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Benoît Mésognon-Gireau

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Limites singulières en faible amplitude pour l'équation des vagues

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Limites singulières en faible amplitude pour l'équation des vagues

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Chapter 1

Introduction

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Cette thèse consiste en l'étude de plusieurs comportements singuliers des équations de Water-Waves dans la limite faible amplitude. Le présent manuscrit consiste en quatre articles rédigés durant cette thèse, dont un a été accepté dans "Annales of IHP" et les trois autres étant en cours de soumission. Dans cette introduction, nous présentons le cadre d'étude dans lequel s'inscrit le travail effectué au cours de cette thèse, ainsi que les champs de recherche associés aux différents résultats démontrés. Nous nous efforçons de retranscrire le cheminement logique adopté pour la rédaction de ce manuscrit, tout en inscrivant chaque article présenté dans son contexte général.

1.1 L'équation des Water-Waves

1.1.1 De Euler à Zakharov-Craig Sulem

L'équation des vagues, plus communément appelée équation des Water-Waves, modélise mathématiquement le mouvement d'un fluide sous l'influence de la gravité, délimité par le bas par un fond fixe, et par le haut par une surface libre qui le sépare de l'air (ou de tout autre fluide à densité négligeable par rapport à la sienne). Usuellement, les hypothèses suivantes sont faites :

- (H1) Le fluide est homogène et non visqueux.
- (H2) Le fluide est incompressible.
- (H3) Le fluide est irrotationnel.
- (H4) La surface et le fond peuvent être paramétrés par des graphes.
- (H5) Les particules de fluide ne traversent pas le fond.
- (H6) Les particules de fluide ne traversent pas la surface.
- (H7) La pression extérieure est constante, et on travaille sans tension de surface.
- (H8) Le fluide est au repos à l'infini.
- (H9) Il existe toujours une hauteur d'eau minimale.

Les hypothèses (H1) et (H2) impliquent que le mouvement du fluide peut être décrit par les équations d'Euler.

L'hypothèse (H3) simplifie l'étude, mais n'est pas nécessaire, et de plus en plus de travaux récents s'intéressent au problème avec vorticit  (voir par exemple [20], [19], [41], [42]; [21], [66], [58]).

L'hypothèse (H4) exclut l'étude des vagues déferlantes. En particulier, on s'attend d'ores et déjà à ce que les équations modélisant le mouvement ne soient pas bien posées globalement en temps dans le cas général. En revanche, il paraît logique que plus l'amplitude des vagues du système étudié est faible, plus le temps d'existence des solutions pour de telles équations est grand. Cette dernière considération est un des tenants importants du travail effectué dans cette thèse. Voir les travaux sur la création de singularités pour l'équation des vagues à ce sujet ([23], [17], [18]).

Les hypothèses (H5) et (H6) apportent aux équations d'Euler des conditions de bord. Ces conditions sont essentielles pour la modélisation mathématique du problème, et jouent un rôle important dans l'étude qui est menée dans cette thèse, comme nous le verrons plus loin.

Le fait que la pression extérieure soit constante (hypothèse (H7)) ou non n'est en fait pas primordial, et de récents travaux sur l'influence de fortes pressions extérieures induites par exemple par des tempêtes ont été menés (voir par exemple [45]). En revanche, nous utiliserons la tension de surface plus tard, mais son introduction sera alors soigneusement expliquée.

Les hypothèses (H8) et (H9) sont raisonnables dans l'étude de domaines de fluide infinis, mais excluent évidemment l'étude de la dynamique des fluides en milieu côtier, qui reste à ce jour un problème ouvert.

Si l'on traduit mathématiquement les hypothèses (H1) à (H9), on obtient les équations dites d'Euler à surface libre. Avant de les introduire, précisons quelques notations:

- On note d la dimension horizontale. En pratique, $d = 1$ ou $d = 2$, et le problème est donc posé dans un espace à $d + 1$ dimensions.
- On note $\Omega_t \subset \mathbb{R}^{d+1}$ le domaine occupé par le fluide à un instant t .

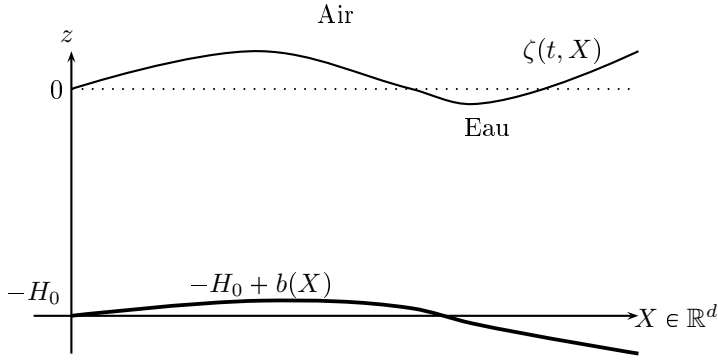


Figure 1.1: Le problème d'écoulement à surface libre

- La vitesse du fluide à un instant t en un point $(X, z) \in \Omega_t$, définie par la vitesse de la particule de fluide située au point (X, z) à l'instant t , est notée $U(t, X, z) \in \mathbb{R}^{d+1}$.
- On note $P(t, X, z) \in \mathbb{R}$ la pression dans le fluide à un instant t et au point (X, z) .
- L'accélération de la pesanteur supposée constante est notée g , et le vecteur normal unitaire dans la direction verticale est noté e_z .
- La densité du fluide supposée constante est notée ρ .

A présent, traduisons mathématiquement les hypothèses (H1) à (H9) (on notera (Hk') la traduction mathématique de l'hypothèse (Hk)).

$$\text{(H1')} \quad \partial_t U + (U \cdot \nabla_{X,z})U = -\frac{1}{\rho} \nabla_{X,z} P - g e_z \text{ dans } \Omega_t,$$

$$\text{(H2')} \quad \operatorname{div} U = 0 \text{ dans } \Omega_t,$$

$$\text{(H3')} \quad \operatorname{rot}(U) = 0 \text{ dans } \Omega_t.$$

L'équation (H1') est obtenue en appliquant le principe fondamental de la dynamique à un cube élémentaire de fluide, puis en faisant tendre la taille de ce cube vers zéro.

L'équation (H2') est obtenue en écrivant que le flux algébrique total de fluide traversant une surface régulière délimitant un volume borné fixe est nul (le fluide ne se comprime ou décompresse pas, donc la quantité de fluide est toujours la même, et "ce qui entre est égal à ce qui sort"). En utilisant alors la formule de Stokes, on obtient que dans tout volume borné la divergence du champ de vitesse est nulle.

L'équation (H3') est la définition de (H3).

Il existe aussi deux fonctions décrivant le fond et la surface, $b : \mathbb{R}^d \rightarrow \mathbb{R}$ et $\zeta : [0; T[\times \mathbb{R}^d \rightarrow \mathbb{R}$ avec $T > 0$ telles que

$$\text{(H4')} \quad \forall t \in [0; T[, \quad \Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

où $H_0 > 0$ est une constante correspondant à la profondeur typique du milieu considéré (qui varie bien entendu selon la dynamique étudiée : mouvement dans un canal, vagues dans la mer, vagues dans un lac...). En notant n le vecteur normal à une surface unitaire orienté vers le haut, on obtient les identités :

$$\text{(H5')} \quad U \cdot n = 0 \text{ sur } \{z = -H_0 + b(X)\},$$

$$\text{(H6')} \quad \partial_t \zeta - \sqrt{1 + |\nabla \zeta|^2} U \cdot n = 0 \text{ sur } \{z = \zeta(t, X)\}.$$

L'équation (H6') peut être obtenue en écrivant qu'une particule de position $(X(t), z(t))$ à la surface reste à la surface : si $z(t) - \zeta(t, X(t)) = 0$ à un instant donné, alors c'est vrai à tout instant, et en dérivant en temps on obtient donc $z'(t) - \partial_t \zeta(t, X(t)) - X'(t) \cdot \nabla \zeta(t, X(t)) = 0$. Comme par définition $U(t, X(t)) = X'(t)$, et que $n(t, X) = {}^t(-\nabla \zeta(t, X), 1)$ on obtient le résultat souhaité pour toute courbe régulière X .

En notant P_0 la pression extérieure supposée constante, on obtient:

(H7') $P = P_0$ sur $\{z = \zeta(t, X)\}$.

(H8') $\forall t \in [0; T[$, $\lim_{(X,z) \in \Omega_t, |(X,z)| \rightarrow \infty} |\zeta(t, X)| + |U(t, X, z)| = 0$.

(H9') $\exists H_{\min} > 0, \forall (t, X) \in [0; T[\times \mathbb{R}^d$, $H_0 + \zeta(t, X) - b(X) \geq H_{\min}$.

Les équations (H1')-(H7') sont appelées "équations d'Euler à surface libre".

D'un point de vue purement "analyse fonctionnelle", les équations d'Euler à surface libre (H1')-(H7') sont difficiles à étudier. En effet, il s'agit d'équations d'évolution définies sur un domaine spatial qui bouge dans le temps. Dans quel espace fonctionnel peut-on alors chercher des solutions à cette équation ? En s'inspirant de l'étude faite pour une équation d'évolution classique, on pourrait chercher des solutions dans des espaces de la forme $C([0; T]; H^N(\Omega_t))$, mais alors la norme de l'espace de Sobolev considéré dépend elle-même du temps. On pourrait considérer un difféomorphisme Σ_t dépendant du temps, tel que si $f : \Omega_t \rightarrow \mathbb{R}^{d+1}$ alors $f \circ \Sigma_t : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ et définir les solutions U, P comme des fonctions telles que $(U, P)(t, \Sigma_t)$ soit dans un espace régulier (par exemple $C([0; T], \mathbb{R}^d)^2$). Voir [63], [22] ou encore [41] pour une approche Lagrangienne de la démonstration du caractère bien posé des équations d'Euler à surface libre.

Dans le dernier chapitre de ce manuscrit, une approche différente à été utilisée pour définir la notion de solutions pour les équations d'Euler à surface libre. En effet, dans notre cadre de travail, le domaine de fluide reste borné dans une bande fixe de la forme $\mathbb{R}^d \times [a; b]$ (notons la S) dans l'intervalle de temps fermé considéré (en réalité, c'est forcément le cas dans un cadre de travail raisonnable : d'après l'hypothèse (H9'), et en supposant raisonnablement que la surface est continue en temps et espace, l'élévation maximale de l'eau reste bornée sur un intervalle de temps fermé). On définit alors une solution d'Euler à surface libre comme une fonction (U, P) définie sur $[0; T] \times S$ telle que la restriction à Ω_t (qui est un ouvert de $[0; T] \times S$) vérifie les équations d'Euler (H1')-(H7'). Nous reviendrons sur cette approche plus loin.

Afin de se diriger vers une formulation mathématique plus simple, c'est-à-dire vers des équations posées sur des domaines fixes, introduisons une formulation équivalente aux équations d'Euler à surface libre, constituant les équations de Bernoulli. En remarquant que le champ de vitesse du fluide U est irrotationnel, on peut écrire ce dernier comme la dérivée d'un potentiel, et l'hypothèse d'incompressibilité implique que ce potentiel est harmonique dans le domaine de fluide considéré. Cette remarque conduit aux équations suivantes :

(H3") $U = \nabla_{X,z} \Phi$.

(H2") $\Delta_{X,z} \Phi = 0$.

(H1") $\partial_t \Phi + \frac{1}{2} |\nabla_{X,z} \Phi|^2 + gz = -\frac{1}{\rho} (P - P_{atm})$ dans Ω_t .

(H5") $\nabla_{X,z} \Phi \cdot n = 0$ sur $\{z = -H_0 + b(X)\}$.

(H6") $\partial_\zeta - \sqrt{1 + |\nabla \zeta|^2} \nabla_{X,z} \Phi = 0$ sur $\{z = \zeta(t, X)\}$.

L'équation (H1") est obtenue par simple intégration de l'équation d'évolution (H1'). A noter que le potentiel Φ est défini à une fonction dépendant uniquement du temps près.

Cette nouvelle formulation ne résout pas le problème d'équations posées sur un domaine dépendant du temps. Ce dernier sera résolu à partir de la fin des années 60, par plusieurs travaux qui allaient donner naissance à la formulation moderne du problème des Water-Waves. Zakaharov fait la remarque fondamentale suivante dans [65] :

Le problème peut être entièrement décrit par deux inconnues : le potentiel à la surface, et la fonction décrivant la surface.

En effet, appelons ψ la valeur du potentiel Φ à la surface $z = \zeta$, et supposons ψ et ζ connus à un instant donné t . En résolvant le problème de Laplace suivant, avec conditions de Dirichlet et de Neumann au bord :

$$\begin{cases} \Delta \Phi = 0 \text{ dans } \Omega_t \\ \Phi|_{z=\zeta} = \psi, \quad \nabla \Phi \cdot n|_{z=-H_0+b} = 0 \end{cases} \quad (1.1.1)$$

on connaît alors la valeur de Φ dans le domaine Ω_t entier. On connaît alors $U = \nabla_{X,x}\Phi$ puis la pression P en utilisant l'équation d'Euler (H1).¹ Fort de cette remarque, Craig-Sulem et Sulem donnent dans [25] et [26] une formulation élégante du problème des Water-Waves en fonction des seules inconnues ζ et ψ :

$$\begin{cases} \partial_t \zeta - \mathcal{G}\psi = 0 \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\mathcal{G}\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0. \end{cases} \quad (1.1.2)$$

Les équations ci-dessus sont posées sur un domaine temps-espace de la forme $[0; T[\times \mathbb{R}^d$ avec $T > 0$, puisque les inconnues ζ et ψ sont définies en espace sur l'espace \mathbb{R}^d tout entier. Le problème fonctionnel des équations d'Euler à surface libre est alors résolu ! Le prix à payer est cependant l'apparition d'un opérateur non local, dit opérateur de Dirichlet-Neumann et associant au potentiel ψ la valeur de la dérivée normale du potentiel à la surface :

$$\mathcal{G}[\zeta, b] : \psi \mapsto \sqrt{1 + |\nabla\zeta|^2} \nabla\Phi \cdot n|_{z=\zeta}.$$

Pour calculer $\mathcal{G}[\zeta, b]\psi$ connaissant ψ et ζ , il faut résoudre le problème de Laplace (1.1.1) pour déterminer la valeur de Φ dans le domaine entier Ω_t , puis calculer sa dérivée normale à la surface. Ainsi, la première équation de (1.1.2) n'est autre que la condition cinématique sur la surface (H6''), et la seconde est la trace de l'équation de Bernoulli (H1'') en $z = \zeta$.

Les équations (1.1.2) constituent ce qu'on appelle la formulation Zakharov-Craig Sulem de l'équation des Water-Waves. Il se trouve que cette équation a une structure hamiltonienne, et que la quantité suivante (appelée hamiltonien) est conservée (les notations correspondent au produit scalaire et à la norme de $L^2(\mathbb{R}^d)$) :

$$H(\zeta, \psi) = \frac{1}{2}(\mathcal{G}\psi, \psi)_{L^2} + \frac{1}{2}|\zeta|_{L^2}^2.$$

1.1.2 Adimensionnement des équations et paramètres pertinents

D'un point de vue physique, la dynamique du fluide étudié dépend grandement de plusieurs grandeurs caractéristiques du milieu considéré :

- (1) La profondeur caractéristique du milieu H_0 ,
- (2) La longueur d'onde typique L_x dans la direction longitudinale,
- (3) La longueur d'onde typique L_y dans la direction transverse (quand $d = 2$),
- (4) L'amplitude typique des vagues considérées a_{surf} ,
- (5) L'amplitude typique de variation du fond a_{bott} .

Il est alors plus pratique d'adimensionner toutes les variables, de manière à faire apparaître des ordres de grandeur typiques dans l'équation. Dans cette optique, on effectue les changements de variables suivants :

$$x' = \frac{x}{L_x}, \quad y' = \frac{y}{L_y}, \quad \zeta' = \frac{\zeta}{a_{surf}}, \quad z' = \frac{z}{H_0}, \quad b' = \frac{b}{a_{bott}},$$

et encore :

$$t' = \frac{t}{t_0}, \quad \Phi' = \frac{\Phi}{\Phi_0},$$

où

$$t_0 = \frac{L_x}{\sqrt{gH_0}}, \quad \Phi_0 = \frac{a_{surf}}{H_0} L_x \sqrt{gH_0}.$$

Si l'adimensionnement des premières variables se comprend aisément, celui de Φ et de t est moins intuitif. Il découle d'une étude du système linéarisé autour de $\zeta = 0, b = 0$ qui permet d'établir une vitesse de déplacement des vagues typique pour ce système, et donc un ordre de grandeur typique

¹Il suffit bien de connaître ψ et ζ à un instant donné pour connaître P à ce même instant : en prenant la divergence de l'équation (H1'), et en utilisant (H2'), on obtient une équation de Laplace $\Delta_{X,z}P = -\rho \operatorname{div}(U \cdot \nabla_{X,z}U)$ avec condition de bord $P(X, \zeta) = P_{atm}$.

pour le potentiel. Nous ne nous étendons pas sur cette étude afin de ne pas surcharger le manuscrit (voir par exemple [37]).

Après changements de variables, quatre paramètres adimensionnés apparaissent dans l'équation. Ils sont :

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma,$$

où $\varepsilon, \mu, \beta, \gamma$, sont respectivement nommés communément paramètre de "non linéarité", de "faible profondeur"², de "topographie" et de "transversalité". Donnons tout de suite quelques ordres de grandeur pour les paramètres qui nous intéressent le plus dans ce manuscrit, à savoir μ et ε :

- En océanographie côtière, pour une houle de longueur d'onde $L_x \sim 100m$, et une amplitude typique de $a_{surf} \sim 1m$ sur un plateau continental de profondeur $H_0 \sim 10m$, on a $\varepsilon \sim 10^{-1}$ et $\mu \sim 10^{-2}$.
- Pour un tsunami où les longueurs d'onde peuvent atteindre $240km$, sur une profondeur de $4km$ et avec une amplitude de vague de $60cm$, on obtient $\varepsilon \sim 10^{-4}$ et $\mu \sim 10^{-4}$.

On voit donc que des situations différentes peuvent donner des paramètres adimensionnés d'ordres de grandeur très différents. A présent, donnons les équations des Water-Waves sous forme adimensionnée (on omet les "primes" pour des questions de clarté) :

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0, \end{cases} \quad (1.1.3)$$

où $G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi$ est l'opérateur de Dirichlet-Neumann adimensionné, donné par :

$$G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = \sqrt{1 + \varepsilon^2 |\nabla^\gamma \zeta|^2} \partial_n \Phi|_{z=\varepsilon \zeta} = (\partial_z \Phi - \mu \nabla^\gamma(\varepsilon \zeta) \cdot \nabla^\gamma \Phi)|_{z=\varepsilon \zeta},$$

où Φ est la solution du problème de Laplace avec conditions de Dirichlet (à la surface) et de Neumann (au fond) suivant :

$$\begin{cases} \Delta^{\mu, \gamma} \Phi = 0 & \text{dans } \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, \quad -1 + \beta b(X) < z < \varepsilon \zeta(X)\} \\ \phi|_{z=\varepsilon \zeta} = \psi, \quad \partial_n \Phi|_{z=-1+\beta b} = 0. \end{cases}$$

Nous avons introduit les notations suivantes :

$$\begin{array}{lll} \nabla^\gamma = {}^t(\partial_x, \gamma \partial_y) & \text{si } d = 2 & \text{et} & \nabla^\gamma = \partial_x & \text{si } d = 1 \\ \Delta^{\mu, \gamma} = \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{si } d = 2 & \text{et} & \Delta^{\mu, \gamma} = \mu \partial_x^2 + \partial_z^2 & \text{si } d = 1 \end{array}$$

et

$$\partial_n \Phi|_{z=-1+\beta b} = \frac{1}{\sqrt{1 + \beta^2 |\nabla^\gamma b|^2}} (\partial_z \Phi - \mu \nabla^\gamma(\beta b) \cdot \nabla^\gamma \Phi)|_{z=-1+\beta b}.$$

Remark 1.1.1 *Le paramètre γ égal au rapport des longueurs d'ondes dans les directions horizontales ne joue aucun rôle dominant dans le présent manuscrit. Sa présence induit néanmoins l'agréable présence d'un "exposant γ " au dessus de tous les opérateurs différentiels... Ceci n'est pas une volonté d'alourdir la lecture, mais de montrer que tous les résultats exposés restent vrais si $\gamma \neq 1$. Une situation typique où cette dernière condition est satisfaite est le cas des ondes faiblement transverses, où $\gamma = \sqrt{\mu}$, qui donne lieu aux systèmes Boussinesq faiblement transverses et à l'équation de Kadomstev-Petviashvili (voir [40]).*

²Le mot anglais étant intraduisible de manière élégante, en général on n'appelle pas ce paramètre quand on parle en français.

1.1.3 Modèles asymptotiques d'écoulement à surface libre

Lorsqu'on étudie l'équation des Water-Waves dans le cas où un ou plusieurs des paramètres adimensionnés est faible, on parle de "régime". Par exemple, lorsque μ est petit, on parle de "régime d'eau peu profonde" (cette appellation simplifiée ne doit cependant pas faire oublier que μ est le rapport entre la longueur d'onde et la profondeur, et qu'une profondeur faible seule ne signifie pas forcément que μ est petit). Il est alors possible de travailler avec des équations simplifiées, qui sont obtenues formellement en effectuant un développement limité en le paramètre de faible valeur considéré, dans les équations de Water-Waves (1.1.3). Nous présentons ici les deux régimes qui nous intéressent dans ce manuscrit.

i) Le régime dit "toit rigide" est atteint pour de faibles valeurs du paramètre ε . Le modèle asymptotique le plus simple est l'équation du toit rigide, qui est obtenu en faisant tendre ε vers 0. Plus précisément, en partant des équations d'Euler à surface libre adimensionnées, on effectue le changement d'échelle en temps suivant :

$$t' = \varepsilon t,$$

puis on fait tendre ε vers 0. Formellement, on obtient les équations suivantes, dites équations du toit rigide (on omet les "primes" pour des questions de clarté) :

$$\left\{ \begin{array}{l} \partial_t V + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\nabla^\gamma P \text{ dans } \Omega \\ \partial_t w + (V^\varepsilon \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -(\partial_z P) \text{ dans } \Omega \\ \underline{w} = 0 \\ \nabla^{\mu, \gamma} \cdot U = 0 \text{ dans } \Omega \\ \text{rot}^{\mu, \gamma}(U) = 0 \text{ dans } \Omega \\ U \cdot e_z = 0 \text{ en } z = -1 \\ P = \zeta \text{ en } z = 0 \end{array} \right. \quad (1.1.4)$$

où Ω est désormais le domaine fixe :

$$\Omega = \{(X, z) \in \mathbb{R}^{d+1}, -1 \leq z \leq 0\},$$

où on a décomposé $U = (V, w)$ en composantes verticales et horizontales, et où pour toute grandeur A , on note $\underline{A} = A(\varepsilon \zeta)$. Nous avons aussi introduit les notations suivantes :

$$\begin{array}{lll} \nabla^{\mu, \gamma} = {}^t(\sqrt{\mu} \partial_x, \gamma \sqrt{\mu} \partial_y, \partial_z) & \text{si } d = 2 & \text{et} & \nabla^{\mu, \gamma} = {}^t(\sqrt{\mu} \partial_x, \partial_z) & \text{si } d = 1 \\ \text{rot}^{\mu, \gamma} = \nabla^{\mu, \gamma} \wedge & \text{si } d = 2 & \text{et} & \text{rot}^{\mu, \gamma} = \nabla^{\mu, \gamma} \perp & \text{si } d = 1. \end{array}$$

Physiquement, les équations du toit rigide modélisent l'écoulement d'un fluide dont la surface serait fixée à 0, comme si on posait un "toit" sur l'ensemble du fluide.

ii) Le régime d'eaux peu profondes est atteint pour de faibles valeurs du paramètre μ . De nombreux modèles asymptotiques existent dans ce cadre de travail. Le plus simple est l'équation de Saint-Venant, aussi appelée équation Shallow-Water, introduit par Lagrange [35] puis Saint-Venant [1] :

$$\left\{ \begin{array}{l} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ \partial_t V + \nabla^\gamma \zeta + \varepsilon (\bar{V} \cdot \nabla^\gamma) \bar{V} = 0. \end{array} \right. \quad (1.1.5)$$

Ces équations sont obtenues formellement en prenant $\mu = 0$ dans les équations de Water-Waves (1.1.3). Bien qu'elles ne constituent pas directement l'objet d'étude de cette thèse, des méthodes employées pour l'étude de cette équation (notamment dans [13]) ont été adaptées dans cette thèse à des modèles plus complexes, telle que l'équation générale des Water-Waves elle-même, ou le modèle Boussinesq-Peregrine introduit ci-dessous.

En effectuant formellement un développement limité à l'ordre 1 en μ , on obtient un modèle rendant mieux compte du caractère dispersif des Water-Waves, appelé équation de Green-Naghdi.

Cependant nous nous intéressons dans ce manuscrit à une version simplifiée de l'équation de Green-Naghdi, dans le cas d'une faible amplitude : $\varepsilon = O(\mu)$. A noter qu'aucune hypothèse de taille n'est faite sur la topographie β). On obtient alors dans ce cas l'équation dite de Boussinesq-Peregrine (voir [11], [10], et [51]) :

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ [1 + \mu \mathcal{T}_b] \partial_t \bar{V} + \varepsilon (\bar{V} \cdot \nabla^\gamma) \bar{V} + \nabla^\gamma \zeta = 0 \end{cases} \quad (1.1.6)$$

où \mathcal{T}_b est l'opérateur suivant :

$$\forall \bar{V}, \quad \mathcal{T}_b \bar{V} = -\frac{1}{3h_b} \nabla^\gamma (h_b^3 \nabla^\gamma \cdot \bar{V}) + \frac{\beta}{2h_b} [\nabla^\gamma (h_b^2 \nabla^\gamma b \cdot \bar{V}) - h_b^2 \nabla^\gamma b \nabla^\gamma \cdot \bar{V}] + \beta^2 \nabla^\gamma b \nabla^\gamma b \cdot \bar{V},$$

avec

$$h_b = 1 - \beta b.$$

Définissons enfin une notion de "consistance" pour un modèle asymptotique, que nous donnons volontairement vaguement ici. On dit qu'une équation asymptotique de la forme (EA=0) est consistante à l'ordre k en μ avec l'équation des Water-Waves, que l'on met sous la forme (EWW=0) si on peut écrire l'équation des Water-Waves sous la forme $EWW = EA + \mu^k R$.

Ainsi, les équations de Shallow-Water (1.1.5) et de Boussinesq-Peregrine (1.1.6) sont consistantes respectivement à l'ordre 1 et à l'ordre 2 en μ avec l'équation des Water-Waves.

1.1.4 Articulation de la thèse

Les modèles asymptotiques sont importants en pratique, car ils permettent une étude simplifiée du problème des Water-Waves. Par étude simplifiée, nous entendons notamment calcul numérique sur des écoulements à surface libre. Par exemple, l'équation de toit rigide (1.1.4) est posée sur un domaine fixe, contrairement aux équations d'Euler à surface libre. L'équation de Boussinesq-Peregrine (1.1.6), quant à elle, ne fait intervenir que des opérateurs différentiels "classiques" (ou tout du moins locaux), tout en permettant l'étude du caractère dispersif des Water-Waves en fond non plat.

Le premier objectif de la thèse était de justifier mathématiquement le modèle toit rigide, en démontrant rigoureusement la convergence quand ε tend vers 0 des équations d'Euler après changement d'échelle en temps $t' = \varepsilon t$. Pour ce faire, plusieurs étapes étaient à franchir :

- (1) donner un temps d'existence de taille $\frac{1}{\varepsilon}$ quelle que soit la taille du fond pour l'équation des Water-Waves ;
- (2) donner une définition rigoureuse des solutions de l'équation d'Euler à surface libre, puis prouver l'équivalence de cette équation avec celle des Water-Waves ;
- (3) étudier la limite toit rigide quand ε tend vers 0.

Le premier point n'est pas anodin. En effet, le théorème d'existence locale le plus précis connu pour l'équation des Water-Waves était celui d'Alvarez-Samaniego Lannes, qui donnait un temps d'existence de taille $\frac{1}{\varepsilon \sqrt{\beta}}$, c'est à dire un temps d'existence $T = \frac{C}{\varepsilon \sqrt{\beta}}$ où C dépend des données initiales, avec

$$a \vee b = \max(a, b).$$

Après changement d'échelle en temps dans le cadre de l'étude du régime toit rigide, $t' = \varepsilon t$, on obtient avec ce Théorème un temps d'existence de taille $\frac{\varepsilon}{\varepsilon \sqrt{\beta}}$, et donc à la limite ε tend vers 0, on obtient des solutions définies... uniquement en zéro, si aucune hypothèse de petitesse n'est faite sur la taille β de la topographie. Un premier article rédigé au cours de cette thèse, portant le doux nom de "The Cauchy problem on large time for the Water Waves equations with large topography variations" et accepté dans le journal "Annales of IHP", démontre un temps d'existence amélioré, de taille $\frac{1}{\varepsilon}$ dans les variables de départ, et ce quelle que soit l'amplitude de variation du fond, pourvu que l'on introduise de la tension de surface. Comme nous le verrons, un temps d'existence de cette taille a également un intérêt théorique dans le cadre de la théorie des systèmes hyperboliques.

Le deuxième point semble apparaître comme tacitement vrai dans la littérature, alors qu'il n'est en réalité que très rarement justifié (voir cependant [3]). Pourtant il existe une véritable question fonctionnelle non triviale qui est de définir mathématiquement un espace de solutions définies sur un domaine qui bouge dans le temps. Une approche originale a été abordée au cours de cette thèse, et sera détaillée plus loin.

Brisons dès à présent le petit mythe crée quelques lignes plus tôt sur l'équation du toit rigide (1.1.4) : cette équation est triviale. Plus précisément, il n'est pas difficile de démontrer que les seules solutions possibles à cette équation sont... nulles. Si ce dernier fait efface instantanément tout intérêt au modèle, il apporte néanmoins une information importante : les solutions de l'équation des Water-Waves dans l'échelle de temps du toit rigide ($t' = \varepsilon t$) doivent converger vers 0 (puisqu'elles convergent vers les solutions de l'équation du toit rigide). La convergence dans le même régime a été étudiée dans [13] dans le cas de l'équation Shallow-Water. La convergence faible y est démontrée. Néanmoins la convergence forte dans un espace de type $L^\infty((0; T); L^2(\mathbb{R}^d))$ n'est pas vraie dans le cas général. Un second objectif au cours de la thèse a été de démontrer qu'un tel défaut de convergence persistait dans le cas de l'équation des Water-Waves complète. C'est l'objet d'un autre article soumis durant cette thèse, intitulé "Singular limit in the rigid lid regime for the Water-Waves equations". Remplir cet objectif nécessitait néanmoins d'établir une estimation de dispersion pour l'équation linéarisée autour d'une surface plate et d'un fond plat, ce qui à notre connaissance n'a pas été fait jusque ici dans le cas des équations en fond fini (voir Ionescu-Pusateri [32], Alazard-Delort [4] et [5] et une large littérature pour l'étude du cas du fond infini, et [45] pour une première étude dans le cas du fond fini). Cette étude constitue un troisième article, intitulé "A dispersive estimate for the linearized Water-Waves equations in finite depth".

Enfin, un dernier objectif était d'adapter la méthode d'existence en temps long utilisée pour l'équation des Water-Waves au modèle asymptotique de Boussinesq-Peregrine. Comme d'aucuns pourront le constater en lisant l'article intitulé "The Cauchy problem on large time for a Boussinesq-Peregrine equation with large topography variations", la structure fort peu pratique de cette équation nous a contraint à envisager un modèle équivalent (en terme de consistance avec l'équation des Water-Waves) se prêtant mieux à la méthode.

Comme nous l'expliquerons plus loin, l'obtention de résultats d'existence en temps long pour les équations de Boussinesq-Peregrine et de Water-Waves en grand fond revient à étudier un problème de perturbation singulière de type anelastique/faible nombre de mach, dans un cadre dispersif.

1.2 Existence en temps longs pour les équations de type hyperboliques

Nous expliquons dans cette partie que l'obtention d'un temps d'existence long de taille $\frac{1}{\varepsilon}$ pour l'équation des Water-Waves (1.1.3) n'est pas une simple amélioration d'un résultat d'existence locale, ni un simple moyen de justifier le modèle toit rigide. Il est en effet intimement lié à la théorie plus générale des systèmes hyperboliques.

1.2.1 Les systèmes hyperboliques

Un système d'équations aux dérivées partielles est dit hyperbolique s'il s'écrit sous la forme

$$\partial_t u + \sum_{j=1}^d \partial_j f_j(u) + g(u) = 0$$

où $u : \mathbb{R}^d \rightarrow \mathbb{R}^p$, $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$ est régulière, et où pour tout j , $A_j = (\partial_1 f_j \ \dots \ \partial_p f_j) \in M_p(\mathbb{R})$, tout $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$, $\alpha_1 A_1 + \dots + \alpha_d A_d$ est diagonalisable à valeurs propres réelles. Un cas particulier intéressant est celui où les matrices A_j sont symétriques, ou symétrisables (ie il existe $S \in M_p(\mathbb{R})$ symétrique définie positive tel que SA_j est symétrique pour tout j), et il existe une large littérature à ce sujet (voir par exemple [61], ou [7] pour ne citer qu'eux). L'équation Shallow-Water (1.1.5) est un cas typique de système hyperbolique symétrisable (multiplier la seconde équation par h pour obtenir un système symétrique).

L'équation des Water-Waves a une structure très proche des équations hyperboliques. Cela apparaît un peu plus clairement sur une formulation équivalente des équations (1.1.3) (voir [37] pour plus de détails) :

$$\begin{cases} \partial_t \zeta + \varepsilon \underline{V} \cdot \nabla \zeta - \frac{1}{\mu} \underline{w} = 0 \\ \partial_t \underline{V} + \underline{a} \nabla \zeta + \varepsilon (\underline{V} \cdot \nabla) \underline{V} = 0, \end{cases}$$

où on rappelle que $\underline{V}, \underline{w}$ sont les composantes respectivement verticales et horizontales de la vitesse à la surface, et où \underline{a} est le coefficient de Rayleigh-Taylor (voir section 2.2.2 du manuscrit pour une définition précise de ce dernier). Bien sûr, le terme \underline{w} est une quantité qui s'exprime de façon non locale en \underline{V} (voir plus loin pour les détails techniques), mais on reconnaît déjà là une forme très proche de la structure hyperbolique.

1.2.2 Méthodes d'énergie pour les systèmes hyperboliques

Une caractéristique des systèmes hyperboliques symétrisables est que leur existence locale se prouve avec des méthodes dites d'estimation d'énergie. Plus précisément, on montre que si on a des solutions, une certaine quantité dépendant des inconnues, qu'on appelle "énergie" est bornée sur un certain intervalle de temps, en démontrant une borne sur cette énergie appelée "estimation d'énergie". Cette énergie donne en général des informations sur les normes des solutions (par exemple sur les normes Sobolev), et par un argument de type point fixe on peut alors réussir à démontrer l'existence locale, dans un espace de la forme $C([0; T]; H^N(\mathbb{R}^d))$ par exemple.

En particulier, pour montrer des résultats d'existence en temps long $\frac{1}{\varepsilon}$, il s'agit de montrer que ces énergies sont bornées sur un intervalle de temps de cette taille. Par exemple, en ce qui concerne l'équation Shallow-Water (1.1.5), il est possible de montrer que la quantité

$$E^N(\zeta, \bar{V}) = (h\bar{V}, \bar{V})_{H^N} + |\zeta|_{H^N}^2$$

est bornée tant que la norme $W^{1,\infty}$ des inconnues n'explose pas, avec $N > d/2 + 1$. Dans le cas de l'équation des Water-Waves, l'énergie considérée est plus complexe, mais comme nous l'avons déjà indiqué plus haut l'hamiltonien suivant est conservé :

$$H(\zeta, \psi) = \frac{1}{2\mu} (\mathcal{G}\psi, \psi)_{L^2} + \frac{1}{2} |\zeta|_{L^2}^2.$$

1.2.3 Temps hyperbolique

Considérons un système quasi-linéaire hyperbolique symétrique de la forme

$$\partial_t u + \varepsilon \sum_{j=1}^d A_j(u) \partial_j u = 0$$

avec les notations ci-dessus ($A_j(u)$ est une matrice symétrique). Il se trouve que le "bon temps" d'existence pour un tel système, appelé souvent "temps hyperbolique", est de taille $\frac{1}{\varepsilon}$. C'est-à-dire qu'il "devrait être" de cette taille, et qu'il ne sera pas plus grand en général. Tentons d'illustrer cette affirmation sur une équation hyperbolique très simple, l'équation de Burgers unidimensionnelle :

$$\begin{cases} \partial_t u + \varepsilon u \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases} \quad (1.2.7)$$

avec des hypothèses raisonnables sur u_0 . En utilisant la méthode des caractéristiques, on prouve que les caractéristiques sont de la forme

$$x(t) = \varepsilon u_0(x_1) t + x_1$$

avec $x_1 \in \mathbb{R}$. Les solutions n'existent pas globalement en temps car les caractéristiques s'intersectent. Estimons le temps auquel elles s'intersectent. Soit donc deux caractéristiques $x(t) = \varepsilon u_0(x_1) t + x_1$ et $\varepsilon u_0(x_2) t + x_2$. Elles se croisent au temps

$$t = -\frac{x_2 - x_1}{\varepsilon(u_0(x_2) - u_0(x_1))}.$$

Par conséquent, le temps d'existence local pour l'équation de Burgers est

$$T = \min_{x_1, x_2 \in \mathbb{R}} - \frac{x_2 - x_1}{\varepsilon(u_0(x_2) - u_0(x_1))} = - \frac{1}{\varepsilon \min_{x \in \mathbb{R}} u'_0(x)}$$

qui est bien de taille $\frac{1}{\varepsilon}$.

1.2.4 Systèmes hyperboliques avec terme non homogène singulier

Nous considérons ici des problèmes hyperboliques dont la forme est très proche de celle de l'équation-modèle suivante :

$$\partial_t u + \varepsilon u \partial_x u + L(\varepsilon u, a(x))u = 0 \tag{1.2.8}$$

avec $u \in H^N(\mathbb{R})$ pour un certain $N > 0$ et où L est un opérateur à coefficients variables, antisymétrique pour le produit scalaire L^2 . Expliquons brièvement l'idée utilisée dans cette thèse permettant d'obtenir des résultats d'existence en temps long pour une équation de cette forme. L'estimation d'énergie usuelle pour une équation hyperbolique consiste à faire commuter des dérivées spatiales avec l'équation, afin d'estimer des énergies "contrôlant" (c'est-à-dire "donnant des informations sur") des normes Sobolev. Dans le cas d'une équation du type (1.2.8), l'estimation d'énergie L^2 est "bonne":

$$\frac{d}{dt} \frac{1}{2} |u(t)|_{L^2}^2 = (\partial_t u, u)_{L^2}$$

et en utilisant l'équation (1.2.8) pour remplacer $\partial_t u$ par son expression, on obtient :

$$\frac{d}{dt} \frac{1}{2} |u(t)|_{L^2}^2 = -\varepsilon (u \partial_x u, u)_{L^2} - (L(\varepsilon u, a(x))u, u)_{L^2}.$$

L'opérateur $L(\varepsilon u, a(x)) + u \partial_x$ étant antisymétrique pour le produit scalaire L^2 , la quantité ci-dessus est nulle, et la norme L^2 de u est conservée dans le temps ! L'estimation H^1 , en revanche, est "singulière", au sens "pas de taille ε ". En effet, dérivons l'équation (1.2.8) une fois par rapport à la variable d'espace :

$$\partial_t \partial_x u + \varepsilon u \partial_x (\partial_x u) + L(\varepsilon u, a(x)) \partial_x u + \varepsilon (\partial_x u) \partial_x u + \varepsilon (\partial_x u) dL_1(\varepsilon u, a(x)) + (\partial_x a) dL_2(\varepsilon u, a(x)) = 0$$

où on a noté dL_i la différentielle de L par rapport à la i -ième composante, pour $i = 1, 2$. Utilisant la même technique que pour l'estimation L^2 ci-dessus, on voit que la quantité $\frac{d}{dt} |u(t)|_{H^1}$ n'est pas de taille ε , à cause du terme en la différentielle dL_2 . En particulier, nous n'avons aucune chance avec cette méthode de montrer que la norme H^1 reste bornée sur un temps de taille $\frac{1}{\varepsilon}$.

La méthode employée principalement dans cette thèse est inspirée de [43], [55] ou encore [13] et consiste à faire commuter des dérivées en temps avec l'équation. Par exemple, pour (1.2.8), en dérivant par rapport au temps, on obtient:

$$\partial_t (\partial_t u) + \varepsilon u \partial_x (\partial_t u) + L(\varepsilon u, a(x)) \partial_t u + \varepsilon (\partial_t u) \partial_x u + \varepsilon (\partial_t u) dL_1(\varepsilon u, a(x)) = 0.$$

Les seuls termes non antisymétriques ont tous un facteur " ε ". Cela reste vrai si on dérive plusieurs fois en temps. On montre alors que les dérivées en temps des inconnues sont bornées sur un intervalle de temps de taille $\frac{1}{\varepsilon}$, et on conclut en utilisant l'équation qui permet de relier les dérivées en temps et les dérivées en espace. Par exemple, si $L(\varepsilon u, a(x))$ est elliptique d'ordre 1, on écrit pour l'équation (1.2.8) :

$$L(\varepsilon u, a(x)) = -\partial_t u - \varepsilon u \partial_x u$$

et en utilisant la bonne estimation pour les dérivées en temps, on obtient, en utilisant l'ellipticité de $L(\varepsilon u, a(x))$:

$$|u(t)|_{H^1} \leq C(|u(t)|_{H^1})\varepsilon t + C_0$$

où C est continue, et C_0 ne dépend que des normes des données initiales. On peut récupérer des estimations similaires pour les normes Sobolev d'ordre supérieur en considérant l'équation sur $(\partial_t)^k u$. Une estimation de la forme

$$|u(t)|_{H^N} \leq C(|u(t)|_{H^N})\varepsilon t + C_0$$

avec N assez grand, permet de montrer par un argument de continuité que u est définie sur un intervalle de temps de taille $\frac{1}{\varepsilon}$.

1.2.5 Résultats obtenus

Pour les équations des Water-Waves et de Boussinesq-Peregrine, la méthode exposée ci-dessus ne s'applique pas directement, et différents problèmes apparaissent. En effet, ces équations font intervenir des termes dispersifs, et la méthode décrite n'a jamais été appliquée à notre connaissance à ce dernier cadre. Par ailleurs, les termes dispersifs en question ont des structures différentes selon l'équation. Par exemple, pour le cas de l'équation des Water-Waves (1.1.3), avec les notations ci-dessus, $u = {}^t(\zeta, \psi)$ et

$$L(\varepsilon u, a(x)) = \begin{pmatrix} 0 & -\frac{1}{\mu} \mathcal{G}[\varepsilon \zeta, \beta b] \\ 1 & 0 \end{pmatrix}.$$

L'équation des Water-Waves n'est pas strictement hyperbolique et l'opérateur L n'est pas elliptique d'ordre un. Dans le cas de l'équation de Boussinesq-Peregrine (1.1.6), la dispersion provient de la présence de l'opérateur $I + \mu \mathcal{T}_b$ devant la dérivée en temps. Ces considérations techniques poussent à développer plusieurs stratégies, qui sont développées dans les articles "The Cauchy problem on large time for the Water Waves equations with large topography variations" et "The Cauchy problem on large time for a Boussinesq-Peregrine equation with large topography variations".

