Algèbres KLR et algèbres de carquois-Schur

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Algèbres KLR et algèbres de carquois-Schur

Résumé

Dans la première partie de la thèse on construit une base de la partie positive $\mathbf{f}$ de l'algèbre quantique de Drinfeld-Jimbo associée à un carquois de Dynkin. Cette base est donnée par des faisceaux de parité en caractéristique quelconque. De plus, on catégorifie la multiplication et la comultiplication dans $\mathbf{f}$ avec des complexes de faisceaux.

Dans la deuxième partie il s'agit de la dualité de Koszul. Stroppel et Webster ont introduit une graduation sur la q-algèbre de Schur cyclotomique $S_d^{\mathbf{s}}$. On démontre que l'algèbre graduée obtenue est Morita équivalente (au sens gradué) à une algèbre de Koszul. La preuve est basée sur le résultat de Rouquier, Shan, Varagnolo et Vasserot qui identifie la catégorie $\mathrm{mod}(S_d^{\mathbf{s}})$ avec une catégorie $\mathcal{O}$ affine parabolique de type A. Cette catégorie admet une graduation de Koszul construite par Shan, Varagnolo et Vasserot. On identifie cette graduation avec la graduation sur $\mathrm{mod}(S_d^{\mathbf{s}})$.

Dans la troisième partie de la thèse on étudie une représentation catégorique dans la catégorie $\mathcal{O}$ parabolique pour la version affine de $\mathfrak{gl}_N$ de niveau $-N-e$. Cette catégorie admet une structure de $\widetilde{\mathfrak{sl}}_e$-représentation catégorique avec des foncteurs E et F. Cette catégorie contient une sous-catégorie plus petite $\mathbf{A}$ qui catégorifie l'espace de Fock de niveau supérieur. On démontre que le foncteur F pour la catégorie $\mathbf{A}$ de niveau $-N-e$ se décompose en terme de composantes du foncteur F pour la catégorie $\mathbf{A}$ de niveau $-N-e-1$. Pour démontrer cela, on construit un isomorphisme entre l'algèbre KLR associée au carquois $A_{e-1}^{(1)}$ et un sous-quotient de l'algèbre KLR associée au carquois $A_e^{(1)}$.

Mots clefs : Algèbres KLR, algèbres de carquois-Schur, algèbres de Hecke, groupes quantiques, faisceaux de parité, base canonique, représentation catégorique.

KLR algebras and quiver Schur algebras

Abstract

In the first part of the thesis we give a construction of a basis of the positive part $\mathbf{f}$ of the Drinfeld-Jimbo quantum enveloping algebra associated with a Dynkin quiver. This basis is given in terms of parity sheaves over fields in any characteristic. We also categorify both the multiplication and the comultiplication of $\mathbf{f}$ via complexes of sheaves.

The second part deals with Koszul duality. Stroppel and Webster have introduced a grading on the cyclotomic q-Schur algebra $S_d^{\mathbf{s}}$. We prove that the obtained graded algebra is graded Morita equivalent to a Koszul algebra. The proof is based on a result of Rouquier, Shan, Varagnolo and Vasserot that identifies the category $\mathrm{mod}(S_d^{\mathbf{s}})$ with a subcategory of an affine parabolic category $\mathcal{O}$ of type A. This subcategory admits a Koszul grading constructed by Shan, Varagnolo and Vasserot. We identify this Koszul grading with the grading on $\mathrm{mod}(S_d^{\mathbf{s}})$.

In the third part of the thesis we study a categorical representation in the parabolic category $\mathcal{O}$ for affine $\mathfrak{gl}_N$ at level $-N-e$. This category admits a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with respect to some functors E and F. This category contains a smaller category $\mathbf{A}$ that caregorifies the higher level Fock space. We prove that the functor F for the category $\mathbf{A}$ at level $-N-e$ can be decomposed in terms of the components of the functor F for the category $\mathbf{A}$ at level $-N-e-1$. To prove this, we construct an isomorphism between the KLR algebra associated with the quiver $A_{e-1}^{(1)}$ and a subquotient of the KLR algebra associated with the quiver $A_e^{(1)}$.

Keywords: KLR algebras, quiver Schur algebras, Hecke algebras, quantum groups, category $\mathcal{O}$, parity sheaves, canonical basis, categorical representation.

Chapitre 1

Introduction

Soit A un anneau. On note $\mathrm{mod}(A)$ la catégorie des A-modules de type fini. Soit A un anneau gradué. On note $\mathrm{grmod}(A)$ la catégorie des A-modules gradués de type fini. On note aussi $\mathrm{grproj}(A)$ la catégorie des modules projectifs dans $\mathrm{grmod}(A)$. Pour une catégorie abélienne $\mathcal{C}$ on note $[\mathcal{C}]$ son groupe de Grothendieck complexifié.

1.1 Les bases canoniques

1.1.1 Le groupe quantique f

Soit $\mathcal{A}$ l'anneau $\mathbb{Z}[q, q^{-1}]$. Soit I un ensemble fini et $(a_{i,j})_{i,j\in I}$ une matrice de Cartan généralisée symétrisable. Pour $n \in \mathbb{N}$ on pose $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathcal{A}$ et $[n]! = \prod_{r=1}^{n} [r]$. Soit $\mathbb{Z}I$ le $\mathbb{Z}$-module libre avec une base paramétrée par les éléments de I. Un élément de $\mathbb{Z}I$ est une combinaison $\mathbb{Z}$-linéaire formelle $\mathbf{d} = \sum_{i\in I} d_i \cdot i$ d'éléments de i. Soit $\mathbb{N}I$ le sous-ensemble de $\mathbb{Z}I$ formé des $\mathbf{d} \in \mathbb{Z}I$ à coefficients dans $\mathbb{N}$.

Soit $'\mathbf{f}$ la $\mathbb{Q}(q)$-algèbre libre engendrée par $\{\theta_i;\ i \in I\}$. L'algèbre $'\mathbf{f}$ est graduée par le monoïde $\mathbb{N}I$ de telle manière que θ_i est de degré i :

$$'\mathbf{f} = \bigoplus_{\mathbf{d}\in\mathbb{N}I} {'\mathbf{f}_{\mathbf{d}}}.$$

Pour $\mathbf{d} = \sum_{i\in I} d_i \cdot i \in \mathbb{N}I$ on pose $|\mathbf{d}| = \sum_{i\in I} d_i$. Pour un élément homogène $x \in \mathbf{f_d}$ on pose aussi $|x| = |\mathbf{d}|$.

Le produit tensoriel $'\mathbf{f} \otimes {'\mathbf{f}}$ (sur $\mathbb{Q}(q)$) a une structure d'algèbre donnée par

$$(x_1 \otimes x_2)(x'_1 \otimes x'_2) = q^{-|x_2||x'_1|} x_1 x'_1 \otimes x_2 x'_2,$$

où $x_1, x_2, x'_1, x'_2 \in {'\mathbf{f}}$ sont homogènes. Cette algèbre est associative. Soit r : $'\mathbf{f} \to {'\mathbf{f}} \otimes {'\mathbf{f}}$ l'unique homomorphisme d'algèbres tel que $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ pour chaque $i \in I$.

Il existe une unique forme bilinéaire $(\bullet, \bullet)\colon {'\mathbf{f}} \times {'\mathbf{f}} \to \mathbb{Q}(q)$ telle que

(a) $(1,1)=1$,
(b) $(\theta_i,\theta_j)=\delta_{ij}(1-q^2)^{-1}$ pour chaque $i,j\in I$,
(c) $(x,y_1y_2)=(r(x),y_1\otimes y_2)$ pour chaque $x,y_1,y_2\in {}'\mathbf{f}$,
(d) $(x_1x_2,y)=(x_1\otimes x_2,y)$ pour chaque $x_1,x_2,y\in {}'\mathbf{f}$,
où la forme bilinéaire $(\bullet,\bullet)$ sur ${}'\mathbf{f}\otimes{}'\mathbf{f}$ est définie par

$$(x_1\otimes x_2,x_1'\otimes x_2')=(x_1,x_1')(x_2,x_2').$$

Cette forme bilinéaire est symétrique. Posons $\theta_i^{(a)}=\theta_i^a/[a]!$.

Lemme 1.1.1. *Le noyau* $\mathrm{rad}(\bullet,\bullet)$ *est linéairement engendré par les éléments suivants*

$$\sum_{a+b=1-a_{i,j}}\theta_i^{(a)}\theta_j\theta_i^{(b)},\qquad i\neq j,\ a\geqslant 0,b\geqslant 0.$$

□

Les propriétés (c), (d) assurent que $\mathrm{rad}(\bullet,\bullet)$ est un idéal bilatère de ${}'\mathbf{f}$.

Définition 1.1.2. L'algèbre $\mathbf{f}$ est le quotient de l'algèbre ${}'\mathbf{f}$ par $\mathrm{rad}(\bullet,\bullet)$.

Soit ${}_{\mathcal{A}}\mathbf{f}$ la $\mathcal{A}$-sous-algèbre de $\mathbf{f}$ engendrée par les éléments $\theta_i^{(a)}$, $i\in I$, $a\in\mathbb{N}$. La graduation de ${}'\mathbf{f}$ induit une graduation sur $\mathbf{f}$ et ${}_{\mathcal{A}}\mathbf{f}$. On pose :

$$\mathbf{f}=\bigoplus_{\mathbf{d}\in\mathbb{N}I}\mathbf{f}_{\mathbf{d}},\qquad {}_{\mathcal{A}}\mathbf{f}=\bigoplus_{\mathbf{d}\in\mathbb{N}I}{}_{\mathcal{A}}\mathbf{f}_{\mathbf{d}}.$$

L'algèbre $\mathbf{f}$ admet un automorphisme $\mathbb{Z}$-linéaire $\overline{\bullet}\colon\mathbf{f}\to\mathbf{f}$ tel que $\overline{\theta}_i=\theta_i$, $\overline{q}=q^{-1}$.

1.1.2 Les bases canoniques

Dans cette section on explique la définition de la base canonique de $\mathbf{f}$. Voir [38, Section 14] pour les détails.

Définition 1.1.3. Soit M un module sur un anneau A. On dit qu'une partie $B\subset M$ est une *base signée* s'il existe une base $B'\subset M$ telle que $B=B'\cup(-B')$.

Considérons l'ensemble

$$\mathcal{B}=\{x\in{}_{\mathcal{A}}\mathbf{f};\ \bar{x}=x,\ (x,x)\in(1+q\mathbb{Z}[[q]])\cap\mathbb{Q}(q)\}.$$

Théorème 1.1.4. *L'ensemble* $\mathcal{B}$ *est une base signée du* $\mathcal{A}$*-module* ${}_{\mathcal{A}}\mathbf{f}$ *et du* $\mathbb{Q}(q)$*-espace vectoriel* $\mathbf{f}$*. Chaque élément de* $\mathcal{B}$ *est homogène, c'est-à-dire que l'on a* $\mathcal{B}=\coprod_{\mathbf{d}\in\mathbb{N}I}\mathcal{B}_{\mathbf{d}}$*, où* $\mathcal{B}_{\mathbf{d}}=\mathcal{B}\cap\mathbf{f}_{\mathbf{d}}$*.* □

Maintenant on définit des sous-ensembles $\mathcal{B}_{i;n}$ et $\mathcal{B}_{i;\geqslant n}$ de $\mathcal{B}$, où $i\in I$, $n\in\mathbb{N}$.

Définition 1.1.5. Pour $i \in I$, $n \in \mathbb{N}$ on pose $\mathcal{B}_{i;\geqslant n} = \mathcal{B} \cap \theta_i^n \mathbf{f}$ et $\mathcal{B}_{i;n} = \mathcal{B}_{i;\geqslant n} - \mathcal{B}_{i;\geqslant n+1}$.

Theorem 1.1.6. (*a*) *L'ensemble $\mathcal{B}_{i;\geqslant n}$ est une base signée du $\mathbb{Q}(q)$-espace vectoriel $\theta_i^n \mathbf{f}$ et du $\mathcal{A}$-module $\sum_{n' : n' \geqslant n} \theta_i^{(n')} {}_{\mathcal{A}}\mathbf{f}$.*

(*b*) *Si $b \in \mathcal{B}_{i;0}$, alors il existe un unique élément $b' \in \mathcal{B}_{i;n}$ tel que $\theta_i^n b - b'$ est une combinaison linéaire d'éléments de $\mathcal{B}_{i;\geqslant n+1}$. De plus, l'application $\pi_{i,n} \colon \mathcal{B}_{i;0} \to \mathcal{B}_{i;n}$, $b \mapsto b'$ est une bijection.*

Maintenant on peut construire la base canonique $\mathbf{B}$. D'abord, on construit récursivement des sous-ensembles $\mathbf{B_d} \subset \mathcal{B}_\mathbf{d}$ en posant

- $\mathbf{B}_0 = \{1\}$,
- $\mathbf{B_d} = \bigcup_{i \in I, n>0; d_i > n} \pi_{i;n}(\mathbf{B}_{\mathbf{d}-ni} \cap \mathcal{B}_{i;0}) \quad \text{si } |\mathbf{d}| > 0.$

Finalement, on pose $\mathbf{B} = \coprod_{\mathbf{d} \in \mathbb{N}I} \mathbf{B_d}$.

Théorème 1.1.7. *L'ensemble $\mathbf{B}$ est une base du $\mathcal{A}$-module ${}_{\mathcal{A}}\mathbf{f}$ et du $\mathbb{Q}(q)$-espace vectoriel $\mathbf{f}$.* □

Cette base $\mathbf{B}$ est appelée la *base canonique* de $\mathbf{f}$.

1.2 Les algèbres de Kac-Moody

1.2.1 Les algèbres de Kac-Moody

Soit I un ensemble. Soit $(a_{i,j})_{i,j \in I}$ une matrice de Cartan généralisée symétrisable.

Définition 1.2.1. L'algèbre de Kac-Moody $\mathfrak{g}_I$ dérivée associée à Γ est l'algèbre de Lie (sur $\mathbb{C}$) de générateurs e_i, f_i, h_i pour $i \in I$ modulo les relations

$$[e_i, f_j] = \delta_{i,j} h_i, \quad [h_i, e_j] = a_{i,j} e_j, \qquad [h_i, f_j] = -a_{i,j} e_j, \qquad [h_i, h_j] = 0,$$

$$\mathrm{ad}(e_i)^{1-a_{i,j}}(e_j) = 0, \qquad \mathrm{ad}(f_i)^{1-a_{i,j}}(f_j) = 0, \qquad \text{pour } i \neq j.$$

Soit $\{\Lambda_i;\ i \in I\}$ l'ensemble des poids fondamentaux de $\mathfrak{g}_I$. Posons $P_I = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$, $P_I^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i$. On considère aussi un $\mathbb{Z}$-module libre $X_I = \bigoplus_{i \in I} \mathbb{Z}\varepsilon_i$ de base $\{\varepsilon_i;\ i \in I\}$ et on pose $X_I^+ = \bigoplus_{i \in I} \mathbb{N}\varepsilon_i$.

Définition 1.2.2. Le groupe quantique $U_q(\mathfrak{g}_I)$ associé à Γ est la $\mathbb{Q}(q)$-algèbre de générateurs E_i, F_i, $K_i^{\pm}$ pour $i \in I$ modulo les relations

$$K_i K_j = K_j K_i, \qquad K_i E_j = q^{a_{i,j}} E_j K_i, \qquad K_i F_j = q^{-a_{i,j}} F_j K_i,$$

$$K_i K_i^{-1} = 1, \qquad [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$\sum_{r=0}^{1-a_{i,j}}(-1)^r E_i^{(r)}E_jE_i^{(1-a_{i,j}-r)}=0 \qquad \text{pour } i\neq j,$$
$$\sum_{r=0}^{1-a_{i,j}}(-1)^r F_i^{(r)}F_jF_i^{(1-a_{i,j}-r)}=0 \qquad \text{pour } i\neq j.$$

On a posé $E_i^{(a)}=E_i^a/[a]!$ et $F_i^{(a)}=F_i^a/[a]!$.

Si $\Lambda\in P_I^+$, soit $L(\Lambda)$ (resp. $L_q(\Lambda)$) le module simple de $\mathfrak{g}_I$ (resp. $U_q(\mathfrak{g}_I)$) de plus haut poids Λ.

1.2.2 Combinatoire

Soit n un entier positif. Une *composition* de n est un uplet $\lambda=(\lambda_1,\cdots,\lambda_k)$ tel que $\sum_{r=1}^k\lambda_r=n$, λ_r est un entier strictement positif pour chaque $r\in[1,k]$. On va noter $\mathcal{C}_n$ l'ensemble des compositions de n.

Une composition $\lambda=(\lambda_1,\cdots,\lambda_k)$ est une *partition* si on a $\lambda_1\geqslant\cdots\geqslant\lambda_k$. On va noter $\mathcal{P}_n$ l'ensemble des partitions de n. Pour $\lambda=(\lambda_1,\cdots,\lambda_k)\in\mathcal{P}_n$ on pose $\ell(\lambda)=k$ et $|\lambda|=n$. On pose aussi $\mathcal{P}=\coprod_{n\in\mathbb{N}}\mathcal{P}_n$ et $\mathcal{C}=\coprod_{n\in\mathbb{N}}\mathcal{C}_n$.

Soit l un entier strictement positif. Une l-*partition* (resp. l-*composition*) de n est un l-uplet $\lambda=(\lambda^{(1)},\cdots,\lambda^{(l)})$ de partitions (resp. compositions) telles que $\sum_{r=1}^l|\lambda^{(r)}|=n$. On va noter $\mathcal{P}_n^l$ (resp. $\mathcal{C}_n^l$) l'ensemble des l-partitions (resp. l-compositions) de n.

Pour une composition $\lambda=(\lambda_1,\cdots,\lambda_k)$ de n notons $\mathfrak{S}_\lambda$ le sous-groupe de Young $\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_k}\subset\mathfrak{S}_n$. Pour une l-composition $\lambda=(\lambda^{(1)},\cdots,\lambda^{(l)})$ de n soit $\mathfrak{S}_\lambda$ le sous-groupe de Young $\mathfrak{S}_{\lambda^{(1)}}\times\cdots\times\mathfrak{S}_{\lambda^{(l)}}\subset\mathfrak{S}_n$.

1.2.3 Représentations de $\widetilde{\mathfrak{sl}}_e$

Dans cette section on suppose que $(a_{i,j})_{i,j\in I}$ est la matrice de Cartan de type $\widehat{A}_{e-1}$. On a $\mathfrak{g}_I=\widetilde{\mathfrak{sl}}_e=\mathfrak{sl}_e\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{1}$. On introduit quelques représentations de l'algèbre de Lie affine $\widetilde{\mathfrak{sl}}_e$. On va identifier l'ensemble I avec $\mathbb{Z}/e\mathbb{Z}$.

Soit V_e un $\mathbb{C}$-espace vectoriel avec une base $\{v_1,\cdots,v_e\}$. Posons $U_e=V_e\otimes\mathbb{C}[t,t^{-1}]$. L'espace vectoriel U_e est muni de la base $\{u_r;\ r\in\mathbb{Z}\}$ telle que $u_{a+eb}=v_a\otimes t^{-b}$ pour $a\in[1,e]$, $b\in\mathbb{Z}$. L'espace vectoriel U_e a une structure de $\widetilde{\mathfrak{sl}}_e$-module de niveau 0 définie par

$$f_i(u_r)=\delta_{i\equiv r}u_{r+1},\quad e_i(u_r)=\delta_{i\equiv r-1}u_{r-1}.$$

Ici on considère des congruences modulo e.

Définition 1.2.3. Soit m un entier. L'*espace de Fock de niveau* 1 est un $\mathbb{C}$-espace vectoriel $\mathcal{F}^m$ linéairement engendrée par les wedges semi-infinis $u_{r_1}\wedge u_{r_2}\wedge u_{r_3}\wedge\cdots$ tels que $r_k=m-k+1$ sauf pour un nombre fini d'indices k. Cet espace vectoriel admet une $\widetilde{\mathfrak{sl}}_e$-action de niveau 1.

À chaque wedge semi-infini on associe une partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ comme suit. Soit t le dernier indice tel que l'on a $r_t > m-t+1$. Pour $k \in [1,t]$ on pose $\lambda_k = r_k - m + k - 1$. Pour telle partition $\lambda = (\lambda_1, \cdots, \lambda_t) \in \mathcal{P}$ on pose $|\lambda, m\rangle = u_{r_1} \wedge u_{r_2} \wedge u_{r_3} \wedge \cdots$. Les éléments $|\lambda, m\rangle$ pour $\lambda \in \mathcal{P}$ forment une base de $\mathcal{F}^m$.

Définition 1.2.4. Soit $\mathbf{m} = (m_1, \cdots, m_l)$ un l-uplet d'entiers. L'*espace de Fock de niveau l* est le $\widetilde{\mathfrak{sl}}_e$-module $\mathcal{F}^{\mathbf{m}} = \mathcal{F}^{m_1} \otimes \cdots \otimes \mathcal{F}^{m_l}$.

Pour une l-partition $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(l)}) \in \mathcal{P}^l$ on pose

$$|\lambda, \mathbf{m}\rangle = |\lambda^{(1)}, m_1\rangle \otimes \cdots \otimes |\lambda^{(l)}, m_l\rangle.$$

Les éléments $|\lambda, \mathbf{m}\rangle$ avec $\lambda \in \mathcal{P}^l$ forment une base de $\mathcal{F}^{\mathbf{m}}$.

L'espace de Fock $\mathcal{F}^m$ de niveau 1 admet une version quantique $\mathcal{F}_q^m$ qui est un $U_q(\widetilde{\mathfrak{sl}}_e)$-module. L'espace de Fock $\mathcal{F}^{\mathbf{m}}$ de niveau l admet deux versions quantiques :

- le produit tensoriel $\mathcal{F}_q^{\mathbf{m}} := \mathcal{F}_q^{m_1} \otimes \cdots \otimes \mathcal{F}_q^{m_l}$,
- la version quantique $\widetilde{\mathcal{F}}_q^{\mathbf{m}}$ introduite par Uglov dans [61] qui dépend d'un paramètre appelé l-charge.

1.3 Les algèbres de Hecke et de Schur

1.3.1 Les algèbres de Hecke

Soit $\mathbf{k}$ un corps commutatif. Fixons $\zeta \in \mathbf{k}$. Soit d un entier positif.

Définition 1.3.1. L'*algèbre de Hecke* ${}^0H_d(\zeta)$ de $\mathfrak{S}_d$ est la $\mathbf{k}$-algèbre engendrée par $T_1, \cdots, T_{d-1}$ modulo les relations

$$\begin{aligned}
&T_rT_s = T_sT_r \text{ si } |r-s| > 1,\\
&T_rT_{r+1}T_r = T_{r+1}T_rT_{r+1}, \quad (T_r - \zeta)(T_r + 1) = 0.
\end{aligned}$$

Définition 1.3.2. L'*algèbre de Hecke affine* $H_d(\zeta)$ est la $\mathbf{k}$-algèbre engendrée par $T_1, \cdots, T_{d-1}$, $X_1, \cdots, X_d$, $X_1^{-1}, \cdots, X_d^{-1}$ modulo les relations

$$\begin{array}{lll}
T_rT_s = T_sT_r \text{ si } |r-s| > 1, & T_rT_{r+1}T_r = T_{r+1}T_rT_{r+1}, & X_rX_r^{-1} = X_r^{-1}X_r = 1,\\
(T_r - \zeta)(T_r + 1) = 0, & T_rX_r = X_rT_r \ \text{ si } |r-s| > 1, & \\
T_rX_{r+1} = X_rT_r + (\zeta - 1)X_{r+1}, & T_rX_r = X_{r+1}T_r - (\zeta - 1)X_{r+1}. &
\end{array}$$

L'algèbre $H_d(\zeta)$ contient l'algèbre ${}^0H_d(\zeta)$ comme une sous-algèbre. Supposons $\zeta \neq 0, 1$. Soit $l > 1$ un entier. Fixons un l-uplet d'entiers $\mathbf{s} = (s_1, \cdots, s_l)$. Pour tout $r \in [1,l]$ on pose $Q_r = \zeta^{s_r}$.

Définition 1.3.3. L'*algèbre de Hecke cyclotomique* $H_d^{\mathbf{s}}(\zeta)$ est le quotient de $H_d(\zeta)$ par l'idéal engendré par $(X_1 - Q_1) \cdots (X_l - Q_l)$.

Pour chaque $d \in \mathbb{N}$ il existe un homomorphisme d'algèbres

$$H_d^{\mathbf{s}}(\zeta) \to H_{d+1}^{\mathbf{s}}(\zeta), \quad X_r \mapsto X_r, \quad T_k \mapsto T_k, \quad \forall r \in [1,d], \quad k \in [1, d-1].$$

Soient

$$E\colon \mathrm{mod}(H_{d+1}^{\mathbf{s}}(\zeta)) \to \mathrm{mod}(H_d^{\mathbf{s}}(\zeta)), \quad F\colon \mathrm{mod}(H_d^{\mathbf{s}}(\zeta)) \to \mathrm{mod}(H_{d+1}^{\mathbf{s}}(\zeta))$$

les foncteurs d'induction et de restriction relatifs à cet homomorphisme.

Soit $\mathscr{F} \subset \mathbf{k}$ la partie formée des éléments de la forme ζ^n avec $n \in \mathbb{Z}$. On peut voir $\mathscr{F}$ comme l'ensemble des sommets d'un carquois avec une flèche $i \to j$ si et seulement si on a $j = \zeta i$. Si ζ est une racine primitive e-ième de l'unité alors on a $\mathfrak{g}_{\mathscr{F}} = \widetilde{\mathfrak{sl}}_e$. On considère le poids $\Lambda = \sum_{r=1}^{l} \Lambda_{s_r}$ de $\mathfrak{g}_{\mathscr{F}}$. Le théorème suivant est démontré par Ariki [1].

Théorème 1.3.4. *Il existe un isomorphisme de $\mathbb{C}$-espaces vectoriels $L(\Lambda) \simeq \bigoplus_{d \in \mathbb{N}} [\mathrm{mod}(H_d^{\mathbf{s}}(\zeta))]$ et des décompositions $E = \bigoplus_{i \in \mathscr{F}} E_i$, $F = \bigoplus_{i \in \mathscr{F}} F_i$ tels que l'action des foncteurs E_i, F_i sur $\bigoplus_{d \in \mathbb{N}} [\mathrm{mod}(H_d^{\mathbf{s}})]$ pour $i \in \mathscr{F}$ s'identifie avec l'action des générateurs e_i et f_i de $\mathfrak{g}_{\mathscr{F}}$ sur $L(\Lambda)$.* □

1.3.2 Les ζ-algèbres de Schur

Pour chaque $r \in [1, d-1]$ notons t_r la transposition $(r, r+1) \in \mathfrak{S}_d$. Pour une permutation $w \in \mathfrak{S}_d$ qui a une expression réduite $w = t_{j_1} \cdots t_{j_r}$ on pose $T_w = T_{j_1} \cdots T_{j_r}$. L'élément T_w ne dépend pas du choix de l'expression réduite de w. Supposons que $\mathbf{a} = (a_1, \cdots, a_l)$ est un l-uplet d'entiers dans $[1, d]$. Posons aussi $u_{\mathbf{a}}^{+} = u_{\mathbf{a},1} u_{\mathbf{a},2} \cdots u_{\mathbf{a},l}$ avec $u_{\mathbf{a},k} = \prod_{r=1}^{a_k} (L_r - Q_k)$. A une l-composition λ de d on associe le l-uplet $\mathbf{a} = (a_1, \cdots, a_l)$ défini par $a_k = \sum_{r=1}^{k-1} |\lambda^{(r)}|$. On pose $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$ et $m_\lambda = u_{\mathbf{a}}^{+} x_\lambda$.

Definition 1.3.5. La *ζ-algèbre de Schur* $S_d^{\mathbf{s}}(\zeta)$ est la $\mathbf{k}$-algèbre

$$\mathrm{End}_{(H_d^{\mathbf{s}}(\zeta))^{\mathrm{op}}}\Big(\bigoplus_{\lambda \in \mathcal{C}_d^l} m_\lambda H_d^{\mathbf{s}}(\zeta)\Big).$$

La ζ-algèbre de Schur est une algèbre quasi-héréditaire. Les objets standards sont appelés les *modules de Weyl*. Ils sont paramétrés par l'ensemble des l-partitions de d.

Remarque 1.3.6. Soit ζ une racine e-ième primitive de l'unité. On peut identifier $\mathscr{F}$ avec $\mathbb{Z}/e\mathbb{Z}$. De même manière que pour les algèbres de Hecke cyclotomiques, pour chaque $d \in \mathbb{N}$ on a des foncteurs de restriction et d'induction

$$E\colon \mathrm{mod}(S_{d+1}^{\mathbf{s}}(\zeta)) \to \mathrm{mod}(S_d^{\mathbf{s}}(\zeta)), \quad F\colon \mathrm{mod}(S_d^{\mathbf{s}}(\zeta)) \to \mathrm{mod}(S_{d+1}^{\mathbf{s}}(\zeta)).$$

Ces foncteurs se décomposent $E = \bigoplus_{i \in \mathscr{F}} E_i$, $F = \bigoplus_{i \in \mathscr{F}} F_i$ et munissent la somme $\bigoplus_{d \in \mathbb{N}} [\mathrm{mod}(S_d^{\mathbf{s}}(\zeta))]$ d'une structure de $\mathfrak{g}_{\mathscr{F}}$-module qui est isomorphe à l'espace de Fock $\mathcal{F}^{\mathbf{s}}$. L'action des foncteurs E_i, F_i s'identifie avec l'action des générateurs e_i, f_i de $\mathfrak{g}_{\mathscr{F}}$.

1.4 Les algèbres KLR et carquois-Schur

1.4.1 Carquois

Soit Γ un carquois sans boucles. Notons I et H l'ensemble des sommets et des flèches de Γ. Pour une flèche $h \in H$ on écrit h', h'' son origine et son but. Pour $i, j \in I$ notons $h_{i,j}$ le nombre de flèches de i dans j. On considère la matrice de Cartan $(a_{i,j})_{i,j\in I}$, définie par

$$a_{i,j} = 2\delta_{i,j} - h_{i,j} - h_{j,i}.$$

Fixons un vecteur de dimension $\mathbf{d} = \sum_{i\in I} d_i \cdot i \in \mathbb{N}I$ et fixons un $\mathbb{C}$-espace vectoriel I-gradué $V = \oplus_{i\in I} V_i$ de dimension graduée $\mathbf{d}$. Notons $E_{\mathbf{d}}$ l'espace des représentations du carquois Γ dans V, i.e., on a

$$E_{\mathbf{d}} = \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''}).$$

Le groupe $G_{\mathbf{d}} = \prod_{i\in I} \mathrm{GL}(V_i)$ agit sur $E_{\mathbf{d}}$. Le résultat suivant est bien connu : supposons que le carquois Γ est de Dynkin (i.e. de type A_n, D_n, E_6, E_7 ou E_8). Alors l'ensemble des $G_{\mathbf{d}}$-orbites dans $E_{\mathbf{d}}$ est fini. Donc pour un carquois de Dynkin les $G_{\mathbf{d}}$-orbites dans $E_{\mathbf{d}}$ forment une stratification algèbrique de $E_{\mathbf{d}}$.

Notons Comp($\mathbf{d}$) l'ensemble des suites $\mu = (\mu_1, \cdots, \mu_k)$ des vecteurs de dimension $\mu_r \in \mathbb{N}I$ tels que $\sum_{r=1}^{k} \mu_r = \mathbf{d}$. Notons $I^{\mathbf{d}}$ (resp. $Y_{\mathbf{d}}$) le sous-ensemble des suites dans Comp($\mathbf{d}$) telles que chaque composante est un élément de I (resp. chaque composante est de la forme $a \cdot i$ pour $a \in \mathbb{Z}_{>0}, i \in I$). On a

$$I^{\mathbf{d}} = \{(i_1, i_2, \cdots, i_d) \in I^d;\ \sum_{r=1}^{d} i_r = \mathbf{d}\},$$

où $d = |\mathbf{d}|$. Pour chaque $\mu = (\mu_1, \cdots, \mu_k) \in \mathrm{Comp}(\mathbf{d})$ considérons l'ensemble F_μ des drapeaux de type μ dans V, c'est-à-dire que F_μ est l'ensemble des drapeaux

$$\mathcal{F} = (0 = V^0 \subset V^1 \subset \cdots \subset V^k = V)$$

tels que pour chaque $r \in [1, k]$ le sous-espace vectoriel V^r de V est homogène par rapport à la décomposition $V = \bigoplus_{i\in I} V_i$ et la dimension graduée de V^r/V^{r-1} est μ_r. On dit que l'élément $x \in E_{\mathbf{d}}$ préserve le drapeau $(0 = V^0 \subset V^1 \subset \cdots \subset V^k = V)$ si on a $x(V^r) \subset V^{r-1}$ pour chaque $r \in [1, k]$. Soit $\widetilde{F_\mu}$ la variété des couples $(x, \phi) \in E_{\mathbf{d}} \times F_\mu$ tels que x préserve ϕ. Soit $\pi_\mu \colon \widetilde{F_\mu} \to E_{\mathbf{d}}$ la projection. Pour tous $\mu, \xi \in \mathrm{Comp}(\mathbf{d})$ on considère la variété

$$Z_{\mu,\xi} = \{(x, \phi_1, \phi_2) \in E_{\mathbf{d}} \times F_\mu \times F_\xi;\ x \text{ préserve } \phi_1 \text{ et } \phi_2\}.$$

1.4.2 Les algèbres KLR

Soient $\Gamma = (I, H)$ un carquois sans boucles et d un entier positif. Pour $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$ on pose $t_r(\mathbf{i}) = (i_1, \cdots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \cdots, i_d)$. On utilise les notations $h_{i,j}$, $a_{i,j}$ de la Section 1.4.1.

Pour $i, j \in I$ on considère le polynôme suivant

$$Q_{i,j}(u,v) = \begin{cases} 0 & \text{si } i = j, \\ (v-u)^{h_{i,j}}(u-v)^{h_{j,i}} & \text{sinon.} \end{cases}$$

Définition 1.4.1. L'algèbre R_d est la $\mathbf{k}$-algèbre engendrée par $\tau_1, \cdots, \tau_{d-1}, x_1, \cdots, x_d, e(\mathbf{i})$ où $\mathbf{i} \in I^d$, modulo les relations suivantes

- $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i})$,
- $\sum_{\mathbf{i} \in I^d} e(\mathbf{i}) = 1$,
- $x_r e(\mathbf{i}) = e(\mathbf{i}) x_r$,
- $\tau_r e(\mathbf{i}) = e(t_r(\mathbf{i}))\tau_r$,
- $x_r x_s = x_s x_r$,
- $\tau_r x_{r+1} e(\mathbf{i}) = (x_r \tau_r + \delta_{i_r, i_{r+1}})e(\mathbf{i})$,
- $x_{r+1}\tau_r e(\mathbf{i}) = (\tau_r x_r + \delta_{i_r, i_{r+1}})e(\mathbf{i})$,
- $\tau_r x_s = x_s \tau_r$ si $s \neq r, r+1$,
- $\tau_r \tau_s = \tau_s \tau_r$ si $|r - s| > 1$,
- $\tau_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{si } i_r = i_{r+1}, \\ Q_{i_r, i_{r+1}}(x_r, x_{r+1})e(\mathbf{i}) & \text{sinon,} \end{cases}$
- $(\tau_r\tau_{r+1}\tau_r - \tau_{r+1}\tau_r\tau_{r+1})e(\mathbf{i}) =$
$\begin{cases} (x_{r+2} - x_r)^{-1}(Q_{i_r,i_{r+1}}(x_{r+2}, x_{r+1}) - Q_{i_r,i_{r+1}}(x_r, x_{r+1}))e(\mathbf{i}) & \text{si } i_r = i_{r+2}, \\ 0 & \text{sinon,} \end{cases}$

pour chaque $\mathbf{i}$, $\mathbf{j}$, r et s. L'algèbre R_d admet une $\mathbb{Z}$-graduation définie par $\deg e(\mathbf{i}) = 0$, $\deg x_r = 2$, $\deg \tau_s e(\mathbf{i}) = -a_{i_s, i_{s+1}}$, pour chaque $1 \leqslant r \leqslant d$, $1 \leqslant s < d$ et $\mathbf{i} \in I^d$.

Soit $\mathbf{d} \in \mathbb{N}I$ un vecteur de dimension tel que $|\mathbf{d}| = d$. On pose $e(\mathbf{d}) = \sum_{\mathbf{i} \in I^{\mathbf{d}}} e(\mathbf{i}) \in R_d$. C'est un idempotent de degré zéro dans R_d. Il y a une décomposition en somme directe d'algèbres unitaires

$$R_d = \bigoplus_{|\mathbf{d}|=d} R_{\mathbf{d}}, \qquad R_{\mathbf{d}} := e(\mathbf{d})R_d.$$

Fixons un poids $\Lambda = \sum_{i \in I} n_i \Lambda_i \in P_I^+$.

Definition 1.4.2. L'algèbre R_d^Λ (resp. $R_{\mathbf{d}}^\Lambda$) est le quotient de R_d (resp. $R_{\mathbf{d}}$) par l'idéal engendré par $x_r^{n_{i_1}} e(\mathbf{i})$ pour chaque $r \in [1, d-1]$, $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$ (resp. $\mathbf{i} \in I^{\mathbf{d}}$).

1.4.3 L'isomorphisme Hecke-KLR

Soit $\mathbf{s} = (s_1, \cdots, s_l)$ un l-uplet d'entiers comme dans la Section 1.3.1. On considère la partie $\mathscr{F} \subset \mathbf{k}$ et l'élément $\Lambda_{\mathbf{s}} = \Lambda_{\zeta^{s_1}} + \cdots + \Lambda_{\zeta^{s_l}} \in P_{\mathscr{F}}^+$. Le résultat suivant est démontré dans [8] et [50].

Theorem 1.4.3. *La* $\mathbf{k}$*-algèbre de Hecke cyclotomique* $H_d^{\mathbf{s}}(\zeta)$ *est isomorphe à l'algèbre KLR cyclotomique* $R_d^{\Lambda_{\mathbf{s}}}$ *associée au carquois* $\mathscr{F}$. □

Remarque 1.4.4. L'algèbre de Hecke cyclotomique $H_d^{\mathbf{s}}(\zeta)$ n'a pas de graduation évidente. Le théorème précédent donne une façon de graduer cette algèbre.

1.4.4 Algèbres KLR et catégorification

Soient $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{N}I$ deux vecteurs de dimension. Posons

$$\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2, \quad |\mathbf{d}_1| = d_1, \quad |\mathbf{d}_2| = d_2, \quad |\mathbf{d}| = d.$$

On considère le produit tensoriel (sur $\mathbf{k}$) $R_{\mathbf{d}_1} \otimes R_{\mathbf{d}_2}$. Il existe un homomorphisme injectif d'algèbres $R_{\mathbf{d}_1} \otimes R_{\mathbf{d}_2} \to R_{\mathbf{d}}$ tel que

$$e(\mathbf{i})\otimes e(\mathbf{j}) \mapsto e(\mathbf{ij}), \qquad x_k e(\mathbf{i})\otimes e(\mathbf{j}) \mapsto x_k e(\mathbf{ij}), \qquad e(\mathbf{i})\otimes x_k e(\mathbf{j}) \mapsto x_{d_1+k} e(\mathbf{ij}),$$

$$\tau_k e(\mathbf{i}) \otimes e(\mathbf{j}) \mapsto \tau_k e(\mathbf{ij}), \qquad e(\mathbf{i}) \otimes \tau_k e(\mathbf{j}) \mapsto \tau_{d_1+k} e(\mathbf{ij}),$$

Ici on note $\mathbf{ij} \in I^{\mathbf{d}}$ la concaténation de $\mathbf{i}$ et $\mathbf{j}$.

Pour un $R_{\mathbf{d}_1}$-module M et un $R_{\mathbf{d}_2}$-module N on note $\mathrm{Ind}_{\mathbf{d}_1,\mathbf{d}_2}^{\mathbf{d}}(M,N)$ l'induction du $R_{\mathbf{d}_1} \otimes R_{\mathbf{d}_2}$-module $M \otimes N$ à $R_{\mathbf{d}}$ par rapport à l'inclusion $R_{\mathbf{d}_1} \otimes R_{\mathbf{d}_2} \subset R_{\mathbf{d}}$. Il est facile de voir que si M, N sont des modules projectifs et de type fini alors $\mathrm{Ind}_{\mathbf{d}_1,\mathbf{d}_2}^{\mathbf{d}}(M,N)$ est aussi projectif et de type fini. Alors l'induction muni le groupe de Grothendieck $\bigoplus_{\mathbf{d}\in\mathbb{N}I} K(\mathrm{grproj}(R_{\mathbf{d}}))$ d'une structure de $\mathcal{A}$-algèbre (associative et unitaire).

Le théorème suivant est démontré par Khovanov-Lauda [35] et Rouquier [50].

Théorème 1.4.5. *Pour chaque* $\mathbf{d} \in \mathbb{N}I$*, il existe un isomorphisme de* $\mathcal{A}$*-algèbres* ${}_{\mathcal{A}}\mathbf{f} \simeq \bigoplus_{\mathbf{d}} K(\mathrm{grproj}(R_{\mathbf{d}}))$. □

La base canonique $\mathbf{B}$ (voir la Section 1.1.2) est une $\mathcal{A}$-base de ${}_{\mathcal{A}}\mathbf{f}$. Considèrons la $\mathbb{Z}$-base $\mathbf{B}_{\mathbb{Z}} = \{bq^n;\ b \in \mathbf{B}, n \in \mathbb{Z}\}$ de ${}_{\mathcal{A}}\mathbf{f}$. Le théorème suivant est démontré par Rouquier [51] et Varagnolo-Vasserot [62].

Théorème 1.4.6. *L'isomorphisme du Théorème 1.4.5 induit une bijection de l'ensemble des classes de modules projectifs indécomposables sur* $\mathbf{B}_{\mathbb{Z}}$. □

Fixons un poids $\Lambda \in P_I^+$. Pour chaque $i \in I$, $\mathbf{d} \in \mathbb{N}I$ il existe un homomorphisme d'algèbres $R_{\mathbf{d}}^{\Lambda} \to R_{\mathbf{d}+i}^{\Lambda}$. Soient

$$E_i \colon \mathrm{mod}(R_{\mathbf{d}+i}^{\Lambda}) \to \mathrm{mod}(R_{\mathbf{d}}^{\Lambda}), \quad F_i \colon \mathrm{mod}(R_{\mathbf{d}}^{\Lambda}) \to \mathrm{mod}(R_{\mathbf{d}+i}^{\Lambda})$$

les foncteurs de restriction et d'induction par rapport à cet homomorphisme. Le théorème suivant est démontré dans [31].

Théorème 1.4.7. *Il existe un isomorphisme de $\mathbb{C}(q)$-espaces vectoriels*

$$L_q(\Lambda) \simeq \bigoplus_{d\in\mathbb{N}} [\mathrm{grmod}(R_d^\Lambda)]$$

tel que l'action des foncteurs E_i, F_i sur $\bigoplus_{d\in\mathbb{N}}[\mathrm{grmod}(R_d^\Lambda)]$ pour $i \in I$ s'identifie avec l'action des générateurs E_i et F_i de $U_q(\mathfrak{g}_I)$ sur $L_q(\Lambda)$. $\square$

En vue du Théorème 1.4.3, le Théorème 1.4.7 est une version quantique du Théorème 1.3.4.

1.4.5 La construction géométrique de l'algèbre KLR

On considère les variétés

$$Z^c_{\mathbf{d}} = \coprod_{\mathbf{i},\mathbf{j}\in I^{\mathbf{d}}} Z_{\mathbf{i},\mathbf{j}}, \quad \widetilde{F}^c = \coprod_{\mathbf{i}\in I^{\mathbf{d}}} \widetilde{F}_{\mathbf{i}}.$$

Notons $H_*^{G_{\mathbf{d}}}(\bullet, \mathbf{k})$ l'homologie de Borel-Moore $G_{\mathbf{d}}$-équivariante à coefficients dans $\mathbf{k}$. L'espace vectoriel $\mathrm{H}_*^{G_{\mathbf{d}}}(Z^c_{\mathbf{d}}, \mathbf{k})$ admet une structure de $\mathbf{k}$-algèbre. Le produit dans $\mathrm{H}_*^{G_{\mathbf{d}}}(Z^c_{\mathbf{d}}, \mathbf{k})$ est le produit de convolution relatif à l'inclusion $Z^c_{\mathbf{d}} \subset \widetilde{F}^c \times \widetilde{F}^c$ (voir [12, Sec. 2.7]).

Le théorème suivant est démontré dans [51] et [62] pour un corps commutatif de caractéristique 0. Ce résultat est étendu au cas d'un corps commutatif quelconque dans la Section 2.2.11.

Théorème 1.4.8. *Il existe un isomorsphisme de $\mathbf{k}$-algèbres $\mathrm{H}_*^{G_{\mathbf{d}}}(Z^c_{\mathbf{d}}, \mathbf{k}) \simeq R_{\mathbf{d}}$.* $\square$

Notons $D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$ la catégorie dérivée bornée $G_{\mathbf{d}}$-équivariante des faisceaux de $\mathbf{k}$-espaces vectoriels sur $E_{\mathbf{d}}$. Pour chaque $\mu \in \mathrm{Comp}(\mathbf{d})$ notons $\mathscr{L}_\mu$ le faisceau $(\pi_\mu)_*\underline{\mathbf{k}}_{\widetilde{F}_\mu} \in D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$. D'après [12, Prop. 8.6.35], l'algèbre de convolution $\mathrm{H}_*^{G_{\mathbf{d}}}(Z^c_{\mathbf{d}}, \mathbf{k})$ est isomorphe à l'algèbre de Yoneda $\mathrm{Ext}^*_{G_{\mathbf{d}}}(\bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}}, \bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}})$. Cela implique le corollaire suivant.

Corollaire 1.4.9. *Il existe un isomorphisme de $\mathbf{k}$-algèbres*

$$R_{\mathbf{d}} \simeq \mathrm{Ext}^*_{G_{\mathbf{d}}}(\bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}}, \bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}}).$$

$\square$

1.4.6 La construction géométrique des bases canoniques

Soit $\Gamma = (I, H)$ un carquois sans boucles. Soit $(a_{i,j})$ la matrice de Cartan généralisée. Soit $\mathbf{f}$ l'algèbre associée à la matrice $(a_{i,j})$. Dans cette section on explique la construction géométrique de Lusztig de la base canonique $\mathbf{B}$ de $\mathbf{f}$. Voir [38] pour les détails.

Soit $\mathrm{Perv}_{\mathbf{d}}$ l'ensemble des faisceaux pervers simples dans $D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$ qui sont des facteurs directs des décalages des faisceaux $\mathscr{L}_y$ pour $y \in Y_{\mathbf{d}}$. Soit $\mathcal{Q}_{\mathbf{d}}$ la sous-catégorie additive pleine de $D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$ formée des sommes directes des décalages des éléments de $\mathrm{Perv}_{\mathbf{d}}$. Pour chaque $\mathbf{i} \in I^{\mathbf{d}}$ on considère le décalage ${}^\delta\mathscr{L}_{\mathbf{i}} = \mathscr{L}_{\mathbf{i}}[\dim \widetilde{F}_{\mathbf{i}}]$ de $\mathscr{L}_{\mathbf{i}}$.

Pour chaque $\mathfrak{L} \in \mathcal{Q}_{\mathbf{d}}$, $\mathfrak{L}' \in \mathcal{Q}_{\mathbf{d}'}$ soit $\mathfrak{L} * \mathfrak{L}'$ le produit d'induction de Lusztig, voir [38, Sec. 9.5.2.]. On pose

$$\mathfrak{L} \circ \mathfrak{L}' = (\mathfrak{L} * \mathfrak{L}')[m_{\mathbf{d},\mathbf{d}'}], \qquad m_{\mathbf{d},\mathbf{d}'} := \sum_{h \in H} d_{h'} d'_{h''} + \sum_{i \in I} d_i d'_i.$$

Pour chaque $y = (a_1 \cdot i_1, \cdots, a_k \cdot i_k) \in Y_{\mathbf{d}}$ on pose $\theta_y = \theta_{i_1}^{(a_1)} \cdots \theta_{i_k}^{(a_k)}$. Soit $K(\mathcal{Q}_{\mathbf{d}})$ le groupe de Grothendieck (scindé) de la catégorie additive $\mathcal{Q}_{\mathbf{d}}$. C'est un $\mathcal{A}$-module libre avec une base $\{[\mathfrak{L}];\ \mathfrak{L} \in \mathrm{Perv}_{\mathbf{d}}\}$, où q agit par le décalage des complexes. On considère le groupe abélien

$$K(\mathcal{Q}) = \bigoplus_{\mathbf{d} \in \mathbb{N}I} K(\mathcal{Q}_{\mathbf{d}}).$$

L'opération $\circ$ donne une structure de $\mathcal{A}$-algèbre associative et unitaire sur cet $\mathcal{A}$-module. Le théorème suivant (voir [38]) décrit cette algèbre.

Théorème 1.4.10. *(a) Il existe un isomorphisme de $\mathcal{A}$-algèbres*

$$\lambda_{\mathcal{A}} : K(\mathcal{Q}) \to {}_{\mathcal{A}}\mathbf{f}, \qquad [{}^\delta\mathfrak{L}_y] \mapsto \theta_y, \quad \forall y \in Y_{\mathbf{d}},\ \mathbf{d} \in \mathbb{N}I.$$

(b) L'ensemble $\{\lambda_{\mathcal{A}}([\mathscr{L}]);\ \mathscr{L} \in \coprod_{\mathbf{d} \in \mathbb{N}I} \mathrm{Perv}_{\mathbf{d}}\}$ coïncide avec la base canonique $\mathbf{B}$ de ${}_{\mathcal{A}}\mathbf{f}$. □

1.4.7 Les algèbres de carquois-Schur

Fixons un entier $e > 1$. Dans cette section on suppose que le carquois Γ est un e-cycle orienté. On identifie l'ensemble I des sommets de Γ avec $\mathbb{Z}/e\mathbb{Z}$. L'ensemble des flèches H est $H = \{i \to i+1;\ i \in I\}$.

La construction suivante est similaire à la construction géométrique de l'algèbre KLR. Cette algèbre, appelée l'algèbre de carquois-Schur, est définie dans [59].

Notons $Z_{\mathbf{d}} = \coprod_{\mu,\xi \in \mathrm{Comp}(\mathbf{d})} Z_{\mu,\xi}$.

Définition 1.4.11. L'*algèbre de carquois-Schur* $A_{\mathbf{d}}$ est l'algèbre $\mathrm{H}_*^{G_{\mathbf{d}}}(Z_{\mathbf{d}}, \mathbf{k})$.

Comme pour les algèbres KLR, l'algèbre de carquois-Schur $A_{\mathbf{d}}$ peut être identifiée avec l'algèbre de Yoneda

$$\mathrm{Ext}^*_{G_{\mathbf{d}}}\Big(\bigoplus_{\mu \in \mathrm{Comp}(\mathbf{d})} \mathscr{L}_\mu, \bigoplus_{\mu \in \mathrm{Comp}(\mathbf{d})} \mathscr{L}_\mu\Big).$$

1.4.8 Les algèbres de carquois-Schur de niveau supérieur

Les algèbres $A_{\mathbf{d}}$ admettent des versions de niveau supérieur. Ces nouvelles algèbres sont définies aussi dans [59].

Soit l un entier > 1. Pour chaque poids ν de $\widetilde{\mathfrak{sl}}_e$ on écrit

$$\nu = \sum_{i \in I} \nu(i)\Lambda_i.$$

On peut voir un poids dominant ν de $\widetilde{\mathfrak{sl}}_e$ comme un vecteur de dimension d'un espace vectoriel I-gradué. Fixons une suite $\underline{\nu} = (\Lambda_{z_1}, \cdots, \Lambda_{z_l})$ de poids et considérons le poids $\nu = \sum_{r=1}^{l} \Lambda_{z_r}$. Pour $k \in [1, l]$ notons aussi $\nu^k = \sum_{r=1}^{k} \Lambda_{z_r}$. En particulier on a $\nu^l = \nu$. Posons

$$E_{\mathbf{d},\nu} = E_{\mathbf{d}} \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, \mathbb{C}^{\nu(i)}).$$

Pour chaque $\mu = (\mu_1, \cdots, \mu_k) \in \mathrm{Comp}(\mathbf{d})$ on pose $\mathbf{d}(\mu) = \mathbf{d}$, $\ell(\mu) = k$. Considérons l'ensemble $\mathrm{Comp} = \coprod_{\mathbf{d} \in \mathbb{N}I} \mathrm{Comp}(\mathbf{d})$. Posons

$$\mathrm{Comp}_l(\mathbf{d}) = \{(\mu^{(0)}, \cdots, \mu^{(l)}) \in \mathrm{Comp}^{l+1};\ \sum_{r=0}^{l} \mathbf{d}(\mu^{(r)}) = \mathbf{d}\}.$$

Soit $\hat{\mu} = (\mu^{(0)}, \cdots, \mu^{(l)})$ un élément de $\mathrm{Comp}_l(\mathbf{d})$. Notons $\mu = \mu^{(0)} \cup \cdots \cup \mu^{(l)} \in \mathrm{Comp}(\mathbf{d})$ la concaténation de ses composantes. Soit $\mathfrak{O}(\hat{\mu}, \underline{\nu})$ l'ensemble des couples $((x, \phi), (\gamma_i)_{i \in I})$ dans $\widetilde{F}_\mu \times \bigoplus_{i \in I} \mathrm{Hom}(V_i, U_i)$ tels que l'on a $\gamma_i(\widetilde{W}_i(k)) \subset \mathbb{C}^{\nu^k(i)}$ pour chaque $i \in I$ et $k \in [1, l]$. Ici $\widetilde{W}(0) \subset \widetilde{W}(1) \subset \cdots \subset \widetilde{W}(l) = V$ est le drapeau I-gradué qui a des composantes parmi les composantes du drapeau ϕ et les dimensions graduées de

$$\widetilde{W}(0),\ \widetilde{W}(1)/\widetilde{W}(0), \cdots,\ \widetilde{W}(l)/\widetilde{W}(l-1)$$

sont $\mathbf{d}(\mu^{(0)}), \mathbf{d}(\mu^{(1)}), \cdots, \mathbf{d}(\mu^{(l)})$ respectivement.

Pour chaque $\hat{\mu} \in \mathrm{Comp}_l(\mathbf{d})$, soit $\pi_{\hat{\mu},\underline{\nu}}$ la projection

$$\pi_{\hat{\mu},\underline{\nu}} \colon \mathfrak{O}(\hat{\mu}, \underline{\nu}) \to E_{\mathbf{d},\nu}, \quad ((x, \phi), (\gamma_i)_{i \in I}) \to (x, (\gamma_i)_{i \in I}).$$

On note $\mathscr{L}_{\hat{\mu},\underline{\nu}}$ le faisceau $(\pi_{\hat{\mu},\underline{\nu}})_* \underline{\mathbf{k}}$ dans $D^b_{G_{\mathbf{d}}}(E_{\mathbf{d},\nu}, \mathbf{k})$. Pour chaque $\hat{\mu}, \hat{\lambda} \in \mathrm{Comp}_l(\mathbf{d})$ on pose $Z^{\underline{\nu}}_{\tilde{\mu},\tilde{\lambda}} = \mathfrak{O}(\hat{\mu}, \underline{\nu}) \times_{E_{\mathbf{d},\nu}} \mathfrak{O}(\hat{\lambda}, \underline{\nu})$. Finalement, on pose

$$Z^{\underline{\nu}}_{\mathbf{d}} = \coprod_{\tilde{\mu},\tilde{\lambda} \in \mathrm{Comp}_l(\mathbf{d})} Z^{\underline{\nu}}_{\hat{\mu},\hat{\lambda}}, \qquad \mathfrak{O}^{\underline{\nu}}_{\mathbf{d}} = \coprod_{\tilde{\lambda} \in \mathrm{Comp}_l(\mathbf{d})} \mathfrak{O}(\hat{\lambda}, \underline{\nu}).$$

Définition 1.4.12. L'algèbre de carquois-Schur (de niveau l) est l'algèbre $\tilde{A}^{\underline{\nu}}_{\mathbf{d}} = H^*_{G_{\mathbf{d}}}(Z^{\underline{\nu}}_{\mathbf{d}}, \mathbf{k})$.

Comme précédemment, on peut identifier A_d^ν avec l'algèbre de Yoneda

$$\mathrm{Ext}^*_{G_V}\Big(\bigoplus_{\hat\mu\in\mathrm{Comp}_l(\mathbf{d})}\mathscr{L}_{\hat\mu,\underline{\nu}},\bigoplus_{\hat\mu\in\mathrm{Comp}_l(\mathbf{d})}\mathscr{L}_{\hat\mu,\underline{\nu}}\Big).$$

Cette algèbre possède des idempotents $e(\hat\mu)$, $\hat\mu\in\mathrm{Comp}_l(\mathbf{d})$, définis comme les classes fondamentales des variétés $Z^{\underline{\nu}}_{\hat\mu,\hat\mu}$.

Les algèbres de carquois-Schur possèdent des versions cyclotomiques. Elles sont définies dans [59].

Définition 1.4.13. L'*algèbre de carquois-Schur cyclotomique* $A_{\mathbf{d}}^{\nu}$ est le quotient de $\widetilde{A}_{\mathbf{d}}^{\nu}$ par l'idéal engendré par tous les idempotents $e(\hat\mu)$, où $\hat\mu=(\mu^{(0)},\cdots,\mu^{(l)})\in\mathrm{Comp}_l(\mathbf{d})$ est tel que $\mu^{(0)}\neq\emptyset$.

Pour $d\in\mathbb{N}$ on pose

$$\widetilde{A}_d^{\nu}=\bigoplus_{|\mathbf{d}|=d}\widetilde{A}_{\mathbf{d}}^{\nu},\quad A_d^{\nu}=\bigoplus_{|\mathbf{d}|=d}A_{\mathbf{d}}^{\nu}.$$

1.4.9 L'isomorphisme ζ-Schur-carquois-Schur

L'article [59] étend le résultat du Théorème 1.4.3 au cas des algèbres de Schur. Soit $\mathbf{s}=(s_1,\cdots,s_l)$ un l-uplet d'entiers. On considère le l-uplet de poids de $\widetilde{\mathfrak{sl}}_e$ donné par $\underline{\nu}=(\Lambda_{\overline{s_1}},\cdots,\Lambda_{\overline{s_l}})$, où $\overline{s_r}$ est le résidu de s_r modulo e. Le théorème suivant est démontré dans [59].

Théorème 1.4.14. *Soit ζ une racine primitive e-ième de l'unité. La ζ-algèbre de Schur $S_d^{\mathbf{s}}(\zeta)$ est isomorphe à l'algèbre de carquois-Schur cyclotomique A_d^{ν}.* □

Remarque 1.4.15. Le papier [59] construit des foncteurs

$$E_i\colon\mathrm{mod}(A^{\nu}_{\mathbf{d}+i})\to\mathrm{mod}(A^{\nu}_{\mathbf{d}}),\quad F_i\colon\mathrm{mod}(A^{\nu}_{\mathbf{d}})\to\mathrm{mod}(A^{\nu}_{\mathbf{d}+i})$$

pour $i\in I$. La somme $\bigoplus_{\mathbf{d}\in\mathbb{N}I}[\mathrm{mod}(A^{\nu}_{\mathbf{d}})]$ est isomorphe à l'espace de Fock $\mathcal{F}_q^{\mathbf{s}}$. L'action des foncteurs E_i, F_i s'identifie avec l'action des générateurs E_i, F_i de $U_q(\widetilde{\mathfrak{sl}}_e)$.

1.4.10 Représentations catégoriques

Fixons un élément invertible $\zeta\neq 1$ dans $\mathbf{k}$. Soit $\mathcal{C}$ une catégorie abélienne $\mathbf{k}$-linéaire.

Définition 1.4.16. Une *pré-représentation catégorique* dans $\mathcal{C}$ est un uplet (E,F,X,T), où (E,F) est un couple de foncteurs biadjoints exacts $\mathcal{C}\to\mathcal{C}$ et $X\in\mathrm{End}(F)^{\mathrm{op}}$, $T\in\mathrm{End}(F^2)^{\mathrm{op}}$ sont des endomorphismes de foncteurs tels que pour chaque $d\in\mathbb{N}$ on a un homomorphisme de $\mathbf{k}$-algèbres $\psi_d\colon H_{R,d}(\zeta)\to\mathrm{End}(F^d)^{\mathrm{op}}$ donné par

$$\begin{aligned}X_r&\mapsto F^{d-r}XF^{r-1}\quad\forall r\in[1,d],\\T_r&\mapsto F^{d-r-1}TF^{r-1}\quad\forall r\in[1,d-1].\end{aligned}$$

Remarque 1.4.17. Par adjonction on a un isomorphisme $\mathrm{End}(E^d) \simeq \mathrm{End}(F^d)^{\mathrm{op}}$. On peut donc donner une définition équivalente en remplaçant les endomorphismes de foncteurs $X \in \mathrm{End}(F)^{\mathrm{op}}$, $T \in \mathrm{End}(F^2)^{\mathrm{op}}$, par des endomorphismes de foncteurs $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$, qui, pour chaque $d \in \mathbb{N}$, induisent un homomorphisme $H_d \to \mathrm{End}(E^d)$.

Dorénavant, on suppose que la catégorie $\mathcal{C}$ est Hom-finie. Soit $\mathscr{F}$ un sous-ensemble de $\mathbf{k}$ tel que $\zeta^{\mathbb{Z}}\mathscr{F}/\zeta^{\mathbb{Z}}$ est fini. On voit $\mathscr{F}$ comme l'ensemble des sommets d'un carquois avec une flèche $i \to j$ si et seulement si $j = \zeta i$.

Pour $i \in \mathscr{F}$ soit $\alpha_i = \varepsilon_i - \varepsilon_{\zeta i} \in X_{\mathscr{F}}$.

Définition 1.4.18. Une représentation catégorique de $\mathfrak{g}_{\mathscr{F}}$ dans $\mathcal{C}$ est une pré-représentation catégorique (E, F, X, T) munie d'une décomposition $\mathcal{C} = \bigoplus_{\mu \in X_{\mathscr{F}}} \mathcal{C}_\mu$ qui satisfait les conditions (a) et (b) ci-dessous. Pour $i \in \mathscr{F}$ soient E_i, F_i les endofoncteurs de $\mathcal{C}$ tels que pour chaque $M \in \mathcal{C}$ les sous-objets $E_i(M) \subset E(M)$, $F_i(M) \subset F(M)$ sont les i-espaces propres généralisés de X (voir aussi la Remarque 1.4.17). On suppose

(a) $F = \bigoplus_{i \in \mathscr{F}} F_i$ et $E = \bigoplus_{i \in \mathscr{F}} E_i$,

(b) $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$ et $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.

Supposons que $\mathscr{F}$ est le carquois cyclique de taille e. Dans ce cas ζ est une racine primitive e-ième de l'unité et $\mathfrak{g}_{\mathscr{F}} = \widetilde{\mathfrak{sl}}_e$. Soit $\Gamma = (I, H)$ le carquois cyclique de taille e. On identifie I avec $\mathbb{Z}/e\mathbb{Z}$. Pour $i \in I$, on pose $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in X_I$.

Definition 1.4.19. Une représentation catégorique (E, F, x, τ) de $\widetilde{\mathfrak{sl}}_e$ dans $\mathcal{C}$ est :

(1) une décomposition $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_\mu$,

(2) un couple de foncteurs biadjoints exacts (E, F) de $\mathcal{C}$ dans $\mathcal{C}$,

(3) des morphismes de foncteurs $x\colon F \to F$, $\tau\colon F^2 \to F^2$,

(4) des décompositions $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$,

tels que les conditions suivantes sont satisfaites.

(a) $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$, $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$,

(b) pour chaque $d \in \mathbb{N}$ il y a un homomorphisme d'algèbres $\psi_d\colon R_d \to \mathrm{End}(F^d)^{\mathrm{op}}$ tel que $\psi_d(e(\mathbf{i}))$ est le projecteur sur $F_{i_d} \cdots F_{i_1}$ et

$$\psi_d(x_r) = F^{d-r} x F^{r-1}, \qquad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1},$$

(c) pour chaque $M \in \mathcal{C}$ l'endomorphisme de $F(M)$ induit par x est nilpotent.

1.5 La catégorie $\mathcal{O}$

1.5.1 L'algèbre de Lie $\widehat{\mathfrak{gl}}_N$

Soient N, e, l des entiers positifs, $e > 1$. Soit

$$\mathbf{g} = \mathfrak{gl}_N(\mathbb{C}), \quad \widehat{\mathbf{g}} = \widehat{\mathfrak{gl}}_N(\mathbb{C}) = \mathbf{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial.$$

Pour $i, j \in [1, N]$, soit $e_{i,j} \in \mathbf{g}$ la matrice élémentaire. Soit $\mathbf{h} \subset \mathbf{g}$ la sous-algèbre de Cartan engendrée par les éléments $e_{i,i}$, $i \in [1, N]$. Soit $(\epsilon_1, \cdots, \epsilon_N)$ la base de $\mathbf{h}^*$ duale de la base $(e_{1,1}, \cdots, e_{N,N})$. Soit P le réseau des poids de $\mathbf{g}$. On a $P = \bigoplus_{i\in[1,N]} \mathbb{Z}\epsilon_i$. On identifie P et $\mathbb{Z}^N$.

Notons $X_e[N]$ l'ensemble des e-uplets $\mu = (\mu_1, \cdots, \mu_e)$ d'entiers positifs tels que $\sum_{r=1}^{e} \mu_r = N$. Fixons un l-uplet $\nu = (\nu_1, \cdots, \nu_l) \in X_l[N]$ et un e-uplet $\mu = (\mu_1, \cdots, \mu_e) \in X_e[N]$.

On considère la sous-algèbre de Cartan $\widehat{\mathbf{h}} = \mathbf{h} \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$ de $\widehat{\mathbf{g}}$. Soient Λ_0 et δ les éléments de $\widehat{\mathbf{h}}^*$ définis par

$$\delta(\partial) = \Lambda_0(\mathbf{1}) = 1, \quad \delta(\mathbf{h} \oplus \mathbb{C}\mathbf{1}) = \Lambda_0(\mathbf{h} \oplus \mathbb{C}\partial) = 0.$$

Soit $(\bullet, \bullet)\colon \widehat{\mathbf{h}}^* \times \widehat{\mathbf{h}}^* \to \mathbb{C}$ la forme bilinéaire telle que $\lambda(\alpha_k^\vee) = (\lambda, \alpha_k), \lambda(\partial) = (\lambda, \Lambda_0)$ pour chaque $k \in [0, N]$ et $\lambda \in \widehat{\mathbf{h}}^*$.

Posons $P_{\mathbb{C}} = P \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^N \simeq \mathbf{h}^*$. Soient

$$\rho = (0, -1, \cdots, -N+1), \quad \rho_\nu = (\nu_1, \nu_1-1 \cdots, 1, \nu_2, \cdots, 1, \cdots, \nu_l, \cdots, 1) \in P.$$

Posons aussi $\widehat{\rho} = \rho + N\Lambda_0$. Pour chaque $\lambda \in P$ on pose

$$\widetilde{\lambda} = \lambda + \tau + z_\lambda\delta + (-N - \kappa)\Lambda_0 \in \widehat{\mathbf{h}}^*,$$

où $z_\lambda = (\lambda, 2\rho + \lambda)/2e$. Pour chaque $\lambda \in P$ soit $\Delta(\lambda)$ le module de Verma dont le plus haut poids est $\widetilde{\lambda}$. On pose $\Delta^\lambda = \Delta(\lambda - \rho)$.

1.5.2 La catégorie $\mathcal{O}$ pour $\widehat{\mathfrak{gl}}_N$

On suppose que le carquois $\Gamma = (I, H)$ est un e-cycle orienté. On identifie I avec $\mathbb{Z}/e\mathbb{Z}$ ou $[1, e]$. On dit que le poids $\lambda \in P$ est *ν-dominant* si $\lambda_r > \lambda_{r+1}$ pour chaque $r \in [1, N-1]\backslash\{\nu_1, \nu_1 + \nu_2, \cdots, \nu_1 + \cdots + \nu_l\}$. Soit $P[\mu]$ le sous-ensemble de P formé des éléments $\lambda \in P$ tels que pour chaque $i \in [1, e]$ le N-uplet λ contient μ_i composantes égales à i modulo e. Soit P^ν l'ensemble des poids ν-dominants dans P. On pose aussi $P^\nu[\mu] = P^\nu \cap P[\mu]$.

Soit $O^\nu_{\mu,-}$ la sous-catégorie de Serre de la catégorie $\mathcal{O}$ parabolique de type ν de $\widehat{\mathbf{g}}$ engendrée par les objets Δ^λ, pour $\lambda \in P^\nu[\mu]$. On pose $O^\nu_{-e} = \bigoplus_{\mu\in X_e[N]} O^\nu_{\mu,-}$.

Pour chaque $i \in I$ on considère les e-uplets $\mu_i^+, \mu_i^- \in \mathbb{Z}^e$ définis par

$$(\mu_i^-)_r = \mu_r + \delta_{r\equiv i+1} - \delta_{r\equiv i}, \quad (\mu_i^+)_r = \mu_r + \delta_{r\equiv i} - \delta_{r\equiv i+1}.$$

On veut décrire la représentation catégorique de $\widetilde{\mathfrak{sl}}_e$ dans O^ν_{-e}. Voir [53, Sec. 5.4] pour les détails. Dans [53] un couple d'endofoncteurs biadjoints E, F de O^ν_{-e} est défini. La catégorie O^ν_{-e}, munie des foncteurs E et F, admet une structure de pré-représentation catégorique. De plus, il y a une décomposition $E = \sum_{i\in I} E_i$, $F = \sum_{i\in I} F_i$ telle que l'on a

$$E_i(O^\nu_{\mu,-}) \subset O^\nu_{\mu_i^-,-}, \qquad F_i(O^\nu_{\mu,-}) \subset O^\nu_{\mu_i^+,-}.$$

Cela donne une structure de représentation catégorique de $\widetilde{\mathfrak{sl}}_e$ dans O^ν_{-e}.

On pose $\wedge^\nu U_e = \wedge^{\nu_1} U_e \otimes \cdots \otimes \wedge^{\nu_l} U_e$. Pour chaque $\lambda \in P^\nu$ soit

$$\wedge^\nu_\lambda = (u_{\lambda_1} \wedge \cdots \wedge u_{\lambda_{\nu_1}}) \otimes \cdots \otimes (u_{\lambda_1+\cdots+\lambda_{l-1}+1} \wedge \cdots \wedge u_{\lambda_{\nu_1}+\cdots+\lambda_{\nu_l}}) \in \wedge^\nu U_e.$$

L'action de $\widetilde{\mathfrak{sl}}_e$ de niveau 0 sur U_e donne une action de $\widetilde{\mathfrak{sl}}_e$ de niveau 0 sur $\wedge^\nu U_e$. On peut voir un élément de $X_e[N]$ comme un poids de $\mathfrak{sl}_e$. Pour chaque $\mu \in X_e[N]$ notons $(\wedge^\nu U_e)_\mu$ l'espace de poids dans $\wedge^\nu U_e$ qui correspond au $\mathfrak{sl}_e$-poids μ.

Le résultat suivant est démontré dans [56, Sec. 6.2].

Théorème 1.5.1. *Il y a un isomorphisme d'espaces vectoriels* $[O^\nu_{\mu,-}] \simeq (\wedge^\nu U)_\mu$. *Cet isomorphisme identifie les foncteurs* F_i, E_i *agissant sur* $[O^\nu_{-e}] = \bigoplus_{\mu \in P_e[N]} [O^\nu_{\mu,-}]$ *avec les générateurs standards* e_i, f_i *de* $\widetilde{\mathfrak{sl}}_e$. □

1.5.3 La catégorie A

Soit $\mathcal{P}^\nu_d \subset \mathcal{P}^l_d$ le sous-ensemble de $\mathcal{P}^l_d$ formé des éléments $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(l)}) \in \mathcal{P}^l_d$ tels que $\ell(\lambda^{(k)}) \leqslant \nu_k$ pour chaque $k \in [1, l]$.

On identifie les l-partitions $\lambda \in \mathcal{P}^\nu_d$ avec les poids

$$(\lambda^1_1, \cdots, \lambda^1_{\ell(\lambda^1)}, 0^{m_1-\ell(\lambda^1)}, \lambda^2_1, \cdots, \lambda^2_{\ell(\lambda^2)}, 0^{m_2-\ell(\lambda^2)}, \cdots, \lambda^l_1, \cdots, \lambda^l_{\ell(\lambda^l)}, 0^{m_l-\ell(\lambda^l)}),$$

Pour $\lambda \in \mathcal{P}^\nu_d$, on pose $\omega(\lambda) = \lambda - \rho + \rho_\nu \in P$.

Definition 1.5.2. Soit $\mathbf{A}^\nu[d] \subset O^\nu_{-e}$ la sous-catégorie de Serre engendrée par les modules $\Delta(\omega(\lambda))$ tels que $\lambda \in \mathcal{P}^\nu_d$.

Pour $\lambda \in \mathcal{P}^\nu_d$ on pose $\Delta[\lambda] = \Delta(\omega(\lambda))$. La restriction du foncteur F à $\mathbf{A}^\nu[d]$ donne un foncteur $F\colon \mathbf{A}^\nu[d] \to \mathbf{A}^\nu[d+1]$. Néanmoins, il n'est pas vrai que $E(\mathbf{A}^\nu[d+1]) \subset \mathbf{A}^\nu[d]$. Il est cependant possible de définir un foncteur $E\colon \mathbf{A}^\nu[d+1] \to \mathbf{A}^\nu[d]$ qui est adjoint à gauche au foncteur $F\colon \mathbf{A}^\nu[d] \to \mathbf{A}^\nu[d+1]$, voir [53, Sec. 5.9].

Les foncteurs E, F admettent des décompositions

$$E = \bigoplus_{i \in I} E_i, \qquad F = \bigoplus_{i \in I} F_i$$

qui viennent des décompositions similaires pour la catégorie $\mathcal{O}$.

On suppose de plus que l'on a la condition suivante

$$\nu_k \geqslant d, \qquad \forall k \in [1, l]. \tag{1.1}$$

Dans ce cas on a $\mathcal{P}^\nu_\alpha = \mathcal{P}^l_\alpha$. Pour $a \in \mathbb{Z}$ posons $\nu + a = (\nu_1 + a, \cdots, \nu_l + a)$. Si ν satisfait (1.1), alors par [53, Lemma 7.6], pour chaque $a \in \mathbb{N}$ on a une équivalence de catégories $\mathbf{A}^\nu[d] \simeq \mathbf{A}^{\nu+a}[d]$, $\Delta[\lambda] \mapsto \Delta[\lambda]$. Pour chaque n-uplet $\nu = (\nu_1, \cdots, \nu_l) \in \mathbb{Z}^l$ on pose $\widetilde{\mathbf{A}}^\nu[d] = \lim_a \mathbf{A}^{\nu+a}[d]$, où les valeurs de a

sont telles que $\nu + a$ satisfait (1.1). De plus, si $\nu - 1$ satisfait (1.1), alors on a les diagrammes commutatifs

$$\begin{tikzcd} \mathbf{A}^{\nu}[d] \arrow[r, "F"] \arrow[d] & \mathbf{A}^{\nu}[d+1] \arrow[d] \\ \mathbf{A}^{\nu+a}[d] \arrow[r, "F"] & \mathbf{A}^{\nu+a}[d+1], \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{A}^{\nu}[d] \arrow[d] & \mathbf{A}^{\nu}[d+1] \arrow[l, "E"'] \arrow[d] \\ \mathbf{A}^{\nu+a}[d] & \mathbf{A}^{\nu+a}[d+1], \arrow[l, "E"'] \end{tikzcd}$$

où les flèches verticales sont les isomorphismes ci-dessus. Cela permet de définir des foncteurs $F_i \colon \widetilde{\mathbf{A}}^{\nu}[d] \to \widetilde{\mathbf{A}}^{\nu}[d+1]$ et $E_i \colon \widetilde{\mathbf{A}}^{\nu}[d+1] \to \widetilde{\mathbf{A}}^{\nu}[d]$.

Le théorème suivant est démontré dans [53].

Théorème 1.5.3. *Il existe un isomorphisme entre le $\mathbb{C}$-espace vectoriel $\bigoplus_{d\in\mathbb{N}}[\widetilde{\mathbf{A}}^{\nu}[d]]$ et l'espace de Fock F^{ν}. Cet isomorphisme identifie l'action des foncteurs E_i, F_i sur $\bigoplus_{d\in\mathbb{N}}[\widetilde{\mathbf{A}}^{\nu}[d]]$ avec l'action des générateurs e_i, f_i de $\widetilde{\mathfrak{sl}}_e$.* □

1.6 Dans cette thèse

1.6.1 Bases canoniques, algèbres KLR et faisceaux de parité

Dans cette partie on étudie les algèbres KLR en caractéristique positive. Premièrement, on étend l'isomorphisme de Rouquier et Varagnolo-Vasserot (Théorème 1.4.8, Corollaire 1.4.9) en caractéristique quelconque.

Théorème 1.6.1. *Soit $\mathbf{k}$ un corps commutatif quelconque. Il existe un isomorphisme de $\mathbf{k}$-algèbres*

$$R_{\mathbf{d}} \simeq \operatorname{Ext}^*_{G_{\mathbf{d}}}(\bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}}, \bigoplus_{\mathbf{i}\in I^{\mathbf{d}}} \mathscr{L}_{\mathbf{i}}).$$

□

En caractéristique zéro le théorème de décomposition (voir [3]) implique que les faisceaux $\mathscr{L}_{\mathbf{i}}$ sont des sommes directes de décalages de faisceaux pervers simples. Ce n'est pas le cas en caractéristique positive. On voudrait trouver une classe de faisceaux qui remplace les faisceaux pervers en caractéristique positive. Les *faisceaux de parité* introduits par Juteau-Mautner-Williamson semblent d'être les bons candidats.

On démontre l'existence de faisceaux de parité sur la variété de carquois pour un carquois de Dynkin. Plus précisément, on démontre le théorème suivant.

Théorème 1.6.2. *Soient $\Gamma = (I, H)$ un carquois de Dynkin (A, D, E), $\mathbf{d} \in \mathbb{N}I$ un vecteur de dimension. Soit $\mathcal{O}$ une $G_{\mathbf{d}}$-orbite dans $E_{\mathbf{d}}$. Alors il existe un unique objet indécomposable $\widetilde{\mathcal{E}}(\mathcal{O}) \in D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$ qui satisfait les propriétés suivantes*

- *Le complexe $\widetilde{\mathcal{E}}(\mathcal{O})$ est constructible par rapport à la stratification par les $G_{\mathbf{d}}$-orbites.*
- *Le support de $\widetilde{\mathcal{E}}(\mathcal{O})$ est dans $\overline{\mathcal{O}}$.*
- *La restriction de $\widetilde{\mathcal{E}}(\mathcal{O})$ à $\mathcal{O}$ est le faisceau constant $\underline{\mathbf{k}}_{\mathcal{O}}$.*
- *On a $\mathcal{H}^r(\widetilde{\mathcal{E}}(\mathcal{O})) = \mathcal{H}^r(\mathbb{D}\widetilde{\mathcal{E}}(\mathcal{O})) = 0$ pour chaque entier impair r.*

Ici $\mathcal{H}^\bullet$ est la cohomologie faisceau, $\mathbb{D}$ est la dualité de Verdier. □

Le complexe décalé $\mathcal{E}(\mathcal{O}) := \widetilde{\mathcal{E}}(\mathcal{O})[\dim_{\mathbb{C}} \mathcal{O}]$ est appelé un faisceau de parité. En caractéristique zéro le complexe $\mathcal{E}(\mathcal{O})$ coïncide avec le complexe d'intersection $\mathrm{IC}(\mathcal{O})$. C'est donc un remplaçant du complexe $\mathrm{IC}(\mathcal{O})$ en caractéristique positive.

Pour démontrer que les complexes $\mathscr{L}_{\mathbf{i}}$ sont des sommes directes de décalages de faisceaux de parité il suffit de voir que les fibres des morphismes $\pi_{\mathbf{i}}$ n'ont pas de groupes de cohomologie impairs. Malheureusement, on ne sait pas si c'est vrai pour les types D et E. Pour le type A on démontre le théorème suivant.

Théorème 1.6.3. *Soit $\Gamma = (I, H)$ un carquois de type A. Alors, pour chaque vecteur de dimension $\mathbf{d} \in \mathbb{N}I$ et chaque $\mathbf{i} \in I^{\mathbf{d}}$, le faisceau $\mathscr{L}_{\mathbf{i}}$ est une somme directe de décalages de faisceaux de parité.* □

Soit Γ un carquois de Dynkin. Soit $\mathrm{Par}_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k}) \subset D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})$ la sous-catégorie additive formée des sommes directes de décalages des faisceaux de parité. Les éléments de cette catégorie sont appelée des *complexes de parité*. Pour $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{N}I$ on a les foncteurs d'induction et de restriction de Lusztig [38, Sec. 9.2],

$$\mathrm{Ind}_{\mathbf{d}_1,\mathbf{d}_2} \colon D^b_{G_{\mathbf{d}_1}}(E_{\mathbf{d}_1}, \mathbf{k}) \times D^b_{G_{\mathbf{d}_2}}(E_{\mathbf{d}_2}, \mathbf{k}) \to D^b_{G_{\mathbf{d}_1+\mathbf{d}_2}}(E_{\mathbf{d}_1+\mathbf{d}_2}, \mathbf{k}),$$

$$\mathrm{Res}_{\mathbf{d}_1,\mathbf{d}_2} \colon D^b_{G_{\mathbf{d}_1+\mathbf{d}_2}}(E_{\mathbf{d}_1+\mathbf{d}_2}, \mathbf{k}) \to D^b_{G_{\mathbf{d}_1} \times G_{\mathbf{d}_2}}(E_{\mathbf{d}_1} \times E_{\mathbf{d}_2}, \mathbf{k}).$$

Les complexes de parité sont stables par la restriction et on obtient un coproduit sur le $\mathcal{A}$-module $\bigoplus_{\mathbf{d} \in \mathbb{N}I} K(\mathrm{Par}_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k}))$. En revanche, il n'est pas clair si les faisceaux de parité sont stables par l'induction. C'est vrai en type A a cause du Théorème 1.6.3. On démontre le résultat suivant.

Théorème 1.6.4. *Soit Γ un carquois de Dynkin. Pour chaque vecteur de dimension $\mathbf{d} \in \mathbb{N}I$ il existe un isomorphisme de $\mathcal{A}$-modules $K(\mathrm{Par}_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})) \simeq {}_{\mathcal{A}}\mathbf{f}_{\mathbf{d}}$. L'isomorphisme de $\mathcal{A}$-modules obtenu $\bigoplus_{\mathbf{d} \in \mathbb{N}I} K(\mathrm{Par}_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \mathbf{k})) \simeq {}_{\mathcal{A}}\mathbf{f}$ est un isomorphisme de cogèbres.* □

Remarque 1.6.5. (a) Le théorème 1.6.4 est un analogue de la construction de Lusztig (voir Section 1.4.6). Les classes des faisceaux de parité donnent une nouvelle base du groupe quantique $\mathbf{f}$.

(b) Pour un carquois de type A, l'isomorphisme du Théorème 1.6.4 est un isomorphisme de bigèbres.

1.6.2 Algèbres de carquois-Schur et dualité de Koszul

La deuxième partie de la thèse étudie la Koszulité de la graduation sur l'algèbre de carquois-Schur.

On suppose que $\Gamma = (I, H)$ est un carquois cyclique de taille e, $e > 1$. Soient ζ une racine primitive e-ième de l'unité, d un entier positif, $\mathbf{s} = (s_1, \cdots, s_l)$ une suite d'entiers. On lui associe la suite $\underline{\nu} = (\Lambda_{\overline{s_1}}, \cdots, \Lambda_{\overline{s_l}})$ de poids fondamentaux de $\widetilde{\mathfrak{sl}}_e$ comme dans la Section 1.4.9. L'algèbre de carquois-Schur $A_d^{\underline{\nu}}$ introduite par Stroppel-Webster (voir la Section 1.4.8) est une version graduée de la ζ-algèbre de Schur $S_d^{\mathbf{s}}(\zeta)$ (voir la Section 1.4.9). Cela nous donne une façon de graduer l'algèbre $S_d^{\mathbf{s}}(\zeta)$.

