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Passivity preserving balanced reduction for the finite and infinite dimensional port Hamiltonian systems

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Passivity preserving balanced reduction for finite and infinite dimensional port Hamiltonian systems

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To my dear wife, for her constant love.

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Chapter 1

General introduction

1.1 Motivation and previous work

This thesis has been done in the context of the French ANR (Agence Nationale de Recherche) sponsored project HAMECMOPSYS (Hamiltonian Methods for the Control of Multi domain Distributed Parameter Systems), reference code ANR-11-BS03-0002. The aim of this project is to use the intrinsic properties of physical systems through the *port Hamiltonian framework* to develop new tools and new theoretical results for the analysis and the control of a large class of complex dynamical systems. The project focuses more specifically on distributed parameter systems through the extension of the port Hamiltonian framework to infinite dimensional systems [78].

As soon as complexity and distribution in space are taken into account during the modeling step, a major concern arising from these approaches is the large scale or infinite dimension nature of the resulting state space. Hence, for control purposes, an important problem is the preservation of the intrinsic structure and passivity property of the original model during the reduction/approximation process. This subject is the core of this thesis.

The port Hamiltonian approach is based on some *energetic* considerations and is well-adapted for the modeling and the control of mechanical, electro-mechanical, multi-physical systems as well as network systems [19, 79, 78, 53, 32, 17]. It is strongly linked to the passivity properties of the considered systems. These passivity properties are very useful for control design [8, 82, 83] and the proposed reduction schemes have then to preserve these properties.

The passivity preserving reduction schemes have been studied for both finite and infinite dimensional systems in many references in the last decade. In the finite dimensional case, the positive real balancing methods have been introduced in [25, 3] for positive real (passive) systems. It has been extended to infinite dimensional systems in [29]. The authors in [38] proposed a passivity preserving reduction method with finite frequency for finite dimensional passive systems. Several other researches tried to insure passivity and structure of the reduced order model. On the one hand, in the finite dimensional case, the Krylov subspaces and moment matching methods have been used to reduce port Hamiltonian systems preserving their passivity [63, 62, 64]. In [26, 27] is discussed the interpolation based model reduction of port Hamiltonian systems. From a geometry point of view, Effort and Flow constraint methods have been proposed in [65]. However, from the balanced reduction point of view, the *lossless* port Hamiltonian systems cannot be reduced by balancing method [76]. The positive

real balancing method is introduced in [61] for *dissipative* port Hamiltonian systems. The Lyapunov balanced method for non linear port Hamiltonian systems has been proposed in [21]. On the other hand, the passivity and Hamiltonian structure preserving approximations of infinite dimensional port Hamiltonian systems have been derived on the basis of the spatial discretization called mixed finite elements method [24, 5, 33], and pseudo-spectral method [52] from the geometry point of view. The Petrov-Galerkin approximation proposed in [34] is concerned with input and output considerations.

Another characteristic of network or multi-physical systems is that the interconnection of subsystems may lead to algebraic constraints. This usually leads to a system of differential algebraic equations (DAE) or an implicit port Hamiltonian system [16, 19]. For instance, in this thesis, a complex micro mechanical manipulator called nanotweezers [7] used for the manipulation of DNA bundles will be formulated as a port Hamiltonian system by interconnecting two mechanical systems. Another example can be found in [10] where a simplified airplane wing model is described through a mechanical beam coupled with a fluid tank. More generally *descriptor state space systems* (also called singular systems/generalized state-space systems) are associated with differential algebraic equations (DAE) systems and have attracted much attention in the last decades [15, 18]. This framework will be used to deal with implicit port Hamiltonian systems arising for example from the interconnection of mechanical subsystems. Moreover balanced reduction schemes proposed for example in [74, 69] will be adapted to implicit port Hamiltonian system.

The above model reduction methods all deal with *open loop* dissipative systems. When control design is concerned, three different approaches can be applied:

- Design a low order controller directly based on the high dimensional system;
- Reduce the high dimensional system by a lower order numerical approximation, then design the controller based on this approximation;
- Apply a model reduction procedure, two cases are considered at the same time: the system and the controller.

The first scheme is always difficult to achieve, so we shall concentrate on the last two methods. The second method have been widely studied for port Hamiltonian systems. Nevertheless it is based on the aforementioned open loop approaches and then on the fact that the open loop system is passive. As a consequence it cannot be applied in case of undamped or weakly damped systems. It is the case for example with hyperbolic systems. Indeed in this case nothing can be used to distinguish some states from the others. As a consequence we shall move to the third approach that consists in designing at the same time the controller and the reduction scheme. The balanced reduction method associated with the LQG control problem is suitable to attain this objective.

In control theory, the *Linear-Quadratic-Gaussian (LQG)* control problem is one of the most fundamental optimal control problems. It concerns uncertain linear systems disturbed by additive white Gaussian noise, having incomplete state information (i.e. not all the state variables are measured and available for feedback) and undergoing control subject to quadratic costs. The solution to the control problem is unique and leads to a linear dynamic feedback that is easy to compute and to implement. Generally speaking, LQG controller is neither stable nor passive. Hence the Hamiltonian structure and passivity properties are not usually preserved in closed loop when LQG control is applied to port Hamiltonian systems. On one hand, a first attempt to preserve these properties in closed loop has been done for finite dimensional positive real linear system in [40, 8].

On the other hand, the balanced reduction method associated with the LQG control problem, called *LQG balanced method*, has been firstly introduced in [36]. This reduction method is perfectly suitable for low order controller design in finite dimension and has been extended to infinite dimensional systems in [12]. However LQG balanced method is not passive preserving in general.

The aim of this doctoral Thesis is:

- To study the reduction of implicit port Hamiltonian systems by using the descriptor system framework.
- To study the LQG based method for passive preserving model reduction of *closed loop* port Hamiltonian system in both finite and infinite dimensional case.

1.2 Organization and contribution of this thesis

This thesis is divided into three chapters. Chapter 2 is devoted to the reduction of the implicit port Hamiltonian systems. The aim of this chapter is to motivate the reduction of network of interconnected systems through physical algebraic constraints. This chapter starts with an introduction on port Hamiltonian systems. A descriptor representation of implicit port Hamiltonian systems, called *port Hamiltonian descriptor system* is then proposed. This port Hamiltonian descriptor formulation and its balanced realization are used to reduce the implicit port Hamiltonian systems.

In Chapter 3 we consider the passivity and structure preserving reduction of closed loop port Hamiltonian systems. More precisely we are interested in the Hamiltonian structure and passivity preserving low order controller design for port Hamiltonian systems, i.e., a low order feedback such that the closed loop system is still a port Hamiltonian system. This is done from the closed loop point of view motivated by the design of reduced order controllers. Hence Chapter 3 may be read independently of Chapter 2. We then define a passive LQG control problem and we suggest a reduction scheme that preserves the Hamiltonian structure in closed loop. The chapter ends with a discussion on the estimation associated to the reduction method.

Chapter 4 proposes to generalize the aforementioned closed loop reduction method to infinite dimensional port Hamiltonian systems.

The thesis ends with some concluding remarks and perspectives of future work in Chapter 5.

The main contributions of this thesis are:

- **Chapter 2:** We use the descriptor state space framework to reformulate the DAE representation of port Hamiltonian systems, called port Hamiltonian descriptor systems. It shows that the port Hamiltonian descriptor systems conserve a Dirac structure. The balanced realization of descriptor systems is used to choose a suitable coordinate system such that the state variables can be separated from the input-output point of view. The Effort and Flow constraint methods are used to reduce the port Hamiltonian descriptor system such that passivity and structure are preserved. At last, we apply this reduction method to a nanotweezers model.
- **Chapter 3:** The closed loop reduction of finite dimensional port Hamiltonian systems. It is proved that LQG controllers can be passive and have a port Hamiltonian realization if the weighting matrices of the associated LQG control problem are chosen in a certain way. These conditions are characterized for port Hamiltonian systems in two different ways to get two

structure preserving dynamical LQG state feedbacks. These two structure preserving LQG controllers are also equivalent to the control of port Hamiltonian systems by interconnection. One of these LQG control problems can not allow us to define a reduction balanced coordinate since the product of the two solutions of the Riccati equations is equal to the identity. Fortunately the other one allows us to define a balanced realization for the reduction. Then we propose to use the Effort constraint method to reduce the LQG balanced port Hamiltonian system and to derive the reduced order controller.

- **Chapter 4**: The closed loop reduction of infinite dimensional port Hamiltonian system. First, we suggest a passive and structure preserving LQG state feedback control design method by a specific choice of the weighting operators. Next we use this passive LQG problem to define a balanced realization. This balanced realization is used to approximate the infinite dimensional port Hamiltonian system by using a Petrov-Galerkin projection method. At last the low order LQG controller can be obtained by using the approximated system and the passive LQG control design. The methods suggested in this chapter can be regarded as the generalization of the method proposed in Chapter 3 from the finite to the infinite dimensional case.

Chapter 2

Port Hamiltonian systems in descriptor form for balanced reduction

2.1 Introduction

In the last decade, a powerful network modeling frame for the compositional modeling of finite and infinite dimensional physical systems called *port Hamiltonian systems* has been developed for electrical, mechanical, electro-mechanical and hydraulic systems which are based on the principle of energy balance equations [19, 78, 17, 42, 46, 32]. In network models the system is considered as the interconnection of energy storing elements via basic physical interconnection laws (e.g. Newton's third law or Kirchhoff's law) together with energy dissipating elements. Port Hamiltonian systems are *passive* if the total energy (*Hamiltonian*) is bounded from below. However the port Hamiltonian modeling of complex networks or multi-physics systems or stemming from the discretization of infinite dimensional port Hamiltonian system may lead to high-order systems. For the purpose of the control of these systems, there might hence appear the need for *model reduction*. Furthermore, in view of keeping the structural properties of the system such as passivity or the interpretation as a network model for the purpose for instance of applying passivity-based control, one may require that the reduced-order model retain the *passivity and port Hamiltonian structure*.

In recent years, different *passivity preserving reduction methods* have been widely studied by many researchers. The positive balanced methods have been used for the passivity preserving reductions in [2, 3, 69, 25], the interpolation methods have been proposed in [20, 71], and the researcher in [38] have proposed a passivity preserving method which deals with the finite frequency H_∞ performance of the error system. For finite dimensional port Hamiltonian systems, some reduction methods have also been proposed. The moment matching methods have been proposed in [64, 63]. In [65], the author proposed a reduction approach from the geometric point of view. Also a reduction method of port Hamiltonian systems based on interpolation method has been suggested in [26].

However in the network models or multi-physics systems, the physical algebraic constraint equations often appear in the interconnection relations between the sub-systems. This may lead to a system of differential algebraic equations (DAE) representation of port Hamiltonian systems or implicit port Hamiltonian system [16, 19] often associated with a Lagrangian multiplier (corresponding

to a so-called *mixed representation* of Dirac structures). Of course one may eliminate the algebraic constraints, but often this is not advantageous as thereby the sparsity of the system is decreased and the interpretation of the model as interconnection of submodels is lost. Therefore this chapter will treat the model reduction of implicit Port Hamiltonian systems, however preserving their DAE representation thereby conserving expression of the physical constraints in the reduced systems

We shall use here the frame of *descriptor systems* (also called singular systems/generalized state-space systems) theory which is an important part in the general field of control systems theory, and has attracted much attention in the last decades [15, 18]. Descriptor systems appear in many fields, such as electrical networks, mechanical systems, and so on. It is a natural way to present the differential algebraic systems which have been made lots contributions in system analysis, control design as well as model reduction. By using some coordinate transformations, the descriptor systems can be represented in the Weierstrass canonical form which separates the system into a slow response part and a fast one [73, 15]. Particularly in model reduction of descriptor systems, there have been major efforts searching in balanced method by T. Stykel [74, 69]. The descriptor systems theory provides us some useful tools to analyze the DAE representation. Hence, we shall use these tools to reduce the constrained port Hamiltonian systems with preserving the Hamiltonian structure as well as the physical constraints in the reduced systems.

The two main contributions of this chapter are given as: a novel representation of port Hamiltonian system so called *port Hamiltonian descriptor system* and a reduction scheme for this port Hamiltonian descriptor system. This novel descriptor representations is obtained by eliminating the Lagrangian multiplier but still conserving the expression of the physical constraints. This allows us to use the descriptor system theory to analyze the port Hamiltonian systems. Hence we shall suggest a reduction scheme by balanced realization of descriptor system and a geometric reduction method to reduce the port Hamiltonian descriptor systems and preserve the passivity and Hamiltonian structure. A micro mechanical system is used to illustrate the effectiveness of the proposed reduction method.

2.1.1 Organization of the chapter

This chapter is organized as follow. In Section 2.2, we recall the basics of port Hamiltonian systems and Dirac structure which are used in the following sections. Section 2.3 derives a port Hamiltonian descriptor system from the linear constrained port Hamiltonian system and its Weierstrass canonical form is obtained. In Section 2.4 the Lyapunov balanced realization of port Hamiltonian descriptor is defined, furthermore a structure preserving reduction procedure is suggested for this balanced system. In Section 2.5, some numerical simulation results are given by applying the proposed method to a micro mechanical actuator model established in FEMTO-ST. Finally the conclusion of this chapter is given in Section 2.6.

2.1.2 Main contributions of the chapter

The main contributions of this chapter are the following.

- In Section 2.3, a class of linear port Hamiltonian system with constraints are defined by the descriptor system is given, namely port Hamiltonian descriptor system. The linear constrained port Hamiltonian system is derived from the constrained Dirac structure which can represent every Dirac structure with some transformation. We show how to get a descriptor form port

Hamiltonian system and its Weierstrass canonical form from constrained representation with some coordinate transformations which is different from the ones given in [45, 16].

- A geometric reduction scheme is given in Section 2.4 for structure preserving of port Hamiltonian descriptor system defined in last section. Firstly we define the Lyapunov balanced realization of port Hamiltonian descriptor system through the controllability and observability Gramians. Then we use the geometric reduction methods so called Flow and Effort constrained methods to reduce the balanced system as well as preserve the Hamiltonian structure. During the reduction scheme, the constraint of the port Hamiltonian system is always conserved.
- In Section 2.5, we apply all the previously proposed concepts to a complex micro mechanical manipulator, called nanotweezer, and developed at the FEMTO-ST laboratory in Besançon (France). The comparison of Bode diagram between the full order system and reduced order system is given. Furthermore the relative H_∞ errors between the full order system and reduced order systems are computed for different orders.

2.2 Dirac structure and port Hamiltonian systems

In this section we shall recall the definition of the port Hamiltonian system. The port Hamiltonian system can be written as a *coordinate-free* representation through a Dirac structure as in the Figure 2.1 by \mathcal{D} . The definition of *Dirac structure* is given as follows:

Definition 2.1. (Dirac structure [11]) Let \mathcal{F} be a linear space with a dual space $\mathcal{E} := \mathcal{F}^*$, and a duality product denoted as $\langle e | f \rangle \in \mathbb{R}$, with $f \in \mathcal{F}$ and $e \in \mathcal{E}$. In vector notation we simply write the duality product as $\langle e | f \rangle = e^T f$. We call \mathcal{F} the space of *flow* variables, and $\mathcal{E} = \mathcal{F}^*$ the space of *effort* variables. Define on $\mathcal{F} \times \mathcal{E}$ the following indefinite bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg = \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle,$$

A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a constant *Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$, where \mathcal{D}^\perp is the orthogonal complement of \mathcal{D} with respect to the indefinite bilinear form $\ll \cdot | \cdot \gg$.

Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ can be represented in *kernel representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0\} \quad (2.1)$$

for linear maps $F : \mathcal{F} \mapsto \mathcal{V}$ and $E : \mathcal{E} \mapsto \mathcal{V}$ satisfying

$$\begin{aligned} EF^* + FE^* &= 0 \\ \text{rank}(F \mid E) &= \dim \mathcal{F} \end{aligned} \quad (2.2)$$

where \mathcal{V} is a linear space with the same dimension as \mathcal{F} , and where $F^* : \mathcal{V}^* \mapsto \mathcal{F}^*$ and $E^* : \mathcal{V}^* \mapsto \mathcal{F}^{**} = \mathcal{F}$ are the adjoint maps of F and E , respectively.

It follows from (2.2) that \mathcal{D} can be also written in *image representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^* \lambda, e = F^* \lambda, \lambda \in \mathcal{V}^*\} \quad (2.3)$$

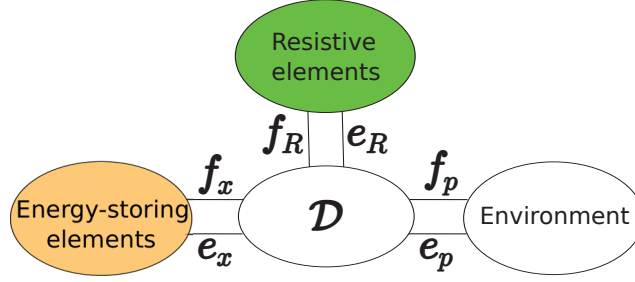


Figure 2.1: Dirac structure

The *port Hamiltonian system* is defined as follows. We start with a Dirac structure on a space involves all the flow and effort variables:

$$\mathcal{D} \subset \mathcal{F}_X \times \mathcal{E}_X \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P.$$

The space $\mathcal{F}_X \times \mathcal{E}_X$ is the space of flow and effort variables corresponding to the energy-storing elements (f_x, e_x) . The space $\mathcal{F}_R \times \mathcal{E}_R$ is defined as the flow and effort variables space of the resistive elements (f_R, e_R) . And the external ports (or sources ports) variables (f_p, e_p) defined on space $\mathcal{F}_P \times \mathcal{E}_P$. Basic property of a Dirac structure is *power-conservation*, it means that the total power associated with the ports variables defined as above is zero, at the ports of the Dirac structure.

The vector of all the flow and effort variables is given as:

$$(f_x, e_x, f_R, e_R, f_p, e_p) \in \mathcal{D} \quad (2.4)$$

with

$$f_x \in \mathcal{F}_X, e_x \in \mathcal{E}_X, f_R \in \mathcal{F}_R, e_R \in \mathcal{E}_R, f_p \in \mathcal{F}_P, e_p \in \mathcal{E}_P.$$

The constitutive relation of the energy-storing elements is defined by the internal ports (f_x, e_x) as follows. We introduce the total energy of the energy-storing elements, the Hamiltonian $H : \mathcal{X} \mapsto \mathbb{R}$ with the state variables vector $x = (x_1, x_2, \dots, x_n)^T$ so called *energy variables*. Hence the Hamiltonian is denoted by $H(x)$. We take the variable space $\mathcal{X} = \mathcal{F}_X$. The flow variables of energy-storing elements are defined as the rate \dot{x} of the energy variables. Furthermore the effort variables of energy-storing elements are given by the co-energy variable $\frac{\partial H}{\partial x}$ as follows:

$$f_x = -\dot{x}, \quad e_x = \frac{\partial H}{\partial x} \quad (2.5)$$

One can imply the balance equation immediately:

$$\frac{d}{dt} H(x) = \left\langle \frac{\partial H}{\partial x} \mid \dot{x} \right\rangle = -e_x^T f_x \quad (2.6)$$

It means that the increase of the total energy $H(x)$ equals to the power $-e_x^T f_x$.

Remark 2.1. In the linear case, the Hamiltonian, total energy $H(x)$ can be represented by the quadratic form:

$$H(x) = \frac{1}{2} x^T Q x \quad (2.7)$$

with the matrix $Q = Q^T > 0$ so called energy matrix.

The constitutive relation of the resistive elements is defined as follows.

$$f_R = -\varphi(e_R) \quad (2.8)$$

with some function φ satisfying

$$e_R^T \varphi(e_R) > 0 \text{ for all } e_R \neq 0 \quad (2.9)$$

Remark 2.2. In the linear case, the resistive elements are given by:

$$f_R = -De_R \text{ with } D = D^T > 0 \quad (2.10)$$

It can be interpreted as the power is always *dissipated* by the resistive elements.

Now we introduce the definition of port Hamiltonian system as follows:

Definition 2.2. (Port Hamiltonian systems) Consider the port variables given in figure 2.1 constrained by the Dirac structure defined in Definition 2.1, the Hamiltonian $H : \mathcal{X} \mapsto \mathbb{R}$ with the constitutive relations of energy storage ports (2.5), and the resistive relation $f_R = -\varphi(e_R)$ as in (2.8). Then the dynamics (2.4) of the resulting port Hamiltonian system is given as

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(t), -\varphi(e_R(t)), e_R(t), f_p(t), e_p(t) \right) \in \mathcal{D} \quad (2.11)$$

From the power-conservation property of Dirac structures, (2.6) and (2.8) that

$$\frac{d}{dt}H(x) = -e_R^T \varphi(e_R) + e_p^T f_p < e_p^T f_p, \quad (2.12)$$

thus showing that the port Hamiltonian system is *passive* if the Hamiltonian H is bounded from below.

An important special case of port Hamiltonian systems as defined above is the class of *input-state-output port Hamiltonian systems*, where there are no algebraic constraints on the state space variables, and the flow and effort variables of resistive, external ports are split into conjugated input-output pairs. Input-state-output port Hamiltonian systems are defined as dynamical systems of the following form

$$\begin{cases} \dot{x} &= J \frac{\partial H}{\partial x} + gu + g_R e_R \\ y &= g^T \frac{\partial H}{\partial x} \\ f_R &= g_R^T \frac{\partial H}{\partial x} \end{cases} \quad (2.13)$$

where (u, y) are the input-output pairs corresponding to the external ports (f_p, e_p) . It should be noted that $y^T u$ equals the power corresponding to the external port. Here $J = -J^T$ is a skew-symmetric matrix. The linear resistive ports (f_R, e_R) respect the resistive relation (2.10), i.e. $f_R = -De_R$ with $D = D^T > 0$ and g_R represents the input matrix corresponding to the resistive port.

2.3 Linear constrained port Hamiltonian system and its descriptor form

In this section we shall consider *constrained* linear port Hamiltonian systems and transform them into descriptor form [15]. We begin by recalling the definition of constrained port Hamiltonian systems. Next we consider the port Hamiltonian systems with constraints which are a particular representation of implicit port Hamiltonian systems defined on Dirac structures [19, 45, 16] which makes explicitly appear constraint equations as well as the associated Lagrangian multipliers. In order to keep the constraint of this type of port Hamiltonian systems, we transform it to the descriptor form so called ***port Hamiltonian descriptor system*** and furthermore to a canonical Weierstrass form. It should be noted that we use a similar coordinate transformation as [45, 16] but without computing the Lagrangian multipliers.

2.3.1 Constrained port Hamiltonian systems

Before introducing the constrained port Hamiltonian systems, we first introduce an alternative representation of Dirac structures called the constrained Dirac structure representation. Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ can be represented as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je + G\lambda, G^T e = 0\}, \quad (2.14)$$

for a skew-symmetric mapping $J : \mathcal{E} \mapsto \mathcal{F}$ and a linear mapping G such that $\text{Im}G = \{f \mid (f, 0) \in \mathcal{D}\}$. Furthermore, $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$. The proof that (2.14) defines a Dirac structure is given in [19].

From the Dirac structure defined in (2.14), any linear port Hamiltonian systems can be represented locally so called *linear constrained port Hamiltonian systems* as follows:

$$\begin{cases} \dot{x} &= JQx + g_c\lambda + gu + g_R e_R \\ 0 &= g_c^T Qx \\ y &= g^T Qx \\ f_R &= g_R^T Qx \end{cases} \quad (2.15)$$

where $x \in \mathbb{R}^n$ is the state vector, $\lambda \in \mathbb{R}^k$ is the vector of Lagrangian multipliers, $H(x) = \frac{1}{2}x^T Qx$ is the Hamiltonian function, $Q \in \mathbb{R}^{n \times n}$ is a positive definite matrix (i.e. $Q = Q^T > 0$) which will be called energy matrix (as an allusion to models of physical systems), $J \in \mathbb{R}^{n \times n}$ is the skew-symmetric Poisson structure matrix, (i.e. $J = -J^T$), the matrices $g_c \in \mathbb{R}^{n \times k}$ and $g \in \mathbb{R}^{n \times m}$ and $g_R \in \mathbb{R}^{n \times m_R}$ describe the input relations of the control ports and the resistive ports respectively. This constrained Port Hamiltonian system will be completed with a linear resistive relation between the port variables (e_R, f_R) such as $e_R = -Df_R$, with $D \in \mathbb{R}^{m_R \times m_R}$ being a symmetric positive matrix (i.e. $D = D^T \geq 0$).

In this way one obtains a dissipative linear Port Hamiltonian system with constraints. Note that the vector $\lambda \in \mathbb{R}^k$ of Lagrangian multipliers is associated with the constraints given by the second equation of (2.15) and that in a mechanical context $g_c\lambda$ may be interpreted as the constraint force associated with constraints in [80].

2.3.2 Elimination of the Lagrangian multipliers

In this section we propose at first to reduce the system by eliminating the Lagrange multipliers. Therefore we define the following coordinate transformation

$$z = \begin{bmatrix} s \\ (g_c^T g_c)^{-1} g_c^T \end{bmatrix} x = Mx$$

where s is a $\mathbb{R}^{(n-k) \times n}$ matrix such that

$$sg_c = 0 \quad \text{and} \quad \text{rank}(s) = n - k$$

Note that this coordinate transformation is inspired from [80]) and differs by the multiplication by $(g_c^T g_c)^{-1}$. If, which is often the case in network models, the matrix $(g_c^T g_c)$ is sparse then the inverse may be computed efficiently. Else, it is preferable to use the transformation in [80]; the results of this paper remain then unchanged when replacing the matrix I_k by the matrix $(g_c^T g_c)$.

In the new coordinates the system (2.15) becomes

$$\begin{cases} \dot{z} &= \bar{J}\bar{Q}z + \bar{g}_c\lambda + \bar{g}u + \bar{g}_R e_R \\ 0 &= \bar{g}_c^T \bar{Q}z \\ y &= \bar{g}^T \bar{Q}z \\ f_R &= \bar{g}_R^T \bar{Q}z \end{cases} \quad (2.16)$$

with: $\bar{J} = MJM^T = -\bar{J}^T$, $\bar{Q} = M^{-T}QM^{-1} > 0$, $\bar{g}_c = Mg_c$, $\bar{g} = Mg$, $\bar{g}_R = Mg_R$. Note that

$$\bar{g}_c = Mg_c = \begin{bmatrix} s \\ (g_c^T g_c)^{-1} g_c^T \end{bmatrix} g_c = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$$

which implies that, decomposing the state vector as follows: $z = [z_1, z_2]^T$, $z_1 \in \mathbb{R}^{n-k}$ and $z_2 \in \mathbb{R}^k$, the system can be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ \bar{J}_{21} & \bar{J}_{22} \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_k \end{bmatrix} \lambda + \begin{bmatrix} \bar{g}_1 \\ \bar{g}_2 \end{bmatrix} u + \begin{bmatrix} \bar{g}_{R1} \\ \bar{g}_{R2} \end{bmatrix} e_R \quad (2.17)$$

$$0 = \begin{bmatrix} 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (2.18)$$

$$y = \begin{bmatrix} \bar{g}_1^T & \bar{g}_2^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (2.19)$$

$$f_R = \begin{bmatrix} \bar{g}_{R1}^T & \bar{g}_{R2}^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (2.20)$$

decomposing the energy matrix into blocks accordingly with the decomposition of the state vector

into vectors of size $(n - k)$ and k

$$\bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \quad (2.21)$$

One may observe by considering the first line of (2.17), that \dot{z}_1 is independent of λ and that the constraint equations (2.18) reduces to

$$\bar{Q}_{21}z_1 + \bar{Q}_{22}z_2 = \frac{\partial \bar{H}}{\partial z_2} = 0 \quad (2.22)$$

with $\bar{H}(z) = \frac{1}{2}z^T \bar{Q}z$.

Since \bar{Q} is positive definite, \bar{Q}_{22} is invertible and then one eliminate the z_2 component of the state vector and obtain an explicit port Hamiltonian system. The second line of (2.17) may be used to compute, if needed, the Lagrangian multipliers, λ can be computed by $\lambda = \dot{z}_2 - \bar{J}_2 \bar{Q}z - \bar{g}_2 u - \bar{g}_{R_2} e_R$ [80].

However we shall not follow this route which might lead to cumbersome calculations and destroying the sparsity of the system's matrices. In the sequel we shall eliminate the Lagrangian multiplier but retain the full state $z \in \mathbb{R}^n$ with the constraint (2.22) and treat it as a descriptor system. The procedure is the same as suggested in [16, p.66] for implicit Hamiltonian systems but detailed for systems with ports and expressed in the coordinates z adapted to the constraints.

2.3.3 Descriptor form of the port Hamiltonian system with constraints

In this section, we shall reformulate the constrained port Hamiltonian system in a generalized state space representation so called descriptor system [73, 15].

2.3.3.1 Descriptor systems and its Weierstrass canonical form

First of all we recall descriptor system in the linear case. Consider a linear time-invariant continuous-time system

$$\begin{aligned} S\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x^0, \\ y(t) &= Cx(t), \end{aligned} \quad (2.23)$$

where $S, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and x^0 is the initial value. The number of state variables n is called the order of system (2.23). If the rank of matrix S is equal to the order of the system, i.e. $\text{rank} S = n$, then the system can be written as a *standard state space system*. Otherwise, i.e. $\text{rank} S = q < n$, the system (2.23) is a descriptor system or generalized state space system or singular system in some references [15, 18].

We assume that the pencil $\lambda S - A$ is regular, i.e. there exist $\lambda \in \mathbb{C}$ such that $\det(\lambda S - A) \neq 0$. In this case $\lambda S - A$ can be reduced to the *Weierstrass canonical form* [73]. There exist nonsingular matrices W and T such that

$$S = W \begin{bmatrix} I_{n-k} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} A_J & 0 \\ 0 & I_k \end{bmatrix} T \quad (2.24)$$

where I_n is the identity matrix of order n , A_J is the Jordan block corresponding to the finite eigenvalues

is a Dirac structure.

Proof. The proof is given in two steps.

First, we have to show that $FE^T + EF^T = 0$ and then that $\text{rank}[F|E] = n + m + m_R$.

From equation (2.26) and the expressions of F and E one can compute:

$$FE^T = \begin{bmatrix} \bar{J}_{11}^T & 0 & \bar{g}_1 & \bar{g}_{R_1} \\ 0 & 0 & 0 & 0 \\ -\bar{g}_1^T & 0 & 0 & 0 \\ -\bar{g}_{R_1}^T & 0 & 0 & 0 \end{bmatrix} \quad (2.28)$$

Since \bar{J} is a skew-symmetric matrix, that is $\bar{J}_{11}^T = -\bar{J}_{11}$, the matrix FE^T is skew-symmetric, hence the condition $FE^T + EF^T = 0$ is verified.

One can define a sub-matrix of $[F|E]$ by the first, third and fourth columns of the matrix F , and the second column of the matrix E . This sub-matrix is of rank $n + m + m_R$, consequently the rank of the matrix $[F|E]$ is $n + m + m_R$. \square

As a conclusion the system (2.25) defines a port Hamiltonian system with respect to the Dirac structure \mathcal{D} in (2.26) and generated by the Hamiltonian function $\bar{H}(z) = \frac{1}{2}z^T\bar{Q}z$. In the sequel we shall call this system a **port Hamiltonian descriptor system**.

Now we shall transform the port Hamiltonian descriptor system (2.25) together with the resistive relation $e_R = -Df_R$, where $D \in \mathbb{R}^{m_R \times m_R}$ is a symmetric positive matrix to the state space descriptor system (2.23). However as only the dynamics \dot{z}_1 is retained in the descriptor formulation, the dissipative relation is restricted to $e_R = -Df_R$. The dissipative matrix \bar{D} can be reformulated as

$$\bar{D} = \begin{bmatrix} \bar{g}_{R_1} \\ 0 \end{bmatrix} D \begin{bmatrix} \bar{g}_{R_1}^T & \bar{g}_{R_2}^T \end{bmatrix} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \\ 0 & 0 \end{bmatrix}.$$

The system (2.25) can be written under its descriptor form:

$$\begin{cases} S\dot{z} &= \bar{E}\bar{Q}z + \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} u \\ y &= \bar{g}^T\bar{Q}z \end{cases} \quad (2.29)$$

with the energy matrix $\bar{Q} \in \mathbb{R}^{n \times n}$ defined in (2.21), $\bar{g} \in \mathbb{R}^{n \times m}$ and

$$\bar{E} = \begin{bmatrix} \bar{J}_{11} - \bar{D}_{11} & \bar{J}_{12} - \bar{D}_{12} \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} = S\bar{g} \quad (2.30)$$

For an easy interpretation as a descriptor system (2.23), we shall also use the following notations

defining input matrix B , the output matrix C as well as the state matrix A

$$\begin{aligned} B &= \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix}; \quad C = \bar{g}^T \bar{Q} = \begin{bmatrix} \bar{g}_1^T & \bar{g}_2^T \end{bmatrix} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \\ A &= \bar{E} \bar{Q} = \begin{bmatrix} \bar{E}_{11} \bar{Q}_{11} + \bar{E}_{12} \bar{Q}_{21} & \bar{E}_{11} \bar{Q}_{12} + \bar{E}_{12} \bar{Q}_{22} \\ & \bar{Q}_{21} & & \bar{Q}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \beta \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \end{aligned}$$

where $\alpha = \bar{E}_{11} \bar{Q}_{11} + \bar{E}_{12} \bar{Q}_{21}$ and $\beta = \bar{E}_{11} \bar{Q}_{12} + \bar{E}_{12} \bar{Q}_{22}$.

2.3.3.3 Canonical form for port Hamiltonian descriptor system

As a complement we shall give the transformation of the descriptor dissipative port Hamiltonian system (2.29) into the canonical Weierstrass form where the state matrix takes a canonical form (2.24) by the following proposition.