Ce manuscrit expose les résultats suivants d'existence en grand temps obtenus durant la thèse (nous ne donnons volontairement pas d'énoncé précis pour ne pas alourdir la lecture, et renvoyons aux énoncés complets) :

Theorem 1.2.1 *L'équation des Water-Waves avec tension de surface est bien posée en temps $\frac{1}{\varepsilon}$ quelle que soit la taille de la topographie pourvu que les effets de tension de surface soient de l'ordre de μ . [Voir Theorem 2.2.2]*

Bien noter qu'aucune hypothèse n'est nécessaire sur la taille du fond, ce qui constitue l'amélioration du résultat de [6]. Comme nous le verrons, la méthode employée repose grandement sur la structure des équations, et par conséquent il a été nécessaire d'introduire la tension de surface pour prouver un tel résultat. Cependant, en pratique, le coefficient de tension de surface est faible, aussi l'hypothèse de l'énoncé est raisonnable si l'on considère des régime où μ est raisonnablement petit. On démontre également les résultats suivants :

Theorem 1.2.2 *L'équation de Boussinesq-Peregrine (1.1.6) est bien posée sur un intervalle de temps ne dépendant pas de μ . [Voir Theorem 3.2.1]*

Aussi étonnant que cela puisse paraître, il n'existe à notre connaissance aucune démonstration de l'existence locale dans la littérature pour cette équation. Pour des raisons techniques dues à la structure de l'équation (typiquement, avec les notations de la section précédente, l'opérateur antisymétrique L n'est pas elliptique d'ordre suffisant par exemple), la méthode de démonstration d'existence en temps long ne s'applique pas bien à l'équation de Boussinesq-Peregrine. Afin de démontrer l'existence en "temps hyperbolique" (donc de taille $\frac{1}{\varepsilon}$), on considère alors modèle un équivalent à (au sens "consistant à l'ordre 2 en μ avec") l'équation :

Theorem 1.2.3 *Il existe une équation consistante à l'ordre 2 en μ avec l'équation de Boussinesq-Peregrine (1.1.6) bien posée en temps $\frac{1}{\varepsilon}$ en dimension 1. [Voir Theorem 3.3.7]*

Ce résultat n'est objectivement pas optimal, car il ne concerne pas l'équation de Boussinesq-Peregrine elle-même, mais un modèle équivalent. De plus, le résultat n'est vrai qu'en dimension 1. Ces défauts proviennent du fait que la structure de l'équation originale (1.1.6) ne semble pas adaptée à l'utilisation de la méthode employée par [13], comme expliqué en détail dans l'article "The Cauchy problem on large time for a Boussinesq-Peregrine equation with large topography variations". En revanche, il constitue un résultat intéressant pour une étude numérique. En effet, les modèles de type Green-Naghdi/Boussinesq permettent de prendre en compte les effets dispersifs de l'équation des Water-Waves. L'équation Green-Naghdi comporte également un terme de la forme $(I + \mu \mathcal{T}_b) \partial_t \bar{V}$, mais avec $h_b = 1 - \beta b$ remplacé par la hauteur d'eau totale $h = 1 + \varepsilon \zeta - \beta b$. Une résolution numérique de l'équation de Green-Naghdi par un schéma de type différences finies en temps nécessite donc à chaque pas de temps l'inversion de l'opérateur $(I + \mu \mathcal{T}_b)$ (qui dépend alors du temps), ce qui est coûteux en calculs. Avoir donc un modèle simplifié de cette équation est alors un bon compromis entre comportement qualitatif des solutions et difficulté d'implémentation pour un schéma numérique.

1.3 Problèmes de limite singulière et défaut de compacité

La démonstration d'un résultat d'existence en grand temps pour des équations d'écoulement à surface libre, et la limite toit rigide qui en découle rentrent dans le cadre des problèmes dits de "limite singulière".

1.3.1 L'échelle de temps en limite toit rigide

Démontrer un résultat d'existence sur un temps de taille $\frac{1}{\varepsilon}$ est équivalent à démontrer un résultat d'existence uniforme par rapport à ε après changement d'échelle en temps

$$t' = \varepsilon t.$$

Précisons : l'équation des Water-Waves (1.1.3) après changement d'échelle prend la forme

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu \varepsilon} \mathcal{G} \psi = 0 \\ \partial_t \psi + \frac{1}{\varepsilon} \zeta + M(\zeta, \psi) = 0 \end{cases} \quad (1.3.9)$$

où $M(\zeta, \psi)$ est un terme bilinéaire en ζ, ψ . Dans l'optique de démontrer des estimations d'énergies valables sur un temps indépendant de ε , on essaye d'éliminer les termes "singuliers" de taille $\frac{1}{\varepsilon}$ dans les estimations. De même, lorsqu'on passe à la limite ε tend vers 0, il convient de traiter les termes de taille $\frac{1}{\varepsilon}$ avec prudence (en pratique, on multiplie l'équation par ε avant de passer à la limite).

1.3.2 Défaut de compacité en limite toit rigide

Dans [13], cette limite dite "singulière" quand ε tend vers 0 en régime toit rigide est étudiée pour l'équation Shallow-Water. La convergence faible des solutions est démontrée, mais on ne s'attend pas en général (du moins si la limite faible est nulle) à avoir convergence forte (sous-entendu "convergence forte globale en espace" dans les espaces de Sobolev³). Une raison très simple à cela est que la quantité

$$(h\bar{V}, \bar{V})_{L^2} + |\zeta|_{L^2}^2$$

est conservée dans le temps.

Dans le cas de l'équation des Water-Waves elle-même, comme indiqué précédemment, la convergence quand ε tend vers 0, si elle a lieu, doit avoir lieu vers zéro (rappelons rapidement que l'équation d'Euler à surface libre tend formellement vers 0 quand ε tend vers 0 vers l'équation du toit rigide, dont les solutions sont nulles). Comme pour le cas Shallow-Water, il n'y a pas convergence forte vers 0 dans L^2 dans le cas général, puisque l'hamiltonien

$$\frac{1}{2\mu} (\mathcal{G}\psi, \psi)_{L^2} + \frac{1}{2} |\zeta|_{L^2}^2$$

est conservé au cours du temps (précisons sans donner de détail technique que \mathcal{G} est symétrique défini positif pour le produit scalaire L^2). Pourtant, l'énergie est bornée quand ε tend vers 0, et cette énergie contrôle des normes de type Sobolev, aussi par un argument de compacité il y a (à sous-extraction près) convergence faible dans L^2 .

1.3.3 Résultats obtenus

Un des objectifs de cette thèse était de mettre en exergue le défaut de compacité pour la limite toit rigide. Exposons brièvement la stratégie utilisée : l'équation linéarisée en $\zeta = 0, b = 0$ est :

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = 0 \\ \partial_t \psi + \zeta = 0, \end{cases} \quad (1.3.10)$$

où l'opérateur $\mathcal{G}_0 = \mathcal{G}[0, 0]$ est donné par

$$\mathcal{G}_0 = \sqrt{\mu} |D| \tanh(|D^\gamma|),$$

³En revanche, la convergence forte locale en espace y est démontrée

avec $\mathcal{F}(g(D)f)(\xi) = g(\xi)\mathcal{F}(f)(\xi)$ où $\mathcal{F}(f)$ désigne la transformée de Fourier de f pour tout $f \in \mathcal{S}'(\mathbb{R}^d)$. Il n'est pas difficile de voir que ζ vérifie alors l'équation

$$\partial_t^2 \zeta + \frac{1}{\mu \varepsilon} \mathcal{G}_0 \zeta = 0$$

et cette équation a une structure proche de celle de l'équation des ondes, hormis le fait qu'à hautes fréquences \mathcal{G}_0 n'est que d'ordre 1. Comme pour l'équation des ondes, on pourrait alors s'attendre à avoir des comportements pour les solutions de (1.3.10) de type composantes partant en $\pm\infty$ en espace, lorsque t tend vers $+\infty$. En réalité, le comportement est très différent de celui obtenu pour l'équation des ondes, du moins en dimension 1, puisqu'on montre un phénomène de dispersion. Plus précisément, on a le résultat suivant :

Theorem 1.3.1 *Les solutions $U = (\zeta, \psi)$ du système linéarisé (1.3.9) pour $d = 1$ vérifient une estimation de dispersion de la forme :*

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |U(t)|_\infty \leq C \left(\frac{1}{\mu^{3/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/3}} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}} \right) (|U(0)|_{H^1} + |x \partial_x U(0)|_2 + |U(0)|_{L^1}).$$

[Voir Theorem 4.2.5]

L'estimation de dispersion est moins bonne que la dispersion en $\frac{1}{t^{1/2}}$ pour l'équation en fond infini (voir par exemple [15]), à cause du comportement à basses fréquences de l'opérateur \mathcal{G}_0 (comme un opérateur d'ordre 2). Bien que \mathcal{G}_0 soit équivalent à basses fréquences à l'opérateur des ondes, son comportement est en fait différent, ne serait-ce que parce que la dérivée seconde de sa transformée de Fourier n'est pas nulle (contrairement à l'opérateur des ondes, justement)! Un tel résultat n'avait pas été obtenu à notre connaissance, pour le cas du fond fini (voir cependant [45] pour une estimation en fond plat dans des espaces différents), et un soin tout particulier a été utilisé pour contrôler précisément la dépendance en μ . Remarquons que si l'on pose $\lambda = \frac{t}{\sqrt{\mu}}$, on retrouve la dispersion du cas en fond infini (voir [15] par exemple) en faisant tendre μ vers $+\infty$: en effet, on écrit $e^{it\omega(D)} = e^{i\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)}$ avec $g(x) \underset{x \rightarrow +\infty}{\sim} \sqrt{|x|}$.

Le Theorem 1.3.1 montre déjà que le défaut de compacité en limite toit rigide existe sur l'équation linéarisée (1.3.10) :

Theorem 1.3.2 *On a, pour tout ζ_0 assez régulière, tout $t > 0$ et pour $d = 1$:*

$$e^{i\frac{t}{\varepsilon}\sqrt{\frac{1}{\mu}\mathcal{G}_0}} \zeta_0 \underset{\varepsilon \rightarrow 0}{\rightarrow} 0$$

et la convergence n'est pas forte dans $L^2(\mathbb{R})$ en général. [Voir Theorem 5.2.6]

Afin de mettre en évidence le même type de comportement sur l'équation globale, on écrit cette dernière sous la forme

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = f \\ \partial_t \psi + \zeta = g, \end{cases}$$

où f et g correspondent aux termes de taille $O(1)$ par rapport à ε présents dans l'équation des Water-Waves (1.3.9). On traite alors les non-linéarités f, g comme des perturbations du système linéarisé (1.3.10), en utilisant l'estimation de dispersion 1.3.1.

Afin d'utiliser l'estimation de dispersion de Theorem 1.3.1, il convenait de démontrer également un résultat d'existence pour l'équation des Water-Waves dans un espace à poids. Nous donnons volontairement un énoncé vague pour ne pas alourdir la lecture :

Theorem 1.3.3 *Pour N assez grand, il existe $k_0 > 0$ tel que si (ζ_0, ψ_0) vérifient*

$$|\zeta_0|_{H^N} + |x \partial_x \zeta_0|_{H^{N-k_0}} + |\psi_0|_{H^{N+1/2}} + |x \partial_x \psi_0|_{H^{N+1/2-k_0}} < \infty,$$

alors les solutions de l'équation des Water-Waves (1.1.3) avec conditions initiales (ζ_0, ψ_0) vérifient la même condition sur leur intervalle d'existence. [Voir Theorem 4.3.11]

La démonstration consiste en une adaptation de la preuve de l'existence locale pour le système (1.1.3) de [37]. Cependant, une difficulté technique non négligeable apparaît, qui est de contrôler de manière optimale le commutateur de l'opérateur de Dirichlet-Neumann \mathcal{G} avec x . Les deux théorèmes ci-dessous constituent l'article "A dispersive estimate for the linearized Water-Waves equations in finite depth", qui sert de support technique à la preuve d'un des résultats principaux de cette thèse, que nous énonçons ici de la façon suivante :

Theorem 1.3.4 *Les solutions de l'équation des Water-Waves non linéaire (1.1.3) pour $d = 1$ avec fond plat convergent fortement quand ε tend vers 0 vers les solutions du système linéarisé (1.3.9), en échelle toit-rigide, dans $L^\infty((0; T); L^2(\mathbb{R}^d))$. [Voir Theorem 5.2.31]*

Ce résultat affirme donc qu'en dimension 1, les solutions de l'équation totale convergent faiblement vers 0, mais pas fortement (à moins d'avoir des conditions initiales nulles). De plus, le défaut de convergence est dû à l'opérateur de l'équation linéaire, qui induit une dispersion des vagues. Expliquons physiquement en quoi consiste la limite toit rigide. Pour obtenir cette limite, on fait d'abord le changement de variable $t' = \varepsilon t$ puis on fait tendre ε vers 0. Dans les variables originales, $t = \frac{t'}{\varepsilon}$ et si on fixe t' , la convergence de ε vers zéro implique que l'on fait tendre l'amplitude des vagues vers 0, tout en "avançant dans le temps" (à la vitesse $\frac{1}{\varepsilon}$). Asymptotiquement, quand t tend vers $+\infty$, toutes les composantes intéressantes des vagues ont été écrasées et il ne reste aucun mouvement (hormis d'éventuels courant rotationnels). On peut se représenter la situation en imaginant que l'on "pose un toit" sur le fluide étudié, écrasant ainsi les vagues.

En revanche la convergence annoncée par ce Théorème ne tient plus si μ est relié à ε (se rappeler de l'estimation de dispersion du Theorem 1.3.1). Ceci se traduit physiquement par le fait qu'on est alors dans un régime où la dispersion n'est plus dominante et où le mécanisme de convergence faible est davantage de type "équation des ondes", c'est-à-dire une séparation de la solution en deux composantes qui partent à l'infini. C'est là le grand intérêt d'avoir une estimation de dispersion avec paramètres. Ceci ouvre des perspectives pour déterminer quels sont les régimes dans lesquels la dispersion va dominer. Typiquement, si $\mu = 1$ et $\varepsilon \ll 1$ alors les effets dispersifs l'emportent (le comportement est proche de l'équation linéaire). Si à l'inverse $\mu = \varepsilon \ll 1$, alors les effets dispersifs sont de même ordre que les effets non-linéaires et on a un comportement proche de celui de l'équation de Korteweg de Vries et le mécanisme est de type "équation des ondes" en plus compliqué. En particulier, on s'attend à ce qu'il ne soit plus possible de traiter les non-linéarités quadratiques du système complet comme de simples perturbations du système linéaire. Si $\varepsilon = \sqrt{\mu} \ll 1$ alors on a un régime où la dispersion est plus faible que la non-linéarité et on s'attend à avoir des vagues déferlantes...

Afin de compléter l'étude de la limite toit rigide entamée, il reste à montrer que les solutions $(\zeta^\varepsilon, \psi^\varepsilon)$ bornées sur $C([0; \frac{T}{\varepsilon}], H^N(\mathbb{R}^d)^2)$ uniformément en ε de l'équation des Water-Waves (1.1.3) fournissent des solutions pour l'équation d'Euler à surface libre. Pour ce faire, nous avons eut l'idée de prolonger $U^\varepsilon = \nabla_{x,z} \Phi^\varepsilon$ (a priori définie uniquement sur Ω_t) à une bande fixe \mathcal{S} de la forme $[a; b] \times \mathbb{R}^d$ dans laquelle Ω_t est inclut pour tout t et tout ε dans l'intervalle d'existence des solutions. On montre alors que le prolongement \tilde{U}^ε est borné dans un espace de la forme $C([0; \frac{T}{\varepsilon}]; H^N(\mathcal{S}))$ uniformément en ε , et qu'il vérifie les équations d'Euler sur le domaine Ω_t . Ceci aboutit au résultat suivant :

Theorem 1.3.5 *L'existence de solutions à l'équation des Water-Waves induit l'existence de solutions aux équations d'Euler à surface libre. [Voir Theorem 5.3.4]*

Par un argument de compacité, on a ensuite à sous-extraction près convergence faible de la suite \tilde{U}^ε quand ε tend vers zéro vers les solutions de l'équation toit rigide, qui sont nulles. La convergence n'est pas forte dans $L^\infty((0; T); L^2(\mathbb{R}^d))$, car elle ne l'est pas pour le système Water-Waves, et les deux systèmes (Euler et Zakharov/Craig-Sulem) sont équivalents.

On peut illustrer une partie du travail effectué durant cette thèse par le diagramme de la Figure 1.2. La convergence forte en régime d'eaux peu profondes de Water-Waves vers Shallow-Water a été démontrée par Alvarez-Samaniego et Lannes [6] et par Iguchi [31]. La flèche de convergence faible de Shallow-Water vers l'équation des lacs (qui est un modèle toit rigide en régime d'eaux peu profondes, voir par exemple [12]) a été justifiée par Bresch-Métivier dans [13]. Nous avons au cours de cette thèse démontré les deux dernières convergences apparaissant sur ce diagramme. Les convergences

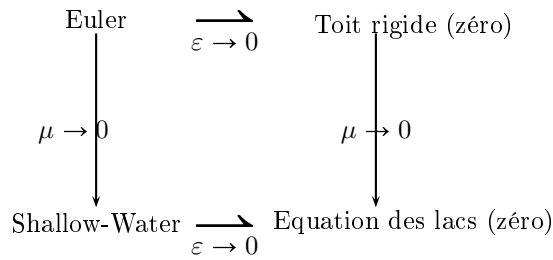


Figure 1.2: Différents modèles

faibles indiquées sur le diagramme ne sont pas fortes en général. Attention : le modèle de l'équation des lacs n'est pas trivial. Il est nul s'il est établi comme limite du modèle toit rigide en régime d'eaux peu profondes. Mathématiquement, le fait que le modèle toit rigide soit trivial provient du fait qu'on "hérite" à la limite faible amplitude des conditions de bord $(H5')$ et $(H6')$ (voir première partie), qui deviennent une condition de nullité sur la composante normale de la vitesse aux bords du domaine. Il est d'ailleurs possible que bien que les solutions soient nulles dans le modèle à rotationnel nul, des courants tournants subsistent dans le cas avec rotationnel.

Les modèles à toit rigide, qu'il s'agisse de l'équation du toit rigide (1.1.4) ou des modèles dérivés (équation des lacs [12] ou des grands lacs [16]) permettent donc de décrire de manière simplifiée la dynamique des courants pour les fluides à surface libre, mais ne disent rien sur la partie irrotationnelle de l'écoulement qui est pourtant la composante fondamentale (et souvent unique) dans l'étude des vagues. Ces modèles sont donc d'un intérêt limité dans ce cadre d'étude.

1.4 Perspectives de recherche

1.4.1 Vers des résultats optimaux

Le premier théorème d'existence en temps long pour l'existence des Water-Waves, Theorem 1.2.1 n'est pas optimal dans le sens où il nécessite la présence de tension de surface. Un résultat identique pour l'équation (1.1.3) sans tension de surface pourrait cependant être envisagé en adoptant une stratégie légèrement différente. On cherche à résoudre sur un temps long de taille $\frac{1}{\varepsilon}$ des équations de la forme

$$\partial_t u + \varepsilon u \cdot \nabla u + L(a(x), \varepsilon u(t, x), t)u = 0 \tag{1.4.11}$$

où L est un opérateur anti-symétrique et elliptique. La stratégie utilisée dans cette thèse consistait à faire commuter avec l'équation des dérivées en temps, de manière à conserver la structure anti-symétrique sur les termes n'étant pas de taille ε . Le problème fondamental pour l'équation des Water-Waves est que L n'est pas elliptique d'ordre 1 : on a L de la forme $\begin{pmatrix} 0 & -\frac{1}{\mu}\mathcal{G} \\ 1 & 0 \end{pmatrix}$ (avec \mathcal{G} bien elliptique d'ordre 1 néanmoins), aussi les dérivées en temps ne sont pas "du même ordre" que les dérivées en espace (dans la deuxième équation de (1.1.3), $\partial_t \psi$ n'est pas d'ordre 1 par rapport à ζ), aussi faire le "lien" entre les dérivées temps-espace ne semble pas être une stratégie viable. Néanmoins, on pourrait envisager de faire commuter des puissances de L avec l'équation : L^2 est en effet elliptique d'ordre 1 ! Une première difficulté est que l'équation des Water-Waves (1.1.3) n'est pas exactement de la forme (1.4.11), et qu'il faudrait donc lui trouver une formulation permettant d'avoir cette structure.

Le deuxième résultat d'existence en temps long prouvé, Theorem 1.2.1 ne semble pas optimal non plus, ne serait-ce que parce qu'il ne marche qu'en dimension 1. Il ne semble cependant pas y avoir

une marche infranchissable pour obtenir le même résultat d'existence en dimension 2 : la difficulté technique consiste essentiellement à estimer correctement le rotationnel de la vitesse moyenne \bar{V} (voir explications dans l'article concerné).

Il pourrait être intéressant de s'attaquer à l'estimation de dispersion pour le problème linéarisé en $\zeta = 0$ dans le cas d'un fond non plat (remplacer $\mathcal{G}_0 = \mathcal{G}[0, 0]$ par $\mathcal{G}[0, \beta b]$ dans l'équation (1.3.9)). Une difficulté technique importante est que l'on ne dispose pas d'expression explicite simple pour l'opérateur $\mathcal{G}[0, \beta b]$, ce qui rend compliquée l'étude d'intégrales oscillantes associées. Dans une optique plus réaliste, l'obtention d'un résultat de dispersion en dimension 2 (avec fond plat) semble à portée de main, dans la mesure où l'opérateur linéaire a une symétrie radiale.

Enfin, la mise en évidence du défaut de compacité apportée par Theorem 1.3.4 pourrait être établie en dimension 2. Une première étape consisterait à démontrer une estimation de dispersion pour l'équation linéarisée (1.3.9) en dimension 2. Ce résultat semble raisonnable à obtenir, dans la mesure où l'opérateur associé à l'équation linéarisée possède une symétrie radiale dans le domaine fréquentiel.

1.4.2 Limite faible contraste de densité dans les modèles multi-couches

La densité des eaux dans l'océan n'étant en fait pas constante, notamment en raison de différences de salinité selon la profondeur, de nombreux phénomènes sous-marins ne pouvant être négligés pour des modèles de circulation océanique à grande échelle ne sont pas pris en compte par le modèle (1.1.3). De nombreux modèles existent pour prendre en compte ce phénomène. Citons par exemple le modèle multi-couche de Saint-Venant. Ce dernier consiste à supposer que le système est constitué de N couches superposées de fluides non miscibles. L'étude du caractère bien posé d'un tel système fait l'objet de nombreux travaux : voir par exemple [28], [47] et [46]. Dans le cadre de ce modèle, un paramètre appelé contraste de densité joue un rôle essentiel. La limite de faible contraste de densité correspond à l'étude du comportement du système lorsque ce contraste tend vers 0 (la densité du fluide le plus lourd s'approche de celle du fluide le plus léger). Elle apparaît comme une limite singulière, non sans rappeler la limite singulière de faible amplitude étudiée au cours de cette thèse. Mieux, dans [29], un défaut de compacité est mis en évidence pour cette même limite : le flot se décompose comme un mode "rapide" s'échappant asymptotiquement vite à l'infini, et un mode "lent" se propageant à vitesse uniformément bornée par rapport au contraste de densité.

Il serait intéressant de combiner les résultats obtenus sur le modèle Shallow-Water multi-couches, avec ceux obtenus dans le cadre de cette thèse sur le modèle "mono-couche" pour Euler complet, afin d'étudier cette même limite "faible contraste de densité" sans supposer le régime d'eaux peu profondes. Une telle étude ne serait néanmoins pas réduite à une simple combinaison des travaux précédemment cités, et de ceux présentés dans ce manuscrit. En effet, une séparation en mode "lent" et "rapide" n'est plus vérifiée lorsque les effets dispersifs (non pris en compte par le modèle de Saint-Venant) entrent en jeu. On s'attend alors à des phénomènes d'interactions entre ces modes (voir par exemple [24]). Un autre aspect, sans parler de limite singulière faible contraste de densité, est que le caractère bien posé du système d'Euler complet n'est pour l'instant établi que dans le cas de deux couches, et sous l'hypothèse de toit rigide (voir [36]).

1.5 Plan de la thèse

Le présent manuscrit consiste, outre cette introduction, en quatre articles. Chaque article est auto-contenu, ce qui implique que les notations et le cadre de travail sont rappelés à chaque chapitre. Aussi est-il possible de lire les chapitres de manière indépendante les uns des autres, bien que leur disposition dans cet ordre constitue l'articulation logique de cette thèse. Nous donnons ici un court plan afin de résumer cette introduction :

1. Le Chapitre 2 est un article intitulé "The Cauchy problem on large time for the Water Waves equations with large topography variations". Il démontre l'existence en temps long pour l'équation des Water-Waves avec tension de surface et grande topographie.
2. Le Chapitre 3 démontre un résultat similaire au précédent pour une équation de type Boussinesq-Peregrine, au travers d'un article intitulé "The Cauchy problem on large time for a Boussinesq-

Peregrine equation with large topography variations".

3. Le Chapitre 4 donne une estimation de dispersion pour l'équation linéarisée en fond plat des Water-Waves, puis un résultat d'existence local pour l'équation complète dans des espaces à poids. Il est constitué de l'article "A dispersive estimate for the linearized Water-Waves equations in finite depth".
4. Le Chapitre 5, conclut l'étude de la limite toit rigide, en utilisant l'estimation de dispersion précédemment évoquée. On prouve dans l'article "The rigid lid limit for the water waves equations" que les solutions du système linéarisé des Water-Waves convergent faiblement, mais pas fortement en général vers 0, en limite toit rigide. On démontre également que les solutions du système total convergent fortement dans cette limite dans $L^\infty((0; T); L^2(\mathbb{R}^d))$ vers les solutions du système linéarisé. Enfin, on y démontre l'équivalence entre l'équation d'Euler à surface libre et l'équation des Water-Waves, prouvant ainsi la convergence faible mais non forte des solutions de l'équation d'Euler vers les solutions du système toit rigide.

Chapter 2

The Cauchy problem on large time for the Water Waves equations with large topography variations

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Abstract

This paper shows that the long time existence of solutions to the Water Waves equations remains true with a large topography in presence of surface tension. More precisely, the dimensionless equations depend strongly on three parameters ε, μ, β measuring the amplitude of the waves, the shallowness and the amplitude of the bathymetric variations respectively. In [6], the local existence of solutions to this problem is proved on a time interval of size $\frac{1}{\max(\beta, \varepsilon)}$ and uniformly with respect to μ . In presence of large bathymetric variations (typically $\beta \gg \varepsilon$), the existence time is therefore considerably reduced. We remove here this restriction and prove the local existence on a time interval of size $\frac{1}{\varepsilon}$ under the constraint that the surface tension parameter must be at the same order as the shallowness parameter μ . We also show that the result of [13] dealing with large bathymetric variations for the Shallow Water equations can be viewed as a particular endpoint case of our result.

2.1 Introduction

We recall here some classical formulations of the Water Waves problem, with and without surface tension. We then shortly introduce the meaningful dimensionless parameters of this problem, and then state the local existence result proved by [6]. We discuss the dependence of the size of the time interval with respect to these parameters and then explain the strategy adopted in this paper to get an improved local existence result.

2.1.1 Formulations of the Water Waves problem

The Water Waves problem puts the motion of a fluid with a free surface into equations. We recall here two equivalent formulations of the Water Waves equations for an incompressible and irrotational fluid. We then introduce the surface tension, and recall a local existence result by [6].

Free surface d -dimensional Euler equations

The motion, for an incompressible, inviscid and irrotational fluid occupying a domain Ω_t delimited below by a fixed bottom and above by a free surface is described by the following quantities:

- the velocity of the fluid $U = (V, w)$, where V and w are respectively the horizontal and vertical components ;
- the free top surface profile ζ ;
- the pressure P .

All these functions depend on the time and space variables t and $(X, z) \in \Omega_t$. There exists a function $b : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the domain of the fluid at the time t is given by

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

where H_0 is the typical depth of the water. The unknowns (U, ζ, P) are governed by the Euler equations:

$$\begin{cases} \partial_t U + U \cdot \nabla_{X,z} U = -\frac{1}{\rho} \nabla P - g e_z & \text{in } \Omega_t \\ \operatorname{div}(U) = 0 & \text{in } \Omega_t \\ \operatorname{curl}(U) = 0 & \text{in } \Omega_t. \end{cases} \quad (2.1.1)$$

We denote here $-g e_z$ the acceleration of gravity, where e_z is the unit vector in the vertical direction, and ρ the density of the fluid. Here, $\nabla_{X,z}$ denotes the $d + 1$ dimensional gradient with respect to both variables X and z .

These equations are completed by boundary conditions:

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - \underline{w} = 0 \\ U \cdot n = 0 & \text{on } \{z = -H_0 + b(X)\} \\ P - P_{atm} = \sigma \kappa(\zeta) & \text{on } \{z = \zeta(t, X)\}. \end{cases} \quad (2.1.2)$$

In these equations, \underline{V} and \underline{w} are the horizontal and vertical components of the velocity evaluated at the surface. The vector n in the last equation stands for the normal upward vector at the bottom ($X, z = -H_0 + b(X)$). We denote P_{atm} the constant pressure of the atmosphere at the surface of the fluid. The first equation states the assumption that the fluid particles do not cross the surface, while the last equation states the assumption that they do not cross the bottom. The last boundary condition on the pressure takes into account the surface tension: σ denotes the surface tension coefficient (take $\sigma = 0$ to get the Euler equations without surface tension), and $\kappa(\zeta)$ is the mean curvature at the surface

$$\kappa(\zeta) = -\nabla \cdot \left(\frac{\nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}} \right).$$

The equations (2.1.1) with boundary conditions (2.1.2) are commonly referred to as the free surface Euler equations with surface tension.

Craig-Sulem-Zakharov formulation

Since the fluid is by hypothesis irrotational, it derives from a scalar potential:

$$U = \nabla_{X,z} \Phi.$$

Zakharov remarked in [65] that the free surface profile ζ and the potential at the surface $\psi = \Phi|_{z=\zeta}$ fully determine the motion of the fluid, and gave an Hamiltonian formulation of the problem. Later, Craig-Sulem, and Sulem ([25] and [26]) gave a formulation of the Water Waves equation involving the Dirichlet-Neumann operator. The following Hamiltonian system is equivalent (see [37] and [3] for more details) to the free surface Euler equations with surface tension (2.1.1) and (2.1.2) (take $\sigma = 0$ to get the Water-Waves equations without surface tension):

$$\begin{cases} \partial_t \zeta - G\psi = 0 \\ \partial_t \psi + g\zeta + \frac{1}{2} |\nabla \psi|^2 - \frac{(G\psi + \nabla \zeta \cdot \nabla \psi)^2}{2(1 + |\nabla \zeta|^2)} = -\frac{\sigma}{\rho} \kappa(\zeta), \end{cases} \quad (2.1.3)$$

where the unknowns are ζ (free top profile) and ψ (velocity potential at the surface) with t as time variable and $X \in \mathbb{R}^d$ as space variable. The fixed bottom profile is b , and G stands for the Dirichlet-Neumann operator, that is

$$G\psi = G[\zeta, b]\psi = \sqrt{1 + |\nabla \zeta|^2} \partial_n \Phi|_{z=\zeta},$$

where Φ stands for the potential, and solves Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$\begin{cases} \Delta_{X,z} \Phi = 0 & \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -H_0 + b(X) < z < \zeta(X)\} \\ \phi|_{z=\zeta} = \psi, & \partial_n \Phi|_{z=-H_0+b} = 0 \end{cases} \quad (2.1.4)$$

with the notation, for the normal derivative

$$\partial_n \Phi|_{z=-H_0+b(X)} = \nabla_{X,z} \Phi(X, -H_0 + b(X)) \cdot n$$

where n stands for the normal upward vector at the bottom ($X, -H_0 + b(X)$). See also [37] for more details.

Dimensionless equations

Since the properties of the solutions depend strongly on the characteristics of the fluid, it is more convenient to non-dimensionalize the equations by introducing some characteristic lengths of the wave motion:

- (1) The characteristic water depth H_0 ;
- (2) The characteristic horizontal scale L_x in the longitudinal direction;
- (3) The characteristic horizontal scale L_y in the transverse direction (when $d = 2$);
- (4) The size of the free surface amplitude a_{surf} ;

(5) The size of bottom topography a_{bott} .

Let us then introduce the dimensionless variables:

$$x' = \frac{x}{L_x}, \quad y' = \frac{y}{L_y}, \quad \zeta' = \frac{\zeta}{a_{surf}}, \quad z' = \frac{z}{H_0}, \quad b' = \frac{b}{a_{bott}},$$

and the dimensionless variables:

$$t' = \frac{t}{t_0}, \quad \Phi' = \frac{\Phi}{\Phi_0},$$

where

$$t_0 = \frac{L_x}{\sqrt{gH_0}}, \quad \Phi_0 = \frac{a_{surf}}{H_0} L_x \sqrt{gH_0}.$$

After re scaling, five dimensionless parameters appear in the equation. They are

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma, \quad B_0 = \frac{\rho g L_x^2}{\sigma},$$

where $\varepsilon, \mu, \beta, \gamma, B_0$ are commonly referred to respectively as "nonlinearity", "shallowness", "topography", "transversality" and "Bond" parameters.

For instance, the Zakharov-Craig-Sulem system (2.1.3) becomes (see [37] for more details) in dimensionless variables (we omit the "primes" for the sake of clarity):

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = -\frac{1}{B_0} \frac{\kappa_\gamma(\varepsilon \sqrt{\mu} \zeta)}{\varepsilon \sqrt{\mu}}, \end{cases} \quad (2.1.5)$$

where $G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi$ stands for the dimensionless Dirichlet-Neumann operator,

$$G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = \sqrt{1 + \varepsilon^2 |\nabla^\gamma \zeta|^2} \partial_n \Phi|_{z=\varepsilon \zeta} = (\partial_z \Phi - \mu \nabla^\gamma(\varepsilon \zeta) \cdot \nabla^\gamma \Phi)|_{z=\varepsilon \zeta},$$

where Φ solves the Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$\begin{cases} \Delta^{\mu, \gamma} \Phi = 0 & \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R} - 1 + \beta b(X) < z < \varepsilon \zeta(X)\} \\ \phi|_{z=\varepsilon \zeta} = \psi, \quad \partial_n \Phi|_{z=-1+\beta b} = 0, \end{cases}$$

and where the surface tension term is

$$\kappa_\gamma(\zeta) = -\nabla^\gamma \cdot \left(\frac{\nabla^\gamma \zeta}{\sqrt{1 + |\nabla^\gamma \zeta|^2}} \right).$$

We used the following notations:

$$\begin{array}{lll} \nabla^\gamma = {}^t(\partial_x, \gamma \partial_y) & \text{if } d = 2 & \text{and} & \nabla^\gamma = \partial_x & \text{if } d = 1 \\ \Delta^{\mu, \gamma} = \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{if } d = 2 & \text{and} & \Delta^{\mu, \gamma} = \mu \partial_x^2 + \partial_z^2 & \text{if } d = 1 \end{array}$$

and

$$\partial_n \Phi|_{z=-1+\beta b} = \frac{1}{\sqrt{1 + \beta^2 |\nabla^\gamma b|^2}} (\partial_z \Phi - \mu \nabla^\gamma(\beta b) \cdot \nabla^\gamma \Phi)|_{z=-1+\beta b}.$$

2.1.2 Main result

Alvarez-Lannes ([6]) proved the following local existence result. We use the notation

$$a \vee b = \max(a, b)$$

Theorem 2.1.1 *Under reasonable assumptions on the initial conditions (ζ^0, ψ^0) , there exists a unique solution (ζ, ψ) of the Water Waves equations (2.1.5) with initial condition (ζ^0, ψ^0) on a time interval $[0; \frac{T}{\varepsilon\sqrt{\beta}}]$, where T only depends on initial data.*

For a precise statement, see section 2.2.3 and see [37] Theorem 9.6 and Chapter 4 for a complete proof. The fact that T does not depend on μ allowed the authors to provide a rigorous justification of most of the Shallow Water models used in the literature for the description, among others, of coastal flows. In these models, one has $\beta = O(\varepsilon)$ and therefore the time scale for the solution is $O(\frac{1}{\varepsilon\sqrt{\beta}}) = O(\frac{1}{\varepsilon})$. See also [31] who also rigorously proved the justification of the Shallow Water models, without a Nash-Moser scheme (which was used in [6]). There exists however some asymptotic models assuming small amplitude surface variation ($\varepsilon = O(\mu)$) and large bottom variation $\beta = O(1)$. This is the case of the well-known Boussinesq-Peregrine model ([11],[10],[51]) that has been used a lot in applications. For such a regime, Theorem 2.1.1 provides an existence time of order $O(1)$ only (see also [48] for a large time of existence on the Boussinesq-Peregrine equation). Our aim is here to improve this result in order to reach an $O(\frac{1}{\varepsilon})$ existence time. We prove the following result (for a complete and precise statement, see later Theorem 2.2.2).

Theorem 2.1.2 *The Water Waves equation with surface tension (2.1.6) admits a solution on a time interval of the form $[0; \frac{T}{\varepsilon}]$ where T only depends on $B_0\mu$ and on the initial data.*

In order to prove Theorem 2.1.2, we use a method inspired by Bresch-Métivier [13] and Métivier-Schochet [43]. The only condition we need is that there is a small amount of surface tension. More precisely, we assume that the capillary parameter $\frac{1}{B_0}$ is at the same order as the shallowness parameter μ .

In the context of the method used by [13], we start by rescaling the time by setting $t' = t\varepsilon$. The Theorem 2.1.1 now gives an existence time of order ε in these new scaled variables. The Craig-Sulem-Zakharov formulation of the Water Waves problem in the newly scaled variables is

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\varepsilon} G\psi = 0 \\ \partial_t \psi + \frac{1}{\varepsilon} \zeta + \frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} \frac{(G\psi + \varepsilon\mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = -\frac{1}{B_0\varepsilon} \frac{\kappa_\gamma(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}, \end{cases} \quad (2.1.6)$$

where G is a shorthand notation for the Dirichlet-Neumann operator $G[\varepsilon\zeta, \beta b]$ (see also Section 2.1.3). The difficulty is therefore to handle the singular $O(\frac{1}{\varepsilon})$ terms of this equation. For the sake of clarity, let us sketch the method of [13], [43] on the example of the Shallow-Water equations, implemented in [13]. The Shallow-Water equations form a typical hyperbolic quasilinear system, and can be read in the present setting variables

$$\begin{cases} \partial_t \zeta + \frac{1}{\varepsilon} \nabla^\gamma \cdot (h\bar{V}) = 0 \\ \partial_t \bar{V} + (\bar{V} \cdot \nabla^\gamma) \bar{V} + \frac{1}{\varepsilon} \nabla^\gamma \zeta = 0. \end{cases} \quad (2.1.7)$$

We denoted h the total height of water:

$$h(t, X) = 1 + \varepsilon\zeta(t, X) - \beta b(X)$$

and

$$\bar{V}(t, X) = \frac{1}{h(t, X)} \int_{-1+\beta b(X)}^{\varepsilon\zeta(t, X)} V(t, X, z) dz$$

the vertical mean of V , the horizontal component of the velocity. The natural energy associated to this equation is

$$E(\zeta, \bar{V}) = \frac{1}{2} |\zeta|_2^2 + \frac{1}{2} (h\bar{V}, \bar{V})_2,$$

where $|\cdot|_2$ is the L^2 norm on \mathbb{R}^d , and $(\cdot, \cdot)_2$ denotes the L^2 scalar product (see also Section 2.1.3). By derivating in time it is easy to check that $E((\varepsilon\partial_t)^k \zeta, (\varepsilon\partial_t)^k \bar{V})$ is uniformly bounded with respect to ε . It is however not the case for H^s norms of these unknowns with $s \geq 1$, because of the commutator

with the bottom parametrization which is of order $\frac{\beta}{\varepsilon}$ and therefore singular if $\beta = O(1)$. Now, to recover an energy uniformly bounded with respect to ε for the spatial derivatives, we use the equation (2.1.7) to write

$$\nabla^\gamma \zeta = \varepsilon \partial_t \bar{V} + \varepsilon R,$$

where $|R|_2 \leq C$ with C independent of ε . Thus $\nabla^\gamma \zeta$ is bounded in L^2 norm uniformly with respect to ε . It allows us to recover a control of $|\zeta|_{H^1}$. For \bar{V} , the equation (2.1.7) gives that $|\nabla^\gamma \cdot (h\bar{V})|_2$ is bounded by $|(\varepsilon \partial_t \zeta)|_2$ (bounded uniformly), and taking the rotational of the second equation, one has that $\text{curl}(\bar{V})$ satisfies a symmetric hyperbolic equation of the form

$$\partial_t \text{curl} \bar{V} + \bar{V} \cdot \nabla^\gamma \text{curl} \bar{V} = R$$

with

$$|R|_{L^2} \leq C,$$

and with C independent of ε , and thus $\text{curl} \bar{V}$ is uniformly bounded in L^2 norm. With an ellipticity argument, one recovers an uniform bound for \bar{V} in H^1 norm. By induction, one can recover the same bound for higher order space derivatives.

We propose here an adaptation of this method to the Water Waves problem. The structure of the equation is important for this method. For example, the Water Waves equation (2.1.6) without surface tension ($\frac{1}{B_0} = 0$) can be read as:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon (G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0. \end{cases} \quad (2.1.8)$$

The equation (2.1.8) still has the structure of quasilinear hyperbolic symmetric systems, with as energy:

$$E(\zeta, \psi) = \frac{1}{2\mu} (G[\varepsilon \zeta, \beta b] \psi, \psi)_2 + \frac{1}{2} |\zeta|_2^2.$$

Recall that $G[\varepsilon \zeta, \beta b]$ is the Dirichlet-Neumann operator (see the definition in Section 2.1.3), and is of order one. Therefore, ψ is controlled by the energy in " $H^{1/2}$ norm", and not in L^2 norm as it was the case for both unknowns of the Shallow-Water equations (2.1.7). For this reason, unlike the Shallow-Water case, space and time derivatives of the unknowns does not have the same "order": for instance, the first equation of (2.1.8) states that $\partial_t \zeta$ is only of order 1/2 with respect to the energy, since it's equal to $G\psi$ and ψ is controlled in $H^{1/2}$ norm by the energy. This consideration justifies the use of surface tension in the Water-Waves equation (2.1.6) used. Note that we pay special attention to the dependence of the surface tension on the existence time; it is indeed important to get a dependence on $B_0 \mu$ and not only on B_0 since this weaken our assumption on the size of the capillary effects.

2.1.3 Notations

We introduce here all the notations used in this paper. All the notations used are coherent with [37].