Soit $\nu = (\nu_1, \cdots, \nu_l)$ une suite d'entiers positifs tels que pour chaque $r \in [1, l]$ on a $s_r = \nu_r \bmod e$ et $\nu_r \geqslant d$. D'après [53] la catégorie $\mathrm{mod}(S_d^{\mathbf{s}}(\zeta))$ est équivalente à la catégorie $\mathbf{A}^\nu[d]$ (voir la Section 1.5.3 pour la définition de $\mathbf{A}^\nu[d]$). Voir la Section A.2 pour des détails sur cette équivalence. D'autre part, l'article [56] construit une graduation de Koszul sur la catégorie $\mathbf{A}^\nu[d]$. Le théorème suivant compare les deux graduations.

Théorème 1.6.6. *L'équivalence de catégories* $\mathrm{mod}(S_d^{\mathbf{s}}(\zeta)) \simeq \mathbf{A}^\nu[d]$ *entrelace la graduation induite par la graduation de l'algèbre de carquois-Schur sur* $\mathrm{mod}(S_d^{\mathbf{d}}(\zeta))$ *avec la graduation de Koszul sur* $\mathbf{A}^\nu[d]$. □

Un point important dans la preuve est de comparer les multiplicités graduées des modules simples dans les modules standards. Ces multiplicités pour la catégorie $\mathcal{O}$ pour $\widehat{\mathfrak{gl}}_N$ sont calculées dans la Section A.1.

1.6.3 Algèbres KLR et actions catégoriques

La troisième partie de la thèse étudie l'action catégorique sur la catégorie $\mathbf{A}$. La motivation initiale est d'identifier les duaux de Koszul des foncteurs E, F sur la catégorie $\mathbf{A}$. Cette partie de thèse contient des résultats utiles pour répondre à cette question.

Le premier résultat fait un lien entre l'algèbre KLR d'un e-cycle et l'algèbre KLR d'un $(e+1)$-cycle. On pose $I = \mathbb{Z}/e\mathbb{Z}$, $\overline{I} = \mathbb{Z}/(e+1)\mathbb{Z}$. On voit I et $\overline{I}$ comme des ensembles de sommets d'un carquois donné par un e-cycle et un $(e+1)$-cycle respectivement. On fixe $k \in [0, e-1]$. Soit $\mathbf{d} = \sum_{i \in I} d_i \cdot i \in \mathbb{N}I$ un vecteur de dimension. On lui associe un vecteur de dimension $\overline{\mathbf{d}} = \sum_{i\in\overline{I}} \overline{d}_i \cdot i$ défini par

$$\overline{d}_i = \begin{cases} d_i & \text{si } i \in [0,k], \\ d_{i-1} & \text{si } i \in [k+1,e], \end{cases}$$

où on identifie I avec $[0, e-1]$, $\overline{I}$ avec $[0, e]$.

On dit qu'une suite $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^{\overline{\alpha}}$ est *bien ordonnée* si pour chaque indice a tel que $i_a = k$ on a $a < d$ et $i_{a+1} = k+1$. On dit qu'une suite $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^{\overline{\alpha}}$ est *désordonnée* s'il existe $r \in [1, d]$ tel que la suite $(i_1, \cdots, i_r)$ contient l'élément $k+1$ plus de fois que l'élément k. On pose $\mathbf{e} = \sum_{\mathbf{i}} e(\mathbf{i}) \in R_\alpha(\overline{\Gamma})$, où on fait la somme sur toutes les suites bien ordonnées dans $\overline{I}^{\overline{\alpha}}$. On démontre le théorème suivant.

Théorème 1.6.7. *Il existe un isomorphisme d'algèbres*

$$R_\alpha(\Gamma) \simeq \mathbf{e}R_{\overline{\alpha}}(\overline{\Gamma})\mathbf{e} / \sum_{\mathbf{i}} \mathbf{e}R_{\overline{\alpha}}(\overline{\Gamma})e(\mathbf{i})R_{\overline{\alpha}}(\overline{\Gamma})\mathbf{e},$$

où on fait la somme sur toutes les suites désordonnées dans $\overline{I}^{\overline{\alpha}}$. □

Le lien entre les algèbres KLR associées à un e-cycle et à un $(e+1)$-cycle obtenu dans le théorème précédent donne un lien entre les $\widetilde{\mathfrak{sl}}_e$-représentations catégoriques et les $\widetilde{\mathfrak{sl}}_{e+1}$-représentations catégoriques. On obtient le résultat suivant.

Soit $\overline{\mathcal{C}}$ une catégorie abélienne Hom-finie et $\mathbf{k}$-linéaire. On suppose que la catégorie $\overline{\mathcal{C}}$ admet une structure de $\widetilde{\mathfrak{sl}}_{e+1}$-représentation catégorique avec des foncteurs

$$\overline{E} = \overline{E}_0 \oplus \overline{E}_1 \oplus \cdots \oplus \overline{E}_e, \qquad \overline{F} = \overline{F}_0 \oplus \overline{F}_1 \oplus \cdots \oplus \overline{F}_e.$$

On suppose de plus que $\overline{\mathcal{C}}_\mu$ est zéro pour chaque $\mu \in X_{\overline{I}} \backslash X_{\overline{I}}^+$.

Pour chaque poids $\mu = \sum_{i\in I} \mu_i \varepsilon_i \in X_I$ de $\widetilde{\mathfrak{sl}}_e$ on considère le poids $\overline{\mu} = \sum_{i=0}^k \mu_i\varepsilon_i + \sum_{i=k+1}^{e-1} \mu_i\varepsilon_{i+1} \in X_{\overline{I}}$ de $\widetilde{\mathfrak{sl}}_{e+1}$. On considère la sous-catégorie $\mathcal{C} \subset \overline{\mathcal{C}}$ définie par

$$\mathcal{C} = \bigoplus_{\mu\in X_I} \overline{\mathcal{C}}_{\overline{\mu}}.$$

On considère les endofoncteurs suivants de $\mathcal{C}$:

$$F_0 = \overline{F}_0\big|_{\mathcal{C}}, \cdots, F_{k-1} = \overline{F}_{k-1}\big|_{\mathcal{C}}, \quad F_k = \overline{F}_{k+1}\overline{F}_k\big|_{\mathcal{C}},$$

$$F_{k+1} = \overline{F}_{k+2}\big|_{\mathcal{C}}, \cdots, F_{e-1} = \overline{F}_e\big|_{\mathcal{C}},$$

$$E_0 = \overline{E}_0\big|_{\mathcal{C}}, \cdots, E_{k-1} = \overline{E}_{k-1}\big|_{\mathcal{C}}, \quad E_k = \overline{E}_k\overline{F}_{k+1}\big|_{\mathcal{C}},$$

$$E_{k+1} = \overline{E}_{k+2}\big|_{\mathcal{C}}, \cdots, E_{e-1} = \overline{E}_e\big|_{\mathcal{C}}.$$

Théorème 1.6.8. *La catégorie $\mathcal{C}$ admet une structure de $\widetilde{\mathfrak{sl}}_e$-représentation catégorique définie par des foncteurs E_i, F_i, $i \in [0, e-1]$.* □

Soit $\mathbf{d} \in \mathbb{N}I$ un vecteur de dimension. On lui associe un vecteur de dimension $\overline{\mathbf{d}} \in \overline{I}$ comme ci-dessus. Soit $\overline{\mathbf{A}}^{\nu}[\mathbf{d}]$ la catégorie définie de même manière que $\mathbf{A}^{\nu}[\mathbf{d}]$ avec $e+1$ au lieu de e. Le théorème 1.6.8 permet de comparer les foncteurs F pour les catégories $\mathbf{A}^{\nu}$ et $\overline{\mathbf{A}}^{\nu}$. On démontre le théorème suivant.

Theorem 1.6.9. *Supposons $\nu_r > |\mathbf{d}|$ pour chaque $r \in [1, l]$ et $e > 2$. Il existe des équivalences de catégories θ_1, θ_2 telles que le diagramme suivant est commutatif*

$$\begin{array}{ccc} \overline{\mathbf{A}}^{\nu}[\overline{\mathbf{d}}] & \xrightarrow{F_{k+1}F_k} & \overline{\mathbf{A}}^{\nu}[\overline{\mathbf{d}}+k+(k+1)] \\ {\scriptstyle \theta_1}\Big\uparrow & & {\scriptstyle \theta_2}\Big\uparrow \\ \mathbf{A}^{\nu}[\mathbf{d}] & \xrightarrow{F_k} & \mathbf{A}^{\nu}[\mathbf{d}+k]. \end{array}$$

□

Chapitre 2

Canonical basis, KLR algebras and parity sheaves

2.1 Introduction

Let $\mathbf{k}$ be a field. Let Γ be a Dynkin quiver with the set of vertices I and let ν be a dimension vector. Let E_V be the space of representations of Γ in a ν-dimensional I-graded vector space V. Let G_V be the group of graded linear automorphisms of V. Let $\mathcal{L}_{V,\mathbf{k}}$ be the Lusztig complex in the bounded G_V-equivariant derived category $D_{G_V}(E_V, \mathbf{k})$ of sheaves of $\mathbf{k}$-vector spaces on E_V whose cohomology sheaves are constructible with respect to the stratification of E_V by G_V-orbits. Shifting the indecomposable direct factors of $\mathcal{L}_{V,\mathbf{k}}$ we construct a new complex ${}^{\delta}\mathcal{L}_{V,\mathbf{k}}$ which is Verdier self-dual. Let $\mathbf{f}$ be the positive part of the Drinfeld-Jimbo quantized enveloping algebra associated with the quiver Γ and let ${}_{\mathcal{A}}\mathbf{f}$ be Lusztig's integral form of $\mathbf{f}$, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$.

The KLR algebras have been recently introduced by Khovanov and Lauda [35] and by Rouquier [50] to give categorification of $\mathbf{f}$. The goal of Section 2.2 is to give a geometric construction of KLR algebras in arbitrary characteristic. The zero characteristic case was done in [51] and [62]. The KLR algebras in positive characteristic were studied in [36].

Let $R_{\nu,\mathbf{k}}$ be the KLR algebra over $\mathbf{k}$ associated with the quiver Γ and the dimension vector ν. Let $\mathcal{Q}_V$ be the full additive subcategory of $D_{G_V}(E_V, \mathbf{k})$ such that the objects of $\mathcal{Q}_V$ are the direct sums of shifts of direct factors of $\mathcal{L}_{V,\mathbf{k}}$. Denote by $\mathrm{proj}(R_{\nu,\mathbf{k}})$ the category of $\mathbb{Z}$-graded projective finitely generated modules over $R_{\nu,\mathbf{k}}$. We prove the following in Theorems 2.2.20, 2.2.21.

Theorem 2.1.1. (1) *The graded* $\mathbf{k}$*-algebras* $R_{\nu,\mathbf{k}}$ *and* $\mathrm{Ext}^*_{G_V}({}^{\delta}\mathcal{L}_{V,\mathbf{k}}, {}^{\delta}\mathcal{L}_{V,\mathbf{k}})$ *are isomorphic.*

(2) *The functor* $\mathcal{Q}_V \to \mathrm{proj}(R_{\nu,\mathbf{k}})$, $\mathcal{F} \mapsto \mathrm{Ext}^*_{G_V}(\mathcal{F}, {}^{\delta}\mathcal{L}_{V,\mathbf{k}})$ *yields an equivalence of categories from the opposite category* $\mathcal{Q}_V^{\mathrm{op}}$ *of* $\mathcal{Q}_V$ *to the category*

$\mathrm{proj}(R_{\nu,\mathbf{k}})$. $\square$

Let $K(R_{\nu,\mathbf{k}})$ be the split Grothendieck group of the additive category $\mathrm{proj}(R_{\nu,\mathbf{k}})$. Set $R_{\mathbf{k}} = \bigoplus_{\nu\in\mathbb{N}I} R_{\nu,\mathbf{k}}$ and $K(R_{\mathbf{k}}) = \bigoplus_{\nu\in\mathbb{N}I} K(R_{\nu,\mathbf{k}})$. Consider also the split Grothendieck group $K(\mathcal{Q}_V)$ of $\mathcal{Q}_V$ and set $\mathcal{Q} = \bigoplus_V \mathcal{Q}_V$, $K(\mathcal{Q}) = \bigoplus_V K(\mathcal{Q}_V)$, where the sum is taken over the isomorphism classes of I-graded finite dimensional $\mathbb{C}$-vector spaces. The $\mathbb{Z}$-modules $K(\mathcal{Q})$ and $K(R_{\mathbf{k}})$ are $\mathcal{A}$-modules such that q acts by the shift of grading.

The induction and restriction functors yield $\mathcal{A}$-algebra and $\mathcal{A}$-coalgebra structures on $K(R_{\mathbf{k}})$. Theorem 2.1.1 yields an $\mathcal{A}$-linear isomorphism $K(\mathcal{Q}) \simeq K(R_{\mathbf{k}})$. So we can transfer the algebra structure from $K(R_{\mathbf{k}})$ to $K(\mathcal{Q})$. This algebra structure coincides with the algebra structure given by Lusztig's induction functor, see Section 2.2.20, yielding an $\mathcal{A}$-algebra isomorphism $\lambda_{\mathcal{A}} \colon K(\mathcal{Q}) \to {}_{\mathcal{A}}\mathbf{f}$, see Theorem 2.2.22.

The construction above is well-studied in the case when the characteristic of $\mathbf{k}$ is zero. Here the decomposition theorem of Beilinson-Bernstein-Deligne is essential. It implies that the indecomposable direct factors of the complex $\mathcal{L}_{V,\mathbf{k}}$ are simple perverse sheaves up to shifts. However, the decomposition theorem fails if the characteristic of $\mathbf{k}$ is positive. Thus it is natural to look for a good replacement for simple perverse sheaves in positive characteristic. Good candidates for this role are parity sheaves introduced recently by Juteau-Mautner-Williamson.

There is no natural geometric construction of the multiplication on $K(\mathcal{Q})$ in positive characteristic. The reason is that the proof of [38, Lem. 9.2.4] uses the property [38, 8.1.6] of perverse sheaves that cannot be helpful in positive characteristic because of the failure of the decomposition theorem. However, if we make some parity assumption on Lusztig complexes then this problem can be fixed by replacing [38, 8.1.6] by Lemma 2.3.9. Unfortunately, Lusztig complexes are not known to be parity in general.

The second goal of this paper is to study the parity sheaves on E_V, see Section 2.3. Let $\mathrm{Par}_{G_V}(E_V)$ be the full additive subcategory of parity complexes in $D_{G_V}(E_V, \mathbf{k})$. Let $K(\mathrm{Par}_{G_V}(E_V))$ be its split Grothendieck group. We set

$$\mathrm{Par} = \bigoplus_V \mathrm{Par}_{G_V}(E_V), \qquad K(\mathrm{Par}) = \bigoplus_V K(\mathrm{Par}_{G_V}(E_V)),$$

where the sum is taken over the isomorphism classes of I-graded finite dimensional $\mathbb{C}$-vector spaces. The $\mathbb{Z}$-module $K(\mathrm{Par})$ is also an $\mathcal{A}$-module. The Lusztig restriction functor Res yields an $\mathcal{A}$-coalgebra structure on $K(\mathrm{Par})$. In Section 2.3 we prove the following result.

Theorem 2.1.2. *There exists an $\mathcal{A}$-coalgebra isomorphism $\beta_{\mathcal{A}} \colon K(\mathrm{Par}) \to {}_{\mathcal{A}}\mathbf{f}$.* $\square$

Theorem 2.1.2 yields an $\mathcal{A}$-basis in ${}_{\mathcal{A}}\mathbf{f}$ in terms of parity sheaves. If the characteristic of $\mathbf{k}$ is zero then this basis coincides with Lusztig's canonical basis. Note that we have no natural geometric construction of the algebra structure on $K(\mathrm{Par})$ induced by $\beta_{\mathcal{A}}$ in positive characteristic, because it is not clear if the convolution of parity complexes is again a parity complex. Lusztig's proof of the fact that the convolution of perverse sheaves in characteristic zero is a perverse sheaf modulo shifts of direct factors is based on the fact that perverse sheaves are direct factors of Lusztig complex $\mathcal{L}_{V,\mathbf{k}}$ modulo shifts. However no such result is known for parity sheaves in positive characteristic. On the other hand the parity sheaves are shifted direct factors of complexes given by Nakajima resolutions, see Sections 2.3.4-2.3.5. But it is not clear what is a convolution of such complexes.

We conjecture that the Lusztig sheaves are parity complexes. To prove the conjecture, the categorification of the multiplication and the comultiplication of $\mathbf{f}$ given by Theorems 2.1.1, 2.1.2 above should help.

Next, we say that the quiver Γ is $\mathbf{k}$-*even* if the Lusztig sheaves are even. We say that Γ is *even* if it is $\mathbf{k}$-even for each field $\mathbf{k}$. If the quiver Γ is $\mathbf{k}$-even then the categories $\mathcal{Q}_V$ and $\mathrm{Par}_{G_V}(E_V)$ coincide and we have a bialgebra isomorphism $\lambda_{\mathcal{A}}\colon K(\mathcal{Q}) \to {}_{\mathcal{A}}\mathbf{f}$, see Section 2.3.10. Note that this is the case if the characteristic of $\mathbf{k}$ is zero. The following theorem is proved in Section 2.3.11.

Theorem 2.1.3. *Each Dynkin quiver of type A is even.* $\square$

However, we have no example of a Dynkin quiver that is not $\mathbf{k}$-even for some field $\mathbf{k}$ of positive characteristic. We conjecture that the statement is still true for types D and E.

Conjecture 2.1.4. *Each Dynkin quiver is even.*

Let Λ_V be the set of G_V-orbits in E_V. For $\lambda \in \Lambda_V$ we will write $\mathcal{O}_\lambda$ for the corresponding orbit. The parity sheaves on E_V and the indecomposable objects in $\mathcal{Q}_V$ (up to shifts of complexes) are both parametrized by the set Λ_V, see Section 2.2.21 and Remark 2.3.19. We denote them respectively $\mathcal{E}(\lambda)$ and Q_λ with $\lambda \in \Lambda_V$. If Conjecture 2.1.4 is true then we have $Q_\lambda \simeq \mathcal{E}(\lambda)$ for each $\lambda \in \Lambda_V$, i.e., the indecomposable direct factors of the Lusztig complex $\mathcal{L}_{V,\mathbf{k}}$ are exactly the parity sheaves. In particular this holds for quivers of type A. Note also that Conjecture 2.1.4 is equivalent to the claim that the category of parity complexes is stable by the convolution and is also equivalent to the claim that the category $\mathcal{Q}$ is stable by the restriction.

The classes of the Q_λ's form an $\mathcal{A}$-basis in $K(\mathcal{Q}_V)$. Thus the elements $\lambda_{\mathcal{A}}([Q_\lambda])$ form an $\mathcal{A}$-basis in ${}_{\mathcal{A}}\mathbf{f}$. On the other hand the classes of the $\mathcal{E}(\lambda)$'s form an $\mathcal{A}$-basis in $K(\mathrm{Par}_{G_V}(E_V))$ and thus the elements $\beta_{\mathcal{A}}([\mathcal{E}(\lambda)])$ form an $\mathcal{A}$-basis in ${}_{\mathcal{A}}\mathbf{f}$. It is natural to compare these bases.

If the quiver Γ is even then we have $Q_\lambda \simeq \mathcal{E}(\lambda)$, see Remark 2.3.19. Then we get $\mathcal{Q}_V = \mathrm{Par}_{G_V}(E_V)$ and $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}}$. This implies $\lambda_{\mathcal{A}}([Q_\lambda]) = \beta_{\mathcal{A}}([\mathcal{E}(\lambda)])$. In particular this is true for type A.

We conjecture that the same is also true for types D and E. We get the following weak version of Conjecture 2.1.4.

Conjecture 2.1.5. *Let $\Gamma = (I, H)$ be an arbitrary Dynkin quiver. For each finite dimensional I-graded $\mathbb{C}$-vector space V and each $\lambda \in \Lambda_V$ we have* $\lambda_{\mathcal{A}}([Q_\lambda]) = \beta_{\mathcal{A}}([\mathcal{E}(\lambda)])$.

A subtle point in the theory of parity sheaves is that they don't always exist. However, they are unique up to isomorphism if they exist. Conjecture 2.1.4 implies the existence of parity sheaves on E_V because all of them can be obtained as direct factors of the Lusztig complex $\mathcal{L}_{V,\mathbf{k}}$. We can prove the existence of parity sheaves without assuming Conjecture 2.1.4 to be true. To do this we use resolutions of singularities for the G_V-orbit closures in E_V coming from Nakajima varieties, see Sections 2.3.4-2.3.5.

A similar construction can be obtained when we replace KLR algebras by quiver Schur algebras associated with a cyclic quiver. In this case an analogue of Conjecture 2.1.4 is easy to prove. See Section 2.3.13 for details.

2.2 Geometric construction of KLR algebras

By a variety we will always mean a reduced and quasi-projective scheme of finite type over $\mathbb{C}$. The symbol A will always denote a commutative ring of finite global dimension. Let $[\bullet]$ denote the shift of the cohomological degree (for complexes) or the shift of grading (for graded modules).

2.2.1 Quivers

Let Γ be a quiver without loops. We denote by I and H the sets of its vertices and arrows respectively. For an arrow $h \in H$ we will write h' and h'' for its source and target respectively. For a dimension vector $\nu = \sum_{i \in I} \nu_i \cdot i \in \mathbb{N}I$ we set

$$E_V = \bigoplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''}), \qquad |\nu| = \sum_{i \in I} \nu_i,$$

where V_i is a $\mathbb{C}$-vector space of dimension ν_i for every $i \in I$. If Γ is a Dynkin quiver, the natural action of $G_V = \prod_{i \in I} GL(V_i)$ on E_V has finitely many orbits. This defines a stratification $\coprod_{\lambda \in \Lambda_V} \mathcal{O}_\lambda$ on E_V, where Λ_V labels all orbits.

Let us introduce some notations for a later use. Let $h_{i,j}$ be the number of arrows from i to j in Γ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

Let Y_ν be the set of all pairs $y = (\mathbf{i}, \mathbf{a})$ where $\mathbf{i} = (i_1, i_2, \cdots, i_m)$ is a sequence of elements of I and $\mathbf{a} = (a_1, a_2, \cdots, a_m)$ is a sequence of nonnegative integers such that $\sum_{l=1}^m a_l i_l = \nu$. We will write $y = (i_1^{(a_1)} \cdots i_m^{(a_m)})$. Let $I^\nu \subset Y_\nu$

be the set of all pairs $y = (\mathbf{i}, \mathbf{a})$ such that $a_l = 1$ for all l. We will abbreviate $\mathbf{i}$ for $y = (\mathbf{i}, \mathbf{a})$ if $y \in I^\nu$. We denote by F_y the variety of all flags

$$\phi = (\{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V)$$

in $V = \bigoplus_{i \in I} V_i$ such that the I-graded vector space V^r/V^{r-1} has graded dimension $a_r \cdot i_r$ for $r \in [1, m]$. We denote by $\widetilde{F}_y$ the variety of pairs $(x, \phi) \in E_V \times F_y$ such that $x(V^r) \subset V^r$ for $r \in \{0, 1, \cdots, m\}$. Let π_y be the natural projection from $\widetilde{F}_y$ to E_V, i.e., $\pi_y : \widetilde{F}_y \to E_V,\ (x, \phi) \mapsto x$. For $y_1, y_2 \in Y_\nu$ we denote by Z_{y_1, y_2} the variety of triples $(x, \phi_1, \phi_2) \in E_V \times F_{y_1} \times F_{y_2}$ such that x preserves ϕ_1 and ϕ_2. For $\mathbf{i} \in I^\nu$ and $l = 1, 2, \cdots, |\nu|$, we define $\mathcal{O}_{\widetilde{F}_\mathbf{i}}(l)$ to be the G_V-equivariant line bundle over $\widetilde{F}_\mathbf{i}$ whose fiber at the flag ϕ is equal to V^l/V^{l-1}.

2.2.2 KLR algebra

Set $m = |\nu|$. The group $\mathfrak{S}_m$ acts on I^ν by $w \cdot (i_1, \cdots, i_m) = (i_{w^{-1}(1)}, \cdots, i_{w^{-1}(m)})$. We denote by s_l the transposition $(l, l+1) \in \mathfrak{S}_m$ for $l \in [1, m-1]$. For each sequence $\mathbf{i} = (i_1, i_2, \cdots, i_m) \in I^\nu$ and each integers $k, l \in [1, m]$, $l \neq m$ we abbreviate

$$h_\mathbf{i}(l) = h_{i_l, i_{l+1}} \text{ if } s_l(\mathbf{i}) \neq \mathbf{i}, \quad h_\mathbf{i}(l) = -1 \text{ if } s_l(\mathbf{i}) = \mathbf{i}.$$

$$a_\mathbf{i}(l) = h_\mathbf{i}(l) + h_{s_l(\mathbf{i})}(l) = -i_l \cdot i_{l+1}.$$

Definition 2.2.1. The *KLR algebra* $R_{\nu,A}$ (or simply R_ν if the ring of coefficients is clear) over the ring A associated with Γ and the dimension vector ν is the A-algebra generated by $1_\mathbf{i}$, $x_\mathbf{i}(k)$, $\tau_\mathbf{i}(l)$ with $\mathbf{i} \in I^\nu$, $1 \leq k, l \leq m$ and $l \neq m$, modulo the following defining relations:

- $1_\mathbf{i} 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}$,
- $\tau_\mathbf{i}(l) = 1_{s_l(\mathbf{i})} \tau_\mathbf{i}(l) 1_\mathbf{i}$,
- $x_\mathbf{i}(k) = 1_\mathbf{i} x_\mathbf{i}(k) 1_\mathbf{i}$,
- $x_\mathbf{i}(k) x_{\mathbf{i}'}(k') = x_{\mathbf{i}'}(k') x_\mathbf{i}(k)$,
- $\tau_{s_l(\mathbf{i})}(l) \tau_\mathbf{i}(l) = Q_{i,l}(x_\mathbf{i}(l), x_\mathbf{i}(l+1))$,
- $\tau_{s_l(\mathbf{i})}(l') \tau_\mathbf{i}(l) = \tau_{s_{l'}(\mathbf{i})}(l) \tau_\mathbf{i}(l')$ if $|l - l'| > 1$,
- $\tau_{s_l s_{l+1}(\mathbf{i})}(l+1) \tau_{s_{l+1}(\mathbf{i})}(l) \tau_\mathbf{i}(l+1) - \tau_{s_{l+1} s_l(\mathbf{i})}(l) \tau_{s_l(\mathbf{i})}(l+1) \tau_\mathbf{i}(l)$
  $= (Q_{\mathbf{i},l}(x_\mathbf{i}(l+2), x_\mathbf{i}(l+1)) - Q_{\mathbf{i},l}(x_\mathbf{i}(l), x_\mathbf{i}(l+1)))(x_\mathbf{i}(l+2) - x_\mathbf{i}(l))^{-1}$
  if $s_{l,l+2}(\mathbf{i}) = \mathbf{i}$ and 0 else,
- $\tau_\mathbf{i}(l) x_\mathbf{i}(k) - x_{s_l(\mathbf{i})}(s_l(k)) \tau_\mathbf{i}(l) = \begin{cases} -1_\mathbf{i} & \text{if } k = l,\ s_l(\mathbf{i}) = \mathbf{i}, \\ 1_\mathbf{i} & \text{if } k = l+1,\ s_l(\mathbf{i}) = \mathbf{i}, \\ 0 & \text{else.} \end{cases}$

Here we have set $s_{l,l+2} = s_l s_{l+1} s_l$ if $l \neq m-1, m$ and

$$Q_{\mathbf{i},l}(u, v) = \begin{cases} (-1)^{h_\mathbf{i}(l)} (u - v)^{a_\mathbf{i}(l)} & \text{if } s_l(\mathbf{i}) \neq \mathbf{i}, \\ 0 & \text{else.} \end{cases}$$

Note that the fraction

$$(Q_{\mathbf{i},l}(x_{\mathbf{i}}(l+2),x_{\mathbf{i}}(l+1))-Q_{\mathbf{i},l}(x_{\mathbf{i}}(l),x_{\mathbf{i}}(l+1)))(x_{\mathbf{i}}(l+2)-x_{\mathbf{i}}(l))^{-1}$$

is indeed an element of $A[x_{\mathbf{i}}(1),\cdots,x_{\mathbf{i}}(m)]$. Note also that $R_{\nu,A} = \bigoplus_{\mathbf{i},\mathbf{j}\in I^\nu} R^{\mathbf{ij}}_{\nu,A}$, where $R^{\mathbf{ij}}_{\nu,A} = 1_{\mathbf{i}}R_{\nu,A}1_{\mathbf{j}}$.

The KLR algebra $R_{\nu,A}$ has a natural $\mathbb{Z}$-grading such that $\deg 1_{\mathbf{i}} = 0$, $\deg x_{\mathbf{i}}(k) = 2$ and $\deg \tau_{\mathbf{i}}(l) = a_{\mathbf{i}}(l)$. The relations in Definition 2.2.1 are homogeneous with respect to this grading.

2.2.3 A faithful representation of $R_{\nu,A}$

Now, we consider the representation of $R_{\nu,A}$ on the space

$$\mathcal{P}ol_{\nu,A} = \bigoplus_{\mathbf{i}\in I^\nu}\mathcal{P}ol_{\mathbf{i},A}, \qquad \mathcal{P}ol_{\mathbf{i},A} = A[x_{\mathbf{i}}(1),\cdots,x_{\mathbf{i}}(m)],$$

which is given by:

- the element $1_{\mathbf{i}}\in R_{\nu,A}$ acts on $\mathcal{P}ol_{\nu,A}$ by the projection on $\mathcal{P}ol_{\mathbf{i},A}$,
- the element $x_{\mathbf{i}}(k)\in R_{\nu,A}$ acts on $\mathcal{P}ol_{\mathbf{j},A}$ by 0 if $\mathbf{i}\neq\mathbf{j}$ and by multiplication by $x_{\mathbf{i}}(k)$ if $\mathbf{i}=\mathbf{j}$,
- the element $\tau_{\mathbf{i}}(l)$ acts on $\mathcal{P}ol_{\mathbf{j},A}$ by 0 if $\mathbf{i}\neq\mathbf{j}$ and it sends $f\in\mathcal{P}ol_{\mathbf{i},A}$ to
  - $-\frac{f-s_l(f)}{x_{\mathbf{i}}(l)-x_{\mathbf{i}}(l+1)}$ if $s_l(\mathbf{i})=\mathbf{i}$,
  - $(x_{s_l(\mathbf{i})}(l)-x_{s_l(\mathbf{i})}(l+1))^{h_{\mathbf{i}}(l)}s_l(f)$ if $s_l(\mathbf{i})\neq\mathbf{i}$.

This representation is faithful, see [50, Sec. 3.2.2]

2.2.4 Reminder on Borel-Moore homology

In this section we suppose that G is an algebraic group which acts on a variety X. Denote by $D_G(X,A)$ the G-equivariant derived category of sheaves of A-modules on X. Let $\mathcal{D}_{X,A}$ be the G-equivariant dualizing complex on X, see [32, Def. 3.1.16] in the non-equivariant case and [10, Def. 3.5.1] in the equivariant case. The space of *G-equivariant Borel-Moore homology* is $\mathrm{H}^G_*(X,A) = \mathrm{H}^*_G(X,\mathcal{D}_{X,A})$ (more precisely, we have $\mathrm{H}^G_k(X,A) = \mathrm{H}^{-k}_G(X,\mathcal{D}_{X,A})$ for each $k\in\mathbb{Z}$). Note also that $\mathcal{D}_{X,A} = \underline{A}_X[2d]$ if X is a smooth G-variety of pure dimension d, where we denote by $\underline{A}_X$ the G-equivariant constant sheaf on X, see [12, Prop. 8.3.4]. In this case we have $\mathrm{H}^G_*(X,A) = \mathrm{H}^*_G(X,\underline{A}_X[2d])$. So we can identify $\mathrm{H}^G_*(X,A)$ with $\mathrm{H}^*_G(X,A)[2d]$. The functor

$$\mathbb{D}=\mathbb{D}_X : D_G(X,A)\to D_G(X,A), \qquad \mathfrak{L}\mapsto \mathscr{H}om(\mathfrak{L},\mathcal{D}_{X,A})$$

has the following properties, see [10, Thm. 3.5.2]:

- $(\mathbb{D}_X)^2 = \mathrm{Id}_{D_G(X,A)}$,
- $\mathbb{D}_Y f_! = f_*\mathbb{D}_X$, $\forall f : X\to Y$ continuous G-map,
- $\mathbb{D}_X f^* = f^!\mathbb{D}_Y$, $\forall f : X\to Y$ continuous G-map.

2.2.5 Algebra $\mathbf{Z}_{V,A}$

Set $G = GL(V)$. Then G_V is a closed reductive subgroup of G. Consider the varieties $Z_V = \coprod_{\mathbf{i},\mathbf{j}\in I^\nu} Z_{\mathbf{i},\mathbf{j}}$ and $\widetilde{F}_V = \coprod_{\mathbf{i}\in I^\nu} \widetilde{F}_{\mathbf{i}}$, see Section 2.2.1. For $\mathbf{i},\mathbf{j} \in I^\nu$ we set

$$\mathbf{Z}_{V,A} = \mathrm{H}_*^{G_V}(Z_V, A), \qquad \mathbf{Z}_{\mathbf{i},\mathbf{j},A} = \mathrm{H}_*^{G_V}(Z_{\mathbf{i},\mathbf{j}}, A),$$

$$\mathbf{F}_{V,A} = \mathrm{H}_*^{G_V}(\widetilde{F}_V, A), \qquad \mathbf{F}_{\mathbf{i},A} = \mathrm{H}_*^{G_V}(\widetilde{F}_{\mathbf{i}}, A).$$

We equip $\mathbf{Z}_{V,A}$ with the convolution product $\star$ relative to the closed embedding $Z_V \subset \widetilde{F}_V \times \widetilde{F}_V$. See [12, Sec. 2.7] for details. In the same way we have the $\mathbf{Z}_{V,A}$-module structure on $\mathbf{F}_{V,A}$. We will denote it also by $\star$.

Denote by F the variety of all complete flags in V. Denote by F_V the variety of all complete flags in V that agree with the grading of V, i.e., $F_V = \coprod_{\mathbf{i}\in I^\nu} F_{\mathbf{i}}$. Then $F_V \subset F$. Let T_V be a maximal torus in G_V. Denote by $\mathfrak{S}_V$ and $\mathfrak{S}$ the Weyl groups of (G_V, T_V) and (G, T_V) respectively. Fix once and for all a T_V-stable flag ϕ_V in F_V. Write

$$\phi_V = (0 \subset D_1 \subset D_1 \oplus D_2 \subset \cdots \subset D_1 \oplus D_2 \oplus \cdots \oplus D_{m-1} \subset D_1 \oplus D_2 \oplus \cdots \oplus D_m = V),$$

where $D_1, \cdots, D_m$ are T_V-stable one-dimensional vector spaces in V. The action of $\mathfrak{S}$ on the set $\{D_1, D_2, \cdots, D_m\}$ identifies $\mathfrak{S}$ with the symmetric group $\mathfrak{S}_m$. Note also that the action of $\mathfrak{S}$ on the set $\{D_1, D_2, \cdots, D_m\}$ induces the action of $\mathfrak{S}$ on the set of T_V-stable flags in F_V. Set $\phi_{V,w} = w(\phi_V)$ for $w \in \mathfrak{S}$. Denote by B and B_V the stabilizers of ϕ_V in G and G_V respectively. Let Δ and Δ_V be the root systems of (G, T_V) and (G_V, T_V) respectively. Let Δ^+ be the subset of positive roots in Δ associated with B. We set

$$\Delta^- = -\Delta^+, \qquad \Delta_V^+ = \Delta^+ \cap \Delta_V, \qquad \Delta_V^- = \Delta^- \cap \Delta_V.$$

Let $\Pi \subset \mathfrak{S}$ be the set of simple reflections. Let $\chi_1, \chi_2, \cdots, \chi_m \in \mathfrak{t}_V^*$ be the weights of the lines $D_1, D_2, \cdots, D_m$ respectively, where $\mathfrak{t}_V = \mathrm{Lie}(T_V)$. The set of simple roots in Δ^+ is $\{\chi_l - \chi_{l+1};\ l = 1, 2, \cdots, m-1\}$. Let $s_l \in \Pi$ denote the reflection with respect to the simple root $\chi_l - \chi_{l+1}$. Under the identification $\mathfrak{S} = \mathfrak{S}_m$ the reflection s_l is a transposition $(l, l+1)$.

2.2.6 Algebras $\mathbf{P}_{V,A}$ and $\mathbf{S}_{V,A}$

Denote by $\mathbf{P}_{V,A}$ the A-algebra $\mathrm{H}^*_{T_V}(\{\mathrm{pt}\}, A)$, where $\{\mathrm{pt}\}$ is the variety containing one point. The A-algebra $\mathbf{P}_{V,A}$ is isomorphic to the algebra of polynomials on $\mathfrak{t}_V$:

$$\mathbf{P}_{V,A} = A[\chi_1, \chi_2, \cdots, \chi_m],$$

where $\chi_1, \cdots, \chi_m$ are as in Section 2.2.5. Let $\mathbf{S}_{V,A}$ be the A-algebra $\mathrm{H}^*_{G_V}(\{\mathrm{pt}\}, A)$. For each positive integer n the fundamental group of $GL_n(\mathbb{C})$

is equal to $\mathbb{Z}$. Thus the fundamental group of G_V does not contain elements of finite order. Hence, by [60, Lem. 1] the torsion index of G_V is equal to 1 (see [26, Sec. 2] for the definition of a torsion index of a Lie group). So by [26, Cor. 2.3] the algebra $\mathbf{S}_{V,A}$ is isomorphic to the algebra $\mathbf{P}_{V,A}^{\mathfrak{S}_V}$ of $\mathfrak{S}_V$-invariant polynomials on $\mathfrak{t}_V$.

2.2.7 Stratification on $F_V \times F_V$

The group G acts diagonally on $F \times F$. The action of the subgroup G_V preserves the subset $F_V \times F_V$. For $x \in \mathfrak{S}$ let O_V^x be the set of all pairs of flags in $F_V \times F_V$ which are in relative position x. More precisely, write $\phi_{V,w',w} = (\phi_{V,w'}, \phi_{V,w})$, $\forall w, w' \in \mathfrak{S}$. Then we set $O_V^x = (F_V \times F_V) \cap G\phi_{V,e,x}$.

2.2.8 Stratification on Z_V

Let Z_V^x be the Zariski closure of the locally closed subset $q^{-1}(O_V^x)$, where q is a natural map

$$q : Z_V \to F_V \times F_V, \ (x, \mathcal{F}_1, \mathcal{F}_2) \mapsto (\mathcal{F}_1, \mathcal{F}_2).$$

Put

$$Z_V^{\leqslant x} = \bigcup_{w \leqslant x} Z_V^w, \qquad \mathbf{Z}_{V,A}^{\leqslant x} = \mathrm{H}_*^{G_V}(Z_V^{\leqslant x}, A), \qquad \mathbf{Z}_{V,A}^{e} = \mathrm{H}_*^{G_V}(Z_V^{\leqslant e}, A).$$

$$Z_V^{< x} = \bigcup_{w < x} Z_V^w, \qquad \mathbf{Z}_{V,A}^{< x} = \mathrm{H}_*^{G_V}(Z_V^{< x}, A),$$

where $\leqslant$ is the Bruhat order.

2.2.9 Generators of $\mathbf{Z}_{V,A}$

Denote by $1_{\mathbf{i}}$ the fundamental class of $Z_{\mathbf{i},\mathbf{i}}$ in $\mathrm{H}_*^{G_V}(Z_{\mathbf{i},\mathbf{i}}, A)$ regarded as an element of $\mathrm{H}_*^{G_V}(Z_V, A)$. Fix a simple reflection $s = s_l$ in Π. The fundamental class of Z_V^s yields an elements $\sigma(l) \in \mathbf{Z}_{V,A}^{\leqslant s}$. For $\mathbf{i}, \mathbf{i}' \in I^\nu$ we write

$$\sigma_{\mathbf{i}',\mathbf{i}}(l) = 1_{\mathbf{i}'} \star \sigma(l) \star 1_{\mathbf{i}} \in \mathbf{Z}_{V,A}^{\leqslant s}.$$

Next, suppose $k \in [1, m]$. The pull-back of the first Chern class of the line bundle $\bigoplus_{\mathbf{i}} \mathcal{O}_{\widetilde{F}_{\mathbf{i}}}(k)$ by the first projection

$$Z_V^e \subset \widetilde{F}_V \times \widetilde{F}_V \xrightarrow{pr_1} \widetilde{F}_V$$

belongs to $\mathrm{H}^*_{G_V}(Z_V^e, A)$, where $\mathcal{O}_{\widetilde{F}_{\mathbf{i}}}(k)$ is as in Section 2.2.1. It yields an element $x(k) \in \mathbf{Z}_{V,A}^e$. We write

$$x_{\mathbf{i}',\mathbf{i}}(k) = 1_{\mathbf{i}'} \star x(k) \star 1_{\mathbf{i}} \in \mathbf{Z}_{V,A}^e.$$

We will need the following standard lemma, see [12, Chap. 6].

Lemma 2.2.2. (a) *The direct image by the closed embedding* $Z_V^{\leqslant x} \subset Z_V$ *gives an injective* A*-module homomorphism* $\mathbf{Z}_{V,A}^{\leqslant x} \to \mathbf{Z}_{V,A}$.

(b) *For* $x, y \in \mathfrak{S}$ *such that* $l(xy) = l(x) + l(y)$ *the convolution product in* $\mathbf{Z}_{V,A}$ *yields an inclusion* $\mathbf{Z}_{V,A}^{\leqslant x} \star \mathbf{Z}_{V,A}^{\leqslant y} \subset \mathbf{Z}_{V,A}^{\leqslant xy}$. *In particular* $\mathbf{Z}_{V,A}^{e}$ *is a subalgebra of* $\mathbf{Z}_{V,A}$.

Proof. All the statements are obvious except, perhaps, the injectivity in (a). It follows from the existence of a decomposition

$$Z_V^{\leqslant x} = \coprod_{w \leqslant x} q^{-1}(O_V^w).$$

We need the following obvious lemma.

Lemma 2.2.3. *For each* G_V*-orbit* $\mathcal{O}$ *in* $F_V \times F_V$ *the map* $p : q^{-1}(\mathcal{O}) \to F_{\mathbf{i}}$ *such that* $(x, \phi_1, \phi_2) \mapsto \phi_2$ *for each* $x \in E_V,\ \phi_1, \phi_2 \in F_V$ *is an affine* G_V*-equivariant fibration, where* $\mathbf{i}$ *is the unique element of* I^ν *such that* $\mathcal{O} \subset F_V \times F_{\mathbf{i}}$. □

Now we can conclude by applying the arguments of [12, Sec. 6.2] and the exact sequence in equivariant Borel-Moore homology. We should just replace [12, Lem. 6.2.5] by Lemma 2.2.3. □

So we can consider $x_{\mathbf{i}',\mathbf{i}}(k)$ and $\sigma_{\mathbf{i}',\mathbf{i}}(l)$ as elements of $\mathbf{Z}_{V,A}$.

2.2.10 The A-algebra structure on $\mathbf{F}_{V,A}$

Let $\mathbf{i} \in I^\nu$. Assigning to a formal variable $x_{\mathbf{i}}(l)$ of degree 2 the first Chern class of G_V-equivariant line bundle $\mathcal{O}_{\widetilde{F}_{\mathbf{i}}}(l)$ for $l = 1, 2, \cdots, m$ yields a graded A-algebra isomorphism

$$A[x_{\mathbf{i}}(1), x_{\mathbf{i}}(2), \cdots, x_{\mathbf{i}}(m)] = \mathbf{F}_{\mathbf{i},A}.$$

The multiplication of polynomials equips each $\mathbf{F}_{\mathbf{i},A}$ with an obvious graded A-algebra structure. Thus, $\mathbf{F}_{V,A} = \bigoplus_{\mathbf{i} \in I^\nu} \mathbf{F}_{\mathbf{i},A}$ is also a graded A-algebra.

Now we can consider $\mathbf{Z}_{V,A}$ as an $\mathbf{F}_{V,A}$-module such that $f(x_{\mathbf{i}}(1), \cdots, x_{\mathbf{i}}(m)) \in \mathbf{F}_{\mathbf{i},A}$ acts on $\mathbf{Z}_{V,A}$ by the operator $z \mapsto f(x_{\mathbf{i}}(1), \cdots, x_{\mathbf{i}}(m)) \star z$. We can interpret this $\mathbf{F}_{V,A}$-action on $\mathbf{Z}_{V,A}$ as follows. Note that

$$\begin{aligned} \widetilde{F}_V &= \{(x, \phi) \in E_V \times F_V;\ x \text{ preserves } \phi\} \\ &\simeq \{(x, \phi, \phi) \in E_V \times F_V \times F_V;\ x \text{ preserves } \phi\} \\ &= Z_V^e. \end{aligned}$$

Then we have $\mathbf{F}_{V,A} \simeq \mathbf{Z}_{V,A}^e$. So the $\mathbf{F}_{V,A}$-action on $\mathbf{Z}_{V,A}$ is the action of the subalgebra $\mathbf{Z}_{V,A}^e$ on $\mathbf{Z}_{V,A}$.

2.2.11 KLR algebra and Borel-Moore homology

For each sequence $\mathbf{i} = (i_1, i_2, \cdots, i_m) \in I^\nu$ and for each integers $k, l \in [1, m]$, $l \neq m$ we consider the following elements $x_{\mathbf{i}}(k) = x_{\mathbf{i},\mathbf{i}}(k)$, $\sigma_{\mathbf{i}}(l) = \sigma_{s_l(\mathbf{i}),\mathbf{i}}(l)$ in $\mathbf{Z}_{V,A}$.

Theorem 2.2.4. *We have an A-algebra isomorphism $R_{\nu,A} \simeq \mathbf{Z}_{V,A}$.*

Proof. First, we claim that there is an A-algebra homomorphism $\Phi_A : R_{\nu,A} \to \mathbf{Z}_{V,A}$ given by

$$\Phi_A(1_{\mathbf{i}}) = 1_{\mathbf{i}}, \qquad \Phi_A(x_{\mathbf{i}}(k)) = x_{\mathbf{i}}(k), \qquad \Phi_A(\tau_{\mathbf{i}}(l)) = (-1)^{h_{\mathbf{i}}(l)}\sigma_{\mathbf{i}}(l).$$

To prove this we must check that the relations from Definition 2.2.1 hold for the elements $1_{\mathbf{i}}$, $x_{\mathbf{i}}(k)$, $(-1)^{h_{\mathbf{i}}(l)}\sigma_{\mathbf{i}}(l)$ in $\mathbf{Z}_{V,A}$. Now according to [62, Thm. 3.6], this holds for $A = \mathbb{C}$. We want to deduce from this that the relations hold for any A. To do this, we first construct an $\mathbf{F}_{V,A}$-basis of $\mathbf{Z}_{V,A}$ using the following lemma, which can be proved in the same way as Lemma 2.2.2.

Lemma 2.2.5. *We have $\mathbf{Z}^{\leqslant x}_{V,A} = \bigoplus_{w \leqslant x} \mathbf{F}_{V,A} \star [Z^w_V]$ and $\mathbf{Z}^{<x}_{V,A} = \bigoplus_{w<x} \mathbf{F}_{V,A} \star [Z^w_V]$ for $w \in \mathfrak{S}$. In particular $\mathbf{Z}_{V,A}$ is a free graded $\mathbf{F}_{V,A}$-module of rank $m!$.* □

By Lemma 2.2.5 the $\mathbb{Z}$-module $\mathbf{Z}_{V,\mathbb{Z}}$ is free. So by the universal coefficient theorem we have an inclusion of rings $j_A \colon \mathbf{Z}_{V,\mathbb{Z}} \to \mathbf{Z}_{V,A}$. By [35, Thm. 2.5] the $\mathbb{Z}$-module $R_{\nu,\mathbb{Z}}$ is free. So we have an inclusion $i_A \colon R_{\nu,\mathbb{Z}} \to R_{\nu,A}$. Since $j_{\mathbb{C}}$ takes the elements $1_{\mathbf{i}}$, $x_{\mathbf{i}}(k)$, $(-1)^{h_{\mathbf{i}}(l)}\sigma_{\mathbf{i}}(l)$ in $\mathbf{Z}_{V,\mathbb{Z}}$ to the corresponding elements in $\mathbf{Z}_{V,\mathbb{C}}$, the defining relations in $R_{\nu,\mathbb{Z}}$ are satisfied by the generators of $\mathbf{Z}_{V,\mathbb{Z}}$. Thus, the ring homomorphism $\Phi_{\mathbb{Z}} : R_{\nu,\mathbb{Z}} \to \mathbf{Z}_{V,\mathbb{Z}}$ is well-defined and we have the following commutative diagram

$$\begin{tikzcd} R_{\nu,\mathbb{C}} \arrow[r, "\Phi_{\mathbb{C}}"] & \mathbf{Z}_{\mathbf{V},\mathbb{C}} \\ R_{\nu,\mathbb{Z}} \arrow[u, "i_{\mathbb{C}}"] \arrow[r, "\Phi_{\mathbb{Z}}"] & \mathbf{Z}_{\mathbf{V},\mathbb{Z}}. \arrow[u, "j_{\mathbb{C}}"'] \end{tikzcd}$$

Now we must prove that $\Phi_{\mathbb{Z}}$ is bijective. Note that $\mathbf{F}_{V,A} = \bigoplus_{\mathbf{i} \in I^\nu} A[x_{\mathbf{i}}(1), x_{\mathbf{i}}(2), \cdots, x_{\mathbf{i}}(m)]$ acts on $R_{\nu,A}$ by multiplication by polynomials. It is easy to see that the map $\Phi_{\mathbb{Z}}$ is $\mathbf{F}_{V,\mathbb{Z}}$-linear. Further, the morphism $\Phi_{\mathbb{Z}}$ is injective because $i_{\mathbb{C}}$ and $\Phi_{\mathbb{C}}$ are. Let us prove that $\Phi_{\mathbb{Z}}$ is surjective. We will need the following lemma.

Lemma 2.2.6. *The following relations hold in $\mathbf{Z}_{V,\mathbb{Z}}$:*
(a) $\sigma_{\mathbf{i},\mathbf{j}}(l) = 0$ *unless* $\mathbf{i} \in \{\mathbf{j}, s_l(\mathbf{j})\}$,
(b) $\sigma_{\mathbf{i},\mathbf{i}}(l) = (x_{\mathbf{i}}(l+1) - x_{\mathbf{i}}(l))^{h_{i_l,i_{l+1}}}$ *if* $s_l(\mathbf{i}) \neq \mathbf{i}$.

Proof. By [62, Lem. 1.8 (a), Prop. 2.22] these relations hold in $\mathbf{Z}_{V,\mathbb{C}}$. Thus, they also hold in $\mathbf{Z}_{V,\mathbb{Z}}$ by injectivity of the map $j_{\mathbb{C}}$. □

The image of $\Phi_{\mathbb{Z}}$ contains the elements $x_{\mathbf{i}}(k)$ and $1_{\mathbf{i}}$ for each $\mathbf{i} \in I^\nu$ and each $k \in [1, m]$. So it contains $\mathbf{F}_{V,\mathbb{Z}} \star [Z_V^e]$ because

$$[Z_V^e] = \sum_{\mathbf{i} \in I^\nu} 1_{\mathbf{i}}, \qquad \mathbf{F}_{V,\mathbb{Z}} = \bigoplus_{\mathbf{i} \in I^\nu} \mathbb{Z}[x_{\mathbf{i}}(1), x_{\mathbf{i}}(2), \cdots, x_{\mathbf{i}}(m)].$$

Next, we prove that the image of $\Phi_{\mathbb{Z}}$ contains $[Z_V^{s_l}]$ for $l \in [1, m-1]$. By Lemma 2.2.6 (a) we have

$$\begin{aligned}
[Z_V^{s_l}] &= \sigma(l) \\
&= \sum_{\mathbf{i},\mathbf{j} \in I^\nu} \sigma_{\mathbf{i},\mathbf{j}}(l) \\
&= \sum_{\mathbf{i} \in I^\nu} \sigma_{s_l(\mathbf{i}),\mathbf{i}}(l) + \sum_{\mathbf{i} \in I^\nu,\ s_l(\mathbf{i}) \neq \mathbf{i}} \sigma_{\mathbf{i},\mathbf{i}}(l) \\
&= \sum_{\mathbf{i} \in I^\nu} \sigma_{\mathbf{i}}(l) + \sum_{\mathbf{i} \in I^\nu,\ s_l(\mathbf{i}) \neq \mathbf{i}} \sigma_{\mathbf{i},\mathbf{i}}(l).
\end{aligned}$$

We know that $\sigma_{\mathbf{i}}(l) = \Phi_{\mathbb{Z}}((-1)^{h_{\mathbf{i}}(l)} \tau_{\mathbf{i}}(l))$. If $s_l(\mathbf{i}) \neq \mathbf{i}$ then by Lemma 2.2.6 (b) we have

$$\sigma_{\mathbf{i},\mathbf{i}}(l) = (x_{\mathbf{i}}(l+1) - x_{\mathbf{i}}(l))^{h_{i_l,i_{l+1}}} = \Phi_{\mathbb{Z}}((x_{\mathbf{i}}(l+1) - x_{\mathbf{i}}(l))^{h_{i_l,i_{l+1}}}).$$

So $[Z_V^{s_l}]$ is in the image of $\Phi_{\mathbb{Z}}$. To conclude we use Lemma 2.2.5 and the following lemma, see [62, Lem. 3.7].

Lemma 2.2.7. *If $l(s_l w) = l(w) + 1$ we have $[Z_V^{s_l}] \star [Z_V^w] = [Z_V^{s_l w}]$ in $\mathbf{Z}_V^{\leqslant s_l w} / \mathbf{Z}_V^{< s_l w}$.* □

This proves that $\Phi_{\mathbb{Z}}$ is bijective. Thus Φ_A is well-defined and bijective for arbitrary commutative ring A because $R_{\nu,A} = R_{\nu,\mathbb{Z}} \otimes A$ and $\mathbf{Z}_{V,A} = \mathbf{Z}_{V,\mathbb{Z}} \otimes A$. □

2.2.12 Stratified varieties

Let X be a G-variety for some connected algebraic group G. We fix a stratification $X = \coprod_{\lambda \in \Lambda} X_\lambda$ of X into smooth connected locally closed G-stable subsets such that the closure of each stratum is a union of strata. Let $D_G(X, A)$ be the bounded G-equivariant derived category of sheaves of A-modules on X constructible with respect to the stratification $X = \coprod_{\lambda \in \Lambda} X_\lambda$.

Lemma 2.2.8. *Suppose that A is either a complete discrete valuation ring or a field. The category $D_G(X, A)$ is a Krull-Schmidt category.*

Proof. By the theorem in the introduction of [37] the category $D_G(X, A)$ is Karoubian because its t-structure is bounded, see [10, Def. 2.2.4, Prop. 2.5.2]. Let $\mathbf{k}$ be the factor of A by its unique maximal ideal. We need to prove that the endomorphism ring $\mathrm{End}(\mathcal{F})$ of each indecomposable object $\mathcal{F}$ in $D_G(X, A)$ is local. Note that under the assumption on the ring A the endomorphism ring $\mathrm{End}(\mathcal{F})$ is local if and only if the ring $\mathbf{k} \otimes_A \mathrm{End}(\mathcal{F})$ is local.

Now, suppose that the ring $\mathbf{k}\otimes_A \mathrm{End}(\mathcal{F})$ is not local. The endomorphism ring of every object of $D_G(X, A)$ is of finite type over A, see the construction of the bounded equivariant derived category in [10, Sec. 2.1-2.2]. Thus the ring $\mathbf{k}\otimes_A\mathrm{End}(\mathcal{F})$ is a finite dimensional $\mathbf{k}$-algebra. In particular the ring $\mathbf{k}\otimes_A \mathrm{End}(\mathcal{F})$ is artinian. Then the identity of $\mathbf{k}\otimes_A\mathrm{End}(\mathcal{F})$ can be written as a sum of two orthogonal idempotents e_1 and e_2 because $\mathbf{k} \otimes_A \mathrm{End}(\mathcal{F})$ is not local. Moreover, by [14, Ex. 6.16] we can lift these idempotents to the idempotents $\widetilde{e}_1, \widetilde{e}_2$ of $\mathrm{End}(\mathcal{F})$ such that $\widetilde{e}_1 + \widetilde{e}_2 = \mathrm{Id}_{\mathcal{F}}$. Thus by Karoubianness the object $\mathcal{F}$ is not indecomposable. □

For each $\lambda \in \Lambda$ we denote by i_λ the inclusion $X_\lambda \to X$ and by d_λ the complex dimension of X_λ. For each $x \in X$ we denote by i_x the inclusion $\{x\} \to X$. For all $\lambda \in \Lambda$, let $\underline{A}_\lambda$ be the G-equivariant constant sheaf on X_λ. For each λ let $\mathrm{Loc}_G(X_\lambda, A)$ be the category of G-equivariant local systems of free A-modules on X_λ. Let $\underline{A}_X$ be the G-equivariant constant sheaf on X.

2.2.13 Lusztig category $\mathcal{Q}_V$

From now on we suppose that Γ is a Dynkin quiver. We have the stratification $E_V = \coprod_{\lambda\in\Lambda_V} \mathcal{O}_\lambda$ on E_V. Here Λ_V is the set of G_V-orbits in E_V. For $\lambda \in \Lambda_V$ we will use the notation i_λ and d_λ as in Section 2.2.12, i.e., the map i_λ is the inclusion $\mathcal{O}_\lambda \to E_V$ and d_λ is the complex dimension of $\mathcal{O}_\lambda$. Consider the following complex in $D_{G_V}(E_V, A)$:

$$\mathcal{L}_{V,A} = \bigoplus_{\mathbf{i}\in I^\nu} \mathcal{L}_{\mathbf{i}}, \qquad \mathcal{L}_{\mathbf{i}} = \pi_{\mathbf{i}*}\underline{A}_{\widetilde{F}_{\mathbf{i}}}, \quad \mathbf{i} \in I^\nu.$$

Consider also the complex in $D_{G_V}(E_V, A)$:

$${}^\delta\mathcal{L}_{V,A} = \bigoplus_{\mathbf{i}\in I^\nu} {}^\delta\mathcal{L}_{\mathbf{i}}, \qquad {}^\delta\mathcal{L}_{\mathbf{i}} = \mathcal{L}_{\mathbf{i}}[\dim_{\mathbb{C}} \widetilde{F}_{\mathbf{i}}], \quad \mathbf{i} \in I^\nu.$$

Definition 2.2.9. Let $\mathcal{Q}_V$ be the full additive subcategory of $D_{G_V}(E_V, A)$ whose objects are the finite direct sums of shifts of direct factors of $\mathcal{L}_{V,A}$.

2.2.14 Local systems

The following lemma is proved in [11, Prop. 2.2.1].

Lemma 2.2.10. *The reductive part of the stabilizer in G_V of an element in E_V is isomorphic to $\prod_i GL_{n_i}(\mathbb{C})$ for some positive integers n_i.* □

We get the following result from Lemma 2.2.10 and [12, Lem. 8.4.11].

Corollary 2.2.11. *Let $\mathcal{O}$ be a G_V-orbit in E_V. Each local system in $\mathrm{Loc}_{G_V}(\mathcal{O})$ is a multiple of the trivial G_V-equivariant local system $\underline{A}_{\mathcal{O}}$ on $\mathcal{O}$.* □

2.2.15 KLR algebra and Yoneda algebras

Consider the G_V-equivariant extension group $\mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$ with the A-algebra structure given by the Yoneda product.

Theorem 2.2.12. *We have the following A-algebra isomorphism $R_{\nu,A} \simeq \mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$.*

Proof. We have proved in Theorem 2.2.4 that there is an A-algebra isomorphism

$$R_{\nu,A} \simeq \mathrm{H}^{G_V}_*(Z_V, A).$$

Now it suffices to show that there exists an A-algebra isomorphism

$$\mathrm{H}^{G_V}_*(Z_V, A) \simeq \mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A}).$$

For each $\mathbf{i}, \mathbf{j} \in I^\nu$ consider the following commutative diagram

$$\begin{array}{ccc} Z_{\mathbf{ij}} & \xrightarrow{p_2} & \widetilde{F}_{\mathbf{j}} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbf{j}}} \\ \widetilde{F}_{\mathbf{i}} & \xrightarrow{\pi_{\mathbf{i}}} & E_V, \end{array}$$

where p_1 and p_2 are the natural projections. For each $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathrm{Ext}^n_{G_V}((\pi_{\mathbf{i}})_*\underline{A}_{\widetilde{F}_{\mathbf{i}}}, (\pi_{\mathbf{j}})_*\underline{A}_{\widetilde{F}_{\mathbf{j}}}) &\simeq \mathrm{Ext}^n_{G_V}(\underline{A}_{\widetilde{F}_{\mathbf{i}}}, (\pi_{\mathbf{i}})^!(\pi_{\mathbf{j}})_*\underline{A}_{\widetilde{F}_{\mathbf{j}}}) \\ &\simeq \mathrm{Ext}^n_{G_V}(\underline{A}_{\widetilde{F}_{\mathbf{i}}}, (p_1)_*(p_2)^!\underline{A}_{\widetilde{F}_{\mathbf{j}}}) \\ &\simeq \mathrm{Ext}^n_{G_V}((p_1)^*\underline{A}_{\widetilde{F}_{\mathbf{i}}}, (p_2)^!\underline{A}_{\widetilde{F}_{\mathbf{j}}}) \\ &\simeq \mathrm{Ext}^n_{G_V}(\underline{A}_{Z_{\mathbf{ij}}}, \mathcal{D}_{Z_{\mathbf{ij}},A}[-2\dim_{\mathbb{C}} \widetilde{F}_{\mathbf{j}}]) \\ &\simeq \mathrm{H}^{n-2\dim_{\mathbb{C}} \widetilde{F}_{\mathbf{j}}}_{G_V}(Z_{\mathbf{ij}}, \mathcal{D}_{Z_{\mathbf{ij}},A}) \\ &\simeq \mathrm{H}^{G_V}_{2\dim_{\mathbb{C}} \widetilde{F}_{\mathbf{j}}-n}(Z_{\mathbf{ij}}, A). \end{aligned}$$

Thus we have a bijection between the sets $\mathrm{H}^{G_V}_*(Z_V, A)$ and $\mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$. Now we need to show that the convolution product on

$\mathrm{H}_*^{G_V}(Z_V, A)$ coincides with the Yoneda product on $\mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$. At first note that this follows from [12, Prop. 8.6.35] for $A = \mathbb{C}$. Now we show this for $A = \mathbb{Z}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_*^{G_V}(Z_V, \mathbb{C}) & \longrightarrow & \mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,\mathbb{C}}, \mathcal{L}_{V,\mathbb{C}}) \\ \uparrow & & \uparrow \\ \mathrm{H}_*^{G_V}(Z_V, \mathbb{Z}) & \longrightarrow & \mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,\mathbb{Z}}, \mathcal{L}_{V,\mathbb{Z}}), \end{array}$$

where the left vertical map is an algebra inclusion, the right vertical map is an algebra homomorphism, the top horizontal map is an algebra isomorphism and the bottom horizontal map is a set bijection. Thus the right vertical map is automatically an algebra inclusion and the bottom horizontal map is automatically an algebra isomorphism. The general case follows from the case $A = \mathbb{Z}$ by extension of scalars. □

Remark 2.2.13. The results of this section are also true for each quiver Γ without loops.

2.2.16 Functor Y'

We consider two different gradings on the algebra $R_{\nu,A}$:

1. the grading introduced in Section 2.2.2,
2. the grading $\bigoplus_{n\in\mathbb{N}} \mathrm{Ext}^n_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$, see Theorem 2.2.12.

Denote by $\mathrm{mod}(R_{\nu,A})$ (resp. $\mathrm{mod}'(R_{\nu,A})$) the category of finitely generated $\mathbb{Z}$-graded $R_{\nu,A}$-modules with respect to the first (resp. second) grading. Denote by $\mathrm{proj}(R_{\nu,A})$ (resp. $\mathrm{proj}'(R_{\nu,A})$) the category of projective finitely generated $\mathbb{Z}$-graded $R_{\nu,A}$-modules with respect to the first (resp. second) grading. Consider the following contravariant functor:

$$Y' : D_{G_V}(E_V, A) \to \mathrm{mod}'(R_{\nu,A}),\ \mathfrak{L} \mapsto \mathrm{Ext}^*_{G_V}(\mathfrak{L}, \mathcal{L}_{V,A}).$$

Theorem 2.2.14. *The restriction of the functor Y' to $\mathcal{Q}_V$ yields an equivalence of categories $\mathcal{Q}_V^{\mathrm{op}} \to \mathrm{proj}'(R_{\nu,A})$.*

Proof. Suppose that $\mathcal{F}$ is a direct factor of the complex $\mathcal{L}_{V,A}$ in $D_{G_V}(E_V, A)$. We have $\mathcal{L}_{V,A} = \mathcal{F} \oplus \mathcal{G}$ for some complex $\mathcal{G}$ in $D_{G_V}(E_V, A)$. We have

$$R_{\nu,A} = \mathrm{Ext}^*_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A}) = \mathrm{Ext}^*_{G_V}(\mathcal{F}, \mathcal{L}_{V,A}) \oplus \mathrm{Ext}^*_{G_V}(\mathcal{G}, \mathcal{L}_{V,A}) = Y'(\mathcal{F}) \oplus Y'(\mathcal{G}).$$

Thus $Y'(\mathcal{F})$ is a projective and finitely generated $R_{\nu,A}$-module. From this we see that for every $X \in \mathcal{Q}_V$ the $R_{\nu,A}$-module $Y'(X)$ is projective and finitely generated. The $R_{\nu,A}$-module $Y'(X)$ is graded by $Y'(X)_r = \mathrm{Ext}^r_{G_V}(X, \mathcal{L}_{V,A})$ for each $r \in \mathbb{Z}$. Now we prove that the functor $Y' \colon \mathcal{Q}_V^{\mathrm{op}} \to \mathrm{proj}' R_{\nu,A}$ is an isomorphism on morphisms. We need to show that for $R, S \in \mathcal{Q}_V$ the morphism

$$Y'_{R,S} : \mathrm{Hom}_{\mathcal{Q}_V}(R, S) \to \mathrm{Hom}_{\mathrm{proj}'(R_{\nu,A})}(\mathrm{Ext}^*_{G_V}(S, \mathcal{L}_{V,A}), \mathrm{Ext}^*_{G_V}(R, \mathcal{L}_{V,A}))$$

is an isomorphism. First, we consider the case $R = S = \mathcal{L}_{V,A}$. Then we have

$$\mathrm{Hom}_{\mathcal{Q}_V}(R, S) = \mathrm{Hom}_{\mathcal{Q}_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A}),$$

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{proj}'(R_{\nu,A})}(\mathrm{Ext}^*_{G_V}(S, \mathcal{L}_{V,A}), \mathrm{Ext}^*_{G_V}(R, \mathcal{L}_{V,A})) &= \mathrm{Hom}_{\mathrm{proj}'(R_{\nu,A})}(R_{\nu,A}, R_{\nu,A}) \\
&= R^0_{\nu,A} \\
&= \mathrm{Ext}^0_{G_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A}) \\
&= \mathrm{Hom}_{\mathcal{Q}_V}(\mathcal{L}_{V,A}, \mathcal{L}_{V,A}).
\end{aligned}$$

Here $R^0_{\nu,A}$ means the homogeneous component of degree 0 in $R_{\nu,A}$ with respect to the second grading. So for $R = S = \mathcal{L}_{V,A}$ the statement is true. Then it is also true if R and S are direct factors of $\mathcal{L}_{V,A}$. Then it is also true if R and S are shifts of direct factors of $\mathcal{L}_{V,A}$. Finally we see that the statement is true for arbitrary $R, S \in \mathcal{Q}_V$.

To conclude we need to show that $Y' \colon \mathcal{Q}_V^{\mathrm{op}} \to \mathrm{proj}'(R_{\nu,A})$ is surjective on isomorphism classes. Let P be a module in $\mathrm{proj}'(R_{\nu,A})$. A choice of a finite number of homogeneous generators of P yields an epimorphism

$$\pi : \bigoplus_{i \in J} R_{\nu,A}[n_i] \to P$$

for some finite set J and some integers n_i. By projectivity this epimorphism splits. Consider the complex $\mathfrak{L} = \bigoplus_{i \in I} \mathcal{L}_{V,A}[-n_i]$. We have

$$Y'(\mathfrak{L}) = \mathrm{Hom}(\mathfrak{L}, \mathcal{L}_{V,A}) = \bigoplus_{i \in J} R_{\nu,A}[n_i].$$

The composition of the natural projection and inclusion

$$Y'(\mathfrak{L}) \overset{\pi_e}{\to} P \overset{i_e}{\to} Y'(\mathfrak{L})$$

yields an idempotent $e \in \mathrm{End}_{\mathrm{proj}'(R_{\nu,A})}(Y'(\mathfrak{L}))$. The functor $Y' \colon \mathcal{Q}_V^{\mathrm{op}} \to \mathrm{proj}'(R_{\nu,A})$ is an isomorphism on morphisms. So we have an idempotent $u \in \mathrm{End}_{\mathcal{Q}_V}(\mathfrak{L})$ such that $Y'(u) = e$. The idempotents u and $1 - u$ split in $D_{G_V}(E_V, A)$ yielding a decomposition $\mathfrak{L} = X \oplus Y$, where $X, Y \in D_{G_V}(E_V, A)$ and u is a composition of the natural projection and the natural inclusion $\mathfrak{L} \overset{\pi_u}{\to} X \overset{i_u}{\to} \mathfrak{L}$. By the definition of $\mathcal{Q}_V$ the objects X, Y belong to $\mathcal{Q}_V$. Now we verify that the map $\pi_e \circ Y'(i_u) : Y'(X) \to P$ is an isomorphism. Let us prove that the map $Y'(\pi_u) \circ i_e : P \to Y'(X)$ is its inverse. We have

$$\begin{aligned}
\pi_e \circ Y'(i_u) \circ Y'(\pi_u) \circ i_e &= \pi_e \circ Y'(u) \circ i_e \\
&= \pi_e \circ e \circ i_e \\
&= (\pi_e \circ i_e) \circ (\pi_e \circ i_e) \\
&= \mathrm{Id}_P
\end{aligned}$$

and

$$\begin{aligned}
Y'(\pi_u) \circ i_e \circ \pi_e \circ Y'(i_u) &= Y'(\pi_u) \circ e \circ Y'(i_u) \\
&= Y'(\pi_u) \circ Y'(u) \circ Y'(i_u) \\
&= Y'(\pi_u \circ u \circ i_u) \\
&= Y'((\pi_u \circ i_u) \circ (\pi_u \circ i_u)) \\
&= Y'(\mathrm{Id}_X) \\
&= \mathrm{Id}_{Y'(X)}.
\end{aligned}$$

□

2.2.17 Induction and restriction

Let $\mathbf{k}$ be a field.

In this section we describe the induction and the restriction functors for representations of KLR algebras. See [35, Sec. 2.6] for more details. Consider $\nu_1, \nu_2 \in \mathbb{N}I$ and set $\nu = \nu_1 + \nu_2$. We set

$$|\nu_1| = m_1, \qquad |\nu_2| = m_2, \qquad |\nu| = m.$$

There is a unique inclusion of graded $\mathbf{k}$-algebras $R_{\nu_1,\mathbf{k}} \otimes R_{\nu_2,\mathbf{k}} \subset R_{\nu,\mathbf{k}}$ such that

$$1_{\mathbf{i}} \otimes 1_{\mathbf{j}} \mapsto 1_{\mathbf{k}}, \qquad x_{\mathbf{i}}(k) \otimes 1_{\mathbf{j}} \mapsto x_{\mathbf{k}}(k), \qquad 1_{\mathbf{i}} \otimes x_{\mathbf{j}}(k) \mapsto x_{\mathbf{k}}(m_1 + k),$$

$$\tau_{\mathbf{i}}(l) \otimes 1_{\mathbf{j}} \mapsto \tau_{\mathbf{k}}(l), \qquad 1_{\mathbf{i}} \otimes \tau_{\mathbf{j}}(l) \mapsto \tau_{\mathbf{k}}(m_1 + l)$$

for each $k, l, \mathbf{i}, \mathbf{j}$. Let $1_{\nu_1,\nu_2}$ be the image of the identity element by this inclusion. We get the following functors:

$$\mathrm{Ind}_{\nu_1,\nu_2} : \mathrm{mod}(R_{\nu_1,\mathbf{k}}) \times \mathrm{mod}(R_{\nu_2,\mathbf{k}}) \to \mathrm{mod}(R_{\nu,\mathbf{k}}), \quad (M_1, M_2) \mapsto R_{\nu,\mathbf{k}} 1_{\nu_1,\nu_2} \bigotimes_{R_{\nu_1,\mathbf{k}} \otimes R_{\nu_2,\mathbf{k}}} (M_1 \otimes M_2),$$

$$\mathrm{Res}_{\nu_1,\nu_2} : \mathrm{mod}(R_{\nu,\mathbf{k}}) \to \mathrm{mod}(R_{\nu_1,\mathbf{k}} \otimes R_{\nu_2,\mathbf{k}}), \qquad M \mapsto 1_{\nu_1,\nu_2} M.$$

These functors take projective modules to projective modules, see [35, Sec. 2.6].

2.2.18 Projective $R_{\nu,\mathbf{k}}$-modules

Denote by $\mathcal{A}$ the ring $\mathbb{Z}[q,q^{-1}]$. For $n \in \mathbb{N},\ n > 0$ consider the following elements of $\mathcal{A}$

$$[n] = \sum_{l=1}^{n} q^{n+1-2l} = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = \prod_{l=1}^{n} [l], \qquad l_n = n(n-1)/2.$$

We also set $[0]! = 1$, $l_0 = 0$. For $\mathbf{n} = (n_1, \cdots, n_k)$, $n_l \in \mathbb{N}$ set

$$[\mathbf{n}]! = \prod_{l=1}^{k} [n_l]!, \qquad l_{\mathbf{n}} = \sum_{l=1}^{k} l_{n_l}.$$

Given a pair $y = (\mathbf{i}, \mathbf{a}) \in Y_\nu$ we define a projective $R_{\nu,\mathbf{k}}$-module R_y as follows

- If $I = \{i\}$, $\nu = mi$, $\mathbf{i} = (i, i, \cdots, i)$ and $y = (i, m)$ we set

  $$\mathbf{P}_y = \mathbf{P}_{i,m} = \mathcal{P}ol_{\nu,A}[l_m].$$

  We have the following isomorphism of left graded $R_{\nu,\mathbf{k}}$-modules, see [62, Example 2.28 (a)]

  $$R_{\nu,\mathbf{k}} \simeq \oplus_{w \in \mathfrak{S}_m} \mathbf{P}_{i,m}[2l(w) - l_m].$$

  So $\mathbf{P}_{i,m}$ is a direct summand of $R_{\nu,\mathbf{k}}[l_m]$. We choose once and for all an idempotent $1_{i,m} \in R_{\nu,\mathbf{k}}$ such that

  $$\mathbf{P}_{i,m} = (R_{\nu,\mathbf{k}} \cdot 1_{i,m})[l_m].$$

- If $y = (\mathbf{i}, \mathbf{a})$ with $\mathbf{i} = (i_1, \cdots, i_k)$, $\mathbf{a} = (a_1, \cdots, a_k)$ we define the idempotent $1_y \in R_{\nu,\mathbf{k}}$ as the image of the element $\otimes_{l=1}^{k} 1_{i_l,a_l}$ by the inclusion of graded $\mathbf{k}$-algebras $\otimes_{l=1}^{k} R_{i_l a_l,\mathbf{k}} \subset R_{\nu,\mathbf{k}}$. Then we set

  $$\mathbf{P}_y = (R_{\nu,\mathbf{k}} \cdot 1_y)[l_{\mathbf{a}}].$$

For an $R_{\mathbf{d}}$-module M and a Laurent polynomial $P(q) = \sum_r a_r q^r \in \mathcal{A}$ we set $P(q)M = \bigoplus_r M[r]^{\oplus a_r}$. We have the following lemma, see [35, Sec. 2.5-2.6].

Lemma 2.2.15. *We have the following graded projective $R_{\nu,\mathbf{k}}$-module isomorphisms.*

(a) $\mathbf{P}_{\mathbf{i}} \simeq [\mathbf{a}]!\mathbf{P}_y$, *where* $y = (i_1^{(a_1)} \cdots i_k^{(a_k)}) \in Y_\nu$, $\mathbf{i} = (i_1^{a_1} \cdots i_k^{a_k}) \in I^\nu$, $\mathbf{a} = (a_1, \cdots, a_k)$,

(b) $\mathrm{Ind}_{\nu_1,\nu_2}(\mathbf{P}_{y_1}, \mathbf{P}_{y_2}) \simeq \mathbf{P}_{y_1 y_2}$, *where* $\nu_1, \nu_2 \in \mathbb{N}I$, $y_1 \in Y_{\nu_1}, y_2 \in Y_{\nu_2}$ *and* $y_1 y_2 \in Y_{\nu_1+\nu_2}$ *is the concatenation of* y_1 *and* y_2. □

2.2.19 The algebra f

We recall the definition and general properties of the positive part $\mathbf{f}$ of the Drinfeld-Jimbo quantized enveloping algebra associated with the quiver Γ. See [38] for more details.

Definition 2.2.16. The algebra $\mathbf{f}$ is the $\mathbb{Q}(q)$-algebra generated by the elements θ_i, $i \in I$, with relations

$$\sum_{a+b=1-i\cdot j} (-1)^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0, \qquad i \neq j,\ a \geqslant 0, b \geqslant 0,$$

where $\theta_i^{(a)} = \theta_i^a/[a]!$.

The assignment $\deg \theta_i = i$ for $i \in I$ defines an $\mathbb{N}I$-grading $\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}I} \mathbf{f}_\nu$. For each homogeneous element $x \in \mathbf{f}_\nu$ we set $|x| = \sum_{i \in I} \nu_i$. The tensor product (over $\mathbb{Q}(q)$) $\mathbf{f} \otimes \mathbf{f}$ has a $\mathbb{Q}(q)$-algebra structure defined by

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{-|x_2||x_1'|} x_1 x_1' \otimes x_2 x_2'$$

where $x_1, x_2, x_1', x_2' \in \mathbf{f}$ are homogeneous. There exists a unique coproduct $r \colon \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$ such that

- $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ for $i \in I$,
- r is a $\mathbb{Q}(q)$-algebra homomorphism.

Definition 2.2.17. Let ${}_\mathcal{A}\mathbf{f}$ be the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by the elements $\theta_i^{(a)}$ with $i \in I, a \in \mathbb{N}$.

2.2.20 Geometric realization of ${}_\mathcal{A}\mathbf{f}$

The construction of this section is very similar to the construction given in Section 2.2.16. Let $K(\mathcal{Q}_V)$ be the split Grothendieck group of the additive category $\mathcal{Q}_V$, i.e., $K(\mathcal{Q}_V)$ is the Abelian group with one generator $[\mathcal{F}]$ for each isomorphism class of objects of $\mathcal{Q}_V$ and with relations $[\mathcal{L}'] + [\mathcal{L}''] = [\mathcal{L}]$ whenever $\mathcal{L}$ is isomorphic to $\mathcal{L}' \oplus \mathcal{L}''$. Set

$$\mathcal{Q} = \bigoplus_V \mathcal{Q}_V, \qquad K(\mathcal{Q}) = \bigoplus_V K(\mathcal{Q}_V),$$

where V runs over the isomorphism classes of finite dimensional I-graded $\mathbf{k}$-vector spaces and

$$R_\mathbf{k} = \bigoplus_{\nu \in \mathbb{N}I} R_{\nu,\mathbf{k}}, \qquad K(R_\mathbf{k}) = \bigoplus_{\nu \in \mathbb{N}I} K(R_{\nu,\mathbf{k}}).$$

The functors $\mathrm{Ind}_{\nu_1,\nu_2}$ induce an associative unital $\mathcal{A}$-algebra structure on $K(R_\mathbf{k})$. The functors $\mathrm{Res}_{\nu_1,\nu_2}$ induce a coassociative counital $\mathcal{A}$-coalgebra structure on $K(R_\mathbf{k})$. We will denote the multiplication and the comultiplication on $K(R_\mathbf{k})$ by Ind and Res respectively.

Remark 2.2.18. There exists a non-degenerate $\mathcal{A}$-bilinear form $(,)\colon K(R_{\mathbf{k}})\times K(R_{\mathbf{k}})\to\mathbb{Q}(q)$ such that

$$(x,\mathrm{Ind}(y_1,y_2))=(\mathrm{Res}(x),y_1\otimes y_2),\qquad \forall x,y_1,y_2\in K(R_{\mathbf{k}}),$$

see [35, Sec. 2.5].

The following theorem is proved in [35, Prop. 3.4, Sec. 3.2].

Theorem 2.2.19. *There is an $\mathcal{A}$-bialgebra isomorphism $\gamma_{\mathcal{A}}: {}_{\mathcal{A}}\mathbf{f}\to K(R_{\mathbf{k}})$ that takes θ_y to $[\mathbf{P}_y]$ for each $y\in Y_\nu$.* □

The grading of $R_{\nu,\mathbf{k}}$ from Section 2.2.2 is different from the Ext-grading, see Section 2.2.16. To avoid this we must modify a bit the complex $\mathcal{L}_{V,\mathbf{k}}$. We get the following theorem, see [62, Thm. 3.6]. Recall the complex ${}^{\delta}\mathcal{L}_{V,\mathbf{k}}$ from Section 2.2.13.

Theorem 2.2.20. *We have the following graded $\mathbf{k}$-algebra isomorphism*

$$R_{\nu,\mathbf{k}}\to\mathrm{Ext}^*_{G_V}({}^{\delta}\mathcal{L}_{V,\mathbf{k}},{}^{\delta}\mathcal{L}_{V,\mathbf{k}}).$$

□

The proof of the following theorem is the same as the proof of Theorem 2.2.14.

Theorem 2.2.21. *We have the following equivalence of categories.*

$$Y\colon\mathcal{Q}_V^{\mathrm{op}}\to\mathrm{proj}(R_{\nu,\mathbf{k}}),\qquad \mathfrak{L}\mapsto\mathrm{Ext}^*_{G_V}(\mathfrak{L},{}^{\delta}\mathcal{L}_{V,\mathbf{k}}).$$

□

So we have an $\mathcal{A}$-linear map $Y\colon K(\mathcal{Q})\to K(R_{\mathbf{k}})$ such that $[\mathcal{F}]\mapsto[Y(\mathcal{F})]$. Moreover, it is clear from the definition of Y that $Y({}^{\delta}\mathcal{L}_{\mathbf{i}})=P_{\mathbf{i}}$ for $\mathbf{i}\in I^\nu$. If $\mathbf{i}\in I^\nu$ is the expansion of the element $y=(\mathbf{j},a)$ in Y_ν then we have ${}^{\delta}\mathcal{L}_{\mathbf{i}}=[a]!{}^{\delta}\mathcal{L}_y$ (see [62, (4.7)]) and $[P_{\mathbf{i}}]=[a]![P_y]$ (see Lemma 2.2.15). Thus, $Y({}^{\delta}\mathcal{L}_y)=P_y$ for each $y\in Y_\nu$. So we have $\gamma_{\mathcal{A}}^{-1}\circ Y([{}^{\delta}\mathcal{L}_y])=\theta_y$ for $y\in Y_\nu$, where $\gamma_{\mathcal{A}}$ is as in Theorem 2.2.19.

