



# Multivariate moment problems : applications of the reconstruction of linear forms on the polynomial ring

Mathieu Collowald

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## THESE

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Discipline: MATHÉMATIQUES

présentée et soutenue par

Mathieu COLLOWALD

**Problèmes multivariés liés aux moments :  
applications de la reconstruction de formes linéaires sur  
l'anneau des polynômes.**

**Multivariate moment problems:  
applications of reconstruction of linear forms on the polynomial  
ring.**

Thèse dirigée par Évelyne HUBERT

soutenue le 18 décembre 2015

Jury :

|                          |    |                        |                       |
|--------------------------|----|------------------------|-----------------------|
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| M. DÉSIDÉRI Jean-Antoine | DR | Inria Sophia Antipolis | Président             |
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| M. MOURRAIN Bernard      | DR | Inria Sophia Antipolis | Co-directeur de thèse |



# Introduction générale

Dans cette thèse, nous considérons le problème général suivant. Soient  $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$  l'anneau des polynômes à  $n$  variables et à coefficients dans un corps  $\mathbb{K}$  de caractéristique zéro et  $\Lambda$  une forme linéaire sur  $\mathbb{K}[x]$  pouvant s'écrire sous la forme d'une combinaison linéaire d'évaluations en des points

$$\Lambda : \mathbb{K}[x] \rightarrow \mathbb{K}, p \mapsto \sum_{j=1}^r a_j p(\xi_j),$$

ou de manière équivalente

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j} \tag{0.1}$$

avec  $r \in \mathbb{N} \setminus \{0\}$ ,  $a_1, \dots, a_r \in \mathbb{K} \setminus \{0\}$  et  $\xi_1, \dots, \xi_r$  des points distincts de  $\mathbb{K}^n$ . Il s'agit de reconstruire cette forme linéaire  $\Lambda$  à partir d'évaluations de celle-ci en des polynômes, c'est-à-dire trouver les valeurs des coefficients  $a_1, \dots, a_r$ , des points  $\xi_1, \dots, \xi_r$  et dans certains cas le nombre de termes  $r$ . Lorsque la forme linéaire  $\Lambda$  est identifiée à l'intégration sur une mesure à support fini, les évaluations en des monômes  $\Lambda(x^k)$  avec  $k \in \mathbb{N}^n$  sont appelés les moments.

Pour certains petits exemples, l'ensemble des calculs peut se faire dans une extension algébrique du corps des rationnels  $\mathbb{Q}$  et donc en arithmétique exacte. Cependant, la majeure partie des exemples intéressants traités ici nécessite la résolution de problèmes d'algèbre linéaire en arithmétique flottante et donc l'utilisation du corps des réels  $\mathbb{R}$ . À noter que, pour certains résultats théoriques, le corps des complexes  $\mathbb{C}$  est utilisé pour une première approche.

Cette formulation à l'aide d'une forme linéaire sur l'anneau des polynômes permet de traiter plusieurs applications. Il s'agit alors dans un premier temps de reformuler ces problèmes. Des exemples sont donnés par la reconstruction de polytopes à partir de leurs moments (*shape-from-moment problem*), la recherche de *cubatures* (analogues multidimensionnels des quadratures), la décomposition de tenseurs (*tensor decomposition*), l'interpolation creuse (*sparse interpolation*) ou l'interpolation exponentielle (*exponential analysis*). Les deux premiers problèmes font l'objet des deux chapitres de cette thèse, tandis que les trois suivants ont déjà été traités dans la littérature avec une approche similaire.

Après reformulation, ces problèmes peuvent être résolus via la méthode de Prony ou une méthode similaire. À l'origine [93], celle-ci est une méthode de résolution pour des problèmes univariés. Son utilisation dans le cadre de problèmes multivariés nécessite alors soit une reformulation du problème pour un traitement univarié, soit une adaptation multivariée à l'aide des opérateurs de Hankel.

## Des problèmes multivariés avec une résolution univariée

La reconstruction de polytopes à partir des moments, les interpolations creuse et exponentielle sont des problèmes multivariés. Une résolution à l'aide de la méthode (univariée) de Prony est cependant possible après avoir reformulé le problème initial. Nous nous attardons ici sur les différentes reformulations et sur l'algorithme que nous proposons pour la reconstruction numérique de polytopes convexes à partir des moments directionnels.

Dans le cadre de l'interpolation creuse, l'algorithme de Ben-Or & Tiwari [11] choisit d'évaluer la fonction à reconstruire en des puissances successives de nombres premiers. Dans le cadre de l'interpolation exponentielle, l'algorithme dans [88] propose d'évaluer la fonction en des points bien choisis d'une grille. Ces choix permettent la reformulation souhaitée. Dans le cadre de la reconstruction de polytopes à partir des moments, nous reprenons l'idée de [47] d'utiliser des moments directionnels suivants des directions bien choisies. La formule de Brion [8] permet alors la reformulation souhaitée.

## Reconstruction numérique de polytopes convexes à partir des moments directionnels

Le premier chapitre de cette thèse est consacré à la reconstruction de polytopes à partir de leurs moments. Plus précisément, nous nous sommes intéressés aux *polytopes convexes* de  $\mathbb{R}^n$  avec  $n \in \mathbb{N}$ , c'est-à-dire aux enveloppes convexes de nombres finis de points. Ces points sont alors appelés les *sommets* du polytope convexe. Ce dernier est noté  $\mathcal{V}$ . Cette définition généralise celle des polygones convexes quand  $n = 2$  et celle des polyèdres convexes quand  $n = 3$ . Les *moments* d'un polytope convexe sont, de manière très générale, une suite de nombres qui caractérisent de manière globale le polytope convexe. Il en existe différents types : les moments géométriques  $(m_\alpha)_{\alpha \in \mathbb{N}^n}$ , directionnels  $(m_k(\delta))_{k \in \mathbb{N}}$  et, quand  $n = 2$ , complexes  $(m_k(1, i))_{k \in \mathbb{N}}$

$$m_\alpha = \int_{\mathcal{V}} x^\alpha dx, \quad m_k(\delta) = \int_{\mathcal{V}} \langle x, \delta \rangle^k dx, \quad m_k(1, i) = \int_{\mathcal{V}} (x + iy)^k dx dy.$$

L'objectif du premier chapitre de cette thèse est de proposer un algorithme pour reconstruire numériquement (en arithmétique flottante) un polytope convexe de  $\mathbb{R}^n$  à partir d'un nombre fini de ses moments directionnels. Il suffit ici de retrouver l'ensemble de ses sommets.

De tels algorithmes ont été proposés, mais dans des cas particuliers. Pour le cas des polygones ( $n = 2$ ) [43, 83], l'algorithme s'articule autour de l'utilisation des moments complexes et de méthodes numériques efficaces. Nous allons tirer profit de ces dernières. Le cas général ( $n$  quelconque) n'a lui été étudié à notre connaissance qu'en arithmétique exacte dans [47]. Cependant, cette étude permet de reformuler le problème de reconstruction de polytopes sous la forme souhaitée (0.1).

**Théorème** (Formule de Brion). *Supposons que les projections orthogonales des  $r$  sommets du polytope convexe  $\mathcal{V}$  sur la direction  $\delta$  sont distinctes, nous avons alors les égalités suivantes*

$$\begin{aligned} \frac{(k+n)!}{k!} m_k(\delta) &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n+k}, \quad k \geq 0, \\ \text{et} \\ 0 &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n-k}, \quad 1 \leq k \leq n, \end{aligned}$$

où  $a_v(\delta)$  dépend de  $\delta$  et des sommets adjacents à  $v$  dans une triangulation de  $\mathcal{V}$ . De plus,

$$a_v(\delta) \neq 0.$$

Ainsi, les moments directionnels du polytope convexe sont reliés aux évaluations d'une forme linéaire univariée (0.1) en les monômes  $x^k$  avec  $k \in \mathbb{N}$

$$\Lambda(x^k) = \begin{cases} 0 & \text{si } k \leq n-1, \\ \frac{(k+n)!}{k!} m_k(\delta) & \text{si } k \geq n. \end{cases}$$

Les points  $\xi_j$  sont ici les projections  $\langle v, \delta \rangle$  des sommets  $v$  sur une droite  $\delta$ , appelée *direction*. La formule de Brion généralise la formule de Davis [28] qui était à la base de la reconstruction des polygones dans [43, 83]. L'algorithme que nous proposons combine les méthodes numériques de la reconstruction de polygones avec la formule théorique pour les dimensions supérieures. Pour obtenir un algorithme robuste numériquement, il y a cependant quelques problèmes sous-jacents à résoudre.

Un premier problème vient du fait que nous travaillons avec des moments directionnels et des projections de sommets. À partir des moments directionnels et de la méthode de Prony, nous pouvons donc retrouver les sommets projetés suivant cette direction. À noter que comme dans [43, 83], nous préférons la variante numériquement plus stable de la méthode de Prony, qui exprime les objets recherchés comme les valeurs propres généralisées d'une paire de matrices. Pour retrouver un sommet à  $n$  coordonnées, nous avons besoin d'au moins  $n$  sommets projetés correspondant au même sommet. Cependant, comme nous récupérons les sommets projetés dans les listes de valeurs propres généralisées, il est impossible *a priori* de décider si une projection est associée à un sommet ou à un autre. Un procédé permettant d'effectuer cette correspondance de manière robuste numériquement est nécessaire : nous avons choisi l'interpolation par intervalles [96].

Un second problème vient du fait qu'en général le nombre de sommets est inconnu. Il s'agit alors d'estimer au mieux ce nombre. Nous proposons ici de construire une matrice suffisamment grande à partir des moments directionnels du polytope et d'en trouver le rang numérique. Ce dernier peut se calculer à l'aide d'une décomposition en valeurs singulières (SVD) de la matrice.

Notre algorithme, qui reprend les grandes lignes de celui dans [47], est constitué principalement de trois étapes :

1. Estimation du nombre de sommets  $r$  à partir d'un majorant  $R$  et des moments jusqu'à l'ordre  $2R + 1 - n$  dans différentes directions.
2. Calcul des projections des sommets suivant au moins  $n + 1$  directions situées relativement proches les unes des autres grâce à une méthode similaire à la méthode de Prony.
3. Correspondance des sommets à partir des projections grâce à l'interpolation par intervalles.

Le premier chapitre de cette thèse se conclut avec des exemples pour les cas  $n = 2$  et  $n = 3$ . Un défi pour nous a été la reconstruction d'une coupe de diamant. En effet, le conditionnement des problèmes de valeurs propres généralisées augmente de manière exponentielle en fonction du nombre de sommets [9, 10]. Ceci nous a conduits à effectuer les calculs en grande précision au lieu de la double précision utilisée pour les autres exemples plus petits.

## Interpolation creuse

Un problème d'interpolation polynomiale, comme l'*interpolation creuse*, consiste à retrouver un polynôme à partir de sa valeur en certains points. Le terme *creuse* fait référence au faible nombre de coefficients non nuls de ce polynôme par rapport à son degré, qui est inconnu. Ce terme s'oppose à l'interpolation *dense* pour laquelle ces deux entiers sont proches. C'est le cas par exemple de l'interpolation lagrangienne pour un polynôme univarié.

La reformulation du problème d'interpolation creuse sous la forme de la reconstruction d'une forme linéaire sur l'anneau des polynômes s'inspire de l'algorithme de Ben-Or & Tiwari [11, 63]. Soit  $f$  un polynôme de  $\mathbb{K}[x]$  défini par

$$f(x) = \sum_{j=1}^r a_j x^{\alpha_j}$$

avec  $a_1, \dots, a_r \in \mathbb{K} \setminus \{0\}$  et  $\alpha_1, \dots, \alpha_r \in \mathbb{N}^n$ . Choisissons un point  $p = (p_1, \dots, p_n)$ , dont les coordonnées  $p_i$  sont des nombres premiers distincts deux à deux et un majorant  $\tau$  du nombre de termes  $r$ . En évaluant le polynôme  $f$  aux puissances de  $p$  suivantes  $p^0, p^1, \dots, p^{2\tau+1}$  avec  $p^k = (p_1^k, \dots, p_n^k)$ , on obtient

$$f(p^k) = \sum_{j=1}^r a_j p^{k\alpha_j} = \sum_{j=1}^r a_j \mathbb{1}_{p^{\alpha_j}}(x^k).$$

Ainsi, en posant  $\xi_j = p^{\alpha_j}$  pour tout  $j = 1, \dots, r$ , on a

$$\Lambda(x^k) = \sum_{j=1}^r a_j \mathbb{1}_{p^{\alpha_j}}(x^k) = f(p^k) \quad \forall k = 0, \dots, 2\tau + 1.$$

Cette reformulation reste valable également si on remplace les nombres premiers  $p_1, \dots, p_n$  par des racines  $p$ -ièmes de l'unité avec  $p$  premier comme dans l'algorithme numérique proposé dans [42].

## Interpolation exponentielle

Dans le même esprit, il est également possible de considérer une somme d'exponentielles, dont on souhaite estimer les paramètres comme dans [13, 87, 88]. L'objectif de l'*interpolation exponentielle*, comme pour l'interpolation polynomiale, est de retrouver ces paramètres à partir de sa valeur en certains points.

La reformulation du problème d'interpolation exponentielle sous la forme de la reconstruction d'une forme linéaire sur l'anneau des polynômes peut se faire de la manière suivante [88]. Soit  $f$  une fonction à valeurs réelles pouvant s'écrire sous la forme d'une somme d'exponentielles

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \sum_{j=1}^r a_j e^{i\omega_j x}$$

avec  $a_1, \dots, a_r \in \mathbb{C} \setminus \{0\}$  et  $\omega_1, \dots, \omega_r \in [-\pi, \pi]^n$ .

Prenons une grille de  $\mathbb{R}^n$  : il existe alors des vecteurs unitaires  $\delta$  indiquant différentes directions de  $\mathbb{R}^n$ . En évaluant la fonction  $f$  aux vecteurs  $k\delta$  avec  $k = 0, 1, \dots, N$  et  $N$  un majorant de  $r$ , on obtient

$$f(k\delta) = \sum_{j=1}^r a_j e^{i\omega_j \delta k} = \sum_{j=1}^r a_j \xi_j^k \quad \forall k = 0, \dots, N.$$

La dernière égalité est obtenue en posant  $\xi_j = e^{i\omega_j \delta}$  pour tout  $j = 1, \dots, r$ . Nous retrouvons alors

$$f(k\delta) = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}(x^k) = \Lambda(x^k) \quad \forall k = 0, \dots, N.$$

Comme pour les méthodes de projection de [18, 47], il suffit ensuite de considérer plusieurs directions pour retrouver les fréquences  $\omega_1, \dots, \omega_r$ .

## Des problèmes multivariés avec une résolution multivariée

La résolution multivariée, c'est-à-dire sans utiliser de méthodes intermédiaires univariées, présentée dans cette thèse a pour point de départ l'étude des opérateurs de Hankel et de la notion d'extension plate. Cette notion est utilisée dans le contexte des matrices de moments dans [24, 25, 71, 72, 73, 74]. Nous montrons que les opérateurs de Hankel permettent de généraliser l'approche par les matrices de moments. Nous pouvons alors travailler avec d'autres bases plus adaptées à nos besoins que des bases monomiales.

Dans la littérature, les opérateurs de Hankel ont été utilisés de manière similaire pour la décomposition de tenseurs [12, 15]. Nous rappelons ici la reformulation du problème initial sous sa forme duale. Dans le deuxième chapitre de cette thèse, nous développons un algorithme pour la recherche de cubatures symétriques à l'aide de cette généralisation des matrices de moments. Notre apport majeur réside dans l'utilisation de la symétrie combinée aux opérateurs de Hankel. Celle-ci permet de diminuer la complexité de l'algorithme. Différentes cubatures symétriques sont données explicitement.

## Une approche par les matrices de moments pour les cubatures symétriques

Le second chapitre de cette thèse est consacré à la recherche de cubatures symétriques. Une cubature est l'analogue multidimensionnel d'une quadrature, c'est-à-dire une forme linéaire sur  $\mathbb{R}[x]$  définie par (0.1) qui satisfait

$$\int p \, d\mu = \Lambda(p) \quad \forall p \in \mathbb{R}[x]_{\leq d}$$

pour une mesure  $\mu$  donnée et un degré  $d$  donné. Nous supposons les mesures positives de Borel sur  $\mathbb{R}^n$  à support compact telles que les fonctions caractéristiques d'un simplex, d'un hypercube, d'une boule ou d'un hexagone. Nous nous concentrons sur les cubatures intérieures à poids positifs, c'est-à-dire celles dont les poids  $a_1, \dots, a_r$  sont strictement positifs et dont les noeuds  $\xi_1, \dots, \xi_r$  sont sur le support de la mesure. Bien que l'existence de telles mesures ait été démontré (théorème de Tchakaloff [7, 89]), certaines questions comme le nombre minimum de noeuds pour un degré fixé et pour une mesure donnée n'ont actuellement pas de réponse. Il existe seulement des bornes sur ce nombre [7, 30, 89]

$$\dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor} \leq r \leq \dim \mathbb{R}[x]_{\leq 2d}.$$

Un point de départ pour la recherche de cubatures est l'article de Radon [92] : l'auteur obtient les 7 noeuds d'une cubature de degré 5 pour les fonctions caractéristiques du triangle, du disque et du carré comme les zéros de trois polynômes orthogonaux de degré 3. L'idée d'obtenir les noeuds de cubatures comme zéros de polynômes orthogonaux s'est poursuivie dans les travaux qui ont suivi (cf. des *reviews* sur le sujet [19, 20, 101]). L'approche que nous avons suivie est celle par les matrices de moments débutée dans [35]. Elle est complétée par l'utilisation de la symétrie pour réduire la taille des matrices et le nombre de paramètres inconnus. L'algorithme que nous proposons détermine en arithmétique exacte l'existence de formules de cubatures et résout ensuite les problèmes de valeurs propres généralisées, dont les solutions nous informent sur les noeuds des cubatures. Un nombre suffisant d'informations permet alors de calculer les noeuds et les poids de la cubature.



La symétrie apparaît naturellement dans la recherche de cubatures de qualité. Les domaines standard d'intégration (simplex, parallélépipède, boule) peuvent être normalisés par transformation affine. Ils présentent alors une symétrie relative à des groupes. La théorie des représentations linéaires pour les groupes finis nous permet d'introduire les *bases orthogonales adaptées à la symétrie* à partir des idées de [22, 39, 94]. Ces bases permettent de travailler avec des matrices d'opérateurs de Hankel diagonales par blocs.

La symétrie nous apporte d'autres informations. En effet, le rang de chaque bloc dans la diagonale est lié à l'organisation des noeuds en types d'orbites. Cette relation est rendue explicite grâce au nouveau concept de *matrice de multiplicités* associée à un groupe fini. Ce concept nous donne des critères préliminaires pour éliminer des organisations de noeuds en types d'orbites et par extension des possibles nombres de noeuds  $r$  de la cubature.

Une contribution supplémentaire est un algorithme sans fraction et sans pivotement pour diagonaliser des matrices symétriques à coefficients polynomiaux en trouvant le lieu où celles-ci sont semidéfinies positives. Cet algorithme en arithmétique exacte est utilisé pour déterminer l'existence de toutes les cubatures d'un certain degré.

Les algorithmes décrits dans le second chapitre sont implémentés en Maple. Ils sont appliqués pour retrouver certaines cubatures connues. De nouvelles cubatures sont également apparues grâce à notre approche, qui permet de trouver toutes les cubatures symétriques d'un certain degré.

## Décomposition de tenseurs

Une telle reformulation pour la *décomposition de tenseurs* existe dans [12, 15]. Nous donnons ici une courte description du cas des tenseurs symétriques à partir de [15]. Soit  $(h_{j_0 \dots j_n})_{0 \leq j_0, \dots, j_n \leq d}$  un tenseur symétrique d'ordre  $d$  et de dimension  $n$ . Il est associé à un polynôme homogène de degré  $d$  à  $n + 1$  variables

$$f(x) = \sum_{j_0 + \dots + j_n = d} h_{j_0 \dots j_n} x_0^{j_0} \dots x_n^{j_n}.$$

Le but de la décomposition de tenseurs (symétriques) est de trouver une réécriture de  $f$  sous la forme

$$f(x) = \sum_{j=1}^r a_j (k_j(x))^d \text{ avec } k_j(x) = k_{j0}x_0 + \dots + k_{jn}x_n. \quad (0.2)$$

avec un nombre minimum de termes  $r$ . Les inconnues sont donc ici le rang du tenseur  $r$ , les coefficients  $a_1, \dots, a_r$  et ceux des polynômes  $k_1, \dots, k_r$ , c'est-à-dire  $k_{10}, \dots, k_{1n}, \dots, k_{r0}, \dots, k_{rn}$ . Cette décomposition (0.2) admet une écriture duale

$$f^* = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j} \text{ avec } k_j \in \mathbb{K}^{n+1}.$$

Celle-ci n'est cependant définie que sur les polynômes homogènes de degré  $d$  à  $n + 1$  variables. En supposant  $k_{j0} = 1$  pour tout  $j = 1, \dots, r$ , cette écriture duale est associée à la forme linéaire sur  $\mathbb{K}[x]$

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j} \text{ avec } \xi_j = (k_{j1}, \dots, k_{jn}).$$

A noter que  $\Lambda$  n'est *a priori* connue que sur  $\mathbb{R}[x]_{\leq d}$ , l'espace des polynômes à  $n$  variables et de degré au plus  $d$ .

Cette thèse est divisée en deux chapitres qui présentent tous deux un exemple de reconstruction de formes linéaires sur l'anneau des polynômes multivariés : soit avec une résolution univariée pour le premier chapitre concernant la reconstruction de polytopes à partir de leurs moments, soit avec une résolution multivariée pour le second chapitre concernant la recherche de cubatures symétriques.



## Chapter I

# Numerical Reconstruction of Convex Polytopes from Directional Moments

### Abstract

We reconstruct an  $n$ -dimensional convex polytope from the knowledge of its directional moments. The directional moments are related to the projection of the polytope vertices on a particular direction. To extract the vertex coordinates from the moment information we combine established numerical algorithms such as generalized eigenvalue computation and linear interval interpolation. Numerical illustrations are given for the reconstruction of 2-d and 3-d convex polytopes.

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# 1 Introduction

The reconstruction of the boundary of a shape from its moments is a problem that has only partially been solved. For instance, when the shape is a polygon [43, 83], or when it defines a quadrature domain in the complex plane [50], it has been proved that its boundary can be reconstructed exactly from the knowledge of its moments. Both results admit no obvious extension to higher dimensions. The technique in [27] is applicable in higher dimensions, but the authors reconstruct a shape's characteristic function, or more generally, a square-integrable function defined on a compact domain. The reconstruction algorithm for polygons is based on Davis' exact integration formula [28] of a function in the complex plane. For polynomial functions, Davis' formula can be seen as a low dimensional case of identities attributed to Brion [8]. Based on the latter, the authors in [47] proposed the reconstruction of an  $n$ -dimensional convex polytope in exact arithmetic.

Brion's integration formula over a polytope does not relate moment information directly to the vertices of the convex polytope, but rather to the projections of these vertices onto some 1-dimensional subspace. To recover the projections, we recognize an inverse problem that arises in several areas [42, 55, 59, 64, 86] and can be solved numerically as a generalized eigenvalue problem.

After recovering the projections of the vertices on various one-dimensional subspaces, there remains the problem of matching different projections (in different directions) of the same vertex, with that vertex. In this paper we describe how to solve this issue, without resorting to exact arithmetic. The problem cannot be solved with ordinary interpolation or least squares approximation. But using an interval interpolation technique [96], we understand why we need  $n + 1$  projections (or more) to solve the matching.

Our method is the result of combining techniques from quite different mathematical disciplines: integer lattices, computer algebra, numerical linear algebra, interval methods, inverse problems. The complete algorithm, the outline of which draws on [47], consists of the following steps:

1. The exact number of vertices  $r$  is computed from an upper bound  $R$  and moments up to order  $2R + 1 - n$ , in a sample of directions.
2. For  $n + 1$  (or more) directions, the projections of the vertices are obtained as the generalized eigenvalues of a structured pair of matrices whose entries are determined from the directional moments up to order  $2r - n - 1$ .
3. Each of these projections is then matched to the corresponding vertex and its coordinates are computed as the coefficients of an  $n$ -dimensional interval interpolant.

We demonstrate the results obtained in dimension 2 and 3 in Section 6 and report on a challenge in Section 7. The different steps in our algorithm involve Hankel matrices, in the singular value decomposition for the computation of  $r$ , as well as in the generalized eigenvalue problem delivering the vertex projections. Structured matrices with real elements have condition numbers that grow exponentially with their size [9, 10], and the size of our matrices is determined by the number of vertices of the polytope. In Section 6 double precision allows us to reconstruct polytopes up to 12 vertices. In Section 7 we are required to use high precision floating-point arithmetic for the polyhedron with 57 vertices that represents a brilliant diamond cut.

The paper is organized as follows. In Section 2 we introduce directional moments together with Brion's formulae. In Section 3 we review Prony's method and the related eigenvalue problem to determine the projections of the vertices from the directional moments. In Section 4 we discuss the determination of the number of vertices. In Section 5 we present an algorithm to solve the matching problem. Numerical illustrations are given in Sections 6 and 7 where we reconstruct 2-d and 3-d convex polytopes.

## 2 Geometric and directional moments

In this section we present identities attributed to Brion. These identities are central in [3] to establish the complexity of the computation of the moments of a polytope. Brion's identities are also at the core of the solution of the inverse problem proposed in [47]. They can actually be seen as a generalisation of Davis' integration formula that was used to solve the shape-from-moment problem in 2-d [43, 83].

We consider a convex polytope in  $\mathbb{R}^n$  determined by the set of its  $r$  vertices  $\mathcal{V}$ . Abusing the notation,  $\mathcal{V}$  also denotes the polytope itself.

The *geometric moments* are

$$m_\alpha = \int_{\mathcal{V}} x^\alpha dx = \int \dots \int_{\mathcal{V}} x_1^{\alpha_1} \dots x_n^{\alpha_n} dx_1 \dots dx_n, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n.$$

The order of the geometric moment  $m_\alpha$  is  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . These moments can be expressed as a multivariate polynomial in the coordinates of the vertices [99]. A complexity analysis for the computation based on Brion's identities is offered in [3] for exact arithmetic.

The *moment in the direction*  $\delta \in \mathbb{R}^n$  of order  $k$  is

$$m_k(\delta) = \int_{\mathcal{V}} \langle x, \delta \rangle^k dx, \quad k \in \mathbb{N},$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual scalar product in  $\mathbb{R}^n$ .

One can obtain any directional moment of order  $k$  from the geometric moments of order  $k$  with the multinomial formula

$$m_k(\delta) = \sum_{|\alpha|=k} \binom{k}{\alpha} m_\alpha \delta^\alpha.$$

Conversely, geometric moments of order  $k$  can be obtained from the directional moments of order  $k$  in  $\binom{n+k-1}{k}$  distinct directions by solving a linear system of equations.

In the context of polygon retrieval ( $n = 2$ ) from tomographic data [83, 43], geometric moments and then complex moments are computed from directional moments. Complex moments can be understood as moments in the direction  $\delta = (1, i)$ . At the core of this shape-from-moments problem is Davis' integration formula for an analytic function  $f$  on the polygon  $\mathcal{V}$  in the complex plane [28]

$$\iint_{\mathcal{V}} f''(x + iy) dx dy = \sum_{v \in \mathcal{V}} a_v f(v),$$

where the vertices  $v \in \mathcal{V}$  are here interpreted as complex numbers. Assuming that  $\check{v}$  and  $\hat{v}$  are the vertices adjacent to  $v$ , the coefficients in Davis' formula are

$$a_v = \frac{V_v}{(v - \check{v})(v - \hat{v})},$$

where  $V_v$  is the oriented area of the parallelogram defined by the vectors with vertices  $\check{v} - v$  and  $\hat{v} - v$ .

This formula bears a generalisation to any dimension, known as Brion's identities. The formula relates the directional moments to the projections of the vertices. This allows us to work directly with directional moments, which are data that can be deduced from tomographic measurements.

**Theorem 2.1.** *[47] Provided that the orthogonal projections of the  $r$  vertices of the convex polytope  $\mathcal{V}$  on the direction  $\delta$  are distinct, we have the following equalities*

$$\begin{aligned} \frac{(k+n)!}{k!} m_k(\delta) &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n+k}, \quad k \geq 0, \\ \text{and} \quad 0 &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n-k}, \quad 1 \leq k \leq n, \end{aligned} \tag{2.1}$$

where the  $a_v(\delta)$  depend on  $\delta$  and the adjacent vertices of  $v$  in a triangulation of  $\mathcal{V}$ .

Moreover

$$a_v(\delta) \neq 0, \quad v \in \mathcal{V}. \tag{2.2}$$

The formula for the coefficients  $a_v(\delta)$  is given in [8, Section 10.3] when  $\mathcal{V}$  is a simple convex polytope. That is, each vertex in the polytope has exactly  $n$  adjacent vertices. Let  $\mathcal{V}_v$  be the set of  $n$  adjacent vertices of  $v$ . The volume  $V_v$  of the parallelepiped determined by  $\mathcal{V}_v$  is obtained through the determinant of the edge vectors of  $\mathcal{V}_v$ . Then

$$a_v(\delta) = \frac{V_v}{\prod_{u \in \mathcal{V}_v} \langle v - u, \delta \rangle}. \tag{2.3}$$

In particular, for a simplex  $\triangle$  with vertices  $v_0, v_1, \dots, v_n$ ,

$$\frac{(k+n)!}{k!} \int_{\triangle} \langle x, \delta \rangle^k dx = V \sum_{i=0}^n \frac{\langle v_i, \delta \rangle^{k+n}}{\prod_{j \neq i} \langle v_i - v_j, \delta \rangle} = V \sum_{k_0 + \dots + k_n = k} \langle v_0, \delta \rangle^{k_0} \dots \langle v_n, \delta \rangle^{k_n} \tag{2.4}$$

where  $V = V_{v_0} = \dots = V_{v_n}$ . Notice that this is actually a polynomial in  $\delta$  though we shall use its more compact rational expression.

For a more general convex polytope, one has to consider a partition of the polytope into simplices that does not introduce any additional vertex [8, Theorem 3.1]. The coefficients  $a_v(\delta)$  for the convex polytope is then a sum of its siblings in the formulae for the simplices. That they do not vanish is proved in [47].

The directions  $\delta \in \mathbb{R}^n$  to which the theorem applies are those for which  $\langle u, \delta \rangle \neq \langle v, \delta \rangle$  for all distinct  $u, v \in \mathcal{V}$ . Those are the *generic directions*. We examine what happens when  $\delta$  fails to be generic in this meaning.

On one hand, Brion's identities are correct for any  $\delta$  that does not make the denominators of  $a_v$  vanish. However, if  $\delta$  is a direction for which the coefficients  $a_v(\delta)$  are well defined but for which there are two distinct vertices  $u, v \in \mathcal{V}$  such that  $\langle u, \delta \rangle = \langle v, \delta \rangle$ , we can write the formula with less than  $r$  terms. The linear recurrence introduced in Section 3 is then of order less than  $r$  and the associated Hankel matrix is of rank less than  $r$ .

In addition, when  $\langle v_0, \delta \rangle = \dots = \langle v_p, \delta \rangle$  for distinct vertices  $v_0, \dots, v_p \in \mathcal{V}$  that belong to the same simplex of any triangulation, there is a formula similar to (2.1) where the  $p+1$  terms  $\langle v_i, \delta \rangle^{n+k}$  are replaced by terms  $\langle v_0, \delta \rangle^{n+k}, (n+k)\langle v_0, \delta \rangle^{n+k-1}, \dots, (n+k)\dots(n+k-p+1)\langle v_0, \delta \rangle^{n+k-p}$ . This can be deduced from the rightmost expression in (2.4). The Hankel matrix constructed in Section 3 is still of rank  $r$  and admits  $\langle v_0, \delta \rangle$  as a generalized eigenvalue of multiplicity  $p+1$ . See for instance [34, 78].

### 3 Recovering the projections of the vertices

In this section we address the problem of retrieving the projections  $\mathcal{V}(\delta) = \{\langle v, \delta \rangle \mid v \in \mathcal{V}\}$  of the vertices of the convex polytope  $\mathcal{V}$  from its directional moments  $m_k(\delta)$ . We recognize an inverse problem that has appeared in several areas [42, 43, 47, 55, 64, 83, 86]. While [47] approach the problem with Prony's method, we favor a formulation in terms of generalized eigenvalues.

The standing assumption is that the projections of the vertices on the direction  $\delta$  are pairwise distinct. Thus  $|\mathcal{V}(\delta)| = |\mathcal{V}| = r$ . Also, we assume in this section that the number of vertices is known. We discuss in next section how this number can be retrieved from the knowledge of the moments.

From the directional moments  $(m_k(\delta))_k$  we introduce the sequence  $(\mu_k(\delta))_{k \in \mathbb{N}}$  of modified directional moments defined by

$$\begin{aligned} \mu_k(\delta) &= 0, & 0 \leq k \leq n-1, \\ \mu_k(\delta) &= \frac{k!}{(k-n)!} m_{k-n}(\delta), & k \geq n. \end{aligned}$$

By Theorem 2.1 there exist  $r$  non-zero real numbers  $a_v(\delta)$  such that this sequence satisfies  $\mu_k(\delta) = \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^k, k \in \mathbb{N}$ . The goal is to retrieve the  $r$  elements  $\langle v, \delta \rangle$  of  $\mathcal{V}(\delta)$  from  $(\mu_k(\delta))_k$  and hence  $(m_k(\delta))_k$ . This is an instance of the following problem.

**Inverse problem:** Consider a sequence  $(\mu_k)_{k \in \mathbb{N}}$  such that for some non-zero real (or complex) numbers  $a_1, \dots, a_r$  and pairwise distinct real (or complex) numbers  $w_1, \dots, w_r$ ,

$$\mu_k = \sum_{i=1}^r a_i w_i^k, \quad \forall k \in \mathbb{N}, \quad (3.1)$$

The problem is to find the  $w_i$  from the knowledge of  $r$  and  $(\mu_k)_{0 \leq k \leq 2r-1}$ . It can be tackled by Prony's method or as a generalized eigenvalue problem<sup>1</sup>.

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<sup>1</sup>The problem can also be addressed through pole estimation or Padé techniques since the generating function  $\sum_{k \geq 0} \mu_k t^k = \sum_{i=1}^r \frac{a_i}{1 - w_i t}$  is a rational function. This is also observed in [48].



In this paper we deal with computed directional moments. In comparison to measured directional moments, we can work with a selected precision and we do not take care of noise effects in the data. When working with measured information and having  $2R - n$  moments available per direction, it is best to replace the square  $r \times r$  Hankel matrices by rectangular  $R \times r$  Hankel matrices and introduce Least Squares or Maximum Likelihood methods to solve this inverse problem [31, 44, 86].

First one observes that the sequence  $(\mu_k)_k$  is a solution of a recurrence equation of order  $r$ , namely

$$\mu_{k+r} = p_{r-1} \mu_{k+r-1} + \dots + p_0 \mu_k, \quad (3.2)$$

where  $(-p_0, \dots, -p_{r-1}, 1)$  are the coefficients of the polynomial

$$p(z) = \prod_{i=1}^r (z - w_i) = z^r - p_{r-1} z^{r-1} - \dots - p_1 z - p_0.$$

Applying (3.2) for  $k = 0, \dots, r-1$  leads to the linear system

$$\underbrace{\begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{r-1} \\ \mu_1 & & & \ddots \\ \vdots & & \ddots & \vdots \\ \mu_{r-1} & \dots & \mu_{2r-2} \end{pmatrix}}_{H_r^{(0)}} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{r-1} \end{pmatrix} = \begin{pmatrix} \mu_r \\ \mu_{r+2} \\ \vdots \\ \mu_{2r-1} \end{pmatrix}. \quad (3.3)$$

From  $(\mu_k)_{0 \leq k \leq 2r-1}$  we can retrieve the characteristic polynomial  $p$  of the underlying recurrence by solving the above linear system<sup>2</sup>. The sought numbers  $w_1, \dots, w_r$  are the roots of this polynomial. Theses two steps (solving the linear system and computing the roots of the entailed polynomial) are known as Prony's method. It was introduced in [93] and is used in the context of the shape-from-moments problem in [47, 83]. The authors of [43, 59] introduce a solution in terms of the generalized eigenvalues of a pencil of matrices. It is based on the following facts.

We can recast (3.3) into the matrix equality:

$$\underbrace{\begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{r-1} \\ \mu_1 & & & \ddots \\ \vdots & & \ddots & \vdots \\ \mu_{r-1} & \dots & \mu_{2r-2} \end{pmatrix}}_{H_r^{(0)}} \underbrace{\begin{pmatrix} 0 & \dots & \dots & 0 & p_0 \\ 1 & \ddots & & \vdots & p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & p_{r-1} \end{pmatrix}}_P = \underbrace{\begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_r \\ \mu_2 & & \ddots & \\ \vdots & & \ddots & \vdots \\ \mu_r & \dots & \mu_{2r-1} \end{pmatrix}}_{H_r^{(1)}}. \quad (3.4)$$

The last column of the matrix on the right hand side comes from Equation (3.3), while the other columns are simply shifts of the columns in the matrix on the left hand side.

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<sup>2</sup>A Hankel system can be changed to a Toeplitz system. Here, the Toeplitz system is the one that arises when looking for the denominator  $1 - p_{r-1} t - \dots - p_0 t^n$  of the generating function of  $(\mu_k)_{k \in \mathbb{N}}$ .

Let us introduce the following notations from [55, Section 7.5] for the matrices arising in the above equality. The  $r \times r$  Hankel matrix with first row given by  $(\mu_d \dots \mu_{r+d-1})$  is denoted by  $H_r^{(d)}$ . The companion matrix of the characteristic polynomial of the recurrence (3.2) is denoted by  $P$ . The matrix equality (3.4) becomes  $H_r^{(0)} P = H_r^{(1)}$ , and more generally we have  $H_r^{(d)} P = H_r^{(d+1)}$  for  $d \geq 0$ .

Since  $w_1, \dots, w_r$  are the roots of  $p(z) = z^r - p_{r-1} z^{r-1} - \dots - p_1 z - p_0$ , we have

$$\underbrace{\begin{pmatrix} 1 & w_1 & \dots & w_1^{r-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & w_r & \dots & w_r^{r-1} \end{pmatrix}}_{W_r} \underbrace{\begin{pmatrix} 0 & \dots & 0 & p_0 \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & p_{r-2} \\ 0 & \dots & 1 & p_{r-1} \end{pmatrix}}_P = \underbrace{\begin{pmatrix} w_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & w_r \end{pmatrix}}_D \underbrace{\begin{pmatrix} 1 & w_1 & \dots & w_1^{r-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & w_r & \dots & w_r^{r-1} \end{pmatrix}}_{W_r}.$$

Let  $D$  and  $W_r$  be respectively the diagonal and the Vandermonde matrices defined by  $w_1, \dots, w_r$  and appearing in the above equality. The latter can thus be written  $W_r P = D W_r$ . The  $w_i$  being pairwise distinct,  $W_r$  is invertible and  $P W_r^{-1} = W_r^{-1} D$ . That is,  $w_1, \dots, w_r$  are the eigenvalues of  $P$  and  $W_r^{-1}$  is a matrix of eigenvectors for  $P$ . From  $H_r^{(1)} = H_r^{(0)} P$  in (3.4) we can deduce

$$H_r^{(1)} W_r^{-1} = H_r^{(0)} W_r^{-1} D,$$

and more generally,  $H_r^{(d+1)} W_r^{-1} = H_r^{(d)} W_r^{-1} D$ , for  $d \in \mathbb{N}$ . Thus  $w_1, \dots, w_r$  are the generalized eigenvalues of the matrix pencils  $(H_r^{(d+1)}, H_r^{(d)})$  and  $W_r^{-1}$  is a matrix of associated generalized eigenvectors.

Computing generalized eigenvalues is a classical problem in numerical linear algebra [29, 45, 57]. The structured problem we consider here is unfortunately known to be potentially ill-conditioned. Following [10] we can give an upper bound for the conditioning of the generalized eigenvalue problem as a constant multiplied by the square of the condition number of the Vandermonde matrix  $W_r$ .

To come back to our initial problem of retrieving  $\mathcal{V}(\delta)$  from  $(\mu_k(\delta))_k$  we introduce the pencil of Hankel matrices  $(H_r^{(1)}(\delta), H_r^{(0)}(\delta))$ . Its generalized eigenvalues are the elements of  $\mathcal{V}(\delta)$ . From those we can construct a matrix of generalized eigenvectors, given by the inverse of the Vandermonde matrix  $W_r(\delta)$ . The condition number of  $W_r(\delta)$  is denoted  $\kappa(\delta)$ . To reduce the conditioning of the generalized eigenvalue problem, which is of order  $\kappa(\delta)^2$ , we consider polytopes lying in the unit ball.

## 4 Estimating the number of vertices

So far, the number  $r$  of vertices has been assumed to be given. But  $r$  can also be an unknown of the problem. In this section, we discuss how to numerically retrieve this number from the Hankel matrices  $H_k^{(0)}(\delta)$  formed from the sequence of modified directional moments  $(\mu_k(\delta))_k$  in a generic direction  $\delta$ .

One first observes that  $H_{r+\ell}^{(0)}(\delta)$  is at most of rank  $r$  for any  $\ell \geq 0$ . Indeed the sequence  $(\mu_k(\delta))_{k \in \mathbb{N}}$  satisfies a recurrence equation (3.1) of order  $r$ . For any  $\ell > 0$ , each of the last  $\ell$  columns of

$H_{r+\ell}^{(0)}(\delta)$  is thus a linear combination of the previous  $r$  columns. Now, noting  $w_1, \dots, w_r$  the elements of  $\mathcal{V}(\delta)$ , we examine the Vandermonde factorisation of the Hankel matrix:

$$H_k^{(0)}(\delta) = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ w_1 & w_2 & \cdots & w_r \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{k-1} & w_2^{k-1} & \cdots & w_r^{k-1} \end{pmatrix}}_{{}^tW_k(\delta)} \underbrace{\begin{pmatrix} a_1(\delta) & 0 & \cdots & 0 \\ 0 & a_2(\delta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_r(\delta) \end{pmatrix}}_{A(\delta)} \underbrace{\begin{pmatrix} 1 & w_1 & \cdots & w_1^{k-1} \\ 1 & w_2 & \cdots & w_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w_r & \cdots & w_r^{k-1} \end{pmatrix}}_{W_k(\delta)}. \quad (4.1)$$

For a generic direction  $\delta$ , the  $r$  elements of  $\mathcal{V}(\delta)$  are pairwise distinct and therefore  $\det H_r^{(0)}(\delta) \neq 0$ . It follows that  $H_{r+\ell}^{(0)}(\delta)$  is exactly of rank  $r$ , for all  $\ell \geq 0$ .

Based on this observation, if a strict upper bound  $R$  for the number of vertices  $r$  is given, then  $r$  can be determined as the rank of  $H_R^{(0)}(\delta)$ . A caveat is that this matrix may be ill-conditioned. The condition number of  $H_r^{(0)}(\delta)$  is determined by the condition number of  $W_r(\delta)$  and  $A(\delta)$  in (4.1). For this we also refer to the discussion in [42] and [65] that examine the situation in the context of sparse interpolation. The condition number of the Vandermonde matrix  $W_r(\delta)$  depends on the distribution of the numbers in  $\mathcal{V}(\delta)$ , the set of the projections of the vertices in the direction  $\delta$  [40]. As for the matrix  $A(\delta)$ , having one of the  $a_i(\delta)$  too small can also lead to an incorrect (numerical) rank for  $H_R^{(0)}$ . Since we can (even randomly) select multiple directions for the projections, we can retain only those directions for which not both  $W_r(\delta)$  and  $A(\delta)$  are too ill-conditioned. Alternatively we could apply the rank estimates for Hankel matrices of [17].

Therefore, if we have an overestimation  $R$  of the number of vertices we can recover the exact number from the analysis of the numerical rank of  $H_R^{(0)}(\delta)$ . In practice we analyze the singular values of  $H_R^{(0)}(\delta)$  computed by a Singular Value Decomposition [45, 29]. This is discussed on specific cases in Section 6 and 7.

## 5 Reconstruction of the vertices from their projections

In this section we show how to retrieve the set of vertices  $\mathcal{V}$  from their projections onto several directions  $\delta$ . A difficulty to overcome is that the sets of projections we start from are not ordered; we have to match elements of different sets as projection of the same vertex. This rules out classical interpolation or least square approximation. Our approach is based on the interval interpolation scheme presented in [96]. To simplify our line of arguments on the relative error we assume that the polytope is already contained in the unit ball, a situation to which we shall reduce in practice.

Let  $\Delta$  be a set of  $s > n$  vectors  $\delta$  in the unit sphere  $S_n$ . The assumption is that for each  $\delta \in \Delta$  we have a set of approximations  $\tilde{\mathcal{V}}(\delta)$  for the values of  $\mathcal{V}(\delta) = \{\langle v, \delta \rangle \mid v \in \mathcal{V}\}$ , the projections of the vertices on the direction  $\delta$ . The result should be a set  $\tilde{\mathcal{V}}$  of  $r$  vectors in  $\mathbb{R}^n$  that consists of approximations for the set of vertices  $\mathcal{V}$  of the polytope. Note that with our assumptions, all the elements of  $\mathcal{V}(\delta)$  are in the interval  $[-1, 1]$ .

As discussed in Section 3, the approximations in  $\tilde{\mathcal{V}}(\delta)$  are obtained from the modified directional moments  $(\mu_k(\delta))_{0 \leq k \leq 2r-1}$ . They are computed as generalized eigenvalues and the conditioning of this problem is given by the square of the condition number  $\kappa(\delta)$  of the Vandermonde matrix

$W_r(\delta)$  made of these generalized eigenvalues. Therefore, an element  $w$  of  $\tilde{\mathcal{V}}(\delta)$  is actually understood as the center of an interval  $[w^-, w^+]$  of size  $2\epsilon(\delta)$  where  $\epsilon(\delta)$  should be taken as

$$\epsilon(\delta) = O\left(\kappa(\delta)^2 \varepsilon_\mu(\delta)\right), \quad (5.1)$$

where  $\varepsilon_\mu(\delta)$  is the bound on the relative error for the input modified directional moments  $(\mu_k(\delta))_{0 \leq k \leq 2r-1}$ .

For an element in  $\tilde{\mathcal{V}}(\delta)$ , we do not know which projection is an approximation of which vertex. So we need to find the correct *labelling* of the projections concurrently with the computation of the vertex coordinates. An algorithm for the reconstruction of the vertex coordinates can be based on the computation of  $r$  linear interval interpolants in  $n$  variables, of the form

$$\begin{aligned} q_u : S_n &\rightarrow [-1, 1] \quad \text{with } u = {}^t(u_1, \dots, u_n) \in \mathbb{R}^n, \\ \delta &\mapsto \langle u, \delta \rangle \end{aligned} \quad (5.2)$$

where  $S_n$  is the unit sphere. The set  $\tilde{\mathcal{V}}$  of  $n$ -tuples of coefficients  $u$  used to define those  $r$  interpolants are the approximations to the set  $\mathcal{V}$  of the polytope vertices. The interpolation condition has to reflect the fact that the  $r$  functions  $q_u$  interpolate exactly one  $w$ -value per  $\tilde{\mathcal{V}}(\delta)$  but  $s$  of those values across the sets  $\tilde{\mathcal{V}}(\delta)$ . Formally, this can be written as

$$\forall u \in \tilde{\mathcal{V}}, \forall \delta \in \Delta, \exists ! w \in \tilde{\mathcal{V}}(\delta) \text{ s.t. } q_u(\delta) \in [w^-, w^+]. \quad (5.3)$$

We remark at this point that any set of  $n$  projections can be interpolated by a function of the form  $q_u(\delta)$ , even if the width of the intervals is zero. At the same time any combination of  $s$  projections can be approximated in the least squares sense by a function of the form  $q_u(\delta)$ . So none of these classical approaches is very useful in figuring out which projections belong to the same vertex  $v \in \mathcal{V}$ . But an interval interpolant through at least  $n+1$  disjoint intervals does the job: the nonzero interval width compensates for overdetermining the linear interpolant by at least one interpolation condition. The interval interpolant is stringing the intervals, containing the projections, like beads on the graph of the interpolating function. Any interval, meaning any projection, through which it passes, is marked as belonging to the same vertex.

An interval interpolation problem does not need *a priori* to have a unique solution: sufficiently small perturbations of the coefficients in the linear form may not violate the interval interpolation conditions. However, [96] provides a method to compute the most robust interval interpolant of the form (5.2). By this we mean the interpolant that stays away as far as possible from violating the interpolation conditions imposed by the interval bounds. This is based on the following line of thoughts: The interpolation conditions given by intervals define a polyhedral cone. Our robust interval interpolant is given by the Chebychev direction of this polyhedral cone. This latter is a one-dimensional subspace which contains the centers of the inscribed balls with maximal radius (maximal for a certain distance given explicitly in [96]). Computing a Chebychev direction is then reduced to a convex optimization problem.

Because of the labelling problem of the approximate projections, the interval interpolation algorithm becomes a 2-step procedure. In a first step we take the subset  $\hat{\Delta}$  of the  $n$  best conditioned directions in  $\Delta$ . Consider the  $n$ -tuples in the Cartesian product  $\prod_{\delta \in \hat{\Delta}} \tilde{\mathcal{V}}(\delta)$  and select those for which the unique (non-interval) interpolant intersects one interval of  $\tilde{\mathcal{V}}(\delta)$  for each of the  $s - n$  remaining directions  $\delta \in \Delta \setminus \hat{\Delta}$ . They reflect a correct labelling of the projections. These better conditioned directions  $\hat{\Delta}$  are usually near to one another and that drastically cuts down the combinatorial aspect of the procedure by a continuity argument.

Since in this first step the coordinates of the vertices are computed from only  $n$  of the  $s$  available directions, the obtained values are not maximally accurate. This leads us to the following second step. After ordering the  $s$  sets  $\tilde{\mathcal{V}}(\delta)$  of  $r$  values according to the vertex they are a projection of, the coordinates of each vertex can be computed to maximal accuracy from the total of its  $s$  projections. To this end the interval interpolation method can be continued with the computation of the most robust interval interpolants satisfying (5.3) following [96].

## 6 Simulations

We now illustrate the proposed approach for the reconstruction of polytopes from their directional moments. For our simulations we consider centered and scaled polytopes: The origin is the center of gravity of the polytope and the vertices lie in the unit ball. This geometric normalisation corresponds to a transformation on the moments as described in [43].

The vertices of the polytope are to be reconstructed from directional moments. The proposed reconstruction of a convex polytope in dimension  $n$  requires directional moments in at least  $n + 1$  generic directions. The order of the moments required in each direction then depends on the number  $r$  of vertices: we need directional moments up to order  $2r - n - 1$ . In dimension 2 this can be compared to [83], where the complex moments up to order  $2r - 3$  are obtained from directional moments up to the same order in  $2r - 2$  directions by inverting a matrix with binomial coefficients. One should note though that this approach removes the necessity of matching the projections. [48] suggests a generalisation of this approach to higher dimensions.

For a given direction  $\delta$ , the directional moments  $m_k(\delta)$  are computed in double precision and used to form the entries of the pair of Hankel matrices  $(H_r^{(1)}(\delta), H_r^{(0)}(\delta))$  described in Section 3. The algorithm consists of 3 main steps:

1. Determine the number  $r$  of vertices by analyzing the singular values of the Hankel matrix  $H_R^{(0)}(\delta)$  for  $R > r$  big enough and a few number of random directions  $\delta$ , as described in Section 4.
2. Compute the generalized eigenvalues for the pair of matrices  $(H_r^{(1)}(\delta), H_r^{(0)}(\delta))$  in at least  $n + 1$  nearby directions  $\delta$  as in Section 3. Determine the condition number  $\kappa(\delta)$  of  $W_r(\delta)$ , the Vandermonde matrix formed with these generalized eigenvalues.
3. Recover the vertices  $\tilde{\mathcal{V}}$  from their approximate projections  $\tilde{\mathcal{V}}(\delta)$  using the interval interpolation technique described in Section 5 with the error estimate based on  $\kappa(\delta)$ .

The first two steps are performed using standard numerical linear algebra routines from the NAG library through the Maple interface. In particular, Step 1 makes use of the implementation of the Singular Value Decomposition and Step 2 the QZ-algorithm [105]. Step 3 is implemented in Matlab. All computations in this Section are performed in double precision.

In step 2, we sample a number of directions and retain those for which the condition number of the Vandermonde matrix  $W_r(\delta)$  is the smallest. The condition number indeed depends on the direction :  $\kappa(\delta)$  depends on the distribution of the projected vertices [40]. In particular it increases when the projections of two vertices get closer to one another.

Furthermore, to cut down on the combinatorial complexity of the interpolation scheme in the last step, it makes sense to select directions reasonably close to one another. Selecting a generic reference direction  $\hat{\delta}$  with a reasonable condition number  $\kappa(\delta)$  and other directions in the neighborhood. Note that here non disjoint intervals for the approximations of the projections on a direction  $\delta$  would induce a poor condition number  $\kappa(\delta)$  of the Vandermonde matrix  $W_r(\delta)$ .

## 6.1 Reconstruction of polygons

We begin our simulations with the reconstruction of 2-dimensional polygons. A direction  $\delta = (\cos \theta, \sin \theta)$  is represented by an angle  $\theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}]$ . The projection of vertex with coordinates  $(v_1, v_2)$  is given by  $v_1 \cos \theta + v_2 \sin \theta$ . Then the interval interpolation problem formulated in terms of  $\theta$  is

$$v_1 \cos \theta + v_2 \sin \theta \in [w^-, w^+], \quad \text{for a single } w \in \tilde{\mathcal{V}}(\theta).$$

where  $w^- = w - \epsilon(\delta)$  and  $w^+ = w + \epsilon(\delta)$  as in Section 5.

### 6.1.1 Reconstruction of a regular hexagon

The regular hexagon ( $r = 6$ ) is presented in Figure 1(a) with its symmetry axes and their bisectors. The symmetry axes correspond to the non generic directions. The condition number  $\kappa(\delta)$  increases drastically when  $\delta$  approaches these directions.

The number  $r$  of vertices is retrieved as the rank of  $H_R^{(0)}(\delta)$ , for  $R$  large enough and arbitrary directions  $\delta$ . Here  $R = 7$  is sufficient to reliably analyse the rank from the singular values. These are plotted in Figure 1(c) for three directions picked at random.

In Figure 1(b) the condition number  $\kappa(\delta)$  is plotted for 300 equidistant generic directions. A minimal value is reached for directions bisecting two consecutive axes of symmetry. This leads us to choose the reference direction  $\hat{\delta}$  with angle  $\hat{\theta} = -\frac{5\pi}{12}$ . It corresponds to one of the bisectors. We then take 4 nearby directions  $\hat{\theta} \pm 0.05, \hat{\theta} \pm 0.10$ . For each of the 5 directions we construct the pair of Hankel matrices  $(H_6^{(1)}(\delta), H_6^{(0)}(\delta))$  and compute their generalized eigenvalues. For these directions  $\kappa(\delta)$  is around 300. We therefore take interval of size  $2 \times 10^{-10}$  according to (5.1).

We compute the interval interpolants from the 5 sets of projections  $\mathcal{V}(\hat{\theta} + k \cdot 0.05), k = -2, -1, 0, 1, 2$ . The relative error on the computed projections is bounded overall by  $4.9 \times 10^{-12}$  for the 5 chosen directions. The coordinates of the 6 different vertices of the regular hexagon are recovered as the coefficients of the interpolants graphed in Figure 1(d). The relative error on each computed coordinate compared to its true value is bounded overall by  $3.1 \times 10^{-12}$ .

### 6.1.2 Reconstruction of a polygon with 12 vertices

In this second simulation, we consider a centered and scaled 12-gon. It is drawn in Figure 2(a).

As in the case of the hexagon above, the number of vertices is retrieved by computing the singular values of  $H_{15}^{(0)}(\delta)$  in 3 directions  $\delta$ . From Figure 2(b) we deduce that the numerical rank is  $r = 12$ .

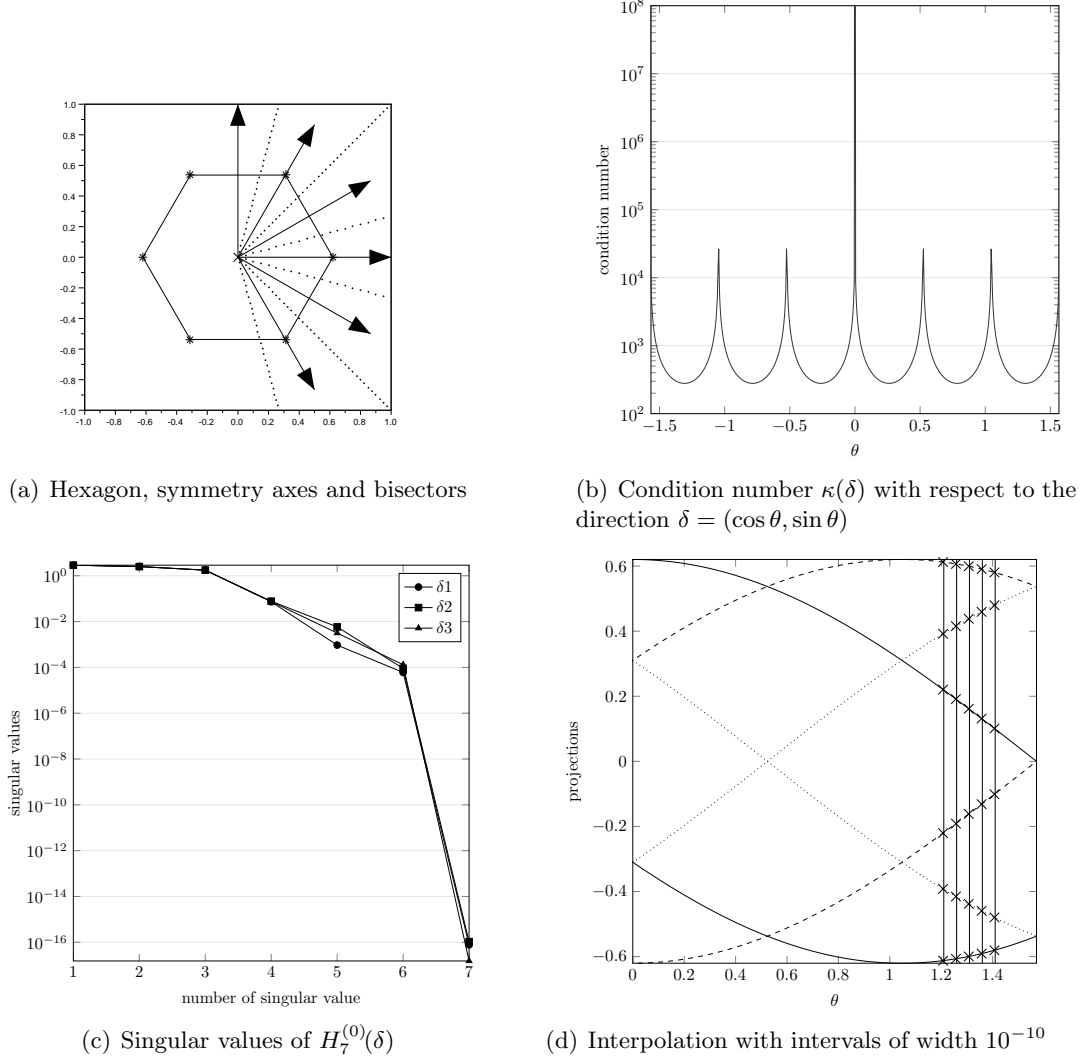
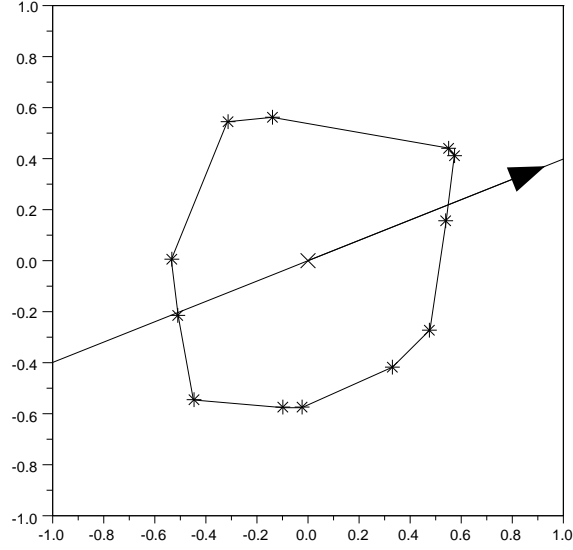


Figure 1: Regular hexagon

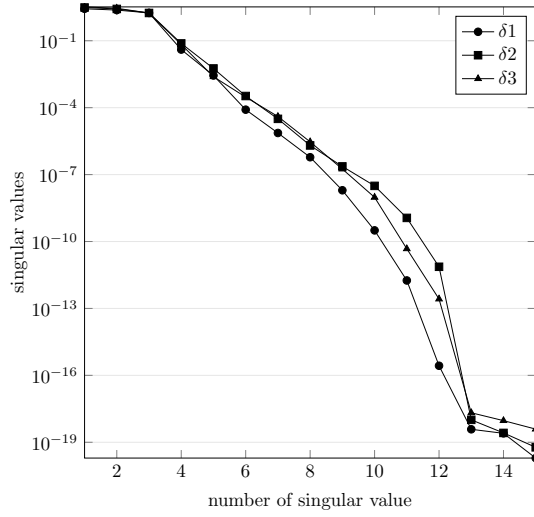
After inspecting some directions, we choose the reference direction  $\hat{\delta}$  with  $\hat{\theta} = 0.379521$  (arrow in Figure 2(a)) and 4 other nearby directions  $\hat{\theta} \pm 0.01, \hat{\theta} \pm 0.02$ . The projections of the vertices on these directions are obtained as the generalized eigenvalues of the pairs of Hankel matrices  $\left(H_{12}^{(1)}(\delta), H_{12}^{(0)}(\delta)\right)$  whose entries are obtained from the respective modified directional moments  $(\mu_k(\delta))_{0 \leq k \leq 21}$ . The condition number  $\kappa(\delta)$  of the matrix of generalized eigenvectors  $W_{12}(\delta)$  is around  $7 \times 10^5$  for all 5 directions. The relative error on the computed projections compared to their true values is bounded by  $5.0 \times 10^{-6}$ .

From these 5 sets  $\mathcal{V}(\hat{\theta} - 0.02), \mathcal{V}(\hat{\theta} - 0.01), \mathcal{V}(\hat{\theta}), \mathcal{V}(\hat{\theta} + 0.01), \mathcal{V}(\hat{\theta} + 0.02)$ , we compute the 12 linear interval interpolants. According to (5.1) we take intervals of size  $2\epsilon = 10^{-4}$ . The relative error in each coordinate is bounded by  $5.3 \times 10^{-5}$ .

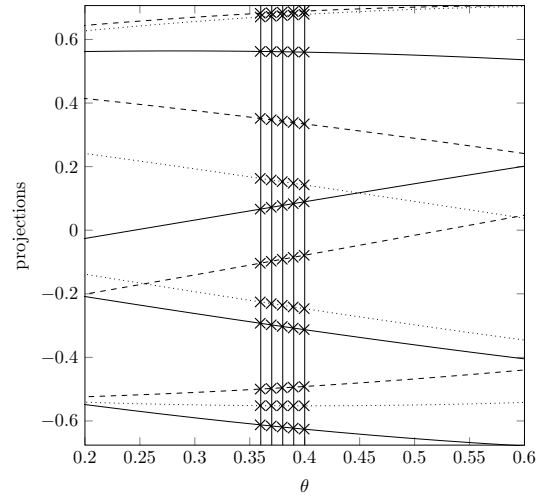
Note that even the two very close vertices in the top right corner in Figure 2(a) are recovered with the accuracy mentioned above. The distance between them is only of the order of  $10^{-2}$ .



(a) Centered and scaled polygon with reference direction



(b) Singular values of  $H_{15}^{(0)}(\delta)$



(c) Interpolation with intervals of width  $10^{-4}$

Figure 2: Polygon with 12 vertices



## 6.2 Reconstruction of polyhedra

We now consider the reconstruction of convex polyhedra in dimension 3. The dimension does not introduce new difficulties in our method. A direction  $\delta$  is represented by a unit vector  $(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta))$  with  $(\theta, \phi)$  lying in  $] -\frac{\pi}{2}, \frac{\pi}{2}] \times ] -\frac{\pi}{2}, \frac{\pi}{2}]$ . The projection of the vertex  $v = (v_1, v_2, v_3)$  on  $\delta = (\theta, \phi)$  equals  $v_1 \cos(\theta)\cos(\phi) + v_2 \sin(\theta)\cos(\phi) + v_3 \sin(\theta)$ . The tuples of coordinates  $(v_1, v_2, v_3)$ , for  $v \in \mathcal{V}$ , are the unknowns in the linear interval interpolation problem  $q_v(\delta) \in [w^-, w^+]$  for a single  $w \in \tilde{\mathcal{V}}(\delta)$ , as described in Section 5.

### 6.2.1 Reconstruction of a polyhedron with well-distributed vertices

We first consider the polyhedron with 10 vertices represented in Figure 3(a).

We retrieve the number of vertices of the polyhedron by computing the numerical rank of the Hankel matrix  $H_{11}^{(0)}(\delta)$  in 3 different directions  $\delta$ . The singular values of  $H_{11}^{(0)}(\delta)$  are plotted in Figure 3(b) for three random directions.

After inspecting several directions, we select  $(\hat{\theta}, \hat{\phi}) = (-1.256637, 0.261799)$  for the reference direction  $\hat{\delta}$  and 4 other nearby directions where the condition number  $\kappa(\delta)$  is of order  $10^4$ .  $\hat{\delta}$  is indicated by an arrow and a dotted line in Figure 3(a). We take the nearby directions as  $(\hat{\theta} + \varepsilon, \hat{\phi})$ ,  $(\hat{\theta}, \hat{\phi} + \varepsilon)$ ,  $(\hat{\theta} - \varepsilon, \hat{\phi} - \varepsilon)$  and  $(\hat{\theta} + \varepsilon, \hat{\phi} - \varepsilon)$  with  $\varepsilon = 0.01$ . For each direction  $\delta$ , the pair of matrices  $(H_{10}^{(1)}(\delta), H_{10}^{(0)}(\delta))$  is built with directional moments up to order 16. The generalized eigenvalues of the pairs  $(H_{10}^{(1)}(\delta), H_{10}^{(0)}(\delta))$  provide the projections of the vertices with a relative error bounded by  $8.2 \times 10^{-8}$ .

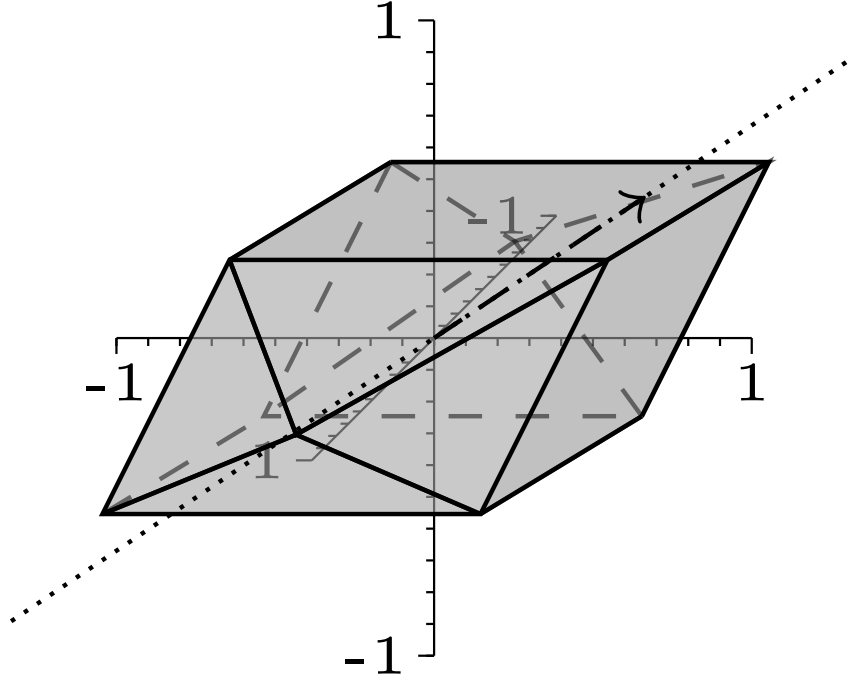
We compute the 10 linear interval interpolants using intervals of width  $2 \times 10^{-7}$  for the projections. In Figure 3(c) we show one of those surfaces. The cross denotes the approximate locus of the 5 projections obtained as generalized eigenvalues. The coordinates of the 10 vertices of the polyhedron are obtained as the coefficients of the interpolants. The relative error on this final result is bounded by  $5.1 \times 10^{-6}$ .

### 6.2.2 Reconstruction of a polyhedron with close vertices

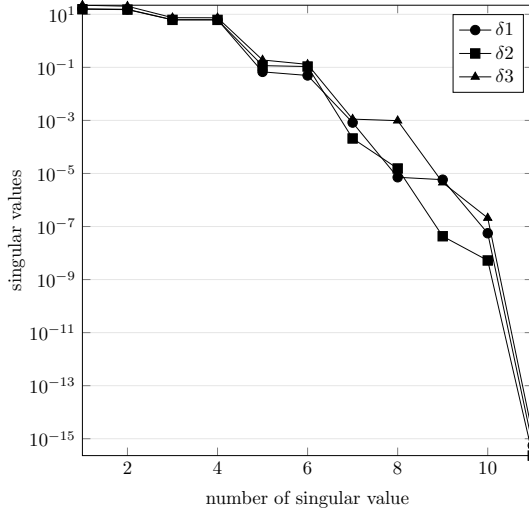
Here we examine a polyhedron with 10 vertices and a triangular face of relatively small size (see the top of Figure 4(a)).

In Figure 4(b) we plot the singular values of  $H_{12}^{(0)}(\delta)$  for 3 random directions  $\delta$ . The gap between the tenth and the eleventh singular value appears for at least one of the directions.

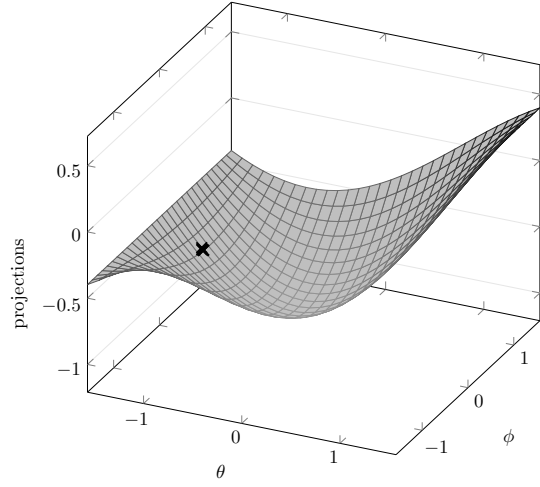
We choose the reference direction  $(\hat{\theta}, \hat{\phi}) = (-0.994838, -0.994838)$  for which the condition number  $\kappa(\delta)$  is  $4.2 \times 10^4$ . It is indicated in Figure 4(a) by an arrow and a dotted line. We additionally pick the nearby directions  $(\hat{\theta} + \varepsilon, \hat{\phi})$ ,  $(\hat{\theta}, \hat{\phi} + \varepsilon)$ ,  $(\hat{\theta} - \varepsilon, \hat{\phi} - \varepsilon)$  and  $(\hat{\theta} + \varepsilon, \hat{\phi} - \varepsilon)$  with  $\varepsilon = 0.01$ . For each direction the pair of matrices  $(H_{10}^{(1)}(\delta), H_{10}^{(0)}(\delta))$  is built with directional moments up to order 16. The projections  $\tilde{\mathcal{V}}(\delta)$  of the vertices on those directions are retrieved as the generalized eigenvalues of  $(H_{10}^{(1)}(\delta), H_{10}^{(0)}(\delta))$  with a relative error bounded by  $5.1 \times 10^{-7}$ .



(a) Centered and scaled polyhedron with reference direction



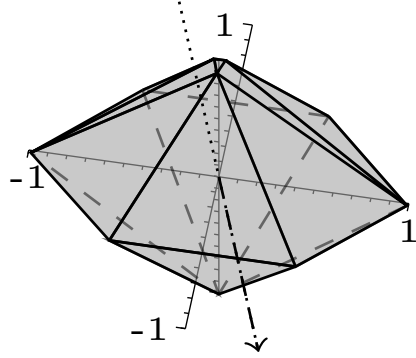
(b) Singular values of  $H_{11}^{(0)}(\delta)$



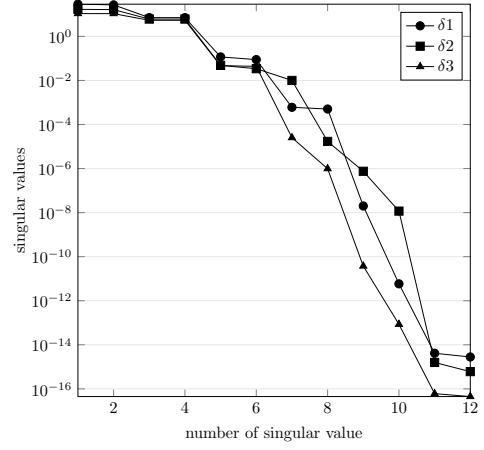
(c) Interpolant for one vertex

Figure 3: First polyhedron with 10 vertices

We determine the 10 interpolants for our sets of projections using intervals of width  $2 \times 10^{-6}$ . The coordinates of the 10 vertices appear as the coefficients of the interpolants. The relative error on these is bounded by  $6.2 \times 10^{-6}$ .



(a) Centered and scaled polyhedron with reference direction

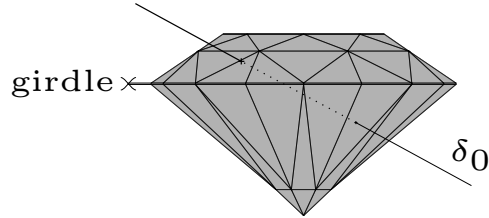


(b) Singular values of  $H_{12}^{(0)}$

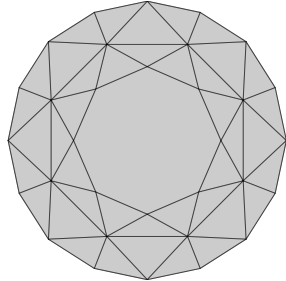
Figure 4: Second polyhedron with 10 vertices

## 7 Diamond

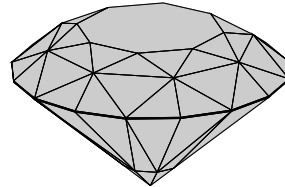
As a challenge we choose an actual brilliant cut of a diamond. It is given as a convex polyhedron with 57 vertices and represented in Figure 5. The stone girdle consists of pairs of vertices very close to one another. The number of vertices and the small distance between the projections of the vertices severely impact the condition number  $\kappa(\delta)$  of the Vandermonde matrix  $W_{57}(\delta)$ : double precision is no longer enough to retrieve sufficiently accurate values for the projections. We rely on the software floats of Maple to provide the needed number of digits for the computations.



(a) Side view and reference direction  $\delta_0$



(b) Top view



(c) In perspective

Figure 5: Diamond

To reliably retrieve the number of vertices, we use a precision of 70 digits. Figure 6 tracks the singular values of  $H_{65}^{(0)}(\delta)$  for 8 random directions  $\delta$  and in different computational precisions.