Operators and quantities

Because of the use of dimensionless variables (see before the "dimensionless equations" paragraph), we use the following twisted partial operators:

$$\begin{array}{llll} \nabla^\gamma = {}^t(\partial_x, \gamma \partial_y) & \text{if } d = 2 & \text{and} & \nabla^\gamma = \partial_x \quad \text{if } d = 1 \\ \Delta^{\mu, \gamma} = \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{if } d = 2 & \text{and} & \Delta^{\mu, \gamma} = \mu \partial_x^2 + \partial_z^2 \quad \text{if } d = 1 \\ \nabla^{\mu, \gamma} = {}^t(\sqrt{\mu} \partial_x, \gamma \sqrt{\mu} \partial_y, \partial_z) & \text{if } d = 2 & \text{and} & {}^t(\sqrt{\mu} \partial_x, \partial_z) \quad \text{if } d = 1 \\ \nabla^{\mu, \gamma} = \sqrt{\mu} \partial_x + \gamma \sqrt{\mu} \partial_y + \partial_z & \text{if } d = 2 & \text{and} & \sqrt{\mu} \partial_x + \partial_z \quad \text{if } d = 1 \\ \text{curl}^{\mu, \gamma} = {}^t(\sqrt{\mu} \gamma \partial_y - \partial_z, \partial_z - \sqrt{\mu} \partial_x, \partial_x - \gamma \partial_y) & \text{if } d = 2. & & \end{array}$$

Remark 2.1.3 *All the results proved in this paper do not need the assumption that the typical wave lengths are the same in both direction, ie $\gamma = 1$. However, if one is not interested in the dependence of γ , it is possible to take $\gamma = 1$ in all the following proofs. A typical situation where $\gamma \neq 1$ is for weakly transverse waves for which $\gamma = \sqrt{\mu}$; this leads to weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili equation (see [40]).*

We use the classical Fourier multiplier

$$\Lambda^s = (1 - \Delta)^{s/2} \text{ on } \mathbb{R}^d$$

defined by its Fourier transform as

$$\mathcal{F}(\Lambda^s u)(\xi) = (1 + |\xi|^2)^{s/2} (\mathcal{F}u)(\xi)$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$. The operator \mathfrak{P} is defined as

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} \quad (2.1.9)$$

where

$$\mathcal{F}(f(D)u)(\xi) = f(\xi)\mathcal{F}(u)(\xi)$$

is defined for any smooth function of polynomial growth f and $u \in \mathcal{S}'(\mathbb{R}^d)$. The pseudo-differential operator \mathfrak{P} acts as the square root of the Dirichlet Neumann operator (see later (2.2.26)).

We denote as before by $G_{\mu,\gamma}$, the Dirichlet-Neumann operator, which is defined as follows in the scaled variables:

$$G_{\mu,\gamma}\psi = G_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi = \sqrt{1 + \varepsilon^2|\nabla^\gamma\zeta|^2}\partial_n\Phi|_{z=\varepsilon\zeta} = (\partial_z\Phi - \mu\nabla^\gamma(\varepsilon\zeta) \cdot \nabla^\gamma\Phi)|_{z=\varepsilon\zeta}.$$

where Φ solves the Laplace equation

$$\begin{cases} \Delta^{\gamma,\mu}\Phi = 0 \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_n\Phi|_{z=-1+\beta b} = 0. \end{cases}$$

For the sake of simplicity, we use the notation $G[\varepsilon\zeta, \beta b]\psi$ or even $G\psi$ when no ambiguity is possible.

The Dirichlet-Neumann problem

In order to study the Dirichlet-Neumann problem (2.1.4), we need to map Ω_t into a fixed domain (and not on a moving subset). For this purpose, we introduce the following fixed strip:

$$\mathcal{S} = \mathbb{R}^d \times (-1; 0)$$

and the diffeomorphism

$$\Sigma_t^\varepsilon : \begin{cases} \mathcal{S} & \rightarrow \Omega_t \\ (X, z) & \mapsto (1 + \varepsilon\zeta(X) - \beta b(X))z + \varepsilon\zeta(X). \end{cases}$$

It is quite easy to check that Φ is the variational solution of (2.1.4) if and only if $\phi = \Phi \circ \Sigma_t^\varepsilon$ is the variational solution of the following problem:

$$\begin{cases} \nabla^{\mu,\gamma} \cdot P(\Sigma_t^\varepsilon)\nabla^{\mu,\gamma}\phi = 0 \\ \phi_{z=0} = \psi, \quad \partial_n\phi_{z=-1} = 0, \end{cases} \quad (2.1.10)$$

and where

$$P(\Sigma_t^\varepsilon) = |\det J_{\Sigma_t^\varepsilon}| J_{\Sigma_t^\varepsilon}^{-1} {}^t(J_{\Sigma_t^\varepsilon}^{-1}),$$

where $J_{\Sigma_t^\varepsilon}$ is the jacobian matrix of the diffeomorphism Σ_t^ε . For a complete statement of the result, and a proof of existence and uniqueness of solutions to these problems, see [37] Chapter 2.

We introduce here the notations for the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\Gamma = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$G[\varepsilon, \beta]: \quad \begin{array}{ccc} \Gamma & \longrightarrow & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \longmapsto & G[\varepsilon \zeta, \beta b] \psi. \end{array}$$

One can prove the differentiability of this mapping. See Appendix 2.A for more details. We denote $d^j G(h, k)\psi$ the j -th derivative of the mapping at (ζ, b) in the direction (h, k) . When we only differentiate in one direction, and no ambiguity is possible, we simply denote $d^j G(h)\psi$ or $d^j G(k)\psi$.

Functional spaces

The standard scalar product on $L^2(\mathbb{R}^d)$ is denoted by $(\cdot, \cdot)_2$ and defined by

$$(f, g)_2 = \int_{\mathbb{R}^d} f(x)g(x)dx$$

and the associated norm $|\cdot|_2$, defined by

$$|f|_2 = \sqrt{\int_{\mathbb{R}^d} |f(x)|^2 dx}.$$

We will denote the norm of the Sobolev spaces $H^s(\mathbb{R}^d)$ by $|\cdot|_{H^s}$, defined by

$$|f|_{H^s} = |\Lambda^s f|_2$$

where Λ^s is the Fourier multiplier defined above.

We introduce the following functional Sobolev-type spaces, or Beppo-Levi spaces:

Definition 2.1.4 We denote $\dot{H}^{s+1}(\mathbb{R}^d)$ the topological vector space

$$\dot{H}^{s+1}(\mathbb{R}^d) = \{u \in L^2_{loc}(\mathbb{R}^d), \nabla u \in H^s(\mathbb{R}^d)\}$$

endowed with the (semi) norm $|u|_{\dot{H}^{s+1}(\mathbb{R}^d)} = |\nabla u|_{H^s(\mathbb{R}^d)}$.

Just remark that $\dot{H}^{s+1}(\mathbb{R}^d)/\mathbb{R}^d$ is a Banach space (see for instance [27]).

The space variables $z \in \mathbb{R}$ and $X \in \mathbb{R}^d$ play different roles in the equations since the Euler formulation (2.1.1) is posed for $(X, z) \in \Omega_t$. Therefore, X lives in the whole space \mathbb{R}^d (which allows to take fractional Sobolev type norms in space), while z is actually bounded. For this reason, we need to introduce the following Banach spaces:

Definition 2.1.5 The Banach space $(H^{s,k}((-1, 0) \times \mathbb{R}^d), |\cdot|_{H^{s,k}})$ is defined by

$$H^{s,k}((-1, 0) \times \mathbb{R}^d) = \bigcap_{j=0}^k H^j((-1, 0); H^{s-j}(\mathbb{R}^d)), \quad |u|_{H^{s,k}} = \sum_{j=0}^k |\Lambda^{s-j} \partial_z^j u|_2.$$

2.2 Main result

This section is dedicated to the proof of Theorem 2.1.2. In Section 2.2.1 we introduce the energy space \mathcal{E}_σ^N used in the Water Waves equations. This energy plays an important role in the proof of the main result, since the key point consists in proving that this energy is uniformly bounded with respect to ε . We also recall in this Subsection the method used to prove the local existence theorem for the Water Waves equation. The proof of the local existence relies on the important assumption that the Rayleigh-Taylor condition holds; this is discussed in Section 2.2.2. The following Section 2.2.3 states the main result of this paper, that is, the precise statement of Theorem 2.1.2. The last Section 2.2.4 is dedicated to the proof of this result.

2.2.1 The energy space

The purpose of this section is to introduce the energy space used in the proof of the local existence result for the Water Waves equations. To this purpose, we explain the strategy of this proof. We adapt here the approach of [37] to the rescaled in time equations (2.2.11), pointing out where the singular terms are. We recall that we rescale the time variable for the equation (2.1.5) by setting

$$t' = t\varepsilon.$$

The Water Waves equations with surface tension (2.1.5) in the newly scaled variables are

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu\varepsilon} G\psi = 0 \\ \partial_t \psi + \frac{1}{\varepsilon} \zeta + \frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} \frac{(G\psi + \varepsilon\mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = -\frac{1}{B_0\varepsilon} \frac{\kappa_\gamma(\varepsilon\sqrt{\mu}\zeta)}{\varepsilon\sqrt{\mu}}, \end{cases} \quad (2.2.11)$$

where as usual, G is a shorthand notation for the Dirichlet-Neumann operator $G[\varepsilon\zeta, \beta b]$ (see also Section 2.1.3).

Remark 2.2.1 *We recall that*

$$\kappa_\gamma(\zeta) = -\nabla^\gamma \cdot \left(\frac{\nabla^\gamma \zeta}{\sqrt{1 + |\nabla^\gamma \zeta|^2}} \right),$$

so the surface tension term that appears on the right hand side of the second equation of (2.2.11) is only of size $\frac{1}{\varepsilon}$ (thought at first sight it seems of size $\frac{1}{\varepsilon^2}$).

The purpose of the proof of an existence time uniform with respect to ε for this equation in the newly time scaled variables, is to get an uniform bound with respect to ε for a good quantity called the energy which controls Sobolev norms of the unknowns. By a continuity argument, one can deduce a time existence independent of ε . For the Water Waves equation with surface tension (2.2.11), a natural quantity appears to act as an "energy". If one looks at the linearized equation around the rest state $\zeta = 0, \psi = 0$, one finds a system of evolution equations

$$\partial_t U + \frac{1}{\varepsilon} \mathcal{A}_\sigma U = 0, \quad \text{with} \quad \mathcal{A}_\sigma = \begin{pmatrix} 0 & -\frac{1}{\mu} G[0, \beta b] \\ 1 - \frac{1}{B_0} \Delta^\gamma & 0 \end{pmatrix}.$$

This system can be made symmetric if we multiply it by the symmetrizer

$$\begin{pmatrix} 1 - \frac{1}{B_0} \Delta^\gamma & 0 \\ 0 & \frac{1}{\mu} G[0, \beta b] \end{pmatrix},$$

where $U = {}^t(\zeta, \psi)$. In [6],[37], $G[0, \beta b]$ is replaced by $G[0, 0]$ in \mathcal{A}_σ . Here, we cannot perform this simplification because the error would be of size $O(\frac{\beta}{\varepsilon})$ and therefore singular since $\beta = O(1)$. This suggests a natural energy of the form

$$|\zeta|_2^2 + \frac{1}{B_0} |\nabla^\gamma \zeta|_2^2 + \left(\frac{1}{\mu} G[0, \beta b] \psi, \psi \right)_2.$$

The last term is uniformly equivalent to $|\mathfrak{P}\psi|_2^2$ where \mathfrak{P} is defined in (2.1.9). See later Remark 2.2.10 for a precise statement. Thus, \mathfrak{P} acts as the square root of the Dirichlet-Neumann operator and is of order 1/2.

This energy has not the sufficient order of derivatives to have a real control of the unknowns. For instance, the product $|\nabla^\gamma \psi|^2$ in the second equation of (2.2.11) is not defined if ψ is only $H^{1/2}(\mathbb{R}^d)$. To recover a control of the unknowns at a higher order, the classical scheme for this kind of method, is to differentiate the equation (2.2.11) in order to get an evolution equation of the unknowns $\partial_{X_i}^k \zeta, \partial_{X_i}^k \psi$. For this purpose, we need to use an explicit shape derivative formula with respect to the surface for the Dirichlet Neumann operator. It is given in Appendix Theorem 2.A.4:

$$dG(h)\psi = -\varepsilon G(h\underline{w}) - \varepsilon\mu \nabla^\gamma \cdot (h\underline{V})$$

with

$$\underline{w} = \frac{G\psi + \varepsilon\mu\nabla^\gamma\zeta \cdot \nabla^\gamma\psi}{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2} \quad \text{and} \quad \underline{V} = \nabla^\gamma\psi - \varepsilon\underline{w}\nabla^\gamma\zeta.$$

See [37] Chapter 3 for a full proof of this formula. By differentiating N times the first equation of (2.2.11), following the approach of [31] (see also [37] Chapter 4) one finds after some computation that, for $|(\alpha, k)| = N$,

$$\begin{aligned} \partial_t \partial^{(\alpha, k)} \zeta + \nabla^\gamma \cdot (\underline{V} \partial^{(\alpha, k)} \zeta) - \left(\frac{1}{\mu\varepsilon} G(\partial^{(\alpha, k)} \psi - \varepsilon\underline{w} \partial^{(\alpha, k)} \zeta) \right. \\ \left. + \frac{1}{\varepsilon} \sum_{j \in \mathbb{N}^*, l_1 + \dots + l_j + \delta = (\alpha, k)} d^j G(\partial^{l_1} b, \dots, \partial^{l_j} b) \partial^\delta \psi \right) = R, \end{aligned}$$

with, for any $d + 1$ -uplet of integers $(\alpha, k) = (\alpha_1, \dots, \alpha_d, k)$, $|(\alpha, k)| = \sum_{i=1}^d \alpha_i + k$, and with the notation

$$\forall (\alpha, k) = (\alpha_1, \dots, \alpha_d, k) \in \mathbb{N}^{d+1}, \quad \partial^{(\alpha, k)} f = \partial_{X_1}^{\alpha_1} \dots \partial_{X_d}^{\alpha_d} (\varepsilon \partial_t)^k f \quad (2.2.12)$$

and where, without entering technical details

$$|R|_{H_\sigma^1} \leq C(|\zeta|_{H_\sigma^N}, |\mathfrak{P}\psi|_{H^N}, \frac{1}{B_0})$$

where:

$$\forall k \in \mathbb{N}, \quad |f|_{H_\sigma^k}^2 = |f|_{H^k}^2 + \frac{1}{B_0} |\nabla^\gamma f|_{H^k}^2 \quad (2.2.13)$$

and C is a non decreasing function of its arguments which does not depend on ε . This evolution equation can be found by differentiating N times the Dirichlet-Neumann operator $G[\varepsilon\zeta, \beta b]\psi$ and thus R contains derivatives of the form

$$dG(\partial^{l_1} \zeta, \dots, \partial^{l_j} \zeta, \partial^{m_1} \beta b, \dots, \partial^{m_k} \beta b) \partial^\delta \psi$$

where the shape derivatives of G with respect to ζ have an ε factor that cancel the $\frac{1}{\varepsilon}$ singularity (see also the definition of dG in Section 2.1.3). Finally, the only singular terms comes from the derivatives of the Dirichlet Neumann operator with respect to the bottom. Up to some extra singular source terms coming from the shape derivatives with respect to the bottom, this equation has the same structure as (2.2.11), with $\partial^{(\alpha, k)} \zeta$ and $\partial^{(\alpha, k)} \psi - \varepsilon\underline{w} \partial^{(\alpha, k)} \zeta$ playing the role of ζ and ψ respectively (recall that $\partial^{(\alpha, k)}$ is defined by (2.2.12)). These quantities are the so called "Alinhac good unknowns" and will be denoted by:

$$\forall |(\alpha, k)| \geq 1 \quad \zeta_{(\alpha, k)} = \partial^{(\alpha, k)} \zeta, \quad \psi_{(\alpha, k)} = \partial^{(\alpha, k)} \psi - \varepsilon\underline{w} \partial^{(\alpha, k)} \zeta. \quad (2.2.14)$$

An evolution equation for the unknown $\psi_{(\alpha, k)}$ in term of the good unknowns can be also obtained (see later Section 2.2.4).

In the surface tension case, the leading order operator is the surface tension term κ_γ . This leads to some technical complications. For instance, in order to control the time derivatives of κ_γ , one has to include the time derivatives of the unknowns in the energy. This method has been used by [52], [36] to study the Water Waves Problem with surface tension. Time derivatives and space derivatives play a different role in this proof, and we use the notation

$$\forall k \in \mathbb{N}, \quad \zeta_{(k)} = (\varepsilon \partial_t)^k \zeta, \quad \text{and} \quad \psi_{(k)} = (\varepsilon \partial_t)^k \psi - \varepsilon\underline{w} (\varepsilon \partial_t)^k \zeta$$

for time derivatives, and

$$\forall \alpha \in \mathbb{N}^d, \forall k \in \mathbb{N}, \quad \zeta_{(\alpha, k)} = (\varepsilon \partial_t)^k \partial^\alpha \zeta \quad \text{and} \quad \psi_{(\alpha, k)} = (\varepsilon \partial_t)^k \partial^\alpha \psi - \varepsilon\underline{w} (\varepsilon \partial_t)^k \partial^\alpha \zeta$$

such that $f_{(\alpha, k)}$ denotes indeed the good unknown defined by (2.2.14) with index the $d + 1$ -uplet (α, k) . See [37] Chapter 9 for more details about how to handle the time derivative in the energy.

All these considerations explain why we do not use, for the local existence result of the Water Waves equations, an energy involving terms of the form $\partial_{X_i}^k \psi$ but rather the following energy:

$$\mathcal{E}_\sigma^N(U) = |\zeta|_2^2 + |\mathfrak{R}\psi|_{H^{t_0+3/2}}^2 + \sum_{(\alpha,k) \in \mathbb{N}^{d+1}, 1 \leq |(\alpha,k)| \leq N} |\zeta_{(\alpha,k)}|_{H_\sigma^1}^2 + |\mathfrak{R}\psi_{(\alpha,k)}|_2^2 \quad (2.2.15)$$

where as before (see (2.2.13))

$$|f|_{H_\sigma^1}^2 = |f|_2^2 + \frac{1}{B_0} |\nabla^\gamma f|_2^2.$$

The choice of N is of course purely technical, and made in particular to have the different products of functions well-defined in the Sobolev Spaces used. Again, it is very important to note that the time derivatives of order less than N appear in the equation.

We consider solutions $U = (\zeta, \psi)$ of the Water Waves equations in the following space:

$$E_{\sigma,T}^N = \{U \in C([0, T]; H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d)), \mathcal{E}_\sigma^N(U(\cdot)) \in L^\infty([0, T])\}.$$

2.2.2 The Rayleigh-Taylor condition

We explained in Subsection 2.2.1 that the Water Waves equations (2.2.11) can be "quasilinearized". In these quasilinearized equations, a quantity appears to play an important role. It is called the "Rayleigh-Taylor coefficient" (see [37] Chapter 4 and also [60] for more details) and is defined by

$$\underline{\mathbf{a}}(\zeta, \psi) = 1 + \varepsilon(\varepsilon \partial_t + \varepsilon \underline{V} \cdot \nabla^\gamma) \underline{\mathbf{w}} = -\varepsilon \frac{P_0}{\rho a g} (\partial_z P)|_{z=\varepsilon\zeta} \quad (2.2.16)$$

where $\underline{\mathbf{w}} = (\partial_z \Phi)|_{z=\varepsilon\zeta}$ and $\underline{V} = (\nabla^\gamma \Phi)|_{z=\varepsilon\zeta}$ are respectively the horizontal and vertical components of the velocity $U = \nabla_{X,z} \Phi$ evaluated at the surface.

The condition for strict hyperbolicity of the Water Waves system appears to be the following "Rayleigh-Taylor condition": $\underline{\mathbf{a}} > 0$. This makes the link with the classical Rayleigh-Taylor criterion

$$\inf_{\mathbb{R}^d} (-\partial_z P)|_{z=\varepsilon\zeta} > 0$$

where P is the dimensionless pressure. See [60] for more details. Ebin ([30]) showed that the Water Waves problem (without surface tension) is ill-posed if the Rayleigh-Taylor condition is not satisfied. Wu [64] proved that this condition is satisfied by any solution of the Water Waves problem in infinite depth. It is proved also in [37] Chapter 4 that this is also true in finite depth for the case of flat bottom. For the surface tension case, the Rayleigh-Taylor condition does not need to be satisfied in order to have well-posedness, but the existence time depends too strongly on the surface tension coefficient and is then too small for most applications to oceanography.

2.2.3 Statement of the result

We now state the main result.

Theorem 2.2.2 *Let $t_0 > d/2, N \geq t_0 + t_0 \vee 2 + 3/2$. Let $U^0 = (\zeta^0, \psi^0) \in E_0^N, b \in H^{N+1 \vee t_0+1}(\mathbb{R}^d)$. Let $\varepsilon, \gamma, \beta$ be such that*

$$0 \leq \varepsilon, \beta, \gamma \leq 1,$$

and moreover assume that:

$$\exists h_{min} > 0, \exists a_0 > 0, \quad 1 + \varepsilon \zeta^0 - \beta b \geq h_{min} \quad \text{and} \quad \underline{\mathbf{a}}(U^0) \geq a_0. \quad (2.2.17)$$

Then, there exists $T > 0$ and a unique solution $U^\varepsilon \in E_{\sigma,T}^N$ to (2.2.11) with initial data U^0 . Moreover,

$$\frac{1}{T} = C_1, \quad \text{and} \quad \sup_{t \in [0; T]} \mathcal{E}_\sigma^N(U^\varepsilon(t)) = C_2$$

with $C_i = C(\mathcal{E}_\sigma^N(U^0), \frac{1}{h_{min}}, \frac{1}{a_0}, |b|_{H^{N+1 \vee t_0+1}}, \mu B_0)$ for $i = 1, 2$.

It is very important to note that T does not depend on ε for small values of ε . The Theorem 2.1.2 gives an existence time of order ε as ε goes to zero, if β is of order 1. We prove here that it is in fact, of order 1. Note that the topography parameter β is fixed in all this study.

Let us now give a result for the Water Waves equation with surface tension (2.1.5) in the initial time variable. The local existence Theorem 2.1.1 provides an existence time of order $\frac{1}{\varepsilon\sqrt{\beta}}$ to the initial Water Waves equation (2.1.5) with surface tension. After the rescaling in time $t' = t\varepsilon$, the Theorem 2.1.1 provides an existence time of order $\frac{\varepsilon}{\varepsilon\sqrt{\beta}} \sim \frac{\varepsilon}{\beta}$ as ε goes to 0. Now, the main result Theorem 2.2.2 claims that the existence time is in fact of order 1 in this variable. It gives then the following result:

Theorem 2.2.3 *Under the assumptions of Theorem 2.2.2, there exists a unique solution (ζ, ψ) of the Water Waves equations (2.1.6) with initial condition (ζ^0, ψ^0) on a time interval $[0; \frac{T}{\varepsilon}]$ where*

$$\frac{1}{T} = C_1, \quad \text{and} \quad \sup_{t \in [0; T]} \mathcal{E}_\sigma^N(U^\varepsilon(t)) = C_2$$

with $C_i = C(\mathcal{E}_\sigma^N(U^0), \frac{1}{h_{\min}}, \frac{1}{a_0}, |b|_{H^{N+1} \vee t_0+1}, \mu B_0)$ for $i = 1, 2$.

This last result gives a gain of an order $\frac{1}{\varepsilon}$ with respect to the time existence provided by Theorem 2.1.1. It is very important to note that the existence time given by Theorem 2.2.2 depends on the constant $B_0\mu$. It implies for instance that in the shallow water limit ($\mu \ll 0$), less on less capillary effects are required (recall that the capillary effects are of order $\frac{1}{B_0}$). This is the reason why in the limit case $\mu = 0$ corresponding to the shallow water equations and investigated in [13], no surface tension is needed. We discuss about the shallow water regime in the Section 2.2.5.

2.2.4 Proof of Theorem 2.2.2

The key point of the proof of Theorem 2.2.2 is the following Proposition:

Proposition 2.2.4 *Let $U^\varepsilon = (\zeta^\varepsilon, \psi^\varepsilon)$ be the unique solution of the equation (2.2.11) on the time interval $[0; T^\varepsilon]$ and*

$$K^\varepsilon = \sup_{t \in [0, T^\varepsilon]} \mathcal{E}_\sigma^N(U^\varepsilon(t)).$$

Then we have, with the previous notations:

$$\forall t \in [0; T^\varepsilon], \quad \mathcal{E}_\sigma^N(U^\varepsilon)(t) \leq C_0 + C_1(K^\varepsilon)(t + \varepsilon) \quad (2.2.18)$$

where $C_0 = C_0(\mathcal{E}_\sigma^N(U^\varepsilon|_{t=0}))$ and $C_1(K^\varepsilon) = C_1(K^\varepsilon, \frac{1}{h_{\min}}, \frac{1}{a_0}, |b|_{H^{N+1} \vee t_0+1}, B_0\mu)$ are non decreasing functions of their arguments.

Proof of Proposition 2.2.4 The quantity ε is fixed throughout the proof. We consider a solution $U^\varepsilon = (\zeta^\varepsilon, \psi^\varepsilon)$ on a time interval $[0; T^\varepsilon]$ of (2.1.5) given by the standard local existence Theorem 2.1.1. To alleviate the notations, we omit, when no ambiguity is possible, the ε in the notation U^ε in the following estimates. Moreover, C will stand for any non decreasing continuous positive function. Let us first sketch the proof.

- (i) The evolution equation for time derivatives of the unknowns is "skew symmetric" with respect to $\frac{1}{\varepsilon}$ terms: these large terms cancel one another in energy estimates. This allows us to get the improved estimate (2.2.18) for time derivatives:

$$|(\varepsilon \partial_t)^k \zeta|_{H^1} + |\mathfrak{P}(\varepsilon \partial_t)^k \psi|_2 \leq C_1(K)t + C_0; \quad k = 0..N. \quad (2.2.19)$$

This is proved in Lemma 2.2.8.

- (ii) To get higher order estimates (with respect to space variables), we use the equation (2.2.11) to get

$$\frac{1}{\mu} G((\varepsilon \partial_t)^k \psi) = (\varepsilon \partial_t)^{k+1} \zeta + \varepsilon R$$

where $|R|_{H_\sigma^1} \leq C_1(K)$. By the first step of the proof, the term $(\varepsilon \partial_t)^{k+1} \zeta$ satisfies the "good" control (2.2.19) in H_σ^1 norm. Since G is of order one and elliptic, this should allow us to recover one space derivative for $\mathfrak{P}(\varepsilon \partial_t)^k \psi$, with the desired control (2.2.18). But there is a little constraint, due to the factor $\frac{1}{B_0}$ in the definition of the H_σ^1 norm (2.2.13):

$$|f|_{H_\sigma^1}^2 = |f|_2^2 + \frac{1}{B_0} |\nabla^\gamma f|_2^2.$$

One has to use precisely the definition of the operator \mathfrak{P} given in (2.1.9) by

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}}$$

and the inequality

$$|\mathfrak{P}\psi|_2 \leq \frac{M}{\mu} (G\psi, \psi)_2$$

(see below (2.2.26)) to get that $|D^\gamma| \mathfrak{P}\psi$ is bounded in H^1 norm by a constant depending on μB_0 . One can use the same technique to control $\Delta^\gamma \zeta$ with $(\varepsilon \partial_t)^{k+1} \psi$ in H_σ^1 norm, and recover again one space derivative for ζ . By finite induction, one recovers the control of the form (2.2.18) for $\zeta_{(k)}$ and $\psi_{(k)}$. This is done in Lemma 2.2.11.

For this proof, we choose (taking smaller time existence if necessary) T^ε such that

$$\forall t \in [0; T^\varepsilon], \quad \underline{\mathbf{a}}(t) \geq \frac{a_0}{2} \quad \text{and} \quad h(t) = 1 + \varepsilon \zeta(t) - \beta b \geq \frac{h_{min}}{2}. \quad (2.2.20)$$

The first condition may be satisfied given the continuity in time of $\underline{\mathbf{a}}$ (see the definition of $\underline{\mathbf{a}}$ in (2.2.16)) on the time interval $[0; T^\varepsilon]$, provided that $\partial_t \underline{\mathbf{a}} \in L^\infty([0; T^\varepsilon]; \mathbb{R}^d)$ (this is proved in the control of A_3 below). The second condition is satisfied by the fact the solution ζ lives in the space $C([0; T^\varepsilon]; H^{t_0+2}(\mathbb{R}^d))$, and the continuous embedding $H^{t_0} \subset L^\infty(\mathbb{R}^d)$ given $t_0 > d/2$. This gives the continuity in time of h (note that b is also in $L^\infty(\mathbb{R}^d)$).

Now let us prove the desired estimate (2.2.18) for time derivatives of ψ and ζ . Because of the energy space introduced in (2.2.15), we want to control quantities like

$$\mathcal{E}_{(\alpha, k)} = |\zeta_{(\alpha, k)}|_{H_\sigma^1}^2 + |\mathfrak{P}\psi_{(\alpha, k)}|_2^2, \quad |(\alpha, k)| \leq N$$

with $\alpha = 0$. Let k be fixed. We look for the equations for the unknown

$$U^k = (\zeta_{(k)}, \psi_{(k)}).$$

We denote the Rayleigh-Taylor coefficient by

$$\underline{\mathbf{a}} = 1 + \varepsilon(\varepsilon \partial_t + \varepsilon \underline{\mathbf{V}} \cdot \nabla^\gamma) \underline{\mathbf{w}}.$$

By differentiating k times the equations (2.2.11) with $(\varepsilon \partial_t)^k$, following the approach of [31] (see also [37] Chapter 4), one finds after some computation the following result:

Lemma 2.2.5 *The unknown $U_{(k)}$ satisfies the following equation:*

$$\partial_t U_{(k)} + \frac{1}{\varepsilon} \mathcal{A}_\sigma[U] U_{(k)} + B[U] U_{(k)} + C_k[U] U_{(k-1)} = {}^t(R_k, S_k) \quad (2.2.21)$$

with the operators

$$\begin{aligned} \mathcal{A}_\sigma[U] &= \begin{pmatrix} 0 & -\frac{1}{\mu} G \\ \underline{\mathbf{a}} - \frac{1}{B_0} \nabla^\gamma \cdot \mathcal{K}(\sqrt{\mu} \varepsilon \nabla^\gamma \zeta) \nabla^\gamma & 0 \end{pmatrix}, \\ \mathcal{B}[U] &= \begin{pmatrix} \underline{\mathbf{V}} \cdot \nabla^\gamma & 0 \\ 0 & \underline{\mathbf{V}} \cdot \nabla^\gamma \end{pmatrix}, \end{aligned}$$

and

$$C_k[U] = \begin{pmatrix} 0 & -\frac{1}{\mu} dG(\varepsilon \partial_t \zeta) \\ \frac{1}{B_0 \varepsilon} \nabla^\gamma \cdot \mathcal{K}_{(k)}[\varepsilon \sqrt{\mu} \nabla^\gamma \zeta] & 0 \end{pmatrix},$$

and where

$$\mathcal{K}(\nabla^\gamma \zeta) = \frac{(1 + |\nabla^\gamma \zeta|^2)I_d - \nabla^\gamma \zeta \otimes \nabla^\gamma \zeta}{(1 + |\nabla^\gamma \zeta|^2)^{3/2}},$$

and

$$\mathcal{K}_{(k)}[\nabla^\gamma \zeta]F = -\nabla^\gamma \cdot \left[d\mathcal{K}(\nabla^\gamma \varepsilon \partial_t \zeta) \nabla^\gamma F + d\mathcal{K}(\nabla^\gamma F) \nabla^\gamma \varepsilon \partial_t \zeta \right].$$

The residual ${}^t(R_k, S_k)$ satisfies the following control:

$$|R_k|_{H_\sigma^1} + |\mathfrak{P}S_k|_2 \leq C_1(K). \quad (2.2.22)$$

Remark 2.2.6 At first sight, C_k seems to involve a singular $\frac{1}{\varepsilon}$ factor. By looking more carefully, it is not the case since $\mathcal{K}_{(k)}[\varepsilon\sqrt{\mu}\nabla^\gamma \zeta]$ has an ε factor which cancels the $\frac{1}{\varepsilon}$ term.

Remark 2.2.7 Let us explain why the residual has to satisfy an estimate of the form (2.2.22). The energy for $\zeta_{(k)}, \psi_{(k)}$ is of the form

$$|\mathfrak{P}\psi_{(k)}|_2^2 + |\zeta_{(k)}|_2^2 + \frac{1}{B_0} |\nabla^\gamma \zeta_{(k)}|_2^2.$$

In order to get energy estimate, we differentiate this energy with respect to time, which leads to the control of terms of the form

$$\frac{1}{\mu} (\partial_t \psi_{(k)}, G\psi_{(k)})_2, \quad (\partial_t \zeta_{(k)}, \zeta_{(k)})_{H_\sigma^1}.$$

In order to control these quantities, we replace $\partial_t(\zeta_{(k)}, \psi_{(k)})$ by their expressions given by the equation (2.2.21). All terms satisfying a control of the form (2.2.22) are harmless for the energy estimate, since they lead to the control of terms such as

$$\frac{1}{\mu} (G\psi_{(k)}, S_k), \quad (\zeta_{(k)}, R_k)_{H_\sigma^1}$$

which is easily done.

Proof of Lemma 2.2.5 The differentiation of the first equation of (2.2.11) takes the form (recall that $G\psi$ stands for $G[\varepsilon\zeta, \beta b]\psi$ and thus any derivative of G with respect to ζ involves an ε factor):

$$\begin{aligned} (\varepsilon \partial_t)^k \partial_t \zeta &= \frac{1}{\mu \varepsilon} G(\varepsilon \partial_t)^k \psi + \frac{1}{\mu \varepsilon} dG((\varepsilon \partial_t)^k \zeta) \psi + \frac{1}{\mu \varepsilon} dG(\varepsilon \partial_t \zeta) \psi_{(k-1)} \\ &+ \frac{1}{\mu \varepsilon} \sum_{\substack{1 \leq j_1 + \dots + j_m + l \leq k \\ 1 \leq l}} \varepsilon^{j_1 + \dots + j_m} dG((\varepsilon \partial_t)^{j_1} \zeta, \dots, (\varepsilon \partial_t)^{j_m} \zeta) (\varepsilon \partial_t)^l \psi. \end{aligned}$$

Using the explicit shape derivative formula with respect to the surface for G given by Proposition 2.A.4, we get that

$$dG((\varepsilon \partial_t)^k \zeta) \psi = -\varepsilon G((\varepsilon \partial_t)^k \zeta \underline{\mathbf{w}}) - \varepsilon \mu \nabla^\gamma \cdot ((\varepsilon \partial_t)^k \zeta \underline{\mathbf{V}}),$$

and thus using the definition of $\zeta_{(k)}, \psi_{(k)}$ given by (2.2.14), one gets the following evolution equation:

$$\begin{aligned} (\varepsilon \partial_t)^k \partial_t \zeta + \nabla^\gamma \cdot (\underline{\mathbf{V}} \zeta_{(k)}) - \frac{1}{\mu \varepsilon} G\psi_{(k)} - \frac{1}{\mu \varepsilon} dG(\varepsilon \partial_t \zeta) \psi_{(k-1)} &= \\ \frac{1}{\mu \varepsilon} \sum_{\substack{1 \leq j_1 + \dots + j_m + l \leq k \\ 1 \leq l}} \varepsilon^{j_1 + \dots + j_m} dG((\varepsilon \partial_t)^{j_1} \zeta, \dots, (\varepsilon \partial_t)^{j_m} \zeta) (\varepsilon \partial_t)^l \psi. \end{aligned}$$

The term $dG(\varepsilon \partial_t \zeta) \psi_{(k-1)}$ is controlled in L^2 norm, but not in H_σ^1 norm. The terms of the right hand side involve derivatives of ψ of order less than $N-2$ and then can be put in a residual R_k with a control (2.2.22), using Proposition 2.A.5. The differentiation of the second equation of (2.2.11) involves the linearization of the surface tension term

$$\frac{1}{\varepsilon \sqrt{\mu}} (\varepsilon \partial_t)^k \kappa_\gamma (\varepsilon \sqrt{\mu} \zeta) = -\nabla^\gamma \cdot \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma (\varepsilon \partial_t)^k \zeta + K_{(k)}[\varepsilon \sqrt{\mu} \nabla^\gamma \zeta] (\varepsilon \partial_t)^{k-1} \zeta + \dots$$

The second order operator $\mathcal{K}_{(k)}$ is not controlled in $H^{1/2}$ norm (or $|\mathfrak{P} \cdot|_2$ norm). The other terms can be put in the residual S_k with a control (2.2.22). See [37] Chapter 9 for a complete proof of the evolution equation in terms of unknowns $\psi_{(k)}, \zeta_{(k)}$.

We now show that the singular terms of size $O(\frac{1}{\varepsilon})$ are transparent in the energy estimates for the evolution equation (2.2.21). This yields the bounds announced in (2.2.19). \square

Proposition 2.2.8 *One has the following estimate for all $0 \leq k \leq N$:*

$$|\zeta_{(k)}|_{H_\sigma^1} + |\mathfrak{P}\psi_{(k)}|_2 \leq C_2(K)t + C_0,$$

where $C_0 = C(\mathcal{E}_\sigma^N(U|_{t=0}))$ and $C_2 = C(\frac{1}{h_{\min}}, \frac{1}{a_0}, |b|_{H^{N+1, \nu_{t_0+1}}})$ are non decreasing functions of their arguments.

Remark 2.2.9 *An evolution equation can also be obtained for space derivatives, and then takes the form*

$$\partial_t U_{(\alpha, k)} + \frac{1}{\varepsilon} \tilde{\mathcal{A}}_\sigma[U] U_{(\alpha, k)} + B[U] \widetilde{U_{(\alpha, k)}} + C_{(\alpha, k)} = {}^t(R_{(\alpha, k)}, S_{(\alpha, k)}) \quad (2.2.23)$$

with

$$\tilde{\mathcal{A}}_\sigma[U] = \begin{pmatrix} \underline{\mathbf{a}} - \frac{1}{B_0} \nabla^\gamma \cdot \mathcal{K}(\sqrt{\mu} \varepsilon \nabla^\gamma \zeta) + \sum_{|\alpha|+|\delta| \leq N} dG(\partial^{\alpha_1} b, \dots, \partial^{\alpha_j} b) \delta^\delta \psi & 0 \\ -\frac{1}{\mu} G & 0 \end{pmatrix},$$

and

$$\widetilde{U_{(\alpha, k)}} = \sum_{j=1}^d U_{(\alpha - e_j, k)} + U_{(\alpha, k-1)}$$

with e_j the unit vector in the j -th direction. This system is then non symmetrizable with respect to $\frac{1}{\varepsilon}$ terms, and the controls are not uniform with respect to ε , due to spatial derivatives of the bottom. This is the reason why we have to control the time derivatives first, and then use the structure of the equation to recover higher order derivatives. In the case of a flat bottom $\beta = 0$, or almost flat bottom $\beta = O(\varepsilon)$, the terms involving space derivatives of b in $\tilde{\mathcal{A}}_\sigma[U]$ can be put in the residual and are easy to control. The proof of Theorem 2.1.1 as considered in [6] gives then a time existence of order $\frac{1}{\varepsilon}$.

Proof of Lemma 2.2.8 The system (2.2.21) can be symmetrized with respect to main order terms of size $\frac{1}{\varepsilon}$ if we multiply it by the operator

$$\mathcal{S}^1[U] = \begin{pmatrix} \underline{\mathbf{a}} - \frac{1}{B_0} \nabla^\gamma \cdot \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma & 0 \\ 0 & \frac{1}{\mu} G \end{pmatrix}. \quad (2.2.24)$$

This suggests to introduce

$$\begin{aligned} E_0 &= \frac{1}{2} |\zeta|_{H_\sigma^1}^2 + \frac{1}{2\mu} (G\psi, \psi)_2 + \frac{1}{2B_0} \left(\frac{\nabla^\gamma \zeta}{\sqrt{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}}, \nabla^\gamma \zeta \right)_2, & k = 0 \\ E_k &= (\mathcal{S}^1[U] U_{(k)}, U_{(k)})_2. & k \neq 0 \end{aligned} \quad (2.2.25)$$

The quantity $\sum_{k=0}^N E_k$ is uniformly equivalent to the energy \mathcal{E}_σ^N introduced in (2.2.15):

Lemma 2.2.10 *There exists $M = C(\frac{1}{h_{\min}}, |\zeta|_{H^{t_0+1}}, |b|_{H_{t_0+1}}, \frac{1}{a_0}, K)$ where C is a non decreasing function of its arguments, such that*

$$\frac{1}{M} \mathcal{E}_\sigma^N \leq \sum_{k=0}^N E_k \leq M \mathcal{E}_\sigma^N.$$

Proof of Lemma 2.2.10 This is proved in [37]; we give here the main steps of the proof for the sake of completeness.

(i) We start to use the following inequalities (see [37] Chapter 3):

$$\left(\psi, \frac{1}{\mu} G\psi \right)_2 \leq M_0 |\mathfrak{P}\psi|_2^2 \quad \text{and} \quad |\mathfrak{P}\psi|_2^2 \leq M_0 \left(\psi, \frac{1}{\mu} G\psi \right)_2 \quad (2.2.26)$$

for all $\psi \in \dot{H}^{1/2}(\mathbb{R}^d)$, where M_0 is a constant of the form $C(\frac{1}{h_{\min}}, |\zeta|_{H^{t_0+1}}, |b|_{H_{t_0+1}})$. The same inequality stands for space and time derivatives.

(ii) Thanks to the Rayleigh-Taylor condition (2.2.20), we have also:

$$\frac{1}{M} \frac{1}{2} |\zeta|^2 \leq \frac{1}{2} (\zeta, \underline{a}\zeta)_2 \leq M \frac{1}{2} |\zeta|^2,$$

with M a constant of the form $C(\frac{1}{a_0}, K)$. The same inequality stands for space and time derivatives.

(iii) At last, $\mathcal{K}(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta)$ is a $d \times d$ symmetric matrix uniformly bounded with respect to time and ε ,

$$\frac{1}{M} |\zeta_{(k)}|_{H_\sigma^1} \leq (\mathcal{K}(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta) \nabla^\gamma \zeta_{(k)}, \nabla^\gamma \zeta_{(k)})_2 \leq M |\zeta_{(k)}|_{H_\sigma^1}$$

where M is a constant of the form $C_1(K)$.