Consider $\nu_1,\nu_2\in\mathbb{N}I$. Let V_1, V_2 be I-graded $\mathbb{C}$-vector spaces with graded dimensions ν_1, ν_2 respectively. Analogically with the case when the characteristic of $\mathbf{k}$ is zero, see [38, Sec. 9.2.7], we have a multiplication

$$\circ\colon\mathcal{Q}_{V_1}\times\mathcal{Q}_{V_1}\to\mathcal{Q}_{V_1\oplus V_2},$$

such that ${}^{\delta}\mathcal{L}_{y_1}\circ{}^{\delta}\mathcal{L}_{y_2}={}^{\delta}\mathcal{L}_{y_1y_2}$ for $y_1\in Y_{\nu_1}$, $y_2\in Y_{\nu_2}$. Thus, $\gamma_{\mathcal{A}}^{-1}\circ Y$ is an algebra homomorphism. We get the following theorem.

Theorem 2.2.22. *There exists an algebra isomorphism $\lambda_{\mathcal{A}}\colon K(\mathcal{Q})\to{}_{\mathcal{A}}\mathbf{f}$ such that for each $y\in Y_\nu$, $\nu\in\mathbb{N}I$ we have $\lambda_{\mathcal{A}}([{}^{\delta}\mathcal{L}_y])=\theta_y$.* □

Remark 2.2.23. To have ${}^{\delta}\mathcal{L}_{y_1}\circ{}^{\delta}\mathcal{L}_{y_2}={}^{\delta}\mathcal{L}_{y_1y_2}$ (and not ${}^{\delta}\mathcal{L}_{y_1}\circ{}^{\delta}\mathcal{L}_{y_2}={}^{\delta}\mathcal{L}_{y_2y_1}$) we should use not exactly the induction diagram from [38] but the opposite one. Note also that the restriction diagram that will be discussed in Section 2.3.6 is also opposite to the restriction diagram in [38].

2.2.21 Indecomposable objects in $\mathcal{Q}_V$

For each $\lambda \in \Lambda_V$ there is $y_\lambda \in Y_\nu$ such that $\widetilde{F}_{y_\lambda} \to \overline{\mathcal{O}_\lambda}$ is a resolution of the orbit closure $\overline{\mathcal{O}_\lambda}$, see [49, Thm. 2.2]. Let i_λ and $\tilde{i}_\lambda$ be the inclusions $i_\lambda \colon \mathcal{O}_\lambda \to E_V$ and $\tilde{i}_\lambda \colon (\pi_{y_\lambda})^{-1}(\mathcal{O}_\lambda) \to \widetilde{F}_{y_\lambda}$. Denote also π'_{y_λ} the morphism $\pi'_{y_\lambda} \colon (\pi_{y_\lambda *})^{-1}(\mathcal{O}_\lambda) \to \mathcal{O}_\lambda$ induced by π_{y_λ}. By the base change theorem we have

$$i_\lambda^* \pi_{y_\lambda *} \underline{\mathbf{k}}_{\widetilde{F}_y} = \pi'_{y_\lambda *} \tilde{i}_\lambda^* \underline{\mathbf{k}}_{\widetilde{F}_y} = \underline{\mathbf{k}}_\lambda.$$

So the complex $\mathcal{L}_{y_\lambda} = \pi_{y_\lambda *} \underline{\mathbf{k}}_{\widetilde{F}_y}$ has a unique indecomposable factor supported on all $\overline{\mathcal{O}_\lambda}$. Denote by Q_{y_λ} the shift by d_λ of this direct factor. We have $i_\lambda^* Q_{y_\lambda} = \underline{\mathbf{k}}_\lambda[d_\lambda]$. Note that Q_{y_λ} is a shift of a direct factor of $\mathcal{L}_{\mathbf{i}}$ for some $\mathbf{i} \in I^\nu$, because if $\mathbf{i} = (i_1^{a_1} \cdots i_k^{a_k})$ is the expansion of $y = (i_1^{(a_1)} \cdots i_k^{(a_k)})$ then we have ${}^\delta\mathcal{L}_{\mathbf{i}} = [a]!\,{}^\delta\mathcal{L}_y$, where $\mathbf{a} = (a_1 \cdots, a_k)$, see Section 2.2.20. Thus, Q_{y_λ} belongs to $\mathcal{Q}_V$. For $\lambda \in \Lambda_V$ we denote by $\mathrm{IC}(\mathcal{O}_\lambda)$ the G_V-equivariant intersection cohomology complex on E_V associated with $\mathcal{O}_\lambda$ and the G_V-equivariant local system $\underline{\mathbf{k}}_\lambda$, see [12, Sec. 8.4].

Lemma 2.2.24. *The complex Q_{y_λ} does not depend on the choice of the element $y_\lambda \in Y_\nu$ such that π_{y_λ} is a resolution of the orbit closure $\overline{\mathcal{O}_\lambda}$.*

Proof. Fix an element y_λ as above for each $\lambda \in \Lambda_V$. Suppose that $\mathbf{k}$ is a field of characteristic zero. Then by the decomposition theorem [3, Thm. 6.2.5] we have $Q_{y_\lambda} = \mathrm{IC}(\mathcal{O}_\lambda)$. Thus, the complex $\mathrm{IC}(\mathcal{O}_\lambda)$ belongs to $\mathcal{Q}_V$. Hence, in view of Corollary 2.2.11 the complexes $Q_{y_\lambda} = \mathrm{IC}(\mathcal{O}_\lambda)$ with $\lambda \in \Lambda_V$, are the representatives of the isomorphism classes of indecomposable objects in $\mathcal{Q}_V$ modulo shifts, and they form an $\mathcal{A}$-basis in $K(\mathcal{Q}_V)$. This yields the equality

$$\dim_{\mathcal{A}}({}_{\mathcal{A}}\mathbf{f}_\nu) = \#(\Lambda_V). \tag{2.1}$$

Now let $\mathbf{k}$ be an arbitrary field. The number of indecomposable objects modulo shifts in $\mathcal{Q}_V$ is equal to $\dim_{\mathcal{A}} K(\mathcal{Q}_V) = \dim_{\mathcal{A}}({}_{\mathcal{A}}\mathbf{f}_\nu)$, see Theorem 2.2.22, and the complexes $Q_{y_\lambda},\ \lambda \in \Lambda_V$ are among them. Thus, (2.1) shows that $Q_{y_\lambda},\ \lambda \in \Lambda_V$, are the representatives of isomorphism classes of indecomposable objects in $\mathcal{Q}_V$ modulo shifts. Hence, each indecomposable object in $\mathcal{Q}_V$ with support equal to $\overline{\mathcal{O}_\lambda}$ is equal to a shift of Q_{y_λ}. From this we see that Q_{y_λ} does not depend on y_λ. □

Definition 2.2.25. Let us write Q_λ instead of Q_{y_λ}.

The following lemma follows directly from the proof of Lemma 2.2.24.

Lemma 2.2.26. *The indecomposable objects in $\mathcal{Q}_V$ are exactly the complexes Q_λ, $\lambda \in \Lambda_V$, modulo shifts.* □

For each I-graded finite dimensional $\mathbb{C}$-vector space V the classes $[Q_\lambda]$ for $\lambda \in \Lambda_V$ form an $\mathcal{A}$-basis in $K(\mathcal{Q}_V)$. Combining this for all V we get an $\mathcal{A}$-basis in $K(\mathcal{Q})$. The homomorphism $\lambda_{\mathcal{A}}$ takes this $\mathcal{A}$-basis to an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}$.

2.3 Parity sheaves

Let A be a complete discrete valuation ring or a field.

2.3.1 Parity sheaves

First, we recall some basic facts about parity sheaves. See [29] for more details. Let G, X be as in Section 2.2.12. Here we will use the notation introduced in Section 2.2.12. We make the following assumption

$$\mathrm{H}^{\mathrm{odd}}_G(X_\lambda, \mathcal{L}) = 0,\ \mathrm{H}^*_G(X_\lambda, \mathcal{L}) \text{ is a free } A\text{-module} \quad \forall \mathcal{L} \in \mathrm{Loc}_G(X_\lambda), \forall \lambda \in \Lambda. \tag{2.2}$$

Definition 2.3.1. A complex $\mathcal{F} \in D_G(X, A)$ is *even* (resp. *odd*) if $\mathcal{H}^r(\mathcal{F}) = \mathcal{H}^r(\mathbb{D}\mathcal{F}) = 0$ for each odd (resp. even) positive integer r, where $\mathcal{H}^\bullet$ means the sheaf cohomology. A complex $\mathcal{F}$ is *parity* if it is a sum of an even complex and an odd complex.

Lemma 2.3.2. *Suppose $\mathcal{F} \in D_G(X, A)$. The following conditions are equivalent.*

(a) $\mathcal{F} \in D_G(X, A)$ is even,

(b) $\mathcal{H}^k(i_\lambda^*\mathcal{F}) = \mathcal{H}^k(i_\lambda^*\mathbb{D}\mathcal{F}) = 0$ for each odd integer k and each $\lambda \in \Lambda$,

(c) $\mathcal{H}^k(i_x^*\mathcal{F}) = \mathcal{H}^k(i_x^*\mathbb{D}\mathcal{F}) = 0$ for each odd integer k and each $x \in X$.

Proof. First we show that (a)⇔(c) By [32, Rem. 2.6.9] we have

$$i_x^*\mathcal{H}^k(\mathcal{F}) = \mathcal{H}^k(i_x^*\mathcal{F}).$$

Now $\mathcal{H}^k(\mathcal{F})$ is zero iff $i_x^*\mathcal{H}^k(\mathcal{F})$ is zero for each $x \in X$. Thus, $\mathcal{H}^k(\mathcal{F}) = 0$ iff $\mathcal{H}^k(i_x^*\mathcal{F}) = 0$ for all $x \in X$. In the same way we prove that $\mathcal{H}^k(\mathbb{D}\mathcal{F}) = 0$ iff $\mathcal{H}^k(i_x^*\mathbb{D}\mathcal{F}) = 0$ for all $x \in X$. Thus, (a)⇔(c). To see that (b)⇔(c) we prove in the same way that $\mathcal{H}^k(i_\lambda^*\mathcal{F}) = 0$ iff $\mathcal{H}^k(i_x^*\mathcal{F}) = 0$ for all $x \in X_\lambda$ and $\mathcal{H}^k(i_\lambda^*\mathbb{D}\mathcal{F}) = 0$ iff $\mathcal{H}^k(i_x^*\mathbb{D}\mathcal{F}) = 0$ for all $x \in X_\lambda$. □

Lemma 2.3.3. *Assume that* (2.2) *holds. Let $\mathcal{F}$ be an indecomposable parity complex. Then*

(1) *the support of $\mathcal{F}$ is of the form $\overline{X_\lambda}$ for some $\lambda \in \Lambda$,*

(2) *the restriction $i_\lambda^*\mathcal{F}$ is isomorphic to $\mathcal{L}[m]$ for some indecomposable object $\mathcal{L} \in \mathrm{Loc}_G(X_\lambda)$ and some integer m,*

(3) *any indecomposable parity complex supported on $\overline{X_\lambda}$ which extends $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$, where $\mathcal{L}$ is as in* (2).

Idea of proof. The lemma is proved in [29, Thm. 2.12]. This proof uses the following claim, which is based on the property (2.2).

Lemma 2.3.4. *Assume that* (2.2) *holds. Let $\mathcal{F}, \mathcal{G} \in D_G(X, A)$. Suppose that $\mathcal{H}^{\mathrm{odd}}(\mathcal{F}) = 0$ and $\mathcal{H}^{\mathrm{odd}}(\mathbb{D}\mathcal{F}) = 0$. Then we have*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}(i_\lambda^*\mathcal{F}, i_\lambda^!\mathcal{G}).$$

□

Definition 2.3.5. A *parity sheaf* is an indecomposable parity complex supported on $\overline{X_\lambda}$ extending $\mathcal{L}[d_\lambda]$ for some indecomposable $\mathcal{L} \in \mathrm{Loc}_G(X_\lambda, A)$ and some $\lambda \in \Lambda$. If such a complex exists, we will denote it by $\mathcal{E}(\lambda, \mathcal{L})$. If $\mathcal{L}$ is the constant sheaf $\underline{A}_\lambda$ we will write $\mathcal{E}(\lambda) = \mathcal{E}(\lambda, \mathcal{L})$.

2.3.2 Parity sheaves on quiver varieties

We still suppose that Γ is a Dynkin quiver. We want to study G_V-equivariant parity sheaves on E_V with respect to the stratification $\coprod_{\lambda\in\Lambda_V} \mathcal{O}_\lambda$, see Section 2.2.13. We must check that the G_V-variety $X = E_V$ satisfies the condition (2.2). This is true by the following lemma, see also Corollary 2.2.11.

Lemma 2.3.6. *For every G_V-orbit $\mathcal{O}$ in E_V we have $\mathrm{H}^{\mathrm{odd}}_{G_V}(\mathcal{O}, \underline{A}_\lambda) = 0$ and $\mathrm{H}^*_{G_V}(\mathcal{O}, \underline{A}_\lambda)$ is a free A-module.*

Proof. Let $x \in \mathcal{O}$. Denote by H the stabilizer of x in G_V. Let H_{red} be the reductive part of H. We have

$$\begin{aligned} \mathrm{H}^*_{G_V}(\mathcal{O}, \underline{A}_{\mathcal{O}}) &= \mathrm{H}^*_{G_V}(G_V/H, A) \\ &= \mathrm{H}^*_H(\bullet, A) \\ &= \mathrm{H}^*_{H_{\mathrm{red}}}(\bullet, A) \\ &= \mathrm{H}^*(BH_{\mathrm{red}}, A). \end{aligned}$$

Note that for each positive integer n the fundamental group of $GL_n(\mathbb{C})$ is equal to $\mathbb{Z}$. Then the fundamental group of H_{red} does not contain elements of finite order, see Lemma 2.2.10. Moreover, the reductive group H_{red} is of type A. Then by [60, Lem. 1] the torsion index of H_{red} is equal to 1, see Section 2.2.6. Thus, the classifying space BH_{red} has no odd cohomology and its cohomology is free over A, see [26, Cor. 2.3]. □

The following lemma is helpful to prove the parity of some complexes on E_V.

Lemma 2.3.7. *Suppose that there exist a smooth G_V-variety Y and a G_V-equivariant proper morphism $\pi\colon Y \to E_V$. Then the complex $\pi_*\underline{A}_Y$ is even if and only if $\mathrm{H}^{\mathrm{odd}}(\pi^{-1}(x), A) = 0$ for each $x \in E_V$.*

Proof. Note that the complex $\pi_*\underline{A}_Y$ is constructible with respect to the stratification of E_V because it is G_V-equivariant and the strata are G_V-orbits. By Lemma 2.3.2 the complex $\pi_*\underline{A}_Y$ is even if and only if for every inclusion $j : \{x\} \hookrightarrow E_V$ we have $\mathrm{H}^{\mathrm{odd}}(j^*\pi_*\underline{A}_Y) = 0$ and $\mathrm{H}^{\mathrm{odd}}(j^*\mathbb{D}\pi_*\underline{A}_Y) = 0$. Let d be the complex dimension of Y. Note that

$$\mathbb{D}\pi_*\underline{A}_Y = \pi_*\mathbb{D}\underline{A}_Y, \tag{2.3}$$

$$\mathbb{D}\underline{A}_Y = \underline{A}_Y[2d], \tag{2.4}$$

because the morphism π is proper and Y is smooth. We set $F = \pi^{-1}(x)$. We have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & E_V \\ {\scriptstyle i}\big\uparrow & & {\scriptstyle j}\big\uparrow \\ F & \xrightarrow{\pi} & \{x\}, \end{array}$$

where i is the inclusion of F to Y. We denote the restriction of π to F again by π. By the base change theorem we have

$$j^*\pi_*\underline{A}_Y = \pi_* i^*\underline{A}_Y, \tag{2.5}$$

$$j^*\pi_*\mathbb{D}\underline{A}_Y = \pi_* i^*\mathbb{D}\underline{A}_Y. \tag{2.6}$$

Now, from (2.3), (2.4), (2.5) and (2.6) we get

$$\mathrm{H}^*(\{x\}, j^*\pi_*\underline{A}_Y) = \mathrm{H}^*(F, A),$$

$$\mathrm{H}^*(\{x\}, j^*\mathbb{D}\pi_*\underline{A}_Y) = \mathrm{H}^{*+2d}(F, A).$$

This completes the proof. □

2.3.3 Extensions of parity complexes

Let $\mathcal{F}$ be a parity complex in $D_{G_V}(E_V, A)$. For each $\lambda \in \Lambda_V$ and $n \in \mathbb{N}$ the sheaf $\mathcal{H}^n(i_\lambda^*\mathcal{F})$ is a direct sum of copies of $\underline{A}_\lambda$, see Corollary 2.2.11. Let us denote the rank of the sheaf $\mathcal{H}^n(i_\lambda^*\mathcal{F})$ by $d_{\lambda,n}(\mathcal{F})$.

Lemma 2.3.8. *Let $\mathcal{F}, \mathcal{G}$ be parity complexes in $D_{G_V}(E_V, A)$. Suppose that we have $d_{\lambda,n}(\mathcal{F}) = d_{\lambda,n}(\mathcal{G})$ for all $\lambda \in \Lambda_V, n \in \mathbb{N}$. Then $\mathcal{F} \simeq \mathcal{G}$.*

Proof. Let us decompose $\mathcal{F}$ and $\mathcal{G}$ into direct sums of shifts of parity sheaves

$$\mathcal{F} = \bigoplus_{\lambda\in\Lambda_V, k\in\mathbb{Z}} \mathcal{E}(\lambda)[k]^{\oplus f_{\lambda,k}}, \qquad \mathcal{G} = \bigoplus_{\lambda\in\Lambda_V, k\in\mathbb{Z}} \mathcal{E}(\lambda)[k]^{\oplus g_{\lambda,k}}.$$

We will prove the statement by induction on the number of indecomposable factors of $\mathcal{F}$. If $\mathcal{F} = 0$ then for each $\lambda \in \Lambda_V, n \in \mathbb{Z}$ we have $d_{\lambda,n}(\mathcal{F}) = d_{\lambda,n}(\mathcal{G}) = 0$. Thus, $\mathcal{F} = \mathcal{G} = 0$. Now suppose that $\mathcal{F} \neq 0$. Define a partial order $\preccurlyeq$ on Λ_V by setting $\mu \preccurlyeq \lambda$ if and only if $\mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda}$. Fix a total order $\leqslant$ on Λ_V that refines $\preccurlyeq$. Let λ_0 be the largest element of Λ_V with respect to $\leqslant$ such that $f_{\lambda_0,k} \neq 0$ for some integer k. Then the decomposition of $\mathcal{F}$ does not contain any shift of the parity sheaves $\mathcal{E}(\mu)$ with $\lambda_0 < \mu$. If for some $\mu, \lambda \in \Lambda_V$ and $n \in \mathbb{N}$ we have $\mathcal{H}^n(i_\mu^*\mathcal{E}(\lambda)) \neq 0$ then $\mu \preccurlyeq \lambda$. Thus we have $d_{\mu,n}(\mathcal{F}) = 0$ for each $\mu > \lambda_0, n \in \mathbb{Z}$. Now, the equality $d_{\mu,n}(\mathcal{G}) = 0$ for each $\mu > \lambda_0, n \in \mathbb{Z}$ assures that the decomposition of $\mathcal{G}$ does not contain any shift of the parity sheaves $\mathcal{E}(\mu)$ with $\lambda_0 < \mu$. Choose n_0 such that $d_{\lambda_0,n_0}(\mathcal{F}) = d_{\lambda_0,n_0}(\mathcal{G}) \neq 0$. It exists because $\mathcal{F}$ has a direct factor of the form $\mathcal{E}(\lambda_0)[k]$ for some $k \in \mathbb{Z}$. We have

$$\mathcal{H}^{n_0}(i^*_{\lambda_0}\mathcal{F}) \simeq \underline{A}_{\lambda_0}^{\oplus f_{\lambda_0,k_0}}, \qquad \mathcal{H}^{n_0}(i^*_{\lambda_0}\mathcal{G}) \simeq \underline{A}_{\lambda_0}^{\oplus g_{\lambda_0,k_0}}$$

for $k_0 = -d_{\lambda_0} - n_0$. This shows that

$$f_{\lambda_0,k_0} = d_{\lambda_0,n_0}(\mathcal{F}) = d_{\lambda_0,n_0}(\mathcal{G}) = g_{\lambda_0,k_0}.$$

Thus, we have

$$\mathcal{F} = \mathcal{F}' \oplus \mathcal{E}(\lambda_0)[k_0]^{\oplus f_{\lambda_0,k_0}}, \qquad \mathcal{G} = \mathcal{G}' \oplus \mathcal{E}(\lambda_0)[k_0]^{\oplus f_{\lambda_0,k_0}}.$$

Thus, we have $d_{\lambda,n}(\mathcal{F}') = d_{\lambda,n}(\mathcal{G}')$ for each $\lambda \in \Lambda_V, n \in \mathbb{Z}$. Thus, by induction hypothesis we have $\mathcal{F}' \simeq \mathcal{G}'$. Hence, $\mathcal{F} \simeq \mathcal{G}$. □

Lemma 2.3.9. *Let $A \to B \to C \xrightarrow{+1}$ be a distinguished triangle in $D_{G_V}(E_V, A)$. Suppose that the complexes A and C are even. Then we have $B \simeq A \oplus C$. In particular the complex B is also even.*

Proof. See also [29, Cor. 2.8]. First we prove that B is an even complex. The duality yields a distinguished triangle

$$\mathbb{D}C \to \mathbb{D}B \to \mathbb{D}A \xrightarrow{+1}.$$

The long exact sequences in cohomology are

$$\to \mathcal{H}^{n-1}(C) \to \mathcal{H}^n(A) \to \mathcal{H}^n(B) \to \mathcal{H}^n(C) \to \mathcal{H}^{n+1}(A) \to,$$

$$\to \mathcal{H}^{n-1}(\mathbb{D}A) \to \mathcal{H}^n(\mathbb{D}C) \to \mathcal{H}^n(\mathbb{D}B) \to \mathcal{H}^n(\mathbb{D}A) \to \mathcal{H}^{n+1}(\mathbb{D}C) \to .$$

For each odd integer n we have

$$\mathcal{H}^n(A) = \mathcal{H}^n(\mathbb{D}A) = \mathcal{H}^n(C) = \mathcal{H}^n(\mathbb{D}C) = 0$$

and thus $\mathcal{H}^n(B) = \mathcal{H}^n(\mathbb{D}B) = 0$. So the complex B is also even. For each $\lambda \in \Lambda_V$ the distinguished triangle

$$i^*_\lambda A \to i^*_\lambda B \to i^*_\lambda C \xrightarrow{+1}$$

yields a long exact sequence in cohomology

$$\to \mathcal{H}^{n-1}(i^*_\lambda C) \to \mathcal{H}^n(i^*_\lambda A) \to \mathcal{H}^n(i^*_\lambda B) \to \mathcal{H}^n(i^*_\lambda C) \to \mathcal{H}^{n+1}(i^*_\lambda A) \to .$$

If n is an even integer then $\mathcal{H}^{n-1}(i^*_\lambda C) = \mathcal{H}^{n+1}(i^*_\lambda A) = 0$, see Lemma 2.3.2. Thus, we have a short exact sequence

$$0 \to \mathcal{H}^n(i^*_\lambda A) \to \mathcal{H}^n(i^*_\lambda B) \to \mathcal{H}^n(i^*_\lambda C) \to 0.$$

Hence, we have

$$d_{\lambda,n}(B) = d_{\lambda,n}(A) + d_{\lambda,n}(C) = d_{\lambda,n}(A \oplus C)$$

for each $\lambda \in \Lambda_V$ and each even integer n and

$$d_{\lambda,n}(A) = d_{\lambda,n}(B) = d_{\lambda,n}(C) = 0$$

for each odd integer n. Thus, by Lemma 2.3.8 we have $B \simeq A \oplus C$. $\square$

2.3.4 Nakajima quiver varieties

Let $\mathbf{k}$ be a field. From now on all sheaves and cohomology groups are to be understood with coefficients in $\mathbf{k}$-vector spaces.

We recall the notion of a (graded) Nakajima quiver variety. We keep the notation of Section 2.2.1 and we still suppose that the quiver Γ is a Dynkin quiver. The notation $i \sim j$ means that there is an arrow $i \to j$ or $j \to i$ in Γ. Fix a function $\xi \colon I \to \mathbb{Z},\ i \mapsto \xi_i$ such that $\xi_i = \xi_j + 1$ if there is an arrows $i \to j$. It is unique modulo a shift by an integer constant. Consider the sets

$$\widehat{I} = \{(i,n) \in I \times \mathbb{Z};\ \xi_i - n \in 2\mathbb{Z}\},$$

$$\widehat{J} = \{(i,n) \in I \times \mathbb{Z};\ (i,n-1) \in \widehat{I}\}.$$

To the quiver Γ we will associate the quiver $\widehat{\Gamma}$ as follows:

- the quiver $\widehat{\Gamma}$ contains two types of vertices:
  - a vertex $w_i(n)$ for each $(i,n) \in \widehat{I}$, we will call them *w-vertices*,
  - a vertex $t_i(n)$ for each $(i,n) \in \widehat{J}$, we will call them *t-vertices*,
- the quiver $\widehat{\Gamma}$ contains three types of edges:
  - an edge $w_i(n) \to t_i(n-1)$ for each $(i,n) \in \widehat{I}$,
  - an edge $t_i(n) \to w_i(n-1)$ for each $(i,n) \in \widehat{J}$,
  - an edge $t_i(n) \to t_j(n-1)$ for each $(i,n), (j,n-1) \in \widehat{J}$ such that $i \sim j$.

Consider an $\widehat{I}$-graded finite dimensional $\mathbb{C}$-vector space $W = \bigoplus_{(i,n)\in\widehat{I}} W_i(n)$ and a $\widehat{J}$-graded finite dimensional $\mathbb{C}$-vector space $T = \bigoplus_{(i,n)\in\widehat{J}} T_i(n)$. Set

$$L^\bullet(T,W) = \bigoplus_{(i,n)\in\widehat{J}} \mathrm{Hom}(T_i(n), W_i(n-1)),$$

$$L^\bullet(W,T) = \bigoplus_{(i,n)\in\widehat{I}} \mathrm{Hom}(W_i(n), T_i(n-1)),$$

$$E^\bullet(T) = \bigoplus_{(i,n)\in\widehat{J}, i\sim j} \mathrm{Hom}(T_i(n), T_j(n-1)),$$

and put
$$M^\bullet(T,W) = E^\bullet(T) \oplus L^\bullet(W,T) \oplus L^\bullet(T,W).$$
Note that $M^\bullet(T,W)$ is just the variety of representations of $\widehat{\Gamma}$ in the $\widehat{I}\coprod\widehat{J}$-graded vector space $T \oplus W$. We will write (B,α,β) for an element of $M^\bullet(T,W)$. The components of B,α,β are
$$B_{ij}(n) \in \mathrm{Hom}(T_i(n), T_j(n-1)),$$
$$\alpha_i(n) \in \mathrm{Hom}(W_i(n), T_i(n-1)),$$
$$\beta_i(n) \in \mathrm{Hom}(T_i(n), W_i(n-1)).$$
Let $\Lambda^\bullet(T,W)$ be the subvariety of the affine space $M^\bullet(T,W)$ defined by the equations
$$\alpha_i(n-1)\beta_i(n) + \sum_{i\sim j} \varepsilon(i,j) B_{ji}(n-1)B_{ij}(n) = 0, \qquad \forall (i,n) \in \widehat{J},$$
where $\varepsilon(i,j) = 1$ (resp. $\varepsilon(i,j) = -1$) if $i \to j$ is an arrow of Γ (resp. $i \to j$ is not an arrow of Γ).

The algebraic group $G_T = \prod_{(i,n)\in\widehat{J}} GL_{T_i(n)}$ acts on $M^\bullet(T,W)$ by $g \cdot (B,\alpha,\beta) = (B',\alpha',\beta')$, where $g = (g_i(n))_{(i,n)\in\widehat{J}}$,
$$B'_{ij}(n) = g_j(n-1)B_{ij}(n)g_i(n)^{-1}, \quad \alpha'_i(n) = g_i(n-1)\alpha_i(n), \quad \beta'_i(n) = \beta_i(n)g_i(n)^{-1}.$$
Consider the categorical quotient
$$\mathfrak{M}_0^\bullet(T,W) = \Lambda^\bullet(T,W)/\!\!/G_T,$$
i.e., the coordinate ring of $\mathfrak{M}_0^\bullet(T,W)$ is the ring of G_T-invariant functions on $\Lambda^\bullet(T,W)$. The points of $\mathfrak{M}_0^\bullet(T,W)$ parametrize the closed G_T-orbits in $\Lambda^\bullet(T,W)$. We will denote by $[B,\alpha,\beta]$ the element of $\mathfrak{M}_0^\bullet(T,W)$ represented by $(B,\alpha,\beta) \in \Lambda^\bullet(T,W)$. We say that a point $(B,\alpha,\beta) \in \Lambda^\bullet(T,W)$ is stable if the following condition holds: if a $\widehat{J}$-graded subspace T' of T is B-invariant and contained in $\mathrm{Ker}\beta$, then $T' = 0$. Let $\Lambda_{\mathrm{s}}^\bullet(T,W)$ be the set of stable points in $\Lambda^\bullet(T,W)$. Consider the set theoretical quotient
$$\mathfrak{M}^\bullet(T,W) = \Lambda_{\mathrm{s}}^\bullet(T,W)/G_T.$$
This quotient coincides with a quotient in the geometric invariant theory (see [44, Sec. 3]). The varieties $\mathfrak{M}^\bullet(T,W)$ and $\mathfrak{M}_0^\bullet(T,W)$ above a called Nakajima quiver varieties. There exists a projective morphism
$$\pi_{T,W} \colon \mathfrak{M}^\bullet(T,W) \to \mathfrak{M}_0^\bullet(T,W),$$
see [44, Sec. 3.18]. For each $\widehat{J}$-graded subspace T' of T we have a closed embedding
$$\mathfrak{M}_0^\bullet(T',W) \subset \mathfrak{M}_0^\bullet(T,W).$$

This allows to define the variety

$$\mathfrak{M}_0^\bullet(\infty, W) = \bigcup_T \mathfrak{M}_0^\bullet(T, W).$$

Let $\mathfrak{M}_0^{\bullet\mathrm{reg}}(T, W)$ be the open subset of $\mathfrak{M}_0^\bullet(T, W)$ parametrizing the closed free G_T-orbits in $\Lambda^\bullet(T, W)$. Denote by $\mathbf{S}(W)$ the set of isomorphism classes of finite dimensional $\widehat{J}$-graded vector spaces T such that $\mathfrak{M}_0^{\bullet\mathrm{reg}}(T, W) \neq \emptyset$. For each W as above we have a decomposition

$$\mathfrak{M}_0^\bullet(\infty, W) = \coprod_{T \in \mathbf{S}(W)} \mathfrak{M}_0^{\bullet\mathrm{reg}}(T, W),$$

see [46, (4.5)].

2.3.5 Resolutions via Nakajima quiver varieties

Definition 2.3.10. Suppose $(i, n), (j, k)$ are in $\widehat{I}$. A *path* from (i, n) to (j, k) is a path in the quiver $\widehat{\Gamma}$ from $w_i(n)$ to $w_j(k)$ that does not contain another w-vertex. To a path p from (i, n) to (j, k) and an element $(B, \alpha, \beta) \in \Lambda^\bullet(T, W)$ we can associate an element $A_p(B, \alpha, \beta) \in \mathrm{Hom}(W_i(n), W_j(k))$ as follows. If

$$p = (w_i(n) \to t_{i_1}(n-1) \to t_{i_2}(n-2) \to \cdots \to t_{i_r}(n-r) \to w_j(k)),$$

where $r = n - k - 1$ (note that $i = i_1$, $j = i_r$), then we set

$$A_p(B, \alpha, \beta) = \beta_j(n-r) B_{i_{r-1} i_r}(n-(r-1)) \cdots B_{i_1 i_2}(n-1) \alpha_i(n) \colon W_i(n) \to W_j(k).$$

The following theorem is proved in [25, Prop. 9.4, Thm. 9.11].

Theorem 2.3.11. *There exists an injective map $\tau\colon I \to \widehat{I}$ (depending on the orientation of Γ) such that for each arrow $i \to j$ in Γ there exists a path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ such that the following property holds: to each finite dimensional I-graded $\mathbb{C}$-vector space V we assign the $\widehat{I}$-graded $\mathbb{C}$-vector space W by setting*

$$W_{\tau(i)} = V_i, \quad \forall i \in I, \qquad W_{\hat{\imath}} = 0 \quad \forall \hat{\imath} \in \widehat{I} \backslash \tau(I).$$

We have an isomorphism

$$\mathfrak{M}_0^\bullet(\infty, W) \to E_V, \qquad [B, \alpha, \beta] \mapsto \bigoplus_{i \to j} A_{i \rightsquigarrow j}(B, \alpha, \beta).$$

□

From now on for each arrow $i \to j$ in Γ we fix a path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ as in Theorem 2.3.11.

Remark 2.3.12. The isomorphism in Theorem 2.3.11 depends on the choice of the path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ for each arrow $i \to j$ in Γ. Given another path p from $\tau(i)$ to $\tau(j)$, the morphisms

$$A_{i\rightsquigarrow j}\colon \mathfrak{M}_0^\bullet(\infty, W) \to \mathrm{Hom}(V_i, V_j), \qquad A_p\colon \mathfrak{M}_0^\bullet(\infty, W) \to \mathrm{Hom}(V_i, V_j)$$

are such that $A_p = \lambda A_{i\rightsquigarrow j}$ for some $\lambda \in \mathbb{C}$. This follows from the following two lemmas.

For each arrow $i \to j$ in Γ we equip the variety $\mathrm{Hom}(V_i, V_j)$ with the G_V-action by $g \cdot \phi = g_j \phi g_i^{-1}$ for $g \in G_V$, $\phi \in \mathrm{Hom}(V_i, V_j)$.

Lemma 2.3.13. *The map*

$$A_p\colon E_V = \mathfrak{M}_0^\bullet(\infty, W) \to \mathrm{Hom}(V_i, V_j)$$

is linear and G_V-equivariant.

Proof. The statement is obvious. □

Lemma 2.3.14. *Let $h_0 = (i \to j)$ be an arrow of Γ. Then each linear G_V-equivariant morphism $f\colon E_V \to \mathrm{Hom}(V_i, V_j)$ is of the form $x \mapsto \lambda x_{h_0}$ for some $\lambda \in \mathbb{C}$.*

Proof. We view E_V and $\mathrm{Hom}(V_i, V_j)$ as linear representations of G_V. The representation E_V is a direct sum of irreducible representations $E_V = \bigoplus_{h\in H} E_h$, where $E_h = \mathrm{Hom}(V_{h'}, V_{h''})$ and h', h'' are as in Section 2.2.1. In particular, we have $E_{h_0} = \mathrm{Hom}(V_i, V_j)$. Now, by Schur's lemma we have

$$\mathrm{Hom}_{G_V}(E_h, E_{h_0}) = \begin{cases} 0 & \text{if } h \neq h_0, \\ \mathbb{C} \cdot \mathrm{Id}_{E_{h_0}} & \text{if } h = h_0. \end{cases}$$

□

The following lemma is proved in [45, Thm. 7.4.1].

Lemma 2.3.15. *Let T and W be as in Section 2.3.4. Then the fibers of $\pi_{T,W}$ have no odd cohomology groups over $\mathbb{Z}$ and their cohomology groups over $\mathbb{Z}$ are torsion free.* □

Let us associate W to V as in Theorem 2.3.11. We will identify E_V with $\mathfrak{M}_0^\bullet(\infty, W)$ as in Theorem 2.3.11. The following lemma is proved in [25, Prop. 9.8]. It interprets the stratification on E_V in terms of Nakajima quiver varieties.

Lemma 2.3.16. *The decomposition $\mathfrak{M}_0^\bullet(\infty, W) = \coprod_{T\in\mathbf{S}(W)} \mathfrak{M}_0^{\bullet\mathrm{reg}}(T, W)$ coincides with the decomposition of E_V into G_V-orbits.* □

Let T be a finite dimensional $\widehat{J}$-graded $\mathbb{C}$-vector space. The morphism $\pi_{T,W}$ defined in Section 2.3.4 together with the inclusion $\mathfrak{M}_0^\bullet(T,W) \subset \mathfrak{M}_0^\bullet(\infty,W)$ yields a morphism

$$\pi_{T,W} \colon \mathfrak{M}^\bullet(T,W) \to \mathfrak{M}_0^\bullet(\infty,W) \simeq E_V.$$

We can consider the following complex in $D_{G_V}(E_V,\mathbf{k})$:

$$\widetilde{\mathscr{L}}_{T,W} = \pi_{T,W*}\underline{\mathbf{k}}_{\mathfrak{M}^\bullet(T,W)}$$

Lemmas 2.3.7, 2.3.15 and 2.3.16 yield the following result.

Corollary 2.3.17. *The complex $\widetilde{\mathscr{L}}_{T,W}$ is an even complex in $D_{G_V}(E_V,\mathbf{k})$.* □

The following lemma assures the existence of the parity sheaves on E_V, see Definition 2.3.5

Lemma 2.3.18. *For each $\lambda \in \Lambda_V$ there exists a parity sheaf $\mathcal{E}(\lambda)$ on $E_V = \coprod_{\lambda\in\Lambda_V} \mathcal{O}_\lambda$.*

Proof. Let T be in $\mathbf{S}(W)$, see Section 2.3.4. By [44, Prop. 3.24] the morphism

$$\pi_{T,W} \colon \mathfrak{M}^\bullet(T,W) \to \mathfrak{M}_0^\bullet(\infty,W)$$

is an isomorphism over $\mathfrak{M}_0^{\bullet\mathrm{reg}}(T,W) \subset \mathfrak{M}_0^\bullet(T,W)$. Its image is contained in $\overline{\mathfrak{M}_0^{\bullet\mathrm{reg}}(T,W)}$ because $\mathfrak{M}^\bullet(T,W)$ is smooth and connected by [45, Thm. 5.5.6] and $\pi_{T,W}^{-1}(\mathfrak{M}_0^{\bullet\mathrm{reg}}(T,W))$ is a dense open subset of $\mathfrak{M}^\bullet(T,W)$. Moreover, the complex $\widetilde{\mathscr{L}}_{T,W}$ is even by Corollary 2.3.17. Thus, the base change argument in Section 2.2.21 shows that each parity sheaf $\mathcal{E}(\lambda)$ for $\lambda \in \Lambda_V$ exists and appears as a direct factor of some shift of $\widetilde{\mathscr{L}}_{T,W}$ for some T. □

Remark 2.3.19.

(a) Lemma 2.3.18 is enough to insure the existence of G_V-equivariant parity sheaves on E_V, see Corollary 2.2.11. Note that in the same way we can prove the existence of G_V-equivariant parity sheaves on E_V over a complete discrete valuation ring.

(b) We have $i_\lambda^* Q_\lambda = i_\lambda^*\mathcal{E}(\lambda) = \underline{\mathbf{k}}_\lambda[d_\lambda]$, see Section 2.2.21. It is natural to ask whether the indecomposable complexes $\mathcal{E}(\lambda)$ and Q_λ are isomorphic. To get this, it is enough to prove that Q_λ is a parity complex, see Lemma 2.3.3.

(c) For $\lambda \in \Lambda_V$ set $\widetilde{\mathcal{E}}(\lambda) = \mathcal{E}(\lambda)[-d_\lambda]$. Let $T \in \mathbf{S}(W)$ be the $\widehat{J}$-graded $\mathbb{C}$-vector space such that $\mathfrak{M}_0^{\bullet\mathrm{reg}}(T,W) \simeq \mathcal{O}_\lambda$. Suppose that the characteristic of $\mathbf{k}$ is zero. Then by decomposition theorem [3, Thm. 6.2.5] the unique indecomposable factor of $\widetilde{\mathscr{L}}_{T,W}$ supported on all $\overline{\mathcal{O}_\lambda}$ is $\mathrm{IC}(\mathcal{O}_\lambda)[-d_\lambda]$, because we have $i_\lambda^*\widetilde{\mathscr{L}}_{T,W} = i_\lambda^*\widetilde{\mathcal{E}}(\lambda) = \underline{\mathbf{k}}_\lambda$. On the other hand $\widetilde{\mathscr{L}_{T,W}}$ is the sum of shifts of parity sheaves and thus the same argument shows that this factor is equal to $\widetilde{\mathcal{E}}(\lambda)$. Thus, we have $\mathrm{IC}(\mathcal{O}_\lambda) \simeq \mathcal{E}(\lambda)$. In particular in characteristic zero the complex $\mathrm{IC}(\mathcal{O}_\lambda)[-d_\lambda]$ is even. Moreover, we have already seen in the proof of Lemma 2.2.24 that $Q_\lambda \simeq \mathrm{IC}(\mathcal{O}_\lambda)$. Thus, in characteristic zero we have $\mathcal{E}(\lambda) \simeq Q_\lambda \simeq \mathrm{IC}(\mathcal{O}_\lambda)$.

2.3.6 Restriction diagrams

In this section we recall the construction of the restriction diagrams for Nakajima quiver varieties and for Lusztig quiver varieties and we compare them. The diagrams considered in this section are opposite to the analogical diagrams in [38] and [47].

Let W be a finite dimensional $\widehat{I}$-graded $\mathbb{C}$-vector space. Let W_1 be an $\widehat{I}$-graded subspace of W and set $W_2 = W/W_1$. Consider the subvariety of $\mathfrak{M}_0^\bullet(\infty, W)$ given by

$$\mathfrak{Z}_0^\bullet(W_1, W_2) = \{[B, \alpha, \beta] \in \mathfrak{M}_0^\bullet(\infty, W);\ \beta B^k \alpha(W_1) \subset W_1\ \forall k \in \mathbb{N}\},$$

where $[B, \alpha, \beta]$ denotes the element of $\mathfrak{M}_0^\bullet(\infty, W)$ represented by $(B, \alpha, \beta) \in \Lambda^\bullet(V, W)$ for some V. Consider the following diagram, see [47, Sec. 3.5] and [63, Sec. 3.5],

$$\mathfrak{M}_0^\bullet(\infty, W_1) \times \mathfrak{M}_0^\bullet(\infty, W_2) \xleftarrow{\kappa'} \mathfrak{Z}_0^\bullet(W_1, W_2) \xrightarrow{\iota'} \mathfrak{M}_0^\bullet(\infty, W),$$

where ι' is the inclusion and κ' is the induced map defined as follows. Consider an element $[B, \alpha, \beta] \in \mathfrak{Z}_0^\bullet(W_1, W_2)$ represented by a tuple $(B, \alpha, \beta) \in \Lambda^\bullet(T, W)$ for some T. Let T' be the maximal ($\widehat{J}$-graded) subspace of T stable by B such that $\beta(T') \subset W_1$. We have $\alpha(W_1) \in T'$ by the definition of $\mathfrak{Z}_0^\bullet(W_1, W_2)$. Then (B, α, β) defines elements of $\Lambda^\bullet(T', W_1)$ and $\Lambda^\bullet(T/T', W_2)$. These elements yield elements of $\mathfrak{M}_0^\bullet(\infty, W_1)$ and $\mathfrak{M}_0^\bullet(\infty, W_2)$.

Now let V be an I-graded finite dimensional $\mathbb{C}$-vector space. Let V_1 be an I-graded subspace of V and set $V_2 = V/V_1$. Consider the subvariety F of E_V consisting of the representations preserving V_1. We consider the following diagram, see [38, Sec. 9.2]

$$E_{V_1} \times E_{V_2} \xleftarrow{\kappa} F \xrightarrow{\iota} E_V,$$

where ι is the inclusion and κ assigns to an elements of F its quotient and restriction. Now assign W to V as in Theorem 2.3.11. The subspace $V_1 \subset V$ yields a subspace $W_1 \subset W$ and the quotient $W_2 = W/W_1$ would correspond to $V_2 = V/V_1$.

Theorem 2.3.20. *We have an isomorphism of diagrams*

$$\begin{tikzcd}
\mathfrak{M}_0^\bullet(\infty, W_1) \times \mathfrak{M}_0^\bullet(\infty, W_2) \arrow[d] & \mathfrak{Z}_0^\bullet(W_1, W_2) \arrow[l, "\kappa'"'] \arrow[r, "\iota'"] \arrow[d] & \mathfrak{M}_0^\bullet(\infty, W), \arrow[d] \\
E_{V_1} \times E_{V_2} & F \arrow[l, "\kappa"'] \arrow[r, "\iota"] & E_V,
\end{tikzcd}$$

where the leftmost and the rightmost isomorphisms are as in Theorem 2.3.11.

Proof. Consider the element $[B, \alpha, \beta] \in \mathfrak{M}_0^\bullet(\infty, W)$ represented by the tuple $(B, \alpha, \beta) \in \Lambda^\bullet(T, W)$ for some T. Let $i \to j$ be an arrow in Γ. Recall that the map $V_i \to V_j$ associated with $[B, \alpha, \beta]$ via the rightmost vertical arrow is

$$\beta_j(n-r)B_{i_{r-1}i_r}(n-(r-1)) \cdots B_{i_1 i_2}(n-1)\alpha_i(n),$$

where $i \rightsquigarrow j = (w_i(n) \to t_{i_1}(n-1) \to t_{i_2}(n-2) \to \cdots \to t_{i_r}(n-r) \to w_j(k))$. So in view of Remark 2.3.12 the condition $[B, \alpha, \beta] \in \mathfrak{Z}_0^\bullet(W_1, W_2)$ means that the corresponding element of E_V via the isomorphism in Theorem 2.3.11 preserves V_1. So we have an isomorphism $\mathfrak{Z}_0^\bullet(W_1, W_2) \simeq F$ making the right part of the diagram in the statement to be commutative. The left part is also commutative because both κ' and κ were defined as the product of the quotient and the restriction. $\square$

The restriction diagram for Lusztig quiver varieties yields a functor

$$\widetilde{\mathrm{Res}}_{V_1,V_2} = \kappa_! \iota^* \colon D_{G_V}(E_V, \mathbf{k}) \to D_{G_{V_1} \times G_{V_2}}(E_{V_1} \times E_{V_2}, \mathbf{k}).$$

Consider also its modification

$$\mathrm{Res}_{V_1,V_2} = \widetilde{\mathrm{Res}}_{V_1,V_2}[M_{V_1,V_2}],$$

where

$$M_{V_1,V_2} = \sum_{h \in H} \dim(V_1)_{h'} \dim(V_2)_{h''} - \sum_{i \in I} \dim(V_1)_i \dim(V_2)_i.$$

2.3.7 Restriction of Nakajima sheaves

Let T be a $\widehat{J}$-graded finite dimensional $\mathbb{C}$-vector space. Let $\mathscr{L}_{T,W}$ be the shift $\mathscr{L}_{T,W} = \widetilde{\mathscr{L}}_{T,W}[\dim_{\mathbb{C}} \mathfrak{M}^\bullet(T, W)]$, where $\widetilde{\mathscr{L}}_{T,W}$ is as in Section 2.3.5. We identify the restriction diagrams for Nakajima quiver varieties and Lusztig quiver varieties as in Theorem 2.3.20. We will write $\iota = \iota', \kappa = \kappa'$ to simplify the notation. Set

$$\widetilde{\mathfrak{Z}}_0^\bullet(T, W_1, W_2) = (\pi_{T,W})^{-1}(\mathfrak{Z}_0^\bullet(W_1, W_2)).$$

We have the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{Z}}^\bullet_0(T, W_1, W_2) & \xrightarrow{\ \tilde{\iota}\ } & \mathfrak{M}^\bullet(T, W) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi_{T,W}} \\ \mathfrak{Z}^\bullet_0(W_1, W_2) & \xrightarrow{\ \iota\ } & \mathfrak{M}^\bullet_0(\infty, W), \end{array}$$

where $\tilde{\iota}$ is the inclusion, p is the restriction of $\pi_{T,W}$.

Set

$$U(T) = \{(\omega_1, \omega_2) \in \mathbb{N}\widehat{J} \times \mathbb{N}\widehat{J};\ \omega_1 + \omega_2 = \mathrm{grdim} T\}.$$

For each pair $(\omega_1, \omega_2) \in U(T)$ set

$$\widetilde{F}(\omega_1, \omega_2) = \{[B, \alpha, \beta] \in \widetilde{\mathfrak{Z}}^\bullet_0(T, W_1, W_2);\ \mathrm{grdim} T' = \omega_1, \mathrm{grdim} T/T' = \omega_2\},$$

where the subset $T' \subset T$ is associated with $[B, \alpha, \beta]$ as in the definition of the restriction diagram for Nakajima quiver varieties (see the definition of the morphism κ'). For each $[B, \alpha, \beta] \in \widetilde{F}(\omega_1, \omega_2)$ the restriction to $T' \oplus W_1$ yields an element of $\Lambda^\bullet_{\mathrm{s}}(T', W_1)$ and the quotient yields an element of $\Lambda^\bullet_{\mathrm{s}}(T/T', W_2)$. This yields a morphism

$$\alpha_{\omega_1,\omega_2} \colon \widetilde{F}(\omega_1, \omega_2) \to \mathfrak{M}^\bullet(T_1, W_1) \times \mathfrak{M}^\bullet(T_2, W_2),$$

where T_1, T_2 are $\widehat{J}$-graded $\mathbb{C}$-vector spaces with dimension vectors ω_1, ω_2 respectively. The morphism $\alpha_{\omega_1,\omega_2}$ is a vector bundle, see also [48, Prop. 3.8]. Let d_{T_1,T_2} be its rank. The following lemma is a positive characteristic analogue of [63, Lem. 4.1].

Lemma 2.3.21. *We have*

$$\widetilde{\mathrm{Res}}_{V_1,V_2}(\widetilde{\mathscr{L}}_{T,W}) = \bigoplus_{T_1 \oplus T_2 \simeq T} \widetilde{\mathscr{L}}_{T_1,W_1} \otimes \widetilde{\mathscr{L}}_{T_2,W_2}[-2d_{T_1,T_2}],$$

where the sum is taken over the isomorphism classes of $\widehat{J}$-graded $\mathbb{C}$-vector spaces T_1, T_2 such that $T_1 \oplus T_2 \simeq T$.

Proof. For each $(\omega_1, \omega_2) \in U(T)$ and each $\widehat{J}$-graded $\mathbb{C}$-vector spaces T_1, T_2 of graded dimensions ω_1, ω_2 we have a commutative diagram

$$\begin{array}{ccc} \widetilde{F}(\omega_1, \omega_2) & \longrightarrow & \widetilde{\mathfrak{Z}}^\bullet_0(T, W_1, W_2) \\ {\scriptstyle \alpha_{\omega_1,\omega_2}}\Big\downarrow & & {\scriptstyle \kappa p}\Big\downarrow \\ \mathfrak{M}^\bullet(T_1, W_1) \times \mathfrak{M}^\bullet(T_2, W_2) & \longrightarrow & \mathfrak{M}^\bullet_0(\infty, W_1) \times \mathfrak{M}^\bullet_0(\infty, W_2). \end{array}$$

The upper horizontal arrow is the obvious inclusion and the lower one is $\pi_{T_1,W_1} \times \pi_{T_2,W_2}$. The varieties $\widetilde{F}(\omega_1, \omega_2)$ with $(\omega_1, \omega_2) \in U(T)$ form a locally closed partition of $\widetilde{\mathfrak{Z}}^\bullet_0(T, W_1, W_2)$. Let us enumerate them $\widetilde{F}_1, \cdots, \widetilde{F}_r$

in such a way that for $j \in [1,r]$ the subvariety $\widetilde{F}_{\leqslant j} = \bigcup_{j' \leqslant j} \widetilde{F}_{j'}$ is closed in $\widetilde{\mathfrak{Z}}_0^\bullet(T, W_1, W_2)$. Let f_j and $f_{\leqslant j}$ be the restrictions of κp

$$f_j \colon \widetilde{F}_j \to \mathfrak{M}_0^\bullet(\infty, W_1) \times \mathfrak{M}_0^\bullet(\infty, W_2), \qquad f_{\leqslant j} \colon \widetilde{F}_{\leqslant j} \to \mathfrak{M}_0^\bullet(\infty, W_1) \times \mathfrak{M}_0^\bullet(\infty, W_2).$$

We also set $\widetilde{F}_{<j} = \widetilde{F}_{\leqslant j-1}$, $f_{<j} = f_{\leqslant j-1}$ for $j \in [2,r]$. Let a_j and b_j be the inclusions

$$a_j \colon \widetilde{F}_j \to \widetilde{F}_{\leqslant j}, \qquad b_j \colon \widetilde{F}_{<j} \to \widetilde{F}_{\leqslant j}.$$

By [3, Sec. 1.4.3], we have a distinguished triangle

$$a_{j!} a_j^* \underline{\mathbf{k}}_{\widetilde{F}_{\leqslant j}} \to \underline{\mathbf{k}}_{\widetilde{F}_{\leqslant j}} \to b_{j!} b_j^* \underline{\mathbf{k}}_{\widetilde{F}_{\leqslant j}} \xrightarrow{+1} .$$

Applying the functor $(f_{\leqslant j})_!$ to this triangle we get the distinguished triangle

$$(f_j)_! \underline{\mathbf{k}}_{\widetilde{F}_j} \to (f_{\leqslant j})_! \underline{\mathbf{k}}_{\widetilde{F}_{\leqslant j}} \to (f_{<j})_! \underline{\mathbf{k}}_{\widetilde{F}_{<j}} \xrightarrow{+1} . \tag{2.7}$$

Now if $\widetilde{F}_j = \widetilde{F}(\omega_1, \omega_2)$ and T_1, T_2 are $\widehat{J}$-graded $\mathbb{C}$-vector spaces with graded dimensions ω_1, ω_2 respectively, we have

$$\alpha_{\omega_1, \omega_2 !} \underline{\mathbf{k}}_{\widetilde{F}_j} = \underline{\mathbf{k}}_{\mathfrak{M}^\bullet(T_1, W_1) \times \mathfrak{M}^\bullet(T_2, W_2)}[-2d_{T_1, T_2}]$$

because $\alpha_{\omega_1, \omega_2}$ is a vector bundle of rank d_{T_1, T_2}. Thus,

$$f_{j!} \underline{\mathbf{k}}_{\widetilde{F}_j} = \widetilde{\mathscr{L}}_{T_1, W_1} \otimes \widetilde{\mathscr{L}}_{T_2, W_2}[-2d_{T_1, T_2}].$$

In particular it is an even complex, see Corollary 2.3.17. Thus, from (2.7) and Lemma 2.3.9 we have by induction

$$\kappa_! p_! \underline{\mathbf{k}}_{\widetilde{\mathfrak{Z}}_0^\bullet(T, W_1, W_2)} = \bigoplus_{T_1 \oplus T_2 \simeq T} \widetilde{\mathscr{L}}_{T_1, W_1} \otimes \widetilde{\mathscr{L}}_{T_2, W_2}[-2d_{T_1, T_2}].$$

This finishes the proof because by the base change theorem we have

$$\kappa_! \iota^* \widetilde{\mathscr{L}}_{T, W} = \kappa_! p_! \underline{\mathbf{k}}_{\widetilde{\mathfrak{Z}}_0^\bullet(T, W_1, W_2)}.$$

□

The split Grothendieck group $K(\mathcal{Q}_V)$ of the additive category $\mathcal{Q}_V$ is given by the $\mathcal{A}$-module structure such that $q[\mathcal{F}] = [\mathcal{F}[1]]$ for $\mathcal{F} \in \mathcal{Q}_V$. We set $K(\mathcal{Q}) = \bigoplus_V K(\mathcal{Q}_V)$. Denote by $\mathrm{Par}_{G_V}(E_V)$ the full additive subcategory of $D_{G_V}(E_V, \mathbf{k})$ whose objects are parity complexes. Similar to the construction above, we have an $\mathcal{A}$-module structure on the split Grothendieck group $K(\mathrm{Par}_{G_V}(E_V))$ of the category $\mathrm{Par}_{G_V}(E_V)$. We also set

$$\mathrm{Par} = \bigoplus_V \mathrm{Par}_{G_V}(E_V), \qquad K(\mathrm{Par}) = \bigoplus_V K(\mathrm{Par}_{G_V}(E_V)).$$

Let $\mathbf{S}(W)$ be as in Section 2.3.4.

Lemma 2.3.22. *The complexes $\widetilde{\mathscr{L}_{T,W}}$, $T \in \mathbf{S}(W)$, form an $\mathcal{A}$-basis in $K(\mathrm{Par}_{G_V}(E_V))$.*

Proof. Note that $i_\lambda^*\widetilde{\mathcal{E}}(\lambda) = \underline{\mathbf{k}}_\lambda$, where $\widetilde{\mathcal{E}}(\lambda)$ is as in Remark 2.3.19. Let $\leqslant$ be the total order on Λ_V as in the proof of Lemma 2.3.8. Denote by T_λ the $\widehat{J}$-graded finite dimensional $\mathbb{C}$-vector space such that $\mathfrak{M}_0^{\bullet\mathrm{reg}}(T_\lambda, W) \simeq \mathcal{O}_\lambda$, see Lemma 2.3.16. We have $\{T_\lambda;\ \lambda \in \Lambda_V\} = \mathbf{S}(W)$. The complex $\widetilde{\mathscr{L}_{T_\lambda,W}}$ is even, see Corollary 2.3.17. By the base change theorem we have $i_\lambda^*\widetilde{\mathscr{L}_{T_\lambda,W}} = \underline{\mathbf{k}}_\lambda$. Thus,

$$\widetilde{\mathscr{L}_{T_\lambda,W}} = \widetilde{\mathcal{E}}(\lambda) \oplus \bigoplus_{\mu<\lambda, d\in\mathbb{Z}} \widetilde{\mathcal{E}}(\mu)[2d]^{\oplus a_{\mu,d}}, \qquad a_{\mu,d} \in \mathbb{N}.$$

Hence, the classes $[\widetilde{\mathscr{L}_{T_\lambda,W}}]$, $\lambda \in \Lambda_V$, form an $\mathcal{A}$-basis in $K(\mathrm{Par}_{G_V}(E_V))$ because the classes $[\widetilde{\mathcal{E}}(\lambda)]$, $\lambda \in \Lambda_V$, form an $\mathcal{A}$-basis in $K(\mathrm{Par}_{G_V}(E_V))$. $\square$

By Lemmas 2.3.21, 2.3.22 the functor $\widetilde{\mathrm{Res}}_{V_1,V_2}$ yields an $\mathcal{A}$-module homomorphism

$$K(\mathrm{Par}_{G_V}(E_V)) \to K(\mathrm{Par}_{G_{V_1}\times G_{V_2}}(E_{V_1} \times E_{V_2})),$$

where $\mathrm{Par}_{G_{V_1}\times G_{V_2}}(E_{V_1} \times E_{V_2})$ is defined as $\mathrm{Par}_{G_V}(E_V)$ using the quiver $\Gamma \coprod \Gamma$. A $G_{V_1} \times G_{V_2}$-orbit in $E_{V_1} \times E_{V_2}$ is of the form $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$. So we identify Λ_V with $\Lambda_{V_1} \times \Lambda_{V_2}$. The complex $\widetilde{\mathcal{E}}(\lambda_1) \otimes \widetilde{\mathcal{E}}(\lambda_2)$ is indecomposable because its endomorphism ring is a tensor product of endomorphism rings of complexes $\widetilde{\mathcal{E}}(\lambda_1)$ and $\widetilde{\mathcal{E}}(\lambda_2)$, thus it is local. It is even, supported on $\overline{\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}}$, and we have

$$i_{\lambda_1\times\lambda_2}^*(\widetilde{\mathcal{E}}(\lambda_1) \otimes \widetilde{\mathcal{E}}(\lambda_2)) = \underline{\mathbf{k}}_{\lambda_1\times\lambda_2}.$$

Thus, by Lemma 2.3.3 we have

$$\widetilde{\mathcal{E}}(\lambda_1) \otimes \widetilde{\mathcal{E}}(\lambda_2) \simeq \widetilde{\mathcal{E}}(\lambda_1 \times \lambda_2).$$

Thus, we have

$$K(\mathrm{Par}_{G_{V_1}\times G_{V_2}}(E_{V_1} \times E_{V_2})) \simeq K(\mathrm{Par}_{G_{V_1}}(E_{V_1})) \otimes_{\mathcal{A}} K(\mathrm{Par}_{G_{V_2}}(E_{V_2}))$$

Finally, the functors $\widetilde{\mathrm{Res}}_{V_1,V_2}$ yield a comultiplication on

$$K(\mathrm{Par}) = \bigoplus_V K(\mathrm{Par}_{G_V}(E_V)),$$

where the sum is taken by isomorphism classes of I-graded finite dimensional $\mathbb{C}$-vector spaces. Thus, the modified restriction functors Res_{V_1,V_2} yield a comultiplication Res on $K(\mathrm{Par})$. From now on we will always consider the coalgebra structure on $K(\mathrm{Par})$ given by Res.

2.3.8 Restriction of Lusztig sheaves

Now we study the restriction of Lusztig sheaves. It is similar to the constructions for Nakajima sheaves in Section 2.3.7. Let

$$E_{V_1} \times E_{V_2} \xleftarrow{\kappa} F \xrightarrow{\iota} E_V$$

be the restriction diagram as in Section 2.3.6. Fix a sequence $(a_1, \cdots, a_k)$. Set $y = (i_1^{(a_1)} \cdots i_k^{(a_k)})$, an element in Y_ν. Set also

$$U(y) = \{(y_1, y_2) \in Y_{\nu_1} \times Y_{\nu_2};\ y_1 = (i_1^{(a'_1)} \cdots i_k^{(a'_k)}), y_2 = (i_1^{(a''_1)} \cdots i_k^{(a''_k)}),$$

$$a'_r, a''_r \in \mathbb{N},\ a'_r + a''_r = a_r\ \forall r \in [1, k]\}.$$

We have a commutative diagram

$$\begin{tikzcd} \widetilde{F} \arrow[r] \arrow[d] & \widetilde{F}_y \arrow[d, "\pi_y"] \\ F \arrow[r] & E_V, \end{tikzcd}$$

where $\widetilde{F} = \pi_y^{-1}(F)$. The variety $\widetilde{F}$ consists of the pairs $(x, \phi) \in E_V \times F_y$ such that x preserves V_1 and x preserves the flag $\phi = (\{0\} = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^k = V)$. For (y_1, y_2), denote by $\widetilde{F}(y_1, y_2)$ the subset of $\widetilde{F}$ consisting of pairs $(x, \phi) \in \widetilde{F}$ such that the flag $(V^0 \cap V_1 \subset V^1 \cap V_1 \subset \cdots \subset V^r \cap V_1)$ has type y_1 and the flag $(V^0/(V^0 \cap V_1) \subset V^1/(V^1 \cap V_1) \subset \cdots \subset V^r/(V^r \cap V_1))$ has type y_2. We have a vector bundle

$$\widetilde{F}(y_1, y_2) \to \widetilde{F}_{y_1} \times \widetilde{F}_{y_2},$$

see [38, Sec. 9.2.4]. Denote its rank by m_{y_1,y_2}.

Lemma 2.3.23. *Suppose that for each $(y_1, y_2) \in U(y)$ the complexes $\mathcal{L}_{y_1}$, $\mathcal{L}_{y_2}$ are even. Then we have*

$$\widetilde{\mathrm{Res}}_{V_1,V_2}(\mathcal{L}_y) = \bigoplus_{(y_1,y_2)\in U(\nu)} \mathcal{L}_{y_1} \otimes \mathcal{L}_{y_2}[-2m_{y_1,y_2}].$$

Proof. The proof is the same as the proof of Lemma 2.3.21. See also [38, Sec. 9.2]. □

Lemma 2.3.24. *For each $\lambda \in \Lambda_V$ let $y_\lambda \in Y_\nu$ be as in Section 2.2.21. The classes $[\mathcal{L}_{y_\lambda}]$, $\lambda \in \Lambda_V$ form an $\mathcal{A}$-basis in $K(\mathcal{Q}_V)$.*

Proof. The proof is the same as the proof of Lemma 2.3.22. □

2.3.9 Coalgebra structure on $K(\mathrm{Par})$

The following lemma follows directly from Lemma 2.3.8.

Lemma 2.3.25. *Let $\mathcal{A}[\Lambda_V \times \mathbb{Z}]$ be a free $\mathcal{A}$-module with basis $1_{\lambda,n}$, $\lambda \in \Lambda_V$, $n \in \mathbb{Z}$. Then there is an $\mathcal{A}$-module inclusion*

$$K(\mathrm{Par}_{G_V}(E_V)) \hookrightarrow \mathcal{A}[\Lambda_V \times \mathbb{Z}], \qquad \mathcal{F} \mapsto \sum_{\lambda,n} d_{\lambda,n}(\mathcal{F}) 1_{\lambda,n},$$

where $d_{\lambda,n}$ are as in Section 2.3.3. □

Theorem 2.3.26. *There exists an $\mathcal{A}$-coalgebra isomorphism $\beta_{\mathcal{A}}\colon K(\mathrm{Par}) \to {}_{\mathcal{A}}\mathbf{f}$.*

Proof. At first suppose $k = \mathbb{C}$. In this case the statement is already known, see [38, Thm. 13.2.11]. Now let $\mathbf{k}$ be an arbitrary field. To avoid confusion we will indicate the field $\mathbf{k}$ or $\mathbb{C}$ in the rest of the proof. By Lemma 2.3.15 the numbers $d_{\lambda,n}(\widetilde{\mathscr{L}}_{T,W,\mathbf{k}})$ do not depend on the field $\mathbf{k}$, where $\lambda \in \Lambda_V$, $n \in \mathbb{N}$, T is a finite dimensional $\widehat{J}$-graded $\mathbb{C}$-vector space. Thus, the image of the inclusion in Lemma 2.3.25 does not depend on the field either. Hence, there exists a linear isomorphism

$$K(\mathrm{Par}_{G_V}(E_V), \mathbf{k}) \to K(\mathrm{Par}_{G_V}(E_V), \mathbb{C})$$

sending $\widetilde{\mathscr{L}}_{T,W,\mathbf{k}}$ to $\widetilde{\mathscr{L}}_{T,W,\mathbb{C}}$ for each T. To conclude we only need to note that this isomorphism preserves the comultiplication because the constants d_{T_1,T_2} from Lemma 2.3.21 are independent of the field. □

Remark 2.3.27. There is an $\mathcal{A}$-basis in $K(\mathrm{Par})$ given by parity sheaves. It corresponds to some $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}$ by $\beta_{\mathcal{A}}$. So Theorem 2.3.26 yields an $\mathcal{A}$-basis in ${}_{\mathcal{A}}\mathbf{f}$ in terms of parity sheaves.

2.3.10 Even quivers

Definition 2.3.28. The quiver Γ is $\mathbf{k}$-*even* if the complex $\mathcal{L}_y$ over $\mathbf{k}$ is even for each $y \in Y_\nu$ and each $\nu \in \mathbb{N}I$. The quiver Γ is *even* if it is $\mathbf{k}$-even for each field $\mathbf{k}$.

In this section we suppose that our Dynkin quiver Γ is even. This hypothesis is necessary to apply Lemma 2.3.23.

Theorem 2.3.29. *Suppose that Γ is a $\mathbf{k}$-even Dynkin quiver. Let V be as above. Then the full subcategories $\mathcal{Q}_V$ and $\mathrm{Par}_{G_V}(E_V)$ of $D_{G_V}(E_V, \mathbf{k})$ coincide. Moreover, there exists an isomorphism of bialgebras $K(\mathcal{Q}) \simeq {}_{\mathcal{A}}\mathbf{f}$, where the algebra structure on $K(\mathcal{Q})$ is as in Section 2.2.20 and the comultiplication is given by* Res.

Proof. The first statement follows from Remark 2.3.19 (b). The second statement is already known if the characteristic of $\mathbf{k}$ is zero, see [38, Thm. 13.2.11]. In positive characteristic we already have the algebra isomorphism

$$\lambda_{\mathcal{A}} \colon {}_{\mathcal{A}}\mathbf{f} \to K(\mathcal{Q})$$

by Theorem 2.2.22. We only need to verify that $\lambda_{\mathcal{A}}$ preserves the coproduct. This follows from the zero characteristic case and Lemma 2.3.23 because the constants m_{y_1,y_2} in Lemma 2.3.23 do not depend on the field. □

2.3.11 Type A

We say that a quiver is of type A if each of its connected components has type A_n for some positive integer n. Now we show that the quivers of type A are even. We start by proving some helpful lemmas.

Definition 2.3.30. Suppose $n, k, s \in \mathbb{N}$, $k \leqslant n$, $s > 0$. Suppose $\mathbf{v}$ and $\mathbf{d}$ are s-tuples of nonnegative integers $\mathbf{v} = (v_1, \cdots, v_s)$, $\mathbf{d} = (d_1, \cdots, d_s)$ such that $v_1 + \cdots + v_s = n$. Let W be an n-dimensional $\mathbb{C}$-vector space and let

$$\phi = (\{0\} = W_0 \subset W_1 \subset \cdots \subset W_s = W)$$

be a flag of type $\mathbf{v}$ in W, i.e., $\dim W_a/W_{a-1} = v_a$ for each a in $[1, s]$. Set

$$X_{\mathbf{v},\mathbf{d},k} = \{U \in \mathrm{Gr}_k(W);\ \dim U \cap W_a = d_a, \forall a \in [1, s]\}.$$

Lemma 2.3.31. *The set $X_{\mathbf{v},\mathbf{d},k}$ is either empty or a variety with an affine cell decomposition.*

Proof. Suppose that $X_{\mathbf{v},\mathbf{d},k}$ is not empty. Let P be the parabolic subgroup in $\mathrm{GL}(W)$ preserving the flag ϕ. The variety $X_{\mathbf{v},\mathbf{d},k}$ is isomorphic to a P-orbit in $\mathrm{Gr}_k(W)$. Let B be a Borel subgroup contained in P. A P-orbit in $\mathrm{Gr}_k(W)$ is a disjoint union of B-orbits. Each of these B-orbits is isomorphic to an affine space. □

Definition 2.3.32. Let W be a finite dimensional $\mathbb{C}$-vector space. Suppose $s, t \in \mathbb{N}$, $s > 0, t > 0$. Let

$$\phi = (\{0\} = W_0 \subset W_1 \subset \cdots \subset W_s = W), \quad \psi = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_t = W)$$

be two partial flags in W. Let $\mathbf{v}$ be a sequence of nonnegative integers $\mathbf{v} = (v_1, \cdots, v_s)$ and let $D = (d_{ij})$ be an $s \times t$-matrix of integers. Set

$$Y_{\phi,\psi,\mathbf{v},D} = \{(\{0\} = U_0 \subset U_1 \subset \cdots \subset U_s = W) :$$
$$\dim U_a/U_{a-1} = v_a, \dim U_a \cap V_b = d_{a,b}, W_a \subset U_a, \quad \forall a \in [1, s], b \in [1, t]\}.$$

Lemma 2.3.33. *The set $Y_{\phi,\psi,\mathbf{v},D}$ is either empty or a variety with an affine cell decomposition.*

Proof. The proof is by induction on s. The case $s=1$ follows from Lemma 2.3.31. Suppose $s>1$. Set

$$\phi' = (\{0\} = W_0 \subset W_2 \subset W_3 \subset \cdots \subset W_s = W), \quad \mathbf{v}' = (v_1+v_2, v_3, \cdots, v_s).$$

Let D' be the matrix that we get from D by erasing the first row. Forgetting the U_1-component of the flag yields a morphism $Y_{\phi,\psi,\mathbf{v},D} \to Y_{\phi',\psi,\mathbf{v}',D'}$. This is a fibration with the fiber of the form $X_\bullet$, see Definition 2.3.30. So $Y_{\phi,\psi,\mathbf{v},D}$ has a decomposition into affine cells by induction hypothesis and Lemma 2.3.31. $\square$

Suppose that the quiver Γ is of type A_n. We enumerate its vertices $I = \{i_1, \cdots, i_n\}$ such that there is an arrow in some direction between i_a and i_{a+1} for $a \in [1, n-1]$. Let V and ν be as in Section 2.2.1. Suppose $y = (j_1^{(a_1)}, \cdots, j_m^{(a_m)}) \in Y_\nu$. Let

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_k = V_{i_n}$$

be a partial flag in V_{i_n}. For each $m \times k$-matrix $D = (d_{ij})$ of integers we set

$$F_D = \{(\{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V) \in \pi_y^{-1}(x) :$$
$$\dim V^r \cap W_s = d_{rs}, \forall r \in [1,m], s \in [1,k]\}.$$

We have the decomposition $\pi_y^{-1}(x) = \coprod_D F_D$.

Lemma 2.3.34. *The set F_D is either empty or a variety with an affine cell decomposition.*

Proof. The proof is by induction on n. For $n=1$ the statement follows from Lemma 2.3.33. Now suppose $n \geqslant 2$. Suppose that the quiver Γ contains the arrow $i_{n-1} \to i_n$. We denote this arrow by h_0. Consider the following flag in $V_{i_{n-1}}$

$$\{0\} = W'_0 \subset W'_1 \subset \cdots \subset W'_{k+1} = V_{i_{n-1}}, \qquad W'_r = x_{h_0}^{-1}(W_{r-1}),\ r \in [1,k],$$

where x_{h_0} is the h_0-component of x. Let Γ' be the quiver that we get from Γ by deleting the vertex i_n. Set $V' = \bigoplus_{r=1}^{n-1} V_{i_r}$, $\nu' = \nu - \nu_{i_n} \cdot i_n$. As before we denote by H the set of arrows in Γ and for an arrow h in H we write h' and h'' for its source and target respectively. Set $H' = H \backslash h_0$. Denote by $E_{V'}$ the variety of representations of Γ' in V', i.e., $E_{V'} = \bigoplus_{h \in H'} \mathrm{Hom}(V_{h'}, V_{h''})$. Set $x' = \bigoplus_{h \in H'} x_h \in E_{V'}$, where x_h is the h-component of $x \in E_V$. Set $y' = (j_1^{(a'_1)}, \cdots, j_m^{(a'_m)}) \in Y_{\nu'}$, where

$$a'_t = \begin{cases} a_t & \text{if } j_t \neq i_n, \\ 0 & \text{if } j_t = i_n, \end{cases} \qquad \forall t \in [1,m].$$

We leave the zero terms in y' to simplify the notation. For each $m\times(k+1)$-matrix of integers $D' = (d'_{rs})$ we set

$$F'_{D'} = \{(\{0\} = V'^0 \subset V'^1 \subset \cdots \subset V'^m = V') \in \pi_{y'}^{-1}(x') :$$
$$\dim V'^r \cap W'_s = d'_{rs}, \forall r \in [1,m], s \in [1,k+1]\}.$$

We have the morphism $e\colon \pi_y^{-1}(x) \to \pi_{y'}^{-1}(x')$ given by the intersection of each component of the flag with V'. Set

$$F_{D,D'} = F_D \cap e^{-1}(F'_{D'}).$$

We have the decomposition $F_D = \coprod_{D'} F_{D,D'}$. So it is enough to construct a decomposition into affine cells for each $F_{D,D'}$. Let us study the following restriction of e

$$e_{D,D'}\colon F_{D,D'} \to F'_{D'}.$$

We want to show that it is a fibration. Let $c_1 < \cdots < c_t$ be the integers in $[1,m]$ such that

$$j_{c_1} = j_{c_2} = \cdots = j_{c_t} = i_n, \qquad j_s \neq i_n \ \forall s \in [1,m]\backslash\{c_1, \cdots, c_t\}.$$

Set also $c_0 = 0$. Let

$$\phi' = (\{0\} = V'^0 \subset V'^1 \subset \cdots \subset V'^m = V')$$

be a flag in $F'_{D'}$. We have

$$\begin{aligned}
e_{D,D'}^{-1}(\phi') &= \{(\{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V) \in F_D : \\
&\quad V^s \cap V' = V'^s, \forall s \in [1,m]\} \\
&\simeq \{(\{0\} = U^0 \subset U^1 \subset \cdots \subset U^m = V_{i_n}) : \\
&\quad (\{0\} = V'^0 \oplus U^0 \subset V'^1 \oplus U^1 \subset \cdots \subset V'^m \oplus U^m = V) \in F_D\} \\
&= \{(\{0\} = U^0 \subset U^1 \subset \cdots \subset U^m = V_{i_n}) : \\
&\quad U^s = U^{s-1}, \forall s \in [1,m]\backslash\{c_1, \cdots, c_t\}, \\
&\quad \dim U^s/U^{s-1} = a_s, \forall s \in \{c_1, \cdots, c_t\}, \\
&\quad x_{h_0}(V'^r \cap V_{i_{n-1}}) \subset U^r, \forall r \in [1,m], \\
&\quad \dim U^r \cap W_p = d_{r,p}, \forall r \in [1,m], p \in [1,k]\}. \\
&\simeq Y_{\phi,\psi,\mathbf{v},D},
\end{aligned}$$

where

$$\phi = (x_{h_0}(V'^0 \cap V_{i_{n-1}}) \subset x_{h_0}(V'^1 \cap V_{i_{n-1}}) \subset \cdots \subset x_{h_0}(V'^{m-1} \cap V_{i_{n-1}}) \subset V_{i_n}),$$

$$\psi = (\{0\} = W_0 \subset W_1 \subset \cdots \subset W_k = V_{i_n}), \qquad \mathbf{v} = (a_1 - a'_1, \cdots, a_m - a'_m),$$

see Definition 2.3.32. Moreover, this fiber does not depend on the choice of $\phi' \in F'_{D'}$ because the types of the flags ϕ, ψ and their relative position do not depend on ϕ'. Really,

$$\begin{aligned}
\dim x_{h_0}(V'^a \cap V_{i_{n-1}}) \cap W_b &= \dim x_{h_0}(V'^a \cap V_{i_{n-1}} \cap x_{h_0}^{-1}(W_b)) \\
&= \dim x_{h_0}(V'^a \cap W'_{b+1}) \\
&= \dim V'^a \cap W'_{b+1} - \dim V'^a \cap W'_{b+1} \cap \mathrm{Ker} x_{h_0} \\
&= \dim V'^a \cap W'_{b+1} - \dim V'^a \cap W'_1 \\
&= d'_{a,b+1} - d'_{a,1}, \qquad \forall a \in [1,m], b \in [1,k], \\
\dim x_{h_0}(V'^a \cap V_{i_{n-1}}) &= \dim x_{h_0}(V'^a \cap V_{i_{n-1}}) \cap W_k \\
&= d'_{a,k+1} - d'_{a,1}, \qquad \forall a \in [1,m].
\end{aligned}$$

So $F_{D,D'}$ has a decomposition into affine cells by induction hypothesis and Lemma 2.3.33.

Now suppose that the quiver Γ contains the arrow $i_n \to i_{n-1}$. Let Γ^{op} be the quiver that we get from Γ by inverting the orientation of each arrow. Consider the I-graded $\mathbb{C}$-vector space $V^* = \bigoplus_{i \in I} V_i^*$ dual to V. Consider the representation $x^* \in E_{V^*}$ dual to x. Let y^{op} be the element Y_ν that we get from y by reversing the order of its components. Then we have an isomorphism of varieties

$$\pi_y^{-1}(x) \to \pi_{y^{\mathrm{op}}}^{-1}(x^*),$$

$$(\{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V) \mapsto (\{0\} = (V^m)^\perp \subset \cdots \subset (V^1)^\perp \subset (V^0)^\perp = V^*).$$

Here $\bullet^\perp$ denotes the annihilator in the dual space V^*. Moreover, this isomorphism preserves the decompositions

$$\pi_y^{-1}(x) = \coprod_D F_D, \quad \pi_{y^{\mathrm{op}}}^{-1}(x^*) = \coprod_D F_D^*$$

(with some permutation of indices). Here F_D^* is defined analogically to F_D with respect to the flag

$$\{0\} = (W_k)^\perp \subset \cdots \subset (W_1)^\perp \subset (W_0)^\perp = V_{i_n}^*,$$

where $\bullet^\perp$ denotes the annihilator in $V_{i_n}^*$ (not in V^*). So we reduce the case $i_n \to i_{n-1}$ to the case $i_{n-1} \to i_n$. □

Theorem 2.3.35. *Let Γ be a quiver of type A (may be not connected). Suppose $y \in Y_\nu$, $x \in E_V$. Then the fiber $\pi_y^{-1}(x)$ is either empty or has a decomposition into affine cells.*

Proof. It is enough to prove the statement in the case when the quiver Γ is connected. In this case the statement follows from Lemma 2.3.34. □

Corollary 2.3.36. *A quiver of type A is even.* □

2.3.12 New basis for small ranks

The goal of this section is to show that the basis described in Remark 2.3.27 coincides with Lusztig's canonical basis for Γ of types A_1 or A_2. As explained in Section 2.3.10, if Γ is an even quiver then we have $\mathcal{Q}_V = \mathrm{Par}_{G_V}(E_V)$. In this situation we have $\beta_{\mathcal{A}} = \lambda_{\mathcal{A}}$. Via the isomorphism $\lambda_{\mathcal{A}}\colon K(\mathcal{Q}) \to {}_{\mathcal{A}}\mathbf{f}$, the basis from Remark 2.3.27 is identified with the basis given by the classes $[\mathcal{E}(\lambda)]$ of parity sheaves, where $\lambda \in \Lambda_V$, V is a finite dimensional I-graded $\mathbb{C}$-vector space. This basis coincides with Lusztig's canonical basis for $\mathbf{k} = \mathbb{C}$ because the parity sheaves over $\mathbb{C}$ are the intersection complexes, see Remark 2.3.19 (c). To prove that this basis coincides with the canonical basis for arbitrary $\mathbf{k}$ we should prove that the placement of the class $[\mathcal{E}(\lambda)]$ in $K(\mathcal{Q}) \simeq {}_{\mathcal{A}}\mathbf{f}$ is independent of $\mathbf{k}$.

We may add to our notation an index $\mathbf{k}$ or $\mathbb{C}$ to specify the field. We want to prove that $[\mathcal{E}(\lambda)_{\mathbf{k}}] = [\mathcal{E}(\lambda)_{\mathbb{C}}]$ in $K(\mathcal{Q}_{\mathbf{k}}) \simeq {}_{\mathcal{A}}\mathbf{f} \simeq K(\mathcal{Q}_{\mathbb{C}})$. This is clear for the type A_1 because in this case E_V is a point for each V.

From now on we assume that the quiver $\Gamma = (I, H)$ is of type A_2. We have $I = \{i, j\}$ and $H = \{i \to j\}$. Take a dimension vector $\nu = ni + mj \in \mathbb{N}I$. We have $E_V = \mathrm{Hom}(V_i, V_j)$, where V_i and V_j are $\mathbb{C}$-vector spaces with dimensions n and m respectively. The G_V-orbits in E_V are labeled by the set $\Lambda_V = [0, \max\{n, m\}]$, where for $k \in \Lambda_V$ the orbit $\mathcal{O}_k \subset E_V$ is the subset of elements of rank k.

Assume first that $n \leqslant m$. For each $k \in [0, n]$ set $y_k = (i^{(k)}, j^{(m)}, i^{(n-k)}) \in Y_\nu$. The morphism $\pi_{y_k}\colon \widetilde{F}_{y_k} \to E_V$ is obviously a resolution of singularities of the orbit closure $\overline{\mathcal{O}_k}$. Let us study the stalks of the complex ${}^\delta\mathcal{L}_{y_k} = (\pi_{y_k})_*\underline{\mathbf{k}}[\dim_{\mathbb{C}} \mathcal{O}_k]$. By the base change theorem, for each $x \in E_V$ we have $H^s(i_x^*({}^\delta\mathcal{L}_{y_k})) \simeq H^{s+\dim_{\mathbb{C}} \mathcal{O}_k}(\pi_{y_k}^{-1}(x))$ because π_{y_k} is a proper morphism. It is easy to see that

$$\pi_{y_k}^{-1}(x) \simeq \begin{cases} \emptyset & \text{if } x \in \mathcal{O}_r, r > k, \\ \mathrm{Gr}_{n-k}(\mathbb{C}^{n-r}) & \text{if } x \in \mathcal{O}_r, r \in [0, k], \end{cases}$$

and that

$$\dim_{\mathbb{C}} \mathcal{O}_k = \dim_{\mathbb{C}} \mathrm{Gr}_k(V_j) + \dim_{\mathbb{C}} \mathrm{Hom}(V_i, \mathbb{C}^k) = k(m-k) + nk = nk + mk - k^2.$$

Recall that $\mathcal{E}(k)$ denotes the parity sheaf supported on $\overline{\mathcal{O}_k}$.