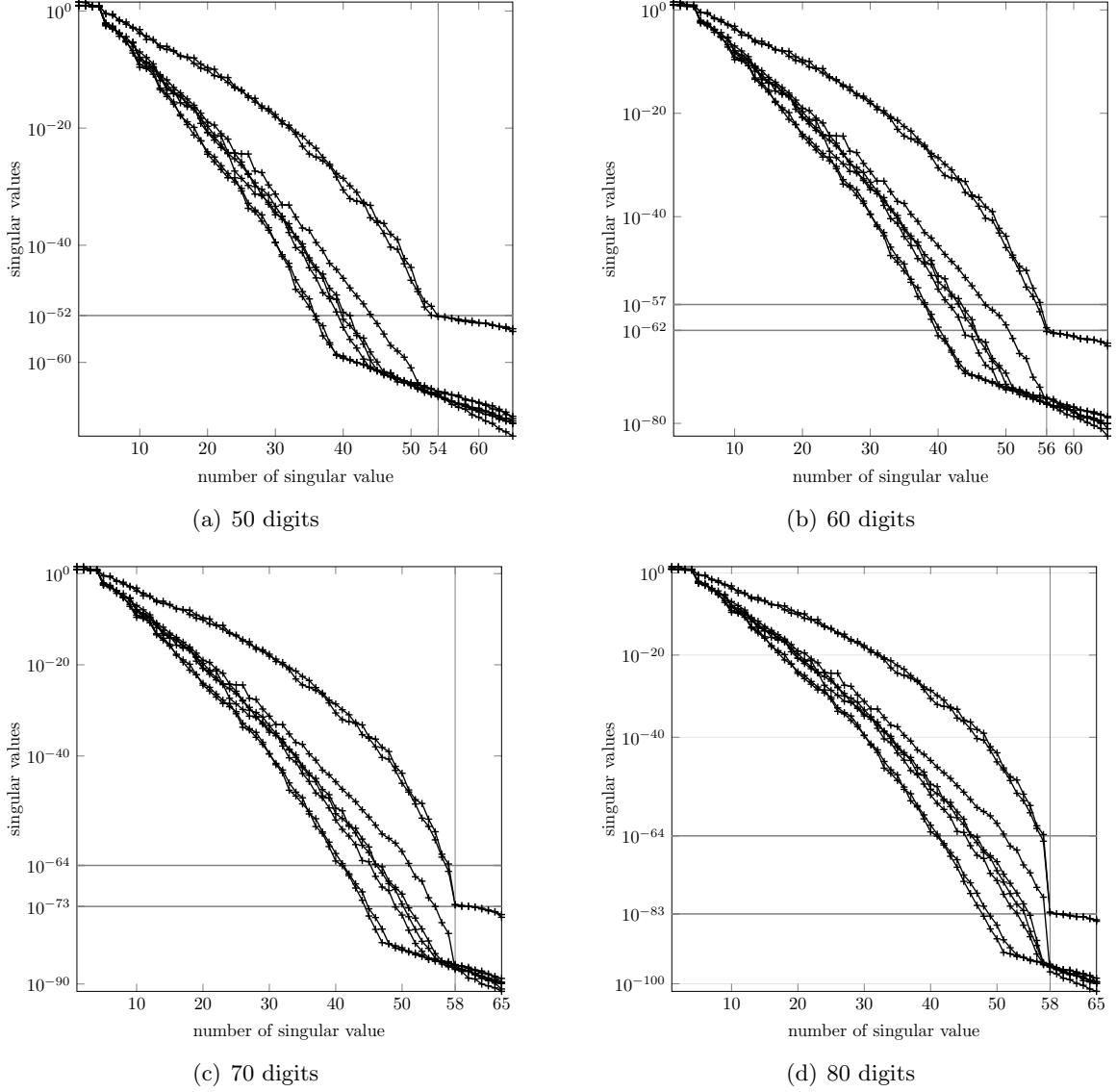


Figure 6: Singular Values of  $H_{65}^{(0)}(\delta)$  for several computational precision

After sampling a rather large number of directions, we select the reference direction  $(\hat{\theta}, \hat{\phi}) = (0.261799, 1.047198)$  shown in Figure 5(a). The condition number  $\kappa(\hat{\delta})$  for this direction is  $1.67 \times 10^{33}$ . We choose 4 nearby directions with a similar condition number  $(\hat{\theta} + \varepsilon, \hat{\phi})$ ,  $(\hat{\theta}, \hat{\phi} + \varepsilon)$ ,  $(\hat{\theta} - \varepsilon, \hat{\phi} - \varepsilon)$  and  $(\hat{\theta} + \varepsilon, \hat{\phi} - \varepsilon)$  where  $\varepsilon = 0.0001$ . Computing with 70 digits we expect to retrieve the projections of the vertices with a relative accuracy of at least  $10^{-3}$ .

For each of the 5 selected directions  $\delta$ , the pair of matrices  $(H_{57}^{(1)}(\delta), H_{57}^{(0)}(\delta))$  is built with the directional moments up to order 110. The projections of the vertices are obtained as the generalized eigenvalues. The relative error is actually bounded by  $8.1 \times 10^{-8}$ .

We compute the 57 interpolants in double precision using intervals of width  $5 \times 10^{-6}$ . The coordinates of the vertices are the coefficients in those interpolants. The relative error is bounded by  $7.8 \times 10^{-5}$ . We plot the error for all the vertices in Figure 7.

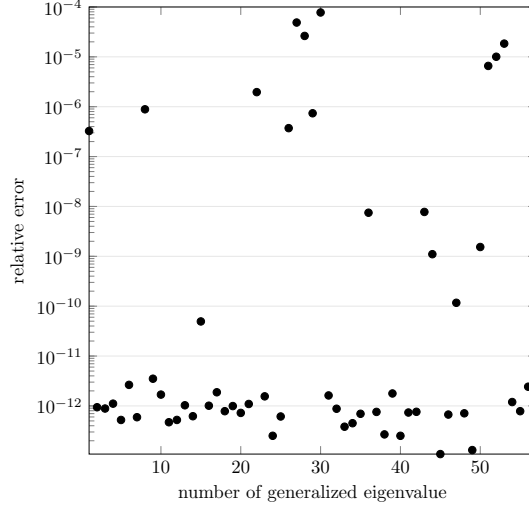
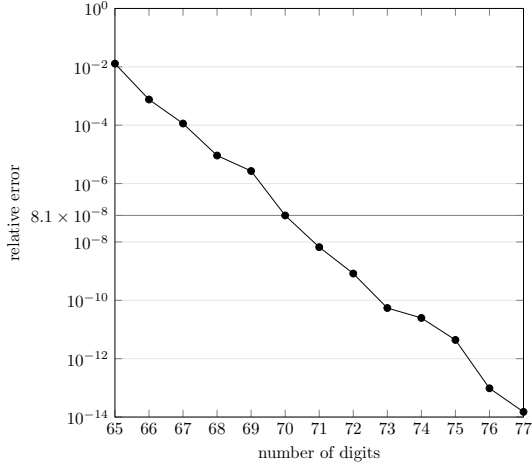
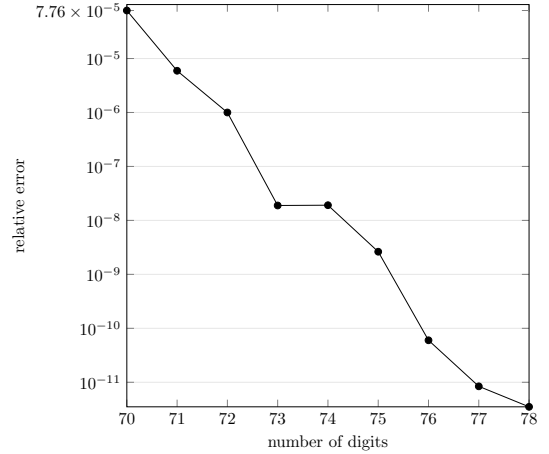


Figure 7: Error observed when operating with 70 digits

In Figure 8 we report on the error using various computational precisions. For computations with less than 65 digits we do not recover all the projections while the complete set of coordinates of the vertices can be retrieved only if we use at least 70 digits.



(a) Error on the retrieved projections



(b) Error on the retrieved vertex coordinates

Figure 8: Error for different computational precisions

## Chapter II

# A Moment Matrix Approach to computing Symmetric Cubatures

### Abstract

A quadrature is an approximation of the definite integral of a univariate function by a weighted sum of function values at specified points, or nodes, within the domain of integration. Gaussian quadratures are constructed to yield exact results for any polynomial of degree  $2r - 1$  or less by a suitable choice of  $r$  nodes and weights. Cubature is a generalization of quadrature in higher dimension. Constructing a cubature amounts to find a linear form

$$\Lambda : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j=1}^r a_j p(\xi_j)$$

from the knowledge of its restriction to  $\mathbb{R}[x]_{\leq d}$ . The unknowns are the number of nodes  $r$ , the weights  $a_1, \dots, a_r$  and the nodes  $\xi_1, \dots, \xi_r$ .

An approach based on moment matrices was proposed in [35]. We give a basis-free version in terms of the Hankel operator  $\hat{\mathcal{H}}$  associated to  $\Lambda$ . The existence of a cubature of degree  $d$  with  $r$  nodes boils down to conditions of ranks and positive semidefiniteness on  $\hat{\mathcal{H}}$ . We then recognize the nodes as the solutions of a generalized eigenvalue problem and the weights as the solutions of a Vandermonde-like linear system.

Standard domains of integration are symmetric under the action of a finite group. It is natural to look for cubatures that respect this symmetry [19, 37, 38]. Introducing adapted bases obtained from representation theory, the symmetry constraint allows to block diagonalize the matrix of the Hankel operator. We then deal with smaller-sized matrices both for securing the existence of the cubature and computing the nodes and the weights. The size of the blocks is furthermore explicitly related to the orbit types of the nodes with the new concept of the matrix of multiplicities of a finite group. This provides preliminary criteria for the existence of a cubature.

The Maple implementation of the presented algorithms allows to determine, with moderate computational efforts, all the symmetric cubatures of a given degree for a given domain. We present new relevant cubatures.

Available Online on: <https://hal.inria.fr/hal-01188290>

Coauthor: E. Hubert

# 1 Introduction

Interpolatory quadratures offer a mean to approximate the integral of a function on an interval by the weighted sum of evaluations at a finite set of *nodes* in the interval. A criterion for the quality of the approximation is the degree of the polynomial functions such that the so obtained value is the exact value of the integral. Gaussian quadratures achieve a degree  $2r - 1$  using  $r$  well-chosen nodes. They are known to be the roots of the orthogonal polynomial of degree  $r$  associated with the considered integral operator. It does not enjoy as clear-cut an answer in higher dimension. Relevant integration schemes are still being investigated for standard domains like the simplex and the square. Those are necessary for the discretization of partial differential equations, whether in high degree for isogeometric analysis [62, 80] or in high dimensions [16, 77]. A simple product of Gaussian quadratures is *curse*d by dimensionality. Our original motivation resides in the use of integration in the geometric modeling of shapes based on a skeleton [60, 61, 106, 107].

The present paper offers a methodology to determine cubatures, that is compute its nodes and weights, in an intrinsically multivariate way. Started by Radon [92] the subject has been overviewed in [19, 20, 101]. We take on the moment matrix approach already described in [35]. The emergence of moment methods in polynomial optimization [68, 75] and their application to computational algebraic geometry [71, 72] have brought out effective techniques to compute cubatures in floating point arithmetic [1]. Our contribution bears on the use of symmetry to reduce the size of the matrices and the number of parameters. We can determine, in exact arithmetic, the existence of high degree cubatures and set up the generalized eigenvalue problems the solutions of which are the nodes of the cubature.

Symmetry occurs naturally in the search for quality cubatures. The standard domains of integration (simplex, parallelepiped, sphere) can be normalized to their unit, and highly symmetric, counterpart by an affine transformation. The degree of the cubature is unchanged by such a transformation. The symmetric cubatures we compute furthermore retain invariance properties of the integral operator being approximated. The key to bring out the structure implied by the symmetry is to stir away from the usual monomial bases of polynomial vector spaces. Building on the ideas in [22, 39, 94], we introduce the *orthogonal symmetry adapted bases*. They allow to block diagonalize the matrix of the Hankel operator, a.k.a. the moment matrix. Further diagonalization of the blocks would provide a basis of (multivariate) orthogonal polynomials.

The rank of the Hankel operator that is central in our approach is equal to the number of nodes of the cubature. In the presence of symmetry we can make further analyses: the rank of the diagonal blocks is related to the organization of the nodes in orbit types. The relation is made explicit thanks to the new concept of the *matrix of multiplicities* associated with the symmetry group. Besides introducing the notion we provide the matrices of multiplicities for the cyclic and dihedral groups. This allows to ascertain that the relation between the ranks and the organization of the nodes in orbit types is one-to-one. Importantly, this concept provides us with preliminary criteria to dismiss certain organizations of nodes in orbit types to build a cubature on.

An additional contribution in the paper is the fraction-free and pivoting free diagonalization of symmetric matrices over their locus of positivity. This is the exact arithmetic algorithm we use to discuss all the symmetric cubatures of a given degree.

The algorithms described in the paper are implemented in Maple. They have been applied to

recover several known cubatures. New cubatures have arisen from the possibility offered in our approach to investigate, with moderate computational efforts, all the symmetric cubatures of a given degree.

The paper is organized as follows. Section 2 is of an introductory nature, reviewing the subject of cubatures and providing an overview of the techniques developed in this paper. Section 3 reviews Hankel operators and their properties. Section 4 describes the fraction and pivoting free algorithm to diagonalize symmetric matrices over their locus of positivity. Section 5 makes precise how the tools exposed so far apply to compute cubatures with a moment matrix approach. The sections afterwards all concern the symmetric case. Section 6 details the construction of orthogonal symmetric bases while Section 7 introduces orbit types and the matrix of multiplicities. This latter section also provides the matrices of multiplicities for the cyclic and dihedral groups. Section 8 details the block diagonal structure of the Hankel operator in the presence of symmetry. In particular we relate the size of the blocks to the organization of the nodes in orbit types. Section 9 provides the detail of the algorithmic content of our approach. The description there introduces how the code has been organized and is used. The following sections offer relevant cubatures for which our approach gave new insight.



## 2 Cubatures

Finding quadratures is a classical problem. The minimal number of nodes to maximize its degree and the construction of Gaussian quadratures thanks to univariate orthogonal polynomials are well-established (see *e.g.* [100, 101] and references herein). The same issues for their multivariate analogues, called *cubatures*, are still open problems in the general case.

To introduce the moment matrix approach, we first present a one-dimensional version of it. We recall then some results on cubatures based on reviews on the subject [19, 20, 101]: existence of cubatures and bounds on the minimal number of nodes (Tchakaloff's and Mysovnikh's theorems), classical approaches that use either product of Gaussian quadratures, or the theory of multivariate orthogonal polynomials. The symmetry of the involved measure appears as a natural ally in order to reduce the complexity of these approaches. We present then a recent approach based on moment matrices [35] and describe our main contribution: how symmetry is taken advantage of in the moment matrix approach to cubatures.

In the following,  $\mathbb{K}$  denotes a field of characteristic zero: the complex numbers  $\mathbb{C}$ , the real numbers  $\mathbb{R}$  or the rational numbers  $\mathbb{Q}$ .  $\mathbb{K}[x]$  denotes the ring of polynomials in the variables  $x = (x_1, \dots, x_n)$  with coefficients in  $\mathbb{K}$ ,  $\mathbb{K}[x]_{\leq \delta}$  the  $\mathbb{K}$ -vector space of polynomials of degree at most  $\delta$  and  $\mathbb{K}[x]_{\delta}$  the  $\mathbb{K}$ -vector space that contains all homogeneous polynomials of degree exactly  $\delta$  and the zero polynomial. For  $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ ,  $x^k$  denotes the monomial  $x_1^{k_1} \dots x_n^{k_n}$ .

### 2.1 Gaussian quadratures

In this section, the classical quadrature problem is introduced (see *e.g.* [100, Chapter 3] or [101, Chapter 1.3] and references herein) and methods to determine Gaussian quadratures are presented. While a classical approach uses the theory of orthogonal polynomials (see *e.g.* [30, Chapter 1]), we give here an alternative approach: the one that we extend to the multidimensional case in the next sections. Here,  $\mathbb{R}[x]$  denotes the  $\mathbb{R}$ -vector space of all polynomials with one variable and coefficients in  $\mathbb{R}$ .

Consider the linear form  $\Omega$  on  $\mathbb{R}[x]$  defined by

$$\Omega : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \int_a^b p(x)\omega(x)dx,$$

where  $[a, b] \subset \mathbb{R}$  is an interval (finite or infinite) and  $\omega$  is a nonnegative *weight function* defined on this interval, that is a function that satisfies the following requirements:

- $\omega$  is measurable on the interval  $[a, b]$  and  $\omega(x) \geq 0$  for all  $x \in [a, b]$ .
- The *moments*  $m_k = \int_a^b x^k \omega(x)dx$  exist and are finite for all  $k \in \mathbb{N}$ .
- For any polynomial  $p \in \mathbb{R}[x]$  that is nonnegative on  $[a, b]$ ,

$$\int_a^b p(x)\omega(x)dx = 0 \quad \Rightarrow \quad p(x) = 0.$$

A measure  $\mu$  is associated with  $\Omega$ . It is defined by

$$d\mu = \mathbb{1}_{[a,b]}\omega(x)dx,$$

where  $\mathbb{1}_{[a,b]}$  is the characteristic function of the interval  $[a, b]$

$$\mathbb{1}_{[a,b]} : \mathbb{R} \rightarrow \{0, 1\}, x \mapsto \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}.$$

A linear form  $\Lambda$  on  $\mathbb{R}[x]$  defined by

$$\Lambda : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j=1}^r a_j p(\xi_j),$$

with  $r > 0$ ,  $a_j \in \mathbb{R} \setminus \{0\}$  and  $\xi_j \in \mathbb{R}$  pairwise distinct, is called a *quadrature of degree  $d$*  for the measure  $\mu$  if it satisfies

$$\int_a^b p(x)\omega(x)dx = \sum_{j=1}^r a_j p(\xi_j) \quad \forall p \in \mathbb{R}[x]_{\leq d}$$

and if this equality does not hold for at least one polynomial  $p$  of degree  $d + 1$ . The points  $\xi_j$  are the *nodes* and the coefficients  $a_j$  are the *weights* of the quadrature. Such a quadrature is called *inside* if the nodes  $\xi_j$  lie on the interval  $[a, b]$  and *minimal* if the number of nodes  $r$  is minimal for a fixed degree  $d$ .

In this univariate case, minimal inside quadratures with positive weights are known and called *Gaussian quadratures* (see *e.g.* [100, Chapter 3.6] for more details). A Gaussian quadrature with  $r$  nodes is of degree  $2r - 1$ .

Several methods are known for the determination of the nodes of a quadrature. Its weights are generally obtained afterwards by solving the Vandermonde linear system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \xi_1 & \xi_2 & \dots & \xi_r \\ \vdots & \vdots & & \vdots \\ \xi_1^{r-1} & \xi_2^{r-1} & \dots & \xi_r^{r-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{r-1} \end{pmatrix} \quad (2.1)$$

obtained from the equations

$$m_k = \Lambda(x^k) = \sum_{j=1}^r a_j \xi_j^k \quad \forall k = 0, \dots, r-1. \quad (2.2)$$

## Moment matrix approach to computing the nodes of a Gaussian quadrature

Assume that a Gaussian quadrature of degree  $2r - 1$  exists for  $\mu$ , that is there exists a linear form

$$\Lambda : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j=1}^r a_j p(\xi_j)$$

with  $a_j > 0$  and  $\xi_j \in [a, b]$  pairwise distinct such that  $m_k = \Lambda(x^k)$  for all  $k \leq 2r - 1$ .

Consider the monic polynomial  $\pi_r$  of degree  $r$  whose roots are the nodes  $\xi_1, \dots, \xi_r$

$$\pi_r(x) = \prod_{j=1}^r (x - \xi_j) = x^r - \tau_{r-1}x^{r-1} - \dots - \tau_0.$$

Since  $\pi_r(\xi_j) = 0$  for all  $j = 1, \dots, r$ , we have  $\Lambda(\pi_r) = 0$  and even

$$\Lambda(p\pi_r) = 0 \quad \forall p \in \mathbb{R}[x]. \quad (2.3)$$

Taking  $p(x) = x^k$  for  $k = 0, \dots, r$ , we deduce by linearity of  $\Lambda$  that

$$\sum_{i=0}^{r-1} \tau_i \Lambda(x^{k+i}) - \Lambda(x^{k+r}) = 0 \quad \forall k = 0, \dots, r.$$

Thus, the vector  $(\tau_0 \ \dots \ \tau_{r-1} \ -1)^t$  is in the kernel of the Hankel matrix

$$H_1^{B^{(r)}} = (\Lambda(x^{i+j-2}))_{1 \leq i, j \leq r+1} = \begin{pmatrix} \Lambda(1) & \Lambda(x) & \dots & \Lambda(x^r) \\ \Lambda(x) & & \ddots & \vdots \\ \vdots & \ddots & & \vdots \\ \Lambda(x^r) & \dots & \dots & \Lambda(x^{2r}) \end{pmatrix}.$$

The latter is the matrix of the symmetric bilinear form

$$\varphi : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}, (p, q) \mapsto \Lambda(pq)$$

restricted to  $\mathbb{R}[x]_{\leq r}$  in the monomial basis  $B^{(r)} = \{1, x, \dots, x^r\}$ .

Alternatively, we observe that

$$H_1^{B^{(r-1)}} M_x^{B^{(r-1)}} = H_x^{B^{(r-1)}} \quad (2.4)$$

with  $H_1^{B^{(r-1)}} = (\Lambda(x^{i+j-2}))_{1 \leq i, j \leq r}$ ,  $H_x^{B^{(r-1)}} = (\Lambda(x^{i+j-1}))_{1 \leq i, j \leq r}$  and where  $M_x^{B^{(r-1)}}$  is the companion matrix of the polynomial  $\pi_r$

$$M_x^{B^{r-1}} = \begin{pmatrix} 0 & \dots & \dots & 0 & \tau_0 \\ 1 & \ddots & & \vdots & \tau_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & \tau_{r-1} \end{pmatrix}.$$

Since the eigenvalues of the companion matrix  $M_x^{B^{(r-1)}}$  are the roots of the polynomial  $\pi_r$ , the generalized eigenvalues of the pair of Hankel matrices  $(H_x^{B^{(r-1)}}, H_1^{B^{(r-1)}})$  are the sought nodes  $\xi_1, \dots, \xi_r$ .

The entries of those Hankel matrices are known since  $\Lambda$  is a cubature of degree  $2r - 1$ . Indeed,  $H_1^{B^{(r-1)}}$  (resp.  $H_x^{B^{(r-1)}}$ ) corresponds to the matrix  $(m_{i+j-2})_{1 \leq i, j \leq r}$  (resp.  $(m_{i+j-1})_{1 \leq i, j \leq r}$ ) of the symmetric bilinear form

$$\begin{aligned} \Phi : \mathbb{R}[x] \times \mathbb{R}[x] &\rightarrow \mathbb{R}, (p, q) \mapsto \int_a^b p(x)q(x)\omega(x)dx \\ (\text{resp. } \Phi_x : \mathbb{R}[x] \times \mathbb{R}[x] &\rightarrow \mathbb{R}, (p, q) \mapsto \int_a^b xp(x)q(x)\omega(x)dx \end{aligned})$$

restricted to  $\mathbb{R}[x]_{\leq r-1}$  in the monomial basis  $B^{(r-1)}$ . The matrix  $H_1^{B^{(r-1)}}$  is therefore called a *moment matrix*.

## A classical approach using orthogonal polynomials

We remark here on the link between quadratures and orthogonal polynomials with respect to the symmetric bilinear form  $\Phi$ . The latter defines actually an inner product on the  $\mathbb{R}$ -vector space  $\mathbb{R}[x]$  and the polynomial  $\pi_r$  constructed above is the monic orthogonal polynomial of degree  $r$ . Indeed, since  $\Lambda$  is a cubature of degree  $2r - 1$  and with the help of (2.3), we get

$$\Phi(p, \pi_r) = \Lambda(p\pi_r) = 0 \quad \forall p \in \mathbb{R}[x]_{\leq r-1}. \quad (2.5)$$

Moreover, since one can construct exactly one monic orthogonal polynomial  $\pi_r$  per degree, the polynomials  $\pi_r$  form a basis  $\Pi$  of  $\mathbb{R}[x]$ . The 'infinite' matrix of  $\Phi$  in the basis  $\Pi$  is therefore a diagonal matrix, whose diagonal elements are  $\Phi(\pi_r, \pi_r)$ , and the 'infinite' matrix of  $\Phi_x$  in the basis  $\Pi$  is tridiagonal since

$$\Phi_x(\pi_{r_1}, \pi_{r_2}) = \Phi(\pi_{r_1}, x\pi_{r_2}) = 0 \text{ if } r_2 \leq r_1 - 2,$$

$$\Phi_x(\pi_{r_1}, \pi_{r_2}) = \Phi(x\pi_{r_1}, \pi_{r_2}) = 0 \text{ if } r_1 \leq r_2 - 2.$$

Taking now the basis  $\tilde{\Pi}$  defined by the orthonormal polynomials  $\tilde{\pi}_r$ , obtained by normalizing the monic orthogonal polynomials  $\pi_r$ , we have that:

- The 'infinite' matrix of  $\Phi$  in the basis  $\tilde{\Pi}$  is the identity.
- The 'infinite' tridiagonal matrix of  $\Phi_x$  in the basis  $\tilde{\Pi}$  is the *Jacobi matrix*

$$J = \begin{pmatrix} a_0 & b_0 & 0 & \dots \\ b_0 & a_1 & b_1 & \ddots \\ 0 & b_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where  $(a_r)_{r \in \mathbb{N}}$  and  $(b_r)_{r \in \mathbb{N}}$  are sequences of real numbers that reflect the recurrence equation of order 2 satisfied by the orthonormal polynomials  $\tilde{\pi}_r$  (see *e.g.* [30, Corollary 1.3.10], [2, Chapter 1]):

$$\begin{cases} x\tilde{\pi}_r(x) = b_r\tilde{\pi}_{r+1}(x) + a_r\tilde{\pi}_r(x) + b_{r-1}\tilde{\pi}_{r-1}(x) & \forall r \geq 1 \\ x\tilde{\pi}_0(x) = b_0\tilde{\pi}_1(x) + a_0\tilde{\pi}_0(x) \end{cases}.$$

- Applying a change of basis in (2.4), the eigenvalues of the  $r \times r$  leading principal submatrix of the Jacobi matrix  $J$  are the roots of the orthonormal polynomial  $\tilde{\pi}_r$ , or equivalently the nodes  $\xi_1, \dots, \xi_r$ . See [100, Theorem 3.6.20] for the classical link between eigenvalues of tridiagonal matrices and roots of orthogonal polynomials.

## 2.2 Cubatures and minimal number of nodes

Cubatures are the multidimensional analogues of quadratures. However, as stated in [101, Chapter 1.3], there is a gap between the construction of minimal quadratures and the construction of minimal cubatures. In particular, the number of minimal nodes is only known in some cases.  $\mathbb{R}[x]$  denotes now the  $\mathbb{R}$ -vector space of the polynomials with  $n$  variables and coefficients in  $\mathbb{R}$  with  $n > 2$ .

Let  $\mu$  be a positive Borel measure on  $\mathbb{R}^n$  and let  $\text{supp } \mu$  be the closed support of the measure  $\mu$ , that is the complement of the biggest open set  $\mathcal{U} \subset \mathbb{R}^n$  with  $\mu(\mathcal{U}) = 0$ . As in [89, Theorem 1], we assume that  $\text{supp } \mu$  is compact. Given any  $k \in \mathbb{N}^n$ , the moment  $m_k$

$$m_k = \int_{\mathbb{R}^n} x^k d\mu(x)$$

therefore exists and is finite. More generally, let  $\Omega$  be the linear form defined from  $\mu$  by

$$\Omega : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \int_{\mathbb{R}^n} p(x) d\mu(x).$$

A linear form  $\Lambda$  defined on  $\mathbb{R}[x]$  by

$$\Lambda : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j=1}^r a_j p(\xi_j)$$

with  $r > 0$ ,  $a_j \in \mathbb{R} \setminus \{0\}$  and  $\xi_j \in \mathbb{R}^n$  pairwise distinct is called a *cubature of degree  $d$*  for  $\mu$  if it satisfies

$$\int_{\mathbb{R}^n} p(x) d\mu(x) = \sum_{j=1}^r a_j p(\xi_j) \quad \forall p \in \mathbb{R}[x]_{\leq d} \quad (2.6)$$

and if this last equality does not hold for at least one polynomial  $p \in \mathbb{R}[x]_{d+1}$ . The points  $\xi_j$  are the *nodes* and the numbers  $a_j$  are the *weights* of the cubature. Such a cubature is called *inside* if the nodes  $\xi_j$  lie on  $\text{supp } \mu$  and *minimal* if the number of nodes  $r$  is minimal.

In our examples, the measure  $\mu$  is the characteristic function of a compact in  $\mathbb{R}^n$ : 1 on  $\text{supp } \mu$  and 0 otherwise. Thus, we often abuse the definition: a cubature for the characteristic function of a compact  $K$  is simply called a cubature for  $K$ .

Given a positive Borel measure with compact support in  $\mathbb{R}^n$ , we are interested in finding inside cubatures with positive weights. Cubatures with these properties are generally numerically more stable than cubatures for which one or both of these properties is lacking [101, Chapter 1]. Their existence is guaranteed with an upper bound on the minimal number of nodes by Tchakaloff's theorem [89, Theorems 1,2],[7, Theorem 2]. Contrary to the one-dimensional case, minimal cubatures are generally not known. A lower bound is provided by [30, Theorem 3.7.1]. Combining those general results, we get

**Theorem 2.1.** *Let  $d$  be a positive integer and let  $\mu$  be a positive Borel measure with compact support in  $\mathbb{R}^n$ . Then there exists an inside cubature of degree  $d$  with positive weights the number of nodes  $r$  of which satisfies*

$$\dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor} \leq r \leq \dim \mathbb{R}[x]_{\leq d}.$$

Notice that Tchakaloff's theorem was originally proved for compact regions in the plane  $\mathbb{R}^2$  as reported in [101, Theorem 3.3-6]. The authors in [7, 89] proved more sophisticated versions of Tchakaloff's theorem with lighter hypotheses on the measure  $\mu$ .

To conclude this section, we shortly discuss the differences between quadratures and cubatures and, as a consequence, the challenge to find minimal cubatures. In [101, Chapter 1.3], the author relates it to two main issues. The first one is a geometric issue, whereas the second one is related to the theory of orthogonal polynomials.

The geometric issue is related to the following property: compact sets being equivalent under an affine transformation have minimal cubatures with the same number of nodes [101, Theorem 1.4-1]. There is essentially one compact set in the one-dimensional case, whereas there are infinitely many distinct compact sets in the multidimensional case. For instance, cubatures for the triangle, the square or the disk are different from each other.

In the multidimensional case, the theory of orthogonal polynomials states that a lower bound on the number of nodes of a cubature of degree  $d = 2k - 1$  is given by  $\dim \mathbb{R}[x]_{\leq k-1}$  [30, Theorem 3.7.1]. As an analogy, a cubature that attains this bound is called a *Gaussian cubature*. However its existence is not guaranteed. Mysovskikh's theorem [30, Theorem 3.7.4] states that a Gaussian cubature of degree  $2k - 1$  exists if and only if the set of orthogonal polynomials of degree  $k$  has exactly  $\dim \mathbb{R}[x]_{\leq k-1}$  common zeros. The author in [69] provides an alternative criterion based on moment matrices for the existence of a Gaussian cubature, which reduces to checking whether a certain overdetermined linear system has a solution. Notice that, if a Gaussian cubature exists, the weights are positive [30, Corollary 3.7.5] and the nodes belong to the interior of the convex hull of  $\text{supp } \mu$  [20, Theorem 7].

Gaussian cubatures are rare as stated in [79, 97], where the authors give an example of such a minimal cubature for every degree.

## 2.3 Construction of cubatures and lower bounds on the number of nodes

As presented in reviews on the topic [19, 101], cubatures have been constructed using several techniques like ones based on product of Gaussian quadratures, on solutions of multivariate nonlinear systems or on zeros of multivariate orthogonal polynomials. More recently, the authors in [35] based their construction on moment matrices. This last approach is the one we choose and a first presentation is given in Section 2.4.

### Techniques based on product of Gaussian quadratures

The techniques based on product of Gaussian quadratures are called either *product formulas* [101, Chapter 2] or *repeated quadratures* [19, Section 4.1]. Consider the compact set  $[a_1, b_1] \times \cdots \times [a_n, b_n]$  with  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ . For each interval  $[a_i, b_i]$ , a Gaussian quadrature  $\Lambda_i$  is chosen

$$\Lambda_i : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j_i=1}^{r_i} a_{j_i} p(\xi_{j_i}).$$

The Gaussian quadratures  $\Lambda_i$  may be the same or distinct ones (of distinct degrees  $d_i$  for instance). The whole cubature  $\Lambda$  for  $[a_1, b_1] \times \cdots \times [a_n, b_n]$  is then

$$\Lambda : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} a_{j_1} \cdots a_{j_n} p(\xi_{j_1}, \dots, \xi_{j_n}). \quad (2.7)$$

The degree  $d$  of such a cubature is between  $\min\{d_1, \dots, d_n\}$  and  $\max\{d_1, \dots, d_n\}$ .

Cubatures for other compact sets are then obtained from (2.7) by performing a suitable change of variables as mentioned in [19, Section 4.2].

The main advantage is that cubatures of every odd degree can be constructed. However, the required number of nodes is then larger than the one of known cubatures obtained in a different way. For instance, there are inside cubatures with positive weights of degree 9, 11, 13, 15 for the square with respectively 17, 24, 33, 44 nodes [21]; whereas cubatures based on product of Gaussian quadratures of the same degree have respectively 25, 36, 49, 64 nodes. Another advantage is that Gaussian quadratures are well-known so that those cubatures remain popular (see *e.g.* [81, 82]).

## Nonlinear systems and simplification using symmetry

The second approach starts by noticing that equalities (2.6) are satisfied for all polynomials of  $\mathbb{R}[x]_{\leq d}$  with  $d$  the degree of the cubature if and only if they are satisfied for all polynomials in any basis  $B^{(d)} = \{b_1, \dots, b_{r_d}\}$  of  $\mathbb{R}[x]_{\leq d}$ . Thus, taking  $p = b_i$  for all  $i = 1, \dots, r_d$  in (2.6), we get the multivariate nonlinear system

$$\sum_{j=1}^r a_j b_i(\xi_j) = \int_{\mathbb{R}^n} b_i(x) d\mu(x) \quad \forall i = 1, \dots, r_d, \quad (2.8)$$

where we assume that the values of the right-hand side can be computed. This is the case for instance for moments of the characteristic function of polytopes [3, 99]. However, since the size of the polynomial system (2.8) increases with the degree  $d$  of the sought cubature and the number of variables  $n$ , a direct resolution is limited to low degrees and small dimensions.

In view of the shape of standard regions of cubatures ( $n$ -dimensional hypercube, simplex or ball), a natural simplification in the resolution of the nonlinear system (2.8) comes from transformations that leave those regions invariant: the *symmetries*.

To give an idea, consider cubatures for the square or the disk that respect the symmetry of the square [91, Tables 1,2]. They satisfy: if  $\xi_j = (x_{1,j}, x_{2,j})$  is a node and  $a_j$  is the corresponding weight, then  $(\pm x_{1,j}, \pm x_{2,j})$  and  $(\pm x_{2,j}, \pm x_{1,j})$  are also nodes associated with the same weight  $a_j$ . The set of nodes is then generated by nodes of the form:  $(0,0)$ ,  $(a,0)$ ,  $(b,b)$ ,  $(c,d)$ , where  $a, b, c, d$  are in  $\mathbb{R}$ . Thus, the nonlinear system (2.8) can be simplified as presented in [91]: some moments of order bigger than the degree of the cubature are zero and smaller subsystems are extracted.

More generally, in any dimension, cubatures for the hypercube or the ball that respect the symmetry of the hypercube are listed in [54, Tables 1,2]. In this case, they satisfy: if  $\xi_j = (x_{1,j}, \dots, x_{n,j})$  is a node and  $a_j$  is the corresponding weight, then  $(\pm x_{\sigma(1),j}, \dots, \pm x_{\sigma(n),j})$  are also nodes associated with the same weight  $a_j$ , where  $\sigma$  is a permutation of the symmetric group

on  $n$  elements. A similar argument as the one given above leads to similar simplifications in the nonlinear system (2.8) as presented in [54].

A more general approach consists in relating the symmetry to the action of a group  $G$  on  $\mathbb{R}^n$ . Cubatures that respect this symmetry are called  $G$ -invariant [19, 37, 38] (a precise definition is given in 8.1). They satisfy the following properties:

- The set of nodes  $\xi_j$  is a union of orbits of this group action.
- The weights  $a_j$  associated with nodes on a same orbit are identical.
- The cubature is of degree  $d$  if it is exact for all  $G$ -invariant polynomials of degree at most  $d$  and if it is not exact for at least one polynomial of degree  $d + 1$  [19, Corollary 5.1 (Sobolev's theorem)].

While the last requirement reduces the number of equations in (2.8), the two first points reduce the number of unknowns as in [54, 91].

Tables 2 and 4 in [37] list  $G$ -invariant cubatures for the square and the triangle with several groups  $G$ : cyclic groups  $C_m$  or dihedral groups  $D_m$  (see Sections 7.3 and 7.4 for a presentation of these groups).  $D_6$ -invariant cubatures for the regular hexagon have been found following this approach in [51].

## Orthogonal polynomials and lower bounds

Radon's work [92] marked a starting point in the search for cubatures thanks to orthogonal polynomials. As stated in [20], the theory developed following this way tries to generalize Radon's approach for higher degrees and dimensions. However, if multivariate orthogonal polynomials provide some results, they do not answer all the questions as in the one-dimensional case.

As in our presentation of the quadrature problem, we start with the computation of the weights  $a_1, \dots, a_r$  once the nodes  $\xi_1, \dots, \xi_r$  are known. Let  $p_1, \dots, p_r$  be polynomials of degree less than or equal to the degree  $d$  of the cubature  $\Lambda$ . The following equations are then satisfied

$$\int p_i d\mu = \Lambda(p_i) = \sum_{j=1}^r a_j p_i(\xi_j) \quad \forall i = 1, \dots, r.$$

The latter generalizes (2.2) for which  $(p_1, p_2, \dots, p_r) = (1, x, \dots, x^{r-1})$ . A linear system like (2.1) is thus deduced

$$\begin{pmatrix} p_1(\xi_1) & p_1(\xi_2) & \cdots & p_1(\xi_r) \\ p_2(\xi_1) & p_2(\xi_2) & \cdots & p_2(\xi_r) \\ \vdots & \vdots & & \vdots \\ p_r(\xi_1) & p_r(\xi_2) & \cdots & p_r(\xi_r) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \Lambda(p_1) \\ \Lambda(p_2) \\ \vdots \\ \Lambda(p_r) \end{pmatrix}. \quad (2.9)$$

The latter is called *Vandermonde-like* since the involved matrix can be seen as a multivariate generalization of a Vandermonde matrix [85]. Choosing the polynomials  $p_1, \dots, p_r$  such that the matrix in this Vandermonde-like linear system is invertible guarantees that the weights  $a_1, \dots, a_r$  are determined uniquely.



Cubatures whose weights are obtained uniquely from the nodes are called *interpolatory*. We focus on these since a subset of the nodes of a non-interpolatory cubature can be used as the set of nodes of an interpolatory one [19, Section 6.1].

Similarly to Section 2.1, a symmetric bilinear form  $\Phi$  on the polynomial space  $\mathbb{R}[x]$

$$\Phi : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}, (p, q) \mapsto \int pq \, d\mu$$

is associated with the measure  $\mu$ . It defines an inner product on  $\mathbb{R}[x]$ . There are thus orthogonal polynomials  $\pi_\varrho$  ( $\varrho \in \mathbb{N}^n$ ) with respect to  $\Phi$  that satisfy the multivariate analogue of (2.5)

$$\Phi(p, \pi_\varrho) = \Lambda(p\pi_\varrho) = 0 \quad \forall p \in \mathbb{R}[x]_{\leq |\varrho|-1}.$$

Notice that there exist  $\dim \mathbb{R}[x]_\delta$  unique orthogonal polynomials of degree  $\delta$  of the form  $x^\varrho + q$  with  $|\varrho| = \delta$  and  $q \in \mathbb{R}[x]_{\leq \delta-1}$  (see (6.3) in [19]). We refer to [30] for explicit expressions of multivariate orthogonal polynomials with respect to different measures and also for theoretical results on multivariate orthogonal polynomials.

The particular case of Gaussian cubatures was already discussed in Section 2.2. Their nodes are characterized as the common zeros of all orthogonal polynomials of degree  $\lfloor \frac{d}{2} \rfloor$ , where  $d$  is the odd degree of the Gaussian cubature (Mysovskikh's theorem). A more general approach involving multivariate orthogonal polynomials (see [19, Section 6], [20, Section 6] and references therein) keeps the idea from algebraic geometry that a set of points in  $\mathbb{R}^n$  corresponds to the variety (set of common zeros) of an ideal generated by an appropriate basis. Methods for the computation of this basis have been developed (see [19, Section 9] and references herein). For instance, as in Radon's work, the common zeros of a selection of three orthogonal polynomials of the same degree have been identified with the nodes of cubatures in the plane in [52, 53]. Notice that symmetry can again simplify the computations [19, Section 9.2].

This theory provides also lower bounds on the minimal number of nodes for a cubature of a fixed degree  $d$  [19, Sections 7,8], [20, Section 5]. Sharp lower bounds are important because they are the starting point of the search for cubatures in several methods. We mention here only two general lower bounds that link directly  $d$  and the number of nodes  $r$ :

- The first one is the one in Theorem 2.1

$$r \geq \dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}.$$

- The second one works for *centrally symmetric* measures, which means that  $m_\alpha = 0$  whenever  $|\alpha|$  is odd, or equivalently that the measure is invariant under the symmetry  $x \mapsto -x$ . When  $n = 2$ , this bound can be written as

$$r \geq \dim \mathbb{R}[x]_{\leq k} + \left\lfloor \frac{k+1}{2} \right\rfloor \quad \text{with } d = 2k+1. \quad (2.10)$$

It is then known as Möller's lower bound [104]. For the general case, we refer to [19, Theorem 8.3], [20, Theorem 13].

There exist thus no Gaussian cubatures of odd degree for centrally symmetric measures when  $n = 2$ .

## 2.4 A moment matrix approach to cubatures

The approach to cubatures we have, devised in [35], incorporates the results overviewed in [74] and additional influences like [70]. The details are given in Sections 3 and 4. In particular we do not restrict ourselves to monomial bases. The presentation we make here prepares for the use of symmetry, where our main contribution resides.

### Main ingredients

The moment matrix approach to cubatures is based on two main ingredients: the characterization of cubatures in terms of positive semidefinite matrices with a prescribed rank (see Sections 3.2 and 3.3) and a flat extension theorem to reconstruct the expected linear form on  $\mathbb{R}[x]$  when working on a restriction to a finite dimensional space (see Section 3.4). We introduce here these ingredients in terms of matrices.

The characterization, the first ingredient, starts with the association of a linear form  $\Lambda$  on  $\mathbb{R}[x]$  with the symmetric bilinear form

$$\varphi : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}, (p, q) \mapsto \Lambda(pq),$$

or equivalently, with the Hankel operator  $\widehat{\mathcal{H}} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]^*$  given by  $\widehat{\mathcal{H}}(p)(q) = \Lambda(pq)$  for all  $p, q \in \mathbb{R}[x]$ . Fix a basis  $B = (b_k)_{k \in \mathbb{N}}$  of  $\mathbb{R}[x]$  such that it is possible to extract a basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$  for every degree  $\delta \in \mathbb{N}$ . Let  $H_1^B$  be the 'infinite' matrix of the symmetric bilinear form  $\varphi$  in the basis  $B$

$$H_1^B = (\Lambda(b_i b_j))_{(i,j) \in \mathbb{N}^2}.$$

When  $B$  is a monomial basis, the matrix  $H_1^B$  is referred to as a *moment matrix*. The linear form  $\Lambda$  is then given by

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j},$$

with  $r > 0$ ,  $a_j > 0$  and  $\xi_j \in \mathbb{R}^n$  pairwise distinct, if and only if the 'infinite' matrix  $H_1^B$  has finite rank  $r$  and is positive semidefinite. The set of nodes  $\{\xi_1, \dots, \xi_r\}$  is then the variety of the ideal  $I_\Lambda$ : the kernel of the Hankel operator  $\widehat{\mathcal{H}}$ .

This variety can be recovered by considering submatrices  $H_1^{B^{(\delta)}}$  of  $H_1^B$ . They are the matrices of the restriction of  $\varphi$  to  $\mathbb{R}[x]_{\leq \delta}$  in a basis  $B^{(\delta)}$ . Take a matrix  $H_1^{B^{(\delta)}}$  such that its submatrix  $H_1^{B^{(\delta-1)}}$  has the same rank  $r$  as the whole matrix. The variety of the kernel of  $H_1^{B^{(\delta)}}$ , which is a zero-dimensional and radical ideal, is then the set of nodes.

The flat extension theorem, the second ingredient, determines in finite terms when the matrix  $H_1^B$  has finite rank. Assume that a linear form  $\Lambda^{(\delta)}$  is known on  $\mathbb{R}[x]_{\leq 2\delta}$  and that the matrix  $H_1^{B^{(\delta)}}$  associated with the bilinear form on  $\mathbb{R}[x]_{\leq \delta}$  has the same rank as the matrix  $H_1^{B^{(\delta-1)}}$  of its restriction to  $\mathbb{R}[x]_{\leq \delta-1}$ . Then there is a unique linear form  $\Lambda$  on the whole polynomial space  $\mathbb{R}[x]$  such that

$$\Lambda(p) = \Lambda^{(\delta)}(p) \quad \forall p \in \mathbb{R}[x]_{\leq 2\delta} \text{ and } \text{rank } H_1^B = r.$$

## Methodology

With the help of the two main ingredients presented above, a first procedure is proposed in Section 5 to determine the existence for the measure  $\mu$  of a cubature  $\Lambda$  of a given degree  $d$ . We give here a short description of it:

- Choose a degree  $\delta \in \mathbb{N}$  such that  $\lfloor \frac{d}{2} \rfloor \leq \delta - 1 \leq d$ .
- Take a monomial basis  $B^{(\delta)} = \{b_1, \dots, b_{r_\delta}\}$  of  $\mathbb{R}[x]_{\leq \delta}$ .
- Construct the moment matrix  $H_1^{B^{(\delta)}} = (\Lambda(b_i b_j))_{1 \leq i, j \leq r_\delta}$  whose entries are:
  - either moments  $\int b_i b_j d\mu$  of the measure  $\mu$  if  $\deg(b_i b_j) \leq d$ ,
  - or parameters  $h_{r_\delta+1}, \dots, h_{r_{2\delta}}$  that stand for the unknown quantities  $\Lambda(b_{r_\delta+1}), \dots, \Lambda(b_{r_{2\delta}})$ .
- Based on the two main ingredients presented above, determine values for  $h_{r_\delta+1}, \dots, h_{r_{2\delta}}$  such that  $H_1^{B^{(\delta)}}$  and  $H_1^{B^{(\delta-1)}}$  are positive semidefinite of rank  $r$  (as small as possible).
- Determine the nodes  $\xi_1, \dots, \xi_r$  of the cubature by solving:
  - either the polynomial system given by the kernel of  $H_1^{B^{(\delta)}}$ : it corresponds to the ideal generated by orthogonal polynomials in Section 2.3;
  - or generalized eigenvalue problems of pairs  $(H_p^r, H_1^r)$  for  $p \in \mathbb{R}[x]_{\leq 1}$  that *separates* the nodes, that is such that  $p(\xi_1), \dots, p(\xi_r)$  are pairwise distinct. The matrix  $H_1^r$  is an invertible  $r \times r$  submatrix of  $H_1^{B^{(\delta-1)}}$ , which is rewritten as  $(\Lambda(\bar{b}_i \bar{b}_j))_{1 \leq i, j \leq r}$  with  $\{\bar{b}_1, \dots, \bar{b}_r\} \subset B^{(\delta-1)}$ . The matrix  $H_p^r$  is then the matrix  $(\Lambda(p \bar{b}_i \bar{b}_j))_{1 \leq i, j \leq r}$ .
- Determine the weights  $a_1, \dots, a_r$  of the cubature by solving the Vandermonde-like linear system (2.9) with  $\{p_1, \dots, p_r\} = \{\bar{b}_1, \dots, \bar{b}_r\}$ .

This procedure can be split in two main parts:

1. The first four points determine the existence for  $\mu$  of a cubature  $\Lambda$  of degree  $d$  with positive weights.
2. The last two points compute the weights and the nodes once the existence is secured.

Theorem 2.1 guarantees the existence of a degree  $\delta$  as in the first point of the procedure, and then values for the unknowns  $h_{r_\delta+1}, \dots, h_{r_{2\delta}}$ . Their determination is a problem of *low rank completion of structured matrices* of the same kind as the one that appears in tensor decomposition [15]. To deal with it, we propose an algorithm based on fraction-free LU-decompositions (see Section 4). This algorithm is applicable to matrices of small size, as the ones we shall obtain when symmetry is taken into account. When the size of the matrices increases, a new exact algorithm has been proposed in [56] to find at least a solution to this issue. They also refer to numerical SemiDefinite Programming solvers for higher dimensions.

There is an optional step between the two main parts, which consists in creating *localizing matrices* and verifying their positive semidefiniteness. It guarantees that the nodes lie on  $\text{supp } \mu$  before their computation.

## 2.5 Contributions

The methodology described above is based on a synthesis of results on moment matrices and truncated moment problems [24, 35, 73, 74]. Our motivation was to tie in symmetry. Related symmetric problems are treated in [22, 39, 94].

Symmetry appears as a natural property to preserve. Many cubatures for standard regions in the plane  $\mathbb{R}^2$  (triangle, square, hexagon, disk) or in the space  $\mathbb{R}^n$  (simplex, hypercube, ball) were computed by imposing a symmetry of the measure to the cubature: in the nonlinear system [37, 51, 54, 91] or in the orthogonal polynomials [52, 53]. As a consequence, constraints are added to the unknowns used in those techniques. The determination of cubatures is thus simplified. This is also the case in the moment matrix approach:

- The matrix  $H_1^{B^{(\delta)}}$  we shall deal with has less parameters  $h_\ell$  than in the procedure presented above (Step 6 in Algorithm 9.6 [Existence of a  $G$ -invariant cubature]).
- This matrix is furthermore block diagonal, when the basis  $B^{(\delta)}$  is well-chosen. Thus, the determination of the conditions on the unknown parameters  $h_\ell$  is done on matrices of smaller size (Step 7 in Algorithm 9.6 [Existence of a  $G$ -invariant cubature]).
- The computation of the nodes as generalized eigenvalues and the computation of the weights as solutions of a Vandermonde-like linear system are also performed on smaller-sized matrices (see Algorithm 9.7 [Weights & Nodes]).

To establish a methodological approach we developed several novel results:

- A basis-free version of the moment matrix approach to cubatures (Sections 3.2, 3.3 and 3.4). It allows us to use another kind of bases than the monomial bases used in [35].
- The introduction of appropriate bases, namely *orthogonal symmetry adapted bases*, such that the matrices  $H_1^{B^{(\delta)}}$  are block diagonal: the size of the blocks and the number of identical blocks are deduced from the computation of those bases (Sections 6.4, 6.5 and 8.1). Previous use of symmetry adapted bases appeared in [39, 94]. However, the induced representations were required to be orthogonal and representations of symmetric groups, that permute the variables, were mostly used. This is not sufficient in our application.
- The introduction of the matrix of multiplicities  $\Gamma_G$  of a finite group  $G$  (Section 7). It is the key to preliminary criteria of existence of symmetric cubatures. Its computation is done for cyclic groups  $C_m$  and dihedral groups  $D_m$  with  $m \geq 2$ .
- The equivalence between the representation on the quotient space  $\mathbb{R}[x]/I_\Lambda$  and the permutation representation on the invariant set of nodes (Section 8.2).
  - The size of the blocks of  $H_1^B$  is related to the organization of the nodes in orbit types and the matrix of multiplicities of the group of symmetry.
  - The distribution of the distinct generalized eigenvalues of the different blocks is known with respect to the organization of the nodes in orbit types.

Possible number of nodes  $r$  for a  $G$ -invariant cubature of fixed degree  $d$  are deduced. They are quantified by the inequalities (9.9) and (9.10) that play the same role as the consistency conditions in [91].

An algorithm for computing  $G$ -invariant cubatures is presented (Section 9) and applied to recover known cubatures (see *e.g.* Section 9.4) and exhibit new cubatures (see *e.g.* Section 10):  $D_6$ -invariant cubatures of degree 13 with 37 nodes for the regular hexagon  $H_2$ .

## 2.6 Related problems

The techniques presented in this paper have other applications than cubatures. Indeed, a number of classical problems can be formulated as we did for quadratures, in the univariate case, and cubatures, in the multivariate case.

In the univariate case, one seeks to retrieve pairs  $\{(a_1, \xi_1), \dots, (a_r, \xi_r)\}$  from the *moments*  $(\mu_k)_{k \in \mathbb{N}}$  or  $(\mu_k)_{0 \leq k \leq R}$  with  $R$  an upper bound of  $r$  under the assumption that

$$\mu_k = \sum_{j=1}^r a_j \xi_j^k.$$

In the case of quadratures,  $\mu_k \in \mathbb{R}$  and we expect  $a_j$  and  $\xi_j$  to be in  $\mathbb{R}$ . In general though  $\mu_k$ ,  $a_j$  and  $\xi_j$  are in  $\mathbb{C}$ . Furthermore, the number of terms  $r$  might be an additional unknown of the problem. As demonstrated above the problem can be solved (uniquely) when  $R = 2r - 1$  moments are available.

In the multivariate case, the input is indexed by  $k = (k_1, \dots, k_n) \in \mathbb{N}^n$  and each  $\xi_j$  belongs to  $\mathbb{C}^n$ . The relationships are given by

$$\mu_k = \sum_{j=1}^r a_j \xi_{j,1}^{k_1} \dots \xi_{j,n}^{k_n}.$$

An additional problem in the multivariate case, even when  $r$  is known, is to find an appropriate  $\mathbb{K}$ -vector space basis of the quotient algebra  $\mathbb{K}[x_1, \dots, x_n]/I_\xi$ , where  $I_\xi$  is the ideal the variety of which is  $\{\xi_1, \dots, \xi_r\} \subset \mathbb{C}^n$ .

### Prony's method for exponential interpolation [93].

One wants to retrieve  $f : \mathbb{C} \rightarrow \mathbb{C}$  as

$$f(x) = \sum_{j=1}^r a_j e^{\lambda_j x}$$

from the knowledge of  $\mu_k = f(k)$  with  $k \in \mathbb{N}$ . One retrieves the above problem by considering  $\xi_j = e^{\lambda_j}$ .

In its multivariate version  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  and

$$f(x) = \sum_{j=1}^r a_j e^{\langle \lambda_j, x \rangle}$$

where  $x = (x_1, \dots, x_n)$ ,  $\lambda_j \in \mathbb{C}^n$  and  $\langle y, x \rangle = y_1 x_1 + \dots + y_n x_n$ . With  $\xi_{j1} = e^{\lambda_{j1}}$ ,  $\dots$ ,  $\xi_{jn} = e^{\lambda_{jn}}$  the problem is then recast into

$$f(x) = \sum_{j=1}^r a_j \xi_j^x.$$

Under some natural assumptions, solutions have been proposed for the univariate case [87], for the multivariate case with a univariate resolution (projection method) [88] and with a multivariate approach [66].

### Sparse interpolation [11].

Assuming a (multivariate) polynomial  $f$  has a small support

$$f(x) = \sum_{j=1}^r a_j x^{\alpha_j},$$

one wishes to retrieve the exponents  $\alpha_j \in \mathbb{N}^n$  from evaluations of the polynomial at chosen points. For  $k \in \mathbb{N}$ , one chooses  $\mu_k = f(p_1^k, \dots, p_n^k)$  where  $p_j$  are distinct prime numbers. From  $\mu_0, \dots, \mu_{2r-1}$  one can retrieve  $\xi_j = p^{\alpha_j} = p_1^{\alpha_{j1}} \dots p_n^{\alpha_{jn}}$  so that the exponent can be found by factorization [11, 63, 64]. Replacing the prime numbers by roots of unity, the authors in [42] proposed a symbolic-numeric solution.

For latest development in Prony's method and sparse interpolation, one can consult the material attached to the 2015 Dagstuhl seminar *Sparse modelling and multi-exponential analysis* (<http://www.dagstuhl.de/15251>).

### Pole estimation, Padé approximant [55].

The input data are the coefficients of the Taylor expansion at  $z = 0$  of a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  and one wishes to find the poles of the function.

$$f(x) = \sum_{j=1}^r \frac{a_j}{1 - \xi_j x} = \sum_{k \in \mathbb{N}} \mu_k x^k.$$

The multivariate version that can be approached is

$$f(x) = \sum_{j=1}^r \frac{a_j}{\langle \xi_j, x \rangle} = \sum_{k \in \mathbb{N}^n} \mu_k x^k.$$

In the univariate case all polynomials can be factored over  $\mathbb{C}$  into degree 1 polynomials. Therefore the problem covers all rational functions with distinct poles. The restriction to denominators that can be factored into linear form is more restrictive in the multivariate case.

### Shape-from-moment problem [47, 83].

The problem consists in recovering the vertices of a convex  $n$ -dimensional polytope  $\mathcal{V}$  from its moments. *Projection methods* have been developed in [18, 47] based on moments in a direction  $\delta \in \mathbb{R}^n$  and Brion's identities

$$\left\{ \begin{array}{ll} \frac{(k+n)!}{k!} \int_{\mathcal{V}} \langle x, \delta \rangle^k dx = \sum_{j=1}^r a_j \langle v_j, \delta \rangle^{n+k} & k \geq 0 \\ 0 = \sum_{j=1}^r a_j \langle v_j, \delta \rangle^{n-k} & 1 \leq k \leq n. \end{array} \right.$$

The coefficients  $a_j$  are nonzero real numbers and  $\langle v_j, \delta \rangle$  are the projections of the vertices  $v_j$  on the direction  $\delta$ . Taking  $\xi_j = \langle v_j, \delta \rangle$ , the formula can be recast into

$$\mu_k = \sum_{j=1}^r a_j \xi_j^k \quad k \in \mathbb{N},$$

where  $\mu_k$  is related to the left hand side of the system of equations above. The set of projected vertices can thus be recovered. Different projections are then required and matching processes are presented in [18, 47]. The case  $n = 2$  treated with complex moments in [31, 43, 83] is linked to this general case in [18].

A multidimensional treatment of the shape-from-moment problem is however impossible due to the lack, up to our knowledge, of an equivalent formula for the moments  $\int_{\mathcal{V}} x^k dx$  with  $k \in \mathbb{N}^n$ .

### Symmetric tensor decomposition [15].

The general (multivariate) problem is to find, for a homogeneous polynomial

$$f(z) = \sum_{|k|=d} \binom{d}{k} \mu_k z^k,$$

the minimal *rank*  $r$  such that there exist  $(a_1, \xi_1), \dots, (a_r, \xi_r) \in \mathbb{C} \times \mathbb{C}^n$  such that  $f$  can be written as a linear combination of linear forms to the  $d^{\text{th}}$  power:

$$f(z) = \sum_{i=1}^r a_i \langle \xi_i, z \rangle^d.$$

Dehomogenizing the binary case ( $n = 2$ ), we obtain an equivalent univariate problem

$$f(z) = \mu_d z^d + \dots + \binom{d}{k} \mu_k z^k + \dots + \mu_0 = \sum_{i=1}^r a_i (1 - \xi_i z)^d$$

initially solved by Sylvester [103].

### Orthogonal polynomials

As seen in the previous section, quadrature is intimately linked with orthogonal polynomials. The nodes of the quadrature of degree  $2d - 1$  are the  $d$  roots of the orthogonal polynomial of degree  $d$ .

One can see that, for a measure  $\mu$  on  $\mathbb{R}^n$ , there is a Gaussian cubature of degree  $2d - 1$  if and only if all the orthogonal polynomials of degree  $d$  admit  $\dim R[x]_{\leq d-1}$  common zeros.

Orthogonal polynomials are obtained, for instance, by diagonalizing the moment matrix, or more generally the Gram matrix for other polynomial bases than monomial bases. In our approach to symmetric cubature, we show that if the linear form  $\Lambda$  is  $G$ -invariant the orthogonal symmetry adapted bases provides a block diagonal Gram matrix. By further refining this diagonalization, we obtain a symmetry adapted basis of orthogonal polynomials.

### 3 Hankel operators and flat extension of moment matrices

We review in this section two major ingredients for a constructive approach to cubatures.

To a linear form  $\Lambda$  on  $\mathbb{K}[x]$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) we associate a Hankel operator  $\mathbb{K}[x] \rightarrow \mathbb{K}[x]^*$ . The Hankel operator allows to study the properties of the linear form  $\Lambda$ . When  $\mathbb{K} = \mathbb{R}$ , the linear form corresponds to a linear combination of evaluations at some points if and only if the associated Hankel operator is of finite rank and positive semidefinite. The kernel of the operator is furthermore the vanishing ideal of those points. It is thus possible to use classical methods from algebraic geometry to obtain information on its variety [23, 32].

Given a linear form  $\Lambda^{(\delta)}$  on  $\mathbb{R}[x]_{\leq 2\delta}$  for a certain degree  $\delta$ , the flat extension criterion provides a sufficient condition for the existence of a unique extension  $\Lambda$  of  $\Lambda^{(\delta)}$  to  $\mathbb{R}[x]$ . Its Hankel operator is of finite rank. It is furthermore positive semidefinite if the one associated with  $\Lambda^{(\delta)}$  is positive semidefinite.

The concept of flat extension was introduced by Curto and Fialkow in the context of moment matrices [24, 25]. Those can be understood as the matrices of the Hankel operators in monomial bases. We recast the results in a basis-free version so that arbitrary bases can be used.

Flat extension were introduced as a mean to answer the classical truncated moment problem [24, 25]. The problem has applications in global optimization through the relaxation techniques introduced by Lasserre [67, 68]. The successful line of developments in this area were surveyed by Laurent [74]. This book chapter presents the foundational results with simplified and algebraic proofs; it served us as reference.

Several other problems can be approached with flat extension techniques, as for instance tensor decomposition [12, 15] and computation of real radicals [70, 71, 72]. The link between cubatures and flat extensions was first expanded on in [35] and leads to new lower bounds on the number of nodes of a minimal cubature. In [69] the flat extension formalism is applied to give a new criterion for the existence of a Gaussian cubature.

#### 3.1 Zero-dimensional ideals

A set of points in  $\mathbb{K}^n$  can be characterized as the variety of a certain zero-dimensional and radical ideal  $I$  in  $\mathbb{K}[x]$ . We recall in this section results that link ideals and varieties. The multiplication operator in the quotient space  $\mathbb{K}[x]/I$  is of particular interest. Its eigenvalues and left eigenvectors are in one-to-one correspondance with the points of the variety of  $I$ .

Given an ideal  $I$  in  $\mathbb{K}[x]$ , its variety is understood in  $\mathbb{C}^n$  and is denoted by  $V_{\mathbb{C}}(I)$ .

**Theorem 3.1** ([23, Finiteness Theorem]). *Let  $I \subset \mathbb{K}[x]$  be an ideal. Then the following conditions are equivalent:*

- *The algebra  $\mathcal{A} = \mathbb{K}[x]/I$  is finite-dimensional as a  $\mathbb{K}$ -vector space.*
- *The variety  $V_{\mathbb{C}}(I) \subset \mathbb{C}^n$  is a finite set.*

An ideal  $I$  satisfying any of the above conditions is said to be *zero-dimensional*. In this case, Theorem 3.2 relates the dimension of the algebra  $\mathbb{K}[x]/I$  to the cardinal of the variety  $V_{\mathbb{C}}(I)$ .



**Theorem 3.2** ([23, Theorem 2.10]). *Let  $I$  be a zero-dimensional ideal in  $\mathbb{K}[x]$  and let  $\mathcal{A} = \mathbb{K}[x]/I$ . Then  $\dim_{\mathbb{K}}(\mathcal{A})$  is greater than or equal to the number of points in  $V_{\mathbb{C}}(I)$ . Equality occurs if and only if  $I$  is a radical ideal.*

Assume that the algebra  $\mathbb{K}[x]/I$  is a finite-dimensional  $\mathbb{K}$ -vector space. The class of any polynomial  $f \in \mathbb{K}[x]$  modulo  $I$  is denoted by  $[f]$ . Given a polynomial  $p \in \mathbb{K}[x]$ , the map

$$\mathcal{M}_p : \mathbb{K}[x]/I \rightarrow \mathbb{K}[x]/I, [f] \mapsto [fp]$$

is a linear map called *multiplication by  $p$* . The following result relates the eigenvalues of the multiplication operators in  $\mathbb{K}[x]/I$  to the variety  $V_{\mathbb{C}}(I)$ . This result underlies the eigenvalue method for solving polynomial equations.

**Theorem 3.3** ([23, Theorem 4.5]). *Let  $I$  be a zero-dimensional ideal in  $\mathbb{K}[x]$  and let  $p \in \mathbb{K}[x]$ . Then, for  $\lambda \in \mathbb{C}$ , the following are equivalent:*

- $\lambda$  is an eigenvalue of the multiplication operator  $\mathcal{M}_p$ .
- $\lambda$  is a value of the function  $p$  on  $V_{\mathbb{C}}(I)$ .

Given a basis  $B$  of  $\mathbb{K}[x]/I$ , the matrix of the multiplication operator  $\mathcal{M}_p$  in the basis  $B$  is denoted by  $M_p^B$ . There is a strong connection between the points of  $V_{\mathbb{C}}(I)$  and the left eigenvectors of the matrix  $M_p^B$  relative to a basis  $B$  of  $\mathbb{K}[x]/I$ . Assume now that  $I$  is radical. Theorem 3.2 implies that  $\mathbb{K}[x]/I$  has dimension  $r$ , where  $r$  is the number of distinct points in  $V_{\mathbb{C}}(I)$ .

**Theorem 3.4.** *Let  $I$  be a zero-dimensional and radical ideal in  $\mathbb{K}[x]$ , let  $\xi_1, \dots, \xi_r$  be the points of  $V_{\mathbb{C}}(I) \subset \mathbb{K}^n$  and let  $b_1, \dots, b_r$  be polynomials in  $\mathbb{K}[x]$ . Then the following conditions are equivalent:*

- The set  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{K}[x]/I$ .
- The matrix  $W = (b_j(\xi_i))_{1 \leq i, j \leq r}$  is invertible.

Assume that any of the above conditions is satisfied. Let  $p$  be a polynomial in  $\mathbb{K}[x]$ , let  $M_p^B$  be the matrix of the multiplication operator  $\mathcal{M}_p$  in the basis  $B$  and let  $D$  be the diagonal matrix whose diagonal elements are  $p(\xi_1), \dots, p(\xi_r)$ . Then

$$WM_p^B = DW.$$

In other words, a left eigenvector associated to the eigenvalue  $p(\xi_i)$  is given by the row vector  $(b_1(\xi_i) \ \dots \ b_r(\xi_i))$ .

*Proof.* Let  $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{K}^r$  such that  $\alpha_1 b_1 + \dots + \alpha_r b_r \equiv 0 \pmod{I}$ . Since  $V_{\mathbb{C}}(I) = \{\xi_1, \dots, \xi_r\}$ , we have

$$\alpha_1 b_1(\xi_i) + \dots + \alpha_r b_r(\xi_i) = 0 \quad \forall i = 1, \dots, r.$$

With the help of the matrix  $W = (b_j(\xi_i))_{1 \leq i, j \leq r}$ , this equality can be rewritten as  $W\alpha = 0$ . Thus, the matrix  $W$  is invertible if and only if  $(\alpha_1, \dots, \alpha_r) = (0, \dots, 0)$ . This means that  $W$  is invertible if and only if  $b_1, \dots, b_r$  are linearly independent modulo  $I$ . Since  $\text{Card } B = \dim \mathbb{K}[x]/I$ , we have that  $W$  is invertible if and only if  $B$  is a basis of  $\mathbb{K}[x]/I$ .

Assume now that  $B$  is a basis of  $\mathbb{K}[x]/I$ . Let  $M_p^B = (m_{ij})_{1 \leq i, j \leq r}$  be the matrix in the basis  $B$  of the multiplication operator  $\mathcal{M}_p$  from  $\mathbb{K}[x]/I$  to itself. By definition of the multiplication operator  $\mathcal{M}_p$ , we have

$$pb_j \equiv \sum_{k=1}^r m_{kj} b_k \pmod{I} \quad \forall j = 1, \dots, r.$$

Since  $\xi_i$  is a point of the variety  $V_{\mathbb{C}}(I)$ , evaluating this equality at a point  $\xi_i$  brings

$$p(\xi_i) b_j(\xi_i) = \sum_{k=1}^r m_{kj} b_k(\xi_i) \quad \forall j = 1, \dots, r,$$

which can be rewritten as

$$p(\xi_i) \begin{pmatrix} b_1(\xi_i) & \dots & b_r(\xi_i) \end{pmatrix} = \begin{pmatrix} b_1(\xi_i) & \dots & b_r(\xi_i) \end{pmatrix} M_p^B.$$

Since this equality holds for every  $\xi_i \in V_{\mathbb{C}}(I)$ , we have then

$$WM_p^B = DW.$$

□

With the help of the matrix  $W$  defined in Theorem 3.4, the next result introduces polynomials  $f_1, \dots, f_r$  that satisfy  $f_j(\xi_i) = \delta_{ij}$  for all  $\xi_i \in V_{\mathbb{C}}(I)$ . The polynomials  $f_1, \dots, f_r$  (resp. the matrix  $W$ ) can be considered as a generalization of the Lagrange polynomials (resp. the Vandermonde matrix) used in polynomial interpolation when  $n = 1$ . Notice that polynomials  $f_1, \dots, f_r$  which satisfy  $f_j(\xi_i) = \delta_{ij}$  are generally not unique.

**Corollary 3.5.** *Let  $I$  be a zero-dimensional and radical ideal in  $\mathbb{K}[x]$ , let  $\xi_1, \dots, \xi_r$  be the points of  $V_{\mathbb{C}}(I)$ , let  $b_1, \dots, b_r$  be polynomials in  $\mathbb{K}[x]$  such that the set  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{K}[x]/I$  and let  $W$  be the matrix  $W = (b_j(\xi_i))_{1 \leq i, j \leq r}$ .*

*Assume that  $V_{\mathbb{C}}(I) \subset \mathbb{K}^n$ . Then there exist  $r$  polynomials  $f_1, \dots, f_r$  in  $\mathbb{K}[x]$  determined by*

$$W^t \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}$$

*and satisfying  $f_i(\xi_j) = \delta_{ij}$ . Moreover, the set  $\{[f_1], \dots, [f_r]\}$  is a basis of  $\mathbb{K}[x]/I$ .*

*Proof.* By Theorem 3.4, since  $B$  is a basis of  $\mathbb{K}[x]/I$ , the matrix  $W$  is invertible so that the polynomials  $f_1, \dots, f_r$  in  $\mathbb{K}[x]$  are determined uniquely by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = W^{-t} \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}. \tag{3.1}$$

Let  $e_j$  be the  $j^{th}$  vector of the canonical basis of  $\mathbb{K}^r$ . By definition of the matrix  $W$ , we have

$$W^t e_j = \begin{pmatrix} b_1(\xi_j) \\ \vdots \\ b_r(\xi_j) \end{pmatrix} \quad \forall j = 1, \dots, r.$$

From the definition of the polynomials  $f_1, \dots, f_r$ , we deduce that they satisfy  $f_i(\xi_j) = \delta_{ij}$ .

Let  $\alpha_1, \dots, \alpha_r \in \mathbb{K}$  such that

$$\alpha_1 f_1 + \dots + \alpha_r f_r \equiv 0 \pmod{I}.$$

Take  $\beta_1, \dots, \beta_r \in \mathbb{K}$  such that  $(\beta_1 \dots \beta_r) = (\alpha_1 \dots \alpha_r) W^{-t}$ . Using the definition of the polynomials  $f_1, \dots, f_r$ , we have then

$$\beta_1 b_1 + \dots + \beta_r b_r \equiv 0 \pmod{I}.$$

Since  $B$  is a basis of  $\mathbb{K}[x]/I$ , we have  $\beta_1 = \dots = \beta_r = 0$ , or equivalently  $\alpha_1 = \dots = \alpha_r = 0$ . The polynomials  $f_1, \dots, f_r$  are then linearly independent modulo  $I$ . Finally, since  $\dim \mathbb{K}[x]/I = r$ , the set  $\{[f_1], \dots, [f_r]\}$  is a basis of  $\mathbb{K}[x]/I$ .  $\square$

### 3.2 Hankel operators of finite rank

After a short description of the dual space of the polynomial space  $\mathbb{K}[x]$ , we associate a Hankel operator to any of its elements. The Hankel operator is our main theoretical object to study the properties of its linear form. The kernel of the Hankel operator is fundamental in characterizing the linear form. The linear operator obtained from the Hankel operator and defined on the quotient of the space  $\mathbb{K}[x]$  by this kernel is of particular interest.

#### Linear forms

The set of  $\mathbb{K}$ -linear forms from  $\mathbb{K}[x]$  to  $\mathbb{K}$  is denoted by  $\mathbb{K}[x]^*$  and called the *dual space* of  $\mathbb{K}[x]$ . Typical examples of linear forms on  $\mathbb{K}[x]$  are the evaluations  $\mathbb{1}_\xi$  at a point  $\xi$  of  $\mathbb{K}^n$ . They are defined by

$$\mathbb{1}_\xi : \mathbb{K}[x] \rightarrow \mathbb{K}, p \mapsto p(\xi).$$

Other examples of linear forms on  $\mathbb{K}[x]$  are given by linear combinations of evaluations

$$\Lambda : \mathbb{K}[x] \rightarrow \mathbb{K}, p \mapsto \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}(p),$$

with  $a_j \in \mathbb{K} \setminus \{0\}$  and  $\xi_j \in \mathbb{K}^n$ , or, when  $\mathbb{K} = \mathbb{R}$ , by the integration over a domain  $\mathcal{D} \subset \mathbb{R}^n$

$$\Omega : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \int_{\mathcal{D}} p(x) dx.$$

Notice that a linear combination of evaluations corresponds to a discrete domain of integration.

#### Construction of Hankel operators

For any linear form  $\Lambda \in \mathbb{K}[x]^*$ , the associated *Hankel operator*  $\widehat{\mathcal{H}}$  is the linear operator

$$\widehat{\mathcal{H}} : \mathbb{K}[x] \rightarrow \mathbb{K}[x]^*, p \mapsto p \star \Lambda,$$

where  $p \star \Lambda : \mathbb{K}[x] \rightarrow \mathbb{K}, q \mapsto \Lambda(pq)$ . The kernel of  $\widehat{\mathcal{H}}$

$$I_\Lambda = \{p \in \mathbb{K}[x] \mid p \star \Lambda = 0\}$$

is an ideal of  $\mathbb{K}[x]$  and its image is included in the subspace  $(I_\Lambda)^\perp = \{\Omega \in \mathbb{K}[x]^* \mid \Omega(p) = 0 \forall p \in I_\Lambda\} \subset \mathbb{K}[x]^*$ .