□

Because the term $\mathcal{C}_k[U]U^{(k-1)}$ which appear in the equation (2.2.21) contains order two derivatives with respect to $\zeta_{(k-1)}$, the time derivatives of the energies E_k are actually not controlled by the energy \mathcal{E}_σ^N . To overcome this problem, we slightly adjust the twisted energy E_k for $k = N$ by defining

$$\begin{aligned} F_k &= \varepsilon (\mathcal{S}^2[U]U_{(k-1)}, U_{(k)})_2 && \text{if } k = N, \\ &= 0 && \text{if } k \neq N, \end{aligned}$$

where

$$\mathcal{S}^2[U] = \begin{pmatrix} \frac{1}{B_0} \mathcal{K}_{(k)}(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta) & 0 \\ 0 & \frac{1}{\mu} dG(\varepsilon\partial_t\zeta) \end{pmatrix}.$$

The presence of the ε in F_k is a consequence of the first time scaling $t' = t\varepsilon$. The matrix operator $\mathcal{S}^2[U]$ symmetrizes the subprincipal term \mathcal{C}_k . One derives with respect to time this "energy". Our goal is to have, for all $0 \leq k \leq N$

$$\frac{d}{dt}(E_k + F_k) \leq C_1(K).$$

We will at the end recover a similar estimate for the energy E_k by using a Young inequality in the control of F_k by the E_j .

Control of $\frac{d}{dt}E_0$

One gets, using the symmetry of G :

$$\frac{dE_0}{dt} = (\partial_t\zeta, \zeta)_2 + \frac{1}{B_0} \left(\frac{\nabla^\gamma\zeta}{\sqrt{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2}}, \nabla^\gamma\partial_t\zeta \right)_2 + \frac{1}{\mu} (G\psi, \partial_t\psi)_2 + A_1,$$

where, the commutator terms $[G, \partial_t]$ and $[\frac{1}{\sqrt{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2}}, \partial_t]$ are

$$A_1 = - \left(\frac{1}{2B_0} \frac{(\varepsilon^2\mu\nabla^\gamma\zeta \cdot \partial_t\nabla^\gamma\zeta)\nabla^\gamma\zeta}{(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)^{3/2}}, \nabla^\gamma\zeta \right)_2 + \frac{1}{2\mu} (dG(\varepsilon\partial_t\zeta)\psi, \psi)_2.$$

Using the equations (2.2.11) to replace $\partial_t\zeta$ and $\partial_t\psi$ in this equality, one can write

$$\begin{aligned} \frac{dE_0}{dt} &= \frac{1}{\mu\varepsilon} (G\psi, \zeta)_2 - \frac{1}{\mu\varepsilon} (G\psi, \zeta)_2 \\ &+ \frac{1}{\mu\varepsilon} \frac{1}{B_0} \left(\frac{\nabla^\gamma\zeta}{\sqrt{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2}}, \nabla^\gamma G\psi \right)_2 - \frac{1}{\mu\varepsilon} \frac{1}{B_0} \left(\frac{\nabla^\gamma\zeta}{\sqrt{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2}}, \nabla^\gamma G\psi \right)_2 \\ &+ A_1 + B_1 + B_2, \end{aligned}$$

where

$$\begin{aligned} B_1 &= -\frac{1}{2\mu} (|\nabla^\gamma\psi|^2, G\psi)_2, \\ B_2 &= \frac{1}{\mu} \left(\frac{(G\psi + \mu\nabla^\gamma(\varepsilon\zeta) \cdot \nabla^\gamma\psi)^2}{2(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)}, \frac{1}{\mu} G\psi \right)_2. \end{aligned}$$

The large terms of order $\frac{1}{\varepsilon}$ cancel one another, thanks to the symmetry of the equation. One must now control A_1, B_1 and B_2 in order to get the desired estimate for E_0 .

- *Control of A_1* Let us start with the first term of A_1 . We use the fact that $|\varepsilon \partial_t \zeta|_{H_\sigma^1}$ and $|\zeta|_{H_\sigma^1}$ are bounded by $C_2(K)$, since $N \geq 2$. Moreover, since $N \geq t_0 + 1$, one has that $\nabla^\gamma \zeta \in L^\infty(\mathbb{R}^d)$ with a control by $C_2(K)$, and thus one can write, using Cauchy-Schwarz inequality,

$$\begin{aligned} \left| - \left(\frac{1}{2B_0} \frac{(\varepsilon^2 \mu \nabla^\gamma \zeta \cdot \partial_t \nabla^\gamma \zeta) \nabla^\gamma \zeta}{(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)^{3/2}}, \nabla^\gamma \zeta \right)_2 \right| &\leq \frac{1}{2} |\varepsilon \partial_t \zeta|_{H_\sigma^1} |\zeta|_{H_\sigma^1} |\nabla^\gamma \zeta|_{L^\infty(\mathbb{R}^d)}^2 \\ &\leq C_2(K). \end{aligned}$$

To control the second term of A_1 , we use the Proposition 2.A.5 in the Appendix with $s = 0$ to write

$$\left| \frac{1}{2\mu} (dG(\varepsilon \partial_t \zeta) \psi, \psi)_2 \right| \leq M_0 |\varepsilon \partial_t \zeta|_{H^{t_0+1}} |\mathfrak{P}\psi|_2 |\mathfrak{P}\psi|_2,$$

where M_0 is a constant of the form $C(\frac{1}{h_{\min}}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$. Moreover, $\varepsilon \partial_t \zeta$ is controlled in H^{t_0+1} norm by $C_2(K)$, since $N \geq t_0 + 2$ and thus, one has

$$\left| \frac{1}{2\mu} (dG(\varepsilon \partial_t \zeta) \psi, \psi)_2 \right| \leq C_2(K).$$

- *Control of B_1* One has to remark that $\nabla^\gamma \psi$ is in $L^\infty(\mathbb{R}^d)$. Indeed, one has

$$|\Lambda^{t_0} \nabla^\gamma \psi|_2 \leq C |\mathfrak{P}\psi|_{H^{t_0+3/2}},$$

where C does not depends on μ nor ψ . Since this last term is controlled by the energy, one has that $\nabla^\gamma \psi \in L^\infty(\mathbb{R}^d)$ with a control by $C_2(K)$. Thus, one can write

$$\begin{aligned} \left| \frac{1}{2\mu} (|\nabla^\gamma \psi|^2, G\psi)_2 \right| &\leq |\nabla^\gamma \psi|_{L^\infty(\mathbb{R}^d)} \frac{1}{\mu} |G\psi|_2 |\nabla^\gamma \psi|_2 \\ &\leq C_2(K). \end{aligned}$$

Now, using the second point of Proposition 2.A.1 with $s = 1/2$, one gets that

$$\left| \frac{1}{2\mu} (|\nabla^\gamma \psi|^2, G\psi)_2 \right| \leq C_2(K) M(3/2) |\mathfrak{P}\psi|_{H^1}$$

where $M(3/2)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{3/2}}, |b|_{H^{3/2}})$. Just note that the first point of Proposition 2.A.1 does not suffice here, since we want a control of $\frac{1}{\mu} G\psi$ and not only of $\frac{1}{\mu^{3/4}} G\psi$. This is the interest of the second point of this Proposition: the Dirichlet-Neumann operator G has to be seen as a $3/2$ order operator in order to be controlled uniformly with respect to μ .

- *Control of B_2* We already noticed in the previous controls that $\nabla^\gamma \zeta$ and $\nabla^\gamma \psi$ were in $L^\infty(\mathbb{R}^d)$ with a $C_2(K)$ control. Moreover, by Proposition 2.A.1 with $s = t_0 + 1/2$, one gets

$$\begin{aligned} \frac{1}{\mu} |\Lambda^{t_0} G\psi|_2 &\leq M(t_0 + 3/2) |\mathfrak{P}\psi|_{H^{t_0+1/2}} \\ &\leq C_2(K) \end{aligned}$$

and thus $\frac{1}{\mu} G\psi$ is in $L^\infty(\mathbb{R}^d)$ with a $C_2(K)$ control. Then, we write, using Cauchy-Schwarz inequality,

$$\begin{aligned} B_2 &\leq \frac{1}{\mu} (|G\psi|_{L^\infty(\mathbb{R}^d)} + \varepsilon \mu |\nabla^\gamma \zeta|_{L^\infty(\mathbb{R}^d)} |\nabla^\gamma \psi|_{L^\infty(\mathbb{R}^d)}) \left(|\nabla^\gamma \psi|_{L^\infty(\mathbb{R}^d)} \varepsilon |\nabla^\gamma \zeta|_2 + |G\psi|_2 \right) \frac{1}{\mu} |G\psi|_2 \\ &\leq C_2(K) \end{aligned}$$

where we used again Proposition 2.A.1 to control $\frac{1}{\mu} G\psi$ in L^2 norm.

- *Synthesis* To conclude, we proved that

$$\frac{dE_0}{dt} \leq C_2(K),$$

which gives, by integrating in time, the following inequality:

$$\forall t \in [0; T^\varepsilon], \quad E_0(t) \leq C_2(K)t + C_0$$

where C_0 only depends on the norm of the initial data.

Control of $\frac{d}{dt}(E_k + F_k)$ for $k \neq 0$

We deal here with the case $k = N$ (if $k < N$, there is no term of order more than N that appears in the derivative of the energy). Recall that for $k = N$, we have

$$E_k + F_k = (\mathcal{S}^1[U]U_{(k)}, U_{(k)})_2 + \varepsilon(\mathcal{S}^2[U]U_{(k-1)}, U_{(k)})_2.$$

Therefore, by derivating in time, and using the symmetry of $\mathcal{S}^1[U]$ we get

$$\begin{aligned} \frac{d}{dt}(E_k + F_k) &= (\mathcal{S}^1[U]U_{(k)}, \partial_t U_{(k)})_2 + \varepsilon(\mathcal{S}^2[U]U_{(k-1)}, \partial_t U_{(k)})_2 \\ &\quad + ([\partial_t, \mathcal{S}^1[U]]U_{(k)}, U_{(k)})_2 + (\partial_t(\mathcal{S}^2[U]U_{(k-1)}), U_{(k)})_2. \end{aligned}$$

We replace $\partial_t U_{(k)}$ by its expression given in the quasilinear system (2.2.21). One gets

$$\begin{aligned} \frac{d}{dt}(E_k + F_k) &= (\mathcal{S}^1[U]U_{(k)}, -\frac{1}{\varepsilon}\mathcal{A}_\sigma[U]U_{(k)} - B[U]U_{(k)} - \mathcal{C}_k[U]U_{(k-1)})_2 \\ &\quad + \varepsilon(\mathcal{S}^2[U]U_{(k-1)}, -\frac{1}{\varepsilon}\mathcal{A}_\sigma[U]U_{(k)} - B[U]U_{(k)} - \mathcal{C}_k[U]U_{(k-1)})_2 \\ &\quad + ([\partial_t, \mathcal{S}^1[U]]U_{(k)}, U_{(k)})_2 + (\partial_t(\mathcal{S}^2[U]U_{(k-1)}), U_{(k)})_2 \\ &\quad + (\mathcal{S}^1[U]U_{(k)}, {}^t(S_k, R_k))_2 + \varepsilon(\mathcal{S}^2[U]U_{(k-1)}, {}^t(S_k, R_k))_2. \end{aligned}$$

Thanks to the symmetry, the large terms of size $\frac{1}{\varepsilon}$ cancel one another, ie

$$\frac{1}{\varepsilon}(\mathcal{S}^1[U]U_{(k)}, \mathcal{A}_\sigma[U]U_{(k)})_2 = 0.$$

This is fundamental and based on the fact that the evolution equation for the unknown $U_{(k)}$ is still symmetrizable with respect to $\frac{1}{\varepsilon}$ terms. Again, it is not the case for the evolution equation in term of spatial derivatives $U_{(\alpha, k)}$. Then, the commutators between $U_{(k)}$ and subprincipal terms of \mathcal{C}_k , which are not controlled by the energy (mainly because of the order two operators) also cancel one another, because of the choice of F_k . More precisely, one gets

$$-(\mathcal{S}^1[U]U_{(k)}, \mathcal{C}_k[U]U_{(k-1)})_2 - (\mathcal{S}^2[U]U_{(k-1)}, \mathcal{A}_\sigma[U]U_{(k)})_2 = 0.$$

One can also check that

$$\varepsilon(\mathcal{S}^2[U]U_{(k-1)}, \mathcal{C}_k[U]U_{(k-1)})_2 = 0.$$

To conclude, we proved that

$$\frac{d}{dt}(E_k + F_k) = A_1 + A_2 + A_3 + A_4 + B_1 + B_2 + B_3 + B_4 + B_5 + B_6,$$

where $([\partial_t, \mathcal{S}^1[U]]U_{(k)}, U_{(k)})_2 = A_1 + A_2 + A_3$ with

$$\begin{aligned} A_1 &= \frac{1}{2\mu}(dG(\varepsilon\partial_t\zeta)\psi_{(k)}, \psi_{(k)})_2, \\ A_2 &= \frac{1}{2B_0}(\partial_t(K(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta))\nabla^\gamma\zeta_{(k)}, \nabla^\gamma\zeta_{(k)})_2, \\ A_3 &= \frac{1}{2}((\partial_t\mathbf{u})\zeta_{(k)}, \zeta_{(k)})_2. \end{aligned}$$

The term $(\partial_t(\mathcal{S}^2[U]U_{(k-1)}), U_{(k)})_2 = A_4$ is given by

$$A_4 = \frac{1}{\mu}(\varepsilon\partial_t(dG(\varepsilon\partial_t\zeta)(\psi_{(k-1)})), \psi_{(k)})_2 + \frac{1}{B_0}(\varepsilon\partial_t\left(\mathcal{K}_{(k)}[\sqrt{\mu}\nabla^\gamma\zeta]\zeta_{(k-1)}\right), \zeta_{(k)})_2.$$

We denote the terms coming from the evolution equation contained in $-(\mathcal{S}^1[U]U_{(k)} - B[U]U_{(k)})_2$ by

$$B_1 = (-\underline{V}\nabla^\gamma \cdot \zeta_{(k)}, \underline{\mathfrak{a}}\zeta_{(k)})_2 - \frac{1}{B_0}(\nabla^\gamma(\underline{V} \cdot \nabla^\gamma\zeta), \mathcal{K}(\sqrt{\mu}\varepsilon\nabla^\gamma\zeta)\nabla^\gamma\zeta_{(k)})_2$$

and

$$B_5 = -(\underline{V} \cdot \nabla^\gamma\psi_{(k)}, \frac{1}{\mu}G\psi_{(k)})_2$$

and the term $-\varepsilon(\mathcal{S}^2[U]U_{(k-1)}, B[U]U_{(k)})_2$ is

$$B_2 = -\varepsilon\frac{1}{\mu}(dG(\partial_t\zeta)\psi_{(k-1)}, \underline{V} \cdot \nabla^\gamma\psi_{(k)})_2 - \frac{1}{B_0}\varepsilon(\mathcal{K}_{(k)}[\sqrt{\mu}\nabla^\gamma\zeta]\zeta_{(k-1)}, \underline{V} \cdot \nabla^\gamma\zeta_{(k)})_2.$$

Finally, the residual terms $(\mathcal{S}^1[U]U_{(k)}, {}^t(S_k, R_k))_2 + \varepsilon(\mathcal{S}^2[U]U_{(k-1)}, {}^t(S_k, R_k))_2$ which are the most easy terms to control are denoted by

$$B_3 = (R_k, \underline{\mathfrak{a}}\zeta_{(k)})_2,$$

$$B_4 = +\varepsilon\frac{1}{\mu}(dG(\partial_t\zeta)\psi_{(k-1)}, S_k)_2 + \varepsilon(\mathcal{K}_{(k)}[\sqrt{\mu}\nabla^\gamma\zeta]\zeta_{(k-1)}, R_k)_2$$

and

$$B_6 = (S_k, \frac{1}{\mu}G\psi_{(k)})_2.$$

We then need to control all these terms by a constant of the form $C_2(K)$ in order to get the desired estimate. These controls requires bounds for quantities such as $dG(h, k)\psi$ or $G\psi$, which can be obtained by the use of Propositions 2.A.1, 2.A.3 and 2.A.5.

-Control of A_1 Using Proposition 2.A.5 with $s = 0$, one gets

$$\begin{aligned} \left|\frac{1}{2\mu}(dG(\varepsilon\partial_t\zeta)\psi_{(k)}, \psi_{(k)})_2\right| &\leq M_0|\varepsilon\partial_t\zeta|_{H^{t_0+1}}|\mathfrak{P}\psi_{(k)}|_2^2 \\ &\leq C_2(K) \end{aligned}$$

since $N \geq t_0 + 2$.

-Control of A_2 We start to check that $\partial_t(K(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta))$ is bounded in $L^\infty(\mathbb{R}^d)$ by $C_2(K)$:

$$\begin{aligned} \partial_t(K(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta)) &= \frac{2\varepsilon^2\mu\nabla^\gamma\zeta \cdot \partial_t\nabla^\gamma\zeta Id - \varepsilon^2\mu(\nabla^\gamma\zeta \otimes \nabla^\gamma\partial_t\zeta + \nabla^\gamma\partial_t\zeta \otimes \nabla^\gamma\zeta)}{(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)^{3/2}} \\ &\quad - \frac{3}{2} \frac{(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)Id - \varepsilon^2\mu\nabla^\gamma\zeta \otimes \nabla^\gamma\zeta}{(1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2)^{5/2}} (2\varepsilon^2\mu\partial_t\nabla^\gamma\zeta \cdot \nabla^\gamma\zeta). \end{aligned}$$

Because of the energy expression given by (2.2.15), one has that $\nabla^\gamma\zeta \in L^\infty(\mathbb{R}^d)$ (since $N \geq t_0 + 1$), $\varepsilon\partial_t\nabla^\gamma\zeta \in L^\infty(\mathbb{R}^d)$ (since $N \geq t_0 + 2$) and thus we get

$$|\partial_t(K(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta))|_{L^\infty(\mathbb{R}^d)} \leq C_2(K).$$

Thus, one can write

$$\begin{aligned} \left|\frac{1}{2B_0}\left(\partial_t(K(\varepsilon\sqrt{\mu}\nabla^\gamma\zeta))\nabla^\gamma\zeta_{(k)}, \nabla^\gamma\zeta_{(k)}\right)_2\right| &\leq C_2(K)\frac{1}{B_0}|\nabla^\gamma\zeta_{(k)}|_2^2 \\ &\leq C_2(K). \end{aligned}$$

-Control of A_3 We prove that

$$|\partial_t\underline{\mathfrak{a}}|_{L^\infty(\mathbb{R}^d)} \leq C_2(K).$$

Recall that

$$\underline{\mathbf{a}} = 1 + \varepsilon(\varepsilon\partial_t + \varepsilon\underline{\mathbf{V}} \cdot \nabla^\gamma)\underline{\mathbf{w}}.$$

By derivating with respect to time, one gets

$$\partial_t \underline{\mathbf{a}} = (\varepsilon\partial_t)^2 \underline{\mathbf{w}} + (\varepsilon\partial_t)\underline{\mathbf{V}} \cdot \nabla^\gamma \underline{\mathbf{w}} + \underline{\mathbf{V}} \cdot \nabla^\gamma (\varepsilon\partial_t)\underline{\mathbf{w}}.$$

We need to use an explicit expression of the horizontal and vertical component of the velocity at the surface $\underline{\mathbf{V}}$ and $\underline{\mathbf{w}}$ here:

$$\underline{\mathbf{w}} = \frac{G\psi + \varepsilon\mu\nabla^\gamma\zeta \cdot \nabla^\gamma\psi}{1 + \varepsilon^2\mu|\nabla^\gamma\zeta|^2}, \quad \text{and} \quad \underline{\mathbf{V}} = \nabla^\gamma\psi - \varepsilon\underline{\mathbf{w}}\nabla^\gamma\zeta.$$

If one takes brutally the $\varepsilon\partial_t$ and $(\varepsilon\partial_t)^2$ derivatives of these expressions, one has to deal with terms of the form

$$d^2G((\varepsilon\partial_t)^2\zeta)\psi, \quad d^2G(\varepsilon\partial_t\zeta, \varepsilon\partial_t\zeta)\psi, \quad dG(\varepsilon\partial_t\zeta)(\varepsilon\partial_t\psi), \quad G(\varepsilon\partial_t)^2\psi.$$

For the first and second terms, we use Proposition 2.A.6 with $s = t_0 + 1/2$ to get

$$\begin{aligned} |d^2G(\varepsilon\partial_t\zeta, \varepsilon\partial_t\zeta)\psi|_{H^{t_0}} &\leq M_0\mu^{3/4}|(\varepsilon\partial_t)^2\zeta|_{H^{t_0+1}}^2 |\mathfrak{P}\psi|_{H^{t_0+1/2}} \\ &\leq C_2(K) \end{aligned}$$

since $N \geq t_0 + 3$. For the third term, we apply the same result to get

$$|dG(\varepsilon\partial_t\zeta)(\varepsilon\partial_t\psi)|_{H^{t_0}} \leq M_0\mu^{3/4}|\varepsilon\partial_t\zeta|_{H^{t_0+1}} |\mathfrak{P}\varepsilon\partial_t\psi|_{H^{t_0+1/2}}.$$

There is a bit more work to achieve in order to control the term $|\mathfrak{P}\varepsilon\partial_t\psi|_{H^{t_0+1/2}}$. We write, noticing that $N \geq t_0 + 3/2$,

$$\begin{aligned} |\mathfrak{P}\varepsilon\partial_t\psi|_{H^{t_0+1/2}} &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1} |\mathfrak{P}\varepsilon\partial_t\partial^\beta\psi|_2 \\ &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1} |\mathfrak{P}\psi_{(\beta,1)} + \mathfrak{P}\varepsilon\underline{\mathbf{w}}\zeta_{(\beta,1)}|_2 \end{aligned}$$

using the definition of $\psi_{(\alpha,k)}$ given by (2.2.14). Using the fact that $|\mathfrak{P}f|_2 \leq C\mu^{-1/4}|f|_{H^{1/2}}$, we get

$$\begin{aligned} |\mathfrak{P}\varepsilon\partial_t\psi|_{H^{t_0+1/2}} &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1} |\mathfrak{P}\psi_{(\beta,1)}|_2 + \mu^{-1/4}\varepsilon|\underline{\mathbf{w}}\zeta_{(\beta,1)}|_{H^{1/2}} \\ &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1} |\mathfrak{P}\psi_{(\beta,1)}|_2 + \mu^{-1/4}\varepsilon|\underline{\mathbf{w}}|_{H^{t_0}} C_2(K) \end{aligned}$$

where we used the Sobolev estimate $|fg|_{1/2} \leq C|f|_{H^{t_0}}|g|_{H^{1/2}}$ to derive the last inequality. At last, we use Proposition 2.A.1 with $\underline{\mathbf{w}}$ (see the remark at the end of this Proposition) and with $s = t_0 + 1/2$ to get

$$\begin{aligned} |\mathfrak{P}\varepsilon\partial_t\psi|_{H^{t_0+1/2}} &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1} |\mathfrak{P}\psi_{(\beta,1)}|_2 + \varepsilon\mu^{1/2}M(t_0+1)|\mathfrak{P}\psi|_{H^{t_0+1}} C_2(K) \\ &\leq C_2(K) \end{aligned}$$

and finally we proved

$$|dG(\varepsilon\partial_t\zeta)(\varepsilon\partial_t\psi)|_{H^{t_0}} \leq C_2(K).$$

It remains $|G(\varepsilon\partial_t)^2\psi|_{H^{t_0}}$ to be controlled. We use Proposition 2.A.1 to write, with $s = t_0 + 1/2$,

$$|G(\varepsilon\partial_t)^2\psi|_{H^{t_0}} \leq \mu^{3/4}M(t_0+1)|\mathfrak{P}(\varepsilon\partial_t)^2\psi|_{H^{t_0+1}}$$

and we use the previous technique to prove that $|\mathfrak{P}(\varepsilon\partial_t)^2\psi|_{H^{t_0+1}} \leq C_2(K)$.

Combining all these results, one can control all term of $\partial_t \underline{\mathbf{a}}$ in L^∞ norm by $C_2(K)$, and one get the desired result:

$$|\partial_t \underline{\mathbf{a}}|_{L^\infty(\mathbb{R}^d)} \leq C_2(K).$$

It is now easy to get

$$\begin{aligned} \left| \frac{1}{2} ((\partial_t \mathbf{a}) \zeta_{(k)}, \zeta_{(k)})_2 \right| &\leq C_2(K) |\zeta_{(k)}|_2^2 \\ &\leq C_2(K). \end{aligned}$$

- *Control of A_4* The fact that we get an $\varepsilon \partial_t$ derivative here, and not only a ∂_t derivative is essential here. The first term involves terms of the form

$$(dG(\zeta_{(j)}) \psi_{(l)}, \psi_{(k)})_2$$

with $j, l \leq N$ and the Proposition 2.A.5 with $s = 0$ allows to control them by $C_2(K)$. The second term of A_4 involves $\frac{1}{B_0} \nabla^\gamma \zeta_{(k)}$ and $\zeta_{(k)}$ terms, which are controlled in L^2 norm, and other L^∞ terms (see the control of A_2 for example). There is no other difficulty than computation, to control A_2 by $C_2(K)$.

- *Control of B_1* The control of the first term requires a classical symmetry trick. We write, by integrating by parts,

$$\begin{aligned} (\underline{V} \cdot \nabla^\gamma \zeta_{(\alpha)}, \underline{\mathbf{a}} \zeta_{(\alpha)})_2 &= -(\zeta_{(\alpha)}, \nabla^\gamma \cdot (\underline{\mathbf{a}} \zeta_{(\alpha)} \underline{V}))_2 \\ &= -(\zeta_{(\alpha)}, \nabla^\gamma \cdot (\underline{\mathbf{a}} \underline{V}) \zeta_{(\alpha)})_2 - (\zeta_{(\alpha)}, \underline{\mathbf{a}} \underline{V} \cdot \nabla^\gamma \zeta_{(\alpha)})_2 \end{aligned}$$

and thus one gets

$$(\underline{V} \cdot \nabla^\gamma \zeta_{(\alpha)}, \underline{\mathbf{a}} \zeta_{(\alpha)})_2 = -\frac{1}{2} (\zeta_{(\alpha)}, \nabla^\gamma \cdot (\underline{\mathbf{a}} \underline{V}) \zeta_{(\alpha)})_2.$$

We can use the same technique as in the control of A_3 to get

$$|\nabla^\gamma \cdot (\underline{\mathbf{a}} \underline{V})|_{L^\infty(\mathbb{R}^d)} \leq C_2(K)$$

and we then get that

$$\begin{aligned} |(\underline{V} \cdot \nabla^\gamma \zeta_{(\alpha)}, \underline{\mathbf{a}} \zeta_{(\alpha)})_2| &\leq |\nabla^\gamma \cdot (\underline{\mathbf{a}} \underline{V})|_{L^\infty(\mathbb{R}^d)} |\zeta_{(k)}|_2^2 \\ &\leq C_2(K). \end{aligned}$$

It is the same trick for the second term. Using the symmetry of \mathcal{K} , we write

$$\frac{1}{B_0} (\underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(k)})_2 = \frac{1}{B_0} (\nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V} \cdot \nabla^\gamma \zeta_{(k)})_2$$

and, by integrating by parts,

$$\begin{aligned} \frac{1}{B_0} (\underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(k)})_2 &= -\frac{1}{B_0} (\nabla^\gamma \zeta_{(k)}, \nabla^\gamma \cdot (\mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V}) \nabla^\gamma \zeta_{(k)})_2 \\ &\quad - \frac{1}{B_0} (\nabla^\gamma \zeta_{(k)}, \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}))_2 \\ &= -\frac{1}{B_0} (\nabla^\gamma \zeta_{(k)}, \nabla^\gamma \cdot (\mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V}) \nabla^\gamma \zeta_{(k)})_2 \\ &\quad - \frac{1}{B_0} (\underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(k)})_2 \end{aligned}$$

and thus

$$\frac{1}{B_0} (\underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(k)})_2 = -\frac{1}{2B_0} (\nabla^\gamma \zeta_{(k)}, \nabla^\gamma \cdot (\mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V}) \nabla^\gamma \zeta_{(k)})_2.$$

We can then use the same type of computation as for the control of A_2 to show that

$$\nabla^\gamma \cdot (\mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \underline{V}) \in L^\infty(\mathbb{R}^d)$$

with a $C_2(K)$ bound, and finally we get, by Cauchy-Schwarz's inequality

$$\begin{aligned} \left| \frac{1}{B_0} (\underline{V} \nabla^\gamma \cdot (\nabla^\gamma \zeta_{(k)}), \mathcal{K}(\varepsilon \sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(k)})_2 \right| &\leq C_2(K) \frac{1}{B_0} |\nabla^\gamma \zeta_{(k)}|^2 \\ &\leq C_2(K). \end{aligned}$$

- *Control of B_2* For the first term, we write

$$(\underline{V} \cdot \nabla^\gamma \psi_{(k)}, \frac{1}{\mu} dG(\varepsilon \partial_t \zeta) \psi_{(k-1)}) = \left(\frac{\nabla^\gamma}{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}} \psi_{(k)}, (1 + \sqrt{\mu} |D^\gamma|)^{1/2} (\underline{V} \frac{1}{\mu} dG(\varepsilon \partial_t \zeta) \psi_{(k-1)}) \right)$$

and we deduce with Cauchy-Schwarz inequality that this quantity is bounded in absolute value by

$$|\underline{V}|_{H^{t_0}} |\mathfrak{P} \psi_{(k)}|_2 \left(\frac{1}{\mu} |dG(\varepsilon \partial_t \zeta) \psi_{(k-1)}|_2 + \mu^{-3/4} |dG(\varepsilon \partial_t \zeta)|_{H^{1/2}} \right).$$

We now use Proposition 2.A.6 to control this term by $C_2(K)$. The second term of B_2 is controlled by using Cauchy-Schwarz inequality.

- *Control of B_3* We use the control $|\underline{a}|_{L^\infty(\mathbb{R}^d)} \leq C_2(K)$ and the control over R_k given by (2.2.22) to get, with Cauchy-Schwarz inequality

$$\begin{aligned} |(R_k, \underline{a} \zeta_{(k)})_2| &\leq |\underline{a}|_{L^\infty(\mathbb{R}^d)} |R_k|_2 |\zeta_{(k)}|_2 \\ &\leq C_2(K). \end{aligned}$$

- *Control of B_4* It is a direct use of Cauchy-Schwarz inequality and Proposition 2.A.5, and the control over R_k and S_k given by (2.2.22).

- *Control of B_5* To control this term, we use a direct application of Proposition 2.A.7 to write

$$\begin{aligned} |(\underline{V} \cdot \nabla^\gamma \psi_{(k)}, \frac{1}{\mu} G \psi_{(k)})_2| &\leq M |\underline{V}|_{W^{1,\infty}} |\mathfrak{P} \psi_{(k)}|_2^2 \\ &\leq C_2(K). \end{aligned}$$

- *Control of B_6* We use Proposition 2.A.3 with $s = 0$ to get

$$\begin{aligned} |(S_k, \frac{1}{\mu} G \psi_{(k)})_2| &\leq \mu M_0 |\mathfrak{P} S_k|_2 |\mathfrak{P} \psi_{(k)}|_2 \\ &\leq C_2(K) \end{aligned}$$

where we used the control over S_k given by (2.2.22) to derive the last inequality.

- *Synthesis* We proved that

$$\frac{d}{dt} (E_k + F_k) \leq C_2(K)$$

and thus we get by integrating in time:

$$\forall t \in [0; T^\varepsilon], \quad (E_k + F_k)(t) \leq C_2(K)t + C_0$$

where C_0 only depends on the initial energy. It is easy to get that

$$|F_k| \leq C_2(K)\varepsilon$$

and thus

$$\forall t \in [0; T^\varepsilon], \quad E_k(t) \leq C_2(K)(t + \varepsilon) + C_0.$$

Thanks to the equivalence between E_k and the initial energy for this problem \mathcal{E}_σ^k , introduced for Theorem 2.2.2 by (2.2.15), proved in Remark 2.2.10, we conclude to the desired result :

$$\forall t \in [0; T^\varepsilon], \forall 0 \leq k \leq N, \quad |\zeta_{(k)}|_{H_\sigma^1} + |\mathfrak{P} \psi_{(k)}|_2 \leq C_2(K)(t + \varepsilon) + C_0$$

□

The Proposition 2.2.8 gives the desired estimate of the form (2.2.18) for time derivatives (more precisely, for $|\zeta_{(k)}|_{H_\sigma^1} + |\mathfrak{P} \psi_{(k)}|_2$). We want to recover the same estimate for space derivatives, using directly the equation (2.2.11). To this purpose, one has to make precisely the link between the H^1 norm of $\mathfrak{P} \psi_{(k)}$ and the H_σ^1 norm of $\zeta_{(k+1)}$. This is the point of the following Lemma.

Lemma 2.2.11 *For all $0 \leq |\alpha| \leq N$, one has:*

$$|\mathfrak{P}\psi_{(\alpha,k)}|_2 + |\zeta_{(\alpha,k)}|_{H_\varepsilon^1} \leq C_1(K)(t + \varepsilon) + C_0$$

where C_0 only depends on the initial energy of the unknowns, for all $0 \leq k \leq N - |\alpha|$.

Proof of Lemma 2.2.11 The proof is done by induction on $|\alpha|$. For $|\alpha| = 0$ it is the Lemma 2.2.8.

Assume that the result is true for $|\alpha| - 1$, with $|\alpha| \geq 1$. Let $0 \leq k \leq N - |\alpha|$. We start to give an evolution equation in term of unknowns $\zeta_{(\alpha,k)}$ and $\psi_{(\alpha,k)}$:

$$\begin{cases} \partial_t \zeta_{(\alpha,k)} + \underline{V} \cdot \nabla^\gamma \zeta_{(\alpha,k)} - \frac{1}{\mu} G \psi_{(\alpha,k)} - \frac{1}{\mu \varepsilon} \sum dG(\partial^{l_1} \beta b, \dots, \partial^{l_i} \beta b, \partial^{m_1} \varepsilon \zeta, \dots, \partial^{m_j} \varepsilon \zeta) \partial^\delta \psi \\ \partial_t \psi_{(\alpha,k)} + \underline{V} \cdot \nabla^\gamma \psi_{(\alpha,k)} + \frac{1}{\varepsilon} \underline{a} \zeta_{(\alpha,k)} - \frac{1}{\varepsilon B_0} \nabla^\gamma \cdot \mathcal{K}(\sqrt{\mu} \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(\alpha,k)} + \mathcal{K}_{(\alpha)}[\varepsilon \sqrt{\mu} \nabla^\gamma \zeta] \check{\zeta}_{(\alpha,k)} = S_k \end{cases} \quad (2.2.27)$$

where the summation in the first equation is over the index $(i, j, l_1, \dots, l_j, m_1, \dots, m_j, \delta)$ satisfying

$$1 \leq i + j, |l_1 + \dots + l_i + m_1 + \dots + m_j| + \delta = |\alpha| + k.$$

We used the notation

$$\check{\zeta}_{(\alpha,k)} = (\zeta_{(\check{\alpha}^1)}, \dots, \zeta_{(\check{\alpha}^d)}, \zeta_{(\alpha,k-1)})$$

with $\check{\alpha}^j = \alpha - e_j$, where e_j denote the unit vector in the j -th direction of \mathbb{R}^d , and where

$$|\mathfrak{P}S_k|_2 \leq C_1(K).$$

This system is very similar to the evolution equation (2.2.21) in term of unknowns $\zeta_{(k)}, \psi_{(k)}$, except that the space derivatives of the bottom b appears from the derivation of $G\psi$, and the same goes for the space derivatives of ζ in the derivation of the surface tension term κ_γ . Now, using (2.2.26), one can write:

$$|\mathfrak{P}\psi_{(\alpha,k)}|_2^2 + |\zeta_{(\alpha,k)}|_{H_\varepsilon^1}^2 \leq C_1(K) \frac{1}{\mu} ((G\psi_{(\alpha,k)}, \psi_{(\alpha,k)})_2 + (\underline{a}\zeta_{(\alpha,k)} - \frac{1}{B_0} \nabla^\gamma \mathcal{K}(\sqrt{\mu} \varepsilon \nabla^\gamma \zeta) \nabla^\gamma \zeta_{(\alpha,k)}, \zeta_{(\alpha,k)})_2)$$

We now use the evolution equation (2.2.27) in order to express $\psi_{(\alpha,k)}$ and $\zeta_{(\alpha,k)}$ with respect to time derivatives plus over terms of size ε :

$$\begin{aligned} |\mathfrak{P}\psi_{(\alpha,k)}|_2^2 + |\zeta_{(\alpha,k)}|_{H_\varepsilon^1}^2 &\leq C_1(K) \left(\varepsilon \partial_t \zeta_{(\alpha,k)} + \varepsilon \underline{V} \cdot \nabla^\gamma \zeta_{(\alpha,k)} \right. \\ &\quad \left. - \frac{1}{\mu} \sum dG(\partial^{l_1} \beta b, \dots, \partial^{l_i} \beta b, \partial^{m_1} \varepsilon \zeta, \dots, \partial^{m_j} \varepsilon \zeta) \partial^\delta \psi, \psi_{(\alpha,k)} \right)_2 \\ &\quad + \left(-\varepsilon \partial_t \psi_{(\alpha,k)} - \varepsilon \underline{V} \cdot \nabla^\gamma \psi_{(k)} - \frac{\varepsilon}{B_0} \mathcal{K}(\alpha, k) [\sqrt{\mu} \nabla^\gamma \zeta] \check{\zeta}_{(\alpha,k)} + S_k, \zeta_{(\alpha,k)} \right)_2. \end{aligned} \quad (2.2.28)$$

Let us control the first term of the r.h.s. of (2.2.28). One has to express this term with respect to $\mathfrak{P}\psi_{(\alpha,k)}$. To this purpose, we assume for convenience that $\alpha^1 \neq 0$ (recall that $|\alpha| \geq 1$). One computes:

$$\begin{aligned} |(\varepsilon \partial_t \zeta_{(\alpha,k)}, \psi_{(\alpha,k)})_2| &= |(\partial^\alpha \zeta_{(k+1)}, \psi_{(\alpha,k)})_2| \\ &= |(\partial^{\alpha - e_1} \zeta_{(k+1)}, \partial^{e_1} \psi_{(\alpha,k)})_2| \\ &\leq (|\zeta_{(\alpha - e_1, k+1)}|, |D^\gamma \psi_{(\alpha,k)}|)_2 \\ &\leq \left| \frac{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}}{(1 + \frac{1}{\sqrt{B_0}} |D^\gamma|)} \left(1 + \frac{1}{\sqrt{B_0}} |D^\gamma|\right) \zeta_{(\alpha - e_1, k+1)} \right|_2 \left| \frac{|D^\gamma|}{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}} \psi_{(\alpha,k)} \right|_2 \end{aligned}$$

Now, remark that

$$\frac{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}}{(1 + \frac{1}{\sqrt{B_0}} |D^\gamma|)} \leq C(B_0 \mu)$$

where $C(B_0\mu)$ is a constant that only depends on $B_0\mu$ (recall here that $0 \leq \gamma \leq 1$ and the definition of $|D^\gamma|$ given in section 2.1.3). It comes:

$$\begin{aligned} (\varepsilon \partial_t \zeta_{(\alpha,k)}, \psi_{(\alpha,k)})_2 &\leq C_1(K) |\zeta_{(\alpha-e_1, k+1)}|_{H_\sigma^1} |\mathfrak{P}\psi_{(\alpha,k)}|_2 \\ &\leq (C_1(K)(t + \varepsilon) + C_0) |\mathfrak{P}\psi_{(\alpha,k)}|_2 \end{aligned}$$

where we used the induction assumption to control $|\zeta_{(\alpha-e_1, k+1)}|_{H_\sigma^1}$, since $|\alpha - e_1| \leq |\alpha| - 1$.

For the control of the third term of the r.h.s. of (2.2.28), one can prove, using Proposition 2.A.5 that for $j \neq 0$, one has (see also [37] Chapter 4 for details):

$$\frac{1}{\mu} (dG(\partial^{l_1} \beta b, \dots, \partial^{l_i} \beta b, \partial^{m_1} \varepsilon \zeta, \dots, \partial^{m_j} \varepsilon \zeta) \partial^\delta \psi, \psi_{(\alpha,k)})_2 = \varepsilon R$$

with

$$R \leq C_1(K).$$

For $j = 0$, we use again Proposition 2.A.5 to write:

$$\frac{1}{\mu} (dG(\partial^{l_1} \beta b, \dots, \partial^{l_i} \beta b) \partial^\delta \psi, \psi_{(\alpha,k)})_2 \leq C_1(K) |\partial^{l_1} \beta b|_{H^{\iota_0}} \dots |\partial^{l_i} \beta b|_{H^{\iota_0}} |\mathfrak{P}\partial^\delta \psi|_2 |\mathfrak{P}\psi_{(\alpha,k)}|_2.$$

Now, recall that $b \in H^{N+1 \vee \iota_0+1}$. In order to prove that this last term is controlled by $C_1(K)(t + \varepsilon) + C_0$, one computes, using the definition of $\psi_{(\alpha)}$ given by (2.2.14):

$$\begin{aligned} |\mathfrak{P}\partial^\delta \psi|_2 &\leq |\mathfrak{P}\psi_{(\delta)}|_2 + \varepsilon |\mathfrak{P}\underline{w}\zeta_{(\delta)}|_2 \\ &\leq |\mathfrak{P}\psi_{(\delta)}|_2 + \varepsilon |\underline{w}|_{H^{\iota_0}} |\zeta_{(\delta)}|_{H^1} \end{aligned}$$

where we used the identity $|\mathfrak{P}f|_2 \leq |f|_{H^1}$. Now, the Proposition 2.A.1 gives $|\underline{w}| \leq \mu^{3/4} C_1(K) |\mathfrak{P}\psi|_{H^{\iota_0+3/2}}$. One can use the induction assumption, because the term $\psi_{(\delta)}$ only contains spatial derivatives of ψ of order less than $|\alpha| - 1$ due to the fact that $i \geq 1$ (the only term with spatial derivative of order $|\alpha|$ of ψ that appears in the system (2.2.27) is $G\psi_{(\alpha,k)}$). One gets :

$$|\mathfrak{P}\partial^\delta \psi|_2 \leq C_1(K)(t + \varepsilon) + C_0,$$

and finally

$$(dG(\partial^{l_1} \beta b, \dots, \partial^{l_i} \beta b, \partial^{m_1} \varepsilon \zeta, \dots, \partial^{m_j} \varepsilon \zeta) \partial^\delta \psi, \psi_{(\alpha,k)})_2 \leq (C_1(K)(t + \varepsilon) + C_0) |\mathfrak{P}\psi_{(\alpha,k)}|.$$

Now, we focus on the most difficult remaining term of (2.2.28), which is $(-\varepsilon \partial_t \psi_{(\alpha,k)}, \zeta_{(\alpha,k)})_2$. One uses the definition of $\psi_{(\alpha,k)}$ given by (2.2.14) $\psi_{(\alpha,k)} = \partial^\alpha (\varepsilon \partial_t)^k \psi - \varepsilon \underline{w} \partial^\alpha (\varepsilon \partial_t)^k \zeta$ to write:

$$\begin{aligned} |(\varepsilon \partial_t \psi_{(\alpha,k)}, \zeta_{(\alpha,k)})_2| &= |(\partial_1 \psi_{(\alpha-e_1, k+1)}, \zeta_{(\alpha,k)})_2 + (\varepsilon (\partial_1 \underline{w}) \zeta_{(\alpha-e_1, k+1)} - (\varepsilon \partial_t) (\varepsilon \underline{w}) \zeta_{(\alpha,k)}, \zeta_{(\alpha,k)})_2| \\ &\leq |\mathfrak{P}\psi_{(\alpha-e_1, k+1)}|_2 \left(\frac{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}}{1 + \frac{1}{\sqrt{B_0}} |D^\gamma|} \left(1 + \frac{1}{\sqrt{B_0}} |D^\gamma| \right) |\zeta_{(\alpha,k)}|_2 + \varepsilon C_1(K) \right). \end{aligned}$$

We now use the induction assumption to control $|\mathfrak{P}\psi_{(\alpha-e_1, k+1)}|_2$ by $C_1(K)(t + \varepsilon) + C_0$, and the identity

$$\frac{(1 + \sqrt{\mu} |D^\gamma|)^{1/2}}{(1 + \frac{1}{\sqrt{B_0}} |D^\gamma|)} \leq C(B_0\mu)$$

where $C(B_0\mu)$ is a constant that only depends on $B_0\mu$. One gets:

$$|(\varepsilon \partial_t \psi_{(\alpha,k)}, \zeta_{(\alpha,k)})_2| \leq (C_1(K)(t + \varepsilon) + C_0) (1 + |\zeta_{(\alpha,k)}|_{H_\sigma^1}).$$

To conclude, we proved:

$$|\psi_{(\alpha,k)}|^2 + |\zeta_{(\alpha,k)}|_{H_\sigma^1}^2 \leq (C_1(K)(t + \varepsilon) + C_0) (1 + |\psi_{(\alpha,k)}|_2 + |\zeta_{(\alpha,k)}|_{H_\sigma^1})$$

and one can use the Young's inequality to get the desired estimate, and the Lemma 2.2.11 is true at rank $|\alpha|$.