Lemma 2.3.37. *We have ${}^\delta\mathcal{L}_{y_k} \simeq \mathcal{E}(k)$.*

Proof. The parity sheaf $\mathcal{E}(k)$ is a direct factor of ${}^\delta\mathcal{L}_{y_k}$ because π_{y_k} is a resolution of singularities of $\overline{\mathcal{O}_k}$. Other direct factors of ${}^\delta\mathcal{L}_{y_k}$ must have supports inside of $\overline{\mathcal{O}_k} \backslash \mathcal{O}_k$. Thus it is enough to show that ${}^\delta\mathcal{L}_{y_k}$ has no direct factors of the form $\mathcal{E}(r)[d]$ for some $r < k$ and some $d \in \mathbb{Z}$.

Assume that ${}^\delta\mathcal{L}_{y_k}$ has such a direct factor $\mathcal{E}(r)[d]$. Then, by auto-duality, the complex ${}^\delta\mathcal{L}_{y_k}$ must also have a direct factor $\mathcal{E}(r)[-d]$. Thus for $x \in \mathcal{O}_r$

the complex $i_x^*({}^\delta\mathcal{L}_{y_k})$ has nonzero cohomology groups $H^s(i_x^*({}^\delta\mathcal{L}_{y_k}))$ with $s = -\dim_{\mathbb{C}} \mathcal{O}_r \pm d$ because $H^{-\dim_{\mathbb{C}} \mathcal{O}_r}(\mathcal{E}(r)) \simeq \mathbf{k}$. Thus, to come to a contradiction, it is enough to show that $H^s(i_x^*({}^\delta\mathcal{L}_{y_k}))$ is zero for $s \geqslant -\dim_{\mathbb{C}} \mathcal{O}_r$.

As explained above we have $H^s(i_x^*({}^\delta\mathcal{L}_{y_k})) = H^{s+\dim_{\mathbb{C}} \mathcal{O}_k}(\mathrm{Gr}_{n-k}(\mathbb{C}^{n-r}))$. Thus it is enough to check that $-\dim_{\mathbb{C}} \mathcal{O}_r + \dim_{\mathbb{C}} \mathcal{O}_k > 2\dim_{\mathbb{C}} \mathrm{Gr}_{n-k}(\mathbb{C}^{n-r})$, i.e., that $-(nr+mr-r^2)+(nk+mk-k^2) > 2(n-k)(k-r)$. This is equivalent to $m+k > n+r$ and thus is true. $\square$

Now we assume that $n \geqslant m$. For $k \in [0, m]$ set $y'_k = (j^{(m-k)}, i^{(n)}, j^{(k)}) \in Y_\nu$. A similar proof to the previous one gives the following lemma.

Lemma 2.3.38. *We have* ${}^\delta\mathcal{L}_{y'_k} \simeq \mathcal{E}(k)$. $\square$

Corollary 2.3.39. *For each* $k \in [0, \min\{m,n\}]$ *we have* $[\mathcal{E}(k)_{\mathbf{k}}] = [\mathcal{E}(k)_{\mathbb{C}}]$ *in* $K(\mathcal{Q}_{\mathbf{k}}) \simeq {}_{\mathcal{A}}\mathbf{f} \simeq K(\mathcal{Q}_{\mathbb{C}})$.

Proof. By Lemmas 2.3.37 and 2.3.38 there is an element $y \in Y_\nu$ (y_k or y'_k) such that ${}^\delta\mathcal{L}_{y,\mathbf{k}} \simeq \mathcal{E}(k)_{\mathbf{k}}$ and ${}^\delta\mathcal{L}_{y,\mathbb{C}} \simeq \mathcal{E}(k)_{\mathbb{C}}$. However, $[{}^\delta\mathcal{L}_{y,\mathbf{k}}]$ and $[{}^\delta\mathcal{L}_{y,\mathbb{C}}]$ correspond both to the same element in ${}_{\mathcal{A}}\mathbf{f}$. $\square$

2.3.13 Quiver Schur algebras

In this section, we apply a similar approach to the quiver Schur algebras in [59]. Let e be an integer, $e > 1$. Let Γ be a quiver with the set of vertices $I = \mathbb{Z}/e\mathbb{Z}$ and the set of arrows $H = \{i \to i+1;\ i \in I\}$. We keep the notation of Section 2.2.1. Let $E_V^n \subset E_V$ be the subvariety of nilpotent representations.

Let us change slightly the notation. Let $\mathrm{VComp}_{\mathrm{e}}(\nu)$ be the set of tuples $\mu = (\mu^{(1)}, \cdots, \mu^{(k)})$ of nonzero elements of $\mathbb{N}I$ such that $\mu^{(1)} + \mu^{(2)} + \cdots + \mu^{(k)} = \nu$. We will call an element of $\mathrm{VComp}_{\mathrm{e}}(\nu)$ a *vector composition*. For μ as above we denote by F_μ the variety of all I-graded flags

$$\phi = (0 = V^0 \subset V^1 \subset \cdots \subset V^k = V)$$

in the I-graded vector space $V = \bigoplus_{i \in I} V_i$ such that the I-graded vector space V^k/V^{k-1} has graded dimension $\mu^{(r)}$ for each $r \in \{1, 2, \cdots, k\}$. Let $\widetilde{F}_\mu$ be the variety of pairs $(x, \phi) \in E_V \times F_\mu$ that are compatible, i.e., we have $x(V^r) \subset V^{r-1}$ for $r \in [1, k]$. Let π_μ be the natural projection from $\widetilde{F}_\mu$ to E_V, i.e.,

$$\pi_\mu : \widetilde{F}_\mu \to E_V,\ (x, \phi) \mapsto x.$$

Let $N \subset \mathfrak{g}$ be the nilpotent cone. Let F be the set of all flags in V (not necessarily complete). Let $F_V \subset F$ be the subset of I-graded flags. Set

$$\widetilde{F} = \{(\phi = (V^0 \subset \cdots \subset V^k = V), x) \in F \times N;\ x(V^r) \subset V^{r-1}\ \forall r \in [1, k]\},$$

$$\widetilde{F}_V = \{(\phi = (V^0 \subset \cdots \subset V^k = V), x) \in F_V \times E_V^n;\ x(V^r) \subset V^{r-1}\ \forall r \in [1, k]\}.$$

Let $\pi\colon \widetilde{F} \to N$, $\pi_V\colon \widetilde{F}_V \to E_V^n$ be natural projections. Set $G = GL(V)$, $\mathfrak{g} = \mathfrak{gl}(V) = \mathrm{Lie}(G)$. Fix a primitive eth root of unity $\xi \in \mathbb{C}$. Consider the element $s \in G$ preserving each V_i and acting by ξ^i on V_i, $i \in I$. Note that the group $G \times \mathbb{C}^*$ acts on $\mathfrak{gl}(V)$ with G acting by the adjoint action and $\mathbb{C}^*$ acting by the multiplication by scalars. The $G \times \mathbb{C}^*$-action on $\mathfrak{g}$ yields a $G \times \mathbb{C}^*$-action on N. Set $\widetilde{s} = (s, \xi^{-1}) \in G \times \mathbb{C}^*$. We have $\mathfrak{g}^{\widetilde{s}} \simeq E_V$ and $N^{\widetilde{s}} \simeq E_V^n$. Here the top index $\widetilde{s}$ always means the set of $\widetilde{s}$-stable points.

Similarly, the G-action on F yields a $G \times \mathbb{C}^*$-action on F, where $\mathbb{C}^*$ acts trivially. The $G \times \mathbb{C}^*$-action on F and N yields a $G \times \mathbb{C}^*$-action on $\widetilde{F}$. We have $F^{\widetilde{s}} \simeq F_V$ and $\widetilde{F}^{\widetilde{s}} \simeq \widetilde{F}_V$. The following lemma is an analogue of Conjecture 2.1.4.

Lemma 2.3.40. *For each $\mu \in \mathrm{VComp_e}(\nu)$ and each $x \in E_V^n$, we have* $\mathrm{H}^{\mathrm{odd}}((\pi_\mu)^{-1}(x), \mathbb{Z}) = 0$.

Proof. The restriction of π to the set of $\widetilde{s}$-stable points yields a morphism $\pi^{\widetilde{s}}\colon \widetilde{F}^{\widetilde{s}} \to N^{\widetilde{s}}$. We have the following commutative diagram

$$\begin{array}{ccc} \widetilde{F}_V & \xrightarrow{\pi_V} & E_V^n \\ \Big\| & & \Big\| \\ \widetilde{F}^{\widetilde{s}} & \xrightarrow{\pi^{\widetilde{s}}} & N^{\widetilde{s}}. \end{array}$$

Then for each $x \in E_V^n$ we can identify the fiber $\pi_V^{-1}(x)$ with $(\pi^{\widetilde{s}})^{-1}(x) = (\pi^{-1}(x))^{\widetilde{s}}$. By [13, Thm. 3.9] we have $\mathrm{H}^{\mathrm{odd}}((\pi^{-1}(x))^{\widetilde{s}}, \mathbb{Z}) = 0$. This implies the statement because for each $\mu \in \mathrm{VComp_e}(\nu)$ the variety $\widetilde{F}_V$ contains a connected component isomorphic to $\widetilde{F}_\mu$ and the restriction of π_V to this component coincides with π_μ. □

It is well-known that the set of G_V-orbits in E_V^n is finite. This observation together with Lemma 2.3.40 motivates us to study the parity sheaves on E_V^n. As before, we fix an arbitrary field $\mathbf{k}$. Let $E_V^n = \coprod_{\lambda \in \Lambda_V^n} \mathcal{O}_\lambda$ be the stratification by G_V-orbits.

Lemma 2.3.41. *There are no nontrivial G_V-equivariant local systems on $\mathcal{O}_\lambda$ for each $\lambda \in \Lambda_V^n$.*

Proof. The proof is the same as the proof of Corollary 2.2.11. □

We can now prove that the stratified variety E_V^n satisfies the condition (2.2) in the same way as in Lemma 2.3.6.

Consider the following complexes in $D_{G_V}(E_V^n, \mathbf{k})$:

$${}^\delta\mathcal{L}_\mu = (\pi_\mu)_* \mathbf{k}_{\widetilde{F}_\mu}[\dim_{\mathbb{C}} F_\mu], \quad {}^\delta\mathcal{L}_{V,\mathbf{k}} = \bigoplus_{\mu \in \mathrm{VComp_e}(\nu)} {}^\delta\mathcal{L}_\mu.$$

Let $\mathcal{Q}_V$ be the full additive subcategory of all direct sums of shifts of direct factors of ${}^\delta\mathcal{L}_{V,\mathbf{k}}$ in $D_{G_V}(E_V^n, \mathbf{k})$.

Lemma 2.3.42. *For each $\lambda \in \Lambda_V^n$, the parity sheaf $\mathcal{E}(\lambda)$ exists. It is contained in $\mathcal{Q}_V$.*

Proof. By Lemma 2.3.40, for each $\mu \in \mathrm{VComp_e}(\nu)$, the morphism π_μ has even fibers. Thus the complex $(\pi_\mu)_*\underline{\mathbf{k}}_{\widetilde{F}_\mu}$ is even, see Lemma 2.3.7. Now, it is enough to prove that for a given λ there exists $\mu \in \mathrm{VComp_e}(\nu)$ such that $\mathcal{E}(\lambda)$ is a direct summand of a shift of $(\pi_\mu)_*\underline{\mathbf{k}}_{\widetilde{F}_\mu}$. To do this, it is enough to find μ such that the image of π_μ is in $\overline{\mathcal{O}}_\lambda$ and such that π_μ yields an isomorphism $\pi_\mu^{-1}(\mathcal{O}_\lambda) \to \mathcal{O}_\lambda$. The existence of such μ is explained in the proof of [59, Prop. 2.7].

□

Let $\mathrm{Par}_{G_V}(E_V^n) \subset D_{G_V}(E_V^n, \mathbf{k})$ be the full subcategory of parity complexes.

Corollary 2.3.43. *The subcategories $\mathcal{Q}_V$ and $\mathrm{Par}_{G_V}(E_V^n)$ of $D_{G_V}(E_V^n, \mathbf{k})$ coincide.* □

Definition 2.3.44. For a given graded dimension vector $\nu \in \mathbb{N}I$ the *quiver Schur algebra* is the graded algebra $A_{\nu,\mathbf{k}} = \mathrm{Ext}^*_{G_V}({}^\delta\mathcal{L}_{V,\mathbf{k}}, {}^\delta\mathcal{L}_{V,\mathbf{k}})$, where the nth graded component is given by the nth extension group.

Remark 2.3.45. It is clear from the definition that, for each $\mu \in \mathrm{VComp_e}(\nu)$, the algebra $A_{\nu,\mathbf{k}}$ contains an idempotent of degree zero 1_μ such that $1_{\mu_1}A_{\nu,\mathbf{k}}1_{\mu_2} = \mathrm{Ext}^*_{G_V}({}^\delta\mathcal{L}_{\mu_2}, {}^\delta\mathcal{L}_{\mu_1})$.

Definition 2.3.46. For each $\mu \in \mathrm{VComp_e}(\nu)$, let P_μ be the graded projective $A_{\nu,\mathbf{k}}$-module $A_{\nu,\mathbf{k}}1_\mu$.

Definition 2.3.47. Let $U_{e,\mathbb{Z}}^-$ be the integral form of the generic nilpotent Hall algebra of the quiver Γ, see [59, Sec. 2.5]. For each $\nu \in \mathbb{N}I$, let $\mathbf{f}_\nu \in U_{e,\mathbb{Z}}^-$ be the characteristic function of the semi-simple representation of Γ with graded dimension vector ν.

Let $\mathrm{proj}(A_{\nu,\mathbf{k}})$ be the category of graded projective finitely generated $A_{\nu,\mathbf{k}}$-modules. Let $K(A_{\nu,\mathbf{k}})$ be the split Grothendieck group of $\mathrm{proj}(A_{\nu,\mathbf{k}})$ viewed as an $\mathcal{A}$-module. Set $K(A_\mathbf{k}) = \bigoplus_{\nu\in\mathbb{N}I} K(A_{\nu,\mathbf{k}})$ and $K(\mathcal{Q}) = \bigoplus_V K(\mathcal{Q}_V)$. For each $\nu_1, \nu_2 \in \mathbb{N}I$, we have an algebra homomorphism $A_{\nu_1,\mathbf{k}} \otimes A_{\nu_2,\mathbf{k}} \to A_{\nu_1+\nu_2,\mathbf{k}}$, see [59, Def. 2.8]. It yields an algebra structure on $K(A_\mathbf{k})$. The following theorem is proved in [59, Prop. 2.12].

Theorem 2.3.48. *There is an $\mathcal{A}$-algebra isomorphism $K(A_\mathbf{k}) \to U_{e,\mathbb{Z}}^-$ such that $[P_\nu] \mapsto \mathbf{f}_\nu$ for each $\nu \in \mathbb{N}I$, where ν is viewed as a trivial vector composition in* $\mathrm{VComp_e}(\nu)$. □

Analogically to Theorem 2.2.21 we get the following result.

Theorem 2.3.49. *The functor $Y\colon \mathcal{Q}_V^{op} \to \mathrm{proj}(A_{\nu,\mathbf{k}})$ such that $\mathcal{F} \mapsto \mathrm{Ext}^*_{G_V}(\mathcal{F}, {}^\delta\mathcal{L}_{V,\mathbf{k}})$ is an equivalence of categories.* □

For $\nu_1, \nu_2 \in \mathbb{N}I$, $\mu_1 \in \mathrm{VComp_e}(\nu_1)$, $\mu_2 \in \mathrm{VComp_e}(\nu_2)$ let $\mu_1 \cup \mu_2 \in \mathrm{VComp_e}(\nu_1 + \nu_2)$ denote the concatenation of μ_1 and μ_2. We can also define an algebra structure on $K(\mathcal{Q})$ as in Section 2.2.20 such that ${}^{\delta}\mathcal{L}_{\mu_1} \circ {}^{\delta}\mathcal{L}_{\mu_2} = {}^{\delta}\mathcal{L}_{\mu_1 \cup \mu_2}$. We get the following result.

Corollary 2.3.50. *There is an $\mathcal{A}$-algebra isomorphism $K(\mathcal{Q}) \to U^-_{e,\mathbb{Z}}$ such that ${}^{\delta}\mathcal{L}_\nu \mapsto \mathbf{f}_\nu$ for each $\nu \in \mathbb{N}I$, where ν is viewed as a trivial vector composition in* $\mathrm{VComp_e}(\nu)$.

Proof. We only need to verify that the algebra structure on $K(\mathcal{Q})$ agrees with the algebra structure on $U^-_{e,\mathbb{Z}}$. This is true because $Y({}^{\delta}\mathcal{L}_\mu) = P_\mu$, $[P_{\mu_1}][P_{\mu_2}] = [P_{\mu_1 \cup \mu_2}]$ (see [59, Prop. 2.9]) and the elements $\mathbf{f}_\nu$ for $\nu \in \mathbb{N}I$ generate $U^-_{e,\mathbb{Z}}$ as an $\mathcal{A}$-algebra. □

In particular Corollary 2.3.50 yields an $\mathcal{A}$-basis of $U^-_{e,\mathbb{Z}}$ in terms of parity sheaves.

Chapitre 3

Quiver Schur algebras and Koszul duality

3.1 Introduction

Let e, l be integers, $e > 1$, $l > 0$. Let Γ be the cyclic quiver of type $\widehat{A}_{e-1}$ and let I be the set of its vertices. Let $q = \zeta \in \mathbb{C}$ be a primitive eth root of unity. With each $d \in \mathbb{N}$ and each tuple $\mathbf{s} \in (\mathbb{Z}/e\mathbb{Z})^l$, Dipper, James and Mathas associate a cyclotomic q-Schur $\mathbb{C}$-algebra $S^{\mathbf{s}}_d$ in [15]. This algebra admits a block decomposition $S^{\mathbf{s}}_d = \bigoplus_{\mathbf{d}} S^{\mathbf{s}}_{\mathbf{d}}$, where $\mathbf{d}$ runs over the set of elements $\mathbf{d} = \sum_{i\in I} d_i \cdot i$ in $\mathbb{N}I$ such that $\sum_{i\in I} d_i = d$. For each $\mathbf{d}$, the block $S^{\mathbf{s}}_{\mathbf{d}}$ is quasi-hereditary and the standard modules are the so-called *Weyl modules*. They are labeled by a subset $\mathcal{P}^l_{\mathbf{d}}$ of the set $\mathcal{P}^l_d$ of l-partitions of d.

In Section 3.3.1 we define a class of gradings on $S^{\mathbf{s}}_{\mathbf{d}}$ called *admissible gradings*. An example of such grading is given in [59]. Fixing an admissible grading of $S^{\mathbf{s}}_{\mathbf{d}}$ yields a grading of the category $\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$.

By [53], we can identify the category $\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$ of finite dimensional $S^{\mathbf{s}}_{\mathbf{d}}$-modules with a highest weight subcategory $\mathbf{A}$ of a parabolic category $\mathcal{O}$ of $\widehat{\mathfrak{gl}}_N$ at negative level (for some positive integer N), see Theorem 3.3.13. The category $\mathbf{A}$ has a Koszul grading by [56, Thm. 6.4]. It is natural to ask whether the grading on $\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$ above coincides with the Koszul grading of $\mathbf{A}$.

In Section 3.3.9 we define a basic graded algebra ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ which is graded Morita equivalent to $S^{\mathbf{s}}_{\mathbf{d}}$. Let $S^{\mathcal{O}}$ be the endomorphism algebra of the minimal projective generator of $\mathbf{A}$ (equipped with the Koszul grading). Our main result is the following theorem, see Theorem 3.3.40 below.

Theorem 3.1.1. *There exists a graded algebra isomorphism* ${}^bS^{\mathbf{s}}_{\mathbf{d}} \simeq S^{\mathcal{O}}$. □

The proof is inspired by [27] and [59]. It is given in Section 3.3. The strategy of the proof is as follows. At first, we want to identify the grading at the level of KLR-algebra. More precisely, we consider a smaller graded non-unitary subalgebra ${}^bR^{\mathbf{s}}_{\mathbf{d}} \subset {}^bS^{\mathbf{s}}_{\mathbf{d}}$ defined as the opposite to the endomorphism algebra of some projective $S^{\mathbf{s}}_{\mathbf{d}}$-module (${}^bR^{\mathbf{s}}_{\mathbf{d}}$ is graded Morita equivalent to a block of the cyclotomic KLR-algebra) and similarly we consider a graded non-unitary subalgebra $R^{\mathcal{O}} \subset S^{\mathcal{O}}$. We want to prove that the algebras ${}^bR^{\mathbf{s}}_{\mathbf{d}}$ and $R^{\mathcal{O}}$ are isomorphic as graded algebras. An essential ingredient of the proof consists in computing the graded multiplicities of the simple modules in the parabolic Verma modules of the category $\mathcal{O}$. This is done in Appendix A.1. To compare the gradings of ${}^bR^{\mathbf{s}}_{\mathbf{d}}$ and $R^{\mathcal{O}}$ we characterize them in terms of radical filtration of projective modules. It becomes possible due to the *very rigidity* of some projective modules in category $\mathbf{A}$ and due to the general technique in Section 3.2.6. Finally, it is easy to lift the graded isomorphism from the KLR algebra level to the Schur algebra level because the algebra ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ can be viewed as the opposite to the endomorphism algebra of the sum of Young modules over ${}^bR^{\mathbf{s}}_{\mathbf{d}}$.

Section 3.2 contains standard results on highest weight categories, Koszul duality and Hecke algebras, to be used in the rest of the text.

After this paper was written we received a copy of Webster's paper [65] where a similar statement to Theorem 3.1.1 is announced.

3.2 Preliminaries

All functors between additive (resp. $\mathbb{C}$-linear) categories are supposed to be additive (resp. $\mathbb{C}$-linear). Let e, l be integers, $l > 0$, $e > 1$. Fix a tuple $\mathbf{s} = (s_1, \cdots, s_l)$ of elements of $\mathbb{Z}/e\mathbb{Z}$.

3.2.1 Graded rings and modules

For any noetherian ring A, let $\mathrm{mod}(A)$ be the category of finitely generated left A-modules. For any noetherian $\mathbb{Z}$-graded ring $A = \bigoplus_{n\in\mathbb{Z}} A_n$, let $\mathrm{grmod}(A)$ be the category of $\mathbb{Z}$-graded finitely generated left A-modules. The morphisms in $\mathrm{grmod}(A)$ are the morphisms which are homogeneous of degree zero. For each $M \in \mathrm{grmod}(A)$ and each $k \in \mathbb{Z}$ denote by M_k the homogeneous component of degree k in M. For $n \in \mathbb{Z}$ let $M\langle n\rangle$ be the nth shift of grading on M, i.e., we have $(M\langle n\rangle)_k = M_{k-n}$. For each $\mathbb{Z}$-graded finite dimensional $\mathbb{C}$-vector space V, let $\dim_q V \in \mathbb{N}[q, q^{-1}]$ be its graded dimension, i.e., $\dim_q V = \sum_{g\in\mathbb{Z}} (\dim V_g) q^g$.

The following lemma is proved in [5, Lem. 2.5.3].

Lemma 3.2.1. *Suppose that M is an indecomposable A-module of finite length. Then, if M admits a graded lift, this lift is unique up to grading shift and isomorphism.* □

The following lemma is proved in [24, Thm. 3.2].

Lemma 3.2.2. *Let A be a graded artin $\mathbb{C}$-algebra. Suppose that $M \in \operatorname{grmod}(A)$ is indecomposable. Then M is also indecomposable in* $\operatorname{mod}(A)$. $\square$

3.2.2 Graded categories

Definition 3.2.3. A *$\mathbb{Z}$-category* (or a *graded category*) is an additive category $\overline{\mathcal{C}}$ with a fixed auto-equivalence $T\colon \overline{\mathcal{C}} \to \overline{\mathcal{C}}$. We call T the shift functor. For each $X \in \overline{\mathcal{C}}$ and $n \in \mathbb{Z}$, we set $X\langle n\rangle = T^n(X)$. A functor of $\mathbb{Z}$-categories is a functor commuting with the shift functor.

For a graded noetherian ring A the category $\operatorname{grmod}(A)$ is a $\mathbb{Z}$-category where T is the shift of grading, i.e., for $M = \oplus_{n\in\mathbb{Z}} M_n \in \operatorname{grmod}(A),\ k \in \mathbb{Z}$, we have $T(M)_k = M_{k-1}$.

Definition 3.2.4. Let $\mathcal{C}$ be an abelian category. We say that an abelian $\mathbb{Z}$-category $\overline{\mathcal{C}}$ is a *graded version* of $\mathcal{C}$ if there exists a functor $F_{\mathcal{C}}\colon \overline{\mathcal{C}} \to \mathcal{C}$ and a graded noetherian ring A such that we have the following commutative diagram, where the horizontal arrows are equivalences of categories and the top horizontal arrow is a functor of $\mathbb{Z}$-categories

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[r] \arrow[d, "F_{\mathcal{C}}"'] & \operatorname{grmod}(A) \arrow[d, "\text{forget}"'] \\ \mathcal{C} \arrow[r] & \operatorname{mod}(A). \end{tikzcd}$$

In the setup of Definition 3.2.4, we say that an object $\overline{X} \in \overline{\mathcal{C}}$ is a *graded lift* of an object $X \in \mathcal{C}$ if we have $F_{\mathcal{C}}(\overline{X}) \simeq X$. For objects $X, Y \in \mathcal{C}$ with fixed graded lifts $\overline{X}, \overline{Y}$ and for each $k \in \mathbb{N}$ the $\mathbb{Z}$-module $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ admits a $\mathbb{Z}$-grading given by $\operatorname{Hom}_{\mathcal{C}}(X, Y)_n = \operatorname{Hom}_{\overline{\mathcal{C}}}(\overline{X}\langle n\rangle, \overline{Y})$. Moreover, since the category $\operatorname{grmod}(A)$ has enough projectives, the decomposition

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\overline{\mathcal{C}}}(\overline{X}\langle n\rangle, \overline{Y})$$

above yields a decomposition

$$\operatorname{Ext}^k_{\mathcal{C}}(X, Y) \simeq \bigoplus_{n\in\mathbb{Z}} \operatorname{Ext}^k_{\overline{\mathcal{C}}}(\overline{X}\langle n\rangle, \overline{Y})$$

for each $k \in \mathbb{N}$. Thus the $\mathbb{Z}$-module $\operatorname{Ext}^k_{\mathcal{C}}(X, Y)$ admits a $\mathbb{Z}$-grading given by $\operatorname{Ext}^k_{\mathcal{C}}(X, Y)_n = \operatorname{Ext}^k_{\overline{\mathcal{C}}}(\overline{X}\langle n\rangle, \overline{Y})$. In the sequel we will often denote the object X and its graded lift $\overline{X}$ by the same symbol.

Definition 3.2.5. For two abelian categories $\mathcal{C}_1$, $\mathcal{C}_2$ with graded versions $\overline{\mathcal{C}_1}$, $\overline{\mathcal{C}_2}$ we say that the functor of $\mathbb{Z}$-categories $\overline{\Phi}\colon \overline{\mathcal{C}}_1 \to \overline{\mathcal{C}}_2$ is a *graded lift* of a functor $\Phi\colon \mathcal{C}_1 \to \mathcal{C}_2$ if $F_{\mathcal{C}_2} \circ \overline{\Phi} = \Phi \circ F_{\mathcal{C}_1}$.

Remark 3.2.6. Assume that a graded finite dimensional $\mathbb{C}$-algebra A admits an anti-involution $a\colon A \to A^{\mathrm{op}}$ that preserves the grading on A. Here $(\bullet)^{\mathrm{op}}$ means the algebra with the opposite product. Then we have a duality $D\colon \mathrm{mod}(A)^{\mathrm{op}} \to \mathrm{mod}(A)$ such that, for each $M \in \mathrm{mod}(A)$, we have $D(M) = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ and the A-action is twisted by a. The duality D has an obvious graded lift $\overline{D}\colon \mathrm{grmod}(A)^{\mathrm{op}} \to \mathrm{grmod}(A)$ such that, for each $M \in \mathrm{grmod}(A)$, we have $\overline{D}(M) = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$, $\overline{D}(M)_n = \mathrm{Hom}_{\mathbb{C}}(M_{-n}, \mathbb{C})$ and the A-action is twisted by a.

3.2.3 Koszul duality

For an abelian category $\mathbf{A}$, let $D^b(\mathbf{A})$ denote the associated bounded derived category. If $\mathbf{A}$ is an abelian category with a graded version $\overline{\mathbf{A}}$, let $[\overline{\mathbf{A}}]$ be the Grothendieck group of $\overline{\mathbf{A}}$, i.e., the $\mathbb{Z}$-module with generators $[M]$, $M \in \overline{\mathbf{A}}$, and relations

$$[X] + [Z] = [Y] \quad \forall \text{ exact sequence } 0 \to X \to Y \to Z \to 0 \text{ in } \overline{\mathbf{A}}.$$

It admits a $\mathbb{Z}[q, q^{-1}]$-module structure given by

$$q[M] = [M\langle 1\rangle] \quad \forall M \in \overline{\mathbf{A}}.$$

Note that $[\overline{\mathbf{A}}]$ coincides with the Grothendieck group of the triangulated category $D^b(\overline{\mathbf{A}})$.

Let $A = \bigoplus_{n\geqslant 0} A_n$ be an $\mathbb{N}$-graded $\mathbb{C}$-algebra such that A_0 is semisimple. We identify A_0 with the left graded A-module $A_0 \simeq A/\oplus_{n>0}A_n$.

Definition 3.2.7. The graded algebra A is *Koszul* if the left graded A-module A_0 admits a projective resolution $\cdots \to P^2 \to P^1 \to P^0 \to A_0$ such that P^k is generated by its degree k component.

If A is Koszul, we consider the graded $\mathbb{C}$-algebra $A^! = \mathrm{Ext}^*_A(A_0, A_0)^{\mathrm{op}}$ and we call it the *Koszul dual* algebra to A. If $\overline{\mathbf{A}} = \mathrm{grmod}(A)$, we write $\overline{\mathbf{A}}^! = \mathrm{grmod}(A^!)$ and we call it the *Koszul dual category* of $\overline{\mathbf{A}}$. We write also $\mathbf{A}^! = \mathrm{mod}(A^!)$.

For a graded noetherian ring A we abbreviate $\overline{D}^b(A) = D^b(\mathrm{grmod}(A))$. Let $[\bullet]$ denote the shift of the cohomological degree. The following is well-known, see [5, Thm. 2.12.5, 2.12.6].

Theorem 3.2.8. *Let A be a Koszul $\mathbb{C}$-algebra. Assume that A and $A^!$ are finite dimensional. Then, the following hold.*

(a) *The algebra $A^!$ is also Koszul and there is a graded algebra isomorphism $(A^!)^! \simeq A$.*

(b) *There is an equivalence of categories $K\colon \overline{D}^b(A) \to \overline{D}^b(A^!)$ such that $K(M\langle n\rangle) = K(M)\langle -n\rangle[-n]$.* □

For a $\mathbb{Z}[q, q^{-1}]$-module L, let $\widetilde{L}$ be the $\mathbb{Z}[q, q^{-1}]$-module isomorphic to L as a $\mathbb{Z}$-module such that q acts on $\widetilde{L}$ in the same way as $-q^{-1}$ acts on L.

Corollary 3.2.9. *Assume that A is Koszul and that A, $A^!$ are finite dimensional. There is a $\mathbb{Z}[q, q^{-1}]$-module isomorphism* $[\mathrm{grmod}(A)] \to \widetilde{[\mathrm{grmod}(A^!)]}$, $[M] \mapsto [K(M)]$, $\forall M \in \overline{D}^b(A)$, *where K is as in Theorem 3.2.8.* □

3.2.4 Graded highest weight categories

Let A be a finite dimensional $\mathbb{C}$-algebra. Denote by $\mathcal{C}$ the abelian category $\mathrm{mod}(A)$. Let $(\Lambda, \leqslant)$ be a finite poset. Let $\Delta = \{\Delta(\lambda); \lambda \in \Lambda\}$ be a family of objects in $\mathcal{C}$.

Definition 3.2.10. The pair $(\mathcal{C}, \Delta)$ is a *highest weight category* if

(a) if $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), M) = 0$ for each $\lambda \in \Lambda$, then $M = 0$,

(b) for each $\lambda \in \Lambda$, there is an indecomposable projective object $P(\lambda) \in \mathcal{C}$ and a surjection $f\colon P(\lambda) \to \Delta(\lambda)$ such that $\mathrm{Ker}(f)$ has a (finite) filtration whose successive quotients are objects of the form $\Delta(\mu)$ with $\mu > \lambda$,

(c) for each $\lambda \in \Lambda$, we have $\mathrm{End}_{\mathcal{C}}(\Delta(\lambda)) = \mathbb{C}$,

(d) for each $\lambda, \mu \in \Lambda$ such that $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) \neq 0$, we have $\lambda \leqslant \mu$.

We call the objects $\Delta(\lambda)$'s the *standard objects*.

For each $\lambda \in \Lambda$, the module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$. Let $\nabla(\lambda)$ be the costandard object with parameter λ, see [53, Prop. 2.1]. The following lemma holds, see [16, Prop. A2.2 (ii)].

Lemma 3.2.11. *For each $k \in \mathbb{N}$ and each $\lambda, \mu \in \Lambda$ we have*

$$\mathrm{Ext}^k_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} \mathbb{C}, & if\ \lambda = \mu, k = 0, \\ 0, & else. \end{cases}$$

□

Now, assume that the $\mathbb{C}$-algebra A is $\mathbb{Z}$-graded. By [24, Cor. 3.4, Prop. 3.5] the modules $L(\lambda)$ and $P(\lambda)$ admit graded lifts. Moreover, the existence of graded lifts for projective modules implies the existence of graded lifts for standard modules $\Delta(\lambda)$ by the proof of [42, Cor. 4]. Finally, by [24, Cor. 3.4] each injective module in $\mathrm{mod}(A)$ admits a graded lift and thus an argument similar to the proof of [42, Cor. 4] implies the existence of graded lifts for costandard modules $\nabla(\lambda)$.

By definition, we have $\dim \mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), L(\lambda)) = \dim \mathrm{Hom}_{\mathcal{C}}(P(\lambda), L(\lambda)) = 1$ for each $\lambda \in \Lambda$. Moreover, we have $\dim \mathrm{Hom}_{\mathcal{C}}(L(\lambda), \nabla(\lambda)) = 1$, see [16, Sec. A.1]. We fix graded lifts of $P(\lambda)$, $\Delta(\lambda)$, $L(\lambda)$, $\nabla(\lambda)$ such that each morphism $P(\lambda) \to \Delta(\lambda)$, $\Delta(\lambda) \to L(\lambda)$, $L(\lambda) \to \nabla(\lambda)$ is homogeneous of degree 0, see also Lemma 3.2.1.

Set $\overline{\mathcal{C}} = \mathrm{grmod}(A)$. Let $M,\ L \in \overline{\mathcal{C}}$, with L irreducible. Let $[M : L]$ be the multiplicity of L in a (graded) Jordan-Hölder series of M. Write

$$[M : L]_q = \sum_{k\in\mathbb{Z}} [M : L\langle k\rangle] q^k \in \mathbb{N}[q, q^{-1}].$$

Let $\overline{\mathcal{C}}^{\Delta}$ be the full subcategory of objects of $\overline{\mathcal{C}}$ that have a finite filtration with subquotients isomorphic to graded modules of the form $\Delta(\lambda)\langle k\rangle$ for $\lambda \in \Lambda$, $k \in \mathbb{Z}$. We will refer to such filtration as a *graded Δ-filtration*. Assume that $M \in \overline{\mathcal{C}}^{\Delta}$. Let $(M : \Delta(\lambda)\langle k\rangle)$ be the multiplicity of $\Delta(\lambda)\langle k\rangle$ in a graded Δ-filtration of M. A standard argument shows that this multiplicity is independent of the choice of a graded Δ-filtration. Set also

$$(M : \Delta(\lambda))_q = \sum_{k\in\mathbb{Z}} (M : \Delta(\lambda)\langle k\rangle) q^k \in \mathbb{N}[q, q^{-1}].$$

Lemma 3.2.12. *An object $M \in \overline{\mathcal{C}}$ is graded Δ-filtered if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ for each $\lambda \in \Lambda$.*

Proof. The only if part follows from [16, Prop. A2.2 (iii)]. Now, let us prove the if part. Assume that $\mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\lambda)) = 0$ for each $\lambda \in \Lambda$. We want to prove that $M \in \overline{\mathcal{C}}^{\Delta}$. We will prove the statement by induction on the length of a Jordan-Hölder series of M. If $M = 0$, the statement is trivial. Now, assume that $M \neq 0$. Let $\lambda_0 \in \Lambda$ be minimal such that $\mathrm{Hom}_{\mathcal{C}}(M, L(\lambda_0)) \neq 0$. Let $k_0 \in \mathbb{Z}$ be such that $\mathrm{Hom}_{\overline{\mathcal{C}}}(M, L(\lambda_0)\langle k_0\rangle) \neq 0$. First, we claim that for each $\mu \leqslant \lambda_0$ we have $\mathrm{Ext}^1_{\mathcal{C}}(M, L(\mu)) = 0$. Really, consider the short exact sequence

$$0 \to L(\mu) \to \nabla(\mu) \to \nabla(\mu)/L(\mu) \to 0$$

in $\mathcal{C}$. It yields an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(M, \nabla(\mu)/L(\mu)) \to \mathrm{Ext}^1_{\mathcal{C}}(M, L(\mu)) \to \mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\mu)).$$

The rightmost term is zero by assumption. Moreover, each simple subquotient of $\nabla(\mu)/L(\mu)$ is of the form $L(\mu')$ with $\mu' < \mu \leqslant \lambda_0$. Thus the leftmost term is also zero by minimality of λ_0. This implies $\mathrm{Ext}^1_{\mathcal{C}}(M, L(\mu)) = 0$.

Let $N(\lambda_0)$ be the radical of $\Delta(\lambda_0)$. Consider the short exact sequence

$$0 \to N(\lambda_0)\langle k_0\rangle \to \Delta(\lambda_0)\langle k_0\rangle \to L(\lambda_0)\langle k_0\rangle \to 0$$

in $\overline{\mathcal{C}}$. It yields the exact sequence

$$\mathrm{Hom}_{\overline{\mathcal{C}}}(M, \Delta(\lambda_0)\langle k_0\rangle) \xrightarrow{\delta} \mathrm{Hom}_{\overline{\mathcal{C}}}(M, L(\lambda_0)\langle k_0\rangle) \to \mathrm{Ext}^1_{\overline{\mathcal{C}}}(M, N(\lambda_0)\langle k_0\rangle).$$

Each simple subquotient of $N(\lambda_0)$ in $\mathcal{C}$ is of the form $L(\mu)$ with $\mu < \lambda_0$. By the previous discussion this implies $\mathrm{Ext}^1_{\mathcal{C}}(M, N(\lambda_0)) = 0$. In particular, we deduce that the rightmost term is zero. Thus the morphism δ is surjective.

Moreover, $\operatorname{Hom}_{\overline{\mathcal{C}}}(M, L(\lambda_0)\langle k_0\rangle) \neq 0$ by the choice of λ_0 and k_0. Let $f \in \operatorname{Hom}_{\overline{\mathcal{C}}}(M, \Delta(\lambda_0)\langle k_0\rangle)$ be such that $\delta(f) \neq 0$. The image of f is not contained in $N(\lambda_0)$. Thus f is surjective. Consider the exact sequence

$$0 \to \operatorname{Ker}(f) \to M \to \Delta(\lambda_0) \to 0$$

in $\mathcal{C}$. For each $\mu \in \Lambda$ it yields an exact sequence

$$\operatorname{Ext}^1_{\mathcal{C}}(M, \nabla(\mu)) \to \operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Ker}(f), \nabla(\mu)) \to \operatorname{Ext}^2_{\mathcal{C}}(\Delta(\lambda_0), \nabla(\mu)).$$

The leftmost term is zero by assumption and the rightmost term is zero by Lemma 3.2.11. Thus, we have $\operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Ker}(f), \nabla(\mu)) = 0$ for each $\mu \in \Lambda$. Applying the induction hypothesis to $\operatorname{Ker}(f)$ we get $\operatorname{Ker}(f) \in \overline{\mathcal{C}}^{\Delta}$. This implies $M \in \overline{\mathcal{C}}^{\Delta}$. □

Remark 3.2.13. [16, Prop. A2.2 (iii)] and Lemma 3.2.12 imply that an object $M \in \overline{\mathcal{C}}$ admits a graded Δ-filtration if and only if it admits a Δ-filtration as an ungraded object.

Corollary 3.2.14. *We have $P(\lambda) \in \overline{\mathcal{C}}^{\Delta}$ for each $\lambda \in \Lambda$.* □

Lemma 3.2.15. *We have $[\nabla(\lambda) : L(\mu)]_{q^{-1}} = (P(\mu) : \Delta(\lambda))_q$ for each $\lambda, \mu \in \Lambda$.*

Proof. For each $\lambda, \mu \in \Lambda$ and each $k \in \mathbb{Z}$, we have $\dim_q \operatorname{Hom}_{\mathcal{C}}(P(\lambda), L(\mu)\langle k\rangle) = \delta_{\lambda,\mu} q^k$. Moreover, the functor $\operatorname{Hom}_{\mathcal{C}}(P(\lambda), \bullet)\colon \overline{\mathcal{C}} \to \operatorname{grmod}(\mathbb{C})$ is exact. Hence, for each $M \in \overline{\mathcal{C}}$, we have $[M : L(\lambda)]_q = \dim_q \operatorname{Hom}_{\mathcal{C}}(P(\lambda), M)$.

Next, we have $\dim_q \operatorname{Hom}_{\mathcal{C}}(\Delta(\lambda)\langle k\rangle, \nabla(\mu)) = \delta_{\lambda,\mu} q^{-k}$, see Lemma 3.2.11. Moreover, the functor $\operatorname{Hom}_{\mathcal{C}}(\bullet, \nabla(\mu))\colon \overline{\mathcal{C}}^{\Delta} \to \operatorname{grmod}(\mathbb{C})$ is exact because $\operatorname{Ext}^1_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) = 0$ for each $\lambda, \mu \in \Lambda$. Thus, for each $M \in \overline{\mathcal{C}}^{\Delta}$, we have $(M : \Delta(\lambda))_q = \dim_{q^{-1}} \operatorname{Hom}_{\mathcal{C}}(M, \nabla(\lambda))$. Therefore, for each $\lambda, \mu \in \Lambda$, we get

$$[\nabla(\lambda) : L(\mu)]_{q^{-1}} = \dim_{q^{-1}} \operatorname{Hom}_{\mathcal{C}}(P(\mu), \nabla(\lambda)) = (P(\mu) : \Delta(\lambda))_q.$$

□

Now, we make the following assumption:

($*$) there is an equivalence of categories $D\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that
- D admits a graded lift $\overline{D}\colon \overline{\mathcal{C}}^{\mathrm{op}} \to \overline{\mathcal{C}}$ that is also an equivalence of categories,
- $D^2 = \mathrm{Id}_{\mathcal{C}}$, $\overline{D}^2 = \mathrm{Id}_{\overline{\mathcal{C}}}$,
- $\overline{D}(L(\lambda)) = L(\lambda)$ for each $\lambda \in \Lambda$,

- $\overline{D}(M\langle 1\rangle) = \overline{D}(M)\langle -1\rangle$ for each $M \in \overline{\mathcal{C}}$.

Corollary 3.2.16. *Assume that* $(*)$ *holds. Then, for each* $\lambda, \mu \in \Lambda$*, we have* $[\Delta(\lambda) : L(\mu)]_q = (P(\mu) : \Delta(\lambda))_q$.

Proof. By [53, Prop. 2.6] the functor D must exchange $\Delta(\lambda)$ and $\nabla(\lambda)$ for each $\lambda \in \Lambda$ because the category $\mathcal{C}^{\mathrm{op}}$ is a highest weight category with standard modules $\nabla(\lambda)$, $\lambda \in \Lambda$, see the discussion after [16, Lem. A3.5]. Thus by the normalization of grading the functor $\overline{D}$ exchanges the graded objects $\Delta(\lambda)$ and $\nabla(\lambda)$. For each $\lambda, \mu \in \Lambda$ and each $k \in \mathbb{Z}$, we have

$$[\Delta(\lambda) : L(\mu)\langle k\rangle] = [\overline{D}(\nabla(\lambda)) : \overline{D}(L(\mu)\langle -k\rangle)] = [\nabla(\lambda) : L(\mu)\langle -k\rangle].$$

This implies $[\Delta(\lambda) : L(\mu)]_q = [\nabla(\lambda) : L(\mu)]_{q^{-1}}$. Now, apply Lemma 3.2.15. □

3.2.5 Highest weight categories with Koszul grading

Let A be a finite dimensional $\mathbb{C}$-algebra equipped with a Koszul grading. Assume that A is basic. Assume also that the category $\mathcal{C} = \mathrm{mod}(A)$ is a highest weight category. We keep the notation from the previous section. For each $\lambda \in \Lambda$ the simple A-module $L(\lambda)$ admits a graded lift concentrated in degree zero because A is $\mathbb{N}$-graded and A_0 is semisimple. As explained in Section 3.2.4 this implies automatically the existence of graded lifts for $P(\lambda)$ and $\Delta(\lambda)$ for each $\lambda \in \Lambda$. These graded lifts are unique if we ask additionally that each morphism $P(\lambda) \to L(\lambda)$ and $\Delta(\lambda) \to L(\lambda)$ is homogeneous of degree zero, see Lemma 3.2.1. We will call such graded lifts of $L(\lambda)$, $\Delta(\lambda)$ and $P(\lambda)$ *natural graded lifts*.

For each $\lambda \in \Lambda$ let e_λ be the primitive idempotent in A_0 that acts by identity on $L(\lambda)$. Then we have a graded A-module isomorphism $P(\lambda) \simeq Ae_\lambda$.

Let Λ' be a subset of Λ such that for each $\lambda \in \Lambda'$ we have $\lambda_0 \in \Lambda'$ for each $\lambda_0 \leqslant \lambda$. Let $\mathcal{C}'$ be the Serre subcategory of $\mathcal{C}$ generated by $L(\lambda)$ for $\lambda \in \Lambda'$. Let A' be the quotient of A by the ideal generated by all e_λ for $\lambda \in \Lambda\backslash\Lambda'$. The algebra A' inherits the $\mathbb{N}$-grading from A. Recall that A is assumed to be basic. Consider the following idempotent $e^!_{\Lambda'} \in A^! \simeq \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Ext}^*_{\mathcal{C}}(L(\lambda), L(\mu))^{\mathrm{op}}$ concentrated in degree zero, $e^!_{\Lambda'} = \sum_{\lambda\in\Lambda'} \mathrm{Id}_{L(\lambda)}$. Set $A^{!'} = e^!_{\Lambda'}A^!e^!_{\Lambda'}$

Lemma 3.2.17. (a) *The category* $\mathcal{C}'$ *is a highest weight category.*

(b) *There is a category equivalence* $\mathcal{C}' \simeq \mathrm{mod}(A')$ *and the algebra* A' *is Koszul.*

(c) *The Koszul dual for* A' *is* $A^{!'}$.

Proof. Part (a) is [16, Prop. A3.4]. Part (b) follows from [56, Lem. 2.2] and its proof and [16, Prop. A3.3]. Finally, part (c) is true because by [16, Prop. A3.3] we have $(A')^! = \bigoplus_{\lambda,\mu\in\Lambda'} \mathrm{Ext}^*(L(\lambda), L(\mu))^{\mathrm{op}} = A^{!\prime}$. Here the first equality is true because the algebra A' is basic. $\square$

Set $\overline{\mathcal{C}} = \mathrm{grmod}(A)$, $\overline{\mathcal{C}}' = \mathrm{grmod}(A')$, $\mathcal{C}^! = \mathrm{mod}(A^!)$, $\overline{\mathcal{C}}^! = \mathrm{grmod}(A^!)$, $(\mathcal{C}')^! = \mathrm{mod}(A^{!\prime})$, $(\overline{\mathcal{C}}')^! = \mathrm{grmod}(A^{!\prime})$.

Remark 3.2.18. (a) The inclusion functor $\mathcal{C}' \to \mathcal{C}$ admits an obvious graded lift $\overline{\mathcal{C}}' \to \overline{\mathcal{C}}$ because A' is a quotient of A by a homogeneous ideal. This graded lift preserves natural graded lifts of $L(\lambda)$ and $\Delta(\lambda)$.

(b) Assume that $A^!$ is a highest weight category with standard modules $\Delta^!(\lambda)$ and simple modules $L^!(\lambda)$, $\lambda \in \Lambda$ with respect to the opposite partial order on Λ. Assume also that $L^!(\lambda)$ corresponds to $P(\lambda)$ via the equivalence in Theorem 3.2.8 (b). Then by [16, Prop. A3.11] the category $(\mathcal{C}')^!$ is a highest weight category. Moreover, it is a quotient category of $\mathcal{C}^!$ with quotient functor

$$Q\colon \mathcal{C}^! = \mathrm{mod}(A^!) \to (\mathcal{C}')^! = \mathrm{mod}(A^{!\prime}),\ M \mapsto e^!_{\Lambda'}M.$$

The standard modules in $(\mathcal{C}')^!$ are $Q(\Delta^!(\lambda))$ for $\lambda \in \Lambda'$. The functor Q has an obvious graded lift $\overline{Q}\colon \overline{\mathcal{C}}^! \to (\overline{\mathcal{C}}')^!$, which preserves natural graded lifts of simple and standard modules.

3.2.6 Morphisms of projective modules

We keep the notation from Section 3.2.4. We assume that for each $\lambda \in \Lambda$ the simple object $L(\lambda)$ admits a graded lift. As explained in Section 3.2.4 this implies automatically the existence of graded lifts for $P(\lambda)$ and $\Delta(\lambda)$ for each $\lambda \in \Lambda$. Let $P = \bigoplus_{\lambda\in\Lambda} P(\lambda)$ be the minimal projective generator of $\mathcal{C}$. Set ${}^bA = \mathrm{End}_{\mathcal{C}}(P)^{\mathrm{op}}$. The grading on projective modules yields a grading on ${}^bA$. The category $\mathcal{C}$ is equivalent to $\mathrm{mod}({}^bA)$ and its graded version $\overline{\mathcal{C}}$ is equivalent to $\mathrm{grmod}({}^bA)$. In this section we see all objects of $\mathcal{C}$ as ${}^bA$-modules. The goal of this section is to give a construction of a graded basis in ${}^bA$. For $\lambda, \mu \in \Lambda$ let $b_{\lambda\mu}(q) = [P(\lambda) : L(\mu)]_q \in \mathbb{N}[q, q^{-1}]$ be the graded multiplicity of the simple object $L(\mu)$ in the projective object $P(\lambda)$. Set $b^s_{\lambda\mu} = [P(\lambda) : L(\mu)\langle s\rangle]$ and $b^{gs}_{\lambda\mu} = [\mathrm{rad}^g P(\lambda)/\mathrm{rad}^{g+1} P(\lambda) : L(\mu)\langle s\rangle]$. We obviously have $b^s_{\lambda\mu} = \sum_{g\in\mathbb{N}} b^{gs}_{\lambda\mu}$ and $b_{\lambda\mu}(q) = \sum_{s\in\mathbb{Z}} q^s b^s_{\lambda\mu}$. Both sums have only a finite number of nonzero terms.

Note that for each $\lambda \in \Lambda$ and each $g \in \mathbb{N}$ we have

$$\mathrm{rad}^g P(\mu)/\mathrm{rad}^{g+1} P(\mu) \simeq \bigoplus_{s\in\mathbb{Z},\lambda\in\Lambda} L(\lambda)\langle s\rangle^{\oplus b^{gs}_{\mu\lambda}}.$$

For each $\lambda, \mu \in \Lambda$, $g \in \mathbb{N}$, $s \in \mathbb{Z}$ and $t \in [1, b^{gs}_{\mu\lambda}]$, we consider the chain of maps

$$\mathrm{rad}^g P(\mu) \to \mathrm{rad}^g P(\mu)/\mathrm{rad}^{g+1} P(\mu) \to L(\lambda)\langle s\rangle,$$

where the right hand arrow is the projection to the tth copy of $L(\lambda)\langle s\rangle$. Since $P(\lambda)$ is projective, there exists a homogeneous map $\theta^{gst}_{\lambda\mu}$ of degree s such that the following diagram commutes

$$\begin{array}{ccc} P(\lambda) & \xrightarrow{\mathrm{Id}} & P(\lambda)\langle s\rangle \\ {\scriptstyle\theta^{gst}_{\lambda\mu}}\downarrow & & \downarrow \\ \mathrm{rad}^g P(\mu) & \longrightarrow & L(\lambda)\langle s\rangle. \end{array}$$

Lemma 3.2.19. *The set* $\Theta=\{\theta^{gst}_{\lambda\mu};\lambda,\mu\in\Lambda,g,s\in\mathbb{N},t\in[1,b^{gs}_{\mu\lambda}]\}$ *is a basis of* ${}^bA$.

Proof. Fix $\lambda,\mu\in\Lambda$. Set $\Theta_{\lambda\mu}=\{\theta^{gst}_{\lambda\mu};g,s\in\mathbb{N},t\in[1,b^{gs}_{\mu\lambda}]\}$. Let $\theta=\sum_{g,s,t}a^{gst}_{\lambda\mu}\theta^{gst}_{\lambda\mu}$ with $a^{gst}_{\lambda\mu}\in\mathbb{C}$ be a nontrivial linear combination of elements of $\Theta_{\lambda\mu}$. Let g_0 be minimal such that $a^{g_0s_0t_0}_{\lambda\mu}\neq 0$ for some s_0,t_0. Then we have $\mathrm{Im}\theta\subset\mathrm{rad}^{g_0}P(\mu)$ and the composition

$$P(\lambda)\xrightarrow{\theta}\mathrm{rad}^{g_0}P(\mu)\to\mathrm{rad}^{g_0}P(\mu)/\mathrm{rad}^{g_0+1}P(\mu)\to L(\lambda)\langle s_0\rangle$$

is nonzero, where the right hand map is the projection on the t_0th copy of $L(\lambda)\langle s_0\rangle$. Thus θ is not zero. Hence, the elements of the set $\Theta_{\lambda\mu}$ are linearly independent. Now, to show that they form a basis of $\mathrm{Hom}_{\mathcal{C}}(P(\lambda),P(\mu))$, we count the dimension. We have

$$|\Theta_{\lambda\mu}|=b_{\mu\lambda}(1)=[P(\mu):L(\lambda)]_1=\dim\mathrm{Hom}_{\mathcal{C}}(P(\lambda),P(\mu)).$$

□

Lemma 3.2.20. *Assume that* $f\in\mathbb{N}$ *and* $\lambda,\mu\in\Lambda$. *The set* $\Theta^{\geqslant f}_{\lambda\mu}=\{\theta^{gst}_{\lambda\mu};g,s\in\mathbb{N},g\geqslant f,t\in[1,b^{gs}_{\mu\lambda}]\}$ *is a basis of* $\mathrm{Hom}_{\mathcal{C}}(P(\lambda),\mathrm{rad}^f P(\mu))$.

Proof. We must check that the image of a linear combination of elements of $\Theta_{\lambda\mu}$ that contains an element of $\Theta_{\lambda\mu}\backslash\Theta^{\geqslant f}_{\lambda\mu}$ with a nonzero coefficient is not in $\mathrm{rad}^f P(\mu)$. Let

$$\theta=\sum_{g,s,t}a^{gst}_{\lambda\mu}\theta^{gst}_{\lambda\mu},\qquad a^{gst}_{\lambda\mu}\in\mathbb{C},$$

be such a linear combination. Let g_0 be minimal such that $a^{g_0s_0t_0}_{\lambda\mu}\neq 0$ for some s_0,t_0. We have $g_0<f$. We have $\mathrm{Im}\theta\subset\mathrm{rad}^{g_0}P(\mu)$. In the same way as in the proof of Lemma 3.2.19 we can show that the composition

$$P(\lambda)\xrightarrow{\theta}\mathrm{rad}^{g_0}P(\mu)\to\mathrm{rad}^{g_0}P(\mu)/\mathrm{rad}^{g_0+1}P(\mu)$$

is nonzero. Thus, we have $\mathrm{Im}\theta\not\subset\mathrm{rad}^{g_0+1}P(\mu)$. This implies $\mathrm{Im}\theta\not\subset\mathrm{rad}^f P(\mu)$ because $f\geqslant g_0+1$. □

Corollary 3.2.21. *Assume that for some* $\lambda,\mu\in\Lambda$ *we have* $b^{gs}_{\lambda\mu}=0$ *for each* $g\neq s$. *Then* $\mathrm{Hom}_{\mathcal{C}}(P(\lambda),\mathrm{rad}^f P(\mu))$ *coincides with the span of homogeneous morphisms of degrees* $\geqslant f$ *in* $\mathrm{Hom}_{\mathcal{C}}(P(\lambda),P(\mu))$. □

3.2.7 Combinatorics

A *composition* of an integer $n \geqslant 0$ is a tuple of positive integers $(\lambda_1, \cdots, \lambda_k)$ such that $\sum_{t=1}^{k} \lambda_t = n$. Note that this is not the usual definition of a composition because we do not allow zero components. We set $\ell(\lambda) = k$, $|\lambda| = n$. A *partition* of an integer $n \geqslant 0$ is a composition $(\lambda_1, \cdots, \lambda_k)$ of n such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. With each partition of n we associate a Young diagram $\mathrm{YD}(\lambda)$ such that $\mathrm{YD}(\lambda)$ has λ_t boxes in the tth row for each $t \in [1, k]$. The empty partition is represented by the empty diagram. An *l-composition* (resp. *l-partition*) of an integer $n \geqslant 0$ is an l-tuple $\lambda = (\lambda^1, \cdots, \lambda^l)$ of compositions (resp. partitions) of integers $n_1, \cdots, n_l \geqslant 0$ such that $\sum_{t=1}^{l} n_t = n$. We associate with each l-partition an l-tuple of Young diagrams $\mathrm{YD}(\lambda) = (\mathrm{YD}(\lambda^1), \cdots, \mathrm{YD}(\lambda^l))$. Let $\mathcal{C}_n^l$ (resp. $\mathcal{P}_n^l$) be the set of l-compositions (resp. l-partitions) of n. Set $\mathcal{P}^l = \coprod_{n \in \mathbb{N}} \mathcal{P}_n^l$.

Let Γ be the quiver with the vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and the arrow set $H = \{i \to i+1; i \in I\}$. Let $(a_{ij})_{i,j \in I}$ be the associated Cartan matrix, i.e., we put $a_{i,j} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$. For $\mathbf{d} = \sum_{i \in I} d_i \cdot i$ in $\mathbb{N}I$ set $|\mathbf{d}| = \sum_{i \in I} d_i$. Let $I^{\mathbf{d}}$ be the set of tuples $\mathbf{i} = (i_1, \cdots, i_{|\mathbf{d}|})$ in $I^{|\mathbf{d}|}$ such that $\mathbf{i}$ has d_i entries equal to i for each $i \in I$. We have $I^d = \coprod_{|\mathbf{d}|=d} I^{\mathbf{d}}$. Let $\mathfrak{S}_d$ be the symmetric group of rank d. It acts on I^d by permutation of the components. This action preserves each subset $I^{\mathbf{d}} \subset I^d$ with $|\mathbf{d}| = d$.

Recall that we have fixed an l-tuple $\mathbf{s} = (s_1, s_2, \cdots, s_l)$. Note that some of the following notations depend on $\mathbf{s}$.

Definition 3.2.22. Let $\lambda \in \mathcal{P}^l$. The *residue* $r(b) \in I$ of the box $b \in \mathrm{YD}(\lambda)$ situated in the spot (i,j) of the pth diagram is $s_p + j - i \bmod e$. Write $\mathcal{P}_{\mathbf{d}}^l = \{\lambda \in \mathcal{P}^l; \sum_{b \in \mathrm{YD}(\lambda)} r(b) = \mathbf{d}\}$, where the sum is taken in $\mathbb{N}I$.

3.2.8 Rigid modules

Let A be a noetherian $\mathbb{C}$-algebra and let M be an A-module of finite length. Let $M \supset \mathrm{rad}^1 M \supset \mathrm{rad}^2 M \supset \cdots \supset \mathrm{rad}^r M = 0$ be the *radical filtration* of M, i.e., $\mathrm{rad}^1 M = \mathrm{rad} M$ and $\mathrm{rad}^{a+1} M = \mathrm{rad}(\mathrm{rad}^a M)$, for $a \geqslant 1$, $\mathrm{rad}^{r-1} M \neq 0$. Let $M = \mathrm{soc}^s M \supset \cdots \supset \mathrm{soc}^2 M \supset \mathrm{soc}^1 M \supset 0$ be the *socle filtration* of M, i.e., $\mathrm{soc}^1 M = \mathrm{soc} M$ and $\mathrm{soc}^{a+1} M$ is the inverse image of $\mathrm{soc}(M/\mathrm{soc}^a M)$ under the natural morphism $M \to M/\mathrm{soc}^a M$ for $a \geqslant 1$, $\mathrm{soc}^{s-1} M \subsetneq M$. An A-module M of finite length is *rigid* if its radical and socle filtrations coincide, i.e., $r = s$ and $\mathrm{rad}^a M = \mathrm{soc}^{r-a} M$ for each $a \in [1, r-1]$.

Now, assume that A is an $\mathbb{N}$-graded $\mathbb{C}$-algebra and let M be a graded A-module of finite length. Let $z \in \mathbb{Z}$ be minimal such that $M_z \neq 0$. The *grading filtration* of M is the filtration $M = \mathcal{G}r_z M \supset \mathcal{G}r_{z+1} M \supset \cdots$ such that $\mathcal{G}r_a M = \bigoplus_{b \geqslant a} M_b$. A graded A-module M of finite length is *very rigid* if its grading filtration coincides with its radical and socle filtrations, i.e., if it is rigid and $\mathrm{rad}^t M = \mathcal{G}r_{z+t}M$ for each $t \in [1, r]$.

We will need the following lemma, see [5, Prop. 2.4.1].

Lemma 3.2.23. *Suppose that A is an $\mathbb{N}$-graded $\mathbb{C}$-algebra, that A_0 is semi-simple and A is generated by A_0 and A_1. Let M be any graded A-module of finite length.*

(a) *The radical filtration of M coincides with the grading filtration of M, up to shift, if $M/\mathrm{rad}M$ is irreducible.*

(b) *The socle filtration of M coincides with the grading filtration of M, up to shift, if $\mathrm{soc}M$ is irreducible.*

Consequently, the module M is very rigid if $\mathrm{soc}M$ and $M/\mathrm{rad}M$ are both irreducible. $\square$

3.2.9 Cyclotomic Hecke and q-Schur algebra

Let $\zeta \in \mathbb{C}$ be an eth primitive root of unity. Set $Q_k = \zeta^{s_k}$ for $k = 1, \cdots, l$. Fix $d \in \mathbb{N}$.

Definition 3.2.24. The *cyclotomic Hecke algebra* $H_d^{\mathbf{s}}$ is the $\mathbb{C}$-algebra with generators $T_0, T_1, \cdots, T_{d-1}$ modulo the following defining relations:

$$\begin{aligned}
(T_0 - Q_1)\cdots(T_0 - Q_l) &= 0, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
(T_k + 1)(T_k - \zeta) &= 0 && \text{for} \quad 1 \leqslant k \leqslant d-1, \\
T_kT_{k+1}T_k &= T_{k+1}T_kT_{k+1} && \text{for} \quad 1 \leqslant k \leqslant d-2, \\
T_iT_j &= T_jT_i && \text{for} \quad 0 \leqslant i \leqslant j-1 \leqslant d-2.
\end{aligned}$$

For each $k \in [1, d]$, set $L_k = \zeta^{1-k}T_{k-1}\cdots T_1T_0T_1\cdots T_{k-1}$. For a composition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of d, let $\mathfrak{S}_\lambda$ be the Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \subset \mathfrak{S}_d$. Similarly, for an l-composition $\lambda = (\lambda^1, \cdots, \lambda^l)$ of d let $\mathfrak{S}_\lambda$ be the Young subgroup $\mathfrak{S}_{\lambda^1} \times \cdots \times \mathfrak{S}_{\lambda^l} \subset \mathfrak{S}_d$. Let $t_1, \cdots, t_{d-1} \in \mathfrak{S}_d$ be the transpositions $(1,2), (2,3), \cdots, (d-1,d)$. For a permutation $w \in \mathfrak{S}_d$ with reduced expression $w = t_{j_1}\cdots t_{j_r}$ we set $T_w = T_{j_1}\cdots T_{j_r}$. Suppose that $\mathbf{a} = (a_1, \cdots, a_l)$ is an l-tuple of integers in $[1, d]$. Set $u_{\mathbf{a}}^+ = u_{\mathbf{a},1}u_{\mathbf{a},2}\cdots u_{\mathbf{a},l}$, where $u_{\mathbf{a},k} = \prod_{r=1}^{a_k}(L_r - Q_k)$. Let λ be an l-composition of d. We associate with it the tuple $\mathbf{a} = (a_1, \cdots, a_l)$ such that $a_k = \sum_{r=1}^{k-1}|\lambda^r|$. Set $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$ and $m_\lambda = u_{\mathbf{a}}^+ x_\lambda$.

Definition 3.2.25. The cyclotomic q-Schur algebra $S_d^{\mathbf{s}}$ is the $\mathbb{C}$-algebra

$$\mathrm{End}_{(H_d^{\mathbf{s}})^{\mathrm{op}}}\Big(\bigoplus_{\lambda\in\mathcal{C}_d^l} m_\lambda H_d^{\mathbf{s}}\Big).$$

Let $\lambda \in \mathcal{P}_d^l$. Let Δ^λ be the Weyl module over $S_d^{\mathbf{s}}$, see, e.g., [41, Def. 4.12]. Let L^λ be the top of Δ^λ and let P^λ be its projective cover in $\mathrm{mod}(S_d^{\mathbf{s}})$. The following is proved in [41, Thm. 4.14].

Lemma 3.2.26. *The category* $\mathrm{mod}(S^{\mathbf{s}}_d)$ *is a highest weight category with standard modules* Δ^λ, *where* $\lambda \in \mathcal{P}^l_d$. □

Remark 3.2.27. Note that [41] uses right modules and we use left modules. However, we can identify right $S^{\mathbf{s}}_d$-modules with left $S^{\mathbf{s}}_d$-modules via the involutory anti-isomorphism of $S^{\mathbf{s}}_d$ defined in [41, Sec. 4.2].

In [41, Sec. 5.1], an idempotent $\mathbf{e} \in S^{\mathbf{s}}_d$ such that $\mathbf{e}S^{\mathbf{s}}_d\mathbf{e} \simeq H^{\mathbf{s}}_d$ is defined. Let $F\colon \mathrm{mod}(S^{\mathbf{s}}_d) \to \mathrm{mod}(H^{\mathbf{s}}_d), M \mapsto \mathbf{e}M$ be the Schur functor. It is a quotient functor, see [22, Prop. III.2.5] and [27, Sec. 2.4]. The *Specht module* S^λ is the $H^{\mathbf{s}}_d$-module $F(\Delta^\lambda)$. The *Young module* Y^λ is the $H^{\mathbf{s}}_d$-module $F(P^\lambda)$. Set $D^\lambda = F(L^\lambda)$. The module D^λ is irreducible or zero because F is a quotient functor. Set $\mathcal{K}^l_d = \{\lambda \in \mathcal{P}^l_d; D^\lambda \neq \{0\}\}$.

There exists a decomposition into a direct sum of unital indecomposable $\mathbb{C}$-algebras $H^{\mathbf{s}}_d = \bigoplus_{\mathbf{d}\in\mathbb{N}I,|\mathbf{d}|=d} H^{\mathbf{s}}_{\mathbf{d}}$, see [39, Thm. A] and [53, Prop. 3.1]. Let $1_{\mathbf{d}}$ be the unit of $H^{\mathbf{s}}_{\mathbf{d}}$. Let $\overline{1}_{\mathbf{d}}$ be the element of $S^{\mathbf{s}}_d$ given by the multiplication by $1_{\mathbf{d}}$. We get a decomposition of the cyclotomic q-Schur algebra $S^{\mathbf{s}}_d = \bigoplus_{|\mathbf{d}|=d} S^{\mathbf{s}}_{\mathbf{d}}$, where $S^{\mathbf{s}}_{\mathbf{d}} = \overline{1}_{\mathbf{d}} S^{\mathbf{s}}_d$.

Lemma 3.2.28. *The algebra* $S^{\mathbf{s}}_{\mathbf{d}}$ *is indecomposable.*

Proof. The statement is proved in [41, Sec. 5.2]. □

For each $\mathbf{d}$, we consider the idempotent $\mathbf{e}_{\mathbf{d}} = \mathbf{e}\overline{1}_{\mathbf{d}} \in S^{\mathbf{s}}_{\mathbf{d}}$. The Schur functor restricts to a quotient functor $F\colon \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}(H^{\mathbf{s}}_{\mathbf{d}}), M \mapsto \mathbf{e}_{\mathbf{d}}M$.

Remark 3.2.29. The algebra $S^{\mathbf{s}}_{\mathbf{d}}$ is quasi-hereditary with the set of standard modules $\{\Delta^\lambda; \lambda \in \mathcal{P}^l_{\mathbf{d}}\}$, see [41, Sec. 5.2].

Lemma 3.2.30. *The functor* F *is fully faithful on projective modules.*

Proof. By [41, Thm. 5.3] the algebras $H^{\mathbf{s}}_d$ and $S^{\mathbf{s}}_d$ satisfy the double centralizer property with respect to the $(H^{\mathbf{s}}_d, S^{\mathbf{s}}_d)$-bimodule $\mathbf{e}S^{\mathbf{s}}_d$. Now, the statement follows from [52, Prop. 4.33]. □

For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, let $\nabla^\lambda \in \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$ be the costandard object with socle L^λ.

3.2.10 Cyclotomic KLR-algebra

Definition 3.2.31. The *KLR-algebra* $\tilde{R}_d$ is the $\mathbb{C}$-algebra with the set of generators $\psi_1, \cdots, \psi_{d-1}, y_1, \cdots, y_d, e(\mathbf{i})$, where $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$, modulo the following defining relations

$$\begin{array}{llllll} e(\mathbf{i})e(\mathbf{j}) & = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}), & \sum_{\mathbf{i}\in I^d} e(\mathbf{i}) & = 1, & & \\ y_r e(\mathbf{i}) & = e(\mathbf{i})y_r, & \psi_r e(\mathbf{i}) & = e(t_r(\mathbf{i}))\psi_r, & y_r y_s & = y_s y_r, \end{array}$$

$$\begin{aligned}
\psi_r y_{r+1} e(\mathbf{i}) &= (y_r\psi_r + \delta_{i_r,i_{r+1}})e(\mathbf{i}), & y_{r+1}\psi_r e(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r,i_{r+1}})e(\mathbf{i}),\\
\psi_r y_s &= y_s\psi_r, & \text{if} \quad & s \neq r, r+1,\\
\psi_r\psi_s &= \psi_s\psi_r, & \text{if} \quad & |r-s| > 1,
\end{aligned}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1},\\ (y_{r+1}-y_r)e(\mathbf{i}), & \text{if } i_r = i_{r+1}-1, e\neq 2,\\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r = i_{r+1}+1, e \neq 2,\\ (y_r-y_{r+1})(y_{r+1}-y_r)e(\mathbf{i}), & \text{if } i_r \neq i_{r+1}, e=2,\\ e(\mathbf{i}), & \textit{otherwise}, \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1}+1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} = i_{r+1}-1, e\neq 2,\\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(\mathbf{i}), & \text{if } i_r = i_{r+2} = i_{r+1}+1, e\neq 2,\\ (\psi_{r+1}\psi_r\psi_{r+1}+y_r-2y_{r+1}+y_{r+2})e(\mathbf{i}), & \text{if } i_r = i_{r+2} \neq i_{r+1}, e=2,\\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}), & \textit{otherwise}, \end{cases}$$

for each $\mathbf{i},\mathbf{j} \in I^d$ and each admissible r and s. The algebra $\tilde{R}_d$ admits a $\mathbb{Z}$-grading such that $\deg e(\mathbf{i}) = 0$, $\deg y_r = 2$, $\deg \psi_s e(\mathbf{i}) = -a_{i_s,i_{s+1}}$, for each $1 \leqslant r \leqslant d$, $1 \leqslant s < d$ and $\mathbf{i} \in I^d$. Here $(a_{i,j})$ is the Cartan matrix defined in Section 3.2.7.

Definition 3.2.32. For each $i \in I$, let c_i be the number of elements of $\mathbf{s}$ equal to i. The *cyclotomic KLR-algebra* $R^{\mathbf{s}}_d$ is the quotient of the algebra $\tilde{R}_d$ by the ideal generated by $y_1^{c_{i_1}}e(\mathbf{i})$ for $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$. The algebra $R^{\mathbf{s}}_d$ inherits the grading from the grading of $\tilde{R}_d$.

For each $\mathbf{d} \in \mathbb{N}I$ such that $|\mathbf{d}| = d$ set $e(\mathbf{d}) = \sum_{\mathbf{i}\in I^{\mathbf{d}}} e(\mathbf{i}) \in \tilde{R}_d$. It is a homogeneous idempotent concentrated in degree zero. We will also denote by $e(\mathbf{d})$ the image of $e(\mathbf{d})$ in $R^{\mathbf{s}}_d$. We have the following decomposition in sums of unital $\mathbb{C}$-algebras $\tilde{R}_d = \bigoplus_{|\mathbf{d}|=d}\tilde{R}_{\mathbf{d}}$, $R^{\mathbf{s}}_d = \bigoplus_{|\mathbf{d}|=d} R^{\mathbf{s}}_{\mathbf{d}}$, where $\tilde{R}_{\mathbf{d}} = e(\mathbf{d})\tilde{R}_d$, $R^{\mathbf{s}}_{\mathbf{d}} = e(\mathbf{d})R^{\mathbf{s}}_d$.

The following theorem is proved in [8, Sec. 4.5] and [50, Sec. 3.2.6].

Theorem 3.2.33. *There is a $\mathbb{C}$-algebra isomorphism $R^{\mathbf{s}}_d \simeq H^{\mathbf{s}}_d$. Moreover, under this isomorphism the idempotent $1_{\mathbf{d}} \in H^{\mathbf{s}}_d$ corresponds to $e(\mathbf{d}) \in R^{\mathbf{s}}_d$. In particular for each $\mathbf{d} \in \mathbb{N}I$ such that $|\mathbf{d}| = d$ we get an algebra isomorphism $R^{\mathbf{s}}_{\mathbf{d}} \simeq H^{\mathbf{s}}_{\mathbf{d}}$.* □

We fix the isomorphism $R^{\mathbf{s}}_d \simeq H^{\mathbf{s}}_d$ as in [9, Thm. 4.1]. It yields a grading on the cyclotomic Hecke algebra $H^{\mathbf{s}}_d$ and on each direct factor $H^{\mathbf{s}}_{\mathbf{d}}$.

The algebra $R^{\mathbf{s}}_d$ admits a graded anti-involution $a' \colon R^{\mathbf{s}}_d \to (R^{\mathbf{s}}_d)^{\mathrm{op}}$ which is the identity on generators. It yields a graded anti-involution $a' \colon R^{\mathbf{s}}_{\mathbf{d}} \to (R^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}}$ for each block $R^{\mathbf{s}}_{\mathbf{d}}$. It yields a duality $D' \colon \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}})$ and its graded lift $\overline{D}' \colon \mathrm{grmod}(R^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{grmod}(R^{\mathbf{s}}_{\mathbf{d}})$, see Remark 3.2.6.

3.2.11 Quiver Schur algebra

Let Γ be the quiver as in Section 3.2.7. Let $V = \bigoplus V_i$ be an I-graded finite dimensional $\mathbb{C}$-vector space with dimension vector $\mathbf{d} = \sum_{i\in I} d_i \cdot i \in \mathbb{N}I$. Set $E_V = \bigoplus_{i\in I} \mathrm{Hom}(V_i, V_{i+1})$ and $|\mathbf{d}| = \sum_{i\in I} d_i$. The group $G_V = \prod_{i\in I} GL(V_i)$ acts on E_V by conjugation.

A *vector composition* of weight $\mathbf{d}$ is a tuple $\mu = (\mu^{(1)}, \cdots, \mu^{(k)})$ of nonzero elements of $\mathbb{N}I$ such that $\mu^{(1)}+\mu^{(2)}+\cdots+\mu^{(k)} = \mathbf{d}$. Let $\mathrm{VComp}(\mathbf{d})$ be the set of vector compositions of weight $\mathbf{d}$. Set also $\mathrm{VComp} = \coprod_{\mathbf{d}\in\mathbb{N}I} \mathrm{VComp}(\mathbf{d})$. For $\mu = (\mu^{(1)}, \cdots, \mu^{(k)}) \in \mathrm{VComp}(\mathbf{d})$ we denote by F_μ the variety of all flags of I-graded vector spaces $\phi = (0 = V^0 \subset V^1 \subset \cdots \subset V^k = V)$ in $V = \bigoplus_{i\in I} V_i$ such that the I-graded vector space V^k/V^{k-1} has graded dimension $\mu^{(r)}$ for $r \in \{1, 2, \cdots, k\}$. Let $\widetilde{F}_\mu$ be the variety of pairs $(x, \phi) \in E_V \times F_\mu$ that are compatible, i.e., we have $x(V^r) \subset V^{r-1}$ for $r \in \{0, 1, \cdots, k\}$. Let π_μ be the projection $\widetilde{F}_\mu \to E_V$ such that $(x, \phi) \mapsto x$.

We call an $(l+1)$-tuple $\tilde{\mu} = (\mu_0, \mu_1, \cdots, \mu_l)$ of elements of VComp a *weight data*. The *weight* of a weight data $\tilde{\mu} = (\mu_0, \mu_1, \cdots, \mu_l)$ is the sum of the weights of $\mu_0, \mu_1, \cdots, \mu_l$. Let $\mathrm{VDat}(\mathbf{d})$ be the set of weight data of weight $\mathbf{d}$. Let $\tilde{\mu} = (\mu_0, \mu_1, \cdots, \mu_l)$ be in $\mathrm{VDat}(\mathbf{d})$. Let μ be the vector composition associated with $\tilde{\mu}$, i.e., μ is the tuple obtained by taking together all the components $\mu_0, \cdots, \mu_l$ of $\tilde{\mu}$ (ordered from μ_0 to μ_l).

For each $\nu \in \mathbb{N}I$ and $i \in I$ we will write $\nu(i)$ for the I-component of ν, i.e., we have $\nu = \sum_{i\in I} \nu(i) \cdot i$. Let $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_l)$ be an l-tuple of elements of $\mathbb{N}I$. Set $\nu = \nu_1 + \cdots + \nu_l$. Let $U = \bigoplus_{i\in I} U_i$ be an I-graded $\mathbb{C}$-vector space with $\dim U_i = \nu(i)$. Fix an I-graded flag $0 = U^0 \subset U^1 \subset \cdots \subset U^l = U$ such that the I-graded vector space U^k/U^{k-1} has graded dimension ν_k for each $k \in [1, l]$. For $i \in I, k \in [0, l]$ let U_i^k be the i-component of U^k, i.e., $U_i^k = U_i \cap U^k$.

Let $\mathfrak{O}(\boldsymbol{\nu}, \tilde{\mu})$ be the set of pairs $((x, \phi), (\gamma_i)_{i\in I})$ in $\widetilde{F}_\mu \times \bigoplus_{i\in I} \mathrm{Hom}(V_i, U_i)$ such that we have $\gamma_i(\widetilde{W}_i(k)) \subset U_i^k$ for each $i \in I$ and $k \in [0, l]$. Here $\widetilde{W}_i(0) \subset \widetilde{W}_i(1) \subset \cdots \subset \widetilde{W}_i(l) = V_i$ is the I-graded flag in V whose components are among the components of the flag ϕ and the graded dimensions of

$$\widetilde{W}_i(0),\ \widetilde{W}_i(1)/\widetilde{W}_i(0), \cdots,\ \widetilde{W}_i(l)/\widetilde{W}_i(l-1)$$

are the weights of the components $\mu_0, \mu_1, \cdots, \mu_l$ of $\tilde{\mu}$ respectively. Set $E_{V,\nu} = E_V \times \bigoplus_{i\in I} \mathrm{Hom}(V_i, U_i)$. For each $\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})$, let $\pi_{\tilde{\mu},\boldsymbol{\nu}}$ be the projection

$$\pi_{\tilde{\mu},\boldsymbol{\nu}} \colon \mathfrak{O}(\boldsymbol{\nu}, \tilde{\mu}) \to E_{V,\nu}, \quad ((x, \phi), (\gamma_i)_{i\in I}) \to (x, (\gamma_i)_{i\in I}).$$

For each $\tilde{\mu}, \tilde{\lambda} \in \mathrm{VDat}(\mathbf{d})$ set $Z^{\boldsymbol{\nu}}_{\tilde{\mu},\tilde{\lambda}} = \mathfrak{O}(\boldsymbol{\nu}, \tilde{\mu}) \times_{E_{V,\nu}} \mathfrak{O}(\boldsymbol{\nu}, \tilde{\lambda})$. Set also

$$Z^{\boldsymbol{\nu}}_{\mathbf{d}} = \coprod_{\tilde{\mu},\tilde{\lambda}\in\mathrm{VDat}(\mathbf{d})} Z^{\boldsymbol{\nu}}_{\tilde{\mu},\tilde{\lambda}}, \qquad \mathfrak{O}^{\boldsymbol{\nu}}_{\mathbf{d}} = \coprod_{\tilde{\lambda}\in\mathrm{VDat}(\mathbf{d})} \mathfrak{O}(\boldsymbol{\nu}, \tilde{\lambda}).$$

Definition 3.2.34. Let $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ be the $\mathbb{C}$-algebra $\mathrm{H}^{G_V}_*(Z^{\boldsymbol{\nu}}_{\mathbf{d}})$, where $\mathrm{H}^{G_V}_*(\bullet)$ is the G_V-equivariant Borel-Moore homology with coefficients in $\mathbb{C}$ and the multiplication on $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ is given by the convolution product with respect to the inclusion $Z^{\boldsymbol{\nu}}_{\mathbf{d}} \subset \mathfrak{O}^{\boldsymbol{\nu}}_{\mathbf{d}} \times \mathfrak{O}^{\boldsymbol{\nu}}_{\mathbf{d}}$, see [12, Sec. 2.7].

Remark 3.2.35. (a) We can identify the $\mathbb{C}$-algebra $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ with the Yoneda extension algebra of

$$\bigoplus_{\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})} (\pi_{\tilde{\mu},\boldsymbol{\nu}})_* \underline{\mathbb{C}}_{\mathfrak{O}(\boldsymbol{\nu},\tilde{\mu})}[\dim \mathfrak{O}(\boldsymbol{\nu},\tilde{\mu})]$$

(in the G_V-equivariant derived category of constructible complexes on $E_{V,\nu}$), see [12, Lem. 8.6.1, Thm. 8.6.7]. Here $\underline{\mathbb{C}}$ is the constant sheaf. We equip $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ with the grading such that the n-graded component is the nth extension group.

(b) In [59, Sec. 4.2], a diagrammatic presentation of $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ is introduced, where $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ is generated by some *diagrams*. The product in $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ corresponds to the vertical concatenation of diagrams (the product of two diagrams is zero if the vertical concatenation is not possible). For each $\mathbf{d}, \mathbf{d}' \in \mathbb{N}I$ and for any vector compositions $\boldsymbol{\nu}$, $\boldsymbol{\nu}'$, there is a bilinear map $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}} \times \tilde{A}^{\boldsymbol{\nu}'}_{\mathbf{d}'} \to \tilde{A}^{\boldsymbol{\nu}\cup\boldsymbol{\nu}'}_{\mathbf{d}+\mathbf{d}'}$, $(a,b) \mapsto a|b$, that puts the diagram of a left to the diagram of b, called a *horizontal multiplication*. Here $\boldsymbol{\nu} \cup \boldsymbol{\nu}'$ is the concatenation of $\boldsymbol{\nu}$ with $\boldsymbol{\nu}'$, see [59, Sec. 4.1]. The horizontal multiplication is well-defined because of the faithful representation of $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ defined in [59, Prop. 4.2], see also [59, Prop. 3.6, Prop. 4.7] for the formulas defining the actions of *splits*, *merges*, *left* and *right shifts* ([59, Def. 3.1, Def. 4.4]) on this representation. The horizontal multiplication is associative and respects the grading, i.e., we have $\deg(a|b) = \deg a + \deg b$.

Let $I^{\boldsymbol{\nu}}_{\mathbf{d}} \subset \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ be the ideal generated by the elements of the form $e_i|a$, where $i \in I$ is such that $\mathbf{d} - i \in \mathbb{N}I$, $a \in \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}-i}$ and e_i is the unit of $\tilde{A}^{\emptyset}_i$. Set $A^{\boldsymbol{\nu}}_{\mathbf{d}} = \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}/I^{\boldsymbol{\nu}}_{\mathbf{d}}$. We call $A^{\boldsymbol{\nu}}_{\mathbf{d}}$ the *cyclotomic quiver Schur algebra*. The ideal $I^{\boldsymbol{\nu}}_{\mathbf{d}}$ is homogeneous because each element $e_i \in \tilde{A}^{\emptyset}_i$ is homogeneous of degree zero and the bilinear map $\tilde{A}^{\emptyset}_i \times \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}-i} \to \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ respects the grading. Thus $A^{\boldsymbol{\nu}}_{\mathbf{d}}$ inherits the grading from $\tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$.

Remark 3.2.36. Diagrammatically, the idempotent $e_i \in \tilde{A}^{\emptyset}_i$ is represented by the diagram consisting of a black line labeled by the element i. Hence, the ideal $I^{\boldsymbol{\nu}}_{\mathbf{d}} \subset \tilde{A}^{\boldsymbol{\nu}}_{\mathbf{d}}$ is generated by the diagrams having a black vertical line on the left labeled by an element of I. In particular, each diagram having a piece of a black line labeled by an element of I on the left of all other lines is in $I^{\boldsymbol{\nu}}_{\mathbf{d}}$.

From now on, we assume that $\boldsymbol{\nu} = \mathbf{s}$. The following is proved in [59, Thm. 6.3].

Theorem 3.2.37. *The algebra $A^{\mathbf{s}}_{\mathbf{d}}$ viewed as an ungraded algebra is isomorphic to the cyclotomic q-Schur algebra $S^{\mathbf{s}}_{\mathbf{d}}$.* □

3.3 Proof of the main theorem

We use the same notation as in the previous section.

3.3.1 Admissible gradings

In this section we equip $H^{\mathbf{s}}_{\mathbf{d}}$ with the grading induced by the isomorphism $H^{\mathbf{s}}_{\mathbf{d}} \simeq R^{\mathbf{s}}_{\mathbf{d}}$ in Section 3.2.10. Recall the duality D' and its graded lift $\overline{D}'$ defined in Section 3.2.10. We want to introduce a class of gradings on $S^{\mathbf{s}}_{\mathbf{d}}$.

Definition 3.3.1. A $\mathbb{Z}$-grading of the algebra $S^{\mathbf{s}}_{\mathbf{d}}$ is *admissible* if the Schur functor $F\colon \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}(H^{\mathbf{s}}_{\mathbf{d}})$ admits a graded lift $\overline{F}$ and if for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, the modules Δ^λ, L^λ have graded lifts such that the following conditions hold :

(a) for each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$, we have $[\Delta^\lambda : L^\xi]_q \in \delta_{\lambda\xi} + q\mathbb{N}[q]$,

(b) there is an equivalence of categories $D\colon \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}} \to \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$ such that

 (b$_1$) D admits a graded lift $\overline{D}\colon \mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}} \to \mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$ that is also an equivalence of categories,

 (b$_2$) $D^2 = \mathrm{Id}_{\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})}$, $\overline{D}^2 = \mathrm{Id}_{\mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})}$,

 (b$_3$) $\overline{D}(L^\lambda) = L^\lambda$ for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$,

 (b$_4$) $\overline{D}(M\langle 1\rangle) = \overline{D}(M)\langle -1\rangle$ for each $M \in \mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$,

 (b$_5$) $D'F = FD$, $\overline{D}'\overline{F} = \overline{F}\,\overline{D}$.

From now on, we fix an admissible grading of $S^{\mathbf{s}}_{\mathbf{d}}$. For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$ the $S^{\mathbf{s}}_{\mathbf{d}}$-modules Δ^λ, ∇^λ, P^λ, L^λ are indecomposable. Thus their graded lifts are unique modulo the shift of grading if they exist, see Lemma 3.2.1. Fix the graded lifts of Δ^λ and L^λ as in Definition 3.3.1. Note that by (a) each morphism $\Delta^\lambda \to L^\lambda$ is homogeneous of degree zero. The projective cover of L^λ in $\mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$ is a graded lift of P^λ, see also Lemma 3.2.2. We will always consider P^λ with this graded lift. We also get graded lifts of D^λ, S^λ and Y^λ obtained by taking the images by $\overline{F}$ of the graded lifts of L^λ, Δ^λ and P^λ. Moreover, the module $\overline{D}(\Delta^\lambda) \in \mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$ is a graded lift of ∇^λ, see [53, Prop. 2.6].

3.3.2 Quiver Schur grading is admissible

The goal of this section is to prove that the graded version $A^{\mathbf{s}}_{\mathbf{d}}$ of $S^{\mathbf{s}}_{\mathbf{d}}$ has an admissible grading. We will see a finite dimensional $\mathbb{Z}$-graded $\mathbb{C}$-vector space $U = \bigoplus_{n\in\mathbb{Z}} U_n$ as a complex with zero differentials $\cdots \to U_{-1} \to U_0 \to U_1 \to \cdots$. The dual $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ is also $\mathbb{Z}$-graded, with $U^*_n = \mathrm{Hom}_{\mathrm{grmod}(\mathbb{C})}(U\langle n\rangle, \mathbb{C})$.

Write $\mathscr{L}_{\tilde{\mu},\mathbf{s}} = (\pi_{\tilde{\mu},\mathbf{s}})_* \underline{\mathbb{C}}_{\mathfrak{O}(\mathbf{s},\tilde{\mu})}[\dim \mathfrak{O}(\mathbf{s},\tilde{\mu})]$. By Remark 3.2.35 we have

$$\tilde{A}^{\mathbf{s}}_{\mathbf{d}} = \mathrm{Ext}^*_{G_V}\Big(\bigoplus_{\tilde{\mu}\in\mathrm{VDat}(\mathbf{d})} \mathscr{L}_{\tilde{\mu},\mathbf{s}}, \bigoplus_{\tilde{\mu}\in\mathrm{VDat}(\mathbf{d})} \mathscr{L}_{\tilde{\mu},\mathbf{s}}\Big).$$

Set $s = \sum_{k=1}^{l} s_l \in \mathbb{N}I$. Let $\Lambda_{V,\mathbf{s}}$ be the set of isomorphism classes of simple G_V-equivariant perverse sheaves on $E_{V,s}$ that appear as shifts of direct summands of $\mathscr{L}_{\tilde{\mu},\mathbf{s}}$ for some $\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})$. For each $\lambda \in \Lambda_{V,\mathbf{s}}$, fix a perverse sheaf IC_λ on $E_{V,s}$ in the class λ. By the decomposition theorem [3, Thm. 6.2.5], for each $\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})$, we have an isomorphism

$$\mathscr{L}_{\tilde{\mu},\mathbf{s}} \simeq \bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda \otimes U_{\tilde{\mu},\mathbf{s},\lambda}$$

for some $\mathbb{Z}$-graded vector spaces $U_{\tilde{\mu},\mathbf{s},\lambda}$. Note that we have a graded $\mathbb{C}$-vector space isomorphism $U_{\tilde{\mu},\mathbf{s},\lambda} \simeq (U_{\tilde{\mu},\mathbf{s},\lambda})^*$, because $\mathscr{L}_{\tilde{\mu},\mathbf{s}}$ is self-dual with respect to the Verdier duality. Set also $\mathscr{L}_{\mathbf{s}} = \bigoplus_{\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})} \mathscr{L}_{\tilde{\mu},\mathbf{s}}$ and $U_{\mathbf{s},\lambda} = \bigoplus_{\tilde{\mu} \in \mathrm{VDat}(\mathbf{d})} U_{\tilde{\mu},\mathbf{s},\lambda}$. Then we have $\mathscr{L}_{\mathbf{s}} = \bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda \otimes U_{\mathbf{s},\lambda}$ and we have a graded $\mathbb{C}$-vector space isomorphism $U_{\mathbf{s},\lambda} \simeq (U_{\mathbf{s},\lambda})^*$.

The graded algebra

$${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}} = \mathrm{Ext}^*_{G_V}(\bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda, \bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda)$$

is Morita equivalent to $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$. The corresponding equivalence of (ungraded) module categories is given by the functor $\mathrm{Y}\colon \mathrm{mod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$, $M \mapsto R \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} M$, where R is the graded $({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}, \tilde{A}^{\mathbf{s}}_{\mathbf{d}})$-bimodule

$$\mathrm{Ext}^*_{G_V}(\bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda \otimes U_{\mathbf{s},\lambda}, \bigoplus_{\lambda \in \Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda).$$

Let $\overline{\mathrm{Y}}\colon \mathrm{grmod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{grmod}({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$ be the obvious graded lift of Y.

The algebra $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$ admits a graded anti-involution $a\colon \tilde{A}^{\mathbf{s}}_{\mathbf{d}} \to (\tilde{A}^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}}$ coming from the Verdier duality.

Lemma 3.3.2. *The composition of a with the projection $\tilde{A}^{\mathbf{s}}_{\mathbf{d}} \to A^{\mathbf{s}}_{\mathbf{d}}$ factors through an anti-involution a of $A^{\mathbf{s}}_{\mathbf{d}}$.*

Proof. Recall that $\tilde{A}^{\mathbf{s}}_{\mathbf{d}} = \mathrm{H}^{G_V}_*(Z^{\mathbf{s}}_{\mathbf{d}})$, where $Z^{\mathbf{s}}_{\mathbf{d}} = \coprod_{\tilde{\mu},\tilde{\lambda} \in \mathrm{VDat}(\mathbf{d})} Z^{\mathbf{s}}_{\tilde{\mu},\tilde{\lambda}}$ and $Z^{\mathbf{s}}_{\tilde{\mu},\tilde{\lambda}} = \mathfrak{O}(\mathbf{s},\tilde{\mu}) \times_{E_{V,s}} \mathfrak{O}(\mathbf{s},\tilde{\lambda})$. Exchanging $\mathfrak{O}(\mathbf{s},\tilde{\mu})$ with $\mathfrak{O}(\mathbf{s},\tilde{\lambda})$ and summing up over all $\tilde{\lambda}, \tilde{\mu} \in \mathrm{VDat}(\mathbf{d})$ yields an automorphism of $Z^{\mathbf{s}}_{\mathbf{d}}$. This automorphism gives an anti-involution of $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$. We can see from the chain of equalities in [12, (8.6.4)] that this anti-involution coincides with a.

Recall the diagrammatic presentation in Remark 3.2.36. The symmetry in the definition of splits and merges [59, Def. 3.1] and of left and right shifts [59, Def. 4.4] implies that the anti-involution a reflects each diagram with respect to a horizontal line in the presentation of $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$ above.

Now, the diagrammatic characterization of the ideal $I^{\mathbf{s}}_{\mathbf{d}}$ is Remark 3.2.36 implies that $I^{\mathbf{s}}_{\mathbf{d}}$ is preserved by a. □

In the same way as above we can define an anti-involution $a\colon {}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}} \to ({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}}$. Now, for each $A \in \{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}, A^{\mathbf{s}}_{\mathbf{d}}, {}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}\}$, we have a duality $D\colon \mathrm{mod}(A)^{\mathrm{op}} \to \mathrm{mod}(A)$ and it has a graded lift $\overline{D}\colon \mathrm{grmod}(A)^{\mathrm{op}} \to \mathrm{grmod}(A)$, see Remark 3.2.6. Consider the graded $(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}, {}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$-bimodule

$$R' = \mathrm{Ext}^*_{G_V}\Big(\bigoplus_{\lambda\in\Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda, \bigoplus_{\lambda\in\Lambda_{V,\mathbf{s}}} \mathrm{IC}_\lambda \otimes U_{\mathbf{s},\lambda}\Big).$$

Lemma 3.3.3. (a) *There is a* $({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}, \tilde{A}^{\mathbf{s}}_{\mathbf{d}})$*-bimodule isomorphism* $R \simeq \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(R', \tilde{A}^{\mathbf{s}}_{\mathbf{d}})$.

(b) *There is an isomorphism* $R \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} \bullet \simeq \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(R', \bullet)$ *of functors* $\mathrm{mod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$.

(c) *The twist by a yields an* $(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}, {}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$*-bimodule structure on R which is isomorphic to* R'.

(d) *The duality D on* $\mathrm{mod}(A^{\mathbf{s}}_{\mathbf{d}})$ *satisfies the properties* (b_1), (b_2), (b_4).

Proof. Parts (a) and (d) are easy. Part (c) follows from the isomorphism $U_{\mathbf{s},\lambda} \simeq (U_{\mathbf{s},\lambda})^*$. Let us concentrate on (b). Assume that $M \in \tilde{A}^{\mathbf{s}}_{\mathbf{d}}$. The $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module R' is projective and finitely generated. Thus, we have $\mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(R', M) \simeq \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(R', \tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} M$. Now, part (b) follows from (a). □

Lemma 3.3.4. *We have isomorphisms of functors* $\mathrm{Y}D \simeq D\mathrm{Y}$ *and* $\overline{\mathrm{Y}}\,\overline{D} \simeq \overline{D}\,\overline{\mathrm{Y}}$.

Proof. For each $M \in \mathrm{mod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$ we have the following chain of ${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module isomorphisms

$$\begin{aligned} D\mathrm{Y}(M) &\simeq \mathrm{Hom}_{\mathbb{C}}(R \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} M, \mathbb{C}) \\ &\simeq \mathrm{Hom}_{\tilde{A}^{\mathbf{s}\,\mathrm{op}}_{\mathbf{d}}}(R, \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})) \\ &\simeq \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(R', \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})) \\ &\simeq R \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C}) \\ &\simeq \mathrm{Y}D(M). \end{aligned}$$

Here, the third isomorphism is Lemma 3.3.3 (c) and the fourth one is Lemma 3.3.3 (b). Thus we get $\mathrm{Y}D \simeq D\mathrm{Y}$. We can prove that $\overline{\mathrm{Y}}\,\overline{D} \simeq \overline{D}\,\overline{\mathrm{Y}}$ in a similar way. □

Lemma 3.3.5. *For each* $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, *the simple* $S^{\mathbf{s}}_{\mathbf{d}}$*-module* L^λ *admits a unique graded lift in* $\mathrm{grmod}(A^{\mathbf{s}}_{\mathbf{d}})$ *which is stable by the duality* $\overline{D}$.

Proof. To prove the existence of such a graded lift, it is enough to check that L^λ has a $\overline{D}$-stable graded lift, where L^λ is viewed as an $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module. Note that we already know (see Section 3.2.4) that L^λ admits a graded lift as an $A^{\mathbf{s}}_{\mathbf{d}}$-module and thus as an $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module. Further, by Lemma 3.3.4, it is enough to show that the ${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module $\mathrm{Y}(L^\lambda)$ admits a $\overline{D}$-stable graded lift. We already know that the ${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module $\mathrm{Y}(L^\lambda)$ admits some graded lift. This graded lift must be concentrated in some degree because ${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$ is non-negatively graded and $\mathrm{Y}(L^\lambda)$ is simple. Thus $\mathrm{Y}(L^\lambda)$ admits a graded lift concentrated in degree zero. This graded lift is obviously $\overline{D}$-stable. The graded lift of L^λ is unique up to a shift of the grading because L^λ is indecomposable. Only one of the graded lifts of L^λ is fixed by $\overline{D}$ by the property (b_4). $\square$

Lemma 3.3.5 shows that condition (b_3) holds. From now on, fix the graded lift on L^λ as in Lemma 3.3.5. Fix also graded lifts on P^λ and Δ^λ normalized as in section 3.2.4. Now, we can check condition (a).

Lemma 3.3.6. *We have $[\Delta^\lambda : L^\xi]_q \in \delta_{\lambda\xi} + q\mathbb{N}[q]$ for each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$.*

Proof. It is enough to prove that $[P^\lambda : L^\xi]_q \in \delta_{\lambda\xi} + q\mathbb{N}[q]$, because Δ^λ is a quotient of P^λ. Let π be the quotient morphism $\pi\colon \tilde{A}^{\mathbf{s}}_{\mathbf{d}} \to A^{\mathbf{s}}_{\mathbf{d}}$. Let $\pi^*\colon \mathrm{grmod}(A^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{grmod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$ be the pull-back by π, and $\pi_!\colon \mathrm{grmod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{grmod}(A^{\mathbf{s}}_{\mathbf{d}})$ be the left adjoint functor to π^*, i.e., we have $\pi_!(N) = A^{\mathbf{s}}_{\mathbf{d}} \otimes_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}} N$ for each N. The grading of the module $Y(\pi^*(L^\lambda))$ is concentrated in degree 0, see the proof of Lemma 3.3.5. Thus there is an idempotent $e_\lambda \in ({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})_0$ such that there is an ${}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$-module isomorphism $Y(\pi^*(L^\lambda)) \simeq e_\lambda({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}/({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})_{>0})$. Consider the projective module $\widetilde{P}^\lambda = Y^{-1}(e_\lambda{}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}}) \in \mathrm{grmod}(\tilde{A}^{\mathbf{s}}_{\mathbf{d}})$. We have $\dim_q \mathrm{Hom}_{A^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\lambda, \pi^*(L^\xi)) = \delta_{\lambda\xi}$ for each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$. We first claim that $\pi_!(\widetilde{P}^\lambda) = P^\lambda$. Note that, since $\pi_!$ is left adjoint of an exact functor, it takes projective modules to projective modules. Now, by adjunction, for each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$, we have

$$\dim_q \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(\pi_!(\widetilde{P}^\lambda), L^\xi) = \dim_q \mathrm{Hom}_{A^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\lambda, \pi^*(L^\xi)) = \delta_{\lambda\xi}.$$

Thus $\pi_!(\widetilde{P}^\lambda)$ is the projective cover of L^λ in $\mathrm{grmod}(A^{\mathbf{s}}_{\mathbf{d}})$. This proves the claim.

The proof of Lemma 3.2.15 implies that $[P^\lambda : L^\xi]_q = \dim_q \mathrm{Hom}_{A^{\mathbf{s}}_{\mathbf{d}}}(P^\xi, P^\lambda)$ and $[\widetilde{P}^\lambda : \pi^*(L^\xi)]_q = \dim_q \mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\xi, \widetilde{P}^\lambda)$. We claim that the functor $\pi_!$ yields a surjection $\mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\xi, \widetilde{P}^\lambda) \to \mathrm{Hom}_{A^{\mathbf{s}}_{\mathbf{d}}}(P^\xi, P^\lambda)$. Indeed, since the right hand side is equal to $\mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\xi, \pi^*\pi_!(\widetilde{P}^\lambda))$ by adjunction and since $\mathrm{Hom}_{\tilde{A}^{\mathbf{s}}_{\mathbf{d}}}(\widetilde{P}^\xi, \bullet)$ is exact, the claim follows from the surjectivity of the unit morphism $\widetilde{P}^\lambda \to \pi^*\pi_!(\widetilde{P}^\lambda)$.

Now, it is enough to prove that $[\widetilde{P}^\lambda : \pi^*(L^\xi)]_q \in \delta_{\lambda\xi} + q\mathbb{N}[[q]]$, or, equivalently, that $[\overline{\mathrm{Y}}(\widetilde{P}^\lambda) : \overline{\mathrm{Y}}(\pi^*(L^\xi))]_q \in \delta_{\lambda\xi} + q\mathbb{N}[[q]]$. We have $({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})_n = 0$ if $n < 0$ and $({}^b\tilde{A}^{\mathbf{s}}_{\mathbf{d}})_0$ is semisimple and basic. The module $\overline{\mathrm{Y}}(\pi^*(L^\lambda))$ has dimension 1 and it is concentrated in degree zero. Thus we get $\overline{\mathrm{Y}}(\widetilde{P}^\lambda)_n = 0$ if $n < 0$ and $\dim Y(\widetilde{P}^\lambda)_0 = 1$. This implies the statement. $\square$

Let $\mathrm{VDat}^0(\mathbf{d})$ be the set of elements $\tilde{\mu} = (\mu_0, \cdots, \mu_l) \in \mathrm{VDat}(\mathbf{d})$ such that $\mu_0, \cdots, \mu_{l-1}$ are empty and each component of μ_l is in I (viewed as a subset of $\mathbb{N}I$). Let $\mathrm{VDat}^1(\mathbf{d})$ be the set of elements $\tilde{\mu} = (\mu_0, \cdots, \mu_l) \in \mathrm{VDat}(\mathbf{d})$ such that each component of $\mu_0, \cdots, \mu_{l-1}, \mu_l$ is in I (viewed as a subset of $\mathbb{N}I$).

The following theorem is proved in [51, Sec. 5.4] and [62, Thm. 3.6].

Theorem 3.3.7. *There is a graded $\mathbb{C}$-algebra isomorphism*

$$\tilde{R}_{\mathbf{d}} \simeq \mathrm{Ext}^*_{G_V}\Big(\bigoplus_{\tilde{\mu}\in \mathrm{VDat}^0(\mathbf{d})} \mathscr{L}_{\tilde{\mu},\mathbf{s}}, \bigoplus_{\tilde{\mu}\in \mathrm{VDat}^0(\mathbf{d})} \mathscr{L}_{\tilde{\mu},\mathbf{s}}\Big).$$

□

Let $\tilde{\mathbf{e}}_{\mathbf{d}}$, $\overline{\mathbf{e}}_{\mathbf{d}}$ be the idempotents of $\tilde{A}^{\mathbf{s}}_{\mathbf{d}}$ concentrated in degree 0 given by:

$$\tilde{\mathbf{e}}_{\mathbf{d}} = \sum_{\tilde{\mu}\in \mathrm{VDat}^0(\mathbf{d})} \mathrm{Id}_{\mathscr{L}_{\tilde{\mu},\mathbf{s}}}, \quad \overline{\mathbf{e}}_{\mathbf{d}} = \sum_{\tilde{\mu}\in \mathrm{VDat}^1(\mathbf{d})} \mathrm{Id}_{\mathscr{L}_{\tilde{\mu},\mathbf{s}}}.$$

Diagrammatically, the idempotent $\tilde{\mathbf{e}}_{\mathbf{d}}$ is the sum of all diagrams containing l red vertical lines left to $|\mathbf{d}|$ black vertical lines such that red lines are labeled by $s_1, \cdots, s_l$ from left to right and exactly d_i black lines have label i for each $i \in I$. Similarly, the idempotent $\overline{\mathbf{e}}_{\mathbf{d}}$ is the sum of all diagrams containing l red vertical lines and $|\mathbf{d}|$ black vertical lines in arbitrary order such that red lines are labeled by $s_1, \cdots, s_l$ from left to right and exactly d_i black lines have label i for each $i \in I$. Let $\mathbf{e}'_{\mathbf{d}}$ and $\mathbf{e}''_{\mathbf{d}}$ be the images of $\tilde{\mathbf{e}}_{\mathbf{d}}$ and $\overline{\mathbf{e}}_{\mathbf{d}}$ respectively in $A^{\mathbf{s}}_{\mathbf{d}}$ by the canonical map. Recall the idempotent $\mathbf{e}_{\mathbf{d}}$ in Section 3.2.9. Theorem 3.3.7 implies that there is a graded algebra isomorphism $\tilde{\mathbf{e}}_{\mathbf{d}}\tilde{A}^{\mathbf{s}}_{\mathbf{d}}\tilde{\mathbf{e}}_{\mathbf{d}} \simeq \tilde{R}_{\mathbf{d}}$. Recall the definition of the c_i in Definition 3.2.32.

Lemma 3.3.8. (a) *Under the identification $S^{\mathbf{s}}_{\mathbf{d}} \simeq A^{\mathbf{s}}_{\mathbf{d}}$, we have $\mathbf{e}_{\mathbf{d}} = \mathbf{e}'_{\mathbf{d}}$.*

(b) *The ideal $\tilde{\mathbf{e}}_{\mathbf{d}}I^{\mathbf{s}}_{\mathbf{d}}\tilde{\mathbf{e}}_{\mathbf{d}}$ of $\tilde{R}_{\mathbf{d}}$ coincides with the ideal generated by $y_1^{c_{i_1}}e(\mathbf{i})$ for $\mathbf{i} = (i_1, \cdots, i_d) \in I^{\mathbf{d}}$. In particular, we have a graded $\mathbb{C}$-algebra isomorphism $\mathbf{e}_{\mathbf{d}}A^{\mathbf{s}}_{\mathbf{d}}\mathbf{e}_{\mathbf{d}} \simeq R^{\mathbf{s}}_{\mathbf{d}}$.*

Proof. Part (a) follows from the proof of [59, Thm. 6.3]. Now, we prove part (b). By [59, Prop. 4.9], the algebra $\mathbf{e}''_{\mathbf{d}}A^{\mathbf{s}}_{\mathbf{d}}\mathbf{e}''_{\mathbf{d}}$ is isomorphic to the *tensor product algebra* with parameters $\mathbf{s}, \mathbf{d}$, which is defined in [64, Sec. 2.1]. In the diagrammatic presentation mentioned in Remark 3.2.36 the element $y_1^{c_{i_1}}\mathbf{e}(\mathbf{i})$ has the following diagram in $\tilde{\mathbf{e}}_{\mathbf{d}}\tilde{A}^{\mathbf{s}}_{\mathbf{d}}\tilde{\mathbf{e}}_{\mathbf{d}} \simeq \tilde{R}_{\mathbf{d}}$: it has from left to right l red vertical lines labeled by $s_1, \cdots, s_l$, a black vertical line labeled by i_1 containing c_{i_1} dots (i.e. containing the polynomial $y_1^{c_{i_1}}$), $d-1$ black vertical lines labeled by $i_2, \cdots, i_d$. Now, we can see $y_1^{c_{i_1}}\mathbf{e}(\mathbf{i})$ as an element of $\mathbf{e}''_{\mathbf{d}}A^{\mathbf{s}}_{\mathbf{d}}\mathbf{e}''_{\mathbf{d}}$. The relations [64, (2.2)] show that the diagram of $y_1^{c_{i_1}}\mathbf{e}(\mathbf{i})$ is equal to a diagram having a piece of a black line labeled by i_1 left to all other lines. So by Remark 3.2.36 we have $y_1^{c_{i_1}}\mathbf{e}(\mathbf{i}) \in \tilde{\mathbf{e}}_{\mathbf{d}}I^{\mathbf{s}}_{\mathbf{d}}\tilde{\mathbf{e}}_{\mathbf{d}}$. So, it suffices to show that both ideals in the lemma have the same codimension in $\tilde{R}_{\mathbf{d}}$ or, equivalently, that $\dim R^{\mathbf{s}}_{\mathbf{d}} = \dim \mathbf{e_d} A^{\mathbf{s}}_{\mathbf{d}} \mathbf{e_d}$. This is true because the algebra $\mathbf{e_d} A^{\mathbf{s}}_{\mathbf{d}} \mathbf{e_d}$ is isomorphic to the cyclotomic Hecke algebra $H^{\mathbf{s}}_{\mathbf{d}}$, see Section 3.2.9. See also Theorem 3.2.33. $\square$

Lemma 3.3.8 implies that the functor $\overline{F}\colon \mathrm{grmod}(A^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{grmod}(R^{\mathbf{s}}_{\mathbf{d}})$, $M \mapsto \mathbf{e_d} M$ is a graded lift of the Schur functor F. Let $a'\colon \tilde{R}_{\mathbf{d}} \to (\tilde{R}_{\mathbf{d}})^{\mathrm{op}}$ be the graded anti-involution coming from the Verdier duality, see Theorem 3.3.7. The same argument as for a shows that a' factors through a graded anti-involution a' of $R^{\mathbf{s}}_{\mathbf{d}}$. It is easy to see that the obtained anti-involution a' of $R^{\mathbf{s}}_{\mathbf{d}}$ coincides with one defined in Section 3.2.10. Hence, condition (b_5) holds.

3.3.3 Lie algebras

Let N be an integer > 0. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be its usual Cartan and Borel subalgebras respectively.

Let $\widehat{\mathfrak{g}}$ be the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$. Consider the Lie subalgebras in $\widehat{\mathfrak{g}}$ given by

$$\widehat{\mathfrak{h}} = \mathbb{C}\partial \oplus \mathfrak{h} \oplus \mathbb{C}\mathbf{1}, \quad \widehat{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathbb{C}\partial \oplus \mathbb{C}\mathbf{1}.$$

We identify $\mathfrak{h}^*$ with the vector subspace of elements of $\widehat{\mathfrak{h}}^*$ that are zero on $\mathbf{1}$ and ∂. Let Π and $\widehat{\Pi}$ be the sets of roots of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ respectively. Denote by $\widehat{\Phi} = \{\alpha_k; k \in [0,N]\}$ the subset of simple roots. For each $\alpha \in \widehat{\Phi}$, let $\alpha^\vee \in \widehat{\mathfrak{h}}$ be the affine coroot associated with α. Let $P = \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_N \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{g}$. Let Λ_0 and δ be the elements of $\widehat{\mathfrak{h}}^*$ defined by $\delta(\partial) = \Lambda_0(\mathbf{1}) = 1$ and $\delta(\mathfrak{h} \oplus \mathbb{C}\mathbf{1}) = \Lambda_0(\mathfrak{h} \oplus \mathbb{C}\partial) = 0$. We have $\widehat{\mathfrak{h}}^* = \mathbb{C}\delta \oplus \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$.

Let $(,)\colon \widehat{\mathfrak{h}}^* \times \widehat{\mathfrak{h}}^* \to \mathbb{C}$ be the $\mathbb{C}$-bilinear form such that $\lambda(\alpha_k^\vee) = (\lambda, \alpha_k)$ and $\lambda(\partial) = (\lambda, \Lambda_0)$ for each $k \in [0,N]$ and each $\lambda \in \widehat{\mathfrak{h}}^*$. The weight $\lambda \in \widehat{\mathfrak{h}}^*$ is *dominant* (resp. *antidominant*) if $\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $k \in [0,N]$ (resp. $-\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $k \in [0,N]$). Let $W = \mathfrak{S}_N$ be the Weyl group of $\mathfrak{g}$. It acts on $\mathfrak{h}^*$ by permuting the elements of the basis $\{\Lambda_1, \cdots, \Lambda_N\}$. Let $\mathbb{Z}\Pi$ be the lattice in $\mathfrak{h}^*$ generated by Π. Let $\widehat{W} = W \ltimes \mathbb{Z}\Pi$ be the Weyl group of $\widehat{\mathfrak{g}}$. The W-action on $\mathfrak{h}^*$ can be extended to the $\widehat{W}$-action on $\widehat{\mathfrak{h}}^*$ as follows, for each $w \in W$, $\tau \in \mathbb{Z}\Pi$, $\lambda \in \mathfrak{h}^*$ we have

$$w(\Lambda_0) = \Lambda_0, \quad w(\delta) = \delta, \quad \tau(\delta) = \delta,$$

$$\tau(\lambda) = \lambda - (\tau, \lambda)\delta, \quad \tau(\Lambda_0) = \tau + \Lambda_0 - (\tau,\tau)\delta/2.$$

3.3.4 Category A

Let N be as above. Fix an l-tuple of positive integers $\mathbf{m} = (m_1, \cdots, m_l)$ such that $\sum_{t=1}^{l} m_t = N$. Let $\widehat{\mathfrak{p}}_{\mathbf{m}}$ be the parabolic subalgebra $\widehat{\mathfrak{b}} \subset \widehat{\mathfrak{p}}_{\mathbf{m}} \subset \widehat{\mathfrak{g}}$ of parabolic type $\mathbf{m}$. Set

$$\widehat{\Phi}_{\mathbf{m}} = \widehat{\Phi}\backslash\{\alpha_k; k = 0, m_1, m_1+m_2, \cdots, m_1+m_2+\cdots+m_l\}.$$

The weight $\lambda \in \widehat{\mathfrak{h}}^*$ is $\mathbf{m}$-*dominant* (resp. $\mathbf{m}$-*antidominant*) if $\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $\alpha_k \in \widehat{\Phi}_{\mathbf{m}}$ (resp. $-\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $\alpha_k \in \widehat{\Phi}_{\mathbf{m}}$). Let $P^{\mathbf{m}}$ be the set of all $\mathbf{m}$-dominant elements in $P \subset \widehat{\mathfrak{h}}^*$.

We will identify the element $\sum_{k=1}^N a_k\Lambda_k$, $a_k \in \mathbb{Z}$, of the weight lattice P with the N-tuple $(a_1, a_2, \cdots, a_N)$. Let $\rho, \rho_{\mathbf{m}} \in P$ be given by

$$\rho = (0, -1, -2, \cdots, -(N-1)),$$

$$\rho_{\mathbf{m}} = (m_1, m_1-1, \cdots, 1, m_2, m_2-1, \cdots, 1, \cdots, m_l, m_l-1, \cdots, 1).$$

Set also $\widehat{\rho} = \rho + N\Lambda_0 \in \widehat{\mathfrak{h}}^*$. For each $\lambda \in \mathfrak{h}^*$, consider the following element $\widetilde{\lambda}^- = z_\lambda\delta + \lambda - (e+N)\Lambda_0 \in \widehat{\mathfrak{h}}^*$, where $z_\lambda = (\lambda, 2\rho+\lambda)/2e$. Note that $\widetilde{\lambda}^-$ is $\mathbf{m}$-dominant if and only if λ is $\mathbf{m}$-dominant.

Definition 3.3.9. Let $\mathcal{O}^{\mathbf{m}}$ be the category of finitely generated $U(\widehat{\mathfrak{g}})$-modules which are semisimple $\widehat{\mathfrak{h}}$-modules such that the $\widehat{\mathfrak{p}}_{\mathbf{m}}$-action is locally finite and the highest weights of each composition factor are of the form $\widetilde{\lambda}^-$ for some λ in $P^{\mathbf{m}}$. Then $\mathcal{O}^{\mathbf{m}}$ is an abelian category.

For each $\lambda \in P^{\mathbf{m}}$, let $M^{\mathbf{m}}(\lambda)$ be the parabolic Verma module in $\mathcal{O}^{\mathbf{m}}$ with highest weight $\widetilde{\lambda}^-$. Let $\mathcal{P}^{\mathbf{m}}$ be the set of l-partitions $\lambda = (\lambda^1, \cdots, \lambda^l)$ such that $|\lambda^a| \leqslant m_a$ for each $a \in [1, l]$.

We identify an l-partition λ as above with the weight

$$(\lambda^1_1, \cdots, \lambda^1_{\ell(\lambda^1)}, 0^{m_1-\ell(\lambda^1)}, \lambda^2_1, \cdots, \lambda^2_{\ell(\lambda^2)}, 0^{m_2-\ell(\lambda^2)}, \cdots, \lambda^l_1, \cdots, \lambda^l_{\ell(\lambda^l)}, 0^{m_l-\ell(\lambda^l)}),$$

where 0^a is the row of a zeros for each $a \in \mathbb{N}$. Consider the map $\omega\colon \mathcal{P}^{\mathbf{m}} \to P^{\mathbf{m}}$ such that $\omega(\lambda) = \lambda - \rho + \rho_{\mathbf{m}}$.

Assume that $\lambda \in \mathcal{P}^{\mathbf{m}}$. Set $\Delta^*_{\mathcal{O}}(\lambda) = M^{\mathbf{m}}(\omega(\lambda)) \in \mathcal{O}^{\mathbf{m}}$. Let λ^* be the l-partition such that $\mathrm{YD}(\lambda^*) = (\mathrm{YD}(\lambda^l)^t, \mathrm{YD}(\lambda^{l-1})^t, \cdots, \mathrm{YD}(\lambda^1)^t)$, where $(\bullet)^t$ means the transposed Young diagram. We abbreviate $\Delta^\lambda_{\mathcal{O}} = \Delta^*_{\mathcal{O}}(\lambda^*)$. (This notation is motivated by Theorem 3.3.13.) For each $\lambda \in \mathcal{P}^{\mathbf{m}}$ let $L^\lambda_{\mathcal{O}}$ be the top of $\Delta^\lambda_{\mathcal{O}}$.

Definition 3.3.10. Let $\mathbf{A}^{\mathbf{m}}$ be the Serre subcategory of $\mathcal{O}^{\mathbf{m}}$ generated by the set $\{L^\lambda_{\mathcal{O}}; \lambda \in \mathcal{P}^{\mathbf{m}}\}$.

The category $\mathbf{A}^{\mathbf{m}}$ is a highest weight category with the set of standard modules $\{\Delta^\lambda_{\mathcal{O}}\,;\, \lambda \in \mathcal{P}^{\mathbf{m}}\}$, see [53, Sec. 5.5]. Fix $\mathbf{d} = \sum_{i\in I} d_i \cdot i \in \mathbb{N}I$ and set $\mathcal{P}^{\mathbf{m}}_{\mathbf{d}} = \mathcal{P}^{\mathbf{m}} \cap \mathcal{P}^l_{\mathbf{d}}$.

Definition 3.3.11. Let $\mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$ be the Serre subcategory of $\mathbf{A}^{\mathbf{m}}$ generated by the set $\{L^\lambda_{\mathcal{O}}; \lambda \in \mathcal{P}^{\mathbf{m}}_{\mathbf{d}}\}$.

Note that we have the following abelian category direct sum decomposition $\mathbf{A}^{\mathbf{m}} = \bigoplus_{\mathbf{d}\in\mathbb{N}I} \mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$. The category $\mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$ is a highest weight category with the set of standard modules $\{\Delta^{\lambda}_{\mathcal{O}}; \lambda \in \mathcal{P}^{\mathbf{m}}_{\mathbf{d}}\}$, see [53, Sec. 5.5]. For each $\lambda \in \mathcal{P}^{\mathbf{m}}_{\mathbf{d}}$, let $\nabla^{\lambda}_{\mathcal{O}}$ be the costandard object in $\mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$ that has simple socle $L^{\lambda}_{\mathcal{O}}$ and let $P^{\lambda}_{\mathcal{O}}$ be the projective cover of $L^{\lambda}_{\mathcal{O}}$ in $\mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$.

3.3.5 The Koszul grading

Choose an l-tuple $\mathbf{m} = (m_1, \cdots, m_l)$ of positive integers such that $m_p \equiv -s_{l+1-p} \bmod e$ and $m_p \geqslant |\mathbf{d}|$ for each $p \in [1, l]$. Set $N = m_1 + \cdots + m_l$.

For simplicity, we abbreviate $\mathbf{A} = \mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$. Note that $\mathcal{P}^{l}_{\mathbf{d}} = \mathcal{P}^{\mathbf{m}}_{\mathbf{d}}$. Set $P_{\mathcal{O}} = \bigoplus_{\lambda\in\mathcal{P}^{l}_{\mathbf{d}}} P^{\lambda}_{\mathcal{O}}$ and $S^{\mathcal{O}} = \mathrm{End}_{\mathbf{A}}(P_{\mathcal{O}})^{\mathrm{op}}$. The functor $\mathrm{Hom}_{\mathbf{A}}(P_{\mathcal{O}}, \bullet)\colon \mathbf{A} \to \mathrm{mod}(S^{\mathcal{O}})$ is an equivalence of categories. We have the following theorem, see [56, Thm. 6.4].

Theorem 3.3.12. *The $\mathbb{C}$-algebra $S^{\mathcal{O}}$ admits a Koszul grading.* □

Here and for the rest of the paper, the grading of $S^{\mathcal{O}}$ is the Koszul grading from Theorem 3.3.12. Set $\overline{\mathbf{A}} = \mathrm{grmod}(S^{\mathcal{O}})$. Let $\lambda \in \mathcal{P}^{l}_{\mathbf{d}}$. The $S^{\mathcal{O}}$-modules $\Delta^{\lambda}_{\mathcal{O}}$, $\nabla^{\lambda}_{\mathcal{O}}$, $P^{\lambda}_{\mathcal{O}}$, $L^{\lambda}_{\mathcal{O}}$ are indecomposable. Thus their graded lifts are unique modulo a shift, see Section 3.2.4 for existence and Lemma 3.2.1 for unicity. Fix natural graded lifts (see Section 3.2.5) on $L^{\lambda}_{\mathcal{O}}$, $P^{\lambda}_{\mathcal{O}}$, $\Delta^{\lambda}_{\mathcal{O}}$. Fix also the graded lift on $\nabla^{\lambda}_{\mathcal{O}}$ such that each morphism $L^{\lambda}_{\mathcal{O}} \to \nabla^{\lambda}_{\mathcal{O}}$ is homogeneous of degree 0.

The following is proved in [53, Prop. 7.9]. See also Appendix A.2 for more details about [53, Prop. 7.9].

Theorem 3.3.13. *Assume that $\mathbf{m}$ satisfies $m_1 \geqslant |\mathbf{d}|$ and $m_{k+1} - m_k \geqslant |\mathbf{d}|$ for each $k \in [1, l-1]$. There is an equivalence of categories $E_{\mathrm{S}}\colon \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}}) \to \mathbf{A}$ that takes L^{λ} to $L^{\lambda}_{\mathcal{O}}$ and Δ^{λ} to $\Delta^{\lambda}_{\mathcal{O}}$ for each $\lambda \in \mathcal{P}^{l}_{\mathbf{d}}$.* □

Now, we explain the link between the category $\mathbf{A}$ and the categories introduced in Appendix A.1. Note that some notation from Appendix A.1 will be used here. Let $\nu \subset \widehat{\Phi}$ be the parabolic type associated with $\mathbf{m}$, i.e.,

$$\nu = \widehat{\Phi}\backslash\{\alpha_s; s = 0, m_1, m_1 + m_2, \cdots, m_1 + \cdots + m_{l-1}\}.$$

Recall that we identify each partition λ with an N-tuple of integers. All weights of the form $\widetilde{\omega(\lambda)}^{-} + \widehat{\rho}$ are in the same W-orbit. Let $o_{\mu,-} + \widehat{\rho}$ be the antidominant weight in this orbit. Then the category $\mathbf{A}$ is a subcategory in $\mathcal{O}^{\nu}_{\mu,-}$. Let us describe the singular type μ of $o_{\mu,-}$. For each $i \in I = \mathbb{Z}/e\mathbb{Z}$ let n'_i be the number of components of residue i modulo e in the N-tuple $\rho_{\mathbf{m}}$. For each $i \in I$ set $n_i = n'_i - d_i + d_{i-1}$. For each $\lambda \in \mathcal{P}^{l}_{\mathbf{d}}$ the N-tuple $\lambda + \rho_{\mathbf{m}}$ has n_i components of residue i modulo e. Set also $k = (\sum_{i\in I} m_i(m_i+1)/2 + |\mathbf{d}| - \sum_{s=1}^{e} sn_s)/e$. Note that here $\sum_{i\in I} m_i(m_i + 1)/2 + |\mathbf{d}|$ is the sum of components in an N-tuple $\lambda + \rho_{\mathbf{m}}$ with $\lambda \in \mathcal{P}^l_{\mathbf{d}}$ and $\sum_{s=1}^e sn_s$ is the sum of components in the N-tuple $(1^{n_1}, 2^{n_2}, \cdots, e^{n_e})$, where s^{n_s} means a row with n_s copies of s. Finally, we have

$$\mu = \widehat{\Phi}\backslash\{\alpha_s; s = -k, -k+n_1, -k+n_1+n_2, \cdots, -k+n_1+\cdots+n_{l-1}\}.$$

Now, having $\mathbf{A} \subset \mathcal{O}^\nu_{\mu,-}$, we can take v large enough such that $\mathbf{A}$ is inside of ${}^v\mathcal{O}^\nu_{\mu,-}$. We get the following result, see also Lemma 3.2.17 and [53, Lem. 5.15].

Lemma 3.3.14. *The category* $\mathbf{A}$ *is a highest weight subcategory of* ${}^v\mathcal{O}^\nu_{\mu,-}$. □

Remark 3.3.15. (a) By Lemma 3.2.17 the Koszul grading on $\mathbf{A}$ is inherited from the Koszul grading on ${}^v\mathcal{O}^\nu_{\mu,-}$. Thus the graded multiplicities of simple modules in Verma modules can be interpreted in terms of corresponding graded multiplicities in ${}^v\mathcal{O}^\nu_{\mu,-}$, see Remark 3.2.18 (a).

(b) Since $\mathbf{A}$ is a highest weight subcategory of ${}^v\mathcal{O}^\nu_{\mu,-}$, the BGG duality yields a duality on $\mathbf{A}$ which fixes the simple modules. Similarly, by Remark 3.2.18 (b) the category $\mathbf{A}^!$ is a Serre quotient of ${}^{v^{-1}}\mathcal{O}^\mu_{\nu,+}$. Thus it admits also a duality. See [56] for details.

3.3.6 Analogue of the cyclotomic KLR-algebra

Set $\mathcal{K}^l_{\mathbf{d}} = \mathcal{K}^l_{|\mathbf{d}|} \cap \mathcal{P}^l_{\mathbf{d}}$ where $\mathcal{K}^l_d$ is as in Section 3.2.9. Now, we define an analogue of the cyclotomic KLR-algebra. Consider the idempotent of $S^{\mathcal{O}}$ given by $\mathbf{e}^{\mathcal{O}} = \bigoplus_{\lambda\in\mathcal{K}^l_{\mathbf{d}}} \mathrm{Id}_{P^\lambda_{\mathcal{O}}}$. It is homogeneous of degree zero. Set $R^{\mathcal{O}} = \mathbf{e}^{\mathcal{O}} S^{\mathcal{O}} \mathbf{e}^{\mathcal{O}}$. Then, $R^{\mathcal{O}}$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra. We have the functor

$$F^{\mathcal{O}} \colon \mathrm{mod}(S^{\mathcal{O}}) \to \mathrm{mod}(R^{\mathcal{O}}), \quad M \mapsto \mathbf{e}^{\mathcal{O}} M.$$

Let $\overline{F}^{\mathcal{O}} : \mathrm{grmod}(S^{\mathcal{O}}) \to \mathrm{grmod}(R^{\mathcal{O}})$ be the obvious graded lift of $F^{\mathcal{O}}$. Consider the modules in $\mathrm{mod}(R^{\mathcal{O}})$ given by $D^\lambda_{\mathcal{O}} = F^{\mathcal{O}}(L^\lambda_{\mathcal{O}})$, $S^\lambda_{\mathcal{O}} = F^{\mathcal{O}}(\Delta^\lambda_{\mathcal{O}})$ and $Y^\lambda_{\mathcal{O}} = F^{\mathcal{O}}(P^\lambda_{\mathcal{O}})$. They admit the graded lifts given by $\overline{F}^{\mathcal{O}}(L^\lambda_{\mathcal{O}})$, $\overline{F}^{\mathcal{O}}(\Delta^\lambda_{\mathcal{O}})$, $\overline{F}^{\mathcal{O}}(P^\lambda_{\mathcal{O}})$. The functor $F^{\mathcal{O}}$ is a quotient functor, see [22, Prop. III.2.5], [27, Sec. 2.4]. Thus the module $D^\lambda_{\mathcal{O}}$ is either zero or irreducible.

By [53, Thm. 5.37], there is a projective module $T \in \mathbf{A}$ such that we have an ungraded algebra isomorphism $\mathrm{End}_{\mathbf{A}}(T)^{\mathrm{op}} \simeq R^{\mathbf{s}}_{\mathbf{d}}$. Consider the functor $F' \colon \mathbf{A} \to \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}})$ given by $M \mapsto \mathrm{Hom}_{\mathbf{A}}(T, M)$. By [53, Prop. 7.9] we have $F' \circ E_S = F$, where E_S is as in Theorem 3.3.13. In particular we get $F'(L^\lambda_{\mathcal{O}}) = D^\lambda$ for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$. This implies that, for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, the projective module $P^\lambda_{\mathcal{O}}$ is a direct factor of T if and only if $\lambda \in \mathcal{K}^l_{\mathbf{d}}$, because $\dim \mathrm{Hom}_{S^{\mathcal{O}}}(P^\lambda_{\mathcal{O}}, L^\mu_{\mathcal{O}}) = \delta_{\lambda,\mu}$. Hence, there is an equivalence of categories

$E_{\mathrm{H}}\colon \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}}) \to \mathrm{mod}(R^{\mathcal{O}})$ such that the following diagram of functors is commutative

$$\begin{tikzcd} \mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}}) \arrow[r, "E_S"] \arrow[d, "F"'] & \mathrm{mod}(S^{\mathcal{O}}) \arrow[d, "F^{\mathcal{O}}"'] \\ \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}}) \arrow[r, "E_H"] & \mathrm{mod}(R^{\mathcal{O}}). \end{tikzcd}$$

Here we identify $\mathbf{A} \simeq \mathrm{mod}(S^{\mathcal{O}})$ as in Section 3.3.5. Moreover, we have

$$E_{\mathrm{H}}(S^\lambda) = S^\lambda_{\mathcal{O}}, \quad E_{\mathrm{H}}(D^\lambda) = D^\lambda_{\mathcal{O}}, \quad E_{\mathrm{H}}(Y^\lambda) = Y^\lambda_{\mathcal{O}}, \quad \forall \lambda \in \mathcal{P}^l_{\mathbf{d}}.$$

Lemma 3.3.16. *The functor $F^{\mathcal{O}}$ is fully faithful on projective modules.*

Proof. The statement follows from Lemma 3.2.30 and from the commutative diagram above. □

3.3.7 Graded BGG reciprocity

We want to show that graded categories $\mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$ and $\overline{\mathbf{A}}$ satisfy condition $(*)$ from Section 3.2.4. This condition for $\mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})$ follows from condition (b) in the definition of an admissible grading.

We must check condition $(*)$ for $\overline{\mathbf{A}}$. We construct the duality functor D on $\mathbf{A}$ in the same way as in [5, Sec. 3.11]. More precisely, for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, let $L^{\mathcal{O}}_\lambda$ be the simple object of $\mathbf{A}^!$ such that $K(P^\lambda_{\mathcal{O}}) = L^{\mathcal{O}}_\lambda$, where K is the equivalence in Theorem 3.2.8. Since $\mathbf{A} \simeq \mathrm{mod}(S^{\mathcal{O}})$ and $S^{\mathcal{O}}$ is a basic algebra, by Theorem 3.2.8 (a) we have a graded algebra isomorphism $S^{\mathcal{O}} \simeq \bigoplus_{\lambda,\mu\in\mathcal{P}^l_{\mathbf{d}}} \mathrm{Ext}^*_{\mathbf{A}^!}(L^{\mathcal{O}}_\lambda, L^{\mathcal{O}}_\mu)^{\mathrm{op}}$. We deduce that there is a graded $\mathbb{C}$-algebra anti-involution $a^{\mathcal{O}}\colon S^{\mathcal{O}} \to (S^{\mathcal{O}})^{\mathrm{op}}$, because the duality on $\mathbf{A}^!$ from Remark 3.3.15 (b) gives an isomorphism $\bigoplus_{\lambda,\mu\in\mathcal{P}^l_{\mathbf{d}}} \mathrm{Ext}^*_{\mathbf{A}^!}(L^{\mathcal{O}}_\lambda, L^{\mathcal{O}}_\mu)^{\mathrm{op}} = \bigoplus_{\lambda,\mu\in\mathcal{P}^l_{\mathbf{d}}} \mathrm{Ext}^*_{\mathbf{A}^!}(L^{\mathcal{O}}_\mu, L^{\mathcal{O}}_\lambda)$. Now, $a^{\mathcal{O}}$ yields a duality $D\colon \mathrm{mod}(S^{\mathcal{O}})^{\mathrm{op}} \to \mathrm{mod}(S^{\mathcal{O}})$ as in Remark 3.2.6. Thus condition $(*)$ holds for $\overline{\mathbf{A}}$.

3.3.8 Decomposition numbers

Recall the graded multiplicities $[\bullet : \bullet]_q$ and $(\bullet : \bullet)_q$ defined in Section 3.2.4.

Definition 3.3.17. For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we set $d_{\lambda\xi}(q) = [\Delta^\lambda : L^\xi]_q$, $c_{\lambda\xi}(q) = [P^\lambda : L^\xi]_q$, $d^{\mathcal{O}}_{\lambda\xi}(q) = [\Delta^\lambda_{\mathcal{O}} : L^\xi_{\mathcal{O}}]_q$ and $c^{\mathcal{O}}_{\lambda\xi}(q) = [P^\lambda_{\mathcal{O}} : L^\xi_{\mathcal{O}}]_q$.

Lemma 3.3.18. *For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we have $[\Delta^\lambda : L^\xi]_q = (P^\xi : \Delta^\lambda)_q$ and $[\Delta^\lambda_{\mathcal{O}} : L^\xi_{\mathcal{O}}]_q = (P^\xi_{\mathcal{O}} : \Delta^\lambda_{\mathcal{O}})_q$.*

Proof. The statement follows from Corollary 3.2.16 and Section 3.3.7. □

Corollary 3.3.19. *We have $c(q) = d(q)^t d(q)$ and $c^{\mathcal{O}}(q) = d^{\mathcal{O}}(q)^t d^{\mathcal{O}}(q)$.* □

Remark 3.3.20. The functors F and $F^{\mathcal{O}}$ are quotient functors. Thus, we have $d_{\lambda\xi}(q) = [S^\lambda : D^\xi]_q$ and $d^{\mathcal{O}}_{\lambda\xi}(q) = [S^\lambda_{\mathcal{O}} : D^\xi_{\mathcal{O}}]_q$ for each $\lambda \in \mathcal{P}^l_{\mathbf{d}}, \xi \in \mathcal{K}^l_{\mathbf{d}}$. We also have $c_{\lambda\xi}(q) = [Y^\lambda : D^\xi]_q$ and $c^{\mathcal{O}}_{\lambda\xi}(q) = [Y^\lambda_{\mathcal{O}} : D^\xi_{\mathcal{O}}]_q$ for each $\lambda, \xi \in \mathcal{K}^l_{\mathbf{d}}$.

Recall that the graded lifts of D^λ, S^λ, Y^λ are the graded modules $\overline{F}(L^\lambda)$, $\overline{F}(\Delta^\lambda)$, $\overline{F}(P^\lambda)$. Another definition of the graded lifts of these modules is given in [9, Thm. 4.11, Sec. 4.10, 5.2]. Denote the graded lifts from [9] by D'^λ, S'^λ, Y'^λ respectively. We want to compare these two choices of graded lifts.

Lemma 3.3.21. *Assume that $\lambda \in \mathcal{K}^l_{\mathbf{d}}$. Then we have the graded $R^{\mathbf{s}}_{\mathbf{d}}$-module isomorphisms $D^\lambda \simeq D'^\lambda$, $S^\lambda \simeq S'^\lambda$ and $Y^\lambda \simeq Y'^\lambda$.*

Proof. The ungraded $R^{\mathbf{s}}_{\mathbf{d}}$-modules D^λ, S^λ, Y^λ are indecomposable because each of them has simple top D^λ, see [66, Prop. 2.2 (3)]. Thus their graded lifts are unique up to a shift of the grading. The graded lifts are normalized such that each morphism $Y'^\lambda \to D'^\lambda$, $S'^\lambda \to D'^\lambda$, $Y^\lambda \to D^\lambda$, $S^\lambda \to D^\lambda$ is homogeneous of degree zero. Hence, it suffices to show that there is a graded $R^{\mathbf{s}}_{\mathbf{d}}$-module isomorphism $D^\lambda \simeq D'^\lambda$. The graded lift D'^λ of D^λ is characterized by stability by the duality $\overline{D}'$, see [9, Thm. 4.11]. On the other hand we have

$$\overline{D}'(D^\lambda) = \overline{D}'\overline{F}(L^\lambda) = \overline{F}\,\overline{D}(L^\lambda) = \overline{F}(L^\lambda) = D^\lambda.$$

Here, the second equality is condition (b_5) in Definition 3.3.1 and the third equality is condition (b_3) in Definition 3.3.1. □

The level l Fock space $F^{\mathbf{s}}$ associated with the parameter $\mathbf{s} = (s_1, \cdots, s_l)$ is a $\mathbb{Q}(q)$-vector space with basis $\{M_\lambda; \lambda \in \mathcal{P}^l\}$ called *standard basis*. It is equipped with a representation of $U_q(\widehat{sl_e})$ defined as in [9, (3.22), (3.23)]. Let $V^{\mathbf{s}}$ be the simple $U_q(\widehat{sl_e})$-module with highest weight $\Lambda_{s_1} + \cdots + \Lambda_{s_l}$. Consider the surjection $F^{\mathbf{s}} \to V^{\mathbf{s}}$ as in [9, (3.28)]. Let M'_λ be the image of M_λ in $V^{\mathbf{s}}$.

For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$ and $\xi \in \mathcal{K}^l_{\mathbf{d}}$ set $d'_{\lambda\xi}(q) = [S'^\lambda : D'^\xi]_q$ and $c'_{\lambda\xi} = [Y'^\lambda : D'^\xi]_q$.

Lemma 3.3.22. *Assume that $\lambda \in \mathcal{P}^l_{\mathbf{d}}$ and $\xi \in \mathcal{K}^l_{\mathbf{d}}$. Then we have $d'_{\lambda\xi}(q) = d^{\mathcal{O}}_{\lambda\xi}(q)$.*

Proof. By [9, (3.35), Cor. 5.15] the elements $d'_{\lambda\xi}(q) \in \mathbb{N}[q, q^{-1}]$ are the coefficients of the decomposition of M'_λ in the canonical basis of $V^{\mathbf{s}}$. Further, [61, Thm. 3.26 (i)] gives an explicit formula for these coefficients in terms of Kazhdan-Lusztig polynomials. Finally, we compute the decomposition numbers in the parabolic category $\mathcal{O}$ with respect to the Koszul grading in Appendix A.1, see also Remark 3.3.15 (a). Now, the statement follows by comparing Lemma A.1.14 with [61, Thm. 3.26 (i)]. □

Remark 3.3.23. The element q in [9] corresponds to q^{-1} in [61]. In particular, the positive canonical basis in [9] corresponds to the negative canonical basis in [61].

Corollary 3.3.24. *Assume that $\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}$. Then we have $c_{\lambda\xi}(q)=c^{\mathcal{O}}_{\lambda\xi}(q)$.*

Proof. By [9, (3.45), Thm. 4.18, Thm. 5.14], for each $\lambda\in\mathcal{K}^l_{\mathbf{d}}$, we have $[Y'^{\lambda}]=\sum_{\xi\in\mathcal{P}^l_{\mathbf{d}}}d'_{\xi\lambda}(q)[S'^{\xi}]$ in $[\mathrm{grmod}(R^{\mathbf{s}}_{\mathbf{d}})]$. Thus, for each $\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}$, we have $c'_{\lambda\xi}=\sum_{\mu\in\mathcal{P}^l_{\mathbf{d}}}d'_{\lambda\mu}d'_{\mu\xi}$. On the other hand, Corollary 3.3.19 implies that $c^{\mathcal{O}}_{\lambda\xi}=\sum_{\mu\in\mathcal{P}^l_{\mathbf{d}}}d^{\mathcal{O}}_{\lambda\mu}d^{\mathcal{O}}_{\mu\xi}$. Hence, by Lemma 3.3.22, we have $c'_{\lambda\xi}=c^{\mathcal{O}}_{\lambda\xi}$. This implies the statement, because Remark 3.3.20 and Lemma 3.3.21 yield $c_{\lambda\xi}=c'_{\lambda\xi}$. □

Lemma 3.3.25. *For each $\lambda,\xi\in\mathcal{P}^l_{\mathbf{d}}$ we have $c^{\mathcal{O}}_{\lambda\xi}(q)\in\mathbb{N}[q]$.*

Proof. The $S^{\mathcal{O}}$-module $L^{\xi}_{\mathcal{O}}$ is concentrated in degree zero. The top of the $S^{\mathcal{O}}$-module $P^{\lambda}_{\mathcal{O}}$ is isomorphic to $L^{\lambda}_{\mathcal{O}}$. Hence, since the grading of $S^{\mathcal{O}}$ is nonnegative, all negative graded components of $P^{\lambda}_{\mathcal{O}}$ are zero. We deduce that $c^{\mathcal{O}}_{\lambda\xi}(q)=[P^{\lambda}_{\mathcal{O}}:L^{\xi}_{\mathcal{O}}]_q\in\mathbb{N}[q]$. □

For each $\lambda,\xi\in\mathcal{P}^l_{\mathbf{d}}$, we write $c_{\lambda\xi}(q)=\sum_{g\in\mathbb{Z}}c^g_{\lambda\xi}q^g$ and $c^{\mathcal{O}}_{\lambda\xi}(q)=\sum_{g\in\mathbb{Z}}c^{\mathcal{O}g}_{\lambda\xi}q^g$.

3.3.9 Identification of grading of the KLR-algebra

We identify $\mathbf{A}\simeq\mathrm{mod}(S^{\mathcal{O}})$ as in Section 3.3.5. Under this identification, we can view the functor E_{S} from Theorem 3.3.13 as a functor $E_{\mathrm{S}}\colon\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})\to\mathrm{mod}(S^{\mathcal{O}})$. Recall that the algebra $S^{\mathbf{s}}_{\mathbf{d}}$ is equipped with an admissible grading, see Section 3.3.1. We consider the graded basic algebra ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ associated with $S^{\mathbf{s}}_{\mathbf{d}}$, i.e., we set

$$ {}^bS^{\mathbf{s}}_{\mathbf{d}}=\bigoplus_{\lambda,\xi\in\mathcal{P}^l_{\mathbf{d}}}\mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^{\lambda},P^{\xi})^{\mathrm{op}}=\bigoplus_{\lambda,\xi\in\mathcal{P}^l_{\mathbf{d}}}\mathrm{Hom}_{R^{\mathbf{s}}_{\mathbf{d}}}(Y^{\lambda},Y^{\xi})^{\mathrm{op}}, $$

see Lemma 3.2.30. The grading of ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ is induced from the grading of the module P^{λ}, see Section 3.3.1. By definition, the categories $\mathrm{mod}({}^bS^{\mathbf{s}}_{\mathbf{d}})$ and $\mathrm{mod}(S^{\mathbf{s}}_{\mathbf{d}})$ are equivalent. Further, we have an ungraded algebra isomorphism ${}^bS^{\mathbf{s}}_{\mathbf{d}}\simeq S^{\mathcal{O}}$, because the categories $\mathrm{mod}({}^bS^{\mathbf{s}}_{\mathbf{d}})$ and $\mathrm{mod}(S^{\mathcal{O}})$ are equivalent and both algebras ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ and $S^{\mathcal{O}}$ are basic.

Now, let ${}^bR^{\mathbf{s}}_{\mathbf{d}}$ be the graded basic algebra associated with $R^{\mathbf{s}}_{\mathbf{d}}$, i.e., we set

$$ {}^bR^{\mathbf{s}}_{\mathbf{d}}=\bigoplus_{\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}}\mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^{\lambda},P^{\xi})^{\mathrm{op}}=\bigoplus_{\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}}\mathrm{Hom}_{R^{\mathbf{s}}_{\mathbf{d}}}(Y^{\lambda},Y^{\xi})^{\mathrm{op}}. $$

The categories $\mathrm{mod}({}^bR^{\mathbf{s}}_{\mathbf{d}})$ and $\mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}})$ are equivalent. Further, we have an ungraded algebra isomorphism ${}^bR^{\mathbf{s}}_{\mathbf{d}}\simeq R^{\mathcal{O}}$, because the categories

$\mathrm{mod}({}^b R^{\mathbf{s}}_{\mathbf{d}})$ and $\mathrm{mod}(R^{\mathcal{O}})$ are equivalent via E_H and both algebras ${}^b R^{\mathbf{s}}_{\mathbf{d}}$ and $R^{\mathcal{O}}$ are basic. Let us prove that these algebras are also isomorphic as graded algebras. To do so, we consider the homogeneous basis of $R^{\mathcal{O}} = \bigoplus_{\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}} \mathrm{End}_{S^{\mathcal{O}}}(P^\lambda_{\mathcal{O}}, P^\xi_{\mathcal{O}})^{\mathrm{op}}$ given by the construction in Section 3.2.6.

For each $\xi \in \mathcal{K}^l_{\mathbf{d}}$ and each $g, s \in \mathbb{N}$, write

$$\begin{aligned} \mathrm{rad}^g P^\xi_{\mathcal{O}}/\mathrm{rad}^{g+1} P^\xi_{\mathcal{O}} &= \bigoplus_{\lambda\in\mathcal{P}^l_{\mathbf{d}}, s\in\mathbb{N}} L^\lambda_{\mathcal{O}}\langle s\rangle^{\oplus c^{\mathcal{O}gs}_{\xi\lambda}}, \\ \mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1} P^\xi &= \bigoplus_{\lambda\in\mathcal{P}^l_{\mathbf{d}}, s\in\mathbb{N}} L^\lambda\langle s\rangle^{\oplus c^{gs}_{\xi\lambda}}. \end{aligned}$$

For each $\lambda, \xi \in \mathcal{K}^l_{\mathbf{d}}$ let $\Theta_{\lambda\xi} = \{\theta^{gst}_{\lambda\xi}; g, s \in \mathbb{N}, t \in [1, c^{\mathcal{O}gs}_{\xi\lambda}]\}$ be the basis of $\mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda_{\mathcal{O}}, P^\xi_{\mathcal{O}})$ constructed as in the proof of Lemma 3.2.19.

Lemma 3.3.26. *For each $\lambda \in \mathcal{K}^l_{\mathbf{d}}$, the graded $S^{\mathcal{O}}$-module $P^\lambda_{\mathcal{O}}$ is very rigid.*

Proof. Recall the projective module $T \in \mathbf{A}$ and the functor $F'\colon \mathbf{A} \to \mathrm{mod}(R^{\mathbf{s}}_{\mathbf{d}})$ defined in Section 3.3.6. The algebra $R^{\mathbf{s}}_{\mathbf{d}}$ is Frobenius by the theorem in the introduction of [40] and the functor F' is 0-faithful (see [52, Sec. 4.2.2]) by [52, Thm. 6.6]. So, by [53, Lem. 2.14], the module T is tilting. Thus it is injective. This implies that, for each $\lambda \in \mathcal{K}^l_{\mathbf{d}}$, the module $P^\lambda_{\mathcal{O}}$ is injective because it is a direct factor of T, see Section 3.3.6. In particular it is autodual with respect to the BGG duality.

Now, by construction, we have $P^\lambda_{\mathcal{O}}/\mathrm{rad}P^\lambda_{\mathcal{O}} = L^\lambda_{\mathcal{O}}$. Since $P^\lambda_{\mathcal{O}}$ is autodual, we deduce that $\mathrm{soc}P^\lambda_{\mathcal{O}}$ is irreducible. Thus $P^\lambda_{\mathcal{O}}$ is very rigid by Lemma 3.2.23. □

Corollary 3.3.27. *For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}, \xi \in \mathcal{K}^l_{\mathbf{d}}$ and each $g, s \in \mathbb{N}$, we have $c^{\mathcal{O}gs}_{\xi\lambda} = c^{\mathcal{O}s}_{\xi\lambda}$ if $g = s$ and $c^{\mathcal{O}gs}_{\xi\lambda} = 0$ else.* □

Further, since $S^{\mathcal{O}}$ and $S^{\mathbf{s}}_{\mathbf{d}}$ are Morita equivalent, we have also the following.

Corollary 3.3.28. *For each $\lambda \in \mathcal{K}^l_{\mathbf{d}}$, the $S^{\mathbf{s}}_{\mathbf{d}}$-module P^λ is rigid.* □

Now, we have the following.

Lemma 3.3.29. *For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we have $c_{\lambda\xi}(q) \in \delta_{\lambda\xi} + q\mathbb{N}[q]$.*

Proof. The statement follows from condition (a) in Definition 3.3.1 and the graded BGG-reciprocity. □

Corollary 3.3.30. *We have $({}^b S^{\mathbf{s}}_{\mathbf{d}})_n = 0$ for each $n < 0$ and the algebra $({}^b S^{\mathbf{s}}_{\mathbf{d}})_0$ is semisimple.*

Proof. The first claim follows from Lemma 3.3.29 because $\dim_q \mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda, P^\xi) = c_{\xi\lambda}(q)$. The second claim is true because by Lemma 3.3.29 we have $({}^b S^{\mathbf{s}}_{\mathbf{d}})_0 = \bigoplus_{\lambda\in\mathcal{P}^l_{\mathbf{d}}} \mathbb{C}\mathrm{Id}_{P^\lambda}$. □

The grading of each simple module L^λ, viewed as a ${}^bS^{\mathbf{s}}_{\mathbf{d}}$-module, is concentrated in degree 0. Hence, since $({}^bS^{\mathbf{s}}_{\mathbf{d}})_0$ is semisimple by Corollary 3.3.30, we have the following.

Corollary 3.3.31. *For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we have $\mathrm{Ext}^1_{\mathrm{grmod}(S^{\mathbf{s}}_{\mathbf{d}})}(L^\lambda, L^\xi) = 0$.* $\square$

Let $\xi \in \mathcal{P}^l_{\mathbf{d}}$. Consider the filtration of P^ξ given by

$$P^\xi = P^\xi(0) \supset P^\xi(1) \supset P^\xi(2) \supset \cdots,$$

where $P^\xi(f) = \sum_\theta \mathrm{Im}\theta$, the map θ runs over the set of homogeneous maps in $\mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda, P^\xi)$ of degree $\geqslant f$ (for the grading in Section 3.3.1) and λ varies in $\mathcal{P}^l_{\mathbf{d}}$. Note that $P^\xi(f)$ is zero for sufficiently large f.

Lemma 3.3.32. *Let $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ and $g, s \in \mathbb{N}$. The following hold.*
(a) *if $s < g$ then $L^\lambda\langle s\rangle$ is not a constituent of $P^\xi(g)$,*
(b) *if $s \geqslant g$ then $L^\lambda\langle s\rangle$ is not a constituent of $P^\xi/P^\xi(g)$.*

Proof. Part (a) follows from the fact that $c_{\xi\lambda}(q) = [P^\xi : L^\lambda]_q$ is in $\mathbb{N}[q]$, see Lemma 3.3.29. Now, we prove (b). Suppose that there exists a submodule $P'^\xi \subset P^\xi$ containing $P^\xi(g)$ such that there exists a nonzero homogeneous morphism $P'^\xi/P^\xi(g) \to L^\lambda\langle s\rangle$. By the projectivity of P^λ, the canonical map $P^\lambda\langle s\rangle \to L^\lambda\langle s\rangle$ factors as follows

$$P^\lambda\langle s\rangle \to P'^\xi \to P'^\xi/P^\xi(g) \to L^\lambda\langle s\rangle.$$

The left arrow yields a morphism $\theta \in \mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda, P'^\xi)$ of degree s such that $\mathrm{Im}\theta \not\subset P^\xi(g)$. This contradicts the definition of $P^\xi(g)$. $\square$

Corollary 3.3.33. *For each $f \in \mathbb{N}$ and each $\xi \in \mathcal{P}^l_{\mathbf{d}}$, we have $\mathrm{rad}^f P^\xi \subset P^\xi(f)$.*

Proof. By Lemma 3.3.32, for each $f \in \mathbb{N}$, any simple subquotient of $P^\xi(f)/P^\xi(f+1)$ is of the form $L^\lambda\langle f\rangle$ for some $\lambda \in \mathcal{P}^l_{\mathbf{d}}$. Thus, by Corollary 3.3.31, the module $P^\xi(f)/P^\xi(f+1)$ is semisimple. This implies the statement. $\square$

Lemma 3.3.34. *For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ and each $g, s \in \mathbb{N}$ such that $s < g$, we have $c^{gs}_{\xi\lambda} = 0$.*

Proof. By Lemma 3.3.32 and Corollary 3.3.33 each simple subquotient of $\mathrm{rad}^g P^\xi$ is of the form $L^\lambda\langle f\rangle$ for some $\lambda \in \mathcal{P}^l_{\mathbf{d}}$ and $f \geqslant g$. This implies the statement. $\square$

Corollary 3.3.35. *For each $\lambda, \xi \in \mathcal{K}^l_{\mathbf{d}}$ we have $[\mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1} P^\xi : L^\lambda]_q = c^g_{\xi\lambda} q^g$.*

Proof. Assume that the statement is false. Let g be maximal such that the statement fails. Thus, we have

$$\begin{aligned}[P^\xi/\mathrm{rad}^{g+1}P^\xi : L^\lambda]_q &= c_{\xi\lambda}(q) - \sum_{t\geqslant g+1}[\mathrm{rad}^t P^\xi/\mathrm{rad}^{t+1}P^\xi : L^\lambda]_q \\ &= c_{\xi\lambda}(q) - \sum_{t\geqslant g+1} c^t_{\xi\lambda}q^t \\ &= \sum_{t=0}^{g} c^t_{\xi\lambda}q^t.\end{aligned}$$

Here the second equality follows from the maximality of g. By Lemma 3.3.34, each power of q in $[\mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1}P^\xi : L^\lambda]_q$ is $\geqslant g$. We deduce that $[\mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1}P^\xi : L^\lambda]_q = c'^g_{\xi\lambda}q^g$ for some $c'^g_{\xi\lambda} < c^g_{\xi\lambda}$. Now, we have

$$c'^g_{\xi\lambda} = [\mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1}P^\xi : L^\lambda]_1 = c^{\mathcal{O}g}_{\xi\lambda} = c^g_{\xi\lambda}.$$

Here the second equality is Corollary 3.3.27. We come to a contradiction. □

Since $[\mathrm{rad}^g P^\xi/\mathrm{rad}^{g+1}P^\xi : L^\lambda]_q = \sum_{s\in\mathbb{N}} c^{gs}_{\xi\lambda}q^s$, we deduce from Corollary 3.3.35 the following.

Corollary 3.3.36. *For each $\lambda,\xi \in \mathcal{K}^l_{\mathbf{d}}$ we have $c^{gs}_{\xi\lambda} = c^s_{\xi\lambda}$ if $g = s$ and $c^{gs}_{\xi\lambda} = 0$ else.* □

Recall the equivalence of categories E_S defined in Theorem 3.3.13. Recall also the definition of the elements $\theta^{gst}_{\lambda\xi} \in \mathrm{Hom}_{\mathbf{A}}(P^\lambda_{\mathcal{O}}, P^\xi_{\mathcal{O}})$. Set

$$v^{gst}_{\lambda\xi} = E_S^{-1}(\theta^{gst}_{\lambda\xi}) \in \mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda, P^\xi) \subset {}^bR^{\mathbf{s}}_{\mathbf{d}}.$$

The set $\{v^{gst}_{\lambda\xi}; \lambda,\xi \in \mathcal{K}^l_{\mathbf{d}}, g,s \in \mathbb{N}, t \in [1, c^{gs}_{\xi\lambda}]\}$ is a basis of ${}^bR^{\mathbf{s}}_{\mathbf{d}}$ because E_S is an equivalence of categories. By Corollaries 3.2.21, 3.3.27 we have $\mathrm{Im}v^{gst}_{\lambda\xi} \subset \mathrm{rad}^g P^\xi$ and $\mathrm{Im}v^{gst}_{\lambda\xi} \not\subset \mathrm{rad}^{g+1}P^\xi$. Thus Corollaries 3.2.21, 3.3.36 imply that the minimal nonzero homogeneous component in $v^{gst}_{\lambda\xi}$ has degree g. Denote this homogeneous component of degree g by $v'^{gst}_{\lambda\xi}$.

Consider the $\mathbb{C}$-linear map $\Phi\colon R^{\mathcal{O}} \to R^{\mathbf{s}}_{\mathbf{d}}$ such that $\theta^{gst}_{\lambda\xi} \mapsto v'^{gst}_{\lambda\xi}$.

Theorem 3.3.37. *The map Φ is a graded $\mathbb{C}$-algebra isomorphism.*

Proof. For each $f \in \mathbb{N}$ set $J^{>f} = \bigoplus_{n>f}({}^bR^{\mathbf{s}}_{\mathbf{d}})_n$. The set $\Theta = \coprod_{\lambda,\xi\in\mathcal{K}^l_{\mathbf{d}}}\Theta_{\lambda\xi}$ forms a basis in $R^{\mathcal{O}}$. By construction, for each $\theta^{gst}_{\lambda\xi} \in \Theta$ we have $\Phi(\theta^{gst}_{\lambda\xi}) \equiv v^{gst}_{\lambda\xi} \bmod J^{>g}$. This implies that for each homogeneous element $\theta \in {}^bR^l_{\mathbf{d}}$ we have $\Phi(\theta) \equiv E_S^{-1}(\theta) \bmod J^{>\deg\theta}$. Now, for each $\theta^{gst}_{\lambda\xi}, \theta^{g's't'}_{\lambda'\xi'} \in \Theta$ we have

$$\begin{aligned}\Phi(\theta^{gst}_{\lambda\xi})\Phi(\theta^{g's't'}_{\lambda'\xi'}) &\equiv v^{gst}_{\lambda\xi}v^{g's't'}_{\lambda'\xi'} \\ &\equiv E_S^{-1}(\theta^{gst}_{\lambda\xi}\theta^{g's't'}_{\lambda'\xi'}) \\ &\equiv \Phi(\theta^{gst}_{\lambda\xi}\theta^{g's't'}_{\lambda'\xi'}) \bmod J^{>g+g'}.\end{aligned}$$

On the other hand the map Φ is homogeneous of degree zero. Thus Φ is a graded algebra isomorphism. $\square$

3.3.10 Identification of grading of the Schur algebra

The goal of this section is to show that the algebras ${}^bS^{\mathbf{s}}_{\mathbf{d}}$ and $S^{\mathcal{O}}$ are isomorphic as graded algebras.

Lemma 3.3.38. *For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, the $R^{\mathbf{s}}_{\mathbf{d}}$-module Y^λ is indecomposable.*

Proof. The $S^{\mathbf{s}}_{\mathbf{d}}$-module P^λ is indecomposable because it has simple top L^λ. Thus the ring $\mathrm{End}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda)$ is local. By Lemma 3.2.30 we have $\mathrm{End}_{R^{\mathbf{s}}_{\mathbf{d}}}(Y^\lambda) = \mathrm{End}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\lambda)$. Thus the module Y^λ is also indecomposable. $\square$

By Theorem 3.3.37 we can identify the graded categories $\mathrm{grmod}(R^{\mathbf{s}}_{\mathbf{d}}) \simeq \mathrm{grmod}(R^{\mathcal{O}})$. For each $\lambda \in \mathcal{P}^l_{\mathbf{d}}$, there is an integer a_λ such that $Y^\lambda = Y^\lambda_{\mathcal{O}}\langle a_\lambda \rangle$, because the modules Y^λ and $Y^\lambda_{\mathcal{O}}$ are indecomposable and are isomorphic as ungraded modules, see the construction of the functor E_H and Lemma 3.3.38.

Lemma 3.3.39. *For each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we have $a_\lambda = a_\xi$.*

Proof. We have

$$\begin{aligned}
c_{\lambda\xi}(q) &= [P^\lambda : L^\xi]_q \\
&= \dim_q \mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^\xi, P^\lambda) \\
&= \dim_q \mathrm{Hom}_{R^{\mathbf{s}}_{\mathbf{d}}}(Y^\xi, Y^\lambda) \\
&= \dim_q \mathrm{Hom}_{R^{\mathcal{O}}}(Y^\xi_{\mathcal{O}}\langle a_\xi \rangle, Y^\lambda_{\mathcal{O}}\langle a_\lambda \rangle) \\
&= \dim_q \mathrm{Hom}_{S^{\mathcal{O}}}(P^\xi_{\mathcal{O}}\langle a_\xi \rangle, P^\lambda_{\mathcal{O}}\langle a_\lambda \rangle) \\
&= q^{a_\lambda - a_\xi}[P^\lambda_{\mathcal{O}} : L^\xi_{\mathcal{O}}]_q \\
&= q^{a_\lambda - a_\xi} c^{\mathcal{O}}_{\lambda\xi}(q)
\end{aligned}$$

Here the second and the sixth equalities follow from the proof of Lemma 3.2.15, the third and the fifth equalities follow from Lemmas 3.2.30 and 3.3.16, the fourth equality follows from the discussion after Lemma 3.3.38. So we have also $c_{\xi\lambda}(q) = q^{a_\xi - a_\lambda} c^{\mathcal{O}}_{\xi\lambda}(q)$. But the matrix $c(q)$ is symmetric by Corollary 3.3.19. Thus, for each λ, ξ in $\mathcal{P}^l_{\mathbf{d}}$ we have $c_{\lambda\xi}(q) = q^{2(a_\xi - a_\lambda)} c_{\lambda\xi}(q)$. Thus $a_\lambda = a_\xi$ whenever $c_{\lambda\xi}(q) \neq 0$. To conclude it is enough to note that for each $\lambda, \xi \in \mathcal{P}^l_{\mathbf{d}}$ we can find a chain $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_n, \xi \in \mathcal{P}^l_{\mathbf{d}}$ such that

$$c_{\lambda\lambda_1}(q) \neq 0, c_{\lambda_1\lambda_2}(q) \neq 0, \cdots, c_{\lambda_{n-1}\lambda_n}(q) \neq 0, c_{\lambda_n\xi}(q) \neq 0$$

because the algebra $S^{\mathbf{s}}_{\mathbf{d}}$ is indecomposable, see Lemma 3.2.28. $\square$

Theorem 3.3.40. *There exists a graded algebra isomorphism* ${}^{b}S^{\mathbf{s}}_{\mathbf{d}} \simeq S^{\mathcal{O}}$.

Proof. We have a chain of graded algebra isomorphisms

$$\begin{aligned}
({}^{b}S^{\mathbf{s}}_{\mathbf{d}})^{\mathrm{op}} &\simeq \bigoplus_{\lambda,\xi\in\mathcal{P}^{l}_{\mathbf{d}}} \mathrm{Hom}_{S^{\mathbf{s}}_{\mathbf{d}}}(P^{\xi}, P^{\lambda}) \\
&\simeq \bigoplus_{\lambda,\xi\in\mathcal{P}^{l}_{\mathbf{d}}} \mathrm{Hom}_{R^{\mathbf{s}}_{\mathbf{d}}}(Y^{\xi}, Y^{\lambda}) \\
&\simeq \bigoplus_{\lambda,\xi\in\mathcal{P}^{l}_{\mathbf{d}}} \mathrm{Hom}_{R^{\mathcal{O}}}(Y^{\xi}_{\mathcal{O}}\langle a_{\xi}\rangle, Y^{\lambda}_{\mathcal{O}}\langle a_{\lambda}\rangle) \\
&\simeq \bigoplus_{\lambda,\xi\in\mathcal{P}^{l}_{\mathbf{d}}} \mathrm{Hom}_{S^{\mathcal{O}}}(P^{\xi}_{\mathcal{O}}, P^{\lambda}_{\mathcal{O}}) \\
&\simeq (S^{\mathcal{O}})^{\mathrm{op}}.
\end{aligned}$$

$\square$

Chapitre 4

KLR algebras and categorical representations

4.1 Introduction

Let O^{ν}_{-e} be the parabolic category $\mathcal{O}$ of the affine Kac-Moody algebra of $\mathfrak{gl}_N$ at level $-N-e$ with parabolic type ν. In [53], a categorical representation of the affine Kac-Moody algebra $\widetilde{\mathfrak{sl}}_e$ in O^{ν}_{-e} is considered. This means that there are exact functors $E_i, F_i \colon O^{\nu}_{-e} \to O^{\nu}_{-e}$ for $i \in [0, e-1]$ which induce a representation of the Lie algebra $\widetilde{\mathfrak{sl}}_e$ on the Grothendieck group $[O^{\nu}_{-e}]$ of O^{ν}_{-e}. The definition of a categorical representation is given in Section 4.3.4. The category O^{ν}_{-e} admits a decomposition

$$O^{\nu}_{-e} = \bigoplus_{\mu \in \mathbb{Z}^e} O^{\nu}_{\mu}$$

that lifts the decomposition of the $\widetilde{\mathfrak{sl}}_e$-module $[O^{\nu}_{-e}]$ in a direct sum of weight spaces.

The category O^{ν}_{μ} is Koszul by [56]. Its Koszul dual is a parabolic category ${}^{+}O^{\mu}_{\nu}$ at level $e-N$ defined similarly to O^{μ}_{ν}. In particular, the Koszul duality exchanges the parameter ν (*the parabolic type*) with the parameter μ (*the singular type*). The Koszul duality yields an equivalence of bounded derived categories $D(O^{\nu}_{\mu}) \simeq D({}^{+}O^{\mu}_{\nu})$. More details about the Koszul duality can be found in [5].

Let $\alpha_0, \cdots, \alpha_{e-1}$ be the simple root of $\widetilde{\mathfrak{sl}}_e$. We have

$$E_i(O^{\nu}_{\mu}) \subset O^{\nu}_{\mu+\alpha_i}, \quad F_i(O^{\nu}_{\mu}) \subset O^{\nu}_{\mu-\alpha_i}.$$

The aim of this paper is to identify the Koszul dual functors

$$D({}^{+}O^{\mu}_{\nu}) \to D({}^{+}O^{\mu+\alpha_i}_{\nu}), \quad D({}^{+}O^{\mu}_{\nu}) \to D({}^{+}O^{\mu-\alpha_i}_{\nu})$$

to the functors

$$E_i\colon D(O^\nu_\mu) \to D(O^\nu_{\mu+\alpha_i}), \quad F_i\colon D(O^\nu_\mu) \to D(O^\nu_{\mu-\alpha_i}).$$

Let $\overline{O}^\nu_\mu$, $\overline{E}_i$, $\overline{F}_i$ be defined in the same way as O^ν_μ, E_i, F_i with e replaced by $e+1$. Let $\overline{\alpha}_0, \cdots, \overline{\alpha}_e$ be the simple roots of $\widetilde{\mathfrak{sl}}_{e+1}$. Fix $k \in [0, e-1]$. For an e-tuple $\mu = (\mu_1, \cdots, \mu_l)$ we set

$$\overline{\mu} = \begin{cases} (\mu_1, \cdots, \mu_k, 0, \mu_{k+1}, \cdots, \mu_e) & \text{if } k \neq 0, \\ (0, \mu_1, \cdots, \mu_e) & \text{if } k = 0. \end{cases}$$

Note that we have $\overline{(\mu - \alpha_k)} = \overline{\mu} - \overline{\alpha}_k - \overline{\alpha}_{k+1}$.

By [19] there is an equivalence of categories $\theta\colon O^\nu_\mu \to \overline{O}^\nu_{\overline{\mu}}$. The direct sum of such equivalences identifies the category O^ν_{-e} with a direct factor of the category $\overline{O}^\nu_{-e-1}$. We want to compare the $\widetilde{\mathfrak{sl}}_e$-action on O^ν_{-e} with the $\widetilde{\mathfrak{sl}}_{e+1}$-action on $\overline{O}^\nu_{-e-1}$. More precisely, we want to prove the following conjecture.

Conjecture 4.1.1. *The following diagram of functors is commutative.*

$$\begin{tikzcd} \overline{O}^\nu_{\overline{\mu}} \arrow[r, "\overline{F}_k"] & \overline{O}^\nu_{\overline{\mu}-\overline{\alpha}_k} \arrow[r, "\overline{F}_{k+1}"] & \overline{O}^\nu_{\overline{\mu}-\overline{\alpha}_k-\overline{\alpha}_{k+1}} \arrow[d, "\theta^{-1}"] \\ O^\nu_\mu \arrow[u, "\theta"] \arrow[rr, "F_k"'] & & O^\nu_{\mu-\alpha_k} \end{tikzcd}$$

Our motivation is the following. Assume that the conjecture holds. One can prove as in [43] that the functor $\overline{F}_k$ is Koszul dual to a parabolic inclusion functor and $\overline{F}_{k+1}$ to a parabolic truncation functor. Then we can deduce that F_k is Koszul dual to a Zuckerman functor (which is the composition of a parabolic inclusion and a parabolic truncation).

It is not hard to see that the diagram from Conjecture 4.1.1 is commutative at the level of Grothendieck groups. In the case of the category $\mathcal{O}$ of $\mathfrak{gl}_N$ (instead of affine $\mathfrak{gl}_N$) this is enough to prove the analogue of Conjecture 4.1.1, using the theory of projective functors. Indeed, [6, Thm. 3.4] implies that two projective functors are isomorphic if their actions on the Grothendieck group coincide. Unfortunately, there is no satisfactory theory of projective functors for the affine case (an attempt to develop such a theory was given in [21]).

We choose another strategy to prove this conjecture. The idea is to relate the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$. This can be done using the following inclusion of Lie algebras $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [0, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$f_r \mapsto \begin{cases} f_r & \text{if } r \in [0, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases}$$

First, we recall the notion of a categorical representation. Let $\mathbf{k}$ be a field. Let $\mathcal{C}$ be an abelian Hom-finite $\mathbf{k}$-linear category that admits a direct sum decomposition $\mathcal{C} = \bigoplus_{\mu \in \mathbb{Z}^e} \mathcal{C}_\mu$. A categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $\mathcal{C}$ is a pair of biadjoint functor $E_i, F_i \colon \mathcal{C} \to \mathcal{C}$ for $i \in [0, e-1]$ satisfying a list of axioms. The main axiom is that for each $d \in \mathbb{N}$ there is an algebra homomorphism $R_d \to \mathrm{End}(F^d)^{\mathrm{op}}$, where R_d is the KLR algebra of rank d associated with the quiver $A^{(1)}_{e-1}$.

Now we explain our main result about categorical representations. Let $\overline{\mathcal{C}}$ be an abelian Hom-finite $\mathbf{k}$-linear category. Assume that $\overline{\mathcal{C}} = \bigoplus_{\mu \in \mathbb{Z}^{e+1}} \overline{\mathcal{C}}_\mu$ has a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$ with respect to functors $\overline{E}_i, \overline{F}_i$ for $i \in [0, e]$. Assume additionally that the subcategory $\overline{\mathcal{C}}_\mu$ is zero whenever μ has a negative entry. For each e-tuple $\mu \in \mathbb{N}^e$ we consider the $(e+1)$-tuple $\overline{\mu}$ as above and we set $\mathcal{C}_\mu = \overline{\mathcal{C}}_{\overline{\mu}}$,

$$\mathcal{C} = \bigoplus_{\mu \in \mathbb{N}^e} \mathcal{C}_\mu.$$

Next, consider the endofunctors of $\mathcal{C}$ given by

$$F_0 = \overline{F}_0\big|_{\mathcal{C}}, \cdots, F_{k-1} = \overline{F}_{k-1}\big|_{\mathcal{C}}, \quad F_k = \overline{F}_{k+1}\overline{F}_k\big|_{\mathcal{C}},$$

$$F_{k+1} = \overline{F}_{k+2}\big|_{\mathcal{C}}, \cdots, F_{e-1} = \overline{F}_e\big|_{\mathcal{C}},$$

$$E_0 = \overline{E}_0\big|_{\mathcal{C}}, \cdots, E_{k-1} = \overline{E}_{k-1}\big|_{\mathcal{C}}, \quad E_k = \overline{E}_k\overline{E}_{k+1}\big|_{\mathcal{C}},$$

$$E_{k+1} = \overline{E}_{k+2}\big|_{\mathcal{C}}, \cdots, E_{e-1} = \overline{E}_e\big|_{\mathcal{C}}.$$

The following theorem holds.

Theorem 4.1.2. *The category $\mathcal{C}$ has a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with respect to the functors $E_0, \cdots, E_{e-1}$, $F_0, \cdots, F_{e-1}$.*

The proof of Theorem 4.1.2 can be reduced to comparing the KLR algebras associated with the quivers of types $A^{(1)}_{e-1}$, $A^{(1)}_e$, denoted by Γ, $\overline{\Gamma}$ respectively. This is done in Section 4.2.

Let $\alpha = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} d_i \alpha_i$ be a dimension vector of the quiver Γ. We consider the dimension vector $\overline{\alpha}$ of $\overline{\Gamma}$ defined in the following way

$$\overline{\alpha} = \sum_{i=0}^{k} d_i \alpha_i + \sum_{i=k+1}^{e} d_{i-1} \alpha_i.$$

Let $R_\alpha(\Gamma)$ (resp. $R_{\overline{\alpha}}(\overline{\Gamma})$) be the KLR algebra associated with the quiver Γ and the dimension vector α (resp. the quiver $\overline{\Gamma}$ and the dimension vector

$\overline{\alpha}$). The algebra $R_{\overline{\alpha}}(\overline{\Gamma})$ contains an idempotent $e(\mathbf{i})$ for each sequence $\mathbf{i}$ of elements of $\mathbb{Z}/(e+1)\mathbb{Z}$ of dimension vector $\overline{\alpha}$. In Section 4.2.4 we consider some subsets of sequences $\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$ and $\overline{I}^{\overline{\alpha}}_{\mathrm{un}}$. Set $\mathbf{e}=\sum_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} e(\mathbf{i})\in R_{\overline{\alpha}}(\overline{\Gamma})$ and

$$S_{\overline{\alpha}}(\overline{\Gamma})=\mathbf{e}R_{\overline{\alpha}}(\overline{\Gamma})\mathbf{e}/\sum_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{un}}}\mathbf{e}R_{\overline{\alpha}}(\overline{\Gamma})e(\mathbf{i})R_{\overline{\alpha}}(\overline{\Gamma})\mathbf{e}.$$

We call $S_{\overline{\alpha}}(\overline{\Gamma})$ a *balanced KLR algebra*. The main result of Section 4.2 is the following theorem.

Theorem 4.1.3. *There is an isomorphism of algebras $R_\alpha(\Gamma)\simeq S_{\overline{\alpha}}(\overline{\Gamma})$.*

Our approach can not be used directly to prove Conjecture 4.1.1 because the $\widetilde{\mathfrak{sl}}_e$-module categorified by O^ν_{-e} is not simple. However, we can obtain a weaker version of Conjecture 4.1.1. The category O^ν_{-e} contains subcategories $\mathbf{A}^\nu[\alpha]$, parameterized by dimension vectors α of Γ. For $i\in\mathbb{Z}/e\mathbb{Z}$ we have $F_i(\mathbf{A}^\nu[\alpha])\subset\mathbf{A}^\nu[\alpha+\alpha_i]$. Let $\overline{\mathbf{A}}^\nu[\overline{\alpha}]$ be defined in the same way as $\mathbf{A}^\nu[\alpha]$ with respect to the parameter $e+1$ instead of e. Let $|\alpha|$ be the height of α. The main result of Section 4.3 is the following theorem.

Theorem 4.1.4. *Assume that $e>2$ and $\nu=(\nu_1,\cdots,\nu_l)$ satisfies $\nu_r>|\alpha|$ for each $r\in[1,l]$. There exists a dimension vector β for $\overline{\Gamma}$ such that for each dimension vector α for Γ there are equivalences of categories $\theta'_\alpha\colon\mathbf{A}^\nu[\alpha]\to\overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}]$ and $\theta'_{\alpha+\alpha_k}\colon\mathbf{A}^\nu[\alpha+\alpha_k]\to\overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}+\overline{\alpha}_k+\overline{\alpha}_{k+1}]$ such that the following diagram is commutative*

$$\begin{tikzcd}
\overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}] \arrow[r, "\overline{F}_{k+1}\overline{F}_k"] & \overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}+\overline{\alpha}_k+\overline{\alpha}_{k+1}] \\
\mathbf{A}^\nu[\alpha] \arrow[u, "\theta'_\alpha"] \arrow[r, "F_k"] & \mathbf{A}^\nu[\beta+\alpha+\alpha_k]. \arrow[u, "\theta'_{\alpha+\alpha_k}"']
\end{tikzcd}$$

□

4.2 KLR algebras and Hecke algebras

For a noetherian ring A we denote by $\mathrm{mod}(A)$ the abelian category of left finitely generated A-modules.

4.2.1 Kac-Moody algebras associated with a quiver

Let $\Gamma=(I,H)$ be a quiver without 1-loops with the set of vertices I and the set of arrows H. For $i,j\in I$ let $h_{i,j}$ be the number of arrows from i to j and set also $a_{i,j}=2\delta_{i,j}-h_{i,j}-h_{j,i}$. Let $\mathfrak{g}_I$ be the Kac-Moody algebra over $\mathbb{C}$ associated with the matrix $(a_{i,j})$. Denote by e_i, f_i for $i\in I$ the Serre generators of $\mathfrak{g}_I$.

For each $i \in I$ let α_i, $\check{\alpha}_i$ be the simple root and coroot corresponding to e_i and let Λ_i be the fundamental weight. Set

$$Q_I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad Q_I^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i, \quad P_I = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i, \quad P_I^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i.$$

Let X_I be the free abelian group with basis $\{\varepsilon_i;\ i \in I\}$. Set also

$$X_I^+ = \bigoplus_{i \in I} \mathbb{N}\varepsilon_i. \tag{4.1}$$

For $\alpha \in Q_I^+$ denote by $|\alpha|$ its height. Set $I^\alpha = \{\mathbf{i} = (i_1, \cdots, i_{|\alpha|}) \in I^{|\alpha|};\ \sum_{r=1}^{|\alpha|} \alpha_{i_r} = \alpha\}$.

Let $\Gamma_\infty = (I_\infty, H_\infty)$ be the quiver with the set of vertices $I_\infty = \mathbb{Z}$ and the set of arrows $H_\infty = \{i \to i+1;\ i \in I_\infty\}$. Assume that $e > 1$ is an integer. Let $\Gamma_e = (I_e, H_e)$ be the quiver with the set of vertices $I_e = \mathbb{Z}/e\mathbb{Z}$ and the set of arrows $H_e = \{i \to i+1;\ i \in I_e\}$. Then $\mathfrak{g}_{I_e}$ is the Lie algebra $\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$.

Assume that $\Gamma = (I, H)$ is a quiver whose connected components are of the form Γ_e, with $e \in \mathbb{N}$, $e > 1$ or $e = \infty$. For $i \in I$ denote by $i+1$ and $i-1$ the (unique) vertices in I such that there are arrows $i \to i+1$, $i-1 \to i$. Let us also consider the following additive map

$$\iota \colon Q_I \to X_I, \quad \alpha_i \mapsto \varepsilon_i - \varepsilon_{i+1}.$$

Fix a formal variable δ and set $X_I^\delta = X_I \oplus \mathbb{Z}\delta$. We can lift the $\mathbb{Z}$-linear map ι to a $\mathbb{Z}$-linear map

$$\iota^\delta \colon Q_I \to X_I^\delta, \quad \alpha_i \mapsto \varepsilon_i - \varepsilon_{i+1} - \delta.$$

Note that the map $\iota^\delta \colon Q_I \to X_I^\delta$ is injective (while ι is not injective). We may omit the symbols ι, ι^δ and write α instead of $\iota(\alpha)$ or $\iota^\delta(\alpha)$.

We will sometimes abbreviate

$$Q_e = Q_{I_e}, \quad X_e = X_{I_e}, \quad X_e^\delta = X_{I_e}^\delta, \quad P_e = P_{I_e}.$$

4.2.2 Doubled quiver

Let $\Gamma = (I, H)$ be a quiver without 1-loops. Fix a decomposition $I = I_0 \sqcup I_1$ such that there are no arrows between the vertices in I_1. In this section we define a *doubled quiver* $\overline{\Gamma} = (\overline{I}, \overline{H})$ associated with (Γ, I_0, I_1). The idea is to "double" each vertex in the set I_1 (we do not touch the vertices from I_0). We replace each vertex $i \in I_1$ by a couple of vertices i^1 and i^2 with an arrow $i^1 \to i^2$. Each arrow entering to i should be replaced by an arrow entering to i^1, each arrow coming from i should be replaced by an arrow coming from i^2.

Now we describe the construction of $\overline{\Gamma} = (\overline{I}, \overline{H})$ formally. Let $\overline{I}_0$ be a set that is in bijection with I_0. Let i^0 be the element of $\overline{I}_0$ associated with an element $i \in I_0$. Similarly, let $\overline{I}_1$ and $\overline{I}_2$ be sets that are in bijection with I_1. Denote by i^1 and i^2 the element of $\overline{I}_1$ and $\overline{I}_2$ respectively that correspond to an element $i \in I_1$. Put $\overline{I} = \overline{I}_0 \sqcup \overline{I}_1 \sqcup \overline{I}_2$. We define $\overline{H}$ in the following way. The set $\overline{H}$ contains 4 types of arrows:

- an arrow $i^0 \to j^0$ for each arrow $i \to j$ in H with $i, j \in I_0$,
- an arrow $i^0 \to j^1$ for each arrow $i \to j$ in H with $i \in I_0, j \in I_1$,
- an arrow $i^2 \to j^0$ for each arrow $i \to j$ in H with $i \in I_1, j \in I_0$,
- an arrow $i^1 \to i^2$ for each vertex $i \in I_1$.

Set $I^\infty = \coprod_{d\in\mathbb{N}} I^d$, $\overline{I}^\infty = \coprod_{d\in\mathbb{N}} \overline{I}^d$, where I^d, $\overline{I}^d$ are the cartesian products. The concatenation yields a monoid structure on I^∞ and $\overline{I}^\infty$. Let $\phi\colon I^\infty \to \overline{I}^\infty$ be the unique morphism of monoids such that for $i \in I \subset I^\infty$ we have

$$\phi(i) = \begin{cases} i^0 & \text{if } i \in I_0, \\ (i^1, i^2) & \text{if } i \in I_1. \end{cases}$$

There is a unique $\mathbb{Z}$-linear map $\phi\colon Q_I \to Q_{\overline{I}}$ such that $\phi(I^\alpha) \subset I^{\phi(\alpha)}$ for each $\alpha \in Q_I^+$. It is given by

$$\phi(\alpha_i) = \begin{cases} \alpha_{i^0} & \text{if } i \in I_0, \\ \alpha_{i^1} + \alpha_{i^2} & \text{if } i \in I_1. \end{cases} \tag{4.2}$$

Let ϕ denote also the unique additive embedding

$$\phi\colon X_I \to X_{\overline{I}}, \quad \varepsilon_i \mapsto \varepsilon_{i'}, \tag{4.3}$$

where

$$i' = \begin{cases} i^0 & \text{if } i \in I_0, \\ i^1 & \text{if } i \in I_1. \end{cases} \tag{4.4}$$

4.2.3 KLR algebras

Let $\mathbf{k}$ be a field. Let $\Gamma = (I, H)$ be a quiver without 1-loops. For $r \in [1, d-1]$ let s_r be the transposition $(r, r+1) \in \mathfrak{S}_d$. For $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$ set $s_r(\mathbf{i}) = (i_1, \cdots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \cdots, i_d)$. For $i, j \in I$ we set

$$Q_{i,j}(u,v) = \begin{cases} 0 & \text{if } i = j, \\ (v-u)^{h_{i,j}}(u-v)^{h_{j,i}} & \text{else.} \end{cases}$$

Definition 4.2.1. The *KLR-algebra* $R_{d,\mathbf{k}}(\Gamma)$ is the $\mathbf{k}$-algebra with the set of generators $\tau_1, \cdots, \tau_{d-1}, x_1, \cdots, x_d, e(\mathbf{i})$ where $\mathbf{i} \in I^d$, modulo the following defining relations

- $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i})$,
- $\sum_{\mathbf{i}\in I^d} e(\mathbf{i}) = 1$,
- $x_r e(\mathbf{i}) = e(\mathbf{i}) x_r$,

- $\tau_r e(\mathbf{i}) = e(s_r(\mathbf{i}))\tau_r$,
- $x_r x_s = x_s x_r$,
- $\tau_r x_{r+1} e(\mathbf{i}) = (x_r \tau_r + \delta_{i_r,i_{r+1}})e(\mathbf{i})$,
- $x_{r+1}\tau_r e(\mathbf{i}) = (\tau_r x_r + \delta_{i_r,i_{r+1}})e(\mathbf{i})$,
- $\tau_r x_s = x_s \tau_r$, if $s \neq r, r+1$,
- $\tau_r \tau_s = \tau_s \tau_r$, if $|r-s| > 1$,
- $\tau_r^2 e(\mathbf{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ Q_{i_r,i_{r+1}}(x_r, x_{r+1})e(\mathbf{i}) & \text{else,} \end{cases}$
- $(\tau_r\tau_{r+1}\tau_r - \tau_{r+1}\tau_r\tau_{r+1})e(\mathbf{i}) =$
$\begin{cases} (x_{r+2}-x_r)^{-1}(Q_{i_r,i_{r+1}}(x_{r+2},x_{r+1}) - Q_{i_r,i_{r+1}}(x_r,x_{r+1}))e(\mathbf{i}) & \text{if } i_r = i_{r+2}, \\ 0 & \text{else,} \end{cases}$

for each $\mathbf{i}$, $\mathbf{j}$, r and s. We may write $R_{d,\mathbf{k}} = R_{d,\mathbf{k}}(\Gamma)$. The algebra $R_{d,\mathbf{k}}$ admits a $\mathbb{Z}$-grading such that $\deg e(\mathbf{i}) = 0$, $\deg x_r = 2$, $\deg \tau_s e(\mathbf{i}) = -a_{i_s,i_{s+1}}$, for each $1 \leqslant r \leqslant d$, $1 \leqslant s < d$ and $\mathbf{i} \in I^d$.

For each $\alpha \in Q_I^+$ such that $|\alpha| = d$ consider the following central idempotent $e(\alpha) = \sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) \in R_{d,\mathbf{k}}$. It is a homogeneous idempotent of degree zero. We have the following decomposition into a sum of unitary $\mathbf{k}$-algebras $R_{d,\mathbf{k}} = \bigoplus_{|\alpha|=d} R_{\alpha,\mathbf{k}}$, where $R_{\alpha,\mathbf{k}} = e(\alpha)R_{d,\mathbf{k}}$.

Let $\mathbf{k}_d^{(I)}$ be the direct sum of copies of the ring $\mathbf{k}_d[x] := \mathbf{k}[x_1, \cdots, x_d]$ labelled by I^d. We write

$$\mathbf{k}_d^{(I)} = \bigoplus_{\mathbf{i}\in I^d} \mathbf{k}_d[x]e(\mathbf{i}), \tag{4.5}$$

where $e(\mathbf{i})$ is the idempotent of the ring $\mathbf{k}_d^{(I)}$ projecting to the component $\mathbf{i}$. A polynomial in $\mathbf{k}_d[x]$ can be considered as an element of $\mathbf{k}_d^{(I)}$ via the diagonal inclusion. For each $i, j \in I$ fix a polynomial $P_{i,j}(u,v)$ such that we have $Q_{i,j}(u,v) = P_{i,j}(u,v)P_{j,i}(v,u)$. Then by [50, Sec. 3.2] the algebra $R_{d,\mathbf{k}}$ has a representation in the vector space $\mathbf{k}_d^{(I)}$ such that the element $e(\mathbf{i})$ acts by the projection to $\mathbf{k}_d^{(I)}e(\mathbf{i})$, the element x_r acts by multiplication by x_r and such that for $f \in \mathbf{k}_d[x]$ we have

$$\tau_r \cdot fe(\mathbf{i}) = \begin{cases} (x_r - x_{r+1})^{-1}(s_r(f) - f)e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ P_{i_r,i_{r+1}}(x_{r+1}, x_r)s_r(f)e(s_r(\mathbf{i})) & \text{otherwise.} \end{cases} \tag{4.6}$$

We will always choose $P_{i,j}$ in the following way:

$$P_{i,j}(u,v) = (u-v)^{h_{j,i}}.$$

Remark 4.2.2. There is an explicit construction of a basis of a KLR algebra. Assume $\mathbf{i}, \mathbf{j} \in I^\alpha$. Set $\mathfrak{S}_{\mathbf{i},\mathbf{j}} = \{w \in \mathfrak{S}_d;\ w(\mathbf{i}) = \mathbf{j}\}$. For each permutation $w \in \mathfrak{S}_{\mathbf{i},\mathbf{j}}$ fix a reduced expression $w = s_{p_1} \cdots s_{p_r}$ and set $\tau_w = \tau_{p_1} \cdots \tau_{p_r}$. Then the vector space $e(\mathbf{j})R_{\alpha,\mathbf{k}}e(\mathbf{i})$ has a basis $\{\tau_w x_1^{a_1} \cdots x_d^{a_d} e(\mathbf{i});\ w \in \mathfrak{S}_{\mathbf{i},\mathbf{j}}, a_1, \cdots, a_d \in \mathbb{N}\}$. Note that the element τ_w depends on the reduced expression of w. Moreover, if we change the reduced expression of w, then the element $\tau_w e(\mathbf{i})$ changes only by a linear combination of monomials of the form $\tau_{q_1}\cdots\tau_{q_t}x_1^{b_1}\cdots x_d^{b_d}e(\mathbf{i})$ with $t<\ell(w)$.

Fix a weight $\Lambda=\sum_{i\in I}n_i\Lambda_i\in P_I^+$.

Definition 4.2.3. The *cyclotomic KLR-algebra* $R_{\alpha,\mathbf{k}}^\Lambda$ is the quotient of $R_{\alpha,\mathbf{k}}$ by the two-sided ideal generated by $x_1^{n_{i_1}}e(\mathbf{i})$ for $\mathbf{i}=(i_1,\cdots,i_d)\in I^\alpha$.

For each $i\in I$ there is an inclusion $R_{\alpha,\mathbf{k}}\subset R_{\alpha+\alpha_i,\mathbf{k}}$ that takes $e(\mathbf{i})$ to $e(\mathbf{i},i)$, x_r to $x_re(\mathbf{i},i)$, τ_r to $\tau_re(\mathbf{i},i)$. It factors through a homomorphism $R_{\alpha,\mathbf{k}}^\Lambda\to R_{\alpha+\alpha_i,\mathbf{k}}^\Lambda$. Let

$$F_i^\Lambda\colon \mathrm{mod}(R_{\alpha,\mathbf{k}}^\Lambda)\to\mathrm{mod}(R_{\alpha+\alpha_i,\mathbf{k}}^\Lambda),\qquad E_i^\Lambda\colon \mathrm{mod}(R_{\alpha+\alpha_i,\mathbf{k}}^\Lambda)\to\mathrm{mod}(R_{\alpha,\mathbf{k}}^\Lambda)$$

be the induction and restriction functors with respect to this algebra homomorphism.

4.2.4 Balanced KLR algebras

Fix a decomposition $I=I_0\sqcup I_1$ as in Section 4.2.2 and consider the quiver $\overline{\Gamma}=(\overline{I},\overline{H})$ as in Section 4.2.2. Recall the decomposition $\overline{I}=\overline{I}_0\sqcup\overline{I}_1\sqcup\overline{I}_2$. In this section we work with the KLR algebra associated with the quiver $\overline{\Gamma}$.

We say that a sequence $\mathbf{i}=(i_1,i_2,\cdots,i_d)\in\overline{I}^d$ is *unordered* if there is an index $r\in[1,d]$ such that the number of elements from $\overline{I}_2$ in the sequence $(i_1,i_2,\cdots,i_r)$ is strictly greater than the number of elements from $\overline{I}_1$. We say that it is *well-ordered* if for each index a such that $i_a=i^1$ for some $i\in I_1$, we have $a<d$ and $i_{a+1}=i^2$. We denote by $\overline{I}^\alpha_{\mathrm{ord}}$ and $\overline{I}^\alpha_{\mathrm{un}}$ the subsets of well-ordered and unordered sequences in $\overline{I}^\alpha$ respectively.

Definition 4.2.4. For $\overline{\alpha}\in Q_{\overline{I}}^+$, the *balanced quotient* $\widetilde{S}_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})$ of $R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})$ is the algebra

$$R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})/\sum_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{un}}}R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})e(\mathbf{i})R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma}).$$

The map ϕ from Section 4.2.2 yields a bijection

$$\phi\colon Q_I^+\to\{\alpha=\sum_{i\in\overline{I}}d_i\alpha_i\in Q_{\overline{I}}^+;\ d_{i^1}=d_{i^2},\ \forall i\in I_1\},\quad \alpha\mapsto\overline{\alpha}.$$

Fix $\alpha\in Q_I^+$. Set $\mathbf{e}=\sum_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}}e(\mathbf{i})\in R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})$. We also denote by $\mathbf{e}$ the image of $\mathbf{e}$ in $\widetilde{S}_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})$.

Definition 4.2.5. For $\alpha\in Q_I^+$, the *balanced KLR algebra* is the algebra

$$S_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})=\mathbf{e}R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})\mathbf{e}/\sum_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{un}}}\mathbf{e}R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})e(\mathbf{i})R_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})\mathbf{e}.$$

In other words we have $S_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma}) = \mathbf{e}\widetilde{S}_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma})\mathbf{e}$. We may write $S_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma}) = S_{\overline{\alpha},\mathbf{k}}$.

Remark 4.2.6. Assume that $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$. Let a be an index such that $i_a \in \overline{I}_1$. We have the relation $\tau_a^2 e(\mathbf{i}) = (x_{a+1} - x_a)e(\mathbf{i})$ in $R_{\overline{\alpha},\mathbf{k}}$. Moreover, we have $\tau_a^2 e(\mathbf{i}) = \tau_a e(s_a(\mathbf{i}))\tau_a e(\mathbf{i})$ and $s_a(\mathbf{i})$ is unordered. Thus we have $x_a e(\mathbf{i}) = x_{a+1}e(\mathbf{i})$ in $S_{\overline{\alpha},\mathbf{k}}$.

4.2.5 The polynomial representation of $S_{\overline{\alpha},\mathbf{k}}$

We assume $\alpha = \sum_{i\in\overline{I}} d_i\alpha_i \in Q_I^+$. Let $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$. Denote by $J(\mathbf{i})$ the ideal of the polynomial ring $\mathbf{k}_d[x]e(\mathbf{i}) \subset \mathbf{k}_d^{(\overline{I})}$ generated by the set

$$\{(x_r - x_{r+1})e(\mathbf{i});\ i_r \in \overline{I}_1\}.$$

Lemma 4.2.7. *Assume that* $\mathbf{i} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$ *and* $\mathbf{j} \in \overline{I}^{\overline{\alpha}}_{\mathrm{un}}$. *Then each element of* $e(\mathbf{i})R_{\overline{\alpha},\mathbf{k}}e(\mathbf{j})$ *maps* $\mathbf{k}_d[x]e(\mathbf{j})$ *to* $J(\mathbf{i})$.

Proof. We will prove by induction that each monomial $\tau_{p_1}\cdots\tau_{p_k}$ such that the permutation $w = s_{p_1}\cdots s_{p_k} \in \mathfrak{S}_d$ satisfies $w(\mathbf{j}) = \mathbf{i}$ maps $\mathbf{k}_d[x]e(\mathbf{j})$ to $J(\mathbf{i})$.

Assume $k = 1$. Write $p = p_1$. Let us write $\mathbf{i} = (i_1, \cdots, i_d)$ and $\mathbf{j} = (j_1, \cdots, j_d)$. Then we have $\mathbf{i} = s_p(\mathbf{j})$. By assumptions we know that there exists $i \in I_1$ such that $i_p = j_{p+1} = i^1$, $i_{p+1} = j_p = i^2$. In this case the statement is obvious because τ_p maps $fe(\mathbf{j}) \in \mathbf{k}_d[x]e(\mathbf{j})$ to $(x_{p+1} - x_p)s_p(f)e(\mathbf{i})$ by (4.6).

Now consider a monomial $\tau_{p_1}\cdots\tau_{p_k}$ such that the permutation $w = s_{p_1}\cdots s_{p_k}$ satisfies $w(\mathbf{j}) = \mathbf{i}$ and assume that the statement is true for all such monomials of smaller length. By assumptions on $\mathbf{i}$ and $\mathbf{j}$ there is an index $r \in [1, d]$ such that $i_r = i^1$ for some $i \in I_1$ and $w^{-1}(r+1) < w^{-1}(r)$. Thus w has a reduced expression of the form $w = s_r s_{r_1}\cdots s_{r_h}$. This implies that $\tau_{p_1}\cdots\tau_{p_k}e(\mathbf{j})$ is equal to a monomial of the form $\tau_r\tau_{r_1}\cdots\tau_{r_h}e(\mathbf{j})$ modulo monomials of the form $\tau_{q_1}\cdots\tau_{q_t}x_1^{b_1}\cdots x_d^{b_d}e(\mathbf{j})$ with $t < k$, see Remark 4.2.2. As the sequence $s_r(\mathbf{i})$ is unordered, the case $k = 1$ and the induction hypothesis imply the statement. □

Lemma 4.2.8. *Assume that* $\mathbf{i}, \mathbf{j} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$. *Then each element of* $e(\mathbf{i})R_{\overline{\alpha},\mathbf{k}}e(\mathbf{j})$ *maps* $J(\mathbf{j})$ *into* $J(\mathbf{i})$.

Proof. Take $y \in \mathbf{e}(\mathbf{i})R_{\overline{\alpha},\mathbf{k}}e(\mathbf{j})$. We must prove that for each $r \in [1, d]$ such that $j_r = i^1$ for some $i \in I_1$ and each $f \in \mathbf{k}_d[x]$ we have $y((x_r - x_{r+1})fe(\mathbf{j})) \in J(\mathbf{i})$. We have $(x_r - x_{r+1})fe(\mathbf{j}) = -\tau_r^2(fe(\mathbf{i}))$. This implies

$$y((x_r - x_{r+1})fe(\mathbf{j})) = -y\tau_r^2(fe(\mathbf{j})) = -y\tau_r e(s_r(\mathbf{j}))(\tau_r(fe(\mathbf{j}))).$$

Thus Lemma 4.2.7 implies the statement because the sequence $s_r(\mathbf{j})$ is unordered. $\square$

The representation of $R_{\overline{\alpha},\mathbf{k}}$ on

$$\mathbf{k}_{\overline{\alpha}}^{(\overline{I})} := \bigoplus_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}} \mathbf{k}_{|\overline{\alpha}|}[x]e(\mathbf{i})$$

yields a representation of $\mathbf{e}R_{\overline{\alpha},\mathbf{k}}\mathbf{e}$ on

$$\mathbf{k}_{\overline{\alpha},\mathrm{ord}}^{(\overline{I})} := \bigoplus_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} \mathbf{k}_{|\overline{\alpha}|}[x]e(\mathbf{i}).$$

Set $J_{\overline{\alpha},\mathrm{ord}} = \bigoplus_{\mathbf{i}\in\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} J(\mathbf{i})$. From Lemmas 4.2.7, 4.2.8 we deduce the following.

Lemma 4.2.9. *The representation of $R_{\overline{\alpha},\mathbf{k}}$ on $\mathbf{k}_{\overline{\alpha}}^{(\overline{I})}$ factors through a representation of $S_{\overline{\alpha},\mathbf{k}}$ on $\mathbf{k}_{\overline{\alpha},\mathrm{ord}}^{(\overline{I})}/J_{\overline{\alpha},\mathrm{ord}}$. This representation is faithful.*

Proof. The faithfulness is proved in the proof of Theorem 4.2.11. $\square$

4.2.6 The comparison of the polynomial representations

Fix $\alpha \in Q_I^+$. Set $d = |\alpha|$ and $\overline{d} = |\overline{\alpha}|$. For each sequence $\mathbf{i} = (i_1, \cdots, i_d) \in I^\alpha$ and $r \in [1, d]$ we denote by r' or $r'_{\mathbf{i}}$ the positive integer such that $r' - 1$ is the length of the sequence $\phi(i_1, \cdots, i_{r-1}) \in \overline{I}^\infty$.

For $r \in [1, d]$ (resp. $r \in [1, d-1]$) consider the element $x_r^* \in S_{\overline{\alpha},\mathbf{k}}$ (resp. $\tau_r^* \in S_{\overline{\alpha},\mathbf{k}}$) such that for each $\mathbf{i} \in I^\alpha$ we have

$$x_r^* e(\phi(\mathbf{i})) = x_{r'} e(\phi(\mathbf{i})),$$

$$\tau_r^* e(\phi(\mathbf{i})) = \begin{cases} \tau_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \tau_{r'}\tau_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \tau_{r'+1}\tau_{r'} e(\phi(\mathbf{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1, \\ \tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r, i_{r+1} \in I_1, i_r \neq i_{r+1}, \\ -\tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r = i_{r+1} \in I_1. \end{cases}$$

For each $\mathbf{i} \in I^\alpha$ we have the algebra isomorphism

$$\mathbf{k}_d[x]e(\mathbf{i}) \simeq \mathbf{k}_{\overline{d}}[x]e(\phi(\mathbf{i}))/J(\phi(\mathbf{i})), \quad x_r e(\mathbf{i}) \mapsto x_{r'} e(\phi(\mathbf{i})).$$

We will always identify $\mathbf{k}_\alpha^{(I)}$ with $\mathbf{k}_{\overline{\alpha},\mathrm{ord}}^{(\overline{I})}/J_{\overline{\alpha},\mathrm{ord}}$ via this isomorphism.

Lemma 4.2.10. *The action of the elements $e(\mathbf{i})$, $x_r e(\mathbf{i})$, $\tau_r e(\mathbf{i})$ of $R_{\alpha,\mathbf{k}}$ on $\mathbf{k}_\alpha^{(I)}$ is the same as the action of the elements $e(\phi(\mathbf{i}))$, $x_r^* e(\phi(\mathbf{i}))$, $\tau_r^* e(\phi(\mathbf{i}))$ of $S_{\overline{\alpha},\mathbf{k}}$ on $\mathbf{k}_{\overline{\alpha},\mathrm{ord}}^{(\overline{I})}/J_{\overline{\alpha},\mathrm{ord}}$.*

Proof. The proof is based on the observation that by construction for each $i \in I_1$ and $j \in I_0$ we have

$$P_{i^1,j^0}(u,v)P_{i^2,j^0}(u,v) = P_{i,j}(u,v), \tag{4.7}$$

$$P_{j^0,i^1}(u,v)P_{j^0,i^2}(u,v) = P_{j,i}(u,v). \tag{4.8}$$

For each $\mathbf{i} \in I^\alpha$, we write $\phi(\mathbf{i}) = (i'_1, i'_2, \cdots, i'_{\overline{d}})$. The only difficult part concerns the operator $\tau_r e(\mathbf{i})$ when at least one of the elements i_r, i_{r+1} is in I_1. Assume that $i_r \in I_1$, $i_{r+1} \in I_0$. In this case we have

$$i'_{r'} = (i_r)^1 \in \overline{I}_1, \quad i'_{r'+1} = (i_r)^2 \in \overline{I}_2, \quad i'_{r'+2} = (i_{r+1})^0 \in \overline{I}_0.$$

In particular, the element $i'_{r'+2}$ is different from $i'_{r'}$ and $i'_{r'+1}$. Then, by (4.6), for each $f \in \mathbf{k}_{\overline{d}}[x]$ the element $\tau^*_r e(\phi(\mathbf{i})) = \tau_{r'}\tau_{r'+1}e(\phi(\mathbf{i}))$ maps $fe(\phi(\mathbf{i})) \in \mathbf{k}^{(\overline{I})}_{\overline{\alpha},\mathrm{ord}}/J_{\overline{\alpha},\mathrm{ord}}$ to

$$\begin{aligned} & P_{i'_{r'},i'_{r'+2}}(x_{r'+1},x_{r'})s_{r'}\left(P_{i'_{r'+1},i'_{r'+2}}(x_{r'+2},x_{r'+1})s_{r'+1}(f)\right)e(s_{r'}s_{r'+1}(\phi(\mathbf{i}))) = \\ = \;& P_{i'_{r'},i'_{r'+2}}(x_{r'+1},x_{r'})P_{i'_{r'+1},i'_{r'+2}}(x_{r'+2},x_{r'})s_{r'}s_{r'+1}(f)e(\phi(s_r(\mathbf{i}))) \\ = \;& P_{i_r,i_{r+1}}(x_{r'+1},x_{r'})s_{r'}s_{r'+1}(f)e(\phi(s_r(\mathbf{i}))), \end{aligned}$$

where the last equality holds by (4.7). Thus we see that the action of $\tau^*_r e(\phi(\mathbf{i}))$ on the polynomial representation is the same as the action of $\tau_r e(\mathbf{i})$. The case when $i_r \in I_0$, $i_{r+1} \in I_1$ can be done similarly.

Assume now that $i_r \neq i_{r+1}$ are both in I_1. By the assumption on the quiver Γ (see Section 4.2.2), there are no arrows in Γ between i_r and i_{r+1}. Thus there are no arrows in $\overline{\Gamma}$ between any of the vertices $(i_r)^1 = i'_{r'}$ or $(i_r)^2 = i'_{r'+1}$ and any of the vertices $(i_{r+1})^1 = i'_{r'+2}$ or $(i_{r+1})^2 = i'_{r'+3}$. Then, by (4.6), for each $f \in \mathbf{k}_{\overline{d}}[x]$ the element $\tau^*_r e(\mathbf{i}) = \tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\mathbf{i}))$ maps $fe(\phi(\mathbf{i}))$ to

$$s_{r'+1}s_{r'+2}s_{r'}s_{r'+1}(f)e(\phi(s_r(\mathbf{i}))).$$

Thus we see that the action of $\tau^*_r e(\phi(\mathbf{i}))$ on the polynomial representation is the same as the action of $\tau_r e(\mathbf{i})$.

Finally, assume that $i_r = i_{r+1} \in I_1$. In this case we have

$$(i_r)^1 = i'_{r'} = (i_{r+1})^1 = i'_{r'+2}, \qquad (i_r)^2 = i'_{r'+1} = (i_{r+1})^2 = i'_{r'+3}.$$

Then, by (4.6), for each $f \in \mathbf{k}_{\overline{d}}[x]$ the element $\tau^*_r e(\phi(\mathbf{i})) = -\tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\mathbf{i}))$ maps $fe(\phi(\mathbf{i}))$ to

$$s_{r'+1}\partial_{r'+2}\partial_{r'}(x_{r'+1} - x_{r'+2})s_{r'+1}(f)e(\phi(s_r(\mathbf{i}))).$$

To prove that this gives the same result as for $\tau_r e(\mathbf{i})$, it is enough to check this on monomials $x_r^n x_{r+1}^m e(\mathbf{i})$. Assume for simplicity that $n \geqslant m$. The situation $n \leqslant m$ can be treated similarly. The element $\tau_r e(\mathbf{i})$ maps this monomial to

$$\partial_r(x_r^n x_{r+1}^m)e(\mathbf{i}) = -\sum_{a=m}^{n-1} x_r^a x_{r+1}^{n+m-1-a} e(\mathbf{i}).$$

Here the symbol $\sum\limits_{a=x}^{y}$ means 0 when $y = x - 1$. The element $\tau_r^* e(\phi(\mathbf{i}))$ maps $x_{r'+1}^n x_{r'+2}^m e(\phi(\mathbf{i}))$ to

$$\begin{aligned}
& s_{r'+1}\partial_{r'+2}\partial_{r'}[x_{r'+1}^{m+1}x_{r'+2}^{n} - x_{r'+1}^{m}x_{r'+2}^{n+1}]e(\phi(\mathbf{i})) = \\
= \;& s_{r'+1}\Big(-(\textstyle\sum_{a=0}^{m} x_{r'}^{a}x_{r'+1}^{m-a})(\sum_{b=0}^{n-1} x_{r'+2}^{b}x_{r'+3}^{n-1-b}) \\
& + (\textstyle\sum_{a=0}^{m-1} x_{r'}^{a}x_{r'+1}^{m-1-a})(\sum_{b=0}^{n} x_{r'+2}^{b}x_{r'+3}^{n-b})\Big)\, e(\phi(\mathbf{i})) \\
= \;& \Big[-(\textstyle\sum_{a=0}^{m} x_{r'}^{a}x_{r'+2}^{m-a})(\sum_{b=0}^{n-1} x_{r'+1}^{b}x_{r'+3}^{n-1-b}) \\
& + (\textstyle\sum_{a=0}^{m-1} x_{r'}^{a}x_{r'+2}^{m-1-a})(\sum_{b=0}^{n} x_{r'+1}^{b}x_{r'+3}^{n-b})\Big]\, e(\phi(\mathbf{i})) \\
= \;& \Big[-x_{r'}^{m}(\textstyle\sum_{b=0}^{n-1} x_{r'+1}^{b}x_{r'+3}^{n-1-b}) \\
& + x_{r'+1}^{n}(\textstyle\sum_{a=0}^{m-1} x_{r'}^{a}x_{r'+2}^{m-1-a})\Big]\, e(\phi(\mathbf{i})) \\
= \;& \Big[-x_{r'+1}^{m}(\textstyle\sum_{b=0}^{n-1} x_{r'+1}^{b}x_{r'+2}^{n-1-b}) \\
& + x_{r'+1}^{n}(\textstyle\sum_{a=0}^{m-1} x_{r'+1}^{a}x_{r'+2}^{m-1-a})\Big]\, e(\phi(\mathbf{i})) \\
= \;& -\Big[\textstyle\sum_{a=m}^{n-1} x_{r'+1}^{a}x_{r'+2}^{m+n-1-a}\Big]\, e(\phi(\mathbf{i})).
\end{aligned}$$

□

4.2.7 Isomorphism Φ

Theorem 4.2.11. *For each $\alpha \in Q_I^+$, there is an isomorphism of algebras $\Phi_{\alpha,\mathbf{k}}\colon R_{\alpha,\mathbf{k}} \to S_{\overline{\alpha},\mathbf{k}}$ such that*

$$e(\mathbf{i}) \mapsto e(\phi(\mathbf{i})),$$

$$x_r e(\mathbf{i}) \mapsto x_r^* e(\phi(\mathbf{i})),$$

$$\tau_r e(\mathbf{i}) \mapsto \tau_r^* e(\phi(\mathbf{i})).$$

Proof. In view of Lemma 4.2.10, it is enough to prove two following facts:

- the elements $e(\phi(\mathbf{i}))$, x_r^*, τ_r^* generate $S_{\overline{\alpha},\mathbf{k}}$,
- the representation $\mathbf{k}_{\overline{\alpha},\mathrm{ord}}^{(\overline{I})}/J_{\overline{\alpha},\mathrm{ord}}$ of $S_{\overline{\alpha},\mathbf{k}}$ is faithful.

Fix $\mathbf{i},\mathbf{j} \in I^\alpha$. Set $\mathbf{i}' = (i'_1,\cdots,i'_d) = \phi(\mathbf{i})$, $\mathbf{j}' = \phi(\mathbf{j})$. Let $\mathbf{B}$ (resp. $\mathbf{B}'$) be the basis of $e(\mathbf{j})R_{\alpha,\mathbf{k}}e(\mathbf{i})$ (resp. $e(\mathbf{j}')R_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}')$) as in Remark 4.2.2. For each element $b = \tau_w x_1^{a_1}\cdots x_d^{a_d}e(\mathbf{i}) \in \mathbf{B}$ we construct an element $b^* \in e(\mathbf{j}')S_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}')$ that acts by the same operator on the polynomial representation. Recall that the definition of b depends on the choice of a reduced expression $w = s_{p_1}\cdots s_{p_k}$. We set

$$b^* = \tau_{p_1}^*\cdots\tau_{p_k}^*(x_1^*)^{a_1}\cdots(x_d^*)^{a_d}e(\mathbf{i}') \in e(\mathbf{j}')S_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}').$$

Let us call the permutation $w \in \mathfrak{S}_{\mathbf{i}',\mathbf{j}'}$ *balanced* if we have $w(a+1) = w(a)+1$ for each a such that $i'_a = i^1$ for some $i \in I$ (and thus $i'_{a+1} = i^2$).

Otherwise we say that w is *unbalanced*. There exists a unique map $u\colon \mathfrak{S}_{\mathbf{i},\mathbf{j}} \to \mathfrak{S}_{\mathbf{i}',\mathbf{j}'}$ such that for each $w \in \mathfrak{S}_{\mathbf{i},\mathbf{j}}$ the permutation $u(w)$ is balanced and $w(r) < w(t)$ if and only if $u(w)(r') < u(w)(t')$ for each $r, t \in [1, d]$, where $r' = r'_{\mathbf{i}}$ and $t' = t'_{\mathbf{i}}$ are as in Section 4.2.6. The image of u is exactly the set of all balanced permutations in $\mathfrak{S}_{\mathbf{i}',\mathbf{j}'}$.

Assume that $w \in \mathfrak{S}_{\mathbf{i}',\mathbf{j}'}$ is unbalanced. We claim that there exists an index a such that $i'_a \in \overline{I}_1$ and $w(a) > w(a+1)$. Really, let J be the set of indices $a \in [1, \overline{d}]$ such that $i'_a \in \overline{I}_1$. As $\mathbf{j}'$ is well-ordered, we have $\sum_{a \in J}(w(a+1) - w(a)) = \#J$. As w is unbalanced, not all summands in this sum are equal to 1. Then one of the summands must be negative. Let $a \in J$ be an index such that $w(a) > w(a+1)$. We can assume that the reduced expression of w is of the form $w = s_{p_1} \cdots s_{p_k} s_a$. In this case the element $\tau_w e(\mathbf{i}')$ is zero in $S_{\overline{\alpha},\mathbf{k}}$ because the idempotent $s_a(\mathbf{i}')$ is unordered.

Assume that $w \in \mathfrak{S}_{\mathbf{i}',\mathbf{j}'}$ is balanced. Thus there exists some $\widetilde{w} \in \mathfrak{S}_{\mathbf{i},\mathbf{j}}$ such that $u(\widetilde{w}) = w$. We choose an arbitrary reduced expression $\widetilde{w} = s_{p_1} \cdots s_{p_k}$ and we choose the reduced expression of $w = s_{q_1} \cdots s_{q_r}$ induced by the reduced expression of $\widetilde{w}$. Then the element $\tau_w e(\mathbf{i}') = \tau_{q_1} \cdots \tau_{q_r} e(\mathbf{i}') \in S_{\alpha,\mathbf{k}}$ is equal to $\pm(\tau_{p_1} \cdots \tau_{p_k} e(\mathbf{i}))^*$.

The discussion above shows that the image of an element $b' \in \mathbf{B}'$ in $e(\mathbf{j}')S_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}')$ is either zero or is of the form $\pm b^*$ for some $b \in \mathbf{B}$. Moreover, each b^* for $b \in \mathbf{B}$ can be obtained in such a way. Now we get the following.

- The elements $e(\phi(\mathbf{i}))$, x_r^*, τ_r^* generate $S_{\overline{\alpha},\mathbf{k}}$ because the image of each element of $\mathbf{B}'$ in $e(\mathbf{j}')S_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}')$ is either zero or a monomial on $e(\phi(\mathbf{i}))$, x_r^*, τ_r^*.
- The representation $\mathbf{k}^{(\overline{I})}_{\overline{\alpha},\mathrm{ord}}/J_{\overline{\alpha},\mathrm{ord}}$ of $S_{\overline{\alpha},\mathbf{k}}$ is faithful because the spanning set $\{b^*;\ b \in \mathbf{B}\}$ of $e(\mathbf{j}')S_{\overline{\alpha},\mathbf{k}}e(\mathbf{i}')$ acts on the polynomial representation by linearly independent operators (because the polynomial representation of $R_{\alpha,\mathbf{k}}$ is faithful).

□

The center of the algebra $R_{\alpha,\mathbf{k}}$ is the ring of symmetric polynomials $\mathbf{k}_d[x]^{\mathfrak{S}_d}$, see [50, Prop. 3.9]. Thus $S_{\overline{\alpha},\mathbf{k}}$ is a $\mathbf{k}_d[x]^{\mathfrak{S}_d}$-algebra under the isomorphism $\Phi_{\alpha,\mathbf{k}}$. Let Σ be the polynomial $\prod_{a<b}(x_a - x_b)^2 \in \mathbf{k}_d[x]^{\mathfrak{S}_d}$. Let $R_{\alpha,\mathbf{k}}[\Sigma^{-1}]$, $S_{\overline{\alpha},\mathbf{k}}[\Sigma^{-1}]$ be the rings of quotients of $R_{\alpha,\mathbf{k}}$, $S_{\overline{\alpha},\mathbf{k}}$ obtained by inverting Σ. We can extend the isomorphism $\Phi_{\alpha,\mathbf{k}}$ from Theorem 4.2.11 to an algebra isomorphism $\Phi_{\alpha,\mathbf{k}}\colon R_{\alpha,\mathbf{k}}[\Sigma^{-1}] \to S_{\overline{\alpha},\mathbf{k}}[\Sigma^{-1}]$.

From now on, we assume that the connected components of the quiver Γ are of the form Γ_a for $a \in \mathbb{N}$, $a > 1$ or $a = \infty$. Note that there is an action of the symmetric group $\mathfrak{S}_d$ on $\mathbf{k}_d^{(I)}$ permuting the variables and the components of $\mathbf{i}$. Consider the following elements in $R_{\alpha,\mathbf{k}}[\Sigma^{-1}]$:

$$\Psi_r e(\mathbf{i}) = \begin{cases} ((x_r - x_{r+1})\tau_r + 1)e(\mathbf{i}) & \text{if } i_{r+1} = i_r, \\ -(x_r - x_{r+1})^{-1}\tau_r e(\mathbf{i}) & \text{if } i_{r+1} = i_r - 1, \\ \tau_r e(\mathbf{i}) & \text{else.} \end{cases}$$

The element $\Psi_r e(\mathbf{i})$ is called *intertwining operator*. Using the formulas (4.6) we can check that $\Psi_r e(\mathbf{i})$ still acts on the polynomial representation and the corresponding operator is equal to $s_r e(\mathbf{i})$. Note also that $\widetilde{\Psi}_r = (x_r - x_{r+1})\Psi_r$ is an element of $R_{\alpha,\mathbf{k}}$.

Lemma 4.2.12. *The images of intertwining operators by the morphism $\Phi_{\alpha,\mathbf{k}}\colon R_{\alpha,\mathbf{k}} \to S_{\overline{\alpha},\mathbf{k}}$ can be described in the following way. For $\mathbf{i} \in I^\alpha$ such that $i_r - 1 \neq i_{r+1}$ we have*

$$\Phi_{\alpha,\mathbf{k}}(\Psi_r e(\mathbf{i})) = \begin{cases} \Psi_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \Psi_{r'}\Psi_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \Psi_{r'+1}\Psi_{r'} e(\phi(\mathbf{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1, \\ \Psi_{r'+1}\Psi_{r'+2}\Psi_{r'}\Psi_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r, i_{r+1} \in I_1. \end{cases}$$

For $\mathbf{i} \in I^\alpha$ such that $i_r - 1 = i_{r+1}$ we have

$$\Phi_{\alpha,\mathbf{k}}(\widetilde{\Psi}_r e(\mathbf{i})) = \begin{cases} \widetilde{\Psi}_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \widetilde{\Psi}_{r'}\Psi_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \Psi_{r'+1}\widetilde{\Psi}_{r'} e(\phi(\mathbf{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1. \end{cases}$$

Here $r' = r'_{\mathbf{i}}$ is as in Section 4.2.6.

Proof. The right hand side in the formulas for $\Phi_{\alpha,\mathbf{k}}(\Psi_r e(\mathbf{i}))$ is an element X in $S_{\overline{\alpha},\mathbf{k}}[\Sigma^{-1}]$ that acts by the same operator that $\Psi_r e(\mathbf{i})$ on the polynomial representation. However, since the polynomial representation of $S_{\overline{\alpha},\mathbf{k}}$ is faithful by Lemma 4.2.9, and since there is a well-defined element of $S_{\overline{\alpha},\mathbf{k}}$ which acts in the same way as X in this polynomial representation, we conclude that X makes sense as an element of $S_{\overline{\alpha},\mathbf{k}}$.

□

The goal of the rest of Section 4.2 is to obtain a deformed version of the isomorphism from Theorem 4.2.11 over some local ring R (see Section 4.2.9). More precisely, the localized Hecke algebra over a field is isomorphic to the localized KLR algebra (see Section 4.2.11). We want to construct an R-algebra homomorphism between a localized Hecke algebra and a balanced analogue of a localized Hecke algebra (see Section 4.2.12 for the choice of parameters and Section 4.2.13 for the definition of a balanced analogue of the cyclotomic Hecke algebra) such that this homomorphism is compatible with the localization of the homomorphism from Theorem 4.2.11 over the residue field of R and over its field of fractions.

4.2.8 Special quivers

From now on we will be interested only in some special types of quivers.

First, consider the quiver $\Gamma = \Gamma_e$, where e is an integer > 1. In particular, from now on we fix $I = \mathbb{Z}/e\mathbb{Z}$. Fix $k \in [0, e-1]$ and set $I_1 = \{k\}$, $I_0 =$

$I\backslash\{k\}$. In this case the quiver $\overline{\Gamma}$ is isomorphic to Γ_{e+1}. More precisely, the decomposition $\overline{I} = \overline{I}_0 \sqcup \overline{I}_2 \sqcup \overline{I}_2$ is such that $\overline{I}_1 = \{k\}$, $\overline{I}_2 = \{k+1\}$. To avoid confusion, for $i \in \overline{I}$ we will write $\overline{\alpha}_i$, $\overline{\varepsilon}_i$ and $\overline{\Lambda}_i$ for α_i, ε_i and Λ_i respectively.

Remark 4.2.13. If Γ is as above, a sequence $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^d$ is well-ordered if for each index a such that $i_a = k$ we have $a < d$ and $i_{a+1} = k+1$. The sequence $\mathbf{i}$ is unordered if there is $r \leqslant d$ such that the subsequence $(i_1, \cdots, i_r)$ contains more elements equal to $k+1$ than elements equal to k.

Let $\Upsilon\colon \mathbb{Z} \to \mathbb{Z}$ be the map given for $a \in \mathbb{Z}, b \in [0, e-1]$ by

$$\Upsilon(ae+b) = \begin{cases} a(e+1)+b & \text{if } b \in [0,k], \\ a(e+1)+b+1 & \text{if } b \in [k+1, e-1]. \end{cases} \tag{4.9}$$

Now, consider the quiver $\widetilde{\Gamma} = (\Gamma_\infty)^{\oplus l}$. Set $\widetilde{\Gamma} = (\widetilde{I}, \widetilde{H})$ and write $\widetilde{\alpha}_i$, $\widetilde{\varepsilon}_i$ and $\widetilde{\Lambda}_i$ for α_i, ε_i and Λ_i respectively for each $i \in \widetilde{I}$. We identify an element of $\widetilde{I}$ with an element $(a,b) \in I \times [1,l]$ in the obvious way. Consider the decomposition $\widetilde{I} = \widetilde{I}_0 \sqcup \widetilde{I}_1$ such that $(a,b) \in \widetilde{I}_1$ if and only if $a \equiv k \bmod e$. In this case the quiver $\overline{\widetilde{\Gamma}}$ is isomorphic to $\widetilde{\Gamma}$. More precisely, in this case we have

$$\begin{aligned} (a,b)^0 &= (\Upsilon(a), b), \\ (a,b)^1 &= (\Upsilon(a), b), \\ (a,b)^2 &= (\Upsilon(a)+1, b). \end{aligned}$$

To distinguish notations, we will always write $\widetilde{\phi}$ for any of the maps $\widetilde{\phi}\colon \widetilde{I}^\infty \to \widetilde{I}^\infty$, $Q_{\widetilde{I}} \to Q_{\widetilde{I}}$, $X_{\widetilde{I}} \to X_{\widetilde{I}}$ in Section 4.2.2.

From now on we write $\Gamma = \Gamma_e$, $\overline{\Gamma} = \Gamma_{e+1}$ and $\widetilde{\Gamma} = (\Gamma_\infty)^{\oplus l}$. Recall that

$$I = I_e = \mathbb{Z}/e\mathbb{Z}, \quad \overline{I} = I_{e+1} = \mathbb{Z}/(e+1)\mathbb{Z}, \quad \widetilde{I} = (I_\infty)^{\oplus l} = \mathbb{Z} \times [1,l].$$

Consider the quiver homomorphism $\pi_e\colon \widetilde{\Gamma} \to \Gamma$ such that

$$\pi_e\colon \widetilde{I} \to I,\ (a,b) \mapsto a \bmod e.$$

Then π_{e+1} is a quiver homomorphism $\pi_{e+1}\colon \widetilde{\Gamma} \to \overline{\Gamma}$. They yield $\mathbb{Z}$-linear maps

$$\pi_e\colon Q_{\widetilde{I}} \to Q_I, \quad \pi_e\colon X_{\widetilde{I}} \to X_I, \quad \pi_{e+1}\colon Q_{\widetilde{I}} \to Q_{\overline{I}}, \quad \pi_{e+1}\colon X_{\widetilde{I}} \to X_{\overline{I}}.$$

The following diagrams are commutative for $\alpha \in Q_I^+$, $\widetilde{\alpha} \in Q_{\widetilde{I}}^+$ such that $\pi_e(\widetilde{\alpha}) = \alpha$,

$$\begin{array}{ccc} Q_{\widetilde{I}} & \xrightarrow{\widetilde{\phi}} & Q_{\widetilde{I}} \\ {\scriptstyle\pi_e}\downarrow & & \downarrow{\scriptstyle\pi_{e+1}} \\ Q_I & \xrightarrow{\phi} & Q_{\overline{I}} \end{array} \qquad \begin{array}{ccc} X_{\widetilde{I}} & \xrightarrow{\widetilde{\phi}} & X_{\widetilde{I}} \\ {\scriptstyle\pi_e}\downarrow & & \downarrow{\scriptstyle\pi_{e+1}} \\ X_I & \xrightarrow{\phi} & X_{\overline{I}} \end{array} \qquad \begin{array}{ccc} \widetilde{I}^{\widetilde{\alpha}} & \xrightarrow{\widetilde{\phi}} & \widetilde{I}^{\widetilde{\phi}(\widetilde{\alpha})} \\ {\scriptstyle\pi_e}\downarrow & & \downarrow{\scriptstyle\pi_{e+1}} \\ I^\alpha & \xrightarrow{\phi} & \overline{I}^{\phi(\alpha)} \end{array}$$

4.2.9 Deformation rings

In this section we introduce some general definitions from [53] for a later use.

We call *deformation ring* $(R, \kappa, \tau_1, \cdots, \tau_l)$ a regular commutative noetherian $\mathbb{C}$-algebra R with 1 equipped with a homomorphism $\mathbb{C}[\kappa, \tau_1, \cdots, \tau_l] \to R$. Let $\kappa, \tau_1, \cdots, \tau_r$ denote also the images of $\kappa, \tau_1, \cdots, \tau_r$ in R. A deformation ring is *in general position* if any two elements of the set

$$\{\tau_u - \tau_v + a\kappa + b, \kappa - c;\ a, b \in \mathbb{Z}, c \in \mathbb{Q}, u \neq v\}$$

have no common non-trivial divisors. A *local deformation ring* is a deformation ring which is a local ring and such that $\tau_1, \cdots, \tau_l, \kappa - e$ belong to the maximal ideal of R. Note that each $\mathbb{C}$-algebra that is a field has a *trivial* local deformation ring structure, i.e., such that $\tau_1 = \cdots = \tau_l = 0$ and $\kappa = e$. We always consider $\mathbb{C}$ as a local deformation ring with a trivial deformation ring structure. A $\mathbb{C}$-algebra R is called *analytic* if it is a localization of the ring of germs of holomorphic functions on some compact polydisc $D \subset \mathbb{C}^d$ for some $d \geqslant 1$.

We will write $\overline{\kappa} = \kappa(e+1)/e$, $\overline{\tau}_r = \tau_r(e+1)/e$. We will abbreviate R for $(R, \kappa, \tau_1, \cdots, \tau_l)$ and $\overline{R}$ for $(R, \overline{\kappa}, \overline{\tau}_1, \cdots, \overline{\tau}_l)$.

Let R be a local analytic deformation ring with residue field $\mathbf{k}$. Consider the elements $q_e = \exp(2\pi\sqrt{-1}/\kappa)$ and $q_{e+1} = \exp(2\pi\sqrt{-1}/\overline{\kappa})$ in R. These elements specialize to $\zeta_e = \exp(2\pi\sqrt{-1}/e)$ and $\zeta_{e+1} = \exp(2\pi\sqrt{-1}/(e+1))$ in $\mathbf{k}$.

4.2.10 Hecke algebras

Let R be a commutative ring with 1. Fix an element $q \in R$.

Definition 4.2.14. The *affine Hecke algebra* $H_{R,d}(q)$ is the R-algebra generated by $T_1, \cdots, T_{d-1}$ and the invertible elements $X_1, \cdots, X_d$ modulo the following defining relations

$$\begin{array}{ll} T_rT_s = T_sT_r \text{ if } |r-s| > 1, & T_rT_{r+1}T_r = T_{r+1}T_rT_{r+1} \\ (T_r - q)(T_r + 1) = 0, & T_rX_r = X_rT_r \ \text{ if } |r-s| > 1, \\ T_rX_{r+1} = X_rT_r + (q-1)X_{r+1}, & T_rX_r = X_{r+1}T_r - (q-1)X_{r+1}. \end{array}$$

Fix elements $Q_1, \cdots, Q_l \in R$.

Definition 4.2.15. The *cyclotomic Hecke algebra* $H^Q_{d,R}(q)$ is the quotient of $H_{d,R}(q)$ by the two-sided ideal generated by $(X_1 - Q_1) \cdots (X_1 - Q_l)$.

Assume that $R = \mathbf{k}$ is a field and $q \neq 0, 1$, $Q_r \neq 0$. The algebra $H_{d,\mathbf{k}}(q)$ has a faithful representation in the vector space $\mathbf{k}[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$ such that $X_r^{\pm 1}$ acts by multiplication by $X_r^{\pm 1}$ and T_r by

$$T_r(P) = qs_r(P) + (q-1)X_{r+1}(X_r - X_{r+1})^{-1}(s_r(P) - P).$$

The following operator acts in $\mathbf{k}[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$ as the reflection s_r

$$\Psi_r = \frac{X_r - X_{r+1}}{qX_r - X_{r+1}}(T_r - q) + 1 = (T_r + 1)\frac{X_r - X_{r+1}}{X_r - qX_{r+1}} - 1.$$

For a future use, consider the element $\widetilde{\Psi}_r \in H_{d,\mathbf{k}}$ given by

$$\widetilde{\Psi}_r = (qX_r - X_{r+1})\Psi_r = (X_r - X_{r+1})T_r + (q-1)X_{r+1}.$$

Now, consider the subset $\mathscr{F} = \mathscr{F}(q, Q)$ of $\mathbf{k}$ given by

$$\mathscr{F}(q, Q) = \bigcup_{r \in \mathbb{Z}, t \in [1,l]} \{q^r Q_t\}. \tag{4.10}$$

We can consider $\mathscr{F}$ as the vertex set of a quiver with an arrow $i \to j$ if and only if $j = qi$. Let M be a finite dimensional $H_{d,\mathbf{k}}^Q(q)$-module. For each $\mathbf{i} = (i_1, \cdots, i_d) \in \mathscr{F}^d$ let $M_\mathbf{i}$ be the generalized eigenspace of $X_1, \cdots, X_d$ with eigenvalues $i_1, \cdots, i_d$. The algebra $H_{d,\mathbf{k}}^Q(q)$ contains an idempotent $e(\mathbf{i})$ which acts on each finite dimensional $H_{d,\mathbf{k}}^Q(q)$-module M by projection to $M_\mathbf{i}$ with respect to $\bigoplus_{\mathbf{j} \neq \mathbf{i}} M_\mathbf{j}$. For $\alpha \in Q_\mathscr{F}^+$ we consider the central idempotent $e(\alpha) = \sum_{\mathbf{i} \in \mathscr{F}^\alpha} e(\mathbf{i})$. The algebra $H_{d,\mathbf{k}}^Q$ decomposes in a direct sum of blocks $H_{\alpha,\mathbf{k}}^Q$ with $\alpha \in Q_\mathscr{F}^+$ where $H_{\alpha,\mathbf{k}}^Q = e(\alpha)H_{d,\mathbf{k}}^Q$. See [8] for more details.

4.2.11 The isomorphism between Hecke and KLR algebras

Assume that $R = \mathbf{k}$ is a field and $q \neq 0, 1$.

First, we define some localized versions of Hecke algebras and KLR algebras. Let $\mathscr{F}$ be a subset of $\mathbf{k}$ such that $q^\mathbb{Z}\mathscr{F}/q^\mathbb{Z}$ is finite. We view $\mathscr{F}$ as the vertex set of a quiver with an arrow $i \to j$ if and only if $j = qi$. Consider the algebra

$$A_1 = \bigoplus_{\mathbf{i} \in \mathscr{F}^d} \mathbf{k}[X_1^{\pm 1}, \cdots, X_d^{\pm 1}][(X_r - X_t)^{-1}, (qX_r - X_t)^{-1};\ r \neq t]e(\mathbf{i}),$$

where $e(\mathbf{i})$ are orthogonal idempotents and X_r commutes with $e(\mathbf{i})$. Let $H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ be the A_1-module given by the extension of scalars from the $\mathbf{k}[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$-module $H_{d,\mathbf{k}}(q)$. It is an A_1-algebra such that

$$T_r e(\mathbf{i}) - e(s_r(\mathbf{i}))T_r = (1-q)X_{r+1}(X_r - X_{r+1})^{-1}(e(\mathbf{i}) - e(s_r(\mathbf{i})))$$

and

$$Z^{-1}T_r = T_r Z^{-1}, \qquad \text{where } Z = \prod_{r<t}(X_r - X_t)^2 \prod_{r \neq t}(qX_r - X_t)^2.$$

In this section the KLR algebras are always defined with respect to the quiver $\mathscr{F}$. We consider the algebra

$$A_2 = \bigoplus_{\mathbf{i}\in\mathscr{F}^d} \mathbf{k}[x_1, \cdots, x_d][S_{\mathbf{i}}^{-1}]e(\mathbf{i}),$$

where

$$S_{\mathbf{i}} = \{(x_r+1), (i_r(x_r+1) - i_t(x_t+1)), (qi_r(x_r+1) - i_t(x_t+1);\ r \neq t)\}.$$

There is a $\mathbf{k}$-algebra structure on $R_{d,\mathbf{k}}^{\mathrm{loc}} = A_2 \otimes_{\mathbf{k}_d^{(\mathscr{F})}} R_{d,\mathbf{k}}$, where $\mathbf{k}_d^{(\mathscr{F})}$ is as in (4.5).

The polynomial representations of $H_{d,\mathbf{k}}(q)$ and $R_{d,\mathbf{k}}$ yield faithful representations of $H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ and $R_{d,\mathbf{k}}^{\mathrm{loc}}$ on A_1 and A_2 respectively. Moreover, there is an isomorphism of $\mathbf{k}$-algebras $A_2 \simeq A_1$ given by $x_r e(\mathbf{i}) \mapsto (i_r^{-1}X_r - 1)e(\mathbf{i})$. This yields the following proposition.

Proposition 4.2.16. *There is an isomorphism of $\mathbf{k}$-algebras $R_{d,\mathbf{k}}^{\mathrm{loc}} \simeq H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ such that*

$$e(\mathbf{i}) \mapsto e(\mathbf{i}),$$

$$x_r e(\mathbf{i}) \mapsto (i_r^{-1}X_r - 1)e(\mathbf{i}),$$

$$\Psi_r e(\mathbf{i}) \mapsto \Psi_r e(\mathbf{i}).$$

□

Now, we consider the subalgebra $\widehat{R}_{d,\mathbf{k}}$ of $R_{d,\mathbf{k}}^{\mathrm{loc}}$ generated by

- the elements of $R_{d,\mathbf{k}}$,
- the elements $(x_r+1)^{-1}$,
- the elements of the form $(i_r(x_r+1) - i_t(x_t+1))^{-1}e(\mathbf{i})$ such that $r \neq t$, $i_r \neq i_t$,
- the elements of the form $(qi_r(x_r+1) - i_t(x_t+1))^{-1}e(\mathbf{i})$ such that $r \neq t$, $qi_r \neq i_t$.

Similarly, consider the subalgebra $\widehat{H}_{d,\mathbf{k}}(q)$ of $H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ generated by

- the elements of $H_{d,\mathbf{k}}(q)$,
- the elements of the form $(X_r - X_t)^{-1}e(\mathbf{i})$ such that $r \neq t$, $i_r \neq i_t$,
- the elements of the form $(qX_r - X_t)^{-1}e(\mathbf{i})$ such that $r \neq t$, $qi_r \neq i_t$.

Note that the element $\Psi_r e(\mathbf{i}) \in H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ belongs to $\widehat{H}_{d,\mathbf{k}}(q)$ if $i_r \neq qi_{r+1}$. We have the following lemma, see also [50, Sec. 3.2].

Proposition 4.2.17. *The isomorphism $R_{d,\mathbf{k}}^{\mathrm{loc}} \simeq H_{d,\mathbf{k}}^{\mathrm{loc}}(q)$ from Proposition 4.2.16 restricts to an isomorphism $\widehat{R}_{d,\mathbf{k}} \simeq \widehat{H}_{d,\mathbf{k}}(q)$.*

Fix $Q = (Q_1, \cdots, Q_l)$, $Q_r \in \mathbf{k}$, $Q_r \neq 0$. Let $\mathscr{F} = \mathscr{F}(q, Q) \subset \mathbf{k}$ be as in (4.10). Consider the weight $\Lambda = \sum_{r=1}^{l} \Lambda_{Q_r} \in P_{\mathscr{F}}^{+}$.

Corollary 4.2.18. *There is an isomorphism between the cyclotomic KLR algebra $R^\Lambda_{d,\mathbf{k}}$ associated with the quiver $\mathscr{F}$ and the cyclotomic Hecke algebra $H^Q_{d,\mathbf{k}}(q)$. Moreover, this isomorphism takes the block $R^\Lambda_{\alpha,\mathbf{k}}$ to the block $H^Q_{\alpha,\mathbf{k}}(q)$.*

Proof. The cyclotomic quotients $R_{d,\mathbf{k}} \to R^\Lambda_{d,\mathbf{k}}$ and $H_{d,\mathbf{k}}(q) \to H^Q_{d,\mathbf{k}}(q)$ factor through

$$\begin{tikzcd} & \widehat{R}_{d,\mathbf{k}} \arrow[dr] & \\ R_{d,\mathbf{k}} \arrow[ur] \arrow[rr] & & R^\Lambda_{d,\mathbf{k}}, \end{tikzcd} \qquad \begin{tikzcd} & \widehat{H}_{d,\mathbf{k}}(q) \arrow[dr] & \\ H_{d,\mathbf{k}}(q) \arrow[ur] \arrow[rr] & & H^Q_{d,\mathbf{k}}(q) \end{tikzcd}$$

because the elements $(i_r(x_r+1) - i_r(x_r+1))e(\mathbf{i})$, for $r \neq t$, $i_r \neq i_t$, etc., are invertible in $e(\mathbf{i})R^\Lambda_{d,\mathbf{k}}e(\mathbf{i})$ (or resp. $e(\mathbf{i})H^Q_{d,\mathbf{k}}(q)e(\mathbf{i})$). Thus the isomorphism $\widehat{R}_{d,\mathbf{k}} \simeq \widehat{H}_{d,\mathbf{k}}(q)$ yields an isomorphism $R^\Lambda_{d,\mathbf{k}} \simeq H^Q_{d,\mathbf{k}}(q)$. $\square$

4.2.12 The choice of the parameters

Let R be a local deformation ring with residue field $\mathbf{k}$ and field of fractions K. Fix a tuple $\nu = (\nu_1, \cdots, \nu_l) \in \mathbb{Z}^l$. Put $q = q_e$ and $Q_r = \exp(2\pi\sqrt{-1}(\nu_r + \tau_r)/\kappa)$ for $r \in [1, l]$. The canonical morphism $R \to \mathbf{k}$ maps q_e to ζ_e and Q_r to $\zeta_e^{\nu_r}$. Let $H^\nu_{d,R}(q_e)$ be the cyclotomic Hecke algebra over R with parameters q, Q_r. Set $H^\nu_{d,\mathbf{k}}(\zeta_e) = \mathbf{k} \otimes_R H^\nu_{d,R}(q_e)$ and $H^\nu_{d,K}(q_e) = K \otimes_R H^\nu_{d,R}(q_e)$.

Fix $k \in [0, e-1]$. To this k we associate a map $\Upsilon \colon \mathbb{Z} \to \mathbb{Z}$ as in (4.9). Consider the tuple

$$\overline{\nu} = (\overline{\nu}_1, \cdots, \overline{\nu}_l) \in \mathbb{Z}^l, \qquad \overline{\nu}_r = \Upsilon(\nu_r)\ \forall r \in [1, l]. \tag{4.11}$$

Consider the cyclotomic Hecke algebra $H^{\overline{\nu}}_{d,\overline{R}}(q_{e+1})$ with parameters $q = q_{e+1}$, $\overline{Q} = (\overline{Q}_1, \cdots, \overline{Q}_l)$, where $\overline{Q}_r = \exp(2\pi\sqrt{-1}(\overline{\nu}_r + \overline{\tau}_r)/\overline{\kappa})$ and $\overline{\kappa}$, $\overline{\tau}_r$ are defined in Section 4.2.9. Set

$$H^{\overline{\nu}}_{d,\overline{\mathbf{k}}}(\zeta_{e+1}) = \overline{\mathbf{k}} \otimes_{\overline{R}} H^{\overline{\nu}}_{d,\overline{R}}(q_{e+1}), \quad H^{\overline{\nu}}_{d,\overline{K}}(q_{e+1}) = \overline{K} \otimes_{\overline{R}} H^{\overline{\nu}}_{d,\overline{R}}(q_{e+1}).$$

4.2.13 The deformation of the homomorphism $\Phi_{\alpha,\mathbf{k}}$

Let $\Gamma = (I, H)$, $\overline{\Gamma} = (\overline{I}, \overline{H})$ and $\widetilde{\Gamma} = (\widetilde{I}, \widetilde{H})$ be as in Section 4.2.8. We have the following isomorphisms of quivers

$$\widetilde{I} \simeq \mathscr{F}(q_e, Q), \quad i = (a, b) \mapsto p_i := \exp(2\pi\sqrt{-1}(a + \tau_b)/\kappa), \tag{4.12}$$

$$\widetilde{I} \simeq \mathscr{F}(q_{e+1}, \overline{Q}), \quad i = (a, b) \mapsto \overline{p}_i := \exp(2\pi\sqrt{-1}(a + \overline{\tau}_b)/\overline{\kappa}), \tag{4.13}$$

where $\mathscr{F}(q,Q)$ is as in (4.10), Q and $\overline{Q}$ are as in Section 4.2.12, $a \in \mathbb{Z}$, $b \in [1,l]$. Notice that $Q_r = p_{(\nu_r,r)}$ and $\overline{Q}_r = \overline{p}_{(\overline{\nu}_r,r)}$ for each r.

The map $R \to \mathbf{k}$ takes $\mathscr{F}(q_e,Q)$ to $\mathscr{F}(\zeta_e,Q)$ and $\mathscr{F}(q_{e+1},\overline{Q})$ to $\mathscr{F}(\zeta_{e+1},\overline{Q})$. We have the following isomorphisms of quivers

$$I \simeq \mathscr{F}(\zeta_e,Q), \quad i \mapsto p_i := \zeta_e^i, \tag{4.14}$$

$$\overline{I} \simeq \mathscr{F}(\zeta_{e+1},\overline{Q}), \quad i \mapsto \overline{p}_i := \zeta_{e+1}^i. \tag{4.15}$$

These isomorphisms yield the following commutative diagrams

$$\begin{tikzcd} \widetilde{I} \arrow[r, "\sim"] \arrow[d, "\pi_e"'] & \mathscr{F}(q_e,Q) \arrow[d] \\ I \arrow[r, "\sim"] & \mathscr{F}(\zeta_e,Q), \end{tikzcd} \qquad \begin{tikzcd} \widetilde{I} \arrow[r, "\sim"] \arrow[d, "\pi_{e+1}"'] & \mathscr{F}(q_{e+1},\overline{Q}) \arrow[d] \\ \overline{I} \arrow[r, "\sim"] & \mathscr{F}(\zeta_{e+1},\overline{Q}). \end{tikzcd}$$

We will identify

$$I \simeq \mathscr{F}(\zeta_e,Q), \quad \overline{I} \simeq \mathscr{F}(\zeta_{e+1},\overline{Q}), \quad \widetilde{I} \simeq \mathscr{F}(q_e,Q), \quad \widetilde{I} \simeq \mathscr{F}(q_{e+1},\overline{Q}) \tag{4.16}$$

as above.

Recall from Section 4.2.10 that the cyclotomic Hecke algebra $H^\nu_{d,\mathbf{k}}(\zeta_e)$ contains some idempotents $e(\mathbf{i})$ for $\mathbf{i} \in I^d$. These idempotents lift to idempotents in the R-algebra $H^\nu_{d,R}(q_e)$ because this algebra is free over R by [41, Thm. 2.2] and R is a local algebra, see, e.g., [14, Ex. 6.16]. We also denote these idempotents $e(\mathbf{i})$.

By base change we have the K-algebra $H^\nu_{d,K}(q_e)$ which contains some idempotents $e(\mathbf{j}) \in H^\nu_{d,K}(q_e)$ for $\mathbf{j} \in \widetilde{I}^d$. The idempotent $e(\mathbf{i}) \in H^\nu_{d,R}(q_e)$ decomposes in $H^\nu_{d,K}(q_e)$ in the following way

$$e(\mathbf{i}) = \sum_{\mathbf{j} \in \widetilde{I}^d, \pi_e(\mathbf{j})=\mathbf{i}} e(\mathbf{j}).$$

We have the decompositions

$$H^\nu_{d,R}(q_e) = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} H^\nu_{\alpha,R}(q_e),$$

$$H^\nu_{d,K}(q_e) = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} H^\nu_{\alpha,K}(q_e),$$

where

$$H^\nu_{\alpha,K}(q_e) = \bigoplus_{\widetilde{\alpha} \in Q_{\widetilde{I}}^+, \pi_e(\widetilde{\alpha})=\alpha} H^\nu_{\widetilde{\alpha},K}(q_e).$$

Our goal is to obtain an analogue of Theorem 4.2.11 over the ring R. First, consider the algebras $\widehat{H}_{d,\mathbf{k}}(\zeta_e)$ and $\widehat{H}_{d,K}(q_e)$ defined in the same way as in Section 4.2.11 with respect to the sets $\mathscr{F}(\zeta_e, Q) \subset \mathbf{k}$ and $\mathscr{F}(q_e, Q) \subset K$, where Q is as in Section 4.2.12. We can consider the R-algebra $\widehat{H}_{d,R}(q_e)$ defined in a similar way with respect to the same set of idempotents as $\widehat{H}_{d,\mathbf{k}}(q_e)$. Sometimes we will write $\widehat{H}^{\nu}_{d,\mathbf{k}}(\zeta_e)$, $\widehat{H}^{\nu}_{d,\mathbf{k}}(q_e)$ and $\widehat{H}^{\nu}_{d,K}(q_e)$ to specify the parameter ν (the l-tuple Q in Section 4.2.12 depends on the l-tuple ν). Note that the algebra $\widehat{H}_d$ (over $\mathbf{k}$, R or K) admits the same block decomposition as the block decomposition for the cyclotomic Hecke algebra H^{ν}_d described above.

Now, we define the algebra $\widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$ that is a Hecke analogue of a localization of the balanced KLR algebra $S_{\overline{\alpha},\mathbf{k}}$. To do so, consider the idempotent $\mathbf{e} = \sum_{\mathbf{i}\in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} e(\mathbf{i})$ in $\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$. We set

$$\widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(q_{e+1}) = \mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e}/\sum_{\mathbf{j}\in \overline{I}^{\overline{\alpha}}_{\mathrm{un}}} \mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})e(\mathbf{j})\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e}.$$

We define the K-algebra $\widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1})$ as a similar quotient of $\mathbf{e}H^{\overline{\nu}}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e}$ by the two-sided ideal generated by $\{e(\mathbf{j});\ \mathbf{j} \in \widetilde{I}^{\overline{\alpha}}_{\mathrm{un}})\}$. Finally, we define the R-algebra $\widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$ as the image in $\widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1})$ of the following composition of homomorphisms

$$\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1}).$$

Remark 4.2.19. By Proposition 4.2.17 we have algebra isomorphisms

$$\widehat{R}_{\alpha,\mathbf{k}}(\Gamma) \simeq \widehat{H}_{\alpha,\mathbf{k}}(\zeta_e), \quad \widehat{R}_{\alpha,K}(\widetilde{\Gamma}) \simeq \widehat{H}_{\alpha,K}(q_e),$$

$$\widehat{R}_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma}) \simeq \widehat{H}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1}), \quad \widehat{R}_{\overline{\alpha},K}(\widetilde{\Gamma}) \simeq \widehat{H}_{\overline{\alpha},\overline{K}}(q_{e+1}),$$

from which we deduce that

$$\widehat{S}_{\overline{\alpha},\mathbf{k}}(\overline{\Gamma}) \simeq \widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1}), \quad \widehat{S}_{\overline{\alpha},K}(\widetilde{\Gamma}) \simeq \widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1}).$$

We may use these isomorphisms without mentioning them explicitly. Using the identifications above between KLR algebras and Hecke algebras, a localization of the isomorphism in Theorem 4.2.11 yields an isomorphism

$$\Phi_{\alpha,\mathbf{k}}\colon \widehat{H}_{\alpha,\mathbf{k}}(\zeta_e) \to \widehat{SH}_{\overline{\alpha},\mathbf{k}}(\zeta_{e+1}). \tag{4.17}$$

In the same way we also obtain an algebra isomorphism

$$\Phi_{\widetilde{\alpha},K}\colon \widehat{H}_{\widetilde{\alpha},K}(q_e) \to \widehat{SH}_{\widetilde{\phi}(\widetilde{\alpha}),K}(q_{e+1})$$

for each $\widetilde{\alpha} \in Q^+_{\widetilde{I}}$. Taking the sum over all $\widetilde{\alpha} \in Q^+_{\widetilde{I}}$ such that $\pi_e(\widetilde{\alpha}) = \alpha$ yields a isomorphism

$$\Phi_{\alpha,K}\colon \widehat{H}_{\alpha,K}(q_e) \to \widehat{SH}_{\overline{\alpha},K}(q_{e+1}). \tag{4.18}$$

Lemma 4.2.20. *The homomorphism* $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})\mathbf{e}$ *factors through a homomorphism* $\widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1}) \to \widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$.

Proof. In Section 4.2.5 we constructed a faithful polynomial representation of $S_{\overline{\alpha},\mathbf{k}}$. Let us call it $\mathcal{P}ol_{\mathbf{k}}$. It is constructed as a quotient of the standard polynomial representation of $\mathbf{e}R_{\overline{\alpha},\mathbf{k}}\mathbf{e}$. After localization we get a faithful representation $\widehat{\mathcal{P}ol}_{\mathbf{k}}$ of $\widehat{S}_{\overline{\alpha},\mathbf{k}}$. Thus the kernel of the algebra homomorphism $\mathbf{e}\widehat{R}_{\overline{\alpha},\mathbf{k}}\mathbf{e} \to \widehat{S}_{\overline{\alpha},\mathbf{k}}$ is the annihilator of the representation $\widehat{\mathcal{P}ol}_{\mathbf{k}}$. We can transfer this to the Hecke side and we obtain that the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$ is the annihilator of the representation $\widehat{\mathcal{P}ol}_{\mathbf{k}}$. Similarly, we can characterize the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1})$ as the annihilator of a similar representation $\widehat{\mathcal{P}ol}_{K}$.

The K-vector space $\widehat{\mathcal{P}ol}_K$ has an R-submodule $\widehat{\mathcal{P}ol}_R$ stable by the action of $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e}$ such that $\mathbf{k}\otimes_R \widehat{\mathcal{P}ol}_R = \widehat{\mathcal{P}ol}_{\mathbf{k}}$ and it is compatible with the algebra homomorphism $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})\mathbf{e}$. By definition of $\widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$ and the discussion above, the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$ is formed by the elements that act by zero on $\widehat{\mathcal{P}ol}_K$. Thus each element of this kernel acts by zero on $\widehat{\mathcal{P}ol}_R$. This implies, that an element of the kernel of $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$ specializes to an element of the kernel of $\mathbf{e}\widehat{H}^{\overline{\nu}}_{\overline{\alpha},\overline{\mathbf{k}}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$. This proves the statement. □

Lemma 4.2.21. *There is a unique algebra homomorphism* $\Phi_{\alpha,R}\colon \widehat{H}_{\alpha,R}(q_e) \to \widehat{SH}_{\overline{\alpha},R}(q_{e+1})$ *such that the following diagram is commutative*

$$\begin{tikzcd}
\widehat{H}_{\alpha,\mathbf{k}}(\zeta_e) \arrow[r, "\Phi_{\alpha,\mathbf{k}}"] & \widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1}) \\
\widehat{H}_{\alpha,R}(q_e) \arrow[r, "\Phi_{\alpha,R}"] \arrow[u] \arrow[d] & \widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1}) \arrow[u] \arrow[d] \\
\widehat{H}_{\alpha,K}(q_e) \arrow[r, "\Phi_{\alpha,K}"] & \widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1}).
\end{tikzcd}$$

Proof. First we consider the algebras $H^{\mathrm{loc}}_{\alpha,\mathbf{k}}(\zeta_e)$, $H^{\mathrm{loc}}_{\alpha,R}(q_e)$ and $H^{\mathrm{loc}}_{\alpha,K}(q_e)$ obtained from $\widehat{H}_{\alpha,\mathbf{k}}(\zeta_e)$, $\widehat{H}_{\alpha,R}(q_e)$ and $\widehat{H}_{\alpha,K}(q_e)$ by inverting

- $(X_r - X_t)$, $(\zeta_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$, $(q_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$, $(q_e X_r - X_t)$ with $r \neq t$

respectively. Let $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{\mathbf{k}}}$ and $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}$ be the localizations of $\widehat{SH}_{\overline{\alpha},\overline{\mathbf{k}}}$ and $\widehat{SH}_{\overline{\alpha},\overline{K}}$ such that the isomorphisms $\Phi_{\alpha,\mathbf{k}}$ and $\Phi_{\alpha,K}$ above induce isomorphisms

$$\Phi_{\alpha,\mathbf{k}}\colon H^{\mathrm{loc}}_{\alpha,\mathbf{k}}(\zeta_e) \to SH^{\mathrm{loc}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$$

$$\Phi_{\alpha,K}\colon H^{\mathrm{loc}}_{\alpha,K}(q_e) \to SH^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}(q_{e+1}).$$

Let $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{R}}$ be the image in $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}$ of the following composition of homomorphisms

$$\mathbf{e}H^{\mathrm{loc}}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \mathbf{e}H^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} \to SH^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}(q_{e+1}).$$

Next, we want to prove that there exists an algebra homomorphism $\Phi_{\alpha,R}\colon H^{\mathrm{loc}}_{\alpha,R}(q_e) \to SH^{\mathrm{loc}}_{\overline{\alpha},\overline{R}}(q_{e+1})$ such that the following diagram is commutative:

$$\begin{tikzcd}
H^{\mathrm{loc}}_{\alpha,\mathbf{k}}(\zeta_e) \arrow[r, "\Phi_{\alpha,\mathbf{k}}"] & SH^{\mathrm{loc}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1}) \\
H^{\mathrm{loc}}_{\alpha,R}(q_e) \arrow[r, "\Phi_{\alpha,R}"] \arrow[u] \arrow[d] & SH^{\mathrm{loc}}_{\overline{\alpha},\overline{R}}(q_{e+1}) \arrow[u] \arrow[d] \\
H^{\mathrm{loc}}_{\alpha,K}(q_e) \arrow[r, "\Phi_{\alpha,K}"] & SH^{\mathrm{loc}}_{\overline{\alpha},\overline{K}}(q_{e+1}).
\end{tikzcd} \tag{4.19}$$

We just need to check that the map $\Phi_{\alpha,K}$ takes an element of $H^{\mathrm{loc}}_{\alpha,R}(q_e)$ to an element of $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{R}}(q_{e+1})$ and that it specializes to the map $\Phi_{\alpha,\mathbf{k}}\colon H^{\mathrm{loc}}_{\alpha,\mathbf{k}}(\zeta_e) \to SH^{\mathrm{loc}}_{\overline{\alpha},\overline{\mathbf{k}}}(\zeta_{e+1})$. We will check this on the generators $e(\mathbf{i})$, $X_r e(\mathbf{i})$ and $\Psi_r e(\mathbf{i})$ of $H^{\mathrm{loc}}_{\alpha,R}(q_e)$.

This is obvious for the idempotents $e(\mathbf{i})$.

Let us check this for $X_r e(\mathbf{i})$. Assume that $\mathbf{i} \in I^\alpha$ and $\mathbf{j} \in \widetilde{I}^{|\alpha|}$ are such that we have $\pi_e(\mathbf{j}) = \mathbf{i}$. Write $\mathbf{i}' = \phi(\mathbf{i})$ and $\mathbf{j}' = \widetilde{\phi}(\mathbf{j})$. Set $r' = r'_{\mathbf{j}} = r'_{\mathbf{i}}$, see the notation in Section 4.2.6. By Theorem 4.2.11 and Proposition 4.2.16 we have

$$\Phi_{\alpha,K}(X_r e(\mathbf{j})) = \overline{p}^{-1}_{j'_{r'}} p_{j_r} X_{r'} e(\mathbf{j}').$$

Since, $\overline{p}^{-1}_{j'_{r'}} p_{j_r}$ depends only on $\mathbf{i}$ and r and $e(\mathbf{i}) = \sum_{\pi_e(\mathbf{j})=\mathbf{i}} e(\mathbf{j})$, we deduce that

$$\Phi_{\alpha,K}(X_r e(\mathbf{i})) = \overline{p}^{-1}_{j'_{r'}} p_{j_r} X_{r'} e(\mathbf{i}').$$

Thus the element $\Phi_{\alpha,K}(X_r e(\mathbf{i}))$ is in $SH^{\mathrm{loc}}_{\overline{\alpha},R}$ and its image in $SH^{\mathrm{loc}}_{\overline{\alpha},\mathbf{k}}$ is $\overline{p}^{-1}_{i'_{r'}} p_{i_r} X_{r'} e(\mathbf{i}') = \Phi_{\alpha,\mathbf{k}}(X_r e(\mathbf{i}))$.

Next, we consider the generators $\Psi_r e(\mathbf{i})$. We must prove that for each $\mathbf{j}$ such that $\pi_e(\mathbf{j}) = \mathbf{i}$ and for each r we have

- $\Phi_{\alpha,K}(\Psi_r e(\mathbf{j})) = \Xi e(\mathbf{j}')$, for some element $\Xi \in H^{\mathrm{loc}}_{\alpha,R}(q_e)$ that depends only on r and $\mathbf{i}$,
- the image of $\Xi e(\mathbf{i}')$ in $SH^{\mathrm{loc}}_{\overline{\alpha},\overline{\mathbf{k}}}(q_{e+1})$ under the specialization $R \to \mathbf{k}$ is $\Phi_{\alpha,\mathbf{k}}(\Psi_r e(\mathbf{i}))$.

This follows from Lemma 4.2.12.

Now we obtain the diagram from the claim of Lemma 4.2.21 as the restriction of the diagram (4.19).

□

4.3 The category O

4.3.1 Affine Lie algebras

Let $N,\, l \in \mathbb{Z}_{>0}$ and $e \in \mathbb{Z}_{>1}$. Let R be a deformation ring, see Section 4.2.9. Set

$$\mathbf{g}_R = \mathfrak{gl}_N(R), \quad \widehat{\mathbf{g}}_R = \widehat{\mathfrak{gl}}_N(R) = \mathfrak{gl}_N(R)[t,t^{-1}] \oplus R\mathbf{1} \oplus R\partial.$$

For $i,j \in [1,N]$ let $e_{i,j} \in \mathbf{g}_R$ denote the matrix unit. Let $\mathbf{h}_R \subset \mathbf{g}_R$ be the Cartan subalgebra generated by the $e_{i,i}$'s, and $\epsilon_1,\cdots,\epsilon_N$ be the basis of $\mathbf{h}_R^*$ dual to $e_{1,1},\cdots,e_{N,N}$. Let $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_N$ be the weight lattice of $\mathbf{g}_R$. We identify P with $\mathbb{Z}^N$. Let $\Pi,\ \widehat{\Pi} \subset \mathbf{h}_R^*$ be the sets of simple roots of $\mathbf{g}_R$ and $\widehat{\mathbf{g}}_R$. Let $W = \mathfrak{S}_N$ be the Weyl group of $\mathbf{g}_R$ and $\widetilde{W} = W \ltimes \mathbb{Z}\Pi$, $\widehat{W} = W \ltimes P$ be the affine and the extended affine Weyl groups.

Recall that $I \in \mathbb{Z}/e\mathbb{Z}$. We still consider the quiver $\Gamma = (I,H)$ as in Section 4.2.8. In particular, we have $X_I = X_e$, $P_I = P_e$, see Section 4.2.1 for the notation. Then we define the element $\mathbf{wt}_e(\lambda) \in X_I$ given by

$$\mathbf{wt}_e(\lambda) = \sum_{s=1}^{N} \varepsilon_{\lambda_r},$$

where we write ε_{λ_r} for $\varepsilon_{(\lambda_r \bmod e)}$. We will abbreviate

$$P[\mu] = \{\lambda \in P;\ \mathbf{wt}_e(\lambda) = \mu\}. \tag{4.20}$$

Similarly, we consider the weight

$$\mathbf{wt}_e^{\delta}(\lambda) = \sum_{r=1}^{N} \varepsilon_{\lambda_r} + (\sum_{r=1}^{N} \lambda_r)\delta \in X_e^{\delta}.$$

Finally, let $X_I[N] \subset X_I$ be the subset given by

$$X_I[N] = \{\mu = \sum_{r=1}^{e} \mu_r \varepsilon_r \in X_I;\ \mu_r \geqslant 0,\ \sum_{r=1}^{e} \mu_r = N\}.$$

We may identify μ with the tuple $(\mu_1, \cdots, \mu_e)$ if no confusion is possible.

Now, consider the Cartan subalgebra $\widehat{\mathbf{h}}_R = \mathbf{h}_R \oplus R\mathbf{1} \oplus R\partial$ of $\widehat{\mathbf{g}}_R$. Let Λ_0 and δ be the elements of $\widehat{\mathbf{h}}_R^*$ defined by

$$\delta(\partial) = \Lambda_0(\mathbf{1}) = 1, \quad \delta(\mathbf{h}_R \oplus R\mathbf{1}) = \Lambda_0(\mathbf{h}_R \oplus R\partial) = 0.$$

Let $(\bullet, \bullet)\colon \widehat{\mathbf{h}}_R^* \times \widehat{\mathbf{h}}_R^* \to R$ be the bilinear form such that

$$\lambda(\alpha_r^\vee) = (\lambda, \alpha_r), \quad \lambda(\partial) = (\lambda, \Lambda_0), \qquad \forall \lambda \in \widehat{\mathbf{h}}_R^*.$$

Set $P_R = P \otimes_{\mathbb{Z}} R$. Given a composition $\nu = (\nu_1, \cdots, \nu_l)$ of N, we define

$$\begin{array}{rcl} \rho & = & (0, -1, \cdots, -N+1), \\ \rho_\nu & = & (\nu_1, \nu_1 - 1 \cdots, 1, \nu_2, \cdots, 1, \cdots, \nu_l, \cdots, 1), \\ \tau & = & (\tau_1^{\nu_1}, \cdots, \tau_l^{\nu_l}), \end{array}$$

where $\tau_r^{\nu_r}$ means ν_r copies of τ_r. Set also

$$\widehat{\rho} = \rho + N\Lambda_0, \quad \widetilde{\lambda} = \lambda + \tau + z_\lambda \delta - (N + \kappa)\Lambda_0, \tag{4.21}$$

where $z_\lambda = (\lambda, 2\rho + \lambda)/2\kappa$. Denote by $\widehat{\mathbf{p}}_{R,\nu}$ the parabolic subalgebra of $\widehat{\mathbf{g}}_R$ of parabolic type ν. For a ν-dominant weight $\lambda \in P$ let $\Delta(\lambda)_R$ be the parabolic Verma module with highest weight $\widetilde{\lambda}$ and $\Delta_R^\lambda = \Delta(\lambda - \rho)_R$. We will also skip the subscript R when $R = \mathbb{C}$.

4.3.2 Extended affine Weyl groups

Assume that $R = \mathbb{C}$. In this section we discuss some combinatorial aspects of the $\widehat{W}$-action on $\widehat{\mathbf{h}}^*$.

The group $\widehat{W}$ is generated by $\{\pi, s_i;\ i \in \mathbb{Z}/N\mathbb{Z}\}$ modulo the relations

$$\begin{array}{ccc} s_i^2 & = & 1, \\ s_i s_j & = & s_j s_i \qquad \forall i \neq j \pm 1, \\ s_i s_{i+1} s_i & = & s_{i+1} s_i s_{i+1}, \\ \pi s_{i+1} & = & s_i \pi. \end{array}$$

Let $\widetilde{W}$ be the subgroup of $\widehat{W}$ generated by $\{s_i;\ i \in \mathbb{Z}/N\mathbb{Z}\}$. The group $\widehat{W}$ acts on P in the following way:

- s_r switches of the rth and $(r+1)$th components of λ if $r \neq 0$,
- $s_0(\lambda_1, \cdots, \lambda_N) = (\lambda_N - e, \lambda_2, \cdots, \lambda_{N-1}, \lambda_1 + e)$,
- $\pi(\lambda_1, \cdots, \lambda_N) = (\lambda_2, \cdots, \lambda_N, \lambda_1 + e)$.

We will call this representation of $\widehat{W}$ on P the *e-action*. We may write $P^{(e)} = P$ to stress that we consider the e-action of $\widehat{W}$ on P. The map

$$P^{(e)} \to \widehat{\mathbf{h}}^*, \quad \lambda \mapsto \widetilde{\lambda - \rho} + \widehat{\rho}$$

is $\widehat{W}$-invariant. This means that the weights $\lambda_1, \lambda_2 \in P$ are in the same $\widehat{W}$-orbit if and only if the highest weights of the Verma modules Δ^{λ_1} and Δ^{λ_2} are linked with respect to the Weyl group $\widehat{W}$, see [56, Sec. 3.2] and [19, Sec. 2.3] for more details about linkage. Note that $P = \coprod_{\mu \in X_I[N]} P[\mu]$ is the decomposition of P into $\widehat{W}$-orbits with respect to the e-action. An element $\lambda \in P$ is *e-anti-dominant* if $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \leqslant \lambda_1 + e$.

Recall the map $\Upsilon\colon \mathbb{Z} \to \mathbb{Z}$ from (4.9). Applying Υ coordinate by coordinate to the elements of P we get a map $\Upsilon\colon P^{(e)} \to P^{(e+1)}$.

Lemma 4.3.1. *The map $\Upsilon\colon P^{(e)} \to P^{(e+1)}$ is $\widehat{W}$-invariant and takes e-anti-dominant weights to $(e+1)$-anti-dominant weights.* □

4.3.3 The standard representation of $\widetilde{\mathfrak{sl}}_e$

Let e_i, f_i, h_i the generators of the complex Lie algebra $\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$. Let V_e be a $\mathbb{C}$-vector spaces with canonical basis $\{v_1, \cdots, v_e\}$ and set $U_e = V_e \otimes \mathbb{C}[z, z^{-1}]$. The vector space U_e has a basis $\{u_r;\ r \in \mathbb{Z}\}$ where $u_{a+eb} = v_a \otimes z^{-b}$ for $a \in [1, e]$, $b \in \mathbb{Z}$. It has a structure of an $\widetilde{\mathfrak{sl}}_e$-module such that

$$f_i(u_r) = \delta_{i \equiv r} u_{r+1}, \quad e_i(u_r) = \delta_{i \equiv r-1} u_{r-1}.$$

Let $\{v'_1, \cdots, v'_{e+1}\}$, $\{u'_r; r \in \mathbb{Z}\}$ denote the bases of V_{e+1} and U_{e+1}.

Fix an integer $0 \leqslant k < e$. Consider the following inclusion of vector spaces

$$V_e \subset V_{e+1},\ v_r \mapsto \begin{cases} v'_r & \text{if } r \leqslant k, \\ v'_{r+1} & \text{if } r > k. \end{cases}$$

It yields an inclusion $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ such that

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [1, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$f_r \mapsto \begin{cases} f_r & \text{if } r \in [1, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$h_r \mapsto \begin{cases} h_r & \text{if } r \in [1, k-1], \\ h_k + h_{k+1} & \text{if } r = k, \\ h_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases}$$

This inclusion lifts uniquely to an inclusion $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ such that

$$e_0 \mapsto \begin{cases} e_0 & \text{if } k \neq 0, \\ [e_0, e_1] & \text{else,} \end{cases}$$

$$f_0 \mapsto \begin{cases} f_0 & \text{if } k \neq 0, \\ [f_1, f_0] & \text{else,} \end{cases}$$

$$h_0 \mapsto \begin{cases} h_0 & \text{if } k \neq 0, \\ h_0 + h_1 & \text{else.} \end{cases}$$

Consider the inclusion $U_e \subset U_{e+1}$ such that $u_r \mapsto u'_{\Upsilon(r)}$.

Lemma 4.3.2. *The embeddings $V_e \subset V_{e+1}$ and $U_e \subset U_{e+1}$ are compatible with the actions of $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ and $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ respectively.* □

4.3.4 Categorical representations

Let R be a $\mathbb{C}$-algebra. Fix an invertible element $q \in R$, $q \neq 1$. Let $\mathcal{C}$ be an exact R-linear category.

Definition 4.3.3. A *representation datum* in $\mathcal{C}$ is a tuple (E, F, X, T) where (E, F) is a pair of exact functors $\mathcal{C} \to \mathcal{C}$ and $X \in \mathrm{End}(F)^{\mathrm{op}}$, $T \in \mathrm{End}(F^2)^{\mathrm{op}}$ are endomorphisms of functors such that for each $d \in \mathbb{N}$, there is an R-algebra homomorphism $\psi_d \colon H_{d,R}(q) \to \mathrm{End}(F^d)^{\mathrm{op}}$ given by

$$\begin{aligned} X_r &\mapsto F^{d-r} X F^{r-1} && \forall r \in [1, d], \\ T_r &\mapsto F^{d-r-1} T F^{r-1} && \forall r \in [1, d-1]. \end{aligned}$$

Now, assume that $R = \mathbf{k}$ is a field. Assume that $\mathcal{C}$ is a Hom-finite abelian category. Let $\mathscr{F}$ be a subset of $\mathbf{k}$ such that $q^{\mathbb{Z}}\mathscr{F}/q^{\mathbb{Z}}$ is finite. We view $\mathscr{F}$ as the vertex set of a quiver with an arrow $i \to j$ if and only if $j = qi$.

Remark 4.3.4. Assume that we have a representation datum in a $\mathbf{k}$-linear category $\mathcal{C}$ such that the functors E and F are biadjoint. Then by adjunction we have an algebra isomorphism $\mathrm{End}(E^d) \simeq \mathrm{End}(F^d)^{\mathrm{op}}$. In particular we get an algebra homomorphism $H_{d,\mathbf{k}} \to \mathrm{End}(E^d)$.

Definition 4.3.5. An $\mathfrak{sl}_{\mathscr{F}}$-categorical representation in $\mathcal{C}$ is the datum of a representation datum (E, F, X, T) and a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_{\mathscr{F}}} \mathcal{C}_\mu$ satisfying the conditions (a) and (b) below. For $i \in \mathscr{F}$ let E_i, F_i be endofunctors of $\mathcal{C}$ such that for each $M \in \mathcal{C}$ the objects $E_i(M)$, $F_i(M)$ are the generalized i-eigenspaces of X acting on $E(M)$ and $F(M)$ respectively, see also Remark 4.3.4. We assume that

(a) $F = \bigoplus_{i \in \mathscr{F}} F_i$ and $E = \bigoplus_{i \in \mathscr{F}} E_i$,

(b) $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$ and $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.

Remark 4.3.6. In this case the homomorphism $\psi_d \colon H_{d,\mathbf{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ extends to a homomorphism $\widehat{H}_{d,\mathbf{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$, where $\widehat{H}_{d,\mathbf{k}}$ is as in Section 4.2.11.

There is an alternative definition of a categorical representation, where the affine Hecke algebra $H_{d,\mathbf{k}}(q)$ is replaced by a KLR algebra. In this section we allow $\Gamma = (I, H)$ be an arbitrary quiver without 1-loops.

Definition 4.3.7. A $\mathfrak{g}_I$-categorical representation (E, F, x, τ) on $\mathcal{C}$ is the following data:

(1) a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_\mu$,
(2) a pair of biadjoint exact endofunctors (E, F) of $\mathcal{C}$,
(3) morphisms of functors $x \colon F \to F$, $\tau \colon F^2 \to F^2$,
(4) decompositions $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$,

satisfying the following conditions.

(a) We have $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$, $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.
(b) For each $d \in \mathbb{N}$ there is an algebra homomorphism $\psi_d \colon R_{d,\mathbf{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ such that $\psi_d(e(\mathbf{i}))$ is the projector to $F_{i_d} \cdots F_{i_1}$, where $\mathbf{i} = (i_1, \cdots, i_d)$ and

$$\psi_d(x_r) = F^{d-r} x F^{r-1}, \qquad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1}.$$

(c) For each $M \in \mathcal{C}$ the endomorphism of $F(M)$ induced by x is nilpotent.

Assume that there is an isomorphism of quivers $I \simeq \mathscr{F}$. Then Definitions 4.3.5, 4.3.7 are equivalent by Proposition 4.2.17.

4.3.5 From $\widetilde{\mathfrak{sl}}_{e+1}$-categorical representations to $\widetilde{\mathfrak{sl}}_e$-categorical representations

Recall that we fix $\Gamma = (I, H)$ and $\overline{\Gamma} = (\overline{I}, \overline{H})$ as in Section 4.2.8. In particular, we have $X_I = X_e$, $X_{\overline{I}} = X_{e+1}$.

Assume that $R = \mathbf{k}$ is a field. Let $\overline{\mathcal{C}}$ be a Hom-finite abelian $\mathbf{k}$-linear category. Let

$$\overline{E} = \overline{E}_0 \oplus \overline{E}_1 \oplus \cdots \oplus \overline{E}_e, \qquad \overline{F} = \overline{F}_0 \oplus \overline{F}_1 \oplus \cdots \oplus \overline{F}_e$$

be endofunctors defining a categorical $\widetilde{\mathfrak{sl}}_{e+1}$-representation in $\overline{\mathcal{C}}$. Let $\overline{\psi}_d \colon R_{d,\mathbf{k}} \to \mathrm{End}(\overline{F}^d)^{\mathrm{op}}$ be the corresponding algebra homomorphism. We set $\overline{F}_{\mathbf{i}} = \overline{F}_{i_d} \cdots \overline{F}_{i_1}$ for any tuple $\mathbf{i} = (i_1, \cdots, i_d) \in \overline{I}^d$ and $\overline{F}_{\overline{\alpha}} = \bigoplus_{\mathbf{i} \in \overline{I}^{\overline{\alpha}}} \overline{F}_{\mathbf{i}}$ for any element $\overline{\alpha} \in Q^+_{\overline{I}}$. If $|\overline{\alpha}| = d$ let $\overline{\psi}_{\overline{\alpha}} \colon R_{\overline{\alpha},\mathbf{k}} \to \mathrm{End}(\overline{F}_{\overline{\alpha}})^{\mathrm{op}}$ be the $\overline{\alpha}$-component of $\overline{\psi}_d$.

Now, recall the notation $X^+_{\overline{I}}$ from (4.1). Assume that

$$\overline{\mathcal{C}}_\mu = 0, \quad \forall \mu \in X_{\overline{I}} \backslash X^+_{\overline{I}}. \tag{4.22}$$

For $\mu \in X^+_I$ set $\mathcal{C}_\mu = \overline{\mathcal{C}}_{\phi(\mu)}$, where the map ϕ is as in (4.3). Let $\mathcal{C} = \bigoplus_{\mu \in X^+_I} \mathcal{C}_\mu$.

Remark 4.3.8. (*a*) $\mathcal{C}$ is stable by $\overline{F}_i$, $\overline{E}_i$ for each $i \neq k, k+1$,

(*b*) $\mathcal{C}$ is stable by $\overline{F}_{k+1}\overline{F}_k$, $\overline{E}_k\overline{E}_{k+1}$,

(*c*) $\overline{F}_{i_d}\overline{F}_{i_{d-1}} \cdots \overline{F}_{i_1}(M) = 0$ for each $M \in \mathcal{C}$ whenever the sequence $(i_1, \cdots, i_d)$ is unordered (see Sections 4.2.4 and 4.2.8).

Consider the following endofunctors of $\mathcal{C}$:

$$F_0 = \overline{F}_0\big|_{\mathcal{C}}, \cdots, F_{k-1} = \overline{F}_{k-1}\big|_{\mathcal{C}}, \quad F_k = \overline{F}_{k+1}\overline{F}_k\big|_{\mathcal{C}},$$
$$F_{k+1} = \overline{F}_{k+2}\big|_{\mathcal{C}}, \cdots, F_{e-1} = \overline{F}_e\big|_{\mathcal{C}},$$
$$E_0 = \overline{E}_0\big|_{\mathcal{C}}, \cdots, E_{k-1} = \overline{E}_{k-1}\big|_{\mathcal{C}}, \quad E_k = \overline{E}_k\overline{E}_{k+1}\big|_{\mathcal{C}},$$
$$E_{k+1} = \overline{E}_{k+2}\big|_{\mathcal{C}}, \cdots, E_{e-1} = \overline{E}_e\big|_{\mathcal{C}}.$$

Similarly to the notations above we set $F_{\mathbf{i}} = F_{i_d} \cdots F_{i_1}$ for any tuple $\mathbf{i} = (i_1, \cdots, i_d) \in I^d$ and $F_\alpha = \bigoplus_{\mathbf{i} \in I^\alpha} F_{\mathbf{i}}$ for any element $\alpha \in Q_I^+$. Note that we have $F_{\mathbf{i}} = \overline{F}_{\phi(\mathbf{i})}\big|_{\mathcal{C}}$ for each $\mathbf{i} \in I^\alpha$.

Let $\alpha \in \mathbb{N}I$ and $\overline{\alpha} = \phi(\alpha)$. Note that we have

$$F_\alpha = \bigoplus_{\mathbf{i} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} \overline{F}_{\mathbf{i}}\big|_{\mathcal{C}}.$$

The homomorphism $\overline{\psi}_{\overline{\alpha}}$ yields a homomorphism $\mathbf{e}R_{\overline{\alpha},\mathbf{k}}\mathbf{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$, where $\mathbf{e} = \sum_{\mathbf{i} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} e(\mathbf{i})$. By (c), the homomorphism $\mathbf{e}R_{\overline{\alpha},\mathbf{k}}\mathbf{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ factors through a homomorphism $S_{\overline{\alpha},\mathbf{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$. Let us call it $\overline{\psi}'_{\overline{\alpha}}$. Then we can define an algebra homomorphism $\psi_\alpha \colon R_{\alpha,\mathbf{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ by setting $\psi_\alpha = \overline{\psi}'_{\overline{\alpha}} \circ \Phi_{\alpha,\mathbf{k}}$.

Now, Theorem 4.2.11 implies the following result.

Lemma 4.3.9. *For each category $\overline{\mathcal{C}}$ as above that satisfies (4.22), we have a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in the subcategory $\mathcal{C}$ of $\overline{\mathcal{C}}$ given by functors F_i, E_i and the algebra homomorphisms $\psi_\alpha \colon R_{\alpha,\mathbf{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$.* □

4.3.6 The category O

Let R be a deformation ring. Fix a composition $\nu = (\nu_1, \cdots, \nu_l)$ of N. First we define an R-deformed version of the category O^ν_{-e}. Recall that we identify the weight lattice P with $\mathbb{Z}^N$. We say that $\lambda \in P$ is *ν-dominant* if $\lambda_r > \lambda_{r+1}$ for each $r \in [1, N-1] \backslash \{\nu_1, \nu_1 + \nu_2, \cdots, \nu_1 + \cdots + \nu_l\}$. Let P^ν be the set of ν-dominant weights of P. Set also $P^\nu[\mu] = P^\nu \cap P[\mu]$, where $P[\mu]$ is as in (4.20). Let O^ν_R be the R-linear abelian category of finitely generated $\widehat{\mathbf{g}}_R$-modules M which are weight $\widehat{\mathbf{h}}_R$-modules, and such that the $\widehat{\mathbf{p}}_{R,\nu}$-action on M is locally finite over R, and the highest weight of any subquotient of M is of the form $\widetilde{\lambda}$ with $\lambda \in P^\nu$, where $\widetilde{\lambda}$ is defined in (4.21). Let $O^\nu_{\mu,R}$ be the Serre subcategory of O^ν_R generated by the modules Δ^λ_R for all $\lambda \in P^\nu[\mu]$. Let $O^{\nu,\Delta}_R \subset O^\nu_R$ be the full subcategory of Δ-filtered modules.

Let U_e, V_e be as in Section 4.3.3. Set $\wedge^\nu U_e = \wedge^{\nu_1} U_e \otimes \cdots \otimes \wedge^{\nu_l} U_e$. For each $\lambda \in P^\nu$ define the following element in $\wedge^\nu U_e$:

$$\wedge^\nu_\lambda = (u_{\lambda_1} \wedge \cdots \wedge u_{\lambda_{\nu_1}}) \otimes \cdots \otimes (u_{\lambda_1 + \cdots + \lambda_{l-1} + 1} \wedge \cdots \wedge u_{\lambda_{\nu_1} + \cdots + \lambda_{\nu_l}}).$$

The obvious $\widetilde{\mathfrak{sl}}_e$-action on U_e yields an $\widetilde{\mathfrak{sl}}_e$-action on $\wedge^\nu U_e$. We identify the abelian group $X_e/\mathbb{Z}(\varepsilon_1+\cdots+\varepsilon_e)$ with the weight lattice of $\mathfrak{sl}_e$. In particular each element $\mu \in X_e$ yields a weight of $\mathfrak{sl}_e$. For each $\mu \in X_e[N]$ let $(\wedge^\nu U_e)_\mu$ be the weight space in $\wedge^\nu U_e$ corresponding to μ.

Set $O^\nu_{-e,R} = \bigoplus_{\mu\in X_e[N]} O^\nu_{\mu,R}$ and similarly for $O^{\nu,\Delta}_{-e,R}$. Now we define a representation datum in the category $O^{\nu,\Delta}_{-e,R}$. See [53, Sec. 5.4] for more details. From now on, we assume that R is either a local analytic deformation ring in general position of dimension $\leqslant 2$ or a field. Let $\mathbf{k}$ and K be its residue field and the field of fractions respectively. For an exact category $\mathcal{C}$ denote by $[\mathcal{C}]$ its complexified Grothendieck group. The following proposition holds, see [53].

Proposition 4.3.10. *There is a pair of exact endofunctors E, F of $O^{\nu,\Delta}_{-e,R}$ such that the following properties hold.*

(*a*) *the functors E, F commute with the base changes $K\otimes_R \bullet$, $\mathbf{k}\otimes_R\bullet$.*
(*b*) *$(O^{\nu,\Delta}_{-e,R}, E, F)$ admits a representation datum structure.*
(*c*) *E, F extend to a pair of biadjoint functors $O^\nu_{-e,R}\to O^\nu_{-e,R}$ if R is a field.*
(*d*) *There are decompositions $E=\bigoplus_{i\in I} E_i$, $F=\bigoplus_{i\in I} F_i$ such that*

$$E_i(O^{\nu,\Delta}_{\mu,R})\subset O^{\nu,\Delta}_{\mu+\alpha_i,R},\qquad F_i(O^{\nu,\Delta}_{\mu,R})\subset O^{\nu,\Delta}_{\mu-\alpha_i,R}.$$

(*e*) *There is a vector space isomorphism $[O^{\nu,\Delta}_{\mu,R}]\simeq(\wedge^\nu U)_\mu$ such that the functors F_i, E_i act on $[O^{\nu,\Delta}_{-e,R}]=\bigoplus_{\mu\in X_e[N]}[O^{\nu,\Delta}_{\mu,R}]$ as the standard generators e_i, f_i of $\widetilde{\mathfrak{sl}}_e$.*
(*f*) *If $R=\mathbf{k}$ with the trivial deformation ring structure, then E_i, F_i yield a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $O^\nu_{-e,\mathbf{k}}$.* □

Fix $k\in[0,e-1]$. Recall the map $\Upsilon\colon P\to P$ from Section 4.3.2 and the map $\phi\colon X_I\to X_{\overline{I}}$ from (4.3). Set $\mu'=\mu-\alpha_k$ and $\overline{\mu}=\phi(\mu)$. Set, $\overline{\mu}^0=\overline{\mu}-\overline{\alpha}_k$ and $\overline{\mu}'=\overline{\mu}-\overline{\alpha}_k-\overline{\alpha}_{k+1}$. Note that $\Upsilon(P[\mu])\subset P[\overline{\mu}]$. For $k\neq 0$ we have

$$\begin{array}{rcllll}
\mu &=& (\mu_1,\cdots,\mu_k, & & \mu_{k+1}, & \cdots,\mu_e),\\
\mu' &=& (\mu_1,\cdots,\mu_k-1, & & \mu_{k+1}+1, & \cdots,\mu_e),\\
\overline{\mu} &=& (\mu_1,\cdots,\mu_k, & 0, & \mu_{k+1}, & \cdots,\mu_e),\\
\overline{\mu}^0 &=& (\mu_1,\cdots,\mu_k-1, & 1, & \mu_{k+1}, & \cdots,\mu_e),\\
\overline{\mu}' &=& (\mu_1,\cdots,\mu_k-1, & 0, & \mu_{k+1}+1, & \cdots,\mu_e).
\end{array}$$

For an e-tuple $\mathbf{a}=(a_1,\cdots,a_e)$ of nonnegative integers we set $1_{\mathbf{a}}=(1^{a_1},\cdots,e^{a_e})$. Note that we have

$$\Upsilon(1_\mu)=1_{\overline{\mu}},\qquad \Upsilon(1_{\mu'})=1_{\overline{\mu}'}. \tag{4.23}$$

Remark 4.3.11. The set $P[\mathbf{a}]$ is a $\widehat{W}$-orbit in $P^{(e)}$. It is a union of $\widetilde{W}$-orbits and each of them contains a unique e-anti-dominant weight. By definition, the weight $1_{\mathbf{a}} \in P[\mathbf{a}]$ is e-anti-dominant. However, there is no canonical way to choose $1_{\mathbf{a}}$. In the case $k = 0$ we need to change our convention and to set $1_{\mathbf{a}} = (0^{a_e}, 1^{a_e}, \cdots, (e-1)^{a_{e-1}})$. This change is necessary to have (4.23).

First, assume that $l = N$ and $\nu = (1, 1, \cdots, 1)$. In this case we will omit the upper index ν.

Lemma 4.3.12. *There is an equivalence of categories $\theta_{\mu}^{\overline{\mu}}\colon O_{\mu,R} \to O_{\overline{\mu},\overline{R}}$ such that $\theta_{\mu}^{\overline{\mu}}(\Delta_R^{\lambda}) \simeq \Delta_{\overline{R}}^{\Upsilon(\lambda)}$. It restricts to an equivalence of categories $\theta_{\mu}^{\overline{\mu}}\colon O_{\mu,R}^{\Delta} \to O_{\overline{\mu},\overline{R}}^{\Delta}$.*

Proof. For each $n \in \mathbb{Z}$ the weight $\pi^n(1_\mu)$ is e-anti-dominant. Let $\mathcal{O}_{\pi^n(1_\mu),R} \subset O_{\mu,R}$ be the Serre subcategory generated by the Verma modules of the form $\Delta_R^{w\pi^n(1_\mu)}$ with $w \in \widetilde{W}$. We have

$$O_{\mu,R} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{\pi^n(1_\mu),R}.$$

The weights $\pi^n(1_\mu) \in P^{(e)}$ and $\pi^n(1_{\overline{\mu}}) \in P^{(e+1)}$ have the same stabilizers in $\widetilde{W}$. Thus by [19, Thm. 11] (see also [53, Prop. 5.24]) we have an equivalence of categories

$$\mathcal{O}_{\pi^n(1_\mu),R} \simeq \mathcal{O}_{\pi^n(1_{\overline{\mu}}),\overline{R}}, \qquad \Delta_R^{w\pi^n(1_\mu)} \mapsto \Delta_{\overline{R}}^{w\pi^n(1_{\overline{\mu}})} \quad \forall w \in \widetilde{W}.$$

Taking the sum by all $n \in \mathbb{Z}$ we get an equivalence of categories

$$\theta_{\mu}^{\overline{\mu}}\colon O_{\mu,R} \simeq O_{\overline{\mu},\overline{R}}, \qquad \Delta_R^{w(1_\mu)} \mapsto \Delta_{\overline{R}}^{w(1_{\overline{\mu}})} \quad \forall w \in \widehat{W}.$$

Recall that we have $\Upsilon(1_\mu) = 1_{\overline{\mu}}$. Thus by $\widehat{W}$-invariance of Υ we get

$$\theta_{\mu}^{\overline{\mu}}(\Delta_R^{\lambda}) \simeq \Delta_{\overline{R}}^{\Upsilon(\lambda)} \quad \forall \lambda \in P[\mu].$$

□

Remark 4.3.13. Notice that [19] yields an equivalence of categories over a field. It is explained in [53] how to get from it an equivalence of categories O_R^{Δ}. First, comparing the endomorphisms of projective generators one gets an equivalence of the abelian categories O_R. Then, comparing the highest weight structure in both sides, we deduce an equivalence of additive categories O_R^{Δ}.

The equivalence $\theta_{\mu}^{\overline{\mu}}$ restricts to equivalences $O_{\mu,R}^{\nu} \simeq O_{\overline{\mu},\overline{R}}^{\nu}$ and $O_{\mu,R}^{\nu,\Delta} \simeq O_{\overline{\mu},\overline{R}}^{\nu,\Delta}$ for each parabolic type ν, see [53, Sec. 5.7.2]. We will also call this equivalence $\theta_{\mu}^{\overline{\mu}}$. We obtain equivalences of categories $\theta_{\overline{\mu}'}^{\mu'}\colon O_{\overline{\mu}',\overline{R}}^{\nu} \simeq O_{\mu',R}^{\nu}$ and $\theta_{\overline{\mu}'}^{\mu'}\colon O_{\overline{\mu}',\overline{R}}^{\nu,\Delta} \simeq O_{\mu',R}^{\nu,\Delta}$ in a similar way.

Conjecture 4.3.14. *There are the following commutative diagrams*

$$\begin{tikzcd}
O^{\nu,\Delta}_{\overline{\mu},\overline{R}} \arrow[r, "\overline{F}_k"] & O^{\nu,\Delta}_{\overline{\mu}^0,\overline{R}} \arrow[r, "\overline{F}_{k+1}"] & \overline{O}^{\nu,\Delta}_{\mu',\overline{R}} \arrow[d, "\theta^{\mu'}_{\overline{\mu}'}"] \\
O^{\nu,\Delta}_{\mu,R} \arrow[u, "\theta^{\overline{\mu}}_{\mu}"] \arrow[rr, "F_k"'] & & O^{\nu,\Delta}_{\mu',R}
\end{tikzcd}$$

and

$$\begin{tikzcd}
O^{\nu,\Delta}_{\overline{\mu},\overline{R}} \arrow[d, "\theta^{\mu}_{\overline{\mu}}"'] & O^{\nu,\Delta}_{\overline{\mu}^0,\overline{R}} \arrow[l, "\overline{E}_k"'] & O^{\nu,\Delta}_{\overline{\mu}',\overline{R}} \arrow[l, "\overline{E}_{k+1}"'] \\
O^{\nu,\Delta}_{\mu,R} & & O^{\nu,\Delta}_{\mu',R} \arrow[ll, "E_k"] \arrow[u, "\theta^{\overline{\mu}'}_{\mu'}"']
\end{tikzcd} .$$

4.3.7 The commutativity in the Grothendieck groups

We have the following commutative diagram of vector spaces

$$\begin{tikzcd}
{[O^{\nu,\Delta}_{-e-1,\overline{R}}]} \arrow[r] & \wedge^{\nu} U_{e+1} \\
{[O^{\nu,\Delta}_{-e,R}]} \arrow[u, "\oplus_\mu \theta^{\overline{\mu}}_{\mu}"] \arrow[r] & \wedge^{\nu} U_e, \arrow[u]
\end{tikzcd}$$

where the horizontal maps are respectively the isomorphisms of $\widetilde{\mathfrak{sl}}_e$-modules and $\widetilde{\mathfrak{sl}}_{e+1}$-modules from Proposition 4.3.10 (e), the right vertical map is given by the injection $U_e \to U_{e+1}$ in Section 4.3.3. Moreover, the right vertical map is a morphism of $\widetilde{\mathfrak{sl}}_e$-modules where $\wedge^\nu U_{e+1}$ is viewed as an $\widetilde{\mathfrak{sl}}_e$-module via the inclusion $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ introduced in Section 4.3.3. Thus $\oplus_\mu \theta^{\overline{\mu}}_{\mu} \colon [O^{\nu,\Delta}_{-e,R}] \to [O^{\nu,\Delta}_{-e-1,\overline{R}}]$ is a morphism of $\widetilde{\mathfrak{sl}}_e$-modules which intertwines

- $[E_r]$ with $[\overline{E}_r]$, $[F_r]$ with $[\overline{F}_r]$ if $r \in [1, k-1]$,
- $[E_k]$ with $[\overline{E}_k \overline{E}_{k+1}] - [\overline{E}_{k+1} \overline{E}_k]$, $[F_k]$ with $[\overline{F}_{k+1} \overline{F}_k] - [\overline{F}_k \overline{F}_{k+1}]$,
- $[E_r]$ with $[\overline{E}_{r+1}]$, $[F_r]$ with $[\overline{F}_{r+1}]$ if $r \in [k+1, e-1]$.

  In particular, we see that the diagrams from Conjecture 4.3.14 commute at the level of Grothendieck groups. Since there is no good notion of projective functors in the affine category $\mathcal{O}$, this is not enough to prove our conjecture.

4.3.8 Partitions

A *partition* of an integer $n \geqslant 0$ is a tuple of positive integers $(\lambda_1, \cdots, \lambda_s)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$ and $\sum_{t=1}^{s} \lambda_t = n$. Denote by $\mathcal{P}_n$ the set of all partitions of n and set $\mathcal{P} = \coprod_{n\in\mathbb{N}} \mathcal{P}_n$. For a partition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of n, we set $|\lambda| = n$ and $\ell(\lambda) = s$. An *l-partition* of an integer $n \geqslant 0$ is an l-tuple $\lambda = (\lambda^1, \cdots, \lambda^l)$ of partitions of integers $n_1, \cdots, n_l \geqslant 0$ such that $\sum_{t=1}^{l} n_t = n$. Let $\mathcal{P}_n^l$ be the set of all l-partitions of n and set $\mathcal{P}^l = \coprod_{n\in\mathbb{N}} \mathcal{P}_n^l$.

A partition λ can be represented by a Young diagram $Y(\lambda)$ and an l-partition $\lambda = (\lambda^1, \cdots, \lambda^l)$ by an l-tuple of Young diagrams $Y(\lambda) = (Y(\lambda^1), \cdots, Y(\lambda^l))$.

Let $\lambda \in \mathcal{P}^l$ be an l-partition. For a box $b \in Y(\lambda)$ situated in the ith row, jth column of the rth component we define its *residue* $\mathrm{Res}_\nu(b) \in I$ as $\nu_r + j - i \bmod e$ and its *deformed residue* $\widetilde{\mathrm{Res}}_\nu(b) \in \widetilde{I}$ as $(\nu_r + j - i, r)$. Set

$$\mathrm{Res}_\nu(\lambda) = \sum_{b\in Y(\lambda)} \alpha_{\mathrm{Res}_\nu(b)} \in Q_I^+, \quad \widetilde{\mathrm{Res}}_\nu(\lambda) = \sum_{b\in Y(\lambda)} \widetilde{\alpha}_{\widetilde{\mathrm{Res}}_\nu(b)} \in Q_{\widetilde{I}}^+.$$

Now for $\alpha \in Q_I^+$ and $\widetilde{\alpha} \in Q_{\widetilde{I}}^+$ set

$$\mathcal{P}_\alpha^l = \{\lambda \in \mathcal{P}^l;\ \mathrm{Res}_\nu(\lambda) = \alpha\}, \qquad \mathcal{P}_{\widetilde{\alpha}}^l = \{\lambda \in \mathcal{P}^l;\ \widetilde{\mathrm{Res}}_\nu(\lambda) = \widetilde{\alpha}\}.$$

This notation depends on ν. We may write $\mathcal{P}_{\alpha,\nu}^l$ and $\mathcal{P}_{\widetilde{\alpha},\nu}^l$ to specify ν. We have decompositions

$$\mathcal{P}_d^l = \bigoplus_{\alpha\in Q_I^+, |\alpha|=d} \mathcal{P}_\alpha^l, \qquad \mathcal{P}_\alpha^l = \bigoplus_{\widetilde{\alpha}\in Q_{\widetilde{I}}^+, \pi_e(\widetilde{\alpha})=\alpha} \mathcal{P}_{\widetilde{\alpha}}^l.$$

4.3.9 The category A

Let $\mathcal{P}_d^\nu \subset \mathcal{P}_d^l$ be the subset of the elements $\lambda = (\lambda^1, \cdots, \lambda^l)$ such that $\ell(\lambda^r) \leqslant \nu_r$ for each $r \in [1, l]$. We can view λ as the weight in P given by

$$(\lambda_1^1, \cdots, \lambda_{\ell(\lambda^1)}^1, 0^{m_1-\ell(\lambda^1)}, \lambda_1^2, \cdots, \lambda_{\ell(\lambda^2)}^2, 0^{m_2-\ell(\lambda^2)}, \cdots, \lambda_1^l, \cdots, \lambda_{\ell(\lambda^l)}^l, 0^{m_l-\ell(\lambda^l)}).$$

Then, we set $\omega(\lambda) = \lambda - \rho + \rho_\nu$ in P.

Definition 4.3.15. Let $\mathbf{A}_R^\nu[d] \subset O_{-e,R}^\nu$ be the Serre subcategory generated by the modules $\Delta(\omega(\lambda))_R$ with $\lambda \in \mathcal{P}_d^\nu$, see Section 4.3.1. Denote by $\mathbf{A}_R^{\nu,\Delta}[d]$ the full subcategory of Δ-filtered modules in $\mathbf{A}_R^\nu[d]$.

We abbreviate $\Delta[\lambda]_R = \Delta(\omega(\lambda))_R$. The restriction of the functor F to the subcategory $\mathbf{A}_R^\nu[d]$ yields a functor $\mathbf{A}_R^\nu[d] \to \mathbf{A}_R^\nu[d+1]$. However, it is not true that $E(\mathbf{A}_R^\nu[d+1]) \subset \mathbf{A}_R^\nu[d]$. Nevertheless, we can define a functor $E\colon \mathbf{A}_R^\nu[d+1] \to \mathbf{A}_R^\nu[d]$ that is left adjoint to $F\colon \mathbf{A}_R^\nu[d] \to \mathbf{A}_R^\nu[d+1]$, see [53, Sec. 5.9].

There is a decomposition $\mathbf{A}_R^\nu[d] = \bigoplus_{\alpha\in Q_I^+, |\alpha|=d} \mathbf{A}_R^\nu[\alpha]$, where $\mathbf{A}_R^\nu[\alpha]$ is the Serre subcategory of $\mathbf{A}_R^\nu[d]$ generated by the Verma modules $\Delta[\lambda]_R$ such that $\lambda \in \mathcal{P}_\alpha^l$. The functors E, F admit decompositions

$$E = \bigoplus_{i\in I} E_i, \qquad F = \bigoplus_{i\in I} F_i$$

such that for each $\alpha \in Q_I^+$ and $i \in I$ we have

$$E_i(\mathbf{A}_R^\nu[\alpha]) \subset \mathbf{A}_R^\nu[\alpha - \alpha_i], \qquad F_i(\mathbf{A}_R^\nu[\alpha]) \subset \mathbf{A}_R^\nu[\alpha + \alpha_i].$$

4.3.10 The change of level for A

For $\lambda_1, \lambda_2 \in P$ we write $\lambda_1 \geqslant \lambda_2$ if $(\lambda_1)_r \geqslant (\lambda_2)_r$ for each $r \in [1, N]$. Here, $(\lambda_i)_r$ is the rth entry of λ_i for each r. We identify Q_I with a sublattice of X_I^δ via the map ι^δ defined in Section 4.2.1.

Lemma 4.3.16. *(a) For each $\lambda_1, \lambda_2 \in P$ we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) \in Q_I$.*
(b) If we also have $\lambda_1 \leqslant \lambda_2$, then $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) \in Q_I^+$.

Proof. It is enough to assume that we have $\lambda_1 = \lambda_2 - \epsilon_r$ for some $r \in [1, N]$. In this case we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) = \alpha_i$, where $i \in I$ is the residue of the integer $(\lambda_1)_r$ modulo e. □

Let us write $\emptyset$ for the empty l-partition. Note that $\Delta[\emptyset]_R = \Delta_R^{\rho_\nu}$ is the Verma module of highest weight $\widetilde{\rho_\nu - \rho}$. Since, ρ_ν lies in $P[\mathbf{wt}_e(\rho_\nu)]$, we have $\Delta[\emptyset]_R \in O^\nu_{\mathbf{wt}_e(\rho_\nu),R}$. More generally, fix an element $\alpha = \sum_{i \in I} d_i \alpha_i$ in Q_I^+. Put $\mu = \mathbf{wt}_e(\rho_\nu) - \alpha \in X_I$. See [53, Sec. 2.3] for the definition of a highest weight category over a local ring. The following proposition holds, see [53, Sec. 5.5].

Proposition 4.3.17. *The category $\mathbf{A}_R^\nu[\alpha]$ is a full subcategory of $O^\nu_{\mu,R}$ that is a highest weight category.* □

For $\lambda \in \mathcal{P}_d^l$ let $P[\lambda]_R$, $\nabla[\lambda]_R$ and $T[\lambda]_R$ be the projective, costandard and the tilting objects in $\mathbf{A}_R^\nu[d]$ with parameter λ, see [53, Prop. 2.1].

Set $\overline{\alpha} = \phi(\alpha) \in Q_{\overline{I}}$, $\overline{\mu} = \phi(\mu) \in X_{\overline{I}}$ and $\beta = \mathbf{wt}_{e+1}^\delta(\rho_\nu) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\rho_\nu))$. By Lemma 4.3.16, we have $\beta \in Q_{\overline{I}}^+$.

Proposition 4.3.18. *The equivalence of categories $\theta_\mu^{\overline{\mu}}$ takes the subcategory $\mathbf{A}_R^\nu[\alpha]$ of $O^\nu_{\mu,R}$ to the subcategory $\mathbf{A}_{\overline{R}}^\nu[\beta + \overline{\alpha}]$ of $O^\nu_{\overline{\mu},\overline{R}}$.* □

Let the map $\phi\colon Q_I \to Q_{\overline{I}}$ be as in Section 4.2.2 (see also Section 4.2.8) and Υ be as in (4.9). First, we prove the following lemma.

Lemma 4.3.19. *If $\lambda_1, \lambda_2 \in P$, then*

$$\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \phi(\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2)).$$

Proof of Lemma 4.3.19. It is enough to prove the statement in the case when we have $\lambda_1 = \lambda_2 - \epsilon_r$ for some $r \in [1, N]$. In this case we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) = \alpha_i$, where i is the residue of $(\lambda_1)_r$ modulo e. If $i \neq k$ then we have $\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \overline{\alpha}_{i'} = \phi(\alpha_i)$, where i' is as in (4.4). If $i = k$ then we have $\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \overline{\alpha}_k + \overline{\alpha}_{k+1} = \phi(\alpha_k)$. □

Proof of Proposition 4.3.18. By definition, $\mathbf{A}^{\nu}_{R}[\alpha] \subset O^{\nu}_{\mu,R}$ is the Serre subcategory of $O^{\nu}_{\mu,R}$ generated by Δ^{λ}_{R} such that the weight $\lambda \in P^{\nu}$ satisfies $\lambda \geqslant \rho_{\nu}$ and $\mathbf{wt}^{\delta}_{e}(\rho_{\nu}) - \mathbf{wt}^{\delta}_{e}(\lambda) = \alpha$. Here $\geqslant$ is the order defined before Lemma 4.3.16.

As $\theta^{\overline{\mu}}_{\mu}(\Delta^{\lambda}_{R})$ is isomorphic to $\Delta^{\Upsilon(\lambda)}_{\overline{R}}$, Lemma 4.3.19 implies that $\theta^{\overline{\mu}}_{\mu}(\mathbf{A}^{\nu}_{R}[\alpha])$ is the Serre subcategory of $O^{\nu}_{\overline{\mu},R}$ generated by $\Delta^{\overline{\lambda}}_{\overline{R}}$ for $\overline{\lambda} \in P^{\nu}$ such that $\overline{\lambda} \geqslant \Upsilon(\rho_{\nu})$ and $\mathbf{wt}^{\delta}_{e+1}(\Upsilon(\rho_{\nu})) - \mathbf{wt}^{\delta}_{e+1}(\overline{\lambda}) = \overline{\alpha}$.

Moreover, for each module $\Delta^{\overline{\lambda}}_{\overline{R}} \in O^{\nu}_{\overline{\mu},\overline{R}}$, the weight $\overline{\lambda}$ has no coordinates that are congruent to $k+1$ modulo $e+1$. Then $\overline{\lambda}$ satisfies $\overline{\lambda} \geqslant \rho_{\nu}$ if and only if it satisfies $\overline{\lambda} \geqslant \Upsilon(\rho_{\nu})$. We have $\mathbf{wt}^{\delta}_{e+1}(\rho_{\nu}) - \mathbf{wt}^{\delta}_{e+1}(\Upsilon(\rho_{\nu})) = \beta$. Thus $\theta^{\overline{\mu}}_{\mu}(\mathbf{A}^{\nu}_{R}[\alpha])$ is the Serre subcategory of $O^{\nu}_{\overline{\mu},\overline{R}}$ generated by the modules $\Delta^{\overline{\lambda}}_{\overline{R}}$ where $\overline{\lambda}$ runs over the set of all $\overline{\lambda} \in P^{\nu}$ such that $\overline{\lambda} \geqslant \rho_{\nu}$ and $\mathbf{wt}^{\delta}_{e+1}(\rho_{\nu}) - \mathbf{wt}^{\delta}_{e+1}(\overline{\lambda}) = \overline{\alpha} + \beta$. This implies $\theta^{\overline{\mu}}_{\mu}(\mathbf{A}^{\nu}_{R}[\alpha]) = \mathbf{A}^{\nu}_{\overline{R}}[\overline{\alpha} + \beta]$. $\square$

4.3.11 The category $\mathcal{A}$

From now on, to avoid cumbersome notation we will use the following abbreviations. First, for each $\alpha \in Q^{+}_{I}$ we set

$$\mathcal{A}^{\nu}_{R}[\alpha] = \mathbf{A}^{\nu}_{\overline{R}}[\beta + \overline{\alpha}], \qquad \mathcal{A}^{\nu}_{R}[d] = \bigoplus_{|\alpha|=d} \mathcal{A}^{\nu}_{R}[\alpha], \qquad \mathcal{A}^{\nu}_{R} = \bigoplus_{d\in\mathbb{N}} \mathcal{A}^{\nu}_{R}[d].$$

Next, we define the endofunctors $E_0, \cdots, E_{e-1}$, $F_0, \cdots, F_{e-1}$ of $\mathcal{A}^{\nu,\Delta}_{R}$ (or of $\mathcal{A}^{\nu}_{R}$ is R is a field) by

$$\begin{gathered} F_0 = \overline{F}_0\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \cdots, F_{k-1} = \overline{F}_{k-1}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \quad F_k = \overline{F}_{k+1}\overline{F}_{k}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \\ F_{k+1} = \overline{F}_{k+2}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \cdots, F_{e-1} = \overline{F}_{e}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \\ E_0 = \overline{E}_0\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \cdots, E_{k-1} = \overline{E}_{k-1}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \quad E_k = \overline{E}_{k}\overline{F}_{k+1}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \\ E_{k+1} = \overline{E}_{k+2}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}, \cdots, E_{e-1} = \overline{E}_{e}\big|_{\mathcal{A}^{\nu,\Delta}_{R}}. \end{gathered} \tag{4.24}$$

By definition, we have $E_i(\mathcal{A}^{\nu,\Delta}_{R}[\alpha]) \subset \mathcal{A}^{\nu,\Delta}_{R}[\alpha - \alpha_i]$ and $F_i(\mathcal{A}^{\nu,\Delta}_{R}[\alpha]) \subset \mathcal{A}^{\nu,\Delta}_{R}[\alpha + \alpha_i]$. Consider the endofunctors $E = \bigoplus_{i\in I} E_i$ and $F = \bigoplus_{i\in I} F_i$ of $\mathcal{A}^{\nu,\Delta}_{R}$. We have $E(\mathcal{A}^{\nu,\Delta}_{R}[d]) \subset \mathcal{A}^{\nu,\Delta}_{R}[d-1]$ and $F(\mathcal{A}^{\nu,\Delta}_{R}[d]) \subset \mathcal{A}^{\nu,\Delta}_{R}[d+1]$.

Let $\theta_{\alpha}\colon \mathbf{A}^{\nu}_{R}[\alpha] \to \mathcal{A}^{\nu}_{R}[\alpha]$ be the equivalence of categories in Proposition 4.3.18. Taking the sum over α's, we get an equivalence $\theta\colon \mathbf{A}^{\nu}_{R} \to \mathcal{A}^{\nu}_{R}$ and an equivalence $\theta_d\colon \mathbf{A}^{\nu}_{R}[d] \to \mathcal{A}^{\nu}_{R}[d]$. Moreover, we have the following commutative diagram of Grothendieck groups

$$\begin{tikzcd} {[\mathbf{A}^{\nu,\Delta}_{R}]} \arrow[r, "F_i"] \arrow[d, "\theta"'] & {[\mathbf{A}^{\nu,\Delta}_{R}]} \arrow[d, "\theta"'] \\ {[\mathcal{A}^{\nu,\Delta}_{R}]} \arrow[r, "F_i"] & {[\mathcal{A}^{\nu,\Delta}_{R}],} \end{tikzcd} \tag{4.25}$$

see Section 4.3.7.

For $\lambda \in \mathcal{P}^l_d$ we set $\overline{\Delta}[\lambda]_R = \Delta_{\overline{R}}^{\Upsilon(\rho_\nu+\lambda)} \in \mathcal{A}^\nu_R[d]$. By construction we have $\theta_d(\Delta[\lambda]_R) \simeq \overline{\Delta}[\lambda]_R$. Let $\overline{P}_R[\lambda]$, $\overline{\nabla}_R[\lambda]$ and $\overline{T}_R[\lambda]$ be the projective, the costandard and the tilting object with parameter λ in $\mathcal{A}^\nu_R[d]$.

4.3.12 The categorical representation in the category O over the field K

In this section we compare the categorical representation in $O^\nu_{-e,K}$ with the representation datum in $O^{\nu,\Delta}_{-e,R}$ introduced above. First, for each $\lambda \in P$ we define the following weight in $X^+_{\widetilde{I}}$

$$\widetilde{\mathbf{wt}}_e(\lambda) = \sum_{r=1}^{l} \left(\sum_{t=\nu_1+\cdots+\nu_{r-1}+1}^{\nu_1+\cdots+\nu_r} \widetilde{\varepsilon}_{(\lambda_t,r)} \right).$$

For $\widetilde{\mu} \in X^+_{\widetilde{I}}$ let $O^\nu_{\widetilde{\mu},K}$ be the Serre subcategory of $O^\nu_{-e,K}$ generated by the Verma modules Δ^λ_K such that $\widetilde{\mathbf{wt}}_e(\lambda) = \widetilde{\mu}$. This decomposition is a refinement of the decomposition $O^\nu_{-e,K} = \bigoplus_{\mu\in X_I} O^\nu_{\mu,K}$ introduced in Section 4.3.6. More precisely, we have

$$O^\nu_{\mu,K} = \bigoplus_{\widetilde{\mu}\in X^+_{\widetilde{I}},\pi_e(\widetilde{\mu})=\mu} O^\nu_{\widetilde{\mu},K}.$$

Similarly, there are decompositions

$$E = \bigoplus_{j\in\widetilde{I}} E_j, \qquad F = \bigoplus_{j\in\widetilde{I}} F_j \tag{4.26}$$

such that E_j and F_j map $O_{\widetilde{\mu},K}$ to $O_{\widetilde{\mu}+\widetilde{\alpha}_j,K}$ and $O_{\widetilde{\mu}-\widetilde{\alpha}_j,K}$ respectively. We set

$$E_i = \bigoplus_{j\in\widetilde{I},\pi_e(j)=i} E_j, \qquad F_i = \bigoplus_{j\in\widetilde{I},\pi_e(j)=i} F_j.$$

We have commutative diagrams

$$\begin{array}{ccc} O^{\nu,\Delta}_{-e,R} & \xrightarrow{E_i} & O^{\nu,\Delta}_{-e,R} \\ {\scriptstyle K\otimes_R\bullet}\downarrow & & \downarrow{\scriptstyle K\otimes_R\bullet} \\ O^{\nu,\Delta}_{-e,K} & \xrightarrow{E_i} & O^{\nu,\Delta}_{-e,K}, \end{array} \qquad \begin{array}{ccc} O^{\nu,\Delta}_{-e,R} & \xrightarrow{F_i} & O^{\nu,\Delta}_{-e,R} \\ {\scriptstyle K\otimes_R\bullet}\downarrow & & \downarrow{\scriptstyle K\otimes_R\bullet} \\ O^{\nu,\Delta}_{-e,K} & \xrightarrow{F_i} & O^{\nu,\Delta}_{-e,K}. \end{array}$$

For each element $\widetilde{\alpha} \in Q^+_{\widetilde{I}}$ let $\mathbf{A}^\nu_K[\widetilde{\alpha}]$ be the Serre subcategory of $\mathbf{A}^\nu_K$ generated by the Verma modules $\Delta[\lambda]_K$ such that $\lambda \in \mathcal{P}^l_{\widetilde{\alpha},\nu}$. Similarly to Section 4.3.9, we have

$$E_j(\mathbf{A}^\nu_K[\widetilde{\alpha}]) \subset \mathbf{A}^\nu_K[\widetilde{\alpha}-\widetilde{\alpha}_j], \quad F_j(\mathbf{A}^\nu_K[\widetilde{\alpha}]) \subset \mathbf{A}^\nu_K[\widetilde{\alpha}+\widetilde{\alpha}_j].$$

See [53, Sec. 7.4] for details.

Similarly, for $j \in \widetilde{I}$ we can define the endofunctor $\overline{E}_j$, $\overline{F}_j$ of the categories $O^{\nu}_{-e-1,\overline{K}}$ and $\overline{\mathbf{A}}^{\nu}_{K}$.

The decomposition $\mathbf{A}^{\nu}_{K}[\alpha] = \bigoplus_{\pi_e(\widetilde{\alpha})=\alpha} \mathbf{A}^{\nu}_{K}[\widetilde{\alpha}]$ yields a decomposition $\mathcal{A}^{\nu}_{K}[\alpha] = \bigoplus_{\pi_e(\widetilde{\alpha})=\alpha} \mathcal{A}^{\nu}_{K}[\widetilde{\alpha}]$. We also consider the endofunctors E_j, F_j of $\mathcal{A}^{\nu}_{K}$ such that for $j = (a,b) \in \widetilde{I}$ we have the following analogue of (4.24):

$$E_j = \begin{cases} \overline{E}_{(\Upsilon(a),b)}\big|_{\mathcal{A}^{\nu}_{K}} & \text{if } \pi_e(j) \neq k, \\ \overline{E}_{(\Upsilon(a),b)}\overline{E}_{(\Upsilon(a)+1,b)}\big|_{\mathcal{A}^{\nu}_{K}} & \text{if } \pi_e(j) = k, \end{cases}$$

$$F_j = \begin{cases} \overline{F}_{(\Upsilon(a),b)}\big|_{\mathcal{A}^{\nu}_{K}} & \text{if } \pi_e(j) \neq k, \\ \overline{F}_{(\Upsilon(a)+1,b)}\overline{F}_{(\Upsilon(a),b)}\big|_{\mathcal{A}^{\nu}_{K}} & \text{if } \pi_e(j) = k. \end{cases}$$

We have $E_j(\mathcal{A}^{\nu}_{K}[\widetilde{\alpha}]) \subset \mathcal{A}^{\nu}_{K}[\widetilde{\alpha} - \widetilde{\alpha}_j]$ and $F_j(\mathcal{A}^{\nu}_{K}[\widetilde{\alpha}]) \subset \mathcal{A}^{\nu}_{K}[\widetilde{\alpha} + \widetilde{\alpha}_j]$.

4.3.13 The modules $T_{\alpha,R}$, $\overline{T}_{\alpha,R}$

Consider the module $T_{\alpha,R} = F_{\alpha}(\Delta[\emptyset]_R)$ in $\mathbf{A}^{\nu}_{R}[\alpha]$ and the module $\overline{T}_{\alpha,R} = F_{\alpha}(\overline{\Delta}[\emptyset]_R)$ in $\mathcal{A}^{\nu}_{R}[\alpha]$. The commutativity of the diagram (4.25) implies that we have the following equality of classes $[\theta_{\alpha}(T_{\alpha,R})] = [\overline{T}_{\alpha,R}]$ in $[\mathcal{A}^{\nu}_{R}]$. The modules $T_{\alpha,R} \in \mathbf{A}^{\nu}_{R}[\alpha]$ and $\overline{T}_{\alpha,R} \in \mathcal{A}^{\nu}_{R}[\alpha]$ are tilting because $\Delta[\emptyset]_R \in \mathbf{A}^{\nu}_{R}[0]$ and $\overline{\Delta}[\emptyset]_R \in \mathcal{A}^{\nu}_{R}[0]$ are tilting and the functors F and $\overline{F}$ preserve tilting modules, see [53, Lem. A.23, Lem. 5.16 (b)]. Since a tilting module is characterized by its class in the Grothendieck group, we deduce that there is an isomorphism of modules

$$\theta_{\alpha}(T_{\alpha,R}) \simeq \overline{T}_{\alpha,R}. \tag{4.27}$$

From now on, we assume that $\nu_r \geqslant |\alpha|$ for each $r \in [1,l]$. Consider the weight

$$\Lambda_{\nu} := \sum_{r=1}^{l} \Lambda_{\nu_r} \in P_I. \tag{4.28}$$

We may abbreviate $\Lambda = \Lambda_{\nu}$. Assume that $R = \mathbf{k}$. The following result is proved in [53, Theorem 5.37].

Proposition 4.3.20. *The homomorphism $R_{\alpha,\mathbf{k}} \to \mathrm{End}(F_{\alpha}(\Delta[\emptyset]_{\mathbf{k}}))^{\mathrm{op}}$ induced by the categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $O^{\nu}_{-e,\mathbf{k}}$ yields an isomorphism $\psi_{\alpha,\mathbf{k}}\colon R^{\Lambda}_{\alpha,\mathbf{k}} \simeq \mathrm{End}(T_{\alpha,\mathbf{k}})^{\mathrm{op}}$.* $\square$

Consider the category $\overline{\mathcal{C}} = O^{\nu}_{-e-1,\overline{\mathbf{k}}}$. Now we must construct a similar isomorphism $\overline{\psi}_{\alpha,\mathbf{k}}\colon R^{\Lambda}_{\alpha,\mathbf{k}} \simeq \mathrm{End}(\overline{T}_{\alpha,\mathbf{k}})^{\mathrm{op}}$ coming from the $\widetilde{\mathfrak{sl}}_{e+1}$-categorical representation in $\overline{\mathcal{C}}$.

Lemma 4.3.9 yields a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in a subcategory $\mathcal{C}$ of $\overline{\mathcal{C}}$. As we have $\overline{\Delta}[\emptyset]_R \in \mathcal{C}$, there is an algebra homomorphism

$$R_{\alpha,\mathbf{k}} \to \mathrm{End}(\overline{T}_{\alpha,\mathbf{k}})^{\mathrm{op}}. \tag{4.29}$$

Lemma 4.3.21. *The homomorphism (4.29) factors through a homomorphism* $\overline{\psi}_{\alpha,\mathbf{k}}\colon R^{\Lambda}_{\alpha,\mathbf{k}} \to \mathrm{End}(\overline{T}_{\alpha,\mathbf{k}})^{\mathrm{op}}$.

Proof. The statement follows from Lemma 4.3.22 below applied to $M = \overline{\Delta}[\emptyset]_{\mathbf{k}}$. □

Lemma 4.3.22. *Let* $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_\mu$ *be a categorical representation of* $\mathfrak{g}_I$ *over* $\mathbf{k}$. *Let* $M \in \mathcal{C}$ *such that there are non-negative integers* t_i *for* $i \in I$ *such that*

- $\mathrm{End}(M) \simeq \mathbf{k}$,
- $E_i F_i(M) \simeq M^{\oplus t_i}, \quad \forall\, i \in I$.

Then, for each $d \in \mathbb{N}$, *the homomorphism* $R_{d,\mathbf{k}} \to \mathrm{End}(F^d(M))^{\mathrm{op}}$ *factors through the cyclotomic quotient with respect to the weight* $\sum_{i\in I} t_i \Lambda_i$.

Proof. It suffices to prove the statement for $d = 1$. By adjunction we have the following isomorphisms of vector spaces

$$\mathrm{Hom}(F_i(M), F_i(M)) \simeq \mathrm{Hom}(E_iF_i(M), M) \simeq \mathrm{Hom}(M, M)^{\oplus t_i} \simeq \mathbf{k}^{t_i}.$$

The image of x in $\mathrm{End}(F_i(M))$ is nilpotent. Thus it must be killed by the t_ith power because $\dim \mathrm{End}(F_i(M)) = t_i$.

□

Remark 4.3.23. The lemma above admits the following equivalent version.

Let $\mathcal{C} = \bigoplus_{\mu \in X_{\mathscr{F}}} \mathcal{C}_\mu$ be a categorical representation of $\mathfrak{g}_{\mathscr{F}}$ over $\mathbf{k}$. Let $M \in \mathcal{C}$ be an object such that there are non-negative integers t_i for $i \in \mathscr{F}$ such that t_i is non-zero only for finitely many i and

- $\mathrm{End}(M) \simeq \mathbf{k}$,
- $E_i F_i(M) \simeq M^{\oplus t_i}, \quad \forall\, i \in \mathscr{F}$.

For each $d \in \mathbb{N}$, the homomorphism $H_{d,\mathbf{k}}(q) \to \mathrm{End}(F^d(M))^{\mathrm{op}}$ factors through the cyclotomic quotient $H^Q_{d,\mathbf{k}}(q)$, where Q is a tuple of elements of $\mathscr{F}$ such that Q contains t_i copies of the element i for each $i \in \mathscr{F}$.

Now we are going to prove that the algebra homomorphism $\overline{\psi}_{\alpha,\mathbf{k}}$ constructed in Lemma 4.3.21 is an isomorphism. To do this, we will use a deformation argument. Recall from Corollary 4.2.18 that the cyclotomic KLR algebra $R^{\Lambda}_{\alpha,\mathbf{k}}$ is isomorphic to the cyclotomic Hecke algebra $H^{\nu}_{\alpha,\mathbf{k}}(\zeta_e)$. We will use the algebra $H^{\nu}_{\alpha,R}(q_e)$ as an R-version of $R^{\Lambda}_{\alpha,\mathbf{k}}$.

Remark 4.3.24. The homomorphism $H_{d,\overline{R}}(q_{e+1}) \to \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\mathrm{op}}$ coming from the representation datum structure commutes with base changes $\mathbf{k} \otimes_R \bullet$ and $K \otimes_R \bullet$, see [53, Prop. 8.14].

Lemma 4.3.25. *For each $d \in \mathbb{N}$ the homomorphism $H_{d,\overline{R}}(q_{e+1}) \to \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\mathrm{op}}$ extends to the algebra $\widehat{H}_{d,\overline{R}}(q_{e+1})$ in such a way that the idempotent $e(\mathbf{i})$ goes to the projector to $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ for each $\mathbf{i}$.*

Proof. We must prove that for each $\mathbf{i} \in \overline{I}^d$ we have the following.

(a_1) The element $(X_r - X_t)e(\mathbf{i}) \in H_{d,\overline{R}}(q_{e+1})$ acts on $F_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ by an invertible operator for each $r \neq t$ such that $i_r \neq i_t$.

(a_2) The element $(q_{e+1}X_r - X_t)e(\mathbf{i}) \in H_{d,\overline{R}}(q_{e+1})$ acts on $F_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ by an invertible operator for each $r \neq t$ such that $i_r + 1 \neq i_t$.

To prove this we need the following standard lemma.

Lemma 4.3.26. *Let M be an R-module. Let $\phi\colon M \to M$ be an endomorphism of M.*

(a) *If M is finitely generated over R and the endomorphism $\mathbf{k}\phi$ of $\mathbf{k}\otimes_R M$ is surjective then ϕ is surjective.*

(b) *If M is free over R and the endomorphism $K\phi$ of $K \otimes_R M$ is injective then ϕ is injective.* □

We have a commutative diagram

$$\begin{tikzcd}
H_{d,\overline{\mathbf{k}}}(\zeta_{e+1}) \arrow[r] & \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_{\mathbf{k}}))^{\mathrm{op}} \\
H_{d,\overline{R}}(q_{e+1}) \arrow[r] \arrow[u] \arrow[d] & \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\mathrm{op}} \arrow[u] \arrow[d] \\
H_{d,\overline{K}}(q_{e+1}) \arrow[r] & \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_K))^{\mathrm{op}}
\end{tikzcd}$$

By Remark 4.3.6 the top and the bottom vertical homomorphisms extend to homomorphisms

$$\widehat{H}_{d,\overline{\mathbf{k}}}(\zeta_{e+1}) \to \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_{\mathbf{k}}))^{\mathrm{op}}$$

and

$$\widehat{H}_{d,\overline{K}}(q_{e+1}) \to \mathrm{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_K))^{\mathrm{op}}.$$

In particular the elements from (a_1) and (a_2) go to elements of $H_{d,\overline{\mathbf{k}}}(\zeta_{e+1})$ and $H_{d,\overline{K}}(q_{e+1})$ that act on $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_{\mathbf{k}})$ and $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_K)$ by invertible operators. To conclude we apply Lemma 4.3.26 to each weight space of the R-module $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ (that is a free R-module of finite rank because $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ is tilting by [53, Lem. A.23, Lem. 5.16 (b)]). □

The homomorphism $\widehat{H}_{d,\overline{R}}(q_{e+1}) \to \mathrm{End}(\overline{F}^d(\Delta[\emptyset]_R))^{\mathrm{op}}$ from Lemma 4.3.25 yields a homomorphism $\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1}) \to \mathrm{End}(\overline{F}_{\overline{\alpha}}(\Delta[\emptyset]_R))^{\mathrm{op}}$ for each $\alpha \in Q_I^+$. Consider the sum $\mathbf{e} = \sum_{\mathbf{i}\in I^{\overline{\alpha}}_{\mathrm{ord}}} e(\mathbf{i})$ in $\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1})$. We get an algebra homomorphism

$$\mathbf{e}\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}} \tag{4.30}$$

because $\overline{T}_{\alpha,R} = \bigoplus_{\mathbf{i}\in I^{\overline{\alpha}}_{\mathrm{ord}}} \overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$.

Lemma 4.3.27. *The homomorphism (4.30) factors through $\widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$.*

Proof. We can construct a K-linear version $\mathbf{e}\widehat{H}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} \to \mathrm{End}(\overline{T}_{\alpha,K})^{\mathrm{op}}$ of the homomorphism (4.30). It factors through a homomorphism $\widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1}) \to \mathrm{End}(\overline{T}_{\alpha,K})^{\mathrm{op}}$ because $\overline{F}_{\mathbf{j}}(\overline{\Delta}[\emptyset]_K) = 0$ for each $\mathbf{j} \in \widetilde{I}^{\overline{\alpha}}_{\mathrm{un}}$. Here we set

$$\widetilde{I}^{\overline{\alpha}}_{\mathrm{un}} = \coprod_{\widetilde{\alpha}\in Q^+_{\widetilde{I}},\pi_{e+1}(\widetilde{\alpha})=\overline{\alpha}} \widetilde{I}^{\widetilde{\alpha}}_{\mathrm{un}}.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{e}\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} & \longrightarrow & \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{e}\widehat{H}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} & \longrightarrow & \mathrm{End}(\overline{T}_{\alpha,K})^{\mathrm{op}}. \end{array} \tag{4.31}$$

By definition, the kernel of the homomorphism

$$\mathbf{e}\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1})$$

is the intersection of the kernel of the homomorphism

$$\mathbf{e}\widehat{H}_{\overline{\alpha},\overline{K}}(q_{e+1})\mathbf{e} \to \widehat{SH}_{\overline{\alpha},\overline{K}}(q_{e+1})$$

with $\mathbf{e}\widehat{H}_{\overline{\alpha},\overline{R}}(q_{e+1})\mathbf{e}$. This proves the statement because the right vertical map in (4.31) is injective because the module $\overline{T}_{\alpha,R}$ is tilting. $\square$

Composing the homomorphism $\widehat{SH}_{\overline{\alpha},\overline{R}}(q_{e+1}) \to \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}}$ in Lemma 4.3.27 with the homomorphism $\Phi_{\alpha,R}$ in Lemma 4.2.21 yield an algebra homomorphism

$$\widehat{H}_{\alpha,R}(q_e) \to \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}} \tag{4.32}$$

Note that there is a surjective algebra homomorphism $\widehat{H}_{\alpha,R}(q_e) \to H^{\nu}_{\alpha,R}(q_e)$ (see also Corollary 4.2.18).

Lemma 4.3.28. *The homomorphism (4.32) factors through a homomorphism $\overline{\psi}_{\alpha,R}\colon H^{\nu}_{\alpha,R}(q_e) \to \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}}$.*

Proof. The proof is similar to the proof of Lemma 4.3.27. The K-linear version $\widehat{H}_{\overline{\alpha},K}(q_e) \to \mathrm{End}(\overline{T}_{\alpha,K})^{\mathrm{op}}$ of this homomorphism factors through the cyclotomic quotient $H^\nu_{d,K}(q_e)$ by Remark 4.3.23. Then we deduce the statement from the commutativity of the following diagram

$$\begin{array}{ccc} \widehat{H}_{\overline{\alpha},R}(q_e) & \longrightarrow & \mathrm{End}(\overline{T}_{\alpha,R})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \widehat{H}_{\overline{\alpha},K}(q_e) & \longrightarrow & \mathrm{End}(\overline{T}_{\alpha,K})^{\mathrm{op}}. \end{array}$$

□

The algebra homomorphism $\overline{\psi}_{\alpha,R}$ in Lemma 4.3.28 is an R-linear version of the homomorphism $\overline{\psi}_{\alpha,\mathbf{k}}$ in Lemma 4.3.21.

4.3.14 The proof of invertibility

The goal of this section is to prove that the homomorphism $\overline{\psi}_{\alpha,R}$ in Lemma 4.3.28 is an isomorphism.

Consider the functors

$$\begin{aligned} \Psi^\nu_\alpha &\colon \mathbf{A}^\nu_R[\alpha] \to \mathrm{mod}(H^\nu_{\alpha,R}(q_e)), \quad M \mapsto \mathrm{Hom}(T_{\alpha,R}, M), \\ \overline{\Psi}^\nu_\alpha &\colon \mathcal{A}^\nu_R[\alpha] \to \mathrm{mod}(H^\nu_{\alpha,R}(q_e)), \quad M \mapsto \mathrm{Hom}(\overline{T}_{\alpha,R}, M), \end{aligned}$$

where $\mathrm{Hom}(T_{\alpha,R}, M)$ and $\mathrm{Hom}(\overline{T}_{\alpha,R}, M)$ are considered as $H^\nu_{d,R}(q_e)$-modules with respect to the homomorphisms $\psi_{\alpha,R}$ and $\overline{\psi}_{\alpha,R}$.

Let us abbreviate $\Psi = \Psi^\nu_\alpha$, $\overline{\Psi} = \overline{\Psi}^\nu_\alpha$ $T_R = T_{\alpha,R}$ and $\overline{T}_R = \overline{T}_{\alpha,R}$. We may write Ψ_R, $\overline{\Psi}_R$ to specify the ring R. For $\lambda \in \mathcal{P}^l_{\alpha,\nu}$ denote by $S[\lambda]_R$ the Specht module of $H^\nu_{\alpha,R}(q_e)$. We will use similar notation for $\mathbf{k}$ or K instead of R. See also [53, Sec. 2.4.3].

Lemma 4.3.29. (*a*) *The homomorphism* $\overline{\psi}_{\alpha,K} \colon H^\nu_{\alpha,K}(q_e) \to \mathrm{End}(\overline{T}_K)^{\mathrm{op}}$ *is an isomorphism.*

(*b*) *For each* $\lambda \in \mathcal{P}^l_{\alpha,\nu}$ *we have* $\overline{\Psi}(\overline{\Delta}[\lambda]_K) \simeq S[\lambda]_K$.

Proof. First, we prove that the homomorphism $\overline{\psi}_{\alpha,K}$ is injective. The algebra $H^\nu_{\alpha,K}(q_e)$ is finite dimensional and semi-simple. Its center is spanned by the idempotents $e(\widetilde{\alpha})$ such that $\widetilde{\alpha} \in Q^+_{\widetilde{I}}$ and $\pi_e(\widetilde{\alpha}) = \alpha$. The idempotent $e(\widetilde{\alpha})$ acts on $T_K = F_\alpha(\overline{\Delta}[\emptyset]_K)$ by projection onto $F_{\widetilde{\alpha}}(\overline{\Delta}[\emptyset]_K)$. Thus, to prove the injectivity of $\overline{\psi}_{\alpha,K}$ we need to check that $F_{\widetilde{\alpha}}(\overline{\Delta}[\emptyset]_K)$ is nonzero whenever $e(\widetilde{\alpha})$ is nonzero. Similarly to the argument in Section 4.3.7, we see that the equivalence $\theta \colon \mathbf{A}^\nu_K \simeq \mathcal{A}^\nu_K$ yields an isomorphism of Grothendieck groups $[\mathbf{A}^\nu_K] \simeq [\mathcal{A}^\nu_K]$ that commutes with functors F_j. Thus the module $F_{\widetilde{\alpha}}(\overline{\Delta}[\emptyset]_K) \in \mathcal{A}^\nu_K$ is nonzero if and only if the module $F_{\widetilde{\alpha}}(\Delta[\emptyset]_K) \in \mathbf{A}^\nu_K$ is nonzero. By [53, Prop. 5.22 (d)], the module $F_{\widetilde{\alpha}}(\Delta[\emptyset]_K) \in \mathbf{A}^\nu_K$ is nonzero whenever $e(\widetilde{\alpha})$ is nonzero. Thus $\overline{\psi}_{\alpha,K}$ is injective.

Thus it is also surjective because

$$\dim_K H^\nu_{\alpha,K}(q_e) = \dim_K \mathrm{End}(T_K)^{\mathrm{op}} = \dim_K \mathrm{End}(\overline{T}_K)^{\mathrm{op}},$$

where the first equality holds by [53, Prop. 5.22 (d)] and the second holds by (4.27). This implies part (a).

The discussion above implies that $\overline{T}_K$ contains each $\overline{\Delta}[\lambda]_K$, $\lambda \in \mathcal{P}^l_\alpha$ as a direct factor. In particular $\overline{T}_K$ is a projective generator of $\mathcal{A}^\nu_K[\alpha]$. Thus $\overline{\Psi}_K$ is an equivalence of categories. It must take $\overline{\Delta}[\lambda]_K$ to $S[\lambda]_K$ because $S[\lambda]_K$ is the unique simple module in the block $\mathrm{mod}(H^\nu_{\tilde{\alpha},K}(q_e))$ of $\mathrm{mod}(H^\nu_{\alpha,K}(q_e))$. □

Lemma 4.3.30. (a) *The homomorphism* $\overline{\psi}_{\alpha,K}\colon H^\nu_{\alpha,R}(q_e) \to \mathrm{End}(\overline{T}_R)^{\mathrm{op}}$ *is an isomorphism.*

(b) *For each* $\lambda \in \mathcal{P}^l_{\alpha,\nu}$ *we have* $\overline{\Psi}(\overline{\Delta}[\lambda]_R) \simeq S[\lambda]_R$.

Proof. Consider the endomorphism u of $H^\nu_{\alpha,R}(q_e)$ obtained from the following chain of homomorphisms

$$u\colon H^\nu_{\alpha,R}(q_e) \overset{\overline{\psi}_{\alpha,R}}{\to} \mathrm{End}_{\mathcal{A}^\nu}(\overline{T}_R)^{\mathrm{op}} \overset{\theta_\alpha^{-1}}{\to} \mathrm{End}_{\mathbf{A}^\nu}(T_R)^{\mathrm{op}} \overset{\psi^{-1}_{\alpha,R}}{\to} H^\nu_{\alpha,R}(q_e).$$

The invertibility of $\overline{\psi}_{\alpha,R}$ is equivalent to the invertibility of u. By [53, Prop. 2.22] to prove that u is an isomorphism it is enough to show that its localization $Ku\colon H^\nu_{\alpha,K}(q_e) \to H^\nu_{\alpha,K}(q_e)$ is an isomorphism and that Ku induces the identity map on Grothendieck groups $[\mathrm{mod}(H^\nu_{\alpha,K}(q_e))] \to [\mathrm{mod}(H^\nu_{\alpha,K}(q_e))]$. The bijectivity of Ku follows from Lemma 4.3.29 (a).

Now we check the condition on the Grothendieck group. We already know from [53, Prop. 5.22 (c)] and the proof of Lemma 4.3.29 that Ψ_K and $\overline{\Psi}_K$ are equivalences of categories. Thus, by semisimplicity of the categories $\mathbf{A}^\nu_K[\alpha]$, $\mathcal{A}^\nu_K[\alpha]$ and $\mathrm{mod}(H^\nu_{\alpha,K}(q_e))$, we have an isomorphism of functors $\Psi_K \simeq \overline{\Psi}_K \circ \theta_\alpha$ because $\Psi_K(M) \simeq \overline{\Psi}_K \circ \theta_\alpha(M)$ for each $M \in \mathbf{A}^\nu_K[\alpha]$. This implies that Ku is the identity on the Grothendieck group. This proves part (a).

Part (b) follows from Lemma 4.3.29 and the characterization of Specht modules, see [53, Sec. 2.4.3]. □

Remark 4.3.31. There is no reason why the automorphism $u\colon H^\nu_{\alpha,R}(q_e) \to H^\nu_{\alpha,R}(q_e)$ in the proof of Lemma 4.3.30 should be identity. Because of this the functor Ψ has no reason to coincide with $\overline{\Psi}\circ\theta_\alpha$. However the automorphism u of $H^\nu_{\alpha,R}(q_e)$ induces an autoequivalence u^* of $\mathrm{mod}(H^\nu_{\alpha,R}(q_e))$ such that we have

$$\Psi = u^* \circ \overline{\Psi} \circ \theta_\alpha. \tag{4.33}$$

Now, specializing to $\mathbf{k}$, we obtain the following.

Corollary 4.3.32. (*a*) *The homomorphism* $\overline{\psi}_{\alpha,\mathbf{k}}\colon R^{\Lambda}_{\alpha,\mathbf{k}} \to \mathrm{End}(\overline{T}_{\mathbf{k}})^{\mathrm{op}}$ *is an isomorphism.*

(*b*) *For each* $\lambda \in \mathcal{P}^l_{\alpha,\nu}$ *we have* $\overline{\Psi}_{\mathbf{k}}(\overline{\Delta}[\lambda]_{\mathbf{k}}) \simeq S[\lambda]_{\mathbf{k}}$. □

4.3.15 The rational Cherednik algebras

Let R be a local commutative $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Let W be a complex reflection group. Denote by $S = S(W)$ and $\mathcal{A}$ the set of pseudo-reflections in W and the set of reflection hyperplanes respectively. Let $\mathfrak{h}$ be the reflection representation of W over R. Let $c\colon S \to R$ be a map which is constant on the W-conjugacy classes.

Denote by $\langle \bullet, \bullet\rangle$ the canonical pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. For each $s \in S$ fix a generator $\alpha_s \in \mathfrak{h}^*$ of $\mathrm{Im}(s\big|_{\mathfrak{h}^*} - 1)$ and a generator $\check{\alpha}_s \in \mathfrak{h}$ of $\mathrm{Im}(s\big|_{\mathfrak{h}} - 1)$ such that $\langle \alpha_s, \check{\alpha}_s\rangle = 2$.

Definition 4.3.33. The *rational Cherednik algebra* $H_c(W,\mathfrak{h})_R$ is the quotient of the smash product $RW \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \qquad [y, y'] = 0, \qquad [y, x] = \langle x, y\rangle - \sum_{s\in S} c_s \langle \alpha_s, y\rangle\langle x, \check{\alpha}_s\rangle s,$$

for each $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. Here $T(\bullet)$ denotes the tensor algebra.

Denote by $\mathcal{O}_c(W,\mathfrak{h})_R$ the category $\mathcal{O}$ of $H_c(W,\mathfrak{h})_R$, see [23, Sec. 3.2] and [53, Sec. 6.1.1]. Let E be an irreducible representation of $\mathbb{C}W$.

Definition 4.3.34. A *Verma module* associated with E is the following module in $\mathcal{O}_c(W,\mathfrak{h})_R$

$$\Delta_R(E) := \mathrm{Ind}^{H_c(W,\mathfrak{h})_R}_{RW\ltimes S(\mathfrak{h})}(RE).$$

Here each element $x \in \mathfrak{h}$ acts on RE by zero.

The category $\mathcal{O}_c(W,\mathfrak{h})_R$ is a highest weight category over R with standard modules $\Delta_R(E)$.

We call a subgroup W' of W *parabolic* if it is a stabilizer of some point of $b \in \mathfrak{h}$. In this case W' is a complex reflection group with reflection representation $\mathfrak{h}/\mathfrak{h}^{W'}$, where $\mathfrak{h}^{W'}$ is the set of W'-stable points in $\mathfrak{h}$. Moreover, the map $c\colon S(W) \to R$ restricts to a map $c\colon S(W') \to R$. There are induction and restriction functors

$${}^{\mathcal{O}}\mathrm{Ind}^W_{W'}\colon \mathcal{O}_c(W',\mathfrak{h}/\mathfrak{h}^{W'})_R \to \mathcal{O}_c(W,\mathfrak{h})_R, \quad {}^{\mathcal{O}}\mathrm{Res}^W_{W'}\colon \mathcal{O}_c(W,\mathfrak{h})_R \to \mathcal{O}_c(W',\mathfrak{h}/\mathfrak{h}^{W'})_R,$$

see [7]. The definitions of these functors depend on b but their isomorphism classes are independent of the choice of b.

The following lemma holds.

Lemma 4.3.35. *Assume that* W' *and* W'' *are conjugated parabolic subgroups in* W. *Let* $P \in \mathcal{O}_c(W,\mathfrak{h})_R$ *be a projective module. Then the following conditions are equivalent*

- *the module P is isomorphic to a direct factor of the module ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}(P')$ for some projective module $P' \in \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R$,*
- *the module P is isomorphic to a direct factor of a module ${}^{\mathcal{O}}\mathrm{Ind}_{W''}^{W}(P'')$ for some projective module $P'' \in \mathcal{O}_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$.*

Proof. Let w be an element of W such that $wW'w^{-1} = W''$. The conjugation by w yields an isomorphism $W' \simeq W''$. Hence, the element w takes $\mathfrak{h}^{W'}$ to $\mathfrak{h}^{W''}$. Thus we get an isomorphism of algebras $H_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \simeq H_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$ and an equivalence of categories $\mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \simeq \mathcal{O}_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$. Moreover, the conjugation by w yields an automorphism t of $H_c(W, \mathfrak{h})_R$ such that for each $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $u \in W$ we have

$$t(x) = w(x), \quad t(y) = w(y), \quad t(u) = wuw^{-1}.$$

The following diagram of functors is commutative up to equivalence of functors

$$\begin{tikzcd}
\mathcal{O}_c(W,\mathfrak{h})_R & \mathcal{O}_c(W,\mathfrak{h})_R \arrow[l, "t^*"'] \\
\mathcal{O}_c(W',\mathfrak{h}/\mathfrak{h}^{W'})_R \arrow[u, "{}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W}"] & \mathcal{O}_c(W'',\mathfrak{h}/\mathfrak{h}^{W''})_R \arrow[l] \arrow[u, "{}^{\mathcal{O}}\mathrm{Ind}_{W''}^{W}"']
\end{tikzcd}$$

To conclude, we only need to prove that the pull-back t^* induces the identity map on the Grothendieck group of $\mathcal{O}_c(W, \mathfrak{h})_R$ (and thus it maps each projective module to an isomorphic one). This is true because t^* maps each Verma module $\Delta(E)_R$ to an isomorphic one because the representation E of W does not change the isomorphism class when we twist the W-action by an inner automorphism. □

4.3.16 The cyclotomic rational Cherednik algebra

From now on, we assume that R is either a local analytic deformation ring in general position of dimension $\leqslant 2$ or a field. Let $\mathbf{k}$ and K be its residue field and the field of fractions respectively.

Let $\Gamma \simeq \mathbb{Z}/l\mathbb{Z}$ be the group of complex lth roots of unity and set $\Gamma_d = \mathfrak{S}_d \ltimes \Gamma^d$. For $\gamma \in \Gamma$, $r \in [1, l]$ denote by γ_r the element of Γ^d having γ at the position r and 1 at other positions. Let $s_{r,t}$ be the transposition in $\mathfrak{S}_d$ exchanging r and t. For $\gamma \in \Gamma$, $r, t \in [1, l]$ set $s_{r,t}^{\gamma} := s_{r,t}\gamma_r\gamma_t^{-1} \in \Gamma_d$. From now on we suppose that the group W is Γ_d and $\mathfrak{h} = R^d$ is the obvious reflection representation of Γ_d. Assume also that $h, h_1, \cdots, h_{l-1}$ are some elements of R and set $h_{-1} = h_{l-1}$. Let us chose the parameter c in the following way

$$c(s_{r,t}^{\gamma}) = -h \qquad \text{for each } r, t \in [1, t],\ r \neq t,\ \gamma \in \Gamma,$$

$$c(\gamma_r) = -\frac{1}{2}\sum_{p=0}^{l-1} \gamma^{-p}(h_p - h_{p-1}) \qquad \text{for each } r \in [1, l],\ \gamma \in \Gamma,\ \gamma \neq 1.$$

Let $\nu_1, \cdots, \nu_l$ be as above. We set

$$h = -1/\kappa, \quad h_p = -(\nu_{p+1} + \tau_{p+1})/\kappa - p/l, \quad p \in [1, l-1].$$

Let us abbreviate $\mathcal{O}^\nu_R[d] = \mathcal{O}_c(\Gamma_d, R^d)_R$. Consider the KZ-functor $\mathrm{KZ}^\nu_d \colon \mathcal{O}^\nu_R[d] \to \mathrm{mod}(H^\nu_{d,R}(q_e))$ introduced in [53, Sec. 6]. Denote by ${}^*\mathcal{O}^\nu_R[d]$ the category defined in the same way as $\mathcal{O}^\nu_R[d]$ with replacement of $(\nu_1, \cdots, \nu_l)$ by $(-\nu_l, \cdots, -\nu_1)$ and $(\tau_1, \cdots, \tau_l)$ by $(-\tau_l, \cdots, -\tau_1)$. Similarly, denote by ${}^*H^\nu_{d,R}(q_e)$ the affine Hecke algebra defined in the same way as $H^\nu_{d,R}(q_e)$ with the replacement of parameters as above. There is also a KZ-functor ${}^*\mathrm{KZ}^\nu_d \colon {}^*\mathcal{O}^\nu_R[d] \to \mathrm{mod}({}^*H^\nu_{d,R}(q_e))$.

The simple $\mathbb{C}\Gamma_d$-modules are labeled by the set $\mathcal{P}^l_d$. We write $E(\lambda)$ for the simple module corresponding to λ. Set $\Delta[\lambda]_R = \Delta(E(\lambda))_R$. Similarly, write $P[\lambda]_R$ and $T[\lambda]_R$ for the projective and tilting object in $\mathcal{O}^\nu_R[d]$ with index λ.

The category ${}^*\mathcal{O}^\nu_R[d]$ is the Ringel dual of the category $\mathcal{O}^\nu_R[d]$, see [53, Sec. 6.2.4]. In particular we have an equivalence between the categories of standardly filtered objects $\mathscr{R} \colon {}^*\mathcal{O}^\nu_R[d]^\Delta \to (\mathcal{O}^\nu_R[d]^\Delta)^{\mathrm{op}}$. Let $\mathrm{proj}(R)$ be the category of projective finitely generated R-modules. There is an algebra isomorphism

$$\iota \colon H^\nu_{d,R}(q_e) \to ({}^*H^\nu_{d,R}(q_e))^{\mathrm{op}}, \quad T_r \mapsto -q_e T_r^{-1}, \ X_r \mapsto X_r^{-1},$$

see [53, Sec. 6.2.4]. It induces an equivalence

$$\mathscr{R}_H = \iota^*(\bullet^\vee) \colon \mathrm{mod}({}^*H^\nu_{d,R}(q_e)) \cap \mathrm{proj}(R) \to (\mathrm{mod}(H^\nu_{d,R}(q_e)) \cap \mathrm{proj}(R))^{\mathrm{op}},$$

where $\bullet^\vee$ is the dual as an R-module. By [53, (6.3)], the following diagram of functors is commutative

$$\begin{tikzcd} {}^*\mathcal{O}^\nu_R[d]^\Delta \arrow[r, "\mathscr{R}"] \arrow[d, "{}^*\mathrm{KZ}^\nu_d"'] & (\mathcal{O}^\nu_R[d]^\Delta)^{\mathrm{op}} \arrow[d, "\mathrm{KZ}^\nu_d"'] \\ \mathrm{mod}({}^*H^\nu_{d,R}(q_e)) \cap \mathrm{proj}(R) \arrow[r, "\mathscr{R}_H"] & (\mathrm{mod}(H^\nu_{d,R}(q_e)) \cap \mathrm{proj}(R))^{\mathrm{op}}. \end{tikzcd} \tag{4.34}$$

Lemma 4.3.36. *Assume that $d = 1$. For each l-partition of 1 λ we have an isomorphism of $H^\nu_{1,R}(q_e)$-modules $\mathrm{KZ}^\nu_1(P[\lambda]_R) \simeq \overline{\Psi}^\nu_1(\overline{T}[\lambda]_R)$.*

Proof. The proof is similar to the proof of [53, Prop. 6.7]. The commutativity of the diagram (4.34) implies

$$\mathrm{KZ}^\nu_1(P[\lambda]_R) = \mathrm{KZ}^\nu_1(\mathscr{R}(T[\lambda]_R)) = \mathscr{R}_H({}^*\mathrm{KZ}^\nu_1(T[\lambda]_R)).$$

To conclude, we just need to compare the highest weight covers $\mathscr{R}_H \circ {}^*\mathrm{KZ}^\nu_1$ and $\overline{\Psi}^\nu_1$ of $H^\nu_{1,R}(q_e)$ using Lemma 4.3.29 (b) and [53, Prop. 2.21]. $\square$

Let $O^+_{\mu,R}$ be the affine parabolic category $\mathcal{O}$ associated with the parabolic type consisting of the single block of size $\nu_1+\cdots+\nu_l$. We define the categories $\mathbf{A}^+_R[d]$, $\mathcal{A}^+_R[d]$ and $O^+_R[d]$ similarly. In this case we will also write the upper index $+$ in the notation of modules and functors (for example $\Delta^+[\lambda]_R$, $T^+_{d,R}$, KZ^+_d, etc.) Let also $H^+_{d,R}(q_e)$ be the cyclotomic Hecke algebra with $l=1$. It is isomorphic to the Hecke algebra of $\mathfrak{S}_d$.

We can prove the following result.

Lemma 4.3.37. *For each* $\lambda\in\mathcal{P}^1_2$ *we have* $\mathrm{KZ}^+_2(P^+[\lambda]_R)\simeq\overline{\Psi}{}^+_2(\overline{T}{}^+[\lambda]_R)$. $\square$

Proof. Similarly to the proof of Lemma 4.3.36 we compare the highest weight covers $\mathscr{R}_H\circ{}^*\mathrm{KZ}^+_2$ and $\overline{\Psi}{}^+_2$ of $H^+_{2,R}(q_e)$ using Lemma 4.3.29 (b) and [53, Prop. 2.21].

$\square$

Denote by $\mathrm{Ind}^{d,\nu}_{d,+}$ the induction functor with respect to the inclusion $H^+_{d,R}(q_e)\subset H^\nu_{d,R}(q_e)$. We will also need the following lemma.

Lemma 4.3.38. *Assume* $\nu_r\geqslant 2$ *for each* $r\in[1,l]$. *Assume also that* $e>2$. *For each* $\lambda\in\mathcal{P}^1_2$ *there exists a tilting module* $\overline{T}_{\lambda,R}\in\mathcal{A}^\nu_R[2]$ *such that* $\Psi^\nu_2(\overline{T}_{\lambda,R})\simeq\mathrm{Ind}^{2,\nu}_{2,+}(\overline{\Psi}{}^+_2(\overline{T}{}^+[\lambda]_R))$.

Proof. Set $\lambda^+=(2)$, $\lambda^-=(1,1)$. We have $\zeta_e\neq-1$ because $e>2$. In this case the algebra $H^+_{2,\mathbf{k}}(\zeta_e)$ is semi-simple. The category $\mathcal{A}^+_{\mathbf{k}}[2]$ is also semi-simple. This implies

$$\overline{T}{}^+_{2,R}\simeq\overline{\Delta}[\lambda^+]_R\oplus\overline{\Delta}[\lambda^-]_R=\overline{T}[\lambda^+]_R\oplus\overline{T}[\lambda^-]_R.$$

By definition, we have $\overline{\Psi}{}^+_2(\overline{T}{}^+_{2,R})\simeq H^+_{2,R}(q_e)$ and $\overline{\Psi}{}^\nu_2(\overline{T}_{2,R})\simeq H^\nu_{2,R}(q_e)$. This implies

$$\overline{\Psi}{}^\nu_2(\overline{T}_{2,R})\simeq\mathrm{Ind}^{2,\nu}_{2,+}(\overline{\Psi}{}^+_2(\overline{T}{}^+_{2,R})).$$

By the proof of [53, Prop. 6.8], the functor Ψ^ν_2 takes indecomposable factors of $T_{2,R}$ to indecomposable modules. Thus, by (4.33), the functor $\overline{\Psi}{}^\nu_2$ takes indecomposable factors of $\overline{T}_{2,R}$ to indecomposable modules. Thus there is a decomposition $\overline{T}_{2,R}=\overline{T}_{\lambda^+,R}\oplus\overline{T}_{\lambda^-,R}$ such that $\overline{T}_{\lambda^+,R}$, $\overline{T}_{\lambda^-,R}$ satisfy the required properties.

$\square$

4.3.17 Proof of Theorem 4.1.4

In this section we finally give a proof of over main result.

A priori there is no reason to have the following isomorphism of functors $\Psi^\nu_\alpha\simeq\overline{\Psi}{}^\nu_\alpha\circ\theta_\alpha$. However, we can modify the equivalence θ_α to make this true.

For $d_1 < d_2$ we have an inclusion $H^\nu_{d_1,R}(q_e) \subset H^\nu_{d_2,R}(q_e)$. Let $\mathrm{Ind}_{d_1}^{d_2}\colon \mathrm{mod}(H^\nu_{d_1,R}(q_e)) \to \mathrm{mod}(H^\nu_{d_2,R}(q_e))$ be the induction with respect to this inclusion. The following lemma can be proved similarly to [53, Lem. 5.41].

Lemma 4.3.39. *Assume that $\nu_r > d$ for each $r \in [1,l]$. Then the following diagram of functors is commutative.*

$$\begin{tikzcd}
\mathcal{A}^\nu_R[d] \arrow[r, "F"] \arrow[d, "\overline{\Psi}^\nu_d"'] & \mathcal{A}^\nu_R[d+1] \arrow[d, "\overline{\Psi}^\nu_{d+1}"'] \\
\mathrm{mod}(H^\nu_{d,R}(q_e)) \arrow[r, "\mathrm{Ind}_d^{d+1}"] & \mathrm{mod}(H^\nu_{d+1,R}(q_e))
\end{tikzcd}$$

□

For a partition λ denote by λ^* the transposed partition. For an l-partition $\lambda = (\lambda_1, \cdots, \lambda_l)$ set $\lambda^* = ((\lambda_l)^*, \cdots, (\lambda_1)^*)$. There is an algebra isomorphism

$$\mathrm{IM}\colon H^\nu_{d,R}(q_e) \to {}^*H^\nu_{d,R}(q_e), \quad T_r \mapsto -q_e T_r^{-1},\ X_r \mapsto X_r^{-1},$$

see [53, Sec. 6.2.4]. Let $\mathrm{IM}^*\colon \mathrm{mod}({}^*H^\nu_{d,R}(q_e)) \to \mathrm{mod}(H^\nu_{d,R}(q_e))$ be the induced equivalence of categories. We have

$$\mathrm{IM}^*(S[\lambda^*]_R) \simeq S[\lambda]_R. \tag{4.35}$$

The following proposition is proved in [53, Thm. 6.9].

Proposition 4.3.40. *Assume that $\nu_r \geqslant d$ for each $r \in [1,l]$. Then there is an equivalence of categories $\gamma_d\colon {}^*\mathcal{O}^\nu_R[d] \simeq \mathbf{A}^\nu_R[d]$ taking $\Delta[\lambda^*]_R$ to $\Delta[\lambda]_R$. Moreover, we have the following isomorphism of functors $\Psi^\nu_d \circ \gamma_d \simeq \mathrm{IM}^* \circ {}^*\mathrm{KZ}^\nu_d$.*

Now, we prove a similar statement for $\mathcal{A}^\nu_R[d]$. For each reflection hyperplane H of the complex reflection group Γ_d let $W_H \subset \Gamma_d$ be the pointwise stabilizer of H.

Proposition 4.3.41. *Assume that $\nu_r \geqslant d$ for each $r \in [1,l]$ and $e > 2$. There is an equivalence of categories $\overline{\gamma}_d\colon {}^*\mathcal{O}^\nu_R[d] \simeq \mathcal{A}^\nu_R[d]$, taking $\Delta[\lambda^*]_R$ to $\overline{\Delta}[\lambda]_R$. Moreover, we have the following isomorphism of functors $\overline{\Psi}^\nu_d \circ \overline{\gamma}_d \simeq \mathrm{IM}^* \circ {}^*\mathrm{KZ}^\nu_d$.*

Proof. The proof is similar to the proof of [53, Thm. 6.9]. We set $\mathcal{C} = {}^*\mathcal{O}^\nu_R[d]$, $\mathcal{C}' = \mathcal{A}^\nu_R[d]$. Consider the following functors

$$\begin{aligned}
&Y\colon \mathcal{C} \to \mathrm{mod}(H^\nu_{d,R}(q_e)), \quad && Y = \mathrm{IM}^* \circ {}^*\mathrm{KZ}^\nu_d,\\
&Y'\colon \mathcal{C}' \to \mathrm{mod}(H^\nu_{d,R}(q_e)), \quad && Y' = \overline{\Psi}^\nu_d.
\end{aligned}$$

By [53, Prop. 2.20] it is enough to check the following four conditions.

(1) We have $Y(\Delta[\lambda^*]_R) \simeq Y'(\Delta[\lambda]_R)$ and the bijection $\Delta[\lambda^*]_R \mapsto \Delta[\lambda]_R$ between the sets of standard objects in $\mathcal{C}$ and $\mathcal{C}'$ respects the highest weight orders.
(2) The functor Y is fully faithful on $\mathcal{C}^\Delta$ and $\mathcal{C}^\nabla$.
(3) The functor Y' is fully faithful on $\mathcal{C}'^\Delta$ and $\mathcal{C}'^\nabla$.
(4) For each reflection hyperplane H of Γ_d and each projective module $P \in \mathcal{O}(W_H)_R$ we have

$$\mathrm{KZ}_d^\nu({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d} P) \in F'(\mathcal{C}'^{\mathrm{tilt}}).$$

It is explained in the proof of [53, Thm. 6.9] that condition (4) announced here implies the fourth condition in [53, Prop. 2.20].

We have $Y(\Delta[\lambda^*]_R) \simeq Y'(\Delta[\lambda]_R)$ by Lemma 4.3.30 (b), [53, Lem. 6.6] and (4.35). The composition of the equivalence $\theta_d\colon \mathbf{A}_R^\nu[d] \to \mathcal{A}_R^\nu[d]$ with the equivalence $\gamma_d\colon {}^*\mathcal{O}_R^\nu[d] \simeq \mathbf{A}_R^\nu[d]$ yields an equivalence of highest weight categories $\mathcal{C} \simeq \mathcal{C}'$ that takes $\Delta[\lambda^*]_R$ to $\Delta[\lambda]_R$. This implies (1).

Condition (2) is already checked in [53, Sec. 6.3.2].

The functor Ψ_d^ν is fully faithful on $\mathbf{A}_R^{\nu,\Delta}[d]$ and $\mathbf{A}_R^{\nu,\nabla}[d]$ by [53, Thm. 5.37 (c)]. Thus the functor $\overline{\Psi}_d^\nu$ is fully faithful on $\mathcal{A}_R^{\nu,\Delta}[d]$ and $\mathcal{A}_R^{\nu,\nabla}[d]$ by (4.33). This implies (3).

Let us check condition (4). There are two possibilities for the hyperplane H.

- The hyperplane is $\mathrm{Ker}(\gamma_r - 1)$ for $r \in [1,d]$. By Lemma 4.3.35, we can assume that $H = \mathrm{Ker}(\gamma_1 - 1)$. By Lemma 4.3.36 there exists a tilting module $\overline{T} \in \mathcal{A}_R^\nu[1]$ such that $\mathrm{KZ}_1^\nu(P) \simeq \overline{\Psi}_1^\nu(\overline{T})$. We get

$$\mathrm{KZ}_d^\nu({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d} P) \simeq \mathrm{Ind}_1^d(\mathrm{KZ}_1^\nu(P)) \simeq \mathrm{Ind}_1^d(\overline{\Psi}_1^\nu(\overline{T})) \simeq \overline{\Psi}_d^\nu(F^{d-1}(\overline{T})).$$

  Here the first isomorphism follows from [53, (6.1)], the third isomorphism follows from Lemma 4.3.39.
- The hyperplane is $\mathrm{Ker}(s_{r,t}^\gamma - 1)$ for $r,t \in [1,d]$, $\gamma \in \Gamma$. By Lemma 4.3.35, we can assume that $H = \mathrm{Ker}(s_{1,2})$. By Lemma 4.3.37 there is a tilting module $\overline{T}^+ \in \mathcal{A}_R^+[2]$ such that $\mathrm{KZ}_2^+(P) \simeq \overline{\Psi}_2^+(\overline{T}^+)$. By Lemma 4.3.38 there is a tilting module $\overline{T} \in \mathcal{A}_R^\nu[2]$ such that $\mathrm{Ind}_{2,+}^{2,\nu}(\overline{\Psi}_2^+(\overline{T}^+)) \simeq \overline{\Psi}_2^\nu(\overline{T})$. Thus we get $\mathrm{Ind}_{2,+}^{2,\nu}\mathrm{KZ}_2^+(P) \simeq \overline{\Psi}_2^\nu(\overline{T})$.

We obtain

$$\mathrm{KZ}_d^\nu({}^{\mathcal{O}}\mathrm{Ind}_{W_H}^{\Gamma_d} P) \simeq \mathrm{Ind}_{2,+}^{d,\nu}(\mathrm{KZ}_2^+(P)) \simeq \mathrm{Ind}_{2,\nu}^{d,\nu}(\overline{\Psi}_2^\nu(\overline{T})) \simeq \Psi_d^\nu(F^{d-2}(\overline{T})).$$

Here the first isomorphism follows from [53, (6.1)], the third isomorphism follows from Lemma 4.3.39. □

Now, composing the equivalences of categories in Propositions 4.3.40, 4.3.41 we obtain the following result.

Corollary 4.3.42. *Assume that $\nu_r \geqslant d$ for each $r \in [1, l]$ and $e > 2$. There is an equivalence of categories $\theta'_d \colon \mathbf{A}^\nu_R[d] \simeq \mathcal{A}^\nu_R[d]$ such that we have the following isomorphism of functors $\overline{\Psi}^\nu_d \circ \theta'_d \simeq \Psi^\nu_d$.* □

For each $\alpha \in Q_I^+$ such that $|\alpha| = d$ the equivalence $\theta'_d \colon \mathbf{A}^\nu_R[d] \simeq \mathcal{A}^\nu_R[d]$ restricts to an equivalence $\theta'_\alpha \colon \mathbf{A}^\nu_R[\alpha] \simeq \mathcal{A}^\nu_R[\alpha]$ such that we have an isomorphism of functors $\overline{\Psi}^\nu_\alpha \circ \theta'_\alpha \simeq \Psi^\nu_\alpha$.

From now on we work over the field $\mathbb{C}$. Recall that we fixed an isomorphism $R^\Lambda_\alpha \simeq H^\nu_\alpha(\zeta_e)$. The following lemma can be proved by the method used in [53, Sec. 5.9].

Lemma 4.3.43. *Assume that $\nu_r > |\alpha|$ for each $r \in [1, l]$. The following diagrams are commutative modulo an isomorphism of functors.*

$$\begin{array}{ccc} \mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathbf{A}^\nu[\alpha+\alpha_k] \\ {\scriptstyle \Psi^\nu_\alpha}\big\downarrow & & {\scriptstyle \Psi^\nu_{\alpha+\alpha_k}}\big\downarrow \\ \mathrm{mod}(R^\Lambda_\alpha) & \xrightarrow{F^\Lambda_k} & \mathrm{mod}(R^\Lambda_{\alpha+\alpha_k}) \end{array}$$

$$\begin{array}{ccc} \mathcal{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathcal{A}^\nu[\alpha+\alpha_k] \\ {\scriptstyle \overline{\Psi}^\nu_\alpha}\big\downarrow & & {\scriptstyle \overline{\Psi}^\nu_{\alpha+\alpha_k}}\big\downarrow \\ \mathrm{mod}(R^\Lambda_\alpha) & \xrightarrow{F^\Lambda_k} & \mathrm{mod}(R^\Lambda_{\alpha+\alpha_k}) \end{array}$$

□

Now, Theorem 4.1.4 follows from the following one.

Theorem 4.3.44. *Assume that $\nu_r > |\alpha|$ for each $r \in [1, l]$ and $e > 2$. Then the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathcal{A}^\nu[\alpha+\alpha_k] \\ {\scriptstyle \theta'_\alpha}\big\uparrow & & {\scriptstyle \theta'_{\alpha+\alpha_k}}\big\uparrow \\ \mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathbf{A}^\nu[\alpha+\alpha_k]. \end{array}$$

Proof. The result follows from Corollary 4.3.42, Lemma 4.3.43 and an argument similar to [54, Lem. 2.4]. □

Annexe A

Appendix to Chapter 3

A.1 Graded decomposition numbers

The goal of this appendix is to calculate the graded decomposition numbers in the affine parabolic category $\mathcal{O}$ needed to prove Lemma 3.3.22.

We keep the notation from Sections 3.3.3, 3.3.4. Let e be an integer > 0. Let $\mathbf{P}$ be the set of proper subsets of $\widehat{\Phi}$. We will refer to elements of $\mathbf{P}$ as *parabolic types*. Suppose $\nu \in \mathbf{P}$. Let $\widehat{\mathfrak{p}}_\nu$ be the unique parabolic subalgebra containing $\widehat{\mathfrak{b}}$ whose set of roots is generated by $\widehat{\Phi} \cup (-\nu)$. We will say that a weight $\lambda \in \widehat{\mathfrak{h}}^*$ is ν-dominant (resp. ν-antidominant) if $\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $\alpha_k \in \nu$ (resp. if $-\lambda(\alpha_k^\vee) \in \mathbb{N}$ for each $\alpha_k \in \nu$).

Let $\mathcal{O}^\nu$ be the category of $\widehat{\mathfrak{g}}$-modules such that $M = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*} M_\lambda$ with $M_\lambda = \{m \in M; xm = \lambda(x)m, \forall x \in \widehat{\mathfrak{h}}\}$ and $U(\widehat{\mathfrak{p}}_\nu)m$ is finite dimensional for each $m \in M$.

Let $\widehat{W}$ be as in Section 3.3.3. Let $\leqslant$ be the Bruhat order on $\widehat{W}$. For each $\lambda \in \widehat{\mathfrak{h}}^*$, $w \in \widehat{W}$ we set $w \cdot \lambda = w(\lambda + \widehat{\rho}) - \widehat{\rho}$. Let S be the set of simple reflections in $\widehat{W}$. Fix $\mu \in \mathbf{P}$. Let $W_\mu \subset \widehat{W}$ be the parabolic subgroup generated by the simple reflection corresponding to μ. Recall from Section 3.3.4 that for $\lambda \in P$ we set $\widetilde{\lambda}^- = z_\lambda \delta + \lambda - (e+N)\Lambda_0$. Set also $\widetilde{\lambda}^+ = z_\lambda \delta + \lambda + (e-N)\Lambda_0$. Let $o_{\mu,-} \in \widehat{\mathfrak{h}}^*$ (resp. $o_{\mu,+} \in \widehat{\mathfrak{h}}^*$) be a weight of the form $\widetilde{\lambda}^-$ (resp. a weight of the form $\widetilde{\lambda}^+$) with $\lambda \in P$ such that $o_{\mu,-} + \widehat{\rho}$ is antidominant (resp. $o_{\mu,+} + \widehat{\rho}$ is dominant) and the stabilizer of $o_{\mu,-}$ (resp. $o_{\mu,+}$) in $\widehat{W}$ under the dot-action is equal to W_μ. Here the notation is as in Section 3.3.4. Let $\leqslant$ be the partial order on $\widehat{\mathfrak{h}}^*$ given by $\lambda_1 \leqslant \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple roots with coefficients in $\mathbb{N}$.

Let $\mathcal{O}^\nu_{\mu,\pm} \subset \mathcal{O}^\nu$ be the full subcategory consisting of the modules such that the highest weight of their simple subquotients are conjugated to $o_{\mu,\pm}$ under the dot-action. For a weight $\lambda \in P$ such that $\widetilde{\lambda}^\pm$ is ν-dominant, let $V^\nu(\widetilde{\lambda}^\pm)$ be the parabolic Verma module with highest weight $\widetilde{\lambda}^\pm$. We write just $V(\widetilde{\lambda}^\pm)$ if $\nu = \emptyset$. Let $L(\widetilde{\lambda}^\pm)$ be the unique simple quotient of $V(\widetilde{\lambda}^\pm)$.

Let $I_{\mu,-}$ (resp. $I_{\mu,+}$) be the set of shortest (resp. longest) representatives of the cosets $\widehat{W}/W_\mu$ in $\widehat{W}$ and $I^\nu_{\mu,-} \subset I_{\mu,-}$ (resp. $I^\nu_{\mu,+} \subset I_{\mu,+}$) be the subset of elements w such that $w \cdot o_{\mu,-}$ (resp. $w \cdot o_{\mu,+}$) is ν-dominant.

Fix $v \in I^\nu_{\mu,-}$ and $u \in I^\nu_{\mu,+}$. Set

$$^vI^\nu_{\mu,-} = \{w \in I^\nu_{\mu,-}; w \leqslant v\}, \quad ^uI^\nu_{\mu,+} = \{w \in I^\nu_{\mu,+}; w \leqslant u\}.$$

Remark A.1.1. There is a bijection $I^\nu_{\mu,-} \to I^\mu_{\nu,+},\ x \mapsto x^{-1}$. If $v \in I^\nu_{\mu,-}$ and $u = v^{-1} \in I^\mu_{\nu,+}$, then there is a bijection $^vI^\nu_{\mu,-} \to {}^uI^\mu_{\nu,+},\ x \mapsto x^{-1}$. See [56, Cor. 3.3].

Let $^v\mathcal{O}^\nu_{\mu,-}$ be the Serre subcategory generated by the modules $L(w \cdot o_{\mu,-})$ with $w \in {}^vI^\nu_{\lambda,-}$ of the subcategory of $\mathcal{O}^\nu_{\mu,-}$ consisting of finitely generated $U(\widehat{\mathfrak{g}})$-modules. Let $^u\mathcal{O}^\nu_{\mu,+}$ be the Serre quotient by the Serre subcategory generated by the modules $L(w \cdot o_{\mu,+})$ with $w \in I^\nu_{\mu,+} \backslash {}^uI^\nu_{\mu,+}$ of the subcategory of $\mathcal{O}^\nu_{\mu,+}$ consisting of finitely generated $U(\widehat{\mathfrak{g}})$-modules.

For each $x \in {}^vI^\nu_{\mu,-}$ and $y \in {}^uI^\nu_{\mu,+}$, let $^vP(x \cdot o_{\mu,-})$ and $^uP(y \cdot o_{\mu,+})$ be the projective covers of $L(x \cdot o_{\mu,-})$ and $L(y \cdot o_{\mu,+})$ in $^v\mathcal{O}^\nu_{\mu,-}$ and $^u\mathcal{O}^\nu_{\mu,+}$ respectively. Set

$$^vP^\nu_{\mu,-} = \bigoplus_{x \in {}^vI^\nu_{\mu,-}} ({}^vP(x \cdot o_{\mu,-})), \quad ^uP^\nu_{\mu,+} = \bigoplus_{x \in {}^uI^\nu_{\mu,+}} ({}^uP(x \cdot o_{\mu,+})).$$

Set $^vA^\nu_{\mu,-} = \mathrm{End}({}^vP^\nu_{\mu,-})^{\mathrm{op}}$ and $^uA^\nu_{\mu,+} = \mathrm{End}({}^uP^\nu_{\mu,+})^{\mathrm{op}}$. We have the equivalences of categories $^v\mathcal{O}^\nu_{\mu,-} \simeq \mathrm{mod}({}^vA^\nu_{\mu,-})$ and $^u\mathcal{O}^\nu_{\mu,+} \simeq \mathrm{mod}({}^uA^\nu_{\mu,+})$. For each $w \in {}^uI^\nu_{\mu,+}$, let $^uV^\nu(w \cdot o_{\mu,+})$ be the image of $V^\nu(w \cdot o_{\mu,+})$ in $^u\mathcal{O}^\nu_{\mu,+}$. The categories $^v\mathcal{O}^\nu_{\mu,-}$, $^u\mathcal{O}^\nu_{\mu,+}$ are highest weight categories with standard modules $\{V^\nu(x \cdot o_{\mu,-}); x \in {}^vI^\nu_{\mu,-}\}$ and $\{{}^uV^\nu(x \cdot o_{\mu,+}); x \in {}^uI^\nu_{\mu,+}\}$ respectively. For each $v \in I^\nu_{\mu,\pm}$ and each $x \in {}^vI^\nu_{\mu,\pm}$, let $^vV^\nu(x \cdot o_{\mu,\pm})^\vee$ be the costandard object in $^v\mathcal{O}^\nu_{\mu,\pm}$ with simple socle $L(x \cdot o_{\mu,\pm})$. We will refer to them as *dual Verma modules*.

By [56, Thm. 3.12] the rings $^vA^\nu_{\mu,-}$ and $^uA^\nu_{\mu,+}$ admit Koszul gradings. Denote by $^v\overline{\mathcal{O}}^\nu_{\mu,-}$ and $^u\overline{\mathcal{O}}^\nu_{\mu,+}$ the corresponding graded versions of $^v\mathcal{O}^\nu_{\mu,-}$ and $^u\mathcal{O}^\nu_{\mu,+}$ respectively, i.e., we have $^v\overline{\mathcal{O}}^\nu_{\mu,-} = \mathrm{grmod}({}^vA^\nu_{\mu,-})$, $^u\overline{\mathcal{O}}^\nu_{\mu,+} = \mathrm{grmod}({}^uA^\nu_{\mu,+})$. All simple modules, Verma modules, dual Verma modules and projective modules in $^v\mathcal{O}^\nu_{\mu,-}$, $^u\mathcal{O}^\nu_{\mu,+}$ have graded lifts, see [42]. These graded lifts are unique up to a shift of the grading because all mentioned modules are indecomposable. Fix the graded lifts of simple modules concentrated in degree zero and fix the graded lifts of projective modules, Verma modules and dual Verma modules that satisfy the same normalization conditions as in Section 3.2.4. In the notation above we will skip the upper index ν if $\nu = \emptyset$.

Remark A.1.2. (a) The graded ring $^vA^\nu_{\mu,\pm}$ is a quotient of $^vA_{\mu,\pm}$ by a homogeneous ideal. See [56, Lem. 5.12] for a proof in the positive level case. The negative level case is similar. In particular, for $v \in I^{\nu}_{\mu,\pm}$ we get the inclusion of Grothendieck groups $[{}^v\overline{\mathcal{O}}^{\nu}_{\mu,\pm}] \subset [{}^v\overline{\mathcal{O}}_{\mu,\pm}]$.

(b) For $v, u \in {}^vI^{\nu}_{\mu,-}$ with $u \leqslant v$ the category inclusion ${}^u\mathcal{O}^{\nu}_{\mu,-} \subset {}^v\mathcal{O}^{\nu}_{\mu,-}$ satisfies the situation of Lemma 3.2.17. Thus the Koszul grading on ${}^u\mathcal{O}^{\nu}_{\mu,-}$ is induced from the Koszul grading on ${}^u\mathcal{O}^{\nu}_{\mu,-}$. Similarly, by Remark 3.2.18 (b) the category ${}^{u^{-1}}\mathcal{O}^{\mu}_{\nu,+}$ is a Serre quotient of ${}^{v^{-1}}\mathcal{O}^{\mu}_{\nu,+}$ and the quotient functor respects the Koszul grading.

A.1.1 Kazhdan-Lusztig polynomials

In this section we follow the notation of Soergel [57]. First, we define the Hecke algebra associated with $(\widehat{W}, S)$. All claims here are true for an arbitrary Coxeter system $(\widehat{W}, S)$. Let $\ell\colon \widehat{W} \to \mathbb{N}$ be the length function.

Definition A.1.3. The *Hecke algebra* $\mathcal{H}$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by the elements H_s, $s \in S$, modulo the following defining relations

- $H_s^2 = 1 + (q^{-1} - q)H_s \quad \forall s \in S$,
- $H_sH_t \cdots H_s = H_tH_s \cdots H_t \quad \forall s, t \in S,\ st \cdots s = ts \cdots t$,
- $H_sH_t \cdots H_t = H_tH_s \cdots H_s \quad \forall s, t \in S,\ st \cdots t = ts \cdots s$.

For $w \in \widehat{W}$ with reduced expression $w = st \cdots r$ set $H_w = H_sH_t \cdots H_r$. The algebra $\mathcal{H}$ has a unique ring homomorphism $\overline{\bullet}\colon \mathcal{H} \to \mathcal{H}$, $H \mapsto \overline{H}$ such that $\overline{q} = q^{-1}$ and $\overline{H_w} = (H_{w^{-1}})^{-1}$. Let $(\underline{H}_x)_{x\in\widehat{W}}$ be the Kazhdan-Lusztig basis of $\mathcal{H}$, see [57, Thm. 2.1]. We write $\underline{H}_x = \sum_{y\in\widehat{W}} h_{y,x}(q)H_y$, where $h_{y,x}(q)$ is a polynomial in $\mathbb{Z}[q]$ for each $x, y \in \widehat{W}$. The polynomial $h_{x,y}(q)$ is nonzero only if $x \leqslant y$.

For each $x, y \in \widehat{W}$, let $h^{x,y}$ be the inverse Kazhdan-Lusztig polynomial, i.e., such that

$$\sum_{z\in\widehat{W}} (-1)^{\ell(x)+\ell(z)} h_{z,x}(q)h^{z,y}(q) = \delta_{x,y}.$$

Fix a subset $f \subset S$ such that the subgroup $W_f \subset \widehat{W}$ generated by f is finite. Let ${}^f\widehat{W}$ be the set of minimal length representatives for the cosets $W_f\backslash\widehat{W}$.

For each $x, y \in {}^f\widehat{W}$, let $n_{x,y}(q)$ be the parabolic Kazhdan-Lusztig polynomial associated with the parabolic subgroup $W_f \subset \widehat{W}$, i.e., we have

$$n_{x,y}(q) = \sum_{z\in W_f} (-q)^{\ell(z)} h_{zx,y}(q).$$

For each $x, y \in {}^f\widehat{W}$, let $n^{x,y}(q)$ be the inverse parabolic Kazhdan-Lusztig polynomial, i.e., we have

$$\sum_{z\in {}^f\widehat{W}} (-1)^{\ell(x)+\ell(z)} n_{z,x}(q)n^{z,y}(q) = \delta_{x,y}.$$

We may write $n^f_{x,y}(q)$, $n^{x,y}_f(q)$ to insist on the parabolic type f.

Now, we recall some basic properties of Kazhdan-Lusztig polynomials.

Lemma A.1.4. *Let $x, y, z \in \widehat{W}$. We have*

(a) $h_{x,y}(q) \in q^{\ell(y)-\ell(x)}(1 + q^{-2}\mathbb{Z}[q^{-2}])$ *if* $x \leqslant y$,

(b) $h_{x,y}(q) = h_{x^{-1},y^{-1}}(q)$, $h^{x,y}(q) = h^{x^{-1},y^{-1}}(q)$,

(c) $n^{x,y}(q) = h^{x,y}(q)$ *if* $x, y \in {}^f\widehat{W}$,

(d) $h^{x,1}(q) = q^{\ell(x)}$,

(e) $n_{xz,y}(q) = q^{\ell(z)} n_{x,y}(q)$ *if* $x, y \in {}^f\widehat{W}$ *and* $\ell(xz) = \ell(x) - \ell(z), \ell(yz) = \ell(y) - \ell(z)$,

(f) $h_{x,y}(-q) = (-1)^{\ell(x)+\ell(y)} h_{x,y}(q)$.

Proof. Part (a) follows from [57, Rem. 2.6] and [34, Lem. 2.6]. Parts (c), (e) are respectively [57, Prop. 3.7 (2)] and [57, Rem. 3.2 (4)]. Part (b) is true because the $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}$ contains an anti-automorphism sending H_w to $H_{w^{-1}}$ for each $w \in \widehat{W}$. The statement (f) follows from (a).

Let us prove (d). The algebra $\mathcal{H}$ has a representation sgn on $\mathbb{Z}[q, q^{-1}]$ such that each $\underline{H}_s$ with $s \in S$ acts by $-q$. We equip $\mathbb{Z}[q, q^{-1}]$ with the involution such that $\overline{p(q)} = p(q^{-1})$ for each $p(q) \in \mathbb{Z}[q, q^{-1}]$. The $\mathcal{H}$-action on $\mathbb{Z}[q, q^{-1}]$ is compatible with the involution. The element $\underline{H}_w$ for $w \in \widehat{W}, w \neq 1$, acts on $\mathbb{Z}[q, q^{-1}]$ by zero, because it acts by some polynomial $p(q) \in q\mathbb{Z}[q]$ such that $p(q) = p(q^{-1})$ because $\underline{H}_w \in H_w + q(\sum_{w'<w} \mathbb{Z}[q]H_{w'})$. Now, consider the equality

$$H_x = \sum_{y \in \widehat{W}, y \leqslant x} (-1)^{\ell(x)+\ell(y)} h^{x,y}(q) \underline{H}_y.$$

Applying sgn to both sides yields $h^{x,1}(q) = q^{\ell(x)}$. □

A.1.2 Plan of the proof

Here we indicate the scheme that we follow to get the graded decomposition numbers in ${}^v\overline{\mathcal{O}}^\nu_{\mu,-}$ and ${}^u\overline{\mathcal{O}}^\nu_{\mu,+}$ (i.e., the graded multiplicities of simple module in parabolic Verma modules in sense of Section 3.2.4).

Step 1. We compute the decomposition numbers of ${}^v\overline{\mathcal{O}}_{\emptyset,-}$ and $v \in \widehat{W}$ using the geometric approach of [55].

Step 2. We compute the decomposition numbers of ${}^v\overline{\mathcal{O}}^\nu_{\emptyset,-}$, for $\nu \in \mathbf{P}$ and $v \in I^\nu_{\emptyset,-}$ using the previous step and a graded version of the BGG-resolution.

Step 3. We compute the decomposition numbers of ${}^u\overline{\mathcal{O}}_{\nu,+}$, for $\nu \in \mathbf{P}$ and $u \in I_{\nu,+}$ using the previous step and the Koszul duality.

Step 4. We compute the decomposition numbers of ${}^u\overline{\mathcal{O}}^\mu_{\nu,+}$, for $\nu \in \mathbf{P}$ and $u \in I^\mu_{\nu,+}$ using the previous step and a graded version of the BGG-resolution.

Step 5. We compute the decomposition numbers of ${}^{v}\overline{\mathcal{O}}^{\nu}_{\mu,-}$, for $\nu, \mu \in \mathbf{P}$ and $v \in I^{\nu}_{\mu,-}$ using the previous step and the Koszul duality.

A.1.3 Step 1

Let $v \in \widehat{W}$. We must count the graded multiplicities

$$[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q = \sum_{g \in \mathbb{Z}} [V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})\langle g \rangle] q^g$$

for each $x, y \in {}^{v}I_{\emptyset,-}$.

Lemma A.1.5. *For each $x, y \in {}^{v}I_{\emptyset,-}$ we have $[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q = h^{x,y}(q)$.*

Proof. Let s be the length of the socle filtration of $V(x \cdot o_{\emptyset,-})$, i.e., the integer s such that $\mathrm{soc}^s V(x \cdot o_{\emptyset,-}) = V(x \cdot o_{\emptyset,-})$ and $\mathrm{soc}^{s-1} V(x \cdot o_{\emptyset,-}) \subsetneq V(x \cdot o_{\emptyset,-})$, see Section 3.2.8. By Lemma 3.2.23 and [18, Lem. 7.3], the grading filtration of $V(x \cdot o_{\emptyset,-})$ (viewed as a ${}^{v}A_{\emptyset,-}$-module) coincides with its socle filtration. Thus the graded multiplicity $[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q$ coincides with

$$[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]^{\mathrm{soc}}_q = \sum_{g=1}^{s} q^{s-g} [\mathrm{soc}^g (V(x \cdot o_{\emptyset,-}))/\mathrm{soc}^{g-1}(V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})].$$

Now, it suffices to show that $[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]^{\mathrm{soc}}_q = h^{x,y}(q)$. This follows from [55]. More precisely, by [55, Prop. 4.2] the Verma modules and the simple ones are in the essential image of the global section functor defined in [55, Sec. 4.3]. Moreover, by [4, Thm. 7.15.6] and [20, Thm. 5.5] the global section functor is fully faithful. Thus $[V(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]^{\mathrm{soc}}_q$ can be interpreted in terms of graded multiplicities for D-modules. This multiplicity can be computed explicitly via Kazhdan-Lusztig polynomials using the formulas [55, (6.1), (6.2)] and the fact that the weight filtration in [55, Sec. 6.1] coincides with the socle filtration, see [2, Lem. 5.2.2] for details. □

A.1.4 Step 2

Let $\mu, \nu \in \mathbf{P}$ and $v \in I^{\nu}_{\mu,-}$.

Lemma A.1.6. *Let $\lambda = w \cdot o_{\mu,-}$ with $w \in \widehat{W}$ and $s \in S$ such that $s \cdot \lambda < \lambda$. Assume that v is large enough such that $V(\lambda)$, $V(s \cdot \lambda) \in {}^{v}\mathcal{O}_{\mu,-}$. Then, each nonzero morphism $\theta \colon V(s \cdot \lambda) \to V(\lambda)$ in ${}^{v}\mathcal{O}_{\mu,-}$ is homogeneous of degree 1.*

Proof. We have $V(o_{\mu,-}) = L(o_{\mu,-})$. It is a submodule of each Verma module $V(x \cdot o_{\mu,-})$, $x \in \widehat{W}$ because the weight $o_{\mu,-} + \widehat{\rho}$ is antidominant. By Lemma A.1.5 and Lemma A.1.4 (d) we have $[V(\lambda) : L(o_{\mu,-})]_q = q^{\ell(w)}$ and $[V(s \cdot \lambda) : L(o_{\mu,-})]_q = q^{\ell(w)-1}$. Thus $V(\lambda)$ and $V(s \cdot \lambda)$ contain a unique submodule isomorphic to $L(o_{\mu,-})$, which is concentrated in degree $\ell(w)$ and $\ell(w) - 1$ respectively. Each nonzero morphism from $V(s \cdot \lambda)$ to $V(\lambda)$ is injective by [30, Sec. 2]. Thus each morphism from $V(s \cdot \lambda)$ to $V(\lambda)$ is homogeneous of degree 1. □

Let $v \in I^{\nu}_{\mu,-}$ and $\lambda = w \cdot o_{\mu,-}$, with $w \in {}^{v}I^{\nu}_{\mu,-}$. See [28, Chap. 6] for the definition of the BGG resolution. Applying the exact functor $U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}}_\nu)} \bullet$ to the BGG resolution of the simple module with highest weight λ of the Levi subalgebra $\widehat{\mathfrak{l}}_\nu \subset \widehat{\mathfrak{p}}_\nu$, we get the parabolic BGG resolution of the parabolic Verma module $V^{\nu}(\lambda)$

$$\cdots \to C_2 \to C_1 \to V(\lambda) \xrightarrow{\epsilon} V^{\nu}(\lambda) \to 0, \tag{A1}$$

where

$$C_t = \bigoplus_{x \in W_\nu, \ell(x) = t} V(x \cdot \lambda).$$

Corollary A.1.7. *Each map except ϵ in* (A1) *is of degree* 1*. The map ϵ is of degree* 0*.*

Proof. We must explain why ϵ is homogeneous of degree 0. By the definition of the graded lifts of (parabolic) Verma modules, the morphisms $V(\lambda) \to L(\lambda)$ and $V^{\nu}(\lambda) \to L(\lambda)$ are homogeneous of degree zero. The morphism $V(\lambda) \to V^{\nu}(\lambda)$ is homogeneous because $\dim \mathrm{Hom}_{\mathcal{O}_{\mu,-}}(V(\lambda), V^{\nu}(\lambda)) = 1$. Its degree is automatically zero. □

From now on we assume that $\mu = \emptyset$.

Corollary A.1.8. *Assume that $v \in I^{\nu}_{\emptyset,-}$. For each $x, y \in {}^{v}I^{\nu}_{\emptyset,-}$ we have*

$$[V^{\nu}(x \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q = \sum_{z \in W_\nu} (-q)^{\ell(z)} h^{zx,y}(q).$$

Proof. By Corollary A.1.7 we have

$$[V^{\nu}(x \cdot o_{\emptyset,-})] = \sum_{z \in W_\nu} (-q)^{\ell(z)} [V(zx \cdot o_{\emptyset,-})]$$

in $[{}^{w_\nu v}\overline{\mathcal{O}}_{\emptyset,-}]$, where w_ν is the longest element in W_ν. Now, apply Lemma A.1.5. □

Corollary A.1.9. *For each $x \in {}^vI^\nu_{\emptyset,-}$ we have the following equality in $[{}^v\overline{\mathcal{O}}^\nu_{\emptyset,-}]$:*

$$[L(x \cdot o_{\emptyset,-})] = \sum_{y \in {}^vI^\nu_{\emptyset,-}} (-1)^{\ell(x)+\ell(y)} h_{y^{-1},x^{-1}}(q)[V^\nu(y \cdot o_{\emptyset,-})].$$

□

Proof. We must prove that

$$\sum_{t \in {}^vI^\nu_{\emptyset,-}} (-1)^{\ell(x)+\ell(t)} h_{t^{-1},x^{-1}}(q)[V^\nu(t \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q = \delta_{x,y}$$

for each $x, y \in {}^vI^\nu_{\emptyset,-}$. We have

$$\begin{aligned} & \sum_{t \in {}^vI^\nu_{\emptyset,-}} (-1)^{\ell(x)+\ell(t)} h_{t^{-1},x^{-1}}(q)[V^\nu(t \cdot o_{\emptyset,-}) : L(y \cdot o_{\emptyset,-})]_q \\ = & \sum_{t \in {}^vI^\nu_{\emptyset,-}, z \in W_\nu} (-q)^{\ell(z)}(-1)^{\ell(x)+\ell(t)} h_{t^{-1},x^{-1}}(q)h^{zt,y}(q) \\ = & \sum_{t \in {}^vI^\nu_{\emptyset,-}, z \in W_\nu} (-1)^{\ell(x)+\ell(t)+\ell(z)} h_{t^{-1}z^{-1},x^{-1}}(q)h^{zt,y}(q) \\ = & \sum_{s \in {}^vI_{\emptyset,-}} (-1)^{\ell(x)+\ell(s)} h_{s^{-1},x^{-1}}(q)h^{s^{-1},y^{-1}}(q) \\ = & \ \delta_{x,y}. \end{aligned}$$

Here the second equality is Lemma A.1.4 (e), the third is Lemma A.1.4 (b). □

A.1.5 Step 3

Let $\nu \in \mathbf{P}$ and $u \in I_{\nu,+}$.

Lemma A.1.10. *For each $x, y \in {}^uI_{\nu,+}$ we have*

$$[{}^uV(x \cdot o_{\mu,+}) : L(y \cdot o_{\mu,+})]_q = h_{x,y}(q).$$

Proof. Set $v = u^{-1}$. As in the proof of Corollary 3.2.16, we deduce from Corollary A.1.9 that we have

$$[L(x \cdot o_{\emptyset,-})] = \sum_{y \in {}^vI^\nu_{\emptyset,-}} (-1)^{\ell(x)+\ell(y)} h_{y^{-1},x^{-1}}(q^{-1})[V^\nu(y \cdot o_{\emptyset,-})^\vee]$$

in $[{}^v\overline{\mathcal{O}}^\nu_{\emptyset,-}]$. By [56, Thm. 3.12] and Corollary 3.2.9 we have a $\mathbb{Z}[q, q^{-1}]$-module isomorphism

$$[{}^v\overline{\mathcal{O}}^\nu_{\emptyset,-}] \to \widetilde{[{}^u\overline{\mathcal{O}}_{\nu,+}]}, \quad [L(x \cdot o_{\emptyset,-})] \mapsto [{}^uP(x^{-1} \cdot o_{\nu,+})],\ [V^\nu(x \cdot o_{\emptyset,-})^\vee] \mapsto [{}^uV(x^{-1} \cdot o_{\nu,+})]\ \forall x \in {}^vI^\nu_{\emptyset,-}.$$

Thus we get

$$[{}^uP(x^{-1}\cdot o_{\nu,+})] = \sum_{y\in {}^vI^{\nu}_{\emptyset,-}} (-1)^{\ell(x)+\ell(y)} h_{y^{-1},x^{-1}}(-q)[{}^uV(y^{-1}\cdot o_{\nu,+})]$$

in $[{}^u\overline{\mathcal{O}}_{\nu,+}]$. Now, the statement follows from Corollary 3.2.16 and Lemma A.1.4 (f). □

A.1.6 Step 4

Let $\mu,\nu\in\mathbf{P}$ and $u,w\in I^{\mu}_{\nu,+}$ with $w\leqslant u$, and set $\lambda = w\cdot o_{\nu,+}$. There is an exact sequence

$$\cdots\to C_2\to C_1\to V(\lambda)\xrightarrow{\epsilon} V^{\mu}(\lambda)\to 0$$

in $\mathcal{O}_{\nu,+}$, where

$$C_t = \bigoplus_{x\in W_\mu,\ell(x)=t} V(x\cdot\lambda),$$

compare Section A.1.4. The image of this exact sequence in ${}^u\mathcal{O}_{\nu,+}$ yields an exact sequence

$$\cdots\to {}^uC_2\to {}^uC_1\to {}^uV(\lambda)\xrightarrow{\epsilon} {}^uV^{\mu}(\lambda)\to 0, \tag{A2}$$

where

$${}^uC_t = \bigoplus_{x\in W_\mu,\ell(x)=t} {}^uV(x\cdot\lambda).$$

Lemma A.1.11. *Let $u\in I_{\nu,+}$ with $x,y\in {}^uI_{\nu,+}$ and $x\geqslant y$. Then*

(a) $\dim \mathrm{Hom}_{{}^u\mathcal{O}_{\nu,+}}({}^uV(x\cdot o_{\nu,+}), {}^uV(y\cdot o_{\nu,+})) = 1$,

(b) *each nonzero morphism ${}^uV(x\cdot o_{\nu,+})\to {}^uV(y\cdot o_{\nu,+})$ is injective,*

(c) $\mathrm{soc}({}^uV(y\cdot o_{\nu,+})) = {}^uV(u\cdot o_{\nu,+})$,

(d) *the inclusion ${}^uV(u\cdot o_{\nu,+})\to {}^uV(y\cdot o_{\nu,+})$ is homogeneous of degree $\ell(u)-\ell(y)$,*

(e) *each nonzero morphism ${}^uV(x\cdot o_{\nu,+})\to {}^uV(y\cdot o_{\nu,+})$ is homogeneous of degree $\ell(x)-\ell(y)$.*

Proof. By the Ringel equivalence we have

$$\mathrm{Hom}_{{}^u\mathcal{O}_{\nu,+}}({}^uV(x\cdot o_{\nu,+}), {}^uV(y\cdot o_{\nu,+})) = \mathrm{Hom}_{\mathcal{O}_{\nu,-}}(V(y\cdot o_{\nu,-}), V(x\cdot o_{\nu,-})),$$

see [56, Prop. 3.9 (b)]. So, to prove (a) it is enough to prove that

$$\dim \mathrm{Hom}_{\mathcal{O}_{\nu,-}}(V(y\cdot o_{\nu,-}), V(x\cdot o_{\nu,-})) = 1.$$

This follows from [18, Lem. 7.3] and [30, Sec. 2], see the proof of [28, Thm. 4.2 (b)].

Now, to prove (b) it is enough to show that there exists one injective morphism ${}^uV(x\cdot o_{\nu,+})\to {}^uV(y\cdot o_{\nu,+})$. By [33, Prop. 3.1] there exists an injective morphism $V(x\cdot o_{\nu,+})\to V(y\cdot o_{\nu,+})$. Its image in ${}^u\mathcal{O}_{\nu,+}$ by the quotient functor (which is exact) must be also injective.

It is clear from (b) that the only simple module that can be contained in $\mathrm{soc}({}^uV(y\cdot o_{\nu,+}))$ is ${}^uV(u\cdot o_{\nu,+})=L(u\cdot o_{\nu,+})$. Hence, by (a), this socle must be equal to ${}^uV(u\cdot o_{\nu,+})$. Part (c) is proved.

By Lemma 3.2.23, the radical filtration of ${}^uV(y\cdot o_{\nu,+})$ coincides with its grading filtration. Here we view ${}^uV(y\cdot o_{\nu,+})$ as a ${}^uA_{\nu,+}$-module. By Lemma A.1.4 (a) and Lemma A.1.10, for all $z\in {}^uI_{\nu,+}$ such that $z\neq u$ and $z\leqslant y$ we have $[{}^uV(y\cdot o_{\nu,+}):L(z\cdot o_{\nu,+})]_q\in q^{\ell(u)-\ell(y)-1}\mathbb{Z}[q^{-1}]$ and $[{}^uV(y\cdot o_{\nu,+}):L(u\cdot o_{\nu,+})]_q\in q^{\ell(u)-\ell(y)}(1+q^{-1}\mathbb{Z}[q^{-1}])$. Thus $\mathrm{rad}^{\ell(u)-\ell(y)}({}^uV(y\cdot o_{\nu,+}))$ coincides with $\mathrm{soc}({}^uV(y\cdot o_{\nu,+}))$. Part (d) follows.

The statement (e) follows from (a), (b), (c), (d).

□

Corollary A.1.12. *Each map in* (A2) *except* ϵ *is homogeneous of degree* 1. *The map* ϵ *is homogeneous of degree* 0. □

Assume that $\nu,\mu\in\mathbf{P}$, $u\in I^{\mu}_{\nu,+}$.

Corollary A.1.13. *For each* $x,y\in {}^uI^{\mu}_{\nu,-}$ *we have*

$$[{}^uV^{\mu}(x\cdot o_{\nu,+}):L(y\cdot o_{\nu,+})]_q=n^{\mu}_{x,y}(q).$$

Proof. By Corollary A.1.12 we have

$$[{}^uV^{\mu}(x\cdot o_{\nu,+})]=\sum_{s\in W_\mu}(-q)^{\ell(s)}[{}^uV(sx\cdot o_{\nu,+})]$$

in $[{}^u\overline{\mathcal{O}}_{\nu,+}]$. Now, from Lemma A.1.10 we get

$$[{}^uV^{\mu}(x\cdot o_{\nu,+}):L(y\cdot o_{\nu,+})]_q=\sum_{s\in W_\mu}(-q)^{\ell(s)}h_{sx,y}(q)=n^{\mu}_{x,y}(q).$$

□

A.1.7 Step 5

Assume that $\nu,\mu\in\mathbf{P}$ and $v\in I^{\mu}_{\nu,-}$.

Lemma A.1.14. *For each* $x\in {}^vI^{\nu}_{\mu,-}$ *we have the following equality in* $[{}^v\overline{\mathcal{O}}^{\nu}_{\mu,-}]$

$$[L(x\cdot o_{\mu,-})]=\sum_{y\in {}^vI^{\nu}_{\mu,-}}(-1)^{\ell(x)+\ell(y)}n^{\mu}_{y^{-1},x^{-1}}(q)[V^{\nu}(y\cdot o_{\mu,-})].$$

Proof. The statement follows from Corollary A.1.13, applying the Koszul duality in the same way as in the proof of Lemma A.1.10. □

Corollary A.1.15. *For each $x, y \in {}^vI^\nu_{\mu,-}$ we have*

$$[V^\nu(x \cdot o_{\mu,-}) : L(y \cdot o_{\mu,-})]_q = \sum_{z \in W_\nu} (-q)^{\ell(z)} h^{zx,y}(q).$$

Proof. We need to check that

$$\sum_{t \in {}^vI^\nu_{\mu,-}, z \in W_\nu} (-q)^{\ell(z)} (-1)^{\ell(x)+\ell(t)} n^\mu_{t^{-1},x^{-1}}(q) h^{zt,y}(q) = \delta_{x,y}.$$

This can be done in the same way as in the proof of Corollary A.1.9. □

A.2 q-Schur algebras and category A

A.2.1 Orders

Let $\mathbf{m}$ be as in Section 3.3.4. Denote by $W_{\mathbf{m}}$ the parabolic subgroup of $W = \mathfrak{S}_N$ of parabolic type $\mathbf{m}$. Let $\widehat{\Pi}^+ \subset \widehat{\Pi}$ be the set of positive roots. Set also $\mathcal{P}^{\mathbf{m}}_d = \mathcal{P}^{\mathbf{m}} \cap \mathcal{P}_d$, see Sections 3.2.7, 3.3.4.

Definition A.2.1. The *linkage ordering* $\leqslant_\ell$ on $\mathcal{P}^{\mathbf{m}}_d$ is the transitive and reflexive closure of the relation $\Uparrow$, where for $\lambda, \mu \in \mathcal{P}^{\mathbf{m}}$ we have $\lambda \Uparrow \mu$ when there exist $w \in W_{\mathbf{m}}$, $\beta \in \widehat{\Pi}^+$ such that such that $\langle \widetilde{\omega(\mu)}^- + \widehat{\rho} : \beta \rangle > 0$ and $ws_\beta \cdot \widetilde{\omega(\mu)}^- = \widetilde{\omega(\lambda)}^-$.

Set $\mathbf{A}^{\mathbf{m}}_d = \bigoplus_{|\mathbf{d}|=d} \mathbf{A}^{\mathbf{m}}_{\mathbf{d}}$, see Section 3.3.4. By [53, Sec. 5.5], the category $\mathbf{A}^{\mathbf{m}}_d$ is a highest weight category with respect to the poset $(\mathcal{P}^{\mathbf{m}}_d, \leqslant_\ell)$. The standard modules are the Verma modules, see Section 3.3.4. If $\mathbf{m}$ satisfies $m_p \geqslant d$ for each $p \in [1, d]$ then we have $\mathcal{P}^{\mathbf{m}}_d = \mathcal{P}^l_d$. In this case $\leqslant_\ell$ is an order on $\mathcal{P}^l_d$.

Definition A.2.2. The *dominance order* $\trianglelefteq$ on $\mathcal{P}^l_d$ is the the order such that for $\lambda, \mu \in \mathcal{P}^l_d$ we have $\lambda \trianglelefteq \mu$ when for each $p \in [1, l-1]$ and $a \in [1, \ell(\lambda^p)]$ we have

$$\sum_{k=1}^{p-1} |\lambda^k| + \sum_{r=1}^{a} \lambda^p_r \leqslant \sum_{k=1}^{p-1} |\mu^k| + \sum_{r=1}^{a} \mu^p_r.$$

Let $S^{\mathbf{s}}_d$ be as in Section 3.2.9. The category $\mathrm{mod}(S^{\mathbf{s}}_d)$ is a highest weight category with respect to the poset $(\mathcal{P}^l_d, \trianglelefteq)$. The standard modules are the Weyl modules, see Section 3.2.9.

Definition A.2.3. Consider the partial order $\preccurlyeq$ on $\mathcal{P}^l_d$ such that for each $\lambda, \mu \in \mathcal{P}^l_d$ we have $\lambda \preccurlyeq \mu$ if and only if either $\lambda = \mu$ or for each $r \in [1, l-1]$ we have

$$|\lambda^1| + |\lambda^2| + \cdots + |\lambda^r| \leqslant |\mu^1| + |\mu^2| + \cdots + |\mu^r|$$

and one of these inequalities is strict.

Recall from Section 3.3.4 the notation λ^* for $\lambda \in \mathcal{P}^l_d$. For $\lambda, \mu \in \mathcal{P}^l_d$ let us write $\mu \trianglelefteq^* \lambda$ when $\mu^* \trianglelefteq \lambda^*$. It is obvious that we can refine the order $\preccurlyeq$ to an order that refines the order $\trianglelefteq$ and to an order that refines the order $\trianglerighteq^*$.

A.2.2 Compare of orders

We say that the l-tuple $\mathbf{m} = (m_1, \cdots, m_l)$ is *dominant* if $m_r - m_{r+1} \geqslant d$ for each $r \in [1, d-1]$. We say that $\mathbf{m}$ is *anti-dominant* if $m_{r+1} - m_r \geqslant d$ for each $r \in [1, d-1]$.

Lemma A.2.4. *Assume that the l-tuple $\mathbf{m}$ satisfies $m_p \geqslant d$ for each $p \in [1, d]$.*

(a) *Assume that $\mathbf{m}$ is dominant. Then the order $\preccurlyeq$ refines the order $\leqslant_\ell$.*

(b) *Assume that $\mathbf{m}$ is anti-dominant. Then the order $\succcurlyeq$ refines the order $\leqslant_\ell$.*

Proof. We prove the statements (a) and (b) together. Assume that $\mathbf{m}$ is either dominant or anti-dominant.

Let $\lambda, \mu \in \mathcal{P}^l_d$ be such that $\lambda \Uparrow \mu$. Assume that $w \in W_{\mathbf{m}}$, $\beta \in \widehat{\Pi}^+$ are such that $(\widetilde{\omega(\mu)}^- + \widehat{\rho}, \beta) > 0$ and $ws_\beta \cdot \widetilde{w(\mu)}^- = \widetilde{w(\lambda)}^-$.

Write $\beta = \alpha_{k,l} + r\delta$. Let λ' be the l-composition of d (maybe not an l-partition) such that $\widetilde{\omega(\lambda')}^- = s_\beta \cdot \widetilde{\omega(\mu)}^-$. The l-partition λ' viewed as a weight in P is equal to

$$s_{k,t}(\mu + \rho_{\mathbf{m}}) + er\alpha_{k,t} - \rho_{\mathbf{m}}.$$

Set $n = (\mu + \rho_{\mathbf{m}}, \alpha_{k,t}) - er = (\widehat{\omega(\mu)} + \widehat{\rho}, \beta) > 0$. We have

$$\begin{array}{rcl}(\lambda' + \rho_{\mathbf{m}})_k &=& (\mu + \rho_{\mathbf{m}})_k - n = (\mu + \rho_{\mathbf{m}})_t + er,\\ (\lambda' + \rho_{\mathbf{m}})_t &=& (\mu + \rho_{\mathbf{m}})_t + n = (\mu + \rho_{\mathbf{m}})_k - er,\\ (\lambda' + \rho_{\mathbf{m}})_s &=& (\mu + \rho_{\mathbf{m}})_s, \quad \forall s \notin \{k, t\}.\end{array}$$

Let us prove that if $\mathbf{m}$ is dominant then $k < t$, if $\mathbf{m}$ is anti-dominant then $k > t$. Assume that $\mathbf{m}$ is dominant and $k > t$ (and thus we also have $p_k > p_t$) or $\mathbf{m}$ is anti-dominant and $k < t$ (and thus we also have $p_k < p_t$). Set $a = m_1 + \cdots + m_{p_t - 1} + \ell(\mu^{(p_t)}) + 1$, $b = m_1 + \cdots + m_{p_t}$. Note that we have $\mu_r = 0$ for $r \in [a, b]$ and thus $(\mu + \rho_{\mathbf{m}})_r = (\rho_{\mathbf{m}})_r$ for each $r \in [a, b]$. By assumption on $\mathbf{m}$, in both cases we have

$$m_{p_t} - m_{p_k} \geqslant d \geqslant \mu_1^{(p_k)} + \ell(\mu^{(p_t)}).$$

Thus we get

$$\begin{array}{rcl}m_{p_k} &\leqslant& m_{p_t} - \mu_1^{(p_k)} - \ell(\mu^{(p_t)})\\ &=& (\rho_{\mathbf{m}})_a - \mu_1^{(p_k)}.\end{array}$$

This implies

$$\begin{aligned}(\mu+\rho_{\mathbf{m}})_k &\leqslant \mu_1^{(p_k)}+m_{p_k}\\ &\leqslant (\rho_{\mathbf{m}})_a\end{aligned}$$

So the inequality

$$(\rho_{\mathbf{m}})_a \geqslant (\mu+\rho_{\mathbf{m}})_k \geqslant (\lambda'+\rho_{\mathbf{m}})_t \geqslant (\rho_{\mathbf{m}})_b$$

implies that there exists an index $c \in [a,b]$ such that $(\lambda'+\rho_{\mathbf{m}})_t = (\rho_{\mathbf{m}})_c$ and we should obviously have $c < t$ as $\lambda'_t > 0$. We get

$$(\lambda'+\rho_{\mathbf{m}})_t = (\rho_{\mathbf{m}})_c = (\mu+\rho_{\mathbf{m}})_c = (\lambda'+\rho_{\mathbf{m}})_c.$$

Thus the stabilizer of $\lambda'+\rho_{\mathbf{m}}$ in $W_{\mathbf{m}}$ is non-trivial. This is absurd.

Note that λ' is obtained from μ by removing n boxes from the component p_k to the component p_t. Moreover, λ has the same number of boxes in each component as λ'. Thus we get $\lambda \preccurlyeq \mu$ when $p_k < p_t$ and $\lambda \succcurlyeq \mu$ when $p_k > p_t$. As proved above, the first situation always holds if $\mathbf{m}$ is dominant and the second situtaion always holds when $\mathbf{m}$ is anti-dominant.

□

We say that two partial orders are *compatible* if there exists an order that refines both of them.

Corollary A.2.5. (a) *Assume that* $\mathbf{m}$ *is dominant. Then the order* $\leqslant_\ell$ *is compatible with the order* $\trianglelefteq$.

(b) *Assume that* $\mathbf{m}$ *is anti-dominant. Then the order* $\leqslant_\ell$ *is compatible with the order* $\trianglelefteq^*$. □

A.2.3 Equivalences of categories

From now on assume that $e > 2$ and that the components of $\mathbf{m}$ have different residues modulo e. From now on we always assume that $\mathbf{m}$ satisfies $m_p \geqslant d$ for each $p \in [1,l]$. Let $\mathbf{s} = (s_1, \cdots, s_l)$ be the l-tuples of residues of elements of $\mathbf{m}$ module e.

Assume that $\mathbf{m}$ is dominant. By [53, Sec. 5.8] we have a highest weight cover

$$\Psi\colon \mathbf{A}_d^{\mathbf{m}} \to \mathrm{mod}(H_d^{\mathbf{s}}), \quad \Delta_{\mathcal{O}}^{\lambda} \mapsto S^{\lambda}.$$

On the other hand we have a highest weight cover

$$F\colon \mathrm{mod}(S_d^{\mathbf{s}}) \to \mathrm{mod}(H_d^{\mathbf{s}}), \quad \Delta^{\lambda} \mapsto S^{\lambda}.$$

Thus by [52, Cor. 4.46] and Corollary A.2.5 (a) we get the following lemma.

Lemma A.2.6. *Assume that* $\mathbf{m}$ *is dominant and* $m_l \geqslant d$. *Then there is an equivalence of categories*

$$\mathbf{A}_d^{\mathbf{m}} \simeq \mathrm{mod}(S_d^{\mathbf{s}}), \quad \Delta_{\mathcal{O}}^{\lambda} \mapsto \Delta^{\lambda}.$$

□

Assume now that $\mathbf{m}$ is anti-dominant. Set $\mathbf{s}^* = (-s_l, \cdots, -s_1)$. By [53, Sec. 6.2.4] there is an isomorphism of algebras $\mathrm{IM}\colon H_d^{\mathbf{s}} \simeq H_d^{\mathbf{s}^*}$ such that its pull-back $\mathrm{IM}^*\colon \mathrm{mod}(H_d^{\mathbf{s}^*}) \to \mathrm{mod}(H_d^{\mathbf{s}})$ satisfies $\mathrm{IM}^*(S^\lambda) = S^{\lambda^*}$.

We have the following highest-weight covers.

$$\Psi\colon \mathbf{A}_d^{\mathbf{m}} \to \mathrm{mod}(H_d^{\mathbf{s}}), \quad \Delta^\lambda \mapsto S^\lambda,$$

$$F\colon \mathrm{mod}(S_d^{\mathbf{s}^*}) \to \mathrm{mod}(H_d^{\mathbf{s}^*}), \quad \Delta_{\mathcal{O}}^\lambda \mapsto S^\lambda.$$

Composing the second one with IM^* we get the heigest-weight cover

$$\mathrm{IM}^* \circ F\colon \mathrm{mod}(S_d^{\mathbf{s}^*}) \to \mathrm{mod}(H_d^{\mathbf{s}}), \quad \Delta^{\lambda^*} \mapsto S^\lambda.$$

Thus by [52, Cor. 4.46] and Corollary A.2.5 (b) we get the following result.

Lemma A.2.7. *Assume that* $\mathbf{m}$ *is anti-dominant and* $m_1 \geqslant d$. *Then there is an equivalence of categories*

$$\mathbf{A}^{\mathbf{m}}\{d\} \simeq \mathrm{mod}(S_d^{\mathbf{s}^*}), \quad \Delta_{\mathcal{O}}^\lambda \mapsto \Delta^{\lambda^*}.$$

□

Remark A.2.8. In this section we assumed that $e > 2$ and that the components of $\mathbf{m}$ have different residues modulo e. These assumption are necessary to use [52, Cor. 4.46]. They can be removed (it is enough to assume only $e \geqslant 2$) using the approach of [53], see [53, Prop. 2.20].

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