Let  $\pi$  be the natural morphism from  $\mathbb{K}[x]$  onto the quotient space  $\mathbb{K}[x]/I_\Lambda$ , which associates to any polynomial  $p \in \mathbb{K}[x]$  its class  $[p]$  modulo  $I_\Lambda$

$$\pi : \mathbb{K}[x] \rightarrow \mathbb{K}[x]/I_\Lambda, p \mapsto [p].$$

The linear operator  $\widehat{\mathcal{H}}$  factors through the morphism  $\mathring{\mathcal{H}} : \mathbb{K}[x]/I_\Lambda \rightarrow \mathbb{K}[x]^*$  defined by  $\widehat{\mathcal{H}} = \mathring{\mathcal{H}} \circ \pi$ . Since  $\mathring{\mathcal{H}}$  is an isomorphism from  $\mathbb{K}[x]/I_\Lambda$  to  $\widehat{\mathcal{H}}(\mathbb{K}[x])$ , the ideal  $I_\Lambda$  is zero-dimensional if and only if the rank of the Hankel operator  $\mathcal{H}$  is finite, in which case  $\dim \mathbb{K}[x]/I_\Lambda = \text{rank } \widehat{\mathcal{H}}$ .

Assume now that the rank of the Hankel operator is finite. The morphism  $\pi$  induces the morphism

$$\pi_* : (\mathbb{K}[x]/I_\Lambda)^* \rightarrow (I_\Lambda)^\perp, \Omega \mapsto \Omega \circ \pi,$$

which is an isomorphism [32, Proposition 7.9]. There exists then a linear operator  $\mathcal{H}$  from the quotient space  $\mathbb{K}[x]/I_\Lambda$  to its dual space  $(\mathbb{K}[x]/I_\Lambda)^*$  defined by

$$\mathcal{H} : \mathbb{K}[x]/I_\Lambda \rightarrow (\mathbb{K}[x]/I_\Lambda)^*, [p] \mapsto (\pi_*)^{-1} \circ \iota \circ \mathring{\mathcal{H}}([p]),$$

where  $\iota$  is the natural inclusion from  $\widehat{\mathcal{H}}(\mathbb{K}[x])$  to  $(I_\Lambda)^\perp$ .

$$\begin{array}{ccccccc} \mathbb{K}[x] & \xrightarrow{\widehat{\mathcal{H}}} & \widehat{\mathcal{H}}(\mathbb{K}[x]) & \xrightarrow{\iota} & (I_\Lambda)^\perp & \xrightarrow{(\pi_*)^{-1}} & (\mathbb{K}[x]/I_\Lambda)^* \\ & \searrow \pi & \nearrow \mathring{\mathcal{H}} & & & \nearrow \mathcal{H} & \\ & & \mathbb{K}[x]/I_\Lambda & & & & \end{array}$$

Figure 9: Commutative diagram representing the construction of the linear operator  $\mathcal{H}$ .

The commutative diagram in Figure 9 helps to check that the linear operator  $\mathcal{H}$  is one-to-one by construction. Since  $\dim \mathbb{K}[x]/I_\Lambda = \dim (\mathbb{K}[x]/I_\Lambda)^*$ ,  $\mathcal{H}$  is furthermore an isomorphism. Notice that this implies that  $\widehat{\mathcal{H}}(\mathbb{K}[x]) = (I_\Lambda)^\perp$ .

## Matrices of Hankel operators

Given a set  $B = \{b_1, \dots, b_r\} \subset \mathbb{K}[x]$  and a polynomial  $p \in \mathbb{K}[x]$ , we introduce the matrix

$$H_p^B = (\Lambda(b_i b_j p))_{1 \leq i, j \leq r}.$$

We relate now the matrix  $H_p^B$ , starting with the case  $p = 1$ , to the linear operator  $\mathcal{H}$  obtained from the Hankel operator  $\widehat{\mathcal{H}}$  following the commutative diagram in Figure 9. For ease of notation, we also denote by  $B$  the set  $\{[b_1], \dots, [b_r]\}$  of classes modulo  $I_\Lambda$ .

**Theorem 3.6.** Assume that  $\text{rank } \widehat{\mathcal{H}} = r < \infty$  and let  $B = \{b_1, \dots, b_r\} \subset \mathbb{K}[x]$ . Then  $B$  is a basis of  $\mathbb{K}[x]/I_\Lambda$  if and only if the matrix  $H_1^B$  is invertible, in which case the matrix  $H_1^B$  is the matrix of the linear operator  $\mathcal{H}$  in the basis  $B$  and its dual basis  $B^*$  in  $(\mathbb{K}[x]/I_\Lambda)^*$ .

*Proof.* Let  $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{K}^r$  such that  $\alpha_1 b_1 + \dots + \alpha_r b_r \equiv 0 \pmod{I_\Lambda}$ . Using the definition of  $I_\Lambda$ , we get

$$\alpha_1 \Lambda(b_1 b_i) + \dots + \alpha_r \Lambda(b_r b_i) = 0 \quad \forall i = 1, \dots, r.$$

With the help of the matrix  $H_1^B = (\Lambda(b_i b_j))_{1 \leq i, j \leq r}$ , this equality can be rewritten as  $H_1^B \alpha = 0$ . Assuming that the matrix  $H_1^B$  is invertible leads to  $(\alpha_1, \dots, \alpha_r) = (0, \dots, 0)$ . Thus, the polynomials  $b_1, \dots, b_r$  are linearly independent modulo  $I_\Lambda$ . Since  $\text{Card } B = \dim \mathbb{K}[x]/I_\Lambda$ , the set  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ .

Conversely, assume that  $B$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ . The  $\mathbb{K}$ -vector space  $\mathbb{K}[x]$  is then the direct sum of the span of  $B$  and the vector space  $I_\Lambda$  so that  $\widehat{\mathcal{H}}(\mathbb{K}[x]) = \widehat{\mathcal{H}}(\text{Span } B)$ . For each  $b_i \in B$ , let  $b_i^*$  be the linear form in the dual space  $(\text{Span } B)^*$  such that  $b_i^*(b_k) = \delta_{ik}$ . We have

$$\widehat{\mathcal{H}}(b_j) = \sum_{i=1}^r \Lambda(b_j b_i) b_i^* \quad \forall b_j \in B \quad (3.2)$$

since

$$\widehat{\mathcal{H}}(b_j)(b_k) = \Lambda(b_j b_k) = \left( \sum_{i=1}^r \Lambda(b_j b_i) b_i^* \right)(b_k) \quad \forall b_k \in B,$$

Equation (3.2) shows that  $\widehat{\mathcal{H}}(\text{Span } B) \subset \text{Span}(b_1^*, \dots, b_r^*)$ . Since  $\text{rank } \widehat{\mathcal{H}} = r$ , the set  $\{b_1^*, \dots, b_r^*\}$  is a basis of  $\widehat{\mathcal{H}}(\mathbb{K}[x])$ , that is a basis of  $(I_\Lambda)^\perp$ . Thanks to the isomorphism  $\pi_*$ , the set  $\{b_1^*, \dots, b_r^*\}$  is identified to the dual basis  $B^*$  in  $(\mathbb{K}[x]/I_\Lambda)^*$  of the basis  $B$  in  $\mathbb{K}[x]/I_\Lambda$ . Thus, the matrix of the linear operator  $\mathcal{H}$  in the basis  $B$  and its dual basis  $B^*$  is the matrix  $H_1^B$ . Since  $\mathcal{H}$  is an isomorphism, the matrix  $H_1^B$  is invertible.  $\square$

Assume that the polynomials  $b_1, \dots, b_r$  are chosen such that  $B$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ . Consider a polynomial  $p \in \mathbb{K}[x]$ . The matrix  $H_p^B = (\Lambda(b_i b_j p))_{1 \leq i, j \leq r}$  is similarly related to the linear operator

$$\mathcal{H}_p = \mathcal{H} \circ \mathcal{M}_p.$$

**Theorem 3.7.** Assume that  $\text{rank } \widehat{\mathcal{H}} = r < \infty$  and that  $B$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ . Then, for any polynomial  $p \in \mathbb{K}[x]$ , the matrix  $H_p^B$  is the matrix of the linear operator  $\mathcal{H} \circ \mathcal{M}_p$  in the basis  $B$  and its dual basis  $B^*$  and

$$H_p^B = H_1^B M_p^B,$$

where  $M_p^B$  is the matrix of the multiplication operator  $\mathcal{M}_p$  in the basis  $B$ .

*Proof.* Let  $p \in \mathbb{K}[x]$ . By definition of the multiplication operator  $\mathcal{M}_p$ , we have

$$\mathcal{H} \circ \mathcal{M}_p([b_j]) = \mathcal{H}([b_j p]) \quad \forall j = 1, \dots, r.$$

As in the proof of Theorem 3.6, we introduce the linear forms  $b_1^*, \dots, b_r^*$  on  $\text{Span } B$ . Equation (3.2) leads to the following equality

$$\widehat{\mathcal{H}}(b_j p) = \sum_{i=1}^r \Lambda(b_i b_j p) b_i^* \quad \forall j = 1, \dots, r. \quad (3.3)$$

Since  $\hat{\mathcal{H}}([b_j p]) = \hat{\mathcal{H}}(b_j p)$  for all  $b_j \in B$  and thanks to the isomorphism  $\pi_*$ , we deduce from (3.3) that the matrix  $H_1^B M_p^B$  of the operator  $\mathcal{H} \circ \mathcal{M}_p$  in the basis  $B$  and its dual basis  $B^*$  is  $(\Lambda(b_i b_j p))_{1 \leq i, j \leq r}$ . The latter is by definition the matrix  $H_p^B$ .  $\square$

The next result shows the change of basis relation for the matrices of the linear operators  $\mathcal{H}_p$ .

**Corollary 3.8.** *Let  $b_1, \dots, b_r$  be polynomials in  $\mathbb{K}[x]$  such that  $B = \{[b_1], \dots, [b_r]\}$  is basis of  $\mathbb{K}[x]/I_\Lambda$  and let  $\tilde{b}_1, \dots, \tilde{b}_r$  be polynomials in  $\mathbb{K}[x]$  given by*

$$\begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_r \end{pmatrix} = Q \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix},$$

where  $Q = (q_{ij})_{1 \leq i, j \leq r}$  is an invertible matrix. Then  $\tilde{B} = \{[\tilde{b}_1], \dots, [\tilde{b}_r]\}$  is a basis of  $\mathbb{K}[x]/I_\Lambda$  and, given a polynomial  $p \in \mathbb{K}[x]$ , the matrix  $H_p^{\tilde{B}}$  satisfies

$$H_p^{\tilde{B}} = Q H_p^B Q^t.$$

*Proof.* On one hand, for every  $i = 1, \dots, r$ , we have by definition  $\tilde{b}_i = \sum_{k=1}^r q_{ik} b_k$ . Applying this to the entries of the matrix  $H_p^{\tilde{B}}$  brings

$$\Lambda(\tilde{b}_i \tilde{b}_j p) = \sum_{k=1}^r q_{ik} \Lambda(b_k \tilde{b}_j p) = \sum_{1 \leq k, \ell \leq r} q_{ik} q_{j\ell} \Lambda(b_k b_\ell p) \quad \forall 1 \leq i, j \leq r.$$

Thus, we have

$$H_p^{\tilde{B}} = \left( \sum_{1 \leq k, \ell \leq r} q_{ik} q_{j\ell} \Lambda(b_k b_\ell p) \right)_{1 \leq i, j \leq r}.$$

On the other hand, we have

$$\begin{aligned} Q H_p^B Q^t &= (q_{ij})_{1 \leq i, j \leq r} (\Lambda(b_i b_j p))_{1 \leq i, j \leq r} (q_{ij})_{1 \leq i, j \leq r}^t \\ &= (q_{ij})_{1 \leq i, j \leq r} \left( \sum_{\ell=1}^r q_{j\ell} \Lambda(b_i b_\ell p) \right)_{1 \leq i, j \leq r} \\ &= \left( \sum_{1 \leq k, \ell \leq r} q_{ik} q_{j\ell} \Lambda(b_k b_\ell p) \right)_{1 \leq i, j \leq r}. \end{aligned}$$

Comparing the two matrices leads to the expected result

$$H_p^{\tilde{B}} = Q H_p^B Q^t.$$

$\square$

For any polynomial  $p \in \mathbb{K}[x]$ , we can deduce from Theorem 3.6 and Theorem 3.7 a way to compute the values of the polynomial  $p$  on the variety  $V_{\mathbb{C}}(I_\Lambda)$ . Indeed, since  $B$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ , Theorem 3.6 says that the matrix  $H_1^B$  is invertible and Theorem 3.7 says that  $H_p^B = H_1^B M_p^B$ . Thus, the eigenvalue problem of the matrix  $M_p^B$  is equivalent to the generalized eigenvalue problem of the pair of matrices  $(H_p^B, H_1^B)$ . Theorem 3.3 leads then to the following result:

**Corollary 3.9.** Assume that  $\text{rank } \widehat{\mathcal{H}} = r < \infty$ . Let  $B$  be a basis of  $\mathbb{K}[x]/I_\Lambda$ , let  $p \in \mathbb{K}[x]$  and let  $H_1^B$  and  $H_p^B$  be the matrices as defined above. Then the generalized eigenvalues of the pair  $(H_p^B, H_1^B)$  are the values of  $p$  on  $V_{\mathbb{C}}(I_\Lambda)$ .

The eigenvalue method for solving polynomial equations uses the eigenvalues of the multiplication operators  $\mathcal{M}_p$  for appropriate polynomials  $p \in \mathbb{K}[x]$ . Similarly, Corollary 3.9 relates the generalized eigenvalues of  $(\mathcal{H}_p, \mathcal{H})$  to the variety  $V_{\mathbb{C}}(I_\Lambda)$ .

Another characterization of the zero-dimensional ideal  $I_\Lambda$  is given in the next result. Its proof is based on the one of Theorem 5.19 in [74].

**Proposition 3.10.** Let  $\Lambda$  be a linear form on  $\mathbb{K}[x]$  with  $\text{rank } \widehat{\mathcal{H}} = r < \infty$ . Let  $b_1, \dots, b_r$  be polynomials in  $\mathbb{K}[x]$  such that  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ . Assume that the polynomials  $b_1, \dots, b_r$  are in  $\mathbb{K}[x]_{\leq \delta-1}$ . Then the ideal generated by the kernel of the restriction  $\mathcal{H}^{(\delta)}$  of  $\widehat{\mathcal{H}}$  to  $\mathbb{K}[x]_{\leq \delta}$  is the ideal  $I_\Lambda$ , that is

$$I_\Lambda = \left( \ker \mathcal{H}^{(\delta)} \right).$$

*Proof.* Since  $\mathcal{H}^{(\delta)}$  is a restriction of the Hankel operator  $\widehat{\mathcal{H}}$ , we have  $\ker \mathcal{H}^{(\delta)} \subset I_\Lambda$ , implying

$$\left( \ker \mathcal{H}^{(\delta)} \right) \subset I_\Lambda.$$

To show the reverse inclusion, we first show using induction on  $|\beta|$  that, for all  $\beta \in \mathbb{N}^n$ ,

$$x^\beta \in \text{Span}(b_1, \dots, b_r) + \left( \ker \mathcal{H}^{(\delta)} \right). \quad (3.4)$$

Since  $b_1, \dots, b_r$  are polynomials in  $\mathbb{K}[x]_{\leq \delta-1}$  such that  $\{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{K}[x]/I_\Lambda$ , those polynomials form a basis of the supplementary of  $\ker \mathcal{H}^{(\delta)}$  in  $\mathbb{K}[x]_{\leq \delta}$ . Thus, if  $|\beta| \leq \delta$ , (3.4) holds. Assume  $|\beta| \geq \delta + 1$ . Then

$$x^\beta = x_j x^\gamma \text{ with } |\gamma| = |\beta| - 1 \text{ and } j \in \{1, \dots, n\}.$$

By the induction assumption,

$$x^\beta = x_j \left( \sum_{i=1}^r \lambda_i b_i + q \right) = \sum_{i=1}^r \lambda_i x_j b_i + x_j q$$

with  $q \in (\ker \mathcal{H}^{(\delta)})$ . Since  $x_j b_i \in \mathbb{K}[x]_{\leq \delta}$  and  $x_j q \in (\ker \mathcal{H}^{(\delta)})$ , we have

$$x^\beta \in \text{Span}(b_1, \dots, b_r) + \left( \ker \mathcal{H}^{(\delta)} \right).$$

Thus, (3.4) holds for all  $\beta \in \mathbb{N}^n$ .

Take  $p \in I_\Lambda$ . In view of (3.4), we can write  $p = u + v$  with  $u \in \text{Span}(b_1, \dots, b_r)$  and  $v \in (\ker \mathcal{H}^{(\delta)}) \subset I_\Lambda$ . Hence,  $p - v \in \text{Span}(b_1, \dots, b_r) \cap I_\Lambda$ , which implies  $p - v = 0$ . Therefore,  $p \in (\ker \mathcal{H}^{(\delta)})$ , which concludes the proof for equality  $I_\Lambda = (\ker \mathcal{H}^{(\delta)})$ .  $\square$

Proposition 3.10 prepares for the *Flat extension theorem* in Section 3.4: it says that the information concerning  $I_\Lambda$ , the kernel of the Hankel operator  $\widehat{\mathcal{H}}$  on the whole polynomial space  $\mathbb{K}[x]$ , is contained in the kernel of  $\mathcal{H}^{(\delta)}$ , the restriction of  $\widehat{\mathcal{H}}$  to  $\mathbb{K}[x]_{\leq \delta}$  for a well-chosen degree  $\delta$ . In other words, if one can choose a degree  $\delta$  as in Proposition 3.10, then the study of  $I_\Lambda$  can be done thanks to the matrix  $H_1^{B^{(\delta)}} = (\Lambda(b_i b_j))_{1 \leq i, j \leq r_\delta}$ , where  $B^{(\delta)} = \{b_1, \dots, b_{r_\delta}\}$  is a basis of  $\mathbb{K}[x]_{\leq \delta}$ .

### 3.3 Kernels of Hankel operators and radical ideals

We show here how to recover a linear form from its Hankel operator, when this latter is of finite rank. We assume now that the zero-dimensional ideal  $I_\Lambda$  is radical.

Thanks to the structure theorem [32, Theorem 7.34], when the ideal  $I_\Lambda$  is zero-dimensional, the space  $(I_\Lambda)^\perp$  is the direct sum of subspaces (of dimension  $\geq 1$ ) and each subspace depends on a point of  $V_{\mathbb{C}}(I_\Lambda)$ . Assume that the ideal is furthermore radical. Since the spaces  $(I_\Lambda)^\perp$  and  $\mathbb{K}[x]/I_\Lambda$  are isomorphic,

$$\dim (I_\Lambda)^\perp = \dim \mathbb{K}[x]/I_\Lambda = r.$$

Thus, each subspace is of dimension 1 and is spanned by the evaluation  $\mathbb{1}_{\xi_j}$  at a point  $\xi_j$  of  $V_{\mathbb{C}}(I_\Lambda)$ . The linear form  $\Lambda$ , which belongs to  $(I_\Lambda)^\perp$ , can then be expressed in a simple way as described in the following result:

**Theorem 3.11.** *Assume that  $\text{rank } \widehat{\mathcal{H}} = r < \infty$ . Then the following conditions are equivalent:*

- *The ideal  $I_\Lambda$  is radical.*
- $\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$  with  $a_j \in \mathbb{C} \setminus \{0\}$  and  $\xi_j \in \mathbb{C}^n$  pairwise distinct.

*If any of the above conditions is satisfied, then  $V_{\mathbb{C}}(I_\Lambda) = \{\xi_1, \dots, \xi_r\}$ .*

*Proof.* Assume that  $\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$  with  $a_j \in \mathbb{C} \setminus \{0\}$  and  $\xi_j \in \mathbb{C}^n$  pairwise distinct. Let

$I(\xi_1, \dots, \xi_r)$  be the ideal of polynomials vanishing at the points  $\xi_1, \dots, \xi_r$ . By the *Strong Nullstellensatz* [23, Chapter 1.4], the ideal  $I(\xi_1, \dots, \xi_r)$  is radical. Let  $f \in I(\xi_1, \dots, \xi_r)$ , then

$$\Lambda(f) = \sum_{j=1}^r a_j f(\xi_j) = 0.$$

Thus,  $I(\xi_1, \dots, \xi_r) \subset I_\Lambda$ . Assume that there exists  $q \in I_\Lambda$  such that  $q \notin I(\xi_1, \dots, \xi_r)$ , which means that there exists  $\xi_j$  such that  $q(\xi_j) \neq 0$ . Since the ideal  $I(\xi_1, \dots, \xi_r)$  is zero-dimensional and radical, Corollary 3.5 implies that there exist polynomials  $f_1, \dots, f_r$  satisfying  $f_i(\xi_j) = \delta_{ij}$ . Then we have  $\Lambda(qf_i) = a_i q(\xi_i) \neq 0$ , which leads to a contradiction. Thus,  $I_\Lambda = I(\xi_1, \dots, \xi_r)$  and the ideal  $I_\Lambda$  is radical.  $\square$

The polynomials introduced in Corollary 3.5 give particular bases of the quotient space  $\mathbb{K}[x]/I_\Lambda$ .

**Corollary 3.12.** *Assume that the ideal  $I_\Lambda$  is zero-dimensional and radical and that  $V_{\mathbb{C}}(I_\Lambda) = \{\xi_1, \dots, \xi_r\} \subset \mathbb{K}^n$ . Let  $f_1, \dots, f_r$  be polynomials as in Corollary 3.5 and denote by  $C$  the basis  $\{[f_1], \dots, [f_r]\}$  of  $\mathbb{K}[x]/I_\Lambda$ . Then, for any polynomial  $p \in \mathbb{K}[x]$ ,*

$$H_p^C = (\Lambda(f_i f_j p))_{1 \leq i, j \leq r} = \begin{pmatrix} a_1 p(\xi_1) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_r p(\xi_r) \end{pmatrix}.$$



The assumption  $V_{\mathbb{C}}(I_{\Lambda}) \subset \mathbb{K}^n$  in this last result leads to  $a_j \in \mathbb{K} \setminus \{0\}$  since  $\Lambda(f_j) = a_j$  for all  $j = 1, \dots, r$ .

In the case  $\mathbb{K} = \mathbb{R}$ , a sufficient condition for  $V_{\mathbb{C}}(I_{\Lambda}) \subset \mathbb{R}^n$  is given in [74, Lemma 5.2] by the positive semidefiniteness of the Hankel operator  $\widehat{\mathcal{H}}$  (see also [24, 73, 84]). Combining this result with Theorem 3.11, we get:

**Proposition 3.13.** *Let  $\Lambda$  be a linear form on  $\mathbb{R}[x]$ . If the Hankel operator  $\widehat{\mathcal{H}}$  is positive semidefinite and  $\text{rank } \widehat{\mathcal{H}} = r < \infty$ , then the ideal  $I_{\Lambda}$  is radical and  $V_{\mathbb{C}}(I_{\Lambda}) \subset \mathbb{R}^n$ . Thus,*

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$$

with  $a_j > 0$  and  $\xi_j \in \mathbb{R}^n$  pairwise distinct.

The fact that the coefficients  $a_1, \dots, a_r$  are positive in Proposition 3.13 is a consequence of Corollary 3.12: taking  $p = 1$ , since the Hankel operator  $\widehat{\mathcal{H}}$  is positive semidefinite, we get  $a_j > 0$  for all  $j = 1, \dots, r$ .

Conversely, if the coefficients  $a_1, \dots, a_r$  are positive, then the Hankel operator  $\widehat{\mathcal{H}}$  is positive semidefinite as well as the linear operators  $\mathcal{H}$  and  $\mathcal{H}_p$  for every  $p \in \mathbb{R}[x]$  such that  $p(\xi_j) \geq 0$  for all  $j = 1, \dots, r$ .

### 3.4 Flat extension

Our approach to cubatures is based on the extension of a linear form on  $\mathbb{R}[x]_{\leq 2\delta}$  to  $\mathbb{R}[x]$ . The existence of such an appropriate extension is secured by *Tchakaloff's theorem* (Theorem 2.1). We present here the criteria that will be used to actually determine them.

Let  $\Lambda^{(\delta)}$  be a linear form on  $\mathbb{R}[x]_{\leq 2\delta}$ . Similarly to Section 3.2, we associate with  $\Lambda^{(\delta)}$  the Hankel operator

$$\mathcal{H}^{(\delta)} : \mathbb{R}[x]_{\leq \delta} \rightarrow \mathbb{R}[x]_{\leq \delta}^*, p \mapsto p \star \Lambda^{(\delta)}.$$

Its matrix in any basis  $B^{(\delta)} = \{b_1, \dots, b_{r_\delta}\}$  of  $\mathbb{R}[x]_{\leq \delta}$  and its dual basis is

$$H_1^{B^{(\delta)}} = (\Lambda(b_i b_j))_{1 \leq i, j \leq r_\delta}.$$

A linear form  $\Lambda^{(\delta+\kappa)}$  on  $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$  is an *extension* of a given linear form  $\Lambda^{(\delta)}$  on  $\mathbb{R}[x]_{\leq 2\delta}$  if its restriction to  $\mathbb{R}[x]_{\leq 2\delta}$  is  $\Lambda^{(\delta)}$ , that is if

$$\Lambda^{(\delta+\kappa)}(p) = \Lambda^{(\delta)}(p) \quad \forall p \in \mathbb{R}[x]_{\leq 2\delta}.$$

A linear form  $\Lambda^{(\delta+\kappa)}$  on  $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$  is a *flat extension* of a given linear form  $\Lambda^{(\delta)}$  if furthermore the rank of its associated Hankel operator  $\mathcal{H}^{(\delta+\kappa)}$  is the rank of the Hankel operator  $\mathcal{H}^{(\delta)}$  associated with the linear form  $\Lambda^{(\delta)}$ . In this case,  $\mathcal{H}^{(\delta+\kappa)}$  is positive semidefinite if and only if  $\mathcal{H}^{(\delta)}$  is positive semidefinite.

The flat extension theorem below was proved in terms of moment matrices, that is matrices  $H_1^{B^{(\delta)}}$  for a monomial basis  $B^{(\delta)} = \{x^\alpha \mid |\alpha| \leq \delta\}$  (see [24, 25, 73, 74]) or in terms of submatrices of those [75].

**Theorem 3.14** (Flat extension theorem). *Let  $\Lambda^{(\delta)}$  be a linear form on  $\mathbb{R}[x]_{\leq 2\delta}$ . Assume that  $\Lambda^{(\delta)}$  is a flat extension of its restriction to  $\mathbb{R}[x]_{\leq 2\delta-2}$ . Then there exists a unique flat extension of  $\Lambda^{(\delta)}$  to  $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$  for all  $\kappa \geq 1$ .*

Assume furthermore the positive semidefiniteness of the Hankel operator  $\mathcal{H}^{(\delta)}$  associated with the linear form  $\Lambda^{(\delta)}$  of the *Flat extension theorem*. The Hankel operator  $\widehat{\mathcal{H}}$  associated with the unique flat extension  $\Lambda$  on  $\mathbb{R}[x]$  of  $\Lambda^{(\delta)}$  is also positive semidefinite. By Proposition 3.13, this linear form  $\Lambda$  takes a particular form. Proposition 3.10 implies that  $V_{\mathbb{C}}(I_{\Lambda}) = V_{\mathbb{C}}(\ker \mathcal{H}^{(\delta)})$ .

**Corollary 3.15.** *Let  $\Lambda^{(\delta)}$  be a linear form on  $\mathbb{R}[x]_{\leq 2\delta}$ . Assume that  $\Lambda^{(\delta)}$  is a flat extension of its restriction to  $\mathbb{R}[x]_{\leq 2\delta-2}$  and that its associated Hankel operator  $\mathcal{H}^{(\delta)}$  is positive semidefinite. Then the linear form  $\Lambda$  on  $\mathbb{R}[x]$  defined by*

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j},$$

*with  $r = \text{rank } \mathcal{H}^{(\delta)}$ ,  $a_j > 0$  and  $\{\xi_1, \dots, \xi_r\} = V_{\mathbb{C}}(\ker \mathcal{H}^{(\delta)}) \subset \mathbb{R}^n$ , is the unique flat extension of  $\Lambda^{(\delta)}$  to  $\mathbb{R}[x]$ .*

In our application, we are interested in linear forms on  $\mathbb{R}[x]$  of the form

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$$

with  $r > 0$ ,  $a_j > 0$  and  $\xi_j \in \mathbb{R}^n$  pairwise distinct. In addition, we expect that the points  $\xi_j$  lie in a prescribed semialgebraic set  $K$  (e.g. a triangle, a square or a disk) defined by

$$K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_s(x) \geq 0\},$$

where  $g_1, \dots, g_s$  are polynomials in  $\mathbb{R}[x]$ .

As described in the next result, this last property can be checked thanks to the positive semidefiniteness of the linear operators  $\mathcal{H}_{g_k}^{(\delta)} : \mathbb{R}[x]_{\leq \delta} \rightarrow \mathbb{R}[x]_{\leq \delta}^*, p \mapsto (g_k p) \star \Lambda^{(\delta)}$  for all  $k = 1, \dots, s$ . This result was proved in terms of *localizing matrices*, that is the matrices of the linear operators  $\mathcal{H}_{g_k}^{(\delta)}$  in a monomial basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$  [74, Theorem 5.23],[25].

**Proposition 3.16.** *Consider the semialgebraic set*

$$K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_s(x) \geq 0\}.$$

*Let  $\kappa \in \mathbb{N}$  be such that  $\kappa \geq 1$  and  $2\kappa \geq \deg g_k$  for all  $k = 1, \dots, s$ . Let  $\Lambda^{(\delta)}$  be a linear form on  $\mathbb{R}[x]_{\leq 2\delta}$  and let  $\Lambda^{(\delta+\kappa)} : \mathbb{R}[x]_{\leq 2\delta+2\kappa} \rightarrow \mathbb{R}$  be a flat extension of  $\Lambda^{(\delta)}$ . Assume that  $\mathcal{H}^{(\delta)}$  and  $\mathcal{H}_{g_k}^{(\delta)}$  are positive semidefinite for all  $k = 1, \dots, s$ . Then the linear form  $\Lambda$  on  $\mathbb{R}[x]$  defined by*

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j},$$

*with  $r = \text{rank } \mathcal{H}^{(\delta)}$ ,  $a_j > 0$  and  $\{\xi_1, \dots, \xi_r\} = V_{\mathbb{C}}(\ker \mathcal{H}^{(\delta)}) \subset \mathbb{R}^n$ , is the unique flat extension of  $\Lambda^{(\delta+\kappa)}$  to  $\mathbb{R}[x]$  and is such that  $\xi_j \in K$ .*

## 4 Fraction-free diagonalization of positive semidefinite matrices

A symmetric matrix whose entries are polynomials in some parameters  $h_1, \dots, h_t$  over a field extension of  $\mathbb{Q}$  is considered. Based on a characterization of positive semidefinite matrices in terms of their principal minors, the goal is to diagonalize the matrix on its locus of semi-positivity. We show that Bareiss' fraction-free elimination scheme can run without pivoting to produce the expected diagonalization. The present approach is for exact arithmetic.

In this section, we present different algorithms that lead step by step to the central computational ingredient for the moment matrix approach to cubature. Starting with a characterization of positive semidefinite matrices in terms of principal minors, we deduce a first recursive procedure to check the positive semidefiniteness of symmetric matrices with real coefficients based on an analysis of the sign of the pivot at each step of a LU-decomposition. Since we are interested in matrices with polynomial entries, a fraction-free LU-decomposition is preferable. We show that Bareiss' scheme for the triangularization of a given matrix [4, 5, 41], and more specifically its revisited version for the diagonalization of a given symmetric matrix [58, 95], is compatible with our first procedure. This leads to a second procedure for symmetric matrices with entries in an integral domain of  $\mathbb{R}$ . In the case of polynomial entries over a field extension of  $\mathbb{Q}$ , we introduce a distinction between the different possible cases in the characterization in terms of principal minors. This leads to a third procedure whose output is a set of triplets: each triplet represents conditions under which the matrix in input is positive semidefinite. The number of triplets grows exponentially with the size of the matrix. When the matrix comes from the search for cubatures, this number is reduced by the problem itself. This leads to the last procedure of the section: Algorithm 4.7 [Diagonalization & Positivity with Rank Constraints].

An alternative approach, based on [6, Chapter 8.2.4], to describe the locus of semi-positivity of a  $c \times c$  symmetric matrix  $A$  with entries in  $\mathbb{R}[h] = \mathbb{R}[h_1, \dots, h_t]$  is to express it as a semialgebraic set. Let  $\chi_A$  be the characteristic polynomial of  $A$

$$\chi_A = X^c - \chi_1 X^{c-1} + \dots + (-1)^k \chi_k X^{c-k} + \dots + (-1)^c \chi_c,$$

where the coefficients  $\chi_k$  are therefore in  $\mathbb{R}[h]$  for all  $k = 1, \dots, c$ . Then the set of points  $\bar{h}$  in  $\mathbb{R}^t$  such that  $A$  is positive semidefinite is the semialgebraic set

$$\{\bar{h} \in \mathbb{R}^t \mid \chi_1(\bar{h}) \geq 0, \dots, \chi_c(\bar{h}) \geq 0\}.$$

If the rank  $r$  is furthermore known, then this semialgebraic set becomes

$$\{\bar{h} \in \mathbb{R}^t \mid \chi_1(\bar{h}) \geq 0, \dots, \chi_{c-r-1}(\bar{h}) \geq 0, \chi_{c-r}(\bar{h}) \geq 0, \chi_{c-r+1}(\bar{h}) = \dots = \chi_c(\bar{h}) = 0\}.$$

Finding the points in such a semialgebraic set is a central question in computational real algebraic geometry [6]. When one is interested in finding a point in this semialgebraic set, a numerical solution is provided by SemiDefinite Programming solvers. They can handle symmetric matrices with a high number of variables. Recently in [56], an exact algorithm has been proposed that decides whether this semialgebraic set is empty or not and, in the negative case, exhibits an algebraic representation of a point in this semialgebraic set. We refer also to [56, Chapter 1.2] for a state-of-the-art on this topic. In comparison, Algorithm 4.7 [Diagonalization & Positivity with Rank Constraints] provides the whole set of points and a change of basis such that the symmetric matrix becomes diagonal. However, this approach is suitable when the size of the symmetric matrix is small or reduced by block diagonalization: this is the case in the proposed search for symmetric cubatures.

## 4.1 Positive semidefinite matrices

A symmetric matrix  $A = (a_{ij})_{1 \leq i, j \leq c}$  is called *positive semidefinite* if

$$X^t A X \geq 0 \quad \forall X \in \mathbb{R}^c.$$

It is called *positive definite* if it satisfies furthermore

$$X^t A X = 0 \Leftrightarrow X = 0.$$

Given any matrix  $A = (a_{ij})_{1 \leq i, j \leq c}$ , the determinant of its submatrix  $A_J = (a_{ij})_{(i,j) \in J^2}$  with  $J \subset \{1, \dots, c\}$  is called a *principal minor*. If the subset  $J$  satisfies furthermore  $J = \{1, \dots, k\}$  with  $1 \leq k \leq c$ , then this determinant is called *leading principal minor*. A characterization of a positive definite (resp. semidefinite) matrix  $A$  is given in terms of its *leading principal minors* (resp. *principal minors*) as follows.

**Theorem 4.1** ([108, Theorem 7.2]). *Let  $A$  be a symmetric matrix. Then*

- *$A$  is positive definite if and only if every leading principal minor of  $A$  is positive.*
- *$A$  is positive semidefinite if and only if every principal minor of  $A$  is nonnegative.*

With the help of Theorem 4.1, a recursive procedure based on Gaussian elimination that checks whether a symmetric matrix is positive semidefinite or not is given by the following result.

**Corollary 4.2.** *Let  $A = (a_{ij})_{1 \leq i, j \leq c}$  be a symmetric matrix.*

1. *If  $A$  is a  $1 \times 1$  matrix, then  $A$  is positive semidefinite if and only if  $a_{11} \geq 0$ .*
2. *Otherwise:*
  - (a) *If  $a_{11} < 0$ , then  $A$  is not positive semidefinite.*
  - (b) *If  $a_{11} = 0$ , then  $A$  is positive semidefinite if and only if  $a_{1j} = 0$  for every  $j = 1, \dots, c$  and the submatrix obtained by deleting the first row and the first column is positive semidefinite.*
  - (c) *If  $a_{11} > 0$ , then for each  $i > 1$  subtract  $\frac{a_{i1}}{a_{11}}$  times row 1 from row  $i$  and delete the first row and the first column. Then  $A$  is positive semidefinite if and only if the resulting matrix is positive semidefinite.*

*Proof.* Since the coefficient  $a_{11}$  is a principal minor,  $a_{11} \geq 0$  is a necessary condition for the positive semidefiniteness of the matrix  $A$ . In the particular case of a  $1 \times 1$  matrix  $A$ , this is also a sufficient condition by Theorem 4.1. It remains now to study the cases 2.(b) and 2.(c).

Assuming that  $a_{11} = 0$ , then we distinguish two cases:

- If  $a_{1j} = 0$  for all  $j = 1, \dots, c$ , then every principal minor  $\det A_J$  with  $1 \in J$  is zero since the coefficients of a row of the matrix  $A_J$  are zero. As a consequence,  $A$  is positive semidefinite if and only if the matrix  $(a_{ij})_{2 \leq i, j \leq c}$  is positive semidefinite.

- If there is a nonzero coefficient in the first row of  $A$ , that is if there exists  $j$  such that  $a_{1j} \neq 0$ , then the principal minor  $\det \begin{pmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj} \end{pmatrix}$  is negative since

$$\det \begin{pmatrix} a_{11} & a_{1j} \\ a_{j1} & a_{jj} \end{pmatrix} = a_{11}a_{jj} - a_{1j}a_{j1} = -a_{1j}^2 < 0.$$

Then  $A$  is not positive semidefinite.

Assuming that  $a_{11} > 0$ , the row operations describe the matrix equality  $A = LU$ , that is

$$\begin{pmatrix} a_{11} & \dots & a_{1c} \\ \vdots & & \vdots \\ a_{c1} & \dots & a_{cc} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{a_{c1}}{a_{11}} & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1c} \\ 0 & \tilde{a}_{22} & \dots & \dots & \tilde{a}_{2c} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & \tilde{a}_{c2} & \dots & \dots & \tilde{a}_{cc} \end{pmatrix}$$

with  $\tilde{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j}$  for all  $(i, j) \in \mathbb{N}^2$  such that  $2 \leq i, j \leq c$ .

Since we have

$$\tilde{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j} = a_{ji} - \frac{a_{1i}}{a_{11}}a_{j1} = \tilde{a}_{ji} \quad \forall 2 \leq i, j \leq c,$$

the matrix  $\tilde{A} = (\tilde{a}_{ij})_{2 \leq i, j \leq c}$  is a symmetric matrix.

For every  $J \subset \{1, \dots, c\}$  with  $1 \in J$ , we also have

$$\det A_J = \det L_J \det U_J \text{ and } \det L_J = 1$$

so that the principal minor  $\det A_J$  of  $A$  is the principal minor  $\det U_J$  of  $U$ .

We have furthermore that

$$\det U_J = a_{11} \det U_{J \setminus \{1\}}.$$

Since  $a_{11}$  is positive, the principal minors of  $U$  are nonnegative if and only if the ones of the matrix  $U$  without the first row and the first column are nonnegative.  $\square$

A direct consequence is a recursive procedure based on Gaussian elimination that checks whether a symmetric matrix  $A = (a_{ij})_{1 \leq i, j \leq c}$  is positive definite or not is given by:

1. If  $A$  is a  $1 \times 1$  matrix, then  $A$  is positive definite if and only if  $a_{11} > 0$ .
2. Otherwise:
  - (a) If  $a_{11} \leq 0$ , then  $A$  is not positive definite.
  - (b) If  $a_{11} > 0$ , then for each  $i > 1$  subtract  $\frac{a_{i1}}{a_{11}}$  times row 1 from row  $i$  and delete the first row and the first column. Then  $A$  is positive definite if and only if the resulting matrix is positive definite.

When the matrix  $A$  depends polynomially on some parameters, one can add a case distinction at each step to provide the conditions on the parameters for the matrix to be positive semidefinite and compute the rank. However the recursive procedure in Corollary 4.2 runs then with rational functions whose numerators and denominators grow fast in degree. We therefore introduce a fraction-free variant in the next section.

## 4.2 Fraction-free triangularization and diagonalization

The fraction-free approach of Bareiss [4, 5, 41] to Gaussian elimination allows a better control of the growth of the entries, whether integers or polynomials, by dividing a known extraneous factor. The intermediate results obtained at each step of elimination can actually be expressed in terms of minors of the original matrix. This gives a clear idea on the growth and the specialization property of the results. The scheme was revisited in [58, 95] for the diagonalization of symmetric matrices. For symmetric positive definite matrices, the algorithm runs without any pivoting. Based on their characterization in terms of non-negativity of the principal minors, we show that for positive semidefinite matrices we can similarly avoid any pivoting. This section was written with the help of G. Labahn.

We deal with  $A = (a_{ij})_{1 \leq i, j \leq c}$  a  $c \times c$  matrix with entries in an integral domain  $R$ . The description of Bareiss' algorithm requires us to introduce the following matrices and minors.

By convention, we write  $A^{(0)} = A$  and  $a_{0,0}^{(-1)} = 1$ . Let  $\ell$  be an integer such that  $1 \leq \ell \leq c - 1$  and let

$$A^{(\ell)} = \begin{pmatrix} a_{1,1}^{(0)} & a_{1,2}^{(0)} & \cdots & \cdots & \cdots & \cdots & a_{1,c}^{(0)} \\ 0 & a_{2,2}^{(1)} & & & & & a_{2,c}^{(1)} \\ \vdots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & a_{\ell,\ell}^{(\ell-1)} & \cdots & \cdots & a_{\ell,c}^{(\ell-1)} \\ \vdots & & & 0 & a_{\ell+1,\ell+1}^{(\ell)} & \cdots & a_{\ell+1,c}^{(\ell)} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{c,\ell+1}^{(\ell)} & \cdots & a_{c,c}^{(\ell)} \end{pmatrix}$$

$$\text{with } a_{i,j}^{(\ell)} = \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,\ell} & a_{1,j} \\ \vdots & & \vdots & \vdots \\ a_{\ell,1} & \cdots & a_{\ell,\ell} & a_{\ell,j} \\ a_{i,1} & \cdots & a_{i,\ell} & a_{i,j} \end{pmatrix} \text{ for all } \ell < i, j \leq c.$$

**Theorem 4.3.** *Let  $\ell$  be an integer such that  $1 \leq \ell \leq c - 1$ .*

1. *If  $a_{k-1,k-1}^{(k-2)} \neq 0$  for all  $k = 1, \dots, \ell$ , then  $A^{(\ell)} = L_\ell \dots L_1 A$  with*

$$L_k = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & 0 & 1 & \ddots & & & \vdots \\ \vdots & & \vdots & -l_{k+1} & l_k & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -l_c & 0 & \cdots & 0 & l_k \end{pmatrix}$$

$$\text{with } l_k = \frac{a_{k,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}} \text{ and } l_j = \frac{a_{j,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}} \text{ for } j = k+1, \dots, c.$$

2. If  $A$  is furthermore symmetric, then the matrix  $\tilde{A}^{(\ell)} = L^{(\ell)} A L^{(\ell)t}$ , with  $L^{(\ell)} = L_\ell \dots L_1$ , is also symmetric. This matrix  $\tilde{A}^{(\ell)}$  is given by

$$\tilde{A}^{(\ell)} = \begin{pmatrix} \tilde{a}_{11}^{(0)} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \tilde{a}_{\ell,\ell}^{(\ell-1)} & 0 & \cdots & 0 \\ & & 0 & \tilde{a}_{\ell+1,\ell+1}^{(\ell)} & \cdots & \tilde{a}_{\ell+1,c}^{(\ell)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & 0 & \tilde{a}_{c,\ell+1}^{(\ell)} & \cdots & \tilde{a}_{c,c}^{(\ell)} \end{pmatrix}, \quad (4.1)$$

with  $\tilde{a}_{i,j}^{(k)} = a_{i,j}^{(k)} a_{k,k}^{(k-1)}$ .

3. If  $R \subset \mathbb{R}$  and  $A$  is furthermore positive semidefinite, then

- $a_{k,k}^{(k-1)} \neq 0$  for all  $k = 1, \dots, \ell$  implies  $a_{k,k}^{(k-1)} > 0$  for all  $k = 1, \dots, \ell$ .
- $a_{\ell+1,\ell+1}^{(\ell)}$  is either positive or zero. In this last case,  $a_{\ell+1,\ell+j}^{(\ell)} = 0$  for all  $1 \leq j \leq c - \ell$ .

*Proof.* 1. The first observation is the basis of Bareiss's algorithm [4, 5, 41].

2. The second observation was presented in [58, 95].

3. Let  $A_J$  be the  $(\ell+2) \times (\ell+2)$  leading principal submatrix of  $A$ . If we apply the adequately truncated  $L^{(\ell)}$  to  $A_J$ , we obtain the  $(\ell+2) \times (\ell+2)$  leading principal submatrix of  $A^{(\ell)}$ . We have then

$$\left(a_{\ell,\ell}^{(\ell-1)}\right)^2 \left(\prod_{k=1}^{\ell-1} a_{k,k}^{(k-1)}\right) \det(A_J) = \left(\prod_{k=1}^{\ell} a_{k,k}^{(k-1)}\right) \left(a_{\ell+1,\ell+1}^{(\ell)} a_{\ell+2,\ell+2}^{(\ell)} - \left(a_{\ell+1,\ell+2}^{(\ell)}\right)^2\right).$$

If  $a_{k,k}^{(k-1)} > 0$  for all  $1 \leq k \leq \ell$  and  $a_{\ell+1,\ell+1}^{(\ell)} = 0$  while  $a_{\ell+1,\ell+2}^{(\ell)} \neq 0$  then  $\det A_J < 0$ . According to Theorem 4.1 this contradicts  $A$  being positive semidefinite.

Notice that we used the fact that

$$\det L_k = \left(\frac{a_{k,k}^{(k-1)}}{a_{k-1,k-1}^{(k-2)}}\right)^{c-k} \quad \text{so that} \quad \det L^{(\ell)} = \left(a_{\ell,\ell}^{(\ell-1)}\right)^{c-\ell} \prod_{k=1}^{\ell-1} a_{k,k}^{(k-1)}$$

and replaced  $c$  by  $\ell+2$  to reflect the truncation.

The same argument, changing  $A_J$  to be another principal matrix, shows that for  $A$  to be positive semidefinite when  $a_{\ell+1,\ell+1}^{(\ell)} = 0$ , it is required that  $a_{\ell+1,\ell+j}^{(\ell)} = 0$  for all  $2 \leq j \leq c - \ell$ .

□

If we consider a symmetric  $c \times c$  matrix  $A$  with entries in  $R \subset \mathbb{R}$  that is positive definite then, according to Theorem 4.1,  $a_{\ell,\ell}^{(\ell-1)} > 0$  for all  $1 \leq \ell \leq c$ . We can thus proceed with elimination steps without any pivoting. After  $c-1$  such steps we have

$$\tilde{A}^{(c-1)} = \begin{pmatrix} a_{0,0}^{(-1)} a_{1,1}^{(0)} & 0 & \cdots & 0 \\ 0 & a_{1,1}^{(0)} a_{2,2}^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{c-1,c-1}^{(c-2)} a_{c,c}^{(c-1)} \end{pmatrix}.$$

In general though one might encounter  $a_{k,k}^{(k-1)}$  that is zero. One then has to introduce a pivoting strategy. This one differs whether one is interested in the triangularization or the diagonalization. See [5, 41] and [58] respectively.

We are here more concerned with positive semidefinite matrices. The third observation in Theorem 4.3 allows us to proceed without pivoting. This result can be restated as follows for an algorithmic point view.

**Proposition 4.4.** *Let  $A$  be a symmetric  $c \times c$  matrix with entries in an integral domain  $R \subset \mathbb{R}$ . Assume that  $a_{k,k}^{(k-1)} > 0$  for  $1 \leq k < \ell$ . If  $a_{\ell,\ell}^{(\ell-1)} = 0$  then*

*$A$  is positive semidefinite of rank  $r \geq \ell - 1$  if and only if  $a_{\ell,\ell+j}^{(\ell-1)} = 0$  for all  $1 \leq j \leq c - \ell$  and the principal submatrix  $A_{\hat{\ell}}$  obtained from  $A$  by removing the  $\ell$ -th row and column is positive semidefinite of rank  $r$ .*

Notice that for  $k \leq \ell$  we have  $(A_{\hat{\ell}})^{(k)} = (A^{(k)})_{\hat{\ell}}$ . So we can start the triangularization process and when we encounter a pivot that is zero, we remove (ignore) that row and column and continue on. The entries of the successive matrices we encounter are thus determinants of a principal submatrix of  $A$ , whose size is the rank of  $A$ .

We give a recursive presentation of the algorithm that will be completed with a branching process in the next section. The algorithm is initially called with  $A$ ,  $P = [1]$  and  $E = []$ . For the algorithmic description we will use the following notation. For a matrix  $A = (a_{ij})_{1 \leq i,j \leq c}$  with entries in  $R$  and  $p \in R$ , we define the first elimination matrix with the last non-zero pivot  $p$  as

$$L(A, p) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\frac{a_{2,1}}{p} & \frac{a_{1,1}}{p} & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\frac{a_{c,1}}{p} & 0 & \dots & 0 & \frac{a_{1,1}}{p} \end{pmatrix}$$

**Algorithm 4.5. Fraction-free diagonalization**

- Input :     $\triangleright$   $A$  a symmetric  $c \times c$  matrix with entries in  $R \subset \mathbb{R}$  that is positive semidefinite.  
              $\triangleright$   $P$  a list of positive elements of  $R$  -  $P$  stands for *pivots* or *positive*.  
              $\triangleright$   $E$  a list of elimination matrices with entries in the quotient field of  $R$ .
- Output:     $\triangleright$   $P$  a list of positive elements of  $R$  with  $\text{Card}(P) = \text{rank } A + 1$ .  
              $\triangleright$   $E$  a list of elimination matrices with entries in the quotient field of  $R$   
                 such that the product  $L$  of the elements in  $E$   
                 satisfies  $L A L^t$  is a diagonal matrix whose non-zero entries  
                 are the product of two consecutive elements of  $P$ .

*If  $c = 0$  then **return**  $P, E$ ; Otherwise:*

*if  $a_{11} > 0$  then*

- Let  $p$  be the last element of  $P$ .
- Append  $a_{1,1}$  to  $P$ .



- Append  $L(A, p)$  to  $E$ .
- Let  $A$  be now the submatrix of  $L(A, p)$  obtained by removing the first row and first column.

elif  $a_{1,1} = 0$  (and therefore  $a_{1,j} = 0$  for any  $2 \leq j \leq c$ ) then

- Let  $A$  be now the submatrix of  $A$  obtained by removing the first row and first column.

Make a recursive call with  $A$ ,  $P$  and  $E$  as obtained above.

### 4.3 Diagonalization over the locus of semi-positivity

Consider now the integral domain  $R$  as the polynomial ring  $\mathbb{K}[h] = \mathbb{K}[h_1, \dots, h_t]$ , where  $\mathbb{K} \subset \mathbb{R}$  is a field extension of  $\mathbb{Q}$ . With the criterion presented in previous section, we can write a fraction-free and pivoting free algorithm that provides a diagonalization of  $A$  over its locus of positivity.

Let us introduce the *specialization*  $\phi : \mathbb{K}[h] \rightarrow \mathbb{R}$ , which is a  $\mathbb{K}$ -morphism. A typical example is given by

$$\phi_h : \mathbb{K}[h] \rightarrow \mathbb{R}, p \mapsto p(\bar{h})$$

with  $\bar{h} = (\bar{h}_1, \dots, \bar{h}_t)$  a point in  $\mathbb{R}^t$ . Given a matrix  $A = (a_{ij})_{1 \leq i, j \leq c}$  with  $a_{ij} \in \mathbb{K}[h]$ , we denote by  $\phi(A)$  the matrix obtained by applying  $\phi$  to the coefficients of  $A$ , that is

$$\phi(A) = (\phi(a_{ij}))_{1 \leq i, j \leq c} = (\bar{a}_{ij})_{1 \leq i, j \leq c}.$$

Likewise

$$\bar{a}_{ij}^{(\ell)} = \det \begin{pmatrix} \bar{a}_{1,1} & \cdots & \bar{a}_{1,\ell} & \bar{a}_{1,j} \\ \vdots & & \vdots & \vdots \\ \bar{a}_{\ell,1} & \cdots & \bar{a}_{\ell,\ell} & \bar{a}_{\ell,j} \\ \bar{a}_{i,1} & \cdots & \bar{a}_{i,\ell} & \bar{a}_{i,j} \end{pmatrix} = \phi \left( a_{ij}^{(\ell)} \right) \quad \forall 1 \leq \ell \leq c-1, \forall \ell < i, j \leq c.$$

We give a recursive description of the algorithm. It is initialized with the full symmetric matrix  $A$ ,  $P = [1]$ ,  $Z = \emptyset$  and  $E = []$ . We will use the following notation. For a matrix  $A = (a_{ij})_{1 \leq i, j \leq c}$  with entries in  $R$  and  $p \in R$ , we define the first elimination matrix with the last non-zero pivot  $p$  as

$$L(A, p) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\frac{a_{2,1}}{p} & \frac{a_{1,1}}{p} & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\frac{a_{c,1}}{p} & 0 & \cdots & 0 & \frac{a_{1,1}}{p} \end{pmatrix}$$

**Algorithm 4.6. Diagonalization & Positivity**

Input :  $\triangleright A = (a_{ij})_{1 \leq i, j \leq c}$  a  $c \times c$  symmetric matrix with entries in  $\mathbb{K}[h]$ .  
 $\triangleright P$  a list of polynomials in  $\mathbb{K}[h]$  -  $P$  stands for *pivots* or *positive*.  
 $\triangleright Z$  a set of polynomials in  $\mathbb{K}[h]$  -  $Z$  stands for *zero*.  
 $\triangleright E$  a list of elimination matrices.

Output:  $\triangleright$  A set  $\mathcal{S}$  of triplets  $[P, Z, E]$ , where  
 $\bullet P$  is a list of polynomials in  $\mathbb{K}[h]$ ,  
 $\bullet Z$  is a set of polynomials in  $\mathbb{K}[h]$ ,  
 $\bullet E$  is a list of elimination matrices in  $\mathbb{K}(h)$   
whose denominators are power products of elements in  $P$ .

The set  $\mathcal{S}$  satisfies the following property: For a specialization  $\phi : \mathbb{K}[h] \rightarrow \mathbb{R}$ ,  $\phi(A)$  is positive semidefinite if and only if there is a triplet  $[P, Z, E]$  in  $\mathcal{S}$  such that

$$\phi(p) > 0 \ \forall p \in P \text{ and } \phi(q) = 0 \ \forall q \in Z.$$

In this case  $\text{Card}(P) - 1$  is the rank of  $\phi(A)$  and, letting  $L$  be the product of the elements of  $E$ ,  $\phi(LAL^t)$  is a diagonal matrix whose non zero entries are the product of two consecutive elements in  $\phi(P) = \{\phi(p), p \in P\}$ .

*If  $c = 0$  then **return**  $[P, Z, E]$ ; Otherwise:*

$\mathcal{S}_1$  and  $\mathcal{S}_2$  initialized to  $\emptyset$ .

*If  $a_{11} \notin \mathbb{K}$  or  $a_{11} > 0$  then*

- $\bullet$  Let  $p$  be the last element of  $P$ .
- $\bullet$  Let  $A_1$  be the submatrix of  $L(A, p)A$  obtained by removing the first row and first column.
- $\bullet$  Append  $a_{11}$  to  $P$  to give  $P_1$ .
- $\bullet$  Append  $L(A, p)$  to  $E$  to give  $E_1$ .
- $\bullet \mathcal{S}_1 := \text{Diagonalization \& Positivity} (A_1, P_1, Z, E_1)$ .

*If  $a_{11} \notin \mathbb{K}$  or  $a_{11} = 0$  then*

- $\bullet Z_2 := \{a_{1,j} \mid 1 \leq j \leq n\} \cup Z$ .
- $\bullet A_2$  is obtained from  $A$  by removing the first row and first column.
- $\bullet \mathcal{S}_2 := \text{Diagonalization \& Positivity} (A_2, P, Z_2, E)$ .

**return**  $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ .

The elements in  $P$  and  $Z$  are principal minors of  $A$  of size up to order  $\text{Card}(P) - 1$ . This determines their degrees in terms of the degrees of the entries of  $A$ . Furthermore, the output set  $\mathcal{S}$  can have as many as  $2^c$  triplets. One can lower this number if we can dismiss certain branches by checking if the semialgebraic set defined by a pair  $(P, Z)$  is consistent. Though this is algorithmic [6], it is no light task. One cannot expect Algorithm 4.6 to run easily on large matrices  $A$  with polynomial entries.

#### 4.4 Diagonalization for determining the existence of cubatures

Taking into account the specificity of the cubature problem, we revisit the procedure in the previous section. There are less cases to be distinguished so that the size of the output is smaller.

Let  $A$  be a  $c \times c$  symmetric matrix whose entries are polynomials of degree at most 1 in the parameters  $h_1, \dots, h_t$ . The cubature problem comes with additional input indices  $c' \leq r \leq c'' \leq c$ . Let  $A'$  (resp.  $A''$ ) be the  $c' \times c'$  (resp.  $c'' \times c''$ ) leading principal submatrix of  $A$ . They satisfy:

- The submatrix  $A'$  has entries in  $\mathbb{K} \subset \mathbb{R}$  a field extension of  $\mathbb{Q}$  and is positive definite.
- The submatrix  $A''$ , under the conditions on  $h_1, \dots, h_t$  we are looking for, is positive semidefinite and has the same rank  $r$  as the matrix  $A$ .

To determine those conditions, Algorithm 4.6 can be applied. The size of its output is reduced thanks to the information on  $A_{J_1}$  and  $A_{J_2}$ : we get less than  $2^c$  triplets.

- Since  $A'$  is positive definite, the algorithm runs without splitting as long as the matrix is of size bigger than  $c - c'$ . The first  $c'$  pivots are added to  $P$ .
- Since  $A''$  is required to have the same rank as  $A$ , the algorithm is stopped when the matrix is of size  $c - c''$ . All its entries are added to  $Z$ .
- Not all the branches created by the algorithm are interesting: only the ones such that  $\text{Card } P = r + 1$  (as we included 1 as first pivot in  $P$ ). There are then  $\binom{c''-c'}{r-c'}$  interesting branches to be studied.

##### Algorithm 4.7. Diagonalization & Positivity with Rank Constraints

Input :   ▷ Integers  $c' \leq r \leq c'' \leq c$ .  
           ▷ A  $c \times c$  symmetric matrix  $A$  with entries in  $\mathbb{K}[h]_{\leq 1}$ ,  
               whose  $c' \times c'$  leading principal submatrix is positive definite with entries in  $\mathbb{K}$ .  
           ▷  $P$  a list of polynomials in  $\mathbb{K}[h]$ .  
           ▷  $Z$  a set of polynomials in  $\mathbb{K}[h]$ .  
           ▷  $E$  a list of elimination matrices.

Output:   ▷ A set  $\mathcal{S}$  of  $\binom{c''-c'}{r-c'}$  triplets  $[P, Z, E]$ , where

- $P$  is a list of polynomials in  $\mathbb{K}[h]$  with  $\text{Card } P = r + 1$ ,
- $Z$  is a set of polynomials in  $\mathbb{K}[h]$ ,
- $E$  is a list of elimination matrices in  $\mathbb{K}(h)$   
       whose denominators are power products of elements in  $P$ .

The set  $\mathcal{S}$  satisfies the following property: For a specialization  $\phi : \mathbb{K}[h] \rightarrow \mathbb{R}$ ,  $\phi(A)$  is positive semidefinite if and only if there is a triplet  $[P, Z, E]$  such that

$$\phi(p) > 0 \quad \forall p \in P \text{ and } \phi(q) = 0 \quad \forall q \in Z.$$

In this case, the rank of  $\phi(A)$  is  $r$  and, letting  $L$  be the product of the elements of  $E$ ,  $\phi(LAL^t)$  is a diagonal matrix whose non zero entries are the product of two consecutive elements in  $\phi(P) = \{\phi(p), p \in P\}$ .

**Remark 4.8.** If the symmetric matrix  $A$  is the matrix of a symmetric bilinear form in a certain basis  $B$  as in the case of the search for cubatures, then the product  $L$  of all elements of  $E$  provides the appropriate change of basis described in Theorem 4.3.2 with  $\ell = c - 1$  that diagonalizes  $A$ .

In addition, under the conditions on  $h_1, \dots, h_t$  given by  $P$  and  $Z$ , this diagonal matrix has exactly  $r$  non zero entries. They correspond to evaluations of the symmetric bilinear form at squares of  $r$  elements of the basis  $B$ . The construction of a basis such that the matrix of the symmetric bilinear form is invertible can then be done without  $E$ . It is sufficient to select the appropriate  $r$  elements in the elements of  $B$ .

## 5 Moment matrix approach to computing cubatures

The algorithms in this section provide the computational solution we propose for the moment matrix approach to cubature. While the authors in [35] lay the foundations of this approach with theoretical results and examples, this section completes them with an algorithmic treatment, which is later enriched for symmetric cubatures.

The proposed solution is divided in three parts. With the help of the last algorithm in the previous section, we first provide a way to determine the existence for a given measure of cubatures of a given degree with positive weights. This issue indeed boils down to find the values of parameters such that a structured symmetric matrix is positive semidefinite of rank  $r$ , the number of nodes. We then show how to guarantee that the nodes lie on the support of the measure. We introduce there an algorithm that extends to any polynomial space a linear form that respects the flat extension and positive semidefiniteness assumptions. Once the existence of a cubature is secured, the third algorithm computes the weights and the coordinates of the nodes by solving generalized eigenvalue problems. Only this last stage resorts to floating point arithmetic.

As an example, we choose the known cubature of degree 5 with 7 nodes for the regular hexagon  $H_2$  in [101]. It can be solved with the present approach, but shows some limitations in the direct application of this algorithm. This computationally intensive example turns rather gentle when taking advantage of symmetry in Section 9.4.

The moment matrix approach to cubature we present is based on Curto-Fialkow's Flat Extension theorem as in [35] and on Hankel operators as in [1]. The search for minimal cubatures is reformulated in a (numerical) SemiDefinite Programming problem in [1]. It consists in minimizing the nuclear norm of Hankel operators associated with possible extensions of the linear form obtained from the known moments. While a SDP solver provides a solution, we are interested in finding all cubatures that satisfy certain assumptions. In addition, in some cases, we are able to provide the exact coordinates of the nodes and the exact weights.

Curto-Fialkow's Flat Extension theorem has also been used in the context of Gaussian cubatures [69]. Their existence is determined by an overdetermined linear system. This is an alternative criterion to the ones known from the theory of multivariate orthogonal polynomials [30, Chapter 3.6] that consist in checking either if the  $n$  multiplication operators  $\mathcal{M}_{x_1}, \dots, \mathcal{M}_{x_n}$  by  $x_1, \dots, x_n$  in  $\mathbb{R}[x]/I_\Lambda$  have exactly  $\dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$  joint eigenvalues, or if they commute pairwise. Notice that the matrices of the multiplication operators in an orthonormal basis are the (multivariate) Jacobi matrices constructed thanks to the three term relation [30, Chapter 3.2]. This link is mentioned in [90], where the author studied cubatures using an operator theory approach.

### 5.1 Existence conditions for a cubature

Let  $\mu$  be a positive Borel measure with compact support in  $\mathbb{R}^n$ . Given a degree  $d$ , we want to determine if there exists a cubature  $\Lambda$  of degree  $d$  with positive weights. The moments of order less than or equal to  $d$  are thus part of the input. They can be computed exactly following [3, 99] when  $\mu$  is the characteristic function of a polytope. The expected number of nodes  $r$  is here fixed. If it is not known, a starting point is given by lower bounds (see Section 2).

### Algorithm 5.1. Existence of a cubature

Input :     $\triangleright$  The degree  $d$  of the expected cubature  $\Lambda$ .  
              $\triangleright$  The moments of order less than or equal to  $d$  for the measure  $\mu$ .  
              $\triangleright$  A number of nodes  $r$  bigger than the lower bound  $\dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$ .  
 Output :     $\triangleright$  A system of equations and inequations that determines the existence for  $\mu$   
                 of a cubature  $\Lambda$  of degree  $d$  with positive weights.

1. Choose a degree  $\delta$  such that  $\dim \mathbb{R}[x]_{\leq \delta-1} \geq r$ .
2. Take the monomial basis  $B^{(\delta)} = \{b_1, \dots, b_c\}$  of  $\mathbb{R}[x]_{\leq \delta}$  following the graded reverse lexicographic order.
3. Construct the moment matrix  $H_1^{B^{(\delta)}} = (\Lambda(b_i b_j))_{1 \leq i, j \leq c}$ .

*This is the matrix of the Hankel operator  $\mathcal{H}^{(\delta)}$  associated with the restriction  $\Lambda^{(\delta)}$  of  $\Lambda$  to  $\mathbb{R}[x]_{\leq 2\delta}$ . Every coefficient  $\Lambda(b_i b_j)$  is:*

- *either the value of the moment  $\int b_i(x) b_j(x) d\mu(x) \in \mathbb{K} \subset \mathbb{R}$  if  $\deg(b_i b_j) \leq d$ ,*
- *or an unknown, denoted by  $h_\ell$ , if the monomial  $b_i b_j$  has degree bigger than  $d$ .*

*There is one distinct parameter  $h_\ell$  per monomial of degree between  $d+1$  and  $2\delta$ . The number  $t$  of distinct parameters is then*

$$t = \dim \mathbb{R}[x]_{\leq 2\delta} - \dim \mathbb{R}[x]_{\leq d}. \quad (5.1)$$

*The  $c' \times c'$  leading principal submatrix of  $H_1^{B^{(\delta)}}$ , with  $c' = \dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$ , has thus entries in  $\mathbb{K}$  and is positive definite.*

4. Find conditions on the parameters  $h_1, \dots, h_t$ , using Algorithm 4.7 [Diagonalization & Positivity with Rank Constraints] on the matrix  $H_1^{B^{(\delta)}}$ , such that the linear form  $\Lambda^{(\delta)}$  is a flat extension of the linear form  $\Lambda^{(\delta-1)}$  and such that the Hankel operator  $\mathcal{H}^{(\delta)}$  is positive semidefinite with rank  $r$ .

*Following Corollary 3.15, those properties are sufficient to prove the existence for the measure  $\mu$  of a cubature of degree  $d$  with  $r$  nodes and with positive weights.*

*The matrix  $H_1^{B^{(\delta)}}$ , under the conditions on the parameters  $h_1, \dots, h_t$ , satisfies then:*

- *Its  $c'' \times c''$  leading principal submatrix, with  $c'' = \dim \mathbb{R}[x]_{\leq \delta-1}$ , has the same rank  $r$  as the whole matrix  $H_1^{B^{(\delta)}}$ .*
- *$H_1^{B^{(\delta)}}$  is positive semidefinite.*

*Thus, Algorithm 4.7 can be used to determine those conditions: each triplet  $[P, Z, E]$  provides a system of equations (from  $Z$ ) and inequations (from  $P$ ) that determines the existence of a cubature.*

Algorithm 5.1 gives thus a system of equations and inequations that determines the existence for  $\mu$  of a cubature of degree  $d$  with positive weights. There is then no guarantee that the nodes lie on the support of  $\mu$ . This property is examined in the next section.

Since this requires generally the determination of additional parameters  $h_\ell$ , we suggest to skip it in practice. The fact that the cubature is inside is then checked after the computation of the nodes.

## 5.2 Existence of an inside cubature

In the following, we assume that the existence of a cubature  $\Lambda$  has been shown, that is we know conditions on parameters  $h_1, \dots, h_t$  such that the assumptions of Corollary 3.15 are satisfied: flat extension and positive semidefiniteness.

To guarantee that the nodes of the sought cubature  $\Lambda$  lie on  $\text{supp } \mu$ , we shall assume that  $\text{supp } \mu$  is semialgebraic

$$\text{supp } \mu = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_s(x) \geq 0\}$$

with  $g_k \in \mathbb{R}[x]$  for all  $k = 1, \dots, s$ . According to Proposition 3.16, it is then sufficient to show that the Hankel operators  $\mathcal{H}_{g_k}$  are positive semidefinite for all  $k = 1, \dots, s$ .

Let  $\kappa \in \mathbb{N}$  be such that  $\kappa \geq 1$  and  $2\kappa \geq \deg g_k$  for all  $k = 1, \dots, s$ . Consider the matrices  $H_{g_k}^{B^{(\delta-1)}} = (\Lambda(g_k b_i b_j))_{1 \leq i, j \leq r_{\delta-1}}$ , where  $B^{(\delta-1)} = \{b_1, \dots, b_{r_{\delta-1}}\}$  is a basis of  $\mathbb{R}[x]_{\leq \delta-1}$ . Its coefficients are numbers in  $\mathbb{K}$ , or polynomials in the parameters  $h_1, \dots, h_t$ , or unknowns<sup>3</sup>. Those unknown coefficients are uniquely determined by the flat extension assumption. Theorem 3.14 implies indeed that the linear form  $\Lambda^{(\delta+\kappa-1)}$  on  $\mathbb{R}[x]_{\leq 2\delta+2\kappa-2}$  is uniquely determined by the conditions on  $h_1, \dots, h_t$ , or equivalently by a triplet  $[P, Z, E]$  using the notations of Algorithm 4.7.

### Algorithm 5.2. Unique Extension

- Input :  $\triangleright$  Integers  $d, \delta$  and  $\kappa$  such that:
- The values  $\Lambda(p)$  are known if  $\deg p \leq d$ .
  - The coefficients of the matrix  $H_1^{B^{(\delta)}}$  of the Hankel operator associated with  $\Lambda$  in a basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$  are either numbers in  $\mathbb{K}$ , or polynomials in  $h_1, \dots, h_t$ .
  - A basis  $B^{(\delta+\kappa-1)}$  of  $\mathbb{R}[x]_{\leq \delta+\kappa-1}$  is taken such that  $B^{(\delta)} \subset B^{(\delta+\kappa-1)}$ .
- $\triangleright$  A system of equations and inequations in the parameters  $h_1, \dots, h_t$  such that the assumptions of Corollary 3.15 are satisfied.
- Output :  $\triangleright$  The matrix  $H_1^{B^{(\delta+\kappa-1)}}$  whose entries are either numbers in  $\mathbb{K}$ , or polynomials in parameters  $h_1, \dots, h_t, h_{t+1}, \dots, h_\tau$ , where  $h_{t+1}, \dots, h_\tau$  are additional parameters defined by  $h_\ell = \Lambda(p_\ell)$  with  $2\delta \leq \deg p_\ell \leq 2\delta + 2\kappa - 2$ .
- $\triangleright$  A system of equations and inequations on  $h_1, \dots, h_\tau$  such that  $\Lambda^{(\delta+\kappa-1)}$  is the unique flat extension of  $\Lambda^{(\delta)}$ .

The main ingredient to determine the system in output of Algorithm 5.2 is Algorithm 4.7. There is however only one triplet  $[P, Z, E]$ . The choice of the unique branch is determined by the system of equations and inequations in the parameters  $h_1, \dots, h_t$ . Indeed, the principal submatrix of  $H_1^{B^{(\delta+\kappa-1)}}$  that corresponds to  $H_1^{B^{(\delta)}}$  has the same rank as  $H_1^{B^{(\delta+\kappa-1)}}$ . Once this submatrix is treated, the remaining entries are then added to  $Z$ .

The entries of the matrices  $H_{g_k}^{B^{(\delta-1)}}$  are then either numbers in  $\mathbb{K}$ , or polynomials in  $h_1, \dots, h_\tau$ . It remains to find conditions on  $h_1, \dots, h_\tau$  such that those matrices are positive semidefinite. This can be done with the help of Algorithm 4.6 or the characteristic polynomial of the matrices  $H_{g_k}^{B^{(\delta-1)}}$  as presented in the introduction of Section 4.

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<sup>3</sup>This last case appears if and only if there exists  $k \in \{1, \dots, s\}$  such that  $\deg g_k > 2$ .

### 5.3 Computation of the weights and the coordinates of the nodes

Assume now that the existence of a cubature has been secured with a solution  $(\hbar_1, \dots, \hbar_t)$  of the polynomial system of equations and inequations output by Algorithm 5.1. To simplify the notations, we do not consider here the additional step in Section 5.2. Algorithm 5.3 then computes the weights and the coordinates of the nodes of the associated cubature.

#### Algorithm 5.3. Weights and Nodes

Input :     $\triangleright$  Input of Algorithm 5.1.  
              $\triangleright$  The parameterized matrix  $H_1^{B^{(\delta)}}$ .  
              $\triangleright$  A value  $\hbar_1, \dots, \hbar_t$  for each parameter  $h_1, \dots, h_t$  introduced in Algorithm 5.1.  
 Output :     $\triangleright$  The coordinates of the nodes  $\xi_1, \dots, \xi_r$ .  
              $\triangleright$  The weights  $a_1, \dots, a_r$ .

1. Take polynomials  $b_1, \dots, b_r$  such that  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{R}[x]/I_\Lambda$ .

*Following Remark 4.8 and Theorem 3.6, the use of Algorithm 4.7 (Step 4 in Algorithm 5.1) gives a way to determine a basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  by selecting the appropriate polynomials in the monomial basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$ .*

2. Compute the invertible matrix  $H_1^B$  of the linear operator  $\mathcal{H}$  associated with the sought cubature  $\Lambda$  in the basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$ . Using the values  $\hbar_1, \dots, \hbar_t$ , this matrix has entries in  $\mathbb{K}$ .

*Since the basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  is obtained from polynomials in  $B^{(\delta)}$ , the matrix  $H_1^B$  is a principal submatrix of  $H_1^{B^{(\delta)}}$  introduced in Step 3 of Algorithm 5.1.*

*The entries are either numbers in  $\mathbb{K}$  or parameters  $h_1, \dots, h_t$ . It is then sufficient to replace the parameters  $h_1, \dots, h_t$  by the values  $\hbar_1, \dots, \hbar_t$ .*

3. Take a polynomial  $p \in \mathbb{R}[x]_{\leq 1}$  that separates the generalized eigenvalues of  $(H_p^B, H_1^B)$ .

*This means that the generalized eigenvalues, which are the values  $p(\xi_1), \dots, p(\xi_r)$  following Corollary 3.9, are distinct.*

4. Compute the matrix  $H_p^B$ .

*Since  $\deg p < 2$ , the entries of  $H_p^B$  are completely determined by the moments of order less than or equal to  $d$  and the values  $\hbar_1, \dots, \hbar_t$  of the parameters  $h_1, \dots, h_t$ .*

5. Compute the left eigenvectors of the matrix  $M_p^B = (H_1^B)^{-1} H_p^B$  and deduce the coordinates of the nodes.

*Following Theorem 3.4, each left eigenvector of  $M_p^B$  contains the evaluations of the elements of  $B$  at a node  $\xi_1, \dots, \xi_r$  (up to a scalar). Assuming that  $[1], [x_1], \dots, [x_n]$  belong to  $B$ , the coordinates of the nodes  $\xi_1, \dots, \xi_r$  can be read in the matrix  $W = (b_j(\xi_i))_{1 \leq i, j \leq r}$  of left eigenvectors (up to a normalization thanks to the presence of  $[1]$  in  $B$ ).*

6. Solve the Vandermonde-like linear system (2.9) with  $(p_1, \dots, p_r) = (b_1, \dots, b_r)$ , that is

$$W^t \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} \Lambda(b_1) \\ \vdots \\ \Lambda(b_r) \end{pmatrix}.$$

The weights  $a_1, \dots, a_r$  are its unique solutions.