□

According to the definition of $\mathcal{E}_\sigma^N(U)$ given by (2.2.15), there is a remaining term to be controlled: it is $|\mathfrak{P}\psi|_{H^{t_0+3/2}}$. The control can be done as follows: let r_0 be such that $t_0 + 3/2 \leq r_0 \leq N - 1$. We have, for all $r \in \mathbb{N}^{d+1}$, $|r| = r_0$,

$$|\mathfrak{P}\partial^r \psi|_2 \leq |\mathfrak{P}\psi_{(r)}|_2 + \varepsilon |\mathfrak{P}\underline{w}\partial^r \zeta|_2$$

with the definition of $\psi_{(r)}$. The first term of the right hand side is controlled by previous estimates. For the second term, we use again the same technique as in the control of A_4 to write

$$\begin{aligned} |\mathfrak{P}\partial^r \psi|_2 &\leq C_1(K)t + C_0 + \varepsilon \mu^{-1/4} |\underline{w}\partial^r \zeta|_{H^{1/2}} \\ &\leq C_1(K)t + C_0 + \varepsilon \mu^{-1/4} |\underline{w}|_{H^{t_0+1/2}} |\zeta|_{H^N} \\ &\leq C_1(K)t + C_0 + \varepsilon \mu^{-1/4} \mu^{3/4} |\mathfrak{P}\psi|_{H^{t_0+1}} \\ &\leq C_1(K)t + C_0 + \varepsilon C_1(K). \end{aligned}$$

The result then comes from the identity

$$|\mathfrak{P}\psi|_{H^{t_0+3/2}} \leq |\mathfrak{P}\psi|_{H^{r_0}} \lesssim |\mathfrak{P}\psi|_2 + \sum_{|r|=r_0} |\mathfrak{P}\partial^r \psi|_2.$$

Finally, we proved

$$\mathcal{E}^N(U)(t) \leq C_0 + C_1(K)(t + \varepsilon)$$

for all t in $[0; T_\varepsilon]$, which ends the proof of Proposition 2.2.4.

□

We can now prove Theorem 2.2.2 by constructing an existence time for all solutions U^ε of the system (2.2.11), which does not depend on ε .

Proof of Theorem 2.2.2 We define

$$\varepsilon_0 = \frac{1}{2C_1(2C_0)}.$$

Let fix a $\varepsilon \leq \varepsilon_0$. Let us consider

$$T_\varepsilon^* = \sup_{t>0} \{t, U^\varepsilon \text{ exists on } [0, t] \text{ and } \mathcal{E}^N(U^\varepsilon)(t) \leq 2C_0, 1 + \varepsilon\zeta(t) - \beta b \geq h_{\min}/2, \underline{a}(t) \geq a_0/2 \text{ on } [0, t]\}.$$

We know that T_ε^* exists and that U^ε , solution to (2.1.5) exists on $[0, T_\varepsilon^*]$. The Proposition 2.2.4 gives the following estimate:

$$\mathcal{E}_\alpha(U^\varepsilon)(t) \leq C_1(K)(t + \varepsilon) + C_0 \quad \forall t \in [0, T_\varepsilon^*].$$

We then consider

$$T_0 = \frac{1}{2C_1(2C_0)} \inf\{1, h_{\min}\}.$$

Let us show that $T_0 \leq T_\varepsilon^*$. Suppose $T_\varepsilon^* < T_0$. First of all, let us prove that the condition over the height $1 + \varepsilon\zeta - \beta b$ is satisfied. One can write for all $0 \leq t \leq T_0$

$$\begin{aligned} (1 + \varepsilon\zeta - \beta b)(t) &= (1 + \varepsilon\zeta - \beta b)(0) + \int_0^t \partial_t(1 + \varepsilon\zeta - \beta b)(s) ds \\ &\geq (1 + \varepsilon\zeta - \beta b)(0) - t \sup_{s \in [0; t]} |\partial_t(1 + \varepsilon\zeta - \beta b)|_{L^\infty(\mathbb{R}^d)} \\ &\geq (1 + \varepsilon\zeta - \beta b)(0) - tC_1(K) \\ &\geq h_{\min} - T_0 C(2C_0) \\ &\geq \frac{h_{\min}}{2}. \end{aligned}$$

One can do the same for the Rayleigh-Taylor condition $\underline{a}(t) \geq \frac{a_0}{2}$. Now, for the energy condition, we would have for all $t \in [0, T_\varepsilon^*]$:

$$\begin{aligned} \mathcal{E}_\alpha(U^\varepsilon)(t) &\leq C_0 + C_1(K)(t + \varepsilon) \\ &\leq C_0 + C_1(2C_0)(T_\varepsilon^* + \frac{1}{2C_1(2C_0)}) \\ &< C_0 + C_1(2C_0)(T_0 + \frac{1}{2C_1(2C_0)}) \\ &< 2C_0 \end{aligned}$$

using the monotony of C and the definition of T_ε^* . We can therefore continue the solution to an interval $[0, \widetilde{T}_\varepsilon^*]$ such that

$$\mathcal{E}_\alpha(U^\varepsilon)(t) \leq 2C_0, \quad \forall t \in [0, \widetilde{T}_\varepsilon^*]$$

which contradicts the definition of T_ε^* . We then have $T_0 \leq T_\varepsilon^*$.

Conclusion: the solution U^ε exists on the time interval $[0; T_0]$.

□

2.2.5 Shallow Water limit

We discuss here the size of the existence time for solutions of the Water Waves equation (2.1.6) when the shallowness parameter μ goes to zero. This regime corresponds to the Shallow Water model:

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h\bar{V}) = 0 \\ \partial_t \bar{V} + \nabla^\gamma \zeta + \varepsilon(\bar{V} \cdot \nabla^\gamma) \bar{V} = 0 \end{cases} \quad (2.2.29)$$

where $h = 1 + \varepsilon\zeta - \beta b$ is the height of the Water. Using the existence time given by Theorem 2.2.2, one can deduce easily the following long time existence result for the Shallow Water problem, proved in [13], in the case of an irrotational initial condition:

Theorem 2.2.12 *Let $t_0 > d/2$, and $N \geq t_0 + t_0 \vee 2 + 3/2$. Let $(\bar{V}_0, \zeta_0) \in H^N(\mathbb{R}^d)^{d+1}$ be such that $\text{curl}(\bar{V}_0) = 0$. Then, there exists $T > 0$ and a unique solution $(\bar{V}, \zeta) \in C([0; \frac{T}{\varepsilon}]; H^{t_0+1}(\mathbb{R}^d)^{d+1})$ to the Shallow Water equation (2.2.29) with initial condition (\bar{V}_0, ζ_0) , with*

$$\frac{1}{T} = C_1, \quad \text{and} \quad \sup_{t \in [0; \frac{T}{\varepsilon}]} |(\zeta, \bar{V})(t)|_{H^{t_0+1}(\mathbb{R}^d)^{d+1}} = C_2$$

where $C_i = C(\frac{1}{h_{\min}}, |(\zeta_0, \bar{V}_0)|_{H^{t_0+1}(\mathbb{R}^d)^{d+1}})$ is a non decreasing function of its arguments.

The result of [13] can therefore be understood as a particular endpoint of our main result, if the initial data \bar{V}_0 is irrotational. For the general case, one should prove a large existence time for the Water-Waves equation with vorticity (see [38]). It is important to notice that the time existence provided here for the solutions of Shallow Water equations does not depend on the bathymetric parameter β .

Proof Let us fix $\varepsilon > 0$ during all the proof. We also set $\mu B_0 = 1$. We define ψ_0 by $\nabla^\gamma \psi^0 = \bar{V}_0$ (which is possible since \bar{V}_0 is irrotational). We recall that Theorem 2.2.2 gives a solution $U^\mu = (\zeta^\mu, \psi^\mu)$ to the Water Waves equation (2.1.6), with initial condition (ζ_0, ψ_0) on a time interval $[0; \frac{T}{\varepsilon}]$ with

$$\frac{1}{T} = C_1 \quad \text{and} \quad \sup_{t \in [0; \frac{T}{\varepsilon}]} \sum_{|(\alpha, k)| \leq N} |\mathfrak{R}\psi_{(\alpha, k)}^\mu|_2 + |\zeta_{(\alpha, k)}^\mu|_2 = C_2 \quad (2.2.30)$$

where $C_i = C(\mathcal{E}_\sigma^N(U^0), \frac{1}{h_{\min}}, \frac{1}{a_0}, |b|_{H^{N+1 \vee t_0+1}}, \mu B_0)$ is a non decreasing function of its arguments.

We now need an asymptotic expansion of $\nabla^\gamma \psi^\mu$ and $G\psi^\mu$ with respect to the vertical mean of the horizontal component of the velocity $V = \nabla^\gamma \Phi$ in shallow water:

$$G\psi^\mu = -\mu\nabla^\gamma \cdot (h^\mu \bar{V}^\mu) \quad \text{and} \quad \nabla^\gamma \psi^\mu = \bar{V}^\mu + \mu R, \quad (2.2.31)$$

with $|R|_{H^{t_0+1}} \leq C_2$ (recall that $N \geq t_0 + t_0 \vee 2 + 3/2$ with $t_0 > d/2$) and $\bar{V}^\mu = \frac{1}{h^\mu} \int_{-1+\beta b}^{\varepsilon \zeta} V^\mu(z) dz$.

For a complete proof of this latter result, see [37] Chapter 3. Now, we take the first equation of (2.1.6), and the gradient of the second equation of (2.1.6) and we replace $G\psi^\mu$ and $\nabla^\gamma \psi^\mu$ by the expressions given by (2.2.31). The surface tension term is of size μ since $\frac{1}{B_0} = \mu$. One can check that we get the following equation, satisfied in the distribution sense of $D'([0; \frac{T}{\varepsilon}] \times \mathbb{R}^d)$:

$$\begin{cases} \partial_t \zeta^\mu + \nabla^\gamma \cdot (h^\mu \bar{V}^\mu) \\ \partial_t \bar{V}^\mu + \varepsilon \bar{V}^\mu \cdot \nabla^\gamma \bar{V}^\mu + \nabla^\gamma \zeta^\mu = \mu R \end{cases} \quad (2.2.32)$$

with $|R|_{H^{t_0+1}} \leq C_2$. It is then easy to show that the sequences $(\bar{V}^\mu)_\mu$ and $(\zeta^\mu)_\mu$ are bounded in $W^{1,\infty}([0; \frac{T}{\varepsilon}; H^{t_0+1}(\mathbb{R}^d))$, using the bound given by (2.2.30). Therefore, up to a subsequence we get the weak convergence of $(\bar{V}^\mu, \zeta^\mu)_\mu$ and $(\zeta^\mu)_\mu$ to an element $(\bar{V}, \zeta) \in C([0; \frac{T}{\varepsilon}; H^{t_0+1}(\mathbb{R}^d)^{d+1})$, with $(\bar{V}, \zeta)(0) = (\bar{V}_0, \zeta_0)$. The weak convergence of the linear terms of the equation (2.2.32) does not raise any difficulty. Since $H^{t_0+1}(\mathbb{R}^d)$ is embedded in $C^1(\mathbb{R}^d)$, we also get the convergence of the non linear terms in the equation (2.2.32). Finally, the limit (\bar{V}, ζ) satisfies the Shallow Water equations (2.2.29) in the distribution sense of $D'([0; \frac{T}{\varepsilon}] \times \mathbb{R}^d)$. The uniqueness is classical for this kind of symmetrizable quasi-linear hyperbolic system, and is done for instance in [61] Chapter XVI. \square

Let us give a qualitative explanation of this latter result. We recall that Proposition 2.2.10 claims that

$$\frac{1}{M_0} |\mathfrak{P}\psi|_2 \leq (\psi, \frac{1}{\mu} G\psi)_2 \leq M_0 |\mathfrak{P}\psi|_2^2$$

where

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}}$$

and therefore $\frac{1}{\mu} G$ acts like an order one operator with respect to ψ . The idea of the proof for Theorem 2.2.2 is to get a "good" energy estimate for time derivatives:

$$\frac{1}{2} |\partial_t \mathfrak{P}\psi|_2 + \frac{1}{2} |\partial_t \zeta|_2 \leq C(K)t\varepsilon + C_0$$

and then use the equation to recover the same estimate for space derivatives. Using the first equation $\partial_t \zeta = \frac{1}{\mu} G\psi$, and the ellipticity of the order one operator $\frac{1}{\mu} G$ would only provide us a gain of half a space derivative for ψ (we already have an estimate for $\mathfrak{P}\psi$ where \mathfrak{P} is of order 1/2). This is why we need surface tension which provides an additional estimate for $\frac{1}{B_0} |\partial_t \nabla^\gamma \zeta|_2$, which leads to recover exactly one space derivative in the estimate of ψ . In the shallow water regime, using (2.2.31), one gets that $\frac{1}{\mu} G\psi \sim -\nabla^\gamma \cdot (h \nabla^\gamma \psi)$ as μ goes to zero. Therefore, when μ goes to zero, $\frac{1}{\mu} G\psi$ goes to an order one operator with respect to $\bar{V} = \nabla^\gamma \psi$.¹ Therefore, if one has a "good" estimate for time derivatives like

$$\frac{1}{2} |\partial_t \zeta|_2 + \frac{1}{2} |\partial_t \bar{V}|_2 \leq Ct\varepsilon + C_0$$

with C independent of ε , one can recover, using the first equation, a "good" estimate for $\nabla^\gamma \cdot (\bar{V})$. Using the second equation to get the same good estimate for $\text{curl} \bar{V}$, one can recover exactly one space derivative of \bar{V} in the estimates. And therefore, we does not need surface tension in this case. Moreover, the surface tension term of size $\frac{1}{B_0} = \mu$ vanishes as μ goes to zero.

2.A The Dirichlet Neumann Operator

Here are for the sake of convenience some technical results about the Dirichlet Neumann operator, and its estimates in Sobolev norms. See [37] Chapter 3 for complete proofs. The first two propositions give a control of the Dirichlet-Neumann operator.

¹Another way to see it is that in the limit $\mu \rightarrow 0$, \mathfrak{P} must be seen as a first order operator ($\mathfrak{P} \sim |D^\gamma|$) and the control of $\mathfrak{P}\psi$ gives the control of a full derivative of ψ .

Proposition 2.A.1 Let $t_0 > d/2$, $0 \leq s \leq t_0 + 3/2$ and $(\zeta, \beta) \in H^{t_0+1} \cap H^{s+1/2}(\mathbb{R}^d)$ such that

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0.$$

(1) The operator G maps continuously $\dot{H}^{s+1/2}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has

$$|G\psi|_{H^{s-1/2}} \leq \mu^{3/4} M(s+1/2) |\mathfrak{P}\psi|_{H^s},$$

where $M(s+1/2)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1/2}}, |b|_{H^{s+1/2}})$.

(2) The operator G maps continuously $\dot{H}^{s+1}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has

$$|G\psi|_{H^{s-1/2}} \leq \mu M(s+1) |\mathfrak{P}\psi|_{H^{s+1/2}},$$

where $M(s+1)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1}}, |b|_{H^{s+1}})$.

Moreover, it is possible to replace $G\psi$ by \underline{w} in the previous result, where $\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}$ (vertical component of the velocity $U = \nabla_{X,z} \Phi$ at the surface).

Remark 2.A.2 In all this paper, we consider the Water-Waves problem in finite depth. This is crucial for all these regularity results on G . For instance, in the linear case $\zeta = b = 0$, the Dirichlet-Neumann operator is $|D^\gamma| \tanh(|D^\gamma|)$ in finite depth, while it is $|D^\gamma|$ in infinite depth. The low frequencies are therefore affected differently.

Proposition 2.A.3 Let $t_0 > d/2$, and $0 \leq s \leq t_0 + 1/2$. Let also $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, 1 + \varepsilon \zeta(X) - \beta b(X) \geq h_0.$$

Then, for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, we have

$$(\Lambda^s G\psi_1, \Lambda^s \psi_2)_2 \leq \mu M_0 |\mathfrak{P}\psi_1|_{H^s} |\mathfrak{P}\psi_2|_{H^s},$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

The second result gives a control of the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\Gamma = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$G[\varepsilon \cdot, \beta \cdot] : \begin{array}{ccc} \Gamma & \longrightarrow & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \longmapsto & G[\varepsilon \zeta, \beta b] \psi. \end{array} \quad (2.A.33)$$

We can prove the differentiability of this mapping. The following Theorem gives a very important explicit formula for the first-order partial derivative of G with respect to ζ :

Theorem 2.A.4 Let $t_0 > d/2$. Let $\Gamma = (\zeta, b) \in \Gamma$ and $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$. Then, for all $h \in H^{t_0+1}(\mathbb{R}^d)$, one has

$$dG(h)\psi = -\varepsilon G(h\underline{w}) - \varepsilon \mu \nabla^\gamma \cdot (h\underline{V}),$$

with

$$\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}, \quad \text{and} \quad \underline{V} = \nabla^\gamma \psi - \varepsilon \underline{w} \nabla^\gamma \zeta.$$

The following result gives estimates of the derivatives of the mapping (2.A.33).

Proposition 2.A.5 Let $t_0 > d/2$, $0 \leq s \leq t_0 + 1/2$ and $(\zeta, \beta) \in H^{t_0+1}(\mathbb{R}^d)$ such that:

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0.$$

Then, for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, for all $(h, k) \in H^{t_0+1}(\mathbb{R}^d)$ one has

$$|(\Lambda^s d^j G(h, k)\psi_1, \Lambda^s \psi_2)| \leq \mu M_0 \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi_1|_s |\mathfrak{P}\psi_2|_s,$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

The following Proposition gives the same type of estimate that the previous one:

Proposition 2.A.6 *Let $t_0 > d/2$ and $(\zeta, b) \in H^{t_0+1}$ be such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0.$$

Then, for all $0 \leq s \leq t_0 + 1/2$,

$$|d^j G(h, k)\psi|_{H^{s-1/2}} \leq M_0 \mu^{3/4} \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^s}$$

The following commutator estimate is useful (see [37] Proposition 3.30):

Proposition 2.A.7 *Let $t_0 > d/2$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^d)$ such that:*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0.$$

For all $\underline{V} \in H^{t_0+1}(\mathbb{R}^d)^2$ and $u \in H^{1/2}(\mathbb{R}^d)$, one has

$$\left((\underline{V} \cdot \nabla^\gamma u), \frac{1}{\mu} Gu \right) \leq M |\underline{V}|_{W^{1,\infty}} |\mathfrak{P}u|_2^2,$$

where M is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+2}}, |b|_{H^{t_0+2}})$.

Chapter 3

The Cauchy problem on large time for a Boussinesq-Peregrine equation with large topography variations

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Abstract

We prove in this paper a long time existence result for a modified Boussinesq-Peregrine equation in dimension 1, describing the motion of Water Waves in shallow water, in the case of a non flat bottom. More precisely, the dimensionless equations depend strongly on three parameters ε, μ, β measuring the amplitude of the waves, the shallowness and the amplitude of the bathymetric variations respectively. For the Boussinesq-Peregrine model, one has small amplitude variations ($\varepsilon = O(\mu)$). We first give a local existence result for the original Boussinesq Peregrine equation as derived by Boussinesq ([11],[10]) and Peregrine ([51]) in all dimensions. We then introduce a new model which has formally the same precision as the Boussinesq-Peregrine equation, and give a local existence result in all dimensions. We finally prove a local existence result on a time interval of size $\frac{1}{\varepsilon}$ in dimension 1 for this new equation, without any assumption on the smallness of the bathymetry β , which is an improvement of the long time existence result for the Boussinesq systems in the case of flat bottom ($\beta = 0$) by [54].

3.1 Introduction

We recall here the context of the Water Waves problem, and introduce all the meaningful dimensionless parameters of this problem. We then present the shallow water regime and more specifically the Boussinesq-Peregrine regime. We finally introduce the different results proved in this paper, which are local existence theorems for the Boussinesq-Peregrine equations on different time scales and with different conditions.

3.1.1 The Water Waves problem

The motion, for an incompressible, inviscid and irrotational fluid occupying a domain Ω_t delimited below by a fixed bottom and above by a free surface is commonly referred to as the Water Waves problem. It is described by the following quantities:

- the velocity of the fluid $U = (V, w)$, where V and w are respectively the horizontal and vertical components ;
- the free top surface profile ζ ;
- the pressure P .

All these functions depend on the time and space variables t and $(X, z) \in \Omega_t$, which is the domain occupied by the water. More precisely, there exists a function $b : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the domain of the fluid at the time t is given by

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

where H_0 is the typical depth of the water. The unknowns (U, ζ, P) are governed by the Euler equations:

$$\begin{cases} \partial_t U + U \cdot \nabla_{X,z} U = -\frac{1}{\rho} \nabla P - g e_z & \text{in } \Omega_t \\ \operatorname{div}(U) = 0 & \text{in } \Omega_t \\ \operatorname{curl}(U) = 0 & \text{in } \Omega_t. \end{cases} \quad (3.1.1)$$

These equations are completed by boundary conditions:

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - \underline{w} = 0 \\ U \cdot n = 0 & \text{on } \{z = -H_0 + b(X)\} \\ P = P_{atm} & \text{on } \{z = \zeta(X)\}. \end{cases} \quad (3.1.2)$$

In these equations, \underline{V} and \underline{w} are the components of the velocity evaluated at the surface. The vector n in the last equation stands for the upward normal vector at the bottom $(X, z = -H_0 + b(X))$, and e_z is the unit upward vector in the vertical direction. We denote P_{atm} the constant pressure of the atmosphere at the surface of the fluid, ρ the (constant) density of the fluid, and g the acceleration of gravity. The first equation of (3.1.2) states the assumption that the fluid particles do not cross the surface, while the second equation of (3.1.2) states the assumption that they do not cross the bottom. The equations (3.1.1) with boundary conditions (3.1.2) are commonly referred to as the free surface Euler equations.

3.1.2 The dimensionless parameters

Since the properties of the solutions of the Water Waves problem depend strongly on the characteristics of the flow, it is more convenient to non-dimensionalize the equations by introducing some characteristic lengths of the wave motion:

- (1) The characteristic water depth H_0 .
- (2) The characteristic horizontal scale L_x in the longitudinal direction.
- (3) The characteristic horizontal scale L_y in the transverse direction (when $d = 2$).
- (4) The size of the free surface amplitude a_{surf} .
- (5) The size of bottom topography a_{bott} .

Let us then introduce the dimensionless variables:

$$x' = \frac{x}{L_x}, \quad y' = \frac{y}{L_y}, \quad \zeta' = \frac{\zeta}{a_{surf}}, \quad z' = \frac{z}{H_0}, \quad b' = \frac{b}{a_{bott}},$$

and the dimensionless variables:

$$t' = \frac{t}{t_0}, \quad P' = \frac{P}{P_0},$$

where

$$t_0 = \frac{L_x}{\sqrt{gH_0}}, \quad P_0 = \rho g H_0.$$

After rescaling, four dimensionless parameters appear in the Euler equation. They are

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma,$$

where $\varepsilon, \mu, \beta, \gamma$ are commonly referred to respectively as "nonlinearity", "shallowness", "topography" and "transversality" parameters. The free surface Euler equations (3.1.1) and (3.1.2) become after rescaling (we omit the "primes" for the sake of clarity):

$$\begin{cases} \partial_t U + \varepsilon(V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) U = -\frac{1}{\varepsilon} \nabla^\gamma P, \\ \partial_t \zeta - \sqrt{1 + \varepsilon^2 |\nabla^\gamma \zeta|^2} U \cdot n = 0 \end{cases} \quad (3.1.3)$$

where we used the following notations:

$$\nabla^\gamma = {}^t(\partial_x, \gamma \partial_y) \quad \text{if } d = 2 \text{ and } \nabla^\gamma = \partial_x \quad \text{if } d = 1,$$

and we recall that the unknown is the velocity $U = (V, w)$ where V and w are respectively the horizontal and vertical components of the velocity. The equations (3.1.3) coupled with boundary conditions on P and U are commonly referred to as the "dimensionless free surface Euler equations".

3.1.3 The Shallow Water regime

When the shallowness parameter $\mu = \frac{H_0^2}{L_x^2}$ is small, it is possible to use a simplified equation in order to study the Water Waves problem. More precisely, at first order with respect to μ , the horizontal velocity V becomes columnar, which means that

$$V = \bar{V} + \mu R \quad (3.1.4)$$

where \bar{V} stands for the vertical average of the horizontal velocity

$$\bar{V}(t, X) = \frac{1}{h(t, X)} \int_{1-\beta b(X)}^{\varepsilon \zeta(t, X)} V(t, X, z) dz$$

and h is the height of the water $h(t, X) = 1 + \varepsilon \zeta(t, X) - \beta b(X)$. We do not give precise estimate for the residual R of (3.1.4) in Sobolev norm here. Lagrange [35], and later Saint-Venant [1] derived

from the Euler equations and under the assumption that the pressure is hydrostatic¹ the following Shallow-Water equation expressed in term of unknowns (\bar{V}, ζ) :

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ \partial_t \bar{V} + \nabla^\gamma \zeta + \varepsilon (\bar{V} \cdot \nabla^\gamma) \bar{V} = 0, \end{cases} \quad (3.1.5)$$

with initial data $(\bar{V}, \zeta)|_{t=0} = (\bar{V}^0, \zeta^0)$. The Shallow-Water equations (3.1.5) are a typical example of quasilinear symmetrizable system (the symmetrization is done by multiplying the second equation by h). The local existence result for such a system is classical, and is done for example in [61] Chapter XVI (see also [7]). The Shallow-Water equation is said to be consistent at the first order in μ with the Water-Waves equations, which means that formally, one has

$$\text{water-waves equation} = \text{shallow-water equation} + O(\mu).$$

It formally means that if one is interested in working with a shallow water (i.e. with small values of μ), one can get rid of all the terms of size μ in the Water-Waves equations and obtain the simplified model of the Shallow-Water equation. Alvarez-Samaniego and Lannes [6], and Iguchi [31] fully justified the Shallow-Water model by proving the mathematical convergence of the Water-Waves equation to the Shallow-Water equation. More precisely, one has the following result:

Theorem 3.1.1 *There exists $N \geq 1$, such that for all $(\bar{V}^0, \zeta^0) \in H^N(\mathbb{R}^d)^{d+1}$, there exists $T > 0$ such that:*

- (1) *There exists a unique solution (ζ^E, U^E) defined on $[0; T[$ to the free surface Euler equation (3.1.1), (3.1.2) such that $(\zeta^E, \bar{V}^E) \in H^N(\mathbb{R}^d)^{d+1}$, and $(\zeta^E, \bar{V}^E)(0) = (\zeta^0, \bar{V}^0)$.*
- (2) *There exists a unique solution $(\zeta^{SW}, \bar{V}^{SW}) \in C([0; T[; H^N(\mathbb{R}^d)^{d+1})$ to the Shallow-Water equation (3.1.5) with initial conditions (ζ^0, \bar{V}^0) .*
- (3) *One has, for all $t \in [0; T[$:*

$$|(\zeta^E, \bar{V}^E)(t) - (\zeta^{SW}, \bar{V}^{SW})(t)|_{H^N} \leq C(|(\zeta^0, \bar{V}^0)|_{H^N}) \mu t.$$

This theorem implies that the error made by using the solutions of the simplified model of Shallow-Water instead of the solutions of the free-surface Euler equations is "of size μ ".

At the second order with respect to μ , one can derive several models for the Water-Waves problem which are formally more precise than the Shallow-Water equation. We consider in this paper the case where

$$\varepsilon = O(\mu),$$

which corresponds to a small amplitude model in the Shallow-Water regime. Boussinesq ([11],[10]) derived the following model for flat bottoms, and later Peregrine [51] for nonflat bottoms:

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ [1 + \mu \mathcal{T}_b] \partial_t \bar{V} + \varepsilon (\bar{V} \cdot \nabla^\gamma) \bar{V} + \nabla^\gamma \zeta = 0 \end{cases} \quad (3.1.6)$$

where \mathcal{T}_b is the following operator :

$$\forall \bar{V}, \quad \mathcal{T}_b \bar{V} = -\frac{1}{3h_b} \nabla^\gamma (h_b^3 \nabla^\gamma \cdot \bar{V}) + \frac{\beta}{2h_b} [\nabla^\gamma (h_b^2 \nabla^\gamma b \cdot \bar{V}) - h_b^2 \nabla^\gamma b \nabla^\gamma \cdot \bar{V}] + \beta^2 \nabla^\gamma b \nabla^\gamma b \cdot \bar{V},$$

with the notation

$$h_b = 1 - \beta b.$$

See also [37] for a complete proof of the formal derivation of this equation. This equation is known as the Boussinesq-Peregrine equation, and has been used a lot in applications to coastal flows. In the case of a flat bottom, one has $\beta = 0$ and $\mathcal{T}_b = -\frac{1}{3} \nabla^\gamma \nabla^\gamma \cdot$. The equation (3.1.6) can then be seen as a particular case of a Boussinesq system (see [8] for the 1d case, and [9] for the 2d case).

¹The pressure is hydrostatic if, in dimensional form, $P(X, z) = P_{atm} - \rho g(z - \zeta)$. This is always true at the leading order in μ . See for instance [37] Section 5.5

The Boussinesq-Peregrine model is a good compromise for numerical simulation between the precision of the approximation of the Water-Waves problem, and the simplicity of the equations. Indeed, the Boussinesq-Peregrine equation preserves the dispersive nature of the Water-Waves equation. To understand this statement, one can look for plane wave solutions $(\zeta, \bar{V}) = (\zeta^0, \bar{V}^0)e^{i(k \cdot X - \omega(k)t)}$ of the linearized Boussinesq-Peregrine equation (3.1.6) and finds solutions with a dispersive relation:

$$\omega(k)^2 = \frac{|k^\gamma|^2}{1 + \frac{\mu}{3}|k^\gamma|^2},$$

with $k^\gamma = {}^t(k_1, \gamma k_2)$. Thus the group velocity $c = \frac{\omega(k)}{|k^\gamma|}$ of the water waves depends on the frequency, which is a definition of dispersion. The Shallow-Water equation, however, is not a dispersive equation in dimension 1, since one would find a group velocity of the water waves equals to 1. One could also derive a even more precise model at the $O(\mu^2)$ order than the Boussinesq-Peregrine model, without any assumption on the smallness of ε , which is called the Green-Naghdi equation (see [57] for the case $d = 1$, [59], [56] for the case of non-flat bottom and also under the name of "fully nonlinear Boussinesq equations" in [62]; see also [44]). It has a very similar form as one of the Boussinesq-Peregrine equation, but with h_b replaced by h in the definition of the operator \mathcal{T}_b . Therefore, though the Green-Naghdi model should be more precise than the Boussinesq-Peregrine equation, its numerical implementation leads to the computation at each time step of the inverse of $I + \mu\mathcal{T}_b$ (which now depends on h , and therefore on the time), which increases the computational complexity².

3.1.4 Long time existence for the Water-Waves models

We are interested in the dependence of the existence time for the solutions of systems like (3.1.5) and (3.1.6) with respect to the parameter ε . For such quasilinear equations with an ε factor on the nonlinearity, the "good" time existence should be of size $\frac{1}{\varepsilon}$. Let us explain this statement by studying the one dimensional Burgers equation:

$$\begin{cases} \partial_t u + \varepsilon u \partial_x u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

where we assume some reasonable regularity on u_0 . Using the method of characteristics to solve it, we find that characteristics are of the form

$$x(t) = \varepsilon u_0(x_1)t + x_1$$

with $x_1 \in \mathbb{R}$. The solutions do not exist globally in the general case because the characteristics intersect themselves. Let us estimate the time at which they intersect. Let us consider two characteristics $x(t) = \varepsilon u_0(x_1)t + x_1$ and $\varepsilon u_0(x_2)t + x_2$. They intersect at time

$$t = -\frac{x_2 - x_1}{\varepsilon(u_0(x_2) - u_0(x_1))}.$$

Therefore, the existence time for the solutions is

$$T = \min_{x_1, x_2 \in \mathbb{R}} -\frac{x_2 - x_1}{\varepsilon(u_0(x_2) - u_0(x_1))} = -\frac{1}{\varepsilon \min_{x \in \mathbb{R}} u'_0(x)}$$

and therefore is of size $\frac{1}{\varepsilon}$.

The standard theory for quasilinear symmetrizable systems gives the local existence of solutions (\bar{V}, ζ) of the Shallow-Water equations (3.1.5) on the space $C([0; T]; H^N(\mathbb{R}^d)^{d+1})$, with N large enough, and gives an explosion criterion: one has $T < \infty$ if and only if $\lim_{t \rightarrow T} |(\bar{V}, \zeta)(t)|_{W^{1, \infty}} = +\infty$. If one could prove an estimate of the form

$$|(\bar{V}, \zeta)(t)|_{H^N} \leq g(|(\bar{V}, \zeta)|_{H^N})t\varepsilon \tag{3.1.7}$$

²Note however that a variant of the Green-Naghdi equations where the operator to invert in time is time independent has been derived recently in [39] to perform 2d computations

for a smooth non decreasing function g , one would get by a continuity argument that the solutions exist on an interval of size $\frac{1}{\varepsilon}$. An L^2 estimate of the form (3.1.7) is easy to obtain, because multiplying the second equation of (3.1.5) symmetrizes both first space derivatives order terms and quantities of size ε . Indeed, one can differentiate with respect to time the "energy"

$$E(\zeta, \bar{V}) = \frac{1}{2}|\zeta|_2^2 + \frac{1}{2}(h\bar{V}, \bar{V})_2$$

to get

$$\frac{dE}{dt} = (\partial_t \zeta, \zeta)_2 + (\partial_t \bar{V}, h\bar{V})_2 + \frac{1}{2}((\partial_t h)\bar{V}, \bar{V})_2$$

and replace $\partial_t(\zeta, \bar{V})$ by their expressions given in the equation (3.1.5). The terms of order 1 are transparent in the energy estimate because they cancel one another, thanks to the "symmetry" of the system, and one gets

$$\frac{dE}{dt} = 0$$

which is even better than needed. However, if one differentiates the equation (3.1.5) with respect to space variable, one finds a system of the form (∂ denotes here any space derivative of order one):

$$\begin{cases} \partial_t \partial \bar{V} + \nabla^\gamma \cdot ((\partial h)\bar{V}) + \nabla^\gamma \cdot (h\partial(\bar{V})) = 0 \\ \partial_t \partial \zeta + \varepsilon(\partial \bar{V}) \cdot \nabla^\gamma \bar{V} + \varepsilon \bar{V} \cdot \nabla^\gamma \partial \bar{V} + \nabla^\gamma \partial \zeta = 0. \end{cases}$$

It is not possible anymore to make this system symmetric in order to cancel the terms that are not of size ε in the energy estimates. Indeed, the term ∂h is not of size ε , since $h = 1 + \varepsilon\zeta - \beta b$, and thus also depends on β . The H^N norms of the unknowns are then not easily controlled by terms of size ε , which prevent us from proving directly an estimate of the form (3.1.7). The long existence for this system is therefore tied to a singular perturbation problem with variable coefficients.

Long time existence results for similar types of equations have been proved for example by Schochet in [55] for the compressible Euler equation in a bounded domain with well-prepared data, or by Schochet-Métivier in [43] for the Non-isentropic Euler equation with general data. Alazard ([2]) proved a long time existence result for the non-isentropic compressible Euler equation, in the case of bounded and unbounded domains. Bresch-Métivier ([13]) proved that for $N > d/2 + 1$ and $(\zeta^0, \bar{V}^0) \in H^N(\mathbb{R}^d)^{d+1}$, there exists a unique solution $(\zeta, \bar{V}) \in C([0; \frac{T}{\varepsilon}]; H^N(\mathbb{R}^d)^{d+1})$ to the equation (3.1.5) with initial data (ζ^0, \bar{V}^0) where T only depends on the norm of the initial data, even if β is not assumed to be small.

In the case of a flat bottom, as explained in Section 3.1.3, the Boussinesq-Peregrine equation can be seen as a particular case of the Boussinesq systems. Saut and Li ([54]) proved the local existence on a large time interval of size $\frac{1}{\varepsilon}$ for most of these systems, when the bottom is flat. There is, in our knowledge, no local existence results in the literature in the case of non flat bottoms. A local existence result for the Boussinesq-Peregrine equation (3.1.6) on a time $\frac{T}{\varepsilon}$ with T independent on μ would be important to fully justify this model, and get a convergence result similar to Theorem 3.1.1.

3.1.5 Main result

In [50], a large time existence result is proved for the Water-Waves equations in presence of large topography, extending the result of [6] which holds for small topography variations ($\beta = O(\varepsilon)$)³. Coupling this result with the one of [13], one can prove that the Shallow-Water equations are consistent at order 1 in μ with the full Water-Waves equations, on a time interval of size $\frac{1}{\varepsilon}$, even in the case of large topography variations ($\beta = O(1)$). A similar result between the Water-Waves equations and the Boussinesq-Peregrine equation (3.1.6) would be a new step in the mathematical justification of the Water-Waves models.

We first prove in this paper a well posedness result for the equation (3.1.6) on a time $O(1)$ (Theorem 3.2.1 below). Though not optimal as we shall see, such a local well posedness result did

³However, this result needs the presence of surface tension.

not seem to be available in the literature for non flat bottoms. The proof consists in a particular adaptation of the proof of local existence for symmetrizable quasilinear systems. The difficulty is that this system is not easily symmetrizable. In the case of the Shallow-Water equations (3.1.5), as explained in Section 3.1.3, one should multiply the second equation by h to get a "symmetric" system. However, in the case of the Boussinesq-Peregrine equation (3.2.11), multiplying the second equation by h indeed symmetrizes the system with respect to order one space derivatives. But the operator $h(I + \mu\mathcal{T}_b)$ is not symmetric. Therefore, $(h(I + \mu\mathcal{T}_b)\partial_t\bar{V}, \bar{V})_2$ is not equal to $\frac{d}{dt}(h(I + \mu\mathcal{T}_b)\bar{V}, \bar{V})_2$. It implies the presence of some commutators terms between $h\mathcal{T}_b$ and ∂_t in the energy estimates which are difficult to control. However, the operator $h_b(I + \mu\mathcal{T}_b)$ is symmetric and we have the equivalence

$$(h_b(I + \mu\mathcal{T}_b)\bar{V}, \bar{V})_2 \sim |\bar{V}|_2^2 + \mu|\nabla^\gamma \cdot \bar{V}|_2^2.$$

Therefore, a "good" energy for the Boussinesq-Peregrine equation seems to be

$$E(\zeta, \bar{V}) = \frac{1}{2}|\zeta|_2^2 + \frac{1}{2}(h_b(I + \mu\mathcal{T}_b)\bar{V}, \bar{V})_2. \tag{3.1.8}$$

But multiplying the second equation by h_b does not properly symmetrize the system with respect to space derivatives. More precisely, it symmetrizes them up to an $\varepsilon\zeta\nabla^\gamma \cdot \bar{V}$ factor. In the Boussinesq-Peregrine regime, one has $\varepsilon = O(\mu)$ and therefore this term is actually controlled by the energy... It yields to a local existence result for a time interval of size $O(1)$ and not $O(\frac{1}{\varepsilon})$. This is the purpose of Theorem 3.2.1.

One then looks for an improved time of existence for the Boussinesq-Peregrine equation. In order to do so, one could try to use an adaptation of the proof of the long time existence result by Bresch-Métivier [13]. The idea of this proof is to have energy estimates of the form

$$E(t) \leq C(E)t\varepsilon + C_0$$

for some constant $C(E)$ which depends on the energy, and C_0 which only depends on initial data, where E is an energy. One can then conclude by a continuity argument that the energy stays bounded on an interval of size $\frac{1}{\varepsilon}$. Let us detail this idea on a simplified model of equation of the form:

$$\partial_t u + \varepsilon(u\partial_x)u + L(\varepsilon u, a(x))u = 0 \tag{3.1.9}$$

where $L(\varepsilon u, a(x)) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear, elliptic, antisymmetric operator of order 1. One computes:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2}|u(t)|_2^2 &= (\partial_t u, u)_2 \\ &= (Lu, u)_2 + \varepsilon((u\partial_x)u, u)_2 \end{aligned}$$

using the equation to replace $\partial_t u$ by its expression. Using the antisymmetry of L and integrating by parts, one can easily see that $\frac{d}{dt} \frac{1}{2}|u(t)|_2^2$ vanishes, which is even better than being of size ε . However, it does not stand true for higher order estimates. Looking for an estimate on $\frac{d}{dt}|u(t)|_{H^N}$ for some $N > 0$, one differentiates the equation (3.1.9) and finds a system of the form:

$$\partial_t \partial_x u + \varepsilon(u\partial_x)\partial_x u + \varepsilon[u\partial_x, \partial_x]u + L(\varepsilon u, a(x))\partial_x u + \varepsilon(\partial_x u)dL_1(\varepsilon u, a(x))u + (\partial_x a)dL_2(\varepsilon u, a(x))u = 0,$$

where we denoted $dL_i(\varepsilon u, a(x))$ the differential of L with respect to the i -th variable, at the point $(\varepsilon u, a(x))$, for $i = 1, 2$. Due to an extra term $(\partial_x a)dL_2(\varepsilon u, a(x))u$, the energy estimates involve terms which are not of size ε . This problem does not appear for time derivatives: if one differentiates the equation (3.1.9) with respect to time, one finds:

$$\partial_t(\partial_t u) + \varepsilon u\partial_x(\partial_t u) + \varepsilon[u\partial_x, \partial_t]u + L(\varepsilon u, a(x))\partial_t u + \varepsilon(\partial_t u)dL_1(\varepsilon u, a(x))u.$$

Therefore, one can find an energy estimate of the form

$$|(\partial_t^k u)(t)|_2 \leq |(\partial_t^k u)(0)|_2 + \varepsilon t C(|(\partial_t^k u)(t)|_2),$$

for all $k \leq N$. In order to find a similar energy estimate in H^N , one uses the equation, which gives an expression of space derivatives with respect to time derivatives:

$$L(\varepsilon u, a(x))u = -\partial_t u - \varepsilon u\partial_x u$$

and using the previous estimate for time derivatives, and the ellipticity of L , one gets:

$$|u(t)|_{H^1} \leq C(|u(t)|_{H^1})\varepsilon t + C_0$$

for some constant C_0 and a non decreasing smooth function C . One can do the same for higher order Sobolev estimates, by considering higher order time derivatives ∂_t^k and using a finite induction on k . By a continuity argument, an estimate of the form

$$|u(t)|_{H^N} \leq C(|u(t)|_{H^1})\varepsilon t + C_0$$

implies that the H^N norm of u stays bounded on an interval of size $\frac{1}{\varepsilon}$.

This technique only works if time and space derivatives have the same "order". More precisely, for the Shallow-Water equation (3.1.5), the time derivatives are equal to sum of terms involving one space derivative. This is not the case for the Boussinesq-Peregrine equation (3.1.6). Indeed, in the second equation, $(I + \mu\mathcal{T}_b)\partial_t\bar{V}$ is equal to one space derivative order terms, while $(I + \mu\mathcal{T}_b)$ is of order two. It leads to issues if one tries to use the equation to control space derivatives by time derivatives and tries to recover an estimate of size ε for the space derivatives. For example, the second equation of (3.1.6) provides

$$\nabla^\gamma \zeta = -(I + \mu\mathcal{T}_b)\partial_t\bar{V} + \varepsilon \times \text{other terms},$$

and \mathcal{T}_b is an order two operator with respect to space. It is therefore not clear that $\mathcal{T}_b\partial_t\bar{V}$ is controlled by the energy (3.1.8).

To overcome this problem, we introduce a modified equation, which is consistent with the Boussinesq-Peregrine equation (3.1.6) at the $O(\mu^2)$ order (and therefore with the Water-Waves equations). Such equation would have a proper structure adapted to the use of the method used by Bresch-Métivier in [13]. The approach of modifying the equation without changing the consistency, in order to improve the structure of the equation has been used for example by Israwi in [33] for the Green-Naghdi equation, or by Saut and Xu ([53]) for a model of full dispersion. In the Boussinesq-Peregrine case, a short study (see later Section 3.3 for more details) leads us to introduce the following modified equation:

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h\bar{V}) = 0 \\ (I + \mu(\mathcal{T}_b - \nabla^\gamma(\frac{1}{h_b}\nabla^\gamma \cdot (h_b \cdot)) - \frac{1}{h_b}\nabla^{\gamma\perp}\nabla^{\gamma\perp}))\partial_t\bar{V} + \varepsilon\bar{V} \cdot \nabla^\gamma\bar{V} + (I - \mu\nabla^\gamma\frac{1}{h_b}\nabla^\gamma \cdot (h_b \cdot))\nabla^\gamma\zeta = 0 \end{cases} \quad (3.1.10)$$

We start to prove a local existence result for this modified equation in dimension 1 and 2. We then prove the main result of this paper (see later Theorem 3.3.7 for a precise statement):

Theorem 3.1.2 *In dimension 1, the equations (3.1.10) admit a unique solution on a time interval of the form $[0; \frac{T}{\varepsilon}]$ where T only depends on the initial data.*

Remark 3.1.3 *For technical reasons which are discussed further below, the Theorem we prove is only true in dimension 1. However, we precisely explain in this paper the difficulties raising for a proof in dimension 2.*

Though this Theorem is proved by adapting the technique used by Bresch-Métivier in [13], its adaptation to the case of a dispersive equation has not been done yet in the literature to our knowledge. As one shall see later in Section 3.3.2, this result is tied to a singular perturbation problem. The plan of the article is the following:

- In Section 3.2, we prove a local existence result for the Boussinesq-Peregrine equation in dimension $d = 1, 2$,
- In Section 3.3 we introduce a modified Boussinesq-Peregrine equation and in Section 3.3.1 we prove its local well-posedness in dimension $d = 1, 2$,
- In Section 3.3.2, we prove the long time existence result for the modified Boussinesq-Peregrine equation, in dimension $d = 1$.

3.1.6 Notations

We introduce here all the notations used in this paper.

Operators and quantities

Because of the use of dimensionless variables (see before the "dimensionless equations" paragraph), we use the following twisted partial operators:

$$\begin{aligned} \nabla^\gamma &= {}^t(\partial_x, \gamma\partial_y) & \text{if } d = 2 & \text{ and} & \nabla^\gamma &= \partial_x & \text{if } d = 1, \\ \nabla^{\gamma^\perp} &= {}^t(-\gamma\partial_y, \partial_x) & \text{if } d = 2 & \text{ and} & \nabla^{\gamma^\perp} &= 0 & \text{if } d = 1. \end{aligned}$$

Remark 3.1.4 *All the results proved in this paper do not need the assumption that the typical wave lengths are the same in both directions, ie $\gamma = 1$. However, if one is not interested in the dependence of γ , it is possible to take $\gamma = 1$ in all the following proofs. A typical situation where $\gamma \neq 1$ is for weakly transverse waves for which $\gamma = \sqrt{\mu}$; this leads to weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili equation (see [40]). A byproduct of our results is therefore a generalization to the case of nonflat bottoms of the results on weakly transverse Boussinesq systems of [40]; this opens new perspectives towards the derivation and justification of Kadomtsev–Petviashvili equations for nonflat bottoms in the spirit of [34] for the KdV equation.*

We define $a \vee b$ for two real numbers a, b by:

$$a \vee b = \max(a, b).$$

For all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we define ∂^α the operator of $\mathcal{S}'(\mathbb{R}^d)$ by:

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}.$$

We use the classical Fourier multiplier

$$\Lambda^s = (1 - \Delta)^{s/2} \text{ on } \mathbb{R}^d$$

defined by its Fourier transform as

$$\mathcal{F}(\Lambda^s u)(\xi) = (1 + |\xi|^2)^{s/2} (\mathcal{F}u)(\xi)$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$. We also use the following operators:

$$\forall \bar{V}, \quad \mathcal{T}_b \bar{V} = -\frac{1}{3h_b} \nabla^\gamma (h_b^3 \nabla^\gamma \cdot \bar{V}) + \frac{\beta}{2h_b} [\nabla^\gamma (h_b^2 \nabla^\gamma b \cdot \bar{V}) - h_b^2 \nabla^\gamma b \nabla^\gamma \cdot \bar{V}] + \beta^2 \nabla^\gamma b \nabla^\gamma \cdot \bar{V}$$

in all this paper, and the operators:

$$\mathcal{A} = \nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot) \right), \quad \mathcal{B} = \left(I + \mu \mathcal{T}_b - \mu \nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot) \right) - \mu \frac{1}{h_b} \nabla^{\gamma^\perp} \nabla^{\gamma^\perp} \cdot \right)$$

in Section 3.3.

Functional spaces

The standard scalar product on $L^2(\mathbb{R}^d)$ is denoted by $(\cdot, \cdot)_2$ and the associate norm $|\cdot|_2$. We will denote the standard scalar product on Sobolev spaces $H^s(\mathbb{R}^d)$ by $(\cdot, \cdot)_{H^s}$ and the associate norm by $|\cdot|_{H^s}$. We denote the norm $W^{k,\infty}(\mathbb{R}^d)$ by $|\cdot|_{W^{k,\infty}}$, and we use the notation $|\cdot|_\infty = |\cdot|_{W^{0,\infty}}$ when no ambiguity is possible.

We also introduce in Section 3.3.2 for all $s \in \mathbb{R}$ the Banach space

$$X^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d)^d, |f|_{X^s} < \infty\}$$

endowed with the norm

$$|f|_{X^s} = |f|_{X^s}^2 = |f|_{H^s}^2 + \mu |\nabla^\gamma \cdot f|_{H^s}^2.$$

3.2 Local existence for the Boussinesq-Peregrine equation

In this section, we prove a local existence result for the Boussinesq-Peregrine equation in dimension $d = 1, 2$, on a time interval independent on μ . We recall that the Boussinesq-Peregrine equation of unknowns \bar{V} and ζ is:

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ [1 + \mu \mathcal{T}_b] \partial_t \bar{V} + \varepsilon (\bar{V} \cdot \nabla^\gamma) \bar{V} + \nabla^\gamma \zeta = 0 \end{cases} \quad (3.2.11)$$

where $h = 1 + \varepsilon \zeta - \beta b$ and where \mathcal{T}_b is the following operator :

$$\forall \bar{V}, \quad \mathcal{T}_b \bar{V} = -\frac{1}{3h_b} \nabla^\gamma (h_b^3 \nabla^\gamma \cdot \bar{V}) + \frac{\beta}{2h_b} [\nabla^\gamma (h_b^2 \nabla^\gamma b \cdot \bar{V}) - h_b^2 \nabla^\gamma b \nabla^\gamma \cdot \bar{V}] + \beta^2 \nabla^\gamma b \nabla^\gamma b \cdot \bar{V}, \quad (3.2.12)$$

with the notation $h_b = 1 - \beta b$, where b describes the variations of the bottom and is known. We also recall (see section 3.1.6) the notation

$$|f|_{X^s} = |f|_{H^s}^2 + \mu |\nabla^\gamma \cdot f|_{H^s}^2.$$

Let us state a local existence result for the Boussinesq-Peregrine equation:

Theorem 3.2.1 *Let $t_0 > d/2$ and $s > t_0 + 1$. Let $b \in H^{s+2}(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that*

$$\inf_{X \in \mathbb{R}^d} h_b(X) \geq h_{\min}.$$

Let ε, β be such that

$$0 \leq \varepsilon, \beta \leq 1.$$

Let $U_0 = (\zeta_0, \bar{V}_0) \in H^s(\mathbb{R}^d) \times X^s(\mathbb{R}^d)$. Then, there exists $\mu_{\max} > 0$ such that for all $0 \leq \mu \leq \mu_{\max}$ with

$$\varepsilon = O(\mu),$$

there exists $T^ > 0$ and a unique solution $U = (\zeta, \bar{V}) \in C([0; T^*]; H^s(\mathbb{R}^d)^{d+1})$ of the equation (3.2.11) with initial condition $U(0) = U_0$.*

Moreover, for all $T < T^$, if one chooses*

$$\lambda \geq \sup_{t \in [0; T]} C_3(|U|_{W^{1,\infty}}, |h_b|_{H^{s+2}}, \mu_{\max})(t),$$

the solution U satisfies the following inequality:

$$\forall t \leq T, \quad \mathcal{E}(t) \leq \frac{\lambda}{C_1(h_{\min})} \int_0^t e^{\frac{\lambda}{C_1(h_{\min})}(t-t')} C_4(|U|_{H^s})(t') dt' + \frac{C_2(|h_b|_{H^{t_0}})}{C_1(h_{\min})} \mathcal{E}(0) e^{\frac{\lambda}{C_1(h_{\min})}t} \quad (3.2.13)$$

with

$$\mathcal{E}(t) = (\mu |\nabla^\gamma \cdot \bar{V}|_{H^s}^2 + |U|_{H^s}^2)(t) \quad (3.2.14)$$

and where C_i are non decreasing smooth functions of their arguments, for $i = 1, 2, 3, 4$.

Remark 3.2.2 – *It is very important to note that the energy estimate (3.2.13) implies that while $|U|_{W^{1,\infty}}(t)$ is bounded, the solution U can be continued. More precisely, if one sets*

$$T = \sup\{t, U \text{ exists on } C([0; t]; H^s(\mathbb{R}^d)^{d+1})\}$$

then if $T < \infty$, one has $\sup_{t \rightarrow T} |U|_{W^{1,\infty}}(t) = +\infty$. Moreover, one has to notice that the energy \mathcal{E} defined in the statement of the Theorem by (3.2.14) controls H^s norms of both U and $\nabla^\gamma \cdot \bar{V}$, while the energy estimate (3.2.13) only requires a bound of \mathcal{E} by the H^s norm of U . In particular, it suffices to bound the H^s norm of U (instead of U and $\nabla^\gamma \cdot \bar{V}$) to use a continuity argument.

– *Note that the time of existence T^* defined by Theorem 3.2.1 is independent on μ . This is crucial in view of the proof of the consistency of the Boussinesq-Peregrine equation with the Water-Waves equations (see Theorem 3.1.1).*

Proof of Theorem 3.2.1 The system (3.2.11) can be put under the form

$$B\partial_t U + \sum_{j=1}^d A_j(U)\partial_j U = 0$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & I + \mu\mathcal{T}_b \end{pmatrix}, \quad A_j(U) = \begin{pmatrix} \varepsilon\bar{V}_j & h \\ 1 & \varepsilon\bar{V}_j I_d \end{pmatrix} \quad \forall 1 \leq j \leq d. \quad (3.2.15)$$

The non linear terms of the equation can be symmetrized if we multiply the system by

$$\tilde{S}(U) = \begin{pmatrix} 1 & 0 \\ 0 & hI_d \end{pmatrix}$$

but for the reasons explained in Section 3.1.5 we use rather the following matrix:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & h_b I_d \end{pmatrix}. \quad (3.2.16)$$

Using this symmetrizer brings one difficulty: the operator $h_b(I + \mu\mathcal{T}_b)$ is symmetric, but multiplying the second equation of (3.2.11) by h_b does not symmetrize the non linear terms of the form $A_j(U)\partial_j U$ defined by (3.2.15), for $j = 1, \dots, d$. The proof of Theorem 3.2.1 is inspired of the classical existence result for quasilinear hyperbolic systems (see [61] Chapter XVI for instance). We follow the following steps:

- Step 1 : We solve a smoothed equation involving a mollifier $(1 - \delta\Delta)$.
- Step 2 : We prove that the existence time of the solution of the mollified equation does not depend on δ , and the uniform bound in H^s norm of this solution.
- Step 3 : We pass to the limit δ goes to zero in the mollified equation to get a solution of the equation (3.2.11).
- Step 4 : We recover regularity for the solution of (3.2.11).

Step 1 We solve the Cauchy problem

$$\begin{cases} (1 - \delta\Delta)SB(1 - \delta\Delta)\partial_t U^\delta + \sum_{i=1}^d SA_i(U^\delta)\partial_i U^\delta = 0 \\ U^\delta(0) = (1 - \delta\Delta)^{-1}U_0, \end{cases} \quad (3.2.17)$$

of unknown U^δ in the Banach space $H^s(\mathbb{R}^d)^{d+1}$. Recall that using the definition of B given in (3.2.15) and S given by (3.2.16), one has

$$SB = \begin{pmatrix} 1 & 0 \\ 0 & h_b(I_d + \mu\mathcal{T}_b) \end{pmatrix}.$$

In order to apply the Cauchy-Lipschitz Theorem, one must check that the application

$$\begin{aligned} H^s(\mathbb{R}^d)^{d+1} &\longrightarrow H^s(\mathbb{R}^d)^{d+1} \\ U &\longmapsto (1 - \delta\Delta)^{-1}(SB)^{-1}(1 - \delta\Delta)^{-1} \sum_{i=1}^d A_i(U)\partial_i U \end{aligned}$$

is well defined and locally Lipschitz. The unique difficulty is to check that $(h_b(I + \mu\mathcal{T}_b))^{-1}$ is well defined from H^s to H^s . It is the point of the following Proposition (see [37] Chapter 5 Lemma 5.44 for a full proof). We first define the Banach space

$$X^s = \{V \in H^s(\mathbb{R}^d)^d, \nabla^\gamma \cdot V \in H^s(\mathbb{R}^d)\}$$

endowed with the norm

$$|V|_{X^s}^2 = |V|_{H^s}^2 + \mu|\nabla^\gamma \cdot V|_{H^s}^2.$$

Proposition 3.2.3 *Let $t_0 > d/2$, $\beta \leq 1$ and $b \in H^{t_0+1}(\mathbb{R}^d)$ be such that there exists h_{\min} such that $h_b = 1 - \beta b \geq h_{\min}$. Then the mapping*

$$h_b(I + \mu\mathcal{T}_b) : X^0 \longrightarrow L^2(\mathbb{R}^d)^d + \nabla^\gamma L^2(\mathbb{R}^d)$$

is well defined, one-to-one and onto. One has, for all $V \in X^0$:

$$C_1(h_{\min})|V|_{X^0}^2 \leq (h_b(I + \mu\mathcal{T}_b)V, V)_2 \leq C_2(|h_b|_{H^{t_0+1}})|V|_{X^0}^2$$

where C_i are non decreasing functions of its arguments. Moreover, for all $s \geq 0$, if $b \in H^{1+s \vee t_0}(\mathbb{R}^d)$, then

$$\forall W \in H^s(\mathbb{R}^d)^d, |(h_b(I + \mu\mathcal{T}_b))^{-1}W|_{X^s} \leq C\left(\frac{1}{h_{\min}}, |b|_{H^{1+s \vee t_0}}\right)|W|_{H^s}$$

where C is a non decreasing function of its arguments. Moreover, one has, for all $s \in \mathbb{R}$:

$$\sqrt{\mu}|(h_b(I + \mu\mathcal{T}_b))^{-1}\nabla^\gamma W|_{H^s} \leq C\left(\frac{1}{h_{\min}}, |b|_{H^{1+|\gamma| \vee t_0}}\right)|W|_{H^s}.$$

Therefore, the Cauchy-Lipschitz Theorem applies and the equation (3.2.17) has a unique solution $U^\delta \in C([0; T^\delta]; H^s(\mathbb{R}^d)^{d+1})$, and if $T^\delta < +\infty$ one has

$$\lim_{t \rightarrow T^\delta} |U^\delta(t)|_{H^s} = +\infty.$$

Step 2 We now check that one can choose T^δ independent of δ by comparing $|U^\delta(t)|_{H^s}$ with a solution of an ODE independent of δ , and using a Gronwall Lemma. We define

$$U_s^\delta = \Lambda^s U^\delta.$$

The unknown U_s^δ satisfies the following system :

$$(1 - \delta\Delta)SB(1 - \delta\Delta)\partial_t U_s^\delta + \sum_{j=1}^d SA_j(U^\delta)\partial_j U_s^\delta = F \quad (3.2.18)$$

where we wrote the commutators under the form

$$F = (1 - \delta\Delta)[SB, \Lambda^s](1 - \delta\Delta)\partial_t U^\delta + \sum_{j=1}^d [SA_j(U^\delta), \Lambda^s]\partial_j U^\delta. \quad (3.2.19)$$

In order to estimate, $|U^\delta(t)|_{H^s}$, recall that

$$SB = \begin{pmatrix} 1 & 0 \\ 0 & h_b(I_d + \mu\mathcal{T}_b) \end{pmatrix}$$

and remark, using Proposition (3.2.3), that

$$|(1 - \delta\Delta)V_s^\delta|_{X^0} \sim (h_b(I + \mu\mathcal{T}_b)(1 - \delta\Delta)V_s^\delta, (1 - \delta\Delta)V_s^\delta)_2$$

where the implicit constant only depend on h_b . Therefore, one computes:

$$\frac{d}{dt} \frac{1}{2} ((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 = ((1 - \delta\Delta)SB(1 - \delta\Delta)\partial_t U_s^\delta, U_s^\delta)_2.$$

Note that the symmetry of SB , and more precisely of $h_b(I + \mu\mathcal{T}_b)$ is crucial here. One uses the equation (3.2.18) to replace $(1 - \delta\Delta)SB(1 - \delta\Delta)\partial_t U_s^\delta$ by its expression. One gets:

$$\frac{d}{dt} \frac{1}{2} ((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 = - \sum_{j=1}^d (SA_j(U^\delta)\partial_j U_s^\delta, U_s^\delta)_2 + (F, U_s^\delta)_2. \quad (3.2.20)$$

Let us check that the first term of the rhs of (3.2.20) has a contribution of order zero to the energy estimate. One uses the definition of A_j given by (3.2.15) to put this matrix under the form $A_j(U^\delta) = \tilde{A}_j(U^\delta) + C(U^\delta)$ with

$$\tilde{A}_j(U^\delta) = \begin{pmatrix} \varepsilon \bar{V}_j^\delta & h_b \\ 1 & \varepsilon \bar{V}_j^\delta I_d \end{pmatrix}, \quad C(U^\delta) = \begin{pmatrix} 0 & \varepsilon \zeta^\delta \\ 0 & 0 \end{pmatrix}, \quad (3.2.21)$$

for $j = 1, \dots, d$. Note that since S is not a true symmetrizer for the equation (3.2.11), the matrix SA_j is not symmetric. The above decomposition allows us to write SA_j under the form of a symmetric matrix $(S\tilde{A}_j)$ plus a rest which we intend to control in the energy estimates. We now write:

$$\sum_{j=1}^d (SA_j(U^\delta) \partial_j U_s^\delta, U_s^\delta)_2 = \sum_{j=1}^d (SC(U^\delta) \partial_j U_s^\delta, U_s^\delta)_2 + \sum_{j=1}^d (S\tilde{A}_j(U^\delta) \partial_j U_s^\delta, U_s^\delta)_2. \quad (3.2.22)$$

Using the definition of $C(U^\delta)$ given by (3.2.21), the first term of the rhs of (3.2.22) is equal to $\int \zeta^\delta \zeta_s^\delta \varepsilon \nabla^\gamma \cdot \bar{V}_s^\delta$ and therefore is controlled, using the Cauchy-Schwarz inequality by

$$\left| \int \varepsilon \zeta^\delta \nabla^\gamma \cdot \bar{V}_s^\delta \zeta_s^\delta \right| \leq |\zeta^\delta|_{W^{1,\infty}} |\mu \nabla^\gamma \cdot \bar{V}_s^\delta|_2 |\zeta_s^\delta|_2 \quad (3.2.23)$$

where we used the Boussinesq regime condition

$$\varepsilon \leq C\mu$$

stated by the Theorem. For the second term of the rhs of (3.2.22), one can write, for all $1 \leq j \leq d$, and using the symmetry of $S\tilde{A}_j(U^\delta)$ (recall the definition of the symmetrizer S given by (3.2.16)):

$$\begin{aligned} (S\tilde{A}_j(U^\delta) \partial_j U_s^\delta, U_s^\delta)_2 &= (\partial_j U_s^\delta, S\tilde{A}_j(U^\delta) U_s^\delta)_2 \\ &= -(U_s^\delta, \partial_j (S\tilde{A}_j(U^\delta) U_s^\delta))_2 \end{aligned}$$

by integrating by parts. Now, one has

$$-(U_s^\delta, \partial_j (S\tilde{A}_j(U^\delta) U_s^\delta))_2 = -(U_s^\delta, \partial_j (S\tilde{A}_j(U^\delta)) U_s^\delta)_2 - (U_s^\delta, S\tilde{A}_j(U^\delta) \partial_j U_s^\delta)_2$$

and thus one has

$$(S\tilde{A}_j(U^\delta) \partial_j U_s^\delta, U_s^\delta)_2 = -\frac{1}{2} (U_s^\delta, \partial_j (S\tilde{A}_j(U^\delta)) U_s^\delta)_2. \quad (3.2.24)$$

Using the definition of S given by (3.2.16), and the definition of \tilde{A}_j given by (3.2.21), one has

$$S\tilde{A}_j = \begin{pmatrix} \varepsilon \bar{V}_j^\delta & h_b \\ h_b & \varepsilon h_b \bar{V}_j^\delta I_d \end{pmatrix}$$

for all $j = 1, \dots, d$ and thus (3.2.24) is controlled, using Cauchy-Schwarz inequality:

$$|(S\tilde{A}_j U^\delta \partial_j U_s^\delta, U_s^\delta)_2| \leq |U_s^\delta|_2^2 c_2 (|U^\delta|_{W^{1,\infty}}, |h_b|_{W^{1,\infty}}), \quad (3.2.25)$$

where c_2 is a non decreasing and smooth function of its arguments.

We now control the second term of the right hand side of the energy estimate (3.2.20). Using the definition of F given by (3.2.19), one has that

$$(F, U_s^\delta)_2 = A_3 + A_4$$

where

$$A_3 = ((1 - \delta\Delta)[SB, \Lambda^s](1 - \delta\Delta) \partial_t U^\delta, U_s^\delta)_2$$

and

$$A_4 = ([SA_j(U^\delta), \Lambda^s] \partial_j U^\delta, U_s^\delta)_2.$$

- *Control of A_3* We start by replacing $(1 - \delta\Delta) \partial_t U^\delta$ by its expression given in the equation (3.2.17):

$$\begin{aligned} &((1 - \delta\Delta)[SB, \Lambda^s](1 - \delta\Delta) \partial_t U^\delta, U_s^\delta) \\ &= - \sum_{j=1}^d ((1 - \delta\Delta)[SB, \Lambda^s](SB)^{-1} (1 - \delta\Delta)^{-1} SA_j(U^\delta) \partial_j U^\delta, U_s^\delta)_2. \end{aligned} \quad (3.2.26)$$

One has to control this term uniformly with respect to δ , and deals with the fact that $(h_b(I + \mu\mathcal{T}_b))^{-1}$ is not optimally estimated. More precisely, that is absolutely not clear that⁴

$$|h_b(I + \mu\mathcal{T}_b)f(h_b(I + \mu\mathcal{T}_b))^{-1}u|_{H^s} \leq |u|_{H^s}$$

for a smooth function f . One has to recall that

$$B = \begin{pmatrix} 1 & 0 \\ 0 & I + \mu\mathcal{T}_b \end{pmatrix}$$

so that

$$[SB, \Lambda^s] = \begin{pmatrix} 0 & 0 \\ 0 & [h_b(I + \mu\mathcal{T}_b), \Lambda^s] \end{pmatrix}.$$

Using the definition of \mathcal{T}_b given by (3.2.12), one writes this operator under the form:

$$I + \mu\mathcal{T}_b = I + \mu(A + B + C + D),$$

where

$$A = -\nabla^\gamma(h_b^3\nabla^\gamma\cdot), \quad B = \beta\nabla^\gamma(h_b^2\nabla^\gamma b\cdot), \quad C = -\beta h_b^2\nabla^\gamma b\nabla^\gamma\cdot, \quad D = \beta^2\nabla^\gamma b\nabla^\gamma\cdot. \quad (3.2.27)$$

One expands the commutator (3.2.26) with respect to A, B, C, D . We set

$$A_{31} = ((1 - \delta\Delta)\mu[A, \Lambda^s](h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta, \bar{V}_s^\delta)_2$$

and we now control this latter term. For all $1 \leq j \leq d$, using the definition of A given by (3.2.27), one has, integrating by parts:

$$A_{31} = ((1 - \delta\Delta)[h_b^3, \Lambda^s]\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta, \sqrt{\mu}\nabla^\gamma \cdot \bar{V}_s^\delta)_2.$$

Using Cauchy-Schwarz inequality and splitting $(1 - \delta\Delta)$, one gets:

$$\begin{aligned} |A_{31}| &\leq \sqrt{\mu}|\nabla^\gamma \cdot \bar{V}_s^\delta|_2 (|[h_b^3, \Lambda^s]\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta|_2 \\ &\quad + \delta|[h_b^3, \Lambda^s]\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta|_{H^2}). \end{aligned}$$

We now use the Kato-Ponce estimate of Proposition 3.A.1 to control $[h_b^3, \Lambda^s]$, using the fact that $s > d/2 + 1$ (and thus $H^{s-1}(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$ are respectively continuously injected into $L^\infty(\mathbb{R}^d)$ and $W^{1,\infty}(\mathbb{R}^d)$):

$$\begin{aligned} |A_{31}| &\leq \sqrt{\mu}|\nabla^\gamma \cdot \bar{V}_s^\delta|_2 C_s(|\nabla^\gamma h_b|_{H^{s+1}})(|\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta|_{H^{s-1}} \\ &\quad + \delta|\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}(1 - \delta\Delta)^{-1}SA_j(U^\delta)\partial_j\bar{V}^\delta|_{H^{s+1}}), \end{aligned} \quad (3.2.28)$$

where C_s is a smooth non decreasing function which only depends on s . We now control the operator $\sqrt{\mu}\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}$ in H^{s-1} and H^{s+1} norms by using the last part of Proposition 3.2.3 and a duality argument. One has, for all $k \geq t_0$ and all $u \in H^k(\mathbb{R}^d)^d$, using the symmetry of $h_b(I + \mu\mathcal{T}_b)$:

$$\begin{aligned} \sqrt{\mu}|\nabla^\gamma \cdot h_b(I + \mu\mathcal{T}_b)^{-1}u|_{H^k} &= \sqrt{\mu} \sup_{\substack{v \in H^{-k}(\mathbb{R}^d) \\ |v|_{H^{-k}}=1}} (\nabla^\gamma \cdot (h_b(I + \mu\mathcal{T}_b))^{-1}u, v)_2 \\ &= \sqrt{\mu} \sup_{\substack{v \in H^{-k}(\mathbb{R}^d) \\ |v|_{H^{-k}}=1}} -(u, (h_b(I + \mu\mathcal{T}_b))^{-1}\nabla^\gamma v)_2 \\ &\leq \sup_{\substack{v \in H^{-k}(\mathbb{R}^d) \\ |v|_{H^{-k}}=1}} C\left(\frac{1}{h_{\min}}, |b|_{H^{k+1}}\right)|u|_{H^k}|v|_{H^{-k}} \\ &\leq C\left(\frac{1}{h_{\min}}, |b|_{H^{k+1}}\right)|u|_{H^k}, \end{aligned} \quad (3.2.29)$$

⁴The operator $h_b(I + \mu\mathcal{T}_b)$ is not technically elliptic of order 1, since its inverse only controls the divergence (and not a full derivative). This is actually a big issue for all the local existence results for the Boussinesq-Peregrine equation (3.2.11).

where C is a smooth non decreasing function of its arguments, and where we used the fact that $k \geq t_0$. Using (3.2.29) with $k = s - 1$ and $k = s + 1$ in (3.2.28) (recall that $s > t_0 + 1$), one gets:

$$\begin{aligned} |A_{31}| &\leq C\sqrt{\mu}|\nabla^\gamma \cdot \bar{V}_s^\delta|_2 C\left(\frac{1}{h_{\min}}, |b|_{H^{s+2}}\right) \left(|(1 - \delta\Delta)^{-1} SA_j(U^\delta) \partial_j \bar{V}^\delta|_{H^{s-1}} \right. \\ &\quad \left. + \delta |(1 - \delta\Delta)^{-1} SA_j(U^\delta) \partial_j \bar{V}^\delta|_{H^{s+1}} \right) \\ &\leq C\sqrt{\mu}|\nabla^\gamma \cdot \bar{V}_s^\delta|_2 C\left(\frac{1}{h_{\min}}, |b|_{H^{s+2}}\right) \left(|SA_j(U^\delta) \partial_j \bar{V}^\delta|_{H^{s-1}} + C\delta \frac{1}{\delta} |SA_j(U^\delta) \partial_j \bar{V}^\delta|_{H^{s-1}} \right) \end{aligned}$$

where we used the estimates $|(1 - \delta\Delta)^{-1} f|_{H^{s-1}} \leq |f|_{H^{s-1}}$ and $|(1 - \delta\Delta)^{-1} f|_{H^{s+1}} \leq \frac{C}{\delta} |f|_{H^{s-1}}$ with C independent on δ . We recall that

$$SA_j(U^\delta) = \begin{pmatrix} \varepsilon \bar{V}_j^\delta & h \\ h_b & \varepsilon h_b \bar{V}_j^\delta \end{pmatrix}$$

for $j = 1, \dots, d$ and we use the Moser estimate of Proposition 3.A.2 and the fact that $s - 1 > d/2$ to conclude:

$$|A_{31}| \leq C\left(\frac{1}{h_{\min}}, |b|_{H^{s+2}}\right) \sqrt{\mu} |\nabla^\gamma \cdot \bar{V}_s^\delta|_2 |\bar{V}_s^\delta|_2^2$$

with C a smooth, non decreasing function of its arguments.

To control the term of (3.2.26) involving B , using the definition of B given by (3.2.27), we write, integrating by parts:

$$\begin{aligned} &\sum_{j=1}^d ((1 - \delta\Delta)\mu[B, \Lambda^s](h_b(I + \mu\mathcal{T}_b))^{-1} (1 - \delta\Delta)^{-1} SA_j(U^\delta) \partial_j \bar{V}^\delta, \bar{V}_s^\delta)_2 \\ &= -\mu((1 - \delta\Delta)[h_b^3 \nabla^\gamma b \cdot, \Lambda^s](h_b(I + \mu\mathcal{T}_b))^{-1} (1 - \delta\Delta)^{-1} SA_j(U^\delta) \partial_j \bar{V}^\delta, \nabla^\gamma \cdot \bar{V}_s^\delta)_2 \end{aligned}$$

and we use exactly the same techniques as used for the control of A_{31} to get the same control. The terms of (3.2.26) involving C and D are easily controlled by

$$C\left(\frac{1}{h_{\min}}, |b|_{H^{s+2}}\right) \sqrt{\mu} |\bar{V}_s^\delta|_2^3.$$

We finally proved that:

$$A_3 \leq c_3 (|U^\delta|_{W^{1,\infty}}, |\nabla^\gamma b|_{H^{s+2}}) |U_s^\delta|_2^2 (|U_s^\delta|_2 + \sqrt{\mu} |\nabla^\gamma \cdot \bar{V}_s^\delta|_2) \quad (3.2.30)$$

where c_3 is a smooth positive non decreasing function of its argument and independent of δ .

- *Control of A_4*

Let us now control A_4 by using the Kato-Ponce estimate of Proposition 3.A.1 and the Moser estimate of Proposition 3.A.2, using again that $s > d/2 + 1$:

$$\begin{aligned} ([SA_j(U^\delta), \Lambda^s] \partial_j U^\delta, U_s^\delta)_2 &\leq C(|\nabla^\gamma SA_j(U^\delta)|_{H^{s-1}} |\partial_j U^\delta|_{L^\infty} + |\nabla^\gamma SA_j(U^\delta)|_{L^\infty} |\partial_j U^\delta|_{H^{s-1}}) |U_s^\delta|_2 \\ &\leq (C(|U^\delta|_\infty) |U^\delta|_{H^s} |\partial_j U^\delta|_\infty |h_b|_{H^s} + C(|U^\delta|_{W^{1,\infty}}, |h_b|_{W^{1,\infty}}) |U^\delta|_{H^s}) |U_s^\delta|_2 \\ &\leq c_4 (|U^\delta|_{W^{1,\infty}}, |h_b|_{H^s}) |U_s^\delta|_2^2. \end{aligned} \quad (3.2.31)$$

If one puts together (3.2.23), (3.2.25), (3.2.30) and (3.2.31), one gets :

$$\frac{d}{dt} ((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 \leq c_5 (|U^\delta|_{W^{1,\infty}}, |h_b|_{H^{s+2}}, \mu_{\max}) (|U_s^\delta|_2^2 + \mu |\nabla^\gamma \cdot \bar{V}_s^\delta|_2^2 + F(|U_s^\delta|_2)) \quad (3.2.32)$$

with c_5 and F some smooth non decreasing functions of their arguments, independent of δ .

At this point, recalling the equivalence $|V|_{X^0} \sim (h_b(I + \mu\mathcal{T}_b)V, V)_2$ stated by Proposition 3.2.3, we proved that

$$\frac{d}{dt} ((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 \leq F((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2$$

where F is a Lipschitz function which does not depend on δ . By Cauchy-Lipschitz theorem, there exists $T^* > 0$ such that the Cauchy problem

$$\begin{cases} \frac{d}{dt}g(t) &= F(g(t)) \\ g(0) &= |U(0)|_{H^s} \end{cases}$$

admits a unique solution g on a time interval $[0; T^*]$. By Gronwall's lemma, one has for all $t < T^*$ that

$$((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 \leq g(t)$$

and consequently, using again the equivalence $|V|_{X^0} \sim (h_b(I + \mu\mathcal{T}_b)V, V)_2$ stated by Proposition 3.2.3:

$$\forall 0 \leq t \leq T^*, \quad |(1 - \delta\Delta)U_s^\delta|_2^2 + \mu|(1 - \delta\Delta)\nabla^\gamma \cdot \bar{V}_s^\delta|_2^2 \leq \frac{1}{C_1(h_{\min})}g(t). \quad (3.2.33)$$

This proves that the H^s norm of U^δ does not explode as t goes to T^* , and then $T^\delta > T^*$, which give us a uniform time of existence for U^δ independent of δ .

We can be more precise for all $0 < T < T^*$ if one chooses

$$\lambda \geq \sup_{t \in [0; T]} c_5(|U^\delta|_{W^{1, \infty}}, |h_b|_{H^{s+2}}, \mu_{\max})(t)$$

then one has the following inequality, using estimate (3.2.32):

$$\frac{d}{dt}((1 - \delta\Delta)SB(1 - \delta\Delta)U_s^\delta, U_s^\delta)_2 \leq \lambda(|U_s^\delta|_2^2 + \mu|\nabla^\gamma \cdot \bar{V}_s^\delta|_2^2 + F(|U_s^\delta|_2))$$

and by integrating in time and using one last time the equivalence $|V|_{X^0} \sim (h_b(I + \mu\mathcal{T}_b)V, V)_2$ stated by Proposition 3.2.3, one gets, for all $0 \leq t \leq T$:

$$\begin{aligned} (\mu|\nabla^\gamma \cdot (1 - \delta\Delta)^{1/2}V_s^\delta|_2^2 + |(1 - \delta\Delta)^{1/2}U_s^\delta|_2^2)(t) &\leq \frac{\lambda}{C_1(h_{\min})} \int_0^t F(|U_s^\delta|_2)(t')dt' \\ &+ \frac{\lambda}{C_1(h_{\min})} \int_0^t (|(1 - \delta\Delta)^{1/2}U_s^\delta|_2^2 + \mu|\nabla^\gamma \cdot (1 - \delta\Delta)^{1/2}\bar{V}_s^\delta|_2^2)(t')dt' \\ &+ \frac{C_2(|h_b|_{H^{t_0}})}{C_1(h_{\min})}\mathcal{E}(0) \end{aligned}$$

where we recall that the energy \mathcal{E} is defined by (3.2.14). One can conclude by Gronwall Lemma that:

$$\begin{aligned} (\mu|\nabla^\gamma \cdot (1 - \delta\Delta)^{1/2}V_s^\delta|_2^2 + |(1 - \delta\Delta)^{1/2}U_s^\delta|_2^2)(t) &\leq \frac{\lambda}{C_1(h_{\min})} \int_0^t e^{\frac{\lambda}{C_1(h_{\min})}(t-t')} F(|U_s^\delta|_2)(t')dt' \\ &+ \frac{C_2(|h_b|_{H^{t_0}})}{C_1(h_{\min})}\mathcal{E}(0)e^{\frac{\lambda}{C_1(h_{\min})}t}. \end{aligned} \quad (3.2.34)$$

Step 3-4 The inequality (3.2.33) and the equation (3.2.17) prove that $(U^\delta)_\delta$ is bounded in the space

$L^\infty([0; T^*]; H^s(\mathbb{R}^d)) \cap W^{1, \infty}([0; T^*]; H^{s-1})$. By compact embedding in $H^{s'}(\mathbb{R}^d)$ for all $s' < s$, one has the strong convergence of $(U^\delta)_\delta$ in $C([0; T^*]; H_{loc}^{s'}(\mathbb{R}^d))$ to a function U . If one chooses s' close enough to s , $H^{s'}(\mathbb{R}^d)$ is embedded in $C^1(\mathbb{R}^d)$ and one can pass to the limit in the non-linear terms of (3.2.17). The linear terms do not raise any difficulty. It gives us a solution U of the problem. A short analysis as in [61] Proposition XVI.1.4 shows that U is in fact $C([0; T^*]; H^s(\mathbb{R}^d))$. One can pass the limit δ goes to zero in the estimate (3.2.34) and recovers the estimate (3.2.13) stated in the Theorem. \square

3.3 Modified equation

As explained in the Introduction, the Boussinesq-Peregrine equation (3.2.11) does not have the proper structure to apply the technique used by Bresch-Métivier in [13]. One time derivative of ζ is not equal to sum of terms of one space derivative order of \bar{V} .

In order to implement the technique used by [13], we modify a bit the equation without changing the consistency with the Water-Waves equation. More precisely, the Boussinesq-Peregrine equation is consistent at order $O(\mu^2)$ with the Water-Waves equation and therefore we look for a new equation consistent with the Boussinesq-Peregrine equation at a $O(\mu^2)$ order. In this new equation, one space derivative of ζ should have the "same order" as one time derivative of \bar{V} . For this purpose, we use the following formal consideration:

$$\partial_t \bar{V} = -(I + \mu \mathcal{T}_b)^{-1} (\varepsilon \bar{V} \cdot \nabla^\gamma \bar{V} + \nabla^\gamma \zeta) = -\nabla^\gamma \zeta + \mu R$$

where R is of order 0 or more in μ (recall that $\varepsilon = O(\mu)$ in the Boussinesq-Peregrine regime). Therefore, one can add to the Boussinesq-Peregrine equation any expression of the form $\mu A \nabla^\gamma \zeta$ and the corresponding term $\mu A \partial_t \bar{V}$, where A is an operator independent on μ , without changing the consistency:

$$\mu A \partial_t \bar{V} = -\mu A \nabla^\gamma \zeta + \mu^2 R.$$

The operator A should respect the following constraints:

- the operator $h_b A$ should be symmetric, since multiplying the second equation by $h_b \bar{V}$ should give the time derivative of a positive quantity, such as $|\bar{V}|_2^2$ or $|\nabla^\gamma \cdot \bar{V}|_2^2$;
- the whole system must conserve a certain symmetry and be of the form: $B \partial_t U + \varepsilon U \cdot \nabla^\gamma U + LU = 0$, where $U = (\zeta, \bar{V})$, B is symmetric and where L is an anti-symmetric operator;
- the operator $I + \mu A$ should be elliptic and of order at least two; therefore, one would have

$$\nabla^\gamma \zeta = -(I + \mu A)^{-1} ((I + \mu \mathcal{T}_b + \mu A) \partial_t \bar{V} + \varepsilon R)$$

and a good control for $\partial_t \bar{V}$ would provide a good control for $\nabla^\gamma \zeta$.

The two first constraints ensure the local existence for the new equation, while the third one ensures the large existence time. A short study shows that one should consider the following operator for A :

$$A \nabla^\gamma W = -\nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b W) \right)$$

and the following symmetrizer for the equation:

$$\mathcal{S} = \begin{pmatrix} I - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \cdot) & 0 \\ 0 & h_b \end{pmatrix}. \quad (3.3.35)$$

However, with the consideration $\partial_t \bar{V} = -\nabla^\gamma \zeta + O(\mu)$, one can for free make the operator $h_b(I + \mu \mathcal{T}_b + \mu A)$ elliptic by addition of the operator $\mu \nabla^{\gamma \perp} \nabla^{\gamma \perp}$, which provides a total control of a full derivative:

$$(h_b(I + \mu \mathcal{T}_b + A) \bar{V}, \bar{V})_2 \sim |\bar{V}|_2^2 + \mu |\nabla^\gamma \cdot \bar{V}|_2^2 + \mu |\nabla^{\gamma \perp} \cdot \bar{V}|_2^2,$$

where

$$\nabla^{\gamma \perp} \cdot \bar{V} = (-\gamma \partial \bar{V}_x + \partial_x \bar{V}_y)$$

if $d = 2$, and $\nabla^{\gamma \perp} \cdot = 0$ if $d = 1$. Remark that $\nabla^{\gamma \perp} \cdot \nabla^\gamma \zeta = 0$, and since $d = 1, 2$, the operator $\nabla^{\gamma \perp}$ acts like the curl operator in dimension 3.

Remark 3.3.1 *The operator $I + \mu A = I - \mu \nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot) \right)$ is not elliptic, but it is invertible and its inverse gives precise control of the H^1 norm of the divergence, which is enough to control $\nabla \zeta$ in H^1 norm, since its curl is zero:*

$$\nabla^\gamma \zeta = -(h_b(I + \mu A))^{-1} (h_b(I + \mu \mathcal{T}_b + \mu A - \nabla^\gamma \nabla^{\gamma \perp} \cdot) \partial_t \bar{V} + \varepsilon R).$$

We are therefore led to consider the following equation:

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot (h \bar{V}) = 0 \\ \left[I + \mu (\mathcal{T}_b - \nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot) \right)) - \frac{1}{h_b} \nabla^{\gamma \perp} \nabla^{\gamma \perp} \cdot \right] \partial_t \bar{V} + \varepsilon \bar{V} \cdot \nabla^\gamma \bar{V} + (I - \mu \nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot) \right)) \nabla^\gamma \zeta = 0 \end{cases} \quad (3.3.36)$$

However, the symmetrizer \mathcal{S} defined by (3.3.35) does not symmetrize properly the equation (3.3.36): there is a residual term in the first equation $\nabla^\gamma \cdot (\varepsilon \zeta \bar{V})$ which is not canceled in the time derivative

of the energy $(SU, U)_2$ (with $U = (\zeta, \bar{V})$). To overcome this problem, we use the following change of variable inspired by Brech-Métivier:

$$q = \frac{1}{\varepsilon} \log\left(1 + \frac{\varepsilon\zeta}{h_b}\right). \quad (3.3.37)$$

The following Proposition is the key point of this change of variable, and states a precise relation between q and ζ :

Proposition 3.3.2 *Let $N > d/2 + 1$. Let also $h_b \in H^N(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that:*

$$\forall X \in \mathbb{R}^d, \quad h_b(X) \geq h_{\min}.$$

Then, for ε small enough, the quantity q defined by (3.3.37) is well defined. Moreover, one has:

$$q = Q(\zeta)\zeta,$$

with $Q(\zeta) > 0$. More precisely, for all $\alpha \in \mathbb{N}^d$, $1 \leq |\alpha| \leq N$, one has

$$\partial^\alpha q = Q_1(\zeta, h_b)\zeta^\alpha + P_\alpha(\zeta, h_b)$$

with

$$Q_1(\zeta, h_b) = \int_0^t \frac{h_b}{h_b + \varepsilon t \zeta} dt$$

and

$$P_\alpha(\zeta, h_b) = \varepsilon \sum_{\beta > \alpha} Q_\beta(\zeta, h_b) + \sum_{0 < \beta \leq \alpha} R_\beta(\zeta, h_b) \partial^\beta h_b$$

with Q_β, R_β smooth functions of their arguments, and

$$|P_\alpha(\zeta, h_b)|_{H^1} \leq C(h_{\min}, |h_b|_{H^N}) |\partial^\alpha \zeta|_2,$$

where C is a smooth non decreasing function of its arguments.

Remark 3.3.3 *The Proposition 3.3.2 states that at the leading order, a derivative of q is equal to a derivative of ζ up to a positive factor. Moreover, if one differentiates q only with respect to time, since $\partial_t h_b = 0$ this equality is true up to an ε factor.*

Proof Just notice that the definition of q (3.3.37) implies

$$q = Q(\zeta)\zeta, \quad (3.3.38)$$

with

$$Q(\zeta) = \int_0^1 \frac{1}{h_b + \varepsilon t \zeta} dt \zeta.$$

For a given $\zeta(0)$, the quantity q is well-defined if there exists $h_{\min} > 0$ such that the following condition is satisfied:

$$\forall X \in \mathbb{R}^d, \quad h_0(X) > h_{\min}.$$

Indeed, for ε small enough one has $h_b + \varepsilon t \zeta(0) > 0$ and this condition stay satisfied for t small enough. Moreover, one has $Q(\zeta) > 0$.

We now differentiate one time (3.3.38) (the notation ∂ stands for any derivative of order one):

$$\partial q = \int_0^t \frac{h_b}{h_b + t\varepsilon\zeta} dt \partial q \quad (3.3.39)$$

which gives the expression of $Q_1(\zeta, h_b)$ given by the Proposition. The end of the proof is done by differentiating (3.3.39).

□

3.3.1 Local existence for the modified Boussinesq-Peregrine equation

We prove in this Section a local existence result for the modified Boussinesq-Peregrine equation (3.3.36) introduced previously. We recall that we consider the change of unknown

$$q = \frac{1}{\varepsilon} \log(1 + \frac{\varepsilon \zeta}{h_b}). \quad (3.3.40)$$

Under the change of unknown (3.3.40), the equation (3.3.36) takes the form:

$$\begin{cases} h_b(\partial_t Q(\zeta)\zeta + \varepsilon \bar{V} \cdot \nabla^\gamma(Q(\zeta)\zeta)) + \nabla^\gamma \cdot (h_b \bar{V}) = 0 \\ h_b(I + \mu(\mathcal{T}_b - \nabla^\gamma(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)) - \frac{1}{h_b} \nabla^{\gamma \perp} \nabla^{\gamma \perp} \cdot)) \partial_t \bar{V} + h_b \varepsilon \bar{V} \cdot \nabla^\gamma \bar{V} \\ \quad + h_b(I - \mu \nabla^\gamma \frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)) \nabla^\gamma \zeta = 0. \end{cases} \quad (3.3.41)$$

For the sake of clarity, we will use the following notations:

$$\mathcal{B} = (I + \mu \mathcal{T}_b - \mu \nabla^\gamma(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)) - \mu \frac{1}{h_b} \nabla^{\gamma \perp} \nabla^{\gamma \perp} \cdot), \quad \mathcal{A} = (I - \mu \nabla^\gamma \frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)), \quad q = Q(\zeta)\zeta. \quad (3.3.42)$$

For all $N \in \mathbb{N}$, we define the following space:

$$\mathcal{E}^N = \{(\bar{V}, \zeta) \in H^N(\mathbb{R}^d)^d \times H^N(\mathbb{R}^d) \mid E^N(\bar{V}, \zeta) < \infty\} \quad (3.3.43)$$

where

$$E^N(\zeta, \bar{V}) = |\zeta|_{H^N} + \sqrt{\mu} |\nabla^\gamma \zeta|_{H^N} + |\bar{V}|_{H^N} + \sqrt{\mu} |\nabla^\gamma \bar{V}|_{H^N}. \quad (3.3.44)$$

We denoted $\nabla^\gamma \bar{V}$ the differential of \bar{V} . The space \mathcal{E}^N endowed with the norm E^N is a Banach space. We prove in this Section the following local existence result:

Theorem 3.3.4 *Let $N \in \mathbb{N}$ be such that $N > d/2 + 2$. Let $h_b \in H^{N+1}(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that*

$$\forall X \in \mathbb{R}^d, h_b(X) \geq h_{\min}.$$

Let $(\zeta_0, \bar{V}_0) \in \mathcal{E}^N$. Then, there exists $T^ > 0$ and a unique solution (\bar{V}, ζ) in $C([0; T^*]; \mathcal{E}^N)$ to the equation (3.3.41). Moreover, one has:*

$$\forall T < T^*, \quad \forall \lambda \geq \sup_{t \in [0; T]} C(\frac{1}{h_b}, |h_b|_{H^{N+1}}, |\zeta, \bar{V}|_{W^{2, \infty}}), \quad E^N(t) \leq E^N(0) e^{\lambda t}. \quad (3.3.45)$$

Even if at first sight the proof seems to follow the lines of the proof of local existence for a standard quasilinear hyperbolic system, as done for the Boussinesq-Peregrine equation in Section 3.2, we give here a detailed proof. Indeed, the energy (3.3.44) is defined in terms of ζ, \bar{V} , while the equation (3.3.41) is expressed in terms of unknowns ζ, q, \bar{V} , and the dependence of q with respect to ζ is not trivial. This leads to technical complications that must be handled carefully.

Remark 3.3.5 *In Theorem 3.2.1 which states the local existence for the initial Boussinesq-Peregrine equation, we used fractional order Sobolev spaces to define the energy of solutions, while we use here integer order Sobolev spaces. The reason is to have a coherent notation with the long time existence Theorem of Section 3.3.2, which can only be proved with an integer number of space derivatives, due to the method used.*

Proof As usual, the local existence follows the steps used for the quasilinear hyperbolic systems:

- Step 1 : We solve a smoothed equation involving a mollifier $(1 - \delta \Delta)$.
- Step 2 : We prove that the existence time of the solution of the mollified equation does not depend on δ , and the uniform bound in H^s norm of this solution.
- Step 3 : We pass to the limit δ goes to zero in the mollified equation to get a solution of the equation (3.3.41).
- Step 4 : We recover regularity for the solution.

Step 1 One sets $\delta \in \mathbb{R}$ be such that $0 < \delta < 1$, and considers the following equation:

$$\begin{cases} (I - \delta\Delta)^2 \partial_t q^\delta + \varepsilon \bar{V} \cdot \nabla^\gamma q^\delta + \frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}) = 0 \\ (I - \delta\Delta) h_b \mathcal{B} (I - \delta\Delta) \partial_t \bar{V} + \varepsilon h_b \bar{V} \cdot \nabla^\gamma \bar{V} + h_b \mathcal{A} \nabla^\gamma \zeta^\delta = 0. \end{cases} \quad (3.3.46)$$

We first solve (3.3.46) in the Banach space \mathcal{E}^N defined by (3.3.43). To this purpose, we note that the linear applications $(I + \delta\Delta)^{-1}$ and $(I + \delta\Delta)^{-1} (h_b \mathcal{B})^{-1} (I + \delta\Delta)^{-1}$ are respectively continuous from H^{N-1} to H^N , and from H^{N-3} to H^N , using the following Proposition:

Proposition 3.3.6 *Let $N \geq 0$, $t_0 > d/2$, and $\beta \leq 1$. Let $b \in H^{N+1}(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that $h_b = 1 - \beta b \geq h_{\min}$. The operator*

$$h_b \mathcal{B} : H^{N+2}(\mathbb{R}^d)^d \longrightarrow H^N(\mathbb{R}^d)^d$$

is one-to-one and onto. One has, for all $V \in H^1(\mathbb{R}^d)^d$:

$$C_1(h_{\min}) |V|_{H^1}^2 \leq (h_b \mathcal{B} V, V)_2 \leq C_2(|h_b|_{H^{t_0+1}}) |V|_{H^1}^2,$$

where the C_i are non decreasing functions of their arguments. Moreover, one has, if $b \in H^{1+N \vee t_0}(\mathbb{R}^d)$ and for all $f \in H^N(\mathbb{R}^d)^d$:

$$|(h_b \mathcal{B})^{-1} f|_{H^N} + \sqrt{\mu} |(h_b \mathcal{B})^{-1} f|_{H^{N+1}} + \mu |(h_b \mathcal{B})^{-1} f|_{H^{N+2}} \leq C\left(\frac{1}{h_b}, |h_b|_{H^{1+N \vee t_0}}\right) |f|_{H^N}$$

where C is a non decreasing continuous function of its arguments.

The proof is an easy adaptation of the proof of the invertibility of the operator $I + \mu \mathcal{T}_b$ stated by Proposition 3.2.3 (see [37] Chapter 5 for a full proof). Just note that if $W \in L^2(\mathbb{R}^d)^d$, $(I + \mu \mathcal{T}_b)^{-1} W$ is only controlled with its divergence in $L^2(\mathbb{R}^d)$ norm, while $(h_b \mathcal{B})^{-1} W$ is controlled in a full H^1 norm, due to the presence of the orthogonal gradient ∇^{γ^\perp} in the operator \mathcal{B} (see definition (3.3.42)).

Therefore, using Cauchy-Lipschitz theorem, there exists $T^\delta > 0$ and a unique solution $(\zeta^\delta, \bar{V}^\delta) \in C([0; T^\delta], E^N)$ to the equation (3.3.46) (just replace ζ^δ by $\frac{h_b}{\varepsilon} (e^{\varepsilon q^\delta} - 1)$ in the second equation, to have an ODE in terms of the unknowns $(q^\delta, \bar{V}^\delta)$). Moreover, $T^\delta < \infty$ if and only if

$$\lim_{t \rightarrow T^\delta} |(\zeta^\delta, \bar{V}^\delta)(t)|_{E^N} = +\infty.$$

Step 2 We now want to bound uniformly with respect to δ the energy E^N defined by (3.3.44) of the unknowns. We use the following notation: for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leq N$, for all distribution f ,

$$f_{(\alpha)} = \partial^\alpha f. \quad (3.3.47)$$

One differentiates the equation (3.3.46) to find the following equation in terms of the unknowns $\zeta_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta$ (recall the notation (3.3.47)):

$$\begin{cases} (I - \delta\Delta)^2 \partial_t q_{(\alpha)}^\delta + \varepsilon \bar{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta + \frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}_{(\alpha)}^\delta) = R_1^\alpha \\ (I - \delta\Delta) h_b \mathcal{B} (I - \delta\Delta) \partial_t \bar{V}_{(\alpha)}^\delta + \varepsilon h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}_{(\alpha)}^\delta + h_b \mathcal{A} \nabla^\gamma \zeta_{(\alpha)}^\delta = R_2^\alpha \end{cases} \quad (3.3.48)$$

where

$$R_1^\alpha = -\varepsilon \sum_{0 < \beta \leq \alpha} \partial^\beta \bar{V}^\delta \cdot \nabla^\gamma \partial^{\alpha-\beta} q^\delta - \sum_{0 < \beta + \nu \leq \alpha} \partial^\beta \left(\frac{1}{h_b}\right) \nabla^\gamma \cdot (\partial^\nu (h_b) \partial^{\alpha-\beta-\nu} \bar{V}^\delta)$$

and

$$R_2^\alpha = (I - \delta\Delta) [h_b \mathcal{B}, \partial^\alpha] (I - \delta\Delta) \partial_t \bar{V}^\delta + \varepsilon [h_b \bar{V}^\delta \cdot \nabla^\gamma, \partial^\alpha] \bar{V}^\delta + [h_b \mathcal{A}, \partial^\alpha] \nabla^\gamma \zeta^\delta.$$

As explained at the beginning of this Section, the system (3.3.48) can be made symmetric by multiplying it by the following operator:

$$\begin{pmatrix} h_b - \mu \nabla^\gamma (h_b \nabla^\gamma) & 0 \\ 0 & I \end{pmatrix}.$$

Note that according to Proposition 3.3.6, one has

$$|\zeta^\delta|_{H^N}^2 + \mu|\nabla^\gamma \zeta^\delta|_{H^N}^2 + (h_b \mathcal{B}V, V)_{H^N} \sim E^N(\zeta^\delta, \bar{V})$$

and thus it is equivalent to control the E^N norm of the unknown and the quantity $|\zeta^\delta|_{H^N}^2 + \mu|\nabla^\gamma \zeta^\delta|_{H^N}^2 + (h_b \mathcal{B}V, V)_{H^N}$. Following these considerations, one takes the L^2 scalar product of the first equation of (3.3.48) with $(h_b - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \cdot))\zeta_{(\alpha)}^\delta$ and the scalar product of the second equation with $\bar{V}_{(\alpha)}^\delta$. We obtain the following equality:

$$(T) + (V) + (Z) = (R_\alpha^1, h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta))_2 + (R_\alpha^2, \bar{V}_{(\alpha)}^\delta)_2, \quad (3.3.49)$$

where the time derivatives are

$$(T) = (h_b \mathcal{B}(I - \delta \Delta) \partial_t \bar{V}_{(\alpha)}^\delta, (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta)_2 + ((I - \delta \Delta) \partial_t q_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta))_2, \quad (3.3.50)$$

the vanishing terms are

$$(V) = \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}_{(\alpha)}^\delta), h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta)\right)_2 + (h_b \mathcal{A} \nabla^\gamma \zeta_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2, \quad (3.3.51)$$

and the terms of order zero to the contribution of the energy estimate:

$$(Z) = \varepsilon (h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2 + \varepsilon (\underline{V} \cdot \nabla^\gamma q_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta))_2. \quad (3.3.52)$$

- *Control of the vanishing terms (V)* All has been made to conserve a certain symmetry in the equation, which is crucial here. Using the definition of \mathcal{A} in the expression (3.3.51), one has

$$\begin{aligned} (V) &= (\nabla^\gamma \cdot (h_b \bar{V}_{(\alpha)}^\delta), \zeta_{(\alpha)}^\delta)_2 + (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2 \\ &\quad - \mu \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}_{(\alpha)}^\delta), \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta)\right)_2 - \mu (\nabla^\gamma \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta)\right)_2, h_b \bar{V}_{(\alpha)}^\delta)_2. \end{aligned}$$

By integrating by parts, the first two terms cancel one another, and the last two terms cancel one another. Therefore, (V) actually vanishes.

- *Control of the terms of order zero (Z)* We start to control the easiest term of (3.3.52), which is the first one, by a classical symmetry trick:

$$\begin{aligned} \varepsilon (h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2 &= \varepsilon \sum_{j=1}^d (\bar{V}_j^\delta \partial_j \bar{V}_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2 \\ &= -\varepsilon \sum_{j=1}^d (\bar{V}_{(\alpha)}^\delta, (\partial_j \underline{V}_j) \bar{V}_{(\alpha)}^\delta)_2 - \varepsilon \sum_{j=1}^d (\bar{V}_{(\alpha)}^\delta, \bar{V}_j^\delta \partial_j \bar{V}_{(\alpha)}^\delta)_2 \end{aligned}$$

by integrating by parts, and therefore

$$\varepsilon (h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2 = -\frac{1}{2} \varepsilon \sum_{j=1}^d (\bar{V}_{(\alpha)}^\delta, (\partial_j \bar{V}_j^\delta) \bar{V}_{(\alpha)}^\delta)_2.$$

Using Cauchy-Schwarz inequality, one gets:

$$|\varepsilon (h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}_{(\alpha)}^\delta, \bar{V}_{(\alpha)}^\delta)_2| \leq \frac{1}{2} \varepsilon |\bar{V}^\delta|_{W^{1,\infty}} |\bar{V}_{(\alpha)}^\delta|_2^2. \quad (3.3.53)$$

Note that the symmetry of this term is crucial here. For the other terms of (Z) given in (3.3.52), the symmetry is less clear, since $q_{(\alpha)}^\delta$ is not exactly $\zeta_{(\alpha)}^\delta$. We use the Proposition 3.3.2 to compute:

$$\varepsilon (\bar{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2 = \varepsilon (\bar{V}^\delta \cdot \nabla^\gamma (Q_1(\zeta^\delta, h_b) \zeta_{(\alpha)}^\delta), h_b \zeta_{(\alpha)}^\delta)_2 + \varepsilon^2 (\bar{V}^\delta \cdot \nabla^\gamma (P_\alpha(\zeta^\delta, h_b)), h_b \zeta_{(\alpha)}^\delta)_2. \quad (3.3.54)$$

The second term of the right hand side of (3.3.54) is bounded using Cauchy-Schwarz inequality and Proposition 3.3.2 by

$$\varepsilon^2 C\left(\frac{1}{h_b}, |h_b|_{H^N}\right) |\zeta_{(\alpha)}^\delta|_2^2 |\overline{V}^\delta|_{\mathcal{C}} |h_b|_{\mathcal{C}},$$

where C is a smooth non decreasing function of its arguments. The first term of the right hand side of (3.3.54) is bounded using the same symmetry trick as for the first term of (V):

$$\begin{aligned} \varepsilon(\overline{V}^\delta \cdot \nabla^\gamma(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta), h_b\zeta_{(\alpha)}^\delta)_2 &= \varepsilon \sum_{j=1}^d (\overline{V}_j^\delta \partial_j(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta), h_b\zeta_{(\alpha)}^\delta)_2 \\ &= \varepsilon \sum_{j=1}^d (\overline{V}_j^\delta (\partial_j Q_1(\zeta^\delta, h_b))\zeta_{(\alpha)}^\delta, h_b\zeta_{(\alpha)}^\delta)_2 \\ &\quad - \varepsilon \sum_{j=1}^d (\zeta_{(\alpha)}^\delta, \partial_j(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta)\overline{V}_j^\delta h_b)_2 \\ &\quad - \varepsilon \sum_{j=1}^d (\zeta_{(\alpha)}^\delta, \partial_j(h_b\overline{V}_j^\delta)Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta)_2 \end{aligned}$$

by integrating by parts. Therefore, using Cauchy-Schwarz's inequality, we get the bound:

$$|\varepsilon(\overline{V}^\delta \cdot \nabla^\gamma(Q_1\zeta_{(\alpha)}^\delta), h_b\zeta_{(\alpha)}^\delta)_2| \leq \frac{\varepsilon}{2} |Q_1(\zeta^\delta, h_b)|_{W^{1,\infty}} |h_b|_{W^{1,\infty}} |\overline{V}^\delta|_{W^{1,\infty}} |\zeta_{(\alpha)}^\delta|_2^2$$

and finally, using the definition of Q_1 given by (3.3.39):

$$\varepsilon(\overline{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta, h_b\zeta_{(\alpha)}^\delta)_2 \leq \varepsilon C\left(\frac{1}{h_b}, |\overline{V}^\delta|_{W^{1,\infty}}, |\zeta^\delta|_{W^{1,\infty}}, |h_b|_{H^N}\right) |\zeta_{(\alpha)}^\delta|_2^2. \quad (3.3.55)$$

The third term of the right hand side of the symmetric term (3.3.52) is controlled with a similar technique:

$$\begin{aligned} \mu(\nabla^\gamma(\overline{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2 &= \mu(\nabla^\gamma(\overline{V}^\delta \cdot \nabla^\gamma(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta)), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2 \\ &\quad + \mu(\nabla^\gamma(\overline{V}^\delta \cdot \nabla^\gamma(\varepsilon P_\alpha(\zeta^\delta, h_b))), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2. \end{aligned} \quad (3.3.56)$$

We recall the identity:

$$\nabla^\gamma(A \cdot B) = (A \cdot \nabla^\gamma)B + (B \cdot \nabla^\gamma)A + B\nabla^{\gamma\perp} \cdot A + A\nabla^{\gamma\perp} \cdot B.$$

The first term of the right hand side of (3.3.56) can be expanded using this last identity:

$$\begin{aligned} \mu(\nabla^\gamma(\overline{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2 &= \mu(\overline{V}^\delta \cdot \nabla^\gamma\nabla^\gamma(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2 \\ &\quad + \mu(\overline{V}^\delta\nabla^{\gamma\perp} \cdot \nabla^\gamma(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2 \\ &\quad + \mu(\nabla^\gamma(Q_1(\zeta^\delta, h_b)\zeta_{(\alpha)}^\delta) \cdot \nabla^\gamma\overline{V}^\delta, h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2. \end{aligned} \quad (3.3.57)$$

The first term of (3.3.57) is a symmetric term, controlled by the same technique as before. The second term vanishes, and the last one with the last term of the right hand side of (3.3.56) are easily controlled, and one gets:

$$|\mu(\nabla^\gamma(\overline{V}^\delta \cdot \nabla^\gamma q_{(\alpha)}^\delta), h_b\nabla^\gamma\zeta_{(\alpha)}^\delta)_2| \leq \mu C(|h_b|_{H^{N+1}}, |\overline{V}^\delta|_{W^{2,\infty}}, |\zeta^\delta|_{W^{2,\infty}}, \frac{1}{h_b}) (|\nabla^\gamma\zeta_{(\alpha)}^\delta|_2 + |\zeta_{(\alpha)}^\delta|_2) |\nabla^\gamma\zeta_{(\alpha)}^\delta|_2. \quad (3.3.58)$$

To conclude, putting together (3.3.53), (3.3.55) and (3.3.58), we proved that

$$|(Z)| \leq \varepsilon C(|h_b|_{H^{N+1}}, |\overline{V}^\delta|_{W^{2,\infty}}, |\zeta^\delta|_{W^{2,\infty}}, \frac{1}{h_b}) E^N(\zeta^\delta, \overline{V}^\delta). \quad (3.3.59)$$

- Control of the time derivatives (T)

The terms of (T) involve time derivatives, and should be, up to terms controlled by the energy E^N , the time derivatives of the energy E^N . The first term of (3.3.50) is already symmetric, using the symmetry of $h_b \mathcal{B}$ (which is crucial here):

$$(h_b \mathcal{B}(I - \delta \Delta) \partial_t \bar{V}_{(\alpha)}^\delta, (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta)_2 = \partial_t \frac{1}{2} (h_b \mathcal{B}(I - \delta \Delta) \bar{V}_{(\alpha)}^\delta, (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta)_2. \quad (3.3.60)$$

For the second term of (3.3.50), we use again Proposition 3.3.2 to write:

$$((I - \delta \Delta)^2 \partial_t q_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2 = ((I - \delta \Delta)^2 \partial_t (Q_1 \zeta_{(\alpha)}^\delta), h_b \zeta_{(\alpha)}^\delta)_2 + ((I - \delta \Delta)^2 \partial_t (\varepsilon P_\alpha \zeta^\delta), h_b \zeta_{(\alpha)}^\delta)_2. \quad (3.3.61)$$

For the first term of the right hand side of (3.3.61), one computes:

$$\begin{aligned} ((I - \delta \Delta)^2 \partial_t (Q_1 \zeta_{(\alpha)}^\delta), h_b \zeta_{(\alpha)}^\delta)_2 &= ((I - \delta \Delta)^2 \partial_t (Q_1) \zeta_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2 + ((I - \delta \Delta)^2 Q_1 \partial_t \zeta_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2 \\ &= ((I - \delta \Delta)^2 \partial_t (Q_1) \zeta_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2 \\ &\quad + ([Q_1, -\delta \Delta] \partial_t \zeta_{(\alpha)}^\delta, (I - \delta \Delta) h_b \zeta_{(\alpha)}^\delta)_2 \\ &\quad + (Q_1 (I - \delta \Delta) \partial_t \zeta_{(\alpha)}^\delta, [-\delta \Delta, h_b] \zeta_{(\alpha)}^\delta)_2 \\ &\quad + \partial_t \frac{1}{2} (Q_1 (I - \delta \Delta) \zeta_{(\alpha)}^\delta, h_b (I - \delta \Delta) \zeta_{(\alpha)}^\delta)_2. \end{aligned} \quad (3.3.62)$$

All these computations are made to obtain the time derivative of a symmetric term with respect to $\zeta_{(\alpha)}^\delta$. The first term of the right hand side of (3.3.62) is easily controlled by

$$|((I - \delta \Delta)^2 \partial_t (Q_1) \zeta_{(\alpha)}^\delta, h_b \zeta_{(\alpha)}^\delta)_2| \leq |\partial_t Q_1|_{W^{1,\infty}} |(I - \delta \Delta) \zeta_{(\alpha)}^\delta|_2^2 |h_b|_{W^{2,\infty}}.$$

In order to control the second term of the right hand side of (3.3.62), we replace $\partial_t \zeta_{(\alpha)}^\delta$ by its expression given by the equation:

$$\partial_t \zeta_{(\alpha)}^\delta = -\partial^\alpha (\nabla^\gamma \cdot (h^\delta \bar{V}^\delta))$$

and we notice that

$$|\delta (I - \delta \Delta)^{-2} [\Delta, Q_1] \partial^\alpha \nabla^\gamma \cdot u|_2 \leq C(|Q_1|_{W^{2,\infty}}) |\partial^\alpha u|_2$$

for all u in H^N . Therefore, one gets:

$$((I - \delta \Delta)^2 \partial_t (Q_1 \zeta_{(\alpha)}^\delta), h_b \zeta_{(\alpha)}^\delta)_2 = \partial_t \frac{1}{2} (Q_1 (I - \delta \Delta) \zeta_{(\alpha)}^\delta, h_b (I - \delta \Delta) \zeta_{(\alpha)}^\delta)_2 + R \quad (3.3.63)$$

with

$$|R| \leq C(|h_b|_{H^N}, \frac{1}{h_b}, |\zeta^\delta|_{W^{2,\infty}}, |\bar{V}^\delta|_{W^{2,\infty}}) (|(I - \delta \Delta) \zeta_{(\alpha)}^\delta|_2 + |\bar{V}_{(\alpha)}^\delta|_2) |(I - \delta \Delta) \zeta_{(\alpha)}^\delta|_2.$$

The same technique can be used for the control of the second term of (3.3.61) and one gets finally, combining (3.3.60) and (3.3.63):

$$(T) = \partial_t \frac{1}{2} (h_b \mathcal{B}(I - \delta \Delta) \bar{V}_{(\alpha)}^\delta, (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta)_2 + \partial_t \frac{1}{2} (Q_1 (I - \delta \Delta) \zeta_{(\alpha)}^\delta, h_b (I - \delta \Delta) \zeta_{(\alpha)}^\delta)_2 + R \quad (3.3.64)$$

with

$$|R| \leq \mu C(|h_b|_{H^N}, \frac{1}{h_b}, |\zeta^\delta|_{W^{2,\infty}}, |\bar{V}^\delta|_{W^{2,\infty}}) E^N ((I - \delta \Delta) \zeta^\delta, (I - \delta \Delta) \bar{V}^\delta). \quad (3.3.65)$$

- *Control of the residual terms* We now control the terms involving the residuals that appear in (3.3.49). One has:

$$R_1^\alpha = -\varepsilon \sum_{0 < \beta \leq \alpha} \partial^\beta \bar{V}^\delta \cdot \nabla^\gamma \partial^{\alpha-\beta} q^\delta - \sum_{0 < \beta + \nu \leq \alpha} \partial^\beta (\frac{1}{h_b}) \nabla^\gamma \cdot (\partial^\nu (h_b) \partial^{\alpha-\beta-\nu} \bar{V}^\delta)$$

and thus one has, using a Kato-Ponce type estimate (of the form of Proposition 3.A.1):

$$\begin{aligned} |R_1^\alpha|_2 + \sqrt{|\mu|} |\nabla^\gamma R_1^\alpha|_2 &\leq C(\frac{1}{h_b}, |h_b|_{H^{N+1}}) (|\zeta^\delta, \bar{V}^\delta|_{W^{2,\infty}} \times \\ &\quad (|\zeta_{(\alpha)}^\delta|_2 + \sqrt{|\mu|} |\nabla^\gamma \zeta_{(\alpha)}^\delta|_2 + |\bar{V}_{(\alpha)}^\delta|_2 + \sqrt{|\mu|} |\nabla^\gamma \bar{V}_{(\alpha)}^\delta|_2)). \end{aligned} \quad (3.3.66)$$

It is very important to have $\nabla^\gamma \cdot (h_b \bar{V}_{(\alpha)}^\delta)$ instead of $\nabla^\gamma \cdot (h \bar{V}_{(\alpha)}^\delta)$ in the equation (3.3.48), because the term $\nabla^\gamma \cdot (\varepsilon \zeta_{(\alpha)}^\delta \bar{V}_{(\alpha)}^\delta)$ would not be properly symmetrized and thus would not be controlled by the energy. One has easily, integrating by parts and using Cauchy-Schwarz inequality:

$$|(R_1^\alpha, h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta))_2| \leq |R_1^\alpha|_2 |h_b|_\infty |\zeta_{(\alpha)}^\delta|_2 + \mu |\nabla^\gamma R_1^\alpha|_2 |h_b|_\infty |\nabla^\gamma \zeta_{(\alpha)}^\delta|_2 \quad (3.3.67)$$

and thus, using (3.3.66), one gets:

$$|(R_1^\alpha, h_b \zeta_{(\alpha)}^\delta - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_{(\alpha)}^\delta))_2| \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}}, |(\zeta^\delta, \bar{V}^\delta)|_{W^{2,\infty}} \right) (E^N)^2. \quad (3.3.68)$$

Recall that

$$R_2^\alpha = (I - \delta \Delta)[h_b \mathcal{B}, \partial^\alpha](I - \delta \Delta) \partial_t \bar{V}^\delta + \varepsilon [h_b \bar{V}^\delta \cdot \nabla^\gamma, \partial^\alpha] \bar{V}^\delta + [h_b \mathcal{A}, \partial^\alpha] \nabla^\gamma \zeta^\delta \quad (3.3.69)$$

To control the first term of (3.3.69), as usual one replaces $(I - \delta \Delta) \partial_t \bar{V}^\delta$ by its expression given by the equation (3.3.48), and uses the definition of \mathcal{T}_b given by (3.2.12):

$$\begin{aligned} & (I - \delta \Delta)[h_b \mathcal{B}, \partial^\alpha](I - \delta \Delta) \partial_t \bar{V}^\delta \\ &= -(I - \delta \Delta)[h_b \mu \mathcal{T}_b, \partial^\alpha](h_b \mathcal{B})^{-1} (I - \delta \Delta)^{-1} (\varepsilon h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}^\delta + h_b \mathcal{A} \nabla^\gamma \zeta^\delta). \end{aligned} \quad (3.3.70)$$

One has:

$$\forall k \geq 2, \quad \mu |(h_b \mathcal{B})^{-1} u|_{H^k} \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) |u|_{H^{k-2}} \quad (3.3.71)$$

using Proposition 3.3.6, for C a smooth non decreasing function of its arguments. One has also:

$$\forall k \geq 0, \quad |(I - \delta \Delta) u|_{H^k} \leq C (|u|_{H^k} + \delta |u|_{H^{k+2}}) \quad (3.3.72)$$

with C independent on δ , and:

$$\forall k \geq 0, \quad \frac{1}{\delta} |(I - \delta \Delta)^{-1} u|_{H^{k+2}} + |(I - \delta \Delta)^{-1} u|_{H^k} \leq |u|_{H^k}. \quad (3.3.73)$$

Using the definition of \mathcal{T}_b (see (3.2.12)), one has:

$$\forall k \geq 0, \quad \mu |[h_b \mathcal{T}_b, \partial^\alpha] u| \leq \mu C (|h_b|_{H^{N+1}}) |u|_{H^{k+\alpha+1}}. \quad (3.3.74)$$

Using successively the identities (3.3.72), (3.3.74), (3.3.71) and (3.3.73), the first term of the right hand side of (3.3.1) is bounded by:

$$|(I - \delta \Delta)[h_b \mu \mathcal{T}_b, \partial^\alpha](h_b \mathcal{B})^{-1} (I - \delta \Delta)^{-1} (\varepsilon h_b \bar{V}^\delta \cdot \nabla^\gamma \bar{V}^\delta)|_2 \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) |\bar{V}_{(\alpha)}^\delta|_2.$$

One has to be more careful for the second term of the right hand side of (3.3.1), because the expression $\mathcal{A} \nabla^\gamma \zeta^\delta$ is of order 2 in $\mu \nabla^\gamma \zeta^\delta$. One writes:

$$\begin{aligned} & |((I - \delta \Delta)[h_b \mu \mathcal{T}_b, \partial^\alpha](h_b \mathcal{B})^{-1} (I - \delta \Delta)^{-1} (h_b \mathcal{A} \nabla^\gamma \zeta^\delta), \bar{V}_{(\alpha)}^\delta)_2| \leq \\ & \sqrt{\mu} |((I - \delta \Delta)[h_b \mu \mathcal{T}_b, \partial^\alpha](h_b \mathcal{B})^{-1} (I - \delta \Delta)^{-1} (h_b \mathcal{A} \nabla^\gamma \zeta^\delta))_{H^{-1}} \sqrt{\mu} |\bar{V}_{(\alpha)}^\delta|_{H^1} \end{aligned}$$

and we use the same controls (3.3.72), (3.3.73), (3.3.71), (3.3.1) as before. Finally, one gets:

$$|(R_2^\alpha, \bar{V}_{(\alpha)}^\delta)| \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) (|\bar{V}_{(\alpha)}^\delta|_2^2 + \mu |\nabla^\gamma \bar{V}_{(\alpha)}^\delta|_2 |\nabla^\gamma \zeta_{(\alpha)}^\delta|_2). \quad (3.3.75)$$

- *Conclusion*

Putting together (3.3.59), (3.3.64), (3.3.65), (3.3.67) and (3.3.75), one gets:

$$\begin{aligned} & \partial_t \frac{1}{2} (h_b \mathcal{B} (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta, (I - \delta \Delta) \bar{V}_{(\alpha)}^\delta)_2 + \partial_t \frac{1}{2} (Q_1 (I - \delta \Delta) \zeta_{(\alpha)}^\delta, h_b (I - \delta \Delta) \zeta_{(\alpha)}^\delta)_2 \\ & \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}}, |(\zeta^\delta, \bar{V}^\delta)|_{W^{2,\infty}} \right) \times E^N ((I - \delta \Delta) \zeta^\delta, (I - \delta \Delta) \bar{V}^\delta). \end{aligned}$$

We recall that using Proposition 3.3.6, one has

$$(h_b \mathcal{B}V, V)_2 \sim |V|_2^2 + \mu |\nabla^\gamma V|_2^2$$

and using Proposition 3.3.2, one has $|Q_1|_\infty \geq C(\frac{1}{h_b})$. Therefore, we obtained:

$$\partial_t E^N((I - \delta\Delta)\zeta^\delta, (I - \delta\Delta)\bar{V}^\delta) \leq C(\frac{1}{h_b}, |h_b|_{H^{N+1}}, |\zeta^\delta, \bar{V}^\delta|_{W^{2,\infty}}) E^N((I - \delta\Delta)\zeta^\delta, (I - \delta\Delta)\bar{V}^\delta) \quad (3.3.76)$$

where C is a non decreasing continuous function of its arguments, independent on δ . Therefore, using Gronwall's Lemma, T^δ does not depends on δ .

Step 3-4 The rest of the proof is exactly the same as for the local existence of the standard Boussinesq-Peregrine equation, and one gets a solution to (3.3.46) on a time interval $[0; T^*]$. One gets however from (3.3.76) that:

$$\forall T < T^*, \quad \forall \lambda \geq \sup_{t \in [0; T]} C(\frac{1}{h_b}, |h_b|_{H^{N+1}}, |\zeta, \bar{V}|_{W^{2,\infty}}), \quad E^N(t) \leq E^N(0)e^{\lambda t}.$$

3.3.2 Long time existence in dimension 1 for the modified Boussinesq-Peregrine equation

We now make the scaling

$$t' = \varepsilon t$$

on the equation (3.3.41), and we obtain the equation (we get rid of the "primes" in the notation t' for the sake of clarity):

$$\begin{cases} \partial_t q + \bar{V} \cdot \nabla^\gamma q + \frac{1}{\varepsilon} \frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}) = 0 \\ h_b \mathcal{B} \partial_t \bar{V} + h_b \bar{V} \cdot \nabla^\gamma \bar{V} + \frac{1}{\varepsilon} h_b \mathcal{A} \nabla^\gamma \zeta = 0. \end{cases} \quad (3.3.77)$$

This change of variable is not necessary on a mathematical point of view, but it allows to highlight the singular terms that must be canceled in the energy estimates in order to prove the result, which are the large terms of size $\frac{1}{\varepsilon}$. Moreover, it allows the equation (3.3.77) to be seen as a singular perturbation problem. Note that a time existence of size $\frac{1}{\varepsilon}$ for the equation (3.3.41) is equivalent to a time existence independent on ε for (3.3.77).

We recall that for all $N \in \mathbb{N}$, we define the following space:

$$\mathcal{E}^N = \{(\bar{V}, \zeta) \in H^N(\mathbb{R}^d)^d \times H^N(\mathbb{R}^d) \mid E^N(\bar{V}, \zeta) < \infty\}$$

where

$$E^N(\zeta, \bar{V}) = |\zeta|_{H^N} + \sqrt{\mu} |\nabla^\gamma \zeta|_{H^N} + |\bar{V}|_{H^N} + \sqrt{\mu} |\nabla^\gamma \bar{V}|_{H^N}.$$

We prove in this section the following result:

Theorem 3.3.7 *Let $d = 1$. Let $N \in \mathbb{N}$ be such that $N > d/2 + 2$. Let $h_b \in H^{N+1}(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that*

$$\forall X \in \mathbb{R}^d, h_b(X) \geq h_{\min}.$$

Let $(\zeta_0, \bar{V}_0) \in \mathcal{E}^N$. Then, there exists $T > 0$ and a unique solution (\bar{V}, ζ) in $C([0; T]; \mathcal{E}^N)$ to the equation (3.3.77), with

$$T = C_1(E^N(\zeta_0, \bar{V}_0), \frac{1}{h_b}, |h_b|_{H^{N+1}}),$$

where C_1 is a non decreasing continuous function of its arguments.

In particular, the time of existence does not depend on ε, μ .

Remark 3.3.8 *It is very important to note that $d = 1$ here. In $d = 2$, there is an extra difficulty due to the need of a good estimate for $\nabla^{\gamma \perp} \cdot \bar{V}$. However, since this is the only difficulty that could prevent a similar result in dimension $d = 2$ to hold, we keep the notations of the multidimensional equation, and we specifically highlight at the end of the proof the difficulty that one must overcome to prove the result in dimension 2.*

Let us consider (\bar{V}, ζ) the unique solution of (3.3.77) given by Theorem 3.3.4 on a time interval $[0; T^\varepsilon]$. We set

$$K = \sup_{t \in [0; T^\varepsilon]} E^N(\zeta, \bar{V}).$$

We use the notation

$$u_k = (\varepsilon \partial_t)^k u$$

for all distribution u (thus u_k corresponds to the time derivative of u in the original time variables).

The idea of the proof is to obtain a "good" energy estimate of the form

$$E(t) \leq C(K)(t + \varepsilon) + C_0,$$

where C is non decreasing and smooth, and where C_0 only depends on the initial data. Such estimate would allow us to get by a continuity argument a time existence uniform with respect to ε . There are two main ideas in the proof:

- The system is still symmetric with respect to singular terms if we differentiate it with respect to time. It allows us to get the "good estimate" for the time derivatives ζ_k, \bar{V}_k .
- Using the equation, one can control the space derivatives by the time derivatives, and recover the "good estimate" for the full energy E^N of the solutions.

The following Proposition states that the time derivatives of the solutions (\bar{V}, ζ) have the same regularity as the space derivatives:

Proposition 3.3.9 *One has, for all $0 \leq k \leq N$,*

$$|(\bar{V}_k, \zeta_k)|_{H^{N-k}} + \sqrt{\mu} |(\bar{V}_k, \zeta_k)|_{H^{N-k+1}} \leq C(K),$$

where C is a smooth, non decreasing function of its argument.

Proof For $k = 0$, it is clear. Suppose it is true for $k \geq 0$. One commutes $(\varepsilon \partial_t)^k$ with the equation (3.3.77). One gets, since $\partial_t h_b = 0$:

$$\begin{cases} q_{k+1} + \varepsilon \sum_{j=0}^k \bar{V}_j \cdot \nabla^\gamma q_{k-j} + \nabla^\gamma \cdot (h_b \bar{V}_k) = 0 \\ \bar{V}_{k+1} = -(h_b \mathcal{B})^{-1} (\varepsilon h_b \sum_{j=0}^k \bar{V}_j \cdot \nabla^\gamma \bar{V}_{k-j} + h_b \mathcal{A} \nabla^\gamma \zeta_k). \end{cases}$$

We only prove the most difficult estimate which is the following, in order to prove that the induction hypothesis is true at rank $k + 1$:

$$|(h_b \mathcal{B})^{-1} (h_b \mathcal{A} \nabla^\gamma \zeta_k)|_{H^{N-k-1}} \leq C(K).$$

We recall that

$$\mathcal{A} = I - \mu \nabla^\gamma \frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)$$

and therefore, using Proposition (3.3.6):

$$\begin{aligned} |(h_b \mathcal{B})^{-1} (h_b \mathcal{A} \nabla^\gamma \zeta_k)|_{H^{N-k-1}} &\leq |(h_b \mathcal{B})^{-1} (h_b \nabla^\gamma \zeta_k)|_{H^{N-k-1}} + \mu |(h_b \mathcal{B})^{-1} (\nabla^\gamma \frac{1}{h_b} \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_k))|_{H^{N-k-1}} \\ &\leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right) |\zeta_k|_{H^{N-k}} \end{aligned}$$

and one gets the desired control by using the induction hypothesis. The other controls are done similarly, using Proposition 3.3.6, and the relation between q_k and ζ_k given by Proposition 3.3.2 and Remark 3.3.3. \square

The key point of the proof of Theorem 3.3.7 is the following Lemma, which states a "good estimate" for the unknowns:

Lemma 3.3.10 *One has*

$$E^N(\zeta, \bar{V}) \leq C(K)(t + \varepsilon) + C_0$$

where C is a non decreasing function of its arguments, and C_0 is a constant which only depends on the initial data.

Proof There are two ideas in the proof of this lemma:

- the time derivatives of the unknowns satisfy a system which is still symmetric with respect to singular terms of size $\frac{1}{\varepsilon}$;
- the space derivatives are related to time derivatives by the equation.

The unknowns (ζ_k, \bar{V}_k) satisfy the following equation:

$$\begin{cases} \partial_t q_k + \bar{V} \cdot \nabla^\gamma q_k + \frac{1}{\varepsilon} \frac{1}{h_b} \nabla^\gamma \cdot (h_b \bar{V}_k) = R_k^1 \\ h_b \mathcal{B} \partial_t \bar{V}_k + h_b \bar{V} \cdot \nabla^\gamma \bar{V}_k + \frac{1}{\varepsilon} h_b \mathcal{A} \nabla^\gamma \zeta_k = R_k^2 \end{cases} \quad (3.3.78)$$

where

$$R_k^1 = [\bar{V}, (\varepsilon \partial_t)^k] q, \quad R_k^2 = [\bar{V} \cdot \nabla^\gamma, (\varepsilon \partial_t)^k] \bar{V}. \quad (3.3.79)$$

The symmetry with respect to large terms of size $\frac{1}{\varepsilon}$ is conserved, which allows to get the following result:

Lemma 3.3.11 *One has, for all $0 \leq k \leq N$,*

$$E^0(\zeta_k, \bar{V}_k) \leq C(K)t + C_0.$$

Proof The equation (3.3.78) is still symmetric with respect to large terms of size $\frac{1}{\varepsilon}$. More precisely, if one multiplies the first equation by $h_b \zeta_k - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_k)$, one finds exactly as in the proof of Theorem 3.3.4 an expression of the form: $(T) + (V) + (Z) = (R_k^1, h_b \zeta_k - \mu \nabla^\gamma \cdot (h_b \nabla^\gamma \zeta_k))_2 + (R_k^2, \bar{V}_k)_2$ with exactly the same terms for (T) , (V) and (Z) as in (3.3.50), (3.3.52) with $(\zeta_{(\alpha)}, \bar{V}_{(\alpha)}^\delta)$ replaced by (ζ_k, \bar{V}_k) . The vanishing terms are exactly ones of size $\frac{1}{\varepsilon}$ and the others are controlled exactly with the same techniques, using Proposition 3.3.9 for the regularity of the time derivatives. \square

Now, we recover the "good estimate" of Lemma 3.3.10 for the space derivatives of the unknowns, using the equation.

Lemma 3.3.12 *One has, for all $0 \leq k \leq N$,*

$$E^{N-k}(\zeta_k, \bar{V}_k) \leq C(K)(t + \varepsilon) + C_0.$$

Proof We prove it by backward finite induction on k . For $k = N$, it is Lemma 3.3.11. Suppose it is true for $k + 1$ with $k \leq N - 1$. Let us prove it is true for k .

$$\nabla^\gamma \zeta_k = -(h_b \mathcal{A})^{-1} (h_b \mathcal{B} \bar{V}_{k+1} + \varepsilon h_b \bar{V} \cdot \nabla^\gamma \bar{V}_k - \varepsilon R_k^2). \quad (3.3.80)$$

Recall that the operator \mathcal{A} is given by

$$\mathcal{A} = (I - \mu \nabla^\gamma \frac{1}{h_b} \nabla^\gamma \cdot (h_b \cdot)).$$

We also recall that we defined $|f|_{X^N}$ for $f \in L^2(\mathbb{R}^d)^d$ by:

$$|f|_{X^N}^2 = |f|_{H^N}^2 + \mu |\nabla^\gamma \cdot f|_{H^N}^2.$$

We used the following Proposition that states the invertibility of $h_b \mathcal{A}$ to derive the equality (3.3.80):

Proposition 3.3.13 *Let $N \in \mathbb{N}$ and let $h_b \in H^N(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that*

$$\forall X \in \mathbb{R}^d, \quad h_b(X) \geq h_{\min}.$$

We set, for all $f \in L^2(\mathbb{R}^d)$:

$$|f|_{X^N}^2 = |f|_{H^N}^2 + \mu |\nabla^\gamma \cdot f|_{H^N}^2.$$

The operator $h_b \mathcal{A}$ is invertible on $H^N(\mathbb{R}^d)^d$. Moreover, the following estimates stand.

(1) *For all $f \in H^N(\mathbb{R}^d)^d$,*

$$|(h_b \mathcal{A})^{-1} f|_{X^N} \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) |f|_{H^N}.$$

(2) For all $g \in H^N(\mathbb{R}^d)$,

$$\sqrt{\mu}|(h_b\mathcal{A})^{-1}\nabla^\gamma g|_{X^N} \leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right)|g|_{H^N}.$$

We postpone the proof of Proposition 3.3.13 to Appendix 3.B for the sake of clarity. Now, in order to use the relation (3.3.80), one takes the H^{N-k-1} scalar product of (3.3.80) with $h_b(I - \mu\nabla^\gamma\nabla^\gamma)\nabla^\gamma\zeta$, and gets, using the notations of Proposition 3.3.13 :

$$|\nabla^\gamma\zeta_k|_{X^{N-k-1}}^2 = -(h_b(I - \mu\nabla^\gamma\nabla^\gamma)\nabla^\gamma\zeta_k, (h_b\mathcal{A})^{-1}(h_b\mathcal{B}\bar{V}_{k+1} + \varepsilon h_b\bar{V} \cdot \nabla^\gamma\bar{V}_k - \varepsilon R_k^2))_{H^{N-k-1}}. \quad (3.3.81)$$

Now, one has by definition of R_k^2 given by (3.3.79) :

$$h_b\bar{V} \cdot \nabla^\gamma\bar{V}_k - R_k^2 = (\varepsilon_t)^k(\bar{V} \cdot \nabla^\gamma\bar{V})$$

and thus this term is sum of terms of the form

$$\bar{V}_l \cdot \nabla^\gamma\bar{V}_{k-l},$$

with $0 \leq l \leq k$ and therefore one has, using Proposition 3.3.13:

$$\begin{aligned} |(h_b\mathcal{A})^{-1}(h_b\bar{V} \cdot \nabla^\gamma\bar{V}_k - R_k^2)|_{X^{N-k-1}} &\leq C\left(\frac{1}{h_{\min}}, |h_b|_{H^{N+1}}\right)|h_b\bar{V} \cdot \nabla^\gamma\bar{V}_k - R_k^2|_{H^{N-k-1}} \\ &\leq C(K). \end{aligned} \quad (3.3.82)$$

We now focus on the control of $((h_b\mathcal{A})^{-1}(h_b\mathcal{B})\bar{V}_{k+1}, h_b(I - \mu\nabla^\gamma\nabla^\gamma)\nabla^\gamma\zeta_k)_{H^{N-k-1}}$. Recall that

$$\mathcal{B} = (I + \mu\mathcal{T}_b - \mu\nabla^\gamma\left(\frac{1}{h_b}\nabla^\gamma \cdot (h_b\cdot)\right) - \mu\frac{1}{h_b}\nabla^\gamma\perp\nabla^\gamma\perp), \quad \mathcal{A} = (I - \mu\nabla^\gamma\frac{1}{h_b}\nabla^\gamma \cdot (h_b\cdot)).$$

Lemma 3.3.14 *One has, for all $V, W \in H^{k+1}(\mathbb{R}^d)^d$, all $0 \leq k \leq N$:*

$$((h_b\mathcal{A})^{-1}h_b\mathcal{B}V, h_b\nabla^\gamma W)_{H^k} \leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right)\sqrt{\mu}|\nabla^\gamma V|_{H^k}|\nabla^\gamma W|_{H^k}.$$

Remark 3.3.15 *This Lemma states that even if $(h_b\mathcal{A})$ is not elliptic (it is essentially $I - \mu\nabla^\gamma\nabla^\gamma$ with variable coefficients), its inverse allows to recover a full derivative if it is applied to a gradient. The quantity $h_b\mathcal{B}$ is essentially composed of gradients, except for the term $\nabla^\gamma\perp\nabla^\gamma\perp$ which vanishes in any scalar product with a gradient.*

Proof We only give the control of the most difficult terms of the quantity to be controlled, which are:

$$\mu((h_b\mathcal{A})^{-1}(\nabla^\gamma(h_b^3\nabla^\gamma \cdot V)), h_b\nabla^\gamma W)_{H^k}, \quad \mu((h_b\mathcal{A})^{-1}(\nabla^\gamma\perp\nabla^\gamma\perp \cdot V), h_b\nabla^\gamma W)_{H^k} \quad (3.3.83)$$

(the other terms from $h_b\mathcal{B}$ are controlled even more easily by similar techniques). For the first term of (3.3.83), one computes:

$$\begin{aligned} \mu((h_b\mathcal{A})^{-1}(\nabla^\gamma(h_b^3\nabla^\gamma \cdot V)), h_b\nabla^\gamma W)_{H^k} &\leq \mu|(h_b\mathcal{A})^{-1}\nabla^\gamma(h_b^3\nabla^\gamma \cdot V)|_{H^k}|h_b\nabla^\gamma W|_{H^k} \\ &\leq \sqrt{\mu}C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right)|h_b^3\nabla^\gamma \cdot V|_{H^k}|\nabla^\gamma W|_{H^k} \end{aligned}$$

where we used Proposition (3.3.13) to derive the last inequality. Finally, one gets:

$$\mu((h_b\mathcal{A})^{-1}(\nabla^\gamma(h_b^3\nabla^\gamma \cdot V)), h_b\nabla^\gamma W)_{H^k} \leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right)\sqrt{\mu}|\nabla^\gamma V|_{H^k}|\nabla^\gamma W|_{H^k}. \quad (3.3.84)$$

For the second term of (3.3.83), one computes, integrating by parts:

$$\begin{aligned} &\mu((h_b\mathcal{A})^{-1}(\nabla^\gamma\perp\nabla^\gamma\perp \cdot V), h_b\nabla^\gamma W)_{H^k} \\ &= -\mu(\nabla^\gamma \cdot h_b(h_b\mathcal{A})^{-1}(\nabla^\gamma\perp\nabla^\gamma\perp \cdot V), W)_{H^k} \\ &+ \mu(\Lambda^{k-1}(h_b\mathcal{A})^{-1}(\nabla^\gamma\perp\nabla^\gamma\perp \cdot V), \Lambda[h_b, \Lambda^k]\nabla^\gamma W)_2 \\ &+ \mu([\Lambda^k, \Lambda^k](h_b\mathcal{A})^{-1}(\nabla^\gamma\perp\nabla^\gamma\perp \cdot V), \Lambda^k\nabla^\gamma W)_2, \end{aligned} \quad (3.3.85)$$

where we recall that $\Lambda = (1 + |D^\gamma|^2)^{1/2}$. One has to notice that for $f \in H^1(\mathbb{R}^d)$, $u = (h_b \mathcal{A})^{-1}(h_b \nabla^\gamma f)$ is a term of the form $\nabla^\gamma g$, since $u = \nabla^\gamma(\frac{1}{h_b} \nabla^\gamma \cdot (h_b u)) + \nabla^\gamma f$, using the definition of \mathcal{A} given by (3.3.42). Therefore, $\nabla^{\gamma \perp} \cdot (h_b \mathcal{A})^{-1}(h_b \nabla^\gamma f) = 0$ for all f , and by duality $\nabla^\gamma \cdot (h_b (h_b \mathcal{A})^{-1} \nabla^{\gamma \perp} w) = 0$ for all w . The first term of the rhs of term (3.3.85) is therefore zero.

For the second term of the rhs of (3.3.85), one easily proves that, for all $f \in H^k(\mathbb{R}^d)$, using the Kato-Ponce estimate of Proposition 3.A.1:

$$|\Lambda[h_b, \Lambda^k]f|_2 \leq C(|h_b|_{H^{N+1}})|f|_{H^k},$$

and thus the second term of the rhs of (3.3.85) is bounded by

$$\mu C(|h_b|_{H^{N+1}})|(h_b \mathcal{A})^{-1}(\nabla^{\gamma \perp} \nabla^{\gamma \perp} \cdot V)|_{H^{k-1}} |\nabla^\gamma W|_{H^k}$$

and using Proposition (3.3.13), one gets the bound

$$\mu |(\Lambda^{k-1}(h_b \mathcal{A})^{-1}(\nabla^{\gamma \perp} \nabla^{\gamma \perp} \cdot V), \Lambda[h_b, \Lambda^k] \nabla^\gamma \nabla^\gamma W)|_2 \leq \mu C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right) |\nabla^\gamma V|_{H^k} |\nabla^\gamma W|_{H^k}. \quad (3.3.86)$$

The third term of (3.3.85) is controlled similarly with the same bound as (3.3.86). Putting together (3.3.84) and (3.3.86), one gets the Lemma. \square

We can now apply Lemma 3.3.84 to get immediately (note that $|\nabla^\gamma \cdot \nabla \zeta|_2 \sim |\nabla^\gamma \nabla^\gamma \zeta|_2$):

$$\begin{aligned} |((h_b \mathcal{A})^{-1}(h_b \mathcal{B}) \bar{V}_{k+1}, h_b(I - \mu \nabla^\gamma \nabla^\gamma \cdot) \nabla^\gamma \zeta_k)_{H^{N-k-1}}| &\leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right) (\sqrt{\mu} |\nabla^\gamma \bar{V}_{k+1}|_{H^{N-k-1}} |\nabla^\gamma \zeta_k|_{H^{N-k-1}} + \sqrt{\mu} |\nabla^\gamma \bar{V}_{k+1}|_{H^{N-k-1}} \sqrt{\mu} |\nabla^\gamma \cdot \nabla^\gamma \zeta_k|_{H^{N-k-1}}) \\ &\leq C\left(\frac{1}{h_b}, |h_b|_{H^{N+1}}\right) \times \\ &\quad (C(K)(t + \varepsilon) + C_0) |\nabla^\gamma \zeta_k|_{X^{N-k-1}} \end{aligned} \quad (3.3.87)$$

using the notations of Proposition (3.3.13), and using the induction hypothesis. Putting (3.3.82) and (3.3.87) into (3.3.80), one gets:

$$|\nabla^\gamma \zeta_k|_{X^{N-k-1}}^2 \leq C(K)\varepsilon + (C(K)(t + \varepsilon) + C_0) |\nabla^\gamma \zeta_k|_{X^{N-k-1}}.$$

By noticing that, for all u smooth enough:

$$|u|_{X^{N-k}} \leq |u|_{X^{N-k-1}} + |\nabla^\gamma u|_{X^{N-k-1}},$$

one finally recovers the "good estimate" k for ζ_k :

$$|\zeta_k|_{H^{N-k}} + \sqrt{\mu} |\nabla^\gamma \zeta_k|_{H^{N-k}} \leq C(K)(t + \varepsilon) + C_0.$$

Now, we get the "good estimate" for \bar{V}_k . The equation (3.3.78) gives:

$$\nabla^\gamma \cdot (h_b \bar{V}_k) = -h_b(q_{k+1} + \varepsilon \bar{V} \cdot \nabla^\gamma q_k)$$

and using induction hypothesis to control q_{k+1} , one easily gets

$$|\nabla^\gamma \cdot (h_b \bar{V}_k)|_{H^{N-k-1}} + \sqrt{\mu} |\nabla^\gamma \nabla^\gamma \cdot (h_b \bar{V}_k)|_{H^{N-k-1}} \leq C(K)(t + \varepsilon) + C_0.$$

If $d = 1$, then we controlled a full derivative of \bar{V}_k , and the induction hypothesis is true for k . \square

Remark 3.3.16 *If $d = 2$, of course, it is not sufficient to control only $\nabla^\gamma \cdot \bar{V}_k$ to recover a good control for \bar{V}_k in norm H^{N-k} . One should look after a good control for $\nabla^{\gamma \perp} \cdot \bar{V}_k$. This is obtained by taking $\nabla^{\gamma \perp}$ of the second equation of (3.3.78):*

$$\partial_t(\nabla^{\gamma \perp} \cdot \bar{V}_k) + \partial_t \mu \nabla^{\gamma \perp} \cdot \nabla^{\gamma \perp}(\nabla^{\gamma \perp} \cdot \bar{V}_k) + (\bar{V} \cdot \nabla^\gamma) \nabla^{\gamma \perp} \cdot \bar{V}_k = \nabla^{\gamma \perp} \cdot (R_2^k) + [\nabla^{\gamma \perp} \cdot, \bar{V}] \bar{V}_k - \nabla^{\gamma \perp} \cdot \mu \mathcal{T}_b \partial_t \bar{V}_k \quad (3.3.88)$$

However, it is difficult to control $\nabla^{\gamma \perp} \cdot \mu \mathcal{T}_b \partial_t \bar{V}_k$. Indeed, $\nabla^{\gamma \perp} \cdot \mathcal{T}_b$ is no longer symmetric, which means that multiplying the equation (3.3.88) by $\nabla^{\gamma \perp} \cdot \bar{V}$ creates a term of the form $(\nabla^{\gamma \perp} \cdot \mathcal{T}_b \partial_t \bar{V}, \nabla^{\gamma \perp} \cdot \bar{V})_2$ which is not the time derivative of a positive quantity.

The key Lemma 3.3.10 is this latter result with $k = 0$. We now end the proof of Theorem 3.3.7. We set

$$\varepsilon_0 = \frac{C_0}{2C(2C_0)}, \quad T_0 = \frac{C_0}{2C(2C_0)}.$$

Let fix an $\varepsilon > 0$ such that $\varepsilon < \varepsilon_0$. There exists T^ε and a unique solution $(\zeta^\varepsilon, \overline{V}^\varepsilon) \in C([0; T^\varepsilon]; \mathcal{E}^N)$ to the equation (3.3.77). We set

$$T_*^\varepsilon = \sup_{t \in [0; T^\varepsilon[} \{t, (\zeta^\varepsilon, \overline{V}^\varepsilon), \text{ exists on } [0; t] \text{ with } \forall s \leq t, E^N(\zeta^\varepsilon, \overline{V}^\varepsilon)(s) \leq 2C_0\}$$

Then, one has $T_*^\varepsilon \geq T_0$. Indeed, suppose it is not true. One has for all $t < T^\varepsilon$, using Lemma 3.3.10:

$$E^N(\zeta^\varepsilon, \overline{V}^\varepsilon)(t) \leq C(K)(t + \varepsilon) + C_0,$$

with

$$K = \sup_{t \in [0; T_*^\varepsilon[} E^N(\zeta^\varepsilon, \overline{V}^\varepsilon).$$

Notice that $K \leq 2C_0$ by definition of T_*^ε . Since C is non decreasing, one has, for all $t \leq T_*^\varepsilon$:

$$\begin{aligned} E^N(\zeta^\varepsilon, \overline{V}^\varepsilon)(t) &\leq C(2C_0)(t + \varepsilon) + C_0 \\ &< C(2C_0)(T_0 + \varepsilon_0) \\ &< 2C_0 \end{aligned}$$

and therefore, by continuity, there exists $\tilde{T}^\varepsilon > T_*^\varepsilon$ such that $(\zeta^\varepsilon, \overline{V}^\varepsilon)$ exists on $[0; \tilde{T}^\varepsilon]$ with $E^N(t) \leq 2C_0$ for all $t \leq \tilde{T}^\varepsilon$. It is absurd, by definition of T_*^ε . Therefore, the solution exists on $[0; T_0]$ which is the result of Theorem 3.3.7. \square

3.A Classical results on Sobolev spaces

We recall here some classical results on Sobolev spaces. Proofs can be found in [61]. The first result is the Kato-Ponce estimate on commutators:

Proposition 3.A.1 (Kato-Ponce) *For all $s \geq 0$ and $f \in H^s \cap W^{1, \infty}$ and $u \in H^{s-1} \cap L^\infty$, one has the following inequality:*

$$|[\Lambda^s, f]u|_2 \leq C(|\nabla f|_{H^{s-1}}|u|_\infty + |\nabla^\gamma f|_\infty|u|_{H^{s-1}})$$

where C is a positive constant independent of f and u .

The following result stands that one can compose any $H^s \cap L^\infty$ function with a smooth function.

Proposition 3.A.2 (Moser) *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, null at zero. Then, for all $s \geq 0$, and all $u \in H^s(\mathbb{R}^d)$, $F(u) \in H^s(\mathbb{R}^d)$ and*

$$|F(u)|_{H^s} \leq c(|u|_\infty)|u|_{H^s}$$

where c is a smooth non decreasing function.

3.B Results on the operator \mathcal{A}

We prove in this section the regularity of the inverse of $h_b \mathcal{A}$ stated by Proposition 3.3.13.

Proposition 3.B.1 *Let $N \in \mathbb{N}$ and let $h_b \in H^N(\mathbb{R}^d)$ be such that there exists $h_{\min} > 0$ such that*

$$\forall X \in \mathbb{R}^d, \quad h_b(X) \geq h_{\min}.$$

We set, for all $f \in L^2(\mathbb{R}^d)$:

$$|f|_{X^N}^2 = |f|_{H^N}^2 + \mu |\nabla^\gamma \cdot f|_{H^N}^2.$$

The operator $h_b \mathcal{A}$ is invertible on $H^N(\mathbb{R}^d)^d$. Moreover, the following estimates stand.

(1) For all $f \in H^N(\mathbb{R}^d)^d$,

$$|(h_b \mathcal{A})^{-1} f|_{X^N} \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) |f|_{H^N}.$$

(2) For all $g \in H^N(\mathbb{R}^d)$,

$$\sqrt{\mu} |(h_b \mathcal{A})^{-1} \nabla^\gamma g|_{X^N} \leq C \left(\frac{1}{h_b}, |h_b|_{H^{N+1}} \right) |g|_{H^N}.$$

Proof Let $N \in \mathbb{N}$. We define

$$X^N = \{V \in L^2(\mathbb{R}^d), \quad \nabla^\gamma \cdot V \in L^2(\mathbb{R}^d)\}.$$

Endowed with the scalar product

$$(\cdot, \cdot)_{X^N} = (\cdot, \cdot)_{H^N} + \mu (\nabla^\gamma \cdot, \nabla^\gamma \cdot)_{H^N},$$

X^N is an Hilbert space with norm

$$|\cdot|_{X^N}^2 = |\cdot|_{H^N}^2 + \mu |\nabla^\gamma \cdot|_{H^N}^2.$$

We start to prove that $h_b \mathcal{A}$ is invertible, by using a Lax Milgram's Theorem with the bilinear form

$$T : (V_1, V_2) \in X^0 \times X^0 \longmapsto (h_b \mathcal{A} V_1, V_2)_2.$$

i) The bilinear form T is continuous:

Indeed, one has, for all $V_1, V_2 \in X^N$:

$$(h_b \mathcal{A} V_1, V_2)_2 = (h_b V_1, V_2)_2 + \mu \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b V_1), \nabla^\gamma \cdot (h_b V_2) \right)_2$$

and therefore one has

$$(h_b \mathcal{A} V_1, V_2)_2 \leq C_1 (h_{\min}, |h_b|_{W^{1,\infty}}) |V_1|_{X^0} |V_2|_{X^0}$$

with C_1 a non decreasing function of its arguments.

ii) The bilinear form T is coercive:

Let us write for all $V \in X^0$:

$$(h_b \mathcal{A} V, V)_2 = (h_b V, V)_2 + \mu \left(\frac{1}{h_b} \nabla^\gamma \cdot (h_b V), \nabla^\gamma \cdot (h_b V) \right)_2$$

which already gives

$$|V|_2^2 \leq \frac{1}{h_{\min}} (h_b V, V)_2. \tag{3.B.89}$$

Moreover, one has

$$\begin{aligned} (h_b V, V)_2 + \left(\frac{1}{h_b} \nabla^\gamma \cdot V, \nabla^\gamma \cdot V \right)_2 &= (h_b \mathcal{A} V, V)_2 - 2\mu (\nabla^\gamma \cdot V, V \cdot \nabla^\gamma h_b)_2 \\ &\quad - \mu \left(\frac{1}{h_b} \nabla^\gamma (h_b) \cdot V, \nabla^\gamma (h_b) \cdot V \right)_2 \\ &\leq (h_b \mathcal{A} V, V)_2 + 2\mu |\nabla^\gamma \cdot V|_2 |V|_2 |h_b|_{W^{1,\infty}} + \frac{1}{h_{\min}} \mu |V|_2^2 |h_b|_{W^{1,\infty}} \end{aligned}$$

and one can conclude using (3.B.89) and Young's inequality that

$$|V|_{X^0}^2 \leq C_2 \left(\frac{1}{h_{\min}}, |h_b|_{W^{1,\infty}} \right) (h_b \mathcal{A} V, V)_2. \tag{3.B.90}$$

Using Lax Milgram's Theorem, for all $f \in L^2(\mathbb{R}^d)$, there exists a unique $V_f \in X^0$ be such that $h_b \mathcal{A} V_f = f$. We now prove the first estimate on V_f stated by the Proposition by induction on N . Taking $V = V_f$ in (3.B.90), one has this estimate for $N = 0$. Let us suppose that the result is true

for $N - 1$ with $N \geq 1$, and let us prove it for N . One has, differentiating N times the relation $h_b \mathcal{A} V_f = f$ (we denote by ∂^N any derivative of order N below):

$$h_b \mathcal{A} \partial^N V_f = \partial^N f + [h_b \mathcal{A}, \partial^N] V_f \quad (3.B.91)$$

and $[h_b \mathcal{A}, \partial^N] V_f$ is sum of terms of the form

$$R_1^N = \mu \partial^{k_1} (h_b) \nabla^\gamma (\partial^{k_2} (\frac{1}{h_b}) \nabla^\gamma \cdot (\partial^{k_3} (h_b) \partial^{k_4} V_f))$$

and

$$R_2^N = \partial^{l_1} (h_b) \partial^{l_2} V_f$$

with $k_1 + k_2 + k_3 + k_4 = N$ and $k_4 < N$, and with $l_1 + l_2 = N$ and $l_2 < N$. Taking the L^2 scalar product of (3.B.91) with $\partial^N V_f$, and noticing that

$$(R_1^N, \partial^N V_f) = \mu (\partial^{k_2} (\frac{1}{h_b}) \nabla^\gamma \cdot (\partial^{k_3} (h_b) \partial^{k_4} V_f), \nabla^\gamma (\partial^{k_1} (h_b) \partial^N V_f))_2,$$

one gets:

$$(h_b \mathcal{A} \partial^N V_f, \partial^N V_f)_2 \leq C (\frac{1}{h_b}, |h_b|_{H^{N+1}}) (|\partial^N V_f|_2 |\partial^{l_2} V_f|_2 + \mu |\nabla^\gamma \cdot \partial^N V_f| (|\nabla^\gamma \cdot \partial^{k_4} V_f|_2 + |\partial^{k_4} V_f|_2)).$$

Using the induction hypothesis, the terms $\nabla^\gamma \cdot \partial^{k_4} V_f$ and $\partial^{l_2} V_f$ are already controlled by

$$C (\frac{1}{h_b}, |h_b|_{H^{N+1}}) |f|_{H^N},$$

since $k_4 < N$ and $l_2 < N$. One finally gets with a Young's inequality that

$$(h_b \mathcal{A} \partial^N V_f, \partial^N V_f)_2 \leq C (\frac{1}{h_{\min}}, |h_b|_{H^{N+1}}) |f|_{H^N} \quad (3.B.92)$$

and combining (3.B.92) with (3.B.90), one gets the estimate of the Theorem by a duality argument.

To prove the second point of the Proposition, one has to notice that for all $f = \sqrt{\mu} \nabla^\gamma g$ with $g \in H^1(\mathbb{R}^d)$ and all $V \in X^0$:

$$(f, V)_2 = -(g, \sqrt{\mu} \nabla^\gamma \cdot (V))_2$$

and one can adapt all the proof of the first point to get the desired result, since $\sqrt{\mu} \nabla^\gamma \cdot (V) \in L^2(\mathbb{R}^d)$.

□

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Chapter 4

A dispersive estimate for the linearized Water-Waves equations in finite depth

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Abstract

We prove a dispersive estimate for the solutions of the linearized Water-Waves equations in dimension 1 in presence of a flat bottom. Adapting the proof from [15] in the case of infinite depth, we prove a decay with respect to time t of order $|t|^{-1/3}$ for solutions with initial data φ such that $|\varphi|_{H^1}$, $|x\partial_x\varphi|_{H^1}$ are bounded. We also give variants to this result with different decays for a more convenient practical use of the dispersive estimate. We then give an existence result for the full Water-Waves equations in weighted spaces for practical uses of the proven dispersive estimates.

4.1 Introduction

We recall here classical formulations of the Water-Waves problem. We then shortly introduce the meaningful dimensionless parameters of this problem, and then present the main results of this paper.

4.1.1 Formulations of the Water-Waves problem

The Water-Waves problem puts the motion of a fluid with a free surface into equations. We recall here two equivalent formulations of the Water Waves equations for an incompressible and irrotationnal fluid.

Free surface d -dimensional Euler equations

The motion, for an incompressible, inviscid and irrotationnal fluid occupying a domain Ω_t delimited below by a fixed bottom and above by a free surface is described by the following quantities:

- the velocity of the fluid $U = (V, w)$, where V and w are respectively the horizontal and vertical components;
- the free top surface profile ζ ;
- the pressure P .

All these functions depend on the time and space variables t and $(X, z) \in \Omega_t$. There exists a function $b : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the domain of the fluid at the time t is given by

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

where H_0 is the typical depth of the water. The unknowns (U, ζ, P) are governed by the Euler equations:

$$\begin{cases} \partial_t U + U \cdot \nabla_{X,z} U = -\frac{1}{\rho} \nabla P - g e_z & \text{in } \Omega_t \\ \operatorname{div}(U) = 0 & \text{in } \Omega_t \\ \operatorname{curl}(U) = 0 & \text{in } \Omega_t. \end{cases} \quad (4.1.1)$$

We denote here $-g e_z$ the acceleration of gravity, where e_z is the unit vector in the vertical direction, and ρ the density of the fluid. Here, $\nabla_{X,z}$ denotes the $d + 1$ dimensional gradient with respect to both variables X and z .

These equations are completed by boundary conditions:

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - \underline{w} = 0 \\ U \cdot n = 0 & \text{on } \{z = -H_0 + b(X)\} \\ P = P_{atm} & \text{on } \{z = \zeta(t, X)\}, \end{cases} \quad (4.1.2)$$

In these equations, \underline{V} and \underline{w} are the horizontal and vertical components of the velocity evaluated at the surface. The vector n in the second equation stands for the normal upward vector at the bottom $(X, z = -H_0 + b(X))$. We denote P_{atm} the constant pressure of the atmosphere at the surface of the fluid. The first equation of (4.1.2) states the assumption that the fluid particles do not cross the surface, while the second equation of (4.1.2) states the assumption that they do not cross the bottom. The equations (4.1.1) with boundary conditions (4.1.2) are commonly referred to as the free surface Euler equations.

Craig-Sulem-Zakharov formulation

Since the fluid is by hypothesis irrotational, it derives from a scalar potential:

$$U = \nabla_{X,z}\Phi.$$

Zakharov remarked in [65] that the free surface profile ζ and the potential at the surface $\psi = \Phi|_{z=\zeta}$ fully determine the motion of the fluid, and gave an Hamiltonian formulation of the problem. Later, Craig-Sulem, and Sulem ([25] and [26]) gave a formulation of the Water Waves equation involving the Dirichlet-Neumann operator. The following Hamiltonian system is equivalent (see [37] and [3] for more details) to the free surface Euler equations (4.1.1) and (4.1.2):

$$\begin{cases} \partial_t \zeta - \mathcal{G}\psi = 0 \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\mathcal{G}\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0, \end{cases} \quad (4.1.3)$$

where the unknowns are ζ (free top profile) and ψ (velocity potential at the surface) with t as time variable and $X \in \mathbb{R}^d$ as space variable. The fixed bottom profile is b , and \mathcal{G} stands for the Dirichlet-Neumann operator, that is

$$\mathcal{G}\psi = \mathcal{G}[\zeta, b]\psi = \sqrt{1 + |\nabla\zeta|^2} \partial_n \Phi|_{z=\zeta},$$

where Φ stands for the potential, and solves a Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$\begin{cases} \Delta_{X,z}\Phi = 0 & \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -H_0 + b(X) < z < \zeta(X)\} \\ \phi|_{z=\zeta} = \psi, & \partial_n \Phi|_{z=-H_0+b} = 0 \end{cases}$$

with the notation, for the normal derivative

$$\partial_n \Phi|_{z=-H_0+b(X)} = \nabla_{X,z}\Phi(X, -H_0 + b(X)) \cdot n$$

where n stands for the normal upward vector at the bottom $(X, -H_0 + b(X))$. See also [37] for more details.

Dimensionless equations

Since the properties of the solutions depend strongly on the characteristics of the fluid, it is more convenient to non-dimensionalize the equations by introducing some characteristic lengths of the wave motion:

- (1) The characteristic water depth H_0 ;
- (2) The characteristic horizontal scale L_x in the longitudinal direction;
- (3) The characteristic horizontal scale L_y in the transverse direction (when $d = 2$);
- (4) The size of the free surface amplitude a_{surf} ;
- (5) The size of bottom topography a_{bott} .

Let us then introduce the dimensionless variables:

$$x' = \frac{x}{L_x}, \quad y' = \frac{y}{L_y}, \quad \zeta' = \frac{\zeta}{a_{surf}}, \quad z' = \frac{z}{H_0}, \quad b' = \frac{b}{a_{bott}},$$

and the dimensionless variables:

$$t' = \frac{t}{t_0}, \quad \Phi' = \frac{\Phi}{\Phi_0},$$

where

$$t_0 = \frac{L_x}{\sqrt{gH_0}}, \quad \Phi_0 = \frac{a_{surf}}{H_0} L_x \sqrt{gH_0}.$$

After re scaling, several dimensionless parameters appear in the equation. They are

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma,$$

where $\varepsilon, \mu, \beta, \gamma$ are commonly referred to respectively as "nonlinearity", "shallowness", "topography" and "transversality" parameters.

For instance, the Zakharov-Craig-Sulem system (4.1.3) becomes (see [37] for more details) in dimensionless variables (we omit the "primes" for the sake of clarity):

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0, \end{cases} \quad (4.1.4)$$

where $\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi$ stands for the dimensionless Dirichlet-Neumann operator,

$$\mathcal{G}_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = \sqrt{1 + \varepsilon^2 |\nabla^\gamma \zeta|^2} \partial_n \Phi|_{z=\varepsilon \zeta} = (\partial_z \Phi - \mu \nabla^\gamma(\varepsilon \zeta) \cdot \nabla^\gamma \Phi)|_{z=\varepsilon \zeta},$$

where Φ solves the Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$\begin{aligned} \Delta^{\mu, \gamma} \Phi = 0 \quad \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R} - 1 + \beta b(X) < z < \varepsilon \zeta(X)\} \\ \phi|_{z=\varepsilon \zeta} = \psi, \quad \partial_n \Phi|_{z=-1+\beta b} = 0. \end{aligned} \quad (4.1.5)$$

We used the following notations:

$$\begin{aligned} \nabla^\gamma &= {}^t(\partial_x, \gamma \partial_y) & \text{if } d=2 & \quad \text{and} & \quad \nabla^\gamma = \partial_x & \text{if } d=1 \\ \Delta^{\mu, \gamma} &= \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{if } d=2 & \quad \text{and} & \quad \Delta^{\mu, \gamma} = \mu \partial_x^2 + \partial_z^2 & \text{if } d=1 \end{aligned}$$

and

$$\partial_n \Phi|_{z=-1+\beta b} = \frac{1}{\sqrt{1 + \beta^2 |\nabla^\gamma b|^2}} (\partial_z \Phi - \mu \nabla^\gamma(\beta b) \cdot \nabla^\gamma \Phi)|_{z=-1+\beta b}.$$

4.1.2 Main result

The linearized Water-Waves equations (4.1.4) in one dimension around a rest state of a flat surface and a zero velocity, in presence of a flat bottom can be read as

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = 0 \\ \partial_t \psi + \zeta = 0 \\ (\zeta, \psi)(0) = (\zeta_0, \psi_0) \end{cases} \quad (4.1.6)$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$. We denote

$$\frac{1}{\mu} \mathcal{G}_0 = \frac{1}{\mu} \mathcal{G}[0, 0]$$

the Dirichlet-Neumann operator in $\zeta = 0$ with a flat bottom, which explicit formulation is given by its Fourier transform

$$\frac{1}{\mu} \widehat{\mathcal{G}_0} f(\xi) = \frac{|\xi| \tanh(\sqrt{\mu} |\xi|)}{\sqrt{\mu}} \widehat{f}(\xi), \quad (4.1.7)$$

for all $f \in \mathcal{S}'(\mathbb{R})$ where μ is the shallowness parameter (see for instance [37] for more details). The equation (4.1.6) leads to the following equation for ζ :

$$\partial_t^2 \zeta + \frac{1}{\mu} \mathcal{G}_0 \zeta = 0$$

which is similar to the wave equation for low frequencies, and to the Water-Wave equation in infinite depth

$$\partial_t^2 \zeta + (-\Delta)^{1/2} \zeta = 0$$

where $\Delta = \partial_x^2$, for high frequencies. In order to study the solutions of the linearized system (4.1.6), we are therefore led to study the decay in time of the operator $e^{it\omega(D)}$ where

$$\omega : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ \xi \longmapsto \sqrt{\frac{|\xi| \tanh(\sqrt{\mu} |\xi|)}{\sqrt{\mu}}}. \end{cases}$$

The dispersive nature of the Water-Waves equations in infinite depth plays a key role in the proof of long time or global time results: see for instance Ionescu-Pusateri [32], Alazard-Delort [4] and [5]. However, there are to our knowledge only few results on decay estimates for the Water-Waves equations in finite depth (see for instance [45]). Recently, Aynur Bulut proved in [15] an L^2 based norm- L^∞ decay estimate for the linear Water-Waves equation in infinite depth:

$$|e^{it(-\Delta)^{1/4}}\varphi|_\infty \leq C(1+|t|)^{-1/2}(|\varphi|_{H^1} + |x\partial_x\varphi|_{L^2}). \quad (4.1.8)$$

As for all oscillatory integrals estimates, the proof of this result relies only on the behaviour of the operator $(-\Delta)^{1/2}$, which is the same as the behaviour of $\frac{1}{\mu}\mathcal{G}_0$ (recall the definition (4.1.7)) for high frequencies. We therefore adapt this proof to get a similar result in the case of a finite depth, with a very special attention given to the dependence in the shallowness parameter μ . We prove in Section 4.2 of this paper the following result:

Theorem 4.1.1 *Let*

$$\omega : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ \xi \longmapsto \sqrt{\frac{|\xi| \tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}}}. \end{cases}$$

Then, there exists $C > 0$ independent on μ such that, for all $\mu > 0$:

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C\left(\frac{1}{\mu^{1/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/8}} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}}\right)(|\varphi|_{H^1} + |x\partial_x\varphi|_2).$$

Though $\sqrt{\mathcal{G}_0}$ and the square root of the wave operator $(-\Delta)^{1/2}$ have the same behaviour for low frequencies, it is not the case for the second order derivatives of these operators. For this reason, one should not be surprised to have a dispersion result for the Water-Waves equations, while the wave equation in dimension 1 is not dispersive.

The decay in $\frac{1}{t^{1/8}}$ given by Theorem 4.1.1 is however very bad. As one might be interested to have a better decay result, we also prove the following result, with different spaces:

Theorem 4.1.2 *With the notations of Theorem 4.1.1, the following estimates hold:*

1. *There exists $C > 0$ independent on μ such that, for all $\mu > 0$:*

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C\left(\frac{1}{\mu^{3/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/3}}|\varphi|_{L^1} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}}(|\varphi|_{H^1} + |x\partial_x\varphi|_2)\right).$$

2. *There exists $C > 0$ independent on μ such that, for all $\mu > 0$:*

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C\left(\frac{1}{\mu^{3/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/3}}|x\varphi|_{L^2} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}}(|\varphi|_{H^1} + |x\partial_x\varphi|_2)\right).$$

As one should remark, the decay given by Theorem 4.1.2 is better than one of Theorem 4.1.1. However, for a practical use of such decay, one should prove that the solutions are bounded in L^1 or in $|x \cdot |_2$ norm, which is more difficult than proving a local existence result in $|x\partial_x \cdot |_2$ -norm. In view of practical use of Theorem 4.1.1, we therefore prove in Section 4.3 and in dimensions $d = 1, 2$ a local existence result for the full Water-Waves equations (4.1.4) in weighted Sobolev spaces. The proof consists in an adaptation of the local existence result by [6], and a technical proof of the commutator $[\mathcal{G}, x]$. The plan of the article is the following:

- In Section 4.2, we prove a dispersive estimate for the linearized Water-Waves equation around a flat bottom and a flat surface in dimension $d = 1$.
- In Section 4.3, we give a local existence result for the full Water-Waves equation (4.1.4) with non flat bottom, in weighted Sobolev spaces and in dimensions $d = 1, 2$.

4.1.3 Notations

We introduce here all the notations used in this paper.

Operators and quantities

Because of the use of dimensionless variables (see before the "dimensionless equations" paragraph), we use the following twisted partial operators:

$$\begin{aligned} \nabla^\gamma &= {}^t(\partial_x, \gamma\partial_y) & \text{if } d = 2 & \text{ and} & \nabla^\gamma &= \partial_x & \text{if } d = 1 \\ \Delta^{\mu,\gamma} &= \mu\partial_x^2 + \gamma^2\mu\partial_y^2 + \partial_z^2 & \text{if } d = 2 & \text{ and} & \Delta^{\mu,\gