\widehat{\alpha},\beta}(1\otimes\lambda^{i/4})\end{aligned}$$

as we have $U^2 = \lambda^{i/4}$. We also have

$$\begin{aligned}X &= \Sigma(U\otimes 1)W\Sigma(1\otimes U)W(1\otimes U^*)\Sigma(U\otimes U)W(U^*\otimes U^*)\\ &= \Sigma(U\otimes 1)W\Sigma(1\otimes U)W\Sigma(1\otimes U)W(U^*\otimes U^*).\end{aligned}$$

We then multiply by $U\otimes U^*$ on the left and by $U\otimes U$ on the right so that the second formula is proved. Finally, we have

$$\begin{aligned}X &= \Sigma(U\otimes 1)(U^*\otimes U^*)\widetilde{V}(U\otimes U)\Sigma(U^*\otimes 1)\widetilde{V}(U\otimes 1)\Sigma\widetilde{V}\\ &= \Sigma(1\otimes U^*)\widetilde{V}(\Sigma(1\otimes U)\widetilde{V})^2.\end{aligned}$$

By multiplying on the left by $\lambda^{i/4}\otimes 1$, the last formula is proved.

Notations 3.1.11.
- Let (S,δ) and $(\widehat{S},\widehat{\delta})$ be the weak Hopf-C*-algebra and the dual weak Hopf-C*-algebra associated with $\mathcal{G}$. We denote $[S\widehat{S}]$ the norm closed linear subspace of $\mathcal{B}(\mathscr{H})$ spanned by the products $L(y)\rho(x)$, for all $y\in S$ and $x\in\widehat{S}$.
- Following [2], if $T\in\mathcal{B}(\mathscr{H}\otimes\mathscr{H})$ we denote $\mathcal{C}(T)$ the norm closure of the linear subspace $\{(\mathrm{id}\otimes\omega)(\Sigma T)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*\}\subset\mathcal{B}(\mathscr{H})$.

Proposition 3.1.12. *$[S\widehat{S}]$ is a C*-algebra.*

Proof. Let us fix $\omega,\psi\in\mathcal{B}(\mathscr{H})_*$. It suffices to show that $\rho(\omega)L(\psi)\in[S\widehat{S}]$. We have

$$\rho(\omega)L(\psi) = (\mathrm{id}\otimes\omega)(V)(\psi\otimes\mathrm{id})(V) = (\psi\otimes\mathrm{id})((\mathrm{id}\otimes\mathrm{id}\otimes\omega)(V_{23}V_{12})).$$

We combine the pentagon identities $V_{23}V_{12} = V_{12}V_{13}V_{23}$ and $V_{12}^*V_{23}V_{12} = V_{13}V_{23}$ to conclude that $V_{23}V_{12} = V_{12}V_{12}^*V_{23}V_{12}$. It then follows that

$$\rho(\omega)L(\psi) = (\psi\otimes\mathrm{id})(VV^*(1\otimes\rho(\omega))V) = (\psi\otimes\mathrm{id})(V(\rho\otimes\rho)(\widehat{\delta}(x))),$$

where $x\in\widehat{S}$ such that $\rho(x) = \rho(\omega)$. Let us write $\psi = \rho(x')\psi'$, where $x'\in\widehat{S}$ and $\psi'\in\mathcal{B}(\mathscr{H})_*$. We then have

$$\rho(\omega)L(\psi) = (\psi'\otimes\mathrm{id})(V(\rho\otimes\rho)(\widehat{\delta}(x))(1\otimes\rho(x'))) = (\psi'\otimes\mathrm{id})(V(\rho\otimes\rho)(\widehat{\delta}(x)(1_{\widehat{S}}\otimes x'))).$$

But $\widehat{\delta}(x)(1_{\widehat{S}}\otimes x')$ is the norm limit of finite sums of the form $\sum_i x_i\otimes x_i'$, where $x_i,x_i'\in\widehat{S}$. Therefore, $\rho(\omega)L(\psi)$ is the norm limit of finite sums of the form

$$\sum_i(\psi'\otimes\mathrm{id})(V(\rho(x_i)\otimes\rho(x_i'))) = \sum_i L(\psi_i)\rho(x_i'),$$

where $\psi_i := \rho(x_i)\psi'\in\mathcal{B}(\mathscr{H})_*$. □

Proposition 3.1.13. $[S\widehat{S}] = [(\mathrm{id}\otimes\omega)(WV)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*]$.

Note that $\{(\mathrm{id}\otimes\omega)(WV)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*\}$ is already a subspace of $\mathcal{B}(\mathscr{H})$.

Proof. Let $\omega\in\mathcal{B}(\mathscr{H})_*$ and $x\in\widehat{S}$, since $W=WW^*W=WVV^*$ we have

$$(\mathrm{id}\otimes\omega)(W)\rho(x)=(\mathrm{id}\otimes\omega)(WVV^*(x\otimes 1))=(\mathrm{id}\otimes\omega')(WVV^*(\rho(x)\otimes L(y))),$$

where $\omega'\in\mathcal{B}(\mathscr{H})_*$ and $y\in S$ are such that $\omega=L(y)\omega'$. Since $V\in\mathcal{M}(\widehat{S}\otimes S)$, $(\mathrm{id}\otimes\omega)(W)$ is the norm limit of finite sums of the form

$$\sum_i(\mathrm{id}\otimes\omega)(WV(\rho(x_i)\otimes L(y_i)))=\sum_i(\mathrm{id}\otimes L(y_i)\omega)(WV)\rho(x_i),\quad x_i\in\widehat{S},\,y_i\in S.$$

Hence, $[S\widehat{S}]\subset[(\mathrm{id}\otimes\omega)(WV)\rho(x)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*,\,x\in\widehat{S}]$. Let us fix $\omega\in\mathcal{B}(\mathscr{H})_*$ and $x\in\widehat{S}$. Let us write $\omega=\rho(x')\omega'$, where $x'\in\widehat{S}$ and $\omega'\in\mathcal{B}(\mathscr{H})_*$. Since $\widehat{\delta}(1)=V^*V$, we have $V=V\widehat{\delta}(1)$. Hence,

$$(\mathrm{id}\otimes\omega)(WV)\rho(x)=(\mathrm{id}\otimes\omega')(WV(\rho\otimes\rho)(\widehat{\delta}(1)(x\otimes x'))).$$

It then follows that $(\mathrm{id}\otimes\omega)(WV)\rho(x)$ is the norm limit of finite sums of the form

$$\sum_i(\mathrm{id}\otimes\omega')(WV(\rho\otimes\rho)(\widehat{\delta}(x_i)(1\otimes x_i')))=\sum_i(\mathrm{id}\otimes\rho(x_i')\omega')(WV(\rho\otimes\rho)\widehat{\delta}(x_i)),\quad x_i,\,x_i'\in\widehat{S}.$$

Hence, $[S\widehat{S}]\subset[(\mathrm{id}\otimes\omega)(WV(\rho\otimes\rho)\widehat{\delta}(x))\,;\,\omega\in\mathcal{B}(\mathscr{H})_*,\,x\in\widehat{S}]$. Now, we have

$$WV(\rho\otimes\rho)\widehat{\delta}(x)=WVV^*(1\otimes\rho(x))V=(1\otimes\rho(x))WV,\quad x\in\widehat{S}.$$

Indeed, since $VV^*=q^{\beta,\alpha}$, $\rho(x)\in\widehat{M}'$ and $\alpha(N)\subset\widehat{M}$ we have $[VV^*,1\otimes\rho(x)]=0$. Moreover, $[W,1\otimes\rho(x)]=0$ since $W\in M\otimes\widehat{M}$. It then follows that

$$(\mathrm{id}\otimes\omega)(WV(\rho\otimes\rho)\widehat{\delta}(x))=(\mathrm{id}\otimes\omega\rho(x))(WV),\quad\omega\in\mathcal{B}(\mathscr{H})_*,\,x\in\widehat{S}.$$

Hence, $[S\widehat{S}]\subset[(\mathrm{id}\otimes\omega)(WV)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*]$. Let us prove the converse inclusion. Let us fix $\omega\in\mathcal{B}(\mathscr{H})_*$. Let us write $\omega=\rho(x')\omega'\rho(x)$, where $x,x'\in\widehat{S}$ and $\omega'\in\mathcal{B}(\mathscr{H})_*$. Let us recall that

$$(1\otimes\rho(x))WV=WV(\rho\otimes\rho)\widehat{\delta}(x).$$

Hence, $(1\otimes\rho(x))WV(1\otimes\rho(x'))=WV(\rho\otimes\rho)(\widehat{\delta}(x)(1\otimes x'))$ is the norm limit of finite sums of the form $\sum_i WVV^*V(\rho(x_i)\otimes\rho(x_i'))=\sum_i WV(\rho(x_i)\otimes\rho(x_i'))$, where $x_i,\,x_i'\in\widehat{S}$. Therefore, we have that $(\mathrm{id}\otimes\omega)(WV)=(\mathrm{id}\otimes\omega')((1\otimes\rho(x))WV(1\otimes\rho(x')))$ is the norm limit of finite sums of the form

$$\sum_i(\mathrm{id}\otimes\omega')(WV(\rho(x_i)\otimes\rho(x_i')))=\sum_i(\mathrm{id}\otimes\rho(x_i')\omega')(WV)\rho(x_i),\quad x_i,x_i'\in\widehat{S}.$$

Hence,

$$(\mathrm{id}\otimes\omega)(WV)\in[(\mathrm{id}\otimes\phi)(WV)\rho(x)\,;\,\phi\in\mathcal{B}(\mathscr{H})_*,\,x\in\widehat{S}].$$

Let $\omega\in\mathcal{B}(\mathscr{H})_*$ and $x\in\widehat{S}$. We write $\omega=L(y)\omega'$ for some $\omega'\in\mathcal{B}(\mathscr{H})_*$ and $y\in S$. We then obtain $(\mathrm{id}\otimes\omega)(WV)\rho(x)=(\mathrm{id}\otimes\omega')(WV(\rho(x)\otimes L(y)))\in[S\widehat{S}]$ by using the fact that $V\in\mathcal{M}(\widehat{S}\otimes S)$. □

The following result is a crucial consequence of Theorem 3.1.6.

Corollary 3.1.14. $[S\widehat{S}] = U^*\mathcal{C}(V)U$. *In particular,* $\mathcal{C}(V)$ *is a* C*-*algebra.*

Proof. We have $WV = \Sigma(1\otimes U^*)\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}\widetilde{V}^*$ (see Corollary 3.1.9 2) and $(1\otimes\widehat{\alpha}(n))\widetilde{V}^* = (\widehat{\beta}(n^{\mathrm{o}})\otimes 1)\widetilde{V}^*$ for all $n \in N$ (see Proposition 2.3.6 3). Therefore, we have

$$\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}\widetilde{V}^* = \sum_{1\leqslant l\leqslant k} \nu_l(\widehat{\beta}(e^{(l)\,\mathrm{o}})\otimes 1)\widetilde{V}^* = (\widehat{\beta}(u)\otimes 1)\widetilde{V}^*, \quad \text{where} \quad u := \sum_{1\leqslant l\leqslant k} \nu_l \cdot e^{(l)\,\mathrm{o}} \in N^{\mathrm{o}}.$$

Since $\widetilde{V} = \Sigma(1\otimes U^*)V(1\otimes U)\Sigma$, we obtain

$$\begin{aligned} WV &= (1\otimes\widehat{\beta}(u))\Sigma(1\otimes U^*)\widetilde{V}^* \\ &= (1\otimes\widehat{\beta}(u))(U^*\otimes U^*)V^*(1\otimes U)\Sigma \\ &= (1\otimes\widehat{\beta}(u)U^*)(U^*\otimes 1)V^*\Sigma(U\otimes 1). \end{aligned}$$

It then follows that

$$(\mathrm{id}\otimes\omega)(WV) = U^*(\mathrm{id}\otimes\widehat{\beta}(u)U^*\omega)(V^*\Sigma)U, \quad \omega\in\mathcal{B}(\mathscr{H})_*.$$

We combine the fact that $\widehat{\beta}(u)U^*$ is unitary with Proposition 3.1.13 to conclude that

$$[S\widehat{S}] = [(\mathrm{id}\otimes\omega)(WV)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*] = U^*[(\mathrm{id}\otimes\omega)(V^*\Sigma)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*]U = U^*\mathcal{C}(V)^*U.$$

We also know that $[S\widehat{S}]$ is a C*-algebra (see Proposition 3.1.12), hence $[S\widehat{S}] = U^*\mathcal{C}(V)U$. In particular, $\mathcal{C}(V) = U[S\widehat{S}]U^*$ is a C*-algebra. □

Remarks 3.1.15. Let us make some useful comments:

1. The C*-subalgebras $[S\widehat{S}]$ and $\mathcal{C}(V)$ of $\mathcal{B}(\mathscr{H})$ are non-degenerate. Note that we also have $[S\widehat{S}] = U\mathcal{C}(V)U^*$.
2. By using the formulas $V^* = (J\otimes\widehat{J})V(J\otimes\widehat{J})$ and $W^* = (\widehat{J}\otimes J)W(\widehat{J}\otimes J)$, we can prove as in [4] that $\mathcal{C}(W)$ and $\mathcal{C}(\widetilde{V})$ are C*-algebras and

$$\mathcal{C}(W) = [R(S)\rho(\widehat{S})], \quad \mathcal{C}(\widetilde{V}) = [L(S)\lambda(\widehat{S})].$$

3.2 Regularity and semi-regularity for measured quantum groupoids

In this section, we introduce the notion of semi-regularity for measured quantum groupoids. Note that the notion of regularity has been generalized for measured quantum groupoids by Enock in [13] and by Timmermann in [21, 22] in the setting of pseudo-multiplicative C*-unitaries. In the case of a colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$, we will prove that the regularity (resp. semi-regularity) of $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ is equivalent to the regularity (resp. semi-regularity) of both the quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. It should be noted that De Commer has provided in [10] an example of a colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ such that $\mathbb{G}_1$ is regular and $\mathbb{G}_2$ is *not* regular.

As already mentioned in Remark 2.2.8, from now on we will make the following convention:

Definition 3.2.1. From now on, if $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ is a measured quantum groupoid we define the dual measured quantum groupoid to be the measured quantum groupoid denoted $\widehat{\mathcal{G}}$ and defined by:

$$\widehat{\mathcal{G}} := (N^{\mathrm{o}}, \widehat{M}', \beta, \widehat{\alpha}, \widehat{\Gamma}^{\mathrm{c}}, \widehat{T}^{\mathrm{c}}, \widehat{T}'^{\mathrm{c}}, \nu^{\mathrm{o}}).$$

Notations 3.2.2. Let $\mathcal{H}$ be a Hilbert space, (N,φ) a von Neumann algebra endowed with a n.s.f. weight. Let $\pi : N \to \mathcal{B}(\mathcal{H})$ and $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{H})$ be unital normal *-representations. In what follows, we will use the notations introduced in §2.1. Following [13], we denote:

$$\mathcal{K}_{\pi,\varphi} := [R^{\pi,\varphi}(\xi)R^{\pi,\varphi}(\eta)^* \,;\, \xi,\eta \in {}_{\varphi}(\pi,\mathcal{H})]$$

$$(\text{resp. } \mathcal{K}_{\gamma,\varphi} := [L^{\gamma,\varphi}(\xi)L^{\gamma,\varphi}(\eta)^* \,;\, \xi,\eta \in (\mathcal{H},\gamma)_{\varphi}]).$$

Note that $\mathcal{K}_{\pi,\varphi}$ (resp. $\mathcal{K}_{\gamma,\varphi}$) is a weakly dense ideal of $\pi(N)'$ (resp. $\gamma(N^{\mathrm{o}})'$) (see Proposition 3 of [6]). In the following, we will denote $\mathcal{K}_{\pi}$ (resp. $\mathcal{K}_{\gamma}$) instead of $\mathcal{K}_{\pi,\varphi}$ (resp. $\mathcal{K}_{\gamma,\varphi}$) since no ambiguity will arise.

Following [1], [2] and [13], we define the notion of semi-regularity and regularity for measured quantum groupoids in general.

Definition 3.2.3. Let $\mathcal{G} = (N, M, \beta, \alpha, \Gamma, T, T', \nu)$ be a measured quantum groupoid, we denote $\mathcal{C}(\mathscr{W}_{\mathcal{G}})$ the norm closure of the linear subspace of $\mathcal{B}(\mathscr{H})$,

$$\{(\mathrm{id} \star \omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) \,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}.$$

The measured quantum groupoid $\mathcal{G}$ is said to be semi-regular (resp. regular) if $\mathcal{K}_{\alpha} \subset \mathcal{C}(\mathscr{W}_{\mathcal{G}})$ (resp. $\mathcal{K}_{\alpha} = \mathcal{C}(\mathscr{W}_{\mathcal{G}})$).

Remarks 3.2.4.

1. By using the intertwining property of $\mathscr{W}_{\mathcal{G}}$ with α (see §2.2), we obtain that $(\mathrm{id} \star \omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) \subset \alpha(N)'$ for all $\omega \in \mathcal{B}(\mathscr{H})_*$. Enock has proved that a measured quantum groupoid $\mathcal{G}$ is always weakly regular in the sense of Definition 4.1 of [13], that is $\{(\mathrm{id} \star \omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) \,;\, \omega \in \mathcal{B}(\mathscr{H})_*\}$ is weakly dense in $\alpha(N)'$. It should be noted that $\mathcal{C}(\mathscr{W}_{\mathcal{G}})$ is a subalgebra of $\mathcal{B}(\mathscr{H})$ (see Proposition 3.10 of [13]).

2. Let us remark that:

$$(\mathrm{id} \star \omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) = (\omega \star \mathrm{id})(\mathscr{W}_{\mathcal{G}}\sigma^{\alpha,\beta}), \quad \omega \in \mathcal{B}(\mathscr{H})_*.$$

Indeed, let $\xi,\eta \in \mathscr{H}$ we have $\lambda_{\eta}^{\beta,\alpha} = \sigma^{\alpha,\beta}\rho_{\eta}^{\alpha,\beta}$ and $\rho_{\xi}^{\alpha,\widehat{\beta}} = \sigma^{\widehat{\beta},\alpha}\lambda_{\xi}^{\widehat{\beta},\alpha}$. Since $(\sigma^{\widehat{\beta},\alpha})^* = \sigma^{\alpha,\widehat{\beta}}$, we then have $(\lambda_{\xi}^{\widehat{\beta},\alpha})^*\sigma^{\alpha,\widehat{\beta}} = (\rho_{\xi}^{\alpha,\widehat{\beta}})^*$. Therefore, we obtain

$$(\lambda_{\xi}^{\widehat{\beta},\alpha})^*\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}\lambda_{\eta}^{\beta,\alpha} = (\rho_{\xi}^{\alpha,\widehat{\beta}})^*\mathscr{W}_{\mathcal{G}}\sigma^{\alpha,\beta}\rho_{\eta}^{\alpha,\beta},$$

for all $\xi,\eta \in \mathscr{H}$ and the result is proved.

Let $\mathcal{G} = (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$ be a measured quantum groupoid on the finite dimensional basis

$$N = \bigoplus_{1\leqslant l\leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$$

endowed with the non-normalized Markov trace $\epsilon = \bigoplus_{1\leqslant l\leqslant k} n_l \cdot \mathrm{Tr}_l$. Let (S,δ) and $(\widehat{S},\widehat{\delta})$ be the weak Hopf-C*-algebra and the dual weak Hopf-C*-algebra of $\mathcal{G}$. Let $\gamma : N^{\mathrm{o}} \to \mathcal{B}(\mathcal{H})$ and $\pi : N \to \mathcal{B}(\mathcal{H})$ be unital *-representations. In that case, for $\xi \in \mathcal{H}$ the operators $R^{\pi,\epsilon}(\xi)$, $L^{\gamma,\epsilon}(\xi) \in \mathcal{B}(\mathscr{H}_{\epsilon},\mathcal{H})$ are given by:

$$R^{\pi,\epsilon}(\xi)\Lambda_{\epsilon}(n) = \pi(n)\xi, \quad L^{\gamma,\epsilon}(\xi)\Lambda_{\epsilon}(n) = \gamma(n^{\mathrm{o}})\xi, \quad n \in N.$$

Since $\mathscr{H}_{\epsilon}$ is a finite dimensional Hilbert space, $L^{\gamma,\epsilon}(\eta)^*$ is a finite rank bounded operator for all $\eta \in \mathscr{H}$ and then so is $L^{\gamma,\epsilon}(\xi)L^{\gamma,\epsilon}(\eta)^*$ for all $\xi,\eta \in \mathcal{H}$. It then follows that $\mathcal{K}_{\gamma}$ is a C*-subalgebra of $\mathcal{K}(\mathcal{H})$. Similarly, we also have that $\mathcal{K}_{\pi}$ is a C*-subalgebras of $\mathcal{K}(\mathcal{H})$.

Lemma 3.2.5. *We have the following formulas:*

$$\mathcal{K}_{\widehat{\beta}} = J\mathcal{K}_\alpha J, \quad \mathcal{K}_{\widehat{\alpha}} = J\mathcal{K}_\beta J, \quad \mathcal{K}_\alpha = \widehat{J}\mathcal{K}_\beta\widehat{J},$$
$$\mathcal{K}_{\widehat{\alpha}} = U^*\mathcal{K}_\alpha U, \quad \mathcal{K}_{\widehat{\beta}} = U^*\mathcal{K}_\beta U.$$

Proof. It suffices to obtain the formulas

$$L^{\widehat{\beta},\epsilon}(J\xi)L^{\widehat{\beta},\epsilon}(J\eta)^* = JR^{\alpha,\epsilon}(\xi)R^{\alpha,\epsilon}(\eta)^*J, \quad R^{\widehat{\alpha},\epsilon}(J\xi)R^{\widehat{\alpha},\epsilon}(J\eta)^* = JL^{\beta,\epsilon}(\xi)L^{\beta,\epsilon}(\eta)^*J,$$
$$R^{\alpha,\epsilon}(\widehat{J}\xi)R^{\alpha,\epsilon}(\widehat{J}\eta)^* = \widehat{J}L^{\beta,\epsilon}(\xi)L^{\beta,\epsilon}(\eta)^*\widehat{J}, \quad \xi,\eta \in \mathscr{H},$$

which follow from the facts that $\widehat{\beta}(n^{\mathrm{o}}) = J\alpha(n^*)J$, $\widehat{\alpha}(n^*) = J\beta(n^{\mathrm{o}})J$ and $\alpha(n^*) = \widehat{J}\beta(n^{\mathrm{o}})\widehat{J}$ for all $n \in N$. For instance, if $\xi \in \mathscr{H}$ we have:

$$JR^{\alpha,\epsilon}(\xi)\Lambda_\epsilon(n^*) = J\alpha(n^*)\xi = \widehat{\beta}(n^{\mathrm{o}})J\xi = L^{\widehat{\beta},\epsilon}(J\xi)\Lambda_\epsilon(n), \quad n \in N.$$

Hence, $JR^{\alpha,\epsilon}(\xi)J_\epsilon = L^{\widehat{\beta},\epsilon}(J\xi)$ for all $\xi \in \mathscr{H}$. Therefore, since $J^* = J$, $J_\epsilon J_\epsilon^* = J_\epsilon^2 = 1$ we have $JR^{\alpha,\epsilon}(\xi)R^{\alpha,\epsilon}(\eta)^*J = L^{\widehat{\beta},\epsilon}(J\xi)L^{\widehat{\beta},\epsilon}(J\eta)^*$ for all $\xi,\eta \in \mathscr{H}$. Moreover, since $J^2 = 1$ we obtain the equality $\mathcal{K}_{\widehat{\beta}} = J\mathcal{K}_\alpha J$. The formulas $\mathcal{K}_{\widehat{\alpha}} = U^*\mathcal{K}_\alpha U$ and $\mathcal{K}_{\widehat{\beta}} = U^*\mathcal{K}_\beta U$ are obtained by combining the previous ones. □

Lemma 3.2.6. *For all $\omega \in \mathcal{B}(\mathscr{H})_*$, we have $(\mathrm{id} \star \omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) = (\mathrm{id} \otimes \omega)(\Sigma W)$. In particular, we have $\mathcal{C}(\mathscr{W}_{\mathcal{G}}) = \mathcal{C}(W)$.*

Proof. In virtue of Remark 3.2.4 2 and the fact that $\sigma^{\beta,\alpha}v^{\beta,\alpha} = v^{\alpha,\beta}\Sigma$, we have

$$(\mathrm{id}\star\omega)(\sigma^{\alpha,\widehat{\beta}}\mathscr{W}_{\mathcal{G}}) = (\omega\star\mathrm{id})(\mathscr{W}_{\mathcal{G}}\sigma^{\alpha,\beta}) = (\omega\otimes\mathrm{id})((v^{\alpha,\widehat{\beta}})^*\mathscr{W}_{\mathcal{G}}v^{\alpha,\beta}\Sigma) = (\omega\otimes\mathrm{id})(W\Sigma) = (\mathrm{id}\otimes\omega)(\Sigma W),$$

for all $\omega \in \mathcal{B}(\mathscr{H})_*$. □

Lemma 3.2.7. *We have $\widehat{J}\mathcal{C}(V)\widehat{J} = \mathcal{C}(W)^*$ and $\mathcal{C}(\widetilde{V}) = U^*\mathcal{C}(W)U$.*

Proof. Firstly, since $\widetilde{V} = (U^* \otimes U^*)W(U \otimes U)$ we have:

$$(\mathrm{id} \otimes \omega)(\Sigma\widetilde{V}) = U^*(\mathrm{id} \otimes \omega_{U^*})(\Sigma W)U, \quad \omega \in \mathcal{B}(\mathscr{H})_*.$$

Hence, $\mathcal{C}(\widetilde{V}) = U^*\mathcal{C}(W)U$ (as $\omega = (\omega_U)_{U^*}$). Secondly, we have $W = \Sigma(U^* \otimes 1)V(U \otimes 1)\Sigma$. Hence, $\Sigma V = (1 \otimes U)W(1 \otimes U^*)\Sigma$. It then follows from the formulas $U = \widehat{J}J$, $U^*\widehat{J} = J$ and $(\widehat{J} \otimes J)W = W^*(\widehat{J} \otimes J)$ that

$$\begin{aligned}(\widehat{J} \otimes 1)\Sigma V(\widehat{J} \otimes 1) &= (1 \otimes U)(\widehat{J} \otimes 1)W(1 \otimes U^*\widehat{J})\Sigma \\ &= (1 \otimes \widehat{J})(\widehat{J} \otimes J)W(1 \otimes J)\Sigma \\ &= (1 \otimes \widehat{J})W^*(\widehat{J} \otimes 1)\Sigma \\ &= (1 \otimes \widehat{J})(\Sigma W)^*(1 \otimes \widehat{J}).\end{aligned}$$

If $\xi,\eta \in \mathscr{H}$, we have:

$$\begin{aligned}\widehat{J}(\mathrm{id} \otimes \omega_{\xi,\eta})(\Sigma V)\widehat{J} &= (\mathrm{id} \otimes \omega_{\xi,\eta})((1 \otimes \widehat{J})(\Sigma W)^*(1 \otimes \widehat{J})) \\ &= (\mathrm{id} \otimes \omega_{\widehat{J}\xi,\widehat{J}\eta})((\Sigma W)^*) \\ &= (\mathrm{id} \otimes \omega_{\widehat{J}\eta,\widehat{J}\xi})(\Sigma W)^*,\end{aligned}$$

since $\overline{\omega}_{\widehat{J}\xi,\widehat{J}\eta} = \omega_{\widehat{J}\eta,\widehat{J}\xi}$. This proves that $\widehat{J}\mathcal{C}(V)\widehat{J} = \mathcal{C}(W)^*$. □

Proposition 3.2.8. *The following statements are equivalent:*

1. $\mathcal{G}$ *is semi-regular (resp. regular), that is* $\mathcal{K}_\alpha \subset \mathcal{C}(W)$ *(resp.* $\mathcal{C}(W) = \mathcal{K}_\alpha$*).*
2. $\widehat{\mathcal{G}}$ *is semi-regular (resp. regular), that is* $\mathcal{K}_\beta \subset \mathcal{C}(V)$ *(resp.* $\mathcal{C}(V) = \mathcal{K}_\beta$*).*
3. $(\mathcal{G}^{\mathrm{o}})^{\mathrm{c}}$ *is semi-regular (resp. regular), that is* $\mathcal{K}_{\widehat{\alpha}} \subset \mathcal{C}(\widetilde{V})$ *(resp.* $\mathcal{C}(\widetilde{V}) = \mathcal{K}_{\widehat{\alpha}}$*).*

Proof. Straightforward consequence of Lemmas 3.2.5, 3.2.6, 3.2.7 and the definitions of V, W and $\widetilde{V}$. □

We give some other equivalent conditions of the semi-regularity (resp. regularity) of $\mathcal{G}$.

Corollary 3.2.9. *The following statements are equivalent:*

1. $\mathcal{G}$ *is semi-regular (resp. regular).*
2. $\mathcal{K}_{\widehat{\beta}} \subset [\widehat{S}S]$ *(resp.* $[\widehat{S}S] = \mathcal{K}_{\widehat{\beta}}$*).*
3. $\mathcal{K}_\alpha \subset [R(S)\widehat{S}]$ *(resp.* $[R(S)\widehat{S}] = \mathcal{K}_\alpha$*).*
4. $\mathcal{K}_{\widehat{\alpha}} \subset [S\lambda(\widehat{S})]$ *(resp.* $[S\lambda(\widehat{S})] = \mathcal{K}_{\widehat{\alpha}}$*).*

In particular, if $\mathcal{G}$ *is regular we have* $[\widehat{S}S] \subset \mathcal{K}(\mathscr{H})$*,* $[R(S)\widehat{S}] \subset \mathcal{K}(\mathscr{H})$ *and* $[S\lambda(\widehat{S})] \subset \mathcal{K}(\mathscr{H})$ *(and also* $\mathcal{C}(V) \subset \mathcal{K}(\mathscr{H})$*,* $\mathcal{C}(W) \subset \mathcal{K}(\mathscr{H})$ *and* $\mathcal{C}(\widetilde{V}) \subset \mathcal{K}(\mathscr{H})$*).*

Proof. The equivalence of the first two statements follows directly from the fact that $\mathcal{G}$ is semi-regular (resp. regular) if and only if $\mathcal{K}_\beta \subset \mathcal{C}(V)$ (resp. $\mathcal{C}(V) = \mathcal{K}_\beta$) and the formulas $U^*\mathcal{C}(V)U = [\widehat{S}S]$, $U^*\mathcal{K}_\beta U = \mathcal{K}_{\widehat{\beta}}$ (see Corollary 3.1.14 and Lemma 3.2.5). The remaining equivalences are immediate consequences of Remark 3.1.15 2 and Proposition 3.2.8. □

In order to investigate the semi-regularity (resp. regularity) of the colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ we will need a technical lemma. First, we have to generalize a notation.

Notations 3.2.10. Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be three Hilbert spaces. If $X \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H}, \mathcal{L} \otimes \mathcal{H})$ (resp. $Y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}, \mathcal{H} \otimes \mathcal{L})$) we will denote $\mathcal{C}(X)$ (resp. $\mathcal{C}(Y)$) the norm closure of the linear subspace

$$\{(\mathrm{id} \otimes \omega)(\Sigma_{\mathcal{L}\otimes\mathcal{H}} X)\,;\, \omega \in \mathcal{B}(\mathcal{H}, \mathcal{L})_*\} \subset \mathcal{B}(\mathcal{K}, \mathcal{H})$$

$$(\text{resp. } \{(\mathrm{id} \otimes \omega)(\Sigma_{\mathcal{H}\otimes\mathcal{L}} Y)\,;\, \omega \in \mathcal{B}(\mathcal{K}, \mathcal{H})_*\} \subset \mathcal{B}(\mathcal{H}, \mathcal{L})).$$

Note that we have:

$$\mathcal{C}(X) = [(\mathrm{id} \otimes \omega_{\xi,\eta})(\Sigma_{\mathcal{L}\otimes\mathcal{H}} X)\,;\, \xi \in \mathcal{L},\, \eta \in \mathcal{H}\,], \quad \mathcal{C}(Y) = [(\mathrm{id} \otimes \omega_{\xi,\eta})(\Sigma_{\mathcal{H}\otimes\mathcal{L}} Y)\,;\, \xi \in \mathcal{H},\, \eta \in \mathcal{K}\,].$$

Lemma 3.2.11. *Let* $\mathcal{H}$*,* $\mathcal{K}$*,* $\mathcal{L}$*,* $\mathcal{E}$ *and* $\mathcal{E}'$ *be Hilbert spaces,* $X \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H}, \mathcal{L} \otimes \mathcal{H})$ *and* $Y \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}, \mathcal{H} \otimes \mathcal{L})$*. The following statements are equivalent:*

1. $\mathcal{K}(\mathcal{H}, \mathcal{L}) \subset \mathcal{C}(Y)$ *(resp.* $\mathcal{K}(\mathcal{K}, \mathcal{H}) \subset \mathcal{C}(X)$*).*
2. $\mathcal{K}(\mathcal{H} \otimes \mathcal{E}, \mathcal{E}' \otimes \mathcal{L}) \subset [(\mathcal{K}(\mathcal{H}, \mathcal{E}') \otimes 1_{\mathcal{L}})Y(1_{\mathcal{H}} \otimes \mathcal{K}(\mathcal{E}, \mathcal{K}))]$
   (resp. $\mathcal{K}(\mathcal{K} \otimes \mathcal{E}', \mathcal{E} \otimes \mathcal{H}) \subset [(\mathcal{K}(\mathcal{L}, \mathcal{E}) \otimes 1_{\mathcal{H}})X(1_{\mathcal{K}} \otimes \mathcal{K}(\mathcal{E}', \mathcal{H}))]$*).*

The equivalences also hold if the inclusions are replaced by equalities.

Proof. A straightforward computation gives:

$$(1_{\mathcal{L}} \otimes \theta_{\zeta',\xi})\Sigma_{\mathcal{H}\otimes\mathcal{L}}\, Y(1_{\mathcal{H}} \otimes \theta_{\eta,\zeta}) = (\mathrm{id} \otimes \omega_{\xi,\eta})(\Sigma_{\mathcal{H}\otimes\mathcal{L}}\, Y) \otimes \theta_{\zeta',\zeta}, \quad \xi \in \mathcal{H},\, \eta \in \mathcal{K},\, \zeta \in \mathcal{E},\, \zeta' \in \mathcal{E}'.$$

It then follows that

$$[(1_{\mathcal{L}} \otimes \mathcal{K}(\mathcal{H}, \mathcal{E}'))\Sigma_{\mathcal{H}\otimes\mathcal{L}}\, Y(1_{\mathcal{H}} \otimes \mathcal{K}(\mathcal{E}, \mathcal{K}))] = [\mathcal{C}(Y) \odot \mathcal{K}(\mathcal{E}, \mathcal{E}')]\ (\subset \mathcal{B}(\mathcal{H} \otimes \mathcal{E}, \mathcal{L} \otimes \mathcal{E}')).$$

The equivalence is then a consequence of the equality

$$\Sigma_{\mathcal{L}\otimes\mathcal{E}'}\,(1_{\mathcal{L}} \otimes \mathcal{K}(\mathcal{H}, \mathcal{E}'))\,\Sigma_{\mathcal{H}\otimes\mathcal{L}} = \mathcal{K}(\mathcal{H}, \mathcal{E}') \otimes 1_{\mathcal{L}}.$$

The equivalence of the parallel statements follows by applying the above equivalence to:

$$Y' := \Sigma_{\mathcal{K}\otimes\mathcal{H}} X^* \Sigma_{\mathcal{H}\otimes\mathcal{L}}, \quad \mathcal{K}' := \mathcal{L}, \quad \mathcal{L}' := \mathcal{K},$$

since we have the equality $\mathcal{C}(Y') = \mathcal{C}(X)^*$. □

From now on, $\mathcal{G}$ is a colinking measured quantum groupoid $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. We will apply the previous lemma to the unitaries:

$$V^i_{rj} : \mathscr{H}_{ij} \otimes \mathscr{H}_{rj} \to \mathscr{H}_{ir} \otimes \mathscr{H}_{rj}, \quad W^k_{ij} : \mathscr{H}_{ij} \otimes \mathscr{H}_{jk} \to \mathscr{H}_{ij} \otimes \mathscr{H}_{ik}, \quad i,j,k,r = 1,2.$$

Lemma 3.2.12. *For all $\omega \in \mathcal{B}(\mathscr{H})_*$ and $i,j,k,r = 1,2$, we have:*

1. $p_{rj}(\mathrm{id} \otimes \omega)(\Sigma V)p_{ij} = (\mathrm{id} \otimes p_{rj}\omega p_{ir})(\Sigma_{ir\otimes rj} V^i_{rj}), \quad p_{rj}\mathcal{K}_\beta p_{ij} = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj})$.
2. $p_{ik}(\mathrm{id} \otimes \omega)(\Sigma W)p_{ij} = (\mathrm{id} \otimes p_{jk}\omega p_{ij})(\Sigma_{ij\otimes ik} W^k_{ij}), \quad p_{ik}\mathcal{K}_\alpha p_{ij} = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik})$.
3. $p_{ji}(\mathrm{id} \otimes \omega)(\Sigma \widetilde{V})p_{ki} = (\mathrm{id} \otimes p_{jk}\omega p_{ki})(\Sigma_{ki\otimes ji} \widetilde{V}^j_{ki}), \quad p_{ji}\mathcal{K}_{\widehat{\alpha}} p_{ki} = \mathcal{K}(\mathscr{H}_{ki}, \mathscr{H}_{ji})$.

Proof. Let $\omega \in \mathcal{B}(\mathscr{H})_*$. Since $(p_{kr} \otimes 1)V(p_{ij} \otimes 1) = \delta^i_k V^i_{rj}$, we have

$$(\mathrm{id} \otimes \omega)(\Sigma V) = \sum_{i,j,k,r=1,2} (\mathrm{id} \otimes \omega)(\Sigma(p_{kr} \otimes 1)V(p_{ij} \otimes 1)) = \sum_{i,j,r=1,2} (\mathrm{id} \otimes p_{rj}\omega p_{ir})(\Sigma_{ir\otimes rj} V^i_{rj}).$$

Hence, $p_{rj}(\mathrm{id} \otimes \omega)(\Sigma V)p_{ij} = (\mathrm{id} \otimes p_{rj}\omega p_{ir})(\Sigma_{ir\otimes rj}\, V^i_{rj})$. We prove the corresponding statements for W and $\widetilde{V}$ in a similar way.

Let $\eta \in \mathscr{H}$, $\zeta \in \mathscr{H}$ and $i,j = 1,2$. For all $n = (n_1, n_2) \in \mathbb{C}^2$, we have

$$\langle L^{\beta,\epsilon}(p_{ij}\eta)^*\zeta,\, n\rangle = \langle \zeta,\, L^{\beta,\epsilon}(p_{ij}\eta)n\rangle = \langle \zeta,\, \beta(n)p_{ij}\eta\rangle = \langle \xi,\, n_j p_{ij}\eta\rangle = \langle \zeta,\, p_{ij}\eta\rangle n_j = \langle\langle p_{ij}\eta, \zeta\rangle\varepsilon_j,\, n\rangle.$$

Hence, $L^{\beta,\epsilon}(p_{ij}\eta)^*\zeta = \langle p_{ij}\eta, \zeta\rangle\varepsilon_j$ for all $\zeta \in \mathscr{H}$. Note that $L^{\beta,\epsilon}(p_{ij}\eta)^* = L^{\beta,\epsilon}(p_{ij}\eta)^* p_{ij}$. Therefore, if $\xi, \eta \in \mathscr{H}$ we have

$$L^{\beta,\epsilon}(p_{rk}\xi)L^{\beta,\epsilon}(p_{ij}\eta)^*\zeta = \langle p_{ij}\eta,\, \zeta\rangle L^{\beta,\epsilon}(p_{rk}\xi)\varepsilon_j = \delta^j_k \langle p_{ij}\eta,\, \zeta\rangle p_{rk}\xi, \quad \zeta \in \mathscr{H}.$$

By sesquilinearity, it then follows that

$$L^{\beta,\epsilon}(\xi)L^{\beta,\epsilon}(\eta)^*\zeta = \sum_{i,j,r,k=1,2} L^{\beta,\epsilon}(p_{rk}\xi)L^{\beta,\epsilon}(p_{ij}\eta)^*\zeta = \sum_{i,j,r,k=1,2} \delta^j_k \langle p_{ij}\eta,\, \zeta\rangle p_{rk}\xi = \sum_{i,j,r=1,2} \langle p_{ij}\eta,\, \zeta\rangle p_{rj}\xi.$$

Hence, $p_{rk}L^{\beta,\epsilon}(\xi)L^{\beta,\epsilon}(\eta)^* p_{ij} = \delta^j_k \theta_{p_{rj}\xi,\, p_{ij}\eta}$ for all $\xi, \eta \in \mathscr{H}$. Hence, $p_{rj}\mathcal{K}_\beta p_{ij} = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj})$. The other statements follow in virtue of Lemma 3.2.5. □

Remark 3.2.13. Actually, both $\mathcal{C}(V)$ and $\mathcal{K}_\beta$ act on $\mathscr{H} = \mathscr{H}_{11} \oplus \mathscr{H}_{21} \oplus \mathscr{H}_{12} \oplus \mathscr{H}_{22}$ as block diagonal matrices in the following way:

$$\mathcal{C}(V) = \left(\begin{array}{c|c} \mathcal{C}_1 & 0 \\ \hline 0 & \mathcal{C}_2 \end{array}\right), \qquad \mathcal{K}_\beta = \left(\begin{array}{c|c} \mathcal{K}_1 & 0 \\ \hline 0 & \mathcal{K}_2 \end{array}\right),$$

where $\mathcal{C}_j := (\mathcal{C}(V^i_{rj}))_{r,i=1,2}$ and $\mathcal{K}_j := \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}) = (\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}))_{r,i=1,2} \subset \mathcal{B}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$ for $j = 1, 2$.

Corollary 3.2.14. *The following statements are equivalent:*

1. $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ *is semi-regular (resp. regular).*
2. *For all* $i, j, r = 1, 2$, *we have* $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}) \subset \mathcal{C}(V^i_{rj})$ (*resp.* $\mathcal{C}(V^i_{rj}) = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj})$).
3. *For all* $i, j, k = 1, 2$, *we have* $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \subset \mathcal{C}(W^k_{ij})$ (*resp.* $\mathcal{C}(W^k_{ij}) = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik})$).
4. *For all* $i, j, k = 1, 2$, *we have* $\mathcal{K}(\mathscr{H}_{ki}, \mathscr{H}_{ji}) \subset \mathcal{C}(\widetilde{V}^j_{ki})$ (*resp.* $\mathcal{C}(\widetilde{V}^j_{ki}) = \mathcal{K}(\mathscr{H}_{ki}, \mathscr{H}_{ji})$).

In particular, if $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ *is regular, then both* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *are regular.*

Proof. This a consequence of Lemma 3.2.12 (see also the above remark) and Proposition 3.2.8. □

Lemma 3.2.15. *For all* $i, j, r, s = 1, 2$, *we have:*

1. $\mathcal{C}(V^i_{rj})\mathscr{H}_{ij} = \mathscr{H}_{rj}$.
2. $\mathcal{C}(V^i_{rj})^* = \mathcal{C}(V^r_{ij})$.
3. $[\mathcal{C}(V^i_{rj})\mathcal{C}(V^s_{ij})] = \mathcal{C}(V^s_{rj})$.

Proof. Note that we have $\mathcal{C}(V) = U[S\widehat{S}]U^* = [R(S)\lambda(\widehat{S})]$ and $p_{rj}S = p_{rj}Sp_{rj}$. By Lemma 3.2.12, we then have

$$\mathcal{C}(V^i_{rj}) = p_{rj}\mathcal{C}(V)p_{ij} = [p_{rj}R(S)\lambda(\widehat{S})p_{ij}] = [R(S)_{rj}E^j_{ri,\lambda}],$$

where we denote $R(S)_{rj} = p_{rj}R(S) \subset \mathcal{B}(\mathscr{H}_{rj})$ (see also Notations 2.5.10). Now, it also follows from the fact that $[S\widehat{S}]$ is a C*-algebra that

$$[L(S)_{ji}E^j_{ir}] = p_{ji}[S\widehat{S}]p_{jr} = p_{ji}[\widehat{S}S]p_{jr} = [E^j_{ir}L(S)_{jr}],$$

where we denote $L(S)_{ij} = L(S)p_{ij} \subset \mathcal{B}(\mathscr{H}_{ij})$. By applying Ad_U to $[L(S)_{ji}E^j_{ir}] = [E^j_{ir}L(S)_{jr}]$, we then obtain

$$[R(S)_{ij}E^j_{ir,\lambda}] = [E^j_{ir,\lambda}R(S)_{rj}], \quad i, j, r = 1, 2.$$

Now, we can prove the three statements of the lemma. Let us fix $i, j, r, s = 1, 2$.
1. Since R is non-degenerate we have $R(S)\mathscr{H} = \mathscr{H}$, hence $R(S)_{ij}\mathscr{H}_{ij} = \mathscr{H}_{ij}$. Moreover, we have $[E^j_{ir,\lambda}\mathscr{H}_{rj}] = \mathscr{H}_{ij}$ (see Remark 2.5.12), hence $\mathcal{C}(V^i_{rj})\mathscr{H}_{ij} = \mathscr{H}_{rj}$.
2. $\mathcal{C}(V^i_{rj})^* = [R(S)_{rj}E^j_{ri,\lambda}]^* = [E^j_{ir,\lambda}R(S)_{rj}] = [R(S)_{ij}E^j_{ir,\lambda}] = \mathcal{C}(V^r_{ij})$. We can also prove it more directly by using the equality $\mathcal{C}(V^i_{rj}) = p_{rj}\mathcal{C}(V)p_{ij}$ and the fact that $\mathcal{C}(V)^* = \mathcal{C}(V)$.
3. By using again Remark 2.5.12, we obtain

$$\begin{aligned}[\mathcal{C}(V^i_{rj})\mathcal{C}(V^s_{ij})] &= [\,[R(S)_{rj}E^j_{ri,\lambda}][R(S)_{ij}E^j_{is,\lambda}]\,] = [R(S)_{rj}E^j_{ri,\lambda}R(S)_{ij}E^j_{is,\lambda}] \\ &= [R(S)_{rj}R(S)_{rj}E^j_{ri,\lambda}E^j_{is,\lambda}] = [R(S)_{rj}E^j_{rs,\lambda}] = \mathcal{C}(V^s_{rj}).\end{aligned}$$

□

Remarks 3.2.16. Since $\mathcal{C}(W) = \widehat{J}\mathcal{C}(V)\widehat{J}$, $\mathcal{C}(\widetilde{V}) = U^*\mathcal{C}(W)U$, we have $\mathcal{C}(W^r_{ij}) = \widehat{J}_{ri}\mathcal{C}(V^j_{ri})\widehat{J}_{ij}$ and $\mathcal{C}(\widetilde{V}^r_{ji}) = U^*_{ri}\mathcal{C}(W^r_{ij})U_{ji}$. It then follows that for all $i, j, r, s = 1, 2$ we have:

- $\mathcal{C}(W^r_{ij})\mathscr{H}_{ij} = \mathscr{H}_{ir}, \quad \mathcal{C}(\widetilde{V}^r_{ji})\mathscr{H}_{ji} = \mathscr{H}_{ri}.$
- $\mathcal{C}(W^r_{ij})^* = \mathcal{C}(W^j_{ir}), \quad \mathcal{C}(\widetilde{V}^r_{ji})^* = \mathcal{C}(\widetilde{V}^j_{ri}).$
- $[\mathcal{C}(W^r_{ji})\mathcal{C}(W^i_{js})] = \mathcal{C}(W^r_{js}), \quad [\mathcal{C}(\widetilde{V}^r_{ij})\mathcal{C}(\widetilde{V}^i_{sj})] = \mathcal{C}(\widetilde{V}^r_{sj}).$

Proposition 3.2.17. *Let us fix $i, j, k, r = 1, 2$.*
a) The following statements are equivalent:

1. $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}) \subset \mathcal{C}(V^i_{rj})$ *(resp.* $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}) = \mathcal{C}(V^i_{rj})$*).*
2. $\mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{ir}, \mathscr{H}_{rj} \otimes \mathscr{H}_{rj}) \subset [\,(\mathcal{K}(\mathscr{H}_{ir}, \mathscr{H}_{rj}) \otimes 1_{\mathscr{H}_{rj}})V^i_{rj}(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{ir}, \mathscr{H}_{rj}))\,]$
   (resp. $\mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{ir}, \mathscr{H}_{rj} \otimes \mathscr{H}_{rj}) = [\,(\mathcal{K}(\mathscr{H}_{ir}, \mathscr{H}_{rj}) \otimes 1_{\mathscr{H}_{rj}})V^i_{rj}(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{ir}, \mathscr{H}_{rj}))\,]$*).*

b) The following statements are equivalent:

1. $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \subset \mathcal{C}(W^k_{ij})$ *(resp.* $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) = \mathcal{C}(W^k_{ij})$*).*
2. $\mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{jk}, \mathscr{H}_{ik} \otimes \mathscr{H}_{ik}) \subset [\,(\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \otimes 1_{\mathscr{H}_{ik}})W^k_{ij}(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{jk}))\,]$
   (resp. $\mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{jk}, \mathscr{H}_{ik} \otimes \mathscr{H}_{ik}) = [\,(\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \otimes 1_{\mathscr{H}_{ik}})W^k_{ij}(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{jk}))\,]$*).*

Proof. The equivalence in a) (resp. b)) follows by applying Lemma 3.2.11 with $X := V^i_{rj}$ (resp. $Y := W^k_{ij}$), $\mathcal{H} := \mathscr{H}_{rj}$ (resp. $\mathscr{H}_{ij}$), $\mathcal{K} := \mathscr{H}_{ij}$ (resp. $\mathscr{H}_{jk}$), $\mathcal{L} := \mathscr{H}_{ir}$ (resp. $\mathscr{H}_{ik}$), $\mathcal{E}' = \mathscr{H}_{ir}$ (resp. $\mathscr{H}_{ik}$) and $\mathcal{E} = \mathscr{H}_{rj}$ (resp. $\mathscr{H}_{jk}$). □

Proposition 3.2.18. *Let us fix $i, j = 1, 2$, the following statements are equivalent:*

1. $\mathcal{K}(\mathscr{H}_{jj}) \subset \mathcal{C}(V^j_{jj})$ *(resp.* $\mathcal{K}(\mathscr{H}_{jj}) = \mathcal{C}(V^j_{jj})$*).*
2. $\mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij}) \subset \mathcal{C}(V^j_{ij})$ *(resp.* $\mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij}) = \mathcal{C}(V^j_{ij})$*).*
3. $\mathbb{G}_j$ *is semi-regular (resp. regular).*

Proof. Since $\mathcal{C}(V^j_{ij})\mathscr{H}_{jj} = \mathscr{H}_{ij}$, the inclusion $[\mathcal{C}(V^j_{ij})\mathcal{K}(\mathscr{H}_{jj})] \subset \mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij})$ is an equality. As a result, the implication $(1 \Rightarrow 2)$ is an immediate consequence of $\mathcal{C}(V^j_{ij}) = [\mathcal{C}(V^j_{ij})\mathcal{C}(V^j_{jj})]$. Conversely, we also have $\mathcal{C}(V^i_{jj})\mathscr{H}_{ij} = \mathscr{H}_{jj}$. The inclusion $[\mathcal{C}(V^i_{jj})\mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij})] \subset \mathcal{K}(\mathscr{H}_{jj})$ is then an equality. Moreover, the second statement is equivalent to $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{jj}) \subset \mathcal{C}(V^i_{jj})$ (resp. $\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{jj}) = \mathcal{C}(V^i_{jj})$) because of $\mathcal{C}(V^j_{ij})^* = \mathcal{C}(V^i_{jj})$. Therefore, the converse implication $(2 \Rightarrow 1)$ follows from the fact that $\mathcal{C}(V^j_{jj}) = [\mathcal{C}(V^i_{jj})\mathcal{C}(V^j_{ij})]$. □

Now, we can state the main result of this paragraph.

Theorem 3.2.19. *Let $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid between two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. The following statements are equivalent:*

1. $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ *is semi-regular (resp. regular).*
2. $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *are semi-regular (resp. regular).*

Proof. The implication $(1 \Rightarrow 2)$ has already been investigated and stated in Corollary 3.2.14. Conversely, let us assume that $\mathbb{G}_1$ and $\mathbb{G}_2$ are semi-regular (resp. regular). In virtue of Proposition 3.2.18, we have

$$\mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij}) \subset \mathcal{C}(V_{ij}^j) \quad (\text{resp. } \mathcal{K}(\mathscr{H}_{jj}, \mathscr{H}_{ij}) = \mathcal{C}(V_{ij}^j)), \quad \text{for all } i, j = 1, 2.$$

By Corollary 3.2.14, the aim is to prove that

$$\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}) \subset \mathcal{C}(V_{rj}^i) \quad (\text{resp. } \mathcal{C}(V_{rj}^i) = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj})), \quad \text{for all } i, j, r = 1, 2.$$

Since $\mathcal{C}(V_{rj}^j)\mathscr{H}_{jj} = \mathscr{H}_{rj}$, we have $[\mathcal{C}(V_{rj}^j)\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{jj})] = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj})$. Therefore, we have

$$\begin{aligned}\mathcal{C}(V_{rj}^i) = [\mathcal{C}(V_{rj}^j)\mathcal{C}(V_{jj}^i)] = [\mathcal{C}(V_{rj}^j)\mathcal{C}(V_{ij}^j)^*] \supset [\mathcal{C}(V_{rj}^j)\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{jj})] = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}) \\ (\text{resp. } = [\mathcal{C}(V_{rj}^j)\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{jj})] = \mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{rj}))\end{aligned}$$

ant the result is proved. □

Corollary 3.2.20. *Let $\mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid between two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Then, $\mathbb{G}_1$ and $\mathbb{G}_2$ are semi-regular (resp. regular) if and only if for all $i, j, k = 1, 2$ we have:*

$$\mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{jk}, \mathscr{H}_{ik} \otimes \mathscr{H}_{ik}) \subset [\,(\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \otimes 1_{\mathscr{H}_{ik}})W_{ij}^k(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{jk}))\,]$$

$$(resp.\ \mathcal{K}(\mathscr{H}_{ij} \otimes \mathscr{H}_{jk}, \mathscr{H}_{ik} \otimes \mathscr{H}_{ik}) = [\,(\mathcal{K}(\mathscr{H}_{ij}, \mathscr{H}_{ik}) \otimes 1_{\mathscr{H}_{ik}})W_{ij}^k(1_{\mathscr{H}_{ij}} \otimes \mathcal{K}(\mathscr{H}_{jk}))\,]).$$

Proof. This result is a straightforward consequence of Theorem 3.2.19, Corollary 3.2.14 and Proposition 3.2.17. □

Remarks 3.2.21. Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be monoidally equivalent locally compact quantum groups.

1. In virtue of Proposition 3.2.18, the condition $\mathcal{K}(\mathscr{H}_{22}, \mathscr{H}_{12}) = \mathcal{C}(V_{12}^2)$ does not imply in general that $\mathbb{G}_1$ is regular. This answers negatively to a question raised in [20].

2. If $\mathbb{G}_1$ and $\mathbb{G}_2$ are semi-regular, the inclusion

   $$\mathcal{K}(\mathscr{H}_{21} \otimes \mathscr{H}_{21}, \mathscr{H}_{21} \otimes \mathscr{H}_{11}) \subset [(1_{\mathscr{H}_{21}} \otimes \mathcal{K}(\mathscr{H}_{11}))(W_{21}^1)^*(\mathcal{K}(\mathscr{H}_{21}) \otimes 1_{\mathscr{H}_{21}})]$$

   will play a crucial role in the equivalence of continuous actions of $\mathbb{G}_1$ and $\mathbb{G}_2$.

Chapter 4

Actions of measured quantum groupoids on a finite basis

In the *whole* chapter, we fix

$$\mathcal{G} := (N, M, \alpha, \beta, \Delta, T, T', \epsilon)$$

a measured quantum groupoid on the finite basis

$$N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C}), \quad \epsilon = \bigoplus_{1 \leqslant l \leqslant k} n_l \cdot \mathrm{Tr}_l.$$

We denote (S, δ) and $(\widehat{S}, \widehat{\delta})$ the weak Hopf-C*-algebra and the dual weak Hopf-C*-algebra associated with $\mathcal{G}$.

4.1 Definition of actions of measured quantum groupoids on a finite basis

We begin this paragraph with the following lemma:

Lemma 4.1.1. *Let A and B be two C*-algebras, $f : A \to \mathcal{M}(B)$ a *-homomorphism and $e \in \mathcal{M}(B)$. The following statements are equivalent:*

1. *There exists an approximate unit $(u_\lambda)_\lambda$ of A such that $f(u_\lambda) \to e$ with respect to the strict topology.*
2. *f extends to a strictly continuous *-homomorphism $f : \mathcal{M}(A) \to \mathcal{M}(B)$, necessarily unique, such that $f(1_A) = e$.*
3. *$[f(A)B] = eB$.*

In that case, e is a self-adjoint projection, for all approximate unit $(v_\mu)_\mu$ of A we have $f(v_\mu) \to e$ with respect to the strict topology and $[Bf(A)] = Be$.

Proof. Let us assume that the assertion 1 holds. It is clear that e is a self-adjoint projection. Let us consider the Hilbert B-module eB. We identify the C*-algebras $\mathcal{L}(B)$ and $\mathcal{M}(B)$. Let us denote $\iota : \mathcal{L}(eB) \to \mathcal{M}(B)$ the faithful *-homomorphism defined by

$$\iota(T)b = T(eb), \quad T \in \mathcal{L}(eB),\, b \in B.$$

Note that ι is strictly continuous and satisfies $\iota(1_{eB}) = e$. It follows immediately from the assumption that $ef(a) = f(a) = f(a)e$ for all $a \in A$. In particular, f induces a *-homomorphism

$\tilde{f} : A \to \mathcal{L}(eB)$ and $f(u_\lambda) \to 1_{eB}$ with respect to the strict topology. Therefore, $\tilde{f}$ is non-degenerate. In particular, $\tilde{f}$ extends to a unital *-homomorphism $\tilde{\tilde{f}} : \mathcal{M}(A) \to \mathcal{L}(eB)$. We have

$$\iota(\tilde{\tilde{f}}(a))b = \tilde{f}(a)(eb) = f(a)eb = f(a)b, \quad a \in A,\, b \in B.$$

Therefore, the *-homomorphism $\iota \circ \tilde{\tilde{f}} : \mathcal{M}(A) \to \mathcal{M}(B)$ is a strictly continuous extension of f and we have $\iota(\tilde{\tilde{f}}(1_A)) = e$. Moreover, since $\tilde{f}$ is non-degenerate, we have $[\tilde{f}(A)eB] = eB$. It then follows that we have the equality $[f(A)B] = eB$. Therefore, we have proved the implications $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$. The implication $(2 \Rightarrow 1)$ is straightforward. Let us prove that $(3 \Rightarrow 1)$ and let us then assume that $[f(A)B] = eB$. In particular, we also have $[Bf(A)] = Be$ since f is stable by the involution. Let $a \in A$, we have $f(a)b \in eB$ and $bf(a) \in Be$ for all $b \in B$. By taking an approximate unit of B, we obtain that $f(a) \in eB \cap Be = eBe$. Hence, $ef(a) = f(a) = f(a)e$ for all $a \in A$. Let $(u_\lambda)_\lambda$ be an approximate unit of A. For all $a \in A$ and $b \in B$, we have $f(u_\lambda)f(a)b = f(u_\lambda a)b \to f(a)b$ with respect to the norm topology. By the assumption, we then have $f(u_\lambda)b = f(u_\lambda)eb \to eb$ with respect to the norm topology for all $b \in B$, which means that $f(u_\lambda) \to e$ with respect to the strict topology and we are done. □

Definition 4.1.2. Let us fix a C*-algebra A. An action of $\mathcal{G}$ on A is a couple (δ_A, β_A) consisting of a faithful *-homomorphism $\delta_A : A \to \mathcal{M}(A \otimes S)$ and a non-degenerate *-homomorphism $\beta_A : N^{\mathrm{o}} \to \mathcal{M}(A)$ such that:

1. δ_A extends to a strictly continuous *-homomorphism from $\mathcal{M}(A)$ to $\mathcal{M}(A \otimes S)$ still denoted δ_A such that $\delta_A(1_A) = q^{\beta_A,\alpha}$.
2. $(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$.
3. $\delta_A(\beta_A(n^{\mathrm{o}})) = q^{\beta_A,\alpha}(1_A \otimes \beta(n^{\mathrm{o}}))$, for all $n \in N$.

We say that the action (δ_A, β_A) is continuous if we have further that

$$[\delta_A(A)(1_A \otimes S)] = q^{\beta_A,\alpha}(A \otimes S).$$

If (δ_A, β_A) is a continuous action of $\mathcal{G}$ on A, we say that the triple (A, δ_A, β_A) is a $\mathcal{G}$-C*-algebra.

Remarks 4.1.3.

- By Lemma 4.1.1, the condition 1 is equivalent to requiring that for some (and then any) approximate unit (u_λ) of A, we have $\delta_A(u_\lambda) \to q^{\beta_A,\alpha}$ with respect to the strict topology of $\mathcal{M}(A \otimes S)$. It is also equivalent to $[\delta_A(A)(A \otimes S)] = q^{\beta_A,\alpha}(A \otimes S)$.
- The condition 1 also implies that the *-homomorphisms $\delta_A \otimes \mathrm{id}_S$ and $\mathrm{id}_A \otimes \delta$ extend uniquely to strictly continuous *-homomorphisms from $\mathcal{M}(A \otimes S)$ to $\mathcal{M}(A \otimes S \otimes S)$ such that
$$(\delta_A \otimes \mathrm{id}_S)(1_{A\otimes S}) = q_{12}^{\beta_A,\alpha}, \quad (\mathrm{id}_A \otimes \delta)(1_{A\otimes S}) = q_{23}^{\beta,\alpha}.$$

Examples 4.1.4. Let us give two basic examples:

- (S, δ, β) is a $\mathcal{G}$-C*-algebra.
- Let us denote $\delta_{N^{\mathrm{o}}} : N^{\mathrm{o}} \to \mathcal{M}(N^{\mathrm{o}} \otimes S)$ the faithful unital *-homomorphism given by $\delta_{N^{\mathrm{o}}}(n^{\mathrm{o}}) = 1_{N^{\mathrm{o}}} \otimes \beta(n^{\mathrm{o}})$ for all $n \in N$. Then, the pair $(\delta_{N^{\mathrm{o}}}, \mathrm{id}_{N^{\mathrm{o}}})$ is an action of $\mathcal{G}$ on N^{o} called the trivial action.

Proposition 4.1.5. *Let (δ_A, β_A) be an action of $\mathcal{G}$ on A. We have:*

i. $(\delta_A \otimes \mathrm{id}_S)\delta_A(1_A) = q_{12}^{\beta_A,\alpha} q_{23}^{\beta,\alpha} = (\mathrm{id}_A \otimes \delta)\delta_A(1_A)$.

ii. $\delta_A(\beta_A(n^{\mathrm{o}})) = (1_A \otimes \beta(n^{\mathrm{o}}))q^{\beta_A,\alpha}$, *for all* $n \in N$.

iii. If (δ_A, β_A) *is continuous, we have* $[(1_A \otimes S)\delta_A(A)] = (A \otimes S)q^{\beta_A,\alpha}$.

Proof. The equality $(\delta_A \otimes \mathrm{id}_S)\delta_A(1_A) = q_{12}^{\beta_A,\alpha} q_{23}^{\beta,\alpha}$ follows easily from $\delta_A(1_A) = q^{\beta_A,\alpha}$ and the condition 3. The second equality $(\mathrm{id}_A \otimes \delta)\delta_A(1_A) = q_{12}^{\beta_A,\alpha} q_{23}^{\beta,\alpha}$ follows from $\delta_A(1_A) = q^{\beta_A,\alpha}$ and the fact that $\delta(\alpha(n)) = (\alpha(n) \otimes 1_S)q^{\beta,\alpha}$, for all $n \in N$.
The statement ii follows from the condition 3 and the fact that α and β commute pointwise. The equality $[(1_A \otimes S)\delta_A(A)] = (A \otimes S)q^{\beta_A,\alpha}$ is an obvious consequence of the fact that $q^{\beta_A,\alpha}$ is self-adjoint. □

Let us provide a more explicit definition of what an action of the dual measured quantum groupoid $\widehat{\mathcal{G}}$ on a C*-algebra B is.

Definition 4.1.6. Let us fix a C*-algebra B. An action of $\widehat{\mathcal{G}}$ on a C*-algebra B is a couple (δ_B, α_B) consisting of a faithful *-homomorphism $\delta_B : B \to \mathcal{M}(B \otimes \widehat{S})$ and a non-degenerate *-homomorphism $\alpha_B : N \to \mathcal{M}(B)$ such that:

1. δ_B extends to a strictly continuous *-homomorphism from $\mathcal{M}(B)$ to $\mathcal{M}(B \otimes \widehat{S})$ still denoted δ_B such that $\delta_B(1_B) = q^{\alpha_B,\beta}$.
2. $(\delta_B \otimes \mathrm{id}_{\widehat{S}})\delta_B = (\mathrm{id}_B \otimes \widehat{\delta})\delta_B$.
3. $\delta_B(\alpha_B(n)) = q^{\alpha_B,\beta}(1_B \otimes \widehat{\alpha}(n))$, for all $n \in N$.

We say that the action (δ_B, α_B) is continuous if we have further that

$$[\delta_B(B)(1_B \otimes \widehat{S})] = q^{\alpha_B,\beta}(B \otimes \widehat{S}).$$

If (δ_B, α_B) is a continuous action of $\widehat{\mathcal{G}}$ on B, we say that the triple (B, δ_B, α_B) is a $\widehat{\mathcal{G}}$-C*-algebra.

Remarks 4.1.7. As for actions of $\mathcal{G}$, we have:

- By Lemma 4.1.1, the condition 1 is equivalent to requiring that for some (and then any) approximate unit $(u_\lambda)_\lambda$ of B, we have $\delta_B(u_\lambda) \to q^{\alpha_B,\beta}$ with respect to the strict topology of $\mathcal{M}(B \otimes \widehat{S})$. It is also equivalent to $[\delta_B(B)(B \otimes \widehat{S})] = q^{\alpha_B,\beta}(B \otimes \widehat{S})$.
- In virtue of 1, the *-homomorphisms $\delta_B \otimes \mathrm{id}_{\widehat{S}}$ and $\mathrm{id}_B \otimes \widehat{\delta}$ extend uniquely to strictly continuous *-homomorphisms from $\mathcal{M}(B \otimes \widehat{S})$ to $\mathcal{M}(B \otimes \widehat{S} \otimes \widehat{S})$ such that

$$(\delta_B \otimes \mathrm{id}_{\widehat{S}})(1_{B \otimes \widehat{S}}) = q_{12}^{\alpha_B,\beta}, \quad (\mathrm{id}_B \otimes \widehat{\delta})(1_{B \otimes \widehat{S}}) = q_{23}^{\widehat{\alpha},\beta}.$$

Examples 4.1.8. Let us give two basic examples:

- $(\widehat{S}, \widehat{\delta}, \widehat{\alpha})$ is a $\widehat{\mathcal{G}}$-C*-algebra.
- Let us denote $\delta_N : N \to \mathcal{M}(N \otimes \widehat{S})$ the faithful unital *-homomorphism given for all $n \in N$ by $\delta_N(n) = 1_N \otimes \widehat{\alpha}(n)$. Then, the pair $(\delta_N, \mathrm{id}_N)$ is an action of $\widehat{\mathcal{G}}$ on N called the trivial action.

Proposition 4.1.9. *Let* (δ_B, α_B) *be an action of* $\widehat{\mathcal{G}}$ *on* B. *We have:*

i. $(\delta_B \otimes \mathrm{id}_{\widehat{S}})\delta_B(1_B) = q_{12}^{\alpha_B,\beta} q_{23}^{\widehat{\alpha},\beta} = (\mathrm{id}_B \otimes \widehat{\delta})\delta_B(1_B)$.

ii. $\delta_B(\alpha_B(n)) = (1_B \otimes \widehat{\alpha}(n))q^{\alpha_B,\beta}$, *for all* $n \in N$.

iii. If (δ_B, α_B) *is continuous, we have* $[(1_B \otimes \widehat{S})\delta_B(B)] = (B \otimes \widehat{S})q^{\alpha_B,\beta}$.

Definition 4.1.10. Let A_i (resp. B_i) be two C*-algebras, for $i = 1, 2$. Let $(\delta_{A_i}, \beta_{A_i})$ (resp. $(\delta_{B_i}, \alpha_{B_i})$) be an action of $\mathcal{G}$ (resp. $\widehat{\mathcal{G}}$) on A_i (resp. B_i), for $i = 1, 2$. A non-degenerate *-homomorphism

$$f : A_1 \to \mathcal{M}(A_2) \quad (\text{resp. } f : B_1 \to \mathcal{M}(B_2))$$

is said to be $\mathcal{G}$-equivariant (resp. $\widehat{\mathcal{G}}$-equivariant) if we have

$$(f \otimes \mathrm{id}_S) \circ \delta_{A_1} = \delta_{A_2} \circ f, \quad f \circ \beta_{A_1} = \beta_{A_2}$$

$$(\text{resp. } (f \otimes \mathrm{id}_{\widehat{S}})\delta_{B_1} = \delta_{B_2} \circ f, \quad f \circ \alpha_{B_1} = \alpha_{B_2}).$$

We denote $\mathrm{Mor}_{\mathcal{G}}(A_1, A_2)$ (resp. $\mathrm{Mor}_{\widehat{\mathcal{G}}}(B_1, B_2)$) the set of $\mathcal{G}$-equivariant (resp. $\widehat{\mathcal{G}}$-equivariant) non-degenerate *-homomorphisms from A_1 (resp. B_1) to $\mathcal{M}(A_2)$ (resp. $\mathcal{M}(B_2)$). We also define the category denoted $\mathcal{G}$-C*-Alg (resp. $\widehat{\mathcal{G}}$-C*-Alg) whose objects are the $\mathcal{G}$-C*-algebras (resp. $\widehat{\mathcal{G}}$-C*-algebras) and whose set of arrows between two $\mathcal{G}$-C*-algebras (resp. $\widehat{\mathcal{G}}$-C*-algebras) $(A_i, \delta_{A_i}, \beta_{A_i})$ (resp. $(B_i, \delta_{B_i}, \alpha_{B_i})$) $i = 1, 2$ is $\mathrm{Mor}_{\mathcal{G}}(A_1, A_2)$ (resp. $\mathrm{Mor}_{\widehat{\mathcal{G}}}(B_1, B_2)$).

4.2 Crossed product

In this section, we define the crossed product of a C*-algebra acted upon by $\mathcal{G}$. First, we introduce some notations. Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. Since the *-homomorphism $L : S \to \mathcal{B}(\mathscr{H})$ (resp. $R : S \to \mathcal{B}(\mathscr{H})$) is non-degenerate the map $\mathrm{id}_A \otimes L$ (resp. $\mathrm{id}_A \otimes R$) extends uniquely to a faithful and unital *-homomorphism from $\mathcal{M}(A \otimes S)$ to $\mathcal{L}(A \otimes \mathscr{H})$ still denoted $\mathrm{id}_A \otimes L$ (resp. $\mathrm{id}_A \otimes R$). Then, let us consider the following faithful *-representations of A on the Hilbert A-module $A \otimes \mathscr{H}$:

$$\pi_L = (\mathrm{id}_A \otimes L)\delta_A, \quad \pi_R = (\mathrm{id}_A \otimes R)\delta_A.$$

Since δ_A extends uniquely to a strictly continuous *-homomorphism δ_A from $\mathcal{M}(A)$ to $\mathcal{M}(A \otimes S)$ such that $\delta_A(1_A) = q^{\beta_A,\alpha}$, π_L (resp. π_R) extends to a faithful strictly continuous *-representation of $\mathcal{M}(A)$ on $A \otimes \mathscr{H}$ still denoted π_L (resp. π_R) such that

$$\pi_L(1_A) = q^{\beta_A,\alpha} \quad (\text{resp. } \pi_R(1_A) = q^{\beta_A,\widehat{\alpha}}).$$

We introduce the following Hilbert A-modules:

$$\mathscr{E}_{A,L} = q^{\beta_A,\alpha}(A \otimes \mathscr{H}), \quad \mathscr{E}_{A,R} = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathscr{H}).$$

• Since $q^{\beta_A,\alpha}\pi_L(m) = \pi_L(m) = \pi_L(m)q^{\beta_A,\alpha}$ for all $m \in \mathcal{M}(A)$, π_L induces a faithful unital strictly continuous *-representation $\pi : \mathcal{M}(A) \to \mathcal{L}(\mathscr{E}_{A,L})$.

• We have $[1_A \otimes m, q^{\beta_A,\alpha}] = 0$ for all $m \in \mathcal{M}(\widehat{S})$ because of $\widehat{S} \subset \widehat{M}' \subset \alpha(N)'$. Therefore, we have a unital strictly continuous *-representation $\widehat{\theta} : \mathcal{M}(\widehat{S}) \to \mathcal{L}(\mathscr{E}_{A,L})$ given by

$$\widehat{\theta}(x) = (1_A \otimes \rho(x))\restriction_{\mathscr{E}_{A,L}}, \quad x \in \mathcal{M}(\widehat{S}).$$

Note that $\widehat{\theta}$ is *not* faithful in general. However, if the fibration map β_A is faithful so is $\widehat{\theta}$. Indeed, let us assume that β_A is faithful and let $x \in \mathcal{M}(\widehat{S})$ such that $\widehat{\theta}(x) = 0$, that is to say

$$\sum_{1 \leqslant l \leqslant k} n_l^{-1} \sum_{1 \leqslant i,j \leqslant n_l} \beta_A(e_{ij}^{(l)\mathrm{o}}) \otimes L(\alpha(e_{ji}^{(l)}))\rho(x) = 0.$$

However, it follows from the assumption that the family $(\beta_A(e_{ij}^{(l)\mathrm{o}}))_{1\leqslant l\leqslant k,\,1\leqslant i,j\leqslant n_l}$ is linearly independent. Hence, $L(\alpha(e_{ji}^{(l)}))\rho(x)=0$ for all $1\leqslant l\leqslant k$ and $1\leqslant i,j\leqslant n_l$. However α is unital, it then follows that

$$\rho(x)=\sum_{1\leqslant l\leqslant k}\sum_{1\leqslant i\leqslant n_l}L(\alpha(e_{ii}^{(l)}))\rho(x)=0.$$

Hence, $x=0$.

Now, we can define the *-representation $\widetilde{\pi}$ of A on the Hilbert A-module $A\otimes\mathscr{H}\otimes\mathscr{H}$:

$$\widetilde{\pi}:=(\pi_L\otimes L)\delta_A=(\mathrm{id}_A\otimes L\otimes L)\delta_A^2:A\to\mathcal{L}(A\otimes\mathscr{H}\otimes\mathscr{H}),$$

where $\delta_A^2:=(\delta_A\otimes\mathrm{id}_S)\delta_A=(\mathrm{id}_A\otimes\delta)\delta_A:A\to\mathcal{M}(A\otimes S\otimes S)$. Moreover, $\widetilde{\pi}$ extends uniquely to a *-representation of $\mathcal{M}(A)$ on $A\otimes\mathscr{H}\otimes\mathscr{H}$ such that $\widetilde{\pi}(1_A)=(\mathrm{id}_A\otimes L\otimes L)(q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha})$.

Definition 4.2.1. We call crossed product of A by the continuous action (δ_A,β_A) of $\mathcal{G}$ on A, the C*-subalgebra $A\rtimes_{\delta_A,\beta_A}\mathcal{G}$ of $\mathcal{L}(\mathscr{E}_{A,L})$ generated by the products of the form $\pi(a)\widehat{\theta}(x)$ for $a\in A$ and $x\in\widehat{S}$.

From now on, we will denote the crossed product $A\rtimes\mathcal{G}$, instead of $A\rtimes_{\delta_A,\beta_A}\mathcal{G}$, since no ambiguity will arise.

Lemma 4.2.2. *The norm closed linear subspace of $\mathcal{L}(A\otimes\mathscr{H})$ spanned by the products of the form $\pi_L(a)(1_A\otimes\rho(\omega))$, $a\in A$, $\omega\in\mathcal{B}(\mathscr{H})_*$, is a C*-algebra.*

Proof. Let us denote $B=[\pi_L(a)(1_A\otimes\rho(\omega))\,;\,a\in A,\,\omega\in\mathcal{B}(\mathscr{H})_*]$. Let $a\in A$ and $\omega\in\mathcal{B}(\mathscr{H})_*$. Let us prove that $(1_A\otimes\rho(\omega))\pi_L(a)\in B$. We have

$$(1_A\otimes\rho(\omega))\pi_L(a)=(\mathrm{id}_A\otimes\mathrm{id}_{\mathscr{H}}\otimes\omega)(V_{23}\pi_L(a)_{12}).\tag{4.2.1}$$

On the one hand, since $(L\otimes L)\delta(y)=V(L(y)\otimes 1)V^*$ for all $y\in S$ we have

$$V_{23}\pi_L(a)V_{23}^*=\widetilde{\pi}(a).\tag{4.2.2}$$

On the other hand, $[\pi_L(a)_{12},V_{23}^*V_{23}]=0$ since $V^*V=q^{\widehat{\alpha},\beta}$ and $\widehat{\alpha}(N)\subset M'$. It then follows that

$$\widetilde{\pi}(a)V_{23}=V_{23}\pi_L(a)_{12}V_{23}^*V_{23}=V_{23}\pi_L(a)_{12},\tag{4.2.3}$$

as V is a partial isometry. We derive from (4.2.1) and (4.2.3) that

$$(1_A\otimes\rho(\omega))\pi_L(a)=(\mathrm{id}_A\otimes\mathrm{id}_{\mathscr{H}}\otimes\omega)(\widetilde{\pi}(a)V_{23}).$$

Since L is non-degenerate, let us write $\omega=\omega' L(s)$ for some $s\in S$ and $\omega'\in\mathcal{B}(\mathscr{H})_*$. First, note that we have $\widetilde{\pi}(a)=\widetilde{\pi}(1_A)\widetilde{\pi}(a)=(\mathrm{id}_A\otimes L\otimes L)(q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha})\widetilde{\pi}(a)=(\mathrm{id}_A\otimes L\otimes L)(q_{12}^{\beta_A,\alpha})\widetilde{\pi}(a)$ in virtue of (4.2.2) and the fact that $q^{\beta,\alpha}=VV^*$. It then follows that

$$\begin{aligned}(\pi_L\otimes L)((1_A\otimes s)\delta_A(a))&=(\pi_L(1_A)\otimes L(s))\widetilde{\pi}(a)\\&=(1_A\otimes 1_{\mathscr{H}}\otimes L(s))(1_A\otimes L\otimes L)(q_{12}^{\beta_A,\alpha})\widetilde{\pi}(a)\\&=(1_A\otimes 1_{\mathscr{H}}\otimes L(s))\widetilde{\pi}(a).\end{aligned}$$

As a result, we have

$$\begin{aligned}(\mathrm{id}_A\otimes\mathrm{id}_{\mathscr{H}}\otimes\omega)(\widetilde{\pi}(a)V_{23})&=(\mathrm{id}_A\otimes\mathrm{id}_{\mathscr{H}}\otimes\omega')((1_A\otimes 1_{\mathscr{H}}\otimes L(s))\widetilde{\pi}(a)V_{23})\\&=(\mathrm{id}_A\otimes\mathrm{id}_{\mathscr{H}}\otimes\omega')((\pi_L\otimes L)((1_A\otimes s)\delta_A(a))V_{23}).\end{aligned}$$

However, $(1_A \otimes s)\delta_A(a)$ is the norm limit of finite sums of the form $\sum_i a_i \otimes s_i$, where $a_i \in A$ and $s_i \in S$. Therefore, $(1_A \otimes \rho(\omega))\pi_L(a)$ is the norm limit of finite sums of the form

$$\begin{aligned}\sum_i (\mathrm{id}_A \otimes \mathrm{id}_{\mathscr{H}} \otimes \omega')((\pi_L(a_i) \otimes L(s_i))V_{23}) &= \sum_i (\mathrm{id}_A \otimes \mathrm{id}_{\mathscr{H}} \otimes \omega')(\pi_L(a_i)_{12}((1 \otimes L(s_i))V)_{23}) \\ &= \sum_i \pi_L(a_i)(1_A \otimes (\mathrm{id} \otimes \omega')((1 \otimes L(s_i))V)) \\ &= \sum_i \pi_L(a_i)(1_A \otimes \rho(\omega' L(s_i))),\end{aligned}$$

where $a_i \in A$ and $s_i \in S$. Hence, $(1_A \otimes \rho(\omega))\pi_L(a) \in B$ for all $a \in A$ and $\omega \in \mathcal{B}(\mathscr{H})_*$. This actually proves that B is a C*-subalgebra of $\mathcal{L}(A \otimes \mathscr{H})$. □

The previous lemma clearly proves that π (resp. $\widehat{\theta}$) defines a faithful unital *-homomorphism (resp. a unital *-homomorphism) $\pi : \mathcal{M}(A) \to \mathcal{M}(A \rtimes \mathcal{G})$ (resp. $\widehat{\theta} : \mathcal{M}(\widehat{S}) \to \mathcal{M}(A \rtimes \mathcal{G})$). The following proposition is also an immediate consequence of this lemma.

Proposition 4.2.3. *We have $\pi(a)\widehat{\theta}(x) \in A \rtimes \mathcal{G}$ for all $a \in A$ and $x \in \widehat{S}$. Moreover, we have*

$$A \rtimes \mathcal{G} = [\, \pi(a)\widehat{\theta}(x)\,;\, a \in A,\, x \in \widehat{S}\,].$$

This proposition allows us to define $\Psi_{L,\rho} : A \rtimes \mathcal{G} \to \mathcal{L}(A \otimes \mathscr{H})$ the faithful strictly continuous *-representation of $A \rtimes \mathcal{G}$ on $A \otimes \mathscr{H}$, given by

$$\Psi_{L,\rho}(\pi(a)\widehat{\theta}(x)) = \pi_L(a)(1_A \otimes \rho(x)), \quad a \in A,\, x \in \widehat{S}.$$

Then, $\Psi_{L,\rho}$ extends uniquely to a faithful *-representation of $\mathcal{M}(A \rtimes \mathcal{G})$ on $A \otimes \mathscr{H}$, still denoted $\Psi_{L,\rho}$, such that $\Psi_{L,\rho}(1_{A \rtimes \mathcal{G}}) = q^{\beta_A,\alpha}$.

Proposition 4.2.4. *We have the following statements:*

1. $\forall a \in A$, $\Psi_{L,\rho}(\pi(a)) = \pi_L(a)$.
2. $\forall x \in \widehat{S}$, $\Psi_{L,\rho}(\widehat{\theta}(x)) = q^{\beta_A,\alpha}(1_A \otimes \rho(x))$.

Proof. Let $(x_\lambda)_\lambda$ be an approximate unit of $\widehat{S}$. Since ρ is non-degenerate, the net $(\rho(x_\lambda))_\lambda$ converges strongly to $1_{\mathscr{H}}$. Hence, $\pi_L(a) = \mathrm{s}-\lim_\lambda \pi_L(a)(1_A \otimes \rho(x_\lambda)) = \mathrm{s}-\lim_\lambda \Psi_{L,\rho}(\pi(a)\widehat{\theta}(x_\lambda))$. Since $\widehat{\theta}$ is non-degenerate, the net $(\widehat{\theta}(x_\lambda))_\lambda$ converges strongly to $1_{A \rtimes \mathcal{G}}$. Therefore, we have $\pi(a) = \mathrm{s} - \lim_\lambda \pi(a)\widehat{\theta}(x_\lambda)$. Hence, $\Psi_{L,\rho}(\pi(a)) = \mathrm{s} - \lim_\lambda \Psi_{L,\rho}(\pi(a)\widehat{\theta}(x_\lambda))$ as $\Psi_{L,\rho}$ is strongly continuous. This proves the first statement. The second statement is proved in a similar way. Indeed, let $(a_\lambda)_\lambda$ be an approximate unit of A. For all $x \in \widehat{S}$, we have

$$q^{\beta_A,\alpha}(1_A \otimes \rho(x)) = \mathrm{s} - \lim_\lambda \pi_L(a_\lambda)(1_A \otimes \rho(x)) = \mathrm{s} - \lim \Psi_{L,\rho}(\pi(a_\lambda)\widehat{\theta}(x)) = \Psi_{L,\rho}(\widehat{\theta}(x))$$

by using $q^{\beta_A,\alpha} = \mathrm{s} - \lim_\lambda \pi_L(a_\lambda)$, the non-degeneracy of π and the fact that $\Psi_{L,\rho}$ is strongly continuous. □

Proposition 4.2.5. *Let $S \rtimes \mathcal{G}$ be the crossed product of S by the action (δ, β) of $\mathcal{G}$ on S. Then, $S \rtimes \mathcal{G}$ is canonically isomorphic to $[S\widehat{S}]$.*

Proof. Since L is non-degenerate, $L \otimes \mathrm{id} : S \otimes \mathcal{K}(\mathscr{H}) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ extends uniquely to a faithful and unital *-homomorphism $L \otimes \mathrm{id} : \mathcal{L}(S \otimes \mathscr{H}) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. Besides, the *-representation $\Psi_{L,\rho} : S \rtimes \mathcal{G} \to \mathcal{L}(S \otimes \mathscr{H})$ is given by

$$\Psi_{L,\rho}(\pi(s)\widehat{\theta}(x)) = (\mathrm{id}_S \otimes L)(\delta(s))(1_S \otimes \rho(x)), \quad s \in S,\, x \in \widehat{S}.$$

We have $(L \otimes L)\delta(s) = W^*(1 \otimes L(s))W$ for all $s \in S$. Moreover, we have $W \in M \otimes \widehat{M}$ and $\rho(\widehat{S}) \subset \widehat{M}'$. It then follows that

$$\begin{aligned}(L \otimes \mathrm{id})\Psi_{L,\rho}(\pi(s)\widehat{\theta}(x)) &= (L \otimes L)(\delta(s))(1 \otimes \rho(x)) = W^*(1 \otimes L(s))W(1 \otimes \rho(x)) \\ &= W^*(1 \otimes L(s)\rho(x))W,\end{aligned}$$

for all $s \in S$ and $x \in \widehat{S}$. We also consider the map $\mathrm{Ad}_W : \mathcal{B}(\mathscr{H} \otimes \mathscr{H}) \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ given by $\mathrm{Ad}_W(x) = WxW^*$ for all $x \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. It is clear that Ad_W is a bounded linear map such that $\mathrm{Ad}_W(x^*) = \mathrm{Ad}_W(x)^*$ for all $x \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. It is worth noting that unfortunately Ad_W is not multiplicative. However, let $x, y \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ such that $[x, q^{\beta,\alpha}] = 0$ or $[y, q^{\beta,\alpha}] = 0$. Then, we have $\mathrm{Ad}_W(xy) = \mathrm{Ad}_W(x)\mathrm{Ad}_W(y)$ since W is a partial isometry and $q^{\beta,\alpha} = W^*W$. In particular, we have

$$\mathrm{Ad}_W(W^*xWW^*yW) = \mathrm{Ad}_W(W^*xW)\mathrm{Ad}_W(W^*yW), \quad \text{for all } x, y \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H}).$$

It then follows that $\Phi = \mathrm{Ad}_W(L \otimes \mathrm{id})\Psi_{L,\rho} : S \rtimes \mathcal{G} \to \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$ is a *-homomorphism. Furthermore, we have $\Phi(\pi(s)\widehat{\theta}(x)) = q^{\alpha,\widehat{\beta}}(1 \otimes L(s)\rho(x))q^{\alpha,\widehat{\beta}} = q^{\alpha,\widehat{\beta}}(1 \otimes L(s)\rho(x))$ for all $s \in S$ and $x \in \widehat{S}$. Indeed, since $\widehat{\beta}(N^{\mathrm{o}}) = M' \cap \widehat{M}$ we have $[\widehat{\beta}(n^{\mathrm{o}}), L(s)] = 0$ and $[\widehat{\beta}(n^{\mathrm{o}}), \rho(x)] = 0$ for all $n \in N$, $s \in S$ and $x \in \widehat{S}$. Hence, $[\widehat{\beta}(n^{\mathrm{o}}), z] = 0$ for all $n \in N$ and $z \in [S\widehat{S}]$.
Moreover, since Ad_W is generally not injective, the faithfulness of Φ is not obvious. However, it suffices to prove that the restriction of Ad_W to the range of $(\mathrm{id} \otimes L)\Psi_{L,\rho}$, that is to say $W^*(1 \otimes [S\widehat{S}])W$, is injective. Assume that $\mathrm{Ad}_W(W^*(1 \otimes z)W) = 0$ for some $z \in [S\widehat{S}]$. Then, we have $q^{\alpha,\widehat{\beta}}(1 \otimes z) = 0$. But α is faithful, it then follows that $\widehat{\beta}(e_{ij}^{(l)\,\mathrm{o}})z = 0$ for all $1 \leqslant l \leqslant k$ and $1 \leqslant i, j \leqslant n_l$. Hence, $z = 0$. Therefore, since $S \rtimes \mathcal{G} = [\pi(s)\widehat{\theta}(x)\,;\, s \in S, x \in \widehat{S}]$, Φ is a *-isomorphism from $S \rtimes \mathcal{G}$ to the C*-algebra $q^{\alpha,\widehat{\beta}}(1 \otimes [S\widehat{S}])$.

It only remains to show that $q^{\alpha,\widehat{\beta}}(1 \otimes [S\widehat{S}])$ is canonically isomorphic to $[S\widehat{S}]$. First, note that there exists $\omega \in \mathcal{B}(\mathscr{H})_*$ such that $(\omega \otimes \mathrm{id})(q^{\alpha,\widehat{\beta}}) = 1$. Indeed, let us consider $\omega : \alpha(N) \to \mathbb{C}$ the map defined by $\omega(\alpha(n)) = \epsilon(n)$ for all $n \in N$. Then, ω is a well-defined normal positive linear functional on the von Neumann algebra $\alpha(N)$. Therefore, it extends to a normal linear functional on $\mathcal{B}(\mathscr{H})$, still denoted ω. Now, since $\omega(\alpha(e_{ij}^{(l)})) = n_l\delta_i^j$ for all $1 \leqslant l \leqslant k$ and $1 \leqslant i, j \leqslant n_l$, we have $(\omega \otimes \mathrm{id})(q^{\alpha,\widehat{\beta}}) = 1$. Let us look at the map $\Theta : [S\widehat{S}] \to q^{\alpha,\widehat{\beta}}(1 \otimes [S\widehat{S}])$ given by $\Theta(z) = q^{\alpha,\widehat{\beta}}(1 \otimes z)$, for all $z \in [S\widehat{S}]$. Then, Θ is a surjective *-homomorphism. Besides, Θ is also injective since $(\omega \otimes \mathrm{id})(\Theta(z)) = (\omega \otimes \mathrm{id})(q^{\alpha,\widehat{\beta}})z = z$ for all $z \in [S\widehat{S}]$. □

Let (B, δ_B, α_B) be a $\widehat{\mathcal{G}}$-C*-algebra. In order to define the crossed product $B \rtimes_{\delta_B,\alpha_B} \widehat{\mathcal{G}}$, we introduce the Hilbert B-module:

$$\mathscr{E}_{B,\lambda} = q^{\alpha_B,\widehat{\beta}}(B \otimes \mathscr{H}).$$

Note that $q^{\alpha_B,\widehat{\beta}} = (\mathrm{id}_B \otimes \lambda)\alpha_B(1_B)$. Exactly as for actions of $\mathcal{G}$, we have:

- a faithful strictly continuous *-representation $\widehat{\pi}_\lambda : B \to \mathcal{L}(B \otimes \mathscr{H})$ given by

$$\widehat{\pi}_\lambda(b) = (\mathrm{id}_B \otimes \lambda)\delta_B(b), \quad b \in B.$$

  Furthermore, the *-homomorphism $\widehat{\pi}_\lambda$ extends uniquely to a faithful strictly continuous *-representation $\widehat{\pi}_\lambda : \mathcal{M}(B) \to \mathcal{L}(B \otimes \mathscr{H})$ such that $\widehat{\pi}_\lambda(1_B) = q^{\alpha_B, \widehat{\beta}}$.

- $\widehat{\pi}_\lambda$ induces a faithful unital strictly continous *-representation $\widehat{\pi} : \mathcal{M}(B) \to \mathcal{L}(\mathscr{E}_{B,\lambda})$.

- We have a unital strictly continuous *-representation $\theta : \mathcal{M}(S) \to \mathcal{L}(\mathscr{E}_{B,\lambda})$ given by $\theta(y) = (1_B \otimes L(y))\restriction_{\mathscr{E}_{B,\lambda}}$, $y \in \mathcal{M}(S)$. If α_B is faithful, so is θ.

Definition 4.2.6. We call crossed product of B by the continuous action (δ_B, α_B) of $\widehat{\mathcal{G}}$ on B, the C*-subalgebra $B \rtimes_{\delta_B,\alpha_B} \widehat{\mathcal{G}}$ of $\mathcal{L}(\mathscr{E}_{B,\lambda})$ generated by the products of the form $\widehat{\pi}(b)\theta(y)$ for $b \in B$ and $y \in S$.

From now on, we will denote the crossed product $B \rtimes \widehat{\mathcal{G}}$, instead of $B \rtimes_{\delta_B,\alpha_B} \widehat{\mathcal{G}}$, since no ambiguity will arise.

As for $\mathcal{G}$-C*-algebras, we have the following lemma:

Lemma 4.2.7. *The norm closed linear subspace of $\mathcal{L}(B \otimes \mathscr{H})$ spanned by the products of the form $\widehat{\pi}_\lambda(b)(1_B \otimes L(\omega))$, where $b \in B$, $\omega \in \mathcal{B}(\mathscr{H})_*$, is a C*-algebra.*

As a result, $\widehat{\pi}$ (resp. θ) defines a faithful unital *-homomorphism (resp. a unital *-homomorphism) $\widehat{\pi} : \mathcal{M}(B) \to \mathcal{M}(B \rtimes \widehat{\mathcal{G}})$ (resp. $\theta : \mathcal{M}(S) \to \mathcal{M}(B \rtimes \widehat{\mathcal{G}})$). The following proposition is also an immediate consequence of this lemma.

Proposition 4.2.8. *We have $\widehat{\pi}(b)\theta(y) \in B \rtimes \widehat{\mathcal{G}}$ for all $b \in B$ and $y \in S$. Moreover, we have*

$$B \rtimes \widehat{\mathcal{G}} = [\, \widehat{\pi}(b)\theta(y)\,;\, b \in B,\, y \in S\,].$$

This proposition allows us to define $\widehat{\Psi}_{\lambda,L} : B \rtimes \widehat{\mathcal{G}} \to \mathcal{L}(B \otimes \mathscr{H})$ the faithful strictly continuous *-representation of $B \rtimes \widehat{\mathcal{G}}$ on $B \otimes \mathscr{H}$, given by

$$\widehat{\Psi}_{\lambda,L}(\widehat{\pi}(b)\theta(y)) = \widehat{\pi}_\lambda(b)(1_B \otimes L(y)), \quad b \in B,\, y \in S.$$

Then, $\widehat{\Psi}_{\lambda,L}$ extends uniquely to a faithful *-representation of $\mathcal{M}(B \rtimes \widehat{\mathcal{G}})$ on $B \otimes \mathscr{H}$, still denoted $\widehat{\Psi}_{\lambda,L}$, such that $\widehat{\Psi}_{\lambda,L}(1_{B \rtimes \widehat{\mathcal{G}}}) = q^{\alpha_B, \widehat{\beta}}$.

Proposition 4.2.9. *We have the following statements:*

1. $\forall b \in B$, $\widehat{\Psi}_{\lambda,L}(\widehat{\pi}(b)) = \widehat{\pi}_\lambda(b)$.

2. $\forall y \in S$, $\widehat{\Psi}_{\lambda,L}(\theta(y)) = q^{\alpha_B, \widehat{\beta}}(1_B \otimes L(y))$.

4.3 Dual action

Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. Since $\widetilde{V} \in \widehat{M}' \otimes M'$ and $\alpha(N) \subset \widehat{M}$, we have $[\widetilde{V}_{23}, q_{12}^{\beta_A,\alpha}] = 0$. Therefore, $\widetilde{V}_{23} \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ restricts to a partial isometry

$$X := \widetilde{V}_{23}{\restriction}_{\mathscr{E}_{A,L}\otimes\mathscr{H}} \in \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H}),$$

whose initial and final projections are given by

$$X^*X = q_{23}^{\widehat{\beta},\alpha}{\restriction}_{\mathscr{E}_{A,L}\otimes\mathscr{H}}, \quad XX^* = q_{23}^{\widehat{\alpha},\beta}{\restriction}_{\mathscr{E}_{A,L}\otimes\mathscr{H}} \in \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H}),$$

since $\widetilde{V}^*\widetilde{V} = q^{\widehat{\beta},\alpha}$ and $\widetilde{V}\widetilde{V}^* = q^{\widehat{\alpha},\beta}$.

Proposition-Definition 4.3.1. *Let $\delta_{A\rtimes\mathcal{G}} : A \rtimes \mathcal{G} \to \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H})$ and $\alpha_{A\rtimes\mathcal{G}} : N \to \mathcal{M}(A \rtimes \mathcal{G})$ be the linear maps given by:*

- $\delta_{A\rtimes\mathcal{G}}(b) := X(b \otimes 1_{\mathscr{H}})X^*$, *for* $b \in A \rtimes \mathcal{G}$.
- $\alpha_{A\rtimes\mathcal{G}}(n) := \widehat{\theta}(\widehat{\alpha}(n)) = (1_A \otimes \rho(\widehat{\alpha}(n)))\restriction_{\mathscr{E}_{A,L}}$, *for* $n \in N$.

*Then, $\delta_{A\rtimes\mathcal{G}}$ is a faithful *-homomorphism and $\alpha_{A\rtimes\mathcal{G}}$ is a non-degenerate *-homomorphism. Moreover, we have the following statements:*

1. $\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x)) = (\pi(a) \otimes 1_{\widehat{S}}) \cdot (\widehat{\theta} \otimes \mathrm{id}_{\widehat{S}})(\widehat{\delta}(x))$, *for all* $a \in A$ *and* $x \in \widehat{S}$. *In particular, $\delta_{A\rtimes\mathcal{G}}$ takes its values in* $\mathcal{M}((A \rtimes \mathcal{G}) \otimes \widehat{S})$.
2. $\alpha_{A\rtimes\mathcal{G}}(n)\pi(a)\widehat{\theta}(x) = \pi(a)\widehat{\theta}(\widehat{\alpha}(n)x)$ *and* $\pi(a)\widehat{\theta}(x)\alpha_{A\rtimes\mathcal{G}}(n) = \pi(a)\widehat{\theta}(x\widehat{\alpha}(n))$ *for all* $n \in N$, $a \in A$ *and* $x \in \widehat{S}$.
3. $\delta_{A\rtimes\mathcal{G}}$ *extends uniquely to a strictly continuous *-homomorphism from* $\mathcal{M}(A \rtimes \mathcal{G})$ *to* $\mathcal{M}((A \rtimes \mathcal{G}) \otimes \widehat{S})$, *still denoted $\delta_{A\rtimes\mathcal{G}}$, and we have* $\delta_{A\rtimes\mathcal{G}}(1_{A\rtimes\mathcal{G}}) = q^{\alpha_{A\rtimes\mathcal{G}},\beta}$.
4. *If β_A is faithful so is $\alpha_{A\rtimes\mathcal{G}}$.*

Proof. It is clear that $\delta_{A\rtimes\mathcal{G}}(b^*) = \delta_{A\rtimes\mathcal{G}}(b)^*$ for all $b \in A \rtimes \mathcal{G}$. Now, let us prove that $\delta_{A\rtimes\mathcal{G}}$ is multiplicative. Since $XX^*X = X$, we only have to show that

$$[b \otimes 1_{\mathscr{H}},\, X^*X] = 0, \quad b \in A \rtimes \mathcal{G}.$$

It suffices to prove that

$$[\pi_L(a)_{12}(1_A \otimes \rho(x) \otimes 1_{\mathscr{H}}),\, \widetilde{V}_{23}^*\widetilde{V}_{23}] = 0, \quad a \in A,\, x \in \widehat{S}.$$

Let $a \in A$ and $x \in \widehat{S}$. This follows immediately from the following facts:

- $[\rho(x) \otimes 1, \widetilde{V}^*\widetilde{V}] = 0$ because of $\widetilde{V}^*\widetilde{V} = q^{\widehat{\beta},\widehat{\alpha}}$, $\widehat{\beta}(N^{\mathrm{o}}) \subset \widehat{M}$ and $\rho(x) \in \widehat{M}'$.
- $[\pi_L(a)_{12}, \widetilde{V}_{23}^*\widetilde{V}_{23}] = 0$ because of $\widehat{\beta}(N^{\mathrm{o}}) \subset M'$ and $L(S) \subset M$.

Let us prove now that the *-representation $\delta_{A\rtimes\mathcal{G}}$ is faithful. According to the previous discussion, we have

$$X^*\delta_{A\rtimes\mathcal{G}}(b) = X^*X(b \otimes 1_{\mathscr{H}})X^* = (b \otimes 1_{\mathscr{H}})X^*XX^* = (b \otimes 1_{\mathscr{H}})X^*, \quad b \in A \rtimes \mathcal{G}.$$

Assume now that $\delta_{A\rtimes\mathcal{G}}(b)=0$ for some $b\in A\rtimes\mathcal{G}$. Hence, $(b\otimes 1_{\mathscr{H}})X^*=0$. It then follows that

$$b(\mathrm{id}_{\mathscr{E}_{A,L}}\otimes\omega)(X^*)=0,\quad \omega\in\mathcal{B}(\mathscr{H})_*.$$

However, we have

$$(\mathrm{id}_{\mathscr{E}_{A,L}}\otimes\omega)(X^*)=(1_A\otimes(\mathrm{id}\otimes\overline{\omega})(\widetilde{V})^*)\restriction_{\mathscr{E}_{A,L}}=(1_A\otimes U(\mathrm{id}\otimes\overline{\omega}_U)(W)^*U^*)\restriction_{\mathscr{E}_{A,L}}\ .$$

Hence, $[(\mathrm{id}_{\mathscr{E}_{A,L}}\otimes\omega)(X^*)\,;\,\omega\in\mathcal{B}(\mathscr{H})_*]=[(1_A\otimes R(y))\restriction_{\mathscr{E}_{A,L}}\,;\,y\in S]$. Therefore, we have $b=0$ by using the non-degeneracy of R.
Let us compute $\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x))$ for all $a\in A$ and $x\in\widehat{S}$. We consider $\mathcal{M}((A\rtimes\mathcal{G})\otimes\widehat{S})$ as a C*-subalgebra of $\mathcal{L}(\mathscr{E}_{A,L}\otimes\mathscr{H})$. First, since $[\pi_L(a)_{12},\widetilde{V}_{23}]=0$ we have $[X,\pi(a)_{12}]=0$. Hence,

$$X(\pi(a)\otimes 1_{\mathscr{H}})X^*=(\pi(a)\otimes 1_{\widehat{S}})XX^*=(\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(q_{23}^{\widehat{\alpha},\beta}).\tag{4.3.1}$$

Furthermore, by using the formula $\widehat{\delta}(x)=\widetilde{V}(x\otimes 1)\widetilde{V}^*$ we obtain

$$X(\widehat{\theta}(x)\otimes 1_{\mathscr{H}})X^*=(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})\widehat{\delta}(x).\tag{4.3.2}$$

We then combine the formulas (4.3.1) and (4.3.2) with the fact that $[m\otimes 1_{\mathscr{H}},\,X^*X]=0$ for all $m\in\mathcal{M}(A\rtimes\mathcal{G})$ to conclude that

$$\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x))=(\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(\widehat{\delta}(x)).$$

In particular, $\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x))\in\mathcal{M}((A\rtimes\mathcal{G})\otimes\widehat{S})$ for all $a\in A$ and $x\in\widehat{S}$. Therefore, $\delta_{A\rtimes\mathcal{G}}$ takes its values in $\mathcal{M}((A\rtimes\mathcal{G})\otimes\widehat{S})$. Now, it is clear that $\delta_{A\rtimes\mathcal{G}}$ is strictly continuous. Therefore, it extends uniquely to a strictly continuous *-homomorhism $\delta_{A\rtimes\mathcal{G}}:\mathcal{M}(A\rtimes\mathcal{G})\to\mathcal{M}((A\rtimes\mathcal{G})\otimes\widehat{S})$ and we have

$$\delta_{A\rtimes\mathcal{G}}(1_{A\rtimes\mathcal{G}})=XX^*=q_{23}^{\widehat{\alpha},\beta}\restriction_{\mathscr{E}_{A,L}\otimes\mathscr{H}}=q^{\alpha_{A\rtimes\mathcal{G}},\beta}.$$

We see immediately that $\alpha_{A\rtimes\mathcal{G}}:N\to\mathcal{M}(A\rtimes\mathcal{G})$ is a non-degenerate *-homomorphism. Let us fix $n\in N$, $a\in A$ and $x\in\widehat{S}$. We have $[\alpha_{A\rtimes\mathcal{G}}(n),\pi(a)]=0$ and $\alpha_{A\rtimes\mathcal{G}}(n)\widehat{\theta}(x)=\widehat{\theta}(\widehat{\alpha}(n)x)$. Hence, $\alpha_{A\rtimes\mathcal{G}}(n)\pi(a)\widehat{\theta}(x)=\pi(a)\widehat{\theta}(\widehat{\alpha}(n)x)$. The formula $\pi(a)\widehat{\theta}(x)\alpha_{A\rtimes\mathcal{G}}(n)=\pi(a)\widehat{\theta}(x\widehat{\alpha}(n))$ is straightforward. As for the last statement of the proposition, we recall that if β_A is faithful so is $\widehat{\theta}$. The faithfulness of $\alpha_{A\rtimes\mathcal{G}}$ follows since $\widehat{\alpha}$ is faithful. □

Proposition 4.3.2. *We have the following statements:*

1. $\forall a\in A$, $\delta_{A\rtimes\mathcal{G}}(\pi(a))=(\pi(a)\otimes 1_{\widehat{S}})(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(q^{\widehat{\alpha},\beta})$.
2. $\forall x\in\widehat{S}$, $\delta_{A\rtimes\mathcal{G}}(\widehat{\theta}(x))=(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(\widehat{\delta}(x))$.

Proof. These statements are straightforward consequences of the formulas (4.3.1) and (4.3.2) of the above proof and the fact that we have $\delta_{A\rtimes\mathcal{G}}(m)=X(m\otimes 1_{\mathscr{H}})X^*$ for all $m\in\mathcal{M}(A\rtimes\mathcal{G})$. □

Theorem 4.3.3. *We have:*

1. *The triple $(A\rtimes\mathcal{G},\delta_{A\rtimes\mathcal{G}},\alpha_{A\rtimes\mathcal{G}})$ is a $\widehat{\mathcal{G}}$-C*-algebra.*
2. *The correspondence*

$$\begin{array}{rccc} -\rtimes\mathcal{G}: & \mathcal{G}\text{-C}^*\text{-Alg} & \longrightarrow & \widehat{\mathcal{G}}\text{-C}^*\text{-Alg} \\ & (A,\delta_A,\beta_A) & \longmapsto & (A\rtimes\mathcal{G},\delta_{A\rtimes\mathcal{G}},\alpha_{A\rtimes\mathcal{G}}) \end{array}$$

is functorial.

Proof. 1. Let us prove that we have

$$[\delta_{A\rtimes\mathcal{G}}(A\rtimes\mathcal{G})(1_{A\rtimes\mathcal{G}}\otimes\widehat{S})] = q^{\alpha_{A\rtimes\mathcal{G}},\beta}((A\rtimes\mathcal{G})\otimes\widehat{S}).$$

Let us fix $a\in A$ and $x,x'\in\widehat{S}$, then $\widehat{\delta}(x)(1_{\widehat{S}}\otimes y')$ is the norm limit of finite sums of the form $\sum_i x_i\otimes x_i'$, where $x_i,x_i'\in\widehat{S}$. Therefore,

$$\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x))(1_{A\rtimes\mathcal{G}}\otimes x') = (\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(\widehat{\delta}(x)(1_{\widehat{S}}\otimes x'))$$

is the norm limit of finite sums of the form $\sum_i \pi(a)\widehat{\theta}(x_i)\otimes x_i'$, with $x_i,x_i'\in\widehat{S}$. It then follows that

$$\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x))(1_{A\rtimes\mathcal{G}}\otimes y)\in(A\rtimes\widehat{S})\otimes\widehat{S}.$$

Consequently, we have $\delta_{A\rtimes\mathcal{G}}(A\rtimes\mathcal{G})(1_{A\rtimes\mathcal{G}}\otimes\widehat{S})\subset(A\rtimes\mathcal{G})\otimes\widehat{S}$. But, since $\delta_{A\rtimes\mathcal{G}}(1_{A\rtimes\mathcal{G}}) = q^{\alpha_{A\rtimes\mathcal{G}},\beta}$ we actually have the inclusion

$$[\delta_{A\rtimes\mathcal{G}}(A\rtimes\mathcal{G})(1_{A\rtimes\mathcal{G}}\otimes\widehat{S})]\subset q^{\alpha_{A\rtimes\mathcal{G}},\beta}((A\rtimes\mathcal{G})\otimes\widehat{S}).$$

For the converse inclusion, let us take $a\in A$, $x,x'\in\widehat{S}$. We have

$$\begin{aligned}(\alpha_{A\rtimes\mathcal{G}}(n')\otimes\beta(n^{\mathrm{o}}))(\pi(a)\widehat{\theta}(x)\otimes x') &= \pi(a)\widehat{\theta}(\widehat{\alpha}(n')x)\otimes\beta(n^{\mathrm{o}})x'\\ &= (\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(\widehat{\alpha}(n')x\otimes\beta(n^{\mathrm{o}})x'),\end{aligned}$$

for all $n,n'\in N$. Hence, $q^{\alpha_{A\rtimes\mathcal{G}},\beta}(\pi(a)\widehat{\theta}(x)\otimes x') = (\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(q^{\widehat{\alpha},\beta}(x\otimes x'))$. Now, $q^{\widehat{\alpha},\beta}(x\otimes x')$ is the norm limit of finite sums of the form $\sum_i\widehat{\delta}(x_i)(1_{\widehat{S}}\otimes x_i')$ where $x_i,x_i'\in\widehat{S}$. Therefore, $q^{\alpha_{A\rtimes\mathcal{G}},\beta}(\pi(a)\widehat{\theta}(x)\otimes x')$ is the norm limit of finite sums of the form

$$\sum_i(\pi(a)\otimes 1_{\widehat{S}})\cdot(\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})(\widehat{\delta}(x_i)(1_{\widehat{S}}\otimes x_i')) = \sum_i\delta_{A\rtimes\mathcal{G}}(\pi(a)\widehat{\theta}(x_i))(1_{A\rtimes\mathcal{G}}\otimes x_i'),$$

which proves that $q^{\alpha_{A\rtimes\mathcal{G}},\beta}((A\rtimes\mathcal{G})\otimes\widehat{S})\subset[\delta_{A\rtimes\mathcal{G}}(A\rtimes\mathcal{G})(1_{A\rtimes\mathcal{G}}\otimes\widehat{S})]$ and then the equality holds.

The formula $\delta_{A\rtimes\mathcal{G}}(\alpha_{A\rtimes\mathcal{G}}(n)) = q^{\alpha_{A\rtimes\mathcal{G}},\beta}(1_{A\rtimes\mathcal{G}}\otimes\widehat{\alpha}(n))$, $n\in N$, follows immediately from

$$\delta_{A\rtimes\mathcal{G}}(\alpha_{A\rtimes\mathcal{G}}(n)) = (\widehat{\theta}\otimes\mathrm{id}_{\widehat{S}})\widehat{\delta}(\widehat{\alpha}(n)),\quad \widehat{\delta}(\widehat{\alpha}(n)) = (1_{\widehat{S}}\otimes\beta(n^{\mathrm{o}}))\widehat{\delta}(1_{\widehat{S}}),\quad n\in N$$

(see Proposition 4.3.2 2). It only remains to prove the coassociativity of $\delta_{A\rtimes\mathcal{G}}$. Since $\delta_{A\rtimes\mathcal{G}}$ and $\widehat{\delta}$ are strictly continuous, we have:

$$(\delta_{A\rtimes\mathcal{G}}\otimes\mathrm{id}_{\widehat{S}})(m) = X_{12}m_{13}X_{12}^*,\quad (\mathrm{id}_{A\rtimes\mathcal{G}}\otimes\widehat{\delta})(m) = \widetilde{V}_{23}m_{12}\widetilde{V}_{23}^*,\quad m\in\mathcal{M}((A\rtimes\mathcal{G})\otimes\widehat{S}).$$

Moreover, since $\widetilde{V}_{12}\widetilde{V}_{13} = \widetilde{V}_{23}\widetilde{V}_{12}\widetilde{V}_{23}^*$ we have $X_{12}X_{13} = \widetilde{V}_{23}X_{12}\widetilde{V}_{23}^*$ in $\mathcal{L}(\mathscr{E}_{A,L}\otimes\mathscr{H}\otimes\mathscr{H})$. We will also need the commutation relation

$$[\widetilde{V}_{23}^*\widetilde{V}_{23},\, X_{12}] = 0\quad\text{in}\quad\mathcal{L}(\mathscr{E}_{A,L}\otimes\mathscr{H}\otimes\mathscr{H}),$$

which follows from the facts that $\widetilde{V}_{23}^*\widetilde{V}_{23} = q_{23}^{\widehat{\beta},\alpha}$, $\widetilde{V}\in\widehat{M}'\otimes M'$ and $\widehat{\beta}(N^{\mathrm{o}})\subset\widehat{M}$. Therefore, for all $b\in A\rtimes\mathcal{G}$ we have

$$\begin{aligned}(\delta_{A\rtimes\mathcal{G}}\otimes\mathrm{id}_{\widehat{S}})\delta_{A\rtimes\mathcal{G}}(b) &= X_{12}\delta_{A\rtimes\mathcal{G}}(b)_{13}X_{12}^*\\ &= X_{12}X_{13}(b\otimes 1_{\mathscr{H}}\otimes 1_{\mathscr{H}})X_{13}^*X_{12}^*\\ &= \widetilde{V}_{23}X_{12}(b\otimes 1_{\mathscr{H}}\otimes 1_{\mathscr{H}})\widetilde{V}_{23}^*\widetilde{V}_{23}X_{12}^*\widetilde{V}_{23}^*\\ &= \widetilde{V}_{23}X_{12}(b\otimes 1_{\mathscr{H}}\otimes 1_{\mathscr{H}})X_{12}\widetilde{V}_{23}^*\widetilde{V}_{23}\widetilde{V}_{23}^*\\ &= \widetilde{V}_{23}X_{12}(b\otimes 1_{\mathscr{H}}\otimes 1_{\mathscr{H}})X_{12}\widetilde{V}_{23}^*\\ &= \widetilde{V}_{23}\delta_{A\rtimes\mathcal{G}}(b)_{12}\widetilde{V}_{23}^*\\ &= (\mathrm{id}_{\widehat{S}}\otimes\widehat{\delta})\delta_{A\rtimes\mathcal{G}}(b).\end{aligned}$$

2. Let (A, δ_A, β_A) and (C, δ_C, β_C) be $\mathcal{G}$-C*-algebras and $f : A \to \mathcal{M}(C)$ a $\mathcal{G}$-equivariant non-degenerate *-homomorphism. Let us denote $\pi_A : A \to \mathcal{L}(\mathscr{E}_{A,L})$, $\pi_C : C \to \mathcal{L}(\mathscr{E}_{C,L})$, $\widehat{\theta}_A : \widehat{S} \to \mathcal{L}(\mathscr{E}_{A,L})$ and $\widehat{\theta}_C : \widehat{S} \to \mathcal{L}(\mathscr{E}_{C,L})$ the canonical *-homomorphisms. There exists a unique *-homomorphism $f_* : A \rtimes \mathcal{G} \to \mathcal{M}(C \rtimes \mathcal{G})$ such that

$$f_*(\pi_A(a)\widehat{\theta}_A(x)) = \pi_C(f(a))\widehat{\theta}_C(x), \quad a \in A,\, x \in \widehat{S}. \tag{4.3.3}$$

Indeed, since f is non-degenerate, we have the following unital strictly continuous *-homomorphism $f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})} : \mathcal{L}(A \otimes \mathscr{H}) \to \mathcal{L}(C \otimes \mathscr{H})$. But since $f \circ \beta_A = \beta_C$, we already have $(f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(\mathcal{L}(\mathscr{E}_{A,L})) \subset \mathcal{L}(\mathscr{E}_{C,L})$ *via* the following identification:

$$\mathcal{L}(\mathscr{E}_{A,L}) = \{T \in \mathcal{L}(A \otimes \mathscr{H})\,;\, Tq^{\beta_A,\alpha} = T = q^{\beta_A,\alpha}T\}$$

and similarly for $\mathcal{L}(\mathscr{E}_{C,L})$. Now, in virtue of the $\mathcal{G}$-equivariance of f we have

$$(f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(\pi_A(a)) = \pi_C(f(a)), \quad a \in A.$$

We also have $(f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(\widehat{\theta}_A(x)) = \widehat{\theta}_C(x)$ for all $x \in \widehat{S}$. Hence, $(f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(A \rtimes \mathcal{G}) \subset C \rtimes \mathcal{G}$. Let us denote $f_* = (f \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})\restriction_{A \rtimes \mathcal{G}}$. Then, $f_* : A \rtimes \mathcal{G} \to \mathcal{M}(C \rtimes \mathcal{G})$ satisfies (4.3.3) and the uniqueness is guaranteed by Proposition 4.2.3.
Let $c \in C$ and $x \in \widehat{S}$, by the non-degeneracy of f we can assume that $c = f(a)c_1$ with $a \in A$ and $c_1 \in C$. Then, we have $\pi_C(c)\widehat{\theta}_C(x) = \pi_C(f(a))\pi_C(c_1)\widehat{\theta}_C(x)$. In virtue of Lemma 4.2.2, $\pi_C(c)\widehat{\theta}_C(x)$ is the norm limit of finite sums of elements of the form $\pi_C(f(a))\widehat{\theta}_C(x_1)\pi_C(c_2)$ with $x_1 \in \widehat{S}$ and $c_2 \in C$. Let us write $x_1 = x_1'x_1''$ with $x_1', x_1'' \in \widehat{S}$. Then $\pi_C(c)\widehat{\theta}_C(x)$ is the norm limit of finite sums of elements of the form $f_*(\pi_A(a)\widehat{\theta}_A(x_1'))\widehat{\theta}_C(x_1'')\pi_C(c_2)$. This proves that f_* is non-degenerate.
Then, f_* extends to a unital strictly continuous *-homomorphism $f_* : \mathcal{M}(A \rtimes \mathcal{G}) \to \mathcal{M}(C \rtimes \mathcal{G})$. Moreover, we have proved that

$$f_* \circ \pi_A = \pi_C \circ f, \quad f_* \circ \widehat{\theta}_A = \widehat{\theta}_C.$$

By the assertions 1 and 2 of Proposition 4.3.1, it follows that

$$\delta_{C \rtimes \mathcal{G}} \circ f_* = (f_* \otimes \mathrm{id}_{\widehat{S}})\delta_{A \rtimes \mathcal{G}}, \quad f_* \circ \alpha_{A \rtimes \mathcal{G}} = \alpha_{C \rtimes \mathcal{G}}.$$

□

Remark 4.3.4. We will need an adaptation of the functoriality of $- \rtimes \mathcal{G} : \mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \widehat{\mathcal{G}}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}$ to possibly *degenerate* (albeit reasonably) $\mathcal{G}$-equivariant *-homomorphisms (*cf.* Lemma 4.1.1). With the notations of the proof of Theorem 4.3.3 2, let us assume that there exists $e \in \mathcal{M}(C)$ such that $[f(A)C] = eC$. We prove in a similar way that there exists a unique $\widehat{\mathcal{G}}$-equivariant *-homomorphism $f_* : A \rtimes \mathcal{G} \to \mathcal{M}(C \rtimes \mathcal{G})$ such that

$$f_*(\pi_A(a)\widehat{\theta}_A(x)) = \pi_C(f(a))\widehat{\theta}_C(x), \quad a \in A,\, x \in \widehat{S}.$$

Moreover, by combining the assumption with Lemma 4.2.2 and Proposition 4.2.3, we obtain the equality $[f_*(A \rtimes \mathcal{G})(C \rtimes \mathcal{G})] = \pi_C(e)(C \rtimes \mathcal{G})$.

Definition 4.3.5. The continuous action $(\delta_{A \rtimes \mathcal{G}}, \alpha_{A \rtimes \mathcal{G}})$ of the measured quantum groupoid $\widehat{\mathcal{G}}$ on the crossed product $A \rtimes \mathcal{G}$ is called the dual action of (δ_A, β_A).

In a similar way, we define the dual action of the measured quantum groupoid $\mathcal{G}$ on the crossed product $B \rtimes \widehat{\mathcal{G}}$, where B is a C*-algebra acted upon by the dual measured quantum groupoid $\widehat{\mathcal{G}}$.

Let (B, δ_B, α_B) be a $\widehat{\mathcal{G}}$-C*-algebra. Since $V \in \widehat{M}' \otimes M$ and $\beta(N^{\mathrm{o}}) \subset \widehat{M}$, we have $[V_{23}, q_{12}^{\alpha_B,\beta}] = 0$. Therefore, $V_{23} \in \mathcal{L}(B \otimes \mathscr{H} \otimes \mathscr{H})$ restricts to a partial isometry

$$Y := V_{23}{\restriction}_{\mathscr{E}_{B,\lambda} \otimes \mathscr{H}} \in \mathcal{L}(\mathscr{E}_{B,\lambda} \otimes \mathscr{H}),$$

whose initial and final projections are given by:

$$Y^*Y = q_{23}^{\widehat{\alpha},\beta}{\restriction}_{\mathscr{E}_{B,\lambda} \otimes \mathscr{H}}, \quad YY^* = q_{23}^{\beta,\alpha}{\restriction}_{\mathscr{E}_{B,\lambda} \otimes \mathscr{H}} \in \mathcal{L}(\mathscr{E}_{B,\lambda} \otimes \mathscr{H}).$$

Let $\delta_{B \rtimes \widehat{\mathcal{G}}} : B \rtimes \widehat{\mathcal{G}} \to \mathcal{L}(\mathscr{E}_{B,\lambda} \otimes \mathscr{H})$ and $\beta_{B \rtimes \widehat{\mathcal{G}}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E}_{B,\lambda})$ be the linear maps given by:

$$\delta_{B \rtimes \widehat{\mathcal{G}}}(c) = Y(c \otimes 1_{\mathscr{H}})Y^*, \quad \beta_{B \rtimes \widehat{\mathcal{G}}}(n^{\mathrm{o}}) = \theta(\beta(n^{\mathrm{o}})) = (1_B \otimes L(\beta(n^{\mathrm{o}})))\restriction_{\mathscr{E}_{B,\lambda} \otimes \mathscr{H}},$$

for all $c \in B \rtimes \widehat{\mathcal{G}}$ and $n \in N$. As for the case of a continuous action of the measured quantum groupoid $\mathcal{G}$, we obtain the formulas:

$$Y(\widehat{\pi}(b) \otimes 1_{\mathscr{H}})Y^* = (\widehat{\pi}(b) \otimes 1_S)(\theta \otimes \mathrm{id}_S)(q^{\beta,\alpha}), \quad Y(\theta(y) \otimes 1_{\mathscr{H}})Y^* = (\theta \otimes \mathrm{id}_S)\delta(y), \quad b \in B,\, y \in S,$$

$$[Y^*Y, c \otimes 1_{\mathscr{H}}] = 0, \quad Y(\widehat{\pi}(b)\theta(y) \otimes 1_{\mathscr{H}})Y^* = (\widehat{\pi}(b) \otimes 1_S)(\theta \otimes \mathrm{id}_S)\delta(y), \quad c \in B \rtimes \widehat{\mathcal{G}}, b \in B, y \in S,$$

$$Y_{12}Y_{13} = V_{23}Y_{12}V_{23}^*, \quad [V_{23}^*V_{23}, Y_{12}] = 0 \quad \text{as operators of} \quad \mathcal{L}(\mathscr{E}_{B,\lambda} \otimes \mathscr{H} \otimes \mathscr{H}).$$

Then, we obtain the following result:

Theorem 4.3.6. *We have:*

1. *The triple $(B \rtimes \widehat{\mathcal{G}}, \delta_{B \rtimes \widehat{\mathcal{G}}}, \beta_{B \rtimes \widehat{\mathcal{G}}})$ is a $\mathcal{G}$-C*-algebra.*
2. *The correspondence*

$$\begin{array}{rccc} - \rtimes \widehat{\mathcal{G}} : & \widehat{\mathcal{G}}\text{-C*-Alg} & \longrightarrow & \mathcal{G}\text{-C*-Alg} \\ & (B, \delta_B, \alpha_B) & \longmapsto & (B \rtimes \widehat{\mathcal{G}}, \delta_{B \rtimes \widehat{\mathcal{G}}}, \beta_{B \rtimes \widehat{\mathcal{G}}}) \end{array}$$

   is functorial.

Note that a generalization of Theorem 4.3.6 2 can be stated as in Remark 4.3.4.

Definition 4.3.7. The continuous action $(\delta_{B \rtimes \widehat{\mathcal{G}}}, \beta_{B \rtimes \widehat{\mathcal{G}}})$ of the measured quantum groupoid $\mathcal{G}$ on the crossed product $B \rtimes \widehat{\mathcal{G}}$ is called the dual action of (δ_B, α_B).

4.4 Takesaki-Takai duality

Let (δ_A, β_A) be a continuous action of the measured quantum groupoid $\mathcal{G}$ on a C*-algebra A. In this paragraph, we investigate the double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ endowed with the bidual action of $\mathcal{G}$. First, we prove that $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ can be canonically identified with a C*-subalgebra D of $\mathcal{L}(A \otimes \mathscr{H})$. We then endow D with a continuous action (δ_D, β_D) of $\mathcal{G}$ obtained by transport of structure.

Let us fix a $\mathcal{G}$-C*-algebra (A, δ_A, β_A). Let us denote $B = A \rtimes \mathcal{G}$. We begin by defining the C*-algebra D.

Proposition-Definition 4.4.1. *Let us denote*

$$D := [\pi_R(a)(1_A \otimes \lambda(x)L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S] \subset \mathcal{L}(A \otimes \mathscr{H}).$$

Then, D is C-subalgebra of $\mathcal{L}(A \otimes \mathscr{H})$. Moreover, we have:*

1. *$dq^{\beta_A,\widehat{\alpha}} = d = q^{\beta_A,\widehat{\alpha}}d$, for all $d \in D$.*
2. *$D(A \otimes \mathscr{H}) = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathscr{H})$.*
3. *There exists a unique faithful strictly continuous *-homomorphism*

$$j_D : \mathcal{M}(D) \to \mathcal{L}(A \otimes \mathscr{H})$$

extending the inclusion map $D \subset \mathcal{L}(A \otimes \mathscr{H})$ such that $j_D(1_D) = q^{\beta_A,\widehat{\alpha}}$.

Proof. Since $[\lambda(\widehat{S})L(S)]$ is a C*-algebra, it is enough to prove that $(1_A \otimes \lambda(x)L(y))\pi_R(a) \in D$, for all $x \in \widehat{S}$, $y \in S$ and $a \in A$. Let us fix $x \in \widehat{S}$, $y \in S$ and $a \in A$. Since $[R(y), L(s)] = 0$ for all $s \in S$, we have

$$\begin{aligned}(1_A \otimes \lambda(x)L(y))\pi_R(a) &= (1_A \otimes U\rho(x)U^*L(y)U)\pi_L(a)(1_A \otimes U^*)\\ &= (1_A \otimes U)(1_A \otimes \rho(x)R(y))\pi_L(a)(1_A \otimes U^*)\\ &= (1_A \otimes U)(1_A \otimes \rho(x))\pi_L(a)(1_A \otimes R(y)U^*)\\ &= (1_A \otimes U)(1_A \otimes \rho(x))\pi_L(a)(1_A \otimes U^*L(y)).\end{aligned}$$

By Lemma 4.2.2, we have that $(1_A \otimes \lambda(x)L(y))\pi_R(a)$ is the norm limit of finite sums of elements of the form

$$(1_A \otimes U)\pi_L(a')(1_A \otimes \rho(x'))(1_A \otimes U^*L(y)) = \pi_R(a')(1_A \otimes \lambda(x')L(y)), \quad x' \in \widehat{S},\ a' \in A.$$

Note that the fact that D is a C*-algebra will also appear as a consequence of Proposition 4.4.3.
1. We have $\pi_R(a)q^{\beta_A,\widehat{\alpha}} = \pi_R(a) = q^{\beta_A,\widehat{\alpha}}\pi_R(a)$, for all $a \in A$. We have $[1_A \otimes \lambda(x), q^{\beta_A,\widehat{\alpha}}] = 0$ for all $x \in \widehat{S}$, since $\widehat{\alpha}(N) \subset \widehat{M}'$ and $\lambda(x) \in \widehat{M}$. But we also have $\widehat{\alpha}(N) \subset M'$, then we obtain $[1_A \otimes L(y), q^{\beta_A,\widehat{\alpha}}] = 0$ for all $y \in S$. This proves that

$$q^{\beta_A,\widehat{\alpha}}\pi_R(a)(1_A \otimes \lambda(x)L(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y))q^{\beta_A,\widehat{\alpha}},$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$ and the first statement is proved.
2. It follows from the second assertion that we have

$$D(A \otimes \mathscr{H}) = q^{\beta_A,\widehat{\alpha}}D(A \otimes \mathscr{H}) \subset q^{\beta_A,\widehat{\alpha}}(A \otimes \mathscr{H}) =: \mathscr{E}_{A,R}.$$

Let $\xi \in \mathscr{E}_{A,R}$, by the non-degeneracy of π_R there exist $a \in A$ and $\xi' \in A \otimes \mathscr{H}$ such that $\xi = \pi_R(a)\xi'$. In particular, ξ is the norm limit of finite sums of elements of the form $\pi_R(a)(b \otimes \eta)$, where $b \in A$ and $\eta \in \mathscr{H}$. By using the non-degeneracy of the C*-algebra $[\lambda(\widehat{S})L(S)] \subset \mathcal{B}(\mathscr{H})$, it follows that ξ is the norm limit of finite sums of elements of the form

$$\pi_R(a)(b \otimes \lambda(x)L(y)\eta') = \pi_R(a)(1_A \otimes \lambda(x)L(y))(b \otimes \eta') \in D(A \otimes \mathscr{H}), \quad x \in \widehat{S},\ y \in S,$$

and the second assertion is proved.
3. This is an immediate consequence of 2. □

Remark 4.4.2. With the notations of §4.2, we have noticed that the crossed products $A \rtimes \mathcal{G}$ and $B \rtimes \widehat{\mathcal{G}}$ are naturally represented in the Hilbert modules $\mathscr{E}_{A,L}$ and $\mathscr{E}_{B,\lambda}$ respectively. In a similar way, it should be noted that the C*-algebra D is naturally represented on the Hilbert A-module $\mathscr{E}_{A,R}$. However, unless mentioned otherwise, we will not do so. Indeed, it will appear more convenient in the computations to work with the representation j_D as defined in Proposition-Definition 4.4.1.

We then prove that we have a canonical *-isomorphism between $B \rtimes \widehat{\mathcal{G}}$ and D.

Proposition 4.4.3. *There exists a unique *-isomorphism $\phi_{B\rtimes\widehat{\mathcal{G}}} : B \rtimes \widehat{\mathcal{G}} \to D$ such that*

$$\phi_{B\rtimes\widehat{\mathcal{G}}}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y)), \quad a \in A,\, x \in \widehat{S},\, y \in S.$$

Proof. Let us consider the operator $Z = 1_A \otimes (1_{\mathscr{H}} \otimes U^*)V(1_{\mathscr{H}} \otimes U) \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$. Let us note that Z is a partial isometry. Besides, since $Z = \Sigma_{23}\widetilde{V}_{23}\Sigma_{23}$, $\widetilde{V}^*\widetilde{V} = q^{\widehat{\beta},\widehat{\alpha}}$ and $\widetilde{V}\widetilde{V}^* = q^{\widehat{\alpha},\beta}$, the initial and final projections of Z are given by:

$$Z^*Z = q_{23}^{\widehat{\alpha},\widehat{\beta}}, \quad ZZ^* = q_{23}^{\beta,\widehat{\alpha}}.$$

Let us prove that we have

$$Z^*(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x))Z = 1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x), \text{ for all } x \in \widehat{S}. \tag{4.4.1}$$

Indeed, since $\lambda(x) = U^*\rho(x)U$ and $(\rho \otimes \rho)\widehat{\delta}(x) = V^*(1 \otimes \rho(x))V$ for all $x \in \widehat{S}$ we have

$$\begin{aligned} 1_A \otimes (\rho \otimes \lambda)(\widehat{\delta}(x)) &= (1_A \otimes 1_{\mathscr{H}} \otimes U^*)(1_A \otimes (\rho \otimes \rho)\widehat{\delta}(x))(1_A \otimes 1_{\mathscr{H}} \otimes U) \\ &= (1_A \otimes (1_{\mathscr{H}} \otimes U^*)V^*)(1_A \otimes 1_{\mathscr{H}} \otimes \rho(x))(1_A \otimes V(1_{\mathscr{H}} \otimes U)) \\ &= Z^*(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x))Z, \end{aligned}$$

for all $x \in \widehat{S}$. Now, let us prove that

$$Z\pi_L(a)_{12}Z^* = (\pi_L \otimes R)\delta_A(a), \text{ for all } a \in A. \tag{4.4.2}$$

We have

$$\begin{aligned} (\pi_L \otimes R)(\delta_A(a)) &= (\mathrm{id}_A \otimes L \otimes R)\delta_A^2(a), \qquad \text{with } \delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A \\ &= (\mathrm{id}_A \otimes (L \otimes R)\delta)(\delta_A(a)), \qquad \text{as } \delta_A^2 = (\mathrm{id}_A \otimes \delta)\delta_A \\ &= (1_A \otimes 1_{\mathscr{H}} \otimes U^*)(\mathrm{id}_A \otimes (L \otimes L)\delta)(\delta_A(a))(1_A \otimes 1_{\mathscr{H}} \otimes U), \end{aligned}$$

since $R(y) = U^*L(y)U$ for all $y \in S$. However, since $(L \otimes L)\delta(y) = V(L(y) \otimes 1)V^*$ for all $y \in S$, we obtain

$$(\mathrm{id}_A \otimes (L \otimes L)\delta)(t) = V_{23}((\mathrm{id}_A \otimes L)(t) \otimes 1_{\mathscr{H}})V_{23}^*,$$

for all $t \in A \otimes S$. The above equality also holds when $t \in \mathcal{M}(A \otimes S)$ since δ is strictly continuous and L is non-degenerate. In particular, we have

$$(\mathrm{id}_A \otimes (L \otimes L)\delta)(\delta_A(a)) = V_{23}((\mathrm{id}_A \otimes L)\delta_A(a) \otimes 1_{\mathscr{H}})V_{23}^* = V_{23}\pi_L(a)_{12}V_{23}^*, \quad a \in A.$$

It then follows that we finally obtain $(\pi_L \otimes R)\delta_A(a) = Z\pi_L(a)_{12}Z^*$ for all $a \in A$ since we have $\pi_L(a)_{12} = (1_A \otimes 1_{\mathscr{H}} \otimes U)\pi_L(a)_{12}(1_A \otimes 1_{\mathscr{H}} \otimes U^*)$. We also have

$$Z(1_A \otimes 1_{\mathscr{H}} \otimes L(y))Z^* = (1_A \otimes 1_{\mathscr{H}} \otimes L(y))ZZ^*, \quad y \in S. \tag{4.4.3}$$

Indeed, by using the facts that $V \in \widehat{M}' \otimes M$ and $R(S) \subset M'$ we have

$$\begin{aligned} Z(1_A \otimes 1_{\mathscr{H}} \otimes L(y))Z^* &= 1_A \otimes [(1 \otimes U^*)V(1 \otimes R(y))V^*(1 \otimes U)] \\ &= (1_A \otimes 1_{\mathscr{H}} \otimes U^*R(y))V_{23}V_{23}^*(1_A \otimes 1_{\mathscr{H}} \otimes U) \\ &= (1_A \otimes 1_{\mathscr{H}} \otimes U^*R(y)U)ZZ^* \\ &= (1_A \otimes 1_{\mathscr{H}} \otimes L(y))ZZ^*, \end{aligned}$$

for all $y \in S$. Since $Z^*Z = q_{23}^{\widehat{\alpha},\widehat{\beta}}$ and $\widehat{\alpha}(N)$, $\widehat{\beta}(N^{\mathrm{o}}) \subset M'$, we have the following commutation relations

$$[\pi_L(a)_{12}, Z^*Z] = 0, \quad [1_A \otimes 1_{\mathscr{H}} \otimes L(y), Z^*Z] = 0, \quad a \in A,\, y \in S. \tag{4.4.4}$$

The first relation gives $Z\pi_L(a)_{12} = Z\pi_L(a)_{12}Z^*Z$ (as $ZZ^*Z = Z$) for all $a \in A$. We then combine this last equality with (4.4.2) to conclude that

$$Z\pi_L(a)_{12} = (\pi_L \otimes R)(\delta_A(a))Z, \quad a \in A. \tag{I}$$

The second one combined with (4.4.3) gives

$$[Z, 1_A \otimes 1_{\mathscr{H}} \otimes L(y)] = 0, \quad y \in S. \tag{II}$$

Since $ZZ^* = q_{23}^{\beta,\widehat{\alpha}}$, $\widehat{\alpha}(N) \subset \widehat{M}'$ and $\lambda(\widehat{S}) \subset \widehat{M}$, we obtain another commutation relation:

$$[ZZ^*, 1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)] = 0, \quad x \in \widehat{S}.$$

It then follows from (4.4.1) that

$$Z(1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x)) = 1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x), \quad x \in \widehat{S}. \tag{III}$$

By applying successively the relations (I), (II) and (III), we obtain

$$\begin{aligned} Z\pi_L(a)_{12}(1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x))(1_A \otimes 1_{\mathscr{H}} \otimes L(y))Z^* = \\ (\pi_L \otimes R)(\delta_A(a))(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)L(y))q_{23}^{\beta,\widehat{\alpha}}, \end{aligned} \tag{$\Diamond$}$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$.

Now, since $B = [\pi(a)\widehat{\theta}(x)\,;\, a \in A, x \in \widehat{S}]$ and $B \rtimes \widehat{\mathcal{G}} = [\widehat{\pi}(b)\theta(y)\,;\, b \in B, y \in S]$ (see Propositions 4.2.3 and 4.2.8), we have

$$B \rtimes \widehat{\mathcal{G}} = [\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)\,;\, a \in A, x \in \widehat{S}, y \in S].$$

Besides, since $\delta_B(\pi(a)\widehat{\theta}(x)) = (\pi(a) \otimes 1_{\widehat{S}})(\widehat{\theta} \otimes \mathrm{id}_{\widehat{S}})(\widehat{\delta}(x))$ for all $a \in A$, $x \in \widehat{S}$, we have

$$\widehat{\pi}_\lambda(\pi(a)\widehat{\theta}(x)) = (\mathrm{id}_B \otimes \lambda)(\delta_B(\pi(a)\widehat{\theta}(x))) = (\pi(a) \otimes 1_{\mathscr{H}})(\widehat{\theta} \otimes \lambda)(\widehat{\delta}(x)),$$

for all $a \in A$ and $x \in \widehat{S}$. Therefore,

$$\widehat{\Psi}_{\lambda,L}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = (\pi(a) \otimes 1_{\mathscr{H}})(\widehat{\theta} \otimes \lambda)(\widehat{\delta}(x))(1_B \otimes L(y)),$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. We then combine this equality with Proposition 4.2.4 to conclude that

$$(\Psi_{L,\rho} \otimes \mathrm{id})\widehat{\Psi}_{\lambda,L}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = \pi_L(a)_{12}(1_A \otimes (\rho \otimes \lambda)(\widehat{\delta}(x)))(1_A \otimes 1_{\mathscr{H}} \otimes L(y)),$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. Indeed, let us denote $u = (\widehat{\theta} \otimes \lambda)(\widehat{\delta}(x))$ for short. We have

$$\begin{aligned}(\Psi_{L,\rho} \otimes \mathrm{id})(u(1_B \otimes L(y))) &= (\Psi_{L,\rho} \otimes \mathrm{id})(u)\Psi_{L,\rho}(1_B)_{12}(1_A \otimes 1_{\mathscr{H}} \otimes L(y)) \\ &= (\Psi_{L,\rho} \otimes \mathrm{id})(u)(1_A \otimes 1_{\mathscr{H}} \otimes L(y))\end{aligned}$$

and we have $(\Psi_{L,\rho} \otimes \mathrm{id})(u) = (\Psi_{L,\rho} \circ \widehat{\theta} \otimes \lambda)\widehat{\delta}(x) = q_{12}^{\beta_A,\alpha}(1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x))$. Let us consider the faithful *-representation $(\Psi_{L,\rho} \otimes \mathrm{id})\widehat{\Psi}_{\lambda,L} : B \rtimes \widehat{\mathcal{G}} \to \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ and let us denote C its range. Therefore, C is a C*-subalgebra of $\mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ and we have proved that

$$C = [\pi_L(a)_{12}(1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x))(1_A \otimes 1_{\mathscr{H}} \otimes L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S].$$

We now consider the map $\mathrm{Ad}_Z : \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H}) \to \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ given by $\mathrm{Ad}_Z(u) = ZuZ^*$ for all $u \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$. It is clear that Ad_Z is a bounded linear map which satisfies $\mathrm{Ad}_Z(u^*) = \mathrm{Ad}_Z(u)^*$ for all $u \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. But Ad_Z is not multiplicative in general. However, we claim that the restriction of Ad_Z to C is multiplicative. This will follow from the fact that

$$uq_{23}^{\widehat{\alpha},\widehat{\beta}} = u = q_{23}^{\widehat{\alpha},\widehat{\beta}}u, \quad u \in C.$$

Indeed, since $Z^*Z = q_{23}^{\widehat{\alpha},\widehat{\beta}}$ and $Z^*ZZ^* = Z$ we then have

$$\mathrm{Ad}_Z(u)\mathrm{Ad}_Z(v) = Zuq_{23}^{\widehat{\alpha},\widehat{\beta}}vZ^* = ZuvZ^* = \mathrm{Ad}_Z(uv),$$

for all $u, v \in C$. Now, we have the following facts:

- $[q_{23}^{\widehat{\alpha},\widehat{\beta}}, \pi_L(a)_{12}] = 0$ for all $a \in A$ and $[q_{23}^{\widehat{\alpha},\widehat{\beta}}, 1_A \otimes 1_{\mathscr{H}} \otimes L(y)] = 0$ for all $y \in S$.
- Let us denote $T = (1 \otimes U^*)V(1 \otimes U) \in \mathcal{B}(\mathscr{H} \otimes \mathscr{H})$. Then, $T = \Sigma\widetilde{V}\Sigma$ is a partial isometry whose initial projection is $T^*T = q^{\widehat{\alpha},\widehat{\beta}}$. Since $(\rho \otimes \rho)\widehat{\delta}(x) = V^*(1 \otimes \rho(x))V$ for all $x \in \widehat{S}$, we have
$$(\rho \otimes \lambda)\widehat{\delta}(x) = T^*(1 \otimes \lambda(x))T, \quad \text{for all } x \in \widehat{S}.$$
Hence, $q^{\widehat{\alpha},\widehat{\beta}}(\rho \otimes \lambda)\widehat{\delta}(x) = (\rho \otimes \lambda)\widehat{\delta}(x) = (\rho \otimes \lambda)\widehat{\delta}(x)q^{\widehat{\alpha},\widehat{\beta}}$ for all $x \in \widehat{S}$. In particular, we have
$$q_{23}^{\widehat{\alpha},\widehat{\beta}}(1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x)) = 1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x) = (1_A \otimes (\rho \otimes \lambda)\widehat{\delta}(x))q_{23}^{\widehat{\alpha},\widehat{\beta}}, \quad x \in \widehat{S}.$$

It then follows that $uq_{23}^{\widehat{\alpha},\widehat{\beta}} = u = q_{23}^{\widehat{\alpha},\widehat{\beta}}u$, for all $u \in C$. Let us prove that the restriction of Ad_Z to C is injective. Let $u \in C$ such that $\mathrm{Ad}_Z(u) = 0$. Then, since $Z^*Z = q_{23}^{\widehat{\alpha},\widehat{\beta}}$ we have

$$0 = \mathrm{Ad}_{Z^*}\mathrm{Ad}_Z(u) = q_{23}^{\widehat{\alpha},\widehat{\beta}}uq_{23}^{\widehat{\alpha},\widehat{\beta}} = uq_{23}^{\widehat{\alpha},\widehat{\beta}} = u.$$

Then, $\mathrm{Ad}_Z(C)$ is a C*-algebra and $\mathrm{Ad}_Z(\Psi_{L,\rho} \otimes \mathrm{id})\widehat{\Psi}_{\lambda,L} : B \rtimes \widehat{\mathcal{G}} \to \mathrm{Ad}_Z(C)$ is a *-isomorphism. In virtue of $(\lozenge)$, we have

$$\mathrm{Ad}_Z(C) = [(\pi_L \otimes R)(\delta_A(a))(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)L(y))q_{23}^{\beta,\widehat{\alpha}}\,;\, a \in A,\, x \in \widehat{S},\, y \in S] \subset \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H}).$$

We have $\mathrm{Ad}_Z(c)q_{12}^{\beta_A,\alpha} = \mathrm{Ad}_Z(c) = q_{12}^{\beta_A,\alpha}\mathrm{Ad}_Z(c)$ for all $c \in C$. By restriction of the faithful *-representation Ad_Z, we obtain a faithful *-representation $\mathrm{Ad}_Z : C \to \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H})$. We now consider the following unitary equivalence of Hilbert A-modules:

$$\begin{aligned}\Xi :\ (A \otimes \mathscr{H}) \otimes_{\pi_L} \mathscr{E}_{A,L} &\longrightarrow \mathscr{E}_{A,L} \otimes \mathscr{H} \\ (a \otimes \xi) \otimes_{\pi_L} \eta &\longmapsto \pi_L(a)\eta \otimes \xi.\end{aligned}$$

Let $x \in \widehat{S}$ and $y \in S$. For all $a \in A$, $\xi \in \mathscr{H}$ and $\eta \in \mathscr{E}_{A,L}$, we have

$$\begin{aligned}(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)L(y))\Xi((a \otimes \xi) \otimes_{\pi_L} \eta) &= \pi_L(a)\eta \otimes \lambda(x)L(y)\xi \\ &= \Xi((a \otimes \lambda(x)L(y)\xi) \otimes_{\pi_L} \eta) \\ &= \Xi((1_A \otimes \lambda(x)L(y)) \otimes_{\pi_L} 1)((a \otimes \xi) \otimes_{\pi_L} \eta).\end{aligned}$$

Hence,

$$(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)L(y))\restriction_{\mathscr{E}_{A,L}\otimes\mathscr{H}} = \Xi((1_A \otimes \lambda(x)L(y)) \otimes_{\pi_L} 1)\Xi^*, \quad x \in \widehat{S},\, y \in S.$$

Let $a \in A$ and $s \in S$. For all $b \in A$, $\xi \in \mathscr{H}$ and $\eta \in \mathscr{E}_{A,L}$, we have

$$\begin{aligned}(\pi_L \otimes R)(a \otimes s)\Xi((b \otimes \xi) \otimes_{\pi_L} \eta) &= \pi_L(a)_{12}(1_A \otimes 1_{\mathscr{H}} \otimes R(s))(\pi_L(b)\eta \otimes \xi) \\ &= \pi_L(ab)\eta \otimes R(s)\xi \\ &= \Xi((ab \otimes R(s)\xi) \otimes_{\pi_L} \eta) \\ &= \Xi((\mathrm{id}_A \otimes R)(a \otimes s) \otimes_{\pi_L} 1)((b \otimes \xi) \otimes_{\pi_L} \eta).\end{aligned}$$

Hence, $(\pi_L \otimes R)(t)\restriction_{\mathscr{E}_{A,L}\otimes\mathscr{H}} = \Xi((\mathrm{id} \otimes R)(t) \otimes_{\pi_L} 1)\Xi^*$ for all $t \in A \otimes S$. This equality also holds when $t \in \mathcal{M}(A \otimes S)$ since the maps $t \in \mathcal{M}(A \otimes S) \mapsto \Xi((\mathrm{id} \otimes R)(t) \otimes_{\pi_L} 1)\Xi^* \in \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H})$ and $\pi_L \otimes R$ are strictly continuous. In particular, we have

$$(\pi_L \otimes R)(\delta_A(a))\restriction_{\mathscr{E}_{A,L}\otimes\mathscr{H}} = \Xi(\pi_R(a) \otimes_{\pi_L} 1)\Xi^*, \quad a \in A.$$

Let $a, b \in A$, $\xi \in \mathscr{H}$ and $\eta \in \mathscr{E}_{A,L}$. We have

$$\begin{aligned}q_{23}^{\beta,\widehat{\alpha}}\Xi((a \otimes \xi) \otimes_{\pi_L} \eta) &= q_{23}^{\beta,\widehat{\alpha}}(\pi_L(a)\eta \otimes \xi) \\ &= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} (\mathrm{id}_A \otimes L)((1_A \otimes \beta(e_{ij}^{(l)\mathrm{o}}))\delta_A(a))\eta \otimes \widehat{\alpha}(e_{ji}^{(l)})\xi.\end{aligned}$$

However, $(1_A \otimes \beta(n^{\mathrm{o}}))\delta_A(a) = \delta_A(\beta_A(n^{\mathrm{o}})a)$ for all $n \in N$, we then have

$$(\mathrm{id}_A \otimes L)((1_A \otimes \beta(n^{\mathrm{o}}))\delta_A(a)) = \pi_L(\beta_A(n^{\mathrm{o}})a), \quad n \in N.$$

It then follows that

$$\begin{aligned}q_{23}^{\beta,\widehat{\alpha}}\Xi((a \otimes \xi) \otimes_{\pi_L} \eta) &= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \pi_L(\beta_A(e_{ij}^{(l)\mathrm{o}})a)\eta \otimes \widehat{\alpha}(e_{ji}^{(l)})\xi \\ &= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \Xi((\beta_A(e_{ij}^{(l)\mathrm{o}})a \otimes \widehat{\alpha}(e_{ji}^{(l)})\xi) \otimes_{\pi_L} \eta) \\ &= \Xi(q^{\beta_A,\widehat{\alpha}} \otimes_{\pi_L} 1)((a \otimes \xi) \otimes_{\pi_L} \eta).\end{aligned}$$

Therefore, we have

$$q_{23}^{\beta,\widehat{\alpha}}\restriction_{\mathscr{E}_{A,L}\otimes\mathscr{H}} = \Xi(q^{\beta_A,\widehat{\alpha}} \otimes_{\pi_L} 1)\Xi^*.$$

Now, let us consider the following map

$$\begin{aligned}\Omega :\ \mathcal{L}(A \otimes \mathscr{H}) &\longrightarrow \mathcal{L}(\mathscr{E}_{A,L} \otimes \mathscr{H}) \\ T &\longmapsto \Xi(T \otimes_{\pi_L} 1)\Xi^*.\end{aligned}$$

Then, Ω is a faithful *-homomorphism (since π_L is faithful). Since $\widehat{\alpha}(N) = M' \cap \widehat{M}'$ we have $[q^{\beta_A,\widehat{\alpha}}, 1_A \otimes L(y)] = 0$ and $[q^{\beta_A,\widehat{\alpha}}, 1_A \otimes \lambda(x)] = 0$ for all $x \in \widehat{S}$ and $y \in S$. In particular, we have

$$\pi_R(a)(1_A \otimes \lambda(x)L(y))q^{\beta_A,\widehat{\alpha}} = \pi_R(a)(1_A \otimes \lambda(x)L(y)), \quad a \in A,\, x \in \widehat{S},\, y \in S.$$

We have proved that

$$\Omega(\pi_R(a)(1_A \otimes \lambda(x)L(y))) = (\pi_L \otimes R)(\delta_A(a))(1_A \otimes 1_{\mathscr{H}} \otimes \lambda(x)L(y))q_{23}^{\beta,\widehat{\alpha}}{}_{\restriction \mathscr{E}_{A,L}\otimes\mathscr{H}},$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. In particular, $\mathrm{Ad}_Z(C) \subset \mathrm{Ran}\,\Omega$. Let us denote $D = \Omega^{-1}\mathrm{Ad}_Z(C)$. Then, D is a C*-subalgebra of $\mathcal{L}(A \otimes \mathscr{H})$ and we have

$$D = [\pi_R(a)(1_A \otimes \lambda(x)L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S].$$

We denote $\phi_{B\rtimes\widehat{\mathcal{G}}} = \Omega^{-1}\mathrm{Ad}_Z(\Psi_{L,\rho} \otimes \mathrm{id})\widehat{\Psi}_{\lambda,L}$. Then, $\phi_{B\rtimes\widehat{\mathcal{G}}}$ is a *-isomorphism from $B \rtimes \widehat{\mathcal{G}}$ onto D and we have proved that

$$\phi_{B\rtimes\widehat{\mathcal{G}}}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) = \pi_R(a)(1_A \otimes \lambda(x)L(y)), \quad a \in A,\, x \in \widehat{S},\, y \in S. \tag{4.4.5}$$

The uniqueness is straightforward since $B \rtimes \widehat{\mathcal{G}} = [\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)\,;\, a \in A, x \in \widehat{S}, y \in S]$. □

Now, we define a continuous action on D by transport of structure:

Notations 4.4.4. Let $\phi_{B\rtimes\widehat{\mathcal{G}}} : B \rtimes \widehat{\mathcal{G}} \to D$ be the *-isomorphism of Proposition 4.4.3. Let us denote:

$$\delta_D := (\phi_{B\rtimes\widehat{\mathcal{G}}} \otimes \mathrm{id}_S) \circ \delta_{B\rtimes\widehat{\mathcal{G}}} \circ \phi^{-1}_{B\rtimes\widehat{\mathcal{G}}} \quad ; \quad \beta_D := \phi_{B\rtimes\widehat{\mathcal{G}}} \circ \beta_{B\rtimes\widehat{\mathcal{G}}}.$$

Proposition 4.4.5. *The pair (δ_D, β_D) is a continuous action of $\mathcal{G}$ on the* C*-*algebra D. Moreover, we have:*

1. *For all $a \in A$, $x \in \widehat{S}$ and $y \in S$,*

$$(j_D \otimes \mathrm{id}_S)\delta_D(\pi_R(a)(1_A \otimes \lambda(x)L(y))) = (\pi_R(a) \otimes 1_S)(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)).$$

2. *For all $n \in N$, $j_D(\beta_D(n^{\mathrm{o}})) = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}})))$.*

Proof. We have that $\phi_{B\rtimes\widehat{\mathcal{G}}}$ extends to a unital *-isomorphism $\phi_{B\rtimes\widehat{\mathcal{G}}} : \mathcal{M}(B \rtimes \widehat{\mathcal{G}}) \to \mathcal{M}(D)$. By using approximate units, we obtain:

$$\phi_{B\rtimes\widehat{\mathcal{G}}}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))) = \pi_R(a)(1_A \otimes \lambda(x)) \quad ; \quad \phi_{B\rtimes\widehat{\mathcal{G}}}(\theta(y)) = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(y)), \tag{4.4.6}$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. Let us fix $a \in A$, $x \in \widehat{S}$ and $y \in S$. Let us denote $d = \pi_R(a)(1_A \otimes \lambda(x)L(y)) \in D$ for short. By (4.4.5) and (4.4.6), we have

$$\begin{aligned}
\delta_D(d) = (\phi_{B\rtimes\widehat{\mathcal{G}}} \otimes \mathrm{id}_S)\delta_{B\rtimes\widehat{\mathcal{G}}}(\phi^{-1}_{B\rtimes\widehat{\mathcal{G}}}(d)) &= (\phi_{B\rtimes\widehat{\mathcal{G}}}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))) \otimes 1_S)(\phi_{B\rtimes\widehat{\mathcal{G}}} \circ \theta \otimes \mathrm{id}_S)(\delta(y)) \\
&= (\pi_R(a)(1_A \otimes \lambda(x)) \otimes 1_S)q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)) \\
&= (\pi_R(a) \otimes 1_S)(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)).
\end{aligned}$$

By definition of δ_D, we have immediately that δ_D is a faithful *-homomorphism satisfying the coassociativity condition $(\delta_D \otimes \mathrm{id}_S)\delta_D = (\mathrm{id}_D \otimes \delta)\delta_D$. It is also clear that the action (δ_D, β_D) is continuous. Let $n \in N$, we have

$$\begin{aligned}
\beta_D(n^{\mathrm{o}})d = \phi_{B\rtimes\widehat{\mathcal{G}}}(\beta_{B\rtimes\widehat{\mathcal{G}}}(n^{\mathrm{o}}))d &= \phi_{B\rtimes\widehat{\mathcal{G}}}(\beta_{B\rtimes\widehat{\mathcal{G}}}(n^{\mathrm{o}})\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(y)) \\
&= \phi_{B\rtimes\widehat{\mathcal{G}}}(\widehat{\pi}(\pi(a)\widehat{\theta}(x))\theta(\beta(n^{\mathrm{o}})y)) \\
&= \pi_R(a)(1_A \otimes \lambda(x)L(\beta(n^{\mathrm{o}})y)) \\
&= q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}})))d,
\end{aligned}$$

as $[\lambda(x), L(\beta(n^{\mathrm{o}}))] = 0$ (because of $\beta(N^{\mathrm{o}}) \subset \widehat{M'}$) and $[\pi_R(a), 1_A \otimes L(\beta(n^{\mathrm{o}}))] = 0$ (because $\widehat{\alpha}$ and β commute pointwise). Hence, $\beta_D(n^{\mathrm{o}})d = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}})))d$, for all $d \in D$. We prove similarly that $d\beta_D(n^{\mathrm{o}}) = dq^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}})))$, for all $d \in D$. Hence, $\beta_D(n^{\mathrm{o}}) = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}})))$. □

In the following, we will provide a description of the action (δ_D, β_D) in terms of the action (δ_A, β_A) and the right regular representation of $\mathcal{G}$.

Notations 4.4.6. Let us introduce some notations:

1. Let us denote $\mathcal{K} := \mathcal{K}(\mathscr{H})$ for short. We consider the flip map $\sigma : S \otimes \mathcal{K} \to \mathcal{K} \otimes S$, given by $\sigma(s \otimes k) = k \otimes s$ for all $s \in S$ and $k \in \mathcal{K}$.

2. Let $\delta_0 : A \otimes \mathcal{K} \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ be the *-homomorphism given by $\delta_0 = (\mathrm{id}_A \otimes \sigma)(\delta_A \otimes \mathrm{id}_{\mathcal{K}})$. Therefore, δ_0 is the unique *-homomorphism from $A \otimes \mathcal{K}$ to $\mathcal{M}(A \otimes \mathcal{K} \otimes S)$ such that

$$\delta_0(a \otimes k) = \delta_A(a)_{13}(1_A \otimes k \otimes 1_S), \quad a \in A,\, k \in \mathcal{K}.$$

   Moreover, δ_0 (resp. $\delta_0 \otimes \mathrm{id}_S$) extends uniquely to a strictly continuous *-homomorphism from $\mathcal{M}(A \otimes \mathcal{K})$ to $\mathcal{M}(A \otimes \mathcal{K} \otimes S)$ (resp. from $\mathcal{M}(A \otimes \mathcal{K} \otimes S)$ to $\mathcal{M}(A \otimes \mathcal{K} \otimes S \otimes S)$) such that

$$\delta_0(1_{A\otimes\mathcal{K}}) = q_{13}^{\beta_A,\alpha} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S),$$

$$(\text{resp. } (\delta_0 \otimes \mathrm{id}_S)(1_{A\otimes\mathcal{K}\otimes S}) = q_{13}^{\beta_A,\alpha} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S \otimes S)).$$

   Similarly, we have a unique strictly continuous *-homomorphism

$$\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta : \mathcal{M}(A \otimes \mathcal{K} \otimes S) \to \mathcal{M}(A \otimes \mathcal{K} \otimes S \otimes S)$$

   extending $\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta$ such that $(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(1_{A\otimes\mathcal{K}\otimes S}) = q_{34}^{\beta,\alpha} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S \otimes S)$.

3. Let us denote $V_0 \in \mathcal{M}(\widehat{S} \otimes S)$ such that $V = (\rho \otimes L)(V_0)$. Then, we denote

$$\mathcal{V} := (\rho \otimes \mathrm{id}_S)(V_0) \in \mathcal{L}(\mathscr{H} \otimes S).$$

   Therefore, $\mathcal{V} \in \mathcal{L}(\mathscr{H} \otimes S)$ is the unique partial isometry such that $(\mathrm{id} \otimes L)(\mathcal{V}) = V$.

4. Let us denote:

$$\delta_{A\otimes\mathcal{K}}(x) := \mathcal{V}_{23}\delta_0(x)\mathcal{V}_{23}^*, \quad x \in A \otimes \mathcal{K} \quad ; \quad \beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}}) := q^{\beta_A,\widehat{\alpha}}(1_A \otimes \beta(n^{\mathrm{o}})), \quad n \in N.$$

Remarks 4.4.7. a) Since $\widehat{\alpha}$ and β commute pointwise, we have that $\beta_{A\otimes\mathcal{K}}$ is a (degenerate) *-homomorphism.

b) $\delta_{A\otimes\mathcal{K}} : A \otimes \mathcal{K} \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ is a bounded linear map and it is also clear that we have $\delta_{A\otimes\mathcal{K}}(x^*) = \delta_{A\otimes\mathcal{K}}(x)^*$, for all $x \in A \otimes \mathcal{K}$. It is worth noting that $\delta_{A\otimes\mathcal{K}}$ is unfortunately *not* multiplicative in general.

c) Note that $\delta_{A\otimes\mathcal{K}}$ is strictly continuous. In particular, $\delta_{A\otimes\mathcal{K}}$ extends uniquely to a strictly continuous linear map $\delta_{A\otimes\mathcal{K}} : \mathcal{M}(A \otimes \mathcal{K}) \to \mathcal{M}(A \otimes \mathcal{K} \otimes S)$ and for all $m \in \mathcal{M}(A \otimes \mathcal{K})$ we have $\delta_{A\otimes\mathcal{K}}(m) = \mathcal{V}_{23}\delta_0(m)\mathcal{V}_{23}^*$.

We will prove that $\delta_{A\otimes\mathcal{K}}$ is coassociative. First, we need the following lemma:

Lemma 4.4.8. *For all $x \in \mathcal{M}(A \otimes \mathcal{K})$, we have*

$$q_{13}^{\beta_A,\alpha}(\delta_0 \otimes \mathrm{id}_S)\delta_0(x) = (\delta_0 \otimes \mathrm{id}_S)\delta_0(x) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_0(x) = q_{34}^{\beta,\alpha}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_0(x).$$

Proof. Since δ_0 is strictly continuous we have $\delta_0(a \otimes 1) = \delta_A(a)_{13}$ for all $a \in A$. By the strict continuity of $\delta_0 \otimes \mathrm{id}_S$ and $\delta_A \otimes \mathrm{id}_S$, it then follows that $(\delta_0 \otimes \mathrm{id}_S)(m_{13}) = (\delta_A \otimes \mathrm{id}_S)(m)_{134}$ for all $m \in \mathcal{M}(A \otimes S)$. Similarly, we have $(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(m_{13}) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(m)_{134}$ for all $m \in \mathcal{M}(A \otimes S)$. We combine these relations with the fact that $(\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A$ to conclude that

$$(\delta_0 \otimes \mathrm{id}_S)\delta_0(a \otimes 1_{\mathscr{H}}) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_0(a \otimes 1_{\mathscr{H}}), \quad a \in A.$$

Now, we also have $\delta_0(1_A \otimes k) = q_{13}^{\beta_A,\alpha}(1_A \otimes k \otimes 1_S)$ for all $k \in \mathcal{K}$. Besides,

$$(\delta_0 \otimes \mathrm{id}_S)(1_A \otimes k \otimes 1_S) = q_{13}^{\beta_A,\alpha}(1_A \otimes k \otimes 1_S \otimes 1_S),$$

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(1_A \otimes k \otimes 1_S) = q_{34}^{\beta,\alpha}(1_A \otimes k \otimes 1_S \otimes 1_S),$$

for all $k \in \mathcal{K}$. We also obtain

$$(\delta_0 \otimes \mathrm{id}_S)(q_{13}^{\beta_A,\alpha}) = q_{13}^{\beta_A,\alpha}q_{34}^{\beta,\alpha} = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(q_{13}^{\beta_A,\alpha}),$$

by using the formulas $\delta_A(\beta_A(n^{\mathrm{o}})) = q^{\beta_A,\alpha}(1_A \otimes \beta(n^{\mathrm{o}}))$ and $\delta(\alpha(n)) = q^{\beta,\alpha}(\alpha(n) \otimes 1_S)$ for all $n \in N$. Therefore,

$$(\delta_0 \otimes \mathrm{id}_S)\delta_0(1_A \otimes k) = q_{13}^{\beta_A,\alpha}q_{34}^{\beta,\alpha}q_{13}^{\beta_A,\alpha}(1_A \otimes k \otimes 1_S \otimes 1_S) = q_{13}^{\beta_A,\alpha}q_{34}^{\beta,\alpha}(1_A \otimes k \otimes 1_S \otimes 1_S),$$

for all $k \in \mathcal{K}$ since $[q_{34}^{\beta,\alpha}, q_{13}^{\beta_A,\alpha}] = 0$. Hence,

$$(\delta_0 \otimes \mathrm{id}_S)\delta_0(1_A \otimes k) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_0(1_A \otimes k), \quad k \in \mathcal{K}.$$

Therefore, $(\delta_0 \otimes \mathrm{id}_S)\delta_0(x) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_0(x)$ for all $x \in A \otimes \mathcal{K}$. This equality also holds for all $x \in \mathcal{M}(A \otimes \mathcal{K})$ because of the strict continuity of δ_0, $\delta_0 \otimes \mathrm{id}_S$ and $\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta$. The other relations follow from the formulas $(\delta_0 \otimes \mathrm{id}_S)(1_{A\otimes\mathcal{K}\otimes S}) = q_{13}^{\beta_A,\alpha}$ and $(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(1_{A\otimes\mathcal{K}\otimes S}) = q_{34}^{\beta,\alpha}$. □

Proposition 4.4.9. *For all $x \in \mathcal{M}(A \otimes \mathcal{K})$, we have*

$$(\delta_{A\otimes\mathcal{K}} \otimes \mathrm{id}_S)\delta_{A\otimes\mathcal{K}}(x) = (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_{A\otimes\mathcal{K}}(x).$$

Proof. We have

$$(\delta_0 \otimes \mathrm{id}_S)(1_A \otimes k \otimes y) = q_{13}^{\beta_A,\alpha}(1_A \otimes k \otimes 1_S \otimes y), \quad k \in \mathcal{K},\, y \in S.$$

Hence, $(\delta_0 \otimes \mathrm{id}_S)(x_{23}) = q_{13}^{\beta_A,\alpha}x_{24}$ for all $x \in \mathcal{K} \otimes S$. This equality actually holds true for all $x \in \mathcal{M}(\mathcal{K} \otimes S)$ because of the strict continuity of δ_0. In particular, up to the identification $\mathcal{M}(\mathcal{K} \otimes S) = \mathcal{L}(\mathscr{H} \otimes S)$, we have $(\delta_0 \otimes \mathrm{id}_S)(\mathcal{V}_{23}) = q_{13}^{\beta_A,\alpha}\mathcal{V}_{24} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S \otimes S)$. Since $(\mathrm{id}_{\mathcal{K}} \otimes \delta)(\mathcal{V}) = \mathcal{V}_{12}\mathcal{V}_{13}$, we have $(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(\mathcal{V}_{23}) = \mathcal{V}_{23}\mathcal{V}_{24}$. Therefore, in virtue of Lemma 4.4.8, we obtain

$$\begin{aligned}
(\delta_{A\otimes\mathcal{K}} \otimes \mathrm{id}_S)\delta_{A\otimes\mathcal{K}}(x) &= \mathcal{V}_{23}(\delta_0 \otimes \mathrm{id}_S)(\delta_{A\otimes\mathcal{K}}(x))\mathcal{V}_{23}^* \\
&= \mathcal{V}_{23}\mathcal{V}_{24}q_{13}^{\beta_A,\alpha}(\delta_0 \otimes \mathrm{id}_S)(\delta_0(x))q_{13}^{\beta_A,\alpha}\mathcal{V}_{24}^*\mathcal{V}_{23}^* \\
&= \mathcal{V}_{23}\mathcal{V}_{24}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)(\delta_0(x))\mathcal{V}_{24}^*\mathcal{V}_{23}^* \\
&= (\mathrm{id}_{A\otimes\mathcal{K}} \otimes \delta)\delta_{A\otimes\mathcal{K}}(x),
\end{aligned}$$

for all $x \in \mathcal{M}(A \otimes \mathcal{K})$. □

Lemma 4.4.10. *We have* $\delta_0(q^{\beta_A,\widehat{\alpha}}) = q_{13}^{\beta_A,\alpha} q_{23}^{\widehat{\alpha},\beta}$. *For all* $n \in N$, *we have*

$$\begin{aligned}\delta_0(\beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}})) &= q_{13}^{\beta_A,\alpha} q_{23}^{\widehat{\alpha},\beta}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S) = q_{13}^{\beta_A,\alpha}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S) q_{23}^{\widehat{\alpha},\beta} \\ &= (1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S) q_{13}^{\beta_A,\alpha} q_{23}^{\widehat{\alpha},\beta}.\end{aligned}$$

Proof. Let $n, n' \in N$. We have

$$\delta_0(\beta_A(n^{\mathrm{o}}) \otimes \widehat{\alpha}(n')) = \delta_A(\beta_A(n^{\mathrm{o}}))_{13}(1_A \otimes \widehat{\alpha}(n') \otimes 1_S) = q_{13}^{\beta_A,\alpha}(1_A \otimes \widehat{\alpha}(n') \otimes \beta(n^{\mathrm{o}})).$$

In particular, we have $\delta_0(q^{\beta_A,\widehat{\alpha}}) = q_{13}^{\beta_A,\alpha} q_{23}^{\widehat{\alpha},\beta}$. Therefore, we have

$$\delta_0(\beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}})) = \delta_0(q^{\beta_A,\widehat{\alpha}})\delta_0(1_A \otimes \beta(n^{\mathrm{o}})) = q_{13}^{\beta_A,\alpha} q_{23}^{\widehat{\alpha},\beta}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S).$$

The other formulas follow from the fact that β commutes pointwise with $\widehat{\alpha}$. □

Proposition 4.4.11. *We have* $\delta_{A\otimes\mathcal{K}}(1_{A\otimes\mathcal{K}}) = q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}$. *For all* $n \in N$, *we have*

$$\delta_{A\otimes\mathcal{K}}(\beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}})) = \delta_{A\otimes\mathcal{K}}(1_{A\otimes\mathcal{K}})(1_{A\otimes\mathcal{K}} \otimes \beta(n^{\mathrm{o}})).$$

Proof. By Proposition 2.3.5 3, we have $\mathcal{V}(1_{\mathscr{H}} \otimes \alpha(n)) = (\widehat{\alpha}(n) \otimes 1_S)\mathcal{V}$ for all $n \in N$. Hence, $\mathcal{V}_{23} q_{13}^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}} \mathcal{V}_{23}$. We then have

$$\delta_{A\otimes\mathcal{K}}(1_{A\otimes\mathcal{K}}) = \mathcal{V}_{23}\delta_0(1_{A\otimes\mathcal{K}})\mathcal{V}_{23}^* = \mathcal{V}_{23} q_{13}^{\beta_A,\alpha} \mathcal{V}_{23}^* = q_{12}^{\beta_A,\widehat{\alpha}} \mathcal{V}_{23}\mathcal{V}_{23}^* = q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}.$$

We recall that $q^{\widehat{\alpha},\beta} = \mathcal{V}^*\mathcal{V}$. Hence, $q^{\widehat{\alpha},\beta}\mathcal{V}^* = \mathcal{V}^*$ since $\mathcal{V}$ is a partial isometry. By Lemma 4.4.10, we then have

$$\begin{aligned}\delta_{A\otimes\mathcal{K}}(\beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}})) &= \mathcal{V}_{23}\delta_0(\beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}}))\mathcal{V}_{23}^* \\ &= \mathcal{V}_{23} q_{13}^{\beta_A,\alpha}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S) q_{23}^{\widehat{\alpha},\beta} \mathcal{V}_{23}^* \\ &= q_{12}^{\beta_A,\widehat{\alpha}} \mathcal{V}_{23}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes 1_S)\mathcal{V}_{23}^* \\ &= q_{12}^{\beta_A,\widehat{\alpha}} \mathcal{V}_{23}\mathcal{V}_{23}^*(1_{A\otimes\mathcal{K}} \otimes \beta(n^{\mathrm{o}})) \qquad (\textit{cf.}\ \text{Proposition 2.3.5 3}) \\ &= q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}(1_{A\otimes\mathcal{K}} \otimes \beta(n^{\mathrm{o}})),\end{aligned}$$

for all $n \in N$. □

We identify $\mathcal{M}(A \otimes \mathcal{K})$ (resp. $\mathcal{M}(A \otimes \mathcal{K} \otimes S)$) with $\mathcal{L}(A \otimes \mathscr{H})$ (resp. $\mathcal{L}(A \otimes \mathscr{H} \otimes S)$).

Proposition 4.4.12. *We have:*

1. *For all* $a \in A$,

$$\begin{aligned}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(\pi_R(a)) &= \Sigma_{23}(\pi_L \otimes R)(\delta_A(a))\Sigma_{23} \\ &= q_{13}^{\beta_A,\alpha}(1_A \otimes U \otimes 1_{\mathscr{H}})\Sigma_{23} V_{23}\pi_L(a)_{12} V_{23}^*\Sigma_{23}(1_A \otimes U^* \otimes 1_{\mathscr{H}}) q_{13}^{\beta_A,\alpha},\end{aligned}$$

$$\delta_{A\otimes\mathcal{K}}(\pi_R(a)) = q_{23}^{\beta,\alpha}(\pi_R(a) \otimes 1_S) = q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}(\pi_R(a) \otimes 1_S).$$

2. *For all* $m \in \mathcal{M}(A)$, $\delta_{A\otimes\mathcal{K}}(\pi_R(m)) = q_{23}^{\beta,\alpha}(\pi_R(m) \otimes 1_S) = q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}(\pi_R(m) \otimes 1_S)$.

3. For all $x \in \widehat{S}$ and $y \in S$,

$$\begin{aligned}\delta_{A\otimes\mathcal{K}}(1_A \otimes \lambda(x)L(y)) &= q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)) \\ &= q_{12}^{\beta_A,\widehat{\alpha}} q_{23}^{\beta,\alpha}(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)).\end{aligned}$$

4. For all $a \in A$, $x \in \widehat{S}$ and $y \in S$,

$$\begin{aligned}\delta_{A\otimes\mathcal{K}}(\pi_R(a)(1_A \otimes \lambda(x)L(y))) &= (\pi_R(a) \otimes 1_S)(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)) \\ &= q_{12}^{\beta_A,\widehat{\alpha}}(\pi_R(a) \otimes 1_S)(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y)).\end{aligned}$$

Proof. Let us consider the partial isometry $Z = 1_A \otimes (1_{\mathscr{H}} \otimes U^*)V(1_{\mathscr{H}} \otimes U) \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$ (see proof of Proposition 4.4.3). Since $(\mathrm{id}_{\mathcal{K}} \otimes L)\sigma(t) = \Sigma(L \otimes \mathrm{id}_{\mathcal{K}})(t)\Sigma$ for all $t \in \mathcal{M}(S \otimes \mathcal{K})$, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(u) = (\mathrm{id}_A \otimes (\mathrm{id}_{\mathcal{K}} \otimes L)\sigma)(\delta_A \otimes \mathrm{id}_{\mathcal{K}})(u) = \Sigma_{23}(\pi_L \otimes \mathrm{id}_{\mathcal{K}})(u)\Sigma_{23},$$

for all $u \in \mathcal{M}(A \otimes \mathcal{K})$. In particular, in virtue of (4.4.2) we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(\pi_R(a)) = \Sigma_{23}(\pi_L \otimes R)(\delta_A(a))\Sigma_{23} = \Sigma_{23}Z\pi_L(a)_{12}Z^*\Sigma_{23},$$

for all $a \in A$. However, we have $\Sigma_{23}Z = \widetilde{V}_{23}\Sigma_{23}$ and $\Sigma_{23}\pi_L(a)_{12}\Sigma_{23} = \pi_L(a)_{13}$ for all $a \in A$. It then follows that

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(\pi_R(a)) = \widetilde{V}_{23}\pi_L(a)_{13}\widetilde{V}_{23}^*, \quad a \in A.$$

Therefore, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)) = V_{23}\widetilde{V}_{23}\pi_L(a)_{13}\widetilde{V}_{23}^*V_{23}^*, \quad a \in A. \tag{4.4.7}$$

According to Corollary 3.1.9 1, we have $V\widetilde{V} = W^*(U^* \otimes 1)\Sigma\mathsf{T}_{\widehat{\beta},\widehat{\alpha}}$. Hence,

$$\begin{aligned}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)) &= W_{23}^*(1_A \otimes U^* \otimes 1_{\mathscr{H}})\mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}\Sigma_{23}\pi_L(a)_{13}\Sigma_{23}\mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}^*(1_A \otimes U \otimes 1_{\mathscr{H}})W_{23} \\ &= W_{23}^*(1_A \otimes U^* \otimes 1_{\mathscr{H}})\mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}\pi_L(a)_{12}\mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}^*(1_A \otimes U \otimes 1_{\mathscr{H}})W_{23},\end{aligned}$$

for all $a \in A$, since $\Sigma_{23}\pi_L(a)_{13}\Sigma_{23} = \pi_L(a)_{12}$ for all $a \in A$ and $\Sigma_{23}\mathsf{T}_{\widehat{\beta},\widehat{\alpha},23} = \mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}\Sigma_{23}$. By using the fact that $\widehat{\alpha}(N) \subset M'$ and the fact that $|\nu_l| = 1$ for all $1 \leqslant l \leqslant k$ (see Notation 3.1.8), we have

$$[\pi_L(a)_{12}, \mathsf{T}_{\widehat{\alpha},\widehat{\beta},23}] = 0, \quad a \in A \quad ; \quad \mathsf{T}_{\widehat{\alpha},\widehat{\beta}}\mathsf{T}_{\widehat{\alpha},\widehat{\beta}}^* = \sum_{1\leqslant l\leqslant k} \widehat{\alpha}(e^{(l)}) \otimes \widehat{\beta}(e^{(l)\mathrm{o}}).$$

It then follows that

$$\begin{aligned}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)) &= W_{23}^* \sum_{1\leqslant l\leqslant k} (1_A \otimes U^*\widehat{\alpha}(e^{(l)}) \otimes \widehat{\beta}(e^{(l)\mathrm{o}}))\pi_L(a)_{12}(1_A \otimes U \otimes 1_{\mathscr{H}})W_{23} \\ &= \sum_{1\leqslant l\leqslant k} 1_A \otimes W^*(\alpha(e^{(l)}) \otimes \widehat{\beta}(e^{(l)\mathrm{o}}))\pi_R(a)_{12}W_{23},\end{aligned}$$

since $U^*\widehat{\alpha}(n) = \alpha(n)U^*$ for all $n \in N$ and $\pi_R(a) = (1_A \otimes U^*)\pi_L(a)(1_A \otimes U)$ for all $a \in A$. However, we have $W^*(\alpha(n) \otimes 1) = W^*(1 \otimes \widehat{\beta}(n^{\mathrm{o}}))$ for all $n \in N$ (see Proposition 2.3.6 1). Hence, $W^*(\alpha(e^{(l)}) \otimes \widehat{\beta}(e^{(l)\mathrm{o}})) = W^*(1 \otimes \widehat{\beta}(e^{(l)\mathrm{o}}))$ for all $1 \leqslant l \leqslant k$. Consequently, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)) = W_{23}^*\pi_R(a)_{12}W_{23}, \quad \text{for all } a \in A.$$

Moreover, $[\pi_R(a)_{12}, W_{23}] = 0$ for all $a \in A$ since $W \in M \otimes \widehat{M}$ and $R(S) \subset M'$. Hence,

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)) = q_{23}^{\beta,\alpha}\pi_R(a)_{12}, \quad \text{for all } a \in A, \tag{4.4.8}$$

since $W^*W = q^{\beta,\alpha}$ and the first statement is proved. Note also that $[q_{23}^{\beta,\alpha}, \pi_R(a)_{12}] = 0$ for all $a \in A$. The second statement follows from the first one and the strict continuity of π_R and $\delta_{A\otimes\mathcal{K}}$. Let $x \in \widehat{S}$ and $y \in S$, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(1_A \otimes \lambda(x)L(y)) = q_{13}^{\beta_A,\alpha}(1_A \otimes \lambda(x)L(y) \otimes 1_{\mathscr{H}}).$$

Since $V(1 \otimes \alpha(n)) = (\widehat{\alpha}(n) \otimes 1)V$ for all $n \in N$ (Proposition 2.3.5 3), we then obtain that $V_{23}q_{13}^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}}V_{23}$. We also have $[V, \lambda(x) \otimes 1] = 0$ (because of $V \in \widehat{M'} \otimes M$ and $\lambda(\widehat{S}) \subset \widehat{M}$) and $V(L(y) \otimes 1)V^* = (L \otimes L)\delta(y)$. Therefore, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(1_A \otimes \lambda(x)L(y)) = q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes \lambda(x) \otimes 1_{\mathscr{H}})(1_A \otimes (L \otimes L)\delta(y)) \tag{4.4.9}$$

and the third statement is proved. Note also that $q_{12}^{\beta_A,\widehat{\alpha}}$ commutes with $1_A \otimes \lambda(x) \otimes 1_{\mathscr{H}}$ and $1_A \otimes (L \otimes L)\delta(y)$. Now, $V_{23}^*V_{23} = q_{23}^{\widehat{\alpha},\beta}$ commutes with $q_{13}^{\beta_A,\alpha}$ (since α and β commute pointwise) and $1_A \otimes \lambda(x)L(y) \otimes 1_{\mathscr{H}}$, for all $x \in \widehat{S}$ and $y \in S$ (since $\widehat{\alpha}(N) = M' \cap \widehat{M'}$). Hence,

$$[(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(1_A \otimes \lambda(x)L(y)), V_{23}^*V_{23}] = 0, \quad \text{for all } x \in \widehat{S},\ y \in S.$$

Therefore, in virtue of (4.4.8) and (4.4.9) we have

$$\begin{aligned}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)&\delta_{A\otimes\mathcal{K}}(\pi_R(a)(1_A \otimes \lambda(x)L(y))) \\ &= (\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a))(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(1_A \otimes \lambda(x)L(y)) \\ &= \pi_R(a)_{12}q_{23}^{\beta,\alpha}(1_A \otimes \lambda(x) \otimes 1_{\mathscr{H}})(1_A \otimes (L \otimes L)\delta(y)),\end{aligned}$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. Moreover, $q_{23}^{\beta,\alpha}$ commutes with $1_A \otimes \lambda(x) \otimes 1_{\mathscr{H}}$ (because of $\beta(N^{\mathrm{o}}) \subset \widehat{M'}$) and we have $q^{\beta,\alpha}(L \otimes L)\delta(y) = (L \otimes L)\delta(y)$ for all $y \in S$ since $q^{\beta,\alpha} = \delta(1_S)$. Hence,

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_{A\otimes\mathcal{K}}(\pi_R(a)(1_A \otimes \lambda(x)L(y))) = \pi_R(a)_{12}(1_A \otimes \lambda(x) \otimes 1_{\mathscr{H}})(1_A \otimes (L \otimes L)\delta(y)),$$

for all $a \in A$, $x \in \widehat{S}$ and $y \in S$. □

Corollary 4.4.13. *The action (δ_D, β_D) of $\mathcal{G}$ on D is given by:*

$$\begin{aligned}(j_D \otimes \mathrm{id}_S)\delta_D(d) = \delta_{A\otimes\mathcal{K}}(d) = \mathcal{V}_{23}\delta_0(d)\mathcal{V}_{23}^*, \quad d \in D, \\ j_D(\beta_D(n^{\mathrm{o}})) = \beta_{A\otimes\mathcal{K}}(n^{\mathrm{o}}) = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}}))), \quad n \in N.\end{aligned}$$

Proof. It is enough to verify the first formula for $d = \pi_R(a)(1_A \otimes \lambda(x)L(y))$, where $a \in A$, $x \in \widehat{S}$ and $y \in S$. Then, the first formula follows from Propositions 4.4.5 1 and 4.4.12 4. The second formula follows from Proposition 4.4.5 2 and the definition of $\beta_{A\otimes\mathcal{K}}$. □

Remark 4.4.14. By strict continuity, we obtain the formulas:

$$(j_D \otimes \mathrm{id}_S)\delta_D(m) = \delta_{A\otimes\mathcal{K}}(j_D(m)) = \mathcal{V}_{23}\delta_0(j_D(m))\mathcal{V}_{23}^*, \quad m \in \mathcal{M}(D).$$

Now, we can state the main theorem of this chapter.

Theorem 4.4.15. *Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. Let us denote*

$$D = [\pi_R(a)(1_A \otimes \lambda(x)L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S] \subset \mathcal{L}(A \otimes \mathscr{H}).$$

Then, D is a C-algebra and we define a continuous action (δ_D, β_D) of $\mathcal{G}$ on D by setting:*

$$(j_D \otimes \mathrm{id}_S)\delta_D(d) = \mathcal{V}_{23}\delta_0(d)\mathcal{V}_{23}^*, \quad d \in D \quad ; \quad j_D(\beta_D(n^{\mathrm{o}})) = q^{\beta_A,\widehat{\alpha}}(1_A \otimes L(\beta(n^{\mathrm{o}}))), \quad n \in N.$$

The double crossed product $((A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}, \delta_{(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}}, \beta_{(A\rtimes\mathcal{G})\rtimes\widehat{\mathcal{G}}})$ is canonically isomorphic to the $\mathcal{G}$-C-algebra (D, δ_D, β_D). Moreover, if $\mathcal{G}$ is regular we have*

$$D = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathcal{K}(\mathscr{H}))q^{\beta_A,\widehat{\alpha}}.$$

For the proof, Proposition 4.4.16 stated below will play a crucial role in the regular case. First, we need to fix some notations. Let us give a concrete description of the G.N.S. construction $(\mathscr{H}_\epsilon, \pi_\epsilon, \Lambda_\epsilon)$ for (N, ϵ):

- $\mathscr{H}_\epsilon = \bigoplus_{1\leqslant l\leqslant k} \mathbb{C}^{n_l} \otimes \overline{\mathbb{C}^{n_l}}$.
- The G.N.S. representation $\pi_\epsilon : N \to \mathcal{B}(\mathscr{H}_\epsilon)$ is given by:

$$\pi_\epsilon(x) = \bigoplus_{1\leqslant l\leqslant k} x_l \otimes 1_{\overline{\mathbb{C}^{n_l}}}, \quad x = (x_l)_{1\leqslant l\leqslant k} \in N.$$

- The G.N.S. map $\Lambda_\epsilon : N \to \mathscr{H}_\epsilon$ is given by:

$$\Lambda_\epsilon(x) = \pi_\epsilon(x)\xi_\epsilon, \quad x \in N, \quad \text{where} \quad \xi_\epsilon = \bigoplus_{1\leqslant l\leqslant k} n_l^{-1/2} \sum_{1\leqslant i\leqslant n_l} \varepsilon_i^{(l)} \otimes \overline{\varepsilon_i^{(l)}}$$

  and $(\varepsilon_i^{(l)})_{1\leqslant i\leqslant n_l}$ is an orthonormal basis of $\mathbb{C}^{n_l}$ for each $1 \leqslant l \leqslant k$. In particular, ξ_ϵ is a cyclic vector for the representation π_ϵ.

Note that if $(e_{ij}^{(l)})$, $1 \leqslant l \leqslant k$, $1 \leqslant i, j \leqslant n_l$, is the system of matrix units of N defined by

$$e_{ij}^{(l)}\varepsilon_r^{(l')} = \delta_{l'}^l \delta_j^r \varepsilon_i^{(l)}, \quad 1 \leqslant l, l' \leqslant k, \quad 1 \leqslant r \leqslant n_{l'}, \quad 1 \leqslant i, j \leqslant n_l,$$

then the family

$$(n_l^{-1/2}\pi_\epsilon(e_{ij}^{(l)})\xi_\epsilon)_{1\leqslant l\leqslant k,\, 1\leqslant i,j\leqslant n_l},$$

is an orthonormal basis of $\mathscr{H}_\epsilon$.

Proposition 4.4.16. *Let $\pi : N \to \mathcal{B}(H)$ and $\gamma : N^{\mathrm{o}} \to \mathcal{B}(K)$ be two unital *-representations. We have:*

1. *For all $\xi, \eta \in H$, we have $q^{\pi,\gamma}(R^{\pi,\epsilon}(\xi)R^{\pi,\epsilon}(\eta)^* \otimes 1_K) = q^{\pi,\gamma}(\theta_{\xi,\eta} \otimes 1_K)q^{\pi,\gamma}$.*
2. *For all $\xi, \eta \in K$, we have $q^{\pi,\gamma}(1_H \otimes L^{\gamma,\epsilon}(\xi)L^{\gamma,\epsilon}(\eta)^*) = q^{\pi,\gamma}(1_H \otimes \theta_{\xi,\eta})q^{\pi,\gamma}$.*

In particular, we have:

$$q^{\pi,\gamma}(\mathcal{K}_\pi \otimes 1_K) = q^{\pi,\gamma}(\mathcal{K}(H) \otimes 1_K)q^{\pi,\gamma}, \quad q^{\pi,\gamma}(1_H \otimes \mathcal{K}_\gamma) = q^{\pi,\gamma}(1_H \otimes \mathcal{K}(K))q^{\pi,\gamma}.$$

Proof. Let us recall that $\theta_{\xi,\eta}$ for $\xi,\eta\in H$ is the rank-one operator on H given for all $\zeta\in H$ by $\theta_{\xi,\eta}(\zeta)=\langle\eta,\,\zeta\rangle\xi$. Note that since $(n_l^{-1/2}\Lambda_\epsilon(e_{rs}^{(l)}))_{1\leqslant l\leqslant k,\,1\leqslant r,s\leqslant n_l}$ is an orthonormal basis of $\mathscr{H}_\epsilon$ (for the system of matrix units defined above), we have

$$\langle x,\,y\rangle_\epsilon=\sum_{1\leqslant l\leqslant k}n_l^{-1}\sum_{1\leqslant r,s\leqslant n_l}\langle x,\,\Lambda_\epsilon(e_{rs}^{(l)})\rangle_\epsilon\langle\Lambda_\epsilon(e_{rs}^{(l)}),\,y\rangle_\epsilon,\quad x,y\in\mathscr{H}_\epsilon. \tag{4.4.10}$$

Let us take $\xi,\eta\in H$. Let $\zeta,\zeta'\in H$ and $\upsilon,\upsilon'\in K$. On one hand, we have

$$\begin{aligned}
&\langle\zeta\otimes\upsilon,\,q^{\pi,\gamma}(R^{\pi,\epsilon}(\xi)R^{\pi,\epsilon}(\eta)^*\otimes 1_K)(\zeta'\otimes\upsilon')\rangle\\
&\quad=\sum_{1\leqslant l\leqslant k}n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\langle\zeta\otimes\upsilon,\,R^{\pi,\epsilon}(\xi)R^{\pi,\epsilon}(\eta)^*\pi(e_{ij}^{(l)})\zeta'\otimes\gamma(e_{ji}^{(l)\mathrm{o}})\upsilon'\rangle,\\
&\qquad\qquad\text{as } R^{\pi,\epsilon}(\xi)R^{\pi,\epsilon}(\eta)^*\in\pi(N)',\\
&\quad=\sum_{1\leqslant l\leqslant k}n_l^{-1}\sum_{1\leqslant i,j\leqslant n_l}\langle R^{\pi,\epsilon}(\xi)^*\zeta,\,R^{\pi,\epsilon}(\eta)^*\pi(e_{ij}^{(l)})\zeta'\rangle_\epsilon\langle\upsilon,\,\gamma(e_{ji}^{(l)\mathrm{o}})\upsilon'\rangle\\
&\quad=\sum_{1\leqslant l,l'\leqslant k}n_l^{-1}n_{l'}^{-1}\sum_{\substack{1\leqslant i,j\leqslant n_l\\1\leqslant r,s\leqslant n_{l'}}}\langle\zeta,\,R^{\pi,\epsilon}(\xi)\Lambda_\epsilon(e_{rs}^{(l')})\rangle\langle R^{\pi,\epsilon}(\eta)\Lambda_\epsilon(e_{rs}^{(l')}),\,\pi(e_{ij}^{(l)})\zeta'\rangle\langle\upsilon,\,\gamma(e_{ji}^{(l)\mathrm{o}})\upsilon'\rangle,\\
&\qquad\qquad\text{in virtue of (4.4.10)},\\
&\quad=\sum_{1\leqslant l,l'\leqslant k}n_l^{-1}n_{l'}^{-1}\sum_{\substack{1\leqslant i,j\leqslant n_l\\1\leqslant r,s\leqslant n_{l'}}}\langle\zeta,\,\pi(e_{rs}^{(l')})\xi\rangle\langle\pi(e_{rs}^{(l')})\eta,\,\pi(e_{ij}^{(l)})\zeta'\rangle\langle\upsilon,\,\gamma(e_{ji}^{(l)\mathrm{o}})\upsilon'\rangle\\
&\quad=\sum_{1\leqslant l\leqslant k}n_l^{-2}\sum_{1\leqslant i,j,s\leqslant n_l}\langle\zeta,\,\pi(e_{is}^{(l)})\xi\rangle\langle\eta,\,\pi(e_{sj}^{(l)})\zeta'\rangle\langle\upsilon,\,\gamma(e_{ji}^{(l)\mathrm{o}})\upsilon'\rangle,\\
&\qquad\qquad\text{since }\pi(e_{rs}^{(l')})^*\pi(e_{ij}^{(l)})=\pi(e_{sr}^{(l')}e_{ij}^{(l)})=\delta_{l'}^l\delta_r^i\pi(e_{sj}^{(l)}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\langle\zeta\otimes\upsilon,\,q^{\pi,\gamma}(\theta_{\xi,\eta}\otimes 1_H)q^{\pi,\gamma}(\zeta'\otimes\upsilon')\rangle\\
&\qquad=\sum_{1\leqslant l,l'\leqslant k}n_l^{-1}n_{l'}^{-1}\sum_{\substack{1\leqslant i,j\leqslant n_l\\1\leqslant r,s\leqslant n_{l'}}}\langle\zeta,\,\pi(e_{ij}^{(l)})\theta_{\xi,\eta}\pi(e_{rs}^{(l')})\zeta'\rangle\langle\upsilon,\,\gamma(e_{ji}^{(l)\mathrm{o}}e_{sr}^{(l')\mathrm{o}})\upsilon'\rangle\\
&\qquad=\sum_{1\leqslant l\leqslant k}n_l^{-2}\sum_{1\leqslant i,j,s\leqslant n_l}\langle\eta,\,\pi(e_{js}^{(l)})\zeta'\rangle\langle\zeta,\,\pi(e_{ij}^{(l)})\xi\rangle\langle\upsilon,\,\gamma(e_{si}^{(l)\mathrm{o}})\upsilon'\rangle.
\end{aligned}$$

Therefore, by exchanging the indices j and s in the last summation we obtain

$$\langle\zeta\otimes\upsilon,\,q^{\pi,\gamma}(R^{\pi,\epsilon}(\xi)R^{\pi,\epsilon}(\eta)^*\otimes 1_K)(\zeta'\otimes\upsilon')\rangle=\langle\zeta\otimes\upsilon,\,q^{\pi,\gamma}(\theta_{\xi,\eta}\otimes 1_K)q^{\pi,\gamma}(\zeta'\otimes\upsilon')\rangle$$

for all $\zeta,\zeta'\in H$ and $\upsilon,\upsilon'\in K$. This proves the first statement. The second one is proved by using similar computations. □

Proof of Theorem 4.4.15. The first part of the theorem is just a restatement of Proposition 4.4.3 and Corollary 4.4.13. Assume that $\mathcal{G}$ is regular. In particular, by Corollary 3.2.9 4 we have $[\lambda(\widehat{S})L(S)]\subset\mathcal{K}$. Since R is non-degenerate we have $R(S)\mathcal{K}=\mathcal{K}$. Therefore, by the continuity of the action (δ_A,β_A) we have $D\subset q^{\beta_A,\widehat{\alpha}}(A\otimes\mathcal{K})$. Hence, $D\subset q^{\beta_A,\widehat{\alpha}}(A\otimes\mathcal{K})q^{\beta_A,\widehat{\alpha}}$ as $q^{\beta_A,\widehat{\alpha}}Dq^{\beta_A,\widehat{\alpha}}=D$ (*cf.* Proposition-Definition 4.4.1 1). Since δ_0 is injective, it suffices to show that

$$\delta_0(q^{\beta_A,\widehat{\alpha}}(A\otimes\mathcal{K})q^{\beta_A,\widehat{\alpha}})\subset\delta_0(D)$$

to prove the converse inclusion. In virtue of the continuity of the action and the fact that $\mathcal{K} = R(S)\mathcal{K}$, we have

$$\begin{aligned} q^{\beta_A,\widehat{\alpha}}(A \otimes \mathcal{K})q^{\beta_A,\widehat{\alpha}} &\subset [q^{\beta_A,\widehat{\alpha}}\pi_R(a)(1_A \otimes k)q^{\beta_A,\widehat{\alpha}}\,;\, a \in A,\, k \in \mathcal{K}] \\ &= [\pi_R(a)q^{\beta_A,\widehat{\alpha}}(1_A \otimes k)q^{\beta_A,\widehat{\alpha}}\,;\, a \in A,\, k \in \mathcal{K}]. \end{aligned}$$

We have $\delta_0(q^{\beta_A,\widehat{\alpha}}) = q_{13}^{\beta_A,\alpha}q_{23}^{\widehat{\alpha},\beta}$ (see Lemma 4.4.10). For all $a \in A$, $k \in \mathcal{K}$, we then have

$$\delta_0(\pi_R(a)q^{\beta_A,\widehat{\alpha}}(1_A \otimes k)q^{\beta_A,\widehat{\alpha}}) = \delta_0(\pi_R(a))q_{13}^{\beta_A,\alpha}q_{23}^{\widehat{\alpha},\beta}(1_A \otimes k \otimes 1_S)q_{23}^{\widehat{\alpha},\beta}q_{13}^{\beta_A,\alpha}$$

since $\delta_0(1_A \otimes k) = q_{13}^{\beta_A,\alpha}(1_A \otimes k \otimes 1_S)$ (see proof of Lemma 4.4.8) and $[q_{13}^{\beta_A,\alpha}, q_{23}^{\widehat{\alpha},\beta}] = 0$. Now, in virtue of the regularity of $\mathcal{G}$, we have $[\lambda(\widehat{S})L(S)] = \mathcal{K}_{\widehat{\alpha}}$ by Corollary 3.2.9 4. By applying Proposition 4.4.16 1 to $\pi := \widehat{\alpha}$ and $\gamma := \beta$ ($H := \mathscr{H} =: K$), we have

$$q^{\widehat{\alpha},\beta}(\mathcal{K} \otimes 1_S)q^{\widehat{\alpha},\beta} = q^{\widehat{\alpha},\beta}(\mathcal{K}_{\widehat{\alpha}} \otimes 1_S) = q^{\widehat{\alpha},\beta}([\lambda(\widehat{S})L(S)] \otimes 1_S).$$

Then, it follows that $\delta_0(\pi_R(a)q^{\beta_A,\widehat{\alpha}}(1_A \otimes k)q^{\beta_A,\widehat{\alpha}})$ belongs to

$$\begin{aligned} &[\delta_0(\pi_R(b))q_{13}^{\beta_A,\alpha}q_{23}^{\widehat{\alpha},\beta}(1_A \otimes \lambda(x)L(y) \otimes 1_S)q_{13}^{\beta_A,\alpha}\,;\, b \in A,\, x \in \widehat{S},\, y \in S] \\ &\qquad = [\delta_0(\pi_R(b))q_{13}^{\beta_A,\alpha}q_{23}^{\widehat{\alpha},\beta}(1_A \otimes \lambda(x)L(y) \otimes 1_S)\,;\, b \in A,\, x \in \widehat{S},\, y \in S], \end{aligned}$$

for all $a \in A$ and $k \in \mathcal{K}$. Finally, we have

$$\delta_0(\pi_R(b))q_{13}^{\beta_A,\alpha}q_{23}^{\widehat{\alpha},\beta}(1_A \otimes \lambda(x)L(y) \otimes 1_S) = \delta_0(\pi_R(b)(1_A \otimes \lambda(x)L(y))) \in \delta_0(D),$$

for all $b \in A$, $x \in \widehat{S}$ and $y \in S$. □

Proposition 4.4.17. *Let (A, δ_A, β_A) and (C, δ_C, β_C) be $\mathcal{G}$-C*-algebras. Let us denote D_A and D_C the $\mathcal{G}$-C*-algebras defined by:*

$$\begin{aligned} D_A &:= [(\mathrm{id}_A \otimes R)\delta_A(a)(1_A \otimes \lambda(x)L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S], \\ D_C &:= [(\mathrm{id}_C \otimes R)\delta_C(c)(1_C \otimes \lambda(x)L(y))\,;\, c \in C,\, x \in \widehat{S},\, y \in S]. \end{aligned}$$

*Let $f : A \to \mathcal{M}(C)$ be a non-degenerate $\mathcal{G}$-equivariant *-homomorphism. Then, there exists a unique non-degenerate $\mathcal{G}$-equivariant *-homomorphism $g : D_A \to \mathcal{M}(D_C)$ such that:*

$$g((\mathrm{id}_A \otimes R)\delta_A(a)(1_A \otimes \lambda(x)L(y))) = (\mathrm{id}_C \otimes R)\delta_C(f(a))(1_C \otimes \lambda(x)L(y)), \quad a \in A,\, x \in \widehat{S},\, y \in S.$$

Proof. This is a straightforward consequence of Theorems 4.3.3 2, 4.3.6 2 and 4.4.15. □

Remark 4.4.18. In virtue of Remark 4.3.4, the above result holds true for $\mathcal{G}$-equivariant *-homomorphisms $f : A \to \mathcal{M}(C)$ such that there exists $e \in \mathcal{M}(C)$ satisfying $[f(A)C] = eC$. Note also that we have $g((\mathrm{id}_A \otimes R)\delta_A(a)) = (\mathrm{id}_C \otimes R)\delta_C(f(a))$ for all $a \in A$.

Chapter 5

Continuous actions of colinking measured quantum groupoids

In the *whole* chapter, we will fix

$$\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$$

a colinking measured quantum groupoid associated with two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$.

5.1 Preliminaries

Notations 5.1.1. For the convenience of the reader, we recall some notations from §2.5:

- Let $\alpha, \beta : \mathbb{C}^2 \to \mathcal{M}(S)$ be the source and target maps of $\mathcal{G}$.
- Let $(\varepsilon_1, \varepsilon_2)$ be the canonical basis of the vector space $\mathbb{C}^2$. For $i, j = 1, 2$, we define the following central self-adjoint projection $p_{ij} = \alpha(\varepsilon_i)\beta(\varepsilon_j) \in \mathcal{Z}(\mathcal{M}(S))$. It follows from $\beta(\varepsilon_1) + \beta(\varepsilon_2) = 1_S$ and $\alpha(\varepsilon_1) + \alpha(\varepsilon_2) = 1_S$ that:

$$\alpha(\varepsilon_i) = p_{i1} + p_{i2}, \quad i = 1, 2 \quad ; \quad \beta(\varepsilon_j) = p_{1j} + p_{2j}, \quad j = 1, 2.$$

- Let us denote $S_{ij} = p_{ij}S$, for $i, j = 1, 2$. Then, S_{ij} is a C*-subalgebra (actually a closed two-sided ideal) of S. In order to provide a description of δ, for all $i, j, k = 1, 2$ we consider

$$\iota_{ij}^k : \mathcal{M}(S_{ik} \otimes S_{kj}) \to \mathcal{M}(S \otimes S)$$

the unique strictly continuous extension of the inclusion $S_{ik} \otimes S_{kj} \subset S \otimes S$ satisfying $\iota_{ij}^k(1_{S_{ik}\otimes S_{kj}}) = p_{ik} \otimes p_{kj}$. Now, let $\delta_{ij}^k : S_{ij} \to \mathcal{M}(S_{ik} \otimes S_{kj})$ be the unique *-homomorphism such that

$$\iota_{ij}^k \circ \delta_{ij}^k(s_{ij}) = (p_{ik} \otimes p_{kj})\delta(s_{ij}), \quad s_{ij} \in S_{ij}.$$

In this paragraph, we will give an equivalent description of the $\mathcal{G}$-C*-algebras in terms of $\mathbb{G}_1$-C*-algebras and $\mathbb{G}_2$-C*-algebras. Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. Then, $\beta_A : \mathbb{C}^2 \to \mathcal{M}(A)$ is a unital *-homomorphism and $\delta_A : A \to \mathcal{M}(A \otimes S)$ is an injective *-homomorphism satisfying the conditions of Definition 4.1.2. Note that:

- the fibration map β_A is central, that is $\beta_A(\mathbb{C}^2) \subset \mathcal{Z}(\mathcal{M}(A))$. Indeed, let $n \in \mathbb{C}^2$. Since $\beta(n) \in \mathcal{Z}(\mathcal{M}(S))$, we have

$$\delta_A(\beta_A(n)a) = \delta_A(1_A)(1_A \otimes \beta(n))\delta_A(a) = \delta_A(a)\delta_A(1_A)(1_A \otimes \beta(n)) = \delta_A(a\beta_A(n)),$$

for all $a \in A$. We then have $[\beta_A(n), a] = 0$ for all $a \in A$ by faithfulness of δ_A. Hence, $[\beta_A(n), m] = 0$ for all $m \in \mathcal{M}(A)$.

- Let us denote $q_i = \beta_A(\varepsilon_i)$ for $i = 1, 2$. Then, q_i is a central self-adjoint projection of $\mathcal{M}(A)$ and $q_1 + q_2 = 1_A$. Let us also denote $A_i = q_i A$, $i = 1, 2$. Then, A_i is a C*-subalgebra (actually a closed two-sided ideal) of A and we have $A = A_1 \oplus A_2$.

- We have
$$q^{\beta_A,\alpha} = q_1 \otimes \alpha(\varepsilon_1) + q_2 \otimes \alpha(\varepsilon_2). \tag{5.1.1}$$
Then, we obtain
$$\delta_A(q_j) = q^{\beta_A,\alpha}(1_A \otimes \beta(\varepsilon_j)) = \sum_{k=1,2} q_k \otimes p_{kj}, \quad j = 1, 2. \tag{5.1.2}$$
Note that $q_j \neq 0$, $j = 1, 2$ (unless $A = \{0\}$). In particular, the fibration map β_A is faithful.

- For $j, k = 1, 2$, we denote $\pi_j^k : \mathcal{M}(A_k \otimes S_{kj}) \to \mathcal{M}(A \otimes S)$ the unique strictly continuous extension of the inclusion $A_k \otimes S_{kj} \subset A \otimes S$ satisfying $\pi_j^k(1_{A_k \otimes S_{kj}}) = q_k \otimes p_{kj}$.

In case of ambiguity, we will denote $\pi_{A,j}^k$ and $q_{A,j}$ instead of π_j^k and q_j.

Lemma 5.1.2. *For all $a \in A$ and $j, k = 1, 2$, we have*
$$(q_k \otimes 1_S)\delta_A(q_j a) = (1_A \otimes \alpha(\varepsilon_k))\delta_A(q_j a) = (q_k \otimes p_{kj})\delta_A(a).$$

Proof. Straightforward consequence of (5.1.2). □

Proposition 5.1.3. *For all $j, k = 1, 2$, there exists a unique faithful non-degenerate *-homomorphism*
$$\delta_{A_j}^k : A_j \to \mathcal{M}(A_k \otimes S_{kj})$$
such that for all $x \in A_j$, we have
$$\pi_j^k \circ \delta_{A_j}^k(x) = (q_k \otimes p_{kj})\delta_A(x) = (q_k \otimes 1_S)\delta_A(x) = (1_A \otimes \alpha(\varepsilon_k))\delta_A(x) = (1_A \otimes p_{kj})\delta_A(x).$$
Moreover, we have:

1. $\delta_A(a) = \sum_{k,j=1,2} \pi_j^k \circ \delta_{A_j}^k(q_j a)$, *for all $a \in A$.*

2. $(\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})\delta_{A_j}^k = (\mathrm{id}_{A_l} \otimes \delta_{lj}^k)\delta_{A_j}^l$, *for all $j, k, l = 1, 2$.*

3. $[\delta_{A_j}^k(A_j)(1_{A_k} \otimes S_{kj})] = A_k \otimes S_{kj}$, *for all $j, k = 1, 2$. In particular, we have*
$$\mathcal{M}(\delta_{A_j}^k(A_j)) \subset \mathcal{M}(A_k \otimes S_{kj}), \quad A_k = [(\mathrm{id}_{A_k} \otimes \omega)\delta_{A_j}^k(a_j)\,;\, a_j \in A_j,\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*].$$

4. $\delta_{A_j}^j : A_j \to \mathcal{M}(A_j \otimes S_{jj})$ *is a continuous action of $\mathbb{G}_j$ on A_j.*

Proof. First, we have $\mathrm{Ran}(\pi_j^k) = (q_k \otimes p_{kj})\mathcal{M}(A \otimes S)$. Indeed, since $\pi_j^k(1_{A_k \otimes S_{kj}}) = q_k \otimes p_{kj}$ we have $\mathrm{Ran}(\pi_j^k) \subset (q_k \otimes p_{kj})\mathcal{M}(A \otimes S)$. Besides, $(q_k \otimes p_{kj})(A \otimes S) = A_k \otimes S_{kj} \subset \mathrm{Ran}(\pi_k^j)$. Then, the inclusion $(q_k \otimes p_{kj})\mathcal{M}(A \otimes S) \subset \mathrm{Ran}(\pi_j^k)$ follows since π_j^k is strictly continuous. Therefore, in virtue of the injectivity of π_j^k, there exists a unique *-homomorphism $\delta_{A_j}^k : A_j \to \mathcal{M}(A_k \otimes S_{kj})$ such that $(q_k \otimes p_{kj})\delta_A(x) = \pi_j^k \circ \delta_{A_j}^k(x)$ for all $x \in A_j$. In virtue of Lemma 5.1.2, the first formulas are then proved. Moreover, by (5.1.2) for all $x \in A_j$ we have
$$\delta_A(x) = \delta_A(q_j x) = \sum_{k=1,2} (q_k \otimes p_{kj})\delta_A(x) = \sum_{k=1,2} \pi_j^k \circ \delta_{A_j}^k(x).$$

Hence, if $a \in A$, we have

$$\delta_A(a) = \delta_A(q_1 a) + \delta_A(q_2 a) = \sum_{j,k=1,2} \pi_j^k \circ \delta_{A_j}^k(q_j a).$$

The first statement is proved. It follows from the non-degeneracy of $\delta_{A_k}^l$ that $\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}}$ extends uniquely to a strictly continuous unital *-homomorphism from $\mathcal{M}(A_k \otimes S_{kj})$ to $\mathcal{M}(A_l \otimes S_{lk} \otimes S_{kj})$ for all $j, k, l = 1, 2$. By strict continuity, we obtain the following formulas:

$$(\pi_k^l \otimes \mathrm{id}_{S_{kj}})(\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})(T) = (q_l \otimes p_{lk} \otimes p_{kj})(\delta_A \otimes \mathrm{id}_S)(\pi_j^k(T)), \quad T \in \mathcal{M}(A_k \otimes S_{kj}),$$

$$(\pi_k^l \otimes \mathrm{id}_{S_{kj}})(\mathrm{id}_{A_l} \otimes \delta_{lj}^k)(T) = (q_l \otimes p_{lk} \otimes p_{kj})(\mathrm{id}_A \otimes \delta)(\pi_j^l(T)), \quad T \in \mathcal{M}(A_l \otimes S_{lj}).$$

Therefore, the second statement follows from these formulas, the coassociativity of δ_A and the faithfulness of $\pi_k^l \otimes \mathrm{id}_{S_{kj}}$. Let $j, k, l = 1, 2$ and $a_j \in A_j$. If $\delta_{A_j}^k(a_j) = 0$, then $\delta_{A_j}^l(a_j) = 0$ for $l = 1, 2$ in virtue of the injectivity of δ_{lj}^k and the equality $(\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})\delta_{A_j}^k = (\mathrm{id}_{A_l} \otimes \delta_{lj}^k)\delta_{A_j}^l$. By using the first statement, we have $\delta_A(a_j) = 0$. Hence, $a_j = 0$ and $\delta_{A_j}^k$ is injective. The third statement is an immediate consequence of the continuity of the action (δ_A, β_A). By the non-degeneracy of $S_{kj} \subset \mathcal{B}(\mathscr{H}_{kj})$, we obtain the equality

$$A_k = [(\mathrm{id}_{A_k} \otimes \omega)(\delta_{A_j}^k(a_j))\,;\, a_j \in A_j,\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*].$$

The equality $[\delta_{A_j}^k(A_j)(1_{A_k} \otimes S_{kj})] = A_k \otimes S_{kj}$ implies that

$$[\delta_{A_j}^k(A_j)(A_k \otimes S_{kj})] = A_k \otimes S_{kj} = [(A_k \otimes S_{kj})\delta_{A_j}^k(A_j)],$$

which proves that $\mathcal{M}(\delta_{A_j}^k(A_j)) \subset \mathcal{M}(A_k \otimes S_{kj})$. The last statement is then straightforward. □

Corollary 5.1.4. *For $j, k = 1, 2$, $j \neq k$, we have:*

1. *If $x \in \delta_{A_j}^k(A_j)$, we have $(\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x) \in \mathcal{M}(\delta_{A_j}^k(A_j) \otimes S_{jj})$.*
2. *The map*
$$\begin{array}{ccc} \delta_{A_j}^k(A_j) & \longrightarrow & \mathcal{M}(\delta_{A_j}^k(A_j) \otimes S_{jj}) \\ x & \longmapsto & (\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x) \end{array}$$
is a continuous action of the quantum group $\mathbb{G}_j$ on the C-algebra $\delta_{A_j}^k(A_j)$.*
3. *Moreover, the map $A_j \to \delta_{A_j}^k(A_j)\,;\, a \mapsto \delta_{A_j}^k(a)$ is a $\mathbb{G}_j$-equivariant *-isomorphism.*

Proof. 1. Let us fix $x \in \delta_{A_j}^k(A_j)$ and let us write $x = \delta_{A_j}^k(a)$, for $a \in A_j$. By Proposition 5.1.3 2, we have $(\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x) = (\delta_{A_j}^k \otimes \mathrm{id}_{S_{jj}})\delta_{A_j}^j(a)$. Moreover, since $\delta_{A_j}^j(a) \in \mathcal{M}(A_j \otimes S_{jj})$ we have

$$(\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x)(\delta_{A_j}^k(a') \otimes y) = (\delta_{A_j}^k \otimes \mathrm{id}_{S_{jj}})(\delta_{A_j}^j(a)(a' \otimes y)) \in \delta_{A_j}^k(A_j) \otimes S_{jj}, \quad a' \in A_j,\, y \in S_{jj}.$$

In a similar way, we have $(\delta_{A_j}^k(a') \otimes y)(\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x) \in \delta_{A_j}^k(A_j) \otimes S_{jj}$.
2. The map $\delta_{A_j}^k(A_j) \to \mathcal{M}(\delta_{A_j}^k(A_j) \otimes S_{jj})\,;\, x \mapsto (\mathrm{id}_{A_k} \otimes \delta_{kj}^j)(x)$ is a faithful *-homomorphism. The coassociativity is an immediate consequence of the formula $(\mathrm{id}_{S_{kj}} \otimes \delta_{jj}^j)\delta_{kj}^j = (\delta_{kj}^j \otimes \mathrm{id}_{S_{jj}})\delta_{kj}^j$. The continuity is a straightforward consequence of the continuity of the action $\delta_{A_j}^j$ and the formula $(\delta_{A_j}^k \otimes \mathrm{id}_{S_{jj}})\delta_{A_j}^j = (\mathrm{id}_{A_k} \otimes \delta_{kj}^j)\delta_{A_j}^j$ (Proposition 5.1.3 2,4).
3. This follows from the faithfulness of $\delta_{A_j}^k$ and the formula $(\delta_{A_j}^k \otimes \mathrm{id}_{S_{jj}})\delta_{A_j}^j = (\mathrm{id}_{A_k} \otimes \delta_{kj}^j)\delta_{A_j}^k$. □

Examples 5.1.5. Let us consider the basic examples:

1. In the case of the trivial action of $\mathcal{G}$ on $N := \mathbb{C}^2$, the C*-algebras A_i are identified to $\mathbb{C}$ and Proposition 5.1.3 4 corresponds exactly to the trivial action of $\mathbb{G}_j$ on $\mathbb{C}$.

2. Let us consider the $\mathcal{G}$-C*-algebra (S, δ, β). Then, we have

$$q_j := \beta(\varepsilon_j) = p_{1j} + p_{2j}, \quad S_j := S_{1j} \oplus S_{2j}, \quad \delta^k_{S_j} := \delta^k_{1j} \oplus \delta^k_{2j}.$$

Remark 5.1.6. In the regular case, we will show that the $\mathbb{G}_j$-C*-algebra $\delta^k_{A_j}(A_j)$ described in Corollary 5.1.4 can be obtained directly by deformation of the $\mathbb{G}_k$-C*-algebra A_k.

From this concrete description of $\mathcal{G}$-C*-algebras we can also give a convenient description of the $\mathcal{G}$-equivariant *-homomorphisms. With the above notations, we have the following result:

Lemma 5.1.7. *Let (A, δ_A, β_A) and (B, δ_B, β_B) be two $\mathcal{G}$-C*-algebras. Let $\iota_k : \mathcal{M}(B_k) \to \mathcal{M}(B)$ be the unique strictly continuous extension of the inclusion map $B_k \subset B$ such that $\iota_k(1_{B_k}) = q_{B,k}$, for $k = 1, 2$.*

1. *Let $f : A \to \mathcal{M}(B)$ be a non-degenerate $\mathcal{G}$-equivariant *-homomorphism. Then, for all $j = 1, 2$, there exists a unique non-degenerate *-homomorphism $f_j : A_j \to \mathcal{M}(B_j)$ such that for $k = 1, 2$ we have:*

$$(f_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{A_j} = \delta^k_{B_j} \circ f_j. \tag{5.1.3}$$

   Moreover, we have $f(a) = \iota_1 \circ f_1(aq_{A,1}) + \iota_2 \circ f_2(aq_{A,2})$ for all $a \in A$.

2. *Conversely, let $f_j : A_j \to \mathcal{M}(B_j)$, for $j = 1, 2$, be non-degenerate *-homomorphisms such that (5.1.3) holds for all $j, k = 1, 2$. Then, the map $f : A \to \mathcal{M}(B)$, given for all $a \in A$ by*

$$f(a) := \iota_1 \circ f_1(aq_{A,1}) + \iota_2 \circ f_2(aq_{A,2}),$$

   *is a non-degenerate $\mathcal{G}$-equivariant *-homomorphism.*

Proof. Since $f \circ \beta_A = \beta_B$ we have $f(q_{A,j}a) = q_{B,j}f(a) \in q_{B,j}\mathcal{M}(B) = \iota_j(\mathcal{M}(B_j))$, $j = 1, 2$. By faithfulness of ι_j, there exists a unique *-homomorphism $f_j : A_j \to \mathcal{M}(B_j)$ such that we have $f(a_j) = \iota_j \circ f_j(a_j)$ for all $a_j \in A_j$. It then follows from the equality $A = A_1 \oplus A_2$ that $f(a) = \iota_1 \circ f_1(q_{A,1}a) + \iota_2 \circ f_2(q_{A,2}a)$ for all $a \in A$. Besides, by Proposition 5.1.3 2, we have $\delta_B \circ \iota_j = \sum_k \pi^k_{B,j} \circ \delta^k_{B_j}$. Therefore, we have

$$\delta_B(f(a_j)) = \delta_B(\iota_j(f_j(a_j))) = \sum_{k=1,2} \pi^k_{B,j} \circ \delta^k_{B_j}(f_j(a_j)), \quad a_j \in A_j.$$

We also have $(f \otimes \mathrm{id}_S) \circ \pi^k_{A,j} = \pi^k_{B,j} \circ (f_k \otimes \mathrm{id}_{S_{kj}})$. Hence, we obtain

$$(f \otimes \mathrm{id}_S)\delta_A(a_j) = \sum_{k=1,2} (f \otimes \mathrm{id}_S) \circ \pi^k_{A,j} \circ \delta^k_{A_j}(a_j) = \sum_{k=1,2} \pi^k_{B,j} \circ (f_k \otimes \mathrm{id}_{S_{kj}})(\delta^k_{A_j}(a_j)), \quad a_j \in A_j.$$

Note that $(q_{B,i} \otimes 1_S)\pi^k_{B,j}(m) = \delta^k_i \pi^k_{B,j}(m)$ for all $i, k = 1, 2$ and $m \in \mathcal{M}(B_k \otimes S_{kj})$. It follows from $(f \otimes \mathrm{id}_S) \circ \delta_A = \delta_B \circ f$ that $\pi^k_{B,j} \circ (f_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{A_j} = \pi^k_{B,j} \circ \delta^k_{B_j} \circ f_j$. Hence, $(f_k \otimes \mathrm{id}_{S_{kj}}) \circ \delta^k_{A_j} = \delta^k_{B_j} \circ f_j$ by faithfulness of $\pi^k_{B,j}$. The second statement is easily verified. □

Therefore, we can state the following result:

Corollary 5.1.8. *With the previous notations, for $i = 1, 2$, the correspondence*

$$\mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \longrightarrow \mathbb{G}_i\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \ ; \ (A, \delta_A, \beta_A) \longmapsto (A_i, \delta^i_{A_i}), \ \ f \in \mathrm{Mor}_{\mathcal{G}}(A, B) \longmapsto f_i \in \mathrm{Mor}_{\mathbb{G}_i}(A_i, B_i)$$

is a functor from the category of $\mathcal{G}$-C-algebras to the category of $\mathbb{G}_i$-C*-algebras.*

In order to prove that the correspondences $\mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_i\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}$, $i = 1, 2$, are equivalences of categories, we will need the following preparatory lemma:

Lemma 5.1.9. *Let A_1 and A_2 be two C*-algebras endowed with non-degenerate faithful *-homomorphisms $\delta^k_{A_j} : A_j \to \mathcal{M}(A_k \otimes S_{kj})$ satisfying the conditions 2 and 3 of Proposition 5.1.3. Let us denote $A = A_1 \oplus A_2$ and let $\pi^k_j : \mathcal{M}(A_k \otimes S_{kj}) \to \mathcal{M}(A \otimes S)$ be the strictly continuous *-homomorphism extending the canonical injection $A_k \otimes S_{kj} \to A \otimes S$. Let $\delta_A : A \to \mathcal{M}(A \otimes S)$ and $\beta_A : \mathbb{C}^2 \to \mathcal{M}(A)$ be the *-homomorphisms given by:*

$$\delta_A(a) = \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{A_j}(a_j), \quad a = (a_1, a_2) \in A \quad ; \quad \beta_A(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad (\lambda, \mu) \in \mathbb{C}^2.$$

Then, (δ_A, β_A) is a continuous action of $\mathcal{G}$ on A.

Proof. First, let us introduce some further notations relevant for this proof. Let us fix $j, k = 1, 2$. Let $p_{A_k} : A \to A_k$ be the canonical surjection, *i.e.* $p_{A_k}(a) = a_k$ for $a = (a_1, a_2) \in A$. Note that p_{A_k} is a non-degenerate *-homomorphism, then it extends to a strictly continuous unital *-homomorphism from $\mathcal{M}(A)$ to $\mathcal{M}(A_k)$. Let $\jmath_{A_k} : \mathcal{M}(A_k) \to \mathcal{M}(A)$ be the unique strictly continuous extension of the canonical injection of A_k into A. We will denote $q_k = \jmath_{A_k}(1_{A_k})$. Let $\jmath_{S_{kj}} : \mathcal{M}(S_{kj}) \to \mathcal{M}(S)$ be the unique strictly continuous extension of the inclusion $S_{kj} \subset S$ such that $\jmath_{S_{kj}}(1_{S_{kj}}) = p_{kj}$.

The fibration map β_A is given by $\beta_A(\lambda, \mu) = \lambda q_1 + \mu q_2$ for all $(\lambda, \mu) \in \mathbb{C}^2$. Then, β_A is a *-homomorphism since q_1 and q_2 are two orthogonal self-adjoint projections and it is also clear that β_A is non-degenerate.
Let us prove that δ_A is injective. We have $(1_A \otimes p_{il})\pi^k_j(x) = \delta^k_i \delta^j_l \pi^k_j(x)$ for all $x \in \mathcal{M}(A_k \otimes S_{kj})$ and $i, j, k, l = 1, 2$. Hence, $(1_A \otimes p_{kj})\delta_A(a) = \pi^k_j(\delta^k_{A_j}(a_j))$ for all $a = (a_1, a_2) \in A$ and $j, k = 1, 2$. In particular, if $\delta_A(a) = 0$ for some $a = (a_1, a_2) \in A$, then we have $\pi^k_j(\delta^k_{A_j}(a_j)) = 0$ for all $j, k = 1, 2$. Therefore, $a_j = 0$ for $j = 1, 2$ by faithfulness of $\delta^k_{A_j}$ and π^k_j. Hence, $a = 0$.
We have $\delta_A = \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{A_j} \circ p_{A_j}$, then δ_A extends uniquely to a strictly continuous *-homomorphism from $\mathcal{M}(A)$ to $\mathcal{M}(A \otimes S)$. Besides, we have

$$\delta_A(m) = \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{A_j}(m_j), \quad m = (m_1, m_2) \in \mathcal{M}(A_1) \oplus \mathcal{M}(A_2).$$

In particular, since $1_A = (1_{A_1}, 1_{A_2})$ we have

$$\delta_A(1_A) = \sum_{k,j=1,2} \pi^k_j(\delta^k_{A_j}(1_{A_j})) = \sum_{k,j=1,2} \jmath_{A_k}(1_{A_k}) \otimes p_{kj} = \sum_{k=1,2} q_k \otimes (p_{k1} + p_{k2}) = \sum_{k=1,2} \beta_A(\varepsilon_k) \otimes \alpha(\varepsilon_k).$$

We also have $\delta_A(\jmath_{A_j}(m_j)) = \sum_{k=1,2} \pi^k_j(\delta^k_{A_j}(m_j))$ for all $m_j \in \mathcal{M}(A_j)$, $j = 1, 2$. In particular,

$$\delta_A(q_j) = \sum_{k=1,2} \pi^k_j(\delta^k_{A_j}(1_{A_j})) = \sum_{k=1,2} \pi^k_j(1_{A_k \otimes S_{kj}}) = \sum_{k=1,2} q_k \otimes p_{kj} = \delta_A(1_A)(1_A \otimes \beta(\varepsilon_j)).$$

Let us now prove the coassociativity of δ_A. We have $(p_{A_i} \otimes \mathrm{id}_S)\pi_j^k(x) = \delta_i^k(\mathrm{id}_{A_k} \otimes \jmath_{S_{kj}})(x)$ for all $x \in \mathcal{M}(A_k \otimes S_{kj})$ and $i, j, k = 1, 2$. We also have

$$(\delta_A \otimes \mathrm{id}_S)(x) = \sum_{i,l=1,2} (\pi_i^l \otimes \mathrm{id}_S)(\delta_{A_i}^l \otimes \mathrm{id}_S)(p_{A_i} \otimes \mathrm{id}_S)(x), \quad x \in \mathcal{M}(A \otimes S).$$

Let $a = (a_1, a_2) \in A$, we then have

$$\begin{aligned}
(\delta_A \otimes \mathrm{id}_S)\delta_A(a) &= \sum_{k,j=1,2} (\delta_A \otimes \mathrm{id}_S)(\pi_j^k(\delta_{A_j}^k(a_j))) \\
&= \sum_{i,j,k,l=1,2} (\pi_i^l \otimes \mathrm{id}_S)(\delta_{A_i}^l \otimes \mathrm{id}_S)(p_{A_i} \otimes \mathrm{id}_S)\pi_j^k(\delta_{A_j}^k(a_j)) \\
&= \sum_{j,k,l=1,2} (\pi_k^l \otimes \jmath_{S_{kj}})(\delta_{A_k}^l \otimes \mathrm{id}_{S_{kj}})\delta_{A_j}^k(a_j) \\
&= \sum_{j,k,l=1,2} (\pi_k^l \otimes \mathrm{id}_S)(\mathrm{id}_{A_l} \otimes (\mathrm{id}_{S_{lk}} \otimes \jmath_{S_{kj}})\delta_{lj}^k)\delta_{A_j}^l(a_j).
\end{aligned}$$

Besides, $\jmath_{A_l} \otimes \jmath_{S_{lk}} = \pi_k^l$ and $\jmath_{S_{lk}} \otimes \jmath_{S_{kj}} = \iota_{lj}^k$. Then, we have

$$\begin{aligned}
(\pi_k^l \otimes \mathrm{id}_S)(\mathrm{id}_{A_l} \otimes (\mathrm{id}_{S_{lk}} \otimes \jmath_{S_{kj}})\delta_{lj}^k)\delta_{A_j}^l(a_j) &= (\jmath_{A_l} \otimes \iota_{lj}^k \circ \delta_{lj}^k)\delta_{A_j}^l(a_j) \\
&= (1_A \otimes p_{lk} \otimes p_{kj})(\jmath_{A_l} \otimes \delta \circ \jmath_{S_{lj}})\delta_{A_j}^l(a_j) \\
&= (1_A \otimes p_{lk} \otimes p_{kj})(\mathrm{id}_A \otimes \delta)\pi_j^l(\delta_{A_j}^l(a_j)).
\end{aligned}$$

However, we have $\delta(p_{lj}) = p_{l1} \otimes p_{1j} + p_{l2} \otimes p_{2j}$ and $(1_A \otimes p_{lj})\pi_j^l(x) = \pi_j^l(x)$ for all $x \in \mathcal{M}(A_l \otimes S_{lj})$. Therefore, we obtain

$$\begin{aligned}
(\delta_A \otimes \mathrm{id}_S)\delta_A(a) &= \sum_{j,k,l=1,2} (1_A \otimes p_{lk} \otimes p_{kj})(\mathrm{id}_A \otimes \delta)\pi_j^l(\delta_{A_j}^l(a_j)) \\
&= \sum_{j,l=1,2} (1_A \otimes \delta(p_{lj}))(\mathrm{id}_A \otimes \delta)\pi_j^l(\delta_{A_j}^l(a_j)) \\
&= \sum_{j,l=1,2} (\mathrm{id}_A \otimes \delta)\pi_j^l(\delta_{A_j}^l(a_j)) \\
&= (\mathrm{id}_A \otimes \delta)\delta_A(a).
\end{aligned}$$

It only remains to show that the action (δ_A, β_A) is continuous. We have:

- If $x \in A \otimes S$, we have

  $$\delta_A(1_A)x = \sum_{k,j=1,2} (q_k \otimes p_{kj})x = \sum_{k,j=1,2} \pi_j^k((p_{A_k} \otimes L_{kj})(x)),$$

  where $L_{kj} : S \to S_{kj}$ is the non-degenerate *-homomorphism given by $L_{kj}(y) = p_{kj}y$, $y \in S$.

- If $a \in A$ and $y \in S$, we have

  $$\delta_A(a)(1_A \otimes y) = \sum_{k,j=1,2} \pi_j^k(\delta_{A_j}^k(a_j)(1_{A_k} \otimes L_{kj}(y))).$$

We combine these statements with the fact that $[\delta_{A_j}^k(A_j)(1_{A_k} \otimes S_{kj})] = A_k \otimes S_{kj}$ for all $j, k = 1, 2$ to conclude that $[\delta_A(A)(1_A \otimes S)] = \delta_A(1_A)(A \otimes S)$. □

5.2 Morita equivalence between $A_1 \rtimes \mathbb{G}_1$ and $A_2 \rtimes \mathbb{G}_2$

Let (A, δ_A, β_A) be a continuous action of $\mathcal{G}$. In this paragraph, we provide a precise description of the crossed product $A \rtimes \mathcal{G}$ and we obtain a canonical Morita equivalence between the crossed products $A_1 \rtimes \mathbb{G}_1$ and $A_2 \rtimes \mathbb{G}_2$.

Since $\delta_A(1_A) = q_1 \otimes \alpha(\varepsilon_1) + q_2 \otimes \alpha(\varepsilon_2)$, we have:

$$\mathscr{E}_{A,L} = \mathscr{E}_{A,1} \oplus \mathscr{E}_{A,2},$$

where $\mathscr{E}_{A,i}$ is the Hilbert A_i-module $A_i \otimes \alpha(\varepsilon_i)\mathscr{H} = (A_i \otimes \mathscr{H}_{i1}) \oplus (A_i \otimes \mathscr{H}_{i2})$, $i = 1, 2$. Let us denote $B = A \rtimes \mathcal{G} \subset \mathcal{L}(\mathscr{E}_{A,L})$. We recall that $B = [\,\pi(a)\widehat{\theta}(x)\,;\, a \in A,\, x \in \widehat{S}\,]$, where $\pi : \mathcal{M}(A) \to \mathcal{M}(B)$ and $\widehat{\theta} : \mathcal{M}(\widehat{S}) \to \mathcal{M}(B)$ are given by:

$$\pi(m) = \pi_L(m)\restriction_{\mathscr{E}_{A,L}}, \quad m \in \mathcal{M}(A) \quad ; \quad \widehat{\theta}(x) = (1_A \otimes \rho(x))\restriction_{\mathscr{E}_{A,L}}, \quad x \in \mathcal{M}(\widehat{S}).$$

It is clear that π (resp. $\widehat{\theta}$) defines by restriction to $\mathscr{E}_{A,i}$ a faithful *-representation

$$\pi_i : A \to \mathcal{L}(\mathscr{E}_{A,i}) \quad (\text{resp. } \widehat{\theta}_i : \widehat{S} \to \mathcal{L}(\mathscr{E}_{A,i}))$$

of A (resp. $\widehat{S}$) on the Hilbert A_i-module $\mathscr{E}_{A,i}$. We have:

$$\pi_i(a) := \pi(a)\restriction_{\mathscr{E}_{A,i}} = \begin{pmatrix} (\mathrm{id}_{A_i} \otimes L_{i1})\delta^i_{A_1}(q_1 a) & 0 \\ 0 & (\mathrm{id}_{A_i} \otimes L_{i2})\delta^i_{A_2}(q_2 a) \end{pmatrix}, \quad a \in A,$$

where $L_{ij} : S_{ij} \to \mathcal{B}(\mathscr{H}_{ij})$ is the faithful *-homomorphism defined by $L_{ij}(y) = p_{ij}L(y)$, $y \in S_{ij}$. We also have:

$$\widehat{\theta}_i(x) := (1_A \otimes \rho(x))\restriction_{\mathscr{E}_{A,i}} = \begin{pmatrix} 1_{A_i} \otimes \widehat{\pi}_i(x_{11}) & 1_{A_i} \otimes \widehat{\pi}_i(x_{12}) \\ 1_{A_i} \otimes \widehat{\pi}_i(x_{21}) & 1_{A_i} \otimes \widehat{\pi}_i(x_{22}) \end{pmatrix}, \quad x \in \widehat{S}.$$

Let us denote π_B the faithful *-representation of B given by the inclusion $B \subset \mathcal{L}(\mathscr{E}_{A,L})$ (that is $\pi_B = \Psi_{L,\rho}$). Then, π_B defines by restriction to $\mathscr{E}_{A,i}$ a faithful *-representation $\pi_{B,i} : B \to \mathcal{L}(\mathscr{E}_{A,i})$ and we have $\pi_{B,i}(\pi(a)\widehat{\theta}(x)) = \pi_i(a)\widehat{\theta}_i(x)$ for all $a \in A$ and $x \in \widehat{S}$. Therefore, we have:

$$\pi_{B,i}(\pi(a)\widehat{\theta}(x)) = \begin{pmatrix} (\mathrm{id}_{A_i} \otimes L_{i1})\delta^i_{A_1}(q_1 a)(1_{A_i} \otimes \widehat{\pi}_i(x_{11})) & (\mathrm{id}_{A_i} \otimes L_{i1})\delta^i_{A_1}(q_1 a)(1_{A_i} \otimes \widehat{\pi}_i(x_{12})) \\ (\mathrm{id}_{A_i} \otimes L_{i2})\delta^i_{A_2}(q_2 a)(1_{A_i} \otimes \widehat{\pi}_i(x_{21})) & (\mathrm{id}_{A_i} \otimes L_{i2})\delta^i_{A_2}(q_2 a)(1_{A_i} \otimes \widehat{\pi}_i(x_{22})) \end{pmatrix},$$

for all $a \in A$ and $x \in \widehat{S}$. Let us denote

$$\mathcal{E}^i_{jk} := [\pi_i(q_j a)\widehat{\theta}_i(x_{jk})\,;\, a \in A,\, x \in \widehat{S}] = [(\mathrm{id}_{A_i} \otimes L_{ij})\delta^i_{A_j}(q_j a)(1_{A_i} \otimes \widehat{\pi}_i(x_{jk}))\,;\, a \in A,\, x \in \widehat{S}].$$

We have $\mathcal{E}^i_{jk} \subset \mathcal{L}(A_i \otimes \mathscr{H}_{ik}, A_i \otimes \mathscr{H}_{ij})$ and $\pi_{B,i}(B) = (\mathcal{E}^i_{jk})_{j,k=1,2}$. Furthermore, we have:

$$\mathcal{E}^i_{ii} = A_i \rtimes_{\delta^i_{A_i}} \mathbb{G}_i, \quad i = 1, 2.$$

Theorem 5.2.1. *For all $i, j, k, l = 1, 2$, we have the following statements:*

1. $(\mathcal{E}^i_{jk})^* = \mathcal{E}^i_{kj}$.
2. $[\mathcal{E}^i_{jk}\mathcal{E}^i_{kl}] = \mathcal{E}^i_{jl}$.
3. $[\mathcal{E}^i_{jk}(A_i \otimes \mathscr{H}_{ik})] = A_i \otimes \mathscr{H}_{ij}$.

For the proof, we will use Proposition 2.5.11 and the following result:

Lemma 5.2.2. *For $i, j, k = 1, 2$, we denote $\mathcal{I}^i_{jk}$ the canonical injection defined by the composition $\mathcal{L}(A_i \otimes \mathscr{H}_{ik}, A_i \otimes \mathscr{H}_{ij}) \hookrightarrow \mathcal{L}(\mathscr{E}_{A,i}) \hookrightarrow \mathcal{L}(\mathscr{E}_{A,L})$. Then, for all $a \in A$ and $x \in \widehat{S}$ we have:*

$$\mathcal{I}^i_{jk}(\widehat{\theta}_i(x_{jk})\pi_i(q_k a)) = (q_i \otimes p_{ij})\widehat{\theta}(x)\pi(a)(q_i \otimes p_{ik}),$$

$$\mathcal{I}^i_{jk}(\pi_i(q_j a)\widehat{\theta}_i(x_{jk})) = (q_i \otimes p_{ij})\pi(a)\widehat{\theta}(x)(q_i \otimes p_{ik}).$$

Proof. This a straightforward computation. □

Proof of Theorem 5.2.1. Let us prove the first statement. Let $a \in A$ and $x \in \widehat{S}$. We have $(\pi_i(q_j a)\widehat{\theta}_i(x_{jk}))^* = \widehat{\theta}_i((x^*)_{kj})\pi_i(q_j a^*)$. By Lemma 5.2.2, we then have

$$\mathcal{I}^i_{kj}((\pi_i(q_j a^*)\widehat{\theta}_i(x_{jk}))^*) = (q_i \otimes p_{ik})\widehat{\theta}(x^*)\pi(a^*)(q_i \otimes p_{ij}).$$

However, it follows from Lemma 4.2.2 that $\widehat{\theta}(x^*)\pi(a^*)$ is the norm limit of finite sums of the form $\sum_s \pi(a_s)\widehat{\theta}(x_s)$, where $a_s \in A$ and $x_s \in \widehat{S}$. By continuity of $\mathcal{I}^i_{kj}$, it then follows that $\mathcal{I}^i_{kj}((\pi_i(q_j a^*)\widehat{\theta}_i(x_{jk}))^*)$ is the norm limit of finite sums of the form

$$\sum_s (q_i \otimes p_{ik})\pi(a_s)\widehat{\theta}(x_s)(q_i \otimes p_{ij}) = \mathcal{I}^i_{kj}\left(\sum_s \pi_i(q_k a_s)\widehat{\theta}_i(x_{s,jk})\right), \quad \text{where} \quad a_s \in A,\, x_s \in \widehat{S}.$$

Since $\mathcal{I}^i_{kj}$ is isometric, it follows that $(\pi_i(q_j a)\widehat{\theta}_i(x_{jk}))^* \in \mathcal{E}^i_{kj}$. Hence, $(\mathcal{E}^i_{jk})^* \subset \mathcal{E}^i_{kj}$. We also have $(\mathcal{E}^i_{kj})^* \subset \mathcal{E}^i_{jk}$, hence $(\mathcal{E}^i_{jk})^* = \mathcal{E}^i_{kj}$.
Let us prove the second statement. Let us fix $i, j, k, l = 1, 2$. Since $\mathcal{E}^i_{kl} = (\mathcal{E}^i_{lk})^*$ and $\mathcal{E}^i_{jl} = (\mathcal{E}^i_{lj})^*$, we have

$$\mathcal{E}^i_{kl} = [\,\widehat{\theta}_i(x_{kl})\pi_i(q_l a)\,;\, x \in \widehat{S},\, a \in A\,], \tag{5.2.1}$$

$$\mathcal{E}^i_{jl} = [\,\widehat{\theta}_i(x_{jl})\pi_i(q_l a)\,;\, x \in \widehat{S},\, a \in A\,]. \tag{5.2.2}$$

Therefore, we have

$$\begin{aligned}
[\mathcal{E}^i_{jk}\mathcal{E}^i_{kl}] &= [\,\pi_i(q_j a)\widehat{\theta}_i(x_{jk})\widehat{\theta}_i(x'_{kl})\pi_i(q_l a')\,;\, a,\, a' \in A,\, x,\, x' \in \widehat{S}\,] && (5.2.1)\\
&\subset [\,\pi_i(q_j a)\widehat{\theta}_i(x_{jl})\pi_i(q_l a')\,;\, a,\, a' \in A,\, x \in \widehat{S}\,] && \text{(Proposition 2.5.11 3)}\\
&\subset [\pi_i(A_j)\mathcal{E}^i_{jl}] = \mathcal{E}^i_{jl}. && (5.2.2)
\end{aligned}$$

For the converse inclusion, let us fix $a \in A$ and $x \in \widehat{S}$. Let us prove that $\pi_i(q_j a)\widehat{\theta}_i(x_{jl}) \in [\mathcal{E}^i_{jk}\mathcal{E}^i_{kl}]$. In virtue of Proposition 2.5.11, we have $[E^i_{jk}E^i_{kl}] = E^i_{jl}$. Therefore, $\widehat{\theta}_i(x_{jl})$ is the norm limit of finite sums of the form $\sum_s \widehat{\theta}_i(x_{s,jk})\widehat{\theta}_i(x'_{s,kl})$, where $x_s,\, x'_s \in \widehat{S}$. Without loss of generality, we can assume that $\widehat{\theta}_i(x_{jl}) = \widehat{\theta}_i(x'_{jk})\widehat{\theta}_i(x''_{kl})$, where $x',\, x'' \in \widehat{S}$. Let us write $a = a_1 a_2$, where $a_1,\, a_2 \in A$. We then have $\pi_i(q_j a)\widehat{\theta}_i(x_{jl}) = \pi_i(q_j a_1)\pi_i(q_j a_2)\widehat{\theta}_i(x'_{jk})\widehat{\theta}_i(x''_{kl})$. Since $\pi_i(q_j a_2)\widehat{\theta}_i(x'_{jk}) \in \mathcal{E}^i_{jk} = (\mathcal{E}^i_{kj})^*$ we finally have $\pi_i(q_j a)\widehat{\theta}_i(x_{jl}) \in [\mathcal{E}^i_{jk}\mathcal{E}^i_{kl}]$.
Let us prove the last statement. The inclusion $[\mathcal{E}^i_{jk}(A_i \otimes \mathscr{H}_{ik})] \subset A_i \otimes \mathscr{H}_{ij}$ is straightforward. By the non-degeneracy of $\pi : A \to \mathcal{L}(\mathscr{E}_{A,L})$, we have $[\pi(A)\mathscr{E}_{A,L}] = \mathscr{E}_{A,L}$. Moreover, by Proposition 2.5.11, we also have $[E^i_{jk}\mathscr{H}_{ik}] = \mathscr{H}_{ij}$. It then follows that

$$[\pi_i(A_j)(A_i \otimes \mathscr{H}_{ij})] = A_i \otimes \mathscr{H}_{ij}, \quad A_i \otimes \mathscr{H}_{ij} = [\widehat{\theta}_i(\beta(\varepsilon_j)\widehat{S}\beta(\varepsilon_k))(A_i \otimes \mathscr{H}_{ik})].$$

We then combine these two formulas to obtain the converse inclusion. □

Corollary 5.2.3. *Let $i, j = 1, 2$. Then, $\mathcal{E}^i_{jj}$ is a C*-algebra and $\mathcal{E}^i_{ij}$ is a Morita equivalence between $\mathcal{E}^i_{ii} = A_i \rtimes_{\delta^i_{A_i}} \mathbb{G}_i$ and $\mathcal{E}^i_{jj}$.*

Proof. It follows from Theorem 5.2.1 that $\mathcal{E}^i_{jj}$ is a non-degenerate C*-subalgebra of $\mathcal{L}(A_i \otimes \mathscr{H}_{ij})$ and we can then consider $\mathcal{M}(\mathcal{E}^i_{jj})$ as a C*-subalgebra of $\mathcal{L}(A_i \otimes \mathscr{H}_{ij}) = \mathcal{M}(A_i \otimes \mathcal{K}(\mathscr{H}_{ij}))$. It also follows from the same theorem that $\mathcal{E}^i_{ij}$ is a $\mathcal{E}^i_{ii}$-$\mathcal{E}^i_{jj}$-bimodule and $\mathcal{E}^i_{ij}$ is a full right (resp. left) Hilbert $\mathcal{E}^i_{jj}$-module (resp. $\mathcal{E}^i_{ii}$-module) whose $\mathcal{E}^i_{jj}$-valued (resp. $\mathcal{E}^i_{ii}$-valued) inner product is given by $\langle \xi, \eta \rangle_{\mathcal{E}^i_{jj}} = \xi^* \circ \eta$ (resp. ${}_{\mathcal{E}^i_{ii}}\langle \xi, \eta \rangle = \xi \circ \eta^*$) for all $\xi, \eta \in \mathcal{E}^i_{ij}$. □

Proposition 5.2.4. *There exists a unique *-isomorphism $\mu_{ji} : \mathcal{E}^i_{jj} \to \mathcal{E}^j_{jj}$ such that*

$$(\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})(x) = (W^j_{ji})^*_{23} \mu_{ji}(x)_{13} (W^j_{ji})_{23}, \quad x \in \mathcal{E}^i_{jj}.$$

Furthermore, we have $\mu_{ji}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) = \pi_j(q_j a)\widehat{\theta}_j(x_{jj})$ for all $a \in A$ and $x \in \widehat{S}$.

Note that $\mu_{jj} = \mathrm{id}$. For the proof, we assemble some important formulas in the following lemma:

Lemma 5.2.5. *For all $i, j, k = 1, 2$, we have:*

1. $(V^i_{jj})_{23} = (W^j_{ki})^*_{12} (V^k_{jj})_{23} (W^j_{ki})_{12}$.
2. $\forall x \in \widehat{S}$, $1_{\mathscr{H}_{ki}} \otimes \widehat{\pi}_i(x_{jj}) = (W^j_{ki})^* (1_{\mathscr{H}_{ki}} \otimes \widehat{\pi}_k(x_{jj})) W^j_{ki}$.
3. $\forall a \in A$, $(\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})\pi_i(q_j a) = (W^j_{ji})^*_{23} \pi_j(q_j a)_{13} (W^j_{ji})_{23}$.

Proof. The first formula is just a restatement of a the first commutation relation of (2.5.2). Let $\omega \in \mathcal{B}(\mathscr{H})_*$, let us denote $x = \rho(\omega) \in \widehat{S}$ and $\omega_{jj} = p_{jj} \omega p_{jj} \in \mathcal{B}(\mathscr{H}_{jj})_*$. By Proposition 2.5.11 1, we have:

$$\widehat{\pi}_i(x_{jj}) = (\mathrm{id} \otimes \omega_{jj})(V^i_{jj}), \quad \widehat{\pi}_k(x_{jj}) = (\mathrm{id} \otimes \omega_{jj})(V^k_{jj}).$$

Then, $1_{\mathscr{H}_{ki}} \otimes \widehat{\pi}_i(x_{jj}) = (W^j_{ki})^* (1_{\mathscr{H}_{ki}} \otimes \widehat{\pi}_k(x_{jj})) W^j_{ki}$ follows directly from the first statement. Let $a \in A$, we have

$$\begin{aligned}
(\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})\pi_i(q_j a) &= (\pi_j \otimes L_{ij})\delta^i_{A_j}(a_j) \\
&= (\mathrm{id}_{A_j} \otimes L_{ji} \otimes L_{ij})(\delta^j_{A_i} \otimes \mathrm{id}_{S_{ij}})\delta^i_{A_j}(q_j a) \\
&= (\mathrm{id}_{A_j} \otimes L_{ji} \otimes L_{ij})(\mathrm{id}_{A_j} \otimes \delta^i_{jj})\delta^j_{A_j}(q_j a) \\
&= (\mathrm{id}_{A_j} \otimes (L_{ji} \otimes L_{ij})\delta^i_{jj})\delta^j_{A_j}(q_j a).
\end{aligned}$$

However, we have $(L_{ji} \otimes L_{ij})\delta^i_{jj}(y) = (W^j_{ji})^* (1_{\mathscr{H}_{ji}} \otimes L_{jj}(y)) W^j_{ji}$ for all $y \in S_{jj}$. Therefore, we have $(\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})\pi_i(q_j a) = (W^j_{ji})^*_{23} \pi_j(q_j a)_{13} (W^j_{ji})_{23}$. □

Proof of Proposition 5.2.4. Let $a \in A$ and $x \in \widehat{S}$. In virtue of the statements 2 and 3 of Lemma 5.2.5 (with $k = j$), we have

$$\begin{aligned}
(\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) &= (\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})(\pi_i(q_j a))(1_{A_j} \otimes 1_{\mathscr{H}_{ji}} \otimes \widehat{\pi}_i(x_{jj})) \\
&= (W^j_{ji})^*_{23} [\pi_j(q_j a)\widehat{\theta}_j(x_{jj})]_{13} (W^j_{ji})^*_{23}.
\end{aligned}$$

Let $\widetilde{\mu}_{ji} : \mathcal{L}(A_i \otimes \mathscr{H}_{ij}) \to \mathcal{L}(A_j \otimes \mathscr{H}_{ji} \otimes \mathscr{H}_{jj})$ be the linear map given for all $x \in \mathcal{L}(A_i \otimes \mathscr{H}_{ij})$ by $\widetilde{\mu}_{ji}(x) = (W^j_{ji})_{23} (\pi_j \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{ij})})(x) (W^j_{ji})^*_{23}$. It is clear that $\widetilde{\mu}_{ji}$ is an injective *-homomorphism.

Moreover, we have proved that there exists a map $\mu_{ji} : \mathcal{E}^i_{jj} \to \mathcal{E}^j_{jj}$ such that $\widetilde{\mu}_{ji}(x) = \mu_{ji}(x)_{13}$ for all $x \in \mathcal{E}^i_{jj}$. In particular, μ_{ij} is an injective *-homomorphism. Furthermore, since

$$\mu_{ji}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) = \pi_j(q_j a)\widehat{\theta}_j(x_{jj}),$$

for all $a \in A$ and $x \in \widehat{S}$, the range of μ_{ji} contains a total subset of $\mathcal{E}^j_{jj}$. However, since μ_{ji} is isometric, the range of μ_{ji} is norm closed in $\mathcal{E}^j_{jj}$. Hence, $\mu_{ji}(\mathcal{E}^i_{jj}) = \mathcal{E}^j_{jj}$. □

As a consequence of Corollary 5.2.3 and Proposition 5.2.4, we finally obtain the main result of this paragraph:

Corollary 5.2.6. *The crossed products $A_1 \rtimes \mathbb{G}_1$ and $A_2 \rtimes \mathbb{G}_2$ are canonically Morita equivalent.*

Now, we will prove that the C*-algebra $\mathcal{E}^i_{jj}$ is endowed with a continuous action of $\widehat{\mathbb{G}}_j$, obtained by restriction of the dual action of (δ_A, β_A) on $B = A \rtimes \mathcal{G}$, such that the *-isomorphism μ_{ji} defined in Proposition 5.2.4 is equivariant.

Proposition-Definition 5.2.7. *For all $i, j = 1, 2$, let us define:*

$$\delta_{\mathcal{E}^i_{jj}}(x) = (\widetilde{V}^j_{ij})_{23}(x \otimes 1_{\mathscr{H}_{ji}})(\widetilde{V}^j_{ij})^*_{23} \in \mathcal{L}(A_i \otimes \mathscr{H}_{ij} \otimes \mathscr{H}_{jj}), \quad x \in \mathcal{E}^i_{jj}.$$

*Then, $\delta_{\mathcal{E}^i_{jj}} : \mathcal{E}^i_{jj} \to \mathcal{M}(\mathcal{E}^i_{jj} \otimes \widehat{S}_{jj})$ is a continuous action of the quantum group $\widehat{\mathbb{G}}_j$. Moreover, $\mu_{ji} : \mathcal{E}^i_{jj} \to A_j \rtimes \mathbb{G}_j$ is a $\widehat{\mathbb{G}}_j$-equivariant *-isomorphism.*

Proof. We already know that $\delta_{\mathcal{E}^j_{jj}} = \delta_{A_j \rtimes \widehat{\mathbb{G}}_j}$ (*cf.* [2]) is a continuous action of $\widehat{\mathbb{G}}_j$ on $\mathcal{E}^j_{jj} = A_j \rtimes \widehat{\mathbb{G}}_j$. It is also clear from the definition that $\delta_{\mathcal{E}^i_{jj}}$ is a *-homomorphism. In virtue of Corollary 2.5.13, we have

$$\widetilde{V}^j_{ij}(\widehat{\pi}_i(x_{jj}) \otimes 1_{\mathscr{H}_{ji}})(\widetilde{V}^j_{ij})^* = (\widehat{\pi}_i \otimes \widehat{\pi}_j)\widehat{\delta}(x_{jj}), \quad x \in \widehat{S}. \tag{5.2.3}$$

Since $[(\widetilde{V}^j_{ij})_{23}, (V^k_{ij})_{12}] = 0$ for $k = 1, 2$ (see the commutation relations (2.5.2)), we have

$$[(\widetilde{V}^j_{ij})_{23}, \pi_i(q_j a) \otimes 1_{\mathscr{H}_{jj}}] = 0, \quad a \in A. \tag{5.2.4}$$

We combine (5.2.3) and (5.2.4), then for all $a \in A$ and $x \in \widehat{S}$ we have

$$\delta_{\mathcal{E}^i_{jj}}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) = (\pi_i(q_j a) \otimes 1_{\mathscr{H}_{jj}})(1_{A_i} \otimes (\widehat{\pi}_i \otimes \widehat{\pi}_j)\widehat{\delta}(x_{jj})) = (\pi_i(q_j a) \otimes 1_{\mathscr{H}_{jj}})(\widehat{\theta}_i \otimes \widehat{\pi}_j)\widehat{\delta}(x_{jj}). \tag{5.2.5}$$

In particular, we have $\delta_{\mathcal{E}^i_{jj}}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) \in \mathcal{M}(\mathcal{E}^i_{jj} \otimes \widehat{S}_{jj})$ for all $a \in A$ and $x \in \widehat{S}$. Hence, $\delta_{\mathcal{E}^i_{jj}}(\mathcal{E}^i_{jj}) \subset \mathcal{M}(\mathcal{E}^i_{jj} \otimes \widehat{S}_{jj})$ and $\delta_{\mathcal{E}^i_{jj}}$ is a non-degenerate *-homomorphism. To complete the proof, it only remains to show that $\delta_{A_j \rtimes \mathbb{G}_j} \circ \mu_{ji} = (\mu_{ji} \otimes \mathrm{id}_{\widehat{S}_{jj}}) \circ \delta_{\mathcal{E}^i_{jj}}$. But, for all $a \in A$ and $x \in \widehat{S}$ we have

$$\begin{aligned}
\delta_{A_j \rtimes \mathbb{G}_j}(\mu_{ji}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj}))) &= (\widetilde{V}^j_{jj})_{23}(\mu_{ji}(\pi_i(q_j a)\widehat{\theta}_i(x_{jj})) \otimes 1_{\mathscr{H}_{jj}})(\widetilde{V}^j_{jj})^*_{23} \\
&= (\widetilde{V}^j_{jj})_{23}(\pi_j(q_j a)\widehat{\theta}_j(x_{jj}) \otimes 1_{\mathscr{H}_{jj}})(\widetilde{V}^j_{jj})^*_{23} && \text{(Proposition 5.2.4)} \\
&= (\pi_j(q_j a) \otimes 1_{\mathscr{H}_{jj}})(\widehat{\theta}_j \otimes \widehat{\pi}_j)\widehat{\delta}(x_{jj}) && \text{(5.2.5) with } i = j \\
&= (\mu_{ji} \otimes \mathrm{id}_{\widehat{S}_{jj}})((\pi_i(q_j a) \otimes 1_{\mathscr{H}_{jj}})(\widehat{\theta}_i \otimes \widehat{\pi}_j)\widehat{\delta}(x_{jj})) \\
&= (\mu_{ji} \otimes \mathrm{id}_{\widehat{S}_{jj}})\delta_{\mathcal{E}^i_{jj}}(\pi_j(q_j a)\widehat{\theta}_j(x_{jj})) && \text{(5.2.5)}.
\end{aligned}$$

□

Remarks 5.2.8. Let us make some comments concerning this paragraph:
1) It should be noted that the Corollary 5.2.6 has been established (with different notations and conventions) by De Commer in [9] in the case of the trivial action of $\mathcal{G}$ on the C*-algebra $A := N^o = \mathbb{C}^2$.
Note also that for this action, the C*-algebras A_j are identified to $\mathbb{C}$, the *-homomorphisms $\delta^k_{A_j} : \mathbb{C} \to \mathcal{M}(\mathbb{C} \otimes S_{kj}) = \mathcal{M}(S_{kj})$ satisfy $\delta^k_{A_j}(1) = p_{kj}$ and we have $\mathcal{E}^i_{jk} = E^i_{jk}$ for all $i, j, k = 1, 2$. Moreover, the crossed product $B = A \rtimes \mathcal{G}$ is canonically isomorphic to $\widehat{S}$. More precisely, $\pi_B : (B, \delta_B, \alpha_B) \to (\widehat{S}, \widehat{\delta}, \widehat{\alpha})$ is a $\widehat{\mathcal{G}}$-equivariant *-isomorphism.
2) If we apply the results of this paragraph to the action (δ, β) of $\mathcal{G}$ on S we obtain that the C*-algebras

$$(S_{11} \rtimes_{\delta^1_{11}} \mathbb{G}_1) \oplus (S_{21} \rtimes_{\delta^1_{21}} \mathbb{G}_1), \quad (S_{12} \rtimes_{\delta^2_{12}} \mathbb{G}_2) \oplus (S_{22} \rtimes_{\delta^2_{22}} \mathbb{G}_2)$$

are Morita equivalent.

5.3 Structure of the double crossed product

In this paragraph, we investigate the double crossed product $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ for a continuous action (δ_A, β_A) of a colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1, \mathbb{G}_2}$ between two monoidally equivalent locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. We will need the notations and the results of the paragraphs 4.4 and 5.1.

Let us fix a $\mathcal{G}$-C*-algebra (A, δ_A, β_A). We will use the notations of §4.4 in order to describe the $\mathcal{G}$-C*-algebra (D, δ_D, β_D) and we take the notations of §5.1 for (A, δ_A, β_A). By the duality theorem (Theorem 4.4.15), we know that the C*-algebra

$$D = [\pi_R(a)(1_A \otimes \lambda(x)L(y)) \,;\, a \in A,\, x \in \widehat{S},\, y \in S]$$

endowed with the continuous action (δ_D, β_D) of $\mathcal{G}$ is canonically $\mathcal{G}$-equivariantly isomorphic to $(A \rtimes \mathcal{G}) \rtimes \widehat{\mathcal{G}}$ endowed with the bidual action. Following the discussion of §5.1, we have:

$$D = D_1 \oplus D_2, \quad D_j := \beta_D(\varepsilon_j)D = (q_j \otimes \beta(\varepsilon_j))D, \quad \text{where} \quad j_D(\beta_D(\varepsilon_j)) = q_j \otimes \beta(\varepsilon_j),\ j = 1, 2.$$

Furthermore, note that δ_D is completely described by the faithful non-degenerate *-homomorphisms $\delta^k_{D_j} : D_j \to \mathcal{M}(D_k \otimes S_{kj})$ given by:

$$\pi^k_j \circ \delta^k_{D_j}(x) = (\beta_D(\varepsilon_k) \otimes p_{kj})\delta_D(x) = (\beta_D(\varepsilon_k) \otimes 1_S)\delta_D(x) = (1_D \otimes p_{kj})\delta_D(x), \quad x \in D_j, \ (5.3.1)$$

where $\pi^k_j : \mathcal{M}(D_k \otimes S_{kj}) \to \mathcal{M}(D \otimes S)$ is the unique strictly continuous extension of the inclusion map $D_k \otimes S_{kj} \subset D \otimes S$ such that $\pi^k_j(1_{D_k \otimes S_{kj}}) = \beta_D(\varepsilon_k) \otimes p_{kj}$.

In order to investigate the C*-algebras D_j and the *-homomorphisms $\delta^k_{D_j}$, we will need some further notations:

Notations 5.3.1. Let $j, k, l = 1, 2$.

- We denote $L_{jk} : S_{jk} \to \mathcal{B}(\mathscr{H}_{jk})$ the faithful *-representation given by $L_{jk}(y) = L(y)\restriction_{\mathscr{H}_{jk}}$ for all $y \in S_{jk}$.
- We recall that $U_{jk} : \mathscr{H}_{jk} \to \mathscr{H}_{kj}$ is the restriction to $\mathscr{H}_{jk}$ of U to the subspace $\mathscr{H}_{jk}$. Let $R_{jk} : S_{jk} \to \mathcal{B}(\mathscr{H}_{kj})$ be the *-representation given by $R_{jk}(y) = U_{jk}L_{jk}(y)U^*_{jk}$, $y \in S_{jk}$.

- Let $\iota_j : \mathcal{M}(D_j) \to \mathcal{M}(D)$ be the unique strictly continuous extension of the inclusion $D_k \subset D$ such that $\iota_j(1_{D_j}) = \beta_D(\varepsilon_j)$, $j = 1, 2$. We have $\iota_j(\mathcal{M}(D_j)) = \beta_D(\varepsilon_j)\mathcal{M}(D)$.
- We consider the following self-adjoint projection $q_{lkj} := q_k \otimes p_{lk} \otimes p_{kj} \in \mathcal{L}(A \otimes \mathscr{H} \otimes \mathscr{H})$.

We finish these preliminaries with a remark about the Hilbert A-module $\mathscr{E}_{A,R} = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathscr{H})$. Since $q^{\beta_A,\widehat{\alpha}} = q_1 \otimes \beta(\varepsilon_1) + q_2 \otimes \beta(\varepsilon_2)$ we have $\mathscr{E}_{A,R} = \mathscr{E}_{A,R,1} \oplus \mathscr{E}_{A,R,2}$, where $\mathscr{E}_{A,R,k}$ is the Hilbert A_k-module $(q_k \otimes \beta(\varepsilon_k))(A \otimes \mathscr{H}) = (A_k \otimes \mathscr{H}_{1k}) \oplus (A_k \otimes \mathscr{H}_{2k})$. Furthermore, since A_1 and A_2 are closed two-sided ideal of A such that $A_1 A_2 = \{0\}$, we have $\mathcal{K}(\mathscr{E}_{A,R}) = \mathcal{K}(\mathscr{E}_{A,R,1}) \oplus \mathcal{K}(\mathscr{E}_{A,R,2})$.

Lemma 5.3.2. *Let $a \in A$, $x \in \widehat{S}$ and $y \in S$. Let us denote $d = \pi_R(a)(1_A \otimes \lambda(x)L(y)) \in D$. Let $d_j := \beta_D(\varepsilon_j)d = (q_j \otimes \beta(\varepsilon_j))d \in D_j$, $j = 1, 2$. We have:*

1. For $j = 1, 2$, we have $d_j = \sum_{l,l'=1,2} d_{ll',j}$, where

$$d_{ll',j} := \pi_R(q_l a)(1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j}) = \pi_R(q_l a)(1_A \otimes \widehat{\pi}^j(\lambda(x_{ll'}))L(y)p_{l'j}), \quad l, l' = 1, 2.$$

2. For $j, k = 1, 2$, we have:

$$(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj})(j_D \otimes L)\delta_D(d_j) = \sum_{l,l'=1,2} q_{lkj} V_{23}(q_k \otimes p_{lj} \otimes p_{kj})\Sigma_{23}(\pi_L \otimes R)(\delta_A(q_l a))\Sigma_{23} \cdots$$
$$\cdots (1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j} \otimes 1_{\mathscr{H}})(q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^* q_{l'kj}.$$

Proof. 1. Let us fix $j = 1, 2$. We have $d_j = (q_j \otimes \beta(\varepsilon_j))d$. Since $\beta_D(\varepsilon_j) = q_j \otimes \beta(\varepsilon_j)$ is central we have $d_j = (q_j \otimes \beta(\varepsilon_j))d(1_A \otimes \beta(\varepsilon_j))$. Since $\beta(\varepsilon_j)$ is central in $\mathcal{M}(S)$ we have $[L(y), \beta(\varepsilon_j)] = 0$. We recall that $\beta(\mathbb{C}^2) \subset \widehat{M}'$ and $\lambda(\widehat{S}) \subset \widehat{M}$, hence $[\lambda(x), \beta(\varepsilon_j)] = 0$. By combining these commutation relations, we obtain

$$d_j = (q_j \otimes \beta(\varepsilon_j))\pi_R(a)(1_A \otimes \beta(\varepsilon_j)\lambda(x)\beta(\varepsilon_j)L(y)) = \sum_{l,l'=1,2} (q_j \otimes \beta(\varepsilon_j))\pi_R(a)(1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)).$$

In virtue of the fact that $\widehat{\alpha} = \beta$ and Lemma 5.1.2, we have

$$\begin{aligned}(q_j \otimes \beta(\varepsilon_j))\pi_R(a)(1_A \otimes p_{lj}) &= (\mathrm{id}_A \otimes R)((q_j \otimes \alpha(\varepsilon_j))\delta_A(a)(1_A \otimes p_{jl}))(1_A \otimes p_{lj}) \\ &= (\mathrm{id}_A \otimes R)((q_j \otimes \alpha(\varepsilon_j)p_{jl})\delta_A(a))(1_A \otimes p_{lj}) \\ &= (\mathrm{id}_A \otimes R)((q_j \otimes p_{jl})\delta_A(a))(1_A \otimes p_{lj}) \\ &= \pi_R(q_l a)(1_A \otimes p_{lj}).\end{aligned}$$

The first statement is then proved since $p_{l'j}L(y) = L(y)p_{l'j}$.

2. Let us fix $j, l, l' = 1, 2$. Let us compute $(j_D \otimes L)\delta_D(d_{ll',j}) = V_{23}(\mathrm{id}_A \otimes L)\delta_0(d_{ll',j})V_{23}^*$ (see Remark 4.4.14). By Proposition 4.4.12, we have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(\pi_R(q_l a)) = \Sigma_{23}(\pi_L \otimes R)(\delta_A(q_l a))\Sigma_{23}.$$

We also have

$$(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j}) = q_{13}^{\beta_A,\alpha}(1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j} \otimes 1_{\mathscr{H}}).$$

By the commutation relation $(1 \otimes \alpha(n))V^* = V^*(\widehat{\alpha}(n) \otimes 1)$ (*cf.* Proposition 2.3.5 3), we have $q_{13}^{\beta_A,\alpha}V_{23}^* = V_{23}^* q_{12}^{\beta_A,\widehat{\alpha}}$. Hence,

$$(j_D \otimes L)\delta_D(d_{ll',j}) = V_{23}\Sigma_{23}(\pi_L \otimes R)(\delta_A(q_l a))\Sigma_{23}(1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j} \otimes 1_{\mathscr{H}})V_{23}^* q_{12}^{\beta_A,\widehat{\alpha}}.$$

By using the fact that the projections $q_k \in \mathcal{M}(A)$, p_{kj}, $p_{jl} \in \mathcal{M}(S)$ are central, we have

$$\begin{aligned}(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj})(j_D \otimes L)\delta_D(d_{ll',j}) \\ = (q_k \otimes \beta(\varepsilon_k) \otimes p_{kj})V_{23}(1_A \otimes p_{lj} \otimes 1_{\mathscr{H}})\Sigma_{23}(\pi_L \otimes R)(\delta_A(q_l a))\Sigma_{23} \cdots \\ \cdots (1_A \otimes p_{lj}\lambda(x)p_{l'j}L(y)p_{l'j} \otimes 1_{\mathscr{H}})(q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^* q_{12}^{\beta_A,\widehat{\alpha}}.\end{aligned}$$

By using the following facts: $p_{lj} = \alpha(\varepsilon_l)p_{lj}$, $[V, \alpha(\varepsilon_l) \otimes 1] = 0$ (see Proposition 2.3.5 2), $\beta(\varepsilon_k)\alpha(\varepsilon_l) = p_{lk}$ and p_{kj} is central; we obtain

$$(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj})V_{23}(1_A \otimes p_{lj} \otimes 1_{\mathscr{H}}) = (q_k \otimes p_{lk} \otimes p_{kj})V_{23}(1_A \otimes p_{lj} \otimes 1_{\mathscr{H}}) = q_{lkj}V_{23}(q_k \otimes p_{lj} \otimes p_{kj}).$$

Similarly, we also have

$$\begin{aligned}(q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^* q_{12}^{\beta_A,\widehat{\alpha}} &= (q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^*(q_k \otimes 1_{\mathscr{H}} \otimes p_{kj})q_{12}^{\beta_A,\widehat{\alpha}} \\ &= (q_k \otimes p_{l'j}\alpha(\varepsilon_{l'}) \otimes p_{kj})V_{23}^*(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj}) \\ &= (q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^*(q_k \otimes \alpha(\varepsilon_{l'})\beta(\varepsilon_k) \otimes p_{kj}) \\ &= (q_k \otimes p_{l'j} \otimes p_{kj})V_{23}^* q_{l'kj}\end{aligned}$$

and the second statement is proved. □

Let us consider the faithful non-degenerate *-homomorphism

$$\pi_{D_j} : D_j \to \mathcal{L}(\mathscr{E}_{A,R,j}) \quad ; \quad \pi_{D_j}(u) = u\restriction_{\mathscr{E}_{A,R,j}}, \quad u \in D_j.$$

If $d \in D$, we have $d = \pi_{D_1}(d_1) + \pi_{D_2}(d_2)$, where $d_j = \beta_D(\varepsilon_j)d \in D_j$. We also consider the faithful non-degenerate *-homomorphism

$$\pi_{D_k} \otimes L_{kj} : D_k \otimes S_{kj} \to \mathcal{L}(\mathscr{E}_{A,R,k} \otimes \mathscr{H}_{kj}).$$

Then, π_{D_j} (resp. $\pi_{D_k} \otimes L_{kj}$) extends uniquely to a faithful unital *-homomorphism

$$\pi_{D_j} : \mathcal{M}(D_j) \to \mathcal{L}(\mathscr{E}_{A,R,j}) \quad (\text{resp. } \pi_{D_k} \otimes L_{kj} : \mathcal{M}(D_k \otimes S_{kj}) \to \mathcal{L}(\mathscr{E}_{A,R,k} \otimes \mathscr{H}_{kj}))$$

and we have:

$$\pi_{D_j}(m) = j_D(\iota_j(m))\restriction_{\mathscr{E}_{A,R,j}}, \quad m \in \mathcal{M}(D_j)$$

$$(\text{resp. } (\pi_{D_k} \otimes L_{kj})(m) = (j_D \otimes L)\pi_j^k(m)\restriction_{\mathscr{E}_{A,R,k} \otimes \mathscr{H}_{kj}}, \quad m \in \mathcal{M}(D_k \otimes S_{kj})).$$

Proposition 5.3.3. *Let $a \in A$, $x \in \widehat{S}$ and $y \in S$. Let us denote $d = \pi_R(a)(1_A \otimes \lambda(x)L(y)) \in D$. Let $d_j := \beta_D(\varepsilon_j)d = (q_j \otimes \beta(\varepsilon_j))d \in D_j$, $j = 1, 2$. We have:*

1. In $\mathcal{L}(\mathscr{E}_{A,R,j}) = \bigoplus_{l,l'=1,2} \mathcal{L}(A_j \otimes \mathscr{H}_{l'j}, A_j \otimes \mathscr{H}_{lj})$, we have

$$\pi_{D_j}(d_j) = \sum_{l,l'=1,2} (\mathrm{id}_{A_j} \otimes R_{jl})\delta_{A_l}^j(q_l a)(1_{A_j} \otimes \widehat{\pi}^j(\lambda(x_{ll'}))L_{l'j}(p_{l'j}y)).$$

2. In $\mathcal{L}(\mathscr{E}_{A,R,k} \otimes \mathscr{H}_{kj}) = \bigoplus_{l,l'=1,2} \mathcal{L}(A_k \otimes \mathscr{H}_{l'k} \otimes \mathscr{H}_{kj}, A_k \otimes \mathscr{H}_{lk} \otimes \mathscr{H}_{kj})$, we have

$$\begin{aligned}(\pi_{D_k} \otimes L_{kj})\delta_{D_j}^k(d_j) = \sum_{l,l'=1,2} (V_{kj}^l)_{23}(\Sigma_{kj\otimes lj})_{23}(\mathrm{id}_{A_k} \otimes L_{kj} \otimes R_{jl})((\delta_{A_j}^k \otimes \mathrm{id}_{S_{jl}})\delta_{A_l}^j(q_l a)) \cdots \\ \cdots (1_{A_k} \otimes 1_{\mathscr{H}_{kj}} \otimes \widehat{\pi}^j(\lambda(x_{ll'}))L_{l'j}(p_{l'j}y))(\Sigma_{l'j\otimes kj})_{23}(V_{kj}^{l'})_{23}^*.\end{aligned}$$

Proof. 1. By applying Lemma 5.3.2 1 and by definition of the *-homomorphisms $\delta^j_{A_l}$ (see Proposition 5.1.3), we have

$$\pi_{D_j}(d_{ll',j}) = (\mathrm{id}_{A_j} \otimes R_{jl})\delta^j_{A_l}(q_l a)(1_{A_j} \otimes \widehat{\pi}^j(\lambda(x_{ll'}))L(y)p_{l'j}) \in \mathcal{L}(A_j \otimes \mathscr{H}_{l'j}, A_j \otimes \mathscr{H}_{lj})$$

and the first statement is proved.
2. We have $(\pi_{D_k} \otimes L_{kj})\delta^k_{D_j}(d_{ll',j}) = (j_D \otimes L)\pi^k_j(\delta^k_{D_j}(d_{ll',j}))\restriction_{\mathscr{E}_{A,R,k}\otimes\mathscr{H}_{kj}}$. Moreover, in virtue of (5.3.1) we have

$$(j_D\otimes L)\pi^k_j(\delta^k_{D_j}(d_{ll',j})) = (j_D(\beta_D(\varepsilon_k))\otimes p_{kj})(j_D\otimes L)\delta_D(d_{ll',j}) = (q_k\otimes\beta(\varepsilon_k)\otimes p_{kj})(j_D\otimes L)\delta_D(d_{ll',j}).$$

We can then apply Lemma 5.3.2 2. We have $q_{lkj}V_{23}(q_k \otimes p_{lj} \otimes p_{kj}) = (V^l_{kj})_{23}(q_k \otimes p_{lj} \otimes p_{kj})$ and $(q_k \otimes p_{l'j} \otimes p_{kj})V^*_{23}q_{l'kj} = (q_k \otimes p_{l'j} \otimes p_{kj})(V^{l'}_{kj})^*_{23}$. Moreover, we also have

$$\begin{aligned}(q_k \otimes p_{lj} \otimes p_{kj})\Sigma_{23}(\pi_L \otimes R)\delta_A(q_l a) &= (q_k \otimes p_{lj} \otimes p_{kj})\Sigma_{23}(\mathrm{id}_A \otimes L \otimes R)((\delta_A \otimes \mathrm{id}_S)\delta_A(q_l a)) \\ &= \Sigma_{23}(\mathrm{id}_A \otimes L \otimes R)((q_k \otimes p_{kj} \otimes p_{jl})(\delta_A \otimes \mathrm{id}_S)\delta_A(q_l a)) \\ &= (\Sigma_{kj\otimes lj})_{23}(\mathrm{id}_{A_k} \otimes L_{kj} \otimes R_{jl})((\delta^k_{A_j} \otimes \mathrm{id}_{S_{jl}})\delta^j_{A_l}(q_l a))\end{aligned}$$

and the second statement is proved. □

The following lemma says in particular that the $\mathbb{G}_j$-C*-algebra $(D_j, \delta^j_{D_j})$ is a linking $\mathbb{G}_j$-C*-algebra (see Definition 1.6.1).

Lemma 5.3.4. *For $j = 1, 2$, we have:*

1. *For $l = 1, 2$, there exists a unique nonzero self-adjoint projection $e_{l,j} \in \mathcal{M}(D_j)$ such that*

$$j_D(\iota_j(e_{l,j})) = q_j \otimes p_{lj}.$$

2. *$\pi_{D_j}(e_{l,j}) = e_{l,A_j}$, where $e_{l,A_j} \in \mathcal{L}(\mathscr{E}_{A,R,j})$ is the orthogonal projection on the Hilbert A_j-module $A_j \otimes \mathscr{H}_{lj}$.*

3. *$e_{1,j} + e_{2,j} = 1_{D_j}$, $\quad [D_j e_{l,j} D_j] = D_j$.*

4. *For $k = 1, 2$, $\delta^k_{D_j}(e_{l,j}) = e_{l,k} \otimes 1_{S_{kj}}$.*

Proof. 1. We have

$$j_D(\mathcal{M}(D)) = \{T \in \mathcal{L}(A \otimes \mathscr{H})\,;\, TD \subset D,\, DT \subset D,\, Tj_D(1_D) = T = j_D(1_D)T\}.$$

Since $j_D(1_D) = q^{\beta_A,\widehat{\alpha}} = \sum_{i=1,2} q_i \otimes \beta(\varepsilon_i) = \sum_{i,k=1,2} q_i \otimes p_{ki}$ and by definition of D we have $q_j \otimes p_{lj} \in j_D(\mathcal{M}(D))$. Since $\beta_D(\varepsilon_j)(q_j \otimes p_{lj}) = q_j \otimes p_{lj}$ and $\beta_D(\varepsilon_j)\mathcal{M}(D) = \iota_j(\mathcal{M}(D_j))$, we actually have $q_j \otimes p_{lj} \in j_D(\iota_j(\mathcal{M}(D_j)))$. This proves that there exists a unique nonzero self-adjoint projection (by faithfulness of j_D and ι_j) $e_{l,j} \in \mathcal{M}(D_j)$ such that $q_j \otimes p_{lj} = j_D(\iota_j(e_{l,j}))$.

2. $\pi_{D_j}(e_{l,j}) = j_D(\iota_j(e_{l,j}))\restriction_{\mathscr{E}_{A,R,j}} = (q_j \otimes p_{lj})\restriction_{\mathscr{E}_{A,R,j}} = e_{l,A_j}$.

3. We have $j_D(\iota_j(1_{D_j})) = j_D(\beta_D(\varepsilon_j)) = q_j \otimes \beta(\varepsilon_j) = j_D(\iota_j(e_{1,j})) + j_D(\iota_j(e_{2,j}))$ because of $\beta(\varepsilon_j) = p_{1j} + p_{2j}$. Hence, $1_{D_j} = e_{1,j} + e_{2,j}$ by faithfulness of j_D and ι_j. Let us prove that $[D_j e_{l,j} D_j] = D_j$. It is equivalent to prove that $[\pi_{D_j}(D_j)e_{l,A_j}\pi_{D_j}(D_j)] = \pi_{D_j}(D_j)$. Then, the result is a consequence of Proposition 5.3.3 1, the fact that $[\lambda(\widehat{S})S]$ is a C*-algebra and the

formulas $[E^j_{ll',\lambda}E^j_{l'l'',\lambda}] = E^j_{ll'',\lambda}$ (see Remark 2.5.12).

4. It amounts to proving that $(j_D \otimes L)\pi^k_j\delta^k_{D_j}(e_{l,j}) = q_{lkj}$. It follows from (5.3.1) that for all $m \in \mathcal{M}(D_j)$, we have $\pi^k_j\delta^k_{D_j}(m) = \delta_D(\iota_j(m))(\beta_D(\varepsilon_k) \otimes p_{kj})$. In particular, we have

$$\begin{aligned}(j_D \otimes L)\pi^k_j\delta^k_{D_j}(e_{l,j}) &= (j_D \otimes L)(\delta_D(\iota_j(e_{l,j})))(j_D(\beta_D(\varepsilon_k)) \otimes p_{kj}) \\ &= (j_D \otimes L)(\delta_D(\iota_j(e_{l,j})))(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj}).\end{aligned}$$

However, we have $(j_D \otimes L)\delta_D(\iota_j(e_{l,j})) = V_{23}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes L)\delta_0(q_j \otimes p_{lj})V^*_{23}$ (*cf.* Remark 4.4.14). We have

$$\begin{aligned}\delta_0(q_j \otimes p_{lj}) &= \delta_A(\beta_A(\varepsilon_j))_{13}(1_A \otimes p_{lj} \otimes 1_S) \\ &= (1_A \otimes 1_{\mathscr{H}} \otimes \beta(\varepsilon_j))q^{\beta_A,\alpha}_{13}(1_A \otimes p_{lj} \otimes 1_S) \\ &= \sum_{r=1,2} q_r \otimes p_{lj} \otimes p_{rj}.\end{aligned}$$

We then obtain

$$\begin{aligned}(j_D \otimes L)\pi^k_j\delta^k_{D_j}(e_{l,j}) &= \sum_{r=1,2} V_{23}(q_r \otimes p_{lj} \otimes p_{rj})V^*_{23}(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj}) \\ &= V_{23}(q_k \otimes p_{lj} \otimes p_{kj})V^*_{23}(q_k \otimes \beta(\varepsilon_k) \otimes p_{kj}).\end{aligned}$$

However, we have $V(p_{lj} \otimes p_{kj})V^*(\beta(\varepsilon_k) \otimes p_{kj}) = \sum_{r=1,2} V(p_{lj} \otimes p_{kj})V^*(p_{rk} \otimes p_{kj})$ and by Proposition 2.5.4 we also have $(p_{lj} \otimes 1)V^*(p_{rk} \otimes 1) = \delta^l_r(p_{lj} \otimes p_{kj})V^*(p_{lk} \otimes p_{kj})$. Hence,

$$V(p_{lj} \otimes p_{kj})V^*(\beta(\varepsilon_k) \otimes p_{kj}) = V(p_{lj} \otimes 1)V^*(p_{lk} \otimes p_{kj}) = \delta(p_{lj})(p_{lk} \otimes p_{kj}) = p_{lk} \otimes p_{kj}.$$

Hence, $(j_D \otimes L)\pi^k_j\delta^k_{D_j}(e_{l,j}) = q_k \otimes p_{lk} \otimes p_{kj}$. □

Notations 5.3.5. Let us denote

$$D_{ll',j} := e_{l,j}D_je_{l',j}, \quad D_{l,j} := D_{ll,j}, \quad j,l,l' = 1,2.$$

The following result is an immediate consequence of Remark 1.6.2, Lemma 5.3.4 and Corollary 5.1.4.

Corollary 5.3.6. *Let us fix $j,l,l' = 1,2$. We have:*

1. *By restriction of the structure of $\mathbb{G}_j$-C*-algebra on D_j, $D_{ll',j}$ is a $\mathbb{G}_j$-equivariant Hilbert $D_{l,j}$-$D_{l',j}$-bimodule.*
2. *If $k \neq j$, we have:*
   a) *The *-homomorphism $(D_j, e_{1,j}, e_{2,j}) \to (\delta^k_{D_j}(D_j), e_{1,k} \otimes 1_{S_{kj}}, e_{2,k} \otimes 1_{S_{kj}})$; $x \mapsto \delta^k_{D_j}(x)$ is a *-isomorphism of linking $\mathbb{G}_j$-C*-algebras.*
   b) *$\delta^k_{D_j}(D_{ll',j})$ is a $\mathbb{G}_j$-equivariant Hilbert $\delta^k_{D_j}(D_{l,j})$-$\delta^k_{D_j}(D_{l',j})$-bimodule. Furthermore, by restriction of the *-isomorphism described in a), the $\mathbb{G}_j$-equivariant Hilbert $D_{l,j}$-$D_{l',j}$-bimodule $D_{ll',j}$ is canonically isomorphic to the Hilbert $\delta^k_{D_j}(D_{l,j})$-$\delta^k_{D_j}(D_{l',j})$-bimodule $\delta^k_{D_j}(D_{ll',j})$ over the *-isomorphisms:*

$$D_{l,j} \to \delta^k_{D_j}(D_{l,j}), \quad D_{l',j} \to \delta^k_{D_j}(D_{l',j}).$$

The case where $\mathbb{G}_1$ and $\mathbb{G}_2$ are regular. In this pararaph, we provide a more precise description of the double crossed product in the regular case. In the following, we assume $\mathbb{G}_1$ and $\mathbb{G}_2$ to be regular. In this situation, we know from Theorem 3.2.19 and the duality theorem (see Theorem 4.4.15) that

$$D = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathcal{K}(\mathscr{H}))q^{\beta_A,\widehat{\alpha}} \subset \mathcal{L}(A \otimes \mathscr{H}).$$

We recall that $\mathscr{E}_{A,R} = q^{\beta_A,\widehat{\alpha}}(A \otimes \mathscr{H}) = \mathscr{E}_{A,R,1} \oplus \mathscr{E}_{A,R,2}$, where $\mathscr{E}_{A,R,j} = A_j \otimes (\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$, for $j = 1, 2$. Since q_j is central, we have

$$D = \bigoplus_{j,l,l'=1,2} (q_j \otimes p_{lj})(A \otimes \mathcal{K}(\mathscr{H}))(q_j \otimes p_{l'j}) = \bigoplus_{j,l,l'=1,2} A_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}).$$

Hence,

$$\pi_{D_j}(D_j) = \bigoplus_{l,l'=1,2} A_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}), \quad j = 1, 2. \tag{5.3.2}$$

In particular, π_{D_j} takes its values in $\mathcal{K}(\mathscr{E}_{A,R,j})$ and $\pi_{D_j} : D_j \to \mathcal{K}(\mathscr{E}_{A,R,j})$ is actually a *-isomorphism of linking algebras.

In the following, we will use the canonical identification $\mathcal{K}(\mathscr{E}_{A,R,j}) = A_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$. Let us introduce some notations that will be useful to describe the linking algebras D_j and the *-homomorphisms $\delta^k_{D_j}$ in this case.

Notations 5.3.7. Let $j, k, l, l' = 1, 2$.

1. We denote:

$$\mathcal{B}_k := \pi_{D_k}(D_k) = A_k \otimes \mathcal{K}(\mathscr{H}_{1k} \oplus \mathscr{H}_{2k}), \quad \mathcal{B}_{ll',k} := A_k \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk}),$$

$$\mathcal{B}_{l,k} := \mathcal{B}_{ll,k} = A_k \otimes \mathcal{K}(\mathscr{H}_{lk}).$$

2. Let $\delta^k_{\mathcal{B}_j} : \mathcal{B}_j \to \mathcal{M}(\mathcal{B}_k \otimes S_{kj})$ be the faithful non-degenerate *-homomorphism given by

$$(\pi_{D_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{D_j} = \delta^k_{\mathcal{B}_j} \circ \pi_{D_j}.$$

3. Let $\delta^k_{\mathcal{B}_j,0} : \mathcal{B}_j \to \mathcal{M}(A_k \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}) \otimes S_{kj})$ be the faithful *-homomorphism given by:

$$\delta^k_{\mathcal{B}_j,0}(a \otimes T) = \delta^k_{A_j}(a)_{13}(1_{A_k} \otimes T \otimes 1_{S_{kj}}), \quad a \in A_j,\ T \in \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}).$$

4. Let $\delta^k_{\mathcal{B}_{ll',j},0} : \mathcal{B}_{ll',j} \to \mathcal{L}(A_k \otimes \mathcal{K}(\mathscr{H}_{l'j}) \otimes S_{kj}, A_k \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) \otimes S_{kj})$ be the linear map given by:

$$\delta^k_{\mathcal{B}_{ll',j},0}(a \otimes T) = \delta^k_{A_j}(a)_{13}(1_{A_k} \otimes T \otimes 1_{S_{kj}}), \quad a \in A_j,\ T \in \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}).$$

5. Let $\delta^k_{\mathcal{B}_{ll',j}} : \mathcal{B}_{ll',j} \to \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj}, \mathcal{B}_{ll',k} \otimes S_{kj})$ be the linear map given by:

$$\delta^k_{\mathcal{B}_{ll',j}}(T) = (V^l_{kj})_{23}\delta^k_{\mathcal{B}_{ll',j},0}(T)(V^{l'}_{kj})^*_{23}, \quad T \in \mathcal{B}_{ll',j}.$$

We also denote:

$$\delta^k_{\mathcal{B}_{l,j}} := \delta^k_{\mathcal{B}_{ll,j}} : \mathcal{B}_{l,j} \to \mathcal{L}(\mathcal{B}_{l,k} \otimes S_{kj}).$$

Proposition 5.3.8. *For all $j,k,l,l' = 1,2$, we have:*

1. *$\mathcal{B}_{l,j}$ is a C*-algebra. Moreover, endowed with the natural actions of $\mathcal{B}_{l,j}$ and $\mathcal{B}_{l',j}$, $\mathcal{B}_{ll',j}$ is a Hilbert $\mathcal{B}_{l,j}$-$\mathcal{B}_{l',j}$-bimodule.*

2. *$\pi_{D_j}(D_{ll',j}) = \mathcal{B}_{ll',j}$.*

3. *For all $\xi \in \mathcal{B}_{ll',j}$, we have*

$$\delta^k_{\mathcal{B}_{ll',j}}(\xi) = (V^l_{kj})_{23}(\mathrm{id}_{A_k} \otimes \sigma)(\delta^k_{A_j} \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj})})(\xi)(V^{l'}_{kj})^*_{23},$$

where $\sigma : S_{kj} \otimes \mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}) \to \mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}) \otimes S_{kj}$; $s \otimes T \mapsto T \otimes s$ is the flip map.

Proof. The first statement follows from the formulas:

$$(\mathcal{B}_{ll',j})^* = \mathcal{B}_{l'l,j}, \quad [\mathcal{B}_{ll',j}\mathcal{B}_{l'l'',j}] = \mathcal{B}_{ll'',j}, \quad j,l,l',l'' = 1,2,$$

which follow from the fact that A_j is a C*-algebra and the following elementary fact: if $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces (with $\mathcal{H}$ non zero) we have $\mathcal{K}(\mathcal{E},\mathcal{K}) = [\mathcal{K}(\mathcal{H},\mathcal{K})\mathcal{K}(\mathcal{E},\mathcal{H})]$. This actually proves that $\mathcal{B}_{ll',j}$ is a Morita equivalence between the C*-algebras $\mathcal{B}_{l,j}$ and $\mathcal{B}_{l',j}$. Note that we also have $[\mathcal{B}_{ll',j}(A_j \otimes \mathscr{H}_{l'j})] = A_j \otimes \mathscr{H}_{lj}$.
The second statement is an immediate consequence of (5.3.2) and Lemma 5.3.4 2. To prove the third one, it is enough to see that the formula holds for $\xi = \pi_{D_j}(d_{ll',j})$ (see the notations of Lemma 5.3.2). However, in that case this is exactly the formula of Proposition 5.3.3 2. □

Proposition 5.3.9. *Let $j,l,l' = 1,2$, we have:*

1. *$\delta^j_{\mathcal{B}_{l,j}} : \mathcal{B}_{l,j} \to \mathcal{M}(\mathcal{B}_{l,j} \otimes S_{jj})$ is a continuous action of the quantum group $\mathbb{G}_j$ on the C*-algebra $\mathcal{B}_{l,j}$.*

2. *Up to the identification $A_j \rtimes \mathbb{G}_j \rtimes \widehat{\mathbb{G}}_j = A_j \otimes \mathcal{K}(\mathscr{H}_{jj})$ (cf. Theorem 1.5.3), the continuous action $\delta^j_{\mathcal{B}_{j,j}}$ of $\mathbb{G}_j$ is the bidual action on the double crossed product $A_j \rtimes \mathbb{G}_j \rtimes \widehat{\mathbb{G}}_j$.*

3. *If $l \neq l'$, then $(\mathcal{B}_{ll',j}, \delta^j_{\mathcal{B}_{ll',j}})$ is a $\mathbb{G}_j$-equivariant Morita equivalence between the $\mathbb{G}_j$-C*-algebras $\mathcal{B}_{l,j} = A_j \otimes \mathcal{K}(\mathscr{H}_{lj})$ and $\mathcal{B}_{l',j} = A_j \otimes \mathcal{K}(\mathscr{H}_{l'j})$.*

Proof. The first statement follows from the fact that $D_{l,j}$ is a $\mathbb{G}_j$-C*-algebra by restriction of the structure of $\mathbb{G}_j$-C*-algebra on D_j. For the second one, there is actually nothing to prove. Finally, it follows from the statements 1 and 2 of Proposition 5.3.8 that the *-isomorphism $\pi_{D_j} : D_j \to A_j \otimes (\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$ induces by restriction to the $\mathbb{G}_j$-equivariant Hilbert $D_{l,j}$-$D_{l',j}$-bimodule $D_{ll',j}$ (see Corollary 5.3.6 1) an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules from $D_{ll',j}$ to $\mathcal{B}_{ll',j}$ over the $\mathbb{G}_j$-equivariant isomorphisms $D_{l,j} \to \mathcal{B}_{l,j}$ and $D_{l',j} \to \mathcal{B}_{l',j}$. □

Notation 5.3.10. For $j,l,l' = 1,2$, we denote

$$\gamma_{ll',j} := (\mathcal{B}_{l,j}, \mathcal{B}_{ll',j}, \mathcal{B}_{l',j})$$

the $\mathbb{G}_j$-equivariant Morita equivalence of the $\mathbb{G}_j$-C*-algebras $\mathcal{B}_{l,j}$ and $\mathcal{B}_{l'j}$ given by the Hilbert bimodule $\mathcal{B}_{ll',j}$ (see Proposition 5.3.9 3).

For the internal tensor product of equivariant bimodules, we refer the reader to [3] (see Proposition 2.10).

Proposition 5.3.11. *For all $j, l, l', l'' = 1, 2$, we have*

$$\gamma_{ll'',j} = \gamma_{ll',j} \otimes_{\mathcal{B}_{l',j}} \gamma_{l'l'',j}.$$

Proof. It is clear that the map

$$\begin{aligned} \pi : \mathcal{B}_{ll',j} \otimes_{\mathcal{B}_{l',j}} \mathcal{B}_{l'l'',j} &\longrightarrow \mathcal{B}_{ll'',j} \\ \xi \otimes_{\mathcal{B}_{l',j}} \eta &\longmapsto \xi \circ \eta \end{aligned}$$

is an isomorphism of Hilbert $\mathcal{B}_{l,j}$-$\mathcal{B}_{l'',j}$-bimodules. Let us show that π is $\mathbb{G}_j$-equivariant. The structure of $\mathbb{G}_j$-equivariant Hilbert $\mathcal{B}_{l'',j}$-module on $\mathcal{B}_{ll',j} \otimes_{\mathcal{B}_{l',j}} \mathcal{B}_{l'l'',j}$ is given by

$$\begin{aligned} \mathcal{B}_{ll',j} \otimes_{\mathcal{B}_{l',j}} \mathcal{B}_{l'l'',j} &\longrightarrow \mathcal{L}(\mathcal{B}_{l'',j} \otimes S_{jj}, (\mathcal{B}_{ll',j} \otimes_{\mathcal{B}_{l',j}} \mathcal{B}_{l'l'',j}) \otimes S_{jj}) \\ \xi \otimes_{\mathcal{B}_{l',j}} \eta &\longmapsto \Delta(\xi, \eta) := (\delta^j_{\mathcal{B}_{ll',j}}(\xi) \otimes_{\mathcal{B}_{l',j} \otimes S_{jj}} \mathrm{id}_{S_{jj}}) \circ \delta^j_{\mathcal{B}_{l'l'',j}}(\eta). \end{aligned}$$

By Proposition 5.3.8 2, we have the formula

$$(\pi \otimes \mathrm{id}_{S_{jj}})(\Delta(\xi, \eta)) = \delta^j_{\mathcal{B}_{ll'',j}}(\pi(\xi \otimes_{\mathcal{B}_{l',j}} \eta)), \quad \xi \in \mathcal{B}_{ll',j},\, \eta \in \mathcal{B}_{l'l'',j}$$

and the result follows. □

Notation 5.3.12. For $j, k = 1, 2$, $j \neq k$, we denote:

$$\mathcal{E}^j_{ll',k} := \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j}) \subset \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj}, \mathcal{B}_{ll',k} \otimes S_{kj}).$$

The following formulas:

$$\begin{aligned} \delta^k_{\mathcal{B}_{l,j}}(a)\delta^k_{\mathcal{B}_{ll',j}}(\xi) &= \delta^k_{\mathcal{B}_{ll',j}}(a\xi), \quad \xi \in \mathcal{B}_{ll',j},\, a \in \mathcal{B}_{l,j}, \\ \delta^k_{\mathcal{B}_{ll',j}}(\xi)\delta^k_{\mathcal{B}_{l',j}}(a) &= \delta^k_{\mathcal{B}_{ll',j}}(\xi a), \quad \xi \in \mathcal{B}_{ll',j},\, a \in \mathcal{B}_{l',j}, \\ \langle \delta^k_{\mathcal{B}_{ll',j}}(\xi),\, \delta^k_{\mathcal{B}_{ll',j}}(\eta) \rangle := \delta^k_{\mathcal{B}_{l'l,j}}(\xi^*)\delta^k_{\mathcal{B}_{ll',j}}(\eta) &= \delta^k_{\mathcal{B}_{l',j}}(\xi^*\eta), \quad \xi, \eta \in \mathcal{B}_{ll',j}, \end{aligned}$$

endow $\mathcal{E}^j_{ll',k}$ with a structure of Hilbert $\delta^k_{\mathcal{B}_{l,j}}(\mathcal{B}_{l,j})$-$\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})$-bimodule. In what follows, we will provide an explicit formula for the action of $\mathbb{G}_j$ on $\mathcal{E}^j_{ll',k}$ obtained from that of $\mathbb{G}_j$ on $\mathcal{B}_{ll',j}$ by transport of structure.

Lemma 5.3.13. *For all $j, k, l, l' = 1, 2$, we have:*

1. *$\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})$ is a non-degenerate C^*-subalgebra of $\mathcal{M}(\mathcal{B}_{l',k} \otimes S_{kj})$. Then, the inclusion map $\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}) \subset \mathcal{M}(\mathcal{B}_{l',k} \otimes S_{kj})$ extends to a unital faithful *-homomorphism*

$$\mathcal{M}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})) \subset \mathcal{M}(\mathcal{B}_{l',k} \otimes S_{kj}).$$

2. *We have the following canonical inclusions:*

$$\begin{aligned} \mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}), \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j})) &\subset \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj}, \mathcal{B}_{ll',k} \otimes S_{kj}), \\ \mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}) \otimes S_{jj}, \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j}) \otimes S_{jj}) &\subset \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj} \otimes S_{jj}, \mathcal{B}_{ll',k} \otimes S_{kj} \otimes S_{jj}). \end{aligned}$$

Proof. By Proposition 5.1.3 3, we have $[\delta^k_{D_j}(D_j)(1_{D_k} \otimes S_{kj})] = D_k \otimes S_{kj}$. It then follows from Lemma 5.3.4 4 that

$$[\delta^k_{D_j}(D_{ll',j})(1_{D_{l',k}} \otimes S_{kj})] = D_{ll',k} \otimes S_{kj}.$$

By composing with $\pi_{D_k} \otimes \mathrm{id}_{S_{kj}}$, we obtain

$$[\delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j})(1_{\mathcal{B}_{l',k}} \otimes S_{kj})] = \mathcal{B}_{ll',k} \otimes S_{kj}.$$

It then follows from the equality $[\mathcal{B}_{ll',k}\mathcal{B}_{l',k}] = \mathcal{B}_{ll',k}$ that $[\delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j})(\mathcal{B}_{l',k} \otimes S_{kj})] = \mathcal{B}_{ll',k} \otimes S_{kj}$, which proves 1 by taking $l = l'$ and gives also a canonical embedding

$$\mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}), \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j})) \subset \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj}, \mathcal{B}_{ll',k} \otimes S_{kj}).$$

The second canonical embedding follows immediately by taking the tensor product by the C*-algebra S_{jj}. □

Notation 5.3.14. By the inclusion $\mathcal{E}^j_{ll',k} := \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j}) \subset \mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}), \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j}))$ and Lemma 5.3.13 2, we obtain an injective linear map

$$\mathcal{E}^j_{ll',k} \to \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj} \otimes S_{jj}, \mathcal{B}_{ll',k} \otimes S_{kj} \otimes S_{jj})\,;\, T \mapsto T_{12}.$$

Therefore, we can consider the linear map

$$\delta^j_{ll',k} : \mathcal{E}^j_{ll',k} \to \mathcal{L}(\mathcal{B}_{l',k} \otimes S_{kj} \otimes S_{jj}, \mathcal{B}_{ll',k} \otimes S_{kj} \otimes S_{jj})$$

defined by the formula

$$\delta^j_{ll',k}(T) = (V^k_{jj})_{23} T_{12} (V^k_{jj})^*_{23}, \quad T \in \mathcal{E}^j_{ll',k}.$$

Proposition 5.3.15. *For $j, k = 1, 2$, $j \neq k$, we have:*

1. *The linear map $\delta^j_{ll',k}$ takes its values in $\mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}) \otimes S_{jj}, \mathcal{E}^j_{ll',k} \otimes S_{jj})$.*

2. *The map*

$$\begin{array}{rcl} (\mathcal{B}_{ll',j}, \delta^j_{\mathcal{B}_{ll',j}}) & \longrightarrow & (\mathcal{E}^j_{ll',k}, \delta^j_{ll',k}) \\ \xi & \longmapsto & \delta^k_{\mathcal{B}_{ll',j}}(\xi) \end{array}$$

   *is an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules over the *-isomorphisms of* C*-*algebras $\delta^k_{\mathcal{B}_{l,j}} : \mathcal{B}_{l,j} \to \delta^k_{\mathcal{B}_{l,j}}(\mathcal{B}_{l,j})$ and $\delta^k_{\mathcal{B}_{l',j}} : \mathcal{B}_{l',j} \to \delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})$.*

Proof. In virtue of Corollary 5.1.4 applied to the continuous bidual action (δ_D, β_D) of $\mathcal{G}$ on the C*-algebra D, we obtain a $\mathbb{G}_j$-equivariant *-isomorphism:

$$\begin{array}{rcl} (D_j, \delta^j_{D_j}) & \longrightarrow & (\delta^k_{D_j}(D_j), \mathrm{id}_{D_k} \otimes \delta^j_{kj}) \\ x & \longmapsto & \delta^k_{D_j}(x). \end{array}$$

By Lemma 5.3.4 2, 4 and Corollary 5.3.6 2 a), we obtain that

$$(\pi_{D_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{D_j} : D_j \to \mathcal{M}(\mathcal{B}_k \otimes S_{kj}) \quad (\mathcal{B}_k := A_k \otimes \mathcal{K}(\mathscr{H}_{1k} \oplus \mathscr{H}_{2k}))$$

is a faithful *-homomorphism of linking algebras and we have

$$(\pi_{D_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{D_j}(d) = \delta^k_{\mathcal{B}_{ll',j}}(\pi_{D_j}(d)), \quad d \in D_{ll',j}. \tag{5.3.3}$$

By Proposition 5.1.3 2, we also have

$$(\mathrm{id}_{D_{ll',k}} \otimes \delta^j_{kj})\delta^k_{D_j}(d) = (\delta^k_{D_j} \otimes \mathrm{id}_{S_{jj}})\delta^j_{D_j}(d), \quad d \in D_{ll',j}.$$

We then compose with $\pi_{D_k} \otimes \mathrm{id}_{S_{kj}} \otimes \mathrm{id}_{S_{jj}}$ and we use (5.3.3) and Proposition 5.3.8 2 to conclude that

$$(\mathrm{id}_{\mathcal{B}_{ll',k}} \otimes \delta^j_{kj})\delta^k_{\mathcal{B}_{ll',j}}(x) = (\delta^k_{\mathcal{B}_{ll',j}} \otimes \mathrm{id}_{S_{jj}})\delta^j_{\mathcal{B}_{ll',j}}(x), \quad x \in \mathcal{B}_{ll',j}.$$

Let us consider the following bijective linear map $\Phi : \mathcal{B}_{ll',j} \to \mathcal{E}^j_{ll',k}\,;\, x \mapsto \delta^k_{\mathcal{B}_{ll',j}}(x)$. Then, we have

$$(\mathrm{id}_{\mathcal{B}_{ll',k}} \otimes \delta^j_{kj})(\xi) = (\Phi \otimes \mathrm{id}_{S_{jj}})\delta^j_{\mathcal{B}_{ll',j}}(\Phi^{-1}(\xi)), \quad \xi \in \mathcal{E}^j_{ll',k}.$$

However, we have $\delta^j_{kj}(s) = V^k_{jj}(s \otimes 1_{S_{jj}})(V^k_{jj})^*$ for all $s \in S_{kj}$. Therefore, for all $\xi \in \mathcal{E}^j_{ll',k}$ we have $(\mathrm{id}_{\mathcal{B}_{ll',k}} \otimes \delta^j_{kj})(\xi) = \delta^j_{ll',k}(\xi)$. It then follows that

$$\delta^j_{ll',k}(\Phi(x)) = (\Phi \otimes \mathrm{id}_{S_{jj}})\delta^j_{\mathcal{B}_{ll',j}}(x), \quad x \in \mathcal{B}^k_{ll',j}.$$

In particular, $(\mathcal{E}^j_{ll',k}, \delta^j_{ll',k})$ is a $\mathbb{G}_j$-equivariant Hilbert bimodule and we have also proved that the map $\Phi : (\mathcal{B}_{ll',j}, \delta^j_{\mathcal{B}_{ll',j}}) \to (\mathcal{E}^j_{ll',k}, \delta^j_{ll',k})$ is an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules over the isomorphisms of C*-algebras:

$$\Psi_l : \mathcal{B}_{l,j} \to \delta^k_{\mathcal{B}_{l,j}}(\mathcal{B}_{l,j})\,;\, x \mapsto \delta^k_{\mathcal{B}_{l,j}}(x), \quad \Psi_{l'} : \mathcal{B}_{l',j} \to \delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})\,;\, x \mapsto \delta^k_{\mathcal{B}_{l',j}}(x).$$

□

Chapter 6

Induction of actions

6.1 Correspondence between the actions of $\mathbb{G}_1$ and $\mathbb{G}_2$

In this paragraph, we fix a colinking measured quantum groupoid

$$\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$$

between two regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ and we continue to use all the notations introduced in §2.4 and §2.5 concerning the objects associated with $\mathcal{G}$.

We have already proved in §5.1 that we can associate to any $\mathcal{G}$-C*-algebra a $\mathbb{G}_i$-C*-algebra $(A_i, \delta^i_{A_i})$ for $i = 1, 2$ in a canonical way. If $\mathcal{G}$ is regular, we will prove that the functor $(A, \delta_A, \beta_A) \to (A_1, \delta^1_{A_1})$ is an equivalence of categories and we will build explicitly the inverse functor $(A_1, \delta_{A_1}) \to (A, \delta_A, \beta_A)$. More precisely, to any $\mathbb{G}_1$-C*-algebra (A_1, δ_{A_1}) we associate a $\mathbb{G}_2$-C*-algebra (A_2, δ_{A_2}) in a canonical way. Then, we will equip the C*-algebra $A := A_1 \oplus A_2$ with a structure of $\mathcal{G}$-C*-algebra (δ_A, β_A). This will allow us to build the inverse functor $(A_1, \delta_{A_1}) \to (A, \delta_A, \beta_A)$. The equivalence of categories $(A_1, \delta_{A_1}) \to (A_2, \delta_{A_2})$ generalizes the correspondence of actions for monoidally equivalent *compact* quantum groups of De Rijdt and Vander Vennet [12]. We recall that an induction procedure has been developed by De Commer in the von Neumann algebraic setting (see [9] §8).

Notations 6.1.1. Let $\delta_{A_1} : A_1 \to \mathcal{M}(A_1 \otimes S_{11})$ be a continuous action of $\mathbb{G}_1$ on a C*-algebra A_1. Let us denote:

$$\delta^1_{A_1} := \delta_{A_1}, \quad \delta^{(2)}_{A_1} := (\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1} : A_1 \to \mathcal{M}(A_1 \otimes S_{12} \otimes S_{21}).$$

Then, $\delta^{(2)}_{A_1}$ is a faithful non-degenerate *-homomorphism. In the following, we will identify S_{21} with $L_{21}(S_{21}) \subset \mathcal{B}(\mathscr{H}_{21})$. We define

$$\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1) := [(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a)\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*] \subset \mathcal{M}(A_1 \otimes S_{12}).$$

Note that $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1) = [(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi,\eta})\delta^{(2)}_{A_1}(a)\,;\, a \in A_1,\, \xi, \eta \in \mathscr{H}_{21}]$.

The main result of this paragraph is the following:

Proposition 6.1.2. *$\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$ is a C*-subalgebra of $\mathcal{M}(A_1 \otimes S_{12})$.*

Proof. Let $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$ and $a \in A_1$. We have

$$(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)(\delta^{(2)}_{A_1}(a))^* = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \overline{\omega})(\delta^{(2)}_{A_1}(a^*)).$$

Therefore, $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1)$ is stable by involution. Let $\xi, \eta, \xi', \eta' \in \mathscr{H}_{21}$, $a, a' \in A_1$. Let us denote $x = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi,\eta})\delta_{A_1}^{(2)}(a)$ and $x' = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi',\eta'})\delta_{A_1}^{(2)}(a')$. We have to prove that

$$xx' = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi,\eta} \otimes \omega_{\xi',\eta'})(\delta_{A_1}^{(2)}(a)_{123}\delta_{A_1}^{(2)}(a')_{124}) \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1).$$

Since $\mathscr{H}_{11}$ and $\mathscr{H}_{21}$ are nonzero Hilbert spaces, we have

$$\mathscr{H}_{21} = [\mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})\mathscr{H}_{11}], \quad \mathscr{H}_{21} = [\mathcal{K}(\mathscr{H}_{21})\mathscr{H}_{21}].$$

We can then assume that $\eta = k\eta_1$ and $\xi' = \ell\xi_1'$, where $\eta_1 \in \mathscr{H}_{21}$, $\xi_1' \in \mathscr{H}_{11}$, $k \in \mathcal{K}(\mathscr{H}_{21})$ and $\ell \in \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$. Since $\omega_{\xi,\eta} = k\omega_{\xi,\eta_1}$ and $\omega_{\xi',\eta'} = \omega_{\xi_1',\eta}\ell^*$, we have

$$xx' = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi,\eta_1} \otimes \omega_{\xi_1',\eta'})(\delta_{A_1}^{(2)}(a)_{123}(k \otimes \ell^*)_{34}\delta_{A_1}^{(2)}(a')_{124}).$$

Note that $\mathbb{G}_1$ and $\mathbb{G}_2$ are regular by assumption and $k \otimes \ell^* \in \mathcal{K}(\mathscr{H}_{21} \otimes \mathscr{H}_{21}, \mathscr{H}_{21} \otimes \mathscr{H}_{11})$. In virtue of Corollary 3.2.20 (see also Remarks 3.2.21 2), we have

$$\mathcal{K}(\mathscr{H}_{21} \otimes \mathscr{H}_{21}, \mathscr{H}_{21} \otimes \mathscr{H}_{11}) \subset [(1_{\mathscr{H}_{21}} \otimes \mathcal{K}(\mathscr{H}_{11}))(W_{21}^1)^*(\mathcal{K}(\mathscr{H}_{21}) \otimes 1_{\mathscr{H}_{21}})].$$

Therefore, xx' is the norm limit of finite sums of the form

$$\begin{aligned}
&\sum_i (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega_{\xi,\eta_1} \otimes \omega_{\xi_1',\eta'})(\delta_{A_1}^{(2)}(a)_{123}((1_{\mathscr{H}_{21}} \otimes k_i)(W_{21}^1)^*(\ell_i \otimes 1_{\mathscr{H}_{21}}))_{34}\delta_{A_1}^{(2)}(a')_{124}) \\
&\quad = \sum_i (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \ell_i\omega_{\xi,\eta_1} \otimes \omega_{\xi_1',\eta'}k_i)(\delta_{A_1}^{(2)}(a)_{123}(W_{21}^1)_{34}^*\delta_{A_1}^{(2)}(a')_{124}) \\
&\quad = \sum_i (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes (W_{21}^1)^*(\omega_{\xi,\eta_1}\ell_i \otimes k_i\omega_{\xi_1',\eta'}))(\delta_{A_1}^{(2)}(a)_{123}(W_{21}^1)_{34}^*\delta_{A_1}^{(2)}(a')_{124}(W_{21}^1)_{34}),
\end{aligned}$$

where $k_i \in \mathcal{K}(\mathscr{H}_{11})$, $\ell_i \in \mathcal{K}(\mathscr{H}_{21})$. Therefore, xx' is the norm limit of finite sums of elements of the form:

$$y = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\delta_{A_1}^{(2)}(a)_{123}(W_{21}^1)_{34}^*\delta_{A_1}^{(2)}(a')_{124}(W_{21}^1)_{34}), \quad \omega \in \mathcal{B}(\mathscr{H}_{21})_*, \, \omega' \in \mathcal{B}(\mathscr{H}_{11})_*.$$

Since $\delta_{21}^1(s) = (W_{21}^1)^*(1 \otimes s)W_{21}^1$ for all $s \in S_{21}$, we have

$$(W_{21}^1)_{34}^*\delta_{A_1}^{(2)}(a')_{124}(W_{21}^1)_{34} = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta_{21}^1)\delta_{A_1}^{(2)}(a').$$

Since $(\mathrm{id}_{S_{12}} \otimes \delta_{21}^1)\delta_{11}^2 = (\delta_{11}^2 \otimes \mathrm{id}_{S_{11}})\delta_{11}^1$ and $(\mathrm{id}_{A_1} \otimes \delta_{11}^1)\delta_{A_1}(a') = (\delta_{A_1} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a')$, we have

$$(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta_{21}^1)\delta_{A_1}^{(2)}(a') = (\delta_{A_1}^{(2)} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a').$$

Hence,

$$y = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\delta_{A_1}^{(2)}(a)_{123}(\delta_{A_1}^{(2)} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a')).$$

Let us write $\omega' = s\omega''$, where $s \in S_{11}$ and $\omega'' \in \mathcal{B}(\mathscr{H}_{11})_*$. By using the continuity of the action δ_{A_1}, $y = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega'')(\, \delta_{A_1}^{(2)}(a)_{123}(\delta_{A_1}^{(2)} \otimes \mathrm{id}_{S_{11}})(\delta_{A_1}(a')(1_{A_1} \otimes s))\,)$ is the norm limit of finite sums of the form

$$\sum_i (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\delta_{A_1}^{(2)}(a)_{123}(\delta_{A_1}^{(2)}(a_i) \otimes s_i)) = \sum_i \omega'(s_i)(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{A_1}^{(2)}(aa_i),$$

where $s_i \in S_{11}$, $a_i \in A_1$. The result is then proved. □

Remarks 6.1.3. 1. Actually, we have even proved that for all $\omega, \omega' \in \mathcal{B}(\mathscr{H}_{21})_*$ we have:

$$(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\delta_{A_1}^{(2)}(m)_{123}\delta_{A_1}^{(2)}(a)_{124}) \in \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1), \quad m \in \mathcal{M}(A_1),\, a \in A_1.$$

This remark will be used several times.

2. Proposition 6.1.2 holds true for strongly continuous actions of semi-regular locally compact quantum groups.

3. The idea of the proof of Proposition 6.1.2 is the same as that of Proposition 5.8 of [4].

Proposition 6.1.4. *Let us denote* $A_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1) \subset \mathcal{M}(A_1 \otimes S_{12})$. *We have:*

1. $[A_2(1_{A_1} \otimes S_{12})] = A_1 \otimes S_{12} = [(1_{A_1} \otimes S_{12})A_2]$. *In particular, the inclusion* $A_2 \subset \mathcal{M}(A_1 \otimes S_{12})$ *defines a faithful non-degenerate *-homomorphism and we have* $\mathcal{M}(A_2) \subset \mathcal{M}(A_1 \otimes S_{12})$.

2. *Let us denote* $\delta_{A_2} := (\mathrm{id}_{A_1} \otimes \delta_{12}^2)\restriction_{A_2}$. *We have* $\delta_{A_2}(A_2) \subset \mathcal{M}(A_2 \otimes S_{22})$ *and* δ_{A_2} *is a continuous action of* $\mathbb{G}_2$ *on* A_2.

3. *The correspondence* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} : \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_2\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}$ *is functorial.*

Proof. 1. Let us prove the inclusion $[A_2(1_{A_1} \otimes S_{12})] \subset A_1 \otimes S_{12}$. Let $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$, $a \in A_1$ and $s \in S_{12}$. Let us denote $x = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{A_1}^{(2)}(a)$. Let us show that $x(1_{A_1} \otimes s) \in A_1 \otimes S_{12}$. Let us write $\omega = s'\omega'$, where $s' \in S_{21}$ and $\omega' \in \mathcal{B}(\mathscr{H}_{21})_*$. We have

$$x(1_{A_1} \otimes s) = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')(\delta_{A_1}^{(2)}(a)(1_{A_1} \otimes s \otimes s')).$$

Since $S_{12} \otimes S_{21} = [\delta_{11}^2(S_{11})(S_{12} \otimes 1_{S_{21}})]$ (see Proposition 2.5.7) and $s \otimes s' \in S_{12} \otimes S_{21}$, it follows that $x(1_{A_1} \otimes s)$ is the norm limit of finite sums of elements of the form

$$\begin{aligned} z &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')(\delta_{A_1}^{(2)}(a)(1_{A_1} \otimes \delta_{11}^2(y))(1_{A_1} \otimes y' \otimes 1_{S_{21}})) \\ &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')(\,(\mathrm{id}_{A_1} \otimes \delta_{11}^2)(\delta_{A_1}(a)(1_{A_1} \otimes y))\,)(1_{A_1} \otimes y'), \end{aligned}$$

where $y \in S_{11}$ and $y' \in S_{12}$. By continuity of the action δ_{A_1}, z is the norm limit of finite sums of the form:

$$\sum_i a_i \otimes (\mathrm{id}_{S_{12}} \otimes \omega')(\delta_{11}^2(y_i))y', \quad a_i \in A_1,\, y_i \in S_{11}.$$

Hence, $x(1_{A_1} \otimes s) \in A_1 \otimes S_{12}$ since $(\mathrm{id}_{S_{12}} \otimes \omega')(\delta_{11}^2(y_i)) \in \mathcal{M}(S_{12})$. In a similar way, we obtain the converse inclusion by applying successively the following formulas (*cf.* Proposition 2.5.7 3):

$$S_{12} = [(\mathrm{id}_{S_{12}} \otimes \omega)(\delta_{11}^2(s))\,;\, s \in S_{11},\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*], \quad A_1 \otimes S_{11} = [\delta_{A_1}(A_1)(1_{A_1} \otimes S_{11})],$$

$$[\delta_{11}^2(S_{11})(1_{S_{12}} \otimes S_{21})] = S_{12} \otimes S_{21}.$$

2. It is clear that δ_{A_2} is a faithful *-homomorphism. Let us prove that δ_{A_2} takes its values in $\mathcal{M}(A_2 \otimes S_{22})$. Let $a \in A_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Let us denote $x = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{A_1}^{(2)}(a)$. Let us prove that $(\mathrm{id}_{A_1} \otimes \delta_{12}^2)(x) \in \mathcal{M}(A_2 \otimes S_{22}) \subset \mathcal{M}(A_1 \otimes S_{12} \otimes S_{21})$. By using the formulas $(\delta_{12}^2 \otimes \mathrm{id}_{S_{21}})\delta_{11}^2 = (\mathrm{id}_{S_{12}} \otimes \delta_{21}^2)\delta_{11}^2$ and $\delta_{21}^2(s) = (W_{22}^1)^*(1 \otimes s)W_{22}^1$ for $s \in S_{21}$, we have

$$\begin{aligned} (\mathrm{id}_{A_1} \otimes \delta_{12}^2 \otimes \mathrm{id}_{S_{21}})\delta_{A_1}^{(2)}(a) &= (\mathrm{id}_{A_1} \otimes \delta_{12}^2 \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta_{11}^2)\delta_{A_1}(a) \\ &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta_{21}^2)(\mathrm{id}_{A_1} \otimes \delta_{11}^2)\delta_{A_1}(a) \\ &= (W_{22}^1)_{34}^* \delta_{A_1}^{(2)}(a)_{124}(W_{22}^1)_{34}. \end{aligned}$$

Since $W^1_{22} \in \mathcal{M}(S_{22} \otimes \mathcal{K}(\mathscr{H}_{21}))$, we have

$$(\mathrm{id}_{A_1} \otimes \delta^2_{12})(x) = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \omega)((W^1_{22})^*_{34}\delta^{(2)}_{A_1}(a)_{124}(W^1_{22})_{34}) \in \mathcal{M}(A_1 \otimes S_{12} \otimes S_{22}).$$

Let $s \in S_{22}$ and let us write $\omega = \omega' u$, where $u \in \mathcal{K}(\mathscr{H}_{21})$ and $\omega' \in \mathcal{B}(\mathscr{H}_{21})_*$. By using again the fact that $W^1_{22} \in \mathcal{M}(S_{22} \otimes \mathcal{K}(\mathscr{H}_{21}))$, it follows that

$$(1_{A_2} \otimes s)(\mathrm{id}_{A_1} \otimes \delta^2_{12})(x) = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \omega')(\,((s \otimes u)(W^1_{22})^*)_{34}\delta^{(2)}_{A_1}(a)_{124}(W^1_{22})_{34}\,)$$

is the norm limit of finite sums of elements of the form

$$y = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \omega'')(\delta^{(2)}_{A_1}(a)_{124}((s' \otimes 1_{\mathscr{H}_{21}})W^1_{22}(1_{S_{22}} \otimes v))_{34}),$$

where $s' \in S_{22}$, $v \in \mathcal{K}(\mathscr{H}_{21})$, $\omega'' \in \mathcal{B}(\mathscr{H}_{21})_*$. Since $\mathbb{G}_2$ is regular, we have (*cf.* (2.5.1) and Proposition 1.3.11):

$$[(S_{22} \otimes 1_{\mathscr{H}_{21}})W^1_{22}(1_{S_{22}} \otimes \mathcal{K}(\mathscr{H}_{21}))] = S_{22} \otimes \mathcal{K}(\mathscr{H}_{21}). \tag{6.1.1}$$

Hence, $(1_{A_1 \otimes S_{12}} \otimes s)(\mathrm{id}_{A_1} \otimes \delta^2_{12})(x) \in A_2 \otimes S_{22}$. We then have $(1_{A_2} \otimes S_{22})\delta_{A_2}(A_2) \subset A_2 \otimes S_{22}$. Then, we also have $\delta_{A_2}(A_2)(1_{A_2} \otimes S_{22}) \subset A_2 \otimes S_{22}$ since δ_{A_2} is involutive. In particular, we have $\delta_{A_2}(A_2) \subset \mathcal{M}(A_2 \otimes S_{22})$.
Let us prove that $[(1_{A_2} \otimes S_{22})\delta_{A_2}(A_2)] = A_2 \otimes S_{22}$. It only remains to prove the inclusion $A_2 \otimes S_{22} \subset [(1_{A_2} \otimes S_{22})\delta_{A_2}(A_2)]$. To do so, we have to follow backward the above argument. Let $a \in A_1$, $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$, $s \in S_{22}$ and let us denote $x = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a)$. Let us write $\omega = v\omega' u$ with $u, v \in \mathcal{K}(\mathscr{H}_{21})$ and $\omega' \in \mathcal{B}(\mathscr{H}_{21})_*$. We have

$$x \otimes s = (\mathrm{id}_{A_1 \otimes S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \omega')((1_{A_1 \otimes S_{12}} \otimes 1_{S_{22}} \otimes u)\delta^{(2)}_{A_1}(a)_{124}(1_{A_1 \otimes S_{12}} \otimes s \otimes v)).$$

By using again the fact that $W^1_{22} \in \mathcal{M}(S_{22} \otimes \mathcal{K}(\mathscr{H}_{21}))$ and (6.1.1), we obtain that $x \otimes s$ is the norm limit of finite sums of elements of the form

$$\begin{aligned} y &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \phi)((1_{A_1 \otimes S_{12}} \otimes s' \otimes u')(W^1_{22})^*_{34}\delta^{(2)}_{A_1}(a)_{124}(W^1_{22})_{34}) \\ &= (1_{A_1} \otimes 1_{S_{12}} \otimes s')(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{22}} \otimes \phi u')((W^1_{22})^*_{34}\delta^{(2)}_{A_1}(a)_{124}(W^1_{22})_{34}), \end{aligned}$$

where $s' \in S_{22}$, $u' \in \mathcal{K}(\mathscr{H}_{21})$ and $\phi \in \mathcal{B}(\mathscr{H}_{21})_*$. We recall (see above) that we have

$$(\mathrm{id}_{A_1} \otimes \delta^2_{12} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1}(a) = (W^1_{22})^*_{34}\delta^{(2)}_{A_1}(a)_{124}(W^1_{22})_{34}.$$

Therefore, we have

$$y = (1_{A_1} \otimes 1_{S_{12}} \otimes s')(\mathrm{id}_{A_1} \otimes \delta^2_{12})((\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \phi u')\delta^{(2)}_{A_1}(a)),$$

which proves that $x \otimes s \in [(1_{A_2} \otimes S_{22})\delta_{A_2}(A_2)]$.

In particular, δ_{A_2} is non-degenerate. By using $(\mathrm{id}_{S_{12}} \otimes \delta^2_{22})\delta^2_{12} = (\delta^2_{12} \otimes \mathrm{id}_{S_{22}})\delta^2_{12}$, we have $(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta^2_{22})(\mathrm{id}_{A_1} \otimes \delta^2_{12})(m) = (\mathrm{id}_{A_1} \otimes \delta^2_{12} \otimes \mathrm{id}_{S_{22}})(\mathrm{id}_{A_1} \otimes \delta^2_{12})(m)$ for all $m \in \mathcal{M}(A_1 \otimes S_{12})$. In particular, we obtain the coassociativity of δ_{A_2}, that is $(\mathrm{id}_{A_2} \otimes \delta^2_{22})\delta_{A_2} = (\delta_{A_2} \otimes \mathrm{id}_{S_{22}})\delta_{A_2}$.

3. Let us consider (A_1, δ_{A_1}) and (B_1, δ_{B_1}) two $\mathbb{G}_1$-C*-algebras. Let us denote $A_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$ and $B_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1)$. Let $f_1 \in \mathrm{Mor}_{\mathbb{G}_1}(A_1, B_2)$ and let $f_1 \otimes \mathrm{id}_{S_{12}}$ be the unital strictly continuous

extension to $\mathcal{M}(A_1 \otimes S_{12})$ of the *-homomorphism $A_1 \otimes S_{12} \to \mathcal{M}(B_1 \otimes S_{12})\,;\, x \mapsto (f_1 \otimes \mathrm{id}_{S_{12}})(x)$. Let $a \in A_1$, $b \in B_1$ and ω, $\phi \in \mathcal{B}(\mathscr{H}_{21})_*$. Since f_1 is $\mathbb{G}_1$-equivariant, it is clear that

$$(f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1} = \delta^{(2)}_{B_1} \circ f_1. \tag{6.1.2}$$

Let us denote $x = (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a)$ and $y = (\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \phi)\delta^{(2)}_{B_1}(b)$. By (6.1.2), we have

$$\begin{aligned}(f_1 \otimes \mathrm{id}_{S_{12}})(x)y &= (\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \phi)(\,((f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1}(a))_{123}\delta^{(2)}_{B_1}(b)_{124}\,)\\ &= (\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \phi)(\delta^{(2)}_{B_1}(f_1(a))_{123}\delta^{(2)}_{B_1}(b)_{124}).\end{aligned}$$

In virtue of Remark 6.1.3 1, we have $(f_1 \otimes \mathrm{id}_{S_{12}})(x)y \in B_2$. Hence, $(f_1 \otimes \mathrm{id}_{S_{12}})(A_2)B_2 \subset B_2$. However, we also have the inclusion $B_2(f_1 \otimes \mathrm{id}_{S_{12}})(A_2) \subset B_2$ since f_1 is stable by the involution. We then obtain $(f_1 \otimes \mathrm{id}_{S_{12}})(A_2) \subset \mathcal{M}(B_2)$. Therefore, $f_1 \otimes \mathrm{id}_{S_{12}}$ restricts to a *-homomorphism $f_2 : A_2 \to \mathcal{M}(B_2)$. Moreover, it is clear that f_2 is non-degenerate. In particular, f_2 extends to a unital strictly continuous *-homomorphism $f_2 : \mathcal{M}(A_2) \to \mathcal{M}(B_2)$. Now, the fact that f_2 is $\mathbb{G}_2$-equivariant is straightforward. Indeed, if $x \in A_2$ we have

$$(f_2 \otimes \mathrm{id}_{S_{12}})\delta_{A_2}(x) = (f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^2_{12})(x) = (\mathrm{id}_{B_1} \otimes \delta^2_{12})(f_1 \otimes \mathrm{id}_{S_{12}})(x) = \delta_{B_2}(f_2(x)).$$

□

Starting from a continuous action of $\mathbb{G}_2$ on a C*-algebra A_2, we define *mutatis mutandis* the following faithful non-degenerate *-homomorphisms:

$$\delta^2_{A_2} := \delta_{A_2}, \quad \delta^{(1)}_{A_2} := (\mathrm{id}_{A_2} \otimes \delta^1_{22})\delta_{A_2} : A_2 \to \mathcal{M}(A_2 \otimes S_{21} \otimes S_{12}).$$

By identifying S_{12} with $L_{12}(S_{12}) \subset \mathcal{B}(\mathscr{H}_{12})$, we define

$$\mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(A_2) := [(\mathrm{id}_{A_2} \otimes \mathrm{id}_{S_{21}} \otimes \omega)\delta^{(1)}_{A_2}(a)\,;\, a \in A_2,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*] \subset \mathcal{M}(A_2 \otimes S_{21}).$$

We have the following result:

Proposition 6.1.5. *Let us denote $A_1 := \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(A_2) \subset \mathcal{M}(A_2 \otimes S_{21})$. We have:*

1. *A_1 is a C*-subalgebra of $\mathcal{M}(A_2 \otimes S_{21})$.*
2. *$[A_1(1_{A_2} \otimes S_{21})] = A_2 \otimes S_{21} = [(1_{A_2} \otimes S_{21})A_1]$. In particular, the inclusion $A_1 \subset \mathcal{M}(A_2 \otimes S_{21})$ defines a faithful non-degenerate *-homomorphism and we have $\mathcal{M}(A_1) \subset \mathcal{M}(A_2 \otimes S_{21})$.*
3. *Let us denote $\delta_{A_1} := (\mathrm{id}_{A_2} \otimes \delta^1_{21})\restriction_{A_1}$. We have $\delta_{A_1}(A_1) \subset \mathcal{M}(A_1 \otimes S_{11})$ and δ_{A_1} is a continuous action of $\mathbb{G}_1$ on A_1.*
4. *The correspondence $\mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2} : \mathbb{G}_2$-C*-Alg $\to \mathbb{G}_1$-C*-Alg is functorial.*

In the following propositions, we investigate the compositions of induction functors:

$$\mathbb{G}_1\text{-C}^*\text{-Alg} \to \mathbb{G}_2\text{-C}^*\text{-Alg} \to \mathbb{G}_1\text{-C}^*\text{-Alg}, \quad \mathbb{G}_2\text{-C}^*\text{-Alg} \to \mathbb{G}_1\text{-C}^*\text{-Alg} \to \mathbb{G}_2\text{-C}^*\text{-Alg}.$$

Proposition 6.1.6. *Let (A_1, δ_{A_1}) be a $\mathbb{G}_1$-C*-algebra. Let us denote $A_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$ endowed with the continuous action $\delta_{A_2} = (\mathrm{id}_{A_1} \otimes \delta^2_{12})\restriction_{A_2}$. Let us consider $C = \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(A_2) \subset \mathcal{M}(A_2 \otimes S_{21})$ endowed with the continuous action $\delta_C = (\mathrm{id}_{A_2} \otimes \delta^1_{21})\restriction_C$. Then, we have:*

1. *$C \subset \mathcal{M}(A_2 \otimes S_{21}) \subset \mathcal{M}(A_1 \otimes S_{12} \otimes S_{21})$ and $C = \delta^{(2)}_{A_1}(A_1)$.*

2. *The map*

$$\begin{aligned}\pi_1 : (A_1, \delta_{A_1}) &\longrightarrow (C, \delta_C)\\ x &\longmapsto \delta^{(2)}_{A_1}(x) := (\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1}(x)\end{aligned}$$

*is a $\mathbb{G}_1$-equivariant *-isomorphism.*

3. *The map*

$$\begin{aligned}\delta^2_{A_1} : A_1 &\longrightarrow \mathcal{M}(A_2 \otimes S_{21})\\ a &\longmapsto \delta^{(2)}_{A_1}(a) := (\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1}(x)\end{aligned}$$

*is a faithful non-degenerate *-homomorphism such that $[\delta^2_{A_1}(A_1)(1_{A_2} \otimes S_{21})] = A_2 \otimes S_{21}$ and $(\delta_{A_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1} = (\mathrm{id}_{A_2} \otimes \delta^2_{21})\delta^2_{A_1}$.*

Proof. By unfolding the C*-algebra C, we obtain

$$C = [(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}} \otimes \omega \otimes \phi)(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta^1_{22} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^2_{12} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1}(a)\,;\; a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*,\, \phi \in \mathcal{B}(\mathscr{H}_{21})_*].$$

In particular, we have $C \subset \mathcal{M}(A_1 \otimes S_{12} \otimes S_{21})$. Moreover, by using coassociativity formulas we obtain

$$\begin{aligned}(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta^1_{22} &\otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^2_{12} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1}(a)\\ &= (\mathrm{id}_{A_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^1_{12} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1}(a)\\ &= (\mathrm{id}_{A_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{11}} \otimes \delta^2_{11})(\mathrm{id}_{A_1} \otimes \delta^1_{11})\delta_{A_1}(a)\\ &= (\mathrm{id}_{A_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{11}} \otimes \delta^2_{11})(\delta_{A_1} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a)\\ &= ((\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1}(a)\\ &= (\delta^{(2)}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{A_1}(a),\end{aligned}$$

for all $a \in A_1$. Therefore, we have

$$C = [\delta^{(2)}_{A_1}(\mathrm{id}_{A_1} \otimes \omega \otimes \phi)\delta^{(2)}_{A_1}(a)\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*,\, \phi \in \mathcal{B}(\mathscr{H}_{21})_*].$$

Since $\delta^2_{11}(s) = (W^1_{12})^*(1 \otimes s)W^1_{12}$ for all $s \in S_{11}$, we have $\delta^{(2)}_{A_1}(a) = (W^1_{12})^*_{23}\delta_{A_1}(a)_{13}(W^1_{12})_{23}$ for all $a \in A_1$. If $\omega \in \mathcal{B}(\mathscr{H}_{12})_*$, $\phi \in \mathcal{B}(\mathscr{H}_{21})_*$ and $a \in A_1$, we have

$$(\mathrm{id}_{A_1} \otimes \omega \otimes \phi)\delta^{(2)}_{A_1}(a) = (\mathrm{id}_{A_1} \otimes W^1_{12}(\omega \otimes \phi)(W^1_{12})^*)(\delta_{A_1}(a)_{13}).$$

Since $W^1_{12} : \mathscr{H}_{12} \otimes \mathscr{H}_{21} \to \mathscr{H}_{12} \otimes \mathscr{H}_{11}$ is unitary, we obtain

$$\begin{aligned}[(\mathrm{id}_{A_1} \otimes \omega \otimes \phi)\delta^{(2)}_{A_1}(a)\,;\, a \in &A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*,\, \phi \in \mathcal{B}(\mathscr{H}_{21})_*]\\ &= [(\mathrm{id}_{A_1} \otimes \omega \otimes \phi)(\delta_{A_1}(a)_{13})\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*,\, \phi \in \mathcal{B}(\mathscr{H}_{21})_*]\\ &= [(\mathrm{id}_{A_1} \otimes \phi)\delta_{A_1}(a)\,;\, a \in A_1,\, \phi \in \mathcal{B}(\mathscr{H}_{21})_*] = A_1.\end{aligned}$$

In particular, $C = \delta^{(2)}_{A_1}(A_1)$. Since the *-homomorphism $\delta^{(2)}_{A_1}$ is faithful, we obtain that π_1 is a *-isomorphism. Let us prove that π_1 is $\mathbb{G}_1$-equivariant. Let $a \in A_1$, we have

$$\begin{aligned}\delta_C(\pi_1(a)) &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta^1_{21})(\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta_{A_1}(a) = (\mathrm{id}_{A_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{11}})(\mathrm{id}_{A_1} \otimes \delta^1_{11})\delta_{A_1}(a)\\ &= (\mathrm{id}_{A_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{11}})(\delta_{A_1} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a) = (\delta^{(2)}_{A_1} \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a) = (\pi_1 \otimes \mathrm{id}_{S_{11}})\delta_{A_1}(a).\end{aligned}$$

The statements 1 and 2 are proved. The last statement follows from the first one, Proposition 6.1.5 2 ($[C(1_{A_2} \otimes S_{21})] = A_2 \otimes S_{21}$) and the formula $(\delta^2_{12} \otimes \mathrm{id}_{S_{21}})\delta^2_{11} = (\mathrm{id}_{S_{12}} \otimes \delta^2_{21})\delta^2_{11}$. □

Starting from a continuous action of $\mathbb{G}_2$ on a C*-algebra A_2, we obtain *mutatis mutandis* the following result:

Proposition 6.1.7. *Let (A_2, δ_{A_2}) be a $\mathbb{G}_2$-C*-algebra. Let us denote $A_1 = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2)$ endowed with the continuous action $\delta_{A_1} = (\mathrm{id}_{A_2} \otimes \delta_{21}^1)\restriction_{A_1}$. Let us consider $D = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1) \subset \mathcal{M}(A_1 \otimes S_{12})$ endowed with the continuous action $\delta_D = (\mathrm{id}_{A_1} \otimes \delta_{12}^2)\restriction_D$. Then, we have:*

1. *$D \subset \mathcal{M}(A_1 \otimes S_{12}) \subset \mathcal{M}(A_2 \otimes S_{21} \otimes S_{12})$ and $D = \delta_{A_2}^{(1)}(A_2)$.*

2. *The map*
$$\begin{aligned} \pi_2 : (A_2, \delta_{A_2}) &\longrightarrow (D, \delta_D) \\ a &\longmapsto \delta_{A_2}^{(1)}(a) := (\mathrm{id}_{A_2} \otimes \delta_{22}^1)\delta_{A_2}(a) \end{aligned}$$
*is a $\mathbb{G}_2$-equivariant *-isomorphism.*

3. *The map*
$$\begin{aligned} \delta_{A_2}^1 : A_2 &\longrightarrow \mathcal{M}(A_1 \otimes S_{12}) \\ a &\longmapsto \delta_{A_2}^{(1)}(a) := (\mathrm{id}_{A_2} \otimes \delta_{22}^1)\delta_{A_2}(a) \end{aligned}$$
*is a faithful non-degenerate *-homomorphism such that $[\delta_{A_2}^1(A_2)(1_{A_1} \otimes S_{12})] = A_1 \otimes S_{12}$ and $(\delta_{A_1} \otimes \mathrm{id}_{S_{12}})\delta_{A_2}^1 = (\mathrm{id}_{A_1} \otimes \delta_{12}^1)\delta_{A_2}^1$.*

Therefore, we have established the following result:

Theorem 6.1.8. *Let $\mathcal{G}$ be a colinking measured quantum groupoid between two regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. The induction functors*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} : \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_2\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \quad ; \quad (A_1, \delta_{A_1}) \mapsto (A_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1), \delta_{A_2} = (\mathrm{id}_{A_1} \otimes \delta_{12}^2)\restriction_{A_2}),$$

$$\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1} : \mathbb{G}_2\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \quad ; \quad (A_2, \delta_{A_2}) \mapsto (A_1 = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2), \delta_{A_1} = (\mathrm{id}_{A_2} \otimes \delta_{21}^1)\restriction_{A_1})$$

are inverse of each other.

Proof. This follows from Propositions 6.1.6 and 6.1.7. It only remains to verify the naturality. Let (A_1, δ_{A_1}) and $(A_1', \delta_{A_1'})$ be two $\mathbb{G}_1$-C*-algebras. Let us denote $(A_2, \delta_{A_2}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1, \delta_{A_1})$, $(A_2', \delta_{A_2'}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1', \delta_{A_1'})$, $(C, \delta_C) = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2, \delta_{A_2})$ and $(C', \delta_{C'}) = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2', \delta_{A_2'})$. Let us then consider $\pi_1 : A_1 \to C$ and $\pi_1' : A_1' \to C'$ the $\mathbb{G}_1$-equivariant *-isomorphisms defined in Proposition 6.1.6 1. Let $f \in \mathrm{Mor}_{\mathbb{G}_1}(A_1, A_1')$, we denote $f_* = (f \otimes \mathrm{id}_{S_{12}})\restriction_{A_2} \in \mathrm{Mor}_{\mathbb{G}_2}(A_2, A_2')$ and $f_{**} = (f_* \otimes \mathrm{id}_{S_{21}})\restriction_C \in \mathrm{Mor}_{\mathbb{G}_1}(C, C')$ (see proof of Propositions 6.1.4 3 and 6.1.5 4). We have

$$\begin{aligned} f_{**} \circ \pi_1 = (f_* \otimes \mathrm{id}_{S_{21}})\pi_1 = (f \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta_{A_1}^{(2)} = (f \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{A_1} \otimes \delta_{11}^2)\delta_{A_1} \\ = (\mathrm{id}_{A_1'} \otimes \delta_{11}^2)(f \otimes \mathrm{id}_{S_{11}})\delta_{A_1} = (\mathrm{id}_{A_1'} \otimes \delta_{11}^2) \circ \delta_{A_1'} \circ f = \delta_{A_1'}^{(2)} \circ f = \pi_1' \circ f. \end{aligned}$$

□

Proposition 6.1.9. *Let $j, k = 1, 2$, $j \neq k$. We have:*

1. *If $y \in S_{jk}$, we have $\delta_{jk}^j(y) \in \mathrm{Ind}_{\mathbb{G}_j}^{\mathbb{G}_k}(S_{jj})$. Moreover, the map*
$$\begin{aligned} (S_{jk}, \delta_{jk}^k) &\longrightarrow \mathrm{Ind}_{\mathbb{G}_j}^{\mathbb{G}_k}(S_{jj}, \delta_{jj}^j) \\ y &\longmapsto \delta_{jk}^j(y) \end{aligned}$$
*is a $\mathbb{G}_k$-equivariant *-isomorphism.*

2. If $y \in S_{jj}$, we have $\delta^k_{jj}(y) \in \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(S_{jk})$. Moreover, the map

$$\begin{array}{ccc}(S_{jj}, \delta^j_{jj}) & \longrightarrow & \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(S_{jk}, \delta^k_{jk}) \\ y & \longmapsto & \delta^k_{jj}(y)\end{array}$$

*is a $\mathbb{G}_j$-equivariant *-isomorphism.*

Proof. 1. By definition, we have

$$\mathrm{Ind}^{\mathbb{G}_k}_{\mathbb{G}_j}(S_{jj}) = [(\mathrm{id}_{S_{jj}} \otimes \mathrm{id}_{S_{jk}} \otimes \omega)\delta^{j(k)}_{jj}(s)\,;\, s \in S_{jj},\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*].$$

However, $\delta^{j(k)}_{jj} := (\mathrm{id}_{S_{jj}} \otimes \delta^k_{jj})\delta^j_{jj} = (\delta^j_{jk} \otimes \mathrm{id}_{S_{kj}})\delta^k_{jj}$. Hence,

$$(\mathrm{id}_{S_{jj}} \otimes \mathrm{id}_{S_{jk}} \otimes \omega)\delta^{j(k)}_{jj}(s) = \delta^j_{jk}(\mathrm{id}_{S_{jk}} \otimes \omega)\delta^k_{jj}(s), \quad s \in S_{jj},\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*.$$

Moreover, we have $S_{jk} = [(\mathrm{id}_{S_{jk}} \otimes \omega)\delta^k_{jj}(s)\,;\, s \in S_{jj},\, \omega \in \mathcal{B}(\mathscr{H}_{kj})_*]$ (see Proposition 2.5.7 3). Therefore, if $y \in S_{jk}$ we have $\delta^j_{jk}(y) \in \mathrm{Ind}^{\mathbb{G}_k}_{\mathbb{G}_j}(S_{jj})$. Actually, we have even proved that $\mathrm{Ind}^{\mathbb{G}_k}_{\mathbb{G}_j}(S_{jj}) = \delta^j_{jk}(S_{jk})$. Since δ^j_{jk} is faithful, the map $S_{jk} \to \mathrm{Ind}^{\mathbb{G}_k}_{\mathbb{G}_j}(S_{jj})\,;\, y \mapsto \delta^j_{jk}(y)$ is a *-isomorphism, which is also $\mathbb{G}_j$-equivariant in virtue of $(\mathrm{id}_{S_{jj}} \otimes \delta^k_{jk})\delta^j_{jk} = (\delta^j_{jk} \otimes \mathrm{id}_{S_{kk}})\delta^k_{jk}$.
2. In a similar way, we prove the second statement by using the formulas:

$$(\mathrm{id}_{S_{jk}} \otimes \delta^j_{kk})\delta^k_{jk} = (\delta^k_{jj} \otimes \mathrm{id}_{S_{jk}})\delta^j_{jk}, \quad S_{jj} = [(\mathrm{id}_{S_{jj}} \otimes \omega)\delta^j_{jk}(y)\,;\, y \in S_{jk},\, \omega \in \mathcal{B}(\mathscr{H}_{jk})_*],$$

$$(\mathrm{id}_{S_{jk}} \otimes \delta^j_{kj})\delta^k_{jj} = (\delta^k_{jj} \otimes \mathrm{id}_{S_{jj}})\delta^j_{jj}.$$

□

By applying the induction procedure established above and Lemma 5.1.9, we will define a correspondence $\mathbb{G}_1$-C*-Alg $\to \mathcal{G}$-C*-Alg inverse of $\mathcal{G}$-C*-Alg $\to \mathbb{G}_1$-C*-Alg (see Corollary 5.1.8).

Notations 6.1.10. Let (B_1, δ_{B_1}) be a $\mathbb{G}_1$-C*-algebra. Let (B_2, δ_{B_2}) be the induced $\mathbb{G}_2$-C*-algebra, that is to say $B_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1)$ and $\delta_{B_2} = (\mathrm{id}_{B_1} \otimes \delta^2_{12})\restriction_{B_2}$. In virtue of Propositions 6.1.6 and 6.1.7, we have four *-homomorphisms:

$$\delta^k_{B_j} : B_j \to \mathcal{M}(B_k \otimes S_{kj}), \quad j,k = 1,2.$$

Let us give a precise description of them. We have denoted $\delta^1_{B_1} := \delta_{B_1}$ and $\delta^2_{B_2} := \delta_{B_2}$. The *-homomorphism $\delta^2_{B_1} : B_1 \to \mathcal{M}(B_2 \otimes S_{21})$ is given by

$$b \in B_1 \mapsto \delta^2_{B_1}(b) := \delta^{(2)}_{B_1}(b) \in \mathcal{M}(B_2 \otimes S_{21}) \quad ; \quad \delta^{(2)}_{B_1}(b) := (\mathrm{id}_{B_1} \otimes \delta^2_{11})\delta^1_{B_1}(b), \quad b \in B_1,$$

whereas the *-homomorphism $\delta^1_{B_2} : B_2 \to \mathcal{M}(B_1 \otimes S_{12})$ is defined by the formula

$$(\pi_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{B_2}(b) = \delta^{(1)}_{B_2}(b), \quad b \in B_2,$$

where

$$\delta^{(1)}_{B_2} := (\mathrm{id}_{B_2} \otimes \delta^1_{22})\delta^2_{B_2}, \quad \pi_1 : B_1 \to \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(B_2)\,;\, b \mapsto \delta^{(2)}_{B_1}(b).$$

Lemma 6.1.11. *For $j,k,l = 1,2$, we have:*

1. $(\delta^l_{B_k} \otimes \mathrm{id}_{S_{kj}})\delta^k_{B_j} = (\mathrm{id}_{B_l} \otimes \delta^k_{lj})\delta^l_{B_j}$.
2. $[\delta^l_{B_k}(B_k)(1_{B_l} \otimes S_{lk})] = B_l \otimes S_{lk}$.

3. $B_l = [(\mathrm{id}_{B_l} \otimes \omega)\delta^l_{B_k}(B_k)\,;\,\omega \in \mathcal{B}(\mathscr{H}_{lk})_*]$.

Proof. 1. We have already proved the coassociativity formulas corresponding to the cases $(j,k,l) = (1,1,1)$, $(j,k,l) = (2,2,2)$ (Proposition 6.1.4 2) and $(j,k,l) = (1,2,2)$ (Proposition 6.1.6 3). The cases $(j,k,l) = (1,1,2)$ and $(j,k,l) = (2,1,2)$ holds by definition of the *-homomorphisms $\delta^2_{B_1}$ and $\delta^1_{B_2}$. Indeed, for the first case we have

$$\begin{aligned}(\delta^2_{B_1} \otimes \mathrm{id}_{S_{11}})\delta^1_{B_1} &= (\mathrm{id}_{B_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{11}})(\delta^1_{B_1} \otimes \mathrm{id}_{S_{11}})\delta^1_{B_1} = (\mathrm{id}_{B_1} \otimes \delta^2_{11} \otimes \mathrm{id}_{S_{11}})(\mathrm{id}_{B_1} \otimes \delta^1_{11})\delta^1_{B_1}\\ &= (\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \delta^1_{21})(\mathrm{id}_{B_1} \otimes \delta^2_{11})\delta^1_{B_1} = (\mathrm{id}_{B_2} \otimes \delta^1_{21})\delta^2_{B_1}.\end{aligned}$$

For the second one, since $\delta^2_{B_1}(b) = \delta^{(2)}_{B_1}(b) = \pi_1(b) \in \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(B_2) \subset \mathcal{M}(B_2 \otimes S_{21})$ for $b \in B_1$, we have

$$(\delta^2_{B_1} \otimes \mathrm{id}_{S_{12}})\delta^1_{B_2} = (\pi_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{B_2} = \delta^{(1)}_{B_2} = (\mathrm{id}_{B_2} \otimes \delta^1_{22})\delta^2_{B_2}.$$

The remaining cases, *i.e.* $(j,k,l) = (1,2,1)$ and $(j,k,l) = (2,1,1)$, are not immediate. In order to check out the first case, we compose by $\delta^2_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}}$ and we use some already established coassociativity formulas as follows:

$$\begin{aligned}(\delta^2_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\delta^1_{B_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{B_1} &= (\mathrm{id}_{B_2} \otimes \delta^1_{22} \otimes \mathrm{id}_{S_{21}})(\delta^2_{B_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{B_1}, \quad (j,k,l) = (2,1,2)\\ &= (\mathrm{id}_{B_2} \otimes \delta^1_{22} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{B_2} \otimes \delta^2_{21})\delta^2_{B_1}, \quad (j,k,l) = (1,2,2)\\ &= (\mathrm{id}_{B_2} \otimes \mathrm{id}_{S_{21}} \otimes \delta^2_{11})(\mathrm{id}_{B_2} \otimes \delta^1_{21})\delta^2_{B_1}\\ &= (\mathrm{id}_{B_2} \otimes \mathrm{id}_{S_{21}} \otimes \delta^2_{11})(\delta^2_{B_1} \otimes \mathrm{id}_{S_{11}})\delta^1_{B_1}, \quad (j,k,l) = (1,1,2)\\ &= (\delta^2_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\mathrm{id}_{B_1} \otimes \delta^2_{11})\delta^1_{B_1}.\end{aligned}$$

Hence, $(\delta^1_{B_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{B_1} = (\mathrm{id}_{B_1} \otimes \delta^2_{11})\delta^1_{B_1}$ by faithfulness of $\delta^2_{B_1}$. We prove in a similar way the last case by composing by $\delta^2_{B_1} \otimes \mathrm{id}_{S_{11}} \otimes \mathrm{id}_{S_{12}}$.
2. In virtue of Proposition 6.1.4 2 and Proposition 6.1.6 3, it only remains to prove that $[\delta^1_{B_2}(B_2)(1_{B_1} \otimes S_{12})] = B_1 \otimes S_{12}$. Let us denote $C = \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(B_2)$ and $D = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(C)$. In virtue of Proposition 6.1.4 1, we have $[D(1_C \otimes S_{12})] = C \otimes S_{12}$. Note that $(\pi_1^{-1} \otimes \mathrm{id}_{S_{12}})\delta^{(1)}_{B_2} = \delta^1_{B_2}$, $D = \delta^{(1)}_{B_2}(B_2)$ (see Proposition 6.1.7 1) and $C = \pi_1(B_1)$ (see Proposition 6.1.6 2). Then, we obtain the result by composing by the *-isomorphism $\pi_1^{-1} \otimes \mathrm{id}_{S_{12}}$.
3. This is a straightforward consequence of the previous statement. □

Therefore, we have the following result:

Theorem 6.1.12. *Let (B_1, δ_{B_1}) be a $\mathbb{G}_1$-C*-algebra. Let $B_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1)$ be the induced $\mathbb{G}_2$-C*-algebra. Let us denote $B = B_1 \oplus B_2$ and let $\pi^k_j : \mathcal{M}(B_k \otimes S_{kj}) \to \mathcal{M}(B \otimes S)$ be the strictly continuous *-homomorphism extending the canonical injection $B_k \otimes S_{kj} \to B \otimes S$. Let us consider the *-homomorphisms $\delta_B : B \to \mathcal{M}(B \otimes S)$ and $\beta_B : \mathbb{C}^2 \to \mathcal{M}(B)$ defined by:*

$$\delta_B(b) := \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{B_j}(b_j), \quad b = (b_1, b_2) \in B \quad ; \quad \beta_B(\lambda,\mu) := \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad (\lambda,\mu) \in \mathbb{C}^2.$$

Therefore, we have:

1. *(δ_B, β_B) is a continuous action of $\mathcal{G}$ on B.*
2. *The correspondence $\mathbb{G}_1$-C*-Alg $\to$ $\mathcal{G}$-C*-Alg $;\, (B_1, \delta_{B_1}) \mapsto (B, \delta_B, \beta_B)$ is functorial.*

Proof. The first statement is an immediate consequence of Lemmas 5.1.9 and 6.1.11. The second one follows from Proposition 6.1.4 3 and Lemma 5.1.7 2. □

Theorem 6.1.13. *The functors*

$$\begin{array}{ccc} \mathcal{G}\text{-C}^*\text{-Alg} & \longrightarrow & \mathbb{G}_1\text{-C}^*\text{-Alg}, \\ (A, \delta_A, \beta_A) & \longmapsto & (A_1, \delta^1_{A_1}) \end{array} \qquad \begin{array}{ccc} \mathbb{G}_1\text{-C}^*\text{-Alg} & \longrightarrow & \mathcal{G}\text{-C}^*\text{-Alg} \\ (B_1, \delta_{B_1}) & \longmapsto & (B, \delta_B, \beta_B) \end{array}$$

are inverse of each other.

For the proof, we will need the following result:

Proposition 6.1.14. *Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. With the notations of Proposition 5.1.3, we denote:*

$$(\widetilde{A}_2, \delta_{\widetilde{A}_2}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1, \delta^1_{A_1}), \quad (\widetilde{A}_1, \delta_{\widetilde{A}_1}) = \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2, \delta^2_{A_2}).$$

Then, we have:

1. *If $x \in A_2$, we have $\delta^1_{A_2}(x) \in \widetilde{A}_2 \subset \mathcal{M}(A_1 \otimes S_{12})$ and the map*

$$\begin{array}{rccc} \widetilde{\pi}_2 : & (A_2, \delta^2_{A_2}) & \longrightarrow & (\widetilde{A}_2, \delta_{\widetilde{A}_2}) \\ & x & \longmapsto & \delta^1_{A_2}(x) \end{array}$$

   *is a $\mathbb{G}_2$-equivariant *-isomorphism.*

2. *If $x \in A_1$, we have $\delta^2_{A_1}(x) \in \widetilde{A}_1 \subset \mathcal{M}(A_2 \otimes S_{21})$ and the map*

$$\begin{array}{rccc} \widetilde{\pi}_1 : & (A_1, \delta^1_{A_1}) & \longrightarrow & (\widetilde{A}_1, \delta_{\widetilde{A}_1}) \\ & x & \longmapsto & \delta^2_{A_1}(x) \end{array}$$

   *is a $\mathbb{G}_1$-equivariant *-isomorphism.*

Proof. Let us prove the first statement since that of the second one is similar. In virtue of Proposition 5.1.3 3, we have $A_2 = [(\mathrm{id}_{A_2} \otimes \omega)\delta^2_{A_1}(a)\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*]$. Let $a \in A_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. By using Proposition 5.1.3 2, we have

$$\begin{aligned} \delta^1_{A_2}(\mathrm{id}_{A_2} \otimes \omega)\delta^2_{A_1}(a) &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)(\delta^1_{A_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1}(a) \\ &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)(\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta^1_{A_1}(a) \\ &= (\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a). \end{aligned}$$

This proves that $\delta^1_{A_2}(x) \in \widetilde{A}_2$ for all $x \in A_2$. Actually, we even have

$$\begin{aligned} \delta^1_{A_2}(A_2) &= [\delta^1_{A_2}(\mathrm{id}_{A_2} \otimes \omega)\delta^2_{A_1}(a)\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*] \\ &= [(\mathrm{id}_{A_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{A_1}(a)\,;\, a \in A_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*] =: \widetilde{A}_2. \end{aligned}$$

Moreover, $\delta^1_{A_2}$ is a faithful *-homomorphism, then $\widetilde{\pi}_2$ is a *-isomorphism. Finally, the fact that $\widetilde{\pi}_2$ is $\mathbb{G}_2$-equivariant is just a restatement of $(\mathrm{id}_{A_1} \otimes \delta^2_{12})\delta^1_{A_2} = (\delta^1_{A_2} \otimes \mathrm{id}_{S_{22}})\delta^2_{A_2}$ (see Proposition 5.1.3 2). □

Proof of Theorem 6.1.13. It is clear that the composition of functors

$$\mathbb{G}_1\text{-C}^*\text{-Alg} \longrightarrow \mathcal{G}\text{-C}^*\text{-Alg} \longrightarrow \mathbb{G}_1\text{-C}^*\text{-Alg}$$

is isomorphic to the identity functor *via* the $\mathbb{G}_1$-equivariant *-isomorphism

$$A_1 \to A_1 \oplus \{0\} \quad ; \quad a \mapsto (a, 0),$$

for each $\mathbb{G}_1$-C*-algebra (A_1, δ_{A_1}). Let us prove that the composition of functors

$$\mathcal{G}\text{-C}^*\text{-Alg} \longrightarrow \mathbb{G}_1\text{-C}^*\text{-Alg} \longrightarrow \mathcal{G}\text{-C}^*\text{-Alg}$$

is isomorphic to the identity functor. Let (A, δ_A, β_A) be a $\mathcal{G}$-C*-algebra. Let us use the notations of §5.1, for $j = 1, 2$ we denote $A_j = q_j A$, where $q_j = \beta_A(\varepsilon_j)$. Let us consider the *-homomorphisms $\delta^k_{A_j} : A_j \to \mathcal{M}(A_k \otimes S_{kj})$ for $j, k = 1, 2$ defined in Proposition 5.1.3. Let us denote:

$$(B_1, \delta_{B_1}) := (A_1, \delta^1_{A_1}), \quad (B_2, \delta_{B_2}) := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1, \delta_{B_1}).$$

Let (B, δ_B, β_B) be the $\mathcal{G}$-C*-algebra associated with the $\mathbb{G}_1$-C*-algebra (B_1, δ_{B_1}) (see Theorem 6.1.12). Let $\widetilde{\pi}_2 : A_2 \to B_2$ be the $\mathbb{G}_2$-equivariant *-isomorphism defined in Proposition 6.1.14 1. Let us consider the map $\tau : A \to B$ given by

$$\tau(a) = (q_1 a, \widetilde{\pi}_2(q_2 a)), \quad a \in A.$$

By proposition 6.1.14 1, it is clear that τ is a *-isomorphism. It it also immediate that we have $\tau \circ \beta_A = \beta_B$. Hence, $\tau(A_j) \subset B_j$ for $j = 1, 2$. Then, let us denote $\tau_j := \tau\restriction_{A_j} = A_j \to B_j$, $j = 1, 2$. Note that $\tau_1 : A_1 \to B_1\,;\, a \mapsto a$ and $\tau_2 = \widetilde{\pi}_2$. Let us prove that $(\tau \otimes \mathrm{id}_S)\delta_A = \delta_B \circ \tau$. It then amounts to proving that

$$(\tau_k \otimes \mathrm{id}_{S_{kj}})\delta^k_{A_j} = \delta^k_{B_j} \circ \tau_j, \quad j, k = 1, 2.$$

Let us proceed by disjunction elimination. If $(j, k) = (1, 1)$ there is nothing to prove. Assume that $(j, k) = (1, 2)$. We have

$$\delta^2_{B_1} \circ \tau_1 = \delta^{(2)}_{B_1} \circ \tau_1 = \delta^{(2)}_{A_1} = (\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta^1_{A_1} = (\delta^1_{A_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1} = (\widetilde{\pi}_2 \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1} = (\tau_2 \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1}.$$

The formula corresponding to the case $(j, k) = (2, 2)$ is just the $\mathbb{G}_2$-equivariance of $\widetilde{\pi}_2$. Let us look at the case $(j, k) = (2, 1)$. We have to prove that $\delta^1_{B_2} \circ \widetilde{\pi}_2 = \delta^1_{A_2}$. By composing by $\pi_1 \otimes \mathrm{id}_{S_{12}}$ (see Notations 6.1.10), we have

$$\begin{aligned}(\pi_1 \otimes \mathrm{id}_{S_{12}}) \circ \delta^1_{B_2} \circ \widetilde{\pi}_2 = \delta^{(1)}_{B_2} \circ \widetilde{\pi}_2 = (\mathrm{id}_{B_2} \otimes \delta^1_{22}) \circ \delta^2_{B_2} \circ \widetilde{\pi}_2 = (\mathrm{id}_{B_2} \otimes \delta^1_{22})(\widetilde{\pi}_2 \otimes \mathrm{id}_{S_{22}})\delta^2_{A_2} \\ = (\widetilde{\pi}_2 \otimes \mathrm{id}_{S_{21}} \otimes \mathrm{id}_{S_{12}})(\mathrm{id}_{A_2} \otimes \delta^1_{22})\delta^2_{A_2} = (\widetilde{\pi}_2 \otimes \mathrm{id}_{S_{21}} \otimes \mathrm{id}_{S_{12}})(\delta^2_{A_1} \otimes \mathrm{id}_{S_{12}})\delta^1_{A_2}\end{aligned}$$

and $(\widetilde{\pi}_2 \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1} = (\delta^1_{A_2} \otimes \mathrm{id}_{S_{21}})\delta^2_{A_1} = (\mathrm{id}_{A_1} \otimes \delta^2_{11})\delta^1_{A_1} = \delta^{(2)}_{A_1} = \pi_1$. Therefore, we have $(\pi_1 \otimes \mathrm{id}_{S_{12}}) \circ \delta^1_{B_2} \circ \widetilde{\pi}_2 = (\pi_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{A_2}$. We conclude that $\delta^1_{B_2} \circ \widetilde{\pi}_2 = \delta^1_{A_2}$ by faithfuless of π_1.

Let us verify the naturality of the isomorphism τ. We keep the previous notations and we introduce analogous notations associated with a second $\mathcal{G}$-C*-algebra $(A', \delta_{A'}, \beta_{A'})$. Let us consider $f : A \to \mathcal{M}(A')$ a $\mathcal{G}$-equivariant non-degenerate *-homomorphism. Let

$$f_1 \in \mathrm{Mor}_{\mathbb{G}_1}(A_1, A'_1), \quad f_2 \in \mathrm{Mor}_{\mathbb{G}_2}(A_2, A'_2) \quad \text{(see Lemma 5.1.7 1)}$$

be the images of f by the functors

$$\mathcal{G}\text{-C}^*\text{-Alg} \longrightarrow \mathbb{G}_1\text{-C}^*\text{-Alg}, \quad \mathcal{G}\text{-C}^*\text{-Alg} \longrightarrow \mathbb{G}_2\text{-C}^*\text{-Alg}.$$

Moreover, let us denote $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1} f_1 \in \mathrm{Mor}_{\mathbb{G}_2}(B_2, B'_2)$ (see proof of Proposition 6.1.4 3, we recall that $B_1 = A_1$ and $B'_1 = A'_1$) the image of f_1 by the functor $\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}$. Finally, let $f_* : B \to \mathcal{M}(B')$ be

the $\mathcal{G}$-equivariant non-degenerate *-homomorphism associated with f_1 and $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1$ (see Lemma 5.1.7 2), that is the image of f by the functor

$$\mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \longrightarrow \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \longrightarrow \mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}.$$

Let $\iota_{B'_j} : \mathcal{M}(B'_j) \to \mathcal{M}(B')$ (resp. $\iota_{A'_j} : \mathcal{M}(A'_j) \to \mathcal{M}(A')$) be the strictly continuous *-homomorphism extending the canonical injection $B'_j \to B'$ (resp. $A'_j \to A'$), $j = 1, 2$. We have

$$f_*(b) = \iota_{B'_1} \circ f_1(b_1) + \iota_{B'_2} \circ \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1(b_2), \quad b = (b_1, b_2) \in B.$$

In particular,

$$f_*(\tau(a)) = \iota_{B'_1} \circ f_1(q_1 a) + \iota_{B'_2} \circ \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1(\widetilde{\pi}_2(q_2 a)), \quad a \in A.$$

However, in virtue of Lemma 5.1.7 1, we have

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1(\widetilde{\pi}_2(q_2 a)) = (f_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{A_2}(q_2 a) = \delta^1_{A'_2} \circ f_2(q_2 a) = \widetilde{\pi}'_2(f_2(q_2 a)), \quad a \in A.$$

Hence, we have

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1 \circ \widetilde{\pi}_2 = \widetilde{\pi}'_2 \circ f_2.$$

By definition of $\tau' : A' \to B'$, we have $\iota_{B'_2} \circ \widetilde{\pi}'_2 = \tau' \circ \iota_{A'_2}$ and $\iota_{B'_1} = \tau' \circ \iota_{A'_1}$. Hence,

$$\iota_{B'_1} \circ f_1(q_1 a) = \tau'(\iota_{A'_1} \circ f_1(q_1 a)), \quad \iota_{B'_2} \circ \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1(\widetilde{\pi}_2(q_2 a)) = \tau'(\iota_{A'_2}(f_2(q_2 a))), \quad a \in A.$$

But $f(a) = \iota_{A'_1} \circ f_1(q_1 a) + \iota_{A'_2} \circ f_2(q_2 a)$ for all $a \in A$. Then, we conclude that $f_* \circ \tau = \tau' \circ f$.□

6.2 Induction and Morita equivalence

In this paragraph, we will prove that the equivalences of categories

$$\mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}, \quad \mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_2\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}, \quad \mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_2\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}$$

exchange the equivariant Morita equivalences.

Let us introduce the following general definition:

Definition 6.2.1. Let $\mathcal{G}$ be a measured quantum groupoid on a finite basis and (S, δ) the associated weak Hopf-C*-algebra. A linking $\mathcal{G}$-C*-algebra (or a $\mathcal{G}$-equivariant Morita equivalence) is a quintuple

$$(J, \delta_J, \beta_J, e_1, e_2)$$

consisting of a $\mathcal{G}$-C*-algebra (J, δ_J, β_J) and nonzero self-adjoint projections $e_1, e_2 \in \mathcal{M}(J)$ such that:

1. $e_1 + e_2 = 1_J$.
2. $[Je_iJ] = J$, $i = 1, 2$.
3. $\delta_J(e_i) = q^{\beta_J,\alpha}(e_i \otimes 1_S)$, $i = 1, 2$.

Remark 6.2.2. As a consequence, we have $[e_i, \beta_J(n^{\mathrm{o}})] = 0$ for all $n \in N$ and $i = 1, 2$. In particular, we have $[q^{\beta_J,\alpha}, e_i \otimes 1_S] = 0$. Indeed, if $n \in N$ and $i = 1, 2$ we have

$$\delta_J(\beta_J(n^{\mathrm{o}})e_i) = \delta_J(\beta_J(n^{\mathrm{o}}))\delta_J(e_i) = q^{\beta_J,\alpha}(1_J \otimes \beta(n^{\mathrm{o}}))q^{\beta_J,\alpha}(e_i \otimes 1_S) = q^{\beta_J,\alpha}(e_i \otimes 1_S)(1_J \otimes \beta(n^{\mathrm{o}})).$$

Therefore, since $\delta_J(1_J) = q^{\beta_J,\alpha}$ we obtain

$$\delta_J(\beta_J(n^{\mathrm{o}})e_i) = q^{\beta_J,\alpha}(e_i \otimes 1_S)(1_J \otimes \beta(n^{\mathrm{o}}))q^{\beta_J,\alpha} = \delta_J(e_i\beta_J(n^{\mathrm{o}})).$$

The result follows by faithfulness of δ_J.

Let $(J, \delta_J, \beta_J, e_1, e_2)$ be a linking $\mathcal{G}$-C*-algebra. Let us use the notations of §5.1:

$$q_j = \beta_J(\varepsilon_j), \quad J_j = q_j J, \quad \delta^k_{J_j} : J_j \to \mathcal{M}(J_k \otimes S_{kj}).$$

Let $\iota_j : \mathcal{M}(J_j) \to \mathcal{M}(J)$ be the strictly continuous extension of the inclusion map $J_j \subset J$ such that $\iota_j(1_J) = q_j$. For all $i, j = 1, 2$ we have $[e_i, q_j] = 0$, then e_iq_j is a self-adjoint projection of $\mathcal{M}(J)$. Besides, we have $e_iq_j \in q_j\mathcal{M}(J) = \iota_j(\mathcal{M}(J_j))$. Then, there exists a unique nonzero self-adjoint projection $e_{i,j} \in \mathcal{M}(J_j)$ such that $\iota_j(e_{i,j}) = e_iq_j$.

Lemma 6.2.3. *With the above notations, for all $i, j, k = 1, 2$ we have:*

1. $e_{1,j} + e_{2,j} = 1_{J_j}$, $[J_je_{i,j}J_j] = J_j$.
2. $\delta^k_{J_j}(e_{i,j}) = e_{i,k} \otimes 1_{S_{kj}}$.

In particular, $(J_j, \delta^j_{J_j}, e_{1,j}, e_{2,j})$ is a linking $\mathbb{G}_j$-C-algebra.*

Proof. 1. $\iota_j(e_{1,j} + e_{2,j}) = e_1q_j + e_2q_j = q_j = \iota_j(1_{J_j})$. Hence, $e_{1,j} + e_{2,j} = 1_{J_j}$.
2. We have $[J_je_{i,j}J_j] \subset J_j$ since $e_{i,j} \in \mathcal{M}(J_j)$. If $x \in J_j \subset J$, x is the norm limit of finite sums of elements of the form ye_iz, where $y, z \in J$. Then, $x = q_jxq_j$ is the norm limit of finite sums of elements of the form $q_jye_izq_j = (q_jy)e_iq_j(q_jz) = \iota_j(y_je_{i,j}z_j)$, where $y, z \in J$ and $y_j = q_jy$, $z_j = q_jz \in J_j$. Then, $x \in [J_je_{i,j}J_j]$ and the result is proved.
3. We have $\delta_J(e_iq_j) = e_iq_1 \otimes p_{1j} + e_iq_2 \otimes p_{2j}$. Hence, $(q_k \otimes 1_S)\delta_J(e_iq_j) = e_iq_k \otimes p_{kj} = \pi^k_j(e_{i,k} \otimes 1_{S_{kj}})$, where $\pi^k_j : \mathcal{M}(J_k \otimes S_{kj}) \to \mathcal{M}(J \otimes S)$ is the strictly continuous extension of the inclusion map $J_k \otimes S_{kj} \subset J \otimes S$. However, we have $(q_k \otimes 1_S)\delta_J(\iota_j(m)) = \pi^k_j \circ \delta^k_{J_j}(m)$ for all $m \in \mathcal{M}(J_j)$ (see Proposition 5.1.3). Hence, $\delta^k_{J_j}(e_{i,j}) = e_{i,k} \otimes 1_{S_{kj}}$ as π^k_j is faithful. □

We have proved that the image of a $\mathcal{G}$-equivariant Morita equivalence by the functor

$$\mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathbb{G}_j\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}\,;\ (J, \delta_J, \beta_J) \mapsto (J_j, \delta^j_{J_j})$$

is a linking $\mathbb{G}_j$-C*-algebra. Conversely, let $(J_1, \delta_{J_1}, e_{1,1}, e_{2,1})$ be a linking $\mathbb{G}_1$-C*-algebra. Let (J, δ_J, β_J) be the image of (J_1, δ_{J_1}) by the functor

$$\mathbb{G}_1\text{-}\mathsf{C}^*\text{-}\mathsf{Alg} \to \mathcal{G}\text{-}\mathsf{C}^*\text{-}\mathsf{Alg}.$$

We recall that we have $J = J_1 \oplus J_2$, where $J_2 = \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(J_1)$ is endowed with the continuous action $\delta_{J_2} = (\mathrm{id}_{J_1} \otimes \delta^2_{12})\restriction_{J_2}$ of $\mathbb{G}_2$. The coproduct is given by the formula:

$$\delta_J(x) = \sum_{k,j=1,2} \pi^k_j \circ \delta^k_{J_j}(x_j), \quad x = (x_1, x_2) \in J,$$

where $\pi^k_j : \mathcal{M}(J_k \otimes S_{kj}) \to \mathcal{M}(J \otimes S)$ are the strictly continuous extensions of the canonical injections $J_k \otimes S_{kj} \to J \otimes S$ and $\delta^k_{J_j} : J_j \to \mathcal{M}(J_k \otimes S_{kj})$ are the *-homomorphisms defined in Notations 6.1.10, $j, k = 1, 2$.

Notation 6.2.4. Let us consider the nonzero self-adjoint projections

$$e_{i,2} := e_{i,1} \otimes 1_{S_{12}} \in \mathcal{M}(J_1 \otimes S_{12}), \quad i = 1, 2.$$

Proposition 6.2.5. *With the above notations,* $(J_2, \delta^2_{J_2}, e_{1,2}, e_{2,2})$ *is a linking* $\mathbb{G}_2$-C*-*algebra. Moreover, for all* $i, j, k = 1, 2$ *we have*

$$\delta^k_{J_j}(e_{i,j}) = e_{i,k} \otimes 1_{S_{kj}}.$$

Proof. Let $i = 1, 2$. Since $\delta_{J_1}(e_{i,1}) = e_{i,1} \otimes 1_{S_{12}}$, we have:

$$e_{i,2}(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(e_{i,1}x),$$

$$(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(x)e_{i,2} = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(xe_{i,1}),$$

for all $x \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. In particular, it follows that $e_{i,2} \in \mathcal{M}(J_2)$ (since $e_{i,1} \in \mathcal{M}(J_1)$) and $e_{1,2} + e_{2,2} = 1_{J_2}$ (since $e_{1,1} + e_{2,1} = 1_{J_1}$). It is clear that

$$\delta^2_{J_j}(e_{i,j}) = e_{i,2} \otimes 1_{S_{2j}}, \quad j = 1, 2. \tag{6.2.1}$$

Let us prove that $[J_2 e_{i,2} J_2] = J_2$. In virtue of the formulas

$$J_2 = [(\mathrm{id}_{J_2} \otimes \omega)\delta^2_{J_1}(x)\,;\, x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*], \quad J_1 = [J_1 e_{i,1} J_1],$$

it amounts to proving that $(\mathrm{id}_{J_2} \otimes \omega)\delta^2_{J_1}(ae_{i,1}b) \in [J_2 e_{i,2} J_2]$ for all $a, b \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Let us fix $a, b \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Let us write $\omega = s\omega'$ with $s \in S_{21}$ and $\omega' \in \mathcal{B}(\mathscr{H}_{21})_*$. By Lemma 6.1.11 2, we have $[\delta^2_{J_1}(J_1)(1_{J_2} \otimes S_{21})] = J_2 \otimes S_{21}$. Therefore,

$$(\mathrm{id}_{J_2} \otimes \omega)\delta^2_{J_1}(ae_{i,1}b) = (\mathrm{id}_{J_2} \otimes \omega')(\delta^2_{J_1}(a)(e_{i,2} \otimes 1_{S_{21}})\delta^2_{J_1}(b)(1_{J_2} \otimes s))$$

is the norm limit of finite sums of elements of the form

$$(\mathrm{id}_{J_2} \otimes \omega')(\delta^2_{J_1}(a)(e_{i,2} \otimes 1_{S_{21}})(y \otimes s')) = (\mathrm{id}_{J_2} \otimes s'\omega')\delta^2_{J_1}(a)e_{i,2}y \in J_2 e_{i,2} J_2, \quad y \in J_2,\, s' \in S_{21}.$$

Finally, we also have

$$\delta^1_{J_j}(e_{i,j}) = e_{i,1} \otimes 1_{S_{1j}}, \quad j = 1, 2.$$

For $j = 1$, there is nothing to prove. For $j = 2$, we compose by the *-isomorphism $\pi_1 \otimes \mathrm{id}_{S_{12}}$, where $\pi_1 : J_1 \to \mathrm{Ind}^{\mathbb{G}_1}_{\mathbb{G}_2}(J_2)\,;\, x \mapsto \delta^{(2)}_{J_1}(x)$ (see Notations 6.1.10). By using the formulas (6.2.1), we have

$$(\pi_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{J_2}(e_{i,2}) = \delta^{(1)}_{J_2}(e_{i,2}) = (\mathrm{id}_{J_2} \otimes \delta^1_{22})\delta^2_{J_2}(e_{i,2}) = e_{i,2} \otimes 1_{S_{21}} \otimes 1_{S_{12}} = (\pi_1 \otimes \mathrm{id}_{S_{12}})(e_{i,1} \otimes 1_{S_{12}}).$$

Hence, $\delta^1_{J_2}(e_{i,2}) = e_{i,1} \otimes 1_{S_{12}}$ and the proposition is proved. □

Notation 6.2.6. Let us denote

$$e_i := (e_{i,1}, e_{i,2}) \in \mathcal{M}(J), \quad i = 1, 2.$$

The following result is then an immediate consequence of Proposition 6.2.5.

Corollary 6.2.7. *With the above notations, the quintuple* $(J, \delta_J, \beta_J, e_1, e_2)$ *is a linking* $\mathcal{G}$-C*-*algebra.*

Therefore, we have proved that the image of a linking $\mathbb{G}_1$-C*-algebra by the functor

$$\mathbb{G}_1\text{-}\mathsf{C^*\text{-}Alg} \to \mathcal{G}\text{-}\mathsf{C^*\text{-}Alg}\,;\, (J_1, \delta_{J_1}) \mapsto (J, \delta_J, \beta_J)$$

is a linking $\mathcal{G}$-C*-algebra.

6.3 Induction of equivariant Hilbert modules

In order to extend the induction procedure to equivariant Hilbert C*-modules (see [3]), we will need to induce *degenerate* equivariant *-homomorphisms. We begin with the following technical lemma:

Lemma 6.3.1. *Let (B_1, δ_{B_1}) and (J_1, δ_{J_1}) be $\mathbb{G}_1$-C*-algebras. Let $f_1 : B_1 \to \mathcal{M}(J_1)$ be a *-homomorphism such that:*

(a) For some approximate unit (u_λ) of B_1, we have $f_1(u_\lambda) \to e_1 \in \mathcal{M}(J_1)$ with respect to the strict topology.

(b) $(f_1 \otimes \mathrm{id}_{S_{11}})\delta_{B_1} = \delta_{J_1} \circ f_1$.

Then, we have:

1. *e_1 is a self-adjoint projection, which does not depend on the approximate unit. More precisely, for all approximate unit $(v_\mu)_\mu$ of B_1 we have $f_1(v_\mu) \to e_1$.*
2. *$\delta_{J_1}(e_1) = e_1 \otimes 1_{S_{11}}$.*
3. *For all $T \in \mathcal{M}(B_1 \otimes S_{12})$, we have*

$$(f_1 \otimes \mathrm{id}_{S_{12}})(T) = (e_1 \otimes 1_{S_{12}})(f_1 \otimes \mathrm{id}_{S_{12}})(T)(e_1 \otimes 1_{S_{12}}).$$

Proof. The assertion 1 and the fact that (b) makes sense follow from (a) and Lemma 4.1.1. In particular, f_1 extends uniquely to a strictly continuous *-homomorphism $f_1 : \mathcal{M}(B_1) \to \mathcal{M}(J_1)$ such that $f_1(1_{B_1}) = e_1$. Moreover, $f_1 \otimes \mathrm{id}_{S_{12}}$ extends uniquely to a strictly continuous *-homomorphism $f_1 \otimes \mathrm{id}_{S_{12}} : \mathcal{M}(B_1 \otimes S_{12}) \to \mathcal{M}(J_1 \otimes S_{12})$, which verifies

$$(f_1 \otimes \mathrm{id}_{S_{12}})(1_{B_1 \otimes S_{12}}) = e_1 \otimes 1_{S_{12}}.$$

It then follows immediately that:

$$(f_1 \otimes \mathrm{id}_{S_{12}})(T) = (e_1 \otimes 1_{S_{12}})(f_1 \otimes \mathrm{id}_{S_{12}})(T)(e_1 \otimes 1_{S_{12}}), \quad T \in \mathcal{M}(B_1 \otimes S_{12}).$$

We have $(f_1 \otimes \mathrm{id}_{S_{11}})\delta_{B_1}(1_{B_1}) = e_1 \otimes 1_{S_{11}}$. Besides, if $(u_\lambda)_\lambda$ is an approximate unit of B_1 we have $(f_1 \otimes \mathrm{id}_{S_{11}})\delta_{B_1}(u_\lambda) = \delta_{J_1}(f_1(u_\lambda)) \to \delta_{J_1}(e_1)$. Hence, $\delta_{J_1}(e_1) = e_1 \otimes 1_{S_{11}}$. □

Proposition 6.3.2. *With the hypotheses and notations of Lemma 6.3.1, we denote:*

$$(B_2, \delta_{B_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1, \delta_{B_1}), \quad (J_2, \delta_{J_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1, \delta_{J_1}), \quad e_2 := e_1 \otimes 1_{S_{12}}.$$

Then, we have:

1. *e_2 is a self-adjoint projection of $\mathcal{M}(J_2)$ and $(f_1 \otimes \mathrm{id}_{S_{12}})(B_2) \subset e_2\mathcal{M}(J_2)e_2$.*

*Therefore, we consider the *-homomorphism $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1 : B_2 \to \mathcal{M}(J_2)$ defined by*

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1(b) = (f_1 \otimes \mathrm{id}_{S_{12}})(b), \quad b \in B_2.$$

We will denote $f_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2} f_1$ for short.

2. *For all approximate unit $(v_\mu)_\mu$ of B_2, we have $f_2(v_\mu) \to e_2$ with respect to the strict topology.*

3. $(f_2 \otimes \mathrm{id}_{S_{22}})\delta_{B_2} = \delta_{J_2} \circ f_2$.

4. Let $F_1 \in e_1\mathcal{M}(J_1)e_1$ *such that:*

- $\delta_{J_1}(F_1) = F_1 \otimes 1_{S_{11}}$.
- $\forall b \in B_1, \quad [F_1, f_1(b)] \in e_1J_1e_1, \quad f_1(b)(F_1 - F_1^*) \in e_1J_1e_1, \quad f_1(b)(F_1^2 - 1) \in e_1J_1e_1$.

We consider $F_2 := F_1 \otimes 1_{S_{12}} \in e_2\mathcal{M}(J_2)e_2$. *Then, we have:*

- $\delta_{J_2}(F_2) = F_2 \otimes 1_{S_{22}}$.
- $\forall b \in B_2, \quad [F_2, f_2(b)] \in e_2J_2e_2, \quad f_2(b)(F_2 - F_2^*) \in e_2J_2e_2, \quad f_2(b)(F_2^2 - 1) \in e_2J_2e_2$.

Proof. 1. It is clear that e_2 is a self-adjoint projection of $\mathcal{M}(J_1 \otimes S_{12})$. By Lemma 6.3.1 2, we have $\delta_{J_1}^{(2)}(e_1) = e_1 \otimes 1_{S_{12}} \otimes 1_{S_{21}}$. Let us take a normal state ϕ on $\mathcal{B}(\mathscr{H}_{21})$. Let $x \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$, we have

$$e_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \phi \otimes \omega)(\delta_{J_1}^{(2)}(e_1)_{123}\delta_{J_1}^{(2)}(x)_{124}).$$

Since $e_1 \in \mathcal{M}(J_1)$ we have $e_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(x) \in J_2$ by Remarks 6.1.3 1. Hence, $e_2J_2 \subset J_2$. But J_2 is stable by the involution and e_2 is self-adjoint. It then follows that $J_2e_2 \subset J_2$ and $e_2 \in \mathcal{M}(J_2)$.
Let us prove that $(f_1 \otimes \mathrm{id}_{S_{12}})(B_2) \subset e_2\mathcal{M}(J_2)e_2$. By Lemma 6.3.1 2, it is enough to prove that $(f_1 \otimes \mathrm{id}_{S_{12}})(B_2) \subset \mathcal{M}(J_2)$. But it also suffices to prove that $(f_1 \otimes \mathrm{id}_{S_{12}})(B_2)J_2 \subset J_2$ since $f_1 \otimes \mathrm{id}_{S_{12}}$ is a *-homomorphism and B_2 and J_2 are stable by the involution. Finally, it then amounts to proving that

$$(f_1 \otimes \mathrm{id}_{S_{12}})((\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{B_1}^{(2)}(b))(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')\delta_{J_1}^{(2)}(x) \in J_2,$$

for all $\omega, \omega' \in \mathcal{B}(\mathscr{H}_{21})_*$, $b \in B_1$, $x \in J_1$. Let us fix $\omega, \omega' \in \mathcal{B}(\mathscr{H}_{21})_*$, $b \in B_1$ and $x \in J_1$. We have

$$\begin{aligned}(f_1 \otimes \mathrm{id}_{S_{12}})((\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{B_1}^{(2)}(b))(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')\delta_{J_1}^{(2)}(x)& \\ = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\,((f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta_{B_1}^{(2)}(b))_{123}\delta_{J_1}^{(2)}(x)_{124}\,).&\end{aligned}$$

Since $(f_1 \otimes \mathrm{id}_{S_{11}})\delta_{B_1} = \delta_{J_1} \circ f_1$, we have $(f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta_{B_1}^{(2)} = \delta_{J_1}^{(2)} \circ f_1$. Then, it follows from the fact that $f_1(b) \in \mathcal{M}(J_1)$ and Remarks 6.1.3 1 that

$$\begin{aligned}(f_1 \otimes \mathrm{id}_{S_{12}})((\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{B_1}^{(2)}(b))(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')\delta_{J_1}^{(2)}(x)& \\ = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega \otimes \omega')(\delta_{J_1}^{(2)}(f_1(b))_{123}\delta_{J_1}^{(2)}(x)_{124}) \in J_2.&\end{aligned}$$

2. Note that we have $e_2f_2(b) = f_2(b) = f_2(b)e_2$ for all $b \in B_2$. By Lemma 4.1.1, we have to prove that $e_2J_2 = [f_2(B_2)J_2]$. The inclusion $[f_2(B_2)J_2] \subset e_2J_2$ is given. Conversely, note that we have

$$e_2J_2 = [(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(f_1(b)x)\,;\, b \in B_1,\, x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*].$$

Indeed, let us fix $x \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Since $\delta_{J_1}^{(2)}(e_1) = e_2 \otimes 1_{S_{21}}$, we have

$$(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(f_1(b)x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(e_1f_1(b)x) = e_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(f_1(b)x).$$

In particular, we have $[(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(f_1(b)x)\,;\, b \in B_1,\, x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*] \subset e_2J_2$. Conversely, let us fix $x \in J_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Let us recall that by assumption we have

$f_1(u_\lambda) \to e_1$ with respect to the strict topology. Hence, $f_1(u_\lambda)x \to e_1 x$ with respect to the norm topology. Therefore,

$$e_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(e_1 x) = \lim_\lambda (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(u_\lambda)x),$$

with respect to the norm topology and the equality is proved. It only remains to show that

$$(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b)x) \in [f_2(B_2)J_2], \quad x \in J_1,\, b \in B_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*.$$

Let us fix $x \in J_1$, $b \in B_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Let us denote $\omega = s\omega'$ with $s \in S_{21}$ and $\omega' \in \mathcal{B}(\mathscr{H}_{21})_*$. Since $(f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{B_1} = \delta^{(2)}_{J_1} \circ f_1$, we have

$$(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b)x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')(\,(f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\delta^{(2)}_{B_1}(b))\delta^{(2)}_{J_1}(x)(1_{J_1 \otimes S_{12}} \otimes s)\,).$$

But, we know that $[\delta^{(2)}_{J_1}(J_1)(1_{J_1 \otimes S_{12}} \otimes S_{21})] = J_2 \otimes S_{21}$ (see Proposition 6.1.6 3). As a result, $(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b)x)$ is the norm limit of finite sums of elements of the form

$$\begin{aligned}(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega')((f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})(\delta^{(2)}_{B_1}(b))(x' \otimes s')) \\ &= (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes s'\omega')((f_1 \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta^{(2)}_{B_1}(b))x' \\ &= (f_1 \otimes \mathrm{id}_{S_{12}})(\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes s'\omega')(\delta^{(2)}_{B_1}(b))x' \\ &= f_2((\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes s'\omega')\delta^{(2)}_{B_1}(b))x' \in f_2(B_2)J_2,\end{aligned}$$

where $x' \in J_2$ and $s' \in S_{21}$.

3. The formula $(f_2 \otimes \mathrm{id}_{S_{22}})\delta_{B_2} = \delta_{J_2} \circ f_2$ is just a straightforward consequence of the definitions of f_2 and the actions δ_{B_2} and δ_{J_2}.

4. Let us prove that $F_2 J_2 \subset J_2$. Let $x \in J_1$, $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$ and ϕ a normal state on $\mathcal{B}(\mathscr{H}_{21})$. Since $\delta^{(2)}_{J_1}(F_1) = F_1 \otimes 1_{S_{12}} \otimes 1_{S_{21}}$, we have

$$F_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(x) = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \phi \otimes \omega)(\delta^{(2)}_{J_1}(F_1)_{123}\delta^{(2)}_{J_1}(x)_{124}) \in J_2$$

by applying Remark 6.1.3 1 to $F_1 \in \mathcal{M}(J_1)$. Similarly, we prove that $F_2^* J_2 \subset J_2$ by applying Remark 6.1.3 1 to $F_1^* \in \mathcal{M}(J_1)$. Therefore, we have $J_2 F_2 \subset J_2$ and $F_2 \in \mathcal{M}(J_2)$. We then have $F_2 = e_2 F_2 e_2 \in e_2 \mathcal{M}(J_2) e_2$. Now, let us prove that $[F_2, f_2(b)] \in e_2 J_2 e_2$ for all $b \in B_2$. It follows from $(f_2 \otimes \mathrm{id}_{S_{21}})\delta^2_{B_1} = \delta^2_{J_1} \circ f_1$ that

$$f_2(B_2) = [(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b))\,;\, b \in B_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*].$$

Then, if we fix $b \in B_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$ we have to prove that

$$[F_2,\, (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b))] \in e_2 J_2 e_2.$$

However, since $\delta^{(2)}_{J_1}(F_1) = F_1 \otimes 1_{S_{12}} \otimes 1_{S_{21}}$ we have

$$[F_2,\, (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b))] = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}([F_1, f_1(b)]).$$

Moreover, we also have

$$e_2 J_2 e_2 = [(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(e_1 x e_1)\,;\, x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*].$$

Indeed, this follows from the fact that, since $\delta^{(2)}_{J_1}(e_1) = e_1 \otimes 1_{S_{12}} \otimes 1_{S_{21}}$, we have

$$e_2(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(x)e_2 = (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(e_1 x e_1), \quad x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*.$$

Therefore, we obtain $[F_2, (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta^{(2)}_{J_1}(f_1(b))] \in e_2 J_2 e_2$ as we have $[F_1, f_1(b)] \in e_1 J_1 e_1$ by assumption. The formula $\delta_{J_2}(F_2) = F_2 \otimes 1_{S_{22}}$ is an immediate consequence of the definitions of F_2 and the action δ_{J_2}. The remaining formulas are proved in a similar way. □

Remark 6.3.3. With the notations and hypotheses of Proposition 6.3.2 4, if for all $b \in B_1$ we have:

$$[F_1, f_1(b)] = 0, \quad f_1(b)(F_1 - F_1^*) = 0, \quad f_1(b)(F_1^2 - 1) = 0,$$

then for all $b \in B_2$ we have:

$$[F_2, f_2(b)] = 0, \quad f_2(b)(F_2 - F_2^*) = 0, \quad f_2(b)(F_2^2 - 1) = 0.$$

Let (B_1, δ_{B_1}) be a $\mathbb{G}_1$-C*-algebra. Let $(\mathscr{E}_1, \delta_{\mathscr{E}_1})$ be a $\mathbb{G}_1$-equivariant Hilbert B_1-module (see §1.6). Let us equip $J_1 := \mathcal{K}(\mathscr{E}_1 \oplus B_1)$ with the continuous action δ_{J_1} of $\mathbb{G}_1$ compatible with the actions $\delta_{\mathscr{E}_1}$ and δ_{B_1} of $\mathbb{G}_1$ on $\mathscr{E}_1$ and B_1. Let us consider the following self-adjoint projections:

$$e_{1,1} = \begin{pmatrix} 1_{\mathscr{E}_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{B_1} \end{pmatrix} \in \mathcal{L}(\mathscr{E}_1 \oplus B_1) = \mathcal{M}(J_1).$$

We recall that $(J_1, \delta_{J_1}, e_{1,1}, e_{2,1})$ is a linking $\mathbb{G}_1$-C*-algebra. In virtue of Proposition 6.2.5, the quintuple $(J_2, \delta_{J_2}, e_{1,2}, e_{2,2})$ where

$$(J_2, \delta_{J_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1, \delta_{J_1}), \quad e_{l,2} := e_{l,1} \otimes 1_{S_{12}} \in \mathcal{M}(J_2), \quad l = 1, 2,$$

is a linking $\mathbb{G}_2$-C*-algebra. In particular, the C*-subalgebra $e_{l,2}J_2e_{l,2}$ of J_2, endowed with the restriction of the action δ_{J_2}, is a $\mathbb{G}_2$-C*-algebra (see Remark 1.6.2).

By definition of the action δ_{J_1}, the canonical *-homomorphism $\iota_{B_1} : B_1 \to J_1$ satisfies the conditions (a) and (b) of Lemma 6.3.1. We have the following result:

Proposition 6.3.4. *Let us denote* $(B_2, \delta_{B_2}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1, \delta_{B_1})$. *Then, for all* $b \in B_2$ *we have* $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}(b) \in e_{2,2}J_2e_{2,2}$. *Furthermore, the map*

$$\begin{array}{ccc} B_2 & \longrightarrow & e_{2,2}J_2e_{2,2} \\ b & \longmapsto & \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}(b) \end{array}$$

is a $\mathbb{G}_2$*-equivariant *-isomorphism.*

Proof. Since ι_{B_1} is faithful so is $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}$. The *-homomorphism $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}$ takes its values in $\mathcal{M}(J_2)$ (and even in $e_{2,2}\mathcal{M}(J_2)e_{2,2}$), we will show that its range actually lies in $e_{2,2}J_2e_{2,2}$. Note that we have

$$e_{2,2}J_2e_{2,2} = [(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(e_{2,1}xe_{2,1})\,;\, x \in J_1,\, \omega \in \mathcal{B}(\mathscr{H}_{21})_*]$$

(see proof of Proposition 6.3.2 4). Moreover, we have $\iota_{B_1}(B_1) = e_{2,1}J_1e_{2,1}$. Let us fix $b \in B_1$ and $\omega \in \mathcal{B}(\mathscr{H}_{21})_*$. Since $\delta_{J_1} \circ \iota_{B_1} = (\iota_{B_1} \otimes \mathrm{id}_{S_{11}})\delta_{B_1}$, we have

$$\begin{aligned}(\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{J_1}^{(2)}(\iota_{B_1}(b)) &= (\mathrm{id}_{J_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)(\iota_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \mathrm{id}_{S_{21}})\delta_{B_1}^{(2)}(b) \\ &= \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}(\mathrm{id}_{B_1} \otimes \mathrm{id}_{S_{12}} \otimes \omega)\delta_{B_1}^{(2)}(b).\end{aligned}$$

Therefore, we have even proved that $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}(B_2) = e_{2,2}J_2e_{2,2}$. It then follows that the map $b \in B_2 \mapsto \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}\iota_{B_1}(b) \in e_{2,2}J_2e_{2,2}$ is a *-isomorphism and is also $\mathbb{G}_2$-equivariant in virtue of Proposition 6.3.2 3. □

From now on, we will identify the $\mathbb{G}_2$-C*-algebras:

$$B_2 = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1), \quad e_{2,2}J_2e_{2,2} = e_{2,2}\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)e_{2,2}.$$

Then, by restriction of the continuous action δ_{J_2} of $\mathbb{G}_2$ (see Remark 1.6.2), we obtain:

Definition 6.3.5. We call the induced Hilbert module of the $\mathbb{G}_1$-equivariant Hilbert B_1-module $\mathscr{E}_1$, the $\mathbb{G}_2$-equivariant Hilbert B_2-module $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1) := e_{1,2}\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)e_{2,2}$.

6.4 Induction and duality

Let us fix a $\mathbb{G}_1$-C*-algebra (A_1, δ_{A_1}). Let us denote $(A_2, \delta_{A_2}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1, \delta_{A_1})$, the induced $\mathbb{G}_2$-C*-algebra. In this paragraph, we will prove that the $\mathbb{G}_2$-C*-algebras $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1 \rtimes \mathbb{G}_1 \rtimes \widehat{\mathbb{G}}_1)$ and $A_2 \rtimes \mathbb{G}_2 \times \widehat{\mathbb{G}}_2$ are canonically $\mathbb{G}_2$-equivariantly Morita equivalent. This result will appear as a consequence of Theorem 4.4.15 applied to the double crossed product $A \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$, where (A, δ_A, β_A) is the image of (A_1, δ_{A_1}) by the functor $\mathbb{G}_1$-C*-Alg $\to \mathcal{G}$-C*-Alg. This result will then be applied to the case of a linking $\mathbb{G}_1$-C*-algebra.

In what follows, we identify the $\mathcal{G}$-C*-algebras $A \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ and (D, β_D, δ_D) (see Theorem 4.4.15 and we use the notations of §5.3 and in particular those of Notations 5.3.7. Let us recall the main results of §5.3.

By identification of the $\mathbb{G}_j$-equivariant Hilbert bimodules

$$D_{ll',j} := e_{l,j} D_j e_{l',j}, \quad \mathcal{B}_{ll',j} := A_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}), \quad j, l, l' = 1, 2,$$

we have for all $j, l, l' = 1, 2$, $l \neq l'$:

- The Hilbert $A_j \otimes \mathcal{K}(\mathscr{H}_{lj})$-$A_j \otimes \mathcal{K}(\mathscr{H}_{l'j})$-bimodule $A_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})$ is a $\mathbb{G}_j$-equivariant Morita equivalence between the $\mathbb{G}_j$-C*-algebras $A_j \otimes \mathcal{K}(\mathscr{H}_{lj})$ and $A_j \otimes \mathcal{K}(\mathscr{H}_{l'j})$.
- The action of $\mathbb{G}_j$ on $A_j \otimes \mathcal{K}(\mathscr{H}_{jj})$ coincide with the bidual action on the double crossed product $A_j \rtimes \mathbb{G}_j \rtimes \widehat{\mathbb{G}}_j$ up to the identification given by the Baaj-Skandalis duality theorem.
- Endowed with the action

$$\begin{array}{rccl} \delta^j_{ll',k} : & \mathcal{E}^j_{ll',k} & \longrightarrow & \mathcal{L}(\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j}) \otimes S_{jj}, \mathcal{E}^j_{ll',k} \otimes S_{jj}), \\ & \xi & \longmapsto & (V^k_{jj})_{23} \xi_{12} (V^k_{jj})^*_{23} \end{array}$$

  $\mathcal{E}^j_{ll',k} := \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j})$ is a $\mathbb{G}_j$-equivariant Hilbert $\delta^k_{\mathcal{B}_{l,j}}(\mathcal{B}_{l,j})$-$\delta^k_{\mathcal{B}_{l',j}}(\mathcal{B}_{l',j})$-bimodule.

Theorem 6.4.1. *Let $j, k, l, l' = 1, 2$, $j \neq k$. We have:*

1. *$\mathcal{E}^j_{ll',k} := \delta^k_{\mathcal{B}_{ll',j}}(\mathcal{B}_{ll',j}) = \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{ll',k})$. In particular, $\delta^k_{\mathcal{B}_{l,j}}(\mathcal{B}_{l,j}) = \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{l,k})$.*
2. *The action $\delta^j_{ll',k}$ of $\mathbb{G}_j$ on the bimodule $\mathcal{E}^j_{ll',k}$ coincides with the action of $\mathbb{G}_j$ induced by that of $\mathbb{G}_k$ on $\mathcal{B}_{ll',k}$.*
3. *The linear map*

$$\begin{array}{rcl} \mathcal{B}_{ll',j} & \longrightarrow & \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{ll',k}) \\ \xi & \longmapsto & \delta^k_{\mathcal{B}_{ll',j}}(\xi) \end{array}$$

   is an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules over the isomorphisms of $\mathbb{G}_j$-C-algebras:*

$$\begin{array}{rcl} \mathcal{B}_{l,j} & \longrightarrow & \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{l,k}), \\ x & \longmapsto & \delta^k_{\mathcal{B}_{l,j}}(x) \end{array} \qquad \begin{array}{rcl} \mathcal{B}_{l',j} & \longrightarrow & \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{l',k}). \\ x & \longmapsto & \delta^k_{\mathcal{B}_{l',j}}(x) \end{array}$$

Proof. By restriction of the $\mathbb{G}_j$-equivariant *-isomorphism

$$\begin{array}{rccl} \widetilde{\pi}_j : & D_j & \longrightarrow & \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_k) \quad \text{(see Proposition 6.1.14)}, \\ & x & \longmapsto & \delta^k_{D_j}(x) \end{array}$$

we obtain the isomorphisms

$$\widetilde{\pi}_{ll',j} : D_{ll',j} \to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{ll',k}), \quad l,l' = 1,2,$$

of $\mathbb{G}_j$-equivariant Hilbert bimodules over the isomorphisms of $\mathbb{G}_j$-C*-algebras

$$\widetilde{\pi}_{ll,j} : D_{l,j} \to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{l,k}), \quad \widetilde{\pi}_{l'l',j} : D_{l',j} \to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{l',k}).$$

Indeed, this follows from the fact that $\delta_{D_j}^k(e_{l,j}) = e_{l,k} \otimes 1_{S_{kj}}$ and $\delta_{D_j}^k(e_{l',j}) = e_{l',k} \otimes 1_{S_{kj}}$ (see Lemma 5.3.4 4). We then have

$$\widetilde{\pi}_{ll',j}(D_{ll',j}) = (e_{l,k} \otimes 1_{S_{kj}})\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_k)(e_{l',k} \otimes 1_{S_{kj}}) = \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{ll',k}).$$

Therefore, the first assertion is proved *via* the identification of the $\mathbb{G}_j$-C*-algebras D_j and $\mathcal{B}_j := A_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$. Moreover, we have

$$\delta_{ll',k}^j(\widetilde{\pi}_{ll',j}(x)) = (V_{jj}^k)_{23}\widetilde{\pi}_{ll',j}(x)_{12}(V_{jj}^k)_{23}^* = (\mathrm{id} \otimes \delta_{kj}^j)\widetilde{\pi}_{ll',j}(x), \quad x \in D_{ll',j},$$

which proves that the action $\delta_{ll',k}^j$ of $\mathbb{G}_j$ on the Hilbert $\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(\mathcal{B}_{l',k})$-module $\mathcal{E}_{ll',k}^j$ is the action of $\mathbb{G}_j$ induced by that of $\mathbb{G}_k$ on the Hilbert $\mathcal{B}_{l',k}$-module $\mathcal{B}_{ll',k}$. The last statement is a consequence of 1, 2 and Proposition 5.3.15 2. □

Corollary 6.4.2. *Let $j,k,l,l' = 1,2$ with $j \neq k$, we have:*

1. *The $\mathbb{G}_j$-C*-algebras $A_j \otimes \mathcal{K}(\mathscr{H}_{lj})$ and $\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k \otimes \mathcal{K}(\mathscr{H}_{lk}))$ are canonically isomorphic. In particular, the $\mathbb{G}_j$-C*-algebras $A_j \otimes \mathcal{K}(\mathscr{H}_{jj})$ and $\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k \otimes \mathcal{K}(\mathscr{H}_{jk}))$ are canonically isomorphic.*

2. *The $\mathbb{G}_j$-equivariant Hilbert $A_j \otimes \mathcal{K}(\mathscr{H}_{lj})$-$A_j \otimes \mathcal{K}(\mathscr{H}_{l'j})$-bimodules $A_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})$ and $\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(A_k \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk}))$ are canonically isomorphic.*

3. *The $\mathbb{G}_2$-C*-algebras $A_2 \rtimes \mathbb{G}_2 \rtimes \widehat{\mathbb{G}}_2$ and $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1 \rtimes \mathbb{G}_1 \rtimes \widehat{\mathbb{G}}_1)$ are canonically $\mathbb{G}_2$-equivariantly Morita equivalent.*

Proof. The first two statements are immediate consequences of Theorem 6.4.1 3. By applying Proposition 5.3.9 3 with $j = 2$, $l = 2$ and $l' = 1$, we obtain that the $\mathbb{G}_2$-C*-algebras $A_2 \otimes \mathcal{K}(\mathscr{H}_{22})$ and $A_2 \otimes \mathcal{K}(\mathscr{H}_{12})$ are canonically $\mathbb{G}_2$-equivariantly Morita equivalent. Moreover, the latter is canonically isomorphic to $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1 \otimes \mathcal{K}(\mathscr{H}_{11}))$ (the first statement with $j = 2$ and $k = 1 = l$). As a result, the last statement is proved in virtue of the Baaj-Skandalis duality theorem. □

The case of a linking $\mathbb{G}_1$-C*-algebra. Let $(J_1, \delta_{J_1}, e_1^1, e_1^2)$ be a linking $\mathbb{G}_1$-C*-algebra. Let $(J_2, \delta_{J_2}) = \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1, \delta_{J_1})$ be the induced $\mathbb{G}_2$-C*-algebra. By Proposition 6.2.5, the quintuple $(J_2, \delta_{J_2}, e_2^1, e_2^2)$, where

$$e_2^1 := e_1^1 \otimes 1_{S_{12}}, \quad e_2^2 := e_1^2 \otimes 1_{S_{12}},$$

is a linking $\mathbb{G}_2$-C*-algebra. Let $(J, \delta_J, \beta_J, e^1, e^2)$ be the associated linking $\mathcal{G}$-C*-algebra, where (J, δ_J, β_J) is the image of (J_1, δ_{J_1}) by the functor $\mathbb{G}_1$-$\mathsf{C^*}$-$\mathsf{Alg} \to \mathcal{G}$-$\mathsf{C^*}$-$\mathsf{Alg}$ and the self-adjoint projections e^1 and e^2 are defined by

$$e^1 := (e_1^1, e_2^1), \quad e^2 := (e_1^2, e_2^2) \quad \text{(see Corollary 6.2.7)}.$$

In what follows, we investigate the structure of linking $\mathcal{G}$-C*-algebra of the double crossed product $J \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ *corresponding to the projections e^1 and e^2*. We will use the notations of §5.3 in which *we replace everywhere A by J*. We also identify $J \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ with the C*-algebra $D \subset \mathcal{L}(J \otimes \mathscr{H})$.

Lemma 6.4.3. *Let $i = 1, 2$, there exists a unique nonzero self-adjoint projection $e^i_D \in \mathcal{M}(D)$ such that*

$$j_D(e^i_D) = q^{\beta_J,\widehat{\alpha}}(e^i \otimes 1_{\mathscr{H}}) \in \mathcal{L}(J \otimes \mathscr{H}) \qquad (\widehat{\alpha} = \beta).$$

Then, $(D, \delta_D, \beta_D, e^1_D, e^2_D)$ is a linking $\mathcal{G}$-C-algebra, that is:*

1. $e^1_D + e^2_D = 1_D, \quad [De^i_D D] = D, \quad i = 1, 2.$
2. $\delta_D(e^i_D) = \delta_D(1_D)(e^i_D \otimes 1_S), \quad i = 1, 2.$

Proof. We recall that we denote $\pi_R := (\mathrm{id}_J \otimes R)\delta_J : J \to \mathcal{L}(J \otimes \mathscr{H})$. We also recall that the C*-algebra D is given by:

$$D = [\pi_R(a)(1_J \otimes \lambda(x)L(y)) \,;\, a \in J,\, x \in \widehat{S},\, y \in S] \subset \mathcal{L}(J \otimes \mathscr{H}).$$

We have $\pi_R(m) = (\mathrm{id}_J \otimes R)\delta_J(m) \in \mathcal{M}(D)$ for all $m \in \mathcal{M}(J)$. However, by the equality

$$j_D(\mathcal{M}(D)) = \{T \in \mathcal{L}(J \otimes \mathscr{H}) \,;\, TD \subset D,\, DT \subset D,\, Tj_D(1_D) = T = j_D(1_D)T\}$$

and the fact that $j_D(1_D) = \pi_R(1_J)$, we have $\pi_R(m) \in j_D(\mathcal{M}(D))$ for all $m \in \mathcal{M}(J)$. In particular, we have

$$q^{\beta_J,\widehat{\alpha}}(e^i \otimes 1_{\mathscr{H}}) = \pi_R(e^i) := (\mathrm{id}_J \otimes R)\delta_J(e^i) \in j_D(\mathcal{M}(D)), \quad i = 1, 2.$$

By the faithfulness of j_D, there exists a unique $e^i_D \in \mathcal{M}(D)$ such that $j_D(e^i_D) = q^{\beta_J,\widehat{\alpha}}(e^i \otimes 1_{\mathscr{H}})$, for $i = 1, 2$. It is also clear that e^1_D and e^2_D are nonzero self-adjoint projections. Now, we have $j_D(e^1_D + e^2_D) = q^{\beta_J,\widehat{\alpha}} = j_D(1_D)$. Hence, $e^1_D + e^2_D = 1_D$. Let us fix $i = 1, 2$ and let us prove that $[De^i_D D] = D$. Let $d \in D$, since D is a C*-algebra we can assume that

$$d = (1_J \otimes L(y')\lambda(x'))\pi_R(a)(1_J \otimes \lambda(x)L(y)), \quad a \in J,\, x, x' \in \widehat{S},\, y, y' \in S.$$

However, we have $[Je^iJ] = J$. Therefore, d is the norm limit of finite sums of elements of the form

$$(1_J \otimes L(y')\lambda(x'))\pi_R(be^ib')(1_J \otimes \lambda(x)L(y)) = (1_J \otimes L(y')\lambda(x'))\pi_R(b)j_D(e^i_D)\pi_R(b')(1_J \otimes \lambda(x)L(y)),$$

where $b, b' \in J$. Hence, $d \in [De^i_D D]$. By Remark 4.4.14, we have

$$(j_D \otimes \mathrm{id}_S)\delta_D(e^i_D) = \delta_{J\otimes\mathcal{K}}(j_D(e^i_D)) = \delta_{J\otimes\mathcal{K}}(\pi_R(e^i)).$$

We recall that $\delta_{J\otimes\mathcal{K}}(\pi_R(m)) = q^{\beta_J,\widehat{\alpha}}_{12} q^{\beta,\alpha}_{23}(\pi_R(m) \otimes 1_S)$, for all $m \in \mathcal{M}(J)$ (see Proposition 4.4.12 2). Furthermore, we have $\pi_R(e^i) = j_D(e^i_D)$ and $(j_D \otimes \mathrm{id}_S)(q^{\beta_D,\alpha}) = q^{\beta_J,\widehat{\alpha}}_{12} q^{\beta,\alpha}_{23}$. Hence,

$$(j_D \otimes \mathrm{id}_S)\delta_D(e^i_D) = (j_D \otimes \mathrm{id}_S)(q^{\beta_D,\alpha}(e^i_D \otimes 1_S)).$$

□

It should be noted that the $\mathcal{G}$-C*-algebra D has *two compatible structures of linking algebras*. The first structure is given by the previous lemma whereas the second one is given by the compatible structure of linking algebra on D_j defined by the projections $e_{l,j}$, $l = 1, 2$ (see Lemma 5.3.4 1). More precisely, we have:

Lemma 6.4.4. *For all $s, j, l = 1, 2$, let $e^s_{l,j} \in \mathcal{M}(D_j)$ be the unique self-adjoint projection such that*

$$\iota_j(e^s_{l,j}) = e^s_D \iota_j(e_{l,j}),$$

where $\iota_j : \mathcal{M}(D_j) \to \mathcal{M}(D)$ is the unique strictly continuous extension of the inclusion map $D_j \subset D$ such that $\iota_j(1_{D_j}) = \beta_D(\varepsilon_j)$. For all $s, j, k, l = 1, 2$, we have

$$\delta^k_{D_j}(e^s_{l,j}) = e^s_{l,k} \otimes 1_{S_{kj}}.$$

Proof. By composing with j_D, it follows immediately from the definitions of the projections that we have the formulas:

$$[e^s_D, \iota_j(e_{l,j})] = 0, \quad e^s_D \iota_j(e_{l,j})\beta_D(\varepsilon_j) = e^s_D \iota_j(e_{l,j})$$

and this proves the existence and uniqueness of the self-adjoint projections $e^s_{l,j}$. Now, let us denote $\pi^k_j : \mathcal{M}(D_k \otimes S_{kj}) \to \mathcal{M}(D \otimes S)$ the strictly continuous extension of the inclusion map $D_k \otimes S_{kj} \subset D \otimes S$ such that $\pi^k_j(1_{D_k \otimes S_{kj}}) = \beta_D(\varepsilon_k) \otimes p_{kj}$. By using the faithfulness of π^k_j and the fact that $\pi^k_j(e^s_{l,k} \otimes 1_{S_{kj}}) = \iota_k(e^s_{l,k}) \otimes p_{kj} = e^s_D \iota_k(e_{l,k}) \otimes p_{kj}$, it suffices to prove that

$$\pi^k_j \delta^k_{D_j}(e^s_{l,j}) = e^s_D \iota_k(e_{l,k}) \otimes p_{kj}.$$

We have

$$\begin{aligned}
\pi^k_j \delta^k_{D_j}(e^s_{l,j}) &= \delta_D(\iota_j(e^s_{l,j}))(\beta_D(\varepsilon_k) \otimes p_{kj}) && \text{(Proposition 5.1.3)} \\
&= \delta_D(e^s_D \iota_j(e_{l,j}))(\beta_D(\varepsilon_k) \otimes p_{kj}) \\
&= \delta_D(1_D)(e^s_D \otimes 1_S)\delta_D(\iota_j(e_{l,j}))(\beta_D(\varepsilon_k) \otimes p_{kj}) && \text{(Lemma 6.4.3 2)} \\
&= \delta_D(1_D)(\beta_D(\varepsilon_k) \otimes p_{kj})(e^s_D \otimes 1_S)\delta_D(\iota_j(e_{l,j}))(\beta_D(\varepsilon_k) \otimes p_{kj}) \\
&= (\beta_D(\varepsilon_k) \otimes p_{kj})(e^s_D \otimes 1_S)\delta_D(\iota_j(e_{l,j}))(\beta_D(\varepsilon_k) \otimes p_{kj}) \\
&= (\beta_D(\varepsilon_k) \otimes p_{kj})(e^s_D \otimes 1_S)\pi^k_j \delta^k_{D_j}(e_{l,j}).
\end{aligned}$$

However, by Lemma 5.3.4 4, we have $\pi^k_j \delta^k_{D_j}(e_{l,j}) = \pi^k_j(e_{l,k} \otimes 1_{S_{kj}}) = \iota_k(e_{l,k}) \otimes p_{kj}$. Hence,

$$\begin{aligned}
\pi^k_j \delta^k_{D_j}(e^s_{l,j}) &= (\beta_D(\varepsilon_k) \otimes p_{kj})(e^s_D \iota_k(e_{l,k}) \otimes p_{kj}) \\
&= (\beta_D(\varepsilon_k) \otimes p_{kj})(\iota_k(e^s_{l,k}) \otimes p_{kj}) \\
&= \iota_k(e^s_{l,k}) \otimes p_{kj} \\
&= e^s_D \iota_k(e_{l,k}) \otimes p_{kj}.
\end{aligned}$$

□

Notations 6.4.5. For $j, l, l', s, s' = 1, 2$, we denote:

$$D^{ss'}_{ll',j} := e^s_{l,j} D_j e^{s'}_{l',j}, \quad D^s_{l,j} := D^{ss}_{ll,j}.$$

Note that $D^{ss'}_{ll',j} = e^s_D D_{ll',j} e^{s'}_D$, $D^s_{l,j} = e^s_D D_{l,j} e^s_D$.

In virtue of Lemma 6.4.4, we have that $D^{ss'}_{ll',j}$ is a $\mathbb{G}_j$-equivariant Hilbert $D^s_{l,j}$-$D^{s'}_{l',j}$-bimodule by restriction of the structure of $\mathbb{G}_j$-C*-algebra on D_j. Moreover, we have the following result:

Proposition 6.4.6. *Let $j,k=1,2$ with $j\neq k$. By restriction of the isomorphism of $\mathbb{G}_j$-C*-algebras*

$$\widetilde{\pi}_j : D_j \to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_k)\,;\, x\mapsto \delta_{D_j}^k(x),$$

we obtain the isomorphisms

$$D_{ll',j}^{ss'} \to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{ll',k}^{ss'}),\quad l,l',s,s'=1,2,$$

of $\mathbb{G}_j$-equivariant Hilbert bimodules over the isomorphisms of $\mathbb{G}_j$-C-algebras*

$$D_{l,j}^{s}\to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{l,k}^{s}),\quad D_{l',j}^{s'}\to \mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{l',k}^{s'}).$$

Proof. It follows from the formulas $\delta_{D_j}^k(e_{l,j}^s)=e_{l,k}^s\otimes 1_{S_{kj}}$ and $\delta_{D_j}^k(e_{l',j}^{s'})=e_{l',k}^{s'}\otimes 1_{S_{kj}}$ (Lemma 6.4.4) that

$$\delta_{D_j}^k(D_{ll',j}^{ss'})=(e_{l,k}^s\otimes 1_{S_{kj}})\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_k)(e_{l',k}^{s'}\otimes 1_{S_{kj}})=\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{ll',k}^{ss'}).$$

In particular, we have $\delta_{D_j}^k(D_{l,j}^s)=\mathrm{Ind}_{\mathbb{G}_k}^{\mathbb{G}_j}(D_{l,k}^s)$. □

In the following, we will obtain the applications of Proposition 6.4.6 to an equivariant Morita equivalence corresponding to an equivariant Hilbert module.

Let (A_1,δ_{A_1}) and (B_1,δ_{B_1}) be two $\mathbb{G}_1$-C*-algebras and $(\mathscr{E}_1,\delta_{\mathscr{E}_1})$ a $\mathbb{G}_1$-equivariant Hilbert A_1-B_1-bimodule. Let us denote:

$$(A_2,\delta_{A_2}):=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1,\delta_{A_1}),\quad (B_2,\delta_{B_2}):=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1,\delta_{B_1}),\quad (\mathscr{E}_2,\delta_{\mathscr{E}_2}):=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1,\delta_{\mathscr{E}_1}).$$

Let us denote:

$$J_1:=\mathcal{K}(\mathscr{E}_1\oplus B_1)=\begin{pmatrix}\mathcal{K}(\mathscr{E}_1) & \mathscr{E}_1\\ \mathscr{E}_1^* & B_1\end{pmatrix}$$

endowed with the continuous action δ_{J_1}, which defines the structure of $\mathbb{G}_1$-equivariant Hilbert B_1-module on $\mathscr{E}_1$. We have

$$J_2:=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1)=\mathcal{K}(\mathscr{E}_2\oplus B_2)=\begin{pmatrix}\mathcal{K}(\mathscr{E}_2) & \mathscr{E}_2\\ \mathscr{E}_2^* & B_2\end{pmatrix}.$$

Let us consider the $\mathcal{G}$-C*-algebra (J,δ_J,β_J) with $J=J_1\oplus J_2$ defined by J_1 and J_2 (see Corollary 6.2.7). We identify the $\mathcal{G}$-C*-algebras $J\rtimes\mathcal{G}\rtimes\widehat{\mathcal{G}}$ and (D,δ_D,β_D) as in Theorem 4.4.15 and we use all the notations of §5.3 and in particular those of Notations 5.3.7.

For $j=1,2$, we identify

$$D_j=\mathcal{B}_j:=J_j\otimes\mathcal{K}(\mathscr{H}_{1j}\oplus\mathscr{H}_{2j})=\mathcal{K}((\mathscr{E}_j\oplus B_j)\otimes(\mathscr{H}_{1j}\oplus\mathscr{H}_{2j})).$$

Note that for all $l,l'=1,2$ we have:

- For $(s,s')=(1,1)$, the $\mathbb{G}_j$-C*-algebra $D_{l,j}^1$ is identified to the $\mathbb{G}_j$-C*-algebra $\mathcal{K}(\mathscr{E}_j)\otimes\mathcal{K}(\mathscr{H}_{lj})$ and the action is given by:

$$x\mapsto (V_{jj}^l)_{23}(\mathrm{id}_{\mathcal{K}(\mathscr{E}_j)}\otimes\sigma)(\delta_{\mathcal{K}(\mathscr{E}_j)}^j\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{lj})})(x)(V_{jj}^l)_{23}^*,\quad x\in\mathcal{K}(\mathscr{E}_j)\otimes\mathcal{K}(\mathscr{H}_{lj}),$$

- For $(s,s')=(2,2)$, the $\mathbb{G}_j$-C*-algebra $D^2_{l,j}$ is identified to the $\mathbb{G}_j$-C*-algebra $B_j\otimes\mathcal{K}(\mathscr{H}_{lj})$ and the action is given by:

$$x\mapsto (V^l_{jj})_{23}(\mathrm{id}_{B_j}\otimes\sigma)(\delta^j_{B_j}\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{lj})})(x)(V^l_{jj})^*_{23},\quad x\in B_j\otimes\mathcal{K}(\mathscr{H}_{lj}),$$

- For $(s,s')=(1,2)$, the $\mathbb{G}_j$-equivariant Hilbert $D^1_{l,j}$-$D^2_{l',j}$-bimodule $D^{12}_{ll',j}$ is identified with the $\mathbb{G}_j$-equivariant Hilbert $\mathcal{K}(\mathscr{E}_j)\otimes\mathcal{K}(\mathscr{H}_{lj})$-$B_j\otimes\mathcal{K}(\mathscr{H}_{l'j})$-bimodule $\mathscr{E}_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj})$ and the action is given by:

$$\xi\mapsto (V^l_{jj})_{23}(\mathrm{id}_{\mathscr{E}_j}\otimes\sigma)(\delta^j_{\mathscr{E}_j}\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj})})(\xi)(V^{l'}_{jj})^*_{23},\quad \xi\in\mathscr{E}_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}),$$

where σ denotes an appropriate flip map.

Theorem 6.4.7. *Let (A_1,δ_{A_1}) and (B_1,δ_{B_1}) be two $\mathbb{G}_1$-C*-algebras and $(\mathscr{E}_1,\delta_{\mathscr{E}_1})$ a $\mathbb{G}_1$-equivariant Hilbert A_1-B_1-bimodule such that the left action of A_1 on $\mathscr{E}_1$ is non-degenerate. Let us denote:*

$$(A_2,\delta_{A_2})=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1,\delta_{A_1}),\quad (B_2,\delta_{B_2})=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(B_1,\delta_{B_1}),\quad (\mathscr{E}_2,\delta_{\mathscr{E}_2})=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1,\delta_{\mathscr{E}_1}).$$

For all $j,k,l,l'=1,2$ with $j\neq k$, the linear map

$$\begin{array}{rcl}\delta^k_{ll',j}:\ \mathscr{E}_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}) & \longrightarrow & \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathscr{E}_k\otimes\mathcal{K}(\mathscr{H}_{l'k},\mathscr{H}_{lk}))\\ \xi & \longmapsto & (V^l_{kj})_{23}(\mathrm{id}_{\mathscr{E}_k}\otimes\sigma)(\delta^k_{\mathscr{E}_j}\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj})})(\xi)(V^{l'}_{kj})^*_{23}\end{array}$$

is an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules over the isomorphisms of C*-*algebras:*

$$A_j\otimes\mathcal{K}(\mathscr{H}_{lj})\to\mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(A_k\otimes\mathcal{K}(\mathscr{H}_{lk}))\,;\,x\mapsto(V^l_{kj})_{23}(\mathrm{id}_{A_k}\otimes\sigma)(\delta^k_{A_j}\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{lj})})(x)(V^l_{kj})^*_{23},$$

$$B_j\otimes\mathcal{K}(\mathscr{H}_{l'j})\to\mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(B_k\otimes\mathcal{K}(\mathscr{H}_{l'k}))\,;\,x\mapsto(V^{l'}_{kj})_{23}(\mathrm{id}_{B_k}\otimes\sigma)(\delta^k_{B_j}\otimes\mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j})})(x)(V^{l'}_{kj})^*_{23},$$

where σ denotes an appropriate flip map.

Proof. It follows from Proposition 6.4.6 that the theorem is already established if we take $\mathcal{K}(\mathscr{E}_j)$ for the $\mathbb{G}_j$-C*-algebra A_j. More precisely, we have an isomorphism of $\mathbb{G}_j$-equivariant Hilbert bimodules

$$\delta^k_{ll',j}:\mathscr{E}_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj})\longrightarrow\mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathscr{E}_k\otimes\mathcal{K}(\mathscr{H}_{l'k},\mathscr{H}_{lk}))$$

over the isomorphisms of C*-algebras

$$\mathcal{K}(\mathscr{E}_j)\otimes\mathcal{K}(\mathscr{H}_{lj})\to\mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathcal{K}(\mathscr{E}_k)\otimes\mathcal{K}(\mathscr{H}_{lk})),\quad B_j\otimes\mathcal{K}(\mathscr{H}_{l'j})\to\mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(B_k\otimes\mathcal{K}(\mathscr{H}_{l'k})).$$

Let us consider the $\mathbb{G}_1$-C*-algebra $\mathcal{K}(\mathscr{E}_1)$, we then have $\mathcal{K}(\mathscr{E}_2)=\mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}\mathcal{K}(\mathscr{E}_1)$. By taking the image of the non-degenerate *-representation $A_1\to\mathcal{L}(\mathscr{E}_1)$ by the induction functor, we obtain a non-degenerate *-representation $A_2\to\mathcal{L}(\mathscr{E}_2)$ (*cf.* §6.3). It then follows that we have a non-degenerate $\mathcal{G}$-equivariant *-homomorphism $A\to\mathcal{L}(\mathscr{E}_1\oplus\mathscr{E}_2)$, where $A=A_1\oplus A_2$. By composing with the canonical *degenerate* *-homomorphism $\mathcal{L}(\mathscr{E}_1\oplus\mathscr{E}_2)\to\mathcal{M}(J)$, we obtain a *-homomorphism $A\to\mathcal{M}(J)$, which extends to a *non* unital strictly continuous *-homomorphism $\mathcal{M}(A)\to\mathcal{M}(J)$. By the compatibility of the actions, the *-homomorphism $\mathcal{L}(\mathscr{E}_1\oplus\mathscr{E}_2)\to\mathcal{M}(J)$ is $\mathcal{G}$-equivariant and so is $A\to\mathcal{M}(J)$. By using the functoriality of the correspondence $-\rtimes\mathcal{G}\rtimes\widehat{\mathcal{G}}$ (see Theorems 4.3.3 2, 4.3.6 2 and Remark 4.3.4), we obtain a $\mathcal{G}$-equivariant *-homomorphism

$$A\rtimes\mathcal{G}\rtimes\widehat{\mathcal{G}}\to\mathcal{M}(J\rtimes\mathcal{G}\rtimes\widehat{\mathcal{G}}).$$

For $j = 1, 2$, let us denote $\mathcal{A}_j := A_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j})$ (*cf.* Notations 5.3.7). By identifying the $\mathcal{G}$-C*-algebras $A \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ and $\mathcal{A}_1 \oplus \mathcal{A}_2$, the above equivariant *-homomorphism restricts to *-homomorphisms:

$$f_j : \mathcal{A}_j \to \mathcal{M}(\mathcal{B}_j), \quad \mathcal{B}_j := J_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}), \quad j = 1, 2,$$

which satisfy $(f_k \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathcal{A}_j} = \delta^k_{\mathcal{B}_j} \circ f_j$ for all $j, k = 1, 2$. By restriction of f_j, we finally obtain non-degenerate $\mathbb{G}_j$-equivariant *-homomorphims

$$A_j \otimes \mathcal{K}(\mathscr{H}_{lj}) \to \mathcal{L}(\mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})), \quad l, l' = 1, 2.$$

Therefore, the equivariance of $\delta^k_{ll',j} : \mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) \to \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathscr{E}_k \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk}))$ follows from Proposition 6.4.6 and the formula $(f_k \otimes \mathrm{id}_{S_{kj}})\delta^k_{\mathcal{A}_j} = \delta^k_{\mathcal{B}_j} \circ f_j$. □

In order to define an equivalence of category between $\mathsf{KK}^{\mathbb{G}_1}$ and $\mathsf{KK}^{\mathbb{G}_2}$ in the next chapter, we will need to specify some notations and give a useful lemma.

Notations 6.4.8. Let us fix $j, k, l, l' = 1, 2$ with $j \neq k$. We have:

1. $\mathcal{B}_{l,j} := J_j \otimes \mathcal{K}(\mathscr{H}_{lj}) = \begin{pmatrix} \mathcal{K}(\mathscr{E}_j) \otimes \mathcal{K}(\mathscr{H}_{lj}) & \mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{lj}) \\ (\mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{lj}))^* & B_j \otimes \mathcal{K}(\mathscr{H}_{lj}) \end{pmatrix}$.

2. The Hilbert $\mathcal{B}_{l,j}$-$\mathcal{B}_{l',j}$-bimodule $\mathcal{B}_{ll',j} := J_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})$ is of the form:

$$\mathcal{B}_{ll',j} = \begin{pmatrix} \mathcal{K}(\mathscr{E}_j) \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) & \mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) \\ (\mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}))^* & B_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) \end{pmatrix}.$$

3. By restriction of $\delta^k_{\mathcal{B}_j}$, we have the following $\mathbb{G}_j$-equivariant isomorphisms:

$$\begin{aligned} \delta^k_{ll',j} : \mathcal{K}(\mathscr{E}_j) \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) &\longrightarrow \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathcal{K}(\mathscr{E}_k) \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk})) \\ x &\longmapsto (V^l_{kj})_{23}(\mathrm{id}_{\mathcal{K}(\mathscr{E}_k)} \otimes \sigma)(\delta^k_{\mathcal{K}(\mathscr{E}_j)} \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})})(x)(V^{l'}_{kj})^*_{23}, \end{aligned}$$

$$\begin{aligned} \delta^k_{ll',j} : \mathscr{E}_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) &\longrightarrow \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(\mathscr{E}_k \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk})) \\ \xi &\longmapsto (V^l_{kj})_{23}(\mathrm{id}_{\mathscr{E}_k} \otimes \sigma)(\delta^k_{\mathscr{E}_j} \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})})(\xi)(V^{l'}_{kj})^*_{23}, \end{aligned}$$

$$\begin{aligned} \delta^k_{ll',j} : B_j \otimes \mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj}) &\longrightarrow \mathrm{Ind}^{\mathbb{G}_j}_{\mathbb{G}_k}(B_k \otimes \mathcal{K}(\mathscr{H}_{l'k}, \mathscr{H}_{lk})) \\ x &\longmapsto (V^l_{kj})_{23}(\mathrm{id}_{B_k} \otimes \sigma)(\delta^k_{B_j} \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H}_{l'j}, \mathscr{H}_{lj})})(x)(V^{l'}_{kj})^*_{23}, \end{aligned}$$

   where σ denotes an appropriate flip map.

Lemma 6.4.9. *For all $j, k, l = 1, 2$ and $T \in \mathcal{M}(\mathcal{K}(\mathscr{E}_l))$, we have*

$$\delta^k_{ll,j}(\mathrm{id}_{\mathcal{K}(\mathscr{E}_j)} \otimes R_{jl})\delta^j_{\mathcal{K}(\mathscr{E}_l)}(T) = (\mathrm{id}_{\mathcal{K}(\mathscr{E}_k)} \otimes R_{kl})\delta^k_{\mathcal{K}(\mathscr{E}_l)}(T) \otimes 1_{S_{kj}}.$$

Proof. This follows immediately from Proposition 4.4.12 2 for the action of $\mathcal{G}$ on J. □

Chapter 7

Application to equivariant KK-theory

In [28], Voigt has established a canonical equivalence of the categories $\mathsf{KK}^{\mathbb{G}_1}$ and $\mathsf{KK}^{\mathbb{G}_2}$ for two monoidally equivalent compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. In this chapter, we generalize this result to the case of two monoidally equivalent regular locally compact quantum groups.

Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent regular locally compact quantum groups. If (A_1, δ_{A_1}) and (B_1, δ_{B_1}) are two $\mathbb{G}_1$-C*-algebras, we consider

$$(A_2, \delta_{A_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1, \delta_{A_1}), \quad (B_2, \delta_{B_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1, \delta_{B_1}),$$

the induced $\mathbb{G}_2$-C*-algebras (see §6.1 and Propositions 6.1.2 and 6.1.4). In this chapter, to any $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule $(\mathscr{E}_1, F_1)$ we associate canonically a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule $(\mathscr{E}_2, F_2)$ and we define an isomorphism of abelian groups

$$\begin{array}{rccl} J_{\mathbb{G}_2,\mathbb{G}_1} : & \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1) & \longrightarrow & \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \\ & x = [(\mathscr{E}_1, F_1)] & \longmapsto & J_{\mathbb{G}_2,\mathbb{G}_1}(x) := [(\mathscr{E}_2, F_2)], \end{array}$$

whose inverse isomorphism $J_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1)$ is obtained in a similar way by exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$.

First, let us recall briefly some results and notations of [3] concerning the equivariant KK-theory for actions of locally compact quantum groups. For more precise information and further details, we refer the reader to [3] and the references therein [16, 17]. Let us fix a regular locally compact quantum group $\mathbb{G}$:

- To any pair of $\mathbb{G}$-C*-algebra (A, B), we associate the set denoted $\mathsf{KK}^{\mathbb{G}}(A, B)$ consisting of all classes of $\mathbb{G}$-equivariant Kasparov A-B-bimodule $(\mathscr{E}, F)$, that is $\mathscr{E}$ is an $\mathbb{G}$-equivariant A-B-bimodule and $F \in \mathcal{L}(\mathscr{E})$ verifies:

$$[F, a] \in \mathcal{K}(\mathscr{E}), \quad a(F - F^*) \in \mathcal{K}(\mathscr{E}), \quad a(F^2 - 1) \in \mathcal{K}(\mathscr{E}), \quad a \in A,$$

$$x(\delta_{\mathcal{K}(\mathscr{E})}(F) - F \otimes 1_{\mathrm{C}_0(\mathbb{G})}) \in \mathcal{K}(\mathscr{E}) \otimes \mathrm{C}_0(\mathbb{G}), \quad x \in A \otimes \mathrm{C}_0(\mathbb{G}).$$

  Note that $\mathsf{KK}^{\mathbb{G}}(A, B)$ becomes an abelian group, with addition given by the direct sum of equivariant Kasparov bimodules.

- Note that if $\mathscr{E}$ is a $\mathbb{G}$-equivariant Hilbert A-B-bimodule and $F_0, F_1 \in \mathcal{L}(\mathscr{E})$ such that $(\mathscr{E}, F_0)$ and $(\mathscr{E}, F_1)$ are $\mathbb{G}$-equivariant Kasparov A-B-bimodules and $a(F_1 - F_0) \in \mathcal{K}(\mathscr{E})$ for all $a \in A$, then we have $[(\mathscr{E}, F_0)] = [(\mathscr{E}, F_1)]$.

- If A, D, B are $\mathbb{G}$-C*-algebras, we have the product (internal Kasparov product):

$$\mathsf{KK}^{\mathbb{G}}(A, D) \times \mathsf{KK}^{\mathbb{G}}(D, B) \to \mathsf{KK}^{\mathbb{G}}(A, B)\,;\, (x, y) \mapsto x \otimes_D y.$$

- Let A be a $\mathbb{G}$-C*-algebra. Then, $(A, 0)$ is a $\mathbb{G}$-equivariant Kasparov A-A-bimodule. We denote:

$$1_A := [(A, 0)] \in \mathsf{KK}^{\mathbb{G}}(A, A).$$

If $x \in \mathsf{KK}^{\mathbb{G}}(A, B)$, we have $1_A \otimes_A x = x$ and $x \otimes_B 1_B = x$.

- Let A be a $\mathbb{G}$-C*-algebra. We identify the $\mathbb{G}$-C*-algebra $A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$ with $A \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}}$. Let us denote:

$$\beta_A := [(A \otimes \mathrm{L}^2(\mathbb{G}), 0)] \in \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})), A),$$
$$\alpha_A := [((A \otimes \mathrm{L}^2(\mathbb{G}))^*, 0)] \in \mathsf{KK}^{\mathbb{G}}(A, A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))).$$

We have $\beta_A \otimes_A \alpha_A = 1_{A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))}$ and $\alpha_A \otimes_{A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))} \beta_A = 1_A$.

- If A and B are $\mathbb{G}$-C*-algebras, for all $x = [(\mathscr{E}, F)] \in \mathsf{KK}^{\mathbb{G}}(A, B)$ we have

$$\beta_A \otimes x \otimes \alpha_B = [(\mathscr{E} \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})), (\mathrm{id}_{\mathcal{K}(\mathscr{E})} \otimes R)\delta_{\mathcal{K}(\mathscr{E})}(F))] \in \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})), B \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})))$$

and the operator $(\mathrm{id}_{\mathcal{K}(\mathscr{E})} \otimes R)\delta_{\mathcal{K}(\mathscr{E})}(F)$ is invariant with respect to the bidual action of $\mathbb{G}$ on $\mathcal{K}(\mathscr{E}) \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))$. Moreover, the map

$$\begin{array}{rcl} \mathsf{KK}^{\mathbb{G}}(A, B) & \longrightarrow & \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G})), B \otimes \mathcal{K}(\mathrm{L}^2(\mathbb{G}))) \\ x & \longmapsto & \beta_A \otimes_A x \otimes_B \alpha_B \end{array}$$

is an isomorphism of abelian groups.

- If $x = [(\mathscr{E}, F)] \in \mathsf{KK}^{\mathbb{G}}(A, B)$, we can assume that the left action $A \to \mathcal{L}(\mathscr{E})$ of A on $\mathscr{E}$ is non-degenerate by replacing x with $1_A \otimes_A x$ if necessary.

- If A_1, A_2, B are $\mathbb{G}$-C*-algebras, a $\mathbb{G}$-equivariant *-homomorphism $f : A_1 \to A_2$ induces a homomorphism of abelian groups $f^* : \mathsf{KK}^{\mathbb{G}}(A_2, B) \to \mathsf{KK}^{\mathbb{G}}(A_1, B)$.

- If A, B_1, B_2 are $\mathbb{G}$-C*-algebras, a $\mathbb{G}$-equivariant *-homomorphism $g : B_1 \to B_2$ induces a homomorphism of abelian groups

$$\begin{array}{rcl} g_* : \mathsf{KK}^{\mathbb{G}}(A, B_1) & \longrightarrow & \mathsf{KK}^{\mathbb{G}}(A, B_2) \\ [(\mathscr{E}, F)] & \longmapsto & [(\mathscr{E} \otimes_g B_2, F \otimes_g 1_{B_2})]. \end{array}$$

- Let A, B be $\mathbb{G}$-C*-algebras and $f : A \to B$ a $\mathbb{G}$-equivariant *-homomorphism. Then, the pair $(B, 0)$ is a Kasparov A-B-bimodule. By abuse of notation, we will denote

$$f := f^*(1_B) = [(B, 0)] \in \mathsf{KK}^{\mathbb{G}}(A, B).$$

Let C, D be $\mathbb{G}$-C*-algebras. We have the formulas:

$$f \otimes_B x = f^*(x), \quad x \in \mathsf{KK}^{\mathbb{G}}(B, C) \quad ; \quad y \otimes_A f = f_*(y), \quad y \in \mathsf{KK}^{\mathbb{G}}(D, A).$$

In particular, if $A, \widetilde{A}, B, \widetilde{B}$ are $\mathbb{G}$-C*-algebras, $f : A \to \widetilde{A}$ and $g : B \to \widetilde{B}$ are $\mathbb{G}$-equivariant *-homomorphisms and $x := [(\mathscr{E}, F)] \in \mathsf{KK}^{\mathbb{G}}(\widetilde{A}, \widetilde{B})$, we have

$$f \otimes_{\widetilde{A}} x \otimes_{\widetilde{B}} g^{-1} = [(\mathscr{E} \otimes_{g^{-1}} B, F \otimes_{g^{-1}} 1_B)] \in \mathsf{KK}^{\mathbb{G}}(A, B),$$

where the left action of A on $\mathscr{E} \otimes_{g^{-1}} B$ is given by $a(\xi \otimes_{g^{-1}} b) = f(a)\xi \otimes_{g^{-1}} b$ for $a \in A$, $\xi \in \mathscr{E}$ and $b \in B$.

Let us recall the following result:

Lemma 7.1. *Let A, B and D be C*-algebras. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be respectively a Hilbert D-module and a Hilbert B-module. Let $\pi : D \to \mathcal{L}(\mathscr{E}_2)$ be a *-homomorphism. Let us denote $\widetilde{\pi} : \mathcal{L}(\mathscr{E}_1) \to \mathcal{L}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$ the *-homomorphism defined by:*

$$\widetilde{\pi}(T) = T \otimes_\pi 1_{\mathscr{E}_2}, \quad T \in \mathcal{L}(\mathscr{E}_1).$$

If $\pi(D) \subset \mathcal{K}(\mathscr{E}_2)$, then we have $\widetilde{\pi}(\mathcal{K}(\mathscr{E}_1)) \subset \mathcal{K}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$.

Since we could not find a proper reference, we also recall the proof:

Proof. For $\xi \in \mathscr{E}_1$, we denote $T_\xi \in \mathcal{L}(\mathscr{E}_2, \mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$ the operator defined by $T_\xi(\eta) = \xi \otimes_\pi \eta$, for all $\eta \in \mathscr{E}_2$. Then, we have

$$T_\xi^*(\zeta_1 \otimes_\pi \zeta_2) = \pi(\langle \xi,\, \zeta_1 \rangle)\zeta_2, \quad \zeta_1 \in \mathscr{E}_1,\, \zeta_2 \in \mathscr{E}_2.$$

A straightforward computation gives

$$T_\xi \pi(x) T_\eta^* = \widetilde{\pi}(\theta_{\xi, \eta x^*}), \quad \xi, \eta \in \mathscr{E}_1,\, x \in D.$$

By assumption, we then have $\widetilde{\pi}(\theta_{\xi,\eta x^*}) \in \mathcal{K}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$, for all $\xi, \eta \in \mathscr{E}_1$ and $x \in D$. By Cohen-Hewitt factorization theorem, we have $\mathscr{E}_1 D = \mathscr{E}_1$. Hence, $\widetilde{\pi}(\theta_{\xi,\eta}) \in \mathcal{K}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$ for all $\xi, \eta \in \mathscr{E}_1$. Therefore, we have $\widetilde{\pi}(\mathcal{K}(\mathscr{E}_1)) \subset \mathcal{K}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$. □

Let us give two technical lemmas:

Lemma 7.2. *Let $\mathbb{G}$ be a locally compact quantum group. Let A, B and D be $\mathbb{G}$-C*-algebras. Let $(\mathscr{E}_1, F_1)$ be a $\mathbb{G}$-equivariant Kasparov A-D-bimodule and $\mathscr{E}_2$ a $\mathbb{G}$-equivariant Hilbert D-B-bimodule. If the left action $D \to \mathcal{L}(\mathscr{E}_2)$ takes its values in $\mathcal{K}(\mathscr{E}_2)$, then the pair $(\mathscr{E}_1 \otimes_D \mathscr{E}_2, F_1 \otimes_D 1_{\mathscr{E}_2})$ is a $\mathbb{G}$-equivariant Kasparov A-B-bimodule and we have*

$$[(\mathscr{E}_1, F_1)] \otimes_D [(\mathscr{E}_2, 0)] = [(\mathscr{E}_1 \otimes_D \mathscr{E}_2, F_1 \otimes_D 1_{\mathscr{E}_2})] \in \mathsf{KK}^{\mathbb{G}}(A, B).$$

Proof. Let us denote $\pi : D \to \mathcal{L}(\mathscr{E}_2)$ the left action of D on $\mathscr{E}_2$, $\mathscr{E} = \mathscr{E}_1 \otimes_\pi \mathscr{E}_2$ and $S = \mathrm{C}_0(\mathbb{G})$. By Lemma 7.1, the hypothesis implies that the *-homomorphism $\mathcal{L}(\mathscr{E}_1) \to \mathcal{L}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)\,;\, x \mapsto x \otimes_\pi 1$ restricts to a *-homomorphism $\mathcal{K}(\mathscr{E}_1) \to \mathcal{K}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2)$. Therefore, we obtain immediately that the pair $(\mathscr{E}, F_1 \otimes_\pi 1)$ is a Kasparov A-B-bimodule. In the following, we will recall some of the notations and identifications used in Proposition 2.10 of [3] and in its proof. Let

$$V_1 \in \mathcal{L}(\mathscr{E}_1 \otimes_{\delta_D} (D \otimes S), \mathscr{E}_1 \otimes S), \quad V_2 \in \mathcal{L}(\mathscr{E}_2 \otimes_{\delta_B} (B \otimes S), \mathscr{E}_2 \otimes S)$$

be the unitaries associated with the actions $\delta_{\mathscr{E}_1}$ and $\delta_{\mathscr{E}_2}$ respectively. We recall that there exists a unitary $\widetilde{V}_2 \in \mathcal{L}(\mathscr{E} \otimes_{\delta_B} (B \otimes S), \mathscr{E}_1 \otimes_{(\pi \otimes \mathrm{id}_S)\delta_D} (\mathscr{E}_2 \otimes S))$ such that

$$\widetilde{V}_2((\xi_1 \otimes_\pi \xi_2) \otimes_{\delta_B} x) = \xi_1 \otimes_{(\pi \otimes \mathrm{id}_S)\delta_D} V_2(\xi_2 \otimes_{\delta_B} x), \quad \xi_1 \in \mathscr{E}_1,\, \xi_2 \in \mathscr{E}_2,\, x \in B \otimes S.$$

We also recall the following identifications:

$$\mathscr{E} \otimes S = (\mathscr{E}_1 \otimes S) \otimes_{\pi \otimes \mathrm{id}_S} (\mathscr{E}_2 \otimes S),$$

$$\mathscr{E}_1 \otimes_{(\pi \otimes \mathrm{id}_S)\delta_D} (\mathscr{E}_2 \otimes S) = (\mathscr{E}_1 \otimes_{\delta_D} (D \otimes S)) \otimes_{\pi \otimes \mathrm{id}_S} (\mathscr{E}_2 \otimes S).$$

Finally, we recall that the unitary associated with the action of $\mathbb{G}$ on $\mathscr{E}$ is

$$V := (V_1 \otimes_{\pi\otimes \mathrm{id}_S} 1)\widetilde{V}_2 \in \mathcal{L}(\mathscr{E} \otimes_{\delta_B} (D \otimes S), \mathscr{E} \otimes S).$$

Let us recall that the action of $\mathbb{G}$ on $\mathcal{K}(\mathscr{E})$ (resp. $\mathcal{K}(\mathscr{E}_1)$) is given by:

$$\delta_{\mathcal{K}(\mathscr{E})}(k) = V(k \otimes_{\delta_B} 1)V^*, \quad k \in \mathcal{K}(\mathscr{E}) \quad (\text{resp. } \delta_{\mathcal{K}(\mathscr{E}_1)}(k) = V_1(k \otimes_{\delta_D} 1)V_1^*, \quad k \in \mathcal{K}(\mathscr{E}_1))$$

(see Remarque 2.8 (a) in [3]). It is clear that we have

$$\widetilde{V}_2((F_1 \otimes_\pi 1) \otimes_{\delta_B} 1) = (F_1 \otimes_{(\pi\otimes \mathrm{id}_S)\delta_D} 1)\widetilde{V}_2, \quad (V_1 \otimes_{\pi\otimes \mathrm{id}_S} 1)(F_1 \otimes_{(\pi\otimes \mathrm{id}_S)\delta_D} 1) = V_1(F_1 \otimes_{\delta_D} 1) \otimes_{\pi\otimes \mathrm{id}_S} 1.$$

However, since $\widetilde{V}_2$ is unitary we then have $\delta_{\mathcal{K}(\mathscr{E})}(F_1 \otimes_\pi 1) = \delta_{\mathcal{K}(\mathscr{E}_1)}(F_1) \otimes_{\pi\otimes \mathrm{id}_S} 1$. By the assumption and the fact that the Kasparov A-D-bimodule $(\mathscr{E}_1, F_1)$ is $\mathbb{G}$-equivariant, we obtain that the Kasparov A-B-bimodule $(\mathscr{E}, F_1 \otimes_\pi 1)$ is $\mathbb{G}$-equivariant.
By the assumption, the pair $(\mathscr{E}_2, 0)$ is a $\mathbb{G}$-equivariant Kasparov D-B-bimodule. The positivity condition is trivially satisfied by the operator $F_1 \otimes_\pi 1$. Therefore, it only remains to show that $F_1 \otimes_\pi 1$ is a 0-connection in $\mathscr{E}_2$. However, in virtue of the assumption it is clear that we have $(x \otimes_\pi 1)(F_1 \otimes_\pi 1) \in \mathcal{K}(\mathscr{E})$ and $(F_1 \otimes_\pi 1)(x \otimes_\pi 1) \in \mathcal{K}(\mathscr{E})$ for all $x \in \mathcal{K}(\mathscr{E}_1)$. □

Lemma 7.3. *Let $\mathbb{G}$ be a locally compact quantum group and A and B two $\mathbb{G}$-C*-algebras. Let $\mathscr{H}$ and $\mathscr{K}$ be two nonzero Hilbert spaces and $\mathscr{E}$ a $\mathbb{G}$-equivariant Hilbert A-B-bimodule. We assume that there exists an operator $F \in \mathcal{L}(\mathscr{E} \otimes \mathcal{K}(\mathscr{K}))$ such that the pair $(\mathscr{E} \otimes \mathcal{K}(\mathscr{K}), F)$ is a $\mathbb{G}$-equivariant Kasparov bimodule. Let us denote[1]:*

$$\begin{aligned} x &:= [(\mathscr{E} \otimes \mathcal{K}(\mathscr{K}), F)] \in \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathscr{K}), B \otimes \mathcal{K}(\mathscr{K})), \\ \gamma_{\mathrm{g}} &:= [(A \otimes \mathcal{K}(\mathscr{K}, \mathscr{H}), 0)] \in \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathscr{H}), A \otimes \mathcal{K}(\mathscr{K})), \\ \gamma_{\mathrm{d}} &:= [(B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), 0)] \in \mathsf{KK}^{\mathbb{G}}(B \otimes \mathcal{K}(\mathscr{K}), B \otimes \mathcal{K}(\mathscr{H})). \end{aligned}$$

Let us denote $\widetilde{F} \in \mathcal{L}(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}))$ the operator acting on $\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})$ by factorization as follows:

$$\widetilde{F}(\xi \otimes kT) = [F(\xi \otimes k)](1_{\mathscr{E}} \otimes T), \quad \xi \in \mathscr{E},\ k \in \mathcal{K}(\mathscr{K}),\ T \in \mathcal{K}(\mathscr{H}, \mathscr{K}).$$

Then, the pair $(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})$ is a $\mathbb{G}$-equivariant Kasparov bimodule and we have

$$x \otimes_{B\otimes\mathcal{K}(\mathscr{K})} \gamma_{\mathrm{d}} = [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})] \in \mathsf{KK}^{\mathbb{G}}(A \otimes \mathcal{K}(\mathscr{K}), B \otimes \mathcal{K}(\mathscr{H})).$$

Henceforth, we assume that the left action of A on $\mathscr{E}$ is non-degegerate and that there exists an operator $F' \in \mathcal{L}(\mathscr{E})$ such that $F = F' \otimes 1_{\mathscr{K}}$ and $(\mathscr{E}, F')$ and $(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}), F' \otimes 1_{\mathscr{H}})$ are $\mathbb{G}$-equivariant Kasparov bimodules. Then, we have

$$\gamma_{\mathrm{g}} \otimes_{A\otimes\mathcal{K}(\mathscr{K})} [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})] = [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}), F' \otimes 1_{\mathscr{H}})].$$

In particular, we have

$$\gamma_{\mathrm{g}} \otimes_{A\otimes\mathcal{K}(\mathscr{K})} x \otimes_{B\otimes\mathcal{K}(\mathscr{K})} \gamma_{\mathrm{d}} = [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}), F' \otimes 1_{\mathscr{H}})].$$

[1]The index "g" (resp. "d") stands for "*gauche*" (resp. "*droite*"), the french words for "left" (resp. "right") and refers to the left (resp. right) action.

Proof. Let us show that $\widetilde{F}$ is well defined. Let us consider an approximate unit $(u_\lambda)_\lambda$ of $\mathcal{K}(\mathscr{K})$. For all $\xi \in \mathscr{E}$, $k \in \mathcal{K}(\mathscr{K})$ and $T \in \mathcal{K}(\mathscr{H}, \mathscr{K})$, we have

$$[F(\xi \otimes k)](1_{\mathscr{E}} \otimes T) = \lim_\lambda [F(\xi \otimes u_\lambda k)](1_{\mathscr{E}_2} \otimes T) = \lim_\lambda [F(\xi \otimes u_\lambda)](1_{\mathscr{E}} \otimes kT).$$

Therefore, if $k, k' \in \mathcal{K}(\mathscr{K})$ and $T, T' \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ are such that $kT = k'T'$, then we have

$$[F(\xi \otimes k)](1_{\mathscr{E}} \otimes T) = [F(\xi \otimes k')](1_{\mathscr{E}} \otimes T').$$

Let us denote $\pi : B \otimes \mathcal{K}(\mathscr{K}) \to \mathcal{L}(B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}))$ the left action on $B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})$. Note that π actually takes its values in $\mathcal{K}(B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}))$. Indeed, let $a \in B$ and $k \in \mathcal{K}(\mathscr{K})$. Since $\mathscr{H} \neq \{0\}$, we have $\mathcal{K}(\mathscr{K}) = [\mathcal{K}(\mathscr{H}, \mathscr{K})\mathcal{K}(\mathscr{K}, \mathscr{H})]$. Then, k is the norm limit of finite sums of the form $\sum_i T_i S_i$, where $T_i \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ and $S_i \in \mathcal{K}(\mathscr{K}, \mathscr{H})$. We also write $a = a_1 a_2$ with a_1, $a_2 \in B$. For all $b \in B$ and $T \in \mathcal{K}(\mathscr{H}, \mathscr{K})$, we have that $\pi(a \otimes k)(b \otimes T) = ab \otimes kT$ is the norm limit of finite sums of the form

$$\sum_i a_1 a_2 b \otimes T_i S_i T = \sum_i (a_1 \otimes T_i)\langle a_2^* \otimes S_i^*, b \otimes T\rangle.$$

By Lemma 7.2, the pair $((\mathscr{E} \otimes \mathcal{K}(\mathscr{K})) \otimes_{B \otimes \mathcal{K}(\mathscr{K})} (B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})), F \otimes_{B \otimes \mathcal{K}(\mathscr{K})} 1)$ is a $\mathbb{G}$-equivariant Kasparov $A \otimes \mathcal{K}(\mathscr{K})$-$B \otimes \mathcal{K}(\mathscr{H})$-bimodule. We also have a $\mathbb{G}$-equivariant unitary equivalence of Hilbert $A \otimes \mathcal{K}(\mathscr{K})$-$B \otimes \mathcal{K}(\mathscr{H})$-bimodules:

$$\begin{array}{rccc} \Omega : & (\mathscr{E} \otimes \mathcal{K}(\mathscr{K})) \otimes_{B \otimes \mathcal{K}(\mathscr{K})} (B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})) & \longrightarrow & \mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}) \\ & (\xi \otimes k) \otimes_{B \otimes \mathcal{K}(\mathscr{K})} (b \otimes T) & \longmapsto & \xi b \otimes kT. \end{array}$$

Moreover, we have $\widetilde{F} = \Omega(F \otimes_{B \otimes \mathcal{K}(\mathscr{K})} 1)\Omega^*$. Therefore, the pair $(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})$ is a $\mathbb{G}$-equivariant Kasparov $A \otimes \mathcal{K}(\mathscr{K})$-$B \otimes \mathcal{K}(\mathscr{H})$-bimodule and we have

$$[(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})] = [((\mathscr{E} \otimes \mathcal{K}(\mathscr{K})) \otimes_{B \otimes \mathcal{K}(\mathscr{K})} (B \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})), F \otimes_{B \otimes \mathcal{K}(\mathscr{K})} 1)].$$

Hence, $x \otimes_{B \otimes \mathcal{K}(\mathscr{K})} \gamma_{\mathrm{d}} = [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})]$ by Lemma 7.2.

Now, let us assume that the left action of A on $\mathscr{E}$ is non-degegerate and that there exists an operator $F' \in \mathcal{L}(\mathscr{E})$ such that $F = F' \otimes 1_{\mathscr{K}}$ and $(\mathscr{E}, F')$ and $(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}), F' \otimes 1_{\mathscr{H}})$ are $\mathbb{G}$-equivariant Kasparov bimodules. Let us prove that

$$\gamma_{\mathrm{g}} \otimes_{A \otimes \mathcal{K}(\mathscr{K})} [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})] = [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}), F' \otimes 1_{\mathscr{H}})].$$

Note that $\widetilde{F} = F' \otimes 1_{\mathcal{K}(\mathscr{H}, \mathscr{K})}$. Since the left action of A on $\mathscr{E}$ is non-degenerate, we have a $\mathbb{G}$-equivariant unitary equivalence of bimodules:

$$\begin{array}{rccc} \Xi : & (A \otimes \mathcal{K}(\mathscr{K}, \mathscr{H})) \otimes_{A \otimes \mathcal{K}(\mathscr{K})} (\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})) & \longrightarrow & \mathscr{E} \otimes \mathcal{K}(\mathscr{H}) \\ & (a \otimes S) \otimes_{A \otimes \mathcal{K}(\mathscr{K})} (\xi \otimes T) & \longmapsto & a\xi \otimes ST. \end{array}$$

The positivity condition is trivially satisfied. We then have to prove that the operator $F' \otimes 1_{\mathscr{H}}$ can be interpreted (*via* the identification Ξ) as a $\widetilde{F}$-connection in the Kasparov product $\gamma_{\mathrm{g}} \otimes_{A \otimes \mathcal{K}(\mathscr{K})} [(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), \widetilde{F})]$. For $x \in A \otimes \mathcal{K}(\mathscr{K}, \mathscr{H})$, we denote

$$T_x \in \mathcal{L}(\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K}), (A \otimes \mathcal{K}(\mathscr{K}, \mathscr{H})) \otimes_{A \otimes \mathcal{K}(\mathscr{K})} (\mathscr{E} \otimes \mathcal{K}(\mathscr{H}, \mathscr{K})))$$

the operator defined by $T_x\xi = x \otimes_{A\otimes\mathcal{K}(\mathscr{K})} \xi$ for all $\xi \in \mathscr{E}\otimes\mathcal{K}(\mathscr{H},\mathscr{K})$. Now, we denote $T'_x = \Xi T_x \in \mathcal{L}(\mathscr{E}\otimes\mathcal{K}(\mathscr{H},\mathscr{K}), \mathscr{E}\otimes\mathcal{K}(\mathscr{H}))$, $x \in A\otimes\mathcal{K}(\mathscr{K},\mathscr{H})$. For $a \in A$ and $S \in \mathcal{K}(\mathscr{K},\mathscr{H})$, we have

$$T'_{a\otimes S}(\xi\otimes T) = a\xi\otimes ST, \quad \xi\in\mathscr{E},\, T\in\mathcal{K}(\mathscr{H},\mathscr{K}).$$

Let $a \in A$ and $S \in \mathcal{K}(\mathscr{K},\mathscr{H})$, we have

$$(F'\otimes 1_{\mathscr{H}})T'_{a\otimes S} - T'_{a\otimes S}\widetilde{F} = [F',a]\otimes S.$$

Let us recall that we have $[F',a] \in \mathcal{K}(\mathscr{E})$ by assumption. Moreover, since $S \in \mathcal{K}(\mathscr{K},\mathscr{H})$ and $\mathcal{K}(\mathscr{K},\mathscr{H}) = [\mathcal{K}(\mathscr{H})\mathcal{K}(\mathscr{K},\mathscr{H})]$ we have $S \in \mathcal{K}(\mathcal{K}(\mathscr{H},\mathscr{K}),\mathcal{K}(\mathscr{H}))$. Therefore, we have $(F'\otimes 1_{\mathscr{H}})T'_{a\otimes S} - T'_{a\otimes S}\widetilde{F} \in \mathcal{K}(\mathscr{E}\otimes\mathcal{K}(\mathscr{H},\mathscr{K}), \mathscr{E}\otimes\mathcal{K}(\mathscr{H}))$. We also prove in a similar way that $(F'^*\otimes 1_{\mathscr{H}})T'_{a\otimes S} - T'_{a\otimes S}\widetilde{F}^* \in \mathcal{K}(\mathscr{E}\otimes\mathcal{K}(\mathscr{H},\mathscr{K}), \mathscr{E}\otimes\mathcal{K}(\mathscr{H}))$. □

Let us fix a colinking measured quantum groupoid $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ between two monoidally equivalent regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let (A_1,δ_{A_1}) and (B_1,δ_{B_1}) be two $\mathbb{G}_1$-C*-algebras, we denote

$$(A_2,\delta_{A_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1,\delta_{A_1}), \quad (B_2,\delta_{B_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1,\delta_{B_1}),$$

the induced $\mathbb{G}_2$-C*-algebras. Let $(\mathscr{E}_1,F_1)$ be a $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule and we assume the left action $A_1 \to \mathcal{L}(\mathscr{E}_1)$ of A_1 on $\mathscr{E}_1$ to be non-degenerate. Let us denote

$$(\mathscr{E}_2,\delta_{\mathscr{E}_2}) := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1,\delta_{\mathscr{E}_1}),$$

the induced $\mathbb{G}_2$-equivariant Hilbert A_2-B_2-bimodule. We also consider the linking $\mathbb{G}_1$-C*-algebra $J_1 := \mathcal{K}(\mathscr{E}_1\oplus A_1)$. With the notations of the previous chapter, we have:

Proposition 7.4. *The pair* $(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)}\otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))$ *is a* $\mathbb{G}_2$*-equivariant Kasparov* $A_2\otimes\mathcal{K}(\mathscr{H}_{12})$*-*$B_2\otimes\mathcal{K}(\mathscr{H}_{12})$*-bimodule.*

Proof. The operator $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{L}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}))$ is invariant for the bidual action of $\mathbb{G}_1$ on $\mathcal{K}(\mathscr{E}_1)\otimes\mathcal{K}(\mathscr{H}_{11})$. Let us denote

$$F' := (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_1)\otimes 1_{S_{12}} \in \mathcal{L}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})\otimes S_{12}).$$

It is clear that the Kasparov $A_1\otimes\mathcal{K}(\mathscr{H}_{11})$-$B_1\otimes\mathcal{K}(\mathscr{H}_{11})$-bimodule

$$(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathscr{E}_1}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_1))$$

satisfies the conditions of Proposition 6.3.2 4. It then follows that the pair

$$(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})), F')$$

is a $\mathbb{G}_2$-equivariant Kasparov $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1\otimes\mathcal{K}(\mathscr{H}_{11}))$-$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1\otimes\mathcal{K}(\mathscr{H}_{11}))$-bimodule. However, by Theorem 6.4.7 and Lemma 6.4.9, we have that $(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})), F')$ is the image of the pair $(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)}\otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))$ by the $\mathbb{G}_2$-equivariant isomorphism $\delta^1_{11,2}$. □

Notations 7.5. For $j,l,l' = 1,2$, we denote $\gamma_{ll',j,\mathrm{g}}$ (resp. $\gamma_{ll',j,\mathrm{d}}$) the $\mathbb{G}_j$-equivariant Morita equivalence $\gamma_{ll',j}$ defined in Notations 5.3.10 corresponding to the $\mathcal{G}$-C*-algebra $A := A_1\oplus A_2$ (resp. $B := B_1\oplus B_2$). By abuse of notation, we will also denote $\gamma_{ll',j,\mathrm{g}}$ (resp. $\gamma_{ll',j,\mathrm{d}}$) the corresponding element of the Kasparov group. More precisely, we have:

$$\gamma_{ll',j,\mathrm{g}} = [(A_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}),0)] \in \mathsf{KK}^{\mathbb{G}_j}(A_j\otimes\mathcal{K}(\mathscr{H}_{lj}), A_j\otimes\mathcal{K}(\mathscr{H}_{l'j})),$$
$$\gamma_{ll',j,\mathrm{d}} = [(B_j\otimes\mathcal{K}(\mathscr{H}_{l'j},\mathscr{H}_{lj}),0)] \in \mathsf{KK}^{\mathbb{G}_j}(B_j\otimes\mathcal{K}(\mathscr{H}_{lj}), B_j\otimes\mathcal{K}(\mathscr{H}_{l'j})).$$

Proposition 7.6. *With the hypotheses and notations of Proposition 7.4, there exists an operator $F_2 \in \mathcal{L}(\mathscr{E}_2)$ such that:*

a) $(\mathscr{E}_2, F_2)$ is a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule.

b) $\beta_{A_2} \otimes_{A_2} [(\mathscr{E}_2, F_2)] \otimes_{B_2} \alpha_{B_2} = \gamma_{21,2,\mathrm{g}} \otimes [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))] \otimes \gamma_{12,2,\mathrm{d}}$.

Moreover, if $F_2, F_2' \in \mathcal{L}(\mathscr{E}_2)$ satisfy the conditions a) and b) above, then we have:

$$[(\mathscr{E}_2, F_2)] = [(\mathscr{E}_2, F_2')] \in \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2).$$

Proof. We have:

$$\gamma_{21,2,\mathrm{g}} = [(A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}), 0)] \in \mathsf{KK}^{\mathbb{G}_2}(A_2 \otimes \mathcal{K}(\mathscr{H}_{22}), A_2 \otimes \mathcal{K}(\mathscr{H}_{12})),$$
$$\gamma_{12,2,\mathrm{d}} = [(B_2 \otimes \mathcal{K}(\mathscr{H}_{22}, \mathscr{H}_{12}), 0)] \in \mathsf{KK}^{\mathbb{G}_2}(B_2 \otimes \mathcal{K}(\mathscr{H}_{12}), B_2 \otimes \mathcal{K}(\mathscr{H}_{22})).$$

We have the following $\mathbb{G}_2$-equivariant unitary equivalences of bimodules:

$$(A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})) \otimes_{A_2 \otimes \mathcal{K}(\mathscr{H}_{12})} (\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12})) = \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}),$$
$$(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})) \otimes_{B_2 \otimes \mathcal{K}(\mathscr{H}_{12})} (B_2 \otimes \mathcal{K}(\mathscr{H}_{22}, \mathscr{H}_{12})) = \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}),$$

which we combine to obtain:

$$(A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})) \otimes_{A_2 \otimes \mathcal{K}(\mathscr{H}_{12})} (\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12})) \otimes_{B_2 \otimes \mathcal{K}(\mathscr{H}_{12})} (B_2 \otimes \mathcal{K}(\mathscr{H}_{22}, \mathscr{H}_{12})) = \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}).$$

Then, let us denote $T_2 \in \mathcal{L}(\mathscr{E}_2)$ an operator such that $(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), T_2)$ is a $\mathbb{G}_2$-equivariant Kasparov $A_2 \otimes \mathcal{K}(\mathscr{H}_{22})$-$B_2 \otimes \mathcal{K}(\mathscr{H}_{22})$-bimodule satisfying

$$\gamma_{21,2,\mathrm{g}} \otimes_{A_2 \otimes \mathcal{K}(\mathscr{H}_{12})} [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))] \otimes_{B_2 \otimes \mathcal{K}(\mathscr{H}_{12})} \gamma_{12,2,\mathrm{d}} = [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), T_2)].$$

By using the isomorphism of abelian groups

$$\begin{array}{ccc} \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) & \longrightarrow & \mathsf{KK}^{\mathbb{G}_2}(A_2 \otimes \mathcal{K}(\mathscr{H}_{22}), B_2 \otimes \mathcal{K}(\mathscr{H}_{22})), \\ x & \longmapsto & \beta_{A_2} \otimes_{A_2} x \otimes_{B_2} \alpha_{B_2} \end{array} \qquad (\Diamond)$$

we obtain an operator $F_2 \in \mathcal{L}(\mathscr{E}_2)$ satisfying the conditions a) and b). If $F_2' \in \mathcal{L}(\mathscr{E}_2)$ satisfies the conditions a) and b), then we have

$$\beta_{A_2} \otimes_{A_2} [(\mathscr{E}_2, F_2)] \otimes_{B_2} \alpha_{B_2} = \beta_{A_2} \otimes_{A_2} [(\mathscr{E}_2, F_2')] \otimes_{B_2} \alpha_{B_2}.$$

Hence, $[(\mathscr{E}_2, F_2)] = [(\mathscr{E}_2, F_2')]$ in virtue of the isomorphism $(\Diamond)$. □

As a corollary, we obtain:

Proposition-Definition 7.7. *For all $x = [(\mathscr{E}_1, F_1)]$ (with a non-degenerate left action), let us denote $J_{\mathbb{G}_2,\mathbb{G}_1}(x) \in \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2)$ the unique element of $\mathsf{KK}^{\mathbb{G}_2}(A_2, B_2)$ satisfying*

$$\beta_{A_2} \otimes_{A_2} J_{\mathbb{G}_2,\mathbb{G}_1}(x) \otimes_{B_2} \alpha_{B_2} = \gamma_{21,2,\mathrm{g}} \otimes [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))] \otimes \gamma_{12,2,\mathrm{d}}.$$

Then, $J_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1) \to \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2)$ is a homomorphism of abelian groups.

Proof. We have to prove that the following map is well defined:

$$\begin{aligned}\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1) &\longrightarrow \mathsf{KK}^{\mathbb{G}_2}(A_2\otimes\mathcal{K}(\mathscr{H}_{12}),B_2\otimes\mathcal{K}(\mathscr{H}_{12}))\\ [(\mathscr{E}_1,F_1)] &\longmapsto [(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}),(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)}\otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))],\end{aligned}$$

i.e. for all $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule $(\mathscr{E}_1,F_1)$ with a non-degenerate left action, $[(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}),(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)}\otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))]\in\mathsf{KK}^{\mathbb{G}_2}(A_2\otimes\mathcal{K}(\mathscr{H}_{12}),B_2\otimes\mathcal{K}(\mathscr{H}_{12}))$ only depends on the class of $(\mathscr{E}_1,F_1)$ in $\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1)$. It amounts to proving that the map

$$\begin{aligned}\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1) &\longrightarrow \mathsf{KK}^{\mathbb{G}_2}(\,\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(A_1\otimes\mathcal{K}(\mathscr{H}_{11})),\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(B_1\otimes\mathcal{K}(\mathscr{H}_{11}))\,)\\ [(\mathscr{E}_1,F_1)] &\longmapsto [(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})),(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_1)\otimes 1_{S_{12}})]\end{aligned}$$

is well defined (see proof of Proposition 7.4). However, it follows from Remark 6.3.3 that if $(\mathscr{E}_1,F_1)$ is degenerate then so is $(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})),(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_1)\otimes 1_{S_{12}})$. Moreover, if we consider an operatorial homotopy $(\mathscr{E}_1,F_t)_{t\in[0,1]}$ of $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodules, then it is clear that $(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11})),(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta_{\mathcal{K}(\mathscr{E}_1)}(F_t)\otimes 1_{S_{12}})_{t\in[0,1]}$ is also an operatorial homotopy. Finally, it is clear that the induction procedure is compatible with the direct sum. Hence, $J_{\mathbb{G}_2,\mathbb{G}_1}$ is a homomorphism of abelian groups. □

Proposition 7.8. *Let $(\mathscr{E}_1,F_1)$ be a $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule such that the left action is non-degenerate and the operator F_1 is invariant. Then, we have*

$$J_{\mathbb{G}_2,\mathbb{G}_1}([(\mathscr{E}_1,F_1)])=[(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1),F_1\otimes 1_{S_{12}})].$$

Proof. By Proposition 6.3.2 4, the pair $(\mathscr{E}_2,F_2)$, where $\mathscr{E}_2:=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)$ and $F_2:=F_1\otimes 1_{S_{12}}$, is a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule and the operator F_2 is invariant. In particular, we have $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)}\otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1)=F_2\otimes 1_{\mathscr{H}_{12}}$. By definition of $J_{\mathbb{G}_2,\mathbb{G}_1}$, we have to prove that

$$\beta_{A_2}\otimes_{A_2}[(\mathscr{E}_2,F_2)]\otimes_{B_2}\alpha_{B_2}=\gamma_{21,2,\mathrm{g}}\otimes[(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}),F_2\otimes 1_{\mathscr{H}_{12}})]\otimes\gamma_{12,2,\mathrm{d}}.$$

However, by Lemma 7.3 we have

$$\gamma_{21,2,\mathrm{g}}\otimes[(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{12}),F_2\otimes 1_{\mathscr{H}_{12}})]\otimes\gamma_{12,2,\mathrm{d}}=[(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{22}),F_2\otimes 1_{\mathscr{H}_{22}})].$$

Moreover, since F_2 is invariant, we have $\beta_{A_2}\otimes_{A_2}[(\mathscr{E}_2,F_2)]\otimes_{B_2}\alpha_{B_2}=[(\mathscr{E}_2\otimes\mathcal{K}(\mathscr{H}_{22}),F_2\otimes 1_{\mathscr{H}_{22}})]$ and the proposition is proved. □

Let us prove that we have a homomorphism of abelian groups:

$$J_{\mathbb{G}_1,\mathbb{G}_2}:\mathsf{KK}^{\mathbb{G}_2}(A_2,B_2)\to\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1).$$

For all $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule $(\mathscr{E}_2,F_2)$, we set:

$$\widetilde{A}_1:=\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2),\quad \widetilde{B}_1:=\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2),\quad \widetilde{\mathscr{E}}_1:=\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(\mathscr{E}_2),\quad J_2:=\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(J_1),\quad \widetilde{J}_1:=\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(J_2).$$

In addition to the $\mathcal{G}$-C*-algebras $A=A_1\oplus A_2$ and $B=B_1\oplus B_2$, we also consider the following $\mathcal{G}$-C*-algebras:

$$\widetilde{A}:=\widetilde{A}_1\oplus A_2,\quad \widetilde{B}:=\widetilde{B}_1\oplus B_2,\quad \widetilde{J}:=\widetilde{J}_1\oplus J_2.$$

First, let us define an auxiliary homomorphism of abelian groups:

$$\widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}:\mathsf{KK}^{\mathbb{G}_2}(A_2,B_2)\to\mathsf{KK}^{\mathbb{G}_1}(\widetilde{A}_1\otimes\mathcal{K}(\mathscr{H}_{21}),\widetilde{B}_1\otimes\mathcal{K}(\mathscr{H}_{21})).$$

By applying Theorem 6.1.12 (and exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$) and Corollary 6.4.2, we note that the action of $\mathbb{G}_1$ on $\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$ (resp. $\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$) is obtained by restriction of that of $\mathcal{G}$ on $\widetilde{A} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ (resp. $\widetilde{B} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$) which also provides a $\mathbb{G}_1$-equivariant isomorphism from $\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$ (resp. $\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$) to $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(A_2 \otimes \mathcal{K}(\mathscr{H}_{22}))$ (resp. $\mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(B_2 \otimes \mathcal{K}(\mathscr{H}_{22}))$).

By exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$ and by applying an adaptation of Theorem 6.4.7 and Lemma 6.4.9 to the linking $\mathcal{G}$-C*-algebra $\widetilde{J}$, we prove:

Proposition 7.9. *For all $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule $(\mathscr{E}_2, F_2)$ with a non-degenerate left action of A_2 on $\mathscr{E}_2$, the pair*

$$(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))$$

is a $\mathbb{G}_1$-equivariant Kasparov $\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$-$\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})$-bimodule.

By applying Proposition 6.1.6 2 and as in the proof of Theorem 6.1.13 (by exchanging the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$), we obtain two $\mathcal{G}$-equivariant *-isomorphisms:

$$f : A \to \widetilde{A}\,;\, (a_1, a_2) \mapsto (\delta^{(2)}_{A_1}(a_1), a_2) \quad ; \quad g : B \to \widetilde{B}\,;\, (b_1, b_2) \mapsto (\delta^{(2)}_{B_1}(b_1), b_2).$$

Note that f (resp. g) acts identically on A_2 (resp. B_2). By using the functoriality of the crossed product (see Theorems 4.3.3 2 and 4.3.6 2), we obtain two $\mathcal{G}$-equivariant *-isomorphisms:

$$f : A \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}} \to \widetilde{A} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}} \quad ; \quad g : B \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}} \to \widetilde{B} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$$

still denoted f and g. We use Proposition 5.3.3 to obtain $\mathbb{G}_1$-equivariant *-isomorphisms:

$$f_{l,1} : A_1 \otimes \mathcal{K}(\mathscr{H}_{l1}) \to \widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{l1}) \quad ; \quad g_{l,1} : B_1 \otimes \mathcal{K}(\mathscr{H}_{l1}) \to \widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{l1}), \quad l = 1, 2.$$

As for Proposition-Definition 7.7 and by using Proposition 7.9, we obtain:

Proposition-Definition 7.10. *For $y = [(\mathscr{E}_2, F_2)] \in \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2)$ (with a non-degenerate left action), we denote $J_{\mathbb{G}_1,\mathbb{G}_2}(y) \in \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1)$ the unique element of the group $\mathsf{KK}^{\mathbb{G}_1}(A_1, B_1)$ satisfying*

$$\beta_{A_1} \otimes_{A_1} J_{\mathbb{G}_1,\mathbb{G}_2}(y) \otimes_{B_1} \alpha_{B_1} = \gamma_{12,1,\mathrm{g}} \otimes f_{2,1} \otimes [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes g^{-1}_{2,1} \otimes \gamma_{21,1,\mathrm{d}}.$$

Then, $J_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1)$ is a homomorphism of abelian groups.

Proof. It is clear that

$$\begin{array}{rcl} \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) & \longrightarrow & \mathsf{KK}^{\mathbb{G}_1}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), \widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})) \\ y = [(\mathscr{E}_2, F_2)] & \longmapsto & [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \end{array}$$

is a well-defined homomorphism of abelian groups and for all $y \in \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2)$ we have

$$\beta_{A_1} \otimes_{A_1} J_{\mathbb{G}_1,\mathbb{G}_2}(y) \otimes_{B_1} \alpha_{B_1} = \gamma_{12,1,\mathrm{g}} \otimes f_{2,1} \otimes \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}(y) \otimes g^{-1}_{2,1} \otimes \gamma_{21,1,\mathrm{d}}.$$

Therefore, $J_{\mathbb{G}_1,\mathbb{G}_2}$ is also a well-defined homomorphism of abelian groups. □

We can state the main result of this chapter:

Theorem 7.11. *$J_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}^{\mathbb{G}_1}(A_1,B_1) \to \mathsf{KK}^{\mathbb{G}_2}(A_2,B_2)$ is an isomorphism of abelian groups and we have $J_{\mathbb{G}_1,\mathbb{G}_2} = (J_{\mathbb{G}_2,\mathbb{G}_1})^{-1}$.*

Proof. Let us prove that $J_{\mathbb{G}_1,\mathbb{G}_2} \circ J_{\mathbb{G}_2,\mathbb{G}_1} = \mathrm{id}_{\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1)}$. Let $(\mathscr{E}_1,F_1)$ be a $\mathbb{G}_1$-equivariant Kasparov A_1-B_1-bimodule (with a non-degenerate left action) and let us denote $\mathscr{E}_2 := \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\mathscr{E}_1)$, the induced $\mathbb{G}_2$-equivariant Hilbert A_2-B_2-bimodule. Let us denote:

$$x_1 := [(\mathscr{E}_1,F_1)], \quad y_2 := J_{\mathbb{G}_2,\mathbb{G}_1}(x_1), \quad x_1' := J_{\mathbb{G}_1,\mathbb{G}_2}(y_2).$$

By Proposition 7.6, there exists an operator $F_2 \in \mathcal{L}(\mathscr{E}_2)$ such that $y_2 = [(\mathscr{E}_2,F_2)]$ and by Proposition-Definition 7.10 we have

$$\beta_{A_1} \otimes_{A_1} x_1' \otimes_{B_1} \alpha_{B_1} = \gamma_{12,1,\mathrm{g}} \otimes f_{2,1} \otimes [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes g_{2,1}^{-1} \otimes \gamma_{21,1,\mathrm{d}}.$$

In order to understand $x_1' := J_{\mathbb{G}_1,\mathbb{G}_2}(y_2)$, let us denote:

$$J_1 := \mathcal{K}(\mathscr{E}_1 \oplus B_1), \quad J_2 := \mathcal{K}(\mathscr{E}_2 \oplus B_2), \quad \widetilde{J_1} := \mathrm{Ind}_{\mathbb{G}_2}^{\mathbb{G}_1}(J_2) = \mathcal{K}(\widetilde{\mathscr{E}_1} \oplus \widetilde{B}_1).$$

By applying again Proposition 6.1.6 2, we have a $\mathcal{G}$-equivariant *-isomorphism between the $\mathcal{G}$-C*-algebras $J := J_1 \oplus J_2$ and $\widetilde{J} := \widetilde{J_1} \oplus J_2$ (acting identically on J_2) given by

$$h : J \to \widetilde{J}\,;\, (x_1,x_2) \mapsto (\delta_{J_1}^{(2)}(x_1),x_2),$$

which induces a $\mathcal{G}$-equivariant *-isomorphism

$$h : J \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}} \to \widetilde{J} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$$

still denoted h, compatible with the $\mathcal{G}$-equivariant *-isomorphisms f and g. By applying Theorem 4.4.15 and Proposition 5.3.3 to h, we obtain

$$f_{2,1} \otimes [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes g_{2,1}^{-1} = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))]. \tag{7.1}$$

Indeed, we first have

$$\begin{aligned} &f_{2,1} \otimes [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes g_{2,1}^{-1} \\ &\quad = [(\,(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21})) \otimes_{g_{2,1}^{-1}} (B_1 \otimes \mathcal{K}(\mathscr{H}_{21})),\, (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes_{g_{2,1}^{-1}} 1_{B_1 \otimes \mathcal{K}(\mathscr{H}_{21})}\,)] \end{aligned}$$

in $\mathsf{KK}^{\mathbb{G}_1}(A_1 \otimes \mathcal{K}(\mathscr{H}_{21}), B_1 \otimes \mathcal{K}(\mathscr{H}_{21}))$, where the left action is given by:

$$a(\xi \otimes_{g_{2,1}^{-1}} b) = f_{2,1}(a)\xi \otimes_{g_{2,1}^{-1}} b, \quad a \in A_1 \otimes \mathcal{K}(\mathscr{H}_{21}),\ \xi \in \widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}),\ b \in B_1 \otimes \mathcal{K}(\mathscr{H}_{21}).$$

We now have the following $\mathbb{G}_1$-equivariant unitary equivalence of bimodules:

$$\Xi : (\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21})) \otimes_{g_{2,1}^{-1}} (B_1 \otimes \mathcal{K}(\mathscr{H}_{21})) \to \mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21}) \quad ; \quad \xi \otimes_{g_{2,1}^{-1}} b \mapsto h^{-1}(\xi)b.$$

We also have

$$\Xi((\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes_{g_{2,1}^{-1}} 1_{B_1 \otimes \mathcal{K}(\mathscr{H}_{21})})\Xi^* = h^{-1}(\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2).$$

By applying Theorem 4.4.15, we identify the $\mathcal{G}$-C*-algebras $J \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ (resp. $\widetilde{J} \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$) and D (resp. $\widetilde{D}$). We recall that we have $h(\mathrm{id}_J \otimes R)\delta_J(x) = (\mathrm{id}_{\widetilde{J}} \otimes R)\delta_{\widetilde{J}}(h(x))$ for all $x \in J$ (*cf.* Remark 4.4.18). It then follows from Proposition 5.3.3 that we have

$$h(\mathrm{id}_{J_j} \otimes R_{jl})\delta^j_{J_l}(\beta_D(\varepsilon_l)x) = (\mathrm{id}_{\widetilde{J}_j} \otimes R_{jl})\delta^j_{\widetilde{J}_l}(\beta_{\widetilde{D}}(\varepsilon_l)h(x)), \quad x \in J,\ j,l = 1,2.$$

Moreover, since h acts identically on J_2 we have

$$h(\mathrm{id}_{J_1} \otimes R_{12})\delta^1_{J_2}(x) = (\mathrm{id}_{\widetilde{J_1}} \otimes R_{12})\delta^1_{J_2}(x), \quad x \in J_2.$$

In particular, we obtain $h(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) = (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$. Hence,

$$\Xi((\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes_{g^{-1}_{2,1}} 1_{B_1 \otimes \mathcal{K}(\mathscr{H}_{21})})\Xi^* = (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$$

and (7.1) is proved. In particular, we have

$$\gamma_{12,1,\mathrm{g}} \otimes f_{2,1} \otimes [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes g^{-1}_{2,1} \otimes \gamma_{21,1,\mathrm{d}}$$
$$= \gamma_{12,1,\mathrm{g}} \otimes [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes \gamma_{21,1,\mathrm{d}}.$$

We have the following $\mathbb{G}_1$-equivariant unitary equivalence of bimodules:

$$(A_1 \otimes \mathcal{K}(\mathscr{H}_{21}, \mathscr{H}_{11})) \otimes_{A_1 \otimes \mathcal{K}(\mathscr{H}_{21})} (\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21})) \otimes_{B_1 \otimes \mathcal{K}(\mathscr{H}_{21})} (B_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})) = \mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}).$$

Hence, there exists an operator $T_1 \in \mathcal{L}(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}))$ such that

$$\gamma_{12,1,\mathrm{g}} \otimes [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes \gamma_{21,1,\mathrm{d}} = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), T_1)].$$

Hence, $\beta_{A_1} \otimes_{A_1} x'_1 \otimes_{B_1} \alpha_{B_1} = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), T_1)]$. It then follows that there exists an operator $F'_1 \in \mathcal{L}(\mathscr{E}_1)$ such that $x'_1 = [(\mathscr{E}_1, F'_1)]$. We have to prove that $x_1 := [(\mathscr{E}_1, F_1)] = [(\mathscr{E}_1, F'_1)] =: x'_1$. By composing with the isomorphism of abelian groups

$$\beta_{A_1} \otimes_{A_1} - \otimes_{B_1} \alpha_{B_1} : \mathsf{KK}^{\mathbb{G}_1}(A_1, B_1) \to \mathsf{KK}^{\mathbb{G}_1}(A_1 \otimes \mathcal{K}(\mathscr{H}_{11}), B_1 \otimes \mathcal{K}(\mathscr{H}_{11})),$$

it amounts to proving that

$$[(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1))] = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1))]$$

in $\mathsf{KK}^{\mathbb{G}_1}(A_1 \otimes \mathcal{K}(\mathscr{H}_{11}), B_1 \otimes \mathcal{K}(\mathscr{H}_{11}))$. Let us recall that:

$$x_1 = [(\mathscr{E}_1, F_1)], \quad y_2 = [(\mathscr{E}_2, F_2)], \quad x'_1 = [(\mathscr{E}_1, F'_1)].$$

We have

$$\beta_{A_2} \otimes_{A_2} y_2 \otimes_{B_2} \alpha_{B_2} = \gamma_{21,2,\mathrm{g}} \otimes_{A_2 \otimes \mathcal{K}(\mathscr{H}_{12})} x_2 \otimes_{B_2 \otimes \mathcal{K}(\mathscr{H}_{12})} \gamma_{12,2,\mathrm{d}}, \tag{1}$$

where $x_2 := [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1))]$ (see Proposition-Definition 7.7). We also have

$$\beta_{A_1} \otimes_{A_1} x'_1 \otimes_{B_1} \alpha_{B_1} = \gamma_{12,1,\mathrm{g}} \otimes_{A_1 \otimes \mathcal{K}(\mathscr{H}_{11})} y_1 \otimes_{A_1 \otimes \mathcal{K}(\mathscr{H}_{11})} \gamma_{21,1,\mathrm{d}}, \tag{2}$$

where $y_1 := [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))]$ (see (7.1) and Proposition-Definition 7.10). We recall that we have:

$$\beta_{A_2} \otimes_{A_2} y_2 \otimes_{B_2} \alpha_{B_2} = [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2))], \tag{7.2}$$

$$\beta_{A_1} \otimes_{A_1} x'_1 \otimes_{B_1} \alpha_{B_1} = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1))]. \tag{7.3}$$

Moreover, in virtue of Proposition 5.3.11, we have

$$\gamma_{12,2,\mathrm{d}} \otimes_{B_2 \otimes \mathcal{K}(\mathscr{H}_{22})} \gamma_{21,2,\mathrm{d}} = \gamma_{11,2,\mathrm{d}} = 1_{B_2 \otimes \mathcal{K}(\mathscr{H}_{12})}.$$

Therefore, by taking the Kasparov product with $\gamma_{21,2,\mathrm{d}}$ on the right in (1) and by using (7.2), we obtain:

$$[(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes_{B_2\otimes\mathcal{K}(\mathscr{H}_{22})} \gamma_{21,2,\mathrm{d}} = \gamma_{21,2,\mathrm{g}} \otimes_{A_2\otimes\mathcal{K}(\mathscr{H}_{22})} x_2. \qquad (1')$$

Actually, (1′) is equivalent to (1) since we also have

$$\gamma_{21,2,\mathrm{d}} \otimes_{B_2\otimes\mathcal{K}(\mathscr{H}_{12})} \gamma_{12,2,\mathrm{d}} = \gamma_{22,2,\mathrm{d}} = 1_{B_2\otimes\mathcal{K}(\mathscr{H}_{22})}.$$

Similarly, we prove by using (7.3) that (2) is equivalent to:

$$\gamma_{21,1,\mathrm{g}} \otimes_{A_1\otimes\mathcal{K}(\mathscr{H}_{11})} [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1'))] = y_1 \otimes_{B_1\otimes\mathcal{K}(\mathscr{H}_{21})} \gamma_{21,1,\mathrm{d}}. \qquad (2')$$

By applying Lemma 7.3 with $\mathbb{G} := \mathbb{G}_2$ (resp. $\mathbb{G}_1$), $\mathscr{E} := \mathscr{E}_2$ (resp. $\mathscr{E}_1$), $\mathscr{H} := \mathscr{H}_{12}$ (resp. $\mathscr{H}_{11}$), $\mathscr{K} := \mathscr{H}_{22}$ (resp. $\mathscr{H}_{21}$), $A := A_2$ (resp. A_1), $B := B_2$ (resp. B_1), $F := (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2)$ (resp. $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$), we obtain:

$$[(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \otimes_{B_2\otimes\mathcal{K}(\mathscr{H}_{22})} \gamma_{21,2,\mathrm{d}} = [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2))] \quad (1'')$$

and

$$[(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))] = y_1 \otimes_{B_1\otimes\mathcal{K}(\mathscr{H}_{21})} \gamma_{21,1,\mathrm{d}} \qquad (2'')$$

respectively, where we still denote $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2)$ (resp. $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$) the same operator but acting on $\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})$ (resp. $\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$) by factorization. For $j = 1, 2$, we use again

$$f_j : \mathcal{A}_j \to \mathcal{M}(\mathcal{B}_j), \quad \mathcal{A}_j := A_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}), \quad \mathcal{B}_j := J_j \otimes \mathcal{K}(\mathscr{H}_{1j} \oplus \mathscr{H}_{2j}),$$

the $\mathbb{G}_j$-equivariant *-homomorphism defined by the action of A_j on $\mathscr{E}_j$ (see proof of Theorem 6.4.7). By combining (1′) and (1″), it then follows that $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2)$ can be interpreted as a $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1)$-connection in the Kasparov product $\gamma_{21,2,\mathrm{g}} \otimes_{A_2\otimes\mathcal{K}(\mathscr{H}_{22})} x_2$, which means that for all $a \in \mathcal{A}_{21,2} := A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})$, we have

$$(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2) f_2(a) - f_2(a)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{K}(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})) \qquad (\mathrm{C}_1)$$

where $f_2(a) \in \mathcal{B}_{21,2} := J_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})$. Let us fix $a \in A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})$ and let us denote

$$d := (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta^2_{\mathcal{K}(\mathscr{E}_2)}(F_2) f_2(a) - f_2(a)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{21})\delta^2_{\mathcal{K}(\mathscr{E}_1)}(F_1).$$

We then have:

- $d \in \mathcal{M}(\mathcal{B}_2)$ and defines an element $d' \in \mathcal{L}(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}))$.
- (C_1) means that $d \in \mathcal{B}_2$, more precisely $d' \in \mathcal{K}(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}), \mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}))$.

Let us denote $c = \delta^1_{\mathcal{B}_2}(d) \in \mathcal{M}(\mathcal{B}_1 \otimes S_{12})$. By Lemma 6.4.9, we have

$$c = ((\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes 1_{S_{12}})(f_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{\mathcal{A}_2}(a) - (f_1 \otimes \mathrm{id}_{S_{12}})\delta^1_{\mathcal{A}_2}(a)((\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \otimes 1_{S_{12}}).$$

However, note that we have $\mathcal{B}_1 = [(\mathrm{id}_{\mathcal{B}_1} \otimes \omega)\delta^1_{\mathcal{B}_2}(b)\,;\, b \in \mathcal{B}_2,\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*]$. Therefore, for all $a \in A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22})$ and $\omega \in \mathcal{B}(\mathscr{H}_{12})_*$ we have

$$(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)f_1(\mathrm{id}_{\mathcal{A}_1} \otimes \omega)\delta^1_{\mathcal{A}_2}(a) - f_1(\mathrm{id}_{\mathcal{A}_1} \otimes \omega)\delta^1_{\mathcal{A}_2}(a)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{B}_1.$$

However, it follows from Proposition 5.1.3 3 and Theorem 6.4.1 3 that

$$A_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21}) = [(\mathrm{id}_{\mathcal{A}_1} \otimes \omega)\delta^1_{\mathcal{A}_2}(a)\,;\, a \in A_2 \otimes \mathcal{K}(\mathscr{H}_{12}, \mathscr{H}_{22}),\, \omega \in \mathcal{B}(\mathscr{H}_{12})_*].$$

As a result, for all $b \in A_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$ we have

$$(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)f_1(b)-f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{K}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}), \mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})). \tag{C$'_1$}$$

In a similar way, it follows from $(2')$ and $(2'')$ that the operator $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$ can be interpreted as a $(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1)$-connection in the Kasparov product

$$\gamma_{21,1,\mathrm{g}} \otimes_{A_1\otimes\mathcal{K}(\mathscr{H}_{11})} [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1))].$$

We then obtain similarly that for all $b \in \mathcal{A}_{21,1} := A_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$, we have:

$$(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)f_1(b)-f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1) \in \mathcal{K}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}), \mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})), \tag{C$_2$}$$

where $f_1(b) \in \mathcal{B}_{21,1} := J_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$. By subtracting (C$_2$) from (C$'_1$), we obtain that for all $b \in A_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$:

$$f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1)-f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)}\otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{K}(\mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}), \mathscr{E}_1\otimes\mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})). \tag{C$'_1$ $-$ C$_2$}$$

Since $\mathscr{H}_{21} \neq \{0\}$, it then follows that for all $b \in A_1 \otimes \mathcal{K}(\mathscr{H}_{11})$ we have:

$$f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1) - f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \in \mathcal{K}(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11})). \qquad (\star)$$

Indeed, we have $\mathcal{K}(\mathscr{H}_{11}) = [\mathcal{K}(\mathscr{H}_{21}, \mathscr{H}_{11})\mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})]$. We can then assume that $b = (1_{A_1} \otimes k)x$, with $k \in \mathcal{K}(\mathscr{H}_{21}, \mathscr{H}_{11})$ and $x \in A_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21})$. Therefore, we have

$$\begin{aligned} &f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) - f_1(b)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) \\ &\qquad = (1_{\mathscr{E}_1} \otimes k)(f_1(x)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1) - f_1(x)(\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1)), \end{aligned}$$

where $1_{\mathscr{E}_1} \otimes k \in \mathcal{L}(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}, \mathscr{H}_{21}), \mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}))$ and $(\star)$ follows from (C$'_1$ $-$ C$_2$). We then have proved that

$$[(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F'_1))] = [(\mathscr{E}_1 \otimes \mathcal{K}(\mathscr{H}_{11}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_1)} \otimes R_{11})\delta^1_{\mathcal{K}(\mathscr{E}_1)}(F_1))]$$

in $\mathsf{KK}^{\mathbb{G}_1}(A_1 \otimes \mathcal{K}(\mathscr{H}_{11}), B_1 \otimes \mathcal{K}(\mathscr{H}_{11}))$. We have proved that $J_{\mathbb{G}_1,\mathbb{G}_2} \circ J_{\mathbb{G}_2,\mathbb{G}_1} = \mathrm{id}_{\mathsf{KK}^{\mathbb{G}_1}(A_1,B_1)}$. This proves that $J_{\mathbb{G}_1,\mathbb{G}_2}$ is surjective and $J_{\mathbb{G}_2,\mathbb{G}_1}$ is injective. In order to prove directly that $J_{\mathbb{G}_1,\mathbb{G}_2}$ is injective, we first note that by Proposition-Definition 7.10 (see also its proof), we have a homomorphism of abelian groups

$$\widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2} : \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}^{\mathbb{G}_1}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), \widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})).$$

It is enough to prove that $\widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}$ is injective. Actually, by composing with the (injective) homomorphism of abelian groups

$$J'_{\mathbb{G}_2,\mathbb{G}_1} : \mathsf{KK}^{\mathbb{G}_1}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21}), \widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})) \to \mathsf{KK}^{\mathbb{G}_2}(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21})), \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})))$$

and by identifying the $\mathbb{G}_2$-C*-algebras $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21}))$ (resp. $\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21}))$) and $A_2 \otimes \mathcal{K}(\mathscr{H}_{22})$ (resp. $B_2 \otimes \mathcal{K}(\mathscr{H}_{22})$), we will prove that the homomorphism $J'_{\mathbb{G}_2,\mathbb{G}_1} \circ \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}$ is the isomorphism ($\Diamond$) of [3]:

$$\mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}^{\mathbb{G}_2}(A_2 \otimes \mathcal{K}(\mathscr{H}_{22}), B_2 \otimes \mathcal{K}(\mathscr{H}_{22})).$$

Let $(\mathscr{E}_2, F_2)$ be a $\mathbb{G}_2$-equivariant Kasparov A_2-B_2-bimodule (with a non-degenerate left action). We recall that we have $\widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}([\mathscr{E}_2, F_2])] = [(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21}), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2))]$ and the operator $(\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2)$ is invariant. By Proposition 7.8, we have

$$J'_{\mathbb{G}_2,\mathbb{G}_1} \circ \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}([(\mathscr{E}_2, F_2)]) = [(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21})), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes 1_{S_{12}})].$$

However, by Theorem 6.4.7 and Lemma 6.4.9, we have that the pair

$$(\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{\mathscr{E}_1} \otimes \mathcal{K}(\mathscr{H}_{21})), (\mathrm{id}_{\mathcal{K}(\widetilde{\mathscr{E}_1})} \otimes R_{12})\delta^1_{\mathcal{K}(\mathscr{E}_2)}(F_2) \otimes 1_{S_{12}})$$

is the image of $(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta_{\mathcal{K}(\mathscr{E}_2)}(F_2))$ by the $\mathbb{G}_2$-equivariant isomorphism $\delta^1_{22,2}$. By making the identifications

$$\mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{A}_1 \otimes \mathcal{K}(\mathscr{H}_{21})) = A_2 \otimes \mathcal{K}(\mathscr{H}_{22}), \quad \mathrm{Ind}_{\mathbb{G}_1}^{\mathbb{G}_2}(\widetilde{B}_1 \otimes \mathcal{K}(\mathscr{H}_{21})) = B_2 \otimes \mathcal{K}(\mathscr{H}_{22}),$$

it then follows that

$$J'_{\mathbb{G}_2,\mathbb{G}_1} \circ \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}([(\mathscr{E}_2, F_2)]) = [(\mathscr{E}_2 \otimes \mathcal{K}(\mathscr{H}_{22}), (\mathrm{id}_{\mathcal{K}(\mathscr{E}_2)} \otimes R_{22})\delta_{\mathcal{K}(\mathscr{E}_2)}(F_2)] = \beta_{A_2} \otimes_{A_2} [(\mathscr{E}_2, F_2)] \otimes_{B_2} \alpha_{B_2}.$$

Therefore, $J'_{\mathbb{G}_2,\mathbb{G}_1} \circ \widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}$ is the isomorphism of abelian groups

$$\beta_{A_2} \otimes_{A_2} - \otimes_{B_2} \alpha_{B_2} : \mathsf{KK}^{\mathbb{G}_2}(A_2, B_2) \to \mathsf{KK}^{\mathbb{G}_2}(A_2 \otimes \mathcal{K}(\mathscr{H}_{22}), B_2 \otimes \mathcal{K}(\mathscr{H}_{22})).$$

Hence, $\widetilde{J}_{\mathbb{G}_1,\mathbb{G}_2}$ is injective and then so is $J_{\mathbb{G}_1,\mathbb{G}_2}$. Therefore, $J_{\mathbb{G}_1,\mathbb{G}_2}$ is bijective and we have $(J_{\mathbb{G}_1,\mathbb{G}_2})^{-1} = J_{\mathbb{G}_2,\mathbb{G}_1}$. □

Chapter 8

Equivariant Hilbert C*-modules

In this chapter, we require that any action (δ_B, β_B) of a measured quantum groupoid $\mathcal{G}$ on a finite basis on a C*-algebra B satisfies $\delta_B(B) \subset \widetilde{\mathcal{M}}(B \otimes S)$ (see the notations at page 1), that is $\delta_B(B)(1_B \otimes S) \subset B \otimes S$. Note that this condition is necessarily satisfied if the action (δ_B, β_B) is continuous.

8.1 Preliminaries

Let us recall some classical notations and elementary facts. Let B be a C*-algebra and $\mathscr{E}$ a Hilbert B-module.

Proposition-Definition 8.1.1. *Let us consider the following maps:*

- $\iota_B : B \to \mathcal{K}(\mathscr{E} \oplus B)$, *the *-homomorphism given by* $\iota_B(b)(\xi \oplus a) = 0 \oplus ba$ *for all* $a, b \in B$ *and* $\xi \in \mathscr{E}$.
- $\iota_{\mathscr{E}} : \mathscr{E} \to \mathcal{K}(\mathscr{E} \oplus B)$, *the bounded linear map given by* $\iota_{\mathscr{E}}(\xi)(\eta \oplus b) = \xi b \oplus 0$ *for all* $b \in B$ *and* $\xi, \eta \in \mathscr{E}$.
- $\iota_{\mathscr{E}^*} : \mathscr{E}^* \to \mathcal{K}(\mathscr{E} \oplus B)$, *the bounded linear map given by* $\iota_{\mathscr{E}^*}(\xi^*)(\eta \oplus b) = 0 \oplus \xi^*\eta$ *for all* $\xi, \eta \in \mathscr{E}$ *and* $b \in B$.
- $\iota_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \mathcal{K}(\mathscr{E} \oplus B)$, *the *-homomorphism given by* $\iota_{\mathcal{K}(\mathscr{E})}(k)(\eta \oplus b) = k\eta \oplus 0$ *for all* $k \in \mathcal{K}(\mathscr{E})$, $\eta \in \mathscr{E}$ *and* $b \in B$.

We have the following statements:

1. $\iota_{\mathscr{E}}(\xi b) = \iota_{\mathscr{E}}(\xi)\iota_B(b)$ *and* $\iota_B(b)\iota_{\mathscr{E}^*}(\xi^*) = \iota_{\mathscr{E}^*}(b\xi^*)$ *for all* $\xi \in \mathscr{E}$ *and* $b \in B$.
2. $\iota_{\mathscr{E}^*}(\xi^*) = \iota_{\mathscr{E}}(\xi)^*$ *and* $\iota_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta}) = \iota_{\mathscr{E}}(\xi)\iota_{\mathscr{E}}(\eta)^*$ *for all* $\xi, \eta \in \mathscr{E}$.
3. $\mathcal{K}(\mathscr{E} \oplus B)$ *is the* C*-*algebra generated by the set* $\iota_B(B) \cup \iota_{\mathscr{E}}(\mathscr{E})$.

Remarks 8.1.2. 1. For $b \in B$, $\xi \in \mathscr{E}$ and $k \in \mathcal{K}(\mathscr{E})$, the operators $\iota_B(b)$, $\iota_{\mathscr{E}}(\xi)$, $\iota_{\mathscr{E}^*}(\xi^*)$ and $\iota_{\mathcal{K}(\mathscr{E})}(k)$ can be denoted as 2-by-2 matrices acting on $\mathscr{E} \oplus B$ as follows:

$$\iota_B(b) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad \iota_{\mathscr{E}}(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \quad \iota_{\mathscr{E}^*}(\xi^*) = \begin{pmatrix} 0 & 0 \\ \xi^* & 0 \end{pmatrix}, \quad \iota_{\mathcal{K}(\mathscr{E})}(k) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, any operator $x \in \mathcal{K}(\mathscr{E} \oplus B)$ can be written in a unique way as follows:

$$x = \begin{pmatrix} k & \xi \\ \eta^* & b \end{pmatrix}, \quad \text{where} \quad k \in \mathcal{K}(\mathscr{E}),\ \xi, \eta \in \mathscr{E},\ b \in B.$$

2. Note that ι_B and $\iota_{\mathcal{K}(\mathscr{E})}$ extend uniquely to strictly continuous unital *-homomorphisms $\iota_{\mathcal{M}(B)} : \mathcal{M}(B) \to \mathcal{L}(\mathscr{E} \oplus B)$ and $\iota_{\mathcal{L}(\mathscr{E})} : \mathcal{L}(\mathscr{E}) \to \mathcal{L}(\mathscr{E} \oplus B)$. Besides, $\iota_{\mathcal{M}(B)}$ and $\iota_{\mathcal{L}(\mathscr{E})}$ are given by
$$\iota_{\mathcal{M}(B)}(m)(\xi \oplus b) = 0 \oplus mb, \quad \iota_{\mathcal{L}(\mathscr{E})}(T)(\xi \oplus b) = T\xi \oplus 0,$$
for all $m \in \mathcal{M}(B)$, $T \in \mathcal{L}(\mathscr{E})$, $\xi \in \mathscr{E}$ and $b \in B$.

3. $\iota_{\mathscr{E}^*}$ admits an extension to a bounded linear map $\iota_{\mathcal{L}(\mathscr{E},B)} : \mathcal{L}(\mathscr{E}, B) \to \mathcal{L}(\mathscr{E} \oplus B)$ in a straightforward way. Similarly, up to the identification $\mathscr{E} = \mathcal{K}(B, \mathscr{E})$, we can also extend $\iota_{\mathscr{E}}$ to a bounded linear map $\iota_{\mathcal{L}(B,\mathscr{E})}$.

4. As in 1, we can denote $\iota_{\mathcal{M}(B)}(m)$, $\iota_{\mathcal{L}(\mathscr{E})}(T)$, $\iota_{\mathcal{L}(B,\mathscr{E})}(S)$ and $\iota_{\mathcal{L}(\mathscr{E},B)}(S^*)$, for $m \in \mathcal{M}(B)$, $T \in \mathcal{L}(\mathscr{E})$ and $S \in \mathcal{L}(B, \mathscr{E})$, as 2-by-2 matrices. Moreover, any operator $x \in \mathcal{L}(\mathscr{E} \oplus B)$ can be written in a unique way as follows:
$$x = \begin{pmatrix} T & S' \\ S^* & m \end{pmatrix}, \quad \text{where} \quad T \in \mathcal{L}(\mathscr{E}),\ S, S' \in \mathcal{L}(B, \mathscr{E}),\ m \in \mathcal{M}(B).$$

By using the matrix notations described above, we derive easily the following useful technical lemma:

Lemma 8.1.3. *Let $x \in \mathcal{L}(\mathscr{E} \oplus B)$ (resp. $x \in \mathcal{K}(\mathscr{E} \oplus B)$). We have the following statements:*

1. *$x \in \iota_{\mathcal{L}(B,\mathscr{E})}(\mathcal{L}(B, \mathscr{E}))$ (resp. $\iota_{\mathscr{E}}(\mathscr{E})$) if and only if $x\iota_{\mathscr{E}}(\xi) = 0$ for all $\xi \in \mathscr{E}$ and $\iota_B(b)x = 0$ for all $b \in B$. In that case, we have $\iota_{\mathcal{M}(B)}(m)x = 0$ for all $m \in \mathcal{M}(B)$.*

2. *$x \in \iota_{\mathcal{L}(\mathscr{E})}(\mathcal{L}(\mathscr{E}))$ (resp. $\iota_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E}))$) if and only if $x\iota_B(b) = 0$ and $\iota_B(b)x = 0$ for all $b \in B$. In that case, we have $x\iota_{\mathcal{M}(B)}(m) = \iota_{\mathcal{M}(B)}(m)x = 0$ for all $m \in \mathcal{M}(B)$.*

Notation 8.1.4. Let $\mathscr{F}$ be a Hilbert B-module. Let $q \in \mathcal{L}(\mathscr{E})$ a self-adjoint projection and $T \in \mathcal{L}(q\mathscr{E}, F)$. Let $\widetilde{T} : \mathscr{E} \to \mathscr{F}$ be the map defined by $\widetilde{T}\xi := Tq\xi$, for all $\xi \in \mathscr{E}$. Therefore, $\widetilde{T} \in \mathcal{L}(\mathscr{E}, \mathscr{F})$ and $\widetilde{T}^* = qT^*$. By abuse of notation, we will still denote T the adjointable operator $\widetilde{T}$.

Let us recall the following definition introduced in [3]:

Definition 8.1.5. Let B and D be two C*-algebras and let $\mathscr{E}$ be a Hilbert C*-module over B. Up to the identification $\mathscr{E} \otimes D = \mathcal{K}(B \otimes D, \mathscr{E} \otimes D)$, we define $\widetilde{\mathcal{M}}(\mathscr{E} \otimes D)$ to be the following subspace of $\mathcal{L}(B \otimes D, \mathscr{E} \otimes D)$
$$\{T \in \mathcal{L}(B \otimes D, \mathscr{E} \otimes D)\,;\, \forall x \in D,\ (1_{\mathscr{E}} \otimes x)T \in \mathscr{E} \otimes D \text{ and } T(1_B \otimes x) \in \mathscr{E} \otimes D\}.$$

Note that $\widetilde{\mathcal{M}}(\mathscr{E} \otimes D)$ is a Hilbert C*-module over $\widetilde{\mathcal{M}}(B \otimes D)$, whose $\widetilde{\mathcal{M}}(B \otimes D)$-valued inner product is given by:
$$\langle \xi, \eta \rangle = \xi^* \eta, \quad \xi, \eta \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes D) \subset \mathcal{L}(B \otimes D, \mathscr{E} \otimes D).$$

We have $\mathcal{K}(\widetilde{\mathcal{M}}(\mathscr{E} \otimes D)) \subset \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes D)$.

8.2 The three pictures

In this paragraph, we introduce a notion of $\mathcal{G}$-equivariant Hilbert C*-module for a measured quantum groupoid $\mathcal{G}$ on a finite basis in the spirit of [3]. Following [3], if (A, δ_A, β_A) is a $\mathcal{G}$-C*-algebra, an action of $\mathcal{G}$ on a Hilbert A-module $\mathscr{E}$ will be defined by three equivalent data.

The proof of the following is straightforward.

Proposition-Definition 8.2.1. *Let A_1 and A_2 be two C*-algebras. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two Hilbert C*-modules over A_1 and A_2 respectively. Assume that $\pi : A_1 \to \mathcal{L}(\mathscr{E}_2)$ is a *-homomorphism and $p \in \mathcal{L}(\mathscr{E}_2)$ is a self-adjoint projection such that for some approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ of A_1 the net $(\pi(u_\lambda))_{\lambda\in\Lambda}$ converges stricly towards p. We consider the π-invariant Hilbert submodule $\widetilde{\mathscr{E}_2} = p\mathscr{E}_2$ of $\mathscr{E}_2$. Therefore, we have a non-degenerate *-homomorphism $\widetilde{\pi} : A_1 \to \mathcal{L}(\widetilde{\mathscr{E}_2})$. There exists a unique unitary $u \in \mathcal{L}(\mathscr{E}_1 \otimes_\pi \mathscr{E}_2, \mathscr{E}_1 \otimes_{\widetilde{\pi}} \widetilde{\mathscr{E}_2})$ such that*

$$u(\xi_1 \otimes_\pi \xi_2) = \xi_1 \otimes_{\widetilde{\pi}} p\xi_2, \quad \xi_1 \in \mathscr{E}_1,\, \xi_2 \in \mathscr{E}_2.$$

Let $\mathcal{G}$ be a measured quantum groupoid on the finite basis $N = \bigoplus_{1\leqslant l\leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$. Let (S, δ) be the weak Hopf-C*-algebra associated with $\mathcal{G}$. Let us fix a $\mathcal{G}$-C*-algebra (A, δ_A, β_A).

Remarks 8.2.2. With the above notations (with $\mathscr{E}_1 = A_1$), we have the following composition of unitary equivalences of Hilbert modules:

$$\begin{array}{ccccc} A_1 \otimes_\pi \mathscr{E}_2 & \longrightarrow & A_1 \otimes_{\widetilde{\pi}} \widetilde{\mathscr{E}_2} & \longrightarrow & \widetilde{\mathscr{E}_2} \\ a_1 \otimes_\pi \xi_2 & \longmapsto & a_1 \otimes_{\widetilde{\pi}} p\xi_2 & \longmapsto & \widetilde{\pi}(a_1)p\xi_2 = \pi(a_1)\xi_2. \end{array}$$

In particular, we have the following unitary equivalences of Hilbert modules:

$$\begin{array}{rcl} A \otimes_{\delta_A} (A \otimes S) & \longrightarrow & q^{\beta_A,\alpha}(A \otimes S) \\ a \otimes_{\delta_A} x & \longmapsto & \delta_A(a)x, \end{array} \tag{8.2.1}$$

$$\begin{array}{rcl} (A \otimes S) \otimes_{\delta_A\otimes \mathrm{id}_S} (A \otimes S \otimes S) & \longrightarrow & q_{12}^{\beta_A,\alpha}(A \otimes S \otimes S) \\ x \otimes_{\delta_A\otimes \mathrm{id}_S} y & \longmapsto & (\delta_A \otimes \mathrm{id}_S)(x)y, \end{array} \tag{8.2.2}$$

$$\begin{array}{rcl} (A \otimes S) \otimes_{\mathrm{id}_A\otimes\delta} (A \otimes S \otimes S) & \longrightarrow & q_{23}^{\beta,\alpha}(A \otimes S \otimes S) \\ x \otimes_{\mathrm{id}_A\otimes\delta} y & \longmapsto & (\mathrm{id}_A \otimes \delta)(x)y. \end{array} \tag{8.2.3}$$

Since the ranges of α and β commute pointwise, we have $[q_{12}^{\beta_A,\alpha}, q_{23}^{\beta,\alpha}] = 0$. In particular, $q_{12}^{\beta_A,\alpha} q_{23}^{\beta,\alpha} \in \mathcal{L}(A \otimes S \otimes S)$ is a self-adjoint projection.

Definition 8.2.3. A $\mathcal{G}$-equivariant Hilbert A-module is a triple $(\mathscr{E}, \delta_\mathscr{E}, \beta_\mathscr{E})$, which consists of a Hilbert A-module $\mathscr{E}$, a linear map $\delta_\mathscr{E} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ and a non-degenerate *-homomorphism $\beta_\mathscr{E} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ such that:

1. For all $a \in A$ and $\xi, \eta \in \mathscr{E}$, we have

$$\delta_\mathscr{E}(\xi a) = \delta_\mathscr{E}(\xi)\delta_A(a), \quad \delta_A(\langle\xi, \eta\rangle) = \langle\delta_\mathscr{E}(\xi), \delta_\mathscr{E}(\eta)\rangle.$$

2. $[\delta_\mathscr{E}(\mathscr{E})(A \otimes S)] = q^{\beta_\mathscr{E},\alpha}(\mathscr{E} \otimes S)$.

3. For all $\xi \in \mathscr{E}$ and $n \in N$, we have $\delta_\mathscr{E}(\beta_\mathscr{E}(n^{\mathrm{o}})\xi) = (1_\mathscr{E} \otimes \beta(n^{\mathrm{o}}))\delta_\mathscr{E}(\xi)$.

4. The linear maps $\delta_{\mathscr{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}} \otimes \delta$ extend to linear maps from $\mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$ to $\mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$. Moreover, we have

$$(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}}(\xi) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)\delta_{\mathscr{E}}(\xi) \in \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S), \quad \xi \in \mathscr{E}.$$

Remarks 8.2.4.

- If the second formula of the condition 1 holds, then $\delta_{\mathscr{E}}$ is isometric (*cf.* [3]). Indeed, we have $\|\langle \delta_{\mathscr{E}}(\xi), \delta_{\mathscr{E}}(\eta)\rangle\| = \|\delta_A(\langle \xi, \eta\rangle)\| = \|\langle \xi, \eta\rangle\|$ for all $\xi, \eta \in \mathscr{E}$. In particular, we have

$$\|\delta_{\mathscr{E}}(\xi)\|^2 = \|\langle \delta_{\mathscr{E}}(\xi), \delta_{\mathscr{E}}(\xi)\rangle\| = \|\langle \xi, \xi\rangle\| = \|\xi\|^2, \quad \xi \in \mathscr{E}.$$

- If the condition 1 holds, then the condition 2 is equivalent to:

$$[\delta_{\mathscr{E}}(\mathscr{E})(1_A \otimes S)] = q^{\beta_{\mathscr{E}},\alpha}(\mathscr{E} \otimes S).$$

  Indeed, if $(u_\lambda)_\lambda$ is an approximate unit of A we have

$$\delta_{\mathscr{E}}(\xi) = \lim_\lambda \delta_{\mathscr{E}}(\xi u_\lambda) = \lim_\lambda \delta_{\mathscr{E}}(\xi)\delta_A(u_\lambda) = \delta_{\mathscr{E}}(\xi) q^{\beta_A,\alpha}, \quad \xi \in \mathscr{E}.$$

  By continuity of the action (δ_A, β_A), the condition 1 of Definition 8.2.3 and the equality $\mathscr{E}A = \mathscr{E}$, we then have $[\delta_{\mathscr{E}}(\mathscr{E})(A \otimes S)] = [\delta_{\mathscr{E}}(\mathscr{E})(1_A \otimes S)]$ and the equivalence follows.

- We will prove (see Remarks 8.2.8) that if $\delta_{\mathscr{E}}$ satisfies the conditions 1 and 2 of Definition 8.2.3, then the extensions of $\delta_{\mathscr{E}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}} \otimes \delta$ always exist and satisfy the formulas:

$$(\mathrm{id}_{\mathscr{E}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)(Tx), \quad (\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) = (\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(Tx),$$

  for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$.

Notation 8.2.5. For $\xi \in \mathscr{E}$, let $T_\xi \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes_{\delta_A} (A \otimes S))$ defined by

$$T_\xi(x) = \xi \otimes_{\delta_A} x, \quad x \in A \otimes S.$$

In the following, we fix a Hilbert A-module $\mathscr{E}$.

Definition 8.2.6. Let $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ be an isometry and $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ a non-degenerate *-homomorphism such that:

1. $\mathscr{V}\mathscr{V}^* = q^{\beta_{\mathscr{E}},\alpha}$.
2. $\mathscr{V}(\beta_{\mathscr{E}}(n^{\mathrm{o}}) \otimes_{\delta_A} 1) = (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}}))\mathscr{V}$ for all $n \in N$.

Then, $\mathscr{V}$ is said to be admissible if we further have:

3. $\mathscr{V}T_\xi \in \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ for all $\xi \in \mathscr{E}$.
4. $(\mathscr{V} \otimes_{\mathbb{C}} \mathrm{id}_S)(\mathscr{V} \otimes_{\delta_A \otimes \mathrm{id}_S} 1) = \mathscr{V} \otimes_{\mathrm{id}_A \otimes \delta} 1 \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A^2} (A \otimes S \otimes S), \mathscr{E} \otimes S \otimes S)$,

where $\delta_A^2 := (\delta_A \otimes \mathrm{id}_S)\delta_A = (\mathrm{id}_A \otimes \delta)\delta_A : A \to \mathcal{M}(A \otimes S \otimes S)$.

The fourth statement in the previous definition makes sense since we have used the canonical identifications thereafter. It follows from the associativity of the internal tensor product, Proposition-Definition 8.2.1, and Remark 8.2.2 that we have the unitary equivalences of Hilbert modules:

$$\begin{array}{rcl}(\mathscr{E}\otimes_{\delta_A}(A\otimes S))\otimes_{\delta_A\otimes\mathrm{id}_S}(A\otimes S\otimes S) & \longrightarrow & \mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)\\ (\xi\otimes_{\delta_A}x)\otimes_{\delta_A\otimes\mathrm{id}_S}y & \longmapsto & \xi\otimes_{\delta_A^2}(\delta_A\otimes\mathrm{id}_S)(x)y,\end{array}\tag{8.2.4}$$

$$\begin{array}{rcl}(\mathscr{E}\otimes_{\delta_A}(A\otimes S))\otimes_{\mathrm{id}_A\otimes\delta}(A\otimes S\otimes S) & \longrightarrow & \mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)\\ (\xi\otimes_{\delta_A}x)\otimes_{\mathrm{id}_A\otimes\delta}y & \longmapsto & \xi\otimes_{\delta_A^2}(\mathrm{id}_A\otimes\delta)(x)y.\end{array}\tag{8.2.5}$$

We also have:

$$\begin{array}{rcl}(\mathscr{E}\otimes S)\otimes_{\delta_A\otimes\mathrm{id}_S}(A\otimes S\otimes S) & \longrightarrow & (\mathscr{E}\otimes_{\delta_A}(A\otimes S))\otimes S\\ (\xi\otimes s)\otimes_{\delta_A\otimes\mathrm{id}_S}(x\otimes t) & \longmapsto & (\xi\otimes_{\delta_A}x)\otimes st,\end{array}\tag{8.2.6}$$

$$\begin{array}{rcl}(\mathscr{E}\otimes S)\otimes_{\mathrm{id}_A\otimes\delta}(A\otimes S\otimes S) & \longrightarrow & q_{23}^{\beta,\alpha}(\mathscr{E}\otimes S\otimes S)\subset\mathscr{E}\otimes S\otimes S\\ \xi\otimes_{\mathrm{id}_A\otimes\delta}y & \longmapsto & (\mathrm{id}_{\mathscr{E}}\otimes\delta)(\xi)y.\end{array}\tag{8.2.7}$$

In particular, we have $\mathscr{V}\otimes_{\delta_A\otimes\mathrm{id}_S}1\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S),(\mathscr{E}\otimes S)\otimes_{\delta_A\otimes\mathrm{id}_S}(A\otimes S\otimes S))$ and $\mathscr{V}\otimes_{\mathbb{C}}\mathrm{id}_S\in\mathcal{L}((\mathscr{E}\otimes S)\otimes_{\delta_A\otimes\mathrm{id}_S}(A\otimes S\otimes S),\mathscr{E}\otimes S\otimes S)$.

The next result provides an equivalence of the definitions 8.2.3 and 8.2.6. Basically, the proof of the following proposition is based on that of Proposition 2.4 of [3].

Proposition 8.2.7. *a) Let $\delta_{\mathscr{E}}:\mathscr{E}\to\widetilde{\mathcal{M}}(\mathscr{E}\otimes S)$ be a linear map and $\beta_{\mathscr{E}}:N^{\mathrm{o}}\to\mathcal{L}(\mathscr{E})$ a non-degenerate *-homomorphism which satisfy the conditions 1, 2, and 3 of Definition 8.2.3. Then, there exists a unique isometry $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A}(A\otimes S),\mathscr{E}\otimes S)$ such that $\delta_{\mathscr{E}}(\xi)=\mathscr{V}T_\xi$ for all $\xi\in\mathscr{E}$. Moreover, the pair $(\mathscr{V},\beta_{\mathscr{E}})$ satisfies the conditions 1, 2, and 3 of Definition 8.2.6.*

*b) Conversely, let $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A}(A\otimes S),\mathscr{E}\otimes S)$ be an isometry and $\beta_{\mathscr{E}}:N^{\mathrm{o}}\to\mathcal{L}(\mathscr{E})$ a non-degenerate *-homomorphism, which satisfy the conditions 1, 2, and 3 of Definition 8.2.6. We consider the map $\delta_{\mathscr{E}}:\mathscr{E}\to\widetilde{\mathcal{M}}(\mathscr{E}\otimes S)$ given by $\delta_{\mathscr{E}}(\xi)=\mathscr{V}T_\xi$ for all $\xi\in\mathscr{E}$. Then, the pair $(\delta_{\mathscr{E}},\beta_{\mathscr{E}})$ satisfies the conditions 1, 2 and 3 of Definition 8.2.3.*

c) Let us assume that the above statements hold, the triple $(\mathscr{E},\delta_{\mathscr{E}},\beta_{\mathscr{E}})$ is a $\mathcal{G}$-equivariant Hilbert A-module if and only if $\mathscr{V}$ is admissible.

Proof. *a)* As in the proof of Proposition 2.4 in [3], there exists a unique isometric $(A\otimes S)$-linear map $\mathscr{V}:\mathscr{E}\otimes_{\delta_A}(A\otimes S)\to\mathscr{E}\otimes S$ such that

$$\mathscr{V}(\xi\otimes_{\delta_A}x)=\delta_{\mathscr{E}}(\xi)(x),\quad \xi\in\mathscr{E},\,x\in A\otimes S.$$

In other words, we have $\mathscr{V}T_\xi=\delta_{\mathscr{E}}(\xi)$ for all $\xi\in\mathscr{E}$. Now, it follows from the second condition of Definition 8.2.3 that the ranges of $\mathscr{V}$ and $q^{\beta_{\mathscr{E}},\alpha}$ are equal. Then, let us consider the range restriction v of $\mathscr{V}$. Therefore, the map $v^{-1}q^{\beta_{\mathscr{E}},\alpha}$ is an adjoint for $\mathscr{V}$. Indeed, for all $x\in\mathscr{E}\otimes S$ and $y\in\mathscr{E}\otimes_{\delta_A}(A\otimes S)$ we have

$$\langle v^{-1}q^{\beta_{\mathscr{E}},\alpha}x,y\rangle=\langle\mathscr{V}v^{-1}(q^{\beta_{\mathscr{E}},\alpha}x),\mathscr{V}y\rangle=\langle q^{\beta_{\mathscr{E}},\alpha}x,\mathscr{V}y\rangle=\langle x,\mathscr{V}y\rangle-\langle(1-q^{\beta_{\mathscr{E}},\alpha})(x),\mathscr{V}y\rangle=\langle x,\mathscr{V}y\rangle,$$

where we used the fact that $\mathscr{V}$ is isometric in the first equality and the fact that $\mathscr{V}y\in\mathrm{Ran}(q^{\beta_{\mathscr{E}},\alpha})$ in the last one. As a result, we have $\mathscr{V}\in\mathcal{L}(\mathscr{E}\otimes_{\delta_A}(A\otimes S),\mathscr{E}\otimes S)$ and then $\mathscr{V}^*\mathscr{V}=1$ and $\mathscr{V}\mathscr{V}^*=\mathscr{V}v^{-1}q^{\beta_{\mathscr{E}},\alpha}=q^{\beta_{\mathscr{E}},\alpha}$.
The conditions 1 and 3 of Definition 8.2.6 are then fulfilled. Now, we have

$\mathscr{V}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\otimes_{\delta_A}1)(\xi\otimes_{\delta_A}x)=\delta_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\xi)(x)=(1_{\mathscr{E}}\otimes\beta(n^{\mathrm{o}}))\delta_{\mathscr{E}}(\xi)(x)=(1_{\mathscr{E}}\otimes\beta(n^{\mathrm{o}}))\mathscr{V}(\xi\otimes_{\delta_A}x)$, for all $\xi\in\mathscr{E}$, $x\in A\otimes S$, and $n\in N$. Hence, the condition 2 of Definition 8.2.6 holds.

b) is straightforward.

c) Let $T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)$. By using the identifications (8.2.3) and (8.2.7) and Notation 8.1.4, we have $T\otimes_{\mathrm{id}_A\otimes\delta}1\in\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$. Now, we can define the extension of $\mathrm{id}_{\mathscr{E}}\otimes\delta$,

$$\mathrm{id}_{\mathscr{E}}\otimes\delta:\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)\to\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$$

by setting

$$(\mathrm{id}_{\mathscr{E}}\otimes\delta)(T):=T\otimes_{\mathrm{id}_A\otimes\delta}1,\quad T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S).$$

We also have $T\otimes_{\delta_A\otimes\mathrm{id}_S}1\in\mathcal{L}(A\otimes S\otimes S,(\mathscr{E}\otimes_{\delta_A}(A\otimes S))\otimes S)$ by using the identifications (8.2.2) and (8.2.6) and Notation 8.1.4. Let us define the extension of $\delta_{\mathscr{E}}\otimes\mathrm{id}_S$,

$$\delta_{\mathscr{E}}\otimes\mathrm{id}_S:\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)\to\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S)$$

by setting

$$(\delta_{\mathscr{E}}\otimes\mathrm{id}_S)(T):=(\mathscr{V}\otimes_{\mathbb{C}}1_S)(T\otimes_{\delta_A\otimes\mathrm{id}_S}1),\quad T\in\mathcal{L}(A\otimes S,\mathscr{E}\otimes S).$$

Therefore, we have

$$(\delta_{\mathscr{E}}\otimes\mathrm{id}_S)\delta_{\mathscr{E}}(\xi)=(\mathscr{V}\otimes_{\mathbb{C}}1_S)(\mathscr{V}\otimes_{\delta_A\otimes\mathrm{id}_S}1)(T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1)\in\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S),$$

$$(\mathrm{id}_{\mathscr{E}}\otimes\delta)\delta_{\mathscr{E}}(\xi)=(\mathscr{V}\otimes_{\mathrm{id}_A\otimes\delta}1)(T_\xi\otimes_{\mathrm{id}_A\otimes\delta}1)\in\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes S\otimes S),$$

for all $\xi\in\mathscr{E}$, where

$$T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1\in\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)),$$

$$T_\xi\otimes_{\mathrm{id}_A\otimes\delta}1\in\mathcal{L}(A\otimes S\otimes S,\mathscr{E}\otimes_{\delta_A^2}(A\otimes S\otimes S)),$$

by using (8.2.2)-(8.2.4) and (8.2.3)-(8.2.5) respectively and Notation 8.1.4. In particular, if $\mathscr{V}$ is admissible the condition 4 of Definition 8.2.3 holds.
Conversely, let us assume that the aforementioned condition is satisfied. In order to show that $\mathscr{V}$ is admissible, we only have to prove that the restrictions of $T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1$ and $T_\xi\otimes_{\mathrm{id}_A\otimes\delta}1$ to the Hilbert submodule $q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha}(A\otimes S\otimes S)$ are surjective.
Let $a\in A$, $x\in A\otimes S$ and $y\in A\otimes S\otimes S$, we set $z=(\delta_A\otimes\mathrm{id}_S)(\delta_A(a)x)y$. It is clear that $z\in q_{12}^{\beta_A,\alpha}(A\otimes S\otimes S)$. Moreover, we have

$$z=(\delta_A\otimes\mathrm{id}_S)\delta_A(a)(\delta_A\otimes\mathrm{id}_S)(x)y=(\mathrm{id}_A\otimes\delta)(\delta_A(a))(\delta_A\otimes\mathrm{id}_S)(x)y$$

and then z also belongs to $q_{23}^{\beta,\alpha}(A\otimes S\otimes S)$. Hence, $z\in q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha}(A\otimes S\otimes S)$. Now, we have

$$(T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1)(\delta_A(a)x\otimes_{\delta_A\otimes\mathrm{id}_S}y)=(\xi\otimes_{\delta_A}\delta_A(a)x)\otimes_{\delta_A\otimes\mathrm{id}_S}y=(\xi a\otimes_{\delta_A}x)\otimes_{\delta_A\otimes\mathrm{id}_S}y.$$

Therefore, we have $(T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1)(z)=\xi a\otimes_{\delta_A^2}(\delta_A\otimes\mathrm{id}_S)(x)y$. Thus, the restriction of $T_\xi\otimes_{\delta_A\otimes\mathrm{id}_S}1$ to $q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha}(A\otimes S\otimes S)$ is surjective thanks to (8.2.4) and the fact that $\mathscr{E}A=\mathscr{E}$. The same statement is obviously true for $T_\xi\otimes_{\mathrm{id}_A\otimes\delta}1$. □

Remarks 8.2.8. In the proof of Proposition 8.2.7, we have shown that:

- By using the identifications (8.2.3) and (8.2.7) and Notation 8.1.4, we have that the linear map $\mathrm{id}_{\mathscr{E}} \otimes \delta : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$ is defined by:

$$(\mathrm{id}_{\mathscr{E}} \otimes \delta)(T) := T \otimes_{\mathrm{id}_A \otimes \delta} 1, \quad T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S).$$

- If $\delta_{\mathscr{E}}$ satisfies the conditions 1 and 2 of Definition 8.2.3, let $\mathscr{V}$ be the isometry associated with $\delta_{\mathscr{E}}$ (see Proposition 8.2.7 a)). By using the identifications (8.2.2) and (8.2.6) and Notation 8.1.4, the linear map $\delta_{\mathscr{E}} \otimes \mathrm{id}_S : \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \to \mathcal{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$ is defined by:

$$(\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T) := (\mathscr{V} \otimes_{\mathbb{C}} 1_S)(T \otimes_{\delta_A \otimes \mathrm{id}_S} 1), \quad T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S).$$

Note that the extensions $\mathrm{id}_{\mathscr{E}} \otimes \delta$ and $\delta_{\mathscr{E}} \otimes \mathrm{id}_S$ satisfy the following formulas:

$$(\mathrm{id}_{\mathscr{E}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) = (\mathrm{id}_{\mathscr{E}} \otimes \delta)(Tx), \quad (\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) = (\delta_{\mathscr{E}} \otimes \mathrm{id}_S)(Tx),$$

for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$.

Let us denote $J := \mathcal{K}(\mathscr{E} \oplus A)$ the linking C*-algebra associated with $\mathscr{E}$.

Definition 8.2.9. An action (δ_J, β_J) of $\mathcal{G}$ on J is said to be compatible with the action (δ_A, β_A) if we have:

1. $\delta_J : J \to \mathcal{M}(J \otimes S)$ is compatible with δ_A, that is to say

$$\iota_{\mathcal{M}(A \otimes S)} \circ \delta_A = \delta_J \circ \iota_A.$$

2. the fibration map β_J is compatible with β_A, that is to say

$$\iota_A(\beta_A(n^{\mathrm{o}})a) = \beta_J(n^{\mathrm{o}})\iota_A(a), \quad n \in N,\, a \in A.$$

Proposition 8.2.10. *Let (δ_J, β_J) be a compatible action of $\mathcal{G}$ on J. There exists a unique non-degenerate *-homomorphism $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ such that*

$$\beta_J(n^{\mathrm{o}}) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix}, \quad n \in N.$$

Moreover, we have

$$q^{\beta_J, \alpha} = \begin{pmatrix} q^{\beta_{\mathscr{E}}, \alpha} & 0 \\ 0 & q^{\beta_A, \alpha} \end{pmatrix}.$$

Proof. Note that since ι_A, β_A and β_J are *-homomorphisms, the condition 2 of Definition 8.2.9 is equivalent to:

$$\iota_A(a\beta_A(n^{\mathrm{o}})) = \iota_A(a)\beta_J(n^{\mathrm{o}}), \quad a \in A,\, n \in N.$$

Therefore, there exists a map $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ necessarily unique such that

$$\beta_J(n^{\mathrm{o}}) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix},$$

for all $n \in N$. Then, it is clear that $\beta_{\mathscr{E}}$ is a non-degenerate *-homomorphism and the last statement is then an immediate consequence. □

Remarks 8.2.11. Note that if β_A is injective then so is β_J. We also have

$$\iota_{\mathcal{K}(\mathscr{E})}(\beta_{\mathscr{E}}(n^{\mathrm{o}})k) = \beta_J(n^{\mathrm{o}})\iota_{\mathcal{K}(\mathscr{E})}(k), \quad n \in N,\, k \in \mathcal{K}(\mathscr{E}).$$

The proof of Proposition 2.7 of [3] is adapted to the needs of our setting.

Proposition 8.2.12. *a) Let us assume that the* C*-*algebra J is endowed with a compatible action (δ_J, β_J) of $\mathcal{G}$. Then, we have the following statements:*

- *There exists a unique linear map $\delta_{\mathscr{E}} : \mathscr{E} \to \widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ such that*

$$\iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}.$$

  *Moreover, $(\mathscr{E}, \delta_{\mathscr{E}}, \beta_{\mathscr{E}})$ is a $\mathcal{G}$-equivariant Hilbert A-module, where $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ is the *-homomorphism defined in Proposition 8.2.10.*

- *There exists a unique faithful *-homomorphism $\delta_{\mathcal{K}(\mathscr{E})} : \mathcal{K}(\mathscr{E}) \to \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S)$ such that*

$$\iota_{\mathcal{L}(\mathscr{E}\otimes S)} \circ \delta_{\mathcal{K}(\mathscr{E})} = \delta_J \circ \iota_{\mathcal{K}(\mathscr{E})}.$$

  Moreover, the pair $(\delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ is an action of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E})$.

*b) Conversely, let $(\mathscr{E}, \delta_{\mathscr{E}}, \beta_{\mathscr{E}})$ be a $\mathcal{G}$-equivariant Hilbert A-module. Then, there exists a faithful *-homomorphism $\delta_J : J \to \widetilde{\mathcal{M}}(J \otimes S)$ such that*

$$\iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}.$$

Moreover, we define a unique action (δ_J, β_J) of $\mathcal{G}$ on J compatible with (δ_A, β_A) by setting

$$\beta_J(n^{\mathrm{o}}) = \begin{pmatrix} \beta_{\mathscr{E}}(n^{\mathrm{o}}) & 0 \\ 0 & \beta_A(n^{\mathrm{o}}) \end{pmatrix}, \quad n \in N.$$

Proof. a) Let us assume that the C*-algebra J is endowed with a compatible action (δ_J, β_J) of $\mathcal{G}$. Let $\beta_{\mathscr{E}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E})$ be the *-homomorphism defined in Proposition 8.2.10. First, let us prove that there exists a unique linear map

$$\delta_{\mathscr{E}} : \mathscr{E} \to \mathcal{L}(A \otimes S, \mathscr{E} \otimes S)$$

such that $\iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)} \circ \delta_{\mathscr{E}} = \delta_J \circ \iota_{\mathscr{E}}$. For all $\xi \in \mathscr{E}$, we have

$$\iota_{\mathcal{L}(\mathscr{E})}(1_{\mathscr{E}})\iota_{\mathscr{E}}(\xi) = \iota_{\mathscr{E}}(\xi), \quad \iota_{\mathscr{E}}(\xi)\iota_{\mathcal{L}(\mathscr{E})}(1_{\mathscr{E}}) = 0.$$

By the second statement of Proposition 8.2.10, we have

$$\delta_J(1_J) = q^{\beta_J,\alpha} = \iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}) + \iota_{\mathcal{M}(A\otimes S)}(\delta_A(1_A))$$

We also have

$$\delta_J(1_J) = \delta_J(\iota_{\mathcal{L}(\mathscr{E})}(1_{\mathscr{E}})) + \delta_J(\iota_{\mathcal{M}(A\otimes S)}(1_A)).$$

Moreover, we have $\iota_{\mathcal{M}(A\otimes S)} \circ \delta_A = \delta_J \circ \iota_{\mathcal{M}(A)}$ since the right-hand and left-hand sides are both strictly continuous *-homomorphisms, which coincide on A. Hence,

$$\delta_J(\iota_{\mathcal{L}(\mathscr{E})}(1_{\mathscr{E}})) = \iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}).$$

Therefore, for all $\xi \in \mathscr{E}$ we have

$$\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha})\delta_J(\iota_{\mathscr{E}}(\xi)) = \delta_J(\iota_{\mathscr{E}}(\xi)), \quad \delta_J(\iota_{\mathscr{E}}(\xi))\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}) = 0.$$

Fix any $\xi \in \mathscr{E}$, then by Lemma 8.1.3 2 we have

$$\iota_{A\otimes S}(x)\delta_J(\iota_{\mathscr{E}}(\xi)) = \iota_{A\otimes S}(x)\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha})\delta_J(\iota_{\mathscr{E}}(\xi)) = 0,$$

for all $x \in A \otimes S$. Now, let $(u_\lambda)_\lambda$ be an approximate unit of A. We have

$$\delta_J(\iota_{\mathscr{E}}(\xi)) = \lim_\lambda \ \delta_J(\iota_{\mathscr{E}}(\xi u_\lambda)) = \lim_\lambda \ \delta_J(\iota_{\mathscr{E}}(\xi))\iota_{\mathcal{M}(A\otimes S)}(\delta_A(u_\lambda)).$$

Hence, $\delta_J(\iota_{\mathscr{E}}(\xi))\iota_{\mathscr{E}\otimes S}(\eta) = 0$ for all $\eta \in \mathscr{E} \otimes S$ by Lemma 8.1.3 1, which proves the statement. Moreover, $\delta_{\mathscr{E}}$ actually takes its values in the subspace $\widetilde{\mathcal{M}}(\mathscr{E} \otimes S)$ of $\mathcal{L}(\mathscr{E} \otimes S)$. Indeed, fix any $\xi \in \mathscr{E}$. For all $s \in S$, we have that

$$\iota_{\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)}((1_{\mathscr{E}} \otimes s)\delta_{\mathscr{E}}(\xi)) = (1_J \otimes s)\delta_J(\iota_{\mathscr{E}}(\xi)), \quad \iota_{\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)}(\delta_{\mathscr{E}}(\xi)(1_A \otimes s)) = \delta_J(\iota_{\mathscr{E}}(\xi))(1_J \otimes s)$$

belong to $\mathcal{K}((\mathscr{E}\otimes S)\oplus(A\otimes S)) = \mathcal{K}(\mathscr{E}\oplus A)\otimes S$ as the range of δ_J lies in $\widetilde{\mathcal{M}}(J\otimes S)$. Consequently, we have $(1_{\mathscr{E}} \otimes s)\delta_{\mathscr{E}}(\xi)$, $\delta_{\mathscr{E}}(\xi)(1_A \otimes s) \in \mathscr{E} \otimes S$. The condition 1 of Definition 8.2.3 derives easily from the compatibility of δ_J while the condition 4 is a straightforward consequence of the coassociativity of δ_J.
The vector subspace of $\delta_J(1_J)((\mathscr{E} \oplus A) \otimes S)$ spanned by the elements of the form

$$\delta_J(\theta_{\xi\oplus a,\eta\oplus b})(\zeta), \quad \xi, \eta \in \mathscr{E},\ a, b \in A,\ \zeta \in (\mathscr{E} \oplus A) \otimes S,$$

is dense. However, we have

$$\delta_J(\theta_{\xi\oplus a,\eta\oplus b})(\zeta) = (\delta_{\mathscr{E}}(\xi) \oplus \delta_A(a))(\delta_{\mathscr{E}}(\eta) \oplus \delta_A(b))^*(\zeta),$$

where $\delta_{\mathscr{E}}(\xi) \oplus \delta_A(a)$, $\delta_{\mathscr{E}}(\eta) \oplus \delta_A(b) \in \mathcal{L}(A \otimes S, \mathscr{E} \otimes S) \oplus \mathcal{L}(A \otimes S) = \mathcal{L}(A \otimes S, (\mathscr{E} \oplus A) \otimes S)$. Therefore, this vector space is in particular spanned by the elements of the form

$$(\delta_{\mathscr{E}}(\xi) \oplus \delta_A(a))(x) = \delta_{\mathscr{E}}(\xi)x \oplus \delta_A(a)x, \quad \xi \in \mathscr{E},\ a \in A,\ x \in A \otimes S.$$

Then, the condition 2 follows since we have

$$\delta_J(1_J)((\mathscr{E} \oplus A) \otimes S) = q^{\beta_{\mathscr{E}},\alpha}(\mathscr{E} \otimes S) \oplus q^{\beta_A,\alpha}(A \otimes S).$$

Now, let us prove that there exists a unique *-homomorphism

$$\delta_{\mathcal{L}(\mathscr{E})} : \mathcal{L}(\mathscr{E}) \to \mathcal{L}(\mathscr{E} \otimes S) = \mathcal{M}(\mathcal{K}(\mathscr{E}) \otimes S)$$

such that $\iota_{\mathcal{L}(\mathscr{E}\otimes S)} \circ \delta_{\mathcal{L}(\mathscr{E})} = \delta_J \circ \iota_{\mathcal{L}(\mathscr{E})}$. We recall that

$$\delta_J(\iota_{\mathcal{L}(\mathscr{E})}(1_{\mathscr{E}})) = \iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}).$$

In addition, we have

$$\iota_{A\otimes S}(x)\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}) = 0, \quad \iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha})\iota_{A\otimes S}(x) = 0, \quad x \in A \otimes S.$$

As a result, it follows that

$$\iota_{A\otimes S}(x)\delta_J(\iota_{\mathcal{L}(\mathscr{E})}(T)) = 0, \quad \delta_J(\iota_{\mathcal{L}(\mathscr{E})}(T))\iota_{A\otimes S}(x) = 0, \quad T \in \mathcal{L}(\mathscr{E}),\ x \in A \otimes S,$$

which means that $\delta_J(\iota_{\mathcal{L}(\mathscr{E})}(T))$ belongs to $\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(\mathcal{L}(\mathscr{E} \otimes S))$ according to Lemma 8.1.3. Finally, the statement is proved since $\iota_{\mathcal{L}(\mathscr{E}\otimes S)}$ is faithful. Since $\iota_{\mathcal{L}(\mathscr{E}\otimes S)}$ is isometric and $\delta_J \circ \iota_{\mathcal{L}(\mathscr{E})}$ is strictly continuous, we have that $\delta_{\mathcal{L}(\mathscr{E})}$ is strictly continuous. Let us denote $\delta_{\mathcal{K}(\mathscr{E})}$ the restriction

of $\delta_{\mathcal{L}(\mathscr{E})}$ to $\mathcal{K}(\mathscr{E})$ so that $(\delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ turns out to be an action of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E})$. Moreover, the map $\delta_{\mathcal{K}(\mathscr{E})}$ does take its values in $\widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S)$ since we have

$$\delta_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta}) = \delta_{\mathscr{E}}(\xi)\delta_{\mathscr{E}}(\eta)^* = \theta_{\delta_{\mathscr{E}}(\xi),\delta_{\mathscr{E}}(\eta)} \in \mathcal{K}(\widetilde{\mathcal{M}}(\mathscr{E} \otimes S)) \subset \widetilde{\mathcal{M}}(\mathcal{K}(\mathscr{E}) \otimes S),$$

for all $\xi, \eta \in \mathscr{E}$, which follows from the fact that $\iota_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta}) = \iota_{\mathscr{E}}(\xi)\iota_{\mathscr{E}}(\eta)^*$.

b) First, it is clear that β_J is a non-degenerate *-homomorphism. It is also clear that β_J is compatible with the fibration map β_A, that is to say $\beta_J(n^{\mathrm{o}})\iota_A(a) = \iota_A(\beta_A(n^{\mathrm{o}})a)$, for all $a \in A$ and $n \in N$. Let $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_A} (A \otimes S), \mathscr{E} \otimes S)$ be the isometry associated with the action $\delta_{\mathscr{E}}$. Let $\imath : q^{\beta_A,\alpha}(A \otimes S) \to A \otimes S$ be the inclusion map. We easily check out that $\imath$ is an $(A \otimes S)$-linear adjointable map and $\imath^* = q^{\beta_A,\alpha}$. In particular, $\imath$ is an isometry as $\imath^*\imath(x) = q^{\beta_A,\alpha}x = x$ for all $x \in q^{\beta_A,\alpha}(A \otimes S)$. Now, let us denote

$$\mathscr{W} = \mathscr{V} \oplus \imath \in \mathcal{L}((\mathscr{E} \otimes_{\delta_A} (A \otimes S)) \oplus q^{\beta_A,\alpha}(A \otimes S), (\mathscr{E} \otimes S) \oplus (A \otimes S)).$$

We have $\mathscr{W}^*\mathscr{W} = 1$ and then $\mathscr{W}$ is an isometry. Henceforth, we will use the following identification (see (8.2.1)):

$$(\mathscr{E} \otimes_{\delta_A} (A \otimes S)) \oplus q^{\beta_A,\alpha}(A \otimes S) = (\mathscr{E} \otimes_{\delta_A} (A \otimes S)) \oplus (A \otimes_{\delta_A} (A \otimes S)) = (\mathscr{E} \oplus A) \otimes_{\delta_A} (A \otimes S).$$

$$(\mathscr{E} \otimes S) \oplus (A \otimes S) = (\mathscr{E} \oplus A) \otimes S.$$

Hence, $\mathscr{W} \in \mathcal{L}((\mathscr{E} \oplus A) \otimes_{\delta_A} (A \otimes S), (\mathscr{E} \oplus A) \otimes S)$. Then, let us define

$$\delta_{\mathcal{L}(\mathscr{E}\oplus A)}(T) := \mathscr{W}(T \otimes_{\delta_A} 1)\mathscr{W}^* \in \mathcal{L}((\mathscr{E} \oplus A) \otimes S), \quad T \in \mathcal{L}(\mathscr{E} \oplus A).$$

In that way, we define a *-homomorphism $\delta_{\mathcal{L}(\mathscr{E}\oplus A)} : \mathcal{L}(\mathscr{E} \oplus A) \to \mathcal{L}((\mathscr{E} \oplus A) \otimes S)$, which is obviously strictly continuous and verifies $\delta_{\mathcal{L}(\mathscr{E}\oplus A)}(1) = \mathscr{W}\mathscr{W}^* = q^{\beta_{\mathscr{E}},\alpha} \oplus q^{\beta_A,\alpha} = q^{\beta_J,\alpha}$. Let us denote δ_J the restriction of $\delta_{\mathcal{L}(\mathscr{E}\oplus A)}$ to $J := \mathcal{K}(\mathscr{E} \oplus A)$.

Let us prove that for all $a \in A$, we have $\iota_{\mathcal{M}(A\otimes S)}(\delta_A(a)) = \delta_J(\iota_A(a))$. It amounts to proving that

$$\iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))\mathscr{W} = \delta_J(\iota_A(a))\mathscr{W}, \quad a \in A,$$

since we have:

$$\begin{aligned}
\iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))\mathscr{W}\mathscr{W}^* &= \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))(\iota_{\mathcal{L}(\mathscr{E}\otimes S)}(q^{\beta_{\mathscr{E}},\alpha}) + \iota_{\mathcal{M}(A\otimes S)}(q^{\beta_A,\alpha})) \\
&= \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a)q^{\beta_A,\alpha}) \qquad \text{(Lemma 8.1.3 2)} \\
&= \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a)) \quad \text{and}
\end{aligned}$$

$$\delta_J(\iota_A(a))\mathscr{W}\mathscr{W}^* = \delta_J(\iota_A(a))\delta_J(1_J) = \delta_J(\iota_A(a)),$$

for all $a \in A$. Therefore, it is enough to prove that $\iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))\mathscr{W} = \mathscr{W}(\iota_A(a) \otimes_{\delta_A} 1)$ for all $a \in A$ because of $\mathscr{W}^*\mathscr{W} = 1$. But, we have

$$\mathscr{W}((\eta \oplus b) \otimes_{\delta_A} x) = \mathscr{V}(\eta \otimes_{\delta_A} x) \oplus \delta_A(b)x = \delta_{\mathscr{E}}(\eta)x \oplus \delta_A(b)x, \quad \eta \in \mathscr{E},\ b \in A,\ x \in A \otimes S.$$

Now, we finally get

$$\begin{aligned}
\mathscr{W}(\iota_A(a) \otimes_{\delta_A} 1)((\eta \oplus b) \otimes_{\delta_A} x) &= \mathscr{W}((0 \oplus ab) \otimes_{\delta_A} x) \\
&= (\mathscr{V} \oplus \imath)(0 \oplus \delta_A(ab)x) \\
&= 0 \oplus \delta_A(ab)x \\
&= \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))(\delta_{\mathscr{E}}(\eta)x \oplus \delta_A(b)x) \\
&= \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))\mathscr{W}((\eta \oplus b) \otimes_{\delta_A} x),
\end{aligned}$$

for all $\eta \in \mathscr{E}$, $a, b \in A$ and $x \in A \otimes S$. By using similar arguments, we can prove that $\iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)}(\delta_{\mathscr{E}}(\xi)) = \delta_J(\iota_{\mathscr{E}}(\xi))$ for all $\xi \in \mathscr{E}$.

By strict continuity, we obtain

$$(\delta_J \otimes \mathrm{id}_S)\iota_{\mathcal{M}(A\otimes S)}(m) = (\iota_{\mathcal{M}(A\otimes S)} \otimes \mathrm{id}_S)(\delta_A \otimes \mathrm{id}_S)(m),$$

$$(\mathrm{id}_J \otimes \delta)\iota_{\mathcal{M}(A\otimes S)}(m) = (\iota_{\mathcal{M}(A)} \otimes \mathrm{id}_S \otimes \mathrm{id}_S)(\mathrm{id}_A \otimes \delta)(m),$$

for all $m \in \mathcal{M}(A \otimes S)$. By compatibility of δ_J with δ_A and coassociativity of δ_A, we then obtain

$$(\delta_J \otimes \mathrm{id}_S)\delta_J(\iota_A(a)) = (\mathrm{id}_J \otimes \delta)\delta_J(\iota_A(a)), \quad a \in A.$$

By using the coassociativity of $\delta_{\mathscr{E}}$ and the formula $\delta_J \circ \iota_{\mathscr{E}} = \iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)} \circ \delta_{\mathscr{E}}$, we prove in a similar way that $(\delta_J \otimes \mathrm{id}_S)\delta_J(\iota_{\mathscr{E}}(\xi)) = (\mathrm{id}_J \otimes \delta)\delta_J(\iota_{\mathscr{E}}(\xi))$, for all $\xi \in \mathscr{E}$. But since $J := \mathcal{K}(\mathscr{E} \oplus A)$ is generated by $\iota_{\mathscr{E}}(\mathscr{E}) \cup \iota_A(A)$ as a C*-algebra, the coassociativity condition holds.

Now, for all $\eta \in \mathscr{E}$, $b \in A$, $x \in A \otimes S$ and $n \in N$, we have

$$\begin{aligned}
\delta_J(\beta_J(n^{\mathrm{o}}))\mathscr{W}((\eta \oplus b) \otimes_{\delta_A} x) &= \mathscr{W}(\beta_J(n^{\mathrm{o}})(\eta \oplus b) \otimes_{\delta_A} x) \\
&= \mathscr{W}((\beta_{\mathscr{E}}(n^{\mathrm{o}})\eta \oplus \beta_A(n^{\mathrm{o}})b) \otimes_{\delta_A} x) \\
&= \delta_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\eta)x \oplus \delta_A(\beta_A(n^{\mathrm{o}})b)x \\
&= (1_J \otimes \beta(n^{\mathrm{o}}))(\delta_{\mathscr{E}}(\eta)x \oplus \delta_A(b)x) \\
&= (1_J \otimes \beta(n^{\mathrm{o}}))\mathscr{W}((\eta \oplus b) \otimes_{\delta_A} x),
\end{aligned}$$

Then, for all $n \in N$ we have

$$\delta_J(\beta_J(n^{\mathrm{o}})) = \delta_J(\beta_J(n^{\mathrm{o}}))\delta_J(1_J) = \delta_J(\beta_J(n^{\mathrm{o}}))\mathscr{W}\mathscr{W}^* = (1_J \otimes \beta(n^{\mathrm{o}}))\mathscr{W}\mathscr{W}^* = (1_J \otimes \beta(n^{\mathrm{o}}))\delta_J(1_J).$$

Therefore, (δ_J, β_J) is an action of $\mathcal{G}$ on J, compatible with (δ_A, β_A). Finally, the uniqueness of δ_J follows from the formulas

$$\iota_{\mathcal{M}(A\otimes S)}(\delta_A(a)) = \delta_J(\iota_A(a)), \quad \iota_{\mathcal{L}(A\otimes S, \mathscr{E}\otimes S)}(\delta_{\mathscr{E}}(\xi)) = \delta_J(\iota_{\mathscr{E}}(\xi)), \quad a \in A,\ \xi \in \mathscr{E},$$

and the fact that $J := \mathcal{K}(\mathscr{E} \oplus A)$ is generated by $\iota_{\mathscr{E}}(\mathscr{E}) \cup \iota_A(A)$ as a C*-algebra. □

As in [3] (see Définition 2.9), we have:

Definition 8.2.13. Let A and B be two $\mathcal{G}$-C*-algebras, $\mathscr{E}$ a $\mathcal{G}$-equivariant Hilbert B-module and $\pi : A \to \mathcal{L}(\mathscr{E})$ a *-representation. We say that π is $\mathcal{G}$-equivariant if we have:

1. $\delta_{\mathscr{E}}(\pi(a)\xi) = (\pi \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathscr{E}}(\xi)$, for all $a \in A$, $\xi \in \mathscr{E}$.
2. $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \pi(a) = \pi(\beta_A(n^{\mathrm{o}})a)$, for all $n \in N$, $a \in A$.

Moreover, if $\mathscr{E}$ is a countably generated B-module, we say that $(\mathscr{E}, \pi)$ (or simply $\mathscr{E}$ if π is understood) is a $\mathcal{G}$-equivariant Hilbert A-B-bimodule.

Remark 8.2.14. Note that a *-homomorphism $\pi : A \to \mathcal{L}(\mathscr{E})$ defines a *-homomorphism $\pi \otimes \mathrm{id}_S : \widetilde{\mathcal{M}}(A \otimes S) \to \mathcal{L}(\mathscr{E} \otimes S)$ (see [3] §1). Indeed, let us denote $\widetilde{A}$ the C*-algebra obtained from A by adjunction of a unit element. Then, π induces a unital *-homomorphism $\widetilde{\pi} : \widetilde{A} \to \mathcal{L}(\mathscr{E})$ defined by $\widetilde{\pi}(a + \lambda) = \pi(a) + \lambda 1_{\mathscr{E}}$, for all $a \in A$ and $\lambda \in \mathbb{C}$. In particular, we have a non-degenerate *-homomorphism $\widetilde{\pi} \otimes \mathrm{id}_S : \widetilde{A} \otimes S \to \mathcal{L}(\mathscr{E} \otimes S)$, which then extends to $\mathcal{M}(\widetilde{A} \otimes S)$. By restriction to $\widetilde{\mathcal{M}}(A \otimes S)$, we obtain a *-homomorphism which extends $\pi \otimes \mathrm{id}_S$.

If π is non-degenerate and satisfies the condition 1 of Definition 8.2.13, then we prove in the following proposition that the condition 2 is necessarily satisfied.

Proposition 8.2.15. *Let A and B be two $\mathcal{G}$-C*-algebras, $\mathscr{E}$ a $\mathcal{G}$-equivariant Hilbert B-module and $\pi : A \to \mathcal{L}(\mathscr{E})$ a non-degenerate *-representation such that*

$$\delta_{\mathscr{E}}(\pi(a)\xi) = (\pi \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathscr{E}}(\xi), \quad \textit{for all } a \in A,\ \xi \in \mathscr{E}.$$

Therefore, we have $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \pi(a) = \pi(\beta_A(n^{\mathrm{o}})a)$ for all $n \in N$, $a \in A$.

Proof. In virtue of the non-degeneracy of π, $\pi \otimes \mathrm{id}_S$ extends to a unital strictly continuous *-homomorphism $\pi \otimes \mathrm{id}_S : \mathcal{M}(A \otimes S) \to \mathcal{L}(\mathscr{E} \otimes S)$. Let $n \in N$, $a \in A$. For all $\xi \in \mathscr{E}$, we have

$$\begin{aligned}
\delta_{\mathscr{E}}(\beta_{\mathscr{E}}(n^{\mathrm{o}})\pi(a)\xi) &= (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}})) \circ \delta_{\mathscr{E}}(\pi(a)\xi) && (8.2.3\ 3)\\
&= (1_{\mathscr{E}} \otimes \beta(n^{\mathrm{o}})) \circ (\pi \otimes \mathrm{id}_S)(\delta_A(a)) \circ \delta_{\mathscr{E}}(\xi) && (8.2.13\ 1)\\
&= (\pi \otimes \mathrm{id}_S)((1_A \otimes \beta(n^{\mathrm{o}}))\delta_A(a)) \circ \delta_{\mathscr{E}}(\xi)\\
&= (\pi \otimes \mathrm{id}_S)(\delta_A(\beta_A(n^{\mathrm{o}})a)) \circ \delta_{\mathscr{E}}(\xi) && (4.1.2\ 3)\\
&= \delta_{\mathscr{E}}(\pi(\beta_A(n^{\mathrm{o}})a)\xi).
\end{aligned}$$

Since $\delta_{\mathscr{E}}$ is injective (isometric), we have $\beta_{\mathscr{E}}(n^{\mathrm{o}})\pi(a)\xi = \pi(\beta_A(n^{\mathrm{o}})a)\xi$ for all $\xi \in \mathscr{E}$. Hence, $\beta_{\mathscr{E}}(n^{\mathrm{o}}) \circ \pi(a) = \pi(\beta_A(n^{\mathrm{o}})a)$. □

If $(\mathscr{E}, \delta_{\mathscr{E}}, \beta_{\mathscr{E}})$ is a $\mathcal{G}$-equivariant Hilbert A-module then the action $(\delta_{\mathcal{K}(\mathscr{E}\oplus A)}, \beta_{\mathcal{K}(\mathscr{E}\oplus A)})$ of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E} \oplus A)$ and the action $(\delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E})$ are *not necessarily continuous*. In the following, we provide a necessary and sufficient condition on $(\mathscr{E}, \delta_{\mathscr{E}}, \beta_{\mathscr{E}})$ so that the triples $(\mathcal{K}(\mathscr{E} \oplus A), \delta_{\mathcal{K}(\mathscr{E}\oplus A)}, \beta_{\mathcal{K}(\mathscr{E}\oplus A)})$ and $(\mathcal{K}(\mathscr{E}), \delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ become $\mathcal{G}$-C*-algebras.

Proposition 8.2.16. *Let $(\mathscr{E}, \delta_{\mathscr{E}}, \beta_{\mathscr{E}})$ be a $\mathcal{G}$-equivariant Hilbert A-module and let us denote $J = \mathcal{K}(\mathscr{E} \oplus A)$. Let us assume further that*

$$[(1_{\mathscr{E}} \otimes S)\delta_{\mathscr{E}}(\mathscr{E})] = (\mathscr{E} \otimes S)q^{\beta_A,\alpha}. \tag{8.2.8}$$

Therefore, the actions (δ_J, β_J) and $(\delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ are continuous, that is to say:

$$[\delta_J(J)(1_J \otimes S)] = q^{\beta_J,\alpha}(J \otimes S), \tag{8.2.9}$$

$$[\delta_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E}))(1_{\mathscr{E}} \otimes S)] = q^{\beta_{\mathscr{E}},\alpha}(\mathcal{K}(\mathscr{E}) \otimes S). \tag{8.2.10}$$

Conversely, let us assume that the action (δ_J, β_J) is continuous. Then, the formulas (8.2.8) and (8.2.10) hold.

Proof. Let us assume that $[(1_{\mathscr{E}} \otimes S)\delta_{\mathscr{E}}(\mathscr{E})] = (\mathscr{E} \otimes S)q^{\beta_A,\alpha}$. First, let us prove that the action $(\delta_{\mathcal{K}(\mathscr{E})}, \beta_{\mathscr{E}})$ is continuous. We have

$$[\delta_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E}))(1_{\mathscr{E}} \otimes S)] = [\delta_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta})(1_{\mathscr{E}} \otimes y)\,;\, \xi, \eta \in \mathscr{E},\ y \in S].$$

However, we have

$$\delta_{\mathcal{K}(\mathscr{E})}(\theta_{\xi,\eta})(1_{\mathscr{E}} \otimes y) = \delta_{\mathscr{E}}(\xi)\delta_{\mathscr{E}}(\eta)^*(1_{\mathscr{E}} \otimes y) = \delta_{\mathscr{E}}(\xi)((1_{\mathscr{E}} \otimes y^*)\delta_{\mathscr{E}}(\eta))^*, \quad \xi, \eta \in \mathscr{E},\ y \in S.$$

In virtue of the assumption and the fact that $\delta_{\mathscr{E}}(\xi)q^{\beta_A,\alpha} = \delta_{\mathscr{E}}(\xi)$ for all $\xi \in \mathscr{E}$, we obtain

$$[\delta_{\mathcal{K}(\mathscr{E})}(\mathcal{K}(\mathscr{E}))(1_{\mathscr{E}} \otimes S)] = [\delta_{\mathscr{E}}(\mathscr{E})(\mathscr{E}^* \otimes S)] = [\delta_{\mathscr{E}}(\mathscr{E})(1_{\mathscr{E}} \otimes S)(\mathscr{E}^* \otimes S)] = q^{\beta_{\mathscr{E}},\alpha}(\mathcal{K}(\mathscr{E}) \otimes S),$$

where we have used $[\delta_{\mathscr{E}}(\mathscr{E})(1_{\mathscr{E}} \otimes S)] = q^{\beta_{\mathscr{E}},\alpha}(\mathscr{E} \otimes S)$, $\mathcal{K}(\mathscr{E}) = [\mathscr{E}\mathscr{E}^*]$ and the fact that any element of S can be written as a product of two elements of S.
Now, let us prove that (δ_J, β_J) is continuous. Let $x \in J$ and $y \in S$. Let us write:

$$x = \begin{pmatrix} k & \xi \\ \eta^* & a \end{pmatrix}, \quad \text{where } a \in A,\ k \in \mathcal{K}(\mathscr{E}),\ \xi, \eta \in \mathscr{E}.$$

Then, we have

$$\begin{aligned}
\delta_J(x)(1_J \otimes y) &= \delta_J(\iota_{\mathcal{K}(\mathscr{E})}(k))(1_J \otimes y) + \delta_J(\iota_{\mathscr{E}}(\xi))(1_J \otimes y) \\
&\quad + \delta_J(\iota_{\mathscr{E}^*}(\eta^*))(1_J \otimes y) + \delta_J(\iota_A(a))(1_J \otimes y) \\
&= \iota_{\mathcal{L}(\mathscr{E}\otimes S)}(\delta_{\mathcal{K}(\mathscr{E})}(k))(1_J \otimes y) + \iota_{\mathcal{L}(A\otimes S,\mathscr{E}\otimes S)}(\delta_{\mathscr{E}}(\xi))(1_J \otimes y) \\
&\quad + \iota_{\mathcal{L}(\mathscr{E}\otimes S,A\otimes S)}(\delta_{\mathscr{E}}(\eta)^*)(1_J \otimes y) + \iota_{\mathcal{M}(A\otimes S)}(\delta_A(a))(1_J \otimes y) \\
&= \iota_{\mathcal{K}(\mathscr{E})\otimes S}(\delta_{\mathcal{K}(\mathscr{E})}(k)(1_{\mathscr{E}} \otimes y)) + \iota_{\mathscr{E}\otimes S}(\delta_{\mathscr{E}}(\xi)(1_A \otimes y)) \\
&\quad + \iota_{\mathscr{E}\otimes S}((1_{\mathscr{E}} \otimes y^*)\delta_{\mathscr{E}}(\eta))^* + \iota_{A\otimes S}(\delta_A(a)(1_A \otimes y)),
\end{aligned}$$

Then, (8.2.9) follows from (8.2.10), the condition 2 of Definition 8.2.3, the assumption (8.2.8) and the continuity of the action (δ_A, β_A). □

Definition 8.2.17. We say that a $\mathcal{G}$-equivariant Hilbert A-module $(\mathscr{E}, \delta_{\mathscr{E}})$ is a $\mathcal{G}$-A-module if it satisfies:

$$[(1_{\mathscr{E}} \otimes S)\delta_{\mathscr{E}}(\mathscr{E})] = (\mathscr{E} \otimes S)q^{\beta_A,\alpha}.$$

We finish this paragraph with some examples:

1) Let $\mathcal{G}$ be a measured quantum groupoid on a finite basis. Let (J, δ_J, e_1, e_2) be a linking $\mathcal{G}$-C*-algebra (see Definition 6.2.1). Let us denote:

$$A := e_2 J e_2, \quad \mathscr{E} := e_1 J e_2.$$

By restriction of the continuous action (δ_J, β_J) of $\mathcal{G}$ on J, we obtain a continuous action (δ_A, β_A) of $\mathcal{G}$ on A and a structure of $\mathcal{G}$-A-module on $\mathscr{E}$.

2) Let us fix a regular colinking measured quantum groupoid $\mathcal{G}$. Let us consider the trivial action of $\mathcal{G}$ on $N := \mathbb{C}^2$. Let us fix $i = 1, 2$, we then consider the following Hilbert N-module:

$$\mathscr{E} := \mathscr{H}_{i1} \oplus \mathscr{H}_{i2}.$$

Let us consider the isometry $\mathscr{V} \in \mathcal{L}(\mathscr{E} \otimes_{\delta_N} (N \otimes S), \mathscr{E} \otimes S)$ and the non-degenerate *-homomorphism $\beta_{\mathscr{E}} : N \to \mathcal{L}(\mathscr{E})$ given by:

$$\mathscr{V}(\xi_{ij} \otimes 1) = \sum_{k=1,2} V^i_{kj}(\xi_{ij} \otimes 1) \quad ; \quad \beta_{\mathscr{E}}(\varepsilon_j) = p_{ij}, \quad j = 1, 2.$$

Then, the pair $(\mathscr{V}, \beta_{\mathscr{E}})$ defines a structure of $\mathcal{G}$-N-module on $\mathscr{E}$.

3) Let $\mathcal{G} := \mathcal{G}_{\mathbb{G}_1,\mathbb{G}_2}$ be a colinking measured quantum groupoid between two monoidally equivalent regular locally compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let us fix a $\mathbb{G}_1$-C*-algebra A_1 and its induced $\mathbb{G}_2$-C*-algebra $A_2 := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(A_1)$. Now, we also consider the $\mathcal{G}$-C*-algebra $A := A_1 \oplus A_2$. Let us fix $\mathscr{E}_1$ a $\mathbb{G}_1$-equivariant Hilbert A_1-module. We denote $\mathscr{E}_2 := \mathrm{Ind}^{\mathbb{G}_2}_{\mathbb{G}_1}(\mathscr{E}_1)$, the induced

$\mathbb{G}_2$-equivariant Hilbert A_2-module. We have the following linking $\mathbb{G}_1$-C*-algebra (resp $\mathbb{G}_2$-C*-algebra):

$$J_1 := \mathcal{K}(\mathscr{E}_1 \oplus A_1) \quad (\text{resp. } J_2 := \mathcal{K}(\mathscr{E}_2 \oplus A_2)).$$

Let $J := J_1 \oplus J_2$ be the associated $\mathcal{G}$-C*-algebra. Let us define $\mathscr{E}$ the Hilbert A-module $\mathscr{E} := \mathscr{E}_1 \oplus \mathscr{E}_2$. By the previous argument, we have proved that $\mathscr{E}$ is a $\mathcal{G}$-A-module. Conversely, we can prove that any $\mathcal{G}$-A-module is of this form.

8.3 The equivariant Hilbert module $\mathscr{E}_{A,R}$

Let $\mathcal{G} = (N, M, \Delta, \alpha, \beta, T, T', \epsilon)$ be a regular measured quantum groupoid on the finite dimensional basis $N = \bigoplus_{1 \leqslant l \leqslant k} \mathsf{M}_{n_l}(\mathbb{C})$. Let (S, δ) and $(\widehat{S}, \widehat{\delta})$ be the associated weak Hopf-C*-algebras.

Let us fix a $\mathcal{G}$-C*-algebra (A, δ_A, β_A).

Notation 8.3.1. We consider the Hilbert A-modules $\mathscr{E}_0 = A \otimes \mathscr{H}$ and $\mathscr{E}_{A,R} = q^{\beta_A, \widehat{\alpha}}\mathscr{E}_0$. Let us denote $V_0 \in \mathcal{M}(\widehat{S} \otimes S)$ such that $V = (\rho \otimes L)(V_0)$. We then define $\mathcal{V} = (\rho \otimes \mathrm{id}_S)(V_0) \in \mathcal{L}(\mathscr{H} \otimes S)$ (*cf.* Notations 4.4.6 3).

Proposition 8.3.2. *There exists a unique linear map $\delta_{\mathscr{E}_0} : \mathscr{E}_0 \to \mathcal{L}(A \otimes S, \mathscr{E}_0 \otimes S)$ such that*

$$\delta_{\mathscr{E}_0}(a \otimes \xi) = \mathcal{V}_{23}\delta_A(a)_{13}(1_A \otimes \xi \otimes 1_S),$$

for all $a \in A$ and $\xi \in \mathscr{H}$.

Remark 8.3.3. We have $\mathcal{V}_{23}$ and $\delta_A(a)_{13} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S)$. Moreover, for $\xi \in \mathscr{E}$, the operator $1_A \otimes \xi \otimes 1_S \in \mathcal{L}(A \otimes S, A \otimes \mathscr{H} \otimes S)$ is defined by $(1_A \otimes \xi \otimes 1_S)(a \otimes s) = a \otimes \xi \otimes s$, $a \in A$ and $s \in S$. We have $(1_A \otimes \xi \otimes 1_S)^* = 1_A \otimes \xi^* \otimes 1_S$, where $\xi^* \in \mathscr{H}^*$ is defined by $\xi^*(\eta) = \langle \xi, \eta \rangle$.

Proof. If B is a C*-algebra and $\mathscr{K}$ a Hilbert space, we will identify $\mathcal{M}(B) \otimes \mathscr{K}$ with a closed vector subspace of $\mathcal{L}(B, B \otimes \mathscr{K})$ by denoting $(m \otimes \xi)(b) = mb \otimes \xi$ where $m \in \mathcal{M}(B)$, $\xi \in \mathscr{K}$ and $b \in B$. For $\xi_0 \in \mathscr{E}_0$, we then have $(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0) \in \mathcal{L}(A \otimes S, A \otimes S \otimes \mathscr{H})$ and we have $(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0)^* = (\delta_A \otimes \mathrm{id}_{\mathscr{H}^*})(\xi_0^*)$ ($\mathscr{E}_0^* = A \otimes \mathscr{H}^*$). Let $\sigma \in \mathcal{L}(S \otimes \mathscr{H}, \mathscr{H} \otimes S)$ be the flip map $\sigma(s \otimes \xi) = \xi \otimes s$, $s \in S$ and $\xi \in \mathscr{H}$. In particular, we have $\sigma_{23} \in \mathcal{L}(A \otimes S \otimes \mathscr{H}, \mathscr{E}_0 \otimes S)$. Now, let us define:

$$\delta_{\mathscr{E}_0}(\xi_0) = \mathcal{V}_{23}\sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0) \in \mathcal{L}(A \otimes S, \mathscr{E}_0 \otimes S), \quad \xi_0 \in \mathscr{E}_0.$$

Therefore, we have a well-defined linear map $\delta_{\mathscr{E}_0} : \mathscr{E}_0 \to \mathcal{L}(A \otimes S, \mathscr{E}_0 \otimes S)$ such that for all $a \in A$ and $\xi \in \mathscr{H}$, $\delta_{\mathscr{E}_0}(a \otimes \xi) = \mathcal{V}_{23}\delta_A(a)_{13}(1_A \otimes \xi \otimes 1_S)$. The uniqueness follows from the continuity of $\delta_{\mathscr{E}_0}$. □

Proposition 8.3.4. *We have the following statements:*

1. $\delta_{\mathscr{E}_0}(\eta_0)^*\delta_{\mathscr{E}_0}(\xi_0) = \delta_A(\langle q^{\beta_A, \widehat{\alpha}}\eta_0, q^{\beta_A, \widehat{\alpha}}\xi_0 \rangle)$, *for all $\xi_0, \eta_0 \in \mathscr{E}_0$.*
2. $\delta_{\mathscr{E}_0}(\xi_0 a) = \delta_{\mathscr{E}_0}(\xi_0)\delta_A(a)$, *for all $\xi_0 \in \mathscr{E}_0$, $a \in A$.*
3. $\delta_{\mathscr{E}_0}(q^{\beta_A, \widehat{\alpha}}\xi_0) = \delta_{\mathscr{E}_0}(\xi_0)$, *for all $\xi_0 \in \mathscr{E}_0$.*
4. $\delta_{\mathscr{E}_0}(\mathscr{E}_0)(A \otimes S) \subset \mathscr{E}_{A,R} \otimes S$.

Proof. 1. Let $\xi_0, \eta_0 \in \mathscr{E}_0$, we have

$$\delta_{\mathscr{E}_0}(\eta_0)^* \delta_{\mathscr{E}_0}(\xi_0) = (\delta_A \otimes \mathrm{id})(\eta_0^*)\sigma_{23}^* \mathcal{V}_{23}^* \mathcal{V}_{23} \sigma_{23}(\delta_A \otimes \mathrm{id})(\xi_0).$$

However, we have

$$\mathcal{V}^*\mathcal{V} = \sum_{1 \leqslant l \leqslant k} n_l^{-1} \sum_{1 \leqslant i,j \leqslant n_l} \rho(\widehat{\alpha}(e_{ij}^{(l)})) \otimes \beta(e_{ji}^{(l)\mathrm{o}}).$$

Therefore, we have

$$\sigma^*\mathcal{V}^*\mathcal{V}\sigma = \sum_{1 \leqslant l \leqslant k} n_l^{-1} \sum_{1 \leqslant i,j \leqslant n_l} \beta(e_{ij}^{(l)\mathrm{o}}) \otimes \rho(\widehat{\alpha}(e_{ji}^{(l)})).$$

For all $n, n' \in N$, we have

$$\begin{aligned}(1_A \otimes \beta(n^{\mathrm{o}}) \otimes \rho(\widehat{\alpha}(n')))(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(a \otimes \xi) &= (1_A \otimes \beta(n^{\mathrm{o}}))\delta_A(a) \otimes \rho(\widehat{\alpha}(n'))\xi \\ &= \delta_A(\beta_A(n^{\mathrm{o}})a) \otimes \rho(\widehat{\alpha}(n'))\xi \\ &= (\delta_A \otimes \mathrm{id}_{\mathscr{H}})((\beta_A(n^{\mathrm{o}}) \otimes \rho(\widehat{\alpha}(n')))(a \otimes \xi)),\end{aligned}$$

for all $a \in A$, $\xi \in \mathscr{H}$. Hence,

$$(1_A \otimes \beta(n^{\mathrm{o}}) \otimes \rho(\widehat{\alpha}(n')))(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0) = (\delta_A \otimes \mathrm{id}_{\mathscr{H}})((\beta_A(n^{\mathrm{o}}) \otimes \rho(\widehat{\alpha}(n')))\xi_0), \quad n, n' \in N.$$

It then follows that

$$\sigma_{23}^* \mathcal{V}_{23}^* \mathcal{V}_{23} \sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0) = (\delta_A \otimes \mathrm{id}_{\mathscr{H}})(q^{\beta_A, \widehat{\alpha}}\xi_0).$$

As a result, we finally have

$$\begin{aligned}\delta_{\mathscr{E}_0}(\eta_0)^* \delta_{\mathscr{E}_0}(\xi_0) &= (\delta_A \otimes \mathrm{id}_{\mathscr{H}^*})(\eta_0^*)(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(q^{\beta_A, \widehat{\alpha}}\xi_0) \\ &= \delta_A(\langle \eta_0, q^{\beta_A, \widehat{\alpha}}\xi_0 \rangle) \\ &= \delta_A(\langle q^{\beta_A, \widehat{\alpha}}\eta_0, q^{\beta_A, \widehat{\alpha}}\xi_0 \rangle),\end{aligned}$$

where the last equality follows from the fact that $q^{\beta_A, \widehat{\alpha}} \in \mathcal{L}(\mathscr{E}_0)$ is a self-adjoint projection.

2. Let $a, b \in A$ and $\xi \in \mathscr{H}$, we have

$$\begin{aligned}\delta_{\mathscr{E}_0}((b \otimes \xi)a) &= \delta_{\mathscr{E}_0}(ba \otimes \xi) \\ &= \mathcal{V}_{23}\delta_A(ba)_{13}(1_A \otimes \xi \otimes 1_S) \\ &= \mathcal{V}_{23}\delta_A(b)_{13}\delta_A(a)_{13}(1_A \otimes \xi \otimes 1_S) \\ &= \mathcal{V}_{23}\delta_A(b)_{13}(1_A \otimes \xi \otimes 1_S)\delta_A(a),\end{aligned}$$

where $\delta_A(a)$ is looked at as an element of $\mathcal{L}(A \otimes S)$ in the last equality. It then follows that

$$\delta_{\mathscr{E}_0}((b \otimes \xi)a) = \delta_{\mathscr{E}_0}(b \otimes \xi)\delta_A(a), \quad a, b \in A,\ \xi \in \mathscr{H},$$

and the statement is proved.

3. For all $b \in A$ and $\eta \in \mathscr{H}$, we have $\delta_{\mathscr{E}_0}(b \otimes \eta) = \mathcal{V}_{23}(1_A \otimes \xi \otimes 1_S)\delta_A(b) = (1_A \otimes \mathcal{V}(\eta \otimes 1_S))\delta_A(b)$. We recall that

$$\mathcal{V}(1_{\mathscr{H}} \otimes \beta(n^{\mathrm{o}})) = \mathcal{V}(\rho(\widehat{\alpha}(n)) \otimes 1_S), \quad n \in N$$

(see Proposition 2.3.6 2). Let $a \in A$ and $\xi \in \mathscr{H}$, we have

$$\begin{aligned}
\delta_{\mathscr{E}_0}(q^{\beta_A,\widehat{\alpha}}(a\otimes\xi)) &= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} \delta_{\mathscr{E}_0}(\beta_A(e_{ij}^{(l)\mathrm{o}})a\otimes\widehat{\alpha}(e_{ji}^{(l)})\xi) \\
&= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} (1_A\otimes\mathcal{V}(\widehat{\alpha}(e_{ji}^{(l)})\xi\otimes 1_S))\delta_A(\beta_A(e_{ij}^{(l)\mathrm{o}})a) \\
&= \sum_{1\leqslant l\leqslant k} n_l^{-1} \sum_{1\leqslant i,j\leqslant n_l} (1_A\otimes\mathcal{V}(\xi\otimes\beta(e_{ji}^{(l)\mathrm{o}})))(1_A\otimes\beta(e_{ij}^{(l)\mathrm{o}}))\delta_A(a) \\
&= \sum_{1\leqslant l\leqslant k} \sum_{1\leqslant i\leqslant n_l} (1_A\otimes\mathcal{V}(\xi\otimes\beta(e_{ii}^{(l)\mathrm{o}})))\delta_A(a) && e_{ij}^{(l)}e_{ji}^{(l)} = e_{ii}^{(l)} \\
&= \sum_{1\leqslant l\leqslant k} (1_A\otimes\mathcal{V}(\xi\otimes\beta(e^{(l)\mathrm{o}})))\delta_A(a) && e^{(l)} = \sum_{1\leqslant i\leqslant n_l} e_{ii}^{(l)} \\
&= \delta_{\mathscr{E}_0}(a\otimes\xi). && \sum_{1\leqslant l\leqslant k} e^{(l)} = 1_N
\end{aligned}$$

4. It suffices to show that $q_{12}^{\beta_A,\widehat{\alpha}}\delta_{\mathscr{E}_0}(\xi_0)x = \delta_{\mathscr{E}_0}(\xi_0)x$ for all $\xi_0 \in \mathscr{E}_0$ and $x \in A\otimes S$. Let us recall that $(\widehat{\alpha}(n)\otimes 1_S)\mathcal{V} = \mathcal{V}(1_{\mathscr{H}}\otimes\alpha(n))$, for all $n \in N$ (see the first formula of Proposition 2.3.5 3). It then follows that $q_{12}^{\beta_A,\widehat{\alpha}}\mathcal{V}_{23} = \mathcal{V}_{23}q_{13}^{\beta_A,\alpha}$. Let $a,b \in A$, $\xi \in \mathscr{H}$ and $s \in S$. Since $q^{\beta_A,\alpha} = \delta_A(1_A)$, we have

$$\begin{aligned}
q_{12}^{\beta_A,\widehat{\alpha}}\delta_{\mathscr{E}_0}(a\otimes\xi)(b\otimes s) &= q_{12}^{\beta_A,\widehat{\alpha}}\mathcal{V}_{23}\delta_A(a)_{13}(b\otimes\xi\otimes s) \\
&= \mathcal{V}_{23}q_{13}^{\beta_A,\alpha}\delta_A(a)_{13}(b\otimes\xi\otimes s) \\
&= \delta_{\mathscr{E}_0}(a\otimes\xi)(b\otimes s)
\end{aligned}$$

and we are done. □

Notation 8.3.5. According to the previous proposition, $\delta_{\mathscr{E}_0}$ restricts to a linear map

$$\delta_{\mathscr{E}_{A,R}} : \mathscr{E}_{A,R} \to \mathcal{L}(A\otimes S, \mathscr{E}_{A,R}\otimes S),$$

which satisfies the following statements:

- $\delta_{\mathscr{E}_{A,R}}(\eta_0)^*\delta_{\mathscr{E}_{A,R}}(\xi_0) = \delta_A(\langle\eta_0,\xi_0\rangle)$, for all $\eta_0,\xi_0 \in \mathscr{E}_{A,R}$.
- $\delta_{\mathscr{E}_{A,R}}(\eta_0 a) = \delta_{\mathscr{E}_{A,R}}(\eta_0)\delta_A(a)$, for all $\eta_0 \in \mathscr{E}_{A,R}$ and $a \in A$.

Proposition 8.3.6. *We have* $\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R}) \subset \widetilde{\mathcal{M}}(\mathscr{E}_{A,R}\otimes S)$.

Proof. To begin with, let us prove that $\delta_{\mathscr{E}_{A,R}}(\xi_0)(1_A\otimes y) \in \mathscr{E}_{A,R}\otimes S$ for all $\xi_0 \in \mathscr{E}_{A,R}$ and $y \in S$. It amounts to proving that $\delta_{\mathscr{E}_0}(\xi_0)(1_A\otimes y) \in \mathscr{E}_0\otimes S$ for all $\xi_0 \in \mathscr{E}_0$ and $y \in S$ thanks to the statements 3 and 4 of Proposition 8.3.4. Let $a \in A$ and $\xi \in \mathscr{H}$, since $\delta_A(a)(1_A\otimes y) \in A\otimes S$, $\delta_{\mathscr{E}_0}(a\otimes\xi)(1_A\otimes y) = (1_A\otimes\mathcal{V}(\xi\otimes 1_S))\delta_A(a)(1_A\otimes y)$ is the limit, with respect to the norm topology, of finite sums of the form:

$$\sum_i a_i\otimes\mathcal{V}(\xi\otimes s_i) \in A\otimes\mathscr{H}\otimes S = \mathscr{E}_0\otimes S, \quad a_i \in A,\ s_i \in S.$$

Hence, $\delta_{\mathscr{E}_0}(a\otimes\xi)(1_A\otimes y) \in \mathscr{E}_0\otimes S$.

Now, let us prove that $(1_{\mathscr{E}_{A,R}} \otimes y)\delta_{\mathscr{E}_{A,R}}(\xi_0) \in \mathscr{E}_{A,R} \otimes S$ for all $\xi_0 \in \mathscr{E}_{A,R}$ and $y \in S$. This also amounts to proving that $(1_{\mathscr{E}_0} \otimes y)\delta_{\mathscr{E}_0}(\xi_0) \in \mathscr{E}_0 \otimes S$ for all $\xi_0 \in \mathscr{E}_0$ and $y \in S$. Let $a \in A$, $\xi \in \mathscr{H}$ and $y \in S$, we have

$$(1_{\mathscr{E}_0} \otimes y)\delta_{\mathscr{E}_0}(a \otimes \xi) = (1_A \otimes (1_{\mathscr{H}} \otimes y)\mathcal{V}(\xi \otimes 1_S))\delta_A(a).$$

There exist $x \in \widehat{S}$ and $\eta \in \mathscr{H}$ such that $\xi = \rho(x)\eta$. Hence,

$$(1_{\mathscr{H}} \otimes y)\mathcal{V}(\xi \otimes 1_S) = (\rho \otimes \mathrm{id}_S)((1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S))(\eta \otimes 1_S).$$

If $(1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S) \in \widehat{S} \otimes S$ for all $y \in S$ and $x \in \widehat{S}$, then we are done. Indeed, if this was true $(1_{\mathscr{H}} \otimes y)\mathcal{V}(\xi \otimes 1_S)$ would belong to $\mathscr{H} \otimes S$. In particular, it would exist $X \in \mathscr{H} \otimes S$ and $s \in S$ such that $(1_{\mathscr{H}} \otimes x)\mathcal{V}(\xi \otimes 1_S) = Xs$. But since $(1_A \otimes s)\delta_A(a) \in A \otimes S$, $(1_{\mathscr{E}_0} \otimes x)\delta_{\mathscr{E}_0}(a \otimes \xi)$ would be the norm limit of finite sums of the form:

$$\sum_i a_i \otimes Xs_i, \quad a_i \in A,\ s_i \in S.$$

Hence, $(1_{\mathscr{E}_0} \otimes y)\delta_{\mathscr{E}_0}(a \otimes \xi) \in \mathscr{E}_0 \otimes S$. Let us prove that $(1_{\widehat{S}} \otimes y)V(x \otimes 1_S) \in \widehat{S} \otimes S$ for all $y \in S$ and $x \in \widehat{S}$ (for the sake of convenience we identify V_0 and V). It is equivalent to showing that $(y \otimes 1)W(1 \otimes \lambda(x)) \in S \otimes \lambda(\widehat{S})$ or else $(x \otimes 1_S)V(1_{\widehat{S}} \otimes y) \in \widehat{S} \otimes S$ at the risk of switching W with V. It suffices to see that $(\mathrm{id} \otimes \omega \otimes \mathrm{id})(V_{12}V_{13}(1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes y)) \in \widehat{S} \otimes S$ for all $\omega \in \mathcal{B}(\mathscr{H})_*$ ($x := \rho(\omega)$, for $\omega \in \mathcal{B}(\mathscr{H})_*$). Let $\omega \in \mathcal{B}(\mathscr{H})_*$ and let us write $\omega = x\omega'$, where $x \in \widehat{S}$ and $\omega' \in \mathcal{B}(\mathscr{H})_*$. Since $V_{12}V_{13} = V_{23}V_{12}V_{23}^*$, we have

$$\begin{aligned}(\mathrm{id} \otimes \omega \otimes \mathrm{id})(V_{12}V_{13}(1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes y)) &= (\mathrm{id} \otimes \omega' \otimes \mathrm{id})(V_{23}V_{12}V_{23}^*(1_{\mathscr{H}} \otimes x \otimes y)) \\ &= (\mathrm{id} \otimes \omega' \otimes \mathrm{id})(V_{23}V_{12}(1_{\mathscr{H}} \otimes V(x \otimes y))).\end{aligned}$$

But $V(x \otimes y) \in \widehat{S} \otimes S$, therefore $(\mathrm{id} \otimes \omega \otimes \mathrm{id})(V_{12}V_{13}(1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes y))$ is the norm limit of finite sums of the form

$$\sum_i (\mathrm{id} \otimes \omega' \otimes \mathrm{id})(V_{23}V_{12}(1_{\mathscr{H}} \otimes x_i \otimes y_i)) = \sum_i (\mathrm{id} \otimes \omega_i' \otimes \mathrm{id})(V_{23}(1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes y_i)V_{12}),$$

where $x_i \in \widehat{S}$, $y_i \in S$ and $\omega_i' := x_i\omega' \in \mathcal{B}(\mathscr{H})_*$. Now let us write $\omega_i' = R(y_i')\omega_i''\lambda(x_i')$, with $x_i' \in \widehat{S}$, $y_i' \in S$ and $\omega_i'' \in \mathcal{B}(\mathscr{H})_*$. Then, since $V \in \widehat{M}' \otimes M$, $\lambda(\widehat{S}) \subset \widehat{M}$ and $R(S) \subset M'$, we have

$$\begin{aligned}(\mathrm{id} \otimes \omega_i' \otimes \mathrm{id})(V_{23}(1_{\mathscr{H}} \otimes 1_{\mathscr{H}} \otimes y_i)V_{12}) &= (\mathrm{id} \otimes \omega_i'' \otimes \mathrm{id})((1_{\mathscr{H}} \otimes V(\lambda(x_i')R(y_i') \otimes y_i))V_{12}) \\ &= (\mathrm{id} \otimes \omega_i'' \otimes \mathrm{id})((1_{\mathscr{H}} \otimes V(U^*\rho(x_i')L(y_i')U \otimes y_i))V_{12})\end{aligned}$$

However, we have the inclusion $[\widehat{S}S] \subset \mathcal{K}(\mathscr{H})$ (Corollary 3.2.9) and $V \in \mathcal{M}(\mathcal{K}(\mathscr{H}) \otimes S)$. Consequently, we have $V(U^*\rho(x_i')L(y_i')U \otimes y_i) \in \mathcal{K}(\mathscr{H}) \otimes S$. Now, if $k \in \mathcal{K}(\mathscr{H})$, $s \in S$ and $\psi \in \mathcal{B}(\mathscr{H})_*$, we have

$$(\mathrm{id} \otimes \psi \otimes \mathrm{id})((1_{\mathscr{H}} \otimes k \otimes s)V_{12}) = (\mathrm{id} \otimes \psi)((1_{\mathscr{H}} \otimes k)V) \otimes s = (\mathrm{id} \otimes \psi k)(V) \otimes s \in \widehat{S} \otimes S$$

and we are done. □

Notation 8.3.7. Let $\beta_{\mathscr{E}_0} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E}_0)$ be the injective *-homomorphism given by

$$\beta_{\mathscr{E}_0}(n^{\mathrm{o}}) = 1_A \otimes \rho(\beta(n^{\mathrm{o}})), \quad n \in N.$$

Since $\widehat{\alpha}(n')\beta(n^{\mathrm{o}}) = \beta(n^{\mathrm{o}})\widehat{\alpha}(n')$ for all $n, n' \in N$, we have $[\beta_{\mathscr{E}_0}(n^{\mathrm{o}}), q^{\beta_A,\widehat{\alpha}}] = 0$ for all $n \in N$. Therefore, $\beta_{\mathscr{E}_0}$ restricts to a non-degenerate faithful *-homomorphism $\beta_{\mathscr{E}_{A,R}} : N^{\mathrm{o}} \to \mathcal{L}(\mathscr{E}_{A,R})$. Since $(\widehat{\alpha}(n) \otimes 1_S)\mathcal{V} = \mathcal{V}(1_{\mathscr{H}} \otimes \alpha(n))$ for all $n \in N$, we have $\mathcal{V}_{23}q_{13}^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}}\mathcal{V}_{23}$. Thus, $[\mathcal{V}_{23}\mathcal{V}_{23}^*, q_{12}^{\beta_A,\widehat{\alpha}}] = 0$ and the operator $\mathcal{V}_{23}\mathcal{V}_{23}^* \in \mathcal{L}(\mathscr{E}_0 \otimes S)$ restricts to a self-adjoint projection $p_{\mathscr{E}_{A,R}}$ of $\mathcal{L}(\mathscr{E}_{A,R} \otimes S)$.

Proposition 8.3.8. *We have the following statements:*

1. $[\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R})(A \otimes S)] = p_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R} \otimes S)$.

2. $p_{\mathscr{E}_{A,R}} = q^{\beta_{\mathscr{E}_{A,R}},\alpha}$.

3. $\delta_{\mathscr{E}_{A,R}}(\beta_{\mathscr{E}_{A,R}}(n^{\mathrm{o}})\eta) = (1_{\mathscr{E}_{A,R}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathscr{E}_{A,R}}(\eta)$, *for all* $\eta \in \mathscr{E}_{A,R}$ *and* $n \in N$.

Proof. 1. Since $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$, we have $\mathcal{V}_{23}\mathcal{V}_{23}^*\delta_{\mathscr{E}_0}(\xi_0) = \delta_{\mathscr{E}_0}(\xi_0)$, for all $\xi_0 \in \mathscr{E}_0$. It then follows that $p_{\mathscr{E}_{A,R}}\delta_{\mathscr{E}_{A,R}}(\xi) = \delta_{\mathscr{E}_{A,R}}(\xi)$, for all $\xi \in \mathscr{E}_{A,R}$. Thus, we have

$$\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R})(A \otimes S) \subset p_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R} \otimes S).$$

Conversely, let $a \in A$, $\xi \in \mathscr{H}$ and $s \in S$. Since $\mathcal{V}_{23}q_{13}^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}}\mathcal{V}_{23}$, we have

$$p_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \xi) \otimes s) = \mathcal{V}_{23}\mathcal{V}_{23}^*q_{12}^{\beta_A,\widehat{\alpha}}(a \otimes \xi \otimes s) = \mathcal{V}_{23}q_{13}^{\beta_A,\alpha}(a \otimes \mathcal{V}^*(\xi \otimes s))$$

and $\mathcal{V}^*(\xi \otimes s) \in \mathscr{H} \otimes S$. In particular, $p_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \xi) \otimes s)$ is the norm limit of finite sums of elements of the form:

$$\mathcal{V}_{23}q_{13}^{\beta_A,\alpha}(a \otimes \xi' \otimes s'), \quad \xi' \in \mathscr{H},\ s \in S.$$

By continuity of (δ_A, β_A), $\mathcal{V}_{23}q_{13}^{\beta_A,\alpha}(a \otimes \xi' \otimes s')$ is the norm limit of finite sums of the form:

$$\sum_i \mathcal{V}_{23}\delta_A(a_i)_{13}(1_A \otimes \xi' \otimes s_i) = \sum_i \delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a_i \otimes \xi'))(1_A \otimes s_i), \quad a_i \in A,\ s_i \in S.$$

As a result, $p_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \xi) \otimes s) \in [\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R})(A \otimes S)]$, for all $a \in A$, $\xi \in \mathscr{H}$ and $s \in S$. Hence, $p_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R} \otimes S) \subset [\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R})(A \otimes S)]$ and the first statement is proved.

2. The statement is a straightforward consequence of the definitions.

3. Let $\eta = q^{\beta_A,\widehat{\alpha}}(a \otimes \xi)$, with $a \in A$ and $\xi \in \mathscr{H}$. We have

$$\beta_{\mathscr{E}_{A,R}}(n^{\mathrm{o}})\eta = (1_A \otimes \rho(\beta(n^{\mathrm{o}})))q^{\beta_A,\widehat{\alpha}}(a \otimes \xi) = q^{\beta_A,\widehat{\alpha}}(a \otimes \rho(\beta(n^{\mathrm{o}}))\xi).$$

Moreover, we have $\mathcal{V}(\rho(\beta(n^{\mathrm{o}})) \otimes 1_S) = (1_{\mathscr{H}} \otimes \beta(n^{\mathrm{o}}))\mathcal{V}$ for all $n \in N$ (see the second formula of Proposition 2.3.5 3). It then follows that

$$\begin{aligned}
\delta_{\mathscr{E}_{A,R}}(\beta_{\mathscr{E}_{A,R}}(n^{\mathrm{o}})\eta) &= \delta_{\mathscr{E}_0}(a \otimes \rho(\beta(n^{\mathrm{o}}))\xi) \\
&= \mathcal{V}_{23}\delta_A(a)_{13}(1_A \otimes \rho(\beta(n^{\mathrm{o}}))\xi \otimes 1_S) \\
&= (1_A \otimes \mathcal{V}(\rho(\beta(n^{\mathrm{o}})) \otimes 1_S))\delta_A(a)_{13}(1_A \otimes \xi \otimes 1_S) \\
&= (1_A \otimes 1_{\mathscr{H}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathscr{E}_0}(a \otimes \xi) \\
&= (1_{\mathscr{E}_{A,R}} \otimes \beta(n^{\mathrm{o}}))\delta_{\mathscr{E}_{A,R}}(\eta)
\end{aligned}$$

and we are done. □

Consequently, $\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S$ and $\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta$ extend to linear maps from $\mathcal{L}(A \otimes S, \mathscr{E}_{A,R} \otimes S)$ to $\mathcal{L}(A \otimes S \otimes S, \mathscr{E}_{A,R} \otimes S \otimes S)$ (*cf.* Remarks 8.2.8) and we have:

$$(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(T)(\delta_A \otimes \mathrm{id}_S)(x) = (\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(Tx),$$

$$(\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(T)(\mathrm{id}_A \otimes \delta)(x) = (\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(Tx)$$

for all $x \in A \otimes S$ and $T \in \mathcal{L}(A \otimes S, \mathscr{E}_{A,R} \otimes S)$.

Proposition 8.3.9. *The linear map $\delta_{\mathscr{E}_{A,R}}$ is coassociative, that is to say*

$$(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)\delta_{\mathscr{E}_{A,R}}(\xi) = (\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)\delta_{\mathscr{E}_{A,R}}(\xi), \quad \xi \in \mathscr{E}_{A,R}.$$

Proof. Let $a \in A$, $\eta \in \mathscr{H}$, $x \in A \otimes S$ and $y \in A \otimes S \otimes S$, we have

$$\begin{aligned}(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))(\delta_A \otimes \mathrm{id}_S)(x)y &= (\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta))x)y \\ &= (\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_0}(a \otimes \eta)x)y \\ &= (\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(\mathcal{V}_{23}(\delta_A(a)x)_{13}(1_A \otimes \eta \otimes 1_S))y.\end{aligned}$$

Now since $\delta_A(a)x \in A \otimes S$, if $b \in A$ and $s \in S$ we have

$$(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(\mathcal{V}_{23}(b \otimes \eta \otimes s)) = (\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(b \otimes \mathcal{V}(\eta \otimes s)).$$

Let $\eta' \in \mathscr{H}$ and $s' \in S$, we have

$$(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(b \otimes \eta' \otimes s') = \delta_{\mathscr{E}_0}(b \otimes \eta') \otimes s' = \mathcal{V}_{23}\delta_A(b)_{13}(1_A \otimes \eta' \otimes 1_S \otimes s').$$

Consequently, we have $(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(b \otimes X) = \mathcal{V}_{23}\delta_A(b)_{13}X_{24} \in \mathcal{L}(A \otimes \mathscr{H} \otimes S \otimes S)$ for all $X \in \mathscr{H} \otimes S$. In particular, we have

$$\begin{aligned}(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(b \otimes \mathcal{V}(\eta \otimes s)) &= \mathcal{V}_{23}\delta_A(b)_{13}\mathcal{V}_{24}(1_A \otimes \eta \otimes 1_S \otimes s) \\ &= \mathcal{V}_{23}\mathcal{V}_{24}\delta_A(b)_{13}(1_A \otimes \eta \otimes 1_S \otimes s).\end{aligned}$$

However, we have $(\mathrm{id}_{\mathscr{H}} \otimes \delta)(\mathcal{V}) = \mathcal{V}_{12}\mathcal{V}_{13}$. Hence, $\mathcal{V}_{23}\mathcal{V}_{24} = (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23})$. Moreover, we have

$$\begin{aligned}\delta_A(b)_{13}(1_A \otimes \eta \otimes 1_S \otimes s) &= (\delta_A(b)_{13} \otimes s)(1_A \otimes \eta \otimes 1_S \otimes 1_S) \\ &= (\delta_{A,13} \otimes \mathrm{id}_S)(b \otimes s)(1_A \otimes \eta \otimes 1_S \otimes 1_S),\end{aligned}$$

for all $b \in A$ and $s \in S$, where $\delta_{A,13} : A \to \mathcal{L}(A \otimes \mathscr{H} \otimes S)$ is the *-homomorphism defined by $\delta_{A,13}(a) = \delta_A(a)_{13}$ for all $a \in A$. As a result, we have

$$(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(\mathcal{V}_{23}Y_{13}(1_A \otimes \eta \otimes 1_S)) = (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23})(\delta_{A,13} \otimes \mathrm{id}_S)(Y)(1_A \otimes \eta \otimes 1_S \otimes 1_S)$$

for all $Y \in A \otimes S$. In particular, we have

$$\begin{aligned}(\delta_{\mathscr{E}_0} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_0}(a \otimes \eta)x)y &= (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23})(\delta_{A,13} \otimes \mathrm{id}_S)(\delta_A(a)x)(1_A \otimes \eta \otimes 1_S \otimes 1_S)y \\ &= (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23})(\delta_{A,13} \otimes \mathrm{id}_S)\delta_A(a)(1_A \otimes \eta \otimes 1_S \otimes 1_S)(\delta_A \otimes \mathrm{id}_S)(x)y.\end{aligned}$$

Besides, we have $(\delta_{A,13} \otimes \mathrm{id}_S)(\delta_A(a)) = (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\delta_A(a)_{13})$. Hence,

$$(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))z = (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23}\delta_A(a)_{13})(1_A \otimes \eta \otimes 1_S \otimes 1_S)z,$$

for all $z \in q_{12}^{\beta_A,\alpha}(A \otimes S \otimes S)$. In particular, if $z \in q_{12}^{\beta_A,\alpha}q_{23}^{\beta,\alpha}(A \otimes S \otimes S)$ we have

$$\begin{aligned}(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))z &= (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\mathcal{V}_{23}\delta_A(a)_{13})(1_A \otimes \eta \otimes q^{\beta,\alpha})z \\ &= (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\delta_{\mathscr{E}_0}(a \otimes \eta))z \\ &= (\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))z,\end{aligned}$$

since for all $x \in A \otimes S$ and $y \in A \otimes S \otimes S$ we have

$$\begin{aligned}(\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))(\mathrm{id}_A \otimes \delta)(x)y &= (\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta))x)y \\ &= (\mathrm{id}_{\mathscr{E}_0} \otimes \delta)(\delta_{\mathscr{E}_0}(a \otimes \eta)x)y.\end{aligned}$$

Therefore, we have $(\delta_{\mathscr{E}_{A,R}} \otimes \mathrm{id}_S)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta))) = (\mathrm{id}_{\mathscr{E}_{A,R}} \otimes \delta)(\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}(a \otimes \eta)))$ for all $a \in A$ and $\eta \in \mathscr{H}$ and then the coassociativity condition holds. □

Now, we can assemble the previous results in the following theorem:

Theorem 8.3.10. *The triple* $(\mathscr{E}_{A,R}, \delta_{\mathscr{E}_{A,R}}, \beta_{\mathscr{E}_{A,R}})$ *is a* $\mathcal{G}$*-equivariant Hilbert A-module.*

In the following, we identify the $\mathcal{G}$-C*-algebras $A \rtimes \mathcal{G} \rtimes \widehat{\mathcal{G}}$ and (D, δ_D, β_D) (see Theorem 4.4.15). We also denote $j_D : D \to \mathcal{L}(\mathscr{E}_{A,R})$ the canonical non-degenerate faithful *-homomorphism (see Proposition-Definition 4.4.1 and also Remark 4.4.2).

Proposition 8.3.11. *The couple* $(\mathscr{E}_{A,R}, j_D)$ *is a* $\mathcal{G}$*-equivariant Hilbert D-A-bimodule.*

Proof. By Proposition 8.2.15, we have to prove that j_D satisfies the following statement (see Definition 8.2.13):

$$\delta_{\mathscr{E}_{A,R}}(j_D(d)\xi) = (j_D \otimes \mathrm{id}_S)(\delta_D(d)) \circ \delta_{\mathscr{E}_{A,R}}(\xi), \quad \text{for all } \xi \in \mathscr{E}_{A,R},\ d \in D.$$

Let us prove it in three steps:

• Let $b \in A$, $x \in \widehat{S}$ and $\eta \in \mathscr{H}$, we have

$$\delta_{\mathscr{E}_0}(b \otimes \lambda(x)\eta) = \mathcal{V}_{23}\delta_A(b)_{13}(1_A \otimes \lambda(x)\eta \otimes 1_S) = (1_A \otimes \mathcal{V}(\lambda(x) \otimes 1_S))\delta_A(b)_{13}(1_A \otimes \eta \otimes 1_S).$$

However, $\mathcal{V}(\lambda(x) \otimes 1_S) = (\lambda(x) \otimes 1_S)\mathcal{V}$ (as $\lambda(\widehat{S}) \subset \widehat{M}$ and $V \in \widehat{M}' \otimes M$). Hence,

$$\delta_{\mathscr{E}_0}(b \otimes \lambda(x)\eta) = (1_A \otimes \lambda(x) \otimes 1_S)\delta_{\mathscr{E}_0}(b \otimes \eta),$$

and then $\delta_{\mathscr{E}_0}((1_A \otimes \lambda(x))\eta_0) = (1_A \otimes \lambda(x) \otimes 1_S)\delta_{\mathscr{E}_0}(\eta_0)$, for all $x \in \widehat{S}$ and $\eta_0 \in \mathscr{E}_0$.

• Let $y \in S$, we have $\mathcal{V}(L(y) \otimes 1_S) = \mathcal{V}q^{\widehat{\alpha},\beta}(L(y) \otimes 1_S)$. Since $\widehat{\alpha}(N) \subset M'$ and $L(y) \in M$, we have

$$\mathcal{V}(L(y) \otimes 1_S) = \mathcal{V}(L(y) \otimes 1_S)q^{\widehat{\alpha},\beta} = \mathcal{V}(L(y) \otimes 1_S)\mathcal{V}^*\mathcal{V} = (L \otimes \mathrm{id}_S)(\delta(y))\mathcal{V}.$$

Let $b \in A$ and $\eta \in \mathscr{H}$, we have

$$\begin{aligned}\delta_{\mathscr{E}_0}((1_A \otimes L(y))(b \otimes \eta)) &= (1_A \otimes \mathcal{V}(L(y) \otimes 1_S))\delta_A(b)_{13}(1_A \otimes \eta \otimes 1_S)\\ &= (1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))\delta_{\mathscr{E}_0}(b \otimes \eta).\end{aligned}$$

Hence, $\delta_{\mathscr{E}_0}((1_A \otimes L(y))\eta_0) = (1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))\delta_{\mathscr{E}_0}(\eta_0)$ for all $y \in S$ and $\eta_0 \in \mathscr{E}_0$.

Thanks to the first two steps, we have

$$\delta_{\mathscr{E}_{A,R}}((1_A \otimes \lambda(x)L(y))\eta_0) = (1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))\delta_{\mathscr{E}_{A,R}}(\eta_0), \quad \eta_0 \in \mathscr{E}_{A,R}.$$

• Let $s \in S$, we have

$$(R(s) \otimes 1)V = (U \otimes 1)\Sigma(1 \otimes L(s))\Sigma(U^* \otimes 1)V = (U \otimes 1)\Sigma(1 \otimes L(s))W\Sigma(U^* \otimes 1).$$

Besides, $(1 \otimes L(s))W = (1 \otimes L(s))WW^*W = WW^*(1 \otimes L(s))W = W\delta(s)$ since we have $WW^* = q^{\alpha,\widehat{\beta}}$ and $L(s) \in M \subset \widehat{\beta}(N^{\mathrm{o}})'$. Therefore, since $(U \otimes 1)\Sigma W = V(U \otimes 1)\Sigma$ we have

$$(R(s) \otimes 1)V = V\Sigma(1 \otimes U)\delta(s)(1 \otimes U^*)\Sigma.$$

Thus, we have

$$(R(s) \otimes 1_S)\mathcal{V} = \mathcal{V}\sigma(\mathrm{id}_S \otimes R)(\delta(s))\sigma^*, \quad \text{for all } s \in S.$$

We then have that $((\mathrm{id}_A \otimes R)(x) \otimes 1_S)\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\mathrm{id}_{A\otimes S} \otimes R)((\mathrm{id}_A \otimes \delta)(x))\sigma_{23}^*$, for all $x \in A \otimes S$. But, since R and δ are strictly continuous this equality also holds for all $x \in \mathcal{M}(A \otimes S)$. In particular, we have

$$\pi_R(a)_{12}\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\mathrm{id}_{A\otimes S} \otimes R)(\delta_A^2(a))\sigma_{23}^*, \quad \text{for all } a \in A,$$

where $\delta_A^2(a) = (\mathrm{id}_A \otimes \delta)\delta_A(a)$. Now, since $\delta_A^2(a) = (\delta_A \otimes \mathrm{id}_S)\delta_A(a)$ we obtain

$$\pi_R(a)_{12}\mathcal{V}_{23} = \mathcal{V}_{23}\sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(\pi_R(a))\sigma_{23}^*, \quad \text{for all } a \in A.$$

It then follows that

$$\begin{aligned}\pi_R(a)_{12}\delta_{\mathscr{E}_0}(\xi_0) &= \mathcal{V}_{23}\sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathcal{K}(\mathscr{H})})(\pi_R(a))(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\xi_0)\\ &= \mathcal{V}_{23}\sigma_{23}(\delta_A \otimes \mathrm{id}_{\mathscr{H}})(\pi_R(a)\xi_0)\\ &= \delta_{\mathscr{E}_0}(\pi_R(a)\xi_0),\end{aligned}$$

for all $a \in A$ and $\xi_0 \in \mathscr{E}_0$. In particular, $\pi_R(a)_{12}\delta_{\mathscr{E}_{A,R}}(\xi_0) = \delta_{\mathscr{E}_{A,R}}(\pi_R(a)\xi_0)$ for all $a \in A$ and $\xi_0 \in \mathscr{E}_{A,R}$.

We have proved that for all $a \in A$, $x \in \widehat{S}$, $y \in S$ and $\xi_0 \in \mathscr{E}_0$, we have

$$\delta_{\mathscr{E}_0}(\pi_R(a)(1_A \otimes \lambda(x)L(y))\xi_0) = \pi_R(a)_{12}(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))\delta_{\mathscr{E}_0}(\xi_0).$$

However, for all $a \in A$, $x \in \widehat{S}$ and $y \in S$ we have

$$(j_D \otimes \mathrm{id}_S)\delta_D(\pi_R(a)(1_A \otimes \lambda(x)L(y))) = \pi_R(a)_{12}(1_A \otimes \lambda(x) \otimes 1_S)(1_A \otimes (L \otimes \mathrm{id}_S)\delta(y))$$

(see Proposition 4.4.5 1). Therefore, if $d = \pi_R(a)(1_A \otimes \lambda(x)L(y)) \in D$, where $a \in A$, $x \in \widehat{S}$ and $y \in S$, we have $\delta_{\mathscr{E}_{A,R}}(j_D(d)\xi) = (j_D \otimes \mathrm{id}_S)(\delta_D(d)) \circ \delta_{\mathscr{E}_{A,R}}(\xi)$, for all $\xi \in \mathscr{E}_{A,R}$. Thus, the statement is proved since $D = [\pi_R(a)(1_A \otimes \lambda(x)L(y))\,;\, a \in A,\, x \in \widehat{S},\, y \in S]$. □

Proposition 8.3.12. *Let $(\delta_{\mathcal{K}(\mathscr{E}_{A,R}\oplus A)}, \beta_{\mathcal{K}(\mathscr{E}_{A,R}\oplus A)})$ be the action of $\mathcal{G}$ on $\mathcal{K}(\mathscr{E}_{A,R} \oplus A)$ associated with the $\mathcal{G}$-equivariant Hilbert A-module $(\mathscr{E}_{A,R}, \delta_{\mathscr{E}_{A,R}}, \beta_{\mathscr{E}_{A,R}})$. Then, the following statements are equivalent:*

1. *$(\delta_{\mathcal{K}(\mathscr{E}_{A,R}\oplus A)}, \beta_{\mathcal{K}(\mathscr{E}_{A,R}\oplus A)})$ is continuous,*
2. *$[(1_{\mathscr{E}_{A,R}} \otimes S)\delta_{\mathscr{E}_{A,R}}(\mathscr{E}_{A,R})] = (\mathscr{E}_{A,R} \otimes S)q^{\beta_A,\alpha}$,*
3. *$[(1_A \otimes (1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S))q_{13}^{\beta_A,\alpha}\,;\, y \in S,\, x \in \widehat{S}] = q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes \widehat{S} \otimes S)q_{13}^{\beta_A,\alpha}$.*

Note that $(1_A \otimes (1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S))q_{13}^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes (1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S))$, for all $y \in S$ and $x \in \widehat{S}$.

Proof. The equivalence of the statements 1 and 2 is given by Proposition 8.2.16. Let us prove that the statements 2 and 3 are equivalent. Note that the statement 2 is equivalent to

$$[(1_{\mathscr{E}_0} \otimes S)\delta_{\mathscr{E}_0}(\mathscr{E}_0)] = q_{12}^{\beta_A,\widehat{\alpha}}(\mathscr{E}_0 \otimes S)q^{\beta_A,\alpha},$$

since we have $\delta_{\mathscr{E}_{A,R}}(q^{\beta_A,\widehat{\alpha}}\xi_0) = \delta_{\mathscr{E}_0}(\xi_0)$, for all $\xi_0 \in \mathscr{E}_0$. Actually, we already have the inclusion $[(1_{\mathscr{E}_0} \otimes S)\delta_{\mathscr{E}_0}(\mathscr{E}_0)] \subset q_{12}^{\beta_A,\widehat{\alpha}}(\mathscr{E}_0 \otimes S)q^{\beta_A,\alpha}$. Let $y \in S$, $a \in A$ and $\xi \in \mathscr{H}$. Let $x \in \widehat{S}$ and $\eta \in \mathscr{H}$ such that $\xi = \rho(x)\eta$. We have

$$(1_{\mathscr{E}_0} \otimes y)\delta_{\mathscr{E}_0}(a \otimes \xi) = (1_A \otimes (1_{\mathscr{H}} \otimes y)\mathcal{V}(\xi \otimes 1_S))\delta_A(a)$$

and $(1_{\mathscr{H}} \otimes y)\mathcal{V}(\xi \otimes 1_S) = (\rho \otimes \mathrm{id}_S)((1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S))(\eta \otimes 1_S)$. Hence, $[(1_{\mathscr{E}_0} \otimes S)\delta_{\mathscr{E}_0}(\mathscr{E}_0)]$ is equal to

$$[(1_A \otimes (\rho \otimes \mathrm{id}_S)((1_{\widehat{S}} \otimes y)V_0(x \otimes 1_S)))\delta_A(a)_{13}(1_A \otimes \eta \otimes 1_S)\,;\, a \in A,\, x \in \widehat{S},\, y \in S,\, \eta \in \mathscr{H}].$$

Now, let $a \in A$, $\xi \in \mathscr{H}$ and $y \in S$. We have

$$q_{12}^{\beta_A,\widehat{\alpha}}(a \otimes \xi \otimes y)q^{\beta_A,\alpha} = q_{12}^{\beta_A,\widehat{\alpha}}((a \otimes y)q^{\beta_A,\alpha})_{13}(1_A \otimes \xi \otimes 1_S).$$

Therefore, by continuity of (δ_A, β_A), $q_{12}^{\beta_A,\widehat{\alpha}}(a \otimes \xi \otimes y)q^{\beta_A,\alpha}$ is the norm limit of finite sums of the form

$$\begin{aligned}\sum_i q_{12}^{\beta_A,\widehat{\alpha}}(1_{\mathscr{E}_0} \otimes y_i)\delta_A(a_i)_{13}(1_A \otimes \xi \otimes 1_S)& \\ = \sum_i (\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_S)(q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes x \otimes y_i)q_{13}^{\beta_A,\alpha})\delta_A(a_i)_{13}(1_A \otimes \eta \otimes 1_S),&\end{aligned}$$

where $a_i \in A$, $y_i \in S$ and $x \in \widehat{S}$, $\eta \in \mathscr{H}$ are such that $\xi = \rho(x)\eta$. Therefore, $q_{12}^{\beta_A,\widehat{\alpha}}(\mathscr{E}_0 \otimes S)q^{\beta_A,\alpha}$ is equal to

$$[(\mathrm{id}_A \otimes \rho \otimes \mathrm{id}_S)(q_{12}^{\beta_A,\widehat{\alpha}}(1_A \otimes x \otimes y)q_{13}^{\beta_A,\alpha})\delta_A(a)_{13}(1_A \otimes \eta \otimes 1_S)\,;\, x \in \widehat{S},\, y \in S,\, \eta \in \mathscr{H}].$$

□

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