#### 5.4 $H_2$ 5 – 1 (Stroud, 1971): a first resolution

We are looking for a cubature of degree 5 with 7 nodes for the regular hexagon  $H_2$  in the plane  $\mathbb{R}^2$ . It is described in [101] under the name  $H_2 : 5 - 1$ .

Let  $H_2$  be the regular hexagon whose vertices are given by  $(\pm 1, 0), (\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$  (see Figure 10(a)). Its moments of order less than or equal to 5 are

$$\begin{aligned} \int_{H_2} 1 dx &= \frac{3\sqrt{3}}{2}, & \int_{H_2} x_1^2 dx &= \int_{H_2} x_2^2 dx = \frac{5\sqrt{3}}{16} \\ \int_{H_2} x_1^4 dx &= \int_{H_2} x_2^4 dx = \frac{21\sqrt{3}}{160}, & \int_{H_2} x_1^2 x_2^2 dx &= \frac{7\sqrt{3}}{160} \end{aligned}$$

and zero otherwise.

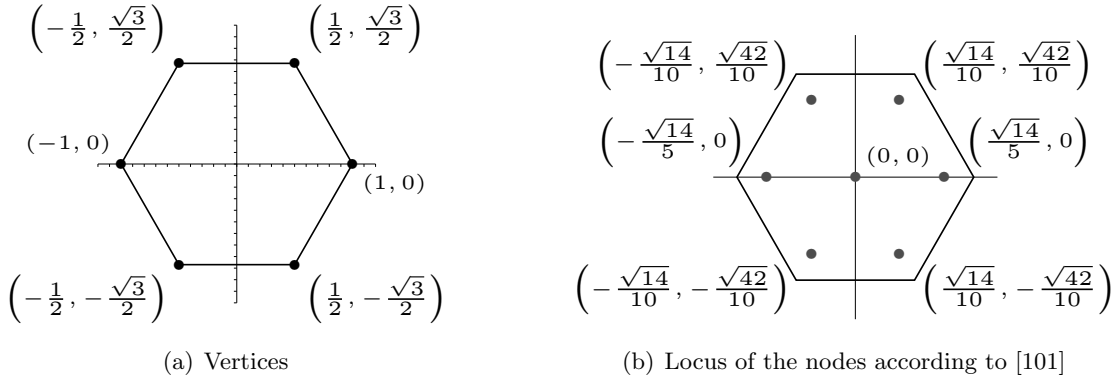


Figure 10: Regular hexagon  $H_2$

#### Existence

Using Algorithm 5.1, we prove the existence of such a cubature. Due to the difficulty of the computation, we do not attempt to find all such cubatures.

1. Take  $\delta = 4$ .
2. The monomial basis  $B^{(4)}$  of  $\mathbb{R}[x]_{\leq 4}$  following the graded reverse lexicographic order is

$$B^{(4)} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4\}.$$

3. The parameterized moment matrix  $H_1^{B^{(4)}}$ , with  $h_{ij} = \Lambda(x_1^i x_2^j)$ , is

$$\left( \begin{array}{cccccc|cccc|cccc|cccc} \frac{3\sqrt{3}}{2} & 0 & 0 & \frac{5\sqrt{3}}{16} & 0 & \frac{5\sqrt{3}}{16} & 0 & 0 & 0 & 0 & \frac{21\sqrt{3}}{160} & 0 & \frac{7\sqrt{3}}{160} & 0 & \frac{21\sqrt{3}}{160} \\ 0 & \frac{5\sqrt{3}}{16} & 0 & 0 & 0 & 0 & \frac{21\sqrt{3}}{160} & 0 & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5\sqrt{3}}{16} & 0 & 0 & 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & \frac{21\sqrt{3}}{160} & 0 & 0 & 0 & 0 & 0 \\ \frac{5\sqrt{3}}{16} & 0 & 0 & \frac{21\sqrt{3}}{160} & 0 & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{33} & h_{24} \\ 0 & 0 & 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{24} & h_{15} \\ \frac{5\sqrt{3}}{16} & 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & \frac{21\sqrt{3}}{160} & 0 & 0 & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{15} & h_{06} \\ \hline 0 & \frac{21\sqrt{3}}{160} & 0 & 0 & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{33} & h_{70} & h_{61} & h_{52} & h_{43} & h_{34} \\ 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{24} & h_{61} & h_{52} & h_{43} & h_{34} & h_{25} \\ 0 & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{15} & h_{52} & h_{43} & h_{34} & h_{25} & h_{16} \\ 0 & 0 & \frac{21\sqrt{3}}{160} & 0 & 0 & 0 & h_{33} & h_{24} & h_{15} & h_{06} & h_{43} & h_{34} & h_{25} & h_{16} & h_{07} \\ \hline \frac{21\sqrt{3}}{160} & 0 & 0 & h_{60} & h_{51} & h_{42} & h_{70} & h_{61} & h_{52} & h_{43} & h_{80} & h_{71} & h_{62} & h_{53} & h_{44} \\ 0 & 0 & 0 & h_{51} & h_{42} & h_{33} & h_{61} & h_{52} & h_{43} & h_{34} & h_{71} & h_{62} & h_{53} & h_{44} & h_{35} \\ \frac{7\sqrt{3}}{160} & 0 & 0 & h_{42} & h_{33} & h_{24} & h_{52} & h_{43} & h_{34} & h_{25} & h_{62} & h_{53} & h_{44} & h_{35} & h_{26} \\ 0 & 0 & 0 & h_{33} & h_{24} & h_{15} & h_{43} & h_{34} & h_{25} & h_{16} & h_{53} & h_{44} & h_{35} & h_{26} & h_{17} \\ \frac{21\sqrt{3}}{160} & 0 & 0 & h_{24} & h_{15} & h_{06} & h_{34} & h_{25} & h_{16} & h_{07} & h_{44} & h_{35} & h_{26} & h_{17} & h_{08} \end{array} \right)$$

There are  $t = \dim \mathbb{R}[x]_{\leq 8} - \dim \mathbb{R}[x]_{\leq 4} = 24$  parameters. The  $6 \times 6$  leading principal submatrix has entries in  $\mathbb{Q}[\sqrt{3}]$  and is positive definite.

4. We look for conditions on those 24 parameters such that the assumptions of Corollary 3.15 are satisfied. We have here  $c' = 6$ ,  $r = 7$ ,  $c'' = 10$  and  $c = 15$  with the notations of Algorithm 4.7. There are then  $\binom{4}{1} = 4$  triplets  $[P, Z, E]$ . We focus here on the one such that the  $7 \times 7$  leading principal submatrix is positive definite.

The number of parameters is here too big for a reasonable use of Algorithm 4.7 on the whole matrix. We first use it for the  $c'' \times c''$  leading principal submatrix. Solving the polynomial system obtained from  $Z$  and respecting the constraints from  $P$ , we get unique values for the involved parameters

$$h_{60} = \frac{539\sqrt{3}}{8000}, h_{51} = 0, h_{42} = \frac{49\sqrt{3}}{8000}, h_{33} = 0, h_{24} = \frac{147\sqrt{3}}{8000}, h_{15} = 0, h_{06} = \frac{441\sqrt{3}}{8000}.$$

Using Algorithm 4.7 on the whole  $c \times c$  matrix with those values, we get the unique values for the remaining parameters

$$h_{80} = \frac{14749\sqrt{3}}{400000}, h_{62} = \frac{343\sqrt{3}}{400000}, h_{44} = \frac{1029\sqrt{3}}{400000}, h_{26} = \frac{3087\sqrt{3}}{400000}, h_{08} = \frac{9261\sqrt{3}}{400000},$$

$$h_{70} = h_{61} = h_{52} = h_{43} = h_{34} = h_{25} = h_{16} = h_{07} = h_{71} = h_{53} = h_{35} = h_{17} = 0.$$

Thus, we know that there exists for  $H_2$  a cubature of degree 5 with 7 nodes and with positive weights. The fact that the nodes lie on  $H_2$  is checked *a posteriori*. Following Remark 4.8, we also have that  $B = \{[1], [x_1], [x_2], [x_1^2], [x_1x_2], [x_2^2], [x_1^3]\}$  is a basis of  $\mathbb{R}[x]/I_\Lambda$ .

Let us emphasize that there could be other cubatures of degree 5 with 7 nodes. Indeed, not all the triplets from the output of Algorithm 4.7 were examined.

## Weights and nodes

With the values of the 24 parameters, we can compute the defining elements of the cubature with Algorithm 5.3.

1. Since the  $7 \times 7$  leading principal submatrix of  $H_1^{B^{(\delta)}}$  was chosen, a basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  is obtained from the polynomials  $1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3$ .
2. With the values of the 24 parameters,

$$H_1^B = \begin{pmatrix} \frac{3\sqrt{3}}{2} & 0 & 0 & \frac{5\sqrt{3}}{16} & 0 & \frac{5\sqrt{3}}{16} & 0 \\ 0 & \frac{5\sqrt{3}}{16} & 0 & 0 & 0 & 0 & \frac{21\sqrt{3}}{160} \\ 0 & 0 & \frac{5\sqrt{3}}{16} & 0 & 0 & 0 & 0 \\ \frac{5\sqrt{3}}{16} & 0 & 0 & \frac{21\sqrt{3}}{160} & 0 & \frac{7\sqrt{3}}{160} & 0 \\ 0 & 0 & 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & 0 \\ \frac{5\sqrt{3}}{16} & 0 & 0 & \frac{7\sqrt{3}}{160} & 0 & \frac{21\sqrt{3}}{160} & 0 \\ 0 & \frac{21\sqrt{3}}{160} & 0 & 0 & 0 & 0 & \frac{539\sqrt{3}}{8000} \end{pmatrix}.$$

3. Taking  $p = x_1 + 5x_2$ , the generalized eigenvalues of  $(H_p^B, H_1^B)$  are all distinct.
4. With the values of the 24 parameters,

$$H_{x_1+5x_2}^B = \begin{pmatrix} 0 & \frac{5\sqrt{3}}{16} & \frac{25\sqrt{3}}{16} & 0 & 0 & 0 & \frac{21\sqrt{3}}{160} \\ \frac{5\sqrt{3}}{16} & 0 & 0 & \frac{21\sqrt{3}}{160} & \frac{7\sqrt{3}}{32} & \frac{7\sqrt{3}}{160} & 0 \\ \frac{25\sqrt{3}}{16} & 0 & 0 & \frac{7\sqrt{3}}{32} & \frac{7\sqrt{3}}{160} & \frac{21\sqrt{3}}{32} & 0 \\ 0 & \frac{21\sqrt{3}}{160} & \frac{7\sqrt{3}}{32} & 0 & 0 & 0 & \frac{539\sqrt{3}}{8000} \\ 0 & \frac{7\sqrt{3}}{32} & \frac{7\sqrt{3}}{160} & 0 & 0 & 0 & \frac{49\sqrt{3}}{1600} \\ 0 & \frac{7\sqrt{3}}{160} & \frac{21\sqrt{3}}{32} & 0 & 0 & 0 & \frac{49\sqrt{3}}{8000} \\ \frac{21\sqrt{3}}{160} & 0 & 0 & \frac{539\sqrt{3}}{8000} & \frac{49\sqrt{3}}{1600} & \frac{49\sqrt{3}}{8000} & 0 \end{pmatrix}.$$

5. The matrix  $W$  of left eigenvectors of the matrix  $M_p^B = (H_1^B)^{-1} H_{x_1+5x_2}^B$  is

$$W = \left( \begin{array}{c|cc|ccc} 1 & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 1 & \frac{\sqrt{14}}{5} & \mathbf{0} & \frac{14}{25} & 0 & 0 & \frac{14\sqrt{14}}{125} \\ 1 & -\frac{\sqrt{14}}{5} & \mathbf{0} & \frac{14}{25} & 0 & 0 & -\frac{14\sqrt{14}}{125} \\ 1 & -\frac{\sqrt{14}}{10} & \frac{\sqrt{42}}{10} & \frac{7}{50} & -\frac{\sqrt{47}}{50} & \frac{21}{50} & -\frac{7\sqrt{14}}{500} \\ 1 & \frac{\sqrt{14}}{10} & -\frac{\sqrt{42}}{10} & \frac{7}{50} & -\frac{\sqrt{47}}{50} & \frac{21}{50} & \frac{7\sqrt{14}}{500} \\ 1 & \frac{\sqrt{14}}{10} & \frac{\sqrt{42}}{10} & \frac{7}{50} & \frac{\sqrt{47}}{50} & \frac{21}{50} & \frac{7\sqrt{14}}{500} \\ 1 & -\frac{\sqrt{14}}{10} & -\frac{\sqrt{42}}{10} & \frac{7}{50} & \frac{\sqrt{47}}{50} & \frac{21}{50} & -\frac{7\sqrt{14}}{500} \end{array} \right).$$

Since  $[x_1]$  and  $[x_2]$  are the second and third classes of polynomials in  $B$ , the second and third columns of  $W$  are the coordinates of the nodes

$$\xi_1 = (0, 0), \quad \xi_2 = \left( \frac{\sqrt{14}}{5}, 0 \right), \quad \xi_3 = \left( -\frac{\sqrt{14}}{5}, 0 \right), \\ \xi_4 = \left( \frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10} \right), \xi_5 = \left( -\frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10} \right), \xi_6 = \left( -\frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10} \right), \xi_7 = \left( \frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10} \right).$$

6. Solving the Vandermonde-like linear system (2.9), we get the weights

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \left( \frac{43\sqrt{3}}{112}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672}, \frac{125\sqrt{3}}{672} \right).$$

We get thus the cubature in [101].

## 6 Symmetry adapted bases

The existence of symmetry in a problem leads to seek a way to reduce the computation by a factor depending on the size of the group describing the symmetry. In the context of the linear action of a finite group  $G$  on a  $\mathbb{C}$ -vector space, representation theory provides the tools to compute a basis of the vector space in which the matrices of a Hermitian form invariant under the group action is block diagonal. Such a basis is called *unitary symmetry adapted basis*. This terminology refers to the fact that the representing matrices in this basis are unitary.

The block diagonalization is obtained thanks to two ingredients: the orthogonality relations for irreducible representations of a finite group and the possibility of using unitary irreducible representations.

In the case of a linear action on a  $\mathbb{R}$ -vector space, we detail how to construct an *orthogonal symmetry adapted basis* from unitary ones. The representing matrices in this basis are orthogonal. The most important property is that the matrices of a symmetric bilinear form invariant under the group action are block diagonal in an orthogonal symmetry adapted basis.

In both cases, there is an additional structure on the blocks. The number of distinct blocks is bounded by the number of distinct irreducible inequivalent representations of the finite group. The size of each distinct block is the multiplicity of the irreducible inequivalent representation. The number of identical blocks per irreducible inequivalent representation is the dimension of the irreducible inequivalent representation.

The basic material on representation of finite groups is taken from [33, 98]. Symmetry adapted bases of  $\mathbb{C}$ -vector spaces are expanded on in [33]. They are used there to block diagonalize linear operators that commutes with a representation [33, The Fundamental Theorem], a consequence of Schur's lemma.

Symmetry adapted bases of  $\mathbb{R}$ -vector spaces were introduced in the context of optimization in [39] and then reused in [94]. In both cases, the authors are interested in the block diagonalization of quadratic forms. For a direct application of [33, The Fundamental Theorem], they restrict their approach to symmetries given by orthogonal representations. Their main applications indeed require mostly permutation representations. This is a strong restriction that needs to be avoided for the present purpose. The symmetries we consider are given by those of polygons, or more generally polytopes. As such they are isometries and therefore given by orthogonal matrices. But the action we need to consider is the one induced on the polynomial ring. The matrices of this induced representation in a monomial basis are not orthogonal.

### 6.1 Linear representations and characters

Let  $G$  be a group, let  $V$  be a  $\mathbb{K}$ -vector space and let  $GL(V)$  be the group of isomorphisms from  $V$  to itself. A *linear representation* of the group  $G$  on the space  $V$  is a group morphism from  $G$  to  $GL(V)$ . In other words, a linear representation  $\mathcal{V}$  assigns to each element  $g$  of the group  $G$  an isomorphism  $\mathcal{V}(g)$  of the group  $GL(V)$  such that

$$\mathcal{V}(g_1)\mathcal{V}(g_2) = \mathcal{V}(g_1g_2) \quad \forall g_1, g_2 \in G.$$

If  $V$  has finite dimension  $n$ , then upon introducing a basis  $B$  in  $V$  the isomorphism  $\mathcal{V}(g)$  can be described by a non-singular  $n \times n$  matrix. This *representing matrix* is denoted by  $\mathcal{V}^B(g)$ .  $V$

is called the *representation space* and  $n$  is the *dimension* (or the *degree*) of the representation  $\mathcal{V}$ . Unless otherwise stated, we always deal with representations  $\mathcal{V}$  of finite dimension.

A linear representation  $\mathcal{V}$  of a group  $G$  on a space  $V$  is said to be *irreducible* provided there is no proper subspace  $W$  of  $V$  with the property that, for every  $g \in G$ , the isomorphism  $\mathcal{V}(g)$  maps every vector of  $W$  into  $W$ . In this case, its representation space  $V$  is also called *irreducible*. A linear representation  $\mathcal{V}$  of a group  $G$  on a space  $V$  is said to be *completely reducible* if its representation space  $V$  is irreducible or if it decomposes into a finite number of irreducible subspaces  $W_1, \dots, W_{\underline{M}}$  such that  $V = W_1 \oplus \dots \oplus W_{\underline{M}}$ .

Two representations  $\mathcal{V}$  and  $\mathcal{W}$  of a group  $G$  respectively on the spaces  $V$  and  $W$  are said to be *equivalent* provided there exists a fixed isomorphism  $\mathcal{T} : V \rightarrow W$  such that

$$\mathcal{W}(g) = \mathcal{T}\mathcal{V}(g)\mathcal{T}^{-1} \quad \forall g \in G.$$

Let  $\mathcal{V}$  be a completely reducible representation of a group  $G$  on a  $\mathbb{K}$ -vector space  $V$ . Let  $\mathcal{V}_j$  ( $j = 1, \dots, \underline{N}$ ) be the irreducible inequivalent  $n_j$ -dimensional representations of  $G$  that appear in  $\mathcal{V}$  with multiplicities  $c_j \geq 1$ . The complete reduction of the representation  $\mathcal{V}$  is denoted by

$$\mathcal{V} = c_1\mathcal{V}_1 \oplus \dots \oplus c_{\underline{N}}\mathcal{V}_{\underline{N}}.$$

Accordingly, its representation space  $V$  decomposes into

$$V = V_1 \oplus \dots \oplus V_{\underline{N}}.$$

Each invariant subspace  $V_j$  is the direct sum of  $c_j$  irreducible subspaces and the restriction of  $\mathcal{V}$  to each one is equivalent to  $\mathcal{V}_j$ . The  $(c_j n_j)$ -dimensional subspaces  $V_j$  of  $V$  are called *isotypic components*. The decomposition of  $V$  into irreducible components is not unique, whereas its decomposition into a direct sum of isotypic components is unique [36, Proposition 1.8] and is called the *isotypic decomposition* of  $V$ .

We recall now important results concerning representations of a finite group on a  $\mathbb{C}$ -vector space. Results over  $\mathbb{R}$  can then be deduced from the ones over  $\mathbb{C}$  thanks to the constructions done in Section 6.4. The first one is given in [33, Chapter 1.11].

**Theorem 6.1.** 1. *Every representation of a finite group is completely reducible.*

2. *For every representation of a finite group, there is a basis such that the representing matrices in this basis are unitary.*

The second one can be deduced from the *orthogonality relations for irreducible representations* (see [33, Chapter 5.1] for the details of the computation).

**Theorem 6.2.** *A finite group  $G$  possesses a finite number of irreducible inequivalent representations.*

The representation space of any representation of a finite group  $G$  admits thus an isotypic decomposition. The latter can be determined in two steps:

1. Determination of all irreducible inequivalent representations  $\mathcal{V}_1, \dots, \mathcal{V}_{\underline{N}}$  of the group  $G$ . They are known for some finite groups (see [33, Chapter 1.9] or [98, Chapter 5] for instance).

2. Computation of the multiplicities  $c_1, \dots, c_N$ , that is the number of times each irreducible inequivalent representation  $\mathcal{V}_j$  occurs in the representation  $\mathcal{V}$ .

This second task can be performed thanks to Theorem 6.3. Let  $\mathcal{V}$  be a representation of an arbitrary group  $G$  on a  $\mathbb{C}$ -vector space  $V$ . The complex-valued function

$$\chi : G \rightarrow \mathbb{C}, g \mapsto \text{Trace}(\mathcal{V}(g))$$

is called the *character* of the representation  $\mathcal{V}$ .

**Theorem 6.3** ([33, Algorithm for computing multiplicities]). *Let  $\mathcal{V}$  (with character  $\chi$ ) be a representation of a finite group  $G$  of order  $|G|$ . Then, for every  $j = 1, \dots, N$ , the irreducible representation  $\mathcal{V}_j$  (with character  $\chi_j$ ) occurs in  $\mathcal{V}$  exactly  $c_j$  times, where*

$$c_j = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \chi(g) \quad c_j \in \{0, 1, 2, \dots\}. \quad (6.1)$$

Notice that  $c_j$  can take here the value 0. This means that the irreducible inequivalent representation  $\mathcal{V}_j$  of the group  $G$  does not occur in the representation  $\mathcal{V}$ . A consequence of Theorem 6.3 is given by the next result. The latter underlines the importance of the character and shows how this function characterizes a representation.

**Theorem 6.4** ([33, Theorem 5.11]). *Two representations of a finite group  $G$  are equivalent to each other if and only if their characters are identical.*

The character gives furthermore an *irreducibility criterion*:

**Theorem 6.5** ([33, Theorem 5.10]). *A representation  $\mathcal{V}$  of a finite group  $G$  of order  $|G|$  is irreducible if and only if the corresponding character  $\chi$  satisfies*

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) = 1.$$

With the help of these results on the representations of a finite group on a  $\mathbb{C}$ -vector space, we can construct bases of their representation spaces that are of special interest.

## 6.2 Unitary symmetry adapted bases

Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $\mathbb{C}$ -vector space  $V$ . Based on [98, Chapters 2.6 & 2.7], we present a way for computing a *unitary symmetry adapted basis* of every isotypic component  $V_j$  of the representation space  $V$ . This notion corresponds to the one of *symmetry adapted basis* given in [39] with the additional property that the matrices of the irreducible representations  $\mathcal{V}_j$  used for its computation are required to be unitary. In this section, we describe also the particular form of the matrices of the representation  $\mathcal{V}$  restricted to an isotypic component in a unitary symmetry adapted basis.

For every inequivalent irreducible  $n_j$ -dimensional representation  $\mathcal{V}_j$  ( $j = 1, \dots, N$ ) of the group  $G$  with character  $\chi_j$ , let  $p_j$  be the map defined on the representation space  $V$  by

$$p_j = \frac{n_j}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \mathcal{V}(g).$$

The latter is the projection of  $V$  onto the isotypic component  $V_j$  associated with the isotypic decomposition of  $V$  [98, Theorem 8]. Thus, the isotypic decomposition of  $V$  can be determined thanks to the projections  $p_1, \dots, p_N$ .

Consider now representing matrices  $\left(r_{\alpha\beta}^j(g)\right)_{1 \leq \alpha, \beta \leq n_j}$  for all irreducible  $n_j$ -dimensional representations  $\mathcal{V}_j$  of  $G$ . For every  $(\alpha, \beta) \in \mathbb{N}^2$  such that  $1 \leq \alpha, \beta \leq n_j$ , let  $p_{j,\alpha\beta} : V \rightarrow V$  be the linear map defined by

$$p_{j,\alpha\beta} = \frac{n_j}{|G|} \sum_{g \in G} r_{\beta\alpha}^j(g^{-1}) \mathcal{V}(g).$$

**Proposition 6.6** ([98, Proposition 8]). *The linear maps  $p_{j,\alpha\beta}$  satisfy the following properties:*

- (1) *For every  $1 \leq \alpha \leq n_j$ , the map  $p_{j,\alpha\alpha}$  is a projection ; it is zero on the isotypic components  $V_k$ ,  $k \neq j$ . Its image  $V_{j,\alpha}$  is contained in the isotypic component  $V_j$  and  $V_j$  is the direct sum of the subspaces  $V_{j,\alpha}$  for  $1 \leq \alpha \leq n_j$ , i.e.*

$$V_j = V_{j,1} \oplus \dots \oplus V_{j,n_j}. \quad (6.2)$$

We have  $p_j = \sum_{\alpha=1}^{n_j} p_{j,\alpha\alpha}$ .

- (2) *For every  $(\alpha, \beta) \in \mathbb{N}^2$  such that  $1 \leq \alpha, \beta \leq n_j$ , the linear map  $p_{j,\alpha\beta}$  is zero on the isotypic components  $V_k$ ,  $k \neq j$ , as well as on the subspaces  $V_{j,\gamma}$  for  $\gamma \neq \beta$  ; it defines an isomorphism from  $V_{j,\beta}$  to  $V_{j,\alpha}$ .*
- (3) *Let  $\zeta_1$  be a nonzero element of  $V_{j,1}$  and let  $\zeta_\alpha = p_{j,\alpha 1}(\zeta_1) \in V_{j,\alpha}$  for all  $\alpha = 1, \dots, n_j$ . For each  $g \in G$ , we have*

$$\mathcal{V}(g)(\zeta_\alpha) = \sum_{\beta=1}^{n_j} r_{\beta\alpha}^j(g) \zeta_\beta \quad \forall \alpha = 1, \dots, n_j.$$

With those properties of the linear maps  $p_{j,\alpha\beta}$ , a *symmetry adapted basis* of every isotypic component  $V_j$  can be computed, that is a basis of  $V_j$  compatible with the decomposition (6.2). If we consider unitary representing matrices  $\left(r_{\alpha\beta}^j(g)\right)_{1 \leq \alpha, \beta \leq n_j}$ , then the same process leads to *unitary symmetry adapted bases*. Since the coefficients of the matrices satisfy then  $\overline{r_{\alpha\beta}^j(g)} = r_{\beta\alpha}^j(g^{-1})$ , the linear maps  $p_{j,\alpha\beta}$  are

$$p_{j,\alpha\beta} = \frac{n_j}{|G|} \sum_{g \in G} \overline{r_{\alpha\beta}^j(g)} \mathcal{V}(g).$$

Let  $\mathcal{V}_j$  be an irreducible  $n_j$ -dimensional representation of the group  $G$  that appears  $c_j$  times ( $c_j \geq 1$ ) in the representation  $\mathcal{V}$ . Take  $\{b_1, \dots, b_{c_j}\}$  a basis of the subspace  $V_{j,1}$  defined as the image of  $V$  by the projection  $p_{j,11}$ . For every  $2 \leq \alpha \leq n_j$ , the linear map  $p_{j,\alpha 1} : V_{j,1} \rightarrow V_{j,\alpha}$  is an isomorphism so that the set  $\{p_{j,\alpha 1}(b_1), \dots, p_{j,\alpha 1}(b_{c_j})\}$  is a basis of  $V_{j,\alpha}$ . A unitary symmetry adapted basis of the isotypic component  $V_j$  is then given by

$$\hat{B}_j = \{b_1, \dots, b_{c_j}, p_{j,21}(b_1), \dots, p_{j,21}(b_{c_j}), \dots, p_{j,n_j 1}(b_1), \dots, p_{j,n_j 1}(b_{c_j})\}.$$

In addition, we have

$$\begin{aligned}\mathcal{V}(g)(b_\gamma) &= r_{11}^j(g)b_\gamma + \sum_{\beta=2}^{n_j} r_{\beta 1}^j(g)p_{j,\beta 1}(b_\gamma) \quad \forall g \in G, \forall \gamma = 1, \dots, c_j, \\ \mathcal{V}(g)(p_{j,\alpha 1}(b_\gamma)) &= r_{1\alpha}^j(g)b_\gamma + \sum_{\beta=2}^{n_j} r_{\beta\alpha}^j(g)p_{j,\beta 1}(b_\gamma) \quad \forall g \in G, \forall \gamma = 1, \dots, c_j, \forall \alpha = 2, \dots, n_j.\end{aligned}$$

Thus, for every  $g \in G$ , the representing matrix of the representation  $\mathcal{V}$  restricted to the isotypic component  $V_j$  in the basis  $\hat{B}_j$  is

$$\left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j} \otimes I_{c_j} = \begin{pmatrix} r_{11}^j(g)I_{c_j} & r_{12}^j(g)I_{c_j} & \dots & r_{1n_j}^j(g)I_{c_j} \\ r_{21}^j(g)I_{c_j} & r_{22}^j(g)I_{c_j} & \dots & r_{2n_j}^j(g)I_{c_j} \\ \vdots & \vdots & & \vdots \\ r_{n_j 1}^j(g)I_{c_j} & r_{n_j 2}^j(g)I_{c_j} & \dots & r_{n_j n_j}^j(g)I_{c_j} \end{pmatrix},$$

where  $\otimes$  denotes the Kronecker product of two matrices and  $I_{c_j}$  the identity matrix of size  $c_j \times c_j$ .

In this construction, the choice of a basis of the subspace  $V_{j,1}$  determines completely the bases  $\hat{B}_j$  of the isotypic component  $V_j$ .

For all isotypic components  $V_j$  of the representation space  $V$ , unitary symmetry adapted bases  $\hat{B}_j$  can be thus computed. The basis  $\hat{B}$ , that consists of the union of the basis  $\hat{B}_j$ , is also called *unitary symmetry adapted*. In this basis  $\hat{B}$ , the matrix of the representation  $\mathcal{V}$  is block diagonal: the matrix of the representation  $\mathcal{V}$  restricted to an isotypic component  $V_j$  in the corresponding basis  $\hat{B}_j$  is a block.

**Remark 6.7.** A basis  $\check{B}_j$ , which is compatible with a decomposition of the isotypic component  $V_j$  into  $c_j$  irreducible subspaces of dimension  $n_j$ , is then given by

$$\check{B}_j = \{b_1, p_{j,21}(b_1), \dots, p_{j,n_j 1}(b_1), \dots, b_{c_j}, p_{j,21}(b_{c_j}), \dots, p_{j,n_j 1}(b_{c_j})\}.$$

The matrix of the representation  $\mathcal{V}$  restricted to  $V_j$  in the basis  $\check{B}_j$  is

$$I_{c_j} \otimes \left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j} = \begin{pmatrix} \left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j} & 0_{n_j} & \dots & 0_{n_j} \\ 0_{n_j} & \left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{n_j} \\ 0_{n_j} & \dots & 0_{n_j} & \left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j} \end{pmatrix},$$

where  $0_{n_j}$  is the  $n_j \times n_j$  zero matrix.

### 6.3 Invariant Hermitian forms

The bases constructed in Section 6.2 give us appropriate bases for studying a Hermitian form invariant under a representation of a finite group. We show that its matrix in these particular bases has a block diagonal structure : each block corresponds to the restriction of the Hermitian



form to an isotypic component. Using unitary symmetry adapted bases for all isotypic components, the corresponding submatrices have an additional block diagonal structure explained in Proposition 6.8.

Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $n$ -dimensional  $\mathbb{C}$ -vector space  $V$ . A map  $\phi : V \times V \rightarrow \mathbb{C}$  is said to be a  $G$ -invariant Hermitian form if it satisfies

$$\begin{aligned} \phi(\lambda u + v, w) &= \lambda \phi(u, w) + \phi(v, w) & \forall \lambda \in \mathbb{C}, \forall u, v, w \in V, \\ \phi(u, \lambda v + w) &= \bar{\lambda} \phi(u, v) + \phi(u, w) & \forall \lambda \in \mathbb{C}, \forall u, v, w \in V, \\ \phi(u, v) &= \overline{\phi(v, u)} & \forall u, v \in V \\ \text{and } \phi(u, v) &= \phi(\mathcal{V}(g)(u), \mathcal{V}(g)(v)) & \forall g \in G, \forall u, v \in V. \end{aligned}$$

The matrix  $\phi^B = (\phi(b_i, b_j))_{1 \leq i, j \leq n}$  of the Hermitian form  $\phi$  in a given basis  $B = \{b_1, \dots, b_n\}$  is Hermitian, that is it satisfies  $\phi^B = \overline{\phi^B}^t$ .

**Proposition 6.8.** *Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $\mathbb{C}$ -vector space  $V$  and let  $\phi : V \times V \rightarrow \mathbb{C}$  be a  $G$ -invariant Hermitian form. Consider  $V_1, \dots, V_{\underline{N}}$  the isotypic components of  $V$ . Then*

$$\phi(u, v) = 0 \quad \forall (u, v) \in V_i \times V_j \text{ with } i \neq j.$$

Hence the matrix  $\phi^{\hat{B}}$  of  $\phi$  in any basis  $\hat{B} = \hat{B}_1 \cup \dots \cup \hat{B}_{\underline{N}}$  that respects the isotypic decomposition of  $V$  is block diagonal.

Assume furthermore that for every isotypic component  $V_j$  associated with an  $n_j$ -dimensional irreducible representation  $\mathcal{V}_j$  that occurs  $c_j$  times in  $\mathcal{V}$ ,

$$\hat{B}_j = \{b_{11}^j, \dots, b_{1c_j}^j, \dots, b_{n_j1}^j, \dots, b_{n_jc_j}^j\}$$

is a unitary symmetry adapted basis. Then the submatrix of  $\phi^{\hat{B}}$  relating to the isotypic component  $V_j$  consists of a diagonal of  $n_j$  identical blocks of size  $c_j$ . It is given by

$$I_{n_j} \otimes \left( \mu_{st}^j \right)_{1 \leq s, t \leq c_j} \quad \text{with } \mu_{st}^j = \phi(b_{1s}^j, b_{1t}^j).$$

*Proof.* We take unitary representing matrices  $\left( r_{\alpha\beta}^j(g) \right)_{1 \leq \alpha, \beta \leq n_j}$  for all irreducible representations  $\mathcal{V}_j$ .

Using Remark 6.7, for every  $s = 1, \dots, c_k$  (resp. for every  $t = 1, \dots, c_\ell$ ), the set  $\{b_{1s}^k, \dots, b_{n_k s}^k\}$  (resp. the set  $\{b_{1t}^\ell, \dots, b_{n_\ell t}^\ell\}$ ) is a basis of an irreducible subspace contained in the isotypic component  $V_k$  (resp.  $V_\ell$ ). We deduce from Proposition 6.6 that

$$\mathcal{V}(g)(b_{\alpha s}^k) = \sum_{\gamma=1}^{n_k} r_{\gamma\alpha}^k(g) b_{\gamma s}^k \quad \forall \alpha = 1, \dots, n_k \quad \text{and} \quad \mathcal{V}(g)(b_{\beta t}^\ell) = \sum_{\delta=1}^{n_\ell} r_{\delta\beta}^\ell(g) b_{\delta t}^\ell \quad \forall \beta = 1, \dots, n_\ell.$$

Since the map  $\phi$  is a  $G$ -invariant Hermitian form, we also have

$$\phi(b_{\alpha s}^k, b_{\beta t}^\ell) = \frac{1}{|G|} \sum_{g \in G} \phi(\mathcal{V}(g)(b_{\alpha s}^k), \mathcal{V}(g)(b_{\beta t}^\ell)) \quad \forall \alpha = 1, \dots, n_k, \forall \beta = 1, \dots, n_\ell.$$

This leads to

$$\phi(b_{\alpha s}^k, b_{\beta t}^\ell) = \sum_{\substack{1 \leq \gamma \leq n_k \\ 1 \leq \delta \leq n_\ell}} \left( \frac{1}{|G|} \sum_{g \in G} r_{\gamma\alpha}^k(g) \overline{r_{\delta\beta}^\ell(g)} \right) \phi(b_{\gamma s}^k, b_{\delta t}^\ell) \quad \forall \alpha = 1, \dots, n_k, \forall \beta = 1, \dots, n_\ell.$$

We deduce then from the *orthogonality relations for irreducible representations of  $G$*  [33, Corollary 5.2] that

$$\phi(b_{\alpha s}^k, b_{\beta t}^\ell) = \begin{cases} \frac{1}{n_k} \sum_{\gamma=1}^{n_k} \phi(b_{\gamma s}^k, b_{\delta t}^\ell) & \text{if } k = \ell \text{ and } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $\phi^{\widehat{B}}$  of the  $G$ -invariant Hermitian form  $\phi$  in the basis  $\widehat{B}$  has thus the expected block diagonal structure.  $\square$

## 6.4 Orthogonal symmetry adapted bases

In the previous sections, we showed that representation theory can be used to compute a basis such that the matrix of a  $G$ -invariant Hermitian form on a  $\mathbb{C}$ -vector space has a block diagonal structure. Similarly, representation theory can be used to compute a basis such that the matrix of any symmetric bilinear form on a  $\mathbb{R}$ -vector space, which is invariant under a linear representation of a finite group, is block diagonal.

Based on [98, Chapter 13.2], we study the linear representations on a  $\mathbb{R}$ -vector space from the linear representations on a  $\mathbb{C}$ -vector space. In fact, any linear representation  $\mathcal{V}$  on a  $\mathbb{R}$ -vector space  $V$  can be considered as a linear representation on the  $\mathbb{C}$ -vector space  $V \otimes_{\mathbb{R}} \mathbb{C}$ , that is the vector space obtained from  $V$  by extending the scalars from the real numbers to the complex numbers.

We recall the three types of irreducible representations on a  $\mathbb{C}$ -vector space distinguished in [98, Chapter 13.2]. Let  $\mathcal{V}_j$  be an irreducible representation of a finite group  $G$  on a  $n_j$ -dimensional  $\mathbb{C}$ -vector space  $V_j$  and let  $\chi_j$  be its character. The three mutually exclusive cases are the following:

1. The representation  $\mathcal{V}_j$  can be realized by matrices having coefficients in  $\mathbb{R}$ , in which case the character  $\chi_j$  is a real-valued function. By restriction of the scalars from the complex numbers to the real numbers,  $\mathcal{V}_j$  defines an irreducible representation  $\mathcal{V}^{(j)}$  on a  $\mathbb{R}$ -vector space of dimension  $n_j$  with character  $\chi_j$ .
2. The character  $\chi_j$  is not real-valued. By restriction of the scalars,  $\mathcal{V}_j$  defines an irreducible representation  $\mathcal{V}^{(j)}$  on a  $\mathbb{R}$ -vector space of dimension  $2n_j$  with character  $\chi_j + \overline{\chi_j}$ .
3. The character  $\chi_j$  is a real-valued function, but the representation  $\mathcal{V}_j$  cannot be realized by matrices having coefficients in  $\mathbb{R}$ . By restriction of the scalars,  $\mathcal{V}_j$  defines an irreducible representation  $\mathcal{V}^{(j)}$  on a  $\mathbb{R}$ -vector space of dimension  $2n_j$  with character  $2\chi_j$ .

An irreducible representation  $\mathcal{V}^{(j)}$  on a  $\mathbb{R}$ -vector space defined thanks to an irreducible representation  $\mathcal{V}_j$  of type 1 (resp. of type 2, of type 3) is called *absolutely irreducible* (resp. of

*complex type*, of *quaternionian type*) [39]. Irreducible representation of quaternionian type are not considered in this paper as they do not arise in our application.

In the case of a representation of a finite group on a  $\mathbb{C}$ –vector space, we presented in Section 6.2 a construction of a unitary symmetry adapted basis for every isotypic component. Similarly, given a representation  $\mathcal{V}$  of a finite group  $G$  on a  $\mathbb{R}$ –vector space  $V$ , we present now a construction of an *orthogonal symmetry adapted basis* for every isotypic component of the  $\mathbb{R}$ –vector space  $V$ ; that is a basis such that, for every  $g \in G$ , the matrix of the representation  $\mathcal{V}$  restricted to this isotypic component in this basis is

$$(r_{\alpha\beta}(g))_{1 \leq \alpha, \beta \leq n} \otimes I_c,$$

where  $n$  is the dimension of the irreducible representation associated with this isotypic component,  $c$  is the number of times it occurs in  $\mathcal{V}$  and  $(r_{\alpha\beta}(g))_{1 \leq \alpha, \beta \leq n}$  is an orthogonal matrix.

The construction of an orthogonal symmetry adapted basis is based on the one of a unitary symmetry adapted basis presented in Section 6.2. It depends on the type of each irreducible representation that occurs in the representation  $\mathcal{V}$  on the  $\mathbb{C}$ –vector space  $V \otimes_{\mathbb{R}} \mathbb{C}$ . Since in our applications we never encounter the case of an irreducible representation of quaternionian type, we do not present here a way of computing a basis in this case.

As in the case of unitary symmetry adapted bases, the union of orthogonal symmetry adapted bases associated with distinct irreducible representations is an orthogonal symmetry adapted basis.

We first give some notations. Given a  $\mathbb{C}$ –vector space  $V$ , we denote by  $\bar{V}$  the  $\mathbb{C}$ –vector space whose vectors are the complex conjugate of the vectors of  $V$ , that is  $\bar{V} = \{\bar{z} \mid z \in V\}$ . Given a linear representation  $\mathcal{V}$  of a group  $G$  on a  $n$ –dimensional  $\mathbb{C}$ –vector space  $V$ , we denote by  $\bar{\mathcal{V}}$  the linear representation of the group  $G$  on the  $n$ –dimensional  $\mathbb{C}$ –vector space  $\bar{V}$  defined by

$$\bar{\mathcal{V}}(g)(\bar{z}) = \overline{\mathcal{V}(g)(z)} \quad \forall g \in G, \forall z \in V.$$

Let  $B = \{v_1, \dots, v_n\}$  be a basis of  $V$  such that, for every  $g \in G$ , the representing matrix  $\mathcal{V}^B(g) = (r_{\alpha\beta}(g))_{1 \leq \alpha, \beta \leq n}$  is unitary. We denote by  $\bar{B}$  the set  $\{\bar{v}_1, \dots, \bar{v}_n\}$ . The latter is a basis of the space  $\bar{V}$ . For every  $g \in G$ , the representing matrix  $\bar{\mathcal{V}}^{\bar{B}}(g)$  is given by  $(\overline{r_{\alpha\beta}(g)})_{1 \leq \alpha, \beta \leq n}$  and is unitary. The characters of the representations  $\mathcal{V}$  and  $\bar{\mathcal{V}}$  are thus complex conjugate. Assuming that the representation  $\mathcal{V}$  is irreducible, the representation  $\bar{\mathcal{V}}$  is therefore irreducible too.

### Absolutely irreducible representation

Let  $\mathcal{V}_k$  be an irreducible  $n_k$ –dimensional representation of type 1 of the group  $G$  that appears  $c_k$  times in the representation  $\mathcal{V}$  on the space  $V \otimes_{\mathbb{R}} \mathbb{C}$  and let  $V_k$  be the isotypic component of type  $\mathcal{V}_k$ . Here, for all  $g \in G$ , we can choose representing matrices  $(r_{\alpha\beta}^k(g))_{1 \leq \alpha, \beta \leq n_k}$  of the irreducible representation  $\mathcal{V}_k$  with coefficients in  $\mathbb{R}$ . Following [33, Chapter 1.11], we can also choose orthogonal representing matrices. We can thus copy the construction of a unitary symmetry adapted basis presented in Section 6.2.

For every  $(\alpha, \beta) \in \mathbb{N}^2$  such that  $1 \leq \alpha, \beta \leq n_k$ , consider the linear map  $p_{k,\alpha\beta} : V \otimes_{\mathbb{R}} \mathbb{C} \rightarrow V \otimes_{\mathbb{R}} \mathbb{C}$

defined here by

$$p_{k,\alpha\beta} = \frac{n_k}{|G|} \sum_{g \in G} r_{\alpha\beta}^k(g) \mathcal{V}(g)$$

and a basis  $\{v_1, \dots, v_n\}$  of the representation space  $V$ . A basis  $\{b_1, \dots, b_{c_k}\}$  of the subspace  $p_{k,11}(V \otimes_{\mathbb{R}} \mathbb{C})$ , where  $b_1, \dots, b_{c_k}$  are real vectors, is then obtained by taking  $c_k$  linearly independent vectors in  $\{p_{k,11}(v_1), \dots, p_{k,11}(v_n)\}$ . A unitary symmetry adapted basis of the isotypic component  $V_k$  is then given by

$$\widehat{B}_k = \{b_1, \dots, b_{c_k}, p_{k,21}(b_1), \dots, p_{k,21}(b_{c_k}), \dots, p_{k,n_k1}(b_1), \dots, p_{k,n_k1}(b_{c_k})\}.$$

The latter has furthermore real vectors. The basis  $\widehat{B}_k$  is then an orthogonal symmetry adapted basis of the space obtained by restricting the scalars of the isotypic component  $V_k$ . It is therefore denoted by  $B_k$ .

### Irreducible representation of complex type

Let  $\mathcal{V}_\ell$  be an irreducible  $n_\ell$ -dimensional representation of type 2 of the group  $G$  with character  $\chi_\ell$ . Assume that  $\mathcal{V}_\ell$  appears  $c_\ell$  times in the representation  $\mathcal{V}$  on the space  $V \otimes_{\mathbb{R}} \mathbb{C}$ . By Theorem 6.3, we have

$$c_\ell = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\ell(g)} \chi(g).$$

Since the character  $\chi$  of the representation  $\mathcal{V}$  is a real-valued function, we have

$$c_\ell = \overline{c_\ell} = \frac{1}{|G|} \sum_{g \in G} \chi_\ell(g) \overline{\chi(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_\ell(g) \chi(g).$$

The representation  $\overline{\mathcal{V}_\ell}$  appears then also  $c_\ell$  times in the representation  $\mathcal{V}$  so that the isotypic component  $\overline{V}_\ell$  of type  $\overline{\mathcal{V}_\ell}$  has the same dimension as the isotypic component  $V_\ell$  of type  $\mathcal{V}_\ell$ . Since the characters  $\chi_\ell$  and  $\overline{\chi_\ell}$  are not identical, the representations  $\mathcal{V}_\ell$  and  $\overline{\mathcal{V}_\ell}$  are inequivalent. The space  $V_\ell \oplus \overline{V}_\ell$  is thus of dimension  $2c_\ell n_\ell$ .

For every  $(\alpha, \beta) \in \mathbb{N}^2$  such that  $1 \leq \alpha, \beta \leq n_\ell$ , let  $p_{\ell,\alpha\beta} : V \otimes_{\mathbb{R}} \mathbb{C} \rightarrow V \otimes_{\mathbb{R}} \mathbb{C}$  be the linear map defined by

$$p_{\ell,\alpha\beta} = \frac{n_\ell}{|G|} \sum_{g \in G} \overline{r_{\alpha\beta}^\ell(g)} \mathcal{V}(g),$$

where  $\left(r_{\alpha\beta}^\ell(g)\right)_{1 \leq \alpha, \beta \leq n_\ell}$  are unitary representing matrices of  $\mathcal{V}_\ell$ . Let  $\{b_1, \dots, b_{c_\ell}\}$  be a basis of the subspace  $p_{\ell,11}(V \otimes_{\mathbb{R}} \mathbb{C})$ , the set

$$\widehat{B}_\ell = \{b_1, \dots, b_{c_\ell}, p_{\ell,21}(b_1), \dots, p_{\ell,21}(b_{c_\ell}), \dots, p_{\ell,n_\ell1}(b_1), \dots, p_{\ell,n_\ell1}(b_{c_\ell})\}$$

is a unitary symmetry adapted basis of the isotypic component  $V_\ell$ . In addition, the basis  $\overline{\widehat{B}_\ell}$  corresponds to the unitary symmetry adapted basis of the space  $\overline{V}_\ell$  computed with the help of the linear maps

$$\overline{p_{\ell,\alpha\beta}} = \frac{n_\ell}{|G|} \sum_{g \in G} r_{\alpha\beta}^\ell(g) \mathcal{V}(g) \quad \forall 1 \leq \alpha, \beta \leq n_\ell.$$

The complex conjugate bases  $\widehat{B}_\ell$  and  $\overline{\widehat{B}_\ell}$  form then a basis of the space  $V_\ell \oplus \overline{V}_\ell$ . In order to obtain a basis  $B_\ell$  of the space  $V_\ell \oplus \overline{V}_\ell$  with real vectors, we can choose the basis

$$B_\ell = \left\{ \frac{1}{2}(b_1 + \overline{b_1}), \dots, \frac{1}{2}(b_{c_\ell} + \overline{b_{c_\ell}}), \frac{1}{2i}(b_1 - \overline{b_1}), \dots, \frac{1}{2i}(b_{c_\ell} - \overline{b_{c_\ell}}), \dots, \right. \\ \left. \frac{1}{2}(p_{\ell, n_\ell 1}(b_1) + \overline{p_{\ell, n_\ell 1}(b_1)}), \dots, \frac{1}{2}(p_{\ell, n_\ell 1}(b_{c_\ell}) + \overline{p_{\ell, n_\ell 1}(b_{c_\ell})}), \right. \\ \left. \frac{1}{2i}(p_{\ell, n_\ell 1}(b_1) - \overline{p_{\ell, n_\ell 1}(b_1)}), \dots, \frac{1}{2i}(p_{\ell, n_\ell 1}(b_{c_\ell}) - \overline{p_{\ell, n_\ell 1}(b_{c_\ell})}) \right\}.$$

Notice that, by Proposition 6.6, we have

$$\frac{1}{2}(p_{\ell, \alpha 1}(b_\gamma) + \overline{p_{\ell, \alpha 1}(b_\gamma)}) = (p_{\ell, \alpha 1} + \overline{p_{\ell, \alpha 1}}) \left( \frac{b_\gamma + \overline{b_\gamma}}{2} \right) \quad \forall \alpha = 1, \dots, n_\ell, \forall \gamma = 1, \dots, c_\ell,$$

$$\frac{1}{2i}(p_{\ell, \alpha 1}(b_\gamma) - \overline{p_{\ell, \alpha 1}(b_\gamma)}) = (p_{\ell, \alpha 1} + \overline{p_{\ell, \alpha 1}}) \left( \frac{b_\gamma - \overline{b_\gamma}}{2i} \right) \quad \forall \alpha = 1, \dots, n_\ell, \forall \gamma = 1, \dots, c_\ell.$$

The basis  $B_\ell$  can then be computed as follows:

- Take a basis  $\{b_1, \dots, b_{c_\ell}\}$  of the subspace  $p_{\ell, 11}(V \otimes_{\mathbb{R}} \mathbb{C})$ .
- Let  $u_\gamma = \operatorname{Re}(b_\gamma)$  and  $v_\gamma = \operatorname{Im}(b_\gamma)$  for all  $\gamma = 1, \dots, c_\ell$ . Compute  $(p_{\ell, \alpha 1} + \overline{p_{\ell, \alpha 1}})(u_\gamma)$  and  $(p_{\ell, \alpha 1} + \overline{p_{\ell, \alpha 1}})(v_\gamma)$  for all  $\alpha = 2, \dots, n_\ell$  and for all  $\gamma = 1, \dots, c_\ell$ .
- The basis  $B_\ell$  is then given by

$$B_\ell = \left\{ u_1, \dots, u_{c_\ell}, v_1, \dots, v_{c_\ell}, \dots, \right. \\ \left. (p_{\ell, n_\ell 1} + \overline{p_{\ell, n_\ell 1}})(u_1), \dots, (p_{\ell, n_\ell 1} + \overline{p_{\ell, n_\ell 1}})(u_{c_\ell}), \right. \\ \left. (p_{\ell, n_\ell 1} + \overline{p_{\ell, n_\ell 1}})(v_1), \dots, (p_{\ell, n_\ell 1} + \overline{p_{\ell, n_\ell 1}})(v_{c_\ell}) \right\}.$$

For every  $g \in G$ , the matrix of the representation  $\mathcal{V}$  restricted to the space  $V_\ell \oplus \overline{V}_\ell$  in the basis  $B_\ell$  is

$$\mathcal{V}^{(\ell)}(g) \otimes I_{c_\ell} = \begin{pmatrix} s_{11}^\ell(g) & t_{11}^\ell(g) & \cdots & s_{1n_\ell}^\ell(g) & t_{1n_\ell}^\ell(g) \\ -t_{11}^\ell(g) & s_{11}^\ell(g) & \cdots & -t_{1n_\ell}^\ell(g) & s_{1n_\ell}^\ell(g) \\ \vdots & \vdots & & \vdots & \vdots \\ s_{n_\ell 1}^\ell(g) & t_{n_\ell 1}^\ell(g) & \cdots & s_{n_\ell n_\ell}^\ell(g) & t_{n_\ell n_\ell}^\ell(g) \\ -t_{n_\ell 1}^\ell(g) & s_{n_\ell 1}^\ell(g) & \cdots & -t_{n_\ell n_\ell}^\ell(g) & s_{n_\ell n_\ell}^\ell(g) \end{pmatrix} \otimes I_{c_\ell}, \quad (6.3)$$

with  $s_{\alpha\beta}^\ell(g) = \frac{1}{2} \left( r_{\alpha\beta}^\ell(g) + \overline{r_{\alpha\beta}^\ell(g)} \right)$  and  $t_{\alpha\beta}^\ell(g) = \frac{1}{2i} \left( r_{\alpha\beta}^\ell(g) - \overline{r_{\alpha\beta}^\ell(g)} \right)$  for all  $(\alpha, \beta) \in \mathbb{N}^2$  such that  $1 \leq \alpha, \beta \leq n_\ell$ . The matrices  $\mathcal{V}^{(\ell)}(g)$  correspond to the representing matrices of an irreducible representation of complex type with character  $\chi_\ell + \overline{\chi}_\ell$ .

**Lemma 6.9.** *The matrices  $\mathcal{V}^{(\ell)}(g)$  are orthogonal for all  $g \in G$ .*

*Proof.* Let  $g \in G$ . Since the matrix  $\left( r_{\alpha\beta}^\ell(g) \right)_{1 \leq \alpha, \beta \leq n_\ell}$  is unitary, we have

$$\left( r_{\alpha\beta}^\ell(g) \right)_{1 \leq \alpha, \beta \leq n_\ell} \left( \overline{r_{\alpha\beta}^\ell(g)} \right)_{1 \leq \alpha, \beta \leq n_\ell}^t = I_{2n_\ell}, \quad (6.4)$$

which is equivalent to the following system of  $n^2$  equations

$$\sum_{\gamma=1}^{n_\ell} r_{\alpha\gamma}^\ell(g) \overline{r_{\beta\gamma}^\ell(g)} = \delta_{\alpha\beta} \quad \forall 1 \leq \alpha, \beta \leq n_\ell.$$

The matrix  $\mathcal{V}^{(\ell)}(g)$  is orthogonal if and only if  $\mathcal{V}^{(\ell)}(g)\mathcal{V}^{(\ell)}(g)^t = I_{2n_\ell}$ . This matrix equation is equivalent to the following system of  $4n_\ell^2$  equations:

$$\begin{cases} \sum_{\gamma=1}^{n_\ell} (s_{\alpha\gamma}^\ell(g) s_{\beta\gamma}^\ell(g) + t_{\alpha\gamma}^\ell(g) t_{\beta\gamma}^\ell(g)) = \delta_{\alpha\beta} & \forall 1 \leq \alpha, \beta \leq 2n_\ell \text{ and } \alpha, \beta \text{ with the same parity} \\ \sum_{\gamma=1}^{n_\ell} (s_{\alpha\gamma}^\ell(g) t_{\beta\gamma}^\ell(g) - s_{\beta\gamma}^\ell(g) t_{\alpha\gamma}^\ell(g)) = 0 & \forall 1 \leq \alpha, \beta \leq 2n_\ell \text{ and } \alpha, \beta \text{ of different parity.} \end{cases}$$

For each equation, we use the definition of the coefficients  $s_{\alpha\beta}^\ell(g)$  and  $t_{\alpha\beta}^\ell(g)$  and expand the left hand side. Thus, this system is equivalent to

$$\begin{cases} \frac{1}{2} \sum_{\gamma=1}^{n_\ell} (\overline{r_{\alpha\gamma}^\ell(g)} r_{\beta\gamma}^\ell(g) + r_{\alpha\gamma}^\ell(g) \overline{r_{\beta\gamma}^\ell(g)}) = \delta_{\alpha\beta} & \forall 1 \leq \alpha, \beta \leq 2n_\ell \text{ and } \alpha, \beta \text{ with the same parity} \\ \frac{1}{2i} \sum_{\gamma=1}^{n_\ell} (\overline{r_{\alpha\gamma}^\ell(g)} r_{\beta\gamma}^\ell(g) - r_{\beta\gamma}^\ell(g) \overline{r_{\alpha\gamma}^\ell(g)}) = 0 & \forall 1 \leq \alpha, \beta \leq 2n_\ell \text{ and } \alpha, \beta \text{ of different parity.} \end{cases}$$

With the help of (6.4), the system is satisfied so that the matrix  $\mathcal{V}^{(\ell)}(g)$  is orthogonal.  $\square$

The basis  $B_\ell$  is then an orthogonal symmetry adapted basis of the space obtained by restricting the scalars of  $V_\ell \oplus \overline{V}_\ell$  from the complex numbers to the real numbers.

## 6.5 Invariant symmetric bilinear forms

In this section, we present an analogue result to the block diagonalization of the matrix of an invariant Hermitian form in a unitary symmetry basis. Orthogonal symmetry adapted bases are now used to block diagonalize matrices of symmetric bilinear forms invariant under a representation of a finite group.

**Remark 6.10.** In what follows, we assume that a representation  $\mathcal{V}$  on a  $\mathbb{R}$ -vector space  $V$  does not contain an irreducible representation of quaternionian type : it contains only absolutely irreducible representations or irreducible representations of complex type.

Consider an orthogonal symmetry adapted basis  $B$  of  $V$  that is computed as described in Section 6.4. The isotypic decomposition of  $V \otimes_{\mathbb{R}} \mathbb{C}$  can be written as follows

$$V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus \cdots \oplus V_M \oplus V_{M+1} \oplus \cdots \oplus V_N \oplus \overline{V_{M+1}} \oplus \cdots \oplus \overline{V_N},$$

where the isotypic component  $V_j$  for  $j = 1, \dots, M$  is associated with an irreducible representation of type 1 and the pair  $(V_j, \overline{V}_j)$  for  $j = M+1, \dots, N$  is associated with an irreducible representation of type 2 and its complex conjugate. The basis  $B$  of  $V$  respects then the following decomposition deduced from the isotypic decomposition of  $V \otimes_{\mathbb{R}} \mathbb{C}$

$$V = U_1 \oplus \cdots \oplus U_M \oplus U_{M+1} \oplus \cdots \oplus U_N, \tag{6.5}$$

where  $U_j$  is the space obtained by restricting the scalars of  $V_j$  for every  $j = 1, \dots, M$  and the space obtained by restricting the scalars of  $V_j \oplus \overline{V}_j$  for every  $j = M+1, \dots, N$ .

Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $n$ -dimensional  $\mathbb{R}$ -vector space  $V$ . A map  $\varphi : V \times V \rightarrow \mathbb{R}$  is said to be a  $G$ -invariant symmetric bilinear form if it satisfies

$$\begin{aligned} \varphi(\lambda u + v, w) &= \lambda \varphi(u, w) + \varphi(v, w) & \forall \lambda \in \mathbb{R}, \forall u, v, w \in V, \\ \varphi(u, \lambda v + w) &= \lambda \varphi(u, v) + \varphi(u, w) & \forall \lambda \in \mathbb{R}, \forall u, v, w \in V, \\ \varphi(u, v) &= \varphi(v, u) & \forall u, v \in V \\ \text{and } \varphi(\mathcal{V}(g)(u), \mathcal{V}(g)(v)) &= \varphi(u, v) & \forall g \in G, \forall u, v \in V. \end{aligned}$$

The matrix  $(\varphi(b_i, b_j))_{1 \leq i, j \leq n}$  of the symmetric bilinear form  $\varphi$  in a given basis  $B = \{b_1, \dots, b_n\}$  is denoted by  $\varphi^B$ .

**Proposition 6.11.** *Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $\mathbb{R}$ -vector space  $V$  and let  $\varphi : V \times V \rightarrow \mathbb{R}$  be a  $G$ -invariant symmetric bilinear form. Consider  $U_1, \dots, U_N$  the components in the decomposition of  $V$  deduced from the isotypic decomposition of  $V \otimes_{\mathbb{R}} \mathbb{C}$ . Then*

$$\varphi(u, v) = 0 \quad \forall (u, v) \in U_i \times U_j \text{ with } i \neq j.$$

Hence the matrix  $\varphi^B$  of  $\varphi$  in any basis  $B = B_1 \cup \dots \cup B_N$  that respects this decomposition of  $V$  is block diagonal.

Assume furthermore that, for every component  $U_j$ ,  $B_j$  is an orthogonal symmetry adapted basis.

1. For every component  $U_j$  associated with an absolutely irreducible  $n_j$ -dimensional representation  $\mathcal{V}^{(j)}$  that appears  $c_j$  times in the representation  $\mathcal{V}$ , the basis  $B_j$  can be written as

$$B_j = \{b_{11}^j, \dots, b_{1c_j}^j, \dots, b_{n_j1}^j, \dots, b_{n_jc_j}^j\}.$$

Then the submatrix of  $\varphi^B$  relating to the component  $U_j$  consists of a diagonal of  $n_j$  identical blocks of size  $c_j$ . It is given by

$$I_{n_j} \otimes \left( \nu_{st}^j \right)_{1 \leq s, t \leq c_j} \quad \text{with } \nu_{st}^j = \varphi(b_{1s}^j, b_{1t}^j).$$

2. For every component  $U_j$  associated with an irreducible  $2n_j$ -dimensional representation  $\mathcal{V}^{(j)}$  of complex type that appears  $c_j$  times in the representation  $\mathcal{V}$ , the basis  $B_j$  can be written as

$$B_j = \{a_{11}^j, \dots, a_{1c_j}^j, b_{11}^j, \dots, b_{1c_j}^j, \dots, a_{n_j1}^j, \dots, a_{n_jc_j}^j, b_{n_j1}^j, \dots, b_{n_jc_j}^j\}.$$

Then the submatrix of  $\varphi^B$  relating to the component  $U_j$  consists of a diagonal of  $n_j$  identical blocks of size  $2c_j$ . It is given by

$$I_{n_j} \otimes \begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix},$$

where  $S_j = \left( \varphi(a_{1s}^j, a_{1t}^j) \right)_{1 \leq s, t \leq c_j} = \left( \varphi(b_{1s}^j, b_{1t}^j) \right)_{1 \leq s, t \leq c_j}$  is a symmetric matrix

and  $A_j = \left( \varphi(a_{1s}^j, b_{1t}^j) \right)_{1 \leq s, t \leq c_j} = - \left( \varphi(b_{1s}^j, a_{1t}^j) \right)_{1 \leq s, t \leq c_j}$  is an antisymmetric matrix.

*Proof.* Let  $\phi$  be the map defined by

$$\phi : (V \otimes_{\mathbb{R}} \mathbb{C}) \times (V \otimes_{\mathbb{R}} \mathbb{C}) \rightarrow \mathbb{C}, (z_1, z_2) \mapsto \varphi(u_1, u_2) + \varphi(v_1, v_2) + i \varphi(v_1, u_2) - i \varphi(u_1, v_2),$$

where  $u_i = \operatorname{Re}(z_i)$  and  $v_i = \operatorname{Im}(z_i)$  for  $i = 1, 2$ . The map  $\phi$  is a  $G$ -invariant Hermitian form that satisfies  $\phi(u, v) = \varphi(u, v)$  for all real vectors  $u$  and  $v$ .

Let  $\widehat{B}$  be a unitary symmetry adapted basis of the  $\mathbb{C}$ -vector space  $V \otimes_{\mathbb{R}} \mathbb{C}$  and let  $\widehat{B}_j \subset \widehat{B}$  be a unitary symmetry adapted basis of an isotypic component  $V_j$  of  $V \otimes_{\mathbb{R}} \mathbb{C}$ . Then, by Proposition 6.8, the submatrix of  $\phi^{\widehat{B}}$  relating to an isotypic component  $V_j$  is

$$I_{n_j} \otimes \left( \mu_{st}^j \right)_{1 \leq s, t \leq c_j},$$

where  $c_j$  is the number of times that the irreducible  $n_j$ -dimensional representation associated with  $V_j$  occurs in the representation  $\mathcal{V}$  on  $V \otimes_{\mathbb{R}} \mathbb{C}$ .

As described in Section 6.4, for each component  $U_j$ , we can construct an orthogonal symmetry adapted basis  $B_j$  from the unitary symmetry adapted bases  $\widehat{B}_j$ . Following this construction and Proposition 6.8, we immediately have

$$\varphi(u, v) = 0 \quad \forall (u, v) \in U_i \times U_j \text{ with } i \neq j.$$

We distinguish now two cases:

1. The component  $U_j$  is associated with an absolutely irreducible  $n_j$ -dimensional representation that occurs  $c_j$  times in the representation  $\mathcal{V}$  on the  $\mathbb{R}$ -vector space  $V$ . Following the construction of the orthogonal symmetry adapted basis  $B_j$  in Section 6.4, the submatrix of  $\varphi^B$  relating to the component  $U_j$  is

$$I_{n_j} \otimes \left( \nu_{st}^j \right)_{1 \leq s, t \leq c_j}.$$

2. The component  $U_j$  is associated with an irreducible  $2n_j$ -dimensional representation of complex type that occurs  $c_j$  times in the representation  $\mathcal{V}$  on the  $\mathbb{R}$ -vector space  $V$ . Let  $z_1, z_2 \in \widehat{B}_j$ . We have then  $u_1, u_2, v_1, v_2 \in B_j$ , where  $u_i = \operatorname{Re}(z_i)$  and  $v_i = \operatorname{Im}(z_i)$  for  $i = 1, 2$ . If  $z_1$  and  $z_2$  do not belong to the same subspace  $V_{j,\alpha}$  in the decomposition (6.2) of the isotypic component  $V_j$  of  $V \otimes_{\mathbb{R}} \mathbb{C}$ , then

$$\phi(u_1, u_2) = \phi(u_1, v_2) = \phi(v_1, u_2) = \phi(v_1, v_2) = 0.$$

Thus, the submatrix of  $\varphi^B$  relating to the component  $U_j$  consists of  $n_j$  identical blocks of size  $2c_j$ . In addition, since  $z_1$  and  $\overline{z_1}$  do not belong to the same isotypic component of  $V \otimes_{\mathbb{R}} \mathbb{C}$ , we have

$$\phi(\overline{z_1}, z_2) = 0,$$

or equivalently

$$\varphi(u_1, u_2) - \varphi(v_1, v_2) - i \varphi(v_1, u_2) - i \varphi(u_1, v_2) = 0.$$

This means that

$$\varphi(u_1, u_2) = \varphi(v_1, v_2) \text{ and } \varphi(v_1, u_2) = -\varphi(u_1, v_2)$$

so that every block of size  $2c_j$  has the expected form  $\begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix}$ .

□



## 7 Matrix of multiplicities

This section is devoted to the introduction of the matrix of multiplicities of a finite group  $G$  and its computation when the group is cyclic or dihedral. It is the key in the determination of the block size of the matrix of the Hankel operator expressed in an orthogonal symmetry adapted basis. Ultimately it provides us with preliminary criteria for the existence of symmetric cubatures.  $\mathbb{K}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ .

The entries of this matrix are the multiplicities of the irreducible inequivalent representations  $\mathcal{V}_1, \dots, \mathcal{V}_N$  of the group  $G$  in specific representations: the *types*  $\mathcal{P}_1, \dots, \mathcal{P}_T$  of the group  $G$ . They are the permutation representations associated with the coset spaces  $G/H_1, \dots, G/H_T$ , where  $H_1, \dots, H_T$  form a maximal family of non-conjugate subgroups of  $G$ . A column of this matrix gives the multiplicities of all irreducible inequivalent representations of  $G$  in a type.

The coset spaces  $G/H_1, \dots, G/H_T$  and the orbits of any action of the group  $G$  are closely linked. It is then possible to describe the permutation representation of  $G$  associated with any invariant finite set: the multiplicity of every irreducible inequivalent representation (over  $\mathbb{R}$  or over  $\mathbb{C}$ ) in the permutation representation can be expressed in terms of the entries of the matrix of multiplicities and a description related to the types  $\mathcal{P}_1, \dots, \mathcal{P}_T$  of the finite set.

In the context of cubatures in the plane, the natural symmetries to consider are the actions of the cyclic groups  $C_m$  or dihedral groups  $D_m$ . The matrices of multiplicities of  $C_m$  and  $D_m$ , with  $m \geq 2$ , are therefore computed.

### 7.1 Permutation representations

Let  $G$  be a finite group with neutral element  $1_G$  acting on a finite set  $\mathcal{O}$ . This means that each element  $g \in G$  permutes the elements of  $\mathcal{O}$  and that the following identities are satisfied

$$1_G \zeta = \zeta, \quad g_1(g_2\zeta) = (g_1g_2)\zeta, \quad \forall g_1, g_2 \in G, \forall \zeta \in \mathcal{O}.$$

Let  $V$  be a  $\mathbb{K}$ -vector space having a basis  $(e_\zeta)_{\zeta \in \mathcal{O}}$  indexed by the elements of  $\mathcal{O}$ . For  $g \in G$ , let  $\mathcal{V}(g)$  be the linear map from  $V$  into  $V$  which sends  $e_\zeta$  to  $e_{g\zeta}$ ; the linear representation of  $G$  thus obtained is called the *permutation representation* associated to  $\mathcal{O}$ . Its representing matrices in the basis  $(e_\zeta)_{\zeta \in \mathcal{O}}$  have the property that each row and each column has exactly one entry 1, the remaining entries being 0. They are called *permutation matrices*. The character  $\chi$  of a permutation representation is

$$\chi(g) = \text{number of elements of } \mathcal{O} \text{ fixed by } g, \quad \forall g \in G.$$

The *regular representation* of a finite group  $G$  (see *e.g.* [98, Chapter 1.2 & Chapter 2.4] [33, Chapter 5.6.1]) is an example of a permutation representation. Let  $|G|$  be the order of  $G$  and let  $V$  be a  $|G|$ -dimensional  $\mathbb{C}$ -vector space with a basis  $(e_h)_{h \in G}$  indexed by the elements  $h$  of the group  $G$ . The regular representation of the group  $G$  is defined by the linear maps

$$\mathcal{V}(g) : V \rightarrow V, e_h \mapsto e_{gh} \quad \forall g \in G.$$

Its character  $\chi$  is given by

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1_G \\ 0 & \text{if } g \neq 1_G. \end{cases}$$

With the help of this character and Theorem 6.3, the number of times that every irreducible inequivalent representation  $\mathcal{V}_j$  of the group  $G$  occurs in the regular representation is given by its dimension  $n_j$ . Thus, the following relation is satisfied

$$|G| = \sum_{j=1}^{\underline{N}} n_j^2,$$

where  $\underline{N}$  is the number of irreducible inequivalent representations.

Similarly, given a subgroup  $H$  of a finite group  $G$ , consider a  $\mathbb{K}$ -vector space  $V$  with a basis  $(e_\zeta)_{\zeta \in G/H}$  indexed by the elements  $\zeta$  of the coset space  $G/H = \{gH \mid g \in G\}$ . The linear maps

$$\mathcal{V}(g) : V \rightarrow V, e_\zeta \mapsto e_{g\zeta} \quad \forall g \in G \quad (7.1)$$

are well-defined and define a permutation representation of the finite group  $G$  on  $V$ .

**Lemma 7.1.** *Assuming that  $H$  is a normal subgroup of  $G$ , that is*

$$ghg^{-1} \in H \quad \forall h \in H, \forall g \in G,$$

*then the character  $\chi_H$  of the permutation representation defined by (7.1) is given by*

$$\chi_H(g) = \begin{cases} |G/H| & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

*Proof.* Since  $\chi_H$  is the character of a permutation representation, we have

$$\chi_H(g) = \text{number of elements of } G/H \text{ fixed by the left multiplication by } g, \quad \forall g \in G.$$

For any fixed element  $g_0H \in G/H$ , we look therefore for conditions on  $g \in G$  such that

$$gg_0H = g_0H,$$

which means that there exist  $h_0, h_1 \in H$  such that

$$gg_0h_0 = g_0h_1.$$

Thus, we have

$$gg_0H = g_0H \text{ if and only if } \exists h_0, h_1 \in H, g = g_0h_1h_0^{-1}g_0^{-1}.$$

Since  $H$  is a normal subgroup of  $G$ , we have

$$gg_0H = g_0H \text{ if and only if } \exists h \in H, g = h.$$

□

Let  $T$  be the number of subgroups of a given finite group  $G$  up to conjugacy and let  $H_1, \dots, H_T$  be subgroups of  $G$  that are not pairwise conjugate. For every  $k = 1, \dots, T$ , the subgroup  $H_k$  is associated with its coset space  $G/H_k = \{gH_k \mid g \in G\}$  and therefore with a permutation representation, called *type* and denoted by  $\mathcal{P}_k$ .

For every irreducible inequivalent representation  $\mathcal{V}_j$  ( $j = 1, \dots, \underline{N}$ ) of  $G$ , its multiplicity  $\gamma_{jk}$  in  $\mathcal{P}_k$  can be computed with Theorem 6.3. We define the *matrix of multiplicities* of a finite group  $G$  as the  $\underline{N} \times T$  integer matrix

$$\Gamma_G = (\gamma_{jk})_{1 \leq j \leq \underline{N}, 1 \leq k \leq T},$$

where the  $j^{th}$  row is associated with the irreducible representation  $\mathcal{V}_j$  of  $G$  and the  $k^{th}$  column corresponds to the type  $\mathcal{P}_k$ .

**Remark 7.2.** Assume that the irreducible representation  $\mathcal{V}_1$  is the one defined by  $\mathcal{V}_1(g) = 1$  for all  $g \in G$ . Then Theorem 6.3 implies that

$$\gamma_{1k} = 1 \quad \forall k = 1, \dots, T.$$

## 7.2 Orbits and isotropy subgroups

Let  $G$  be a finite group acting on a  $\mathbb{K}$ -vector space  $V$ . For every point  $\zeta \in V$ , the set

$$G_\zeta = \{g \in G \mid g\zeta = \zeta\}$$

is called the *isotropy subgroup* of  $\zeta$ . Points on the same orbit have conjugate isotropy subgroups [46, Chapter XIII, Lemma 1.1]. More precisely,

$$G_{g\zeta} = gG_\zeta g^{-1} \quad \forall g \in G, \forall \zeta \in V. \quad (7.2)$$

The set of all points of  $V$  that have conjugate isotropy subgroups is called an *orbit type* of the action.

The *orbit-stabilizer theorem* [46, Chapter XIII, Proposition 1.2] shows that there is a bijection between the orbit  $\mathcal{O}_\zeta = \{g\zeta \in V \mid g \in G\}$  of a point  $\zeta \in V$  and the coset space  $G/G_\zeta = \{gG_\zeta \mid g \in G\}$  of the subgroup  $G_\zeta$  of  $G$  given by

$$\mathring{f}_\zeta : \mathcal{O}_\zeta \rightarrow G/G_\zeta, g\zeta \mapsto gG_\zeta.$$

This bijection is furthermore an *equivariant map*, that is it satisfies

$$\mathring{f}_\zeta(gy) = g\mathring{f}_\zeta(y) \quad \forall g \in G, \forall y \in \mathcal{O}_\zeta.$$

For every point  $\zeta \in V$ , the permutation representations associated with  $\mathcal{O}_\zeta$  and  $G/G_\zeta$  are then equivalent. The next result is a direct consequence.

**Proposition 7.3.** *Let  $\mathcal{O}$  be a union of orbits of the action of a finite group  $G$  on a  $\mathbb{C}$ -vector space (resp. on a  $\mathbb{R}$ -vector space). Assume that there are  $m_k$  orbits associated with the type  $\mathcal{P}_k$ . Then, for every  $j = 1, \dots, \underline{N}$  (resp. for every  $j = 1, \dots, N$ ), the multiplicity  $c_j$  of the irreducible representation  $\mathcal{V}_j$  (resp.  $\mathcal{V}^{(j)}$ ) in the permutation representation associated with  $\mathcal{O}$  is given by*

$$c_j = \sum_{k=1}^T m_k \gamma_{jk},$$

where  $T$  is the number of types of  $G$  and  $(\gamma_{jk})_{1 \leq j \leq \underline{N}, 1 \leq k \leq T}$  is the matrix of multiplicities of  $G$ .

## 7.3 The cyclic group $C_m$ with $m \geq 2$

**Presentation** (see *e.g.* [98, Chapter 5.1] [33, Chapter 1.9.1])

The cyclic group  $C_m$  is the group of order  $m$  consisting of the powers  $1, g, \dots, g^{m-1}$  of an element  $g$  such that  $g^m = 1$ . It can be realized as the group of rotations of the plane (resp. of the space) around a fixed point (resp. a fixed axis) through angles  $\frac{2\ell\pi}{m}$  with  $\ell = 0, \dots, m-1$ . It is an abelian group. The irreducible representations of  $C_m$  are therefore one-dimensional. There are  $m$  irreducible inequivalent 1-dimensional representations  $\mathcal{V}_j$  of characters  $\chi_j$  defined by

$$\chi_j(g^\ell) = e^{(j-1)\ell \frac{2i\pi}{m}} \quad \forall \ell = 0, \dots, m-1.$$

## Subgroups

Every subgroup of the cyclic group  $C_m$  is isomorphic to the cyclic group  $C_a$  with  $a$  a divisor of  $m$ . We consider that the subgroup consisting of the neutral element is isomorphic to  $C_1$ . Moreover, for every divisor  $a$  of  $m$ , there is a subgroup of  $C_m$  of order  $a$  generated by the element  $g^{\frac{m}{a}}$  of  $C_m$ :

$$\{1, g^{\frac{m}{a}}, \dots, g^{(a-1)\frac{m}{a}}\}.$$

The latter is the unique subgroup (up to conjugacy) isomorphic to  $C_a$ .

## Coset spaces

Let  $a$  be a divisor of  $m$ . For ease of notation, we identify  $C_a$  with the unique subgroup of  $C_m$  of order  $a$ . The cyclic group  $C_a$  is abelian. It is therefore a normal subgroup. Thus, the coset space

$$C_m/C_a = \{[1], [g], \dots, [g^{\frac{m}{a}-1}]\} = \{[1], [g^{\frac{m}{a}}-1], \dots, [g^{(\frac{m}{a}-1)(\frac{m}{a}-1)}]\}$$

is a group of order  $\frac{m}{a}$ , generated by  $[g^{\frac{m}{a}-1}]$  and therefore isomorphic to  $C_{\frac{m}{a}}$ . Notice that each class has exactly  $a$  elements of  $C_m$ .

## Types

Let  $a_1, \dots, a_T$  be the divisors of  $m$ . The subgroups of  $C_m$  are isomorphic to  $C_{a_1}, \dots, C_{a_T}$ . Each one is unique up to conjugacy and is therefore associated with a type  $\mathcal{P}_1, \dots, \mathcal{P}_T$ : the permutation representation associated with the coset space isomorphic to  $C_{\frac{m}{a_1}}, \dots, C_{\frac{m}{a_T}}$ . For any  $k = 1, \dots, T$ , the type  $\mathcal{P}_k$  is defined by its character  $\chi^{(k)} : C_m \rightarrow \mathbb{C}$ . Since  $C_{a_k}$  is a normal subgroup of  $C_m$ , Lemma 7.1 implies

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{m}{a_k} & \text{if } \frac{m}{a_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \forall \ell = 0, \dots, m-1.$$

## Matrix of multiplicities $\Gamma_{C_m} = (\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$

Let  $j = 1, \dots, m$  (here  $\underline{N} = m$ ) and let  $k = 1, \dots, T$ . The multiplicity  $\gamma_{jk}$  can be computed thanks to Theorem 6.3

$$\gamma_{jk} = \frac{1}{|C_m|} \sum_{g \in C_m} \chi^{(k)}(g) \overline{\chi_j(g)} = \frac{1}{m} \sum_{\ell=0}^{a_k-1} \chi^{(k)}(g^{\ell \frac{m}{a_k}}) \overline{\chi_j(g^{\ell \frac{m}{a_k}})} = \frac{1}{a_k} \sum_{\ell=0}^{a_k-1} e^{-\frac{(j-1)\ell}{a_k} 2i\pi}.$$

If  $a_k$  divides  $j-1$ , then

$$e^{-(j-1)\frac{\ell}{a_k} 2i\pi} = 1.$$

If  $a_k$  does not divide  $j-1$ , then

$$\sum_{\ell=0}^{a_k-1} e^{-(j-1)\frac{\ell}{a_k} 2i\pi} = 0.$$

Thus,

$$\gamma_{jk} = \begin{cases} 1 & \text{if } a_k \text{ divides } j - 1 \\ 0 & \text{otherwise .} \end{cases}$$

**Example : matrix of multiplicities  $\Gamma_{C_6}$**

Let  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$  be the permutation representations respectively associated with the coset spaces isomorphic to  $C_1, C_2, C_3, C_6$ . The matrix of multiplicities is then given by

$$\Gamma_{C_6} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**An action of  $C_m$  on  $\mathbb{R}^2$ : realization of  $C_m$  as the rotations around a fixed point**

Only  $C_1$  and  $C_m$  appear as isotropy subgroups for the action of  $C_m$  on  $\mathbb{R}^2$  by rotation: the only point fixed by  $C_m$  is the origin. The corresponding submatrix of  $\Gamma_{C_m}$  is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.$$

**Remark on reflection**

Consider the action of  $C_2$  on  $\mathbb{R}^2$  whose generator is given by the reflection through an axis. The isotropy subgroups are again only  $C_1$  and  $C_2$ . Notice that, in this example, the points fixed by  $C_2$  is not reduced to the origin (or to a unique point), but it consists of every point on the axis of reflection.

## 7.4 The dihedral group $D_m$ with $m \geq 2$

**Presentation** (see *e.g.* [98, Chapter 5.3] [33, Chapter 1.9.2])

The dihedral group  $D_m$  is the group of rotations and reflections of the plane that preserve a regular polygon with  $m$  vertices. It contains  $m$  rotations, which form a subgroup isomorphic to  $C_m$ , and  $m$  reflections. Its order is  $2m$ . If we denote by  $g$  the rotation through an angle  $\frac{2\pi}{m}$  and if  $h$  is any reflection, we have:

$$g^m = 1, \quad h^2 = 1, \quad hgh = g^{-1}.$$

Each element of  $D_m$  can be written uniquely, either in the form  $g^\ell$ , with  $\ell = 0, \dots, m-1$  (if it is a rotation), or in the form  $g^\ell h$ , with  $\ell = 0, \dots, m-1$  (if it is a reflection).

## Subgroups

Every subgroup of the dihedral group  $D_m$  is either isomorphic to the cyclic group  $C_a$ , or isomorphic to the dihedral group  $D_a$ , where  $a$  is a divisor of  $m$ . Notice that  $D_1$  is isomorphic to  $C_2$ .

If  $m$  is odd, then, for all  $\ell = 0, \dots, m-1$ , the reflections  $g^\ell h$  are conjugate (see in [46, Chapter XII.5(b)]). Thus, for every  $a$  that divides  $m$ , there are up to conjugacy a unique subgroup isomorphic to the cyclic group  $C_a$  and a unique subgroup isomorphic to the dihedral group  $D_a$ .

If  $m$  is even, then, for all  $\ell = 0, \dots, \frac{m}{2} - 2$ , all reflections  $g^{2\ell} h$  are conjugate, all reflections  $g^{2\ell+1} h$  are conjugate, but  $h$  and  $gh$  are not conjugate (see [46, Chapter XII.5(b)]). Thus, for every  $a$  that divides  $m$ , there are up to conjugacy a unique subgroup isomorphic to the cyclic group  $C_a$  ( $a \neq 2$ ) and one or two subgroups isomorphic to the dihedral group  $D_a$  ( $a \neq 1$ ).

More precisely, there are two subgroups isomorphic to  $D_a$  that are not conjugate if  $\frac{m}{a}$  is even ( $a \neq 1$ ) and there is one subgroup isomorphic to  $D_a$  if  $\frac{m}{a}$  is odd. Indeed, consider the two following subgroups of  $D_m$ :

$$D_a(h) = \{1, g^{\frac{m}{a}}, \dots, g^{(a-1)\frac{m}{a}}, h, g^{\frac{m}{a}}h, \dots, g^{(a-1)\frac{m}{a}}h\}$$

and

$$D_a(gh) = \{1, g^{\frac{m}{a}}, \dots, g^{(a-1)\frac{m}{a}}, gh, g^{\frac{m}{a}+1}h, \dots, g^{(a-1)\frac{m}{a}+1}h\}.$$

If  $\frac{m}{a}$  is even,  $D_a(h)$  only contains reflections  $g^\ell h$  with  $\ell$  even, whereas  $D_a(gh)$  only contains reflections  $g^\ell h$  with  $\ell$  odd. Thus, the two subgroups are not conjugate. If  $\frac{m}{a}$  is odd, both subgroups contain reflections  $g^\ell h$  with  $\ell$  odd and reflections  $g^\ell h$  with  $\ell$  even. Thus, the two subgroups are conjugate.

In addition, if  $m$  is even, there are up to conjugacy three subgroups isomorphic to  $C_2$  (or  $D_1$ ):

$$\{1, g^{\frac{m}{2}}\}, \quad D_1(h) = \{1, h\}, \quad D_1(gh) = \{1, gh\}.$$

## Coset spaces

Let  $a$  be a divisor of  $m$ . In the following, for ease of notation, we identify  $C_a$  with the unique subgroup of  $D_m$  isomorphic to  $C_a$ ,  $D_a$  with the subgroup of  $D_m$  isomorphic to  $D_a$  if its unique and we keep the notation  $D_a(h)$  and  $D_a(gh)$  introduced above if there are two subgroups isomorphic to  $D_a$ .

$C_a$  is a normal subgroup of  $D_m$ . The coset space

$$D_m/C_a = \{[1], [g^{\frac{m}{a}-1}], \dots, [g^{(\frac{m}{a}-1)(\frac{m}{a}-1)}], [h], [g^{\frac{m}{a}-1}h], \dots, [g^{(\frac{m}{a}-1)(\frac{m}{a}-1)}h]\}$$

is then a group of order  $2\frac{m}{a}$ , generated by  $[g^{\frac{m}{a}-1}]$  and  $[h]$ . Since its elements are either in the form  $[g^{\frac{m}{a}-1}]^\ell$  with  $\ell = 0, \dots, \frac{m}{a}$ , or in the form  $[g^{\frac{m}{a}-1}]^\ell[h]$  with  $\ell = 0, \dots, \frac{m}{a}$ , the coset space  $D_m/C_a$  is isomorphic to the dihedral group  $D_{\frac{m}{a}}$ . Notice that each class has exactly  $a$  elements of  $D_m$ .

The coset spaces  $D_m/D_a$ ,  $D_m/D_a(h)$  and  $D_m/D_a(gh)$  can all be written as

$$\{[1], [g], \dots, [g^{\frac{m}{a}-1}]\}.$$

Each class has exactly  $2a$  elements of  $D_m$ . However, those elements are different according to the coset space. For instance, for  $D_m/D_a$  and  $D_m/D_a(h)$ ,

$$[1] = \{1, g^{\frac{m}{a}}, \dots, g^{(a-1)\frac{m}{a}}, h, g^{\frac{m}{a}}h, \dots, g^{(a-1)\frac{m}{a}}h\},$$

whereas for  $D_m/D_a(gh)$ ,

$$[1] = \{1, g^{\frac{m}{a}}, \dots, g^{(a-1)\frac{m}{a}}, gh, g^{\frac{m}{a}+1}h, \dots, g^{(a-1)\frac{m}{a}+1}h\}.$$

#### 7.4.1 Case $m$ odd

There are 2 irreducible inequivalent representations of dimension 1, denoted by  $\mathcal{V}_1, \mathcal{V}_2$ , and  $\frac{m-1}{2}$  irreducible inequivalent representations of dimension 2, denoted by  $\mathcal{V}_3, \dots, \mathcal{V}_{2+\frac{m-1}{2}}$  [98, Chapter 5.3]. They are defined by:

$$\begin{aligned} \mathcal{V}_1(g) &= 1 & \mathcal{V}_1(h) &= 1, \\ \mathcal{V}_2(g) &= 1 & \mathcal{V}_2(h) &= -1, \\ \mathcal{V}_j(g) &= \begin{pmatrix} \cos(\frac{2(j-2)\pi}{m}) & -\sin(\frac{2(j-2)\pi}{m}) \\ \sin(\frac{2(j-2)\pi}{m}) & \cos(\frac{2(j-2)\pi}{m}) \end{pmatrix} & \mathcal{V}_j(h) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \forall j = 3, \dots, 2 + \frac{m-1}{2}. \end{aligned}$$

### Types

Let  $a_1, \dots, a_t$  (here  $t = \frac{T}{2}$ ) be the divisors of  $m$ . The subgroups of  $D_m$  are isomorphic to  $D_{a_1}, \dots, D_{a_t}, C_{a_1}, \dots, C_{a_t}$ . Each one is unique up to conjugacy and is therefore associated with a type  $\mathcal{P}_1, \dots, \mathcal{P}_T$ : the permutation representation associated with the coset space  $D_m/D_{a_1}, \dots, D_m/D_{a_t}, D_m/C_{a_1}, \dots, D_m/C_{a_t}$ .

**Lemma 7.4.** *The type  $\mathcal{P}_k$  is defined by its character  $\chi^{(k)} : D_m \rightarrow \mathbb{C}$ .*

- If  $k = 1, \dots, t$  and  $\ell = 0, \dots, m-1$ , then

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{m}{a_k} & \text{if } \frac{m}{a_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = 1.$$

- If  $k = t+1, \dots, T$  and  $\ell = 0, \dots, m-1$ , then let  $\alpha_k = a_{k-t}$

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{2m}{\alpha_k} & \text{if } \frac{m}{\alpha_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = 0.$$

*Proof.* • If  $k = t+1, \dots, T$ , then  $\mathcal{P}_k$  is the permutation representation associated with the coset space  $D_m/C_{\alpha_k}$ , where  $\alpha_k = a_{k-t}$ . Since  $C_{\alpha_k}$  is a normal subgroup of  $D_m$ , Lemma 7.1 implies for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} |D_m/C_{\alpha_k}| & \text{if } g^\ell \in C_{\alpha_k} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2m}{\alpha_k} & \text{if } \frac{m}{\alpha_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \chi^{(k)}(g^\ell h) = 0.$$

• If  $k = 1, \dots, t$ , then  $\mathcal{P}_k$  is the permutation representation associated with the coset space  $D_m/D_{a_k}$  and

$$\chi^{(k)}(g_0) = \text{number of elements in } D_m/D_{a_k} \text{ fixed by the left multiplication by } g_0 \in D_m.$$

We look therefore for the number of elements  $[g^j] \in D_m/D_{a_k}$  with  $j = 0, \dots, \frac{m}{a_k} - 1$  that satisfy

$$g_0[g^j] = [g^j].$$

Since the left multiplication is here well-defined, we have for every  $j = 0, \dots, \frac{m}{a_k} - 1$

$$g_0[g^j] = [g^j] \text{ if and only if } g_0g^j \in [g^j].$$

If  $g_0$  is a rotation of  $D_m$ , that is there is  $\ell \in \{0, \dots, m-1\}$  such that  $g_0 = g^\ell$ , then for every  $j = 0, \dots, \frac{m}{a_k} - 1$

$$g^\ell g^j \in [g^j] \text{ if and only if } \frac{m}{a_k} \text{ divides } \ell.$$

If  $g_0$  is a reflection of  $D_m$ , that is there is  $\ell \in \{0, \dots, m-1\}$  such that  $g_0 = g^\ell h$ , then for every  $j = 0, \dots, \frac{m}{a_k} - 1$

$$\begin{aligned} g^\ell h g^j \in [g^j] & \text{ if and only if } g^\ell g^{-j} h \in [g^j] \\ & \text{ if and only if } \exists k_0, g^{\ell-j} h = g^{j+k_0 \frac{m}{a_k}} h \\ & \text{ if and only if } \frac{m}{a_k} \text{ divides } \ell - 2j. \end{aligned}$$

Thus, we have for every  $\ell = 0, \dots, \frac{m}{a_k} - 1$

$$\chi^{(k)}(g^\ell h) = \text{number of multiples of } \frac{m}{a_k} \text{ in } \mathcal{S}_\ell = \left\{ \ell - 2j \mid j = 0, \dots, \frac{m}{a_k} - 1 \right\}. \quad (7.3)$$

Since  $\mathcal{S}_\ell$  is included in an interval of length  $2 \left( \frac{m}{a_k} - 1 \right)$ , which is smaller than  $2 \frac{m}{a_k}$ , the number of multiples of  $\frac{m}{a_k}$  in  $\mathcal{S}_\ell$  is at most 2. Since  $\frac{m}{a_k}$  is odd, two consecutive multiples of  $\frac{m}{a_k}$  are even and odd. Since  $\mathcal{S}_\ell$  contains either only odd integers if  $\ell$  is odd or only even integers if  $\ell$  is even, there is a unique multiple of  $\frac{m}{a_k}$  in  $\mathcal{S}_\ell$ . This implies that

$$\chi^{(k)}(g^\ell h) = 1 \quad \forall \ell = 0, \dots, \frac{m}{a_k} - 1.$$

□

**Matrix of multiplicities**  $\Gamma_{D_m} = (\gamma_{jk})_{1 \leq j \leq \underline{N}, 1 \leq k \leq T}$

Let  $j = 1, \dots, \underline{N}$  and let  $k = 1, \dots, t$ . The multiplicity  $\gamma_{jk}$  can be computed thanks to Theorem 6.3

$$\gamma_{jk} = \frac{1}{|D_m|} \sum_{g \in D_m} \chi^{(k)}(g) \overline{\chi_j(g)} = \frac{1}{2m} \left( \sum_{\ell=0}^{a_k-1} \chi^{(k)}(g^{\ell \frac{m}{a_k}}) \overline{\chi_j(g^{\ell \frac{m}{a_k}})} + \sum_{\ell=0}^{m-1} \chi^{(k)}(g^\ell h) \overline{\chi_j(g^\ell h)} \right).$$

For  $j = 1$ ,

$$\gamma_{1k} = \frac{1}{2m} \left( \sum_{\ell=0}^{a_k-1} \frac{m}{a_k} \cdot 1 + \sum_{\ell=0}^{m-1} 1 \cdot 1 \right) = 1.$$



For  $j = 2$ ,

$$\gamma_{2k} = \frac{1}{2m} \left( \sum_{\ell=0}^{a_k-1} \frac{m}{a_k} \cdot 1 + \sum_{\ell=0}^{m-1} 1 \cdot (-1) \right) = 0.$$

For  $j = 3, \dots, 2 + \frac{m-1}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{a_k-1} \frac{m}{a_k} \cdot 2 \cos \left( \frac{2(j-2)\pi}{m} \frac{lm}{a_k} \right) + \sum_{\ell=0}^{m-1} 1 \cdot 0 \right) = \begin{cases} 1 & \text{if } a_k \text{ divides } j-2 \\ 0 & \text{otherwise} \end{cases}.$$

Let  $j = 1, \dots, \underline{N}$  and let  $k = t+1, \dots, T$ . The multiplicity  $\gamma_{jk}$  can be computed thanks to Theorem 6.3

$$\begin{aligned} \gamma_{jk} &= \frac{1}{|D_m|} \sum_{g \in D_m} \chi^{(k)}(g) \overline{\chi_j(g)} \\ &= \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \chi^{(k)}(g^{\ell \frac{m}{\alpha_k}}) \overline{\chi_j(g^{\ell \frac{m}{\alpha_k}})} + \sum_{\ell=0}^{\alpha_k-1} \chi^{(k)}(g^{\ell \frac{m}{\alpha_k}} h) \overline{\chi_j(g^{\ell \frac{m}{\alpha_k}} h)} \right). \end{aligned}$$

For  $j = 1$ ,

$$\gamma_{1k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 1 + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot 1 \right) = 1.$$

For  $j = 2$ ,

$$\gamma_{2k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 1 + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot (-1) \right) = 1.$$

For  $j = 3, \dots, 2 + \frac{m-1}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 2 \cos \left( \frac{2(j-2)\pi}{m} \frac{lm}{\alpha_k} \right) + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot 0 \right) = \begin{cases} 2 & \text{if } \alpha_k \text{ divides } j-2 \\ 0 & \text{otherwise} \end{cases}.$$

### Example : matrix of mutliplicities $\Gamma_{D_3}$

Let  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$  be the permutation representations respectively associated with the coset spaces isomorphic to  $C_1, C_3, D_1, D_3$ . The matrix of multiplicities is then given by

$$\Gamma_{D_3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

### Action of $D_m$ on $\mathbb{R}^2$ that fix a polygon with $m$ vertices

This action implies a representation  $\mathcal{W}$  of  $D_m$  on  $\mathbb{R}^2$ : it is generated by

$$\mathcal{W}(g) = \begin{pmatrix} \cos(\frac{2\pi}{m}) & -\sin(\frac{2\pi}{m}) \\ \sin(\frac{2\pi}{m}) & \cos(\frac{2\pi}{m}) \end{pmatrix} \text{ and } \mathcal{W}(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and is therefore given by

$$[\mathcal{W}(g)^0, \mathcal{W}(g)^1, \dots, \mathcal{W}(g)^{m-1}, \mathcal{W}(g)^0 \mathcal{W}(h), \mathcal{W}(g)^1 \mathcal{W}(h), \dots, \mathcal{W}(g)^{m-1} \mathcal{W}(h)].$$

The isotropy subgroups associated with this action of  $D_m$  on  $\mathbb{R}^2$  are  $C_1$ ,  $D_1$  and  $D_m$ . The unique point fixed by  $D_m$  is the center of gravity of the polygon and the points fixed by a subgroup isomorphic to  $D_1$  are the ones on the symmetry axis of the polygon (see [46, Chapter XIII.5] for more details).

The corresponding submatrix of  $\Gamma_{D_m}$  is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 2 \end{pmatrix}.$$

In the case  $m = 3$ , the orbit types are then characterized by the 3 cases in Figure 11.

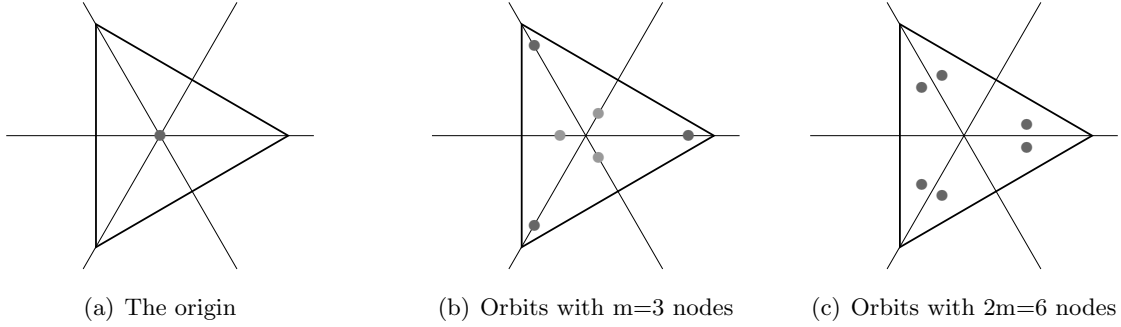


Figure 11: Orbit types for the action of  $D_3$  on  $\mathbb{R}^2$

#### 7.4.2 Case $m$ even

There are 4 irreducible inequivalent representations of dimension 1, denoted by  $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$ , and  $\frac{m}{2} - 1$  irreducible inequivalent representations of dimension 2, denoted by  $\mathcal{V}_5, \dots, \mathcal{V}_{3+\frac{m}{2}}$  if  $m \neq 2$  [33, Chapter 1.9] [98, Chapter 5.3]. Notice that if  $m = 2$ , there is no irreducible inequivalent representation of dimension 2. The irreducible representations  $\mathcal{V}_j$  are defined by:

$$\begin{aligned} \mathcal{V}_1(g) &= 1 & \mathcal{V}_1(h) &= 1, \\ \mathcal{V}_2(g) &= 1 & \mathcal{V}_2(h) &= -1, \\ \mathcal{V}_3(g) &= -1 & \mathcal{V}_3(h) &= 1, \\ \mathcal{V}_4(g) &= -1 & \mathcal{V}_4(h) &= -1, \\ \mathcal{V}_j(g) &= \begin{pmatrix} \cos(\frac{2(j-4)\pi}{m}) & -\sin(\frac{2(j-4)\pi}{m}) \\ \sin(\frac{2(j-4)\pi}{m}) & \cos(\frac{2(j-4)\pi}{m}) \end{pmatrix} & \mathcal{V}_j(h) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \forall j = 5, \dots, 3 + \frac{m}{2}. \end{aligned}$$

## Types

Let  $a_1, \dots, a_t$  be the divisors of  $m$ . The subgroups of  $D_m$  are isomorphic to  $D_{a_1}, \dots, D_{a_t}, C_{a_1}, \dots, C_{a_t}$  (with  $C_2 \simeq D_1$ ). They are not necessarily unique up to conjugacy (see above the distinction between  $D_a(h)$  and  $D_a(gh)$  in the case  $\frac{m}{a}$  even). Let  $\tau$  be the number of non-conjugate subgroups isomorphic to  $D_{a_1}, \dots, D_{a_t}$ : we have then  $T = \tau + t$  non-conjugate subgroups of  $D_m$ . Each one is associated with a type  $\mathcal{P}_1, \dots, \mathcal{P}_T$ .

**Lemma 7.5.** *For every  $k = 1, \dots, T$ , the type  $\mathcal{P}_k$  is defined by its character  $\chi^{(k)} : D_m \rightarrow \mathbb{C}$ .*

- If  $k = 1, \dots, \tau$  and if the type  $\mathcal{P}_k$  is the permutation representation associated with a coset space  $D_m/D_a$ , where  $a$  is the appropriate divisor of  $m$ , then for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{m}{a} & \text{if } \frac{m}{a} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = 1.$$

- If  $k = 1, \dots, \tau$  and if the type  $\mathcal{P}_k$  is the permutation representation associated with a coset space  $D_m/D_a(h)$ , where  $a$  is the appropriate divisor of  $m$ , then for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{m}{a} & \text{if } \frac{m}{a} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = \begin{cases} 2 & \text{if } \ell \text{ even} \\ 0 & \text{if } \ell \text{ odd} \end{cases}.$$

- If  $k = 1, \dots, \tau$  and if the type  $\mathcal{P}_k$  is the permutation representation associated with a coset space  $D_m/D_a(gh)$ , where  $a$  is the appropriate divisor of  $m$ , then for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{m}{a} & \text{if } \frac{m}{a} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = \begin{cases} 0 & \text{if } \ell \text{ even} \\ 2 & \text{if } \ell \text{ odd} \end{cases}.$$

- If  $k = \tau + 1, \dots, T$ , let  $\alpha_k = a_{k-\tau}$ , then for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{2m}{\alpha_k} & \text{if } \frac{m}{\alpha_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi^{(k)}(g^\ell h) = 0.$$

*Proof.* • If  $k = \tau + 1, \dots, T$ , then  $\mathcal{P}_k$  is the permutation representation associated with the coset space  $D_m/C_{\alpha_k}$ , where  $\alpha_k = a_{k-\tau}$ . Since  $C_{\alpha_k}$  is a normal subgroup of  $D_m$ , Lemma 7.1 implies for every  $\ell = 0, \dots, m-1$

$$\chi^{(k)}(g^\ell) = \begin{cases} |D_m/C_{\alpha_k}| & \text{if } g^\ell \in C_{\alpha_k} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2m}{\alpha_k} & \text{if } \frac{m}{\alpha_k} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } \chi^{(k)}(g^\ell h) = 0.$$

- If  $k = 1, \dots, \tau$ , then  $\mathcal{P}_k$  is the permutation representation associated with a coset space of the form  $D_m/D_a$ ,  $D_m/D_a(h)$  or  $D_m/D_a(gh)$ , where  $a$  is the corresponding divisor of  $m$ , and

$$\chi^{(k)}(g_0) = \text{number of elements in the coset space fixed by the left multiplication by } g_0 \in D_m.$$

Similarly to the proof of Lemma 7.4, if  $g_0$  is a rotation given by  $g_0 = g^\ell$  with  $\ell \in \{0, \dots, m-1\}$ , we get

$$\chi^{(k)}(g^\ell) = \begin{cases} \frac{2m}{a} & \text{if } \frac{m}{a} \text{ divides } \ell \\ 0 & \text{otherwise} \end{cases}.$$

If  $g_0$  is a reflection given by  $g_0 = g^\ell h$  with  $\ell \in \{0, \dots, m-1\}$ , (7.3) is still satisfied if the coset space is of the form  $D_m/D_a$  or  $D_m/D_a(h)$  and it becomes

$$\chi^{(k)}(g^\ell h) = \text{number of multiples of } \frac{m}{a} \text{ in } \mathcal{S}'_\ell = \left\{ \ell - 2j - 1 \mid j = 0, \dots, \frac{m}{a} - 1 \right\}$$

if the coset space is of the form  $D_m/D_a(gh)$ .

- If the coset space is of the form  $D_m/D_a$ ,  $\frac{m}{a}$  is odd and, using the same ideas as in the proof of Lemma 7.4, we get

$$\chi^{(k)}(g^\ell h) = 1.$$

- If the coset space is of the form  $D_m/D_a(h)$ ,  $\frac{m}{a}$  is even. Two consecutive multiples of  $\frac{m}{a}$  are then even. Since  $\mathcal{S}_\ell$  contains either only even integers if  $\ell$  is even or only odd integers if  $\ell$  is odd, there are either 2 multiples of  $\frac{m}{a}$  in  $\mathcal{S}_\ell$  if  $\ell$  is even or 0 multiple of  $\frac{m}{a}$  in  $\mathcal{S}_\ell$  if  $\ell$  is odd. Thus, we get

$$\chi^{(k)}(g^\ell h) = \begin{cases} 2 & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases}.$$

- If the coset space is of the form  $D_m/D_a(gh)$ ,  $\frac{m}{a}$  is even. Two consecutive multiples of  $\frac{m}{a}$  are then even. Since  $\mathcal{S}'_\ell$  contains either only even integers if  $\ell$  is odd or only odd integers if  $\ell$  is even, there are either 2 multiples of  $\frac{m}{a}$  in  $\mathcal{S}_\ell$  if  $\ell$  is odd or 0 multiple of  $\frac{m}{a}$  in  $\mathcal{S}_\ell$  if  $\ell$  is even. Thus, we get

$$\chi^{(k)}(g^\ell h) = \begin{cases} 0 & \text{if } \ell \text{ is even} \\ 2 & \text{if } \ell \text{ is odd} \end{cases}.$$

□

**Matrix of multiplicities**  $\Gamma_{D_m} = (\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$

Let  $j = 1, \dots, N$  and let  $k = 1, \dots, \tau$ . The multiplicity  $\gamma_{jk}$  of the type  $\mathcal{P}_k$  associated with a coset space  $D_m/D_a$ ,  $D_m/D_a(h)$  or  $D_m/D_a(gh)$  can be computed thanks to Theorem 6.3

$$\begin{aligned} \gamma_{jk} &= \frac{1}{|D_m|} \sum_{g \in D_m} \chi^{(k)}(g) \overline{\chi_j(g)} \\ &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \chi^{(k)}(g^{\ell \frac{m}{a}}) \overline{\chi_j(g^{\ell \frac{m}{a}})} + \sum_{\ell=0}^{m-1} \chi^{(k)}(g^\ell h) \overline{\chi_j(g^\ell h)} \right) \end{aligned}$$

- If  $\mathcal{P}_k$  is associated with a coset space  $D_m/D_a$ , then

$$\begin{aligned} \gamma_{1k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{m-1} 1 \cdot 1 \right) = 1. \\ \gamma_{2k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{m-1} 1 \cdot (-1) \right) = 0. \\ \gamma_{3k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{m-1} 1 \cdot (-1)^\ell \right) = 0. \\ \gamma_{4k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{m-1} 1 \cdot (-1)^{\ell+1} \right) = 0. \end{aligned}$$

For  $j = 5, \dots, 3 + \frac{m}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 2 \cos \left( \frac{2(j-4)\pi \ell m}{m} \right) + \sum_{\ell=0}^{m-1} 1 \cdot 0 \right) = \begin{cases} 2 & \text{if } a \text{ divides } j-4 \\ 0 & \text{otherwise} \end{cases}.$$

- If  $\mathcal{P}_k$  is associated with a coset space  $D_m/D_a(h)$ , then

$$\begin{aligned} \gamma_{1k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot 1 \right) = 1. \\ \gamma_{2k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1) \right) = 0. \\ \gamma_{3k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1)^{2\ell} \right) = 1. \\ \gamma_{4k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1)^{2\ell+1} \right) = 0. \end{aligned}$$

For  $j = 5, \dots, 3 + \frac{m}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 2 \cos \left( \frac{2(j-4)\pi \ell m}{m} \right) + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot 0 \right) = \begin{cases} 2 & \text{if } a \text{ divides } j-4 \\ 0 & \text{otherwise} \end{cases}.$$

- If  $\mathcal{P}_k$  is associated with a coset space  $D_m/D_a(gh)$ , then

$$\begin{aligned} \gamma_{1k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot 1 \right) = 1. \\ \gamma_{2k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 1 + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1) \right) = 0. \\ \gamma_{3k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1)^{2\ell+1} \right) = 0. \\ \gamma_{4k} &= \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot (-1)^{\ell \frac{m}{a}} + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot (-1)^{2\ell+2} \right) = 1. \end{aligned}$$

For  $j = 5, \dots, 3 + \frac{m}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{a-1} \frac{m}{a} \cdot 2 \cos \left( \frac{2(j-4)\pi \ell m}{m} \right) + \sum_{\ell=0}^{\frac{m}{2}-1} 2 \cdot 0 \right) = \begin{cases} 2 & \text{if } a \text{ divides } j-4 \\ 0 & \text{otherwise} \end{cases}.$$

Let  $j = 1, \dots, \underline{N}$  and let  $k = \tau + 1, \dots, T$ . The multiplicity  $\gamma_{jk}$  can be computed thanks to Theorem 6.3

$$\begin{aligned} \gamma_{jk} &= \frac{1}{|D_m|} \sum_{g \in D_m} \chi^{(k)}(g) \overline{\chi_j(g)} \\ &= \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \chi^{(k)}(g^{\ell \frac{m}{\alpha_k}}) \overline{\chi_j(g^{\ell \frac{m}{\alpha_k}})} + \sum_{\ell=0}^{\alpha_k-1} \chi^{(k)}(g^{\ell \frac{m}{\alpha_k}} h) \overline{\chi_j(g^{\ell \frac{m}{\alpha_k}} h)} \right). \end{aligned}$$

For  $j = 1$ ,

$$\gamma_{1k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 1 + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot 1 \right) = 1.$$

For  $j = 2$ ,

$$\gamma_{2k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 1 + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot (-1) \right) = 1.$$

For  $j = 3$ ,

$$\gamma_{3k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot (-1)^{\ell \frac{m}{\alpha_k}} + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot (-1)^{\ell \frac{m}{\alpha_k}} \right) = \begin{cases} 1 & \text{if } \alpha_k \text{ odd} \\ 0 & \text{if } \alpha_k \text{ even} \end{cases}.$$

For  $j = 4$ ,

$$\gamma_{4k} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot (-1)^{\ell \frac{m}{\alpha_k}} + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot (-1)^{\ell \frac{m}{\alpha_k} + 1} \right) = \begin{cases} 1 & \text{if } \alpha_k \text{ odd} \\ 0 & \text{if } \alpha_k \text{ even} \end{cases}.$$

For  $j = 5, \dots, 3 + \frac{m}{2}$ ,

$$\gamma_{jk} = \frac{1}{2m} \left( \sum_{\ell=0}^{\alpha_k-1} \frac{2m}{\alpha_k} \cdot 2 \cos \left( \frac{2(j-4)\pi}{m} \frac{lm}{\alpha_k} \right) + \sum_{\ell=0}^{\alpha_k-1} 0 \cdot 0 \right) = \begin{cases} 2 & \text{if } \alpha_k \text{ divides } j-4 \\ 0 & \text{otherwise} \end{cases}.$$

### Example : matrix of multiplicities $\Gamma_{D_6}$

Let  $\mathcal{P}_1, \dots, \mathcal{P}_{10}$  be the permutation representations respectively associated with the coset spaces  $D_6/D_6, D_6/D_3(h), D_6/D_3(gh), D_6/D_2, D_6/D_1(h), D_6/D_1(gh), D_6/C_6, D_6/C_3, D_6/C_2, D_6/C_1$ . The matrix of multiplicities is then given by

$$\Gamma_{D_6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

### Action of $D_m$ on $\mathbb{R}^2$ that fix a polygon with $m$ vertices

This action implies a representation  $\mathcal{W}$  of  $D_m$  on  $\mathbb{R}^2$ : it is generated by

$$\mathcal{W}(g) = \begin{pmatrix} \cos(\frac{2\pi}{m}) & -\sin(\frac{2\pi}{m}) \\ \sin(\frac{2\pi}{m}) & \cos(\frac{2\pi}{m}) \end{pmatrix} \text{ and } \mathcal{W}(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and is therefore given by

$$[\mathcal{W}(g)^0, \mathcal{W}(g)^1, \dots, \mathcal{W}(g)^{m-1}, \mathcal{W}(g)^0 \mathcal{W}(h), \mathcal{W}(g)^1 \mathcal{W}(h), \dots, \mathcal{W}(g)^{m-1} \mathcal{W}(h)].$$

The isotropy subgroups of this action of  $D_m$  on  $\mathbb{R}^2$  are  $C_1, D_1(h), D_1(gh), D_m$ . The unique point fixed by  $D_m$  is the center of gravity of the polygon and the points fixed by a subgroup

isomorphic to  $D_1$  are the ones on the symmetry axis of the polygon: if the symmetry axis goes through the vertices,  $D_1(h)$  fix its points, whereas if the symmetry axis goes through the middle of the edges,  $D_1(gh)$  fix its points (see [46, Chapter XIII.5] for more details).

The corresponding submatrix of  $\Gamma_{D_m}$  is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

In the case  $m = 6$ , the orbit types are then characterized by the 4 cases in Figure 12.

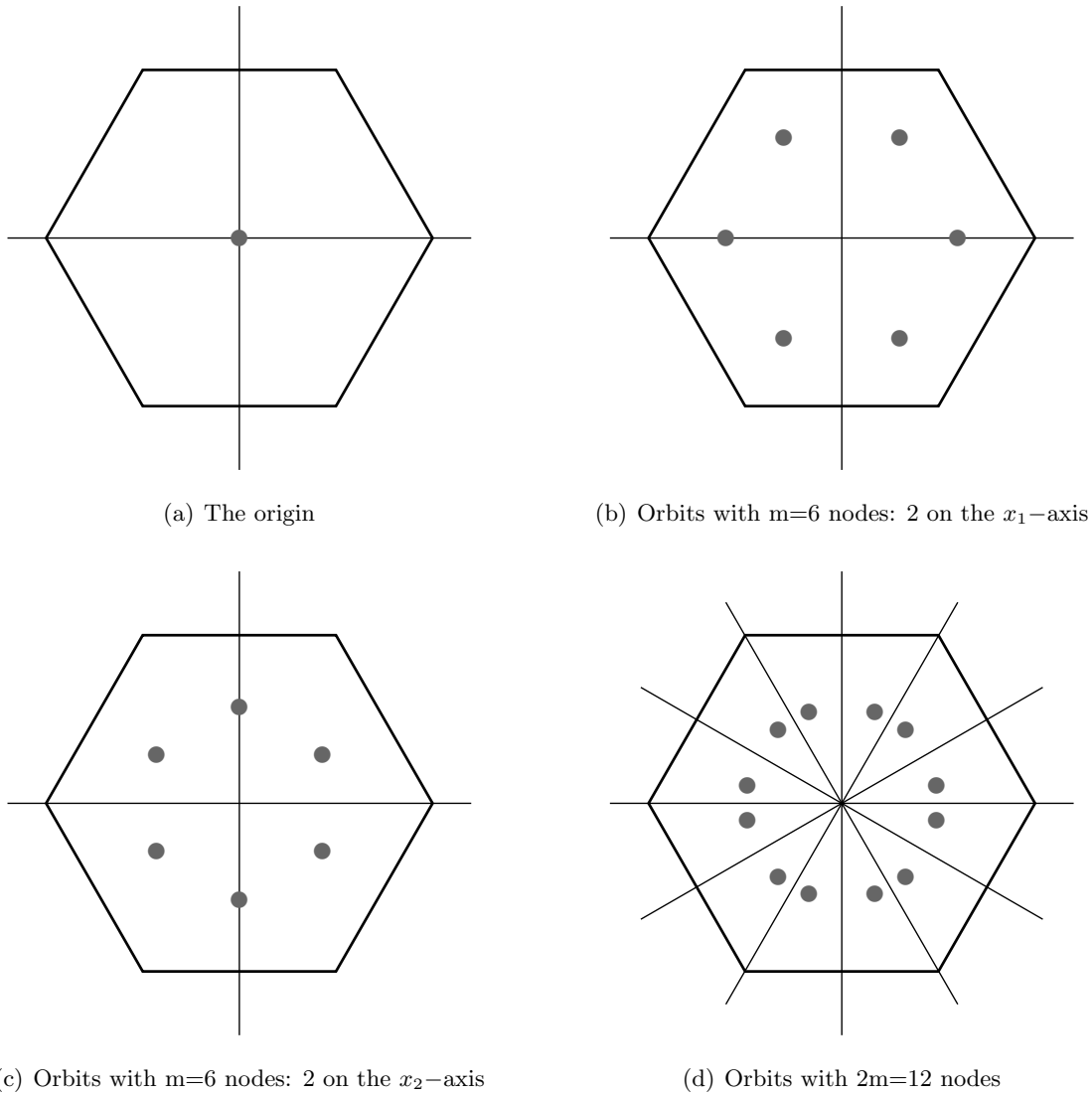


Figure 12: Orbit types for the action of  $D_6$  on  $\mathbb{R}^2$

## 8 Hankel operators and symmetry

In Section 3, the Hankel operator associated with a linear form on  $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$  is used to make explicit the properties and the defining elements of the linear form. This section is devoted to furthering this special connection in the presence of symmetry.

When the linear form is invariant under the linear action of a finite group, applying Section 6, we can determine a polynomial basis such that the matrix of the Hankel operator in this basis is block diagonal. This is also the case of the matrices of the linear operators  $\mathcal{H}_p$  on the quotient space when  $p$  is a  $G$ -invariant polynomial. The generalized eigenvalue problem that provides information on elements of the linear form is therefore transformed in several ones on smaller matrices.

In the special case of a  $G$ -invariant linear form defined by a linear combination of evaluations, the set of nodes is a union of orbits. With the matrix of multiplicities of the group  $G$ , we show how the types of the nodes in the orbits characterize the blocks of the linear operators on the quotient space: their size and the number of identical generalized eigenvalues per block.

In the cubature problem, it is not necessary to consider the generalized eigenvalue problems for all the blocks. The required ones are determined by the existence of a unimodular submatrix in the matrix of multiplicities.

### 8.1 Block diagonalization

The notions of  $G$ -invariant polynomials and linear forms are now defined. It is then proved that, for any  $G$ -invariant polynomial  $p \in \mathbb{R}[x]$ , the matrix of the linear operator  $\mathcal{H}_p$  associated with a  $G$ -invariant linear form  $\Lambda$  on  $\mathbb{R}[x]$  in an orthogonal symmetry adapted basis of  $\mathbb{R}[x]/I_\Lambda$  has a specific block diagonal structure.

Let  $\mathcal{W}$  be a linear representation of a finite group  $G$  on the space  $\mathbb{R}^n$ . A linear representation  $\hat{\mathcal{V}}$  of the group  $G$  on the polynomial space  $\mathbb{R}[x]$  can be constructed as follows

$$\hat{\mathcal{V}}(g) : \mathbb{R}[x] \rightarrow \mathbb{R}[x], p(x) \mapsto p(\mathcal{W}(g^{-1})(x)). \quad (8.1)$$

Notice that  $\hat{\mathcal{V}}(g)$  is an algebra morphism since  $\hat{\mathcal{V}}(g)(pq) = \hat{\mathcal{V}}(g)(p) \hat{\mathcal{V}}(g)(q)$  for all  $p, q \in \mathbb{R}[x]$ .

A polynomial  $p \in \mathbb{R}[x]$  is said to be a  $G$ -invariant polynomial if it satisfies

$$\hat{\mathcal{V}}(g)(p) = p \quad \forall g \in G.$$

Let  $\pi_G$  be the projection of  $\mathbb{R}[x]$  onto the component associated with the one-dimensional absolutely irreducible representation with character  $\chi$  defined by  $\chi(g) = 1$  for all  $g \in G$ . Since for every polynomial  $p \in \mathbb{R}[x]$

$$\pi_G(p) = \frac{1}{|G|} \sum_{g \in G} \hat{\mathcal{V}}(g)(p)$$

defines a  $G$ -invariant polynomial, this component corresponds to the ring of all  $G$ -invariant polynomials  $\mathbb{R}[x]^G$ . The projection  $\pi_G$  is called *Reynolds operator* [102, Chapter 2.1].

A linear form  $\Lambda$  on  $\mathbb{R}[x]$  is said to be  $G$ -invariant if it satisfies

$$\Lambda(\hat{\mathcal{V}}(g)(p)) = \Lambda(p) \quad \forall g \in G, \forall p \in \mathbb{R}[x].$$



This implies that any  $G$ -invariant linear form  $\Lambda$  on  $\mathbb{R}[x]$  satisfies

$$\Lambda(p) = \Lambda(\pi_G(p)) \quad \forall p \in \mathbb{R}[x]. \quad (8.2)$$

A  $G$ -invariant linear form on  $\mathbb{R}[x]$  is therefore determined by its values on  $\mathbb{R}[x]^G$ .

In order to prove the main result of this section, we first present two lemmas.

**Lemma 8.1.** *Let  $\widehat{\mathcal{V}}$  be the representation of a finite group  $G$  on the space  $\mathbb{R}[x]$  induced by a representation  $\mathcal{W}$  on  $\mathbb{R}^n$ . Let  $\Lambda$  be a  $G$ -invariant linear form on  $\mathbb{R}[x]$  and let  $I_\Lambda$  be the kernel of its associated Hankel operator. Then, for every  $g \in G$ , the subspace  $I_\Lambda \subset \mathbb{R}[x]$  (resp. the variety  $V_{\mathbb{C}}(I_\Lambda)$ ) is invariant under the isomorphism  $\widehat{\mathcal{V}}(g)$  (resp. the isomorphism  $\mathcal{W}(g)$ ).*

*Proof.* Let  $g \in G$ , let  $p \in I_\Lambda$  and let  $q \in \mathbb{R}[x]$ . Since the linear form  $\Lambda$  is  $G$ -invariant, we have  $\Lambda(\widehat{\mathcal{V}}(g)(p)q) = \Lambda(p\widehat{\mathcal{V}}(g^{-1})(q))$ . The latter is zero since  $p \in I_\Lambda$ . Thus,  $\widehat{\mathcal{V}}(g)(p) \in I_\Lambda$ . The subspace  $I_\Lambda$  is then invariant under the isomorphism  $\widehat{\mathcal{V}}(g)$ .

Let  $\xi \in V_{\mathbb{C}}(I_\Lambda)$ . By definition of the representation  $\widehat{\mathcal{V}}$ , we have  $\widehat{\mathcal{V}}(g)(p)(\xi) = p(\mathcal{W}(g^{-1})(\xi))$ . Since  $\widehat{\mathcal{V}}(g)(p) \in I_\Lambda$ , we have  $\mathcal{W}(g^{-1})(\xi) \in V_{\mathbb{C}}(I_\Lambda)$ . The variety  $V_{\mathbb{C}}(I_\Lambda)$  is then invariant under the isomorphism  $\mathcal{W}(g^{-1})$ .  $\square$

Let  $g \in G$  and  $q_1, q_2 \in \mathbb{R}[x]$  such that  $q_1 \equiv q_2 \pmod{I_\Lambda}$ . Thanks to Lemma 8.1, we have then  $\widehat{\mathcal{V}}(g)(q_1) \equiv \widehat{\mathcal{V}}(g)(q_2) \pmod{I_\Lambda}$ . This implies that the linear operator

$$\mathcal{V}(g) : \mathbb{R}[x]/I_\Lambda \rightarrow \mathbb{R}[x]/I_\Lambda, [p] \mapsto [\widehat{\mathcal{V}}(g)(p)] \quad (8.3)$$

is well-defined. Thus,  $\mathcal{V}(g)$  is an isomorphism from  $\mathbb{R}[x]/I_\Lambda$  to itself and  $\mathcal{V}$  is a linear representation of the finite group  $G$  on the quotient space  $\mathbb{R}[x]/I_\Lambda$ .

**Lemma 8.2.** *Let  $\widehat{\mathcal{V}}$  be the representation of a finite group  $G$  on the space  $\mathbb{R}[x]$  induced by a representation  $\mathcal{W}$  on  $\mathbb{R}^n$ , let  $\Lambda$  be a  $G$ -invariant linear form on  $\mathbb{R}[x]$  and let  $p \in \mathbb{R}[x]$  be a  $G$ -invariant polynomial. Assume that the rank of the Hankel operator  $\widehat{\mathcal{H}}$  associated with the linear form  $\Lambda$  is finite. Then the map  $\varphi_p$  defined by*

$$\varphi_p : \mathbb{R}[x]/I_\Lambda \times \mathbb{R}[x]/I_\Lambda \rightarrow \mathbb{R}, ([q_1], [q_2]) \mapsto \mathcal{H}_p([q_1])([q_2]) \quad \forall q_1, q_2 \in \mathbb{R}[x]$$

*is a  $G$ -invariant symmetric bilinear form.*

*Proof.* Let  $b_1, \dots, b_r$  be polynomials in  $\mathbb{R}[x]$  such that  $B$  is a basis of  $\mathbb{R}[x]/I_\Lambda$ . By Theorem 3.7, the matrix  $H_p^B = (\Lambda(pb_i b_j))_{1 \leq i, j \leq r}$  is the matrix of the linear operator  $\mathcal{H}_p$  in the basis  $B$  and its dual basis  $B^*$ . Thus, we have

$$\mathcal{H}_p([b_i])([b_j]) = \Lambda(pb_i b_j) = \mathcal{H}_p([b_j])([b_i]) \quad \forall 1 \leq i, j \leq r.$$

The map  $\varphi_p$  is therefore a symmetric bilinear form and  $H_p^B$  is also its matrix in the basis  $B$ .

It remains to show that  $\varphi_p$  is  $G$ -invariant. Let  $g \in G$  and let  $q_1, q_2 \in \mathbb{R}[x]$ . By definition, we have

$$\begin{aligned} \varphi_p(\mathcal{V}(g)([q_1]), \mathcal{V}(g)([q_2])) &= \varphi_p([\widehat{\mathcal{V}}(g)(q_1)], [\widehat{\mathcal{V}}(g)(q_2)]) \\ &= \Lambda(p \widehat{\mathcal{V}}(g)(q_1) \widehat{\mathcal{V}}(g)(q_2)). \end{aligned}$$

Since the polynomial  $p$  is  $G$ -invariant, we have

$$\begin{aligned}\Lambda(p \widehat{\mathcal{V}}(g)(q_1) \widehat{\mathcal{V}}(g)(q_2)) &= \Lambda(\widehat{\mathcal{V}}(g)(p) \widehat{\mathcal{V}}(g)(q_1) \widehat{\mathcal{V}}(g)(q_2)) \\ &= \Lambda(\widehat{\mathcal{V}}(g)(pq_1q_2)).\end{aligned}$$

Since the linear form  $\Lambda$  is  $G$ -invariant, we have

$$\Lambda(\widehat{\mathcal{V}}(g)(pq_1q_2)) = \Lambda(pq_1q_2).$$

From the latter, we deduce that

$$\varphi_p(\mathcal{V}([q_1]), \mathcal{V}([q_2])) = \varphi_p([q_1], [q_2]).$$

□

In fact, Lemma 8.2 shows that a  $G$ -invariant linear form  $\Lambda$  on  $\mathbb{R}[x]$  such that  $\text{rank } \widehat{\mathcal{H}} < \infty$  defines a  $G$ -invariant symmetric bilinear form on  $\mathbb{R}[x]/I_\Lambda$  for any  $G$ -invariant polynomial  $p \in \mathbb{R}[x]$ . With the help of Proposition 6.11, we obtain the following result:

**Theorem 8.3.** *Let  $\widehat{\mathcal{V}}$  be the representation of a finite group  $G$  on the space  $\mathbb{R}[x]$  induced by a representation  $\mathcal{W}$  on  $\mathbb{R}^n$ . Let  $\Lambda$  be a  $G$ -invariant linear form on  $\mathbb{R}[x]$  and assume that the rank of the Hankel operator  $\widehat{\mathcal{H}}$  associated with the linear form  $\Lambda$  is finite. Let  $\mathcal{V}$  be the induced representation on  $\mathbb{R}[x]/I_\Lambda$ .*

*Consider  $U_1, \dots, U_N$  the components in the decomposition of  $\mathbb{R}[x]/I_\Lambda$  deduced from the isotypic decomposition of  $\mathbb{R}[x]/I_\Lambda \otimes_{\mathbb{R}} \mathbb{C}$ . Then, for any  $G$ -invariant polynomial  $p \in \mathbb{R}[x]$  and for any basis  $B = B_1 \cup \dots \cup B_N$  that respects this decomposition of  $\mathbb{R}[x]/I_\Lambda$ , the matrix  $H_p^B$  is block diagonal.*

*Assume furthermore that, for every component  $U_j$ ,  $B_j$  is an orthogonal symmetry adapted basis.*

1. *For every component  $U_j$  associated with an absolutely irreducible  $n_j$ -dimensional representation  $\mathcal{V}^{(j)}$  that appears  $c_j$  times in the representation  $\mathcal{V}$ , the basis  $B_j$  can be written as*

$$B_j = \{b_{11}^j, \dots, b_{1c_j}^j, \dots, b_{n_j1}^j, \dots, b_{n_jc_j}^j\}.$$

*Then the submatrix of  $H_p^B$  relating to the component  $U_j$  consists of a diagonal of  $n_j$  identical blocks of size  $c_j$ . It is given by*

$$I_{n_j} \otimes \left( \nu_{st}^j \right)_{1 \leq s, t \leq c_j} \quad \text{with } \nu_{st}^j = \Lambda(\pi_G(pb_{1s}^j b_{1t}^j)).$$

2. *For every component  $U_j$  associated with an irreducible  $2n_j$ -dimensional representation  $\mathcal{V}^{(j)}$  of complex type that appears  $c_j$  times in the representation  $\mathcal{V}$ , the basis  $B_j$  can be written as*

$$B_j = \{a_{11}^j, \dots, a_{1c_j}^j, b_{11}^j, \dots, b_{1c_j}^j, \dots, a_{n_j1}^j, \dots, a_{n_jc_j}^j, b_{n_j1}^j, \dots, b_{n_jc_j}^j\}.$$

*Then the submatrix of  $H_p^B$  relating to the component  $U_j$  consists of a diagonal of  $n_j$  identical blocks of size  $2c_j$ . It is given by*

$$I_{n_j} \otimes \begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix},$$

where  $S_j = \left( \Lambda(\pi_G(pa_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = \left( \Lambda(\pi_G(pb_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j}$  is a symmetric matrix and  $A_j = \left( \Lambda(\pi_G(pa_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = - \left( \Lambda(\pi_G(pb_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j}$  is an antisymmetric matrix.

Since the basis  $B$  in Theorem 8.3 does not depend on the  $G$ -invariant polynomial  $p$ , the following result is a consequence of Theorem 3.6 and Theorem 3.7.

**Corollary 8.4.** *With the hypotheses of Theorem 8.3, the matrix of the multiplication operator  $M_p$  in the basis  $B$  has the block diagonal structure described in Theorem 8.3.*

*Proof.* Let  $p \in \mathbb{R}[x]$  be a  $G$ -invariant polynomial. By Theorem 8.3, the matrices  $H_1^B$  and  $H_p^B$  have the same block diagonal structure. By Theorem 3.6, the matrix  $H_1^B$  is invertible. Moreover, the matrix  $(H_1^B)^{-1}$  has the same block diagonal structure as the matrix  $H_1^B$ . By Theorem 3.7, the matrix  $M_p^B$  is the product of the matrices  $(H_1^B)^{-1}$  and  $H_p^B$ . Thus, the matrix  $M_p^B$  has the same block diagonal structure as  $H_1^B$ .  $\square$

Notice that a similar result for symmetry adapted bases that do not need to be orthogonal is shown in [22, Corollary 5] for the fields  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

## 8.2 Block size

Consider the special case of a linear form  $\Lambda$  on  $\mathbb{R}[x]$  defined by

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$$

with  $r > 0$ ,  $a_j \in \mathbb{R} \setminus \{0\}$  and  $\xi_j \in \mathbb{R}^n$  pairwise distinct. Let  $\mathcal{W}$  be a representation of a finite group  $G$  on  $\mathbb{R}^n$  and assume that the linear form  $\Lambda$  is  $G$ -invariant. Let  $\mathcal{V}$  be the induced representation on  $\mathbb{R}[x]/I_\Lambda$ . By Theorem 8.3, we can construct block diagonal matrices of the linear operators  $\mathcal{H}$  and  $\mathcal{H}_p$  associated with  $\Lambda$  when  $p \in \mathbb{R}[x]$  is a  $G$ -invariant polynomial. In this section, we relate the organisation of the nodes in orbit types with the size of the blocks and the multiplicities of the generalized eigenvalues of the linear operators  $\mathcal{H}_p$  and  $\mathcal{H}$  thanks to the matrix of multiplicities  $\Gamma_G$ .

By Lemma 8.1, for every  $g \in G$ , the variety  $V_{\mathbb{C}}(I_\Lambda) = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$  is invariant under the isomorphism  $\mathcal{W}(g)$ . It is thus a union of orbits  $\mathcal{O}_1, \dots, \mathcal{O}_{r_1}$  of the action defined by  $G \times \mathbb{R}^n \rightarrow \mathbb{R}^n, (g, \xi) \mapsto \mathcal{W}(g)(\xi)$ . The linear form  $\Lambda$  can thus be rewritten as

$$\Lambda = \sum_{\alpha=1}^{r_1} \check{a}_\alpha \sum_{\zeta_\alpha \in \mathcal{O}_\alpha} \mathbb{1}_{\zeta_\alpha}, \quad (8.4)$$

where  $\check{a}_\alpha > 0$  is the common weight of the nodes  $\zeta_\alpha$  lying on the orbit  $\mathcal{O}_\alpha$ .

A permutation representation is therefore associated with  $V_{\mathbb{C}}(I_\Lambda)$ . It is closely related to the representation  $\mathcal{V}$  on  $\mathbb{R}[x]/I_\Lambda$  defined by (8.3) as shown in the next result.

**Theorem 8.5.** *Let  $\Lambda$  be the  $G$ -invariant linear form on  $\mathbb{R}[x]$  defined by (8.4). The induced representation  $\mathcal{V}$  on  $\mathbb{R}[x]/I_\Lambda$  is equivalent to the permutation representation associated with the invariant set  $V_{\mathbb{C}}(I_\Lambda)$ .*

*Proof.* The ideal  $I_\Lambda$  is zero-dimensional and radical. Following Corollary 3.5, we introduce polynomials  $f_1, \dots, f_r$  that satisfy  $f_i(\xi_j) = \delta_{ij}$  and such that  $C = \{[f_1], \dots, [f_r]\}$  is a basis of  $\mathbb{R}[x]/I_\Lambda$ .

Since the variety  $V_{\mathbb{C}}(I_\Lambda)$  is furthermore an invariant set of  $\mathbb{R}^n$ , there exists a permutation  $\sigma(g)$  for every  $g \in G$  such that

$$\mathcal{W}(g)(\xi_j) = \xi_{\sigma(g)(j)} \quad \forall \xi_j \in V_{\mathbb{C}}(I_\Lambda).$$

Let  $g \in G$  and let  $i \in \{1, \dots, r\}$ . Then, for every  $\xi_j \in V_{\mathbb{C}}(I_\Lambda)$ , we have

$$\widehat{\mathcal{V}}(g)(f_i)(\xi_j) = f_i(\mathcal{W}(g^{-1})(\xi_j)) = f_i(\xi_{\sigma(g^{-1})(j)}) = \delta_{i\sigma(g^{-1})(j)}.$$

Since  $\delta_{i\sigma(g^{-1})(j)} = \delta_{\sigma(g)(i)j}$  for all  $j = 1, \dots, r$ , we have

$$\widehat{\mathcal{V}}(g)(f_i)(\xi_j) = f_{\sigma(g)(i)}(\xi_j) \quad \forall \xi_j \in V_{\mathbb{C}}(I_\Lambda).$$

Hence the polynomial  $\widehat{\mathcal{V}}(g)(f_i) - f_{\sigma(g)(i)}$  vanishes on  $V_{\mathbb{C}}(I_\Lambda)$ . It follows from the *Strong Nullstellensatz* [23, Chapter 1.4] that  $[\widehat{\mathcal{V}}(g)(f_i)] = [f_{\sigma(g)(i)}]$ . In other words,  $\mathcal{V}(g)([f_i]) = [f_{\sigma(g)(i)}]$ .

For every  $g \in G$ , the representing matrix  $\mathcal{V}^C(g)$  is then the matrix of the permutation  $\sigma(g)$ .  $\square$

Proposition 7.3 and Theorem 8.5 can be used to express the multiplicities of the irreducible representations that appear in  $\mathcal{V}$  in terms of the orbit types of the elements of  $V_{\mathbb{C}}(I_\Lambda)$ .

**Corollary 8.6.** *Let  $\Lambda$  be the  $G$ -invariant linear form on  $\mathbb{R}[x]$  defined by (8.4) and let  $\mathcal{V}$  be the representation on  $\mathbb{R}[x]/I_\Lambda$  defined by (8.3).*

*Let  $(\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$  be the submatrix of the matrix of multiplicities  $\Gamma_G$  of the group  $G$  and, for every  $k = 1, \dots, T$ , let  $m_k$  be the number of distinct orbits in the invariant set  $V_{\mathbb{C}}(I_\Lambda)$  associated with the type  $\mathcal{P}_k$ . The multiplicity  $\gamma_j$  of the irreducible representation  $\mathcal{V}^{(j)}$  of  $G$  in the representation  $\mathcal{V}$  is*

$$\gamma_j = \sum_{k=1}^T m_k \gamma_{jk}. \quad (8.5)$$

As described in Theorem 8.3, the multiplicity  $\gamma_j$  of an irreducible representation  $\mathcal{V}^{(j)}$  is closely related to the size  $r_j$  of the blocks of the matrix  $H_p^B$  for any  $G$ -invariant polynomial  $p \in \mathbb{R}[x]$  and any orthogonal symmetry adapted basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$ . Equation (8.5) gives thus an important information on those blocks

$$r_j = \begin{cases} \gamma_j & \text{if } 1 \leq j \leq M \\ 2\gamma_j & \text{if } M+1 \leq j \leq N \end{cases}.$$

**Remark 8.7.** If  $\mathcal{V}^{(1)}$  is the trivial representation of  $G$ , that is  $\mathcal{V}^{(1)}(g) = 1$  for all  $g \in G$ , then using Remark 7.2 the size of the block associated with  $\mathcal{V}^{(1)}$  is

$$r_1 = \gamma_1 = \sum_{k=1}^T m_k,$$

that is the number of distinct orbits in  $V_{\mathbb{C}}(I_\Lambda)$ .

For every  $j = 1, \dots, N$ , let  $H_p^{(j)}$  be one of the identical blocks of the matrix  $H_p^B$  associated with the irreducible representation  $\mathcal{V}^{(j)}$ . By Corollary 3.9, the generalized eigenvalues of the pair of matrices  $(H_p^B, H_1^B)$  are the values of the  $G$ -invariant polynomial  $p$  on the invariant variety  $V_{\mathbb{C}}(I_{\Lambda})$ . Since the block diagonal structure is the same for all  $G$ -invariant polynomials, the computation of those generalized eigenvalues can be done using only the pairs of blocks  $(H_p^{(j)}, H_1^{(j)})$ . Theorem 8.5 and Corollary 8.6 can therefore be used to express their multiplicities in terms of the orbit types of the elements of  $V_{\mathbb{C}}(I_{\Lambda})$ .

**Corollary 8.8.** *Let  $\Lambda$  be the  $G$ -invariant linear form on  $\mathbb{R}[x]$  defined by (8.4).*

*Let  $B$  be an orthogonal symmetry adapted basis of  $\mathbb{R}[x]/I_{\Lambda}$ , let  $p \in \mathbb{R}[x]$  be a  $G$ -invariant polynomial and let  $\lambda$  be a generalized eigenvalue of the pair of matrices  $(H_p^B, H_1^B)$ . For every  $j = 1, \dots, N$ , let  $\mathring{c}_j$  be the multiplicity of  $\lambda$  as a generalized eigenvalue of the pair of matrices  $(H_p^{(j)}, H_1^{(j)})$  and, for every  $k = 1, \dots, T$ , let  $\mathring{m}_k$  be the number of distinct orbits  $\mathcal{O}_{\alpha}$  associated with the type  $\mathcal{P}_k$  such that*

$$\exists \zeta_{\alpha} \in \mathcal{O}_{\alpha} \text{ with } \lambda = p(\zeta_{\alpha}).$$

*Two cases are to be distinguished:*

1. *Assume that the pair of blocks  $(H_p^{(j)}, H_1^{(j)})$  is relating to an absolutely irreducible representation  $\mathcal{V}^{(j)}$ . Then*

$$\mathring{c}_j = \sum_{k=1}^T \mathring{m}_k \gamma_{jk} \quad (1 \leq j \leq M).$$

2. *Assume that the pair of blocks  $(H_p^{(j)}, H_1^{(j)})$  is relating to an irreducible representation  $\mathcal{V}^{(j)}$  of complex type. Then*

$$\mathring{c}_j = 2 \sum_{k=1}^T \mathring{m}_k \gamma_{jk} \quad (M+1 \leq j \leq N).$$

**Remark 8.9.** If the pair of blocks  $(H_p^{(1)}, H_1^{(1)})$  in Corollary 8.8 is relating to the trivial representation  $\mathcal{V}^{(1)}$ , then using Remark 7.2

$$\mathring{c}_1 = \sum_{k=1}^T \mathring{m}_k \geq 1.$$

The distinct generalized eigenvalues of the pair of matrices  $(H_p^B, H_1^B)$  are generalized eigenvalues of this pair of blocks  $(H_p^{(1)}, H_1^{(1)})$ .

For convenience, we introduce  $\mathring{\mathring{c}}_j$  by

$$\mathring{\mathring{c}}_j = \begin{cases} \mathring{c}_j & \text{if } 1 \leq j \leq M \\ 2\mathring{c}_j & \text{if } M+1 \leq j \leq N \end{cases}.$$

Conversely, the numbers  $m_k$  in Corollary 8.6 (resp.  $\mathring{m}_k$  in Corollary 8.8) can be found from the multiplicities  $\gamma_j$  (resp.  $\mathring{\mathring{c}}_j$ ) and the submatrix  $(\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$  of the matrix of multiplicities

$\Gamma_G$ . This is equivalent to solving over the integers the linear system

$$\begin{aligned} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_N \end{pmatrix} &= \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1T} \\ \vdots & & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NT} \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_T \end{pmatrix} \\ \text{(resp. } \begin{pmatrix} \underline{\mathcal{C}}_1 \\ \vdots \\ \underline{\mathcal{C}}_M \\ \underline{\mathcal{C}}_{M+1} \\ \vdots \\ \underline{\mathcal{C}}_N \end{pmatrix} &= \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1T} \\ \vdots & & \vdots \\ \gamma_{M1} & \cdots & \gamma_{MT} \\ \gamma_{M+11} & \cdots & \gamma_{M+1T} \\ \vdots & & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NT} \end{pmatrix} \begin{pmatrix} \mathring{m}_1 \\ \vdots \\ \mathring{m}_T \end{pmatrix} \text{)}. \end{aligned} \quad (8.6)$$

However, those linear systems may have more than a unique solution: two subgroups that are not conjugated may be associated with equivalent permutation representations [49].

In our application, we encounter matrices of multiplicities that have unimodular submatrices (determinant  $\pm 1$ ). Given the multiplicities, the above linear systems have at most one solution.

## 9 Algorithm for computing symmetric cubatures

In this section, a detailed description of our procedure for finding  $G$ –invariant cubatures is given. In contrast with Section 5, in this symmetric case, the size of the matrices in input of Algorithm 4.7 and the number of parameters are reduced. This is due to the block diagonalization of the matrix of the Hankel operator in an orthogonal symmetry adapted basis.

The problem of finding a symmetric cubature with a moment matrix approach can now be formulated as follows. Let  $\mu$  be a positive Borel measure with compact support in  $\mathbb{R}^n$  such that

$$\text{supp } \mu = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_s(x) \geq 0\},$$

with  $g_k(x) \in \mathbb{K}[x]$  for all  $k = 1, \dots, s$ . Let  $\Omega$  be the linear form on  $\mathbb{R}[x]$  defined from  $\mu$  by

$$\Omega : \mathbb{R}[x] \rightarrow \mathbb{R}, p \mapsto \int_{\mathbb{R}^n} p(x) d\mu(x).$$

Assume furthermore that this linear form is  $G$ –invariant. This is the case for instance when  $\mu$  is the characteristic function of the triangle ( $G = D_3$ ), the square ( $G = D_4$ ) or the hexagon ( $G = D_6$ ). Notice that, if  $\Omega$  is  $G$ –invariant, it is also invariant for all the subgroups of  $G$  so that when  $\mu$  is the characteristic function of the disk,  $G$  may be any cyclic group  $C_m$  or any dihedral group  $D_m$ .

Given a degree  $d$ , we want to find for  $\mu$  a  $G$ –invariant inside cubature  $\Lambda$  of degree  $d$  with positive weights. The invariant set of  $r$  nodes is partitioned into  $r_1$  orbits  $\mathcal{O}_1, \dots, \mathcal{O}_{r_1}$ . The sought cubature  $\Lambda$  can thus be written as

$$\Lambda = \sum_{\alpha=1}^{r_1} \check{a}_\alpha \sum_{\zeta_\alpha \in \mathcal{O}_\alpha} \mathbb{1}_{\zeta_\alpha},$$

where  $\check{a}_\alpha > 0$  is the common weight of the nodes  $\zeta_\alpha$  lying on the orbit  $\mathcal{O}_\alpha$ .

The proposed procedure retains the structure of the one presented in Section 5:

1. Determination of the existence of such a  $G$ –invariant cubature (Section 9.2).
2. Computation of the coordinates of the nodes  $\zeta_\alpha$  and the common weights  $\check{a}_\alpha$  for all the orbits  $\mathcal{O}_\alpha$  with  $\alpha = 1, \dots, r_1$  (Section 9.3).

In the first step, the computations are done using exact arithmetic in  $\mathbb{K} \subset \mathbb{R}$  a field extension of  $\mathbb{Q}$  such that the moments of  $\mu$  are in  $\mathbb{K}$ . For instance,  $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$  if  $\mu$  is the characteristic function of a square,  $\mathbb{K} = \mathbb{Q}[\sqrt{3}]$  if  $\mu$  is the characteristic function of an equilateral triangle or a regular hexagon. We recall that moments of polytopes can be computed exactly following [3, 99].

We conclude this section with a new treatment of example of Section 5.4. In this symmetric case, the core of the computation of the nodes is reduced to dividing two numbers in  $\mathbb{Q}[\sqrt{3}]$  and taking the square root of a number in  $\mathbb{Q}$ .

## 9.1 Initialization and subroutines

Before describing the procedure itself, we first introduce an algorithmic representation of the tools from representation theory: the irreducible inequivalent representations  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(N)}$  of a finite group  $G$ , any representation of  $G$  on  $\mathbb{R}^n$ , an orthogonal symmetry adapted basis of a polynomial space (Algorithm 9.3 [Symmetry Adapted Polynomial Basis]),.... We describe then subroutines (Algorithm 9.4 [Symmetric Hankel Blocks] and Algorithm 9.5 [Parametrization Hankel Blocks]) used for the computation of the parameterized blocks in the matrix of the Hankel operator associated with the expected cubature  $\Lambda$  in an orthogonal symmetry adapted basis.

### Finite group and irreducible representations

Let  $G$  be a finite group of order  $|G|$  and let  $1_G$  be its neutral element. The order of the elements of the group  $G$  is fixed

$$\{1_G, g_1, \dots, g_{|G|-1}\}. \quad (9.1)$$

Based on Section 6, the group  $G$  has a finite number of irreducible inequivalent representations, denoted by  $\mathcal{V}_1, \dots, \mathcal{V}_{\underline{N}}$  on  $\mathbb{C}$ -vector spaces (Theorem 6.2) and  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(N)}$  on  $\mathbb{R}$ -vector spaces (see the constructions done in Section 6.4).

Focusing on the ones on  $\mathbb{R}$ -vector spaces, we denote by  $M$  the number of absolutely irreducible representations. There are therefore  $N - M$  irreducible representations of complex type since we assume that there is no irreducible representation of quaternionian type (Remark 6.10). Irreducible representations of complex type are obtained from 2 complex conjugate irreducible representations of type 2 so that:  $\underline{N} = M + 2(N - M)$ .

- Each of the  $M$  absolutely irreducible representation  $\mathcal{V}^{(j)}$ , if  $1 \leq j \leq M$ , is given by an ordered list of  $|G|$  orthogonal  $n_j \times n_j$  matrices (with entries in  $\mathbb{R}$ ):

$$[\mathcal{V}^{(j)}(1_G), \mathcal{V}^{(j)}(g_1), \dots, \mathcal{V}^{(j)}(g_{|G|-1})].$$

The latter are representing matrices of the representation  $\mathcal{V}^{(j)}$ , or equivalently of  $\mathcal{V}_j$ .

- Each of the  $N - M$  irreducible representation  $\mathcal{V}^{(j)}$  of complex type, if  $M + 1 \leq j \leq N$ , is given by an ordered list of  $|G|$  unitary  $n_j \times n_j$  matrices (with entries in  $\mathbb{C}$ ):

$$[\mathcal{V}^{(j)}(1_G), \mathcal{V}^{(j)}(g_1), \dots, \mathcal{V}^{(j)}(g_{|G|-1})].$$

The latter are representing matrices of one of the two complex conjugate irreducible representations  $\mathcal{V}_j$  of type 2 used to define  $\mathcal{V}^{(j)}$ .

The set of irreducible representations of the group  $G$  is given by a pair  $[L_{abs}, L_{com}]$ , where  $L_{abs}$  is a list of  $M$  absolutely irreducible inequivalent representations and  $L_{com}$  is a list of  $N - M$  irreducible inequivalent representations of complex type.

### Symmetry adapted basis

Let  $\mathcal{V}$  be a representation of a finite group  $G$  on a  $\mathbb{R}$ -vector space  $V$ . It is given by an ordered list of  $|G|$  representing matrices that respects the order of the elements of the group  $G$  in (9.1). The size of the matrices is the dimension of the representation space  $V$ .



Such a representation  $\mathcal{V}$  can be completely reduced using the irreducible representations  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(N)}$ . Their multiplicity  $c_1, \dots, c_N$  in  $\mathcal{V}$  can be computed thanks to Theorem 6.3.

Similarly to the set of irreducible inequivalent representations, any orthogonal symmetry adapted basis  $B$  is given by a pair  $B = [[B_1, \dots, B_M], [B_{M+1}, \dots, B_N]]$ . Each orthogonal symmetry adapted basis  $B_j$  is then given by a list

$$B_j = [B_{j1}, \dots, B_{jn_j}]$$

with

$$1. \quad B_{jk} = [b_{k1}^j, \dots, b_{kc_j}^j] \quad \text{if } 1 \leq j \leq M, \quad (9.2)$$

$$2. \quad B_{jk} = [a_{k1}^j, \dots, a_{kc_j}^j, b_{k1}^j, \dots, b_{kc_j}^j] \quad \text{if } M+1 \leq j \leq N. \quad (9.3)$$

The orthogonal symmetry adapted bases  $B_1, \dots, B_N$  are computed following Section 6.4. This provides the algorithm

**Algorithm 9.1. Symmetry Adapted Basis**

- Input :  $\triangleright$  A representation of the finite group  $G$ .  
 $\triangleright$  The set of irreducible representations of the group  $G$ .  
Output :  $\triangleright$  An orthogonal symmetry adapted basis  $B = [[B_1, \dots, B_M], [B_{M+1}, \dots, B_N]]$  of the representation space with  $B_j = [B_{j1}, \dots, B_{jn_j}]$  and  $B_{jk}$  is
- either a list of  $c_j$  vectors if  $1 \leq j \leq M$ ,
  - or a list of  $2c_j$  vectors if  $M+1 \leq j \leq N$ .

Assume now that the finite group  $G$  acts linearly on  $\mathbb{R}^n$ . This implies a representation  $\mathcal{W}$  of  $G$  on  $\mathbb{R}^n$ . Let  $\widehat{\mathcal{V}}$  be the induced representation of  $G$  on the polynomial space  $\mathbb{R}[x]$  of all polynomials in  $n$  variables as in (8.1). For every  $g \in G$ , the image by the isomorphism  $\widehat{\mathcal{V}}(g)$  of a homogeneous polynomial in  $\mathbb{R}[x]$  is a homogeneous polynomial in  $\mathbb{R}[x]$  of the same degree. It is therefore possible to consider the restriction of  $\widehat{\mathcal{V}}$  to the space  $\mathbb{R}[x]_{\leq \delta}$  for any degree  $\delta \in \mathbb{N}$ . Its representing matrices in a basis that respects the degree are block diagonal: one block per degree.

Those representing matrices are thus computed degree by degree. It is sufficient to:

1. Take any (monomial) basis  $B_{\tilde{\delta}}$  of  $\mathbb{R}[x]_{\tilde{\delta}}$ , the space of homogeneous polynomials of degree exactly  $\tilde{\delta}$ , for every  $0 \leq \tilde{\delta} \leq \delta$ .
2. Express  $\widehat{\mathcal{V}}(g)(b)(x) = b(\mathcal{W}(g^{-1})(x))$  in the basis  $B_{\tilde{\delta}}$  for every  $g \in G$  and  $b \in B_{\tilde{\delta}}$ . This gives the columns of the representing matrices.

This procedure describes the following algorithm.

**Algorithm 9.2. Induced Representation**

- Input :  $\triangleright$  A representation of a finite group  $G$  on  $\mathbb{R}^n$ .  
 $\triangleright$  A degree  $\delta$ .  
Output :  $\triangleright$  The induced representation of  $G$  on  $\mathbb{R}[x]_{\leq \delta}$ .

Combining Algorithm 9.1 and Algorithm 9.2 leads to the computation of an orthogonal symmetry adapted basis of the space  $\mathbb{R}[x]_{\leq \delta}$  for a degree  $\delta \in \mathbb{N}$ :

**Algorithm 9.3. Symmetry Adapted Polynomial Basis**

- Input :     $\triangleright$  A representation of a finite group  $G$  on  $\mathbb{R}^n$ .  
              $\triangleright$  The set of irreducible representations of the group  $G$ .  
              $\triangleright$  A degree  $\delta$ .
- Output :    $\triangleright$  An orthogonal symmetry adapted basis  $[[B_1, \dots, B_M], [B_{M+1}, \dots, B_N]]$  of  $\mathbb{R}[x]_{\leq \delta}$  with  $B_j = [B_{j1}, \dots, B_{jn_j}]$  and where
- $B_{jk} = [b_{k1}^j, \dots, b_{kc_j}^j]$  is a list of  $c_j$  polynomials if  $1 \leq j \leq M$ ,
  - $B_{jk} = [a_{k1}^j, \dots, a_{kc_j}^j, b_{k1}^j, \dots, b_{kc_j}^j]$  is a list of  $2c_j$  polynomials if  $M+1 \leq j \leq N$ .

Since the computation of the bases  $B_{jk}$  is based on Algorithm 9.2, polynomials of the same degree are together in the list  $B_{jk}$ . We choose to order them such that

$$\deg a_{k\alpha}^j \leq \deg a_{k\beta}^j \quad \forall 1 \leq k \leq n_j, \forall M+1 \leq j \leq N, \forall 1 \leq \alpha < \beta \leq c_j,$$

$$\deg b_{k\alpha}^j \leq \deg b_{k\beta}^j \quad \forall 1 \leq k \leq n_j, \forall 1 \leq j \leq N, \forall 1 \leq \alpha < \beta \leq c_j.$$

Notice that by construction

$$\deg a_{k\ell}^j = \deg b_{k\ell}^j \quad \forall 1 \leq k \leq n_j, \forall 1 \leq j \leq N, \forall 1 \leq \ell \leq c_j.$$

**Parametrization of the matrix of the Hankel operator associated with a  $G$ -invariant linear form**

Let  $\widehat{\mathcal{V}}$  be the representation of a finite group  $G$  on  $\mathbb{R}[x]$  induced by a representation on  $\mathbb{R}^n$ , let  $\Lambda$  be a  $G$ -invariant linear form on  $\mathbb{R}[x]$  and let  $\Lambda^{(\delta)}$  be the restriction of the linear form  $\Lambda$  to  $\mathbb{R}[x]_{\leq 2\delta}$  for a certain degree  $\delta \in \mathbb{N}$ . In an orthogonal symmetry adapted basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$ , the matrix  $H_1^{B^{(\delta)}}$  of the Hankel operator associated with  $\Lambda^{(\delta)}$  has the block diagonal structure described in Proposition 6.11. In particular, for every  $j = 1, \dots, N$ , its submatrix in the basis  $B_j$  consists of  $n_j$  identical blocks: it is therefore sufficient to focus on only one of them, denoted by  $H^{(j)}$ .

Since  $\Lambda$  is  $G$ -invariant, it is determined by its values on  $\mathbb{R}[x]^G$ , that is (8.2) is satisfied. With the notations of (9.2) and (9.3) and with the help of Reynolds operator  $\pi_G$ , the matrices  $H^{(1)}, \dots, H^{(N)}$  are then

- for  $1 \leq j \leq M$

$$H^{(j)} = \left( \Lambda(\pi_G(b_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j}, \quad (9.4)$$

- for  $M+1 \leq j \leq N$

$$H^{(j)} = \begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix}$$

with  $S_j = \left( \Lambda(\pi_G(a_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = \left( \Lambda(\pi_G(b_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j}$     (9.5)

and  $A_j = \left( \Lambda(\pi_G(a_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = - \left( \Lambda(\pi_G(b_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j}.$

The coefficients of the matrices  $H^{(1)}, \dots, H^{(N)}$  are therefore values of  $\Lambda$  at  $G$ -invariant homogeneous polynomials of  $\mathbb{R}[x]_{\leq 2\delta}^G$ . Thus, we first compute the matrices  $H^{(1)}, \dots, H^{(N)}$  with only those  $G$ -invariant homogeneous polynomials in the following algorithm.

**Algorithm 9.4. Symmetric Hankel Blocks**

- Input :  $\triangleright$  A representation of a finite group  $G$  on  $\mathbb{R}^n$ .  
 $\triangleright$  An orthogonal symmetry adapted basis of the space  $\mathbb{R}[x]_{\leq \delta}$  for a certain degree  $\delta$ .
- Output :  $\triangleright$  Matrices  $H^{(1)}, \dots, H^{(N)}$  with:
- $H^{(j)} = \left( \pi_G(b_{1s}^j b_{1t}^j) \right)_{1 \leq s, t \leq c_j}$  if  $1 \leq j \leq M$ ,
  - $H^{(j)} = \begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix}$  if  $M+1 \leq j \leq N$   
with  $S_j = \left( \pi_G(a_{1s}^j a_{1t}^j) \right)_{1 \leq s, t \leq c_j}$  and  $A_j = \left( \pi_G(a_{1s}^j b_{1t}^j) \right)_{1 \leq s, t \leq c_j}$ .

Given a degree  $d \in \mathbb{N}$ , assume furthermore that there is a linear form  $\Omega$  on  $\mathbb{R}[x]$  such that

$$\Lambda(p) = \Omega(p) \quad \forall p \in \mathbb{R}[x]_{\leq d}. \quad (9.6)$$

The coefficients of the matrices  $H^{(1)}, \dots, H^{(N)}$  are then either real values determined by (9.6) or the unknown evaluations of the sought  $\Lambda$  at  $G$ -invariant homogeneous polynomials of degree between  $d+1$  and  $2\delta$ .

Let  $\{p_1, \dots, p_t\}$  be a basis of the supplementary of  $\mathbb{R}[x]_{\leq d}^G$  in  $\mathbb{R}[x]_{\leq 2\delta}^G$ . The unknown coefficients of the matrices  $H^{(1)}, \dots, H^{(N)}$  can be expressed as linear combinations of parameters  $h_1, \dots, h_t$  with

$$h_\ell = \Lambda(p_\ell) \quad \forall \ell = 1, \dots, t.$$

Similarly to Step 3 in Algorithm 5.1, in which we only worked with monomials, all the parameters  $h_1, \dots, h_t$  are required to express the unknown entries of  $H^{(1)}, \dots, H^{(N)}$ . Thus, the number of unknown parameters to be introduced is

$$t = \dim \mathbb{R}[x]_{\leq 2\delta}^G - \dim \mathbb{R}[x]_{\leq d}^G.$$

This is less than in Algorithm 5.1.

**Algorithm 9.5. Parametrization Hankel Blocks**

- Input :  $\triangleright$  Matrices  $H^{(1)}, \dots, H^{(N)}$   
whose coefficients are homogeneous polynomials of  $\mathbb{R}[x]_{\leq 2\delta}^G$ .  
 $\triangleright$  A linear form  $\Omega$  on  $\mathbb{R}[x]_{\leq d}$ .
- Output :  $\triangleright$  A list  $[p_1, \dots, p_t]$  of polynomials among the entries of  $H_1, \dots, H_N$   
such that  $\{p_1, \dots, p_t\}$  is a basis of a supplementary of  $\mathbb{R}[x]_{\leq d}^G$  in  $\mathbb{R}[x]_{\leq 2\delta}^G$ .  
 $\triangleright$  A list  $[h_1, \dots, h_t]$  of parameters.  
 $\triangleright$  Matrices  $\tilde{H}^{(1)}, \dots, \tilde{H}^{(N)}$  obtained  
by applying the linear map  $\psi : \mathbb{R}[x]_{\leq 2\delta} \rightarrow \mathbb{R}[h_1, \dots, h_t]$  defined by
- $\psi(p) = \Omega(p)$  if  $\deg p \leq d$ ,
  - $\psi(p_\ell) = h_\ell$  for all  $\ell = 1, \dots, t$ ,
- to the coefficients of the matrices  $H^{(1)}, \dots, H^{(N)}$ .

## 9.2 Existence conditions for a $G$ -invariant cubature

The first step of our procedure is described in Algorithm 9.6 [Existence of a  $G$ -invariant cubature]. It provides conditions on some parameters such that the expected  $G$ -invariant cubature  $\Lambda$  exists. In comparison to Algorithm 5.1, the use of a symmetry adapted basis allows to introduce less parameters (Step 5) and to deal with smaller-sized matrices in input of Algorithm 4.7 (Step 6).

### Algorithm 9.6. Existence of a $G$ -invariant cubature

- Input :
- ▷ The degree  $d$  of the expected  $G$ -invariant cubature.
  - ▷ The values  $\Omega(p) = \int p \, d\mu$  for all  $p \in \mathbb{R}[x]_{\leq d}^G$ .
  - ▷ A representation of the finite group  $G$  on  $\mathbb{R}^n$ .
  - ▷ The set of irreducible representations of the group  $G$ .
  - ▷ The matrix of multiplicities  $\Gamma_G = (\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$ .
  - ▷ Integers  $m_1, \dots, m_T$ , where  $m_k$  is the number of orbits of type  $\mathcal{P}_k$  in the invariant set of nodes.
- Output :
- ▷ A system of equations and inequations that determines the existence for  $\mu$  of a  $G$ -invariant inside cubature  $\Lambda$  of degree  $d$  with positive weights.

Notice that the number of nodes  $r$  of the expected  $G$ -invariant cubature  $\Lambda$  is then

$$r = \sum_{j=1}^M n_j \sum_{k=1}^T \gamma_{jk} m_k + 2 \sum_{j=M+1}^N n_j \sum_{k=1}^T \gamma_{jk} m_k, \quad (9.7)$$

where the dimension of the irreducible representation  $\mathcal{V}^{(j)}$  is  $n_j$  if  $1 \leq j \leq M$  and  $2n_j$  if  $M+1 \leq j \leq N$ .

1. Compute the expected multiplicities  $\gamma_1, \dots, \gamma_N$  and the expected ranks  $r_1, \dots, r_N$ .

*Consider the kernel  $I_\Lambda$  of the Hankel operator associated with the expected cubature  $\Lambda$  and the induced representation  $\mathcal{V}$  of the finite group  $G$  on the quotient space  $\mathbb{R}[x]/I_\Lambda$ . The expected multiplicity of every irreducible representation  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(N)}$  of  $G$  in  $\mathcal{V}$  is computed from the integers  $m_1, \dots, m_T$  and the matrix of multiplicities  $\Gamma_G$  (Corollary 8.6) and is denoted respectively by  $\gamma_1, \dots, \gamma_N$*

$$\gamma_j = \sum_{k=1}^T \gamma_{jk} m_k. \quad (9.8)$$

*Consider the matrix  $H_1^B$  of the linear operator  $\mathcal{H}$  associated with  $\Lambda$  in an orthogonal symmetry adapted basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$ . Following Theorem 8.3,  $H_1^B$  has a block diagonal structure. For every  $1 \leq j \leq N$ , the size of each identical block associated with the irreducible representation  $\mathcal{V}^{(j)}$  is:*

$$r_j = \begin{cases} \gamma_j & \text{if } 1 \leq j \leq M \\ 2\gamma_j & \text{if } M+1 \leq j \leq N. \end{cases}$$

2. Verify that the expected multiplicities  $\gamma_1, \dots, \gamma_N$  satisfy

$$\gamma_j \geq \underline{c}_j' \quad \forall j = 1, \dots, N, \quad (9.9)$$

where  $c_j'$  is the multiplicity of the irreducible representation  $\mathcal{V}^{(j)}$  of  $G$  in the induced representation on  $\mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$ .

Since the moments up to order  $d$  are known, the matrix of the Hankel operator associated with the sought cubature  $\Lambda$  in a basis of  $\mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$  has entries in  $\mathbb{K}$  and has full rank (positive definite). Assuming that this basis is orthogonal symmetry adapted, we get the inequalities (9.9).

3. Choose a degree  $\delta$  such that the following inequalities are satisfied

$$\gamma_j \leq c_j'' \quad \forall j = 1, \dots, N, \quad (9.10)$$

where  $c_j''$  is the multiplicity of the irreducible representation  $\mathcal{V}^{(j)}$  of  $G$  in the induced representation on  $\mathbb{R}[x]_{\leq \delta-1}$ .

Following Corollary 3.15, the cubature  $\Lambda$  exists if and only if its restriction  $\Lambda^{(\delta)}$  to  $\mathbb{R}[x]_{\leq 2\delta}$  for a certain degree  $\delta \in \mathbb{N}$  satisfies the following properties:

- $\Lambda^{(\delta)}$  is a flat extension of its restriction  $\Lambda^{(\delta-1)}$  to  $\mathbb{R}[x]_{\leq 2\delta-2}$ .
- Its associated Hankel operator  $\mathcal{H}^{(\delta)}$  is positive semidefinite of rank  $r$ .

With the help of Theorem 3.6, this implies that polynomials  $b_1, \dots, b_r$  such that  $B = \{[b_1], \dots, [b_r]\}$  is a basis of  $\mathbb{R}[x]/I_\Lambda$  can be chosen in  $\mathbb{R}[x]_{\leq \delta-1}$ . Assuming furthermore that  $B$  is an orthogonal symmetry adapted basis of  $\mathbb{R}[x]/I_\Lambda$ , using (9.7) and (9.8), a necessary condition on the degree  $\delta$  is then given by (9.10).

4. Compute an orthogonal symmetry adapted basis  $B^{(\delta)} = [[B_1^{(\delta)}, \dots, B_M^{(\delta)}], [B_{M+1}^{(\delta)}, \dots, B_N^{(\delta)}]]$  of  $\mathbb{R}[x]_{\leq \delta}$ .

See Algorithm 9.3 [Symmetry Adapted Polynomial Basis] and the paragraph **Symmetry adapted basis**.

5. Construct the distinct blocks  $H^{(1)}, \dots, H^{(N)}$ .

Consider the matrix  $H_1^{B^{(\delta)}}$  of the Hankel operator associated with  $\Lambda^{(\delta)}$  in the orthogonal symmetry adapted basis  $B^{(\delta)}$  of  $\mathbb{R}[x]_{\leq \delta}$ . Following Proposition 6.11,  $H_1^{B^{(\delta)}}$  has  $N$  distinct blocks on its main diagonal. We denote them by  $H^{(1)}, \dots, H^{(N)}$ .

See Algorithm 9.4 [Symmetric Hankel Blocks] and the paragraph **Parametrization of the matrix of Hankel operators associated with  $G$ -invariant linear forms** for more details on the computation of  $H^{(1)}, \dots, H^{(N)}$ .

6. Parameterize the unknown coefficients of the matrices  $H^{(1)}, \dots, H^{(N)}$ .

Every coefficient of the matrices  $H^{(1)}, \dots, H^{(N)}$  is:

- either a value  $\int p d\mu \in \mathbb{K} \subset \mathbb{R}$  if  $p \in \mathbb{R}[x]_{\leq d}^G$ ,
- or a linear combination of unknown parameters  $h_1, \dots, h_t$  otherwise.

See Algorithm 9.5 [Parametrization Hankel Blocks] and the paragraph **Parametrization of the matrix of Hankel operators associated with  $G$ -invariant linear forms** for more details on the determination of the parameters  $h_1, \dots, h_t$ .

Each matrix  $H^{(j)}$  has a  $c_j' \times c_j'$  principal submatrix whose entries are in  $\mathbb{K}$  and that is positive definite, where  $c_j'$  is the dimension of the component of  $\mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$  associated with the irreducible representation  $\mathcal{V}^{(j)}$ .

7. Find conditions on the parameters  $h_1, \dots, h_t$ , using Algorithm 4.7 [Diagonalization & Positivity with Rank Constraints] on each matrix  $H^{(1)}, \dots, H^{(N)}$ , such that the linear form  $\Lambda^{(\delta)}$  is a flat extension of the linear form  $\Lambda^{(\delta-1)}$  and such that the Hankel operator  $\mathcal{H}^{(\delta)}$  is positive semidefinite with rank  $r$ .

Following Corollary 3.15, those properties on  $\mathcal{H}^{(\delta)}$  are sufficient to prove the existence for the measure  $\mu$  of a cubature of degree  $d$  with positive weights. Contrary to Step 4 in Algorithm 5.1 [Existence of a cubature], those properties are verified on the smaller distinct blocks  $H^{(1)}, \dots, H^{(N)}$  of  $H_1^{B^{(\delta)}}$ . Each block  $H^{(j)}$ , under the conditions on the parameters  $h_1, \dots, h_t$ , satisfies then:

- Its  $c_j'' \times c_j''$  principal submatrix that corresponds to the restriction of  $\mathcal{H}^{(\delta)}$  to  $\mathbb{R}[x]_{\leq \delta-1}$  has the same rank  $r_j$  as the whole matrix  $H^{(j)}$ .
- $H^{(j)}$  is positive semidefinite.

Thus, up to permutations of rows and columns, Algorithm 4.7 can be used to determine those conditions. We get  $\binom{c_j''-c_j'}{\gamma_j-c_j'}$  triplets  $[P_j, Z_j, E_j]$  for each matrix  $H^{(j)}$  if  $1 \leq j \leq N$ .

Choosing one triplet per matrix  $H^{(j)}$ , this provides a system of equations (from  $Z_1, \dots, Z_N$ ) and inequations (from  $P_1, \dots, P_N$ ) that determines the existence of a cubature.

8. Find conditions such that the Hankel operators  $\mathcal{H}_{g_k}^{(\delta)}$  are positive semidefinite for all  $k = 1, \dots, s$ . (optional)

Following Proposition 3.16, this guarantees that the nodes lie on  $\text{supp } \mu$ .

See Section 5.2 - **Existence of an inside cubature** for the computations.

At the end of Step 7, Algorithm 9.6 gives a system of equations and inequations that determines the existence for  $\mu$  of a  $G$ -invariant cubature of degree  $d$  with positive weights. There is then no guarantee that the nodes lie on the support of  $\mu$ . This property is provided by Step 8.

Since this last step requires the computation of new matrices (that are generally not block diagonal), we often skip it in practice. The fact that the cubature is an inside cubature is then checked after the computation of the nodes.

### 9.3 Computation of the weights and the coordinates of the nodes

Assume now that the existence of a  $G$ -invariant cubature has been secured with a solution  $h_1, \dots, h_t$  of the polynomial system of equations and inequations output by Algorithm 9.6. Algorithm 9.7 then computes the weights and the coordinates of the nodes of the associated  $G$ -invariant cubature.

#### Algorithm 9.7. Weights & Nodes

- Input :
- ▷ Input of Algorithm 9.6.
  - ▷ An orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq \delta}$  (Step 4 in Algorithm 9.6).
  - ▷ The parameterized distinct blocks  $H^{(1)}, \dots, H^{(N)}$  (Step 6 in Algorithm 9.6).
  - ▷ A value  $h_1, \dots, h_t$  for every parameter  $h_1, \dots, h_t$  introduced in Algorithm 9.6.
- Output :
- ▷ The coordinates of the nodes  $\zeta_\alpha$  for all orbits  $\mathcal{O}_\alpha$  with  $\alpha = 1, \dots, r_1$ .
  - ▷ The common weights  $\tilde{a}_1, \dots, \tilde{a}_{r_1}$ .

1. Take polynomials  $b_1, \dots, b_r$  such that  $B = \{[b_1], \dots, [b_r]\}$  is an orthogonal symmetry adapted basis of  $\mathbb{R}[x]/I_\Lambda$ .

Following Remark 4.8 and Theorem 3.6, the use of Algorithm 4.7 (Step 7 in Algorithm 9.6) gives a way to determine a basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  by selecting the appropriate polynomials in the orthogonal symmetry adapted basis  $B^{(\delta-1)}$  of  $\mathbb{R}[x]_{\leq \delta-1}$  with  $B^{(\delta-1)} = B^{(\delta)} \cap \mathbb{R}[x]_{\leq \delta-1}$ . The set of selected polynomials in each orthogonal symmetry adapted basis  $B_j^{(\delta-1)} = B_j^{(\delta)} \cap B^{(\delta-1)}$  is denoted by  $B_j = \{b_1^j, \dots, b_{n_j r_j}^j\}$ , where  $r_j$  is the expected rank computed in Step 1 of Algorithm 9.6.

2. Compute the distinct invertible blocks  $H_1^{(1)}, \dots, H_1^{(N)}$  of the matrix  $H_1^B$  of the linear operator  $\mathcal{H}$  associated with the sought cubature  $\Lambda$  in the orthogonal symmetry adapted basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$ . Using the values  $h_1, \dots, h_t$ , those blocks have entries in  $\mathbb{K}$ .

Since the orthogonal symmetry adapted basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  is obtained from polynomials in  $B^{(\delta)}$ , the matrices  $H_1^{(1)}, \dots, H_1^{(N)}$  are principal submatrices of the blocks  $H^{(1)}, \dots, H^{(N)}$  introduced in Step 5 of Algorithm 9.6. We have

$$H_1^{(j)} = \left( \Lambda(\pi_G(b_s^j b_t^j)) \right)_{1 \leq s, t \leq r_j} \quad \forall 1 \leq j \leq N.$$

The entries are either numbers in  $\mathbb{K}$  or linear combinations of  $h_1, \dots, h_t$ . It is then sufficient to replace the parameters  $h_1, \dots, h_t$  by the values  $\bar{h}_1, \dots, \bar{h}_t$ .

3. Take a separating set  $\{p_1, \dots, p_\eta\}$  of  $G$ -invariant polynomials.

This means that the polynomial system (9.11), defined below, has a unique solution.

4. Construct the distinct blocks  $H_{p_\nu}^{(1)}, \dots, H_{p_\nu}^{(N)}$  for every polynomial  $p_\nu$  in the separating set of Step 3.

By Theorem 8.3, the matrices  $H_{p_\nu}^B$  have the same block diagonal structure as the matrix  $H_1^B$ . For every  $1 \leq j \leq N$ , take the set  $B_j$  of polynomials introduced in Step 1 and compute

$$H_{p_\nu}^{(k)} = \left( \Lambda(\pi_G(p_\nu b_s^j b_t^j)) \right)_{1 \leq s, t \leq r_j},$$

where  $\pi_G$  is Reynolds operator.

The coefficients of those matrices are then numbers in  $\mathbb{K}$ , or polynomials in  $h_1, \dots, h_t$ , or unknowns. This last case appears if there exists  $\nu \in \{1, \dots, \eta\}$  such that  $\deg p_\nu > 2$ . As in Section 5.2 - **Existence of an inside cubature**, those unknown coefficients are then uniquely determined thanks to Algorithm 5.2 with  $\kappa \geq 1$  and  $2\kappa \geq \deg p_\nu$  for all  $\nu = 1, \dots, \eta$ . In addition, since we chose values  $\bar{h}_1, \dots, \bar{h}_t$ , Algorithm 5.2 gives also unique values  $\bar{h}_{t+1}, \dots, \bar{h}_\tau$  for the additional parameters  $h_{t+1}, \dots, h_\tau$ .

5. For every  $\nu = 1, \dots, \eta$  and every  $\alpha = 1, \dots, r_1$ , find the value  $\lambda_{\nu\alpha}$  of every polynomial  $p_\nu$  in the separating set of Step 3 on the orbit  $\mathcal{O}_\alpha$ .

Consider the generalized eigenvalue problems for every pair of matrices  $(H_{p_\nu}^B, H_1^B)$ . Following Corollary 3.9, they are the values of the polynomial  $p_\nu$  on the invariant set of nodes. The latter is a union of  $r_1$  distinct orbits  $\mathcal{O}_1 \cup \dots \cup \mathcal{O}_{r_1}$  (Remark 8.7). Since the polynomial  $p_\nu$  is  $G$ -invariant, the values  $p_\nu(\zeta_\alpha)$  are the same for the nodes  $\zeta_\alpha$ . They are therefore denoted by  $\lambda_{\nu\alpha}$ : one per  $G$ -invariant polynomial  $p_\nu$  and per orbit  $\mathcal{O}_\alpha$ .

Moreover, since the matrices  $H_{p_\nu}^B$  and  $H_1^B$  have the same block diagonal structure, it is here sufficient to consider the generalized eigenvalue problem for the pairs of distinct

blocks  $(H_{p_\nu}^{(1)}, H_1^{(1)}), \dots, (H_{p_\nu}^{(N)}, H_1^{(N)})$ . The occurrence of a generalized eigenvalue  $\lambda_{\nu\alpha}$  for a given pair of blocks is given in Corollary 8.8.

Using Remark 8.9, it is sufficient to consider the generalized eigenvalues of the pairs  $(H_{p_\nu}^{(1)}, H_1^{(1)})$ . The computation runs then as follows:

- (a) Take a first  $G$ -invariant polynomial  $p_1$  in the separating set.
- (b) Compute the  $r_1$  generalized eigenvalues  $\lambda_{1\alpha}$  of the pair  $(H_{p_1}^{(1)}, H_1^{(1)})$  and their associated generalized eigenvector  $\omega_\alpha$ . They satisfy

$$H_{p_1}^{(1)}\omega_\alpha = \lambda_{1\alpha}H_1^{(1)}\omega_\alpha.$$

- (c) Compute  $H_1^{(1)}\omega_\alpha$  and  $H_{p_\nu}^{(1)}\omega_\alpha$  for each other  $G$ -invariant polynomial  $p_\nu$  in the separating set.
- (d) The generalized eigenvalues  $\lambda_{\nu\alpha}$  are then obtained as the only values such that

$$H_{p_\nu}^{(1)}\omega_\alpha = \lambda_{\nu\alpha}H_1^{(1)}\omega_\alpha \quad \forall \nu = 2, \dots, \eta.$$

- (e) Create the set of generalized eigenvalues  $\{\lambda_{1\alpha}, \dots, \lambda_{\eta\alpha}\}$  associated with the generalized eigenvector  $\omega_\alpha$ , and therefore with the orbit  $\mathcal{O}_\alpha$ .

Notice that it is sometimes more suitable to compute the generalized eigenvalues of other pairs of blocks, especially

- if those blocks are smaller than the blocks  $(H_{p_\nu}^{(1)}, H_1^{(1)})$ , or
- if those blocks do not contain unknown coefficients (see Step 4) whereas the block  $H_{p_\nu}^{(1)}$  does.

6. Solve the polynomial system for every orbit  $\mathcal{O}_\alpha$  with  $\alpha = 1, \dots, r_1$

$$p_\nu(x) = \lambda_{\nu\alpha} \quad \forall \nu = 1, \dots, \eta \quad (9.11)$$

to get the coordinates of a node  $\zeta_\alpha$  per orbit. The other nodes in the orbit  $\mathcal{O}_\alpha$  are then computed using the group action of  $G$  on  $\mathbb{R}^n$ .

7. Solve the Vandermonde-like linear system (9.13).

Given a  $G$ -invariant polynomial  $q$ ,

$$\Lambda(q) = \sum_{\alpha=1}^{r_1} \check{a}_\alpha \sum_{\zeta_\alpha \in \mathcal{O}_\alpha} q(\zeta_\alpha) = \sum_{\alpha=1}^{r_1} |\mathcal{O}_\alpha| q(\zeta_\alpha) \check{a}_\alpha. \quad (9.12)$$

Taking  $B_1 = \{b_1^1, \dots, b_{r_1}^1\} \subset B$  the orthogonal symmetry adapted basis associated with the absolutely irreducible representation  $\mathcal{V}^{(1)}$  defined by  $\mathcal{V}^{(1)}(g) = 1$  for all  $g \in G$ , then

$$\begin{pmatrix} b_1^1(\zeta_1) & \cdots & b_1^1(\zeta_{r_1}) \\ \vdots & & \vdots \\ b_{r_1}^1(\zeta_1) & \cdots & b_{r_1}^1(\zeta_{r_1}) \end{pmatrix} \begin{pmatrix} |\mathcal{O}_1| & & \\ & \ddots & \\ & & |\mathcal{O}_{r_1}| \end{pmatrix} \begin{pmatrix} \check{a}_1 \\ \vdots \\ \check{a}_{r_1} \end{pmatrix} = \begin{pmatrix} \Lambda(b_1^1) \\ \vdots \\ \Lambda(b_{r_1}^1) \end{pmatrix}. \quad (9.13)$$



#### 9.4 $H_2 : 5 - 1$ (Stroud, 1971)

We look for a cubature of degree 5 for the regular hexagon  $H_2$  in the plane  $\mathbb{R}^2$ . It is described in [101] under the name  $H_2 : 5 - 1$  and it was examined in Section 5.4. Here, we recover it as the unique  $D_6$ -invariant cubature such that the nodes are organized as: the origin and an orbit whose 6 nodes lie on the symmetry axes that go through the vertices of the hexagon (see Section 7.4.2).

Let  $H_2$  be the regular hexagon whose vertices are given by  $(\pm 1, 0), (\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$  (see Figure 13(a)). The dihedral group  $D_6$  of order 12 (see Section 7.4 for a description of  $D_6$ ) leaves the hexagon  $H_2$  invariant under its classical action on the plane  $\mathbb{R}^2$ . The representation of  $D_6$  on  $\mathbb{R}^2$  is given by the list of matrices

$$\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right].$$

There are 6 irreducible inequivalent representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \mathcal{V}^{(3)}, \mathcal{V}^{(4)}, \mathcal{V}^{(5)}, \mathcal{V}^{(6)}$  of the group  $D_6$  (see Section 7.4.2). The submatrix of the matrix of multiplicities  $\Gamma_{D_6}$  that corresponds to this group action is (see Section 7.4.2)

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

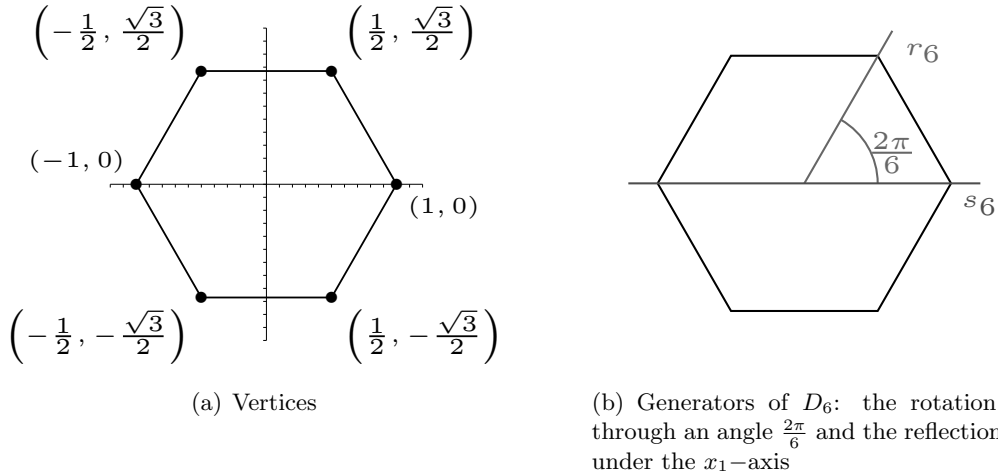


Figure 13: Regular hexagon  $H_2$

## Existence conditions

1. Since the two expected orbits are associated with the types of the first and the second column of  $\Gamma$ , the expected multiplicities  $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$  of the irreducible representations  $\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(N)}$  of  $D_6$  on  $\mathbb{R}[x]/I_\Lambda$  are

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = (2, 0, 1, 0, 1, 1), \text{ i.e. } \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_6 \end{pmatrix} = \Gamma \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since all irreducible representations of  $D_6$  are absolutely irreducible, the expected multiplicities  $\gamma_j$  are the expected ranks  $r_j$ .

2. The inequalities (9.9) are satisfied since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \mathcal{V}^{(3)}, \mathcal{V}^{(4)}, \mathcal{V}^{(5)}, \mathcal{V}^{(6)}$  of  $D_6$  in the induced representation on  $\mathbb{R}[x]_{\leq 2}$  are

$$(\underline{c}'_1, \underline{c}'_2, \underline{c}'_3, \underline{c}'_4, \underline{c}'_5, \underline{c}'_6) = (2, 0, 0, 0, 1, 1).$$

3. The inequalities (9.10) are satisfied by choosing  $\delta = 4$  since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \mathcal{V}^{(3)}, \mathcal{V}^{(4)}, \mathcal{V}^{(5)}, \mathcal{V}^{(6)}$  of  $D_6$  in the induced representation on  $\mathbb{R}[x]_{\leq 3}$  are

$$(\underline{c}''_1, \underline{c}''_2, \underline{c}''_3, \underline{c}''_4, \underline{c}''_5, \underline{c}''_6) = (2, 0, 1, 1, 2, 1).$$

4. An orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq 4}$  is

$$\begin{bmatrix} [1, x_1^2 + x_2^2, x_1^4 + 2x_1^2x_2^2 + x_2^4], [ ], [x_1^3 - 3x_1x_2^2], [x_1^2x_2 - \frac{1}{3}x_2^3], \\ [x_1, x_1^3 + x_1x_2^2], [x_2, x_1^2x_2 + x_2^3], \\ [x_1^2 - x_2^2, x_1^4 - x_2^4, x_1^2x_2^2 - \frac{1}{3}x_2^4], [2x_1x_2, 2x_1^3x_2 + 2x_1x_2^3, x_1^3x_2 - \frac{1}{3}x_1x_2^3] \end{bmatrix}.$$

5. The distinct blocks  $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}, H^{(5)}, H^{(6)}$  are then computed.
6. After the parametrization, the distinct blocks  $H^{(1)}, H^{(2)}, H^{(3)}, H^{(4)}, H^{(5)}, H^{(6)}$  are

$$\begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{5\sqrt{3}}{8} & \frac{7\sqrt{3}}{20} \\ \frac{5\sqrt{3}}{8} & \frac{7\sqrt{3}}{20} & \mathbf{h}_1 + 6\mathbf{h}_2 \\ \frac{7\sqrt{3}}{20} & \mathbf{h}_1 + 6\mathbf{h}_2 & \mathbf{h}_4 \end{pmatrix}, \begin{pmatrix} \end{pmatrix}, \begin{pmatrix} \mathbf{h}_1 \end{pmatrix}, \begin{pmatrix} \frac{2}{3}\mathbf{h}_2 \end{pmatrix}, \\ \begin{pmatrix} \frac{5\sqrt{3}}{16} & \frac{7\sqrt{3}}{40} \\ \frac{7\sqrt{3}}{40} & \frac{1}{2}\mathbf{h}_1 + 3\mathbf{h}_2 \end{pmatrix}, \begin{pmatrix} \frac{7\sqrt{3}}{40} & \frac{1}{2}\mathbf{h}_1 + 3\mathbf{h}_2 & \mathbf{h}_2 \\ \frac{1}{2}\mathbf{h}_1 + 3\mathbf{h}_2 & \frac{1}{2}\mathbf{h}_4 & \mathbf{h}_3 \\ \mathbf{h}_2 & \mathbf{h}_3 & \frac{1}{3}\mathbf{h}_3 \end{pmatrix}.$$

7. The parameters  $h_1, h_2, h_3, h_4$  are determined using Algorithm 4.7 on each block. There is only one possible set of values for the parameters

$$\hbar_1 = \frac{49\sqrt{3}}{250}, \hbar_2 = 0, \hbar_3 = 0, \hbar_4 = \frac{343\sqrt{3}}{3125}$$

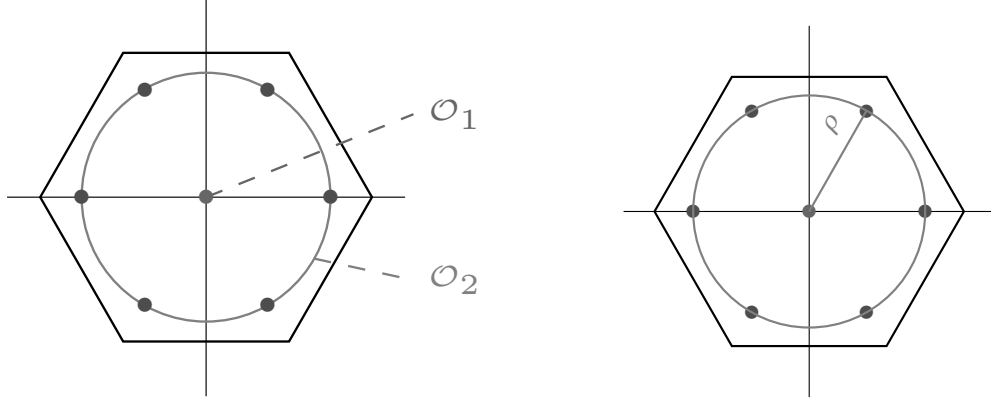
such that the linear form  $\Lambda^{(4)}$  is a flat extension of the linear form  $\Lambda^{(3)}$  and such that the Hankel operator  $\mathcal{H}^{(4)}$  is positive semidefinite with rank 7.

Thus, there exists for the hexagon  $H_2$  a single  $D_6$ -invariant cubature of degree 5 with positive weights and with this organization of nodes in orbit types.

We do not prove here that the nodes lie on the hexagon  $H_2$ . This is shown *a posteriori* once the coordinates of the nodes are computed.

## Computation of the weights and the nodes

The computation of the coordinates of the nodes is based on the knowledge obtained from the organization of the nodes in orbit types (see Figure 14(a)). Indeed, the only missing information is the radius  $\rho$  of the circle where the nodes lie on, except the origin (see Figure 14(b)). This is obtained as the square root of the evaluation of the  $D_6$ -invariant polynomial  $p = x_1^2 + x_2^2$  at any node on this circle.



(a) Expected organization of the nodes in orbits:  $\mathcal{O}_1$  the origin and  $\mathcal{O}_2$  an orbit with 6 nodes

(b) The coordinates of the nodes are determined by the radius  $\rho$  with  $\rho^2 = \frac{14}{15}$

Figure 14: Nodes of the cubature

1. An orthogonal symmetry adapted basis  $B$  of  $\mathbb{R}[x]/I_\Lambda$  is given by the selected polynomials

$$\begin{bmatrix} [[1, x_1^2 + x_2^2]], [[ ]], [[x_1^3 - 3x_1x_2^2]], [[ ]], \\ [[x_1], [x_2]], \\ [[x_1^2 - x_2^2], [2x_1x_2]] \end{bmatrix}$$

2. The distinct invertible blocks  $H_1^{(1)}, H_1^{(2)}, H_1^{(3)}, H_1^{(4)}, H_1^{(5)}, H_1^{(6)}$  with the values  $\hbar_1, \hbar_2, \hbar_3, \hbar_4$  are

$$\left( \begin{pmatrix} \frac{3\sqrt{3}}{2} & \frac{5\sqrt{3}}{8} \\ \frac{5\sqrt{3}}{8} & \frac{7\sqrt{3}}{20} \end{pmatrix}, () , \left( \frac{49\sqrt{3}}{250} \right), () , \left( \frac{5\sqrt{3}}{16} \right), \left( \frac{7\sqrt{3}}{40} \right) \right).$$

3. The separating set contains only one  $D_6$ -invariant polynomial

$$p = x_1^2 + x_2^2.$$

4. The distinct blocks  $H_p^{(1)}, H_p^{(2)}, H_p^{(3)}, H_p^{(4)}, H_p^{(5)}, H_p^{(6)}$  are

$$\left( \begin{pmatrix} \frac{5\sqrt{3}}{8} & \frac{7\sqrt{3}}{20} \\ \frac{7\sqrt{3}}{20} & \frac{49\sqrt{3}}{250} \end{pmatrix}, () , \left( \frac{343\sqrt{3}}{3125} \right), () , \left( \frac{7\sqrt{3}}{40} \right), \left( \frac{49\sqrt{3}}{500} \right) \right).$$

In this case, there is no additional unknown coefficients.

5. Instead of computing the generalized eigenvalues for the pair of  $2 \times 2$  blocks  $(H_p^{(1)}, H_1^{(1)})$ , we can compute here the one of any other pair of  $1 \times 1$  blocks  $(H_p^{(j)}, H_1^{(j)})$  with  $j \in \{3, 5, 6\}$ . The generalized eigenvalue problem is thus reduced to a single division.

For instance, if we take the fifth blocks, the generalized eigenvalue, that is the evaluation of the polynomial  $p$  at any node (except the origin), is given by

$$\frac{7}{40}\sqrt{3} \div \frac{5}{16}\sqrt{3} = \frac{14}{25}.$$

If we would have solved the generalized eigenvalue problem for the pair of  $2 \times 2$  blocks  $(H_p^{(1)}, H_1^{(1)})$ , we would get the evaluations of the polynomial  $p$  at the distinct orbits, that is 0 since the origin is an orbit and  $\frac{14}{25}$  the one that corresponds to  $\rho^2$ .

6. The polynomial system for the orbit with 6 nodes is here reduced to a single equation

$$x_1^2 + x_2^2 = \frac{14}{25}.$$

Since it is here sufficient to know the radius  $\rho$ , it is even reduced to the computation of the square root of the computed generalized eigenvalue. The coordinates of the nodes are then the following (see Figure 14(b))

$$(0, 0), \left(\frac{\sqrt{14}}{5}, 0\right), \left(\frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10}\right), \left(-\frac{\sqrt{14}}{10}, \frac{\sqrt{42}}{10}\right), \left(-\frac{\sqrt{14}}{5}, 0\right), \left(-\frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10}\right), \left(\frac{\sqrt{14}}{10}, -\frac{\sqrt{42}}{10}\right).$$

7. The computation of the weights is reduced to the Vandermonde-like linear system (9.13) with a  $2 \times 2$  matrix

$$\begin{pmatrix} 1 & 6 \\ 0 & \frac{84}{25} \end{pmatrix} \begin{pmatrix} \check{a}_1 \\ \check{a}_2 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{5\sqrt{3}}{8} \end{pmatrix}$$

and the distinct weights are

$$\check{a}_1 = \frac{43\sqrt{3}}{112}, \check{a}_2 = \frac{125\sqrt{3}}{672}.$$

The expected  $D_6$ -invariant inside cubature with positive weights of degree 5 for the hexagon  $H_2$  is then determined thanks to Table 1.

|                 | Type<br>(number of nodes) | Weight                    | A node per orbit                      |
|-----------------|---------------------------|---------------------------|---------------------------------------|
| $\mathcal{O}_1$ | $\mathcal{P}_1$ (1)       | $\frac{43\sqrt{3}}{112}$  | (0,0)                                 |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ (6)       | $\frac{125\sqrt{3}}{672}$ | $\left(\frac{\sqrt{14}}{5}, 0\right)$ |

Table 1: Weights and nodes of the cubature

## 10 $D_6$ –invariant cubatures of degree 13 for the regular hexagon

In this section, we detail how to search for inside  $D_6$ –invariant cubatures for the regular hexagon  $H_2$  with positive weights. In [51, Section 6 - Rotational-symmetric formulas], the author looked for cubatures whose nodes are union of orbits with 1 node or 6 nodes. With the inequalities (9.9), we explain why it is impossible to find an inside  $D_6$ –invariant cubature of degree bigger than 11 with positive weights and without an orbit with 12 nodes. We prove then that there exist  $D_6$ –invariant cubatures of degree 13 using Algorithm 9.6 [Existence of a  $G$ –invariant cubature] and that there is no such cubature with less than 37 nodes. We conclude with the construction of cubatures with 37 nodes using Algorithm 9.7.

### 10.1 Existence of cubatures with at least 37 nodes

We first introduce the input of Algorithm 9.6 [Existence of a  $G$ –invariant cubature] and discuss the different possible organizations of the nodes of the sought cubatures in orbit types. Applying Algorithm 9.6, we show that only three organizations provide cubatures.

In this search, we are looking for  $D_6$ –invariant cubatures of degree 13 for the regular hexagon  $H_2$ :  $(\pm 1, 0)$ ,  $(\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$ . The moments for  $H_2$  of order less than or equal to 13 are computed thanks to the formulas in [99]. Any value  $\int p \, d\mu$  with  $p \in \mathbb{R}[x]_{\leq 13}^G$  is then a linear combination of those moments.

Taking  $m = 6$  in Section 7.4, we get:

- ▷ A representation  $\mathcal{W}$  of  $D_6$  on  $\mathbb{R}^2$  deduced from the action of  $D_6$  that leaves  $H_2$  invariant.

$$\left[ \begin{array}{l} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right].$$

- ▷ The set  $[L_{abs}, L_{com}]$  of irreducible representations  $\nu^{(1)}, \nu^{(2)}, \nu^{(3)}, \nu^{(4)}, \nu^{(5)}, \nu^{(6)}$ . Since they are all absolutely irreducible,  $L_{com}$  is empty and  $L_{abs}$  is

$$\left[ \begin{array}{l} [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \\ [1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1] \\ [1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1] \\ [1, -1, 1, -1, 1, -1, -1, 1, -1, 1, -1, 1] \\ \left[ \begin{array}{l} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right] \\ \left[ \begin{array}{l} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right] \end{array} \right]$$

- ▷ The matrix of multiplicities  $\Gamma_{D_6}$ , and more precisely the submatrix corresponding to this group action

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

The different possibilities for the integers  $(m_1, m_2, m_3, m_4)$  are now discussed, where:

- $m_1 = 1$  if the origin is a node and  $m_1 = 0$  otherwise.
- $m_2$  is the number of orbits with 6 nodes such that 2 nodes lie on the  $x_1$ -axis.
- $m_3$  is the number of orbits with 6 nodes such that 2 nodes lie on the  $x_2$ -axis.
- $m_4$  is the number of orbits with 12 nodes.

They satisfy

$$r = m_1 + 6(m_2 + m_3) + 12m_4,$$

where  $r$  is the number of nodes. The solutions of this equation over the nonnegative integers are given in Table 2. Only the cases  $r = 31$ ,  $r = 36$  and  $r = 37$  are considered since

- $r = 31$  corresponds to Möller's lower bound (2.10).
- $m_1 \in \{0, 1\}$  so that  $r \equiv 0 \pmod{6}$  or  $r \equiv 1 \pmod{6}$ . The cases  $r = 32$ ,  $r = 33$ ,  $r = 34$  and  $r = 35$  are therefore impossible.
- We shall show that there is no cubature with less than 37 nodes.

|          |   |
|----------|---|
| 31 nodes | (1, 5, 0, 0), (1, 4, 1, 0), (1, 3, 2, 0), (1, 2, 3, 0), (1, 1, 4, 0), (1, 0, 5, 0)<br>(1, 3, 0, 1), ( <b>1, 2, 1, 1</b> ), ( <b>1, 1, 2, 1</b> ), (1, 0, 3, 1),<br>(1, 1, 0, 2), (1, 0, 1, 2)   |
| 36 nodes | (0, 6, 0, 0), (0, 5, 1, 0), (0, 4, 2, 0), (0, 3, 3, 0), (0, 2, 4, 0), (0, 1, 5, 0), (0, 0, 6, 0)<br>(0, 4, 0, 1), ( <b>0, 3, 1, 1</b> ), ( <b>0, 2, 2, 1</b> ), ( <b>0, 1, 3, 1</b> ), (0, 0, 4, 1),<br>(0, 2, 0, 2), (0, 1, 1, 2), (0, 0, 2, 2)                            |
| 37 nodes | (1, 6, 0, 0), (1, 5, 1, 0), (1, 4, 2, 0), (1, 3, 3, 0), (1, 2, 4, 0), (1, 1, 5, 0), (1, 0, 6, 0)<br>(1, 4, 0, 1), ( <b>1, 3, 1, 1</b> ), ( <b>1, 2, 2, 1</b> ), ( <b>1, 1, 3, 1</b> ), (1, 0, 4, 1),<br>( <b>1, 2, 0, 2</b> ), ( <b>1, 1, 1, 2</b> ), ( <b>1, 0, 2, 2</b> ) |

Table 2: Possible values for the integers  $(m_1, m_2, m_3, m_4)$

The number of possible cases in Table 2 can be reduced thanks to the inequalities (9.9) in Step 2 of Algorithm 9.6. The expected multiplicities  $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$  are linked to the integers  $m_1, m_2, m_3, m_4$  thanks to (9.8) by

$$\gamma_1 = m_1 + m_2 + m_3 + m_4, \quad \gamma_2 = m_4, \quad \gamma_3 = m_2 + m_4, \quad \gamma_4 = m_3 + m_4,$$

$$\gamma_5 = m_2 + m_3 + 2m_4, \quad \gamma_6 = m_2 + m_3 + 2m_4.$$

Here  $\lfloor \frac{d}{2} \rfloor = 6$  and the multiplicities  $\underline{c}'_1, \underline{c}'_2, \underline{c}'_3, \underline{c}'_4, \underline{c}'_5, \underline{c}'_6$  are

$$(\underline{c}'_1, \underline{c}'_2, \underline{c}'_3, \underline{c}'_4, \underline{c}'_5, \underline{c}'_6) = (5, 1, 2, 2, 4, 5).$$

According to (9.9),

$$\gamma_j \geq \underline{c}'_j \quad \forall j = 1, \dots, 6,$$

thus:

- Since  $\gamma_2 = m_4$  and  $\underline{c}'_2 = 1$ , (9.9) implies that there is at least 1 orbit with 12 nodes. The cases with  $m_4 = 0$  are therefore impossible.

This explains why no inside cubature of degree 13 with positive weights can be found in [51]. A  $D_6$ -invariant cubature of degree bigger than 13 has at least 1 orbit with 12 nodes.

- Since  $\underline{c}'_1 = 5$  and  $\gamma_1 = m_1 + m_2 + m_3 + m_4$  (Remark 8.7), (9.9) implies that there are at least 5 orbits. The cases with  $m_4 = 2$  for 31 nodes and 36 nodes in Table 2 are therefore impossible.
- Since  $\underline{c}'_3 = \underline{c}'_4 = 2$ ,  $\gamma_3 = m_2 + m_4$  and  $\gamma_4 = m_3 + m_4$ , (9.9) allows us to discard the cases in Table 2 with  $m_2 = 0$  and  $m_4 = 1$  or with  $m_3 = 0$  and  $m_4 = 1$ .

The remaining cases are then the ones in bold in Table 2.

With the help of Algorithm 9.6 (until Step 7), we find the systems of equations and inequations that determine the existence for  $H_2$  of  $D_6$ -invariant cubatures of degree 13 with positive weights<sup>4</sup>. The only cases for which there exist such cubatures are in Table 3. The choice of the degree  $\delta$  (Step 3) and the corresponding values of  $(c''_1, c''_2, c''_3, c''_4, c''_5, c''_6)$  and  $(c_1, c_2, c_3, c_4, c_5, c_6)$  are indicated. Notice that the irreducible representations of  $D_6$  are absolutely irreducible so that

$$\gamma_j = r_j, \quad \underline{c}'_j = c'_j, \quad \underline{c}''_j = c''_j \quad \forall j = 1, \dots, 6.$$

| 37 nodes       | $\delta$ | $(c''_1, c''_2, c''_3, c''_4, c''_5, c''_6)$ | $(c_1, c_2, c_3, c_4, c_5, c_6)$ |
|----------------|----------|--|----------------------------------|
| Case (1,3,1,1) | 10       | (7, 2, 5, 5, 10, 8)                          | (9, 3, 5, 5, 10, 12)             |
| Case (1,2,2,1) | 9        | (7, 2, 3, 3, 7, 8)                           | (7, 2, 5, 5, 10, 8)              |
| Case (1,1,3,1) | 10       | (7, 2, 5, 5, 10, 8)                          | (9, 3, 5, 5, 10, 12)             |

Table 3: Values for the integers  $(m_1, m_2, m_3, m_4)$  such that cubatures exist

The last column informs us on the size of the matrices we deal with in this symmetric approach. In comparison, if symmetry were not taken into account as in Section 5, the size of the matrices would have been:  $\dim \mathbb{R}[x]_{\leq 10} = 66$  in the cases (1,3,1,1) and (1,1,3,1) and  $\dim \mathbb{R}[x]_{\leq 9} = 55$  in the case (1,2,2,1).

The systems of the cases (1,3,1,1) and (1,1,3,1) have both a unique solution, whereas the systems of the case (1,2,1,1) have two solutions.

<sup>4</sup>Due to the size of the matrices, we do not give here the details of the computations.

## 10.2 Computation of the weights and the nodes of the 4 cubatures

With the help of the 4 solutions found thanks to Algorithm 9.6 [Existence of a  $G$ -invariant cubature], we are now able to compute the 4 associated cubatures using Algorithm 9.7 [Weights & Nodes]. We first need to determine a separating set  $\{p_1, \dots, p_\eta\}$  of  $G$ -invariant polynomials and then choose if it is preferable to solve the generalized eigenvalues for the pairs of blocks  $(H_{p_\nu}^{(1)}, H_1^{(1)})$  or for other blocks (as in the example of Section 9.4). Both depend on the organization of nodes in orbit types (see Figures 15(a), 16(a), 17(a) and 18(a)).

Thanks to the values of the integers  $(m_1, m_2, m_3, m_4)$  in those cases (Table 3), we know that:

- The origin is a node: this orbit is denoted by  $\mathcal{O}_1$ .
- There are 4 orbits  $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$  and  $\mathcal{O}_5$  with 6 nodes (see Figures 15(b), 16(b), 17(b) and 18(b)): the missing information is the radii  $\rho_2, \rho_3, \rho_4$  and  $\rho_5$  of the circles where the nodes lie on. They are obtained thanks to the  $D_6$ -invariant polynomial  $p_1 = x_1^2 + x_2^2$ .
- There is 1 orbit  $\mathcal{O}_6$  with 12 nodes (see Figures 15(c), 16(c), 17(c) and 18(c)). For this orbit, knowing the radius  $\rho_6$  is not enough. The additional required information is the angle  $\theta_6$  of one of the node on  $\mathcal{O}_6$ . We choose the unique node such that  $0 < \theta_6 < \frac{\pi}{6}$ . It is possible to compute it thanks to a second  $D_6$ -invariant polynomial given in [46, Chapter XII.4] by

$$p_2 = (x_1 + ix_2)^6 + (x_1 - ix_2)^6 = 2x_1^6 - 30x_1^4x_2^2 + 30x_1^2x_2^4 - 2x_2^6.$$

Indeed, assuming the evaluation  $p_2(\zeta_6)$  of  $p_2$  at any node  $\zeta_6$  on  $\mathcal{O}_6$  is known,  $\theta_6$  is then the unique solution of the equation in  $\theta$

$$p_2(\zeta_6) = 2\rho_6^6 \cos(6\theta) \quad (10.1)$$

under the constraint  $0 < \theta < \frac{\pi}{6}$ .

The separating set in Step 3 of Algorithm 9.7 is then

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^6 - 30x_1^4x_2^2 + 30x_1^2x_2^4 - 2x_2^6\}.$$

As noticed in Step 5 of Algorithm 9.7, it is sometimes more suitable to take other blocks than the first blocks. We choose here to consider the second, third and fourth blocks. This corresponds to the choice of a  $3 \times 3$  unimodular submatrix of  $\Gamma$ . Indeed, since the first column of  $\Gamma$  informs on the presence of the origin as a node and since we already have this information, a  $4 \times 4$  unimodular submatrix of  $\Gamma$  is not required to solve uniquely the linear systems (8.6).

The strategy for Steps 5 and 6 of Algorithm 9.7 is then the following:

1. Compute the generalized eigenvalues of  $(H_{p_1}^{(2)}, H_1^{(2)})$ ,  $(H_{p_1}^{(3)}, H_1^{(3)})$  and  $(H_{p_1}^{(4)}, H_1^{(4)})$ . The unique generalized eigenvalue of the pair  $(H_{p_1}^{(2)}, H_1^{(2)})$  is  $\rho_6^2$ . It is also a generalized eigenvalue of the pairs  $(H_{p_1}^{(3)}, H_1^{(3)})$  and  $(H_{p_1}^{(4)}, H_1^{(4)})$ . The other generalized eigenvalues are  $\rho_2^2, \rho_3^2, \rho_4^2$  and  $\rho_5^2$ . This determines the orbits  $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$  and  $\mathcal{O}_5$ .
2. Compute the generalized eigenvalue of the pair  $(H_{p_2}^{(2)}, H_1^{(2)})$ . Solving (10.1), we get then  $\theta_6$ . This determines the orbit  $\mathcal{O}_6$ .



This strategy has the following advantages:

- The second, third and fourth blocks are smaller than the first ones since

$$\gamma_1 = m_1 + m_2 + m_3 + m_4 = 6$$

whereas

$$\gamma_2 = m_4 = 1, \quad \gamma_3 = m_2 + m_4 \leq 4, \quad \gamma_4 = m_3 + m_4 \leq 4.$$

- The matrix  $H_{p_2}^{(1)}$  contains coefficients that have not been determined yet, whereas the blocks  $H_{p_1}^{(2)}, H_{p_1}^{(3)}, H_{p_1}^{(4)}, H_{p_2}^{(2)}$  do not.

### Case 37 nodes (1,3,1,1)

Among the different triplets  $[P, Z, E]$  output by Algorithm 9.6, only one triplet has a solution. This is checked by computing a Groebner basis of the polynomials in each set  $Z$ : if it is reduced to  $\{1\}$ , then there is no solution. Solving the polynomial system obtained from this set  $Z$ , we get 2 distinct sets of values for the 14 unknown parameters in the blocks  $H^{(1)}, \dots, H^{(6)}$  introduced in Step 6 of Algorithm 9.6. But only 1 of these distinct sets of values satisfies the positivity constraints from  $P$ . Using this remaining set of values, we apply Algorithm 9.7.

1. An orthogonal symmetry adapted basis of the second, third and fourth component of  $\mathbb{R}[x]/I_\Lambda$  is given by selecting the following polynomials in the second, third and fourth component of  $\mathbb{R}[x]_{\leq 9}$

$$\begin{aligned} & [x_1^5 x_2 - \frac{10}{3} x_1^3 x_2^3 + x_1 x_2^5], \\ & [x_1^3 - 3x_1 x_2^2, x_1^5 - 2x_1^3 x_2^2 - 3x_1 x_2^4, x_1^7 - x_1^5 x_2^2 - 5x_1^3 x_2^4 - 3x_1 x_2^6, x_1^9 - 6x_1^5 x_2^4 - 8x_1^3 x_2^6 - 3x_1 x_2^8], \\ & [x_1^2 x_2 - \frac{1}{3} x_2^3, x_1^4 x_2 + \frac{2}{3} x_1^2 x_2^3 - \frac{1}{3} x_2^5]. \end{aligned}$$

2. The invertible blocks  $H_1^{(2)}, H_1^{(3)}$  and  $H_1^{(4)}$  are respectively

$$\begin{pmatrix} \frac{31\sqrt{3}}{34496} \end{pmatrix}, \begin{pmatrix} \frac{309\sqrt{3}}{2240} & \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} \\ \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & \hbar_1 \\ \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & \hbar_1 & \hbar_2 \\ \frac{32289\sqrt{3}}{560560} & \hbar_1 & \hbar_2 & \hbar_3 \end{pmatrix}, \begin{pmatrix} \frac{3\sqrt{3}}{320} & \frac{33\sqrt{3}}{5600} \\ \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} \end{pmatrix}$$

with

- $\hbar_1 = \frac{a_1}{b_1} \sqrt{3} + \frac{c_1}{d_1} \sqrt{17144267591348974794021990}$ , where  $a_1 = 19\ 186\ 284\ 158\ 106\ 782\ 360\ 533\ 739\ 463\ 657\ 337\ 817\ 123\ 801$ ,  $b_1 = 418\ 401\ 904\ 389\ 034\ 302\ 548\ 727\ 436\ 218\ 159\ 228\ 313\ 795\ 248$ ,  $c_1 = 5\ 567\ 090\ 272\ 441\ 677\ 908\ 851\ 774\ 900\ 194\ 238\ 557$ ,  $d_1 = 841\ 929\ 568\ 618\ 188\ 155\ 306\ 432\ 227\ 859\ 273\ 914\ 412\ 831\ 615\ 499\ 878\ 072$ ;
- $\hbar_2 = \frac{a_2}{b_2} \sqrt{3} + \frac{c_2}{d_2} \sqrt{17144267591348974794021990}$ , where  $a_2 = 100\ 644\ 786\ 976\ 659\ 682\ 070\ 463\ 039\ 891\ 774\ 035\ 345\ 240\ 805\ 103\ 944\ 907\ 979\ 378\ 573\ 344\ 459\ 912\ 746\ 782\ 918\ 911$ ,  $b_2 = 2\ 704\ 980\ 476\ 033\ 486\ 556\ 489\ 785\ 064\ 780\ 349\ 140\ 562\ 806\ 614\ 191\ 813\ 930\ 605\ 564\ 527\ 233\ 948\ 504\ 807\ 144\ 477\ 272$ ,  $c_2 = 2\ 692\ 747\ 051\ 260\ 847\ 118\ 697\ 851\ 888\ 062\ 234\ 830\ 635\ 134\ 922\ 176\ 183\ 333\ 837\ 316\ 013\ 944\ 129\ 422\ 605$ ,  $d_2 = 111\ 083\ 655\ 886\ 817\ 745\ 301\ 726\ 914\ 106\ 003\ 422\ 756\ 088\ 032\ 763\ 135\ 997\ 877\ 992\ 203\ 328\ 157\ 474\ 585\ 305\ 760\ 353\ 810\ 740\ 892$ ;

- $\hbar_3 = \frac{a_3}{b_3}\sqrt{3} + \frac{c_3}{d_3}\sqrt{17144267591348974794021990}$ , where  $a_3 = 7\ 283\ 689\ 389\ 530\ 020\ 687\ 327\ 879\ 287\ 363\ 947\ 837\ 479\ 130\ 940\ 966\ 369\ 197\ 834\ 056\ 038\ 784\ 443\ 739\ 447\ 843\ 426\ 948\ 705\ 675\ 601\ 589\ 918\ 194\ 245\ 146\ 220\ 917\ 649\ 105$ ,  $b_3 = 237\ 928\ 944\ 478\ 830\ 793\ 763\ 337\ 210\ 016\ 263\ 094\ 342\ 851\ 319\ 049\ 025\ 571\ 460\ 275\ 675\ 526\ 275\ 599\ 800\ 626\ 064\ 134\ 483\ 668\ 500\ 122\ 886\ 339\ 856\ 563\ 412\ 032\ 025\ 094\ 728$ ,  $c_3 = 1\ 241\ 436\ 209\ 175\ 718\ 349\ 395\ 214\ 969\ 806\ 564\ 819\ 809\ 454\ 926\ 485\ 085\ 828\ 176\ 115\ 515\ 396\ 204\ 763\ 573\ 238\ 272\ 305\ 862\ 722\ 038\ 062\ 295\ 489\ 152\ 070\ 911\ 905$ ,  $d_3 = 23\ 675\ 573\ 565\ 914\ 340\ 619\ 610\ 870\ 129\ 744\ 084\ 654\ 082\ 590\ 667\ 785\ 100\ 505\ 000\ 310\ 812\ 665\ 988\ 891\ 402\ 293\ 048\ 479\ 659\ 862\ 166\ 950\ 222\ 395\ 055\ 486\ 727\ 951\ 870\ 690\ 172\ 583\ 454$ .

3. The separating set is

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^6 - 30x_1^4x_2^2 + 30x_1^2x_2^4 - 2x_2^6\}.$$

4. The distinct blocks  $H_{p_1}^{(2)}$ ,  $H_{p_1}^{(3)}$ ,  $H_{p_1}^{(4)}$  and  $H_{p_2}^{(2)}$  are respectively

$$(\hbar_4), \begin{pmatrix} \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & \hbar_1 \\ \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & \hbar_1 & \hbar_2 \\ \frac{32289\sqrt{3}}{560560} & \hbar_1 & \hbar_2 & \hbar_3 \\ \hbar_1 & \hbar_2 & \hbar_3 & \hbar_5 \end{pmatrix}, \begin{pmatrix} \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} \\ \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} \end{pmatrix}, (\hbar_6)$$

with

- $\hbar_4 = \frac{a_4}{b_4}\sqrt{3} + \frac{c_4}{d_4}\sqrt{17144267591348974794021990}$ , where  $a_4 = 9\ 455$ ,  $b_4 = 18\ 403\ 616$ ,  $c_4 = 31$ ,  $d_4 = 571\ 361\ 519\ 791\ 841\ 856$ ;
- $\hbar_5 = \frac{a_5}{b_5}\sqrt{3} + \frac{c_5}{d_5}\sqrt{17144267591348974794021990}$ , where  $a_5 = 255\ 046\ 343\ 368\ 657\ 902\ 032\ 035\ 810\ 584\ 989\ 396\ 180\ 758\ 522\ 278\ 323\ 088\ 786\ 814\ 384\ 588\ 121\ 792\ 142\ 231\ 366\ 698\ 352\ 182\ 106\ 512\ 356\ 391\ 354\ 154\ 109\ 916\ 513\ 250\ 420\ 859\ 737\ 118\ 041\ 674\ 388\ 085\ 063\ 549\ 809\ 716\ 817\ 197\ 395$ ,  $b_5 = 20\ 504\ 879\ 343\ 462\ 084\ 258\ 987\ 829\ 821\ 461\ 719\ 328\ 722\ 616\ 605\ 023\ 822\ 361\ 842\ 992\ 680\ 724\ 544\ 792\ 997\ 979\ 674\ 389\ 719\ 434\ 382\ 521\ 396\ 555\ 528\ 421\ 509\ 283\ 704\ 410\ 354\ 725\ 643\ 019\ 794\ 602\ 699\ 099\ 189\ 322\ 392\ 194\ 883\ 907\ 072$ ,  $c_5 = 4\ 576\ 894\ 285\ 553\ 975\ 518\ 305\ 535\ 223\ 630\ 459\ 761\ 168\ 220\ 718\ 266\ 886\ 005\ 032\ 577\ 650\ 137\ 651\ 108\ 792\ 249\ 894\ 623\ 688\ 027\ 596\ 430\ 503\ 882\ 513\ 971\ 127\ 137\ 510\ 545\ 693\ 051\ 677\ 164\ 647\ 182\ 982\ 585\ 772\ 352\ 668\ 925$ ,  $d_5 = 35\ 085\ 850\ 298\ 044\ 533\ 172\ 867\ 654\ 356\ 342\ 901\ 088\ 780\ 624\ 369\ 783\ 916\ 703\ 845\ 920\ 672\ 391\ 742\ 891\ 346\ 291\ 148\ 336\ 384\ 113\ 426\ 888\ 123\ 568\ 677\ 963\ 230\ 440\ 693\ 759\ 232\ 902\ 912\ 809\ 977\ 532\ 342\ 648\ 595\ 129\ 944\ 742\ 775\ 149\ 349\ 759\ 008$ ;
- $\hbar_6 = \frac{a_6}{b_6}\sqrt{3} + \frac{c_6}{d_6}\sqrt{17144267591348974794021990}$ , where  $a_6 = 77\ 781\ 325$ ,  $b_6 = 187\ 074\ 235\ 502$ ,  $c_6 = 355\ 858\ 765$ ,  $d_6 = 1\ 463\ 599\ 811\ 939\ 162\ 046\ 290\ 064$ .

The computation is now done in floating point arithmetic.

5. The generalized eigenvalue of the pair  $(H_{p_1}^{(2)}, H_1^{(2)})$  is

$$\rho_6^2 = 0.7160263383987789.$$

The generalized eigenvalues of the pair  $(H_{p_1}^{(3)}, H_1^{(3)})$ , distinct from  $\rho_6^2$ , are

$$\rho_2^2 = 0.5402717232537627, \rho_3^2 = 0.1688260881819940, \rho_4^2 = 0.8696140693752899.$$

The generalized eigenvalue of the pair  $(H_{p_1}^{(4)}, H_1^{(4)})$ , distinct from  $\rho_6^2$ , is

$$\rho_5^2 = 0.4273663513856634.$$

The generalized eigenvalue of the pair  $(H_{p_2}^{(2)}, H_1^{(2)})$  is

$$2\rho_6^6 \cos(6\theta_6) = -0.1841200190295809.$$

6. We first deduce the coordinates of a node per orbit (Table 4) from the equations above and then the coordinates of the whole set of nodes from the group action and the orbit type (see Figure 15(a)).
7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Table 4).

A  $D_6$ -invariant cubature of degree 13 for the hexagon  $H_2$  is thus determined (Table 4).

|                 | Type<br>(number of nodes) | Common weight $\check{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_1$ (1)       | 0.1581420400555712               | (0,0)  |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ (6)       | 0.0784240699308296               | (0.7350317838391498,0)                               |
| $\mathcal{O}_3$ | $\mathcal{P}_2$ (6)       | 0.1344412904126819               | (0.4108845192776117,0)                               |
| $\mathcal{O}_4$ | $\mathcal{P}_2$ (6)       | 0.0252992190340063               | (0.9325310018306576,0)                               |
| $\mathcal{O}_5$ | $\mathcal{P}_3$ (6)       | 0.0945409138972820               | (0,0.6537326298921168)                               |
| $\mathcal{O}_6$ | $\mathcal{P}_4$ (12)      | 0.0369751009707455               | (0.8073714597089485,0.2533331096525298)              |

Table 4: Weights and nodes of the cubature in the case (1,3,1,1)

### Case 37 nodes (1,2,2,1)

Among the different triplets  $[P, Z, E]$  output by Algorithm 9.6, only one triplet has a solution. This is checked by computing a Groebner basis of the polynomials in each set  $Z$ : if it is reduced to  $\{1\}$ , then there is no solution. Solving the polynomial system obtained from this set  $Z$ , we get 4 distinct sets of values <sup>5</sup> for the 10 unknown parameters in the blocks  $H^{(1)}, \dots, H^{(6)}$  introduced in Step 6 of Algorithm 9.6. But only 2 of these distinct sets of values satisfy the positivity constraints from  $P$ . Using this remaining set of values, we apply Algorithm 9.7.

1. An orthogonal symmetry adapted basis of the second, third and fourth component of  $\mathbb{R}[x]/I_\Lambda$  is given by selecting the following polynomials in the second, third and fourth component of  $\mathbb{R}[x]_{\leq 8}$

$$\begin{aligned}
& [x_1^5 x_2 - \frac{10}{3} x_1^3 x_2^3 + x_1 x_2^5], \\
& [x_1^3 - 3x_1 x_2^2, x_1^5 - 2x_1^3 x_2^2 - 3x_1 x_2^4, x_1^7 - x_1^5 x_2^2 - 5x_1^3 x_2^4 - 3x_1 x_2^6], \\
& [x_1^2 x_2 - \frac{1}{3} x_2^3, x_1^4 x_2 + \frac{2}{3} x_1^2 x_2^3 - \frac{1}{3} x_2^5, x_1^6 x_2 + \frac{5}{3} x_1^4 x_2^3 + \frac{1}{3} x_1^2 x_2^5 - \frac{1}{3} x_2^7].
\end{aligned}$$

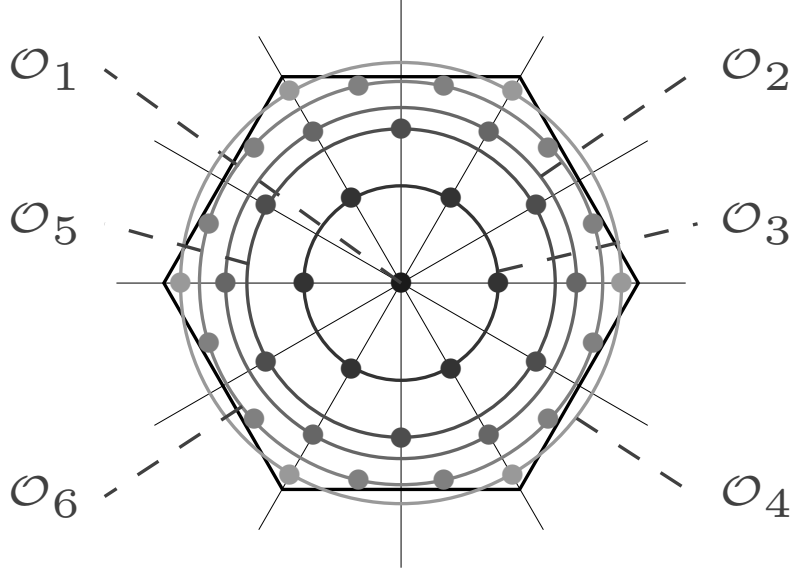
2. The invertible blocks  $H_1^{(2)}$ ,  $H_1^{(3)}$  and  $H_1^{(4)}$  are respectively

$$\begin{pmatrix} \frac{31\sqrt{3}}{34496} \end{pmatrix}, \begin{pmatrix} \frac{309\sqrt{3}}{2240} & \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} \\ \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} \\ \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & h_1 \end{pmatrix}, \begin{pmatrix} \frac{3\sqrt{3}}{320} & \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} \\ \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} \\ \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & h_2 \end{pmatrix}.$$

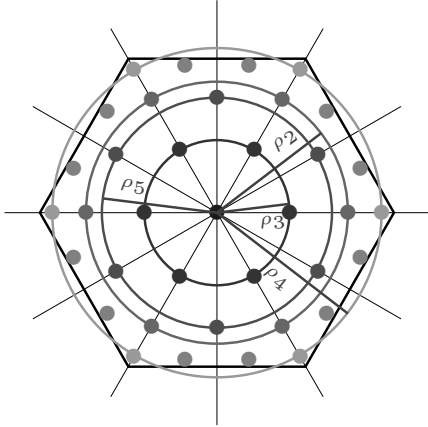
Since the exact value of  $h_1$  and  $h_2$  is too large to appear here, we give here an approximation

$$\begin{aligned}
\text{First solution:} \quad & h_1 = 0.0792022183895574, \quad h_2 = 0.0031602798204254. \\
\text{Second solution:} \quad & h_1 = 0.0792484380582109, \quad h_2 = 0.0031582155999142.
\end{aligned}$$

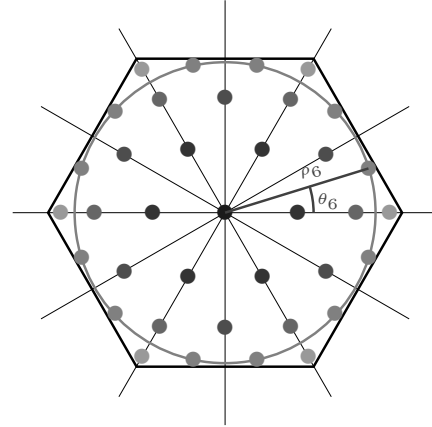
<sup>5</sup>In fact, 8 distinct sets of values, but 4 are in  $\mathbb{C} \setminus \mathbb{R}$ .



(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the origin,  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  and  $\mathcal{O}_4$  the orbits with 6 nodes and 2 nodes on the  $x_1$ -axis,  $\mathcal{O}_5$  the orbit with 6 nodes and 2 nodes on the  $x_2$ -axis and  $\mathcal{O}_6$  the orbit with 12 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  are determined by the radii  $\rho_2$ ,  $\rho_3$ ,  $\rho_4$  and  $\rho_5$



(c) The coordinates of the nodes in  $\mathcal{O}_6$  are determined by the radius  $\rho_6$  and the angle  $\theta_6$

Figure 15: Nodes of the cubature in the case (1,3,1,1)

3. The separating set is

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^6 - 30x_1^4x_2^2 + 30x_1^2x_2^4 - 2x_2^6\}.$$

4. The distinct blocks  $H_{p_1}^{(2)}$ ,  $H_{p_1}^{(3)}$ ,  $H_{p_1}^{(4)}$  and  $H_{p_2}^{(2)}$  are respectively

$$(\tilde{h}_3), \begin{pmatrix} \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} \\ \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} & \tilde{h}_1 \\ \frac{32289\sqrt{3}}{560560} & \tilde{h}_1 & \tilde{h}_4 \end{pmatrix}, \begin{pmatrix} \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} \\ \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & \tilde{h}_2 \\ \frac{113\sqrt{3}}{43120} & \tilde{h}_2 & \tilde{h}_5 \end{pmatrix}, (\tilde{h}_6).$$

Since the exact value of  $\hbar_3$ ,  $\hbar_4$ ,  $\hbar_5$  and  $\hbar_6$  is too large to appear here, we give here an approximation

$$\begin{aligned} \text{First solution: } \quad & \hbar_3 = 0.0012635157417496, \quad \hbar_4 = 0.0638006574002314, \\ & \hbar_5 = 0.0022579544688321, \quad \hbar_6 = 0.0006922999031606. \\ \text{Second solution: } \quad & \hbar_3 = 0.0012892840704738, \quad \hbar_4 = 0.0639268521092782, \\ & \hbar_5 = 0.0022540005061904, \quad \hbar_6 = 0.0012137720826728. \end{aligned}$$

The computation is now done in floating point arithmetic.

5. The generalized eigenvalue of the pair  $(H_{p_1}^{(2)}, H_1^{(2)})$  is

$$\begin{aligned} \text{First solution: } \quad & \rho_6^2 = 0.8117589301751710. \\ \text{Second solution: } \quad & \rho_6^2 = 0.8283140630210464. \end{aligned}$$

The generalized eigenvalues of the pair  $(H_{p_1}^{(3)}, H_1^{(3)})$ , distinct from  $\rho_6^2$ , are

$$\begin{aligned} \text{First solution: } \quad & \rho_2^2 = 0.4895274642961736, \quad \rho_3^2 = 0.8373579271553270. \\ \text{Second solution: } \quad & \rho_2^2 = 0.5060570527375981, \quad \rho_3^2 = 0.1247012858056488. \end{aligned}$$

The generalized eigenvalues of the pair  $(H_{p_1}^{(4)}, H_1^{(4)})$ , distinct from  $\rho_6^2$ , is

$$\begin{aligned} \text{First solution: } \quad & \rho_4^2 = 0.5884676559118877, \quad \rho_5^2 = 0.1876538077028625. \\ \text{Second solution: } \quad & \rho_4^2 = 0.2412793258525374, \quad \rho_5^2 = 0.6068420380215161. \end{aligned}$$

The generalized eigenvalue of the pair  $(H_{p_2}^{(2)}, H_1^{(2)})$  is

$$\begin{aligned} \text{First solution: } \quad & 2\rho_6^6 \cos(6\theta_6) = 0.4447753282217330. \\ \text{Second solution: } \quad & 2\rho_6^6 \cos(6\theta_6) = 0.7798005950781460. \end{aligned}$$

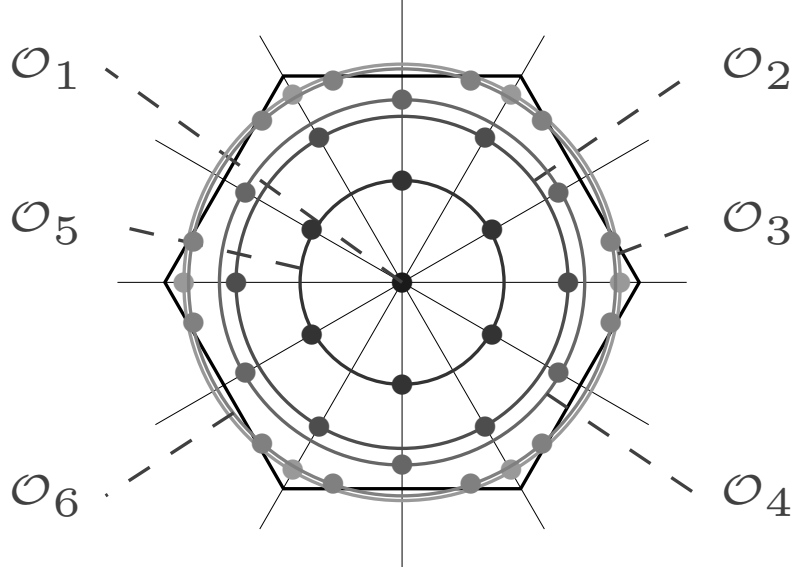
6. We first deduce the coordinates of a node per orbit (Tables 5 and 6) from the equalities above and then the coordinates of the whole set of nodes from the group action and the orbit type (see Figures 16(a) and 17(a)).

7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Tables 5 and 6).

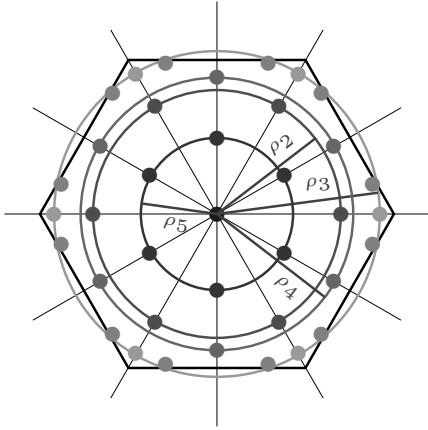
$D_6$ -invariant cubatures of degree 13 for the hexagon  $H_2$  are thus determined (Tables 5 and 6).

|                 | Type<br>(number of nodes) | Common weight $\check{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_1$ (1)       | 0.1743980715348907               | (0,0)  |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ (6)       | 0.1083603943222180               | (0.6996623930786290,0)                               |
| $\mathcal{O}_3$ | $\mathcal{P}_2$ (6)       | 0.0207256103020582               | (0.9150726337802696,0)                               |
| $\mathcal{O}_4$ | $\mathcal{P}_3$ (6)       | 0.0843685971535321               | (0,0.7671164552592022)                               |
| $\mathcal{O}_5$ | $\mathcal{P}_3$ (6)       | 0.1510318603712617               | (0,0.4331902673476431)                               |
| $\mathcal{O}_6$ | $\mathcal{P}_4$ (12)      | 0.0197299472436671               | (0.8847052130152165,0.1704570803272869)              |

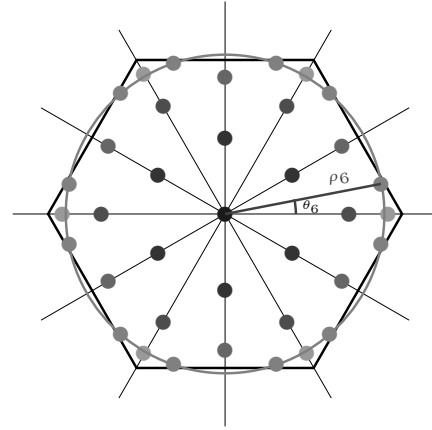
Table 5: Weights and nodes of the cubature in the case (1,2,2,1): first solution



(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the origin,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  the orbits with 6 nodes and 2 nodes on the  $x_1$ -axis,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  the orbits with 6 nodes and 2 nodes on the  $x_2$ -axis and  $\mathcal{O}_6$  the orbit with 12 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  are determined by the radii  $\rho_2$ ,  $\rho_3$ ,  $\rho_4$  and  $\rho_5$

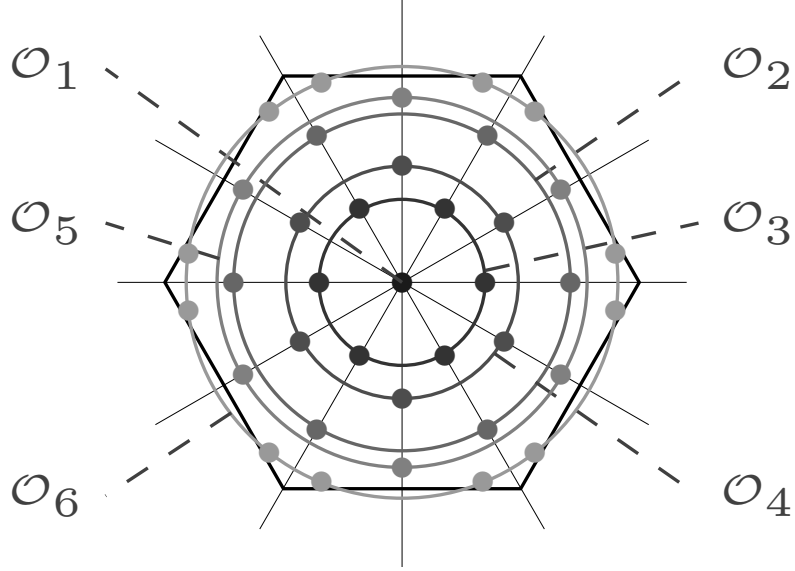


(c) The coordinates of the nodes in  $\mathcal{O}_6$  are determined by the radius  $\rho_6$  and the angle  $\theta_6$

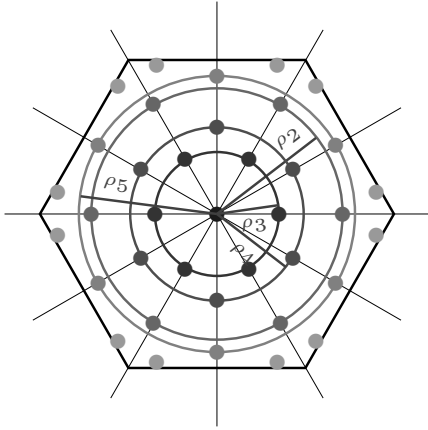
Figure 16: Nodes of the cubature in the case (1,2,2,1): first solution

|                 | Type<br>(number of nodes) | Common weight $\check{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_1$ (1)       | 0.1492131124137626               | (0,0)  |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ (6)       | 0.1043930407971358               | (0.7113768710890690,0)                               |
| $\mathcal{O}_3$ | $\mathcal{P}_2$ (6)       | 0.0615527088469823               | (0.3531306917173286,0)                               |
| $\mathcal{O}_4$ | $\mathcal{P}_3$ (6)       | 0.1071155341363768               | (0,0.4912019196316788)                               |
| $\mathcal{O}_5$ | $\mathcal{P}_3$ (6)       | 0.0804531873973581               | (0,0.7790006662549782)                               |
| $\mathcal{O}_6$ | $\mathcal{P}_4$ (12)      | 0.0273146893227030               | (0.9017400519011072,0.1232028482491464)              |

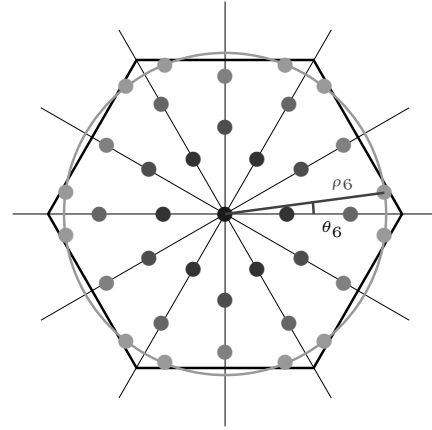
Table 6: Weights and nodes of the cubature in the case (1,2,2,1): second solution



(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the origin,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  the orbits with 6 nodes and 2 nodes on the  $x_1$ -axis,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  the orbits with 6 nodes and 2 nodes on the  $x_2$ -axis and  $\mathcal{O}_6$  the orbit with 12 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  are determined by the radii  $\rho_2$ ,  $\rho_3$ ,  $\rho_4$  and  $\rho_5$



(c) The coordinates of the nodes in  $\mathcal{O}_6$  are determined by the radius  $\rho_6$  and the angle  $\theta_6$

Figure 17: Nodes of the cubature in the case (1,2,2,1): second solution



### Case 37 nodes (1,1,3,1)

Among the different triplets  $[P, Z, E]$  output by Algorithm 9.6, only one triplet has a solution. This is checked by computing a Groebner basis of the polynomials in each set  $Z$ : if it is reduced to  $\{1\}$ , then there is no solution. Solving the polynomial system obtained from this set  $Z$ , we get 2 distinct sets of values for the 14 unknown parameters in the blocks  $H^{(1)}, \dots, H^{(6)}$  introduced in Step 6 of Algorithm 9.6. But only 1 of these distinct sets of values satisfies the positivity constraints from  $P$ . Using this remaining set of values, we apply Algorithm 9.7.

1. An orthogonal symmetry adapted basis of the second, third and fourth component of  $\mathbb{R}[x]/I_\Lambda$  is given by selecting the following polynomials in the second, third and fourth component of  $\mathbb{R}[x]_{\leq 9}$

$$\begin{aligned} & [x_1^5 x_2 - \frac{10}{3} x_1^3 x_2^3 + x_1 x_2^5], \\ & [x_1^3 - 3x_1 x_2^2, x_1^5 - 2x_1^3 x_2^2 - 3x_1 x_2^4], \\ & [x_1^2 x_2 - \frac{1}{3} x_2^3, x_1^4 x_2 + \frac{2}{3} x_1^2 x_2^3 - \frac{1}{3} x_2^5, x_1^6 x_2 + \frac{5}{3} x_1^4 x_2^3 + \frac{1}{3} x_1^2 x_2^5 - \frac{1}{3} x_2^7, x_1^8 x_2 + \frac{14}{3} x_1^4 x_2^5 - \frac{8}{9} x_1^2 x_2^7 - \frac{19}{81} x_2^9]. \end{aligned}$$

2. The invertible blocks  $H_1^{(2)}$ ,  $H_1^{(3)}$  and  $H_1^{(4)}$  are respectively

$$\begin{pmatrix} \frac{31\sqrt{3}}{34496} \end{pmatrix}, \begin{pmatrix} \frac{309\sqrt{3}}{2240} & \frac{1661\sqrt{3}}{16800} \\ \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} \end{pmatrix}, \begin{pmatrix} \frac{3\sqrt{3}}{320} & \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} & \frac{13\sqrt{3}}{6160} \\ \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & \frac{152h_1}{27} + h_2 \\ \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & 8h_1 + h_2 & h_3 \\ \frac{13\sqrt{3}}{6160} & \frac{152h_1}{27} + h_2 & h_3 & h_4 \end{pmatrix}.$$

Since the exact values of  $h_1$ ,  $h_2$ ,  $h_3$  and  $h_4$  are too large to appear here, we give here an approximation

$$h_1 = 0.0002347083678754, h_2 = 0.0012829189683071,$$

$$h_3 = 0.0019029768903960, h_4 = 0.0012473869646381.$$

3. The separating set is

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^6 - 30x_1^4 x_2^2 + 30x_1^2 x_2^4 - 2x_2^6\}.$$

4. The distinct blocks  $H_{p_1}^{(2)}$ ,  $H_{p_1}^{(3)}$ ,  $H_{p_1}^{(4)}$  and  $H_{p_2}^{(2)}$  are respectively

$$(h_2), \begin{pmatrix} \frac{1661\sqrt{3}}{16800} & \frac{2743\sqrt{3}}{36960} \\ \frac{2743\sqrt{3}}{36960} & \frac{32289\sqrt{3}}{560560} \end{pmatrix}, \begin{pmatrix} \frac{33\sqrt{3}}{5600} & \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & \frac{152}{27}h_1 + h_2 \\ \frac{13\sqrt{3}}{3360} & \frac{113\sqrt{3}}{43120} & 8h_1 + h_2 & h_3 \\ \frac{113\sqrt{3}}{43120} & 8h_1 + h_2 & h_3 & h_5 \\ \frac{152}{27}h_1 + h_2 & h_3 & h_5 & h_6 \end{pmatrix}, (h_7).$$

Since the exact values of  $h_5$ ,  $h_6$  and  $h_7$  are too large to appear here, we give here an approximation

$$h_5 = 0.0014224327872072, h_6 = 0.0009661526368457, h_7 = 0.0012045336983775.$$

The computation is now done in floating point arithmetic.

5. The generalized eigenvalue of the pair  $(H_{p_1}^{(2)}, H_1^{(2)})$  is

$$\rho_6^2 = 0.8242247364265768.$$

The generalized eigenvalues of the pair  $(H_{p_1}^{(3)}, H_1^{(3)})$ , distinct from  $\rho_6^2$ , is

$$\rho_2^2 = 0.4905623338439775.$$

The generalized eigenvalues of the pair  $(H_{p_1}^{(4)}, H_1^{(4)})$ , distinct from  $\rho_6^2$ , are

$$\rho_3^2 = 0.5855752916340393, \rho_4^2 = 0.1877507115737364, \rho_5^2 = 0.7139224749993048.$$

The generalized eigenvalue of the pair  $(H_{p_2}^{(2)}, H_1^{(2)})$  is

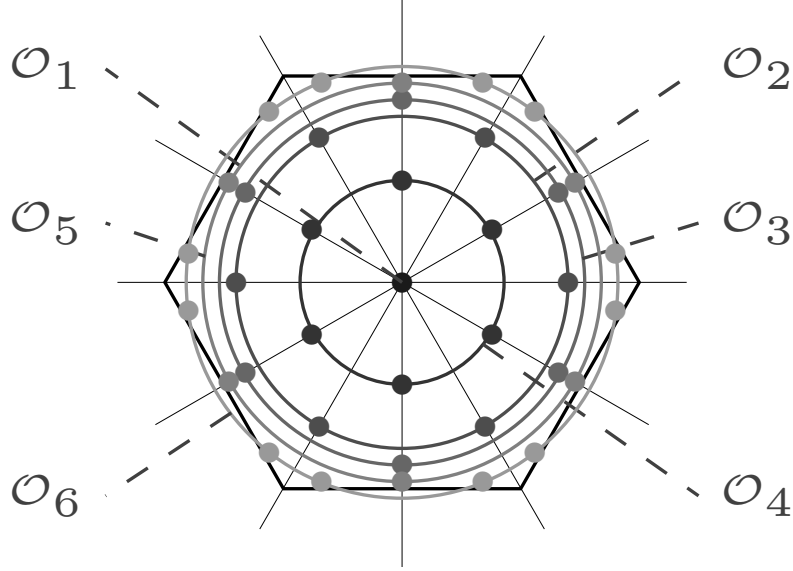
$$2\rho_6^6 \cos(6\theta_6) = 0.7738652982675917.$$

6. We first deduce the coordinates of a node per orbit (Table 7) from the equalities above and then the coordinates of the whole set of nodes from the group action and the orbit type (see Figure 18(a)).
7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Table 7).

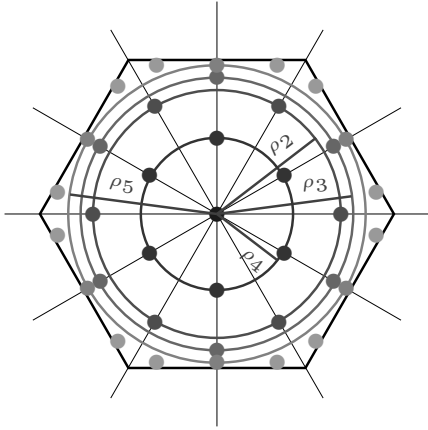
A  $D_6$ -invariant cubature of degree 13 for the hexagon  $H_2$  is thus determined (Table 7).

|                 | Type<br>(number of nodes) | Common weight $\tilde{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_1$ (1)       | 0.1744622899297936               | (0,0)  |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ (6)       | 0.1086813753508295               | (0.7004015518572025,0)                               |
| $\mathcal{O}_3$ | $\mathcal{P}_3$ (6)       | 0.0813509550506738               | (0,0.7652289145308346)                               |
| $\mathcal{O}_4$ | $\mathcal{P}_3$ (6)       | 0.1511466131316961               | (0,0.4333021019724418)                               |
| $\mathcal{O}_5$ | $\mathcal{P}_3$ (6)       | 0.0057447253848289               | (0,0.8449393321412519)                               |
| $\mathcal{O}_6$ | $\mathcal{P}_4$ (12)      | 0.0285059923262794               | (0.8996509650134044,0.1218723823391791)              |

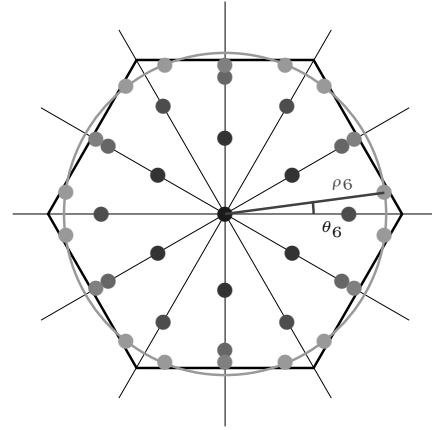
Table 7: Weights and nodes of the cubature in the case (1,1,3,1)



(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the origin,  $\mathcal{O}_2$  the orbit with 6 nodes and 2 nodes on the  $x_1$ -axis,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  the orbits with 6 nodes and 2 nodes on the  $x_2$ -axis and  $\mathcal{O}_6$  the orbit with 12 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  and  $\mathcal{O}_5$  are determined by the radii  $\rho_2$ ,  $\rho_3$ ,  $\rho_4$  and  $\rho_5$



(c) The coordinates of the nodes in  $\mathcal{O}_6$  are determined by the radius  $\rho_6$  and the angle  $\theta_6$

Figure 18: Nodes of the cubature in the case (1,1,3,1)

## 11 $D_3$ –invariant cubatures of degree 7 for the equilateral triangle

With the example in this section, we emphasize that the proposed procedure provides all  $G$ –invariant inside cubatures with positive weights for a given measure  $\mu$ , degree  $d$  and organization of nodes in orbit types  $(m_1, \dots, m_T)$ . With the help of Table 4 in [37], we want to recover the  $D_3$ –invariant inside cubatures with positive weights for a triangle. We focus here on the one of degree 7 with 15 nodes whose weights and nodes are given in [76]<sup>6</sup>. Applying the procedure in Section 9, we show the existence of 2 such cubatures with the same organization of nodes in orbit types: 1 orbit with 3 nodes and 2 orbits with 6 nodes. Up to our knowledge, only the cubature in [76] is known: it corresponds to the second cubature (Table 9 and Figure 20).

### 11.1 Existence of 2 cubatures with 15 nodes

We first introduce the input of Algorithm 9.6 [Existence of a  $G$ –invariant cubature] and apply this algorithm for the same organization of nodes in orbit types as in [76].

In this search, we look for  $D_3$ –invariant cubatures of degree 7 for the equilateral triangle  $T_2$ :  $(1, 0)$ ,  $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ . The values  $\int p d\mu$  with  $p \in \mathbb{R}[x]_{\leq 7}^G$  are computed explicitly by

$$\int p d\mu = \int_{T_2} p(x_1, x_2) dx_1 dx_2 = \int_{-1}^1 \left( \int_{-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x_1}^{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}x_1} p(x_1, x_2) dx_2 \right) dx_1.$$

Taking  $m = 3$  in Section 7.4, we get:

- ▷ A representation  $\mathcal{W}$  of  $D_3$  on  $\mathbb{R}^2$  deduced from the action of  $D_3$  that leaves  $T_2$  invariant.

$$\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \right].$$

- ▷ The set  $[L_{abs}, L_{com}]$  of irreducible representations  $\nu^{(1)}, \nu^{(2)}, \nu^{(3)}$ . Since they are all absolutely irreducible,  $L_{com}$  is empty and  $L_{abs}$  is

$$\left[ \begin{array}{c} [1, 1, 1, 1, 1, 1] \\ [1, 1, 1, -1, -1, -1] \\ \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \right] \end{array} \right].$$

- ▷ The matrix of multiplicities  $\Gamma_{D_3}$ , and more precisely the submatrix corresponding to this group action

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

- ▷ The integers  $m_1, m_2, m_3$  are 0, 1, 2 since there are 1 orbit with 3 nodes and 2 orbits with 6 nodes in [76].

---

<sup>6</sup>The one of degree 7 with 13 nodes in Table 4 in [37] refers to a cubature with negative weights.

## Existence conditions

1. The expected multiplicities  $\gamma_1, \gamma_2, \gamma_3$  are linked to the integers  $m_1, m_2, m_3$  thanks to (9.8) by

$$\gamma_1 = m_1 + m_2 + m_3 = 3, \quad \gamma_2 = m_3 = 2, \quad \gamma_3 = m_2 + 2m_3 = 5.$$

Since all irreducible representations of  $D_3$  are absolutely irreducible, the expected multiplicities  $\gamma_j$  are the expected ranks  $r_j$ .

2. The inequalities (9.9) are satisfied since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \mathcal{V}^{(3)}$  of  $D_3$  in the induced representation on  $\mathbb{R}[x]_{\leq 3}$  are

$$(\underline{c}'_1, \underline{c}'_2, \underline{c}'_3) = (3, 1, 3).$$

3. The inequalities (9.10) are satisfied by choosing  $\delta = 6$  since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \mathcal{V}^{(3)}$  of  $D_3$  in the induced representation on  $\mathbb{R}[x]_{\leq 5}$  are

$$(\underline{c}''_1, \underline{c}''_2, \underline{c}''_3) = (5, 2, 7).$$

4. An orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq 6}$  is

$$\left[ \begin{array}{l} [[1, x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2, x_1^4 + 2x_1^2x_2^2 + x_2^4, x_1^5 - 2x_1^3x_2^2 - 3x_1x_2^4, \\ x_1^6 + 5x_1^2x_2^4 + \frac{2}{3}x_2^6, x_1^4x_2^2 - \frac{2}{3}x_1^2x_2^4 + \frac{1}{9}x_2^6]], \\ [x_1^2x_2 - \frac{1}{3}x_2^3, x_1^4x_2 + \frac{2}{3}x_1^2x_2^3 - \frac{1}{3}x_2^5, x_1^5x_2 - \frac{10}{3}x_1^3x_2^3 + x_1x_2^5], \\ [x_1, x_1^2 - x_2^2, x_1^3 + x_1x_2^2, x_1^4 - x_2^4, x_1^2x_2^2 - \frac{1}{3}x_2^4, x_1^5 + \frac{5}{3}x_1x_2^4, x_1^3x_2^2 - \frac{1}{3}x_1x_2^4, \\ x_1^6 - \frac{5}{3}x_1^2x_2^4 - \frac{2}{3}x_2^6, x_1^4x_2^2 + \frac{2}{3}x_1^2x_2^4 - \frac{1}{3}x_2^6], \\ [x_2, -2x_1x_2, x_1^2x_2 + x_2^3, -2x_1^3x_2 - 2x_1x_2^3, -x_1^3x_2 + \frac{1}{3}x_1x_2^3, \frac{10}{3}x_1^2x_2^3 + \frac{2}{3}x_2^5, \\ \frac{1}{2}x_1^4x_2 - \frac{2}{3}x_1^2x_2^3 + \frac{1}{6}x_2^5, -x_1^5x_2 - \frac{10}{3}x_1^3x_2^3 - \frac{7}{3}x_1x_2^5, -x_1^5x_2 - \frac{2}{3}x_1^3x_2^3 + \frac{1}{3}x_1x_2^5] \end{array} \right].$$

5. The distinct blocks  $H^{(1)}, H^{(2)}, H^{(3)}$  are then computed.

6. After the parametrization, the distinct blocks  $H^{(1)}, H^{(2)}, H^{(3)}$  are

$$\left( \begin{array}{cccccc} \frac{3\sqrt{3}}{4} & \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{33\sqrt{3}}{896} & \frac{3\sqrt{3}}{4480} \\ \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} & \mathbf{h}_1 + 6\mathbf{h}_4 & \mathbf{h}_4 \\ \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{21\sqrt{3}}{112} & \frac{3\sqrt{3}}{112} & \mathbf{h}_1 & \mathbf{h}_2 - 6\mathbf{h}_3 & 2\mathbf{h}_3 \\ \frac{3\sqrt{3}}{40} & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} & \mathbf{h}_1 + 9\mathbf{h}_4 & \mathbf{h}_2 & -\frac{1}{2}\mathbf{h}_5 + 3\mathbf{h}_6 & -\frac{1}{4}\mathbf{h}_5 + \frac{1}{2}\mathbf{h}_6 \\ \frac{3\sqrt{3}}{70} & \frac{8960}{112} & \mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_5 & 2\mathbf{h}_7 - 4\mathbf{h}_8 & 2\mathbf{h}_8 \\ \frac{33\sqrt{3}}{896} & \mathbf{h}_1 + 6\mathbf{h}_4 & \mathbf{h}_2 - 6\mathbf{h}_3 & -\frac{1}{2}\mathbf{h}_5 + 3\mathbf{h}_6 & 2\mathbf{h}_7 - 4\mathbf{h}_8 & \mathbf{h}_{11} & 6\mathbf{h}_9 + \mathbf{h}_{10} \\ \frac{3\sqrt{3}}{4480} & \mathbf{h}_4 & 2\mathbf{h}_3 & -\frac{1}{4}\mathbf{h}_5 + \frac{1}{2}\mathbf{h}_6 & 2\mathbf{h}_8 & 6\mathbf{h}_9 + \mathbf{h}_{10} & \mathbf{h}_9 \end{array} \right) \left( \begin{array}{ccc} \frac{3\sqrt{3}}{4480} & \mathbf{h}_4 & 2\mathbf{h}_3 \\ \mathbf{h}_4 & -\frac{1}{4}\mathbf{h}_5 + \frac{1}{2}\mathbf{h}_6 & 2\mathbf{h}_8 \\ 2\mathbf{h}_3 & 2\mathbf{h}_8 & \mathbf{h}_{10} \end{array} \right) \left( \begin{array}{cccccc} \frac{3\sqrt{3}}{32} & \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & 0 & \frac{33\sqrt{3}}{17920} & \frac{9\sqrt{3}}{17920} & \frac{3\sqrt{3}}{224} & 0 \\ \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & \frac{3\sqrt{3}}{140} & \frac{87\sqrt{3}}{4480} & \frac{9\sqrt{3}}{8960} & \frac{3\sqrt{3}}{224} & 0 & \mathbf{h}_1 + 3\mathbf{h}_4 & \frac{3\mathbf{h}_4}{2} \\ \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & \frac{87\sqrt{3}}{4480} & \frac{3\sqrt{3}}{224} & 0 & \frac{\mathbf{h}_1}{2} + 3\mathbf{h}_4 & \frac{3\mathbf{h}_4}{4} & \mathbf{h}_2 & 0 \\ \frac{3\sqrt{3}}{140} & \frac{87\sqrt{3}}{4480} & \frac{3\sqrt{3}}{224} & \frac{\mathbf{h}_1}{2} + \frac{9\mathbf{h}_4}{2} & \frac{3\mathbf{h}_4}{2} & \frac{\mathbf{h}_2}{2} & 0 & -\frac{\mathbf{h}_5}{4} + \frac{3\mathbf{h}_6}{2} & -\frac{3\mathbf{h}_5}{8} + \frac{3\mathbf{h}_6}{4} \\ 0 & \frac{8960}{112} & 0 & \frac{3\mathbf{h}_4}{2} & \frac{\mathbf{h}_4}{2} & \mathbf{h}_3 & -\frac{\mathbf{h}_3}{2} & -\frac{\mathbf{h}_5}{4} + \frac{\mathbf{h}_6}{2} & -\frac{\mathbf{h}_5}{8} + \frac{\mathbf{h}_6}{4} \\ \frac{33\sqrt{3}}{8960} & \frac{3\sqrt{3}}{224} & \frac{\mathbf{h}_1}{2} + 3\mathbf{h}_4 & \frac{\mathbf{h}_2}{2} & \mathbf{h}_3 & \mathbf{h}_6 & -\frac{\mathbf{h}_5}{8} + \frac{\mathbf{h}_6}{4} & -\frac{\mathbf{h}_5}{32} + \frac{\mathbf{h}_6}{16} & \mathbf{h}_7 & \mathbf{h}_8 \\ \frac{9\sqrt{3}}{17920} & 0 & \frac{3\mathbf{h}_4}{2} & 0 & -\frac{\mathbf{h}_3}{2} & -\frac{\mathbf{h}_5}{8} + \frac{\mathbf{h}_6}{4} & -\frac{\mathbf{h}_5}{32} + \frac{\mathbf{h}_6}{16} & \frac{\mathbf{h}_8}{2} & \mathbf{h}_8 \\ \frac{3\sqrt{3}}{224} & \frac{\mathbf{h}_1}{2} + 3\mathbf{h}_4 & \frac{\mathbf{h}_2}{2} & -\frac{\mathbf{h}_5}{8} + \frac{3\mathbf{h}_6}{4} & -\frac{\mathbf{h}_5}{8} + \frac{\mathbf{h}_6}{4} & \mathbf{h}_7 & \mathbf{h}_8 & \frac{\mathbf{h}_{10}}{2} + \frac{\mathbf{h}_{11}}{2} & \mathbf{h}_9 & \mathbf{h}_{10} \\ 0 & \frac{3\mathbf{h}_4}{2} & 0 & -\frac{3\mathbf{h}_5}{8} + \frac{3\mathbf{h}_6}{4} & -\frac{\mathbf{h}_5}{8} + \frac{\mathbf{h}_6}{4} & \mathbf{h}_8 & -\frac{\mathbf{h}_8}{2} & \frac{9\mathbf{h}_9}{2} + \frac{\mathbf{h}_{10}}{2} & \frac{9\mathbf{h}_9}{2} + \frac{\mathbf{h}_{10}}{2} \end{array} \right).$$

7. The parameters  $h_1, \dots, h_{11}$  are determined using Algorithm 4.7 on each block. Each block  $H^{(1)}, H^{(2)}$  provides 1 triplet  $[P_1, Z_1, E_1], [P_2, Z_2, E_2]$ , whereas the block  $H^{(3)}$  provides 6 triplets  $[P_3, Z_3, E_3]$ . There are thus 6 systems of equations and inequations. However, only one of them has a solution. This is checked by computing a Groebner basis of the polynomials in each set  $Z = \{Z_1, Z_2, Z_3\}$ : if it is reduced to  $\{1\}$ , then there is no solution. Solving the polynomial system obtained from this set  $Z$ , we get 2 distinct sets of real values for the 11 unknown parameters in the blocks  $H^{(1)}, H^{(2)}, H^{(3)}$ . Both sets satisfy the positivity constraints from  $P = [P_1, P_2, P_3]$ .

This shows that for the equilateral triangle  $T_2$  there exist exactly 2  $D_3$ -invariant cubatures of degree 7 with positive weights and with this organization of the 15 nodes in orbit types.

8. It is here easier to check *a posteriori* that we have an inside cubature.

## 11.2 Computation of the weights and the nodes of the 2 cubatures

With the help of the 2 solutions found thanks to Algorithm 9.6 [Existence of a  $G$ -invariant cubature], we are now able to compute the 2 associated cubatures using Algorithm 9.7. We first need to determine a separating set  $\{p_1, \dots, p_\eta\}$  of  $G$ -invariant polynomials. It depends on the organization of nodes in orbit types (see Figures 19(a) and 20(a)).

In this example, we look for 3 orbits:  $\mathcal{O}_1$  with 3 nodes (see Figures 19(b) and 20(b)),  $\mathcal{O}_2$  and  $\mathcal{O}_3$  with 6 nodes (see Figures 19(c) and 20(c)). A missing information is the radii  $\rho_1, \rho_2, \rho_3$  of the circles where the nodes lie on. They are obtained thanks to the  $D_3$ -invariant polynomial  $p_1 = x_1^2 + x_2^2$ . This information is not enough. The additional required information is the angles  $\theta_1, \theta_2, \theta_3$  of a node  $\zeta_1, \zeta_2, \zeta_3$  on each orbit  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ . We choose the unique node such that  $0 \leq \theta_1, \theta_2, \theta_3 \leq \frac{\pi}{3}$ . It is possible to compute it thanks to a second  $D_3$ -invariant polynomial given in [46, Chapter XII.4] by

$$p_2 = (x_1 + ix_2)^3 + (x_1 - ix_2)^3 = 2x_1^3 - 6x_1x_2^2.$$

Indeed, assuming the evaluation  $p_2(\zeta_\alpha)$  is known,  $\theta_\alpha$  is then the unique solution of the equation in  $\theta$

$$p_2(\zeta_\alpha) = 2\rho_\alpha^3 \cos(3\theta) \quad \forall \alpha = 1, 2, 3$$

under the constraint  $0 \leq \theta \leq \frac{\pi}{3}$ . Notice that  $\theta_1 \in \{0, \frac{\pi}{3}\}$  and  $0 < \theta_2, \theta_3 < \frac{\pi}{3}$ .

The separating set in Step 3 of Algorithm 9.7 is then

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^3 - 6x_1x_2^2\}.$$

For the 2 cubatures, it is sufficient to consider the first blocks  $H_1^{(1)}, H_{p_1}^{(1)}, H_{p_2}^{(1)}$  as noticed in Step 5 of Algorithm 9.7. The latter is therefore simplified:

1. An orthogonal symmetry adapted basis of the first component of  $\mathbb{R}[x]/I_\Lambda$  is given by selecting the following polynomials in the first component of  $\mathbb{R}[x]_{\leq 5}$

$$[1, x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2].$$

2. The invertible block  $H_1^{(1)}$  is

$$\begin{pmatrix} \frac{3\sqrt{3}}{4} & \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} \\ \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} \\ \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{21\sqrt{3}}{640} \end{pmatrix}.$$

3. The separating set is

$$\{p_1, p_2\} = \{x_1^2 + x_2^2, 2x_1^3 + 6x_1x_2^2\}.$$

4. The distinct blocks  $H_{p_1}^{(1)}, H_{p_2}^{(1)}$  are respectively

$$H_{p_1}^{(1)} = \begin{pmatrix} \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} \\ \frac{3\sqrt{3}}{40} & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} \\ \frac{3\sqrt{3}}{70} & \frac{3\sqrt{3}}{112} & h_1 \end{pmatrix}, \quad H_{p_2}^{(1)} = \begin{pmatrix} \frac{3\sqrt{3}}{20} & \frac{3\sqrt{3}}{35} & \frac{21\sqrt{3}}{320} \\ \frac{3\sqrt{3}}{35} & \frac{3\sqrt{3}}{56} & 2h_1 \\ \frac{21\sqrt{3}}{320} & 2h_1 & h_2 \end{pmatrix}$$

with

$$\text{First solution: } h_1 = 0.0364045528321075, \quad h_2 = 0.0578230474283380.$$

$$\text{Second solution: } h_1 = 0.0366481235855616, \quad h_2 = 0.0588850622813001.$$

5. Applying the steps (a)-(e), we get the different values  $\lambda_{\nu\alpha}$  with  $\nu = 1, 2$  and  $\alpha = 1, 2, 3$ .  
First solution:

$$\begin{aligned} \lambda_{11} = \rho_1^2 &= 0.0730202430083909, & \lambda_{21} = 2\rho_1^3 \cos(3\theta_1) &= 0.033324853191846, \\ \lambda_{12} = \rho_2^2 &= 0.2527220147370857, & \lambda_{22} = 2\rho_2^3 \cos(3\theta_2) &= 0.047793211645650, \\ \lambda_{13} = \rho_3^2 &= 0.6435998129921440, & \lambda_{23} = 2\rho_3^3 \cos(3\theta_3) &= 1.023593517287630. \end{aligned}$$

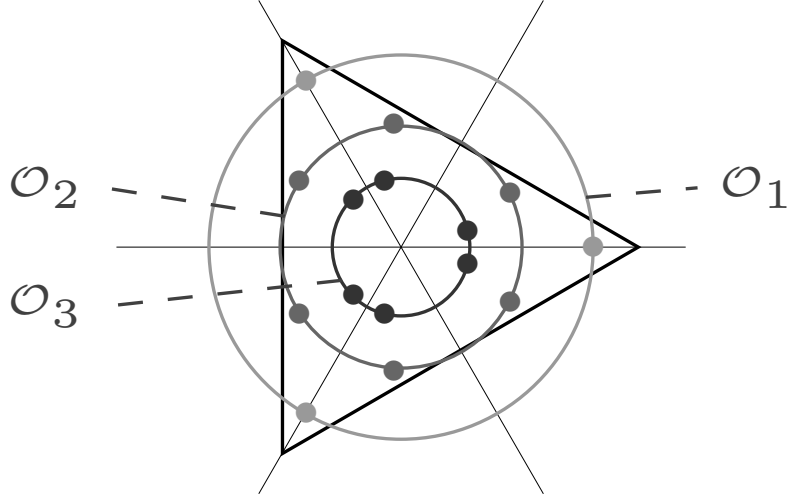
Second solution:

$$\begin{aligned} \lambda_{11} = \rho_1^2 &= 0.6483606648970231, & \lambda_{21} = 2\rho_1^3 \cos(3\theta_1) &= 1.044130985690171, \\ \lambda_{12} = \rho_2^2 &= 0.2697235135960804, & \lambda_{22} = 2\rho_2^3 \cos(3\theta_2) &= 0.047793211645650, \\ \lambda_{13} = \rho_3^2 &= 0.0815049434976416, & \lambda_{23} = 2\rho_3^3 \cos(3\theta_3) &= 0.032636335919029. \end{aligned}$$

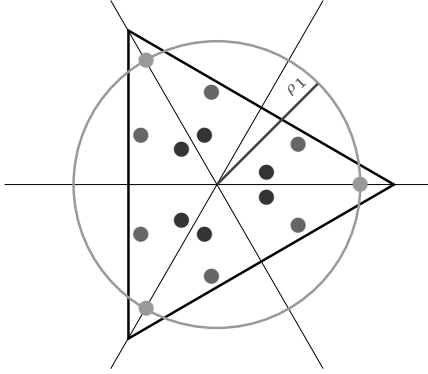
6. Solving the systems formed by each row above, we get the coordinates of a node per orbit (Tables 8 and 9) and then the coordinates of the whole set of nodes from the group action and the orbit type (see Figures 19(a) and 20(a)).

7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Tables 8 and 9).

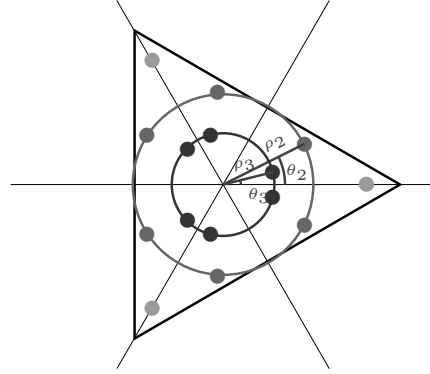
2  $D_3$ -invariant cubatures of degree 7 for the equilateral triangle  $T_2$  are thus determined (Tables 8 and 9).



(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the orbit with 3 nodes,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  the orbits with 6 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_1$  are determined by the radius  $\rho_1$  and the angle  $\theta_1 = 0$



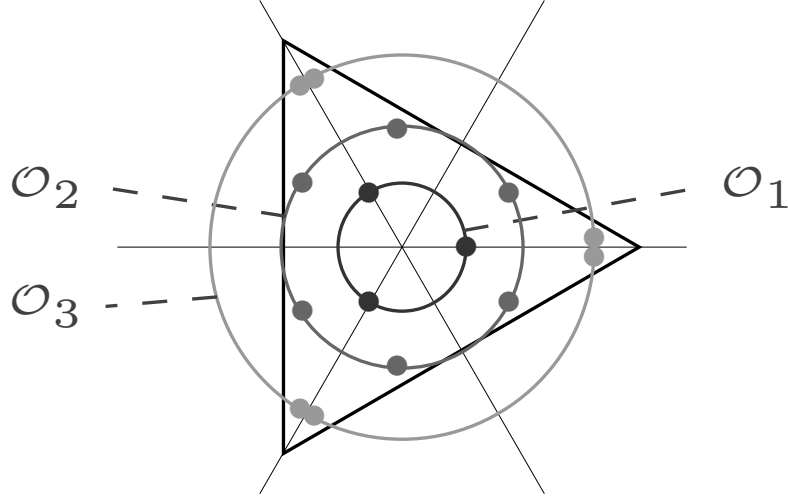
(c) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  are determined by the radii  $\rho_2$ ,  $\rho_3$  and the angles  $\theta_2$ ,  $\theta_3$

Figure 19: Nodes of the first cubature

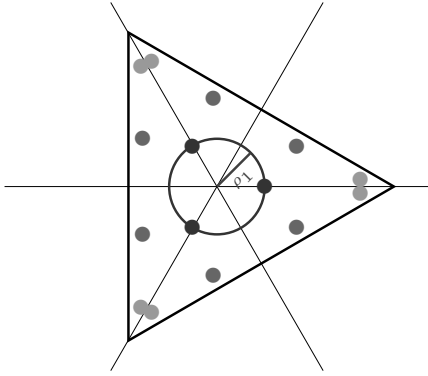
|                 | Type<br>(number of nodes) | Common weight $\tilde{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_2$ (3)       | 0.1628910742849014               | (0.2702225804931746,0)                               |
| $\mathcal{O}_2$ | $\mathcal{P}_3$ (6)       | 0.0991248412090212               | (0.4459621387678824,0.2320340180289498)              |
| $\mathcal{O}_3$ | $\mathcal{P}_3$ (6)       | 0.0359359725946377               | (0.8014638082178955,0.0354341235112252)              |

Table 8: Weights and nodes of the first cubature

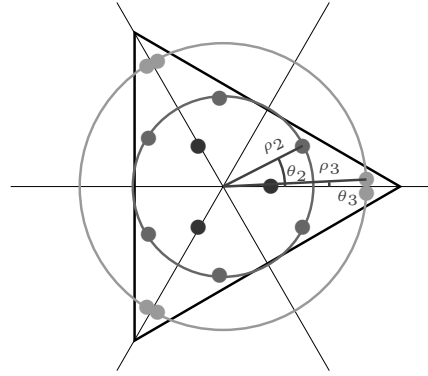




(a) Organization of the nodes in orbits:  $\mathcal{O}_1$  the orbit with 3 nodes,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  the orbits with 6 nodes



(b) The coordinates of the nodes in  $\mathcal{O}_1$  are determined by the radius  $\rho_1$  and the angle  $\theta_1 = 0$



(c) The coordinates of the nodes in  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  are determined by the radii  $\rho_2$ ,  $\rho_3$  and the angles  $\theta_2$ ,  $\theta_3$

Figure 20: Nodes of the second cubature

|                 | Type<br>(number of nodes) | Common weight $\tilde{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|---------------------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_2$ (3)       | 0.0689500870910645               | (0.8052084605225054,0)                               |
| $\mathcal{O}_2$ | $\mathcal{P}_3$ (6)       | 0.0899904517797997               | (0.4638660157340427,0.2335633383969007)              |
| $\mathcal{O}_3$ | $\mathcal{P}_3$ (6)       | 0.0920408556207777               | (0.2755599086039839,0.0746436887339156)              |

Table 9: Weights and nodes of the second cubature

## 12 $C_3$ -invariant cubatures of degree 7 for the triangle

For degree 7 the Gaussian lower bound is 10. In Section 11 we achieved a  $D_3$ -invariant cubature with 15 nodes. Was it the smallest possible number of nodes ? We examine here invariance with respect to a subgroup of  $D_3$ : the group  $C_3$  of rotations. We retrieve the cubature of [37] as the unique  $C_3$ -invariant cubature with 12 nodes.

### 12.1 Existence of cubatures with 12 nodes

We first introduce the input of Algorithm 9.6 [Existence of a  $G$ -invariant cubature] and apply this algorithm for the same organization of nodes in orbit types as in [37]: 4 orbits with 3 nodes.

In this search, we look for all  $C_3$ -invariant cubatures of degree 7 with 12 nodes for the equilateral triangle  $T_2$ :  $(1, 0)$ ,  $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$ . Indeed, the organization of nodes in orbit types we consider is the only possible with this number of nodes. The values  $\int p d\mu$  with  $p \in \mathbb{R}[x]_{\leq 7}^G$  are computed explicitly by

$$\int p d\mu = \int_{T_2} p(x_1, x_2) dx_1 dx_2 = \int_{-1}^1 \left( \int_{-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}x_1}^{\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}x_1} p(x_1, x_2) dx_2 \right) dx_1.$$

Taking  $m = 3$  in Section 7.3, we get:

- ▷ A representation  $\mathcal{W}$  of  $C_3$  on  $\mathbb{R}^2$  deduced from the realization of  $C_3$  as the group of rotations of the plane around the origin through angles  $\frac{2\ell\pi}{3}$  with  $\ell = 0, 1, 2$ .

$$\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right].$$

- ▷ The set  $[L_{abs}, L_{com}]$  of irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ .  $\mathcal{V}^{(1)}$  is absolutely irreducible and  $\mathcal{V}^{(2)}$  is of complex type <sup>7</sup>.

$$L_{abs} = [[1, 1, 1]] \quad \text{and} \quad L_{com} = \left[ \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \right] \right].$$

- ▷ The matrix of multiplicities  $\Gamma_{C_3}$ , and more precisely the submatrix corresponding to this group action

$$\Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- ▷ The integers  $m_1, m_2$  are 0, 4 since there are 4 orbits with 3 nodes in [76].

### Existence conditions

1. The expected multiplicities  $\gamma_1, \gamma_2$  are linked to the integers  $m_1, m_2$  thanks to (9.8) by

$$\gamma_1 = m_1 + m_2 = 4, \quad \gamma_2 = m_2 = 4.$$

---

<sup>7</sup>Over  $\mathbb{C}$  there are 3 1-dimensional irreducible representations  $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ .  $\mathcal{V}_2$  and  $\mathcal{V}_3$  are complex conjugate

The expected ranks  $r_1, r_2$  are given by

$$r_1 = \gamma_1 = 4, \quad r_2 = 2\gamma_2 = 8.$$

2. The inequalities (9.9) are satisfied since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$  of  $C_3$  in the induced representation on  $\mathbb{R}[x]_{\leq 3}$  are

$$(\underline{c}'_1, \underline{c}'_2) = (4, 3).$$

3. The inequalities (9.10) are satisfied by choosing  $\delta = 5$  since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$  of  $C_3$  in the induced representation on  $\mathbb{R}[x]_{\leq 4}$  are

$$(\underline{c}''_1, \underline{c}''_2) = (5, 5).$$

4. An orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq 5}$  is

$$\left[ \begin{array}{l} [[1, x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2, x_1^2x_2 - \frac{1}{3}x_2^3, x_1^4 + 2x_1^2x_2^2 + x_2^4, \\ x_1^5 - 2x_1^3x_2^2 - 3x_1x_2^4, x_1^4x_2 + \frac{2}{3}x_1^2x_2^3 - \frac{1}{3}x_2^5]]], \\ \\ [[x_1, x_1^2 - x_2^2, x_1^3 + x_1x_2^2, x_1^4 - \frac{6}{5}x_1^2x_2^2 - \frac{3}{5}x_2^4, x_1^3x_2, \\ x_1^5 + \frac{10}{11}x_1^3x_2^2 + \frac{15}{11}x_1x_2^4, x_1^4x_2 - \frac{6}{7}x_1^2x_2^3 + \frac{3}{7}x_2^5], \\ [-x_2, 2x_1x_2, -x_1^2x_2 - x_2^3, \frac{4}{5}x_1^3x_2 + \frac{12}{5}x_1x_2^3, -\frac{1}{8}x_1^4 - \frac{3}{4}x_1^2x_2^2 + \frac{3}{8}x_2^4, \\ -\frac{5}{11}x_1^4x_2 - \frac{30}{11}x_1^2x_2^3 - \frac{9}{11}x_2^5, \frac{1}{7}x_1^5 + 2x_1^3x_2^2 - \frac{3}{7}x_1x_2^4]]] \end{array} \right].$$

5. The distinct blocks  $H^{(1)}, H^{(2)}$  are then computed.

6. After the parametrization, the distinct blocks  $H^{(1)}$  and  $H^{(2)} = \begin{pmatrix} S_2 & A_2 \\ -A_2 & S_2 \end{pmatrix}$  are given by

$$H^{(1)} = \begin{pmatrix} \frac{3\sqrt{3}}{4} & \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & 0 & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & 0 \\ \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & 0 & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} & 0 \\ \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{21\sqrt{3}}{640} & 0 & \frac{3\sqrt{3}}{112} & \mathbf{h}_1 & \mathbf{h}_4 \\ 0 & 0 & 0 & \frac{3\sqrt{3}}{4480} & 0 & \mathbf{h}_4 & \mathbf{h}_3 \\ \frac{3\sqrt{3}}{40} & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} & 0 & \mathbf{h}_1 + 9\mathbf{h}_3 & \mathbf{h}_2 & \mathbf{h}_5 \\ \frac{3\sqrt{3}}{70} & \frac{3\sqrt{3}}{112} & \mathbf{h}_1 & \mathbf{h}_4 & \mathbf{h}_2 & \mathbf{h}_6 & \frac{77\mathbf{h}_8}{36} \\ 0 & 0 & \mathbf{h}_4 & \mathbf{h}_3 & \mathbf{h}_5 & \frac{77\mathbf{h}_8}{36} & \mathbf{h}_{10} \end{pmatrix}$$

$$S_2 = \begin{pmatrix} \frac{3\sqrt{3}}{32} & \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & 0 & \frac{93\sqrt{3}}{4928} & 0 \\ \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & \frac{51\sqrt{3}}{2800} & 0 & \frac{3\sqrt{3}}{224} & 0 \\ \frac{3\sqrt{3}}{80} & \frac{3\sqrt{3}}{140} & \frac{87\sqrt{3}}{4480} & \frac{3\sqrt{3}}{224} & 0 & \frac{\mathbf{h}_1}{2} + \frac{81\mathbf{h}_3}{22} & \frac{3\mathbf{h}_4}{7} \\ \frac{3\sqrt{3}}{140} & \frac{51\sqrt{3}}{2800} & \frac{3\sqrt{3}}{224} & \frac{\mathbf{h}_1}{2} + \frac{81\mathbf{h}_3}{50} & \frac{9\mathbf{h}_4}{20} & \mathbf{h}_7 & \frac{54\mathbf{h}_5}{35} + \frac{11\mathbf{h}_9}{7} \\ 0 & 0 & 0 & \frac{9\mathbf{h}_4}{20} & \frac{\mathbf{h}_1}{128} + \frac{81\mathbf{h}_3}{128} & \frac{81\mathbf{h}_5}{88} + \frac{5\mathbf{h}_9}{8} & \frac{27\mathbf{h}_2}{56} - \frac{55\mathbf{h}_7}{56} \\ \frac{93\sqrt{3}}{4928} & \frac{3\sqrt{3}}{224} & \frac{\mathbf{h}_1}{2} + \frac{81\mathbf{h}_3}{22} & \mathbf{h}_7 & \frac{81\mathbf{h}_5}{88} + \frac{5\mathbf{h}_9}{8} & \frac{\mathbf{h}_6}{2} + \frac{729\mathbf{h}_{10}}{242} & \mathbf{h}_8 \\ 0 & 0 & \frac{3\mathbf{h}_4}{7} & \frac{54\mathbf{h}_5}{35} + \frac{11\mathbf{h}_9}{7} & \frac{27\mathbf{h}_2}{56} - \frac{55\mathbf{h}_7}{56} & \mathbf{h}_8 & \frac{\mathbf{h}_6}{98} + \frac{81\mathbf{h}_{10}}{98} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{3\sqrt{3}}{1120} & 0 & -\frac{57\sqrt{3}}{15680} \\ 0 & 0 & 0 & 0 & \frac{57\sqrt{3}}{17920} & 0 & -\frac{3\sqrt{3}}{1568} \\ 0 & 0 & 0 & 0 & \frac{3\sqrt{3}}{1792} & -\frac{3\mathbf{h}_4}{11} & -\frac{\mathbf{h}_1}{14} - \frac{27\mathbf{h}_3}{14} \\ 0 & 0 & 0 & 0 & \frac{\mathbf{h}_1}{16} + \frac{81\mathbf{h}_3}{80} & -\mathbf{h}_9 & -\frac{6\mathbf{h}_2}{7} + \frac{11\mathbf{h}_7}{7} \\ -\frac{3\sqrt{3}}{1120} & -\frac{57\sqrt{3}}{17920} & -\frac{3\sqrt{3}}{1792} & -\frac{\mathbf{h}_1}{16} - \frac{81\mathbf{h}_3}{80} & 0 & -\frac{3\mathbf{h}_2}{8} + \frac{5\mathbf{h}_7}{8} & \frac{27\mathbf{h}_5}{56} + \frac{55\mathbf{h}_9}{56} \\ 0 & 0 & \frac{3\mathbf{h}_4}{11} & \mathbf{h}_9 & \frac{3\mathbf{h}_2}{8} - \frac{5\mathbf{h}_7}{8} & 0 & -\frac{\mathbf{h}_6}{14} - \frac{243\mathbf{h}_{10}}{154} \\ \frac{57\sqrt{3}}{15680} & \frac{3\sqrt{3}}{1568} & \frac{\mathbf{h}_1}{14} + \frac{27\mathbf{h}_3}{14} & \frac{6\mathbf{h}_2}{7} - \frac{11\mathbf{h}_7}{7} & -\frac{27\mathbf{h}_5}{56} - \frac{55\mathbf{h}_9}{56} & \frac{\mathbf{h}_6}{14} + \frac{243\mathbf{h}_{10}}{154} & 0 \end{pmatrix}$$

7. The parameters  $h_1, \dots, h_{10}$  are determined using Algorithm 4.7 on each block. To be able to use it on the block  $H^{(2)}$ , we should first permute its rows and columns. This permutation is equivalent to consider, instead of the orthogonal symmetry adapted basis computed in Step 4, the basis

$$\begin{bmatrix} [[1, x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2, x_1^2x_2 - \frac{1}{3}x_2^3, x_1^4 + 2x_1^2x_2^2 + x_2^4, \\ x_1^5 - 2x_1^3x_2^2 - 3x_1x_2^4, x_1^4x_2 + \frac{2}{3}x_1^2x_2^3 - \frac{1}{3}x_2^5]]], \\ [[x_1, -x_2, x_1^2 - x_2^2, 2x_1x_2, x_1^3 + x_1x_2^2, -x_1^2x_2 - x_2^3, \\ x_1^4 - \frac{6}{5}x_1^2x_2^2 - \frac{3}{5}x_2^4, \frac{4}{5}x_1^3x_2 + \frac{12}{5}x_1x_2^3, x_1^3x_2, -\frac{1}{8}x_1^4 - \frac{3}{4}x_1^2x_2^2 + \frac{3}{8}x_2^4, \\ x_1^5 + \frac{10}{11}x_1^3x_2^2 + \frac{15}{11}x_1x_2^4, -\frac{5}{11}x_1^4x_2 - \frac{30}{11}x_1^2x_2^3 - \frac{9}{11}x_2^5, \\ x_1^4x_2 - \frac{6}{7}x_1^2x_2^3 + \frac{3}{7}x_2^5, \frac{1}{7}x_1^5 + 2x_1^3x_2^2 - \frac{3}{7}x_1x_2^4]]] \end{bmatrix}.$$

Then, the block  $H^{(1)}$  provides 1 triplet  $[P_1, Z_1, E_1]$ , whereas the block  $H^{(2)}$  provides 2 triplets  $[P_2, Z_2, E_2]$ . There are thus 2 systems of equations and inequations. However, only one of them has a solution. This is checked by computing a Groebner basis of the polynomials in each set  $Z = \{Z_1, Z_2, Z_3\}$ : if it is reduced to  $\{1\}$ , then there is no solution. Solving the polynomial system obtained from this set  $Z$ , we get 3 distinct sets of real values for the 10 unknown parameters in the blocks  $H^{(1)}, H^{(2)}$ . Only 2 sets satisfy the positivity constraints from  $P = [P_1, P_2, P_3]$ . The corresponding values of the parameters  $h_1, \dots, h_{10}$  are

$$\begin{aligned} h_1 &= \frac{17427\sqrt{3}}{824320}, h_2 = \frac{3531\sqrt{3}}{206080}, h_3 = \frac{149\sqrt{3}}{824320}, h_4 = -\frac{9a}{824320}, h_5 = -\frac{a}{206080}, \\ h_6 &= \frac{259341\sqrt{3}}{18959360}, h_7 = \frac{202893\sqrt{3}}{23699200}, h_8 = -\frac{1737a}{364967680}, h_9 = \frac{513a}{260691200}, h_{10} = \frac{8471\sqrt{3}}{170634240} \\ &\text{with } a \in \left\{ \frac{\sqrt{4893}}{7}, -\frac{\sqrt{4893}}{7} \right\}. \end{aligned}$$

This shows that for the equilateral triangle  $T_2$  there exist 2  $C_3$ -invariant cubatures of degree 7 with positive weights and with 12 nodes.

8. It is here easier to check *a posteriori* that we have an inside cubature.

Notice that it is also possible to show that there is no  $C_3$ -invariant Gaussian cubature of degree 7 for the triangle  $T_2$ . We give here the outline of the proof using Algorithm 9.6 [Existence of a  $G$ -invariant cubature].

Such a cubature would have had 10 nodes. The unique organization of nodes in orbit types is the origin and 3 orbits with 3 nodes. It implies  $\gamma_1 = 4$  and  $\gamma_2 = 3$ . The inequalities (9.9) and (9.10) are satisfied since  $(c'_1, c'_2) = (c''_1, c''_2) = (4, 3)$ . The blocks  $H^{(1)}$  and  $H^{(2)}$  to be considered are obtained by taking the  $5 \times 5$  leading principal submatrices of  $H^{(1)}$ ,  $S_2$  and  $A_2$  above. Only 3 unknown parameters are here required. But, using Algorithm 4.7 on each block with a similar permutation of rows and columns as above on  $H^{(2)}$ , we do not find values for these parameters such that the blocks  $H^{(1)}$  and  $H^{(2)}$  have respectively rank 4 and 3. The existence of a  $C_3$ -invariant Gaussian cubature of degree 7 for the triangle  $T_2$  is therefore impossible.

## 12.2 Computation of the weights and the nodes of the 2 cubatures

With the help of the above 2 sets of parameter values found thanks to Algorithm 9.6 [Existence of a  $G$ -invariant cubature], we are now able to compute the 2 associated cubatures using Algo-

rithm 9.7. We first need to determine a separating set  $\{p_1, \dots, p_\eta\}$  of  $G$ -invariant polynomials. It depends on the organization of nodes in orbit types (see Figures 21 and 22).

In this example, we look for 4 orbits  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$  with 3 nodes. A missing information is the radii  $\rho_1, \rho_2, \rho_3, \rho_4$  of the circles where the nodes lie on. They are obtained thanks to the  $C_3$ -invariant polynomial  $p_1 = x_1^2 + x_2^2$ . This information is not enough. The additional required information is the angles  $\theta_1, \theta_2, \theta_3, \theta_4$  of a node  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  on each orbit  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ . We choose the unique node such that  $-\frac{\pi}{3} \leq \theta_1, \theta_2, \theta_3, \theta_4 \leq \frac{\pi}{3}$ . It is possible to compute it thanks to the  $C_3$ -invariant polynomials<sup>8</sup>

$$p_2 = x_1^3 + 3x_1x_2^2 \text{ and } p_3 = x_1^2x_2 + \frac{1}{3}x_2^3.$$

Indeed, assuming the evaluations  $p_2(\zeta_\alpha)$  and  $p_3(\zeta_\alpha)$  are known,  $\theta_\alpha$  is then the unique solution of the equations in  $\theta$

$$p_2(\zeta_\alpha) = \rho_\alpha^3 \cos(3\theta) \text{ and } p_3(\zeta_\alpha) = \frac{1}{3}\rho_\alpha^3 \sin(3\theta) \quad \forall \alpha = 1, 2, 3, 4$$

under the constraint  $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$ .

The separating set in Step 3 of Algorithm 9.7 is then

$$\{p_1, p_2, p_3\} = \{x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2, x_1^2x_2 - \frac{1}{3}x_2^3\}.$$

For the 2 cubatures, it is sufficient to consider the first blocks  $H_1^{(1)}, H_{p_1}^{(1)}, H_{p_2}^{(1)}, H_{p_3}^{(1)}$  as noticed in Step 5 of Algorithm 9.7. The latter is therefore simplified:

1. An orthogonal symmetry adapted basis of the first component of  $\mathbb{R}[x]/I_\Lambda$  is given by selecting the following polynomials in the first component of  $\mathbb{R}[x]_{\leq 5}$

$$[1, \underbrace{x_1^2 + x_2^2}_{p_1}, \underbrace{x_1^3 - 3x_1x_2^2}_{p_2}, \underbrace{x_1^2x_2 - \frac{1}{3}x_2^3}_{p_3}].$$

2. The invertible block  $H_1^{(1)}$  is

$$\begin{pmatrix} \frac{3\sqrt{3}}{4} & \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & 0 \\ \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & 0 \\ \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{21\sqrt{3}}{640} & 0 \\ 0 & 0 & 0 & \frac{3\sqrt{3}}{4480} \end{pmatrix}.$$

3. The separating set is

$$\{p_1, p_2, p_3\} = \{x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2, x_1^2x_2 - \frac{1}{3}x_2^3\}.$$

---

<sup>8</sup>They are found in the orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq 5}$  computed previously in Step 4 of Algorithm 9.6.

4. The distinct blocks  $H_{p_1}^{(1)}, H_{p_2}^{(1)}, H_{p_3}^{(1)}$  are respectively

$$H_{p_1}^{(1)} = \begin{pmatrix} \frac{3\sqrt{3}}{16} & \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & 0 \\ \frac{3\sqrt{3}}{40} & \frac{87\sqrt{3}}{2240} & \frac{3\sqrt{3}}{112} & 0 \\ \frac{3\sqrt{3}}{70} & \frac{3\sqrt{3}}{112} & \hbar_1 & \hbar_4 \\ 0 & 0 & \hbar_4 & \hbar_3 \end{pmatrix}, \quad H_{p_2}^{(1)} = \begin{pmatrix} \frac{3\sqrt{3}}{40} & \frac{3\sqrt{3}}{70} & \frac{21\sqrt{3}}{640} & 0 \\ \frac{3\sqrt{3}}{70} & \frac{3\sqrt{3}}{112} & \hbar_1 & \hbar_4 \\ \frac{21\sqrt{3}}{640} & \hbar_1 & -\frac{51\hbar_2}{4} + \frac{55\hbar_7}{2} & \frac{27\hbar_5}{4} + \frac{55\hbar_9}{6} \\ 0 & \hbar_4 & \frac{27\hbar_5}{4} + \frac{55\hbar_9}{6} & \frac{55\hbar_2}{36} - \frac{55\hbar_7}{18} \end{pmatrix},$$

$$H_{p_3}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \frac{3\sqrt{3}}{4480} \\ 0 & 0 & \hbar_4 & \hbar_3 \\ 0 & \hbar_4 & \frac{27\hbar_5}{4} + \frac{55\hbar_9}{6} & \frac{55\hbar_2}{36} - \frac{55\hbar_7}{18} \\ \frac{3\sqrt{3}}{4480} & \hbar_3 & \frac{55\hbar_2}{36} - \frac{55\hbar_7}{18} & -\frac{23\hbar_5}{36} - \frac{55\hbar_9}{54} \end{pmatrix}.$$

The values  $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_7, \hbar_9$  refer to the values of the corresponding parameters found above thanks to Algorithm 9.6.

5. Applying the steps (a)-(e), we get the different values  $\lambda_{\nu\alpha}$  with  $\nu = 1, 2, 3$  and  $\alpha = 1, 2, 3, 4$ .  
First solution:

$$\begin{aligned} \lambda_{11} = \rho_1^2 = 0.6482859840598436, & \quad \lambda_{21} = \rho_1^3 \cos(3\theta_1) = 0.5219036254941061, \\ & \quad \lambda_{31} = \rho_1^3 \sin(3\theta_1) = -0.0028831746621002, \\ \lambda_{12} = \rho_2^2 = 0.0787565135268152, & \quad \lambda_{22} = \rho_2^3 \cos(3\theta_2) = 0.0173884388064068, \\ & \quad \lambda_{32} = \rho_2^3 \sin(3\theta_2) = 0.0045477275754564, \\ \lambda_{13} = \rho_3^2 = 0.2963353570658749, & \quad \lambda_{23} = \rho_3^3 \cos(3\theta_3) = 0.0378314308570911, \\ & \quad \lambda_{33} = \rho_3^3 \sin(3\theta_3) = 0.0522721156592267, \\ \lambda_{14} = \rho_4^2 = 0.2423226284392470, & \quad \lambda_{24} = \rho_4^3 \cos(3\theta_4) = 0.0128786052414025, \\ & \quad \lambda_{34} = \rho_4^3 \sin(3\theta_4) = -0.0395297192910533. \end{aligned}$$

Second solution:

$$\begin{aligned} \lambda_{11} = \rho_1^2 = 0.6482859840598436, & \quad \lambda_{21} = \rho_1^3 \cos(3\theta_1) = 0.5219036254941061, \\ & \quad \lambda_{31} = \rho_1^3 \sin(3\theta_1) = 0.0028831746621002, \\ \lambda_{12} = \rho_2^2 = 0.0787565135268152, & \quad \lambda_{22} = \rho_2^3 \cos(3\theta_2) = 0.0173884388064068, \\ & \quad \lambda_{32} = \rho_2^3 \sin(3\theta_2) = -0.0045477275754564, \\ \lambda_{13} = \rho_3^2 = 0.2963353570658749, & \quad \lambda_{23} = \rho_3^3 \cos(3\theta_3) = 0.0378314308570911, \\ & \quad \lambda_{33} = \rho_3^3 \sin(3\theta_3) = -0.0522721156592267, \\ \lambda_{14} = \rho_4^2 = 0.2423226284392470, & \quad \lambda_{24} = \rho_4^3 \cos(3\theta_4) = 0.0128786052414025, \\ & \quad \lambda_{34} = \rho_4^3 \sin(3\theta_4) = 0.0395297192910533. \end{aligned}$$

6. Solving the systems formed by each row above, we get the coordinates of a node per orbit (Tables 10 and 11) and then the coordinates of the whole set of nodes from the group action and the orbit type (see Figures 21 and 22).
7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Tables 10 and 11).

Two  $C_3$ -invariant cubatures of degree 7 for the equilateral triangle  $T_2$  are thus determined (Tables 10 and 11). In fact, since one is obtained from the other one by an affine transformation, that is the reflection through the  $x_1$ -axis, these 2 cubatures describe the same cubature. Thus, there is a unique  $C_3$ -invariant cubature of degree 7 for the equilateral triangle  $T_2$ .

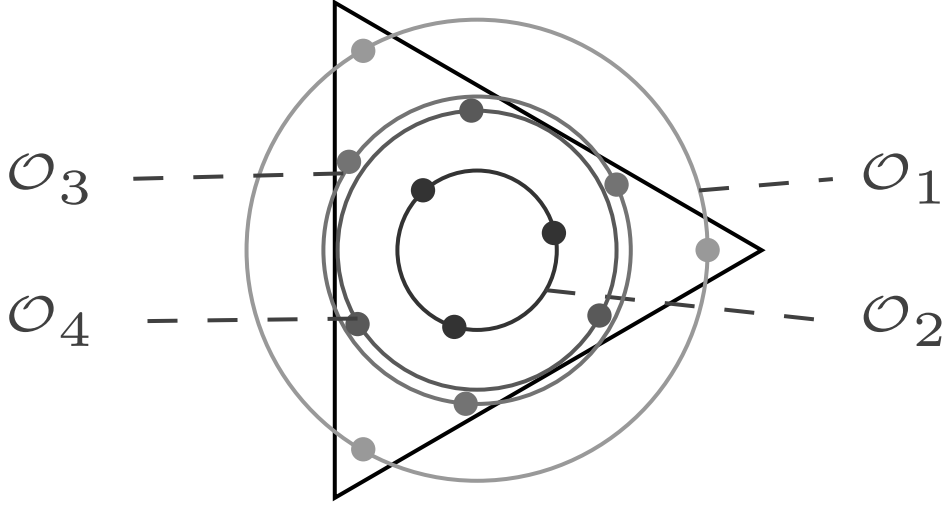


Figure 21: Nodes of the first cubature

|                 | Type            | Common weight $\check{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|-----------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_2$ | 0.0688932600516325               | (0.8051498017475222,-0.0044475617780204)             |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ | 0.1753524435986101               | (0.2737635015303786,0.0617256733996609)              |
| $\mathcal{O}_3$ | $\mathcal{P}_2$ | 0.0747597541403278               | (0.4914237942800738,0.2341751726405909)              |
| $\mathcal{O}_4$ | $\mathcal{P}_2$ | 0.1140072441016490               | (0.4349080742366314,-0.2306026786553698)             |

Table 10: Weights and nodes of the first cubature

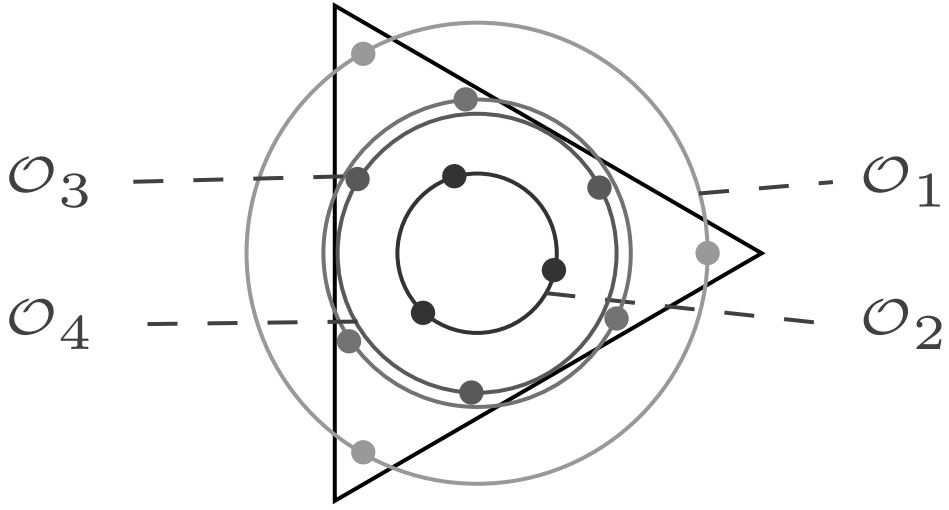


Figure 22: Nodes of the second cubature

|                 | Type            | Common weight $\check{a}_\alpha$ | A node $\zeta_\alpha$ per orbit $\mathcal{O}_\alpha$ |
|-----------------|-----------------|----------------------------------|--|
| $\mathcal{O}_1$ | $\mathcal{P}_2$ | 0.0688932600516325               | (0.8051498017475222, 0.0044475617780204)             |
| $\mathcal{O}_2$ | $\mathcal{P}_2$ | 0.1753524435986101               | (0.2737635015303786, -0.0617256733996609)            |
| $\mathcal{O}_3$ | $\mathcal{P}_2$ | 0.0747597541403278               | (0.4914237942800738, -0.2341751726405909)            |
| $\mathcal{O}_4$ | $\mathcal{P}_2$ | 0.1140072441016490               | (0.4349080742366314, 0.2306026786553698)             |

Table 11: Weights and nodes of the second cubature



## 13 Gaussian $C_2$ –invariant cubatures of degree 4 for the square

The starting point of this example is the Gaussian cubature for the square of degree 4 with 6 nodes presented in Example 1.6 in [35]. To find a Gaussian cubature, the authors fix the values of three parameters seemingly arbitrarily.

The cubature found in [35] is  $C_2$ –invariant. Indeed, the nodes are symmetric with respect to a symmetry axis of the square: this generates a group isomorphic to  $C_2$ . We consider the square  $\mathcal{C}_2$  whose vertices are  $(\pm 1, \pm 1)$  and the cyclic group  $C_2$  generated by the reflection through the  $x_1$ –axis. The 6 nodes we look for are then of the form either  $(a, 0)$  or  $(b, \pm c)$  with  $a, b, c \in [-1, 1]$ .

We find all Gaussian  $C_2$ –invariant cubatures (with respect to this group action): they form a one-parameter family of cubatures. The value of the parameter determines if the cubature is inside. It is therefore interesting to apply Section 5.2. The criteria of Section 5.2 allows to select the intervals for which this parameter defines an inside cubature.

In this section, we show that there exist Gaussian  $C_2$ –invariant cubatures of degree 4 for the square  $\mathcal{C}_2$  with respect to the group action generated by the reflection through the  $x_1$ –axis. Since the rotation of angle  $\frac{\pi}{2}$  leaves the square  $\mathcal{C}_2$  invariant, there exist then Gaussian  $C_2$ –invariant cubatures of degree 4 for the square  $\mathcal{C}_2$  with respect to the group action generated by the reflection through the  $x_2$ –axis. Moreover, we can check with a simple computation (Steps 1 and 2 of Algorithm 9.6 [Existence of a  $G$ –invariant cubature]) that there do not exist such  $C_2$ –invariant cubatures with respect to the group action generated by the rotation through an angle  $\pi$ .

### 13.1 Existence of a family of cubatures with 6 nodes

We first introduce the input of Algorithm 9.6 [Existence of a  $G$ –invariant cubature] and apply this algorithm for the same organization of nodes in orbit types as in [35].

In this search, we look for  $C_2$ –invariant cubatures of degree 4 for the square  $\mathcal{C}_2$ :  $(\pm 1, \pm 1)$ . The values  $\int p \, d\mu$  with  $p \in \mathbb{R}[x]_{\leq 4}^G$  are computed explicitly by

$$\int p \, d\mu = \int_{\mathcal{C}_2} p(x_1, x_2) dx_1 dx_2 = \int_{-1}^1 \left( \int_{-1}^1 p(x_1, x_2) dx_2 \right) dx_1.$$

Taking  $m = 2$  in Section 7.3, we get:

- ▷ A representation  $\mathcal{W}$  of the group  $C_2$  on  $\mathbb{R}^2$  deduced from the action of the group  $C_2$  generated by the reflection through the  $x_1$ –axis.

$$\left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

- ▷ The set  $[L_{abs}, L_{com}]$  of irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$ . Since they are all absolutely irreducible,  $L_{com}$  is empty and  $L_{abs}$  is

$$[[1, 1], [1, -1]].$$

- ▷ The matrix of multiplicities  $\Gamma_{C_2}$ , and more precisely the submatrix corresponding to this group action

$$\Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \Gamma_{C_2}.$$

- ▷ The integers  $m_1, m_2$  are 2, 2 since there are 2 orbits with 1 node on the  $x_1$ -axis and 2 orbits with 2 nodes in [35].

## Existence conditions

1. The expected multiplicities  $\gamma_1, \gamma_2$  are linked to the integers  $m_1, m_2$  thanks to (9.8) by

$$\gamma_1 = m_1 + m_2 = 4, \quad \gamma_2 = m_2 = 2.$$

Since all irreducible representations of the group  $C_2$  are absolutely irreducible, the expected multiplicities  $\gamma_j$  are the expected ranks  $r_j$ .

2. The inequalities (9.9) are satisfied since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$  of the group  $C_2$  in the induced representation on  $\mathbb{R}[x]_{\leq 2}$  are

$$(\underline{c}'_1, \underline{c}'_2) = (4, 2).$$

All Gaussian  $C_2$ -invariant cubatures of degree 4 for the square have nodes organized with  $m_1 = m_2 = 1$ . The fact that  $\underline{c}'_j = \gamma_j$  for all  $j$  is characteristic of a Gaussian cubature.

3. The inequalities (9.10) are satisfied by choosing  $\delta = 3$  since the multiplicities of the irreducible representations  $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}$  of the group  $C_2$  in the induced representation on  $\mathbb{R}[x]_{\leq 2}$  are

$$(\underline{c}''_1, \underline{c}''_2) = (4, 2).$$

4. An orthogonal symmetry adapted basis of  $\mathbb{R}[x]_{\leq 3}$  is

$$B^{(3)} = \left[ \begin{array}{c} [1, x_1, x_1^2, x_2^2, x_1^3, x_1 x_2^2], \\ [x_2, x_1 x_2, x_1^2 x_2, x_2^3] \end{array} \right].$$

5. The distinct blocks  $H^{(1)}, H^{(2)}$  are then computed.

$$H^{(1)} = \begin{pmatrix} 1 & x_1 & x_1^2 & x_2^2 & x_1^3 & x_1 x_2^2 \\ x_1 & x_1^2 & x_1^3 & x_1 x_2^2 & x_1^4 & x_1^2 x_2^2 \\ x_1^2 & x_1^3 & x_1^4 & x_1^2 x_2^2 & x_1^5 & x_1^3 x_2^2 \\ x_2^2 & x_1 x_2^2 & x_1^2 x_2^2 & x_2^4 & x_1^3 x_2^2 & x_2^2 x_1^2 \\ x_1^3 & x_1^4 & x_1^5 & x_1^3 x_2^2 & x_1^6 & x_1^4 x_2^2 \\ x_1 x_2^2 & x_1^2 x_2^2 & x_1^3 x_2^2 & x_2^4 x_1 & x_1^4 x_2^2 & x_1^2 x_2^4 \end{pmatrix}, H^{(2)} = \begin{pmatrix} x_2^2 & x_1 x_2^2 & x_1^2 x_2^2 & x_2^4 \\ x_1 x_2^2 & x_1^2 x_2^2 & x_1^3 x_2^2 & x_2^2 x_1^2 \\ x_1^2 x_2^2 & x_1^3 x_2^2 & x_1^4 x_2^2 & x_1^2 x_2^4 \\ x_2^4 & x_2^2 x_1^2 & x_1^2 x_2^4 & x_2^6 \end{pmatrix}$$

6. After the parametrization, the distinct blocks  $H^{(1)}, H^{(2)}$  are

$$H^{(1)} = \begin{pmatrix} 4 & 0 & \frac{4}{3} & \frac{4}{3} & 0 & 0 \\ 0 & 4 & \frac{4}{3} & 0 & \frac{4}{5} & \frac{4}{9} \\ 4 & \frac{4}{3} & 4 & 4 & \mathbf{h}_1 & \mathbf{h}_5 \\ \frac{4}{3} & 0 & \frac{4}{5} & \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_7 \\ \frac{4}{3} & 0 & \frac{4}{9} & \frac{4}{5} & \mathbf{h}_5 & \mathbf{h}_7 \\ 0 & 4 & \mathbf{h}_1 & \mathbf{h}_5 & \mathbf{h}_2 & \mathbf{h}_6 \\ 0 & \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_7 & \mathbf{h}_6 & \mathbf{h}_4 \end{pmatrix}, H^{(2)} = \begin{pmatrix} \frac{4}{3} & 0 & \frac{4}{9} & \frac{4}{5} \\ 0 & \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_7 \\ \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_6 & \mathbf{h}_4 \\ \frac{4}{5} & \mathbf{h}_7 & \mathbf{h}_4 & \mathbf{h}_3 \end{pmatrix}.$$

7. The parameters  $h_1, \dots, h_7$  are determined using Algorithm 4.7 on each block. Each block  $H^{(1)}, H^{(2)}$  provides 1 triplet  $[P_1, Z_1, E_1], [P_2, Z_2, E_2]$ . There is thus 1 system of equations and inequations. The corresponding set  $Z$  and list  $P$  are respectively

$$Z = \left\{ \begin{array}{l} \frac{4096}{6075}h_2 - \frac{16384}{50625} - \frac{256}{135}h_1^2 - \frac{256}{135}h_5^2, \frac{4096}{6075}h_6 - \frac{16384}{91125} - \frac{256}{135}h_1h_5 - \frac{256}{135}h_5h_7, \\ \frac{4096}{6075}h_4 - \frac{16384}{164025} - \frac{256}{135}h_5^2 - \frac{256}{135}h_7^2, \frac{16}{27}h_6 - \frac{64}{729} - \frac{4}{3}h_5^2, \\ \frac{16}{27}h_4 - \frac{64}{405} - \frac{4}{3}h_5h_7, \frac{16}{27}h_3 - \frac{64}{225} - \frac{4}{3}h_7^2 \end{array} \right\},$$

$$P = \left[ 4, \frac{16}{3}, \frac{256}{135}, \frac{4096}{6075}, \frac{4}{3}, \frac{16}{27} \right].$$

Notice that the positivity constraints in  $P$  do not depend on the parameters  $h_1, \dots, h_7$ . This is due to the fact that we look for Gaussian cubatures. Those constraints are therefore satisfied.

The system of equations obtained from  $Z$  can be solved. However, it would be difficult to manipulate the expression of the parameters  $h_1, \dots, h_7$ . That is why we divide the system of equations in two parts : first the one obtained from  $Z_2$ , then the one obtained from  $Z_1$ .

$$Z_2 = \left\{ \frac{16}{27}h_6 - \frac{64}{729} - \frac{4}{3}h_5^2, \frac{16}{27}h_4 - \frac{64}{405} - \frac{4}{3}h_5h_7, \frac{16}{27}h_3 - \frac{64}{225} - \frac{4}{3}h_7^2 \right\}.$$

A solution of the system obtained from  $Z_2$  is given by

$$\left\{ h_3 = \frac{12}{25} + \frac{9}{4}h_7^2, h_4 = \frac{9}{4}h_5h_7 + \frac{4}{15}, h_6 = \frac{4}{27} + \frac{9}{4}h_5^2 \right\},$$

where  $h_5$  and  $h_7$  are still free parameters. With this solution, the set  $Z_1$  becomes a triangular system

$$\begin{cases} 102400\mathbf{h}_2 - 288000h_1^2 - 288000h_5^2 = 49152 \\ 1555200\mathbf{h}_1h_5 + 1555200h_5h_7 - 1244160h_5^2 = -65536 \\ 1555200\mathbf{h}_5^2 - 1244160\mathbf{h}_5\mathbf{h}_7 + 1555200\mathbf{h}_7^2 = 65536 \end{cases}.$$

The last entry provides the equation of an ellipse. It can be parameterized taking

$$h_5 = a \cos t + b \sin t, h_7 = a \cos t - b \sin t \text{ with } a = \frac{8\sqrt{10}}{135}, b = \frac{8\sqrt{210}}{945}, t \in ]-\pi, \pi].$$

Solving the system of equations obtained from  $Z_1$  and  $Z_2$  using this parametrization, we get the values of the parameters  $h_1, \dots, h_7$  with respect to the parameter  $t$

$$\begin{aligned} h_1 &= -\frac{8\sqrt{10}}{4725} \frac{15\sqrt{21} - 168 \cos t \sin t + 34\sqrt{21} \cos^2 t}{\sqrt{21} \cos t + 3 \sin t}, \\ h_2 &= \frac{4}{14175} \frac{6003 + 3522\sqrt{21} \cos t \sin t + 16932 \cos^2 t - 288\sqrt{21} \cos^3 t \sin t - 3776 \cos^4 t}{3 + 2\sqrt{21} \cos t \sin t + 4 \cos^2 t}, \\ h_3 &= \frac{2428}{4725} - \frac{64\sqrt{21}}{2835} \cos t \sin t + \frac{128}{2835} \cos^2 t, \\ h_4 &= \frac{189}{44} + \frac{64}{567} \cos^2 t, \\ h_5 &= \frac{8\sqrt{10}}{945} (7 \cos t + \sqrt{21} \sin t), \\ h_6 &= \frac{172}{945} + \frac{64\sqrt{21}}{2835} \cos t \sin t + \frac{128}{2835} \cos^2 t, \\ h_7 &= \frac{8\sqrt{10}}{945} (7 \cos t - \sqrt{21} \sin t). \end{aligned}$$

This solution is available for  $t \in ]-\pi, \pi] \setminus \{\arctan(-\frac{1}{3}\sqrt{21}), \arctan(-\frac{1}{3}\sqrt{21}) + \pi\}$ .

This shows that for the square  $\mathcal{C}_2$  there exist a one-parameter family of  $C_2$ -invariant cubatures of degree 4 with positive weights and with this organization of the 6 nodes in orbit types.

8. In this example, we check that the nodes lie on the square following Section 5.2. The square  $\mathcal{C}_2$  is first expressed as a semialgebraic set

$$\mathcal{C}_2 = \{x \in \mathbb{R}^2 \mid -x_1 + 1 \geq 0, x_1 + 1 \geq 0, -x_2^2 + 1 \geq 0\}.$$

We construct then the matrices  $H_{-x_1+1}^{B^{(2)}}, H_{x_1+1}^{B^{(2)}}, H_{-x_2^2+1}^{B^{(2)}}$ , where  $B^{(2)}$  is the orthogonal symmetry adapted basis obtained as  $B^{(2)} = B^{(3)} \cap \mathbb{R}[x]_{\leq 2}$ .

$$\begin{aligned} H_{-x_1+1}^{B^{(2)}} &= \left( \begin{array}{cccc|cc} 4 & -\frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & 0 \\ -\frac{4}{3} & \frac{4}{3} & -\frac{4}{5} & -\frac{4}{9} & 0 & 0 \\ \frac{4}{3} & -\frac{4}{5} & -\mathbf{h}_1 + \frac{4}{5} & -\mathbf{h}_5 + \frac{4}{9} & 0 & 0 \\ \frac{4}{3} & -\frac{4}{9} & -\mathbf{h}_5 + \frac{4}{9} & -\mathbf{h}_7 + \frac{4}{5} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{4}{3} & -\frac{4}{9} \\ 0 & 0 & 0 & 0 & -\frac{4}{9} & -\mathbf{h}_5 + \frac{4}{9} \end{array} \right), \\ H_{x_1+1}^{B^{(2)}} &= \left( \begin{array}{cccc|cc} 4 & \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 0 & 0 \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{5} & \frac{4}{9} & 0 & 0 \\ \frac{4}{3} & \frac{4}{5} & \mathbf{h}_1 + \frac{4}{5} & \mathbf{h}_5 + \frac{4}{9} & 0 & 0 \\ \frac{4}{3} & \frac{4}{9} & \mathbf{h}_5 + \frac{4}{9} & \mathbf{h}_7 + \frac{4}{5} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{4}{3} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{4}{9} & \mathbf{h}_5 + \frac{4}{9} \end{array} \right), \\ H_{-x_2^2+1}^{B^{(2)}} &= \left( \begin{array}{cccc|cc} \frac{8}{3} & 0 & \frac{8}{9} & \frac{8}{15} & 0 & 0 \\ 0 & \frac{8}{9} & -\mathbf{h}_5 & -\mathbf{h}_7 & 0 & 0 \\ \frac{8}{9} & -\mathbf{h}_5 & -\mathbf{h}_6 + \frac{4}{5} & -\mathbf{h}_4 + \frac{4}{9} & 0 & 0 \\ \frac{8}{15} & -\mathbf{h}_7 & -\mathbf{h}_4 + \frac{4}{9} & -\mathbf{h}_3 + \frac{4}{5} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{8}{15} & -\mathbf{h}_7 \\ 0 & 0 & 0 & 0 & -\mathbf{h}_7 & -\mathbf{h}_4 + \frac{4}{9} \end{array} \right). \end{aligned}$$

Using Algorithm 4.7 on each block of  $H_{-x_1+1}^{B^{(2)}}, H_{x_1+1}^{B^{(2)}}, H_{-x_2^2+1}^{B^{(2)}}$ , we get a unique triplet  $[P, Z, E]$ .  $Z$  is here an empty set. Solving the polynomial inequations obtained from  $P$ , we get:

The matrices  $H_{-x_1+1}^{B^{(2)}}, H_{x_1+1}^{B^{(2)}}, H_{-x_2^2+1}^{B^{(2)}}$  are positive semidefinite if and only if

$$\begin{aligned} \arctan\left(-\frac{5\sqrt{21}}{3} - \frac{2\sqrt{154}}{3}\right) &\leq t \leq \arctan\left(-\frac{5\sqrt{21}}{3} + \frac{2\sqrt{154}}{3}\right) - \pi, \\ \arctan\left(-\frac{5\sqrt{21}}{3} - \frac{2\sqrt{154}}{3}\right) + \pi &\leq t \leq \arctan\left(-\frac{5\sqrt{21}}{3} + \frac{2\sqrt{154}}{3}\right). \end{aligned} \tag{13.1}$$

Approximatively, (13.1) is

$$\begin{aligned} -2.575485786518756 &\leq t \leq -1.508028268370882, \\ 0.5661068670710366 &\leq t \leq 1.633564385218911. \end{aligned}$$

The intervals defined above describe the values of the parameter  $t$  such that there exists an inside cubature.

### 13.2 Computation of the weights and the nodes for several cubatures

With the help of the solutions found thanks to Algorithm 9.6 [Existence of a  $G$ -invariant cubature], we are now able to compute the associated cubatures for a selection of values of the parameter  $t$  using Algorithm 9.7 [Weights & Nodes]. We first determine a separating set  $\{p_1, \dots, p_\eta\}$  of  $G$ -invariant polynomials. Here, it is sufficient to look at the orthogonal symmetry adapted basis of the first component of  $\mathbb{R}[x]_{\leq 3}$  to understand that a separating set is given by

$$\{p_1, p_2\} = \{x_1, x_2^2\}.$$

1. Since we look for Gaussian cubatures, an orthogonal symmetry adapted basis of  $\mathbb{R}[x]/I_\Lambda$  is given by taking the classes of an orthonormal basis of  $\mathbb{R}[x]_{\leq 2}$

$$[[1, x_1, x_1^2, x_2^2], [x_2, x_1 x_2]].$$

2. The invertible blocks  $H_1^{(1)}, H_1^{(2)}$  are respectively

$$H_1^{(1)} = \begin{pmatrix} 4 & 0 & \frac{4}{3} & \frac{4}{3} \\ 0 & \frac{4}{3} & 0 & 0 \\ \frac{4}{3} & 0 & \frac{4}{5} & \frac{4}{9} \\ \frac{4}{3} & 0 & \frac{4}{9} & \frac{4}{5} \end{pmatrix}, \quad H_1^{(2)} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{9} \end{pmatrix}.$$

3. The separating set is

$$\{p_1, p_2\} = \{x_1, x_2^2\}.$$

4. The distinct blocks  $H_{p_1}^{(1)}, H_{p_1}^{(2)}, H_{p_2}^{(1)}, H_{p_2}^{(2)}$  are respectively

$$H_{p_1}^{(1)} = \begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ \frac{4}{3} & 0 & \frac{4}{5} & \frac{4}{9} \\ 0 & \frac{4}{5} & \mathbf{h}_1 & \mathbf{h}_5 \\ 0 & \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_7 \end{pmatrix}, \quad H_{p_1}^{(2)} = \begin{pmatrix} 0 & \frac{4}{9} \\ \frac{4}{9} & \mathbf{h}_5 \end{pmatrix},$$

$$H_{p_2}^{(1)} = \begin{pmatrix} \frac{4}{3} & 0 & \frac{4}{9} & \frac{4}{5} \\ 0 & \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_7 \\ \frac{4}{9} & \mathbf{h}_5 & \mathbf{h}_6 & \mathbf{h}_4 \\ \frac{4}{5} & \mathbf{h}_7 & \mathbf{h}_4 & \mathbf{h}_3 \end{pmatrix}, \quad H_{p_2}^{(2)} = \begin{pmatrix} \frac{4}{5} & \mathbf{h}_7 \\ \mathbf{h}_7 & \mathbf{h}_4 \end{pmatrix}.$$

5. The parameters  $h_1, h_2, h_3, h_4, h_5, h_6, h_7$  are known with respect to the parameter  $t$ . We take several values of  $t$  in the two different intervals in (13.1). Then we compute and match together the generalized eigenvalues of the matrices  $(H_{p_1}^{(1)}, H_1^{(1)})$ ,  $(H_{p_1}^{(2)}, H_1^{(2)})$ ,  $(H_{p_2}^{(2)}, H_1^{(2)})$ .
6. Since  $p_1 = x_1$ , the generalized eigenvalues of  $(H_{p_1}^{(1)}, H_1^{(1)})$  and  $(H_{p_1}^{(2)}, H_1^{(2)})$  are directly the first coordinates of the nodes. Since  $p_2 = x_2^2$ , the generalized eigenvalues of  $(H_{p_2}^{(2)}, H_1^{(2)})$  are the square of the nonzero second coordinates of the nodes. We get thus the coordinates of the nodes (see Tables 12 and 13 and Figures 23 and 24).
7. The solutions of the Vandermonde-like linear system (9.13) are the distinct weights (Tables 12 and 13).

Several  $C_2$ -invariant cubatures of degree 4 for the square  $\mathcal{C}_2$  are thus determined (Tables 12 and 13).

| Parameter $t$ | Common weights $\check{a}_\alpha$   | Nodes $\zeta_\alpha$   |
|---------------|---|--|
| -2.5          | 0.4116480473530860<br>1.311259163323216<br>0.2662078471075957<br>0.8723385475542534 | (0.9847953274692768,0)<br>(-0.4546279561226131,0)<br>(-0.8812404876448891, $\pm 0.8672370904283691$ )<br>(0.3782546739587111, $\pm 0.7312414613234438$ ) |
| -2.3          | 0.4617799514368910<br>1.305007359913070<br>0.3063944067771868<br>0.8102119375478327 | (0.9519966724282465,0)<br>(-0.4041574671761068,0)<br>(-0.8699323764465240, $\pm 0.8156747805538032$ )<br>(0.3831715456952224, $\pm 0.7557954170542742$ ) |
| -2.1          | 0.4993638678071540<br>1.277059037157444<br>0.3665075118569558<br>0.7452810356607448 | (0.9295200025216641,0)<br>(-0.3384120492184044,0)<br>(-0.8454142531487735, $\pm 0.7606056050370451$ )<br>(0.3942840235917741, $\pm 0.7810361207955825$ ) |
| -1.9          | 0.5170436921225884<br>1.225183988483287<br>0.4506371350469478<br>0.6782490246501143 | (0.9194639275958594,0)<br>(-0.2486288422003721,0)<br>(-0.8093616796039842, $\pm 0.7063402593247500$ )<br>(0.4118471898699617, $\pm 0.8071163140964325$ ) |
| -1.7          | 0.4960974500619558<br>1.163129499989356<br>0.5603090669586502<br>0.6100774580156940 | (0.9314119617888543,0)<br>(-0.1185304767466817,0)<br>(-0.7642236136719319, $\pm 0.6575176254464650$ )<br>(0.4361725120370695, $\pm 0.8340840476632787$ ) |

Table 12: Weights and nodes of the cubatures for values of  $t$  in the first interval

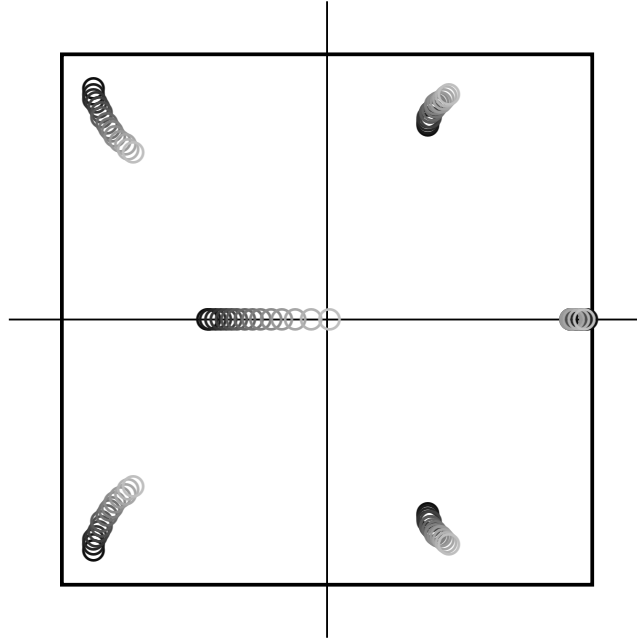


Figure 23: Nodes of the cubatures with the values of  $t$  in the first interval. The higher is the value of the parameter  $t$ , the lighter are the nodes.

| Parameter $t$ | Common weights $\check{a}_\alpha$   | Nodes $\zeta_\alpha$   |
|---------------|---|--|
| 0.6           | 0.4000036381868590<br>1.310384396759925<br>0.2599615286290525<br>0.8848444538975553 | (-0.9929669002961246,0)<br>(0.4637069421155644,0)<br>(0.8818439900189286, $\pm 0.8771738179203162$ )<br>(-0.3779958100368506, $\pm 0.7262048106814606$ ) |
| 0.8           | 0.4522386314691690<br>1.307909376441351<br>0.2965415374569065<br>0.8233844585878337 | (-0.9579694533571518,0)<br>(0.4157066067281893,0)<br>(0.8734116149243455, $\pm 0.8268177747528796$ )<br>(-0.3816451803909277, $\pm 0.7506384897529334$ ) |
| 1             | 0.4928493136505840<br>1.284887456334569<br>0.3521487113182183<br>0.7589829036892052 | (-0.9333043132636291,0)<br>(0.3536943566874154,0)<br>(0.8515312070228974, $\pm 0.7721431623028911$ )<br>(-0.3914516938242643, $\pm 0.7757219508086625$ ) |
| 1.2           | 0.5156420468076848<br>1.237679523103433<br>0.4310185215913971<br>0.6923206934530438 | (-0.9202501854149314,0)<br>(0.2699240577533140,0)<br>(0.8176987587893692, $\pm 0.7173309826586556$ )<br>(-0.4076480852518896, $\pm 0.8016187465996738$ ) |
| 1.4           | 0.5054107417663664<br>1.174983214283699<br>0.5355162981033686<br>0.6242867238715990 | (-0.9260461566779571,0)<br>(0.1501179501027602,0)<br>(0.7742169493717551, $\pm 0.6670000724540969$ )<br>(-0.4305425418596418, $\pm 0.8284064659242651$ ) |

Table 13: Weights and nodes of the cubatures for values of  $t$  in the second interval

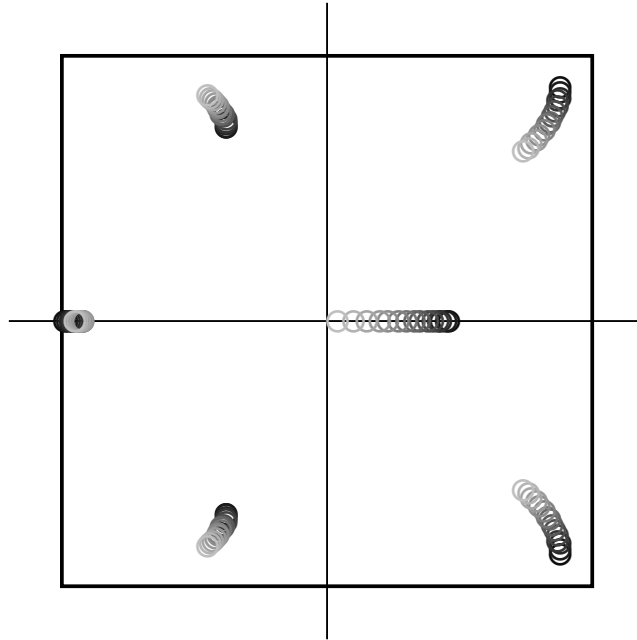


Figure 24: Nodes of the cubatures with the values of  $t$  in the second interval. The higher is the value of the parameter  $t$ , the lighter are the nodes.





# Conclusion générale

Dans cette thèse, nous avons étudié le problème de reconstruction de formes linéaires sur l'anneau des polynômes suivant : Soient  $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$  l'anneau des polynômes à  $n$  variables et à coefficients dans un corps  $\mathbb{K}$  de caractéristique zéro et  $\Lambda$  une forme linéaire sur  $\mathbb{K}[x]$  pouvant s'écrire sous la forme d'une combinaison linéaire d'évaluations en des points

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$$

avec  $a_1, \dots, a_r \in \mathbb{K} \setminus \{0\}$  et  $\xi_1, \dots, \xi_r$  des points distincts de  $\mathbb{K}^n$ . Celui-ci a plusieurs applications : la reconstruction de polytopes à partir des moments et la recherche de cubatures symétriques en font partie et font l'objet des deux chapitres de cette thèse. La résolution que nous proposons généralise la méthode de Prony pour des problèmes univariés : soit par une méthode de projection, soit par une adaptation multivariée à l'aide des opérateurs de Hankel et de théorèmes d'extension plate.

La symétrie, qui apparaît naturellement dans les problèmes multivariés, permet de simplifier les calculs dans l'algorithme proposé pour la recherche de cubatures. L'introduction de la matrice de multiplicités d'un groupe fini permet de quantifier des conditions d'existence de cubatures et de réduire le nombre de problèmes de valeurs propres généralisées à résoudre lors du calcul des coordonnées des noeuds.

Les algorithmes proposés pour chaque application permettent de repousser certaines limites qui existaient dans la littérature. Une reconstruction numérique de polytopes convexes à partir de moments est désormais possible quelque soit la dimension. Une coupe de diamant - un polytope de dimension 3 avec 57 sommets - a ainsi été reconstruite. De nouvelles cubatures ont été trouvées et notre méthode permet de certifier que la totalité des cubatures présentant une symétrie est trouvée pour une mesure, un degré et un nombre de noeuds donnés. Ainsi, nous avons montré qu'il existe pour l'hexagone régulier  $H2$  exactement 4 cubatures  $D_6$ -invariantes intérieures à poids positifs de degré 13.

L'algorithme de reconstruction de polytopes que nous proposons présente cependant des limites. En effet, la précision requise lorsque le nombre de sommets augmente (au-delà de 12 sommets, la double précision n'est en général plus suffisante) semble indiquer que la réalisation de cette méthode en pratique est peu probable. Il reste également à étudier la sensibilité au bruit de cet algorithme : les données mesurées étant en général perturbées.

Concernant la recherche de cubatures symétriques, nous sommes persuadés que l'algorithme proposé peut permettre de trouver de nombreuses autres cubatures. En particulier, des cubatures pour les groupes de symétrie de polyèdres réguliers (tétraèdre et cube par exemple) semblent des exemples intéressants au vu de l'ordre de ces groupes.

De plus, comme suggéré dans l'introduction, la méthode proposée, qui allie opérateurs de Hankel et extensions plates, pourrait être utilisé pour d'autres applications utilisant la méthode de Prony comme moyen de résolution. On peut penser aux interpolations creuse et exponentielle, mais aussi à la reconstruction de polytopes à la condition de trouver une formule équivalente à la formule de Brion pour les moments géométriques. L'utilisation de la symétrie dans ces contextes est également à étudier.



# Résumé

Le problème général étudié dans cette thèse est le suivant : Soient  $\mathbb{K}[x] = \mathbb{K}[x_1, \dots, x_n]$  l'anneau des polynômes à  $n$  variables et à coefficients dans un corps  $\mathbb{K}$  de caractéristique zéro et  $\Lambda$  une forme linéaire sur  $\mathbb{K}[x]$  pouvant s'écrire sous la forme d'une combinaison linéaire d'évaluations en des points

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j} \quad (13.2)$$

avec  $a_1, \dots, a_r \in \mathbb{K} \setminus \{0\}$  et  $\xi_1, \dots, \xi_r$  des points distincts de  $\mathbb{K}^n$ . Il s'agit de reconstruire cette forme linéaire  $\Lambda$  à partir d'évaluations de celle-ci en des polynômes, c'est-à-dire trouver les valeurs des coefficients  $a_1, \dots, a_r$ , des points  $\xi_1, \dots, \xi_r$  et dans certains cas le nombre de termes  $r$ .

## Chapitre 1 : Reconstruction numérique de polytopes convexes à partir des moments directionnels

Une première application de ce problème général est la reconstruction de polytopes à partir des moments. Nous proposons un algorithme numérique pour la reconstruction de polytopes convexes en dimension quelconque à partir de leurs moments directionnels. Ces derniers font intervenir le produit scalaire usuel  $\langle \cdot, \cdot \rangle$  sur  $\mathbb{R}^n$  et sont définis par

$$m_k(\delta) = \int_{\mathcal{V}} \langle x, \delta \rangle^k dx \quad \forall k \in \mathbb{N},$$

où  $\mathcal{V}$  est le polytope convexe de  $\mathbb{R}^n$  et  $\delta$  un vecteur unitaire de  $\mathbb{R}^n$ , appelé *direction*. L'algorithme proposé est construit à partir des algorithmes numériques (en arithmétique flottante) développés pour le cas des polygones ( $n = 2$ ) [31, 43, 83] et de la formule de Brion, qui a permis aux auteurs de [47] de proposer un algorithme en arithmétique exacte de reconstruction de polytopes convexes en dimension quelconque.

La formule de Brion [8, 47] est l'ingrédient principal qui permet de reformuler le problème sous la forme (13.2).

**Théorème 1** (Formule de Brion). *Supposons que les projections orthogonales des  $r$  sommets du polytope convexe  $\mathcal{V}$  sur la direction  $\delta$  sont distinctes, nous avons alors les égalités suivantes*

$$\begin{aligned} \frac{(k+n)!}{k!} m_k(\delta) &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n+k}, \quad k \geq 0, \\ \text{et} \\ 0 &= \sum_{v \in \mathcal{V}} a_v(\delta) \langle v, \delta \rangle^{n-k}, \quad 1 \leq k \leq n, \end{aligned}$$

où les coefficients  $a_v(\delta)$  dépendent de  $\delta$  et des sommets adjacents à  $v$  dans une triangulation de  $\mathcal{V}$ . De plus,

$$a_v(\delta) \neq 0.$$

Pour une direction  $\delta$  fixée, on considère alors une suite de nombres réels  $(\mu_k(\delta))_{k \in \mathbb{N}}$  liés aux moments directionnels par

$$\begin{cases} \mu_k(\delta) = 0 & 0 \leq k \leq n-1, \\ \mu_k(\delta) = \frac{k!}{(k-n)!} m_{k-n}(\delta), & k \geq n. \end{cases}$$

Lorsque le nombre de sommets  $r$  est connu, la méthode de Prony permet alors à partir de cette suite  $(\mu_k)_{k \in \mathbb{N}}$  de retrouver l'ensemble  $\mathcal{V}(\delta) = \{ \langle v_j, \delta \rangle \}_{j=1, \dots, r}$  des sommets  $v_1, \dots, v_r$  projetés suivant la direction  $\delta$ . A la méthode de Prony classique [93], nous lui préférons sa version plus stable numériquement. Celle-ci affirme que les sommets projetés sont les valeurs propres généralisées des matrices de Hankel  $(H_r^{(0)}, H_r^{(1)})$  avec

$$H_r^{(0)} = (\mu_{i+j-2})_{1 \leq i, j \leq r} \text{ et } H_r^{(1)} = (\mu_{i+j-1})_{1 \leq i, j \leq r}.$$

Cette dernière version est appelée *pencil method* [43, 59].

Comme chaque sommet a  $n$  coordonnées, nous avons besoin de connaître  $n$  projections de chaque sommet pour les décrire. Par conséquent, nous avons besoin d'au moins  $n$  directions pour obtenir ces projections comme valeurs propres généralisées. Une direction supplémentaire est cependant nécessaire pour faire correspondre les valeurs propres généralisées aux projections d'un même sommet. Nous avons choisi une méthode numériquement robuste : l'interpolation par intervalles [96]. Le but de cette méthode est de trouver pour chaque sommet le meilleur interpolant qui respecte notre estimation de l'erreur sur les valeurs propres généralisées.

L'interpolation par intervalles se déroule ici en trois étapes. Dans un premier temps, on sélectionne  $n$  listes de valeurs propres généralisées, qui correspondent donc aux projections des sommets pour  $n$  directions, et on calcule tous les interpolants linéaires  $q_u$  de la forme

$$q_u : S_n \rightarrow [-1, 1], \delta \mapsto \langle u, \delta \rangle,$$

où  $S_n \subset \mathbb{R}^n$  est la sphère unité et  $u$  un vecteur de  $\mathbb{R}^n$ , qui passent par les projections sélectionnées. Dans un deuxième temps, on ajoute une liste de valeurs propres généralisées et on conserve les interpolants  $q_u$  qui passent à proximité des projections pour cette direction. Cette proximité est mesurée à l'aide de l'erreur relative  $\xi_\mu(\delta)$  sur les entrées et du conditionnement des problèmes de valeurs propres généralisées que nous majorons par  $\kappa(\delta)^2$ , où  $\kappa(\delta)$  est le conditionnement de la matrice de vecteurs propres généralisés. Dans un troisième temps, le calcul du meilleur interpolant est réalisé en tenant compte de toutes les valeurs propres généralisées associées à un même sommet.

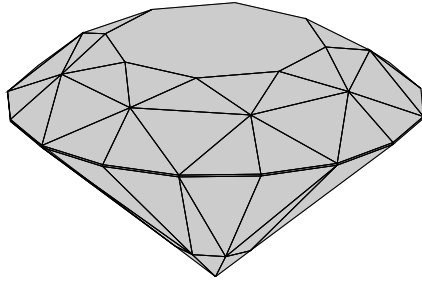
Au bout de la deuxième étape, si le problème est bien posé, il n'y a qu'un unique interpolant  $q_u$  pour chaque sommet. Dans le cas contraire, on peut agir soit sur le choix des  $n+1$  directions, soit sur la précision du calcul. Nous avons obtenu de meilleurs résultats avec des directions proches les unes des autres.

Le nombre de sommets peut également faire partie des inconnues du problème. Pour l'estimer, nous proposons de trouver le rang de la matrice  $H_R^{(0)}$ , où  $R$  est un majorant de  $r$  et est donnée initialement. En effet, le rang de cette matrice est le nombre de sommets du polytope. Ce calcul de rang peut être réalisé en analysant les valeurs singulières de la matrice  $H_R^{(0)}$ .

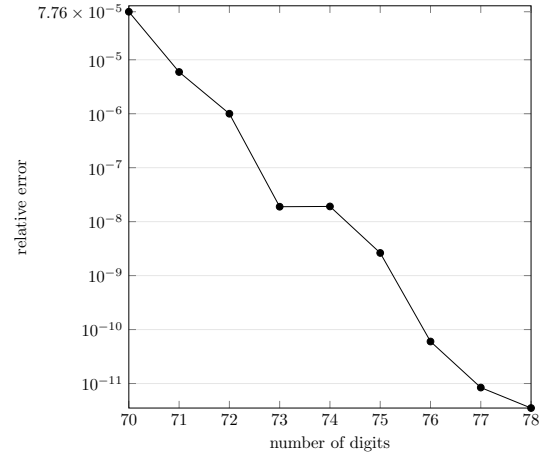
Nous proposons un algorithme en 3 étapes pour reconstruire un polytope convexe à partir de ses moments directionnels :

1. Estimation du nombre de sommets  $r$  à partir d'un majorant  $R$  et d'une analyse des valeurs singulières.
2. Pour  $n+1$  directions (ou plus), les projections des  $r$  sommets sont obtenues comme valeurs propres généralisées d'une paire de matrices structurées construites à partir des moments directionnels.
3. Chaque projection est associée à son sommet avec une interpolation par intervalles et les coordonnées du sommet sont alors calculées.

Cet algorithme a été implémenté d'une part avec le logiciel Maple et d'autre part avec le logiciel Matlab. Nous l'avons appliqué à différents exemples de polygones ( $n = 2$ ) et de polyèdres ( $n = 3$ ). Un défi pour nous a été la reconstruction d'une coupe de diamant (Figure 25(a)). La précision du calcul requise est très grande (Figure 25(b)). Ceci est dû au grand nombre de sommets : 57. En effet, le conditionnement  $\kappa(\delta)$  croît de manière exponentielle avec la taille de la matrice de vecteurs propres généralisés [9, 10], qui correspond ici au nombre de sommets. Généralement, la double précision suffit pour reconstruire des polytopes avec au plus 12 sommets.



(a) Coupe de diamant à 57 sommets



(b) Erreur relative maximale entre les sommets réels et calculés en fonction de la précision demandée pour les calculs (1 Digit = 1 décimale)

Figure 25: Diamant

## Chapitre 2 : Une approche par les matrices de moments pour la recherche de cubatures symétriques

Une seconde application du problème général est la recherche de cubatures. Nous proposons une méthode pour trouver des cubatures à l'aide des matrices de moments [74] - plus généralement les matrices des opérateurs de Hankel [12, 15] - et de théorèmes d'extension plate [24, 25, 73, 74]. Cette méthode est ensuite complétée pour la recherche de cubatures symétriques par des résultats relatifs aux groupes et à la théorie des représentations linéaires [33, 98]. L'utilisation de la symétrie permet de diagonaliser par blocs les matrices des opérateurs de Hankel lorsque des

bases appropriées sont choisies : nous les appelons *bases orthogonales adaptées à la symétrie* en référence aux bases adaptées à la symétrie dans [33, 39, 94]. L'adjectif *orthogonales* signifie que ces bases sont calculées à partir de matrices orthogonales pour les représentations irréductibles. La symétrie rend notre méthode plus efficace et nous permet de trouver toutes les cubatures pour une mesure, un degré et un nombre de noeuds donnés. En particulier, nous donnons explicitement de nouvelles cubatures.

Une cubature  $\Lambda$  pour une certaine mesure positive de Borel sur  $\mathbb{R}^n$  est une forme linéaire sur l'anneau des polynômes à  $n$  variables  $\mathbb{R}[x]$  définie par (13.2) et satisfaisant

$$\Lambda(p) = \int p \, d\mu \quad \forall p \in \mathbb{R}[x]_{\leq d},$$

où  $d$  est le degré de la cubature  $\Lambda$ . Les cubatures sont les analogues multidimensionnels des quadratures. Les coefficients  $a_1, \dots, a_r$  et les points  $\xi_1, \dots, \xi_r$  de  $\mathbb{R}^n$  sont appelés respectivement les poids et les noeuds de la cubature  $\Lambda$ . Les cubatures qui nous intéressent ici sont les cubatures intérieures à poids positifs, c'est-à-dire celles telles que

$$a_j > 0 \quad \text{et} \quad \xi_j \in \text{supp } \mu \quad \forall j = 1, \dots, r,$$

où  $\text{supp } \mu$  est le support de la mesure  $\mu$ . Les théorèmes de Tchakaloff [7, 89] et Mysovskikh [30, Chapter 3.7.1] nous garantissent l'existence de telles cubatures dans le cas où le support  $\text{supp } \mu$  est compact.

**Théorème 2.** *Soit  $d$  un entier naturel et  $\mu$  une mesure positive de Borel à support compact dans  $\mathbb{R}^n$ . Alors il existe une cubature intérieure à poids positifs de degré  $d$ , dont le nombre de noeuds  $r$  satisfait*

$$\dim \mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor} \leq r \leq \dim \mathbb{R}[x]_{\leq d}.$$

Il existe une large littérature concernant la recherche de cubatures ; nous ne ferons mention ici que des *reviews* [19, 20, 101] et d'apports récents sur le sujet [14, 26].

La recherche de cubature à l'aide des matrices de moments a été initiée dans [35, 69]. Une généralisation est possible à l'aide des opérateurs de Hankel [12, 15]. A toute forme linéaire  $\Lambda$  sur  $\mathbb{K}[x]$ , avec  $\mathbb{K} = \mathbb{R}$  ou  $\mathbb{C}$ , nous lui associons l'opérateur (linéaire) de Hankel

$$\widehat{\mathcal{H}} : \mathbb{K}[x] \rightarrow \mathbb{K}[x]^*, f \mapsto f \star \Lambda,$$

où  $f \star \Lambda : \mathbb{K}[x] \rightarrow \mathbb{K}, g \mapsto \Lambda(fg)$ . Son noyau, noté  $I_\Lambda$ , est un idéal de  $\mathbb{K}[x]$  ; il est zéro-dimensionnel si et seulement si le rang de  $\widehat{\mathcal{H}}$  est fini. Dans ce cas, nous construisons l'opérateur linéaire  $\mathcal{H}$  défini sur  $\mathbb{K}[x]/I_\Lambda$  par le diagramme commutatif en Figure 26.

$\mathcal{H}$  est un isomorphisme et la matrice  $H_1^B = (\Lambda(b_i b_j))_{1 \leq i, j \leq r}$  est inversible si et seulement si  $B = \{b_1, \dots, b_r\}$  est une base de  $\mathbb{K}[x]/I_\Lambda$ . Dans ce cas,  $H_1^B$  est la matrice de  $\mathcal{H}$  dans la base  $B$  et sa duale  $B^*$ . Pour tout polynôme  $p \in \mathbb{K}[x]$ , nous définissons alors l'opérateur  $\mathcal{H}_p = \mathcal{H} \circ \mathcal{M}_p$ , où  $\mathcal{M}_p$  est l'opérateur de multiplication par  $p$  dans  $\mathbb{K}[x]/I_\Lambda$ . Sa matrice dans la base  $B$  et sa duale  $B^*$  est  $H_p^B = (\Lambda(p b_i b_j))_{1 \leq i, j \leq r}$ . Nous en déduisons que les valeurs propres généralisées des matrices  $(H_p^B, H_1^B)$  sont les valeurs de  $p$  en les points de la variété  $V_{\mathbb{C}}(I_\Lambda)$  associée à l'idéal  $I_\Lambda$ . A noter que si  $B$  est une base monomiale, la matrice  $H_p^B$  est une matrice de moments.

$$\begin{array}{ccccc}
\mathbb{K}[x] & \xrightarrow{\widehat{\mathcal{H}}} & \widehat{\mathcal{H}}(\mathbb{K}[x]) & \xrightarrow{\iota} & (I_\Lambda)^\perp \xrightarrow{(\pi_*)^{-1}} (\mathbb{K}[x]/I_\Lambda)^* \\
& \searrow \pi & \nearrow \mathring{\mathcal{H}} & \nearrow \mathcal{H} & \\
& & \mathbb{K}[x]/I_\Lambda & & 
\end{array}$$

Figure 26: Diagramme commutatif représentant la construction de l'opérateur linéaire  $\mathcal{H}$ .  $\pi$  est la surjection canonique de  $\mathbb{K}[x]$  sur  $\mathbb{K}[x]/I_\Lambda$ ,  $\iota$  l'inclusion naturelle de l'image de  $\mathbb{K}[x]$  par l'opérateur de Hankel  $\widehat{\mathcal{H}}$  dans  $(I_\Lambda)^\perp$  l'orthogonal du noyau  $I_\Lambda$ ,  $\pi_*$  est un isomorphisme construit dans [32].

Supposons que l'opérateur de Hankel  $\widehat{\mathcal{H}}$  est de plus semidéfini positif, alors l'idéal  $I_\Lambda$  est également radical,  $V_{\mathbb{C}}(I_\Lambda) \subset \mathbb{R}^n$  et

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j}$$

avec  $a_j > 0$  et  $\xi_j \in \mathbb{R}^n$  distincts deux à deux.

Un théorème fondamental dans la recherche de cubatures dans [35] est celui d'extension plate [24, 25, 73, 74].

**Théorème 3.** [Théorème d'extension plate de Curto-Fialkow] Soit  $\Lambda^{(\delta)}$  une forme linéaire sur  $\mathbb{R}[x]_{\leq 2\delta}$ . Supposons que  $\Lambda^{(\delta)}$  soit une extension plate de sa restriction sur  $\mathbb{R}[x]_{\leq 2\delta-2}$ , c'est-à-dire que les rangs des opérateurs de Hankel associés à  $\Lambda^{(\delta)}$  et sa restriction sont égaux. Alors il existe une unique extension plate de  $\Lambda^{(\delta)}$  à  $\mathbb{R}[x]_{\leq 2\delta+2\kappa}$  pour tout  $\kappa \geq 1$ .

**Corollaire 4.** Soit  $\Lambda^{(\delta)}$  une forme linéaire sur  $\mathbb{R}[x]_{\leq 2\delta}$ . Supposons que  $\Lambda^{(\delta)}$  soit une extension plate de sa restriction sur  $\mathbb{R}[x]_{\leq 2\delta-2}$  et que l'opérateur de Hankel associé  $\mathcal{H}^{(\delta)}$  soit semidéfini positif. Alors la forme linéaire  $\Lambda$  définie sur  $\mathbb{R}[x]$  par

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j},$$

avec  $r = \text{rank } \mathcal{H}^{(\delta)}$ ,  $a_j > 0$  et  $\{\xi_1, \dots, \xi_r\} = V_{\mathbb{C}}(\ker \mathcal{H}^{(\delta)}) \subset \mathbb{R}^n$ , est l'unique extension plate de  $\Lambda^{(\delta)}$  à  $\mathbb{R}[x]$ .

Nous constatons alors que la forme linéaire  $\Lambda$  du corollaire précédent satisfait à toutes les propriétés requises pour la recherche de cubatures intérieures à poids positifs, sauf  $\xi_1, \dots, \xi_r \in \text{supp } \mu$ . Cette dernière est vérifiée à l'aide des matrices de localisation en supposant que  $\text{supp } \mu$  est un ensemble semialgébrique [25, 74].

**Proposition 5.** Considérons l'ensemble semialgébrique

$$K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_s(x) \geq 0\}.$$

Soit  $\kappa \in \mathbb{N}$  tel que  $\kappa \geq 1$  et  $2\kappa \geq \deg g_k$  pour tout  $k = 1, \dots, s$ . Soit  $\Lambda^{(\delta)}$  une forme linéaire sur  $\mathbb{R}[x]_{\leq 2\delta}$  et soit  $\Lambda^{(\delta+\kappa)} : \mathbb{R}[x]_{\leq 2\delta+2\kappa} \rightarrow \mathbb{R}$  une extension plate de  $\Lambda^{(\delta)}$ . Supposons que  $\mathcal{H}^{(\delta)}$  et

$\mathcal{H}_{g_k}^{(\delta)}$  soient semidéfinies positives pour tout  $k = 1, \dots, s$ . Alors la forme linéaire  $\Lambda$  définie sur  $\mathbb{R}[x]$  par

$$\Lambda = \sum_{j=1}^r a_j \mathbb{1}_{\xi_j},$$

avec  $r = \text{rank } \mathcal{H}^{(\delta)}$ ,  $a_j > 0$  et  $\{\xi_1, \dots, \xi_r\} = V_{\mathbb{C}}(\ker \mathcal{H}^{(\delta)}) \subset \mathbb{R}^n$ , est l'unique extension de  $\Lambda^{(\delta+\kappa)}$  à  $\mathbb{R}[x]$  telle que  $\xi_j \in K$ .

A l'aide de ces résultats, nous pouvons écrire un algorithme pour la recherche de cubatures.

Dans un premier temps, celui-ci détermine des conditions d'existence d'une cubature  $\Lambda$  à poids positifs pour une mesure  $\mu$  et un degré de cubature  $d$  donnés. A partir du Corollaire 4, une telle cubature n'existe que si l'opérateur de Hankel associé est semidéfini positif et s'il existe un degré  $\delta$  tel que sa restriction  $\Lambda^{(\delta)}$  est une extension plate de sa restriction  $\Lambda^{(\delta-1)}$ . Ces conditions sont exprimées dans notre algorithme sous la forme d'un système d'équations et d'inéquations en des paramètres. Ceux-ci représentent les moments inconnus : ceux d'ordre compris entre  $d$  et  $2\delta$ . Pour réaliser cette première étape, nous décrivons une procédure qui diagonalise une matrice symétrique à coefficients polynomiaux et qui définit des conditions selon lesquelles cette matrice est semidéfinie positive. Cette procédure est construite autour d'une décomposition LU sans fraction et sans pivotement.

Dans un deuxième temps, des conditions pour l'existence d'une cubature intérieure sont recherchées à partir des matrices de localisation. Cette deuxième étape est optionnelle : une vérification *a posteriori* est souvent moins coûteuse.

Dans un troisième temps, le calcul des noeuds et des poids est opéré. Concernant les noeuds, il s'agit de calculer les vecteurs propres généralisés pour les matrices  $(H_p^B, H_1^B)$ , où  $p$  est un polynôme séparent de degré 1. En supposant que les polynômes  $1, x_1, \dots, x_n$  ont été utilisés pour construire la base  $B$  de  $\mathbb{R}[x]/I_{\Lambda}$ , ces vecteurs propres généralisés contiennent les coordonnées des noeuds. Les poids sont ensuite calculés comme solution d'un système linéaire de type Vandermonde.

A l'aide d'un exemple - la cubature  $H2 : 5 - 1$  dans [101] - nous avons constaté que cette méthode par le calcul exact montre rapidement ses limites : si le degré de cubature  $d$  est trop important, le nombre de paramètres et la taille des matrices deviennent trop importants. La symétrie, qui est un ingrédient naturel pour la recherche de cubatures, joue alors un rôle important dans la réduction du nombre de paramètres et de la taille des matrices considérées.

L'utilisation de la symétrie se fait ici au travers de la théorie des représentations linéaires pour les groupes finis [33, 98]. Cette théorie permet de décomposer un  $\mathbb{K}$ -espace vectoriel ( $\mathbb{K} = \mathbb{R}$  ou  $\mathbb{C}$ ) en sous-espaces vectoriels invariants selon l'action d'un groupe fini  $G$ . Cette décomposition, réalisée à partir des représentations irréductibles inéquivalentes du groupe fini  $G$ , est appelée isotypique dans le cas  $\mathbb{K} = \mathbb{C}$ . La théorie des représentations fournit également un algorithme pour le calcul de bases respectant cette décomposition. Cependant, les bases adaptées à la symétrie et le théorème pour les endomorphismes équivariants introduits dans [33] ne sont ici pas appropriés. En effet, une condition nécessaire serait l'orthogonalité des matrices pour les représentations sur l'anneau des polynômes comme dans [39, 94].



En partant de matrices orthogonales pour les représentations irréductibles du groupe fini, nous définissons les *bases orthogonales adaptées à la symétrie* pour un  $\mathbb{R}$ -espace vectoriel. L'algorithme permettant de calculer les bases orthogonales adaptées à la symétrie est détaillé au Chapitre 2. Dans ces bases, nous prouvons que la matrice d'une forme bilinéaire symétrique  $G$ -invariante a une structure par blocs particulière. Cette dernière est conservée pour le cas de la matrice de l'opérateur de Hankel associé à une forme linéaire  $G$ -invariante.

**Théorème 6.** *Soit  $\widehat{\mathcal{V}}$  la représentation d'un groupe fini  $G$  sur l'espace  $\mathbb{R}[x]$  induite par une représentation  $\mathcal{W}$  sur  $\mathbb{R}^n$ . Soit  $\Lambda$  une forme linéaire  $G$ -invariante sur  $\mathbb{R}[x]$  et supposons que le rang de l'opérateur de Hankel  $\widehat{\mathcal{H}}$  associé à la forme linéaire  $\Lambda$  est fini. Soit  $\mathcal{V}$  la représentation induite sur  $\mathbb{R}[x]/I_\Lambda$ .*

*Considérons  $U_1, \dots, U_N$  les composantes dans la décomposition de  $\mathbb{R}[x]/I_\Lambda$  issue de la décomposition isotypique de  $\mathbb{R}[x]/I_\Lambda \otimes_{\mathbb{R}} \mathbb{C}$ . Alors, pour tout polynôme  $G$ -invariant  $p \in \mathbb{R}[x]$  et pour toute base  $B = B_1 \cup \dots \cup B_N$  qui respecte cette décomposition de  $\mathbb{R}[x]/I_\Lambda$ , la matrice  $H_p^B$  est diagonale par blocs.*

*Supposons qu'en plus, pour chaque composante  $U_j$ ,  $B_j$  soit orthogonale adaptée à la symétrie. Notons  $\pi_G$  l'opérateur de Reynolds, c'est-à-dire la projection sur l'espace vectoriel des polynômes  $G$ -invariants.*

1. *Pour chaque composante  $U_j$  associée à une représentation absolument irréductible  $\mathcal{V}^{(j)}$  de dimension  $n_j$  qui apparaît  $c_j$  fois dans la représentation  $\mathcal{V}$ , la base  $B_j$  peut s'écrire sous la forme*

$$B_j = \{b_{11}^j, \dots, b_{1c_j}^j, \dots, b_{n_j 1}^j, \dots, b_{n_j c_j}^j\}.$$

*Alors la sous-matrice de  $H_p^B$  relative à cette composante  $U_j$  consiste en une diagonale de  $n_j$  blocs identiques de taille  $c_j$ . Elle est donnée par*

$$I_{n_j} \otimes \left( \nu_{st}^j \right)_{1 \leq s, t \leq c_j} \text{ avec } \nu_{st}^j = \Lambda(\pi_G(pb_{1s}^j b_{1t}^j)).$$

2. *Pour chaque composante associée à une représentation irréductible de type complexe  $\mathcal{V}^{(j)}$  de dimension  $2n_j$  qui apparaît  $c_j$  fois dans la représentation  $\mathcal{V}$ , la base  $B_j$  peut s'écrire sous la forme*

$$B_j = \{a_{11}^j, \dots, a_{1c_j}^j, b_{11}^j, \dots, b_{1c_j}^j, \dots, a_{n_j 1}^j, \dots, a_{n_j c_j}^j, b_{n_j 1}^j, \dots, b_{n_j c_j}^j\}.$$

*Alors la sous-matrice de  $H_p^B$  relative à la composante  $U_j$  consiste en une diagonale de  $n_j$  blocs identiques de taille  $2c_j$ . Elle est donnée par*

$$I_{n_j} \otimes \begin{pmatrix} S_j & -A_j \\ A_j & S_j \end{pmatrix},$$

*où  $S_j = \left( \Lambda(\pi_G(pa_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = \left( \Lambda(\pi_G(pb_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j}$  est une matrice symétrique et  $A_j = \left( \Lambda(\pi_G(pa_{1s}^j b_{1t}^j)) \right)_{1 \leq s, t \leq c_j} = - \left( \Lambda(\pi_G(pb_{1s}^j a_{1t}^j)) \right)_{1 \leq s, t \leq c_j}$  est une matrice anti-symétrique.*

La théorie des groupes permet également de simplifier l'approche par matrices de moments pour la recherche de cubatures. Les noeuds d'une cubature  $G$ -invariante forment une union d'orbites pour l'action du groupe  $G$  sur  $\mathbb{R}^n$ . Chaque noeud est associé à un type d'orbites, c'est-à-dire au sous-groupe d'isotropie du noeud à conjugaison près. La taille des blocs, ou de manière équivalente les multiplicités  $c_j$  des représentations irréductibles  $\mathcal{V}^{(j)}$  du groupe  $G$  (cf. Théorème 6), peut être liée à l'organisation des noeuds en types d'orbites.

**Théorème 7.** *Soit  $\Lambda$  une forme linéaire  $G$ -invariante définie sur  $\mathbb{R}[x]$  par (13.2). La représentation induite  $\mathcal{V}$  sur  $\mathbb{R}[x]/I_\Lambda$  est équivalente à la représentation par permutations associée à l'ensemble invariant  $V_{\mathbb{C}}(I_\Lambda)$ .*

Ce lien est explicité grâce à l'introduction de la matrice de multiplicités  $\Gamma_G = (\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$ . Ses colonnes contiennent les multiplicités  $\gamma_{jk}$  des  $N$  représentations irréductibles  $\mathcal{V}_j$  (sur  $\mathbb{C}$ ) du groupe  $G$  dans les  $T$  types  $\mathcal{P}_k$  du groupe  $G$ . Ces derniers sont définis comme les représentations par permutations des cosets  $G/H_k$ , où  $H_1, \dots, H_T$  sont les sous-groupes de  $G$  à conjugaison près.

**Corollaire 8.** *Soit  $\Lambda$  une forme linéaire  $G$ -invariante définie sur  $\mathbb{R}[x]$  par (13.2) et soit  $\mathcal{V}$  la représentation induite sur  $\mathbb{R}[x]/I_\Lambda$ .*

*Soit  $\Gamma_G = (\gamma_{jk})_{1 \leq j \leq N, 1 \leq k \leq T}$  la matrice de multiplicités du groupe  $G$  et, pour tout  $k = 1, \dots, T$ , soit  $m_k$  le nombre d'orbites distinctes dans l'ensemble invariant  $V_{\mathbb{C}}(I_\Lambda)$  associées avec le type  $\mathcal{P}_k$ . La multiplicité  $\gamma_j$  de la représentation irréductible  $\mathcal{V}^{(j)}$  de  $G$  dans la représentation  $\mathcal{V}$  est*

$$\gamma_j = \sum_{k=1}^T m_k \gamma_{jk}.$$

Le calcul des matrices de multiplicités  $\Gamma_G$  est réalisé pour les groupes cycliques  $C_m$  et diédraux  $D_m$ .

Nous pouvons alors compléter notre algorithme à l'aide des résultats relatifs à la symétrie.

Premièrement, nous pouvons réduire le nombre possible de noeuds  $r$  pour une mesure  $\mu$  et un degré  $d$  donnés. Il suffit de calculer toutes les organisations de noeuds en types d'orbites. Pour chacune d'elles, Corollaire 8 donne les multiplicités souhaitées  $\gamma_j$  pour chaque représentations irréductibles  $\mathcal{V}^{(j)}$  avec  $j = 1, \dots, N$ . En les comparant avec les multiplicités  $c'_j$  des représentations irréductibles  $\mathcal{V}^{(j)}$  dans  $\mathbb{R}[x]_{\leq \lfloor \frac{d}{2} \rfloor}$ , nous obtenons une première condition nécessaire à l'existence d'une cubature sous la forme d'inégalités. La vérification de cette première condition nécessaire, pouvant se faire avant tout autre calcul, est un atout puissant pour la recherche du nombre possible de noeuds  $r$  pour une cubature symétrique.

Deuxièmement, le choix du degré  $\delta$  tel que la forme linéaire  $\Lambda^{(\delta)}$  sur  $\mathbb{R}[x]_{\leq 2\delta}$  soit une extension plate de sa restriction  $\Lambda^{(\delta-1)}$  sur  $\mathbb{R}[x]_{\leq 2\delta-2}$  est fait grâce au calcul des multiplicités  $c''_j$  des représentations irréductibles  $\mathcal{V}^{(j)}$  dans la représentation sur  $\mathbb{R}[x]_{\leq \delta-1}$ . À noter que, comme la matrice de l'opérateur de Hankel est diagonale par blocs, nous pouvons appliquer à chaque bloc la procédure de diagonalisation au lieu de l'appliquer à toute la matrice (cf. ci-dessus la description de l'algorithme sans symétrie). Ceci réduit grandement le nombre de sorties possibles pour cette procédure. De plus, les paramètres inconnus sont moins nombreux, car

chacun est associé à un polynôme  $G$ -invariant dont le degré est compris entre  $d$  et  $2\delta$ . Par conséquent, l'utilisation de la symétrie que nous proposons permet de simplifier les conditions d'existence d'une cubature symétrique en comparaison avec l'algorithme proposé initialement ou [35].

Troisièmement, une fois l'existence d'une cubature établie, le calcul des noeuds et des poids peut être effectué à l'aide uniquement d'un bloc des matrices  $H_1^B, H_{p_1}^B, \dots, H_{p_\eta}^B$ , où  $\{p_1, \dots, p_\eta\}$  est un ensemble séparent de polynômes  $G$ -invariants. La résolution des problèmes d'algèbre linéaire sous-jacents (valeurs propres généralisées et système linéaire de type Vandermonde) est donc de taille réduite. Une résolution en arithmétique exacte est même possible dans certains cas.

Grâce à ce nouvel algorithme pour la recherche de cubatures symétriques, nous avons retrouvé des cubatures existantes parmi celles reportées dans [37, 51, 101] (Figure 27). D'autres sont apparues pendant notre recherche grâce à notre méthode qui permet de retrouver toutes les cubatures symétriques d'une mesure  $\mu$  et d'un degré  $d$  donnés (Figures 28 et 29). Nous avons aussi pu expliciter des cubatures  $D_6$ -invariantes intérieures et à poids positifs pour l'hexagone régulier  $H2$  de degré 13 (Figure 30), alors que le degré maximal de telles cubatures était jusqu'alors 11 [51].

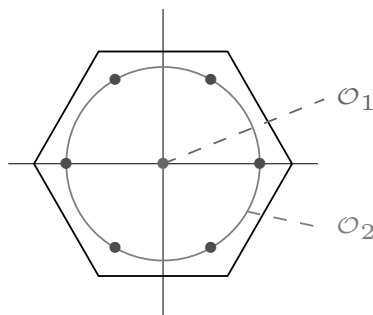
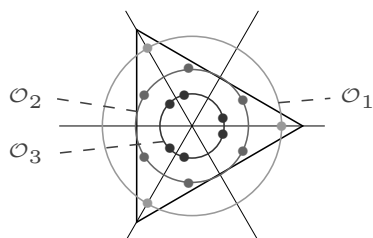
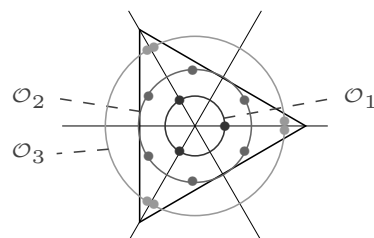


Figure 27: Organisation des noeuds en orbites pour la cubature  $H2 : 5 - 1$

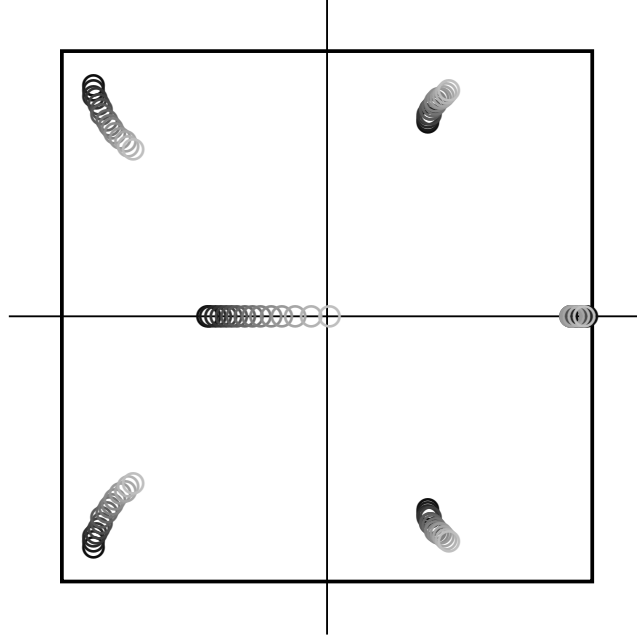


(a) Organisation des noeuds en orbites :  $\mathcal{O}_1$  l'orbite avec 3 noeuds,  $\mathcal{O}_2$  et  $\mathcal{O}_3$  les orbites avec 6 noeuds

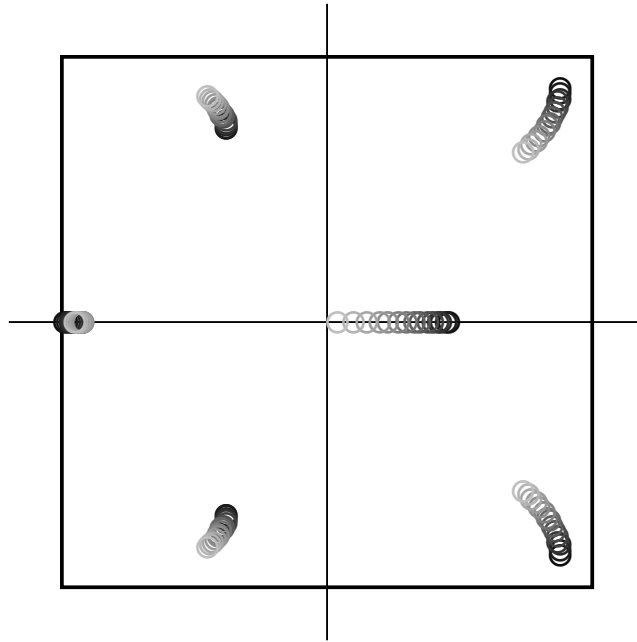


(b) Organisation des noeuds en orbites :  $\mathcal{O}_1$  l'orbite avec 3 noeuds,  $\mathcal{O}_2$  et  $\mathcal{O}_3$  les orbites avec 6 noeuds

Figure 28: Noeuds des deux cubatures  $D_3$ -invariantes de degré 7 pour le triangle. Celle de droite est connue dans [76] et celle de gauche ne figure pas dans la littérature à notre connaissance.

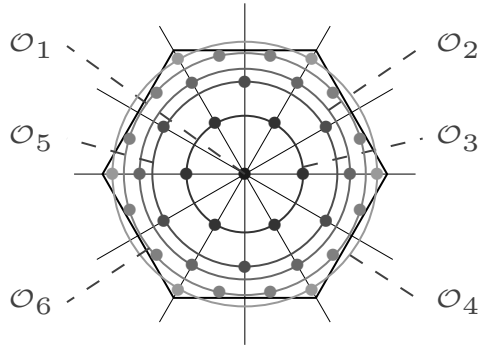


(a) Noeuds des cubatures pour différentes valeurs du degré de liberté dans le premier intervalle

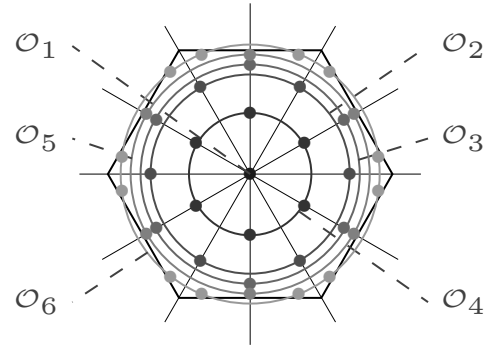


(b) Noeuds des cubatures pour différentes valeurs du degré de liberté dans le deuxième intervalle

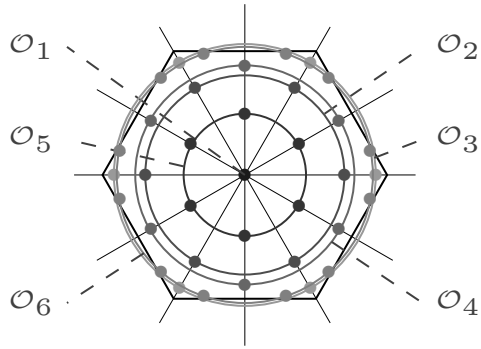
Figure 29: Noeuds des cubatures  $C_2$ —invariantes de degré 4 pour le carré. Dans [35], les auteurs signalent l'existence de nombreuses cubatures, mais n'en donnent explicitement qu'une seule en donnant des valeurs à 3 paramètres. Ces valeurs semblent choisies de manière arbitraire.



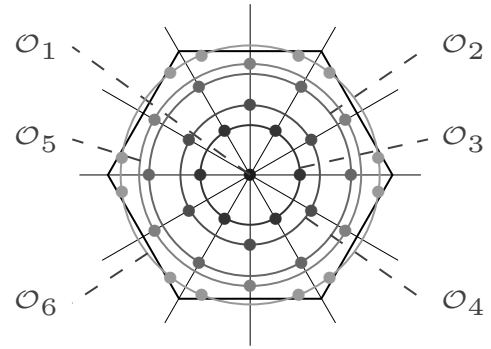
(a) Organisation des noeuds en types d'orbites :  $\mathcal{O}_1$  l'origine,  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  et  $\mathcal{O}_4$  les orbites avec 6 noeuds dont 2 noeuds sur l'axe des abscisses,  $\mathcal{O}_5$  l'orbite à 6 noeuds dont 2 noeuds sur l'axe des ordonnées et  $\mathcal{O}_6$  l'orbite à 12 noeuds



(b) Organisation des noeuds en types d'orbites :  $\mathcal{O}_1$  l'origine,  $\mathcal{O}_2$  l'orbite à 6 noeuds dont 2 noeuds sur l'axe des abscisses,  $\mathcal{O}_3$ ,  $\mathcal{O}_4$  et  $\mathcal{O}_5$  les orbites à 6 noeuds dont 2 noeuds sur l'axe des ordonnées et  $\mathcal{O}_6$  l'orbite à 12 noeuds



(c) Organisation des noeuds en types d'orbites :  $\mathcal{O}_1$  l'origine,  $\mathcal{O}_2$  et  $\mathcal{O}_3$  les orbites avec 6 noeuds dont 2 noeuds sur l'axe des abscisses,  $\mathcal{O}_4$  et  $\mathcal{O}_5$  les orbites à 6 noeuds dont 2 noeuds sur l'axe des ordonnées et  $\mathcal{O}_6$  l'orbite à 12 noeuds



(d) Organisation des noeuds en types d'orbites :  $\mathcal{O}_1$  l'origine,  $\mathcal{O}_2$  et  $\mathcal{O}_3$  les orbites à 6 noeuds dont 2 noeuds sur l'axe des abscisses,  $\mathcal{O}_4$  et  $\mathcal{O}_5$  les orbites à 6 noeuds dont 2 noeuds sur l'axe des ordonnées et  $\mathcal{O}_6$  l'orbite à 12 noeuds

Figure 30: Noeuds des cubatures de degré 13 pour l'hexagone régulier  $H2$



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