Proposition 2.2. *Considering the equivalence transformation defined by a (right) transform corresponding to the change of coordinates $z = R\hat{z}$ and the left transform defined by the matrix L as a (left) multiplier or combination matrix with*

$$L = \begin{bmatrix} I_{n-k} & -\beta \bar{Q}_{22}^{-1} \\ 0 & L_2^{-1} \end{bmatrix}, \quad R = \begin{bmatrix} I_{n-k} & 0 \\ -\bar{Q}_{22}^{-1} \bar{Q}_{21} & L_2^{-T} \end{bmatrix},$$

here L_2 is an invertible triangular matrix corresponding to the Cholesky factorization of \bar{Q}_{22}

$$\bar{Q}_{22} = L_2 L_2^T, \quad (2.31)$$

the port Hamiltonian descriptor system (2.29) is equivalent to the following descriptor Hamiltonian system:

$$\begin{cases} \hat{S} \dot{z} &= \hat{E} \hat{Q} z + \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} u \\ y &= \hat{g}^T \hat{Q} z \end{cases} \quad (2.32)$$

where

$$\hat{S} = S; \quad \hat{E} = \begin{bmatrix} \bar{J}_{11} - \bar{D}_{11} & 0 \\ 0 & I_k \end{bmatrix}; \quad \hat{Q} = \begin{bmatrix} \bar{Q}_s & 0 \\ 0 & I_k \end{bmatrix} \quad (2.33)$$

$\bar{Q}_s = \bar{Q}_{11} - \bar{Q}_{12} \bar{Q}_{22}^{-1} \bar{Q}_{21}$ is the Schur complement of the matrix \bar{Q} and

$$\hat{g}^T = [\bar{g}_1^T \quad \bar{g}_1^T \bar{Q}_{12} L_2^{-T} + \bar{g}_2^T L_2]$$

The descriptor port Hamiltonian system (2.32) is a canonical Weierstrass form of the system (2.29).

Proof. One can see the matrix S in the system (2.29) is already in the diagonal-bloc form and the

matrix A is under the form as

$$A = \bar{E}\bar{Q} = \begin{bmatrix} \bar{E}_{11}\bar{Q}_{11} + \bar{E}_{12}\bar{Q}_{21} & \bar{E}_{11}\bar{Q}_{12} + \bar{E}_{12}\bar{Q}_{22} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix}$$

where $\alpha = \bar{E}_{11}\bar{Q}_{11} + \bar{E}_{12}\bar{Q}_{21}$ and $\beta = \bar{E}_{11}\bar{Q}_{12} + \bar{E}_{12}\bar{Q}_{22}$. To obtain the Weierstrass canonical form (2.32), we carry out the following two steps.

Firstly, we shall transform the matrix A to the diagonal-bloc form. To get this, we can use the Schur complement. Recall that the matrix \bar{Q}_{22} is invertible, we introduce two matrix:

$$\bar{L} = \begin{bmatrix} I_{n-k} & -\beta\bar{Q}_{22}^{-1} \\ 0 & I_k \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} I_{n-k} & 0 \\ -\bar{Q}_{22}^{-1}\bar{Q}_{21} & I_k \end{bmatrix}$$

such that the matrix

$$\tilde{A} = \bar{L}A\bar{R} = \begin{bmatrix} \alpha - \beta\bar{Q}_{22}^{-1}\bar{Q}_{21} & 0 \\ 0 & \bar{Q}_{22} \end{bmatrix}$$

is in the diagonal-bloc form.

Secondly, we shall transform the sub-matrix \bar{Q}_{22} of matrix \tilde{A} to an identity matrix without touching the other parts of matrix \tilde{A} . Because the sub-matrix \bar{Q}_{22} is symmetric positive definite, thus there exists a Cholesky factorization:

$$\bar{Q}_{22} = \tilde{L}_2\tilde{L}_2^T$$

where $\tilde{L}_2, \tilde{L}_2^T$ are the triangular invertible matrices and their diagonal elements are the eigenvalues of the matrix \bar{Q}_{22} , then we can get:

$$\tilde{L}_2^{-1}\bar{Q}_{22}\tilde{L}_2^{-T} = I_k.$$

So we can define two matrices as:

$$\tilde{L} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & \tilde{L}_2^{-1} \end{bmatrix} \quad \text{and} \quad \tilde{R} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & \tilde{L}_2^{-T} \end{bmatrix}$$

such that:

$$\hat{A} = \tilde{L}\tilde{A}\tilde{R} = \begin{bmatrix} \alpha - \beta\bar{Q}_{22}^{-1}\bar{Q}_{21} & 0 \\ 0 & I_k \end{bmatrix}.$$

After these two steps, we can get

$$\check{A} = \tilde{L}\tilde{A}\tilde{R} = \tilde{L}\tilde{L}A\tilde{R}\tilde{R}$$

Then we have:

$$L = \tilde{L}\bar{L} = \begin{bmatrix} I_{n-k} & -\beta\bar{Q}_{22}^{-1} \\ 0 & \tilde{L}_2^{-1} \end{bmatrix} \quad \text{and} \quad R = \bar{R}\tilde{R} = \begin{bmatrix} I_{n-k} & 0 \\ -\bar{Q}_{22}^{-1}\bar{Q}_{21} & \tilde{L}_2^{-T} \end{bmatrix}$$

Now we consider a change of state variables $z = R\hat{z}$ and premultiply L at the two sides of the

system (2.29}), the system becomes:

$$\begin{cases} \hat{S}\dot{\hat{z}} &= \hat{A}\hat{z} + \hat{B}u \\ y &= \hat{C}\hat{z} \end{cases} \quad (2.34)$$

where $\hat{S} = LSR$, $\hat{A} = LAR$, $\hat{B} = LB$, $\hat{C} = CR$, $\hat{E} = L\bar{E}R^{-T}$, $\hat{Q} = R^T\bar{Q}R$. We have

$$\hat{S} = LSR = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \bar{E}_{11}(\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{21}) & 0 \\ 0 & I_k \end{bmatrix},$$

$$\hat{B} = LB = \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \bar{g}_1^T(\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{21}) & \bar{g}_1^T\bar{Q}_{12}\tilde{L}_2^{-T} + \bar{g}_2^T\tilde{L}_2 \end{bmatrix}.$$

Finally the port Hamiltonian descriptor system (2.29) can be written in the form as

$$\begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} \end{cases} \quad (2.35)$$

with the following notations

$$\begin{aligned} \hat{A}_1 &= \bar{J}_{11}(\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{21}); & \hat{B}_1 &= \bar{g}_1; \\ \hat{C}_1 &= \bar{g}_1^T(\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{21}); & \hat{C}_2 &= \bar{g}_1^T\bar{Q}_{12}\tilde{L}_2^{-T} + \bar{g}_2^T\tilde{L}_2. \end{aligned} \quad (2.36)$$

We can say this is the Weierstrass canonical form of the port Hamiltonian descriptor system (2.29). \square

Remark 2.3. By the proof of the proposition 2.2, we can see that the pencil $\lambda S - A$ is regular if the sub-matrix \bar{Q}_{22} is invertible.

Remark 2.4. In the Weierstrass canonical form of the port Hamiltonian descriptor system (2.32), the linear constrained port Hamiltonian system is separated to the *continuous and impulse subsystems* corresponding [15] to the state variables \hat{z}_1 and \hat{z}_2 respectively.

2.4 Geometric model reduction of dissipative descriptor PHS

In this section we suggest a procedure for a structure preserving reduction of the constrained port Hamiltonian system (2.15) using the procedure to compute a balanced realization of descriptor systems suggested in [74] and then instead of reduction by truncation, adapt the flow constraint method suggested in [65] for the effective calculation of a reduced port Hamiltonian system in descriptor form (2.25).

2.4.1 Controllability and observability

Before starting the reduction procedure, we shall recall the controllability and observability of descriptor systems [15, 74].

Definition 2.3. Descriptor system (2.23) with the triplet (S, A, B) are called *controllable on the reachable set (R-controllable)* if

$$\text{rank} [\lambda S - A, B] = n, \quad \text{for all finite } \lambda \in \mathbb{C}. \quad (2.37)$$

Descriptor system (2.23) with the triplet (S, A, B) are called *impulse controllable (I-controllable)* if

$$\text{rank} [S, AK_S, B] = n, \quad \text{where the columns of } K_S \text{ spanker } S. \quad (2.38)$$

Descriptor system (2.23) with the triplet (S, A, B) are called *completely controllable (C-controllable)* if (2.37) holds and

$$\text{rank} [S, B] = n. \quad (2.39)$$

Observability is a dual property of controllability.

Definition 2.4. Descriptor system (2.23) with the triplet (S, A, B) are called *observability on the reachable set (R-observability)* if

$$\text{rank} \begin{bmatrix} \lambda S - A \\ C \end{bmatrix} = n, \quad \text{for all finite } \lambda \in \mathbb{C}. \quad (2.40)$$

Descriptor system (2.23) with the triplet (S, A, B) are called *impulse observability (I-observability)* if

$$\text{rank} \begin{bmatrix} S \\ K_{S^T}^T A \\ C \end{bmatrix} = n, \quad \text{where the columns of } K_{S^T} \text{ spanker } S^T. \quad (2.41)$$

Descriptor system (2.23) with the triplet (S, A, B) are called *completely observability (C-observability)* if (2.40) holds and

$$\text{rank} \begin{bmatrix} S \\ C \end{bmatrix} = n. \quad (2.42)$$

Let us now characterize the observability and controllability properties of the system according to the Definition 2.3 and 2.4. Observe by inspection of the Weierstrass canonical form (2.32) of port Hamiltonian descriptor system, that it is not Completely controllable as

$$\text{rank} \left[\hat{S}, \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} \right] = n - k < n.$$

Next we can compute that $K = \ker \hat{S} = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ and the system (2.32) is Impulse controllable since

$$\text{rank} \left[\hat{S}, \hat{A} \begin{bmatrix} 0 \\ I_k \end{bmatrix}, \begin{bmatrix} \hat{g}_1 \\ 0 \end{bmatrix} \right] = n.$$

It is R-controllable if and only if

$$\text{rank} [\lambda I_{n-k} - (\bar{J}_{11} - \bar{D}_{11}) \bar{Q}_s, \hat{g}_1 \bar{Q}_s] = n.$$

In the sequel, for the sake of simplicity we shall assume that the system (2.32) is R-controllable.

Concerning the observability, it is seen immediately that the system is Impulse observable as

$$\text{rank} \begin{bmatrix} \hat{S} \\ K'^T \hat{A} \\ \hat{C} \end{bmatrix} = n$$

by choosing the left kernel of \hat{S} as $K' = \begin{bmatrix} 0 & I_k \end{bmatrix}$.

The Complete observability condition $\text{rank} \begin{bmatrix} \hat{S} \\ \hat{C} \end{bmatrix} = n$ is equivalent with $\text{rank} (\bar{g}_1^T \bar{Q}_{12} L_2^{-T} + \bar{g}_2^T L_2) =$

k . The R-observability condition

$$\text{rank} \begin{bmatrix} \lambda \hat{S} - \hat{A} \\ \hat{C} \end{bmatrix} = n$$

reduces to the same condition as R-controllability.

Note that as by hypothesis the matrix \bar{Q}_s is positive definite, then the assumption of controllability of the proper subsystem implies also the observability using the port conjugated variable defined by the output matrix $C_c = \bar{g}_1^T \bar{Q}_s$ [81], hence the system is also R-observable and R-minimal.

2.4.2 Balanced realization

In this part, we shall discuss the balanced realization of the port Hamiltonian descriptor system. Because the system (2.33) has already been reformulated by the Weierstrass canonical form, in the sequel we consider only the *impulse part* of this system corresponding to state variables \hat{z}_1 . Thus we introduce the *proper controllability and observability Gramians* as the following form,

$$\mathcal{G}_{pc} = \begin{bmatrix} G_{1c} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{G}_{po} = \begin{bmatrix} G_{1o} & 0 \\ 0 & 0 \end{bmatrix} \quad (2.43)$$

where $G_{1c} \in \mathbb{R}^{(n-k) \times (n-k)}$ and $G_{1o} \in \mathbb{R}^{(n-k) \times (n-k)}$ satisfy the standard continuous time Lyapunov equations:

$$\begin{aligned} \hat{A}_1 G_{1c} + G_{1c} \hat{A}_1^T &= -\hat{B}_1 \hat{B}_1^T \\ \hat{A}_1^T G_{1o} + G_{1o} \hat{A}_1 &= -\hat{C}_1^T \hat{C}_1 \end{aligned} \quad (2.44)$$

Because $\hat{A}_1 = (\bar{J}_{11} - \bar{D}_{11}) (\bar{Q}_{11} - \bar{Q}_{12} \bar{Q}_{22}^{-1} \bar{Q}_{21})$, $\hat{B}_1 = \hat{g}_1$ and $\hat{C}_1 = \hat{g}_1^T \bar{Q}_s$, and by taking the notations of equation (2.32), the equations become:

$$\begin{aligned} \bar{E}_{11} \bar{Q}_s G_{1c} + G_{1c} \bar{Q}_s^T \bar{E}_{11}^T &= -\hat{g}_1 \hat{g}_1^T \\ \bar{Q}_s^T \bar{E}_{11}^T G_{1o} + G_{1o} \bar{E}_{11} \bar{Q}_s &= -\bar{Q}_s^T \hat{g}_1 \hat{g}_1^T \bar{Q}_s \end{aligned} \quad (2.45)$$

Remark 2.5. In the following results, we consider only the *continuous part* of this system, thus proper controllability and observability Gramians are called as *controllability and observability Gramians* for short.

Generally speaking, the product of controllability and observability Gramians $\mathcal{G}_{pc} \mathcal{G}_{po}$ is not invariant. However from [74], the matrix product $\mathcal{G}_{pc} S^T \mathcal{G}_{po} S$ is coordinate invariant. As the matrix S in the system (2.33) has already been reformulated by the Weierstrass canonical form, i.e.

$$S = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix}, \quad (2.46)$$

hence we define the Hankel singular values of (2.33) as follows

Definition 2.5. The square roots of the eigenvalues of the matrix $G_{1c}G_{1o}$ denoted by ς_j are called the Hankel singular values of port Hamiltonian descriptor system (2.33), and the Hankel singular values are ordered decreasingly, i.e.

$$\varsigma_i = \sqrt{\lambda_i(G_{1c}G_{1o})}, \text{ with } \varsigma_1 \geq \varsigma_2 \geq \dots \geq \varsigma_{n-k} \geq 0 \quad (2.47)$$

For the reduction objective, we shall introduce the balanced realization of port Hamiltonian descriptor system (2.33):

Definition 2.6. The descriptor system is balanced if and only if:

$$G_{1c} = G_{1o} = \Sigma \quad \text{with} \quad \Sigma = \text{diag}(\varsigma_1, \varsigma_2, \dots, \varsigma_{n-k}) \quad (2.48)$$

where G_{1c} and G_{1o} are computed by Lyapunov equations (2.45), ς_i are the Hankel singular values.

Because the system (2.32) is R-controllable and R-observable, thus G_{1c} and G_{1o} are symmetric and positive definite, and there exists the following Cholesky factorization:

$$G_{1c} = R_p R_p^T, \quad G_{1o} = L_p^T L_p \quad (2.49)$$

where the matrices $R_p^T, L_p \in \mathbb{R}^{(n-k) \times (n-k)}$ are upper triangular Cholesky factors.

Let

$$L_p R_p = U_p \Sigma V_p^T, \quad (2.50)$$

be singular value decomposition of $L_p R_p$, where U_p, V_p are orthogonal ($V_p^T U_p = I_{(n-k)}$). Consider the matrices :

$$W_b = \begin{bmatrix} L_p^T U_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} W_{b1} & 0 \\ 0 & I_k \end{bmatrix} \quad (2.51)$$

and

$$T_b = \begin{bmatrix} R_p V_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} T_{b1} & 0 \\ 0 & I_k \end{bmatrix} \quad (2.52)$$

Then $(T_b)^T W_b = I$, and we can transform the coordinate by

$$\hat{z} = T_b x_b, \quad (2.53)$$

and premultiply by matrix W_b on the two sides of the system, we obtain a balanced system from the system (2.32) with these new matrices: $\bar{E}_b = W_b^T \hat{E} T_b^{-T} = W_b^T \hat{E} W_b$, $Q_b = T_b^T \hat{Q} T_b > 0$ and $g_b^T = \hat{g}^T T_b^{-T} = \hat{g}^T W_b$.

Then the balanced system can also be written as the same port-Hamiltonian descriptor with (2.32) and (2.33) in which the sub-matrices of the balanced system are $J_b = W_{b1}^T \bar{J}_{11} W_{b1} = -J_b^T$, $D_b = W_{b1}^T \bar{D}_{11} W_{b1} = D_b^T \geq 0$, $Q_{b1} = T_{b1}^T \hat{Q} T_{b1} > 0$. The balanced system can be written as follows:

$$\begin{cases} S_b \dot{x}_b &= \bar{E}_b Q_b x_b + B_b u \\ y &= g_b^T Q_b x_b \end{cases} \quad (2.54)$$

The matrices of system (2.54) are given by

$$\begin{aligned} S_b &= W_b^T \hat{S} T_b = \begin{bmatrix} \Sigma^{-1/2} U_p^T L_p & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_p V_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \\ \bar{E}_b &= W_b^T \hat{E} W_b = \begin{bmatrix} \Sigma^{-1/2} U_p^T L_p & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{E}_{11} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} L_p^T U_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} \bar{E}_b & 0 \\ 0 & I_k \end{bmatrix} \\ Q_b &= T_b^T \hat{Q} T_b = \begin{bmatrix} \Sigma^{-1/2} V_p^T R_p^T & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{Q}_s & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} R_p V_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} Q_{b_1} & 0 \\ 0 & I_k \end{bmatrix} \\ B_b &= W_b^T \hat{b} = \begin{bmatrix} \Sigma^{-1/2} U_p^T L_p & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} \bar{g}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} g_{b_1} \\ 0 \end{bmatrix} \\ g_b^T &= \hat{g}^T T_b^{-T} = \hat{g}^T W_b = \begin{bmatrix} \bar{g}_1^T & \bar{g}_1^T \bar{Q}_{12} \bar{L}_2^{-T} + \bar{g}_2^T \bar{L}_2 \end{bmatrix} \begin{bmatrix} L_p^T U_p \Sigma^{-1/2} & 0 \\ 0 & I_k \end{bmatrix} = \begin{bmatrix} g_{b_1}^T & g_{b_2}^T \end{bmatrix} \end{aligned}$$

Then the balanced system can be written as:

$$\begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_{b_1} \\ \dot{x}_{b_2} \end{bmatrix} = \begin{bmatrix} J_b - D_b & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} Q_{b_1} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} + \begin{bmatrix} g_{b_1} \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} g_{b_1}^T & g_{b_2}^T \end{bmatrix} \begin{bmatrix} Q_{b_1} & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} \end{cases} \quad (2.55)$$

By taking $f_b = -\dot{x}_b$, $e_b = Q_b x_b$, $f_p = y$, $e_p = u$, $f_R = -De_R$ the balanced system can be reformulated as the following structure:

$$\underbrace{\begin{bmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}}_{F_b} \begin{bmatrix} f_{x_{b_1}} \\ f_{x_{b_2}} \\ f_p \\ f_R \end{bmatrix} + \underbrace{\begin{bmatrix} J_b & 0 & g_{b_1} & g_{R_{b_1}} \\ 0 & I_k & 0 & 0 \\ g_{b_1}^T & g_{b_2}^T & 0 & 0 \\ g_{R_b}^T & 0 & 0 & 0 \end{bmatrix}}_{E_b} \begin{bmatrix} e_{x_{b_1}} \\ e_{x_{b_2}} \\ e_p \\ e_R \end{bmatrix} = 0 \quad (2.56)$$

with $F_c, E_c \in \mathbb{R}^{(n+m) \times (n+m)}$.

Proposition 2.3. Define the flow variables $f_b^T := (f_{x_b}, f_p, f_R)^T$ and the effort variables $e_b^T := (e_{x_b}, e_p, e_R)^T$ in the bond space $\mathcal{F} \times \mathcal{E} = \mathbb{R}^{(n+m+m_R) \times (n+m+m_R)} \times \mathbb{R}^{(n+m+m_R) \times (n+m+m_R)}$, the structure matrices:

$$F_b = \begin{bmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}, \quad E_b = \begin{bmatrix} J_b & 0 & g_{b_1} & g_{R_b} \\ 0 & I_k & 0 & 0 \\ g_{b_1}^T & g_{b_2}^T & 0 & 0 \\ g_{R_b}^T & 0 & 0 & 0 \end{bmatrix} \quad (2.57)$$

then the subspace of $\mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \{f_b \in \mathcal{F}, e_b \in \mathcal{E} | F_b f_b + E_b e_b = 0\} \quad (2.58)$$

is a Dirac structure.

Proof. The proof is given in two steps.

First, we have to show that $F_b E_b^T + E_b F_b^T = 0$ and then that $\text{rank}[F_b | E_b] = n + m + m_R$.

From equation (2.26) and the expressions of F_b and E_b one can compute:

$$F_b E_b^T = \begin{bmatrix} J_b^T & 0 & g_{b_1}^T & g_{R_b}^T \\ 0 & 0 & 0 & 0 \\ -g_{b_1} & 0 & 0 & 0 \\ -g_{R_b} & 0 & 0 & 0 \end{bmatrix} \quad (2.59)$$

Since \bar{J} is a skew-symmetric matrix, that is $J_b^T = -J_b$, the matrix $F_b E_b^T$ is skew-symmetric, hence the condition $F_b E_b^T + E_b F_b^T = 0$ is verified.

One can define a sub-matrix of $[F_b | E_b]$ by the first, third and fourth columns of the matrix F_b , and the second column of the matrix E_b . This sub-matrix is

$$\begin{bmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & g_{b_2}^T & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}$$

with rank $n + m + m_R$, consequently the rank of the matrix $[F_b | E_b]$ is $n + m + m_R$. \square

2.4.3 Reduction by the flow and effort constraints methods

In this section we shall reduce the initial system of order n to a system of order $r < n$ by using the flow constraint method proposed by [65] instead of the truncation method proposed by [74]. Using the flow constraint method, ensures the conservation of the port Hamiltonian structure of the reduced system.

The idea of the reduction by flow and effort constraint is to “cut” the interconnection

$$\dot{x}_{b2} = -f_{xb2}, \quad \frac{\partial H}{\partial x_{b2}} = e_{xb2} \quad (2.60)$$

between the energy storage corresponding to the x_{b2} and the Dirac structure, in such a way that no energy is transferred. Hence the exchange of energy between the energy storage and the other system elements through the Dirac structure happens only via port associated with x_{b1} , x_{b1} being the reduced order state vector as shown in Figure (2.2).

The energy flow through the interconnection (2.60) is set equal to zero by making both products

$$\left(\frac{\partial H}{\partial x_{b2}} \right)^T \dot{x}_{b2} = - (e_{xb2})^T f_{xb2} = 0 \quad (2.61)$$

This can be done in the two following canonical way:

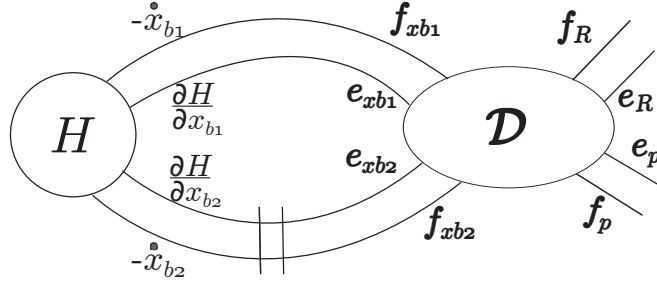


Figure 2.2: Flow/effort constraints model reduction scheme

- Flow variables constraint:

$$\dot{x}_{xb2} = 0, \quad \text{and} \quad f_{xb2} = 0 \quad (2.62)$$

- Effort variables constraint:

$$\frac{\partial H}{\partial x_{b2}} = 0, \quad \text{and} \quad e_{xb2} = 0 \quad (2.63)$$

We recall the Dirac structure associated with the balanced descriptor Hamiltonian system is given in the last section (2.56):

$$\begin{bmatrix} I_r & 0 & 0 & 0 & 0 \\ 0 & I_{n-k-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & 0 & -I_{m_R} \end{bmatrix} \begin{bmatrix} f_{xb_{11}} \\ f_{xb_{12}} \\ f_{xb_2} \\ f_p \\ f_R \end{bmatrix} + \begin{bmatrix} J_{b_{11}} & J_{b_{12}} & 0 & g_{b_{11}} & g_{R_{b_1}} \\ J_{b_{21}} & J_{b_{22}} & 0 & g_{b_{12}} & g_{R_{b_2}} \\ 0 & 0 & I_k & 0 & 0 \\ g_{b_{11}}^T & g_{b_{12}}^T & g_{b_2}^T & 0 & 0 \\ g_{R_{b_1}}^T & g_{R_{b_2}}^T & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{xb_{11}} \\ e_{xb_{12}} \\ e_{xb_2} \\ e_p \\ e_R \end{bmatrix} = 0 \quad (2.64)$$

where $f_{x_b} = \begin{bmatrix} f_{xb_{11}} \\ f_{xb_{12}} \\ f_{xb_2} \end{bmatrix} = \dot{x}_b$ and $e_{x_b} = \begin{bmatrix} e_{xb_{11}} \\ e_{xb_{12}} \\ e_{xb_2} \end{bmatrix} = Q_b x_b$. The variables we want to reduce are $f_{xb_{12}}$ and $e_{xb_{12}}$.

In the following, we shall use the flow constraint method to reduce the port Hamiltonian descriptor systems.

2.4.3.1 Flow constraint method for port Hamiltonian descriptor systems

In order to apply the flow constraint method of [65], we impose a constraint on the flow variables as follows:

$$f_{xb_{12}} = 0$$

Then

$$\begin{bmatrix} f_{xb_{11}} \\ 0 \\ f_{xb_2} \\ f_p \\ f_R \end{bmatrix} = M_f \begin{bmatrix} f_{xb_{11}} \\ f_{xb_2} \\ f_p \\ f_R \end{bmatrix} \quad (2.65)$$

where $M_f \in \mathbb{R}^{n+m+m_R \times r+k+m+m_R}$ is defined as follows:

$$M_f = \begin{bmatrix} I_r & 0_{r \times k} & 0_{r \times m} & 0_{r \times m_R} \\ 0_{n-k-r \times r} & 0_{n-k-r \times k} & 0_{n-k-r \times m} & 0_{n-k-r \times m_R} \\ 0_{k \times r} & I_k & 0_{k \times m} & 0_{k \times m_R} \\ 0_{m \times r} & 0_{m \times k} & I_m & 0_{m \times m_R} \\ 0_{m_R \times r} & 0_{m_R \times k} & 0_{m_R \times m} & I_{m_R} \end{bmatrix} \quad (2.66)$$

Let define a matrix $L^f \in \mathbb{R}^{(r+k+m+m_R) \times (n+m+m_R)}$ such that:

$$L^f \begin{bmatrix} J_{b_{12}} \\ J_{b_{22}} \\ 0 \\ g_{b_{12}}^T \\ g_{R_{b_2}}^T \end{bmatrix} = 0 \quad \text{with} \quad \text{rank}(L^f) = r + k + m + m_R$$

Then if $J_{b_{22}}$ is invertible, one can define:

$$L^f = \begin{bmatrix} I_r & -J_{b_{12}} J_{b_{22}}^{-1} & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 & 0 \\ 0 & -g_{b_{12}}^T J_{b_{22}}^{-1} & 0 & I_m & 0 \\ 0 & -g_{R_{b_2}}^T J_{b_{22}}^{-1} & 0 & 0 & I_{m_R} \end{bmatrix}$$

By premultiplying (2.64) by L^f and considering equation (2.65), one can define the reduced matrices F_r and E_r :

$$\begin{aligned} F_r &= L^f F M_f \\ &= \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix} \end{aligned} \quad (2.67)$$

$$\begin{aligned} E_r &= L^f E M_f \\ &= \begin{bmatrix} J_{b_s} & 0 & \alpha & \beta \\ 0 & I_k & 0 & 0 \\ \alpha^T & g_{b_2}^T & \gamma & \mu \\ \beta^T & 0 & -\mu^T & \zeta \end{bmatrix} \end{aligned} \quad (2.68)$$

where :

$$\begin{aligned} J_{b_s} &= J_{b_{11}} - J_{b_{12}} J_{b_{22}}^{-1} J_{b_{21}} & \alpha &= g_{b_{11}} - J_{b_{12}} J_{b_{22}}^{-1} g_{b_{12}} \\ \beta &= g_{R_{b_1}} - J_{b_{12}} J_{b_{22}}^{-1} g_{R_{b_2}} & \gamma &= -g_{b_{12}}^T J_{b_{22}}^{-1} g_{b_{12}} \\ \mu &= -g_{b_{12}}^T J_{b_{22}}^{-1} g_{R_{b_2}} & \zeta &= -g_{R_{b_2}}^T J_{b_{22}}^{-1} g_{R_{b_2}} \end{aligned} \quad (2.69)$$

$F_r, E_r \in \mathbb{R}^{r+k+m \times r+k+m}$ are square matrices where $\gamma = -\gamma^T$ and $\zeta = -\zeta^T$.

These reduced matrices define a reduced Dirac structure as stated in the following proposition.

Proposition 2.4. *Define the flow variables*

$$f_r^T := (f_{x_{b_{11}}}, f_{x_{b_2}}, f_p, f_R)^T$$

and the effort variables

$$e_r^T := (e_{x_{b_{11}}}, e_{x_{b_2}}, e_p, e_R)^T$$

in the bond space $\mathcal{F} \times \mathcal{E} = \mathbb{R}^{N_r} \times \mathbb{R}^{N_r}$ with $N_r = (r + k + m + m_R)$, and the structure matrices F_r and E_r defined in (2.67) and (2.68) respectively. The linear subspace \mathcal{D} of $\mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \{f_r \in \mathcal{F}, e_r \in \mathcal{E} | F_r f_r + E_r e_r = 0\} \quad (2.70)$$

is a Dirac structure.

Proof. To prove this proposition, we consider the following two conditions:

Firstly, we prove that $F_r E_r^T + E_r F_r^T = 0$:

$$\begin{aligned} F_r E_r^T &= \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix} \begin{bmatrix} J_{b_s}^T & 0 & \alpha & \beta \\ 0 & I_k & g_{b_2} & 0 \\ \alpha^T & 0 & \gamma^T & -\mu \\ \beta^T & 0 & \mu^T & \zeta^T \end{bmatrix} \\ &= \begin{bmatrix} J_{b_s}^T & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 \\ -\alpha^T & 0 & -\gamma^T & \mu \\ -\beta^T & 0 & -\mu^T & -\zeta^T \end{bmatrix} \end{aligned} \quad (2.71)$$

Since J_{b_s} , γ and ζ are skew-symmetric, the matrix $F_r E_r^T$ is skew-symmetric, then the condition $F_r E_r^T + E_r F_r^T = 0$ is verified.

Second we prove that $\text{rank}[F_r | E_r] = r + k + m + m_R$: A sub-matrix of $[F_r | E_r]$ is defined by the first, third and fourth columns of the matrix F_r , and the second column of the matrix E_r , so this sub-matrix is:

$$\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_{m_R} & 0 \end{bmatrix}$$

This sub-matrix is of rank $r + k + m + m_R$, consequently the rank of the matrix $[F_r | E_r]$ is $r + k + m + m_R$. The proof is similar to the one of Proposition (2.1).

Hence the proposition 2.4 is proved. \square

To obtain the Hamiltonian of the reduced system, we use the relation $f_b = -\dot{x}_b$, $e_b = Q_b x_b$, i.e.

$$\begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_{12}}} \\ e_{x_{b_2}} \end{bmatrix} = \begin{bmatrix} Q_{b_{11}} & Q_{b_{12}} & 0 \\ Q_{b_{21}} & Q_{b_{22}} & 0 \\ 0 & 0 & I_k \end{bmatrix} \begin{bmatrix} x_{b_{11}} \\ x_{b_{12}} \\ x_{b_2} \end{bmatrix} \quad (2.72)$$

Since the flow variables $f_{x_{b_{12}}}$ are constrained, i.e. $\dot{x}_{b_{12}} = 0$ it implies $x_{b_{12}} = \text{constant}$. By choosing

$x_{b_{12}} = 0$ one can get the effort variables $e_{x_{b_{11}}} = Q_{b_{11}} x_{b_{11}}$, $e_{x_{b_2}} = x_{b_2}$. The reduced energy matrix Q_r is then given by:

$$Q_r = \begin{bmatrix} Q_{b_{11}} & 0 \\ 0 & I_k \end{bmatrix} \quad (2.73)$$

and the Hamiltonian of the reduced system is $H_r = \frac{1}{2} x_r^T Q_r x_r$. One can formulate the reduced Dirac structure in an explicit descriptor Hamiltonian system.

2.4.3.2 Effort constraint method for port Hamiltonian descriptor systems

We impose a constraint on the effort variables as follows:

$$e_{x_{b_{12}}} = 0$$

Then

$$\begin{bmatrix} e_{x_{b_{11}}} \\ 0 \\ e_{x_{b_2}} \\ e_p \\ e_R \end{bmatrix} = M_e \begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_2}} \\ e_p \\ e_R \end{bmatrix} \quad (2.74)$$

where $M_e \in \mathbb{R}^{n+m+m_R \times r+k+m+m_R}$ is defined as follows:

$$M_e = \begin{bmatrix} I_r & 0_{r \times k} & 0_{r \times m} & 0_{r \times m_R} \\ 0_{n-k-r \times r} & 0_{n-k-r \times k} & 0_{n-k-r \times m} & 0_{n-k-r \times m_R} \\ 0_{k \times r} & I_k & 0_{k \times m} & 0_{k \times m_R} \\ 0_{m \times r} & 0_{m \times k} & I_m & 0_{m \times m_R} \\ 0_{m_R \times r} & 0_{m_R \times k} & 0_{m_R \times m} & I_{m_R} \end{bmatrix} \quad (2.75)$$

Let us define a matrix $L^e \in \mathbb{R}^{(r+k+m+m_R) \times (n+m+m_R)}$ such that:

$$L^e \begin{bmatrix} 0 \\ I_{n-k-r} \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \quad \text{with} \quad \text{rank}(L^e) = r + k + m + m_R.$$

We can define the simplest L^e as:

$$L^e = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & I_{m_R} \end{bmatrix} \quad (2.76)$$

By premultiplying (2.64) by L^e and considering equation (2.74), one can define the reduced ma-

trices F_r^e and E_r^e :

$$\begin{aligned} F_r^e &= L^e F M_e \\ &= \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix} \end{aligned} \quad (2.77)$$

$$\begin{aligned} E_r^e &= L^e E M_e \\ &= \begin{bmatrix} J_{b_{11}} & 0 & g_{b_{11}} & g_{R_{b_1}} \\ 0 & I_k & 0 & 0 \\ g_{b_{11}}^T & g_{b_2}^T & 0 & 0 \\ g_{R_{b_1}}^T & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (2.78)$$

$F_r^e, E_r^e \in \mathbb{R}^{r+k+m \times r+k+m}$ are square matrices.

These reduced matrices define a reduced Dirac structure as stated in the following proposition.

Proposition 2.5. *Define the flow variables*

$$f_r^T := (f_{x_{b_{11}}}, f_{x_{b_2}}, f_p, f_R)^T$$

and the effort variables

$$e_r^T := (e_{x_{b_{11}}}, e_{x_{b_2}}, e_p, e_R)^T$$

in the bond space $\mathcal{F} \times \mathcal{E} = \mathbb{R}^{N_r} \times \mathbb{R}^{N_r}$ with $N_r = (r + k + m + m_R)$, and the structure matrices F_r^e and E_r^e defined in (2.77) and (2.78) respectively. The linear subspace \mathcal{D} of $\mathcal{F} \times \mathcal{E}$ defined by:

$$\mathcal{D} = \{f_r^e \in \mathcal{F}, e_r^e \in \mathcal{E} | F_r^e f_r + E_r e_r = 0\} \quad (2.79)$$

is a Dirac structure.

Proof. To prove this proposition, we consider the following two conditions:

Firstly, we prove that $F_r^e E_r^{eT} + E_r^e F_r^{eT} = 0$:

$$\begin{aligned} F_r^e E_r^{eT} &= \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix} \begin{bmatrix} J_{b_{11}}^T & 0 & g_{b_{11}} & g_{R_{b_1}} \\ 0 & I_k & g_{b_2} & 0 \\ g_{b_{11}}^T & 0 & 0 & 0 \\ g_{R_{b_1}}^T & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} J_{b_{11}}^T & 0 & g_{b_{11}} & g_{R_{b_1}} \\ 0 & 0 & 0 & 0 \\ -g_{b_{11}}^T & 0 & 0 & 0 \\ -g_{R_{b_1}}^T & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (2.80)$$

Since $J_{b_{11}}$ is skew-symmetric, the matrix $F_r^e E_r^{eT}$ is skew-symmetric, then the condition $F_r^e E_r^{eT} + E_r^e F_r^{eT} = 0$ is verified.

Second we prove that $\text{rank}[F_r^e | E_r^e] = r + k + m + m_R$: A sub-matrix of $[F_r^e | E_r^e]$ is defined by the first, third and fourth columns of the matrix F_r^e , and the second column of the matrix E_r^e , so this

sub-matrix is:

$$\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_{m_R} & 0 \end{bmatrix}$$

This sub-matrix is of rank $r + k + m + m_R$, consequently the rank of the matrix $[F_r^e | E_r^e]$ is $r + k + m + m_R$. The proof is similar to the one of Proposition (2.1).

Hence the proposition 2.5 is proved. \square

To obtain the Hamiltonian of the effort constraint reduced system, we use the relation $f_b = -x_b$, $e_b = Q_b x_b$, i.e.

$$\begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_{12}}} \\ e_{x_{b_2}} \end{bmatrix} = \begin{bmatrix} Q_{b_{11}} & Q_{b_{12}} & 0 \\ Q_{b_{21}} & Q_{b_{22}} & 0 \\ 0 & 0 & I_k \end{bmatrix} \begin{bmatrix} x_{b_{11}} \\ x_{b_{12}} \\ x_{b_2} \end{bmatrix} \quad (2.81)$$

Since the effort variables $e_{x_{b_{12}}}$ are constrained, i.e. $Q_{b_{21}} x_{b_{11}} + Q_{b_{22}} x_{b_{12}} = 0$, and the matrix $Q_{b_{22}}$ is invertible, thus one can compute that

$$x_{b_{12}} = -Q_{b_{22}}^{-1} Q_{b_{21}} x_{b_{11}}. \quad (2.82)$$

Hence by using the relation $e_{x_{b_{11}}} = Q_{b_{11}} x_{b_{11}} + Q_{b_{12}} x_{b_{12}}$, we can get the reduced effort variables as

$$\begin{bmatrix} e_{x_{b_{11}}} \\ e_{x_{b_2}} \end{bmatrix} = \begin{bmatrix} Q_s & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} x_{b_{11}} \\ x_{b_2} \end{bmatrix} \quad (2.83)$$

where $Q_s = Q_{b_{11}} - Q_{b_{12}} Q_{b_{22}}^{-1} Q_{b_{21}}$ is the Schur complement of matrix $\begin{bmatrix} Q_{b_{11}} & Q_{b_{12}} \\ Q_{b_{21}} & Q_{b_{22}} \end{bmatrix}$.

The reduced Hamiltonian is

$$H_r^e = \frac{1}{2} x_r^T \begin{bmatrix} Q_s & 0 \\ 0 & I_k \end{bmatrix} x_r \quad (2.84)$$

We can formulate the reduced Dirac structure in an explicit descriptor Hamiltonian system.

2.5 Application to nano-tweezer

We use the Nano-tweezers established in FEMTO-ST [7] to illustrate the effectiveness of the proposed method. The nano-tweezers are modeled as a Timoshenko beam, and its infinite dimensional Hamiltonian formula can be find in [42, 43, 41]. The Timoshenko beam has been discretized by a mixed finite elements method [66, 24, 33, 5].

2.5.1 Application representation

In this section, we shall apply the proposed model reduction method to a Port Hamiltonian model of the nano-tweezers of the FEMTO-ST laboratory [7]. Consider the simplified model of a silicon nano-tweezers used for DNA manipulation given in Figure (2.3). The tweezers is made up with a

flexible arm that can be modeled as a Timoshenko beam clamped to a transverse suspension system. The trapped DNA bundle is approximated by a spring/damper-mass-spring/damper system attached at the tip of Timoshenko beam.

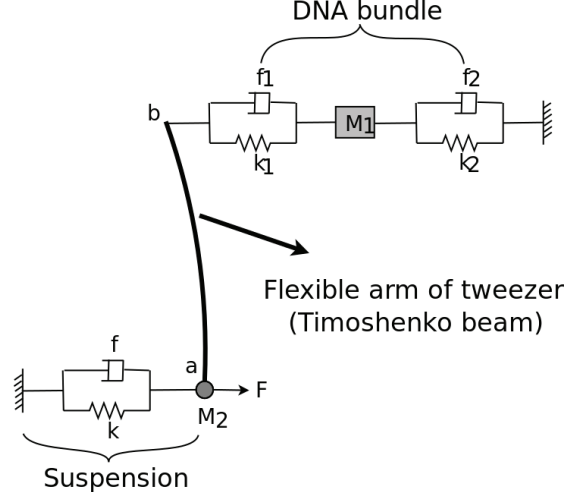


Figure 2.3: Suspension-nano-tweezers-DNA

The Timoshenko beam model may be expressed as an infinite dimensional Port Hamiltonian system [37]:

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_1} \frac{\partial}{\partial z} \begin{bmatrix} Kx_1 \\ \frac{1}{\rho}x_2 \\ EIx_3 \\ \frac{1}{I_\rho}x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P_0} \begin{bmatrix} Kx_1 \\ \frac{1}{\rho}x_2 \\ EIx_3 \\ \frac{1}{I_\rho}x_4 \end{bmatrix} \quad (2.85)$$

The matrices P_1 and P_0 define the skew-symmetric differential operator of order 1 acting on the state space $X = L_2(a; b; \mathbb{R}^4)$. The energy of the beam is expressed in terms of the energy variables,

$$H = \frac{1}{2} \int_a^b (Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2) dz \quad (2.86)$$

where the state (energy) variables are: the shear displacement x_1 , the transverse momentum distribution x_2 , the angular displacement x_3 and the angular momentum distribution x_4 . The coefficients ρ , I_ρ , E , I and K are the mass per unit length, the rotary moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively.

Using the mixed finite element semi-discretization method suggested in [24], one obtains a finite dimensional explicit Port Hamiltonian system such as:

$$\begin{aligned} \dot{x} &= (J - R) \frac{\partial H}{\partial x} + Bu \\ y &= B^T \frac{\partial H}{\partial x} \end{aligned} \quad (2.87)$$

where $J = -J^T$, $R = R^T > 0$, H is the Hamiltonian function. According to [41, 66] the discretization of the Timoshenko beam model, leads to the following structure matrices:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & M & 0 & 0 \\ M^T & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & M^T & 0 \end{bmatrix}}_{\tilde{P}_1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & -\Phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi^T & 0 & 0 & 0 \end{bmatrix}}_{\tilde{P}_0} \begin{bmatrix} \frac{\partial E}{\partial x_1} \\ \frac{\partial E}{\partial x_2} \\ \frac{\partial E}{\partial x_3} \\ \frac{\partial E}{\partial x_4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & B_1 & 0 \\ B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & B_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\
y &= \begin{bmatrix} 0 & B_2^T & 0 & 0 \\ 0 & 0 & 0 & B_2^T \\ B_1^T & 0 & 0 & 0 \\ 0 & 0 & B_1^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial E}{\partial x_1} \\ \frac{\partial E}{\partial x_2} \\ \frac{\partial E}{\partial x_3} \\ \frac{\partial E}{\partial x_4} \end{bmatrix}
\end{aligned} \tag{2.88}$$

where the sub-matrices are:

$$M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{bmatrix} \quad \text{with } M \in \mathbb{R}^{N \times N} \tag{2.89}$$

$$\Phi = \text{diag}(\beta, \dots, \beta) \quad \text{with } \Phi \in \mathbb{R}^{N \times N} \tag{2.90}$$

$$B_1 = \begin{bmatrix} 0 \\ 0_{N-2} \\ 1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} -1 \\ 0_{N-2} \\ 0 \end{bmatrix} \tag{2.91}$$

where β is the distance to the infinitesimal section.

The inputs and outputs of the system are the velocities in translation v and rotation ω as well as the forces F and torques T at the boundaries a and b :

$$\begin{aligned}
u &= \begin{bmatrix} v(b) & \omega(b) & F(a) & T(a) \end{bmatrix}^T = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \\
y &= \begin{bmatrix} F(b) & T(b) & -v(a) & -\omega(a) \end{bmatrix}^T = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T
\end{aligned} \tag{2.92}$$

The DNA bundle and the suspension system can be modeled as two simple finite dimensional port Hamiltonian systems like (2.87). The matrices of DNA bundle are defined as follows:

$$\begin{cases} \dot{v}_b &= (J_b - R_b) \frac{dE_b}{dv_b} + g_b u_b \\ y_b &= g_b^T \frac{dE_b}{dv_b} + S_b u_b \end{cases} \tag{2.93}$$

where the sub-matrices are:

$$\begin{aligned} J_b = -J_b^T &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, & R_b = R_b^T &= \begin{bmatrix} \frac{1}{f_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_2 \end{bmatrix} \\ g_b^T &= \begin{bmatrix} \frac{1}{f_1} & 0 & 0 \\ 0 & 0 & f_\theta \end{bmatrix}, & S_b &= \begin{bmatrix} \frac{1}{f_1} & 0 \\ 0 & f_\theta \end{bmatrix} \end{aligned} \quad (2.94)$$

With the energy of the DNA bundle is given as:

$$H_b = \frac{1}{2} \left(k_1(x_{c2} - x_{c1})^2 + k_2x_{c2}^2 + \frac{1}{M}(p_{c2})^2 \right) \quad (2.95)$$

where M is the mass of DNA bundle, x_{c1} and x_{c2} are the relative positions of point b and mass M and $p_{c2} = M\dot{x}_{c2}$ is its momentum. k_1 , k_2 , f_1 and f_2 represent the constants of the springs and the viscous dampers of the DNA bundle respectively. f_θ is the rotation damper of the DNA bundle in point b . The suspension system is also modeled as system (2.87), in which the matrices are given as:

$$J_a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R_a = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}, \quad g_a^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (2.96)$$

with the energy of the suspension system is given as:

$$H_a = \frac{1}{2} (kx_a^2 + \frac{1}{M_2}(p_a)^2) \quad (2.97)$$

where x_a is the relative position of point a , M_2 is the mass of point a , $p_a = M_2\dot{x}_a$ is its momentum, k and f represent the constant of the springs and the viscous damper of suspension system respectively.

The interconnection relations of the tweezers arm and the suspension system are:

$$u_a = y_3 \quad \text{and} \quad u_3 = -y_a \quad (2.98)$$

where

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = y_b \quad \text{and} \quad u_b = - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Since the arm of the tweezers is clamped to the suspension system, we consider the additional constraint:

$$y_4 = \omega(a) = 0 \quad (2.99)$$

With the above interconnections, one can express the total system with the constraint as (2.15) where the dissipation port is closed and the total energy of the system $H_t = H + H_b + H_a$.

$$\begin{aligned} \dot{x}_t &= (J_t - R_t) \frac{\partial E_t}{\partial x_t} + Bu + B_c u_c \\ y &= B^T \frac{\partial E_t}{\partial x_t} \\ 0 = y_c &= B_c^T \frac{\partial E_t}{\partial x_t} \end{aligned} \quad (2.100)$$

where the variables are $x_t = \begin{bmatrix} x & v_b & v_a \end{bmatrix}^T$, and

L	Length	$5150\mu m$
l	Width	$150\mu m$
e	Thickness	$50\mu m$
E	Young's modulus	$190Gpa$
I	Area moment of inertia	$1.4 \times 10^{-17}m^4$
G	Shear modulus	$80GPa$
ρ	Mass density	$2330KG/m^3$

Table 2.1: The parameters of tweezers arm [7]

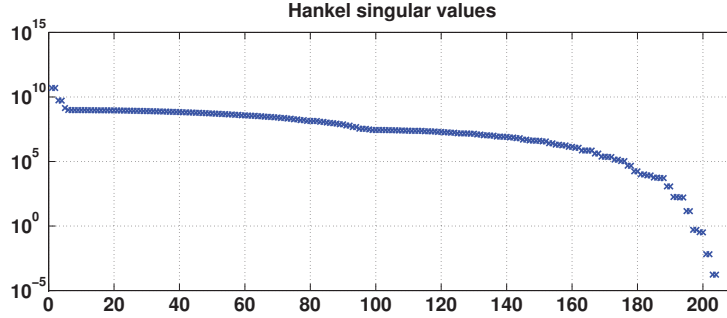


Figure 2.4: Hankel singular values of the nano-tweezers

The sub-matrices are defined as:

$$J_t = \begin{bmatrix} J & G_1 & G_2 \\ -G_1^T & J_b & 0 \\ -G_2^T & 0 & J_a \end{bmatrix}, R_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_b & 0 \\ 0 & 0 & R_a \end{bmatrix}, G_1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{B_2}{f_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & B_1 \\ \frac{B_2}{f_1} & 0 & 0 \\ 0 & \frac{B_2}{f_\theta} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

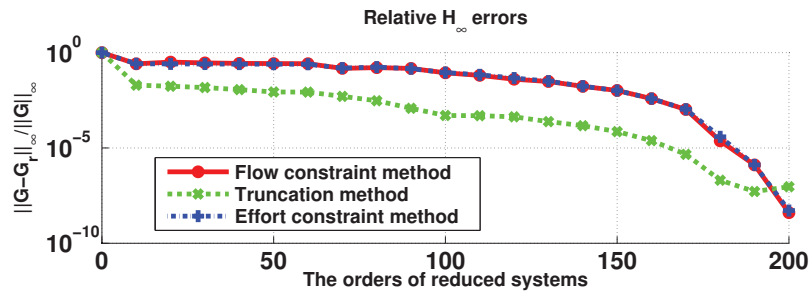
The system (2.100) has the same form as the system (2.15) except the dissipation port is closed. So one can use the proposed reduction method to reduce the order of the system.

We have chosen the order of the discrete Timoshenko beam model $n_T = 200$, the orders of the DNA bundle and the suspension system are $n_D = 3$ and $n_S = 2$ respectively. The total system order is $n = 205$. The parameters of tweezers arm are given in table (2.1).

2.5.2 Simulation results

Firstly, we shall show the Hankel singular values of the given system as in Figure (2.4).

Figure (2.5) gives the relative H_∞ norms of the input/output systems for the dimensions of the reduced order models r from 0 to 190 by using the flow constraint, effort constraint and truncation methods respectively. We can observe that the reduced order systems by using the classic truncation

Figure 2.5: Relative H_∞ norm errors

Reduced order system by	H_∞ norm
Balanced truncation method	5.0052×10^{-4}
Flow constraint method	0.08656
Effort constraint method	0.09006

Table 2.2: H_∞ norm errors for $r = 100$

method have the better performance in H_∞ norm errors than the flow or effort constraint method. However we should always remain that the flow/effort constraint methods can preserve the passivity and Hamiltonian structure of the system, but the classic truncation method can't. Furthermore, we can see the errors evaluate as the Hankel singular values in Figure (2.4).

We will show in the following some simulation results for the reduced system given in order 100 by different methods. In the table 2.2, we shall compare the H_∞ norm of the error systems obtained by using the balanced truncation method, flow and effort constraint method. It shows the same result as in Figure (2.5).

In the Figure (2.6), we show the H_∞ norm errors between the full order system and the reduced order systems ($r = 100$) which are obtained by the different methods, flow, effort constraint methods and balanced truncation method. One can observe that in the low frequency, the flow and effort methods have better performance than the balanced truncation method, specially by using the flow constraint method, while in the high frequency, the balanced truncation method does a better job than the others.

Figure (2.7) gives the comparative bode plots of discretized systems with 200 or 100 elements, and the reduced system with 100 states which is reduced from the discretized system with 200 elements. It shows the intrinsic advantage of our proposed reduction scheme comparing to size equivalent discretization method.

2.6 Conclusion

In this chapter, we have derived a reduction method for linear constrained port Hamiltonian systems which preserves the passivity and the geometric (Hamiltonian) structure of the original system. The linear constrained port Hamiltonian system is first transformed into a descriptor form, called in the sequel, *port Hamiltonian descriptor system*, by using a coordinate transformation and the elimination of the Lagrangian multiplier. This procedure is similar to the geometric reduction of constrained systems proposed in [16], but differs from this work as it preserves the algebraic constraints not

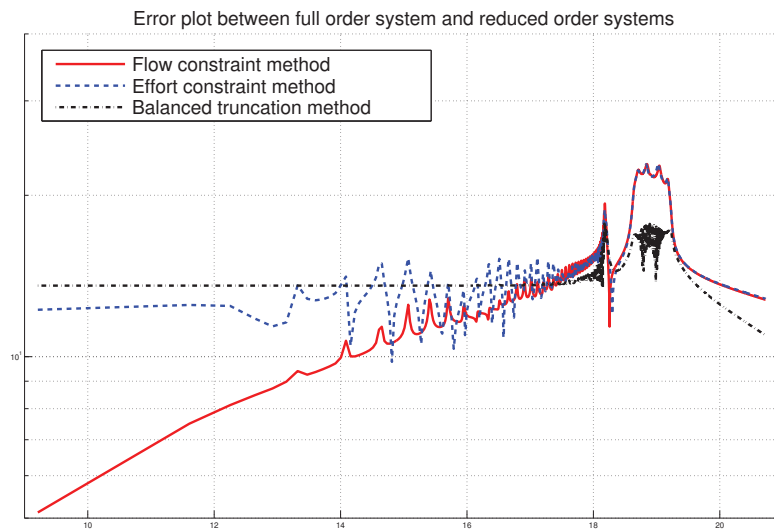


Figure 2.6: H_∞ norm errors between the full order system and the reduced order systems obtained by different methods

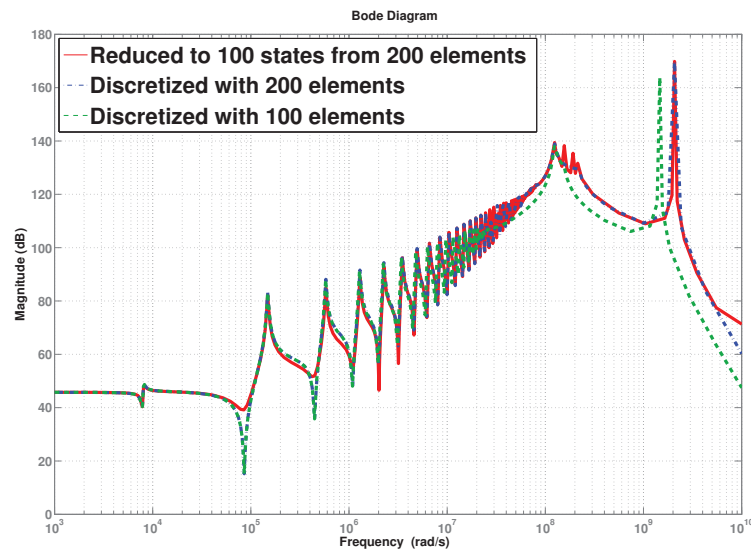


Figure 2.7: Bode diagram of discretized systems with 200 or 100 elements and the reduced system with 100 states from 200 elements

involving the Lagrangian multipliers.

This port Hamiltonian descriptor representation allows us to combine the tools for the system's analysis of descriptor systems with the structure properties of port Hamiltonian systems. Thus the port Hamiltonian representation of the Weierstrass canonical form of the port Hamiltonian descriptor system has been introduced. This canonical form allows to split the descriptor system into two parts, one associated with slow dynamics and the other one associated with fast dynamics while retaining the port Hamiltonian structure of the system. The balanced port Hamiltonian realization is derived, and the reduction scheme using flow or effort constraint method is used to provide a geometric reduction scheme that preserves the passivity and the Hamiltonian structure of the original system.

Finally the reduction procedure has been illustrated on the example of a micro mechanical manipulator (nano-tweezers) under development for the manipulation of DNA bundles in order to illustrate the effectiveness of the proposed method by numerical simulation. We show the errors between the full order system and the reduced order system by the relative H_∞ norm. The results show that the Effort and Flow constraint method leads to errors of higher H_∞ norm than with the use of the truncation method, but with the proposed approach the passivity and structure of the original system are conserved. At last we show that the balanced structure preserving reduction method provides a better approximation in the high frequency than the spatial discretization method.

Chapter 3

Modified LQG method for the structure preserving reduction of port Hamiltonian systems

3.1 Introduction

The port Hamiltonian systems are fundamentally a structured representation for dynamical systems where the energetic properties (energy exchanges, dissipation and accumulation) are explicitly coded. It induces naturally some geometric properties (Dirac structures, Casimir invariants) and a passivity of the system. Thus it seems interesting to develop a reduction method which aims to preserve the structure and the passivity of the original port Hamiltonian systems. The passivity and structure preserving reduction methods of port Hamiltonian systems have been proposed by using different methods. We can first cite the positive real balancing method proposed in [2, 61] which is formulated as a balancing of two positive real “gramians” given as a solution of two Riccati equations. We have also the moment matching method defined as a projection on the Kylov subspaces [63, 84, 64]. Furthermore, the moment matching based reduction method using interpolation points has been proposed in [27, 26] which reduces the port Hamiltonian system by using the projection based method which allows to interpolate the transfer function at a fixed frequency. From the geometric point of view, in [65] the authors introduce the effort and flow constraint methods which consists to force either a flow variable or effort variable to zero. All these methods consider the open loop system for the reduction. However the open loop based reduction schemes do not apply to a conservative (lossless) port Hamiltonian systems since there is no possible balancing of state space variables in the sense of input-output contributions. This can be also viewed as an undamped systems where all the poles are on the imaginary axis.

In this chapter we consider a closed loop based reduction of finite dimensional port Hamiltonian systems. This closed loop based reduction by using balancing method has been firstly proposed in [36] by E. A. Jonckheere and L. M. Silverman in 1983 named *LQG balanced method* associated with the LQG control problem. This model reduction aims to reduce at same time model of the system and the LQG controller. It consists in balancing the LQG “Gramians” which are the solutions of the

optimal control Riccati equation and the optimal filter Riccati equation. The balancing leads to a new basis for the state space where the realization of the model is said to be balanced and where vector of the state variables is sorted according to their importance in the closed loop systems. In the balanced realization the LQG Gramians are then equal and diagonal where the diagonal elements are called singular values. This singular values are positive constants, state invariant and sorted in decreasing order. Then the classic truncation method can be used to reduce the system and we get a reduced model and LQG controller which stabilizes the full order system.

The LQG balancing for *lossless* port Hamiltonian systems was studied by A. van der Schaft in [76] and he shows that the LQG balancing method cannot be used for reducing the lossless port Hamiltonian systems since the product of the solutions of the two Riccati equations are equal to identity. That means that all the state variables are equivalent for the control design and no possible separation of the states variables. From the other side, the LQG controller or the reduced LQG controller are unfortunately neither stable nor passive in general case. It can be stable if the weighting matrices and covariance matrices of the optimal control and filtering problem are under certain conditions [31]. However we can't guarantee the passivity and the Hamiltonian structure in the closed loop systems.

The passive LQG control design method for the linear positive real (passive) systems have been introduced by R. Lozano-Leal and S. M. Joshi in [40]. In this LQG control design, the authors have considered covariance matrices of the optimal filtering problem which depend on the weighting matrices of the optimal control problem. From the other hand, it is well known that the port Hamiltonian systems are suited for passivity based control (PBC) [57], or methods like Interconnection and Damping Assignment Passivity Based Control (IDA-PBC) which aims to shape the energy and to assign a new interconnection and damping structure for the system in closed loop [59, 77]. Because the LQG controller is in general not passive, no Hamiltonian structure is guaranteed for the closed loop systems.

In this chapter we develop closed loop passive preserving LQG reduction method for finite dimensional port Hamiltonian systems. We develop first an extended version of the passive LQG control design given in R. Lozano-Leal and S. M. Joshi in [40] but which still be useless for balancing. We develop then a *passivity preserving* LQG control design method where the vector of state variables is separable and make then possible to write a balanced realization and to reduce the model. Next, we will give a conditions under which the LQG controller can be written in port Hamiltonian form. This have the advantage to formulate the controller as a control interconnection.

The considered control problem in this chapter is the stabilization of a port Hamiltonian using a passive and reduced LQG controller. We shall take a non-standard and linked weighting matrices for the LQG optimal control and filtering problem which is different from the one presented in [36].

Since the classic truncation based reduction method can't preserve the structure and the passivity of port Hamiltonian systems during the reduction, we use the geometric effort constrained reduction method to derive a reduced port Hamiltonian model from the balanced realization of the original full state model.

3.1.1 Organization of the chapter

This chapter is organized as follows. In Section 3.2, we recall the LQG control problem for LTI (Linear Time-Invariant) systems and the passive LQG control design method for linear positive real

systems. Based on this passive LQG control design method, we will suggest a passive LQG controller which is equivalent to the control by interconnection of port Hamiltonian systems. However, this LQG problem can't allow us to reduce the model because all the state variables have the same importance in the control point of view. Furthermore, a novel passive LQG problem will be suggested which is also equivalent to the control of port Hamiltonian systems by interconnection and permit us to reduce the port Hamiltonian systems. In Section 3.3, we will derive a reduction scheme by the LQG balanced realization defined in the last section and the effort constraint method in order to preserve the structure and passivity of the system. In Section 3.4, we will give a characterization of the error of the proposed reduction method. In Section 3.5, we give a summary of the proposed methods using some illustration schemes. In Section 3.6, some numerical simulation results are given by applying the proposed methods to a classic mechanical mass-spring-damper system in order to show the effectiveness of the proposed methods. Finally, we give conclusion in Section 3.7.

3.1.2 Main contributions of the chapter

The main contributions of this chapter are the following.

- In Section 3.2, a new passive LQG control problem is suggested to the port Hamiltonian systems in Theorem 3.2 by inspiring the positive real LQG control design of the positive real system. We show that this new passive LQG control is equivalent to the control of port Hamiltonian systems by interconnection. However, it turns out this LQG problem can't allow us to reduce the system because all the singular values are equal to one and the state variables are not splittable from this LQG control problem point of view.
- Since the first passive LQG control does not allow us to reduce the port Hamiltonian system, we suggest a novel passive LQG control problem from the conjugated inputs outputs point of view in Theorem 3.3. We show that in this passive LQG problem, if the weighting matrices of optimal control and the covariance matrices of filtering problems are related by some conditions, then the LQG control of port Hamiltonian system is passive and the solutions of control and filtering Riccati equations are related through the energy matrix Q of the port Hamiltonian system. Contrariwise to the work in [4], we propose the dynamical LQG control which is equivalent to control by interconnection of two port Hamiltonian systems for the model reduction objective of the closed loop system.
- The second passive LQG control problem proposed by Theorem 3.3 can define a balanced realization of the port Hamiltonian system given in Definition 3.4. This implies that the state variables can be separated by their importance in the control design point of view, meanwhile we can reduce the port Hamiltonian system that is written in the balanced realization.

3.2 Passive LQG control design for port Hamiltonian systems

In this section, firstly we recall briefly the LQG control design method for LTI (Linear Time-Invariant) systems and recall under which conditions such a controller is positive real when the control system is itself positive real [8, Chap. 6]. Secondly, we suggest two design methods for the LQG controller [85] which are equivalent to a control by interconnection of two PHS [60]. However differ from the

interconnection control given in [60, 58], these proposed LQG controllers are the dynamical states feedback. Then we will discuss their suitability for the structure preserving reduction of port Hamiltonian systems.

3.2.1 Reminders of LQG control for LTI systems and positive real (passive) systems

In this section we shall recall the LQG control for LTI systems as given by the standard reference [2] and we apply this controller design method to the positive (passive) real systems then we recall how to design the passive LQG controller for positive real systems.

3.2.1.1 LQG control for LTI systems

First consider the following LTI dynamic system:

$$\begin{cases} \dot{x} = Ax + Bu + v \\ y = Cx + w \end{cases} \quad (3.1)$$

where $x \in \mathbb{R}^n$ represents the state variables of the system, u the vector of control inputs and y the vector of outputs. Both additive white Gaussian system's noise v and additive white Gaussian measurement noise w affect the system.

We recall that the LQG control problem for LTI system (3.1) is simply the combination of a Kalman filter, i.e. a *linear quadratic estimator* (LQE) with a *linear quadratic regulator* (LQR) as show in Figure (3.1).

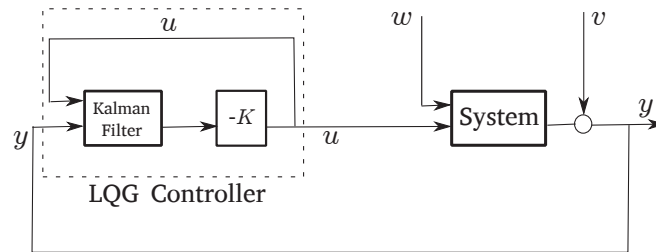


Figure 3.1: LQG controller design

The *separation principle* guarantees that the estimation and control can be designed and computed independently.

First we can design the Kalman filter to estimate the state variables of the system containing the noises. The additive Gaussian system's and measurement noises are assumed to be independent white Gaussian processes with the two covariance matrices Q_v , R_w

$$\begin{aligned} E[w w^T] &= R_w \delta & E[v v^T] &= Q_v \delta, \\ E[w v^T] &= 0 \end{aligned}$$

where δ is the Dirac delta function and

$$R_w = R_w^T > 0 \quad \text{and} \quad Q_v = Q_v^T > 0.$$

If the pair $(A, Q_v^{1/2})$ is stabilizable and the pair (A, C) detectable, thus the linear quadratic estimator is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}) \quad (3.2)$$

with

$$F = P_f C^T R_w^{-1} \quad (3.3)$$

where P_f is the unique solution of the filter Riccati equation CARE:

$$AP_f + P_f A^T - P_f C^T R_w^{-1} C P_f + Q_v = 0 \quad (3.4)$$

with $P_f = P_f^T > 0$.

Second we design a linear quadratic regulator which minimizes the cost function

$$J_c = \lim_{T \rightarrow \infty} E \left[\int_0^T (x^T \tilde{Q} x + u^T \tilde{R} u) dt \right] \quad (3.5)$$

where \tilde{Q} and \tilde{R} are two weighting matrices with $\tilde{Q} = \tilde{Q}^T \geq 0$ and $\tilde{R} = \tilde{R}^T > 0$. If the pair (A, B) is stabilizable and the pair $(A, \tilde{Q}^{1/2})$ detectable, thus the optimal control is given as

$$u = -K\hat{x} \quad (3.6)$$

with

$$K = \tilde{R}^{-1} B^T P_c \quad (3.7)$$

where P_c is the unique solution of control Riccati equation (CARE):

$$A^T P_c + P_c A - P_c B \tilde{R}^{-1} B^T P_c + \tilde{Q} = 0 \quad (3.8)$$

with $P_c = P_c^T > 0$.

By using the two steps and from the *separation principle*, one can design a dynamic controller so-called LQG controller and its dynamic can be presented as:

$$\begin{cases} \dot{\hat{x}} &= (A - B\tilde{R}^{-1}B^T P_c - P_f C^T R_w^{-1} C) \hat{x} + P_f C^T R_w^{-1} u_c \\ y_c &= \tilde{R}^{-1} B^T P_c \hat{x} \end{cases} \quad (3.9)$$

with $u_c = y$ and $u = -y_c$.

3.2.1.2 LQG balanced realization and truncation reduction

We will recall in this part the LQG balanced realization proposed in [36]. We call it standard LQG balanced realization to avoid confusing with the other proposed LQG balanced realization in the next section.

This LQG balanced realization is related with LQG problem with the following choice of weighting and covariance matrices:

$$\begin{aligned} \tilde{Q} &= C^T C, & Q_v &= B B^T, \\ \tilde{R} &= R_w = I. \end{aligned} \quad (3.10)$$

Then the filter Riccati equation and the control Riccati equation become:

$$AP_f^s + P_f^s A^T - P_f^s C^T C P_f^s + BB^T = 0 \quad (3.11)$$

and

$$A^T P_c^s + P_c^s A - P_c^s BB^T P_c^s + C^T C = 0 \quad (3.12)$$

The Standard LQG balanced realization is defined by the following Definition.

Definition 3.1. The LTI (3.1) admits a standard LQG balanced realization if the Gramians P_f^s and P_c^s of the Standard LQG problem are equal and diagonal:

$$P_c^s = P_f^s = \Sigma^s = \text{diag}(\mu_1^s, \mu_2^s, \dots, \mu_n^s) \quad (3.13)$$

where, denoting by $\lambda_i(P)$ the i -th eigenvalue of a matrix P ,

$$\mu_i^s = \sqrt{\lambda_i(P_c^s P_f^s)} \text{ and } \mu_1^s > \mu_2^s > \dots > \mu_n^s > 0 \quad (3.14)$$

with μ_i^s called the standard LQG singular values.

Because the Gramians P_f and P_c of the standard LQG problem in (3.11) and (3.12) are positive, then there exists the following Cholesky factorization:

$$P_c = R_p R_p^T, P_f = L_p^T L_p \quad (3.15)$$

where the matrices $R_p^T, L_p \in \mathbb{R}^n$ are upper triangular Cholesky factors.

Let

$$L_p R_p = U_p \Sigma V_p^T \quad (3.16)$$

be the singular value decomposition of $L_p R_p$, where U_p, V_p are orthogonal. Thus we can define the transformation matrices as

$$W_b = L_p^T U_p \Sigma^{-1/2}, T_b = R_p V_p \Sigma^{-1/2} \quad (3.17)$$

and $(T_b)^T W_b = I_n$. We can transform the coordinate by

$$x = T_b x_b \quad (3.18)$$

and premultiply by matrix W_b on the two sides of the system (3.1), we obtain a standard LQG balanced realization of the port Hamiltonian system which is denoted as follows

$$\begin{cases} \dot{x}_b &= A_b x_b + B_b u \\ y &= C_b x_b \end{cases} \quad (3.19)$$

where $A_b = W_b^T A T_b$, $B_b = W_b^T B$, $C_b = C T_b$.

Since the standard LQG singular values of the balanced realization are in the decreasing order, we can eliminate some of the state variables, which correspond to the smaller parts of standard LQG singular values and less important for the LQG control design, by using the truncation method. Let

partition the corresponding matrices as follows:

$$A_b = \begin{bmatrix} A_{b11} & A_{b12} \\ A_{b21} & A_{b22} \end{bmatrix}, \quad B_b = \begin{bmatrix} B_{b1} \\ B_{b2} \end{bmatrix}, \quad C_b = \begin{bmatrix} C_{b1} & C_{b2} \end{bmatrix}. \quad (3.20)$$

Definition 3.2. [2, Def 7.8] The system

$$\begin{cases} \dot{x}_{b1} &= A_{b11}x_{b1} + B_{b1}u \\ y &= C_{b1}x_{b1} \end{cases} \quad (3.21)$$

is the reduced order system obtained from (3.19) by *balanced truncation*.

It is well known that the standard LQG truncation method can't preserve the passivity of the passive systems. Moreover, in the case of port Hamiltonian systems, the truncation method can't preserve the structure of the systems [61]. Hence we shall consider other reduction methods instead of the truncation method to preserve the passivity and the structure of the port Hamiltonian systems.

3.2.1.3 Passive LQG control design of Positive real (passive) system

It is well-known that LQG controller given as (3.9) are in general neither stable nor passive [31]. In this part, we shall briefly recall how to design a passive LQG controller for positive real systems [8]. Consider a minimal realization of positive real system expressed by LTI system (3.1) with holding the following equations:

$$A + A^T = -Q_A \leq 0 \quad (3.22)$$

and

$$B = C^T. \quad (3.23)$$

The above conditions are equivalent to the Positive Real (Kalman-Yakubovich-Popov) lemma [1, 8, Lemma 3.1]. The LQG controller for the system (3.1), such that (3.22) and (3.23) hold, is given by

$$\begin{cases} \dot{\hat{x}} &= \left(A - B\tilde{R}^{-1}B^T P_c - P_f B R_w^{-1} B^T \right) \hat{x} + P_f B R_w^{-1} u_c \\ y_c &= \tilde{R}^{-1} B^T P_c \hat{x} \end{cases} \quad (3.24)$$

where $P_f = P_f^T > 0$, $P_c = P_c^T > 0$ are the solutions of the filter Riccati equation (3.4) and control Riccati equation (3.8) respectively.

Theorem 3.1. [8, Chap. 3] Consider the Positive real system defined by (3.1), (3.22) and (3.23) and the LQG controller of the form (3.24) through (3.4) and (3.8). If \tilde{Q} , \tilde{R} , Q_v , R_w are such that:

$$Q_v = Q_A + B\tilde{R}^{-1}B^T \quad (3.25)$$

$$R_w = \tilde{R} \quad (3.26)$$

and

$$\tilde{Q} - B\tilde{R}^{-1}B^T \triangleq Q_B > 0 \quad (3.27)$$

then the controller is positive real.

This above result states that, if the weighting matrices for the optimal regulator and the filter

satisfy the relations (3.25), (3.26) and (3.27), the resulting LQG controller is positive real (passive). However, it should be noted that this controller would not be optimal with respect to some actual noise covariance matrices. The noise covariance matrix is used herein simply as controller design parameters and has no statistical meaning. It may also be noted that the positivity condition (3.27) is not particularly restrictive as it amounts to choose a positive matrix Q_B and define the weighting matrix of the LQR performance index as $\tilde{Q} = B\tilde{R}^{-1}B^T + Q_B$ which is equivalent to introducing an additional term $y^T\tilde{R}^{-1}y$ to an arbitrary quadratic state weight.

3.2.2 LQG control applied to port Hamiltonian systems

In this section, we shall consider the LQG problem for port Hamiltonian systems and study how to design a passive LQG type controller for this class of systems. Therefore let us consider a class of linear port Hamiltonian system (PHS) [8, Chap. 6] with some system's and measurement noises in order to be able to correspond to a LQG problem .

Definition 3.3. A linear dissipative port Hamiltonian system (PHS) with state variable $x \in \mathbb{R}^n$, input variable $u \in \mathbb{R}^m$, output variable $y \in \mathbb{R}^m$ and additive system's measurement noises v, w is defined as follows:

$$\begin{cases} \dot{x} &= (J - R)Qx + Bu + v \\ y &= B^T Qx + w \end{cases} \quad (3.28)$$

where $J = -J^T \in \mathbb{R}^{n \times n}$ is the skew-symmetric structure matrix, $R = R^T \in \mathbb{R}^{n \times n}$ is the symmetric semi-positive definite dissipation matrix and $Q = Q^T \in \mathbb{R}^{n \times n}$ is the symmetric and positive definite energy matrix and $B \in \mathbb{R}^{n \times m}$ is the input matrix. The system's and measurement noises v, w are assumed to be independent white Gaussian processes with the two covariance matrices Q_v, R_w

$$\begin{aligned} E[w w^T] &= R_w \delta & E[v v^T] &= Q_v \delta, \\ E[w v^T] &= 0 \end{aligned}$$

with δ is the Dirac delta function and

$$R_w = R_w^T > 0 \quad \text{and} \quad Q_v = Q_v^T > 0$$

The LQG control problem consists in finding a control which minimizes the following cost function:

$$J_c = \lim_{T \rightarrow \infty} E \left[\int_0^T (x^T \tilde{Q}x + u^T \tilde{R}u) dt \right] \quad (3.29)$$

where \tilde{Q} and \tilde{R} are two weighting matrices with $\tilde{Q} = \tilde{Q}^T > 0$ and $\tilde{R} = \tilde{R}^T > 0$.

The solution of this problem may be decomposed into the two following steps applied to the PHS (3.28).

- **First step:** If the pair $((J - R)Q, Q_v^{1/2})$ is stabilizable and the pair $((J - R)Q, B^T Q)$ detectable, one can estimate the state x by the classic Kalman filter equation, i.e.

$$\dot{\hat{x}} = (J - R)Q\hat{x} + Bu + F(y - B^T Q\hat{x}) \quad (3.30)$$

with

$$F = P_f Q B R_w^{-1} \quad (3.31)$$

where P_f is the unique solution of the filter Riccati equation

$$(J - R)Q P_f + P_f Q (J - R)^T - P_f Q B R_w^{-1} B^T Q P_f + Q_v = 0 \quad (3.32)$$

with $P_f = P_f^T > 0$.

- **Second step:** If the pair $((J - R)Q, B)$ is stabilizable and the pair $((J - R)Q, \tilde{Q}^{1/2})$ detectable, the optimal control is:

$$u = -K \hat{x} \quad (3.33)$$

with

$$K = \tilde{R}^{-1} B^T P_c \quad (3.34)$$

where P_c is the unique solution of control Riccati equation:

$$Q(J - R)^T P_c + P_c (J - R)Q - P_c B \tilde{R}^{-1} B^T P_c + \tilde{Q} = 0 \quad (3.35)$$

with $P_c = P_c^T > 0$.

Remark 3.1. Following [49], we call P_f and P_c the *LQG Gramians* of the port Hamiltonian system (3.28).

The LQG controller may be expressed as the feedback interconnection of the system (3.28) with the dynamical controller:

$$\begin{cases} \dot{\hat{x}} &= [(J - R)Q - BK - FB^T Q] \hat{x} + F u_c \\ y_c &= K \hat{x} \end{cases} \quad (3.36)$$

As the matrix Q is assumed to be invertible, using the expressions (3.31) and (3.34) and defining the matrix

$$R_c = R + B \tilde{R}^{-1} B^T P_c Q^{-1} + P_f Q B R_w^{-1} B^T, \quad (3.37)$$

the controller may be written in the form:

$$\begin{cases} \dot{\hat{x}} &= (J - R_c)Q \hat{x} + P_f Q B R_w^{-1} u_c \\ y_c &= (\tilde{R}^{-1} B^T P_c Q^{-1})Q \hat{x} \end{cases} \quad (3.38)$$

In this expression, the state matrix of the controller is decomposed into the product $(J - R_c)Q$ with the energy matrix Q of the port Hamiltonian system (3.28), the skew symmetric matrix J and R_c the matrix defined by (3.37). Without other assumption, this matrix has no positivity property nor symmetry. Furthermore the input and output matrices are not conjugated with respect to the energy.

3.2.3 Minimizes the energy dissipation LQG control design

In this part we shall write conditions on the LQG problem such that the controller (3.38) has a port Hamiltonian realization. Since the port Hamiltonian systems are naturally positive real, inspired

by the positive real LQG controller design method of Theorem 3.1, we derive a passive LQG type controller design of port Hamiltonian which can be summarized in the following theorem.

Theorem 3.2. *Consider the LQG problem with the following choice of the weighting matrices:*

$$Q_v = 2R + BR_w^{-1}B^T \quad (3.39)$$

$$\tilde{R} = R_w = R_w^T > 0 \quad (3.40)$$

and

$$\tilde{Q} = QQ_vQ = \tilde{Q}^T > 0 \quad (3.41)$$

then the LQG controller is passive and the closed loop system can be written as the feedback interconnection of the port Hamiltonian system (3.28) with the port Hamiltonian realization of the LQG controller.

Proof. Consider the filtering Riccati equation (3.32), we factorize P_f from the two sides of equation, so:

$$P_f^{-1}(J - R)Q + Q(J - R)^T P_f^{-1} - QBR_w^{-1}B^TQ + P_f^{-1}Q_vP_f^{-1} = 0 \quad (3.42)$$

then we subtract this equation by equation (3.35), and by using the conditions (3.40) and (3.41) in the proposition (3.2), we obtain:

$$\begin{aligned} & (P_c - P_f^{-1})(J - R)Q + Q(J - R)^T(P_c - P_f^{-1}) \\ & - P_cBR_w^{-1}B^TP_c + QBR_w^{-1}B^TQ + QQ_vQ - P_f^{-1}Q_vP_f^{-1} = 0 \end{aligned} \quad (3.43)$$

As a consequence the LQG Gramians depend on the energy matrix Q :

$$P_f = Q^{-1} \quad \text{and} \quad P_c = Q \quad (3.44)$$

The LQG controller (3.38) obtained with this solution may be written:

$$\begin{cases} \dot{\hat{x}} &= (J - R_c)Q\hat{x} + BR_w^{-1}u_c \\ y_c &= \tilde{R}^{-1}B^TQ\hat{x} \end{cases} \quad (3.45)$$

with the added dissipation matrix

$$R_c = R + B\tilde{R}^{-1}B^T + BR_w^{-1}B^T, \quad (3.46)$$

which, using (3.40) and the symmetry of R_w is immediately seen to be symmetric, positive definite $R_c = R_c^T > 0$. Using (3.40) it may also be seen that $\tilde{R}^{-1}B^T = R_w^{-1}B^T$ and the output y_c is the power conjugated of the input u_c .

As consequence, the LQG control with the weighting matrices conditions in the proposition (3.2), may be equivalently written as a control by interconnection of port Hamiltonian systems. \square

The above theorem gives us a passive LQG controller design method, and the closed loop system with this controller can be represented as interconnection control of port Hamiltonian system if the weighting matrices for the regulator and the filters are chosen in a certain manner. However, as the

positive real LQG controller design, the noise covariance matrices are used herein merely as controller design parameters and have no statistical meaning.

Let us write the cost function in terms of the weighting matrices Q_v and \tilde{Q} of the theorem 3.2

$$\begin{aligned} J_c &= \lim_{T \rightarrow \infty} E[\int_0^T (x^T Q(2R + BR_w^{-1}B^T)Qx + u^T \tilde{R}u)dt] \\ &= \lim_{T \rightarrow \infty} E[\int_0^T (2x^T QRQx + y^T R_w^{-1}y + u^T \tilde{R}u)dt] \end{aligned} \quad (3.47)$$

Hence the optimization problem may be interpreted as the minimization of the sum of dissipation of the energy of the port Hamiltonian system with the norm and the dissipation on the input and output variables with respect to the weighting matrices R_w^{-1} and \tilde{R} respectively.

Remark 3.2. If we consider a lossless port Hamiltonian system, i.e. $R = 0$, and $\tilde{R} = R_w = I$. Then the weighting matrices become $Q_v = BB^T$ and $\tilde{Q} = QQ_vQ = QBB^TQ$, and the LQG problem of Theorem 3.2 becomes the standard LQG problem [36]. In this case, one recovers the result of [76] that the product of the LQG Gramians is equal to the identity.

However in view of the balanced reduction of the port Hamiltonian system, the LQG problem suggested in Theorem 3.2, is not useful as the product of the LQG Gramians is

$$P_f P_c = I$$

and does not allow to split the state space coordinates into two subsets associated with small or big singular values. So for the objective of balanced reduction, we should consider another LQG problem which can be used to design a passive LQG controller and meanwhile allow to split the state space coordinates, i.e. $P_f P_c \neq I$.

3.2.4 Q-conjugated LQG control design

In this part, we suggest another passive LQG controller design method which is also equivalent to control by interconnection of port Hamiltonian systems. Let recall the LQG controller (3.38) in quasi-Hamiltonian form because that the inputs and outputs are not conjugated generally, thus we shall introduce a special choice of covariance matrices and weighting matrices such that the inputs and outputs are conjugated, the main result is given in the following theorem.

Theorem 3.3 (Q-conjugated LQG control design). *Denote the LQG Gramians P_f , solution of the filter Riccati equation (3.32) and P_c , solution of the control Riccati equation (3.35). Consider the LQG problem with the following relation between the covariance matrix R_w of the output noise and the weighting matrix \tilde{R} of the cost functional (3.29)*

$$R_w = \tilde{R}. \quad (3.48)$$

and the covariance matrix Q_v of the state noise and the weighting matrix \tilde{Q} of the cost functional (3.29) are related by

$$Q_v = Q^{-1}(2QJ^T P_c + 2P_c JQ + \tilde{Q})Q^{-1} \quad (3.49)$$

Then

$$P_c Q^{-1} = Q P_f, \quad (3.50)$$

is a choice allowing to satisfy the two Riccati equations (3.32) and (3.35).

Furthermore, assuming that the port Hamiltonian system is stable, then the control Riccati equation (3.35) and the filter Riccati equation (3.32) admit a unique solution, the LQG controller is passive and the closed loop system can be written as the feedback interconnection of the port Hamiltonian system (3.28) with the port Hamiltonian realization of the LQG controller.

Proof. Assume Q is invertible since it is symmetric and positive definite. The filter Riccati equation (3.32) can be written as:

$$Q(J-R)QP_fQ + QP_fQ(J-R)^TQ - QP_fQBR_w^{-1}B^TQP_fQ + QQ_vQ = 0 . \quad (3.51)$$

By using the condition (3.49), the above equation becomes

$$Q(J-R)QP_fQ + QP_fQ(J-R)^TQ - QP_fQBR_w^{-1}B^TQP_fQ + 2QJ^TP_c + 2P_cJQ + \tilde{Q} = 0 \quad (3.52)$$

Then subtracting the equation (3.52) to the control Riccati equation (3.35) and considering condition (3.48)

$$R_w = \tilde{R}, \quad (3.53)$$

we can get

$$QJ(QP_fQ - P_c) + (QP_fQ - P_c)J^TQ - (QP_fQ - P_c)BR_w^{-1}B^T(P_c - QP_fQ) = 0 \quad (3.54)$$

Finally one possible choice is

$$P_cQ^{-1} = QP_f. \quad (3.55)$$

One can check that this choice allows to satisfy the two Riccati equation (3.32) and (3.35).

Using the Hautus criterion, the detectability condition is given as follows:

$$\text{Rank} \begin{bmatrix} (J-R)Q - \lambda I \\ \tilde{Q}^{1/2} \end{bmatrix} = n \quad \text{for} \quad \text{Re}(\lambda) \geq 0 \quad (3.56)$$

the stabilizability Hautus criterion becomes

$$\text{Rank} \begin{bmatrix} (J-R)Q - \lambda I & Q_v^{1/2} \end{bmatrix} = n \quad \text{for} \quad \text{Re}(\lambda) \geq 0 \quad (3.57)$$

It is obvious that for asymptotically stable port Hamiltonian system, these conditions are met for any matrix \tilde{Q} and Q_v and both filter and control Riccati equations admit a unique solution.

Finally, we will show that the LQG controller is passive and it can be considered as a controller of port Hamiltonian system by interconnection. Firstly by using the condition (3.50) and the condition (3.48), the output of controller (3.38) becomes:

$$\begin{aligned} y_c &= (\tilde{R}^{-1}B^TP_cQ^{-1})Qx_c \\ \Leftrightarrow y_c &= (R_w^{-1}B^TQP_f)Qx_c \end{aligned} \quad (3.58)$$

which means that the output of the controller (3.38) y_c is port-conjugated to the input u_c . Secondly, if

we take the condition (3.50) and the condition (3.48), one can easily check that the matrix R_c defined in equation (3.37) is symmetric by computing $R_c^T - R_c = 0$. Our controller (3.38) is designed by LQG method, it is well known that the closed loop system by this type of controller is stable. Consequently by Chetaev's theorem [48, Thm 2.5], all the eigenvalues of the closed loop system must be in the left of the complex plane. Then the LQG controller is also stable and the eigenvalues of this controller are negative, thus matrix R_c is positive definite since the eigenvalues of the controller are the eigenvalues of matrix R_c . \square

The Theorem 3.3 gives us another passive LQG controller design method which is also equivalent to the control by interconnection of port Hamiltonian systems. We shall call the LQG problem defined by this theorem *the Q-conjugated LQG problem* since the two LQG Gramians P_c and P_f are related by the energy matrix Q .

It should be noticed that the parameters Q_v , R_w , \tilde{Q} and \tilde{R} are related. That means that on one hand we can firstly choose the optimal control criterion, i.e. the weighting matrices \tilde{Q} and \tilde{R} , then the filtering problem is fixed, since we compute the covariance matrices Q_v and R_w , according to the theorem. On the other hand, we can firstly concentrate on the noises of the given system, i.e. the covariance matrices Q_v and R_w , we can obtain the weighting matrix \tilde{Q} and \tilde{R} .

Remark 3.3. If we choose the observation problem first, i.e. the covariance matrices Q_v and R_w are given, then we can resolve the filter Riccati equation (3.32) to get the unique positive defined solution P_f , then the weighting matrices can be computed as

$$\tilde{R} = R_w \quad (3.59)$$

and

$$\tilde{Q} = Q(2JQP_f + 2P_fQJ^T + Q_v)Q. \quad (3.60)$$

This means that the LQR control problem criterion are used just for passive LQG controller design parameters and may be not the optimal control problem as we desire.

Remark 3.4. The other advantage of this LQG control design method is by using the specific choice of weighting and covariance matrices (3.49) and (3.48), the two solutions of FARE (3.32) and CARE (3.35) are related by

$$P_cQ^{-1} = QP_f. \quad (3.61)$$

This means that we just need to resolve one Riccati equation instead of two Riccati equations. For the numerical example, this method will be more effective than the other LQG methods.

Furthermore in this passive LQG controller design method problem, the product of the LQG Gramians is:

$$P_fP_c = P_fQP_fQ \neq I.$$

Because the product of the LQG Gramians is not equal to identity, the importance of the state variables for the control design in the Q-conjugated sense is different. This property allows us to find a balanced base in which we can separate the state variables to two parts. One part is more important for Q-conjugated LQG control design and other part is less important. Because of this point, we can reduce the system which we shall discuss in the next section.

3.3 Q-conjugated LQG balanced reduction of PHS with preserving the structure and passivity

In the last section, the Q-conjugated LQG control design method has been proposed. The LQG controller obtained by this method is also a port Hamiltonian system such that the passivity and the Hamiltonian structure are guaranteed in the closed loop system. However, this LQG controller has the same dimension as the system itself. In order to derive the lower order LQG controllers, we will consider a Q-conjugated LQG balanced reduction for open loop port Hamiltonian systems, then the low order LQG controller can be designed by the reduced order systems.

To get the reduced order system, we will consider the following steps. In the first step, we shall use the Q-conjugated LQG method in order to derive a balanced realization of the open loop port Hamiltonian system (3.28), called Q-conjugated LQG balanced realization. This balanced realization is defined in a specific coordinate system such that the state variables of the open loop port Hamiltonian system can be separated into two parts. One part is more important for the control design and another part is less important. In the second step, we will use the Effort constraint method to reduce the Q-conjugated LQG balanced realization in order to derive the reduced order port Hamiltonian system.

3.3.1 Balanced realization associated with the Q-conjugated LQG problem

Inspired the standard LQG balanced realization in Definition 3.1, we can define a *Q-conjugated LQG balanced realization* as follows:

Definition 3.4. The port Hamiltonian system (3.28) admits a *Q-conjugated LQG balanced realization* if the Gramians P_f^B and P_c^B of the Q-conjugated LQG problem of Theorem 3.3, are diagonal:

$$P_c^B = P_f^B = \Sigma = \text{diag}(\mu_1, \mu_2, \dots, \mu_n) \quad (3.62)$$

where, denoting by $\lambda_i(P)$ the i -th eigenvalue of a matrix P ,

$$\mu_i = \sqrt{\lambda_i(P_c^B P_f^B)} \text{ and } \mu_1 > \mu_2 > \dots > \mu_n > 0 \quad (3.63)$$

and P_c^B and P_f^B are the solutions of the Riccati equations of the balanced realization.

In this novel balanced system, the *Q-conjugated LQG* singular values μ_i are in the decrease order. This means in the sense of closed loop performance of this *Q-conjugated LQG* problem, that the importance of the state variables in novel balanced coordinate is different.

Remark 3.5. Following the Theorem 3.3, we have $P_c = QP_fQ$, then

$$P_c P_f = QP_f Q P_f.$$

Hence it is sufficient to diagonalize the matrix QP_f for the derivation of the Q-conjugated balanced realization.

This balanced realization can be got by the transformation matrices W_b and T_b defined as (3.17).

We first transform the coordinate by

$$x = T_b x_b \quad (3.64)$$

and then premultiply by matrix W_b on the two sides of the first equation of system (3.28), we obtain a Q-conjugated LQG balanced realization of the port Hamiltonian system which is denoted as follows

$$\begin{cases} \dot{x}_b &= (J_b - R_b)Q_b x_b + B_b u + v_b \\ y &= B_b^T Q_b x_b + w_b \end{cases} \quad (3.65)$$

where $J_b = W_b^T J W_b$, $R_b = W_b^T R W_b$, $Q_b = T_b^T Q T_b$, $B_b = W_b^T B$, $v_b = W_b^T v$ and $w_b = w$.

The Gramian of the balanced realization (3.65) Σ is given by diagonalizing the Gramians P_f and P_c of the Q-conjugated LQG problem with the transformation matrices W_b and T_b are given as:

$$P_c^B = P_f^B = T_b^T P_c T_b = W_b^T P_f W_b = \Sigma. \quad (3.66)$$

In the Q-conjugated LQG balanced coordinate system, the state variables are related to the Q-conjugated LQG singular values for the closed loop performance. It means that the state variables with the larger singular value play the more important role in the Q-conjugated LQG controller design. Hence this coordinate system can be used to reduce the system, it will be discussed in the next part.

3.3.2 Reduction by the Effort constraint method

In this sub-section we shall use the Q-conjugated balanced realization in order to reduce the system. The classical balanced reduction method is the truncation method. However, such method does not preserve the Hamiltonian structure and the passivity of the port Hamiltonian system [61]. Hence we shall use a geometric reduction method so-called *Effort constraint method* suggested in [65] to reduce the system. This method has the advantage of preserving the port Hamiltonian structure and thereby also the passivity properties. However we shall in the sequel adapt this method to explicit port Hamiltonian systems, as, in the original work, it is based on the definition of implicit port Hamiltonian systems [81, 19] defined with respect to a Dirac structure (Defined in Chapter 2). The implicit port Hamiltonian systems are defined as follows:

Definition 3.5. [19] An implicit linear dissipative port Hamiltonian system (PHS) with state variable $x \in \mathbb{R}^n$, port variables $(f_p, e_p) \in \mathbb{R}^m \times \mathbb{R}^m$, dissipation port variables $(f_R, e_R) \in \mathbb{R}^{m_R} \times \mathbb{R}^{m_R}$ with respect to the Dirac structure (2.1) is defined as follows

$$F \begin{bmatrix} \frac{dx}{dt} \\ f_p \\ f_R \end{bmatrix} + E \begin{bmatrix} Qx \\ e_p \\ e_R \end{bmatrix} = 0$$

completed with the dissipative closure relation $e_R = -R f_R$ with $R = R^T \in \mathbb{R}^{r \times r}$ being positive.

Now assume that the Q-conjugated LQG singular values (3.63) are split into two sets $(\mu_i)_{i=1, \dots, r} \gg (\mu_i)_{i=r+1, \dots, n}$ and have the objective to reduce the system in such a way to retain only the states associated with the first r singular values. By using the flow and effort variables $f_b = -\dot{x}_b$, $e_b = Q_b x_b$, splitting these vectors according to the two sets of singular values (e.g. $x_b^T = (x_{b1}^T, x_{b2}^T) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$),

the resistive relations $R_b = g_{rb}\hat{R}g_{rb}^T$, $e_R = -\hat{R}f_R$, and port variables $u = e_p$ and $y = f_p$, where $f_b, e_b \in \mathbb{R}^n$, $f_p, e_p \in \mathbb{R}^m$, $f_R, e_R \in \mathbb{R}^{m_R}$, the Q-conjugated balanced realization (3.65) may be expressed as an implicit port Hamiltonian system with the following parametrization:

$$F = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_{m_R} \end{bmatrix}$$

$$E = \begin{bmatrix} J_{b11} & J_{b12} & B_{b1} & g_{br1} \\ J_{b21} & J_{b22} & B_{b2} & g_{br2} \\ B_{b1}^T & B_{b2}^T & 0 & 0 \\ g_{br1}^T & g_{br2}^T & 0 & 0 \end{bmatrix}$$

By using the effort constraint method [65], the reduced model is obtained by imposing the constraint

$$e_{b2} = 0$$

Following [65] it can be shown that the reduced system with state variable x_{b1} , can then be again expressed as an implicit port Hamiltonian system with respect to the Dirac structure defined by the reduced matrices F_r and E_r

$$F_r = L^e F M_e = \begin{bmatrix} I_r & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & -I_{m_R} \end{bmatrix} \quad (3.67)$$

$$E_r = L^e E M_e = \begin{bmatrix} J_{11} & B_{b1} & g_{br1} \\ B_{b1}^T & 0 & 0 \\ g_{br1}^T & 0 & 0 \end{bmatrix} \quad (3.68)$$

where the matrix $M_e \in \mathbb{R}^{(n+m+m_R) \times (r+m+m_R)}$ is defined as:

$$M_e = \begin{bmatrix} I_r & 0_{r \times m} & 0_{r \times m_R} \\ 0_{(n-r) \times r} & 0_{(n-r) \times m} & 0_{(n-r) \times m_R} \\ 0_{m \times r} & I_m & 0_{m \times m_R} \\ 0_{m_R \times r} & 0_{m_R \times m} & I_{m_R} \end{bmatrix} \quad (3.69)$$

and the projector matrix $L^e \in \mathbb{R}^{(r+m+m_R) \times (n+m+m_R)}$ is given by:

$$L^e = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_{m_R} \end{bmatrix} \quad (3.70)$$

and generated by the energy matrix Q_b which is the Schur complement $Q_s = Q_{b11} - Q_{b12}Q_{b22}^{-1}Q_{b21}$.

It appears that this reduced port Hamiltonian system is actually again an explicit port Hamiltonian

system of the form:

$$\begin{cases} \dot{x}_{b1} &= (J_{11} - R_{11})Q_s x_{b1} + B_{b1}u \\ y &= B_{b1}^T Q_s x_{b1} \end{cases} \quad (3.71)$$

By using the reduced system (3.71) and Theorem 3.3, one can design a reduced order controller for the system in order to stabilize the full order system [36]. The *error estimation* between the full order port Hamiltonian system (3.28) and the reduced order port Hamiltonian system (3.71) should be considered, but there are two difficulties: the first one is that we use a modified LQG balanced realization and the second one is that we use the effort constraint method instead of the classic truncation method to reduce the system. The error estimation will be suggested in the next section (Section 3.4).

To show the effectiveness of the proposed model and controller reduction method, we shall illustrate the method on a mass-spring-damper system in Section 3.6.

3.4 Error estimation

In this sub section, we shall discuss how to characterize the error of the Q-conjugated balanced method associate with the effort constraint method. The error estimation is related to the two following parts. Firstly the effort constraint method, and secondly the Q-conjugated balanced realization.

3.4.1 Error estimation of the effort constraint method

To determinate the error of effort constraint method, we will first discuss the reduction error estimation on the balanced coordinate associated with controllability and observability gramians which are the unique solutions of two Lyapunov equations.

Consider the Lyapunov balanced realization of port Hamiltonian system as

$$\begin{cases} \dot{x}_b &= \underbrace{(J_b - R_b)}_{F_b} Q_b x_b + B_b u \\ y &= B_b^T Q_b x_b \end{cases} \quad (3.72)$$

associated with the following controllability and observability Lyapunov equations:

$$\begin{aligned} A_b \Sigma + \Sigma A_b^T + B_b B_b^T &= 0 \\ A_b^T \Sigma + \Sigma A_b + Q_b B_b B_b^T Q_b &= 0 \end{aligned} \quad (3.73)$$

where $A_b = F_b Q_b$, $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$.

We use the effort constraint method to reduce the balanced port Hamiltonian system (3.72) while preserving the Hamiltonian structure (More details was presented in Section 2.4 and Section 3.3). The reduced port Hamiltonian system is given as

$$\begin{cases} \dot{x}_1 &= F_{11} Q_s x_1 + B_1 u \\ y &= B_1^T Q_s x_1 \end{cases} \quad (3.74)$$

with $Q_s = Q_{b11} - Q_{b12} Q_{b22}^{-1} Q_{b21}$ is the Schur complement of Q_b .

In the literature, an error bound is given for the case of truncated based reduction method used for the Lyapunov balanced realization. This error is expressed as twice the sum of the truncated singular values.

In order to characterize the error bound of the effort constraint method, we will relate the reduced order system (3.74) to a reduced order system obtained by using truncation method from the input output equivalent point of view. We will show this relation in the following Lemma.

Lemma 3.1. *Consider the Lyapunov balanced realization of port Hamiltonian system (3.72):*

$$\begin{cases} \dot{x}_b &= \underbrace{(J_b - R_b)}_{F_b} Q_b x_b + B_b u \\ y &= B_b^T Q_b x_b \end{cases} \quad (3.75)$$

and its representation in the new coordinates

$$\begin{cases} \dot{\tilde{x}} &= \tilde{F} \tilde{x} + \tilde{B} u \\ y &= \tilde{B}^T \tilde{x} \end{cases} \quad (3.76)$$

where $\tilde{F} = S F_b S^T$, $\tilde{B} = S B_b$ and $\tilde{x} = S x_b$ with

$$S = \begin{bmatrix} Q_s^{\frac{1}{2}} & 0 \\ Q_{b22}^{-\frac{1}{2}} Q_{b21} & Q_{b22}^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \quad (3.77)$$

is the a decomposition of Q_b such

$$Q_b = S^T S. \quad (3.78)$$

Then the effort constraint reduced system (3.72):

$$\begin{cases} \dot{x}_1 &= F_{11} Q_s x_1 + B_1 u \\ y &= B_1^T Q_s x_b \end{cases} \quad (3.79)$$

with $Q_s = Q_{b11} - Q_{b12} Q_{b22}^{-1} Q_{b21}$, and the reduced system of (3.76) by using truncation method

$$\begin{cases} \dot{\tilde{x}}_1 &= \tilde{F}_{11} \tilde{x}_1 + \tilde{B}_1 u \\ y &= \tilde{B}_1^T \tilde{x}_1 \end{cases} \quad (3.80)$$

are input-output equivalent by coordinate change $\tilde{x}_1 = Q_s^{\frac{1}{2}} x_{b1}$.

Proof. We can easily check $Q_b = S^T S$ by compute by that

$$\begin{bmatrix} Q_s^{\frac{1}{2}} & Q_{b21}^T Q_{b22}^{-\frac{1}{2}} \\ 0 & Q_{b22}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} Q_s^{\frac{1}{2}} & 0 \\ Q_{b22}^{-\frac{1}{2}} Q_{b21} & Q_{b22}^{\frac{1}{2}} \end{bmatrix} = Q_b \quad (3.81)$$

Consider the state matrix \tilde{F} of the truncation reduced order system in block form as

$$\tilde{F} = \begin{bmatrix} Q_s^{\frac{1}{2}} F_{b11} Q_s^{\frac{1}{2}T} & * \\ * & * \end{bmatrix} \quad (3.82)$$

and input matrix

$$\tilde{B} = \begin{bmatrix} Q_s^{\frac{1}{2}} B_{b1} \\ * \end{bmatrix} \quad (3.83)$$

The reduced order system by using truncation method is

$$\begin{cases} \dot{\tilde{x}}_1 &= \underbrace{Q_s^{\frac{1}{2}} F_{b11} Q_s^{\frac{1}{2}T}}_{\tilde{F}_{11}} \tilde{x}_1 + \underbrace{Q_s^{\frac{1}{2}} B_{b1} u}_{\tilde{B}_1} \\ y &= \underbrace{B_{b1}^T Q_s^{\frac{1}{2}T}}_{\tilde{B}_1^T} \tilde{x}_1 \end{cases} \quad (3.84)$$

By using the coordinate change $\tilde{x}_1 = Q_s^{\frac{1}{2}} x_{b1}$, the above system becomes as

$$\begin{cases} \dot{x}_{b1} &= F_{b11} Q_s x_{b1} + B_{b1} u \\ y &= B_{b1}^T Q_s x_{b1} \end{cases} \quad (3.85)$$

As consequence the truncation reduced system (3.80) is input-output equivalent to the effort constraint reduced system (3.74). \square

From the Lemma 3.1, one can change the error estimation of reduced order system (3.74) obtained by using the effort constraint method to the truncation reduced one (3.80) such as

$$\| G(s) - G_r^E(s) \| = \| G(s) - G_r^T(s) \| \quad (3.86)$$

where the $G(s)$ is the transfer function of full order system:

$$G(s) = B_b^T Q_b (sI - F_b Q_b)^{-1} B_b = \tilde{B}^T (sI - \tilde{F}) \tilde{B}, \quad (3.87)$$

$G_r^E(s)$ is the transfer function of reduced order system by using the Effort constraint method:

$$G_r^E(s) = B_{b1}^T Q_s (sI - F_{b11} Q_s)^{-1} B_{b1}, \quad (3.88)$$

and $G_r^T(s)$ is the transfer function of truncation reduced order system:

$$G_r^T(s) = \tilde{B}_1^T (sI - \tilde{F}_{11})^{-1} \tilde{B}_1. \quad (3.89)$$

To characterize the error bound and simplify the demonstration, we first introduce the following notation for system (3.76):

$$\tilde{F} = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } \tilde{B} = B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \quad (3.90)$$

Then the transfer function of system (3.76) and the reduced system (3.80) can be written as follows:

$$G(s) = B^T (sI - A)^{-1} B \quad (3.91)$$

and

$$G_r^E(s) = B_1^T (sI - A_{11})^{-1} B_1. \quad (3.92)$$

respectively. Controllability and observability Lyapunov equations associated to system of (3.91) are:

$$\begin{cases} AP_c + P_c A^T + BB^T = 0 \\ A^T P_f + P_f A + BB^T = 0 \end{cases}. \quad (3.93)$$

The solutions P_c and P_f can be related to the Hankel matrix Σ by coordinate change matrix S (given in (3.77)) as follows:

$$\begin{cases} P_c = S\Sigma S^T \\ P_f = S^{-T}\Sigma S^{-1} \end{cases} \quad (3.94)$$

Let us introduce the following notations:

$$\begin{aligned} \phi(s) &:= (sI - A_{11})^{-1} \\ \psi(s) &:= sI - A_{22} - A_{21}\phi(s)A_{12} \\ \bar{B}(s) &:= A_{21}\phi(s)B_1 + B_2 \\ \bar{C}(s) &:= B_1^T\phi(s)A_{12} + B_2^T \end{aligned} \quad (3.95)$$

Proposition 3.1. *Consider the port Hamiltonian system (3.72) with its transfer function $G(s)$ and its reduced order system by using the effort constraint method (3.74) with its transfer function $G_r^E(s)$, then*

$$\|G(s) - G_r^E(s)\|_\infty \leq \lambda_{\max}^{1/2} ([L + \psi^{-1}(j\omega)L^*(j\omega)] \cdot [M + \psi^{-*}(j\omega)M^*\psi(j\omega)]) \quad (3.96)$$

with

$$L = S_{22}\Sigma_2 S_{22} + S_{21}\Sigma_1 [S_{21}^T + S_{11}\phi^*(j\omega)A_{21}^T] \quad (3.97)$$

and

$$M = S_{22}^{-1}\Sigma_2 S_{22}^{-1} [I - S_{21}S_{11}^{-1}\phi(j\omega)A_{12}]. \quad (3.98)$$

and where $\lambda_{\max}^{1/2}$ means the maximal frequency eigenvalues of $G(s) - G_r^E(s)$.

Proof. From the Lemma 3.1, The reduced order system (3.74) and the truncation reduced system (3.80) are input output equivalent. Then the error estimation of the effort constraint method can be computed by using the well known error bound of the truncation method such as:

$$G(s) - G_r^E(s) = G(s) - G_r^T(s) \quad (3.99)$$

By using the same procedure used for truncation method used in [86, Sec 2.3] and with the notations given in (3.95), we can obtain:

$$G(s) - G_r(s) = \bar{C}(s)\psi^{-1}(s)\bar{B}(s) \quad (3.100)$$

computing this quantity on the imaginary axis to get the maximal singular value of the frequency response of the error transfer function

$$\bar{\sigma}[G(j\omega) - G_r(j\omega)] = \lambda_{\max}^{1/2} [\psi^{-1}(j\omega)\bar{B}(j\omega)\bar{B}^*(j\omega)\psi^{-*}(j\omega)\bar{C}^*(j\omega)\bar{C}(j\omega)] \quad (3.101)$$

with

$$\begin{aligned} \bar{B}(j\omega)\bar{B}^*(j\omega) &= A_{21}\phi(j\omega)B_1B_1^T\phi^*(j\omega)A_{21}^T + A_{21}\phi(j\omega)B_1B_2^T \\ &\quad + B_2B_1^T\phi^*(j\omega)A_{21}^T + B_2B_2^T \end{aligned} \quad (3.102)$$

and

$$\begin{aligned} \bar{C}^*(j\omega)\bar{C}(j\omega) &= A_{12}^T\phi^*(j\omega)B_1B_1^T\phi(j\omega)A_{12} + A_{12}^T\phi^*(j\omega)B_1B_2^T \\ &\quad + B_2B_1^T\phi(j\omega)A_{12} + B_2B_2^T \end{aligned} \quad (3.103)$$

We denote that $T = S^{-1}$ as

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \quad (3.104)$$

with $Q_b^{-1} = TT^T$. Then the Gramian matrices P_c and P_f can be written under the block form

$$\begin{aligned} P_c &= \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} S_{11}^T & S_{21}^T \\ 0 & S_{22}^T \end{bmatrix} \\ &= \begin{bmatrix} S_{11}\Sigma_1S_{11}^T & S_{11}\Sigma_1S_{21}^T \\ S_{21}\Sigma_1S_{11}^T & S_{21}\Sigma_1S_{21}^T + S_{22}\Sigma_2S_{22}^T \end{bmatrix} \\ &= \begin{bmatrix} P_c^{11} & P_c^{12} \\ P_c^{21} & P_c^{22} \end{bmatrix} \end{aligned} \quad (3.105)$$

$$\begin{aligned} P_f &= T^T\Sigma T \\ &= \begin{bmatrix} T_{11}^T & T_{21}^T \\ 0 & T_{22}^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} T_{11}^T\Sigma_1T_{11} + T_{21}^T\Sigma_2T_{21} & T_{21}^T\Sigma_2T_{22} \\ T_{22}^T\Sigma_2T_{21} & T_{22}^T\Sigma_2T_{22} \end{bmatrix} \\ &= \begin{bmatrix} P_f^{11} & P_f^{12} \\ P_f^{21} & P_f^{22} \end{bmatrix} \end{aligned} \quad (3.106)$$

Then the controllability Lyapunov equation of system (3.76) is

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} P_c^{11} & P_c^{12} \\ P_c^{21} & P_c^{22} \end{bmatrix} + \begin{bmatrix} P_c^{11} & P_c^{12} \\ P_c^{21} & P_c^{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} = 0 \quad (3.107)$$

We can write it in terms of partitioned form as follows

$$\begin{cases} A_{11}P_c^{11} + A_{12}P_c^{21} + P_c^{11}A_{11}^T + P_c^{12}A_{12}^T + B_1B_1^T = 0 \\ A_{11}P_c^{12} + A_{12}P_c^{22} + P_c^{11}A_{21}^T + P_c^{21}A_{22}^T + B_1B_2^T = 0 \\ A_{21}P_c^{11} + A_{22}P_c^{21} + P_c^{21}A_{11}^T + P_c^{22}A_{12}^T + B_2B_1^T = 0 \\ A_{21}P_c^{12} + A_{22}P_c^{22} + P_c^{21}A_{21}^T + P_c^{22}A_{22}^T + B_2B_2^T = 0 \end{cases} \quad (3.108)$$

and the observability Lyapunov equation of system (3.76) can be written in the same way

$$\begin{cases} A_{11}^T P_f^{11} + A_{12}^T P_f^{21} + P_f^{11} A_{11} + P_f^{12} A_{12} + B_1 B_1^T = 0 \\ A_{11}^T P_f^{12} + A_{12}^T P_f^{22} + P_f^{11} A_{21} + P_f^{21} A_{22} + B_1 B_2^T = 0 \\ A_{21}^T P_f^{11} + A_{22}^T P_f^{21} + P_f^{21} A_{11} + P_f^{22} A_{12} + B_2 B_1^T = 0 \\ A_{21}^T P_f^{12} + A_{22}^T P_f^{22} + P_f^{21} A_{21} + P_f^{22} A_{22} + B_2 B_2^T = 0 \end{cases} \quad (3.109)$$

Now we can compute $\bar{B}(j\omega)\bar{B}^*(j\omega)$ by equations (3.108). The result is given as follows

$$\bar{B}(j\omega)\bar{B}^*(j\omega) = [P_c^{22} + A_{21}\phi(j\omega)P_c^{12}] \psi^*(j\omega) + \psi(j\omega) [P_c^{22} + P_c^{21}\phi^*(j\omega)A_{21}^T] \quad (3.110)$$

We can also compute $\bar{C}^*(j\omega)\bar{C}(j\omega)$ by equations (3.109). The result is given as follows

$$\bar{C}^*(j\omega)\bar{C}(j\omega) = [P_f^{22} + A_{12}^T\phi^*(j\omega)P_f^{12}] \psi(j\omega) + \psi^*(j\omega) [P_f^{22} + P_f^{21}\phi(j\omega)A_{12}] \quad (3.111)$$

Hence we can get

$$\begin{aligned} & \psi^{-1}(j\omega)\bar{B}(j\omega)\bar{B}^*(j\omega)\psi^{-*}(j\omega)\bar{C}^*(j\omega)\bar{C}(j\omega) \\ = & \left[(P_c^{22} + P_c^{21}\phi^*(j\omega)A_{21}^T) + \psi^{-1}(j\omega)(P_c^{22} + A_{21}\phi(j\omega)P_c^{12})\psi^*(j\omega) \right] \\ & \left[(P_f^{22} + P_f^{21}\phi(j\omega)A_{12}) + \psi^{-*}(j\omega)(P_f^{22} + A_{12}^T\phi^*(j\omega)P_f^{12})\psi(j\omega) \right] \end{aligned} \quad (3.112)$$

We use the equations (3.105) and (3.106) to relate the above equation with the singular values. Then we can obtain

$$\begin{aligned} & \psi^{-1}(j\omega)\bar{B}(j\omega)\bar{B}^*(j\omega)\psi^{-*}(j\omega)\bar{C}^*(j\omega)\bar{C}(j\omega) \\ = & [L + \psi^{-1}(j\omega)L^*(j\omega)] \cdot [M + \psi^{-*}(j\omega)M^*\psi(j\omega)] \end{aligned} \quad (3.113)$$

with

$$L = S_{22}\Sigma_2S_{22} + S_{21}\Sigma_1S_{21}^T [I + S_{21}^{-T}S_{11}\phi^*(j\omega)A_{21}^T] \quad (3.114)$$

and

$$M = S_{22}^{-1}\Sigma_2S_{22}^{-1} [I - S_{21}S_{11}^{-1}\phi(j\omega)A_{12}] \quad (3.115)$$

Finally we obtain the following result

$$\begin{aligned} \|G(s) - G_r(s)\|_\infty & \leq \bar{\sigma} [G(j\omega) - G_r(j\omega)] = \lambda_{\max}^{1/2} [\psi^{-1}(j\omega)\bar{B}(j\omega)\bar{B}^*(j\omega)\psi^{-*}(j\omega)\bar{C}^*(j\omega)\bar{C}(j\omega)] \\ & = \lambda_{\max}^{1/2} ([L + \psi^{-1}(j\omega)L^*(j\omega)] \cdot [M + \psi^{-*}(j\omega)M^*\psi(j\omega)]) \end{aligned} \quad (3.116)$$

□

From the Proposition 3.1 we can see that the error of the effort constraint based reduction method is completely different from the one obtained using truncation method, where the error depends only on Σ_2 . The error of the effort constraint method depends at the same time on Σ_2 and Σ_1 . Hence the computing of the upper error bound (maximal value over ω of the error) for the effort constraint method is still an open problem. Moreover, the equation (3.112) is only the error bound on the Lyapunov balanced coordinate. In the next sub section we will discuss how to relate the Q-conjugated balanced coordinate to the Lyapunov balanced one.

3.4.2 Error estimation on the Q-conjugated balanced coordinate

In order to determinate the error on the Q-conjugated balanced coordinate, we shall first recall the error estimation of the standard LQG method.

Consider the system:

$$\begin{cases} \dot{x} & = Ax + Bu \\ y & = Cx \end{cases} \quad (3.117)$$

and its transfer function

$$G(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = C(sI - A)^{-1}B \quad (3.118)$$

The standard LQG problem is associated with the two following Riccati equations:

- Control Riccati equation

$$A^T P_c + P_c A + C^T C - P_c B B^T P_c = 0 \quad (3.119)$$

- Filtering Riccati equation

$$A P_f + P_f A^T + B B^T - P_f C^T C P_f = 0 \quad (3.120)$$

where $P_c = P_c^T > 0$ and $P_f = P_f^T > 0$. The feedback gain is $K = B^T P_c$ and the filter gain is $F = P_f C$.

The idea of the error estimation of reduction associated with the above standard LQG problem is to find the relation of this standard LQG reduction with the Lyapunov balanced reduction method. This relation is found by the normalized right coprime factorization of transfer function (3.118) $G(s)$ which is given as:

$$\begin{bmatrix} \tilde{M}(s) \\ \tilde{N}(s) \end{bmatrix} = \begin{bmatrix} A - BK & B \\ K \\ C \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A_{rcf} & B_{rcf} \\ C_{rcf} & F_{rcf} \end{bmatrix} \quad (3.121)$$

where

$$\begin{aligned} \tilde{M}(s) &= C[sI - A_{rcf}]^{-1}B & \tilde{N}(s) &= K[sI - A_{rcf}]^{-1}B + I \\ G(s) &= \tilde{N}(s)\tilde{M}^{-1}(s) \end{aligned} \quad (3.122)$$

The Lyapunov equations of the right coprime factorization system (3.121) are given as:

$$\begin{aligned} A_{rcf}M + MA_{rcf}^T + B_{rcf}B_{rcf}^T &= 0 \\ A_{rcf}^T P + PA_{rcf} + C_{rcf}^T C_{rcf} &= 0 \end{aligned} \quad (3.123)$$

with M and P , the symmetric positive definite matrices, are the unique solutions of the Lyapunov equations. The relations between the solutions of Riccati equation P_f and P_c and the solutions of Lyapunov equations M and P are given by the following Lemma:

Lemma 3.2. [47] *The controllability and observability Gramians of $\begin{bmatrix} \tilde{M}(s) \\ \tilde{N}(s) \end{bmatrix}$, M and P respectively, are given by*

$$\begin{aligned} M &= (I + P_f P_c)^{-1} P_f \\ P &= P_c \end{aligned} \quad (3.124)$$

where P_c, P_f are the unique positive definite solutions to CARE (3.119) and FARE (3.120) respectively.

Then following this Lemma, we can find the relation between the LQG singular values and the Hankel singular values of the right coprime factorization system (3.121). The LQG singular values are

$$\mu_i = \lambda_i \sqrt{P_f P_c} \quad (3.125)$$

and the Hankel singular values are

$$\sigma_i = \lambda_i \sqrt{MP} \quad (3.126)$$

where $\lambda_i(H)$ are the eigenvalues of the matrix H . From Lemma 3.2, we can get

$$\begin{aligned} \sigma_i^2 &= \lambda_i(MP) \\ &= \lambda_i \left((I + P_f P_c)^{-1} P_f P_c \right) \\ &= \frac{\lambda_i(P_f P_c)}{1 + \lambda_i(P_f P_c)} \\ &= \frac{\mu_i^2}{1 + \mu_i^2} \end{aligned} \quad (3.127)$$

then

$$\sigma_i = \sqrt{\frac{\mu_i^2}{1 + \mu_i^2}} \quad (3.128)$$

It is well known that the error bound of Lyapunov balanced reduction is

$$\|G - G_r^{LYP}\|_\infty \leq 2 \sum_{i=r+1}^n \sigma_i \quad (3.129)$$

Then the error bound of standard LQG balanced reduction can be presented as:

$$\|G - G_r^{SLQG}\|_\infty \leq 2 \sum_{i=r+1}^n \frac{\mu_i}{\sqrt{1 + \mu_i^2}} \quad (3.130)$$

Now we will discuss how to get the error bound of the Q-conjugated LQG balanced reduction. Let recall the two Riccati equations associated with the Q-conjugated LQG problem:

$$AP_f + P_f A^T - P_f C^T R_w^{-1} C P_f + Q_v = 0 \quad (3.131)$$

and

$$A^T P_c + P_c A - P_c B \tilde{R}^{-1} B^T P_c + \tilde{Q} = 0 \quad (3.132)$$

with $A = (J - R)Q$, $C = B^T Q$. We will find a new system for which observability Lyapunov equation is equivalent to CARE (3.132) of port Hamiltonian system which have the same solution, i.e. $P = P_c$:

$$\underbrace{\left[A - B \tilde{R}^{-1} B^T P_c \right]^T}_{A_n^T} P + P \underbrace{\left[A - B \tilde{R}^{-1} B^T P_c \right]}_{A_n} + \underbrace{P_c B \tilde{R}^{-1} B^T P_c + \tilde{Q}}_{C_n^T C_n} = 0 \quad (3.133)$$

Then the new system can be written as:

$$\begin{bmatrix} A - B \tilde{R}^{-1} B^T P_c & B_n \\ \begin{bmatrix} \tilde{R}^{1/2} B^T P_c \\ \tilde{Q}^{1/2} \end{bmatrix} & D_n \end{bmatrix} \quad (3.134)$$

This new system looks like the right coprime factorization system (3.121) but the output matrix is different, so we call it right coprime factorization like system. And its controllability Lyapunov

equation is written as

$$\underbrace{\left[A - B\tilde{R}^{-1}B^T P_c \right]}_{A_n} M + M \underbrace{\left[A - B\tilde{R}^{-1}B^T P_c \right]^T}_{A_n^T} + B_n B_n^T = 0 \quad (3.135)$$

By using the observability Lyapunov equation (3.133) of right coprime factorization like system and the CARE (3.132) of port Hamiltonian system, we can find the relation of the two solutions $P = P_c$. But how to find the relation between the solutions of the FARE (3.131) and the controllability Lyapunov equation (3.135) P_f and M is still a difficulty.

We shall consider another way to get a coprime factorization like system for relating the solutions of Lyapunov and Riccati equations.

Proposition 3.2. *Consider the port Hamiltonian system:*

$$\begin{cases} \dot{x} &= (J - R)Qx + Bu \\ y &= B^T Qx \end{cases} \quad (3.136)$$

associated with the Q -conjugated LQG problem and two Riccati equations:

$$\begin{aligned} (J - R)QP_f + P_f Q(J - R)^T - P_f QBR_w^{-1}B^T QP_f + Q_v &= 0 \\ Q(J - R)^T P_c + P_c(J - R)Q - P_c B\tilde{R}^{-1}B^T P_c + \tilde{Q} &= 0 \end{aligned} \quad (3.137)$$

with a specific choice $\tilde{Q} = QBR_w^{-1}B^T Q$ and a new system

$$\begin{cases} \dot{\bar{x}} &= A_{n1}\bar{x} + B_{n1}u \\ y &= C_{n1}\bar{x} \end{cases} \quad (3.138)$$

associated with the controllability and observability Lyapunov equations:

$$\begin{aligned} A_{n1}^T P + P A_{n1} + C_{n1}^T C_{n1} &= 0 \\ A_{n1} M + M A_{n1}^T + B_{n1} B_{n1}^T &= 0 \end{aligned} \quad (3.139)$$

with

$$\begin{aligned} A_{n1} &= A - B\tilde{R}^{-1}B^T P_c - LP_c, \quad B_{n1} = K^{1/2} \\ C_{n1} &= \begin{bmatrix} Q_v^{1/2} P_c \\ \tilde{Q}^{1/2} \end{bmatrix} \end{aligned} \quad (3.140)$$

and

$$\begin{aligned} L &= J^T QP_f + P_f QJ \\ K &= Q_v - (I + P_f P_c)^{-1} P_f P_c L (I + P_c P_f)^{-1} - (I + P_f P_c)^{-1} L P_c P_f (I + P_c P_f)^{-1} \end{aligned} \quad (3.141)$$

Then the solutions of Riccati equations and Lyapunov equations are related by:

$$\begin{aligned} P &= P_c \\ M &= P_f (I + P_c P_f)^{-1} \end{aligned} \quad (3.142)$$

Proof. First we recall the Q -conjugated LQG controller is obtained under the following conditions:

$$\begin{aligned}\tilde{R} &= R_w \\ Q_v &= Q^{-1} \left[2QJ^T P_c + 2P_c JQ + \tilde{Q} \right] Q^{-1} \\ P_c Q^{-1} &= QP_f\end{aligned}$$

The third condition shows that Q_v and \tilde{Q} are related. In practice we can fix one weighting matrix and calculate the second one using this condition.

Let us consider the *specific Q-conjugated LQG controller* where we fix \tilde{Q} :

$$\tilde{Q} = C^T R_w^{-1} C = QBR_w^{-1} B^T Q, \quad C = B^T Q \quad (3.143)$$

with this specific choice of \tilde{Q} , Q_v can be expressed as follows:

$$Q_v = B\tilde{R}^{-1}B^T + 2\underbrace{(J^T QP_f + P_f QJ)}_L \quad (3.144)$$

Thus the Riccati equations are rewritten as:

$$AP_f + P_f A^T - P_f C^T R_w^{-1} C P_f = -Q_v \quad (3.145)$$

$$A^T P_c + P_c A - P_c B\tilde{R}^{-1} B^T P_c = -C^T R_w^{-1} C \quad (3.146)$$

We replace the expression of $-C^T R_w^{-1} C$ taken from the filtering Riccati equation (3.145) in the control Riccati equation (3.146):

$$AP_f + P_f A^T + P_f A^T P_c P_f + P_f P_c A P_f - P_f P_c B\tilde{R}^{-1} B^T P_c P_f = -Q_v \quad (3.147)$$

$$\iff (I + P_f P_c) A P_f + P_f A^T (I + P_c P_f) - P_f P_c B\tilde{R}^{-1} B^T P_c P_f = -Q_v \quad (3.148)$$

The control Riccati equation can be written as:

$$\begin{aligned}\underbrace{\left[A - B\tilde{R}^{-1} B^T P_c - L P_c \right]^T P_c}_{A_{n1}^T} &+ P_c \underbrace{\left[A - B\tilde{R}^{-1} B^T P_c - L P_c \right]}_{A_{n1}} \\ &+ \underbrace{P_c B\tilde{R}^{-1} B^T P_c + 2P_c L P_c + QBR_w^{-1} B^T Q}_{C_{n1}^T C_{n1}} = 0\end{aligned} \quad (3.149)$$

or

$$\underbrace{\left[A - B\tilde{R}^{-1} B^T P_c - L P_c \right]^T P_c}_{A_{n1}^T} + P_c \underbrace{\left[A - B\tilde{R}^{-1} B^T P_c - L P_c \right]}_{A_{n1}} + \underbrace{P_c Q_v P_c + QBR_w^{-1} B^T Q}_{C_{n1}^T C_{n1}} = 0 \quad (3.150)$$

which is equivalent to the following Lyapunov equation:

$$A_{n1}^T P_c + P_c A_{n1} + C_{n1}^T C_{n1} = 0 \quad (3.151)$$

where

$$\begin{aligned}A_{n1} &= A - B\tilde{R}^{-1} B^T P_c - L P_c \\ C_{n1} &= \begin{bmatrix} Q_v^{1/2} P_c \\ R_w^{1/2} B^T Q \end{bmatrix} = \begin{bmatrix} Q_v^{1/2} P_c \\ \tilde{Q}^{1/2} \end{bmatrix}\end{aligned} \quad (3.152)$$

Inspired the demonstration of standard LQG balanced reduction in [47], we consider the following equality:

$$\begin{aligned}
-(I + P_f P_c) Q_v (I + P_c P_f) &= -Q_v - P_f P_c Q_v - Q_v P_c P_f - P_f P_c Q_v P_c P_f \\
&= -Q_v - P_f P_c \left[B\tilde{R}^{-1}B^T + 2L \right] - \left[B\tilde{R}^{-1}B^T + 2L \right] P_c P_f \\
&\quad - P_f P_c \left[B\tilde{R}^{-1}B^T + 2L \right] P_c P_f
\end{aligned} \tag{3.153}$$

Then we replace Q_v by its expression from the filter Riccati equation (3.145):

$$\begin{aligned}
-(I + P_f P_c) Q_v (I + P_c P_f) &= \left[(I + P_f P_c) A P_f + P_f A^T (I + P_c P_f) - P_f P_c B\tilde{R}^{-1}B^T P_c P_f \right] \\
&\quad - P_f P_c \left[B\tilde{R}^{-1}B^T + 2L \right] - \left[B\tilde{R}^{-1}B^T + 2L \right] P_c P_f \\
&\quad - P_f P_c \left[B\tilde{R}^{-1}B^T + 2L \right] P_c P_f \\
&= (I + P_f P_c) A P_f + P_f A^T (I + P_c P_f) \\
&\quad - (I + P_f P_c) \left(B\tilde{R}^{-1}B^T P_c + L P_c \right) P_f \\
&\quad - P_f \left(P_c B\tilde{R}^{-1}B^T + P_c L \right) (I + P_c P_f) - P_f P_c L - L P_c P_f \\
&= (I + P_f P_c) \left[A - B\tilde{R}^{-1}B^T P_c - L P_c \right] P_f \\
&\quad + P_f \left[A - B\tilde{R}^{-1}B^T P_c - L P_c \right]^T (I + P_c P_f) \\
&\quad - P_f P_c L - L P_c P_f
\end{aligned} \tag{3.154}$$

Finally we get:

$$\begin{aligned}
&\underbrace{A_{n1} P_f (I + P_c P_f)^{-1}}_M + \underbrace{(I + P_f P_c)^{-1} P_f A_{n1}}_M \\
&+ Q_v - (I + P_f P_c)^{-1} P_f P_c L (I + P_c P_f)^{-1} \\
&- (I + P_f P_c)^{-1} L P_c P_f (I + P_c P_f)^{-1} = 0
\end{aligned} \tag{3.155}$$

If we take

$$K = Q_v - (I + P_f P_c)^{-1} P_f P_c L (I + P_c P_f)^{-1} - (I + P_f P_c)^{-1} L P_c P_f (I + P_c P_f)^{-1} \tag{3.156}$$

then we can get a new system as

$$\begin{bmatrix} A_{n1} & B_{n1} \\ C_{n1} & D_{n1} \end{bmatrix} \tag{3.157}$$

with

$$\begin{aligned}
A_{n1} &= A - B\tilde{R}^{-1}B^T P_c - L P_c, \quad B_{n1} = K^{1/2} \\
C_{n1} &= \begin{bmatrix} Q_v^{1/2} P_c \\ \tilde{Q}^{1/2} \end{bmatrix}
\end{aligned} \tag{3.158}$$

and associate the controllability and observability Lyapunov equations:

$$\begin{aligned}
A_{n1}^T P + P A_{n1} + C_{n1}^T C_{n1} &= 0 \\
A_{n1} M + M A_{n1}^T + B_{n1} B_{n1}^T &= 0
\end{aligned} \tag{3.159}$$

Then the solutions of the Riccati equations of port Hamiltonian system P_c and P_f have the

following relation between the solutions of the Lyapunov equations of the system (3.157):

$$\begin{aligned} P &= P_c \\ M &= P_f (I + P_c P_f)^{-1} \end{aligned} \tag{3.160}$$

□

The Proposition 3.2 gives us the same relation (3.160) of the Riccati and Lyapunov equation solutions for the Q-conjugated LQG balanced reduction as the standard LQG balanced reduction. Since in standard LQG balanced method, we use the relation between the Lyapunov equations of right coprime factorization and the Riccati equations of the system itself because the equivalence of the transfer functions. But the *problem* here is that how to find the relation of the reduction between the port Hamiltonian system and the the new system (3.157).

For the standard LQG problem, the transfer function of right coprime factorization is the same as the system itself. Hence the error bound of the standard LQG balanced reduction can be characterized by the Lyapounov balanced reduction of the right coprime factorization. However, it is to difficult to find the relation between the transfer function of the new system (3.138) given in the Proposition 3.2 and the port Hamiltonian system.

As a summary, to characterize the error of the passive preserving Q-conjugated LQG balanced reduction method, still has two difficulties. First, it is difficult to compute the upper error bound of the Effort constraint method even by using the Lyapunov balanced reduction method. Secondly, to characterize the error on the Q-conjugated balanced coordinate is still a very difficult problem. These problems will be considered in a future work.

3.5 Summary of proposed methods

In this section, we shall give a summary and comparison between the proposed structure preserving LQG method with the standard LQG method from the control design and closed loop reduction points of view respectively.

In Figure 3.2 the different LQG methods for control design are illustrated. On the left, we use the standard LQG method where the weighting matrices are $\tilde{Q} = QBB^TQ$ and $\tilde{R} = I$, the covariance matrices are $Q_v = BB^T$ and $R_w = I$. This method can not preserve the Hamiltonian structure in the closed loop system. The Q-conjugated LQG method and Minimize dissipation energy LQG method (Min-energy LQG), in which the weighting and covariance matrices are under the condition of Theorem 3.3 and Theorem 3.2 are presented in middle and right part of the figure, respectively. These two methods can preserve the Hamiltonian structure for the closed loop systems.

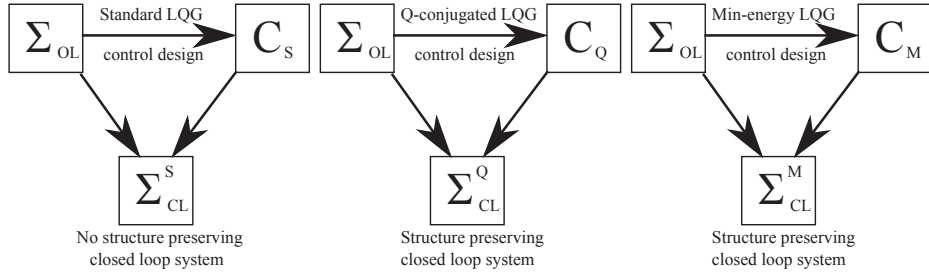


Figure 3.2: Full order closed loop systems by using different LQG methods

In Figure 3.3, we show the reduction schemes of the closed loop systems by using the NO structure preserving (Standard) and the structure preserving (Q-conjugated) LQG method.

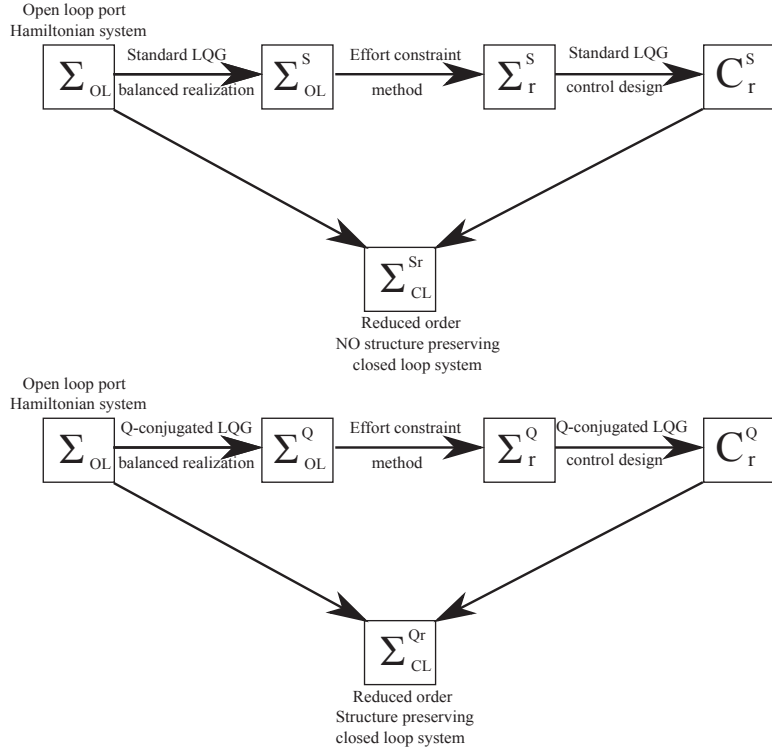


Figure 3.3: Reduced order closed loop systems by using different LQG methods

3.6 Illustration on the mass-spring-damper system

In this section, we consider the benchmark example of a mass-spring-damper system treated in [65] which can be interpreted as the spatially discretized model of a robotic flexible link of a robots or as a multi-mass system for vibration absorbers. We first compare the open loop performances of the full order system with the different reduced one obtained by the different reduction methods on Q-conjugated balanced coordinate and standard LQG balanced coordinate respectively. Secondly the performances of the closed loop systems shall be compared by using the different controllers.

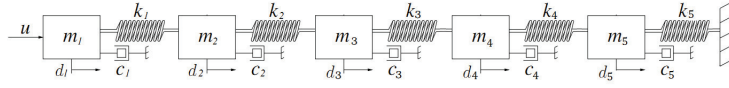


Figure 3.4: Mass-spring-damper system

The mass-spring-damper system represented on Figure 3.4 may be formulated as a port Hamiltonian system (3.28). The state variable $x \in \mathbb{R}^{2N}$ contains the displacement d and the momentum p variables of the N masses, the input of system is the force F on the mass m_1 and its dual output is its velocity.

$$x = [d_1, p_1, d_2, p_2, d_3, p_3, \dots, d_N, p_N]^T \quad (3.161)$$

$$u = F \quad (3.162)$$

$$y = v_1 \quad (3.163)$$

The physical parameters of this system are: the masses $m_i = 2$, the elasticity coefficients of the spring $k_i = 4$, the friction coefficients $c_i = 0.1$. Denoting $\alpha_i = k_i + k_{i-1}$, $\beta_i = -k_i$ and $\gamma_i = \frac{1}{m_i}$, the structure matrix $J \in \mathbb{R}^{2N \times 2N}$, the dissipation matrix $R \in \mathbb{R}^{2N \times 2N}$, the energy matrix $Q \in \mathbb{R}^{2N \times 2N}$ and the input matrix $B \in \mathbb{R}^{2N}$ are given as

$$J = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \quad (3.164)$$

$$R = \text{diag}[0 \ c_1 \ 0 \ c_2 \ 0 \ c_3 \ \cdots \ 0 \ c_N] \quad (3.165)$$

$$Q = \begin{bmatrix} \alpha_1 & 0 & \beta_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \beta_1 & 0 & \alpha_2 & 0 & \beta_2 & \ddots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \gamma_2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \beta_2 & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \alpha_{N-1} & 0 & \beta_{N-1} & 0 \\ \vdots & \ddots & 0 & \ddots & \ddots & 0 & 0 & \gamma_{N-1} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & \beta_{N-1} & 0 & \alpha_N & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & \gamma_N \end{bmatrix} \quad (3.166)$$

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T \quad (3.167)$$

In the numerical simulation, we choose $N = 20$, it means that the system state is $x \in \mathbb{R}^{40}$.

Before proceeding to the reduction of the system, let us compare the singular values obtained from the standard LQG balancing and the positive real as well as the Q-conjugated balancing in Figure 3.5 . The singular values (depicted in green in Figure 3.5) obtained by the LQG balancing method of proposition 3.2 which minimizes a dissipation rate, are all equal to 1. This is in accordance with the fact that product of the LQG Gramians obtained from Theorem 3.2 is equal to the identity. Secondly, one may observe that the singular values (depicted by red circles in Figure 3.5) obtained by the Q-conjugated LQG balancing have a much bigger decay rate than the ones obtained with standard LQG balancing method (depicted by blue circles in Figure 3.5). Because we can choose a more important weighting matrix for the optimal control. In the standard LQG method, the optimal control weighting matrix of states equal to CC^T , but in the Q-conjugated LQG method, we choose the states weighting matrix of the optimal control is equal to $\tilde{Q} = 5CC^T$. That is why the singular values obtained by the Q-conjugated LQG balancing have a much bigger decay rate. It should be noted that in Q-conjugated LQG balancing method, the choices of weighting matrices and covariance matrices are related. In this numerical simulation, we fix the optimal control weighting matrices, thus the covariance matrices are also fixed for getting a passive LQG controller by Theorem 3.3.

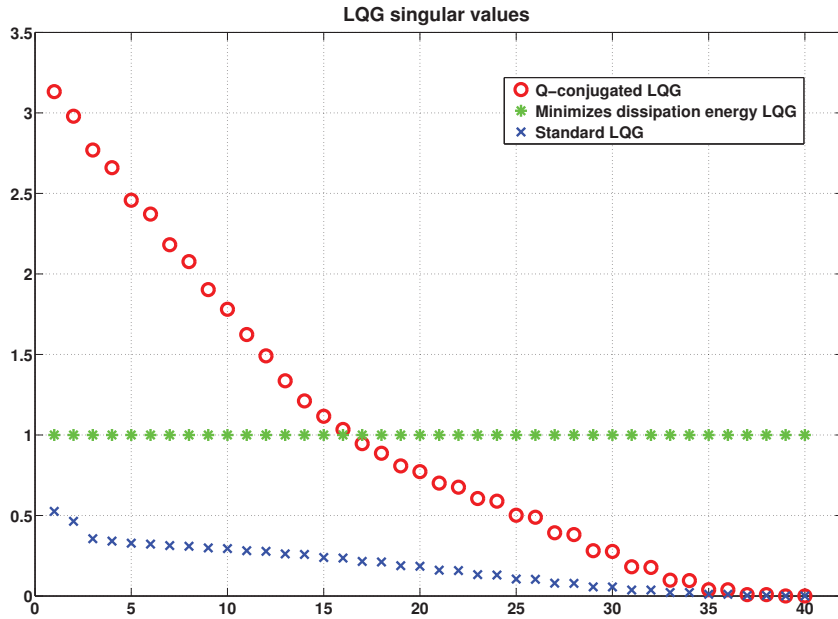


Figure 3.5: Singular values associated with different LQG problems

3.6.1 Comparisons of open loop systems

Next we shall compare in Table 3.1 and 3.2 the relative errors between the Q-conjugated LQG balanced coordinate and the standard LQG balanced coordinate by using *Effort constraint method* in H_∞ norm ($\frac{\|G - G_r\|_\infty}{\|G\|_\infty}$) and H_2 norm ($\frac{\|G - G_r\|_2}{\|G\|_2}$) respectively. We recall that the error bound of LQG balanced method is given as

$$\|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^n \frac{\mu_i}{\sqrt{1 + \mu_i^2}} \tag{3.168}$$

Order of the approximation	4	6	8	10	12	14	16	18
Q-conjugated LQG	19.76	14.76	10.85	7.93	5.27	5.92	4.51	2.50
Standard LQG	7.68	2.69	2.02	2.54	2.57	2.07	1.53	1.93

Table 3.1: Relative H_∞ errors of the different balanced realizations using the effort constraint reduction method

Order of the approximation	4	6	8	10	12	14	16	18
Q-conjugated LQG	4.08	3.47	3.04	2.69	2.32	2.41	2.16	1.95
Standard LQG	2.74	1.69	1.38	1.46	1.45	1.33	1.19	1.15

Table 3.2: Relative H_2 errors of the different balanced realizations using the effort constraint reduction method

One can observe that compared with standard LQG method, the reduced order systems by using Q-conjugated LQG balanced reduction method have bigger H_∞ or H_2 error. Because the singular values given by Q-conjugated balanced method are bigger than the standard LQG method shown in Figure 3.5. But we should always remind that the controller designed by Q-conjugated method is passive and the closed loop system is also a port Hamiltonian system.

We shall compare the errors of reduced order systems by Effort constraint method and Truncation method on the same coordinate-Q-conjugated balanced realization. The errors will be shown by relative H_∞ and H_2 norms on Table 3.3 and 3.4 respectively.

Order of the approximation	4	6	8	10	12	14	16	18
Effort constraint method	19.76	14.76	10.85	7.93	5.27	5.92	4.51	2.50
Truncation method	1.64	2.12	8.01	7.52	7.03	6.49	1.98	1.64

Table 3.3: Relative H_∞ errors of the different reduction methods on the Q-conjugated balanced realization

Order of the approximation	4	6	8	10	12	14	16	18
Effort constraint method	4.08	3.47	3.04	2.69	2.32	2.41	2.16	1.95
Truncation method	Inf	Inf	Inf	Inf	Inf	Inf	Inf	Inf

Table 3.4: Relative H_2 errors of the different methods on the Q-conjugated balanced realization

We can see that the reduced order systems obtained by effort constraint method have the larger H_∞ errors than the reduced ones by truncation method. However the truncation method can preserve neither the Hamiltonian structure nor the passivity of the port Hamiltonian system. Sometimes, the truncation method may even lose the stability of the system as it may be seen in the Table 3.4, where the H_2 errors of reduced order systems obtained by using truncation method are infinite, that means the reduced systems obtained by using truncation method are unstable. Hence using the Q-conjugated balanced coordinate, the truncation method loses not only the *passivity* but also the *stability* of the system.

Next, we shall compare the open loop reduction performances in a reduced order given by Bode diagrams. Here we consider the reduced order is $r = 20$.

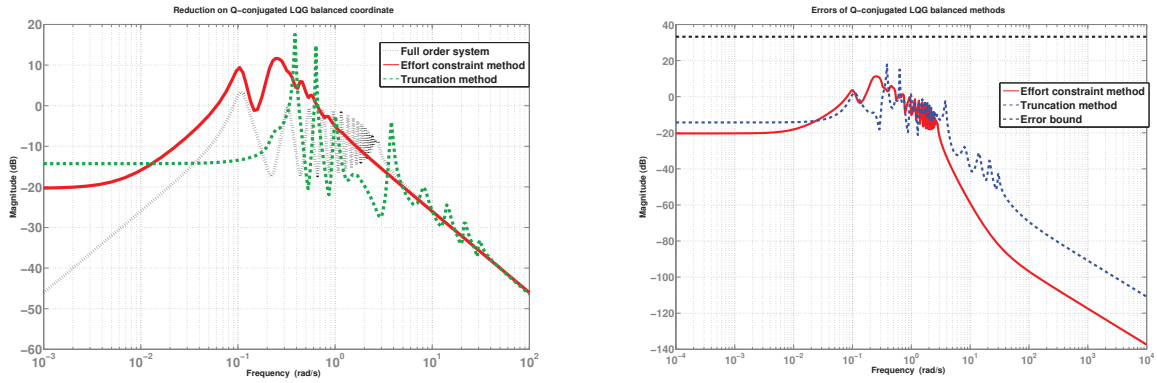


Figure 3.6: Bode diagrams by Effort constraint method and truncation method on Q-conjugated LQG balanced coordinate (Left) and their errors (Right)

In Figure 3.6 we compare the reduced order systems by Effort constraint method (the red curve) and truncation method (the green curve) on the Q-conjugated LQG balanced realization with the full order system (the black curve). We can see that the Effort constraint method preserves a better performance in the high frequencies and the truncation method has the better performance in the low frequencies which is the similar results as given in [61]. The errors of those reduction methods do not exceed the error bound given by equation (3.168).

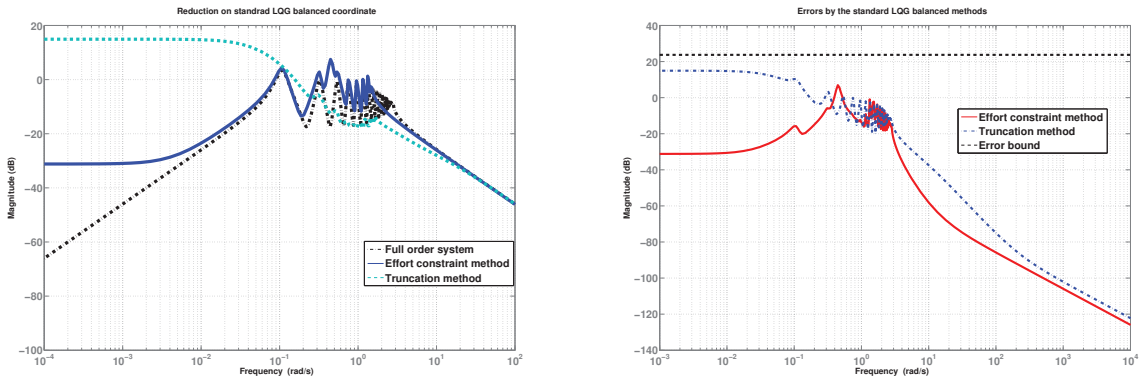


Figure 3.7: Bode diagrams by Effort constraint method and truncation method on Standard LQG balanced coordinate (Left) and their errors (Right)

Figure 3.7 is given the Bode diagrams of reduced order systems obtained by Effort constraint method (the blue curve) and truncation method (the green curve) respectively on the Standard LQG balanced realization with the full order system (the blue curve). One can see that the two errors don't exceed the error bound defined by equation (3.168), and the two reduced order systems have almost the same performances at high frequencies, however the reduced system given by Effort constraint method has very bad performances in the low frequency and the truncation method have a better performance. But we should always remind that the Effort constraint method always preserves the Hamiltonian structure and passivity of the port Hamiltonian system and the truncation method doesn't, and in this case the truncation method, the reduced order system loses the stability of the

port Hamiltonian system.

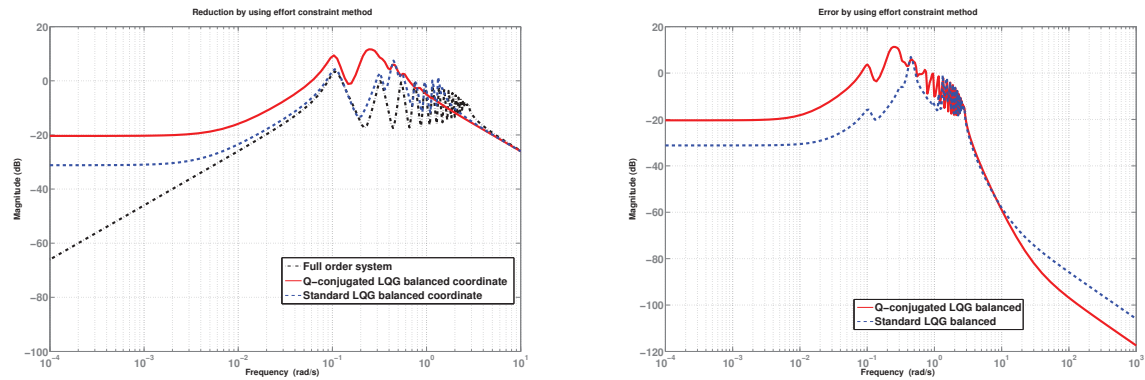


Figure 3.8: Bode diagrams by using the Effort constraint method on the different balanced coordinates (Left) and their errors (Right)

As said before, the Effort constraint method can preserve the Hamiltonian structure and the passivity of the port Hamiltonian system. The we compare the reduced order systems obtained by Effort constraint method on the Q-conjugated LQG balanced coordinate (the red curve) and on the standard LQG balanced coordinate (the blue curve) respectively in Figure 3.8 with the full order system (the blue curve). We can see that the reduced order system on the standard LQG balanced coordinate has the better performance which we can also find in Tables 3.1 and 3.2. In the next subsection, we will discuss the closed loop behaviors of the different methods.

3.6.2 Comparisons of closed loop systems

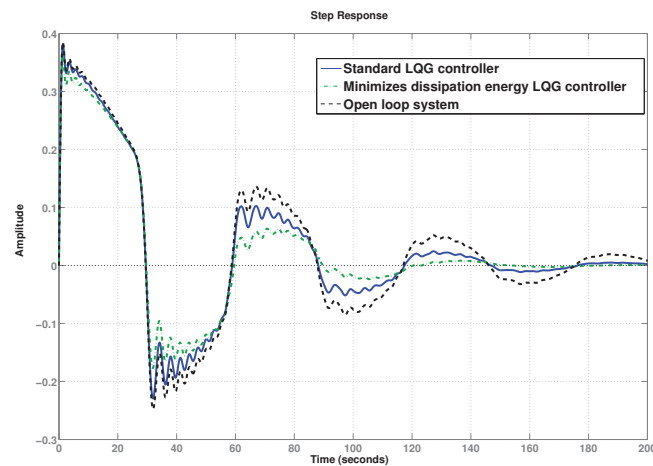


Figure 3.9: Comparison of the standard LQG controller and modified LQG controller minimizing dissipation energy

In Figure 3.9, we are comparing the step responses of the closed loop system stabilized by the controllers by the minimizes the dissipation energy LQG method (the green curve) and standard LQG method (the blue curve) with open loop system (the black curve) respectively. The step responses have almost the same performance. But it should be noted that the closed loop system with the Minimizes dissipation energy LQG method is also a port Hamiltonian system, i.e. we preserve the Hamiltonian structure in the closed loop system. However, the Hamiltonian structure is lost in the closed loop system with the standard LQG controller.

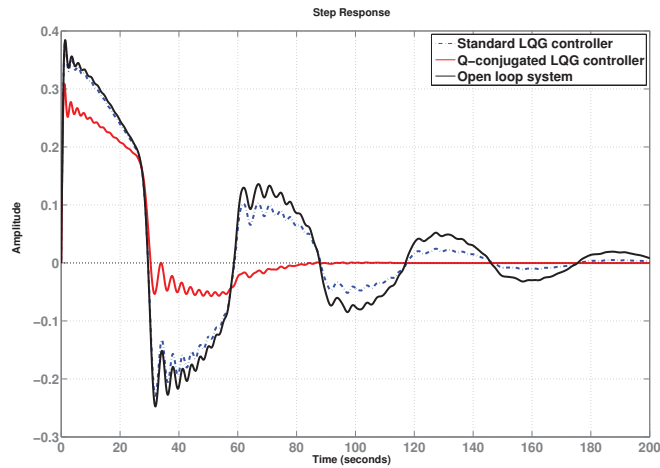


Figure 3.10: Comparison of the closed loop systems with standard LQG controller and Q-conjugated LQG controller

In Figure 3.10, we compare closed loop system stabilized by Q-conjugated LQG controller (the red curve) and standard LQG controller (the blue curve) with the open loop system (the black curve) respectively. We can see the closed loop system with Q-conjugated LQG controller has the much better performance. And it should be reminded that the closed loop system with Q-conjugated LQG controller is still a port Hamiltonian system and also the choice of the weighting matrices or covariance matrices is free, i.e. we can define the optimal control or the filter problem as we want.

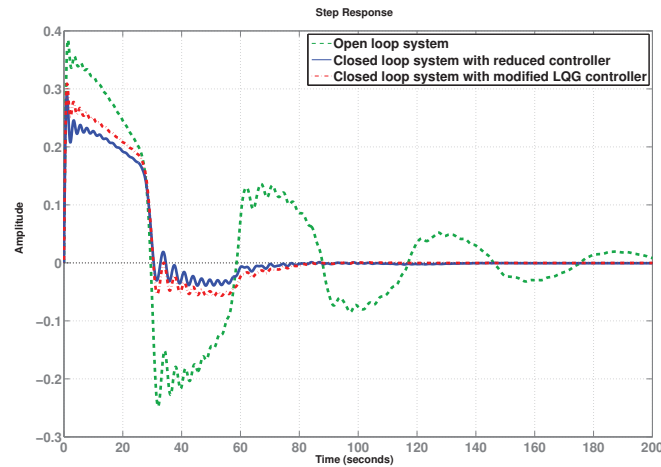


Figure 3.11: Comparison of the closed loop systems with Q-conjugated full order and reduced order controllers

Because the Q-conjugated LQG method does not only allow us to design a passive Hamiltonian controller, but also allows us to reduce the system and its controller. Thus in Figure 3.11, we show the step responses of closed loop systems with the full order Q-conjugated LQG controller (the blue curve) and reduced order Q-conjugated controller (the red curve) respectively. The reduced order is chosen as $r = 20$, since we can see in Figure 3.5, the first 20 Q-conjugated LQG singular values are much bigger than the other ones, it means that the first 20 state variables are more important than the others. It is also why we can find the closed loop system with the full order and reduced order Q-conjugated LQG controllers have almost the same performance.

3.7 Conclusion

In this chapter, we have suggested two passive LQG controller design methods for port Hamiltonian systems. Furthermore one of these method allows us to reduce the port Hamiltonian system and its passive LQG controller.

We firstly propose a passive LQG control design method for port Hamiltonian system by inspiring from the positive real LQG control design method of positive real system proposed in [8]. This passive LQG method is interpreted as a controller which minimizes the dissipated energy LQG method because it considers the sum of dissipation of the energy of the port Hamiltonian system with the norm of the input and output variables with the weighting matrices. And the covariance matrices have no statistic meaning and are only considered as the controller design parameters. However, this passive LQG control design method doesn't allow us to reduce the port Hamiltonian system because all the singular values are equal to one, we can't separate the state variables.

Thus the second passive LQG method Q-conjugated LQG method was proposed. In this passive LQG control method, the LQG Gramians were related in order to get a port Hamiltonian formulation of LQG controller. Hence, the optimal control problem and optimal filtering problem are related in this Q-conjugated LQG method, if the optimal control problem is given, then the covariance matrices of optimal filtering problem are just the control design parameters and have no statistic meaning.

In contrast, if the covariance matrices are given, i.e. there are the real noises perturbing the state variables and output, the weighting matrices of optimal control problem are related to those noise and should be considered as the control design parameters. Furthermore, Q-conjugated LQG method allow us to reduce the port Hamiltonian systems since the singular values are generally different, it means the state variables have the different importance in the closed loop behavior. Then we define a new LQG balanced realization based on the Q-conjugated LQG method and the effort constraint method is used to get reduced order system with preserving the Hamiltonian structure and passivity of port Hamiltonian system. The reduced order system can be used to design the low order LQG controller with the proposed LQG method.

Finally we have used a classical mechanical mass-spring-damper system to illustrate the effectiveness of the proposed method. The errors between the full order system and reduced order systems obtained by different methods on the standard LQG balanced and Q-conjugated LQG balanced coordinates are shown by the relative H_∞ norm and H_2 norm. The reduced order systems obtained by Q-conjugated LQG method have bigger relative error than the ones obtained by standard LQG method however we should always remember that the Q-conjugated LQG method can preserve the passivity and Hamiltonian structure of the system but the standard LQG can't. Furthermore the closed loop performances of the different LQG controllers are shown by the step responses at next and we find the one by using Q-conjugated LQG controller has better performance than standard LQG method. And the reduced order Q-conjugated LQG controller has almost the same performance as the full order one in the closed loop systems.

Chapter 4

Hamiltonian LQG method for structure preserving reduction of infinite dimensional port Hamiltonian system

4.1 Introduction

In this chapter we consider the model reduction and the control of infinite dimensional systems driven by linear PDEs. As in the case of large scale systems there exists an extensive literature on this topic. Nevertheless as soon as passivity is concerned, the reduction and control schemes have to preserve the original passivity properties of the system. For example a passivity preserving reduction scheme based on positive real balanced truncation has been proposed in [29]. The passivity and (Hamiltonian) structure preserving approximation of infinite dimensional port Hamiltonian systems has been derived on the basis of the spatial discretization [24, 5, 33] through mixed finite elements method, using a pseudo-spectral approximation [52] and a Petrov-Galerkin approximation [34] for the *open-loop* systems. The main drawback of such methods is that they cannot be applied on a large class of distributed parameter systems *i.e.* the power preserving systems (hyperbolic systems), for which the poles are all located on the imaginary axis. In this case all the state variables have the same weight and the aforementioned reduction technics fail in providing a model suitable for control design purposes. This is particularly true in the infinite dimensional setting. Even if such approach has been considered for the reduction of large scale finite dimensional systems (cf last chapter and [85]) it has been hardly considered in the infinite dimensional case. As it was the case in the previous chapter the reduction we use is mainly based on LQG control and balanced realization and truncation.

In the infinite dimensional case, the model reduction of (open loop) linear systems by using the balanced method has been widely studied in literature. The balanced realization and truncation method for infinite dimensional system has been firstly introduced by [54, 22, 23] in the last century. In recent years, this method has been generalized to the systems with nuclear Hankel operators [13]

and the error of approximation evaluated in [30, 68, 28]. The nuclear Hankel operators have summable sequence of Hankel singular values, i. e. the singular values converge to zero and their sum is bounded. This property will be used to separate the state spaces since in the balanced realization, the state operator can be regarded as infinite matrix. The truncation method of finite dimensional system is then naturally generalized to the infinite dimensional case.

With the same idea in mind, the LQG balanced method has been generalized to the infinite dimensional linear systems associated with two operator Riccati equations by [12]. These primary works have been generalized to discrete-time and continuous-time infinite dimensional linear systems in [56, 55]. In [9, 70] the authors have compared the LQG balanced method to the balanced method for the controller design objective. The simulation results in those papers show that the results obtained from LQG balanced method are better than the one obtained using balanced method.

In [6], a robust reduced order controllers designed by LQG balanced method has been proposed. The authors have shown that the LQG balanced method can be used to design not only the LQG type control defined as in [36, 12] but also other reduced order controllers associated with the modified Riccati equations. A difficult point of infinite dimensional LQG balanced method is how to solve the operator Riccati equations. In the literature, most of the authors use a discretization method like finite elements method to approximate the system for solving the operator Riccati equation. It has been shown in [51, 50] from the graph topology point of view, that under some conditions the solutions of the finite dimensional Riccati equations converge to the solutions of the operator Riccati equations, and that the finite dimensional controller converges to the infinite dimensional one. However, the use of standard LQG balanced method defined in [12] even for infinite dimensional port Hamiltonian systems, does not preserve the passivity properties in open loop nor in closed loop because the LQG control design is not a passive control design method in general.

Hence this chapter aims to find a passive reduced order LQG controller design method for infinite dimensional port Hamiltonian (IDPHS) with distributed control in the domain and no boundary energy exchange. This approach is also used to develop a LQG based model reduction with *closed-loop* considerations [9, 6, 12]. We shall first derive the conditions under which a LQG controller is passive and equivalent to a control by interconnection of two port Hamiltonian systems [60, 58, 44]. From this LQG approach we shall define a balanced basis in which the LQG Hankel operator is nuclear with summable singular values. The reduction procedure will use the Petrov-Galerkin approximation [34] preserving both the port Hamiltonian structure and the passivity of the system.

4.1.1 Organization of this chapter

This chapter is organized as follows. First in Section 4.2, we recall some important definitions and theorems of infinite dimensional balanced and LQG balanced methods. In Section 4.3, we introduce a class of infinite dimensional passive systems which can be represented in the port Hamiltonian framework. In Section 4.4 we apply the LQG method to this infinite dimensional port Hamiltonian system and then propose a LQG controller which is passive and equivalent to the control of the port Hamiltonian system by interconnection. In the next section (Section 4.5) we use this LQG problem in order to derive a reduction scheme that preserves the port Hamiltonian structure. In the last section we illustrate and compare the full order LQG controllers and reduced order LQG controller on the example of Timoshenko beam.

4.1.2 Main contributions of this chapter

The main contributions of this chapter are the following.

- In Section 4.4, a novel passive LQG control design method for infinite dimensional port Hamiltonian systems is proposed. More precisely Theorem 4.4 generalizes the passive LQG control design method proposed for finite dimensional port Hamiltonian systems in the previous chapter. We show this novel passive LQG control design method is equivalent to the control of infinite dimensional port Hamiltonian systems by interconnection. However, the LQG controller designed by this method still has an infinite dimension.
- In Section 4.5, Theorem 4.5 used the proposed passive LQG design method to define a balanced realization by a specific choice of the weighting operators acting on the state. The nuclearity of the LQG Hankel operator is proven and used to separate the state space as explained in the introduction of this chapter.
- Because the state space of the infinite dimensional port Hamiltonian system can be separated to one finite dimensional part and one infinite dimensional residue of finite sum, we propose in Theorem 4.6 a finite dimensional approximation of the port Hamiltonian system by using a Petrov-Galerkin projection method.
- In Section 4.6, we shall apply the proposed approximation scheme to a Timoshenko beam model. The reduced finite dimensional controller performances will be discussed and we will compare the reduced order controller with high order controller.

4.2 Reminders of balanced methods for infinite dimensional linear system

This sub section is devoted to the balanced realization of bounded well-posed linear systems with nuclear Hankel operator and finite dimensional input and output spaces.

4.2.1 Infinite dimensional linear system with nuclear Hankel operator

We consider in this chapter infinite dimensional systems of the form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u \quad (4.1)$$

where $x(t, z)$ is the state space defined on a real Hilbert space, \mathcal{A} a linear differential operator with domain $D(\mathcal{A})$ and \mathcal{B} a linear (bounded) input mapping. We are interested in this work to the existence and the properties of the solutions of (4.1). For that purpose we use the semigroup theory and more generally the functional analysis [14, 72, 75] to derive our results. A semigroup $\mathfrak{A}(t)_{t \geq 0}$ defined on a Hilbert space is an operator that generalizes the e^{At} operator used to derive the solutions of linear finite dimensional systems to the infinite dimensional ones. In a first instance we recall some important definitions and properties of semigroups.

Definition 4.1 (Strongly continuous semigroup (C_0 -semigroup) [14]). Let X be a Hilbert space. $\mathfrak{A}(t)_{t \geq 0}$ is called a *strongly continuous semigroup* (C_0 -semigroup for short) if the following holds:

1. For all $t \geq 0$, $\mathfrak{A}(t)$ is a bounded linear operator on X , i.e., $\mathfrak{A}(t) \in \mathcal{L}(X)$;
2. $\mathfrak{A}(0) = I$;
3. $\mathfrak{A}(t+s) = \mathfrak{A}(t)\mathfrak{A}(s)$ for all $t, s \geq 0$;
4. For all $x_0 \in X$, we have that $\|\mathfrak{A}(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$, i.e., $t \mapsto \mathfrak{A}(t)$ is strongly continuous at zero.

Definition 4.2 (Generator [14]). The operator $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ defined by

$$D(\mathcal{A}) = \left\{ x \in X \mid \lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}(t)x - x) \in X \right\}, \quad \mathcal{A}x = \lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}(t)x - x), \quad (4.2)$$

is called the *generator* of the semigroup $\mathfrak{A}_{t \geq 0}$.

The set of all the bounded linear operators from X to Y is denoted by $\mathcal{L}(X; Y)$ where X and Y are the Hilbert spaces. If $X = Y$, then we normally write $\mathcal{L}(X)$ instead of $\mathcal{L}(X; X)$.

We called X the *state space*, and its elements *states*. The easiest example of a strongly continuous semigroup is the exponential of a matrix. That is, let A be an $n \times n$ matrix, the matrix-valued function $\mathfrak{A}(t) = e^{At}$ defines a strongly continuous semigroup on the Hilbert space \mathbb{R}^n .

We introduce the two following notations. The *kernel* (null-space) and *range* of the operator $\mathcal{A} \in \mathcal{L}(X, Y)$ is denote as:

$$\text{Ker } \mathcal{A} = \{x \in X \mid \mathcal{A}x = 0\}, \quad \text{Ran } \mathcal{A} = \{\mathcal{A}x \mid x \in X\}. \quad (4.3)$$

Definition 4.3 (Dissipative operator [75]). The operator $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ is called *dissipative* if

$$\text{Re} \langle \mathcal{A}x, x \rangle \leq 0 \quad \text{for all } x \in D(\mathcal{A}). \quad (4.4)$$

Definition 4.4 (Maximal dissipative operator [75]). A dissipative operator $\mathcal{A} : D(\mathcal{A}) \rightarrow X$ is called *maximal dissipative* (*m-dissipative*) if

1. $\text{Ran}(sI - \mathcal{A}) = X$ for all $s \in \mathbb{C}_0$;
2. $D(\mathcal{A})$ is dense.

Definition 4.5 (Contraction semigroup [75]). A strongly continuous semigroup $\mathfrak{A}(t)_{t \geq 0}$ on X is called a *contraction semigroup* if

$$\|\mathfrak{A}(t)\| \leq 1 \quad \text{for all } t \geq 0. \quad (4.5)$$

Theorem 4.1. [75] *If \mathcal{A} is the generator of a contraction semigroup on X , then \mathcal{A} is m-dissipative.*

An example of the above definitions can be found in Section 4.6. A mechanical Timoshenko beam model will be used to illustrate these operators.

Definition 4.6 (Compact operator [75]). Let X and Y be Hilbert spaces, An operator $T \in \mathcal{L}(X, Y)$ is called compact if there exists a sequence (T_n) in $\mathcal{L}(X, Y)$ such that

$$\dim \text{Ran } T_n < \infty, \quad \lim T_n = T. \quad (4.6)$$

A compact operator $T \in \mathcal{L}(X, Y)$ acting between two Hilbert spaces X and Y is known to admit a singular value decomposition

$$Tx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle_X v_n \quad (4.7)$$

for some nonincreasing null sequence of $(\sigma_n)_{n \in \mathbb{N}}$ in \mathbb{R}_+ and orthonormal systems $(u_n)_{n \in \mathbb{N}}$ in X and $(v_n)_{n \in \mathbb{N}}$ in Y [67, p. 203]. The numbers σ_n are called *singular values* and (u_n, v_n) is called a *Schmidt pair* associated to σ_n .

Definition 4.7 (Nuclear operator [13]). A compact operator is nuclear if

$$\sum_{i=n}^{\infty} (\sigma_n)_{n \in \mathbb{N}} < \infty \quad (4.8)$$

where $(\sigma_n)_{n \in \mathbb{N}}$ are the singular values in decreasing order with ℓ_1 is the space of summable sequences.

If the compact operator $T \in \mathcal{L}(X, Y)$ is *nuclear*, a singular value decomposition of T can also be written as,

$$T = V \Sigma U^* \quad (4.9)$$

where U^* is the adjoint of the operator U . The operators $\Sigma \in \mathcal{L}(\ell_2)$, $U \in \mathcal{L}(\ell_2; X)$, $V \in \mathcal{L}(\ell_2; Y)$ defined by

$$\Sigma(x_n)_{n \in \mathbb{N}} = (\sigma_n x_n)_{n \in \mathbb{N}} \quad (4.10)$$

and

$$U(x_n)_{n \in \mathbb{N}} := \sum_{n=0}^{\infty} x_n u_n, \quad V(x_n)_{n \in \mathbb{N}} := \sum_{n=0}^{\infty} x_n v_n. \quad (4.11)$$

Now we denote the input, state and output spaces by \mathcal{U} , X , \mathcal{Y} which are Hilbert spaces. In the following we consider the infinite dimensional linear system defined in [68, 72] of the form:

$$\begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}u \\ y &= \mathcal{C}x + \mathcal{D}u \end{cases} \quad (4.12)$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ is a generator of a bounded strongly continuous semigroup \mathfrak{A} on X , $\mathcal{B} \in \mathcal{L}(\mathcal{U}; X)$, $\mathcal{C} : D(\mathcal{C}) \subset X \rightarrow \mathcal{Y}$ is linear with $D(\mathcal{C}) \subset X$, $\mathcal{D} \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$.

Let input and output spaces \mathcal{U} and \mathcal{Y} be finite dimensional Hilbert spaces and the state space X an arbitrary Hilbert space. The four linear operators \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , \mathfrak{D} are

$$\begin{aligned} \mathfrak{A} : X &\rightarrow X & \mathfrak{B} : L^2(\mathbb{R}_-; \mathcal{U}) &\rightarrow X \\ x(0) &\mapsto x(t), & u(\cdot) &\mapsto \int_{-\infty}^0 \mathfrak{A}(-\tau) \mathcal{B}u(\tau) d\tau, \\ \mathfrak{C} : X &\rightarrow L^2(\mathbb{R}_+; \mathcal{Y}) & \mathfrak{D} : L^2(\mathbb{R}; \mathcal{U}) &\rightarrow L^2(\mathbb{R}; \mathcal{Y}) \\ x &\mapsto C \mathfrak{A}(\cdot) x, & u(\cdot) &\mapsto C \int_{-\infty}^{\cdot} \mathfrak{A}(\cdot - \tau) \mathcal{B}u(\tau) d\tau + \mathcal{D}u(\cdot), \end{aligned} \quad (4.13)$$

with state space X , input and output spaces \mathcal{U} and \mathcal{Y} such that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ forms a stable well-posed linear system in the sense of [72, Definition 2.2.1] if it satisfies the conditions of Definition ???. If $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ defines the well-posed linear systems via (4.13), then the quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is called the generators of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$.

Definition 4.8 (Hankel operator, Gramians [68, Definition 2.2]). For a bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathcal{U}, X, \mathcal{Y})$,

1. $\mathfrak{H} = \mathfrak{C}\mathfrak{B} \in \mathcal{L}(L^2(\mathbb{R}_-; \mathcal{U}); L^2(\mathbb{R}_+; \mathcal{Y}))$ is called the *Hankel operator*,
2. $P = \mathfrak{B}\mathfrak{B}^* \in \mathcal{L}(X)$ is called the *controllability Gramian*,
3. $Q = \mathfrak{C}^*\mathfrak{C} \in \mathcal{L}(X)$ is called the *observability Gramian*.

Remark 4.1. The nuclearity of the Hankel operator gives rise to the existence of its singular value decomposition, that is,

$$\mathfrak{H} = V\Sigma U^*, \quad (4.14)$$

with diagonal operator $\Sigma \in \mathcal{L}(\ell_2)$ as in (4.10). The elements of the nonincreasing sequence $(\sigma_n)_{n \in \mathbb{N}} \in \ell_1$ are called *Hankel singular values*.

4.2.2 Balanced realization of infinite dimensional linear systems

We now introduce the balanced realization of well-posed linear systems. As in the finite dimensional case, this involves that both Gramians are equal to some diagonal operator Σ .

Definition 4.9 (Balanced systems [68, Definition 4.1]). A bounded well-posed linear system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ on $(\mathcal{U}, X, \mathcal{Y})$ is called *balanced* if $X = \ell_2$ and there exists a positive and nonincreasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that the Gramians P and Q are both equal to the diagonal operator $\Sigma \in \mathcal{L}(\ell_2)$ defined in (4.10). In other words

$$P = \mathfrak{B}\mathfrak{B}^* = Q = \mathfrak{C}^*\mathfrak{C} = \Sigma \quad (4.15)$$

Remark 4.2. The sequence $(\sigma_n)_{n \in \mathbb{N}}$ in the above definition consists indeed of the Hankel singular values of the system.

Let $S \in \mathcal{L}(X_s; X)$ and $L \in \mathcal{L}(X_L; X)$ be two operators, X_s and X_L are two Hilbert spaces such that the controllability and observability Gramians satisfy

$$P = \mathfrak{B}\mathfrak{B}^* = LL^* \quad \text{and} \quad Q = \mathfrak{C}^*\mathfrak{C} = SS^*. \quad (4.16)$$

Then the Hankel operator is given by

$$\mathfrak{H} = S^*L \quad (4.17)$$

and it is nuclear [68, Thm 5.1]. The singular value decomposition of Hankel operator can be written as

$$S^*L = V\Sigma U^* \quad (4.18)$$

Next we show that we can get the infinite dimensional balanced realization of the system by using the singular decomposition of the nuclear Hankel operator in the following theorem.

Theorem 4.2. [68, Theorem 5.5] *Let consider the system (4.12) with the nuclear Hankel operator. With $S \in \mathcal{L}(X_S; X)$ and $L \in \mathcal{L}(X_L; X)$ as in (4.16) and the notations given in (4.10), (4.11), let (4.18) be the singular value decomposition of the operator $S^*L \in \mathcal{L}(X_L; X_S)$. Then the balanced realization is derived from the following mappings*

$$\begin{aligned} T &:= \Sigma^{-1/2} V^* S^* \subset X \mapsto \ell_2, \\ T^+ &:= L U \Sigma^{-1/2} \subset \ell_2 \mapsto X \end{aligned} \quad (4.19)$$

and leads to:

$$\begin{cases} \dot{x}_b &= \mathcal{A}_b x_b + \mathcal{B}_b u \\ y &= \mathcal{C}_b x_b + \mathcal{D} u \end{cases} \quad (4.20)$$

where $\mathcal{A}_b x_b = T A T^+ x_b$, $\mathcal{B}_b u = T B u$ and $\mathcal{C}_b x_b = C T^+ x_b$ with $x_b \in \ell_2$.

Thus system can be reduced using the truncation of its balanced realization (4.20) leading to a finite dimensional system as stated in Theorem 4.3.

Theorem 4.3. [68, Theorem 5.6] *Let $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b, \mathcal{D})$ be the operators of the balanced realization of the infinite dimensional linear system (4.12) and the canonical unit vector $e_i = (\delta_1, \delta_2, \dots) \in \ell_2$ for all $i \in \mathbb{N}$. The finite dimensional truncated balanced realization of this system is given by*

$$\begin{cases} \dot{x}_r &= A_r x_r + B_r u \\ y &= C_r x_r + \mathcal{D} u \end{cases} \quad (4.21)$$

where the matrices

$$\begin{aligned} A_r &= \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \in \mathbb{C}^{r \times r}, \quad B_r = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \in \mathcal{L}(\mathcal{U}; \mathbb{C}^r) \\ C_r &= \begin{bmatrix} c_1 & \cdots & c_r \end{bmatrix} \in \mathcal{L}(\mathbb{C}^r; \mathcal{Y}) \end{aligned} \quad (4.22)$$

are defined from

$$\begin{aligned} a_{ij} &= \langle \mathcal{A}_b e_j, e_i \rangle_{\ell_2} \in \mathbb{C} \\ b_i &= \langle \mathcal{B}_b, e_i \rangle_{\ell_2} \in \mathcal{L}(\mathcal{U}; \mathbb{C}) \\ c_j &= \mathcal{C}_b e_j \in \mathcal{Y} \end{aligned} \quad (4.23)$$

An immediate consequence of this is that a truncated balanced system can indeed be obtained by truncating the operator $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b, \mathcal{D})$ of a balanced realization of the infinite dimensional linear system (4.12).

4.2.3 LQG balanced realization of infinite dimensional linear systems

Let now discuss the balanced truncation of port Hamiltonian systems. if the infinite dimensional port Hamiltonian system is lossless hyperbolic case, i.e. the system has not dissipation, its eigenvalues are on the imaginary axis. Once this system has been written in its balanced realization, every state variable of the system has the same importance. In other words, all the singular values are the same and equal to one [76]. Hence we can't separate the state space of the infinite dimensional port Hamiltonian systems by using the balanced truncation defined in Theorem 4.3. It points out the

interest of using some considerations before reduction. For that purpose let recall the LQG balancing for infinite dimensional linear system (4.12) [12, 55] associated with the following Riccati equations:

$$(\mathcal{A}^*P_c^S + P_c^S\mathcal{A} - P_c^S\mathcal{B}\mathcal{B}^*P_c^S + \mathcal{C}^*\mathcal{C})x = 0, \forall x \in D(\mathcal{A}) \quad (4.24)$$

$$(\mathcal{A}P_f^S + P_f^S\mathcal{A}^* - P_f^S\mathcal{C}^*\mathcal{C}P_f^S + \mathcal{B}\mathcal{B}^*)x = 0, \forall x \in D(\mathcal{A}^*) \quad (4.25)$$

These two Riccati equations correspond to the optimal control problem with the cost functional

$$J_{co} = \frac{1}{2} \int_0^\infty \|y\|^2 + \|u\|^2 dt \quad (4.26)$$

and its dual filter problem. We assume the system (4.12) is stabilizable and detectable, i.e. the operators $\mathcal{A}_{P_c} = \mathcal{A} - \mathcal{B}\mathcal{B}^*P_c$ and $\mathcal{A}_{P_f} = \mathcal{A} - P_f\mathcal{C}^*\mathcal{C}$ generate two exponentially stable semigroups on state space.

The LQG balanced realization is based on the operators P_c^S and P_f^S who are the unique, self adjoint and nonnegative definite solutions of the control and filter Riccati equations (4.24), (4.25) respectively.

Definition 4.10 (LQG balanced realization [12, Definition 4.5]). The infinite dimensional linear system (4.12) is called LQG balanced if there exist two self adjoint, nonnegative solutions P_c, P_f to its control and filter Riccati equations such that

$$P_c^S = P_f^S = \Lambda \quad (4.27)$$

where Λ is a diagonal operator.

To reduce the system (4.12) using truncation based on the nuclearity of the operator defined in Theorems 4.2 and 4.3, one has to use the relationship between the exponentially stabilizable and detectable state linear system and their normalized right-coprime factorization defined as

$$\Sigma \left(\mathcal{A}_{P_c^S}, \mathcal{B}, \begin{bmatrix} \mathcal{C} \\ -\mathcal{B}^*P_c^S \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix} \right)$$

given as follows.

Lemma 4.1. [12, Lemma 4.6] *If system $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ (4.12) is an exponentially stabilizable and detectable system with finite rank input and output operators, its normalized right coprime factor system $\Sigma \left(\mathcal{A}_{P_c^S}, \mathcal{B}, \begin{bmatrix} \mathcal{C} \\ -\mathcal{B}^*P_c^S \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix} \right)$ is an exponentially stable linear system with nuclear Hankel operator. And furthermore, the solutions of the Riccati equations (4.24), (4.25) P_c^S, P_f^S and $P_c^S P_f^S$ are nuclear operators. The eigenvalues of $P_c^S P_f^S$ are positive and they are system invariant.*

The above lemma carries the finite dimensional LQG balanced reduction to the infinite dimensional one. That is the eigenvalues of $\sqrt{P_c^S P_f^S}$ are positive and they are system invariant. Hence control and filter Riccati equations of the LQG balanced realization are satisfied for

$$P_c^S = P_f^S = \Lambda = \text{diag}(\sigma_n)_{n \in \mathbb{N}} \in \mathcal{L}(\ell_2) \quad (4.28)$$

where $(\sigma_n)_{n \in \mathbb{N}}$ are the eigenvalues of $\sqrt{P_c^S P_f^S}$.

Hence we can use the truncation method defined in (4.3) on the LQG balanced realization since the operators of the LQG balanced realization have infinite matrix representation on ℓ_2 space.

Furthermore, the port Hamiltonian systems are passive which is an important property for control design. So in the reduction method, we aim to preserve the passive and Hamiltonian structure of the system. However, if we use the truncation method defined in Theorem 4.3 to reduce the infinite dimensional port Hamiltonian systems, the passivity can not be preserved for the reduced system. So we shall consider a novel balanced realization which can take the closed loop structure of the systems into account. We shall also propose a new passive and structure preserving approximation scheme for this novel balanced realization of infinite dimensional port Hamiltonian systems.

4.3 A class of infinite dimensional port Hamiltonian system

In this section we consider the class of linear infinite-dimensional dissipative systems [34] defined as follows:

Definition 4.11. A linear infinite-dimensional system of the form:

$$\begin{cases} \dot{x}(t) &= \mathcal{M}Qx(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{B}^*Qx(t) \end{cases} \quad (4.29)$$

is called a linear infinite-dimensional dissipative port-Hamiltonian system (IDPHS) if it satisfies

- $x(t) \in X$, X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$ and norm $\| \cdot \|_X^2$;
- $\mathcal{M} : D(\mathcal{M}) \subset X$, the domain of the operator \mathcal{M} is a densely definite maximal dissipative (m -dissipative) linear operator;
- $\mathcal{Q} : X \mapsto X$ is a bounded linear operator that is self-adjoint ($\mathcal{Q}^* = \mathcal{Q}$) and coercive ($\langle \mathcal{Q}h, h \rangle_X \geq \alpha \|h\|_X^2 \forall h \in X$ with $\alpha > 0$);
- The input operator $\mathcal{B} : \mathbb{C}^p \mapsto X$ is bounded and $\{0\} \neq \text{Im}(\mathcal{B}) \subset X$.
- The inputs u and outputs y have the same dimension.

The operator $\mathcal{M}\mathcal{Q}$ is dissipative with respect to the inner product $\langle g, h \rangle_{\mathcal{Q}} = \langle g, \mathcal{Q}h \rangle_X$, $g, h \in X$. In addition, $\text{Ran}(\lambda I - \mathcal{M}\mathcal{Q}) = X$ is satisfied for some $\lambda \in \mathbb{C}_0$, because \mathcal{M} is m -dissipative and \mathcal{Q} is bijective. Hence $\mathcal{M}\mathcal{Q}$ is m -dissipative and therefore generates a contraction C_0 -semigroup [39, Thm. 1.2.3].

The *total energy* of the system is defined through the *Hamiltonian* (Energy storage equation) as

$$H(x(t)) = \frac{1}{2} \langle \mathcal{Q}x(t), x(t) \rangle_X \quad (4.30)$$

Then the power balance equation of the system is given by

$$\begin{aligned} \frac{d}{dt}H(x(t)) &= \frac{1}{2} \langle \mathcal{Q}x(t), \dot{x}(t) \rangle_X + \frac{1}{2} \langle \mathcal{Q}\dot{x}(t), x(t) \rangle_X \\ &= \frac{1}{2} \langle \mathcal{Q}x(t), \dot{x}(t) \rangle_X + \frac{1}{2} \langle \dot{x}(t), \mathcal{Q}^*x(t) \rangle_X. \end{aligned} \quad (4.31)$$

Due to the relation $\mathcal{Q}^* = \mathcal{Q}$, we have

$$\operatorname{Re} \langle \mathcal{Q}\dot{x}(t), x(t) \rangle_X = \operatorname{Re} \langle \dot{x}(t), \mathcal{Q}x(t) \rangle_X = \operatorname{Re} \langle \mathcal{Q}x(t), \dot{x}(t) \rangle_X. \quad (4.32)$$

Thus by taking the equation (4.29) into account, one get

$$\begin{aligned} \frac{d}{dt}H(x(t)) &= \operatorname{Re} \langle \mathcal{Q}x(t), \dot{x}(t) \rangle_X \\ &= \operatorname{Re} \langle \mathcal{Q}x(t), \mathcal{M}\mathcal{Q}x(t) + \mathcal{B}u(t) \rangle_X \end{aligned} \quad (4.33)$$

and

$$\langle \mathcal{Q}x(t), \mathcal{B}u(t) \rangle_X = \langle \mathcal{B}^* \mathcal{Q}x(t), u(t) \rangle_{\mathbb{C}^p} = y^T(t)u(t). \quad (4.34)$$

Considering the m -dissipativity of \mathcal{M} we have that $\operatorname{Re} \langle \mathcal{Q}x(t), \mathcal{M}\mathcal{Q}x(t) \rangle_X \leq 0$ and then from (4.33) and (4.34), we have:

$$\frac{d}{dt}H(x(t)) \leq y^T(t)u(t) \quad (4.35)$$

The Hamiltonian $H(x(t))$ represents the total energy stored in the system (4.29) and from the physical point view, we can interpret that the variation of total energy $dH(x(t))/dt$ is lower than or equal to the power $y^T(t)u(t)$ fed into the system. Therefore, the system does not contain any internal sources. If we consider that H admits a lower bound, due to this property, the infinite dimensional system (4.29) can be regarded as *passive* [77].

Assumption 1. *Through out this chapter, we suppose the domain of operator \mathcal{M} equals to the domain of \mathcal{M}^* , i.e.,*

$$D(\mathcal{M}^*) = D(\mathcal{M}) \quad (4.36)$$

By using Assumption 1, system (4.29) can be written as:

$$\begin{cases} \dot{x}(t) &= (\mathcal{J} - \mathcal{R})\mathcal{Q}x(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{B}^* \mathcal{Q}x(t) \end{cases} \quad (4.37)$$

where

$$\mathcal{J} = \frac{1}{2}(\mathcal{M} - \mathcal{M}^*) \quad \text{and} \quad \mathcal{R} = -\frac{1}{2}(\mathcal{M} + \mathcal{M}^*) \quad (4.38)$$

with $D(\mathcal{J}) = D(\mathcal{R}) = D(\mathcal{M}) \subset X$. Hence the system (4.29) can be regarded as an infinite dimensional port Hamiltonian system (IDPHS) defined in [78]. Here the operator $\mathcal{J} = -\mathcal{J}^* \in \mathcal{L}(X)$ is a skew-adjoint differential operator which present the energy exchange in the domain, and the operator $\mathcal{R} = \mathcal{R}^* \in \mathcal{L}(X)$ is a semi positive definite self-adjoint differential operator which represents the energy dissipation in the domain.

It should be noted that the input mapping $\mathcal{B} \in \mathcal{L}(\mathbb{C}^p; X)$ is bounded and $\{0\} \neq \operatorname{Im}\mathcal{B} \subset X$, it means that there are not energy exchanges at the boundary of the spatial domain for this class of port Hamiltonian systems. In other words, this class of systems is closed, and is controlled in the spatial domain and not at the boundary. This is excluded from the work [78] which presents a large class of boundary controlled infinite dimensional port Hamiltonian systems, i.e., where the boundary control and input operators \mathcal{B} are unbounded. This class of systems is not considered in this chapter, it shall be studied in future works.

In the next sections we shall discuss how to reduce this class of port Hamiltonian systems by taking

its *closed loop* behavior into account. Indeed, as it has been previously met it is difficult to separate the state space of the open loop system when the port Hamiltonian system is lossless or with little energy loss.

4.4 Passive LQG control design of infinite dimensional PHS

In the finite dimensional case, the passive LQG control design has been proposed for positive real system in [8]. In the precedent chapter, we have introduced a passive LQG control design method for the finite dimensional port Hamiltonian system which is equivalent to the control by interconnection [85].

In this section, we shall consider the LQG control design of infinite dimensional port Hamiltonian systems and formulate it as a port Hamiltonian control by interconnection [60, 58]. It shall be noted that the LQG controller has the same dimension as the system itself, i.e., the controller is still infinite dimensional.

4.4.1 LQG control of infinite dimensional port Hamiltonian systems

To apply the LQG method, let us first make the following assumption about the stabilizability and the detectability of the IDPHS (4.37).

Assumption 2. *The IDPHS (4.37) with the m -dissipative operator $\mathcal{M}\mathcal{Q}$ is exponentially stabilizable, i.e., there exists an operator $K \in \mathcal{L}(X, \mathbb{C}^p)$ such that $\mathcal{M}\mathcal{Q} - \mathcal{B}K$ generates an exponentially stable semi-group. We also consider that it is exponentially detectable, i.e., there exists an operator $F \in \mathcal{L}(\mathbb{C}^p, X)$ such that the operator $\mathcal{M}\mathcal{Q} - F\mathcal{B}^*\mathcal{Q}$ generates an exponentially stable semi-group.*

The LQG control problem of IDPHS (4.37) is considered as follows:

Problem 4.1 (LQG control problem). Let $\tilde{\mathbf{Q}}, \mathbf{Q}_v \in \mathcal{L}(X)$ be self-adjoint positive definite operators, $\tilde{\mathbf{R}}, \mathbf{R}_w \in \mathcal{L}(\mathbb{C}^p)$ also be self-adjoint strictly positive definite operators and $x \in D(\mathcal{M}) = D(\mathcal{M}^*)$. Then state feedback gain $K = \tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c$ with P_c the unique positive-definite solution to the operator Riccati equation:

$$\left(\mathcal{Q}\mathcal{M}^*P_c + P_c\mathcal{M}\mathcal{Q} - P_c\mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c + \tilde{\mathbf{Q}} \right) x = 0 \quad (4.39)$$

and $\mathcal{M}\mathcal{Q} - \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c$ generates an exponentially stable semi-group. The filter gain is $F = P_f\mathcal{B}\mathbf{R}_w^{-1}$ where P_f is the unique positive definite solution to

$$\left(\mathcal{M}\mathcal{Q}P_f + P_f\mathcal{Q}\mathcal{M}^* - P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^*\mathcal{Q}P_f + \mathbf{Q}_v \right) x = 0 \quad (4.40)$$

and $\mathcal{M}\mathcal{Q} - P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^*\mathcal{Q}$ generates an exponentially stable semi-group.

The control design problem remains to solve the Riccati equation (4.39) and (4.40) in order to minimize the control cost function:

$$J_{co} = \int_0^\infty \left\langle x, \tilde{\mathbf{Q}}x \right\rangle_X + \left\langle u, \tilde{\mathbf{R}}u \right\rangle_{\mathbb{C}^p} dt \quad (4.41)$$

and the estimation error:

$$e(t) = x(t) - x_c(t). \quad (4.42)$$

where x_c is the estimation of the state variables and also the state variables of the dynamical LQG controller.

As a consequence the dynamic controller associated with the LQG control problem (4.1) can be written as:

$$\begin{cases} \dot{x}_c &= \left(\mathcal{M}\mathcal{Q} - \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c - P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^*\mathcal{Q} \right) x_c + P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}u_c \\ y_c &= \tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c x_c \end{cases} \quad (4.43)$$

The closed loop system with the above LQG controller is stable but not in general passive, because the Hamiltonian structure is lost in the closed loop formulation. We should discuss how to reformulate a passive LQG controller in order to preserve the passivity and Hamiltonian structures in closed loop.

4.4.2 LQG formulation of control by interconnection

We first reformulate the above LQG controller under a *quasi-Hamiltonian form* by choosing the energy operator of the controller equal to \mathcal{Q} . This leads to:

$$\begin{aligned} \dot{x}_c &= \left(\mathcal{M} - \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} - P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^* \right) \mathcal{Q}x_c + P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}u_c \\ &= (\mathcal{J} - \mathcal{R}_c)\mathcal{Q}x_c + P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}u_c \\ y_c &= \tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1}\mathcal{Q}x_c \end{aligned} \quad (4.44)$$

with

$$\mathcal{R}_c = \mathcal{R} + \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} + P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^* \quad (4.45)$$

In this expression the state operator is decomposed into the product $(\mathcal{J} - \mathcal{R}_c)\mathcal{Q}$ with the energy operator \mathcal{Q} defined in (4.37). However the operator $\mathcal{R}_c = \mathcal{R} + \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} + P_f\mathcal{Q}\mathcal{B}\mathbf{R}_w^{-1}\mathcal{B}^*$ is in general neither self-adjoint nor positive.

Next we shall derive the conditions on the LQG control Problem 4.1 such that the controller (4.44) has a port Hamiltonian realization, which are summarized in the following theorem.

Theorem 4.4 (Hamiltonian LQG method). *Consider the LQG problem with the following relation between the operator $\tilde{\mathbf{R}}$ and \mathbf{R}_w :*

$$\tilde{\mathbf{R}} = \mathbf{R}_w. \quad (4.46)$$

If the weighting operators $\tilde{\mathcal{Q}}$ and \mathcal{Q}_v are related by:

$$\mathcal{Q}_v z = \mathcal{Q}^{-1} \left(2\mathcal{Q}\mathcal{J}^*P_c + 2P_c\mathcal{J}\mathcal{Q} + \tilde{\mathcal{Q}} \right) \mathcal{Q}^{-1}z \quad \text{with } z \in X \quad (4.47)$$

then the operator equations (4.39) and (4.40) admit a unique solution, P_c and P_c respectively. These two solutions are related by:

$$\mathcal{Q}^{-1}P_c = P_f\mathcal{Q} \quad (4.48)$$

Furthermore the LQG controller is passive called Hamiltonian LQG controller and the closed loop system can be written as the feedback interconnection of the IDPHS (4.29) with the port Hamiltonian realization of the LQG regulator.

Proof. By using condition (4.48) and \mathcal{Q} is invertible as it is coercive, positive definite, the filter Riccati equation (4.40) is equivalent to:

$$(\mathcal{M}P_c\mathcal{Q}^{-1} + \mathcal{Q}^{-1}P_c\mathcal{M}^* - \mathcal{Q}^{-1}P_c\mathcal{B}\mathcal{R}_w^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} + Q_v)z = 0. \quad (4.49)$$

With condition (4.46) and factorizing \mathcal{Q}^{-1} from the two sides of left of this equation (4.49) this equation becomes:

$$(\mathcal{Q}\mathcal{M}P_c + P_c\mathcal{M}^*\mathcal{Q} - P_c\mathcal{B}\mathcal{R}_w^{-1}\mathcal{B}^*P_c + \mathcal{Q}Q_v\mathcal{Q})z = 0 \quad (4.50)$$

Subtracting the equation (4.50) by the control Riccati equation (4.39) leads to:

$$\left[\mathcal{Q}\mathcal{M}^*P_c + P_c\mathcal{M}\mathcal{Q} + \tilde{\mathbf{Q}} - (\mathcal{Q}\mathcal{M}P_c + P_c\mathcal{M}^*\mathcal{Q} + \mathcal{Q}Q_v\mathcal{Q}) \right] z = 0$$

because $D(\mathcal{M}^*) = D(\mathcal{M})$ and $\mathcal{M}z = (\mathcal{J} - \mathcal{R})z$, so:

$$\begin{aligned} & \left[\mathcal{Q}(\mathcal{M}^* - \mathcal{M})P_c + P_c(\mathcal{M} - \mathcal{M}^*)\mathcal{Q} + \tilde{\mathbf{Q}} - \mathcal{Q}Q_v\mathcal{Q} \right] z = 0 \\ & \Leftrightarrow \mathcal{Q}Q_v\mathcal{Q}z = \left(2\mathcal{Q}\mathcal{J}^*P_c + 2P_c\mathcal{J}\mathcal{Q} + \tilde{\mathbf{Q}} \right) z \end{aligned}$$

\mathcal{Q} being invertible, then:

$$\mathbf{Q}_v z_1 = \mathcal{Q}^{-1} \left(2\mathcal{Q}\mathcal{J}^*P_c + 2P_c\mathcal{J}\mathcal{Q} + \tilde{\mathbf{Q}} \right) \mathcal{Q}^{-1} z_1 \quad \text{with } z_1 \in X$$

Hence, the equation (4.47) is verified.

By using Assumption 2, the IDPHS (4.37) is exponentially stabilisable and exponentially detectable, this assumption is met for any operator $\tilde{\mathbf{Q}}$ and \mathbf{Q}_v and both filter and control Riccati equations admit a unique solution.

We now show that the LQG controller is passive and it can be considered as a controller of the IDPHS by interconnection. First by using conditions (4.48) and (4.46), the controller output (4.44) becomes:

$$\begin{aligned} y_c &= \left(\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} \right) \mathcal{Q}x_c \\ \Leftrightarrow y_c &= \left(\mathbf{R}_w^{-1}\mathcal{B}^*QP_f \right) \mathcal{Q}x_c \end{aligned} \quad (4.51)$$

which means that the output mapping of controller (4.44) y_c is the power conjugate of its input mapping u_c .

Secondly, if we consider conditions (4.48) and (4.46), one can easily check that the operator \mathcal{R}_c defined in equation (4.45)

$$\mathcal{R}_c = \mathcal{R} + \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} + \mathcal{Q}^{-1}P_c\mathcal{Q}\mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^* \quad (4.52)$$

is self-adjoint, i.e.,

$$\mathcal{R}_c = \mathcal{R}_c^*. \quad (4.53)$$

Furthermore the operator

$$\begin{aligned} \mathcal{M}_c\mathcal{Q} &= \left(\mathcal{M} - \mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^*P_c\mathcal{Q}^{-1} - \mathcal{Q}^{-1}P_c\mathcal{B}\tilde{\mathbf{R}}^{-1}\mathcal{B}^* \right) \mathcal{Q} \\ &= (\mathcal{J} - \mathcal{R}_c)\mathcal{Q} \end{aligned} \quad (4.54)$$

generates an exponentially stable semi-group and the operator Q being bijective, the operator \mathcal{R}_c is positive definite.

As a consequence, the LQG controller has a port Hamiltonian realization, then the closed loop system can be regraded as the interconnection of two infinite dimensional port Hamiltonian systems. \square

Remark 4.3. The choice of the control design parameters Q_v or \tilde{Q} is free by using the Theorem 4.4, but the two parameters are related by (4.47). That means that on one hand we can firstly choose the weighting operator \tilde{Q} and then compute the co-variance operator Q_v as Theorem 4.4. On the other hand we can define the co-variance operator Q_v , and derive the weight operator \tilde{Q} . However, the parameters Q_v , R_w and \tilde{Q} , \tilde{R} depend on each others. If we choose the optimal control problem first, i.e., weighting operators \tilde{Q} and \tilde{R} , then the co-variance operators Q_v and R_w are just the control design parameters and have no statistic meaning in the filter design problem. On the other hand, if we consider the filter design problem first, then the weighting operators \tilde{Q} and \tilde{R} have no sense in the optimal control problem.

Theorem 4.4 gives us a passive LQG control design method and we call it *Hamiltonian LQG method* because this passive LQG controller is obtained in a port Hamiltonian realization. The closed loop system by using *Hamiltonian LQG controller* can be regarded as the control by interconnection of two port Hamiltonian systems. Hence the structure and passivity are conserved in the closed loop system.

In the next section, we will discuss the passivity and structure preserving reduction for the closed loop port Hamiltonian system through the *Hamiltonian LQG method*.

4.5 Hamiltonian LQG reduction of infinite dimensional PHS with preserving the passivity

In this section, we define a LQG balanced realization for the IDPHS (4.37) by using its Hamiltonian LQG formula of Theorem 4.4 and find an approximation of this LQG balanced IDPHS. To do so, we should discuss the product of the Hamiltonian LQG Gramians given as:

$$P_f P_c = P_f Q P_f Q$$

that is different from the identity. It allows us to find a balanced transformation in order to truncate the state space in a meaningful way with respect to closed loop performance. Before studying the reduction procedure, we define the balanced realization of the IDPHS (4.29) through its Hamiltonian LQG form.

4.5.1 Hamiltonian LQG balanced realization of IDPHS

Before defining the Hamiltonian LQG Balanced realization, we introduce the *Hamiltonian LQG Hankel operator* of IDPHS (4.37).

Definition 4.12. Consider two operators $S \in \mathcal{L}(X_s; X)$ and $L \in \mathcal{L}(X_L; X)$ where X_S and X_L are

Hilbert spaces such that the Hamiltonian LQG Gramians satisfy

$$P_c = SS^* \text{ and } P_f = LL^* \quad (4.55)$$

Then

$$\mathcal{H}_{LQG} = S^*L \in \mathcal{L}(X) \quad (4.56)$$

is called an *Hamiltonian LQG Hankel operator* of IDPHS (4.37).

Next we shall discuss the nuclearity of the Hamiltonian Gramians P_f , P_c and the Hamiltonian LQG Hankel operator \mathcal{H}_{LQG} .

Lemma 4.2. *If we chose the weighting operator \tilde{Q} as*

$$\tilde{Q} = \gamma^2 C^* C = \gamma^2 Q B \tilde{R}^{-1} B^* Q, \quad \forall \gamma \in \mathbb{R}. \quad (4.57)$$

Then P_c and P_f , the two solutions to the operator Riccati equations (4.39) and (4.40) of the LQG control problem 4.1 associated with Theorem 4.4, are nuclear. Furthermore, The Hamiltonian LQG Hankel operator \mathcal{H}_{LQG} is also nuclear.

Proof. To prove the nuclearity of the solution P_c of Riccati equation (4.39), we consider the Lyapunov equation of the closed loop system:

$$\begin{aligned} & \underbrace{[(\mathcal{M}Q - \tilde{B}\tilde{R}^{-1}B^*P_c)^* L_o + L_o (\mathcal{M}Q - \tilde{B}\tilde{R}^{-1}B^*P_c)]}_{\mathcal{M}_{co}^*} \\ & \quad + P_c \tilde{B} \tilde{R}^{-1} B^* P_c + \tilde{Q} z = 0 \end{aligned} \quad (4.58)$$

with $z \in D(\mathcal{M})$. By developing this Lyapunov equation, we can get:

$$(\mathcal{Q}\mathcal{M}^*L_o + L_o\mathcal{M}Q - P_c\tilde{B}\tilde{R}^{-1}B^*L_o - L_o\tilde{B}\tilde{R}^{-1}B^*P_c + P_c\tilde{B}\tilde{R}^{-1}B^*P_c + \tilde{Q})z = 0 \quad (4.59)$$

with $z \in D(\mathcal{M})$ Then

$$L_o = P_c, \quad (4.60)$$

and (4.59) is equivalent to the Riccati equation (4.39).

The closed loop system is formulated as

$$\left[\begin{array}{c|c} \begin{array}{c} \mathcal{M}_{co} \\ \tilde{Q}^{1/2} \\ \tilde{R}^{1/2} B^* P_c \end{array} & \begin{array}{c} \mathcal{B} \\ \mathcal{D}_{co} \end{array} \end{array} \right] \quad (4.61)$$

with $\mathcal{M}_{co} = \mathcal{M}Q - \tilde{B}\tilde{R}^{-1}B^*P_c$, and \mathcal{B} , \mathcal{D}_{co} are the input and feedthrough operators of this closed loop system. If we can prove L_o is nuclear, then $P_c = L_o$ is also nuclear.

The solution of Lyapunov equation (4.58), L_o is the observability Gramian of the system (4.61). By using the Theorem of [13, Thm 3.1], if the following conditions

1. The operator \mathcal{M}_{co} is a generator of a exponentially stable C_0 semi-group;

2. The system (4.61) has a finite rank output space

hold, then $P_c = L_o$ is nuclear.

From Assumption 2, the operator \mathcal{M}_{co} is the generator of an exponentially stable C_0 -semigroup, thus the first condition holds. To satisfy the second condition, we can define the weighting operator:

$$\tilde{\mathbf{Q}} = \gamma^2 \mathbf{Q} \mathbf{B} \mathbf{B}^* \mathbf{Q} \quad (4.62)$$

with $\gamma \in \mathbb{R}$. Then we get

$$\tilde{\mathbf{Q}}^{1/2} = \gamma \mathbf{B}^* \mathbf{Q} \in \mathcal{L}(X, \mathbb{C}^p). \quad (4.63)$$

The operator

$$\tilde{\mathbf{R}}^{1/2} \mathbf{B}^* P_c \in \mathcal{L}(X, \mathbb{C}^p). \quad (4.64)$$

Hence the output operator of system (4.61)

$$\mathcal{C}_{co} = \begin{bmatrix} \tilde{\mathbf{Q}}^{1/2} \\ \tilde{\mathbf{R}}^{1/2} \mathbf{B}^* P_c \end{bmatrix} \in \mathcal{L}(X, \mathbb{C}^{2p}) \quad (4.65)$$

maps from the state space X to a finite rank space \mathbb{C}^{2p} , i.e., the system (4.61) has a finite rank output space, the second condition holds too. As a consequence the LQG Gramian P_c is *nuclear*. Similarly we can prove that $P_f = \mathbf{Q} P_c \mathbf{Q}$ and Hamiltonian LQG Hankel operator \mathcal{H}_{LQG} are also *nuclear*. \square

Following Lemma 4.2, the Hamiltonian LQG Hankel operator \mathcal{H}_{LQG} is nuclear. We can then define the balanced realization with respect to the Hamiltonian LQG Gramians P_f and P_c of the LQG control problem 4.1 associated with Theorem 4.4.

Definition 4.13. The IDPHS is called *Hamiltonian LQG balanced* if $X = \ell_2$ and there exists a positive and non-increasing sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that the Hamiltonian LQG Gramians P_f and P_c are both equal to the diagonal operator:

$$\Sigma = \text{diag}(\sigma_n)_{n \in \mathbb{N}} \in \mathcal{L}(\ell_2). \quad (4.66)$$

In other words:

$$P_f = P_c = \Sigma. \quad (4.67)$$

As we discussed in Lemma 4.2, the Hamiltonian LQG Hankel operator $\mathcal{H}_{LQG} = S^* L \in \mathcal{L}(X)$ is nuclear with $S, L \in \mathcal{L}(X)$ and admits a singular value decomposition:

$$S^* L = V \Sigma U^* \quad (4.68)$$

where

$$\Sigma = \text{diag}(\sigma_n)_{n \in \mathbb{N}} \in \mathcal{L}(\ell_2)$$

with the positive sequence of Hamiltonian LQG Hankel singular values (σ_n) . And $V, U \in \mathcal{L}(\ell_2; X)$ are isometrics onto their ranges, i.e.,

$$V^* V = I, \quad U^* U = I. \quad (4.69)$$

We define the transformation operators :

$$\begin{aligned} T &:= \Sigma^{-1/2}V^*S^* \subset X \mapsto \ell_2; \\ T^+ &:= LU\Sigma^{-1/2} \subset \ell_2 \mapsto X. \end{aligned} \quad (4.70)$$

Theorem 4.5. *Suppose that the IDPHS (4.29) and the transformation operators T and T^+ defined in (4.70), then a balanced realization from (4.29) is given as*

$$\begin{cases} \dot{x}_b(t) &= M_b Q_b x_b(t) + B_b u(t) \\ y(t) &= B_b^* Q_b x_b(t) \end{cases} \quad (4.71)$$

with

$$M_b = TMT^* \quad Q_b = T^{+*}QT^+ \quad B_b = TB \quad (4.72)$$

The state space of the balanced IDPHS is $x_b \in \ell_2$.

Proof. The two solutions of Riccati equations have the following decomposition:

$$P_c = SS^* \text{ and } P_f = LL^*. \quad (4.73)$$

We compute

$$TP_fT^* = \Sigma^{-1/2}V^*S^*LL^*SV\Sigma^{-1/2}, \quad (4.74)$$

with the SVD of Hamiltonian LQG Hankel operator:

$$S^*L = V\Sigma U^*, \quad (4.75)$$

then

$$TP_fT^* = \Sigma. \quad (4.76)$$

We can compute

$$T^+P_cT^{+*} = \Sigma \quad (4.77)$$

in the same way. As a consequence, we get the Hamiltonian LQG Gramians of the system (4.71) as follows:

$$TP_fT^* = T^+P_cT^{+*} = \Sigma. \quad (4.78)$$

The system (4.71) is the Hamiltonian LQG balanced realization of the IDPHS (4.29). \square

By using the above proposition, the original IDPHS is reformulated into its Hamiltonian LQG balanced realization (4.71). This balanced realization is defined on an ℓ_2 space, and the state variables are separated and arranged in decreasing order according to their importance in the closed-loop system defined from the Hamiltonian LQG singular values. In other words, the state variables associated with large singular values are more important for the Hamiltonian LQG control design than the other ones. Hence from the closed loop point of view, this balanced realization gives us the good choice of state space to reduce the IDPHS. This reduction method is derived in the next sub-section.

4.5.2 Approximation of IDPHS

From the Proposition 4.5 and the transformation operators defined by (4.70), we introduced the Hamiltonian LQG balanced realization of IDPHS (4.29) as system (4.71). To reduce this balanced system, we can of course use the truncation method given in Theorem 4.3. However, the passivity and Hamiltonian structure are not preserved with this finite dimensional approximation scheme. Hence, we should consider another approximation scheme to preserve the passivity and Hamiltonian structure during the reduction procedure.

To do so, we use the Petrov-Galerkin projection method [34] to get a finite dimensional approximation of the balanced system (4.71). We consider the decomposition of the state variables $x(t) = x_n(t) + x_r(t)$ with $x(t) \in X$, $x_n(t) \in V$ and $x_r(t) \in W^\perp$, where $V = \text{span}\{v_1, \dots, v_n\}$ and $W = \text{span}\{w_1, \dots, w_n\}$ with v_i in the state operator domain ($v_i \in D(\mathcal{M}\mathcal{Q})$ in system (4.29)) and $w_i \in X$ the n -dimensional subspaces of X , and W^\perp the orthogonal complement of W . This decomposition exists and is unique if $V \cap W^\perp = \{0\}$. The linear operators $\mathcal{V} : \mathbb{C}^n \mapsto X$ and $\mathcal{W} : X \mapsto \mathbb{C}^n$ defined by:

$$\mathcal{V}\alpha = \sum_{i=1}^n v_i \alpha_i \quad \mathcal{W}h = \begin{bmatrix} \langle h, w_1 \rangle_X \\ \vdots \\ \langle h, w_n \rangle_X \end{bmatrix} \quad (4.79)$$

for all $\alpha \in \mathbb{C}^n$, $h \in X$ are such that $V \cap W^\perp = \{0\}$. This property can be easily verified by using $\det(\mathcal{W}\mathcal{V}) \neq 0$ where $\mathcal{W}\mathcal{V} \in \mathbb{C}^{n \times n}$. Thus one can use the internal direct sum decomposition $X = V \oplus W^\perp$ if the choices of W and V are such that $\det(\mathcal{W}\mathcal{V}) \neq 0$. In order to determine a finite-dimensional model that describes the dynamics of x_n it is advantageous to introduce the projection $\mathcal{P} : X \mapsto V$ of X onto V along W^\perp , yielding the relation $x_n(t) = \mathcal{P}X(t)$. This projection can be expressed as $\mathcal{P} = \mathcal{V}(\mathcal{W}\mathcal{V})^{-1}\mathcal{W}$ and satisfies $\mathcal{P} = \mathcal{P}^2$. Its range and null space satisfy:

$$\text{Ran}\mathcal{P} = V; \quad \text{Ker}\mathcal{P} = W^\perp$$

To preserve the passivity of the port Hamiltonian system (4.29) by using the Petrov-Galerkin projection method, a special choice of operator \mathcal{V} and \mathcal{W} is given in [34]. In this method, the authors did not give the choice of vectors v_i .

Inspired from this method we shall introduce a choice of vectors v_i which defines the projection operator \mathcal{V} and \mathcal{W} to preserve the passivity and Hamiltonian structure through the *balanced reduction* of (4.71):

Theorem 4.6. Define $\mathcal{V} : \mathbb{C}^n \mapsto \ell_2$ by

$$\mathcal{V}z = \sum_{i=1}^n v_i z_i \quad \forall z_i \in \mathbb{C}^n, \quad i \in \mathbb{N} \quad (4.80)$$

with $v_i = (\delta_{i,1}, \delta_{i,2}, \dots) \in \ell_2$ is the canonical unit vector. Consider the special choice $\mathcal{W} = \mathcal{V}^*Q_b$. Then a structure preserving approximation of the infinite-dimensional DPHS is a linear DPHS:

$$\begin{cases} \dot{x}_n &= M_n Q_n x_n + B_n u \\ y &= B_n^* Q_n x_n \end{cases} \quad (4.81)$$

with

$$\begin{aligned} M_n &= \mathcal{V}^* Q_b M_b Q_b \mathcal{V} & Q_n &= (\mathcal{V}^* Q_b \mathcal{V})^{-1} \\ B_n &= \mathcal{V}^* Q_b B_b \end{aligned} \quad (4.82)$$

Proof. First we can show that

$$\mathcal{P} = \mathcal{V}(\mathcal{W}\mathcal{V})^{-1}\mathcal{W} = \mathcal{P}^2$$

is a projection.

Next we choose $x_b \approx \mathcal{V}z_n$ and premultiply (4.71) by the operator $\mathcal{W} = \mathcal{V}^* Q_b$. The finite-dimensional approximation becomes

$$\begin{cases} \mathcal{V}^* Q_b \mathcal{V} \dot{z}_n &= \mathcal{V}^* Q_b M_b Q_b \mathcal{V} z_n + \mathcal{V}^* Q_b B_b u(t) \\ y(t) &= B_b^* Q_b \mathcal{V} z_n \end{cases} \quad (4.83)$$

We choose $x_n = \mathcal{V}^* Q_b \mathcal{V} z_n$. The matrix M_n can be separated in two parts. One part is skew symmetric and the other part is symmetric positive definite because

$$M_n + M_n^* \leq 0$$

and

$$J_n = \frac{1}{2}(M_n - M_n^*) \text{ and } R_n = -\frac{1}{2}(M_n + M_n^*) \quad (4.84)$$

□

Remark 4.4. In this projection method, the operator \mathcal{V} is used to separate the state space X , and the special choice of $\mathcal{W} = \mathcal{V}^* Q_b$ can make sure that the finite dimensional approximation still has the port Hamiltonian structure and preserves the passivity.

By using the finite dimensional PHS (4.81) and the LQG control Problem 4.1 associated with Theorem 4.4, one can then design a finite dimensional controller in order to stabilize the IDSHP (4.37).

4.6 Application to the control of a Timoshenko Beam

In this section, we consider a 1-dimensional Timoshenko Beam model with damping in the spatial coordinate $z \in [0, 1]$ under a port Hamiltonian framework [35, 43] as follows:

$$\begin{aligned} \dot{x} &= \mathcal{M}Qx + \mathcal{B}u(t) \\ y &= \mathcal{B}^*Qx \end{aligned} \quad (4.85)$$

with the operators:

$$\mathcal{M} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 & -1 \\ \frac{\partial}{\partial z} & -R_p & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 1 & 0 & \frac{\partial}{\partial z} & -R_{rp} \end{bmatrix} \quad (4.86)$$

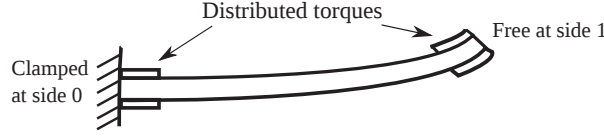


Figure 4.1: Example: Clamped Timoshenko beam

$$\mathcal{Q} = \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{bmatrix}. \quad (4.87)$$

The energy of the beam is expressed in terms of the energy variables:

$$H = \frac{1}{2} \int_a^b (Kx_1^2 + \frac{1}{\rho}x_2^2 + EIx_3^2 + \frac{1}{I_\rho}x_4^2) dz \quad (4.88)$$

where x_1 is the shear displacement, x_2 is the transverse momentum distribution, x_3 is the angular displacement and x_4 is the angular momentum distribution. The coefficients ρ , I_ρ , E , I and K are the mass per length unit, the rotary moment of inertia of a cross section, the elasticity Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively, and the state space $X = L_2(0; 1; \mathbb{R}^4)$. The beam is clamped at the extremity $z = 0$, and free at $z = 1$ as shown in Figure 4.1, i.e., $\frac{1}{\rho}x_2(0, t) = \frac{1}{I_\rho}x_4(0, t) = 0 \forall t \geq 0$ and $Kx_1(1, t) = EIx_3(1, t) = 0 \forall t \geq 0$. The domain of the operator \mathcal{M} is

$$D(\mathcal{M}) = \{x \in H_1(0, 1; \mathbb{R}^n) \mid x_2(0, t) = x_4(0, t) = 0, x_1(1, t) = x_3(1, t) = 0, \forall t \geq 0\} \subset X \quad (4.89)$$

then one can easily check that the domain of the adjoint operator \mathcal{M}^* is the same as \mathcal{M} .

The operator \mathcal{Q} is self-adjoint and coercive. We consider that the beam is actuated by a distributed torque $b_1(z)u_1(t)$ on the small interval $I_{b_1} = [0, 0.1]$, with $b_1(z) = 1$ for $z \in I_{b_1}$ and $b_1(z) = 0$ elsewhere, and also by another distributed torque $b_2(z)u_2(t)$ on the small interval $I_{b_2} = [0.9, 1]$, with $b_2(z) = 1$ for $z \in I_{b_2}$ and $b_2(z) = 0$ elsewhere. As output, we consider the angular velocities mean value $y_1 = \int_0^1 b_1(z) \frac{1}{I_\rho} x_4 dz$ and $y_2 = \int_0^1 b_2(z) \frac{1}{I_\rho} x_4 dz$. As a consequence the input operator is:

$$\mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ b_1(z) & b_2(z) \end{bmatrix} \quad (4.90)$$

where $\mathcal{B} : \mathbb{C}^2 \mapsto X$. The output is the conjugated to this input, i.e.,

$$y = \mathcal{B}^* \mathcal{Q}x. \quad (4.91)$$

We can easily verify that the scalar inner product

$$\operatorname{Re}\langle \mathcal{M}z, z \rangle \leq 0 \quad \forall z \in D(\mathcal{M}) \quad (4.92)$$

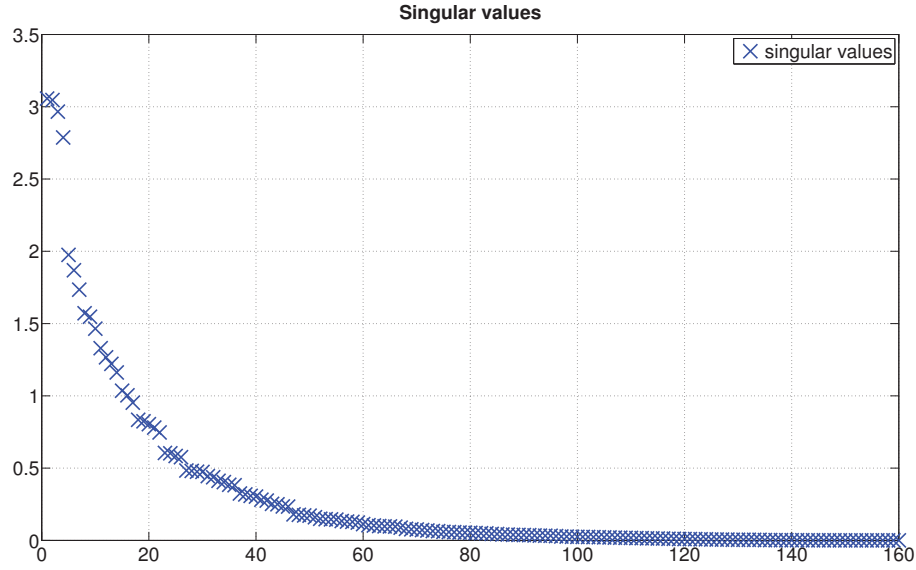


Figure 4.2: LQG singular values

The operator \mathcal{M} is dissipative, and \mathcal{M} generates a C_0 semi-group. Furthermore there exists a $0 < \lambda_0 \in \rho(\mathcal{M})$ such that $\text{Ran}(\lambda_0 I - \mathcal{M}) = X$ [14, Lemma 2.1.11], then \mathcal{M} is m-dissipative. It is a generator of a contraction semigroup.

As a consequence, the system (4.85) satisfies all the same conditions as system (4.37), then one can get the reduced system by using Theorem 4.4.

All the numerical parameters of the Timoshenko beam have been selected equal to one in the simulations. To solve the operator Riccati equation and for the reduced controller design, we use the mixed-finite element method [24, 5] to discretize the Timoshenko beam. The beam has been divided in 40 infinitesimal subsections, hence the system has 160 state variables.

Firstly, we use the proposed Theorem (4.4) to design the controller and reduce the system by taking the weighting operator $\tilde{\mathbf{Q}} = \mathbf{Q}\mathbf{B}\mathbf{B}^*\mathbf{Q}$. Figure (4.2) shows the LQG singular values. One can observe that the first four singular values are larger than the others (there is 0.8 difference between 4th and 5th singular values). That means that the first four states of the balanced system play the most important role in the closed loop system. As consequence we reduce the balanced system to order 4 using Theorem 4.6, then we design a reduced order controller on this reduced system.

We get two closed loop systems by interconnecting the full order controller ($x_c \in \mathbb{R}^{160}$, the black curve) and reduced order controller ($x_{cr} \in \mathbb{R}^4$, the red solid curve) with the open loop system (The blue curve). Figure 4.3 shows the step response of the open loop system, the closed loop systems with full order controller and reduced order controller respectively. The inputs are a torque on the clamped side of the beam and the output is the angular velocity of the free side of the beam. One can observe that the closed loop systems with full order and reduced order have almost the same performances but have a little gap (the difference is less than 0.01) in the overshoot peaks as shown in the sub Figure 4.3 that is due to the fact that the reduced controller have only an order 4. It means that even if we conserved the most important state variables in the reduced system, but we still lost some

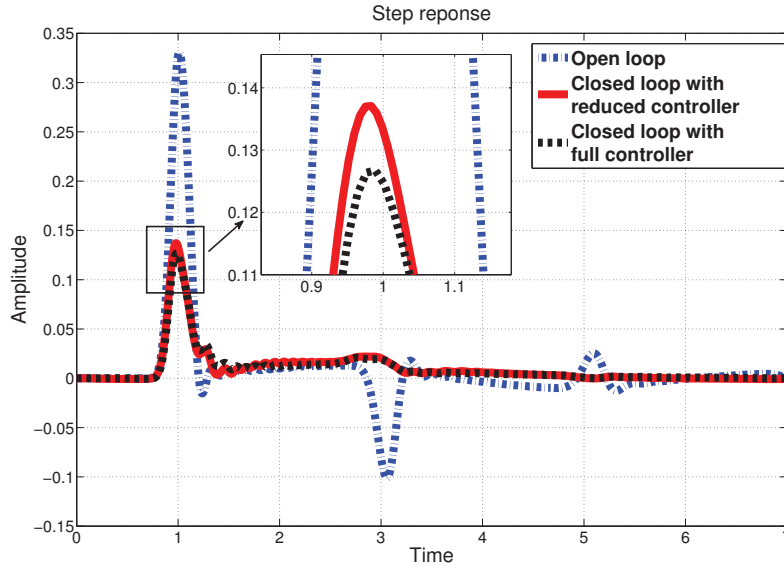


Figure 4.3: Compare the step response of low order open loop systems (160) with its closed loop systems by using full order controller (160) and reduced order controller (4)

information for the control design.

In Figure 4.4 we show the poles location of the two closed loop systems. The blue one are the poles of the open loop system. We can see that these poles are very near to the imaginary axis because the system have small dissipation. The four poles on the left of the closed loop system with the reduced order controller (red) are not far away from the ones of the closed loop system with the full order controller (black), especially the ones equal to -26 .

Next we will show the reduced order controller is also efficient for the even higher order system. To do so we discretize the Timoshenko beam with 80 infinitesimal subsections, that means there are 320 state variables (high order system), which is much more close to the infinite dimensional Timoshenko beam.

Figure 4.5 is a comparison of the high order ($x \in \mathbb{R}^{320}$, black curve) and low order ($x \in \mathbb{R}^{160}$, blue solid curve) open loop system. We can observe in the figure, the step responses of the low order open loop system and the high order open loop system are almost the same and no much different.

In Figure 4.6, we show that the full order LQG controller ($x_c \in \mathbb{R}^{160}$, black curve) and the reduced order controller ($x_c \in \mathbb{R}^4$, red solid curve) designed on the low order system are also stabilize the high order system and have almost same performances. We call the full order LQG controller but its states variables is $x_c \in \mathbb{R}^{160}$ but not as the same order as the high order system $x \in \mathbb{R}^{320}$ because the controller is designed from low order system ($x \in \mathbb{R}^{160}$). The reduced order controller is also obtained by the reduced order system which is the approximation of the low order system.

Even the performance of the two different closed loop systems are almost the same, however the step responses have a small gap between the overshoot peaks (less than 0.01). The reason is the same as the low order closed loop systems. We conserve just 4 order in the reduced controller, that means that some information is lost during the reduction procedure.

We show in Figure 4.7 the poles location of the high order closed loop systems by interconnecting

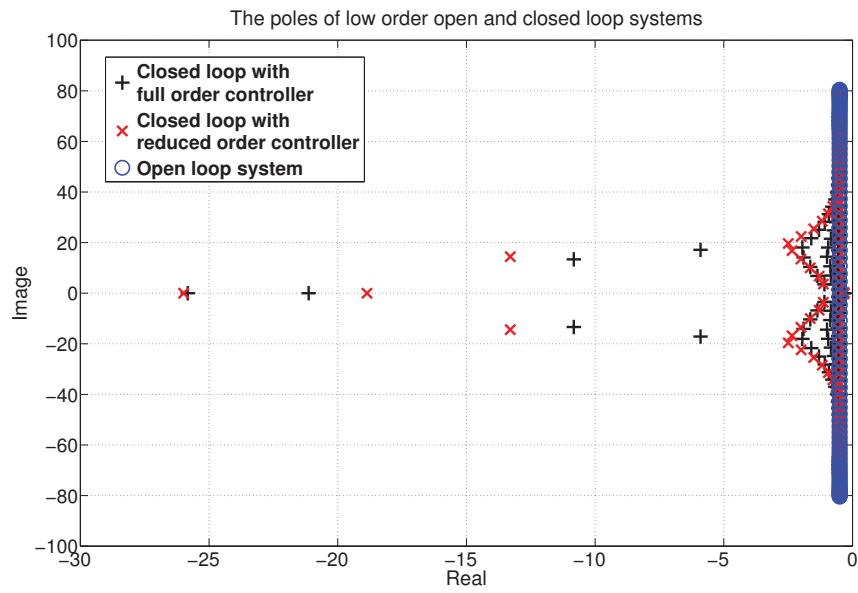


Figure 4.4: Compare the poles of low order open loop systems ($x \in \mathbb{R}^{160}$) with its closed loop systems by using full order controller ($x_c \in \mathbb{R}^{160}$) and reduced order controller ($x_{cr} \in \mathbb{R}^4$)

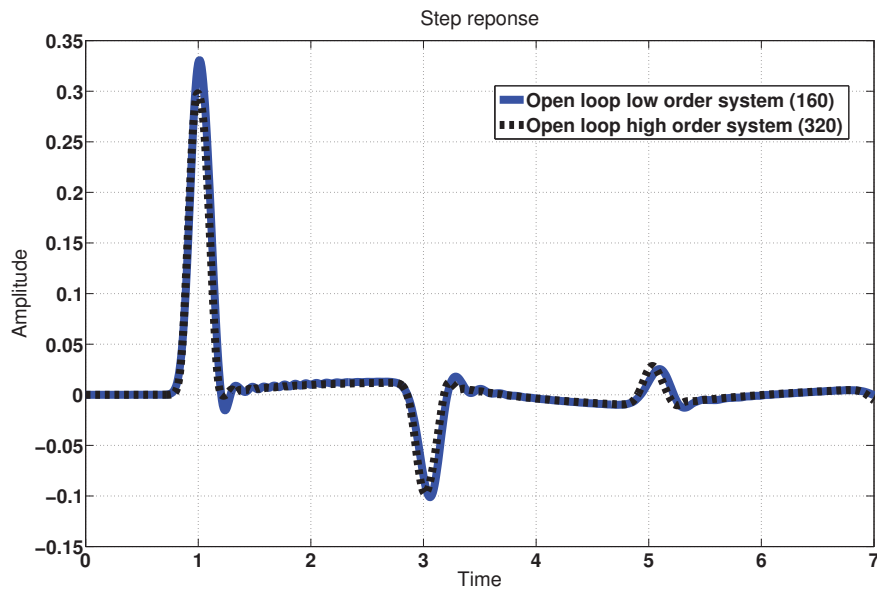


Figure 4.5: Compare the step response of low order open loop system (160) with high order one (320)

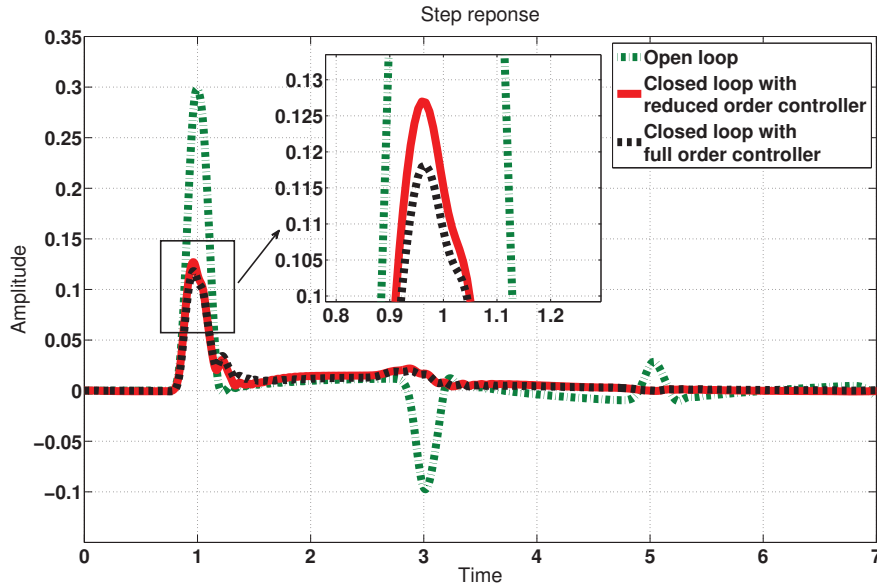


Figure 4.6: Compare the step response of High order open loop systems (320) with its closed loop systems by using full order controller (160) and reduced order controller (4)

the full order ($x_c \in \mathbb{R}^{160}$, black ones) controller and reduced order ($x_{cr} \in \mathbb{R}^4$, red ones) respectively (The blue ones are the poles of open loop system). As shown in the low order systems, on the left side, the four poles of the two different closed loop systems are near to each other, especially the ones in the left side. Because the reduced order system conserve the first four important state variables in the control design point of view, thus the reduced order controller has almost the same performance as the full order one.

4.7 Conclusion

In this chapter, we have suggested a passive LQG control design method for IDPHS called Hamiltonian LQG method. Furthermore based on this control design method, a passivity and Hamiltonian structure preserving reduction method for IDPHS has been proposed.

By taking the special choice of the weighting operators of optimal control problem and filter design problem, the Hamiltonian LQG method gives us a *passive LQG controller* for IDPHS and moreover it is a *Hamiltonian structure preserving* LQG controller design method. In the other words, the Hamiltonian structure is preserved in the *closed loop system* since the Hamiltonian LQG controller itself is also a port Hamiltonian system, and the closed loop system can be regarded as the control by interconnection of two port Hamiltonian systems. The choices of the weighting operators are special but not unique. We can choose the optimal control problem or the filter (observer) design problem as we want but the parameters for two problems are related. because the choice of these parameters has to make sure the LQG controller to be passive and has the port Hamiltonian realization. In the other words, if we define the optimal control problem, it means the optimal control weighting operators $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ are decided, thus the observer design weighting operators \mathbf{Q}_v and \mathbf{R}_w are also fixed which may

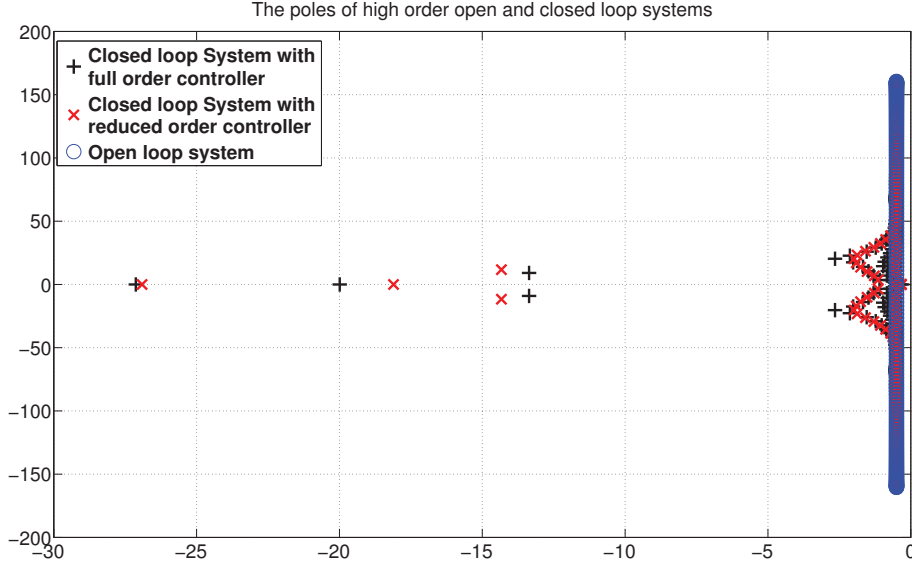


Figure 4.7: Compare the poles of High order open loop systems (320) with its closed loop systems by using full order controller (160) and reduced order controller (4)

be not the optimal observer design weighting operators but only the controller design parameters.

However, the passive LQG controller obtained by using the *Hamiltonian LQG method* is still infinite dimensional as the system itself. Hence we suggested a reduction-then-design scheme to get a finite dimensional controller for the IDPHS. Then we take a special choice of the weighting operator $\tilde{\mathbf{Q}}$, the Hamiltonian LQG method allows us to define a balanced realization in which the sum of Hamiltonian LQG singular values is less infinity and in the decreasing order because the LQG Hankel operator is nuclear. That means the state variables related to the bigger Hamiltonian LQG singular values are more important for the control design. This balanced realization gives us a choice to separate the state variables from the closed loop point of view.

To approximate the Hamiltonian LQG balanced realization, we suggested to use the Petrov-Galerkin projection method with the projection operator \mathcal{V} to separate the state space and a special choice of operator $\mathcal{W} = \mathcal{V}^* \mathcal{Q}$ to conserve the Hamiltonian structure and passivity of the IDPHS. By using this method, we can get a finite dimensional port Hamiltonian approximation of IDPHS and we design the finite dimensional passive LQG controller by this finite dimensional approximation.

In the last part, the proposed approximation scheme is illustrated by the Timoshenko beam on the port Hamiltonian framework. The control of this Timoshenko beam is on the domain and no energy exchange on the boundary. To apply our method, we first discretize our IDPHS by mixed finite elements method [24, 5], because how to solve the operator Riccati equations is still an open problem and rarely trade in the literature. We compared the reduced order controller ($x_{cr} \in \mathbb{R}^4$) and the full order controller by using the step responses and the poles placements. The simulation results are satisfactory and the two controllers have not so much difference. Furthermore, we applied this reduced order controller to a higher order system which can be considered closer to the infinite dimensional system. The results are also satisfactory, The closed loop system by using the reduced order controller has almost the same performance as the one using the full order controller.

Chapter 5

Conclusion

5.1 General conclusion

This thesis deals with passivity and structure preserving model reduction of port Hamiltonian systems with different perspectives.

In chapter 2, a passivity preserving reduction method has been proposed for the reduction of implicit port Hamiltonian systems. The implicit port Hamiltonian systems stem from physical algebraic constraints arising from network modeling and interconnection. These systems have been redefined under the descriptor state space framework, called port Hamiltonian descriptor systems, by using a coordinate transformation and the elimination of the Lagrangian multipliers. This descriptor realization preserves the Dirac structure. This representation then allows to combine tools for the descriptor system's analysis and Dirac structures which define the port Hamiltonian systems. In this context Weierstrass canonical form is used to separate the slow and fast dynamics. It is then possible to put the system under a balanced form by using the controllability and observability Lyapunov equations. This balanced realization allows to choose which part of state variables has to be conserved in the reduced system through the Hankel singular values decomposition. Instead of traditional truncation method, the effort and flow constraint methods have been applied to reduce the balanced port Hamiltonian descriptor system in order to preserve both its passivity and its geometric structure. This model reduction method has been proposed to reduce implicit port Hamiltonian systems from the input/output point of view. The main advantage of this method is that not only the passivity and geometric structure are preserved, but also the fast response of the sub-systems. At last, a micro mechanical manipulator (nano-tweezers) under development for the manipulation of DNA bundles has been used to illustrate the proposed model reduction method. It is shown with this example that the method preserves the structure and the passivity of the original system. This is a great advantage with respect to truncation methods and size equivalent spatial discretization methods (in the high frequencies).

Chapter 3 is concerned with passivity and structure preserving LQG design and closed loop reduction of port Hamiltonian systems. First it is shown that up to some assumptions on the constitutive matrices LQG controller can be recast under passive port Hamiltonian formulation in Theorem 3.2. The closed loop system can then be regarded as the control by interconnection of the two systems and the passivity and Hamiltonian structure of the systems are preserved in the closed loop. From

this reformulation it appears that the only control design parameter is the covariance matrices that have no more statistical meaning. An important issue of such formulation is that the product of the two solutions of the LQG associated Riccati equations is equal to identity, i.e. this LQG method cannot be used to define a balanced realization suitable for model reduction i.e., all the state variables have the same importance for control design. Another passive LQG control design method, called Q-conjugated LQG control method, has been proposed in Theorem 3.3 to overcome this drawback. It remains to link the cost weighting matrices to the covariance matrices. In this case the resulting closed loop system can also be regarded as a control by interconnection and the passivity and Hamiltonian structure are preserved in closed loop. One advantage of this formulation is that one can freely choose the weighting matrices or the covariance matrices of the control problem. Moreover the product of the solutions of the two associated Riccati equations will not be equal to one allowing to reduce the system.

As a consequence this Q-conjugated LQG control design method allows to define a closed loop LQG balanced realization in which the state variables are separated, since the LQG singular values are in the decreasing order in this LQG coordinates. Effort constraint method is then used to reduce this LQG balanced realization with preserving the passivity and geometric structure of the original system. The reduced order controller is then derived from this reduction step. It has a port Hamiltonian form and its passivity and geometric structure are preserved in the closed loop systems. Those proposed methods have been illustrated with the control of a mass-spring-damper system. It shows that in closed loop the Q-conjugated full order and reduced order controllers lead to better results than the standard one and also preserve the passivity and the geometric structure.

In chapter 4, the passive LQG method, called Hamiltonian LQG method, has been considered for the control design and the approximation of infinite dimensional port Hamiltonian systems with control within the spatial domain (bounded input mapping). The port Hamiltonian formulation of the infinite dimensional passive LQG control problem has been proposed in Theorem 4.3. As in the finite dimensional case, it implies some conditions in the choice of the weighting and covariance operators. Once again this control design can be interpreted as the control by interconnection. However the resulting passive LQG controller has the same dimensions as the system itself, i.e. it is infinite dimensional. Hence to get the finite dimensional controller, one has to proceed to model reduction (in closed loop in order to deal with weakly or undamped systems).

It is shown that the port Hamiltonian LQG formulation associated with a specific choice of the weighting and covariance operators, leads to a *nuclear* LQG Hankel operator (defined as the square root of the product of the solutions of the Riccati equations) . From this property the system is put under balanced realization by an appropriate change of variables. The state space of this balanced realization is a ℓ_2 space. The LQG Hankel singular values are in the decreasing order and converge to zero. This property allows us to separate the state space and to reduce the system using a Petrov-Galerkin projection method. The finite dimensional (low order) LQG controller is derived from this approximation. Finally, the proposed approximation scheme has been illustrated on the control of a Timoshenko beam. The mixed finite elements method is used to discretize the system and to solve the operator Riccati equations. The simulation results shows that the low order LQG controller derived using this approximation scheme can perfectly stabilize the infinite dimensional system as well as the infinite dimensional Hamiltonian LQG controller.

5.2 Future research

There are several perspectives for this work. The most significant are listed below:

- In the finite dimensional case:
 - the error estimation of Q-conjugated LQG reduction method is still an open problem. Differently from the standard LQG method, the optimal control problem and optimal filtering problem are related in Q-conjugated LQG method, it is then very difficult to compute the error of estimation.
 - in this work we have considered a closed loop balanced reduction method for *linear* port Hamiltonian systems. It would be interesting to study the non linear closed loop reduction. Indeed the open loop balanced reduction method has been generalized to *non linear* port Hamiltonian systems in [21] and its extension to closed loop reduction remains challenging.
- In the infinite dimensional case:
 - the main difficulty of the method proposed in this thesis is the resolution of the operator Riccati equations. It has been done by using a finite approximation of the considered operators but the effects of such a scheme remains to be studied. From a theoretical point of view it is a major concern to deal with.
 - as in the finite dimensional case the estimation of the error due to the approximation has to be characterized in the future.
 - only input control within the domain with bounded operators has been considered. All the proposed result still have to be generalized to boundary control, with all the theoretical difficulties associated with the use of unbounded operators. These perspectives shall be considered in the future research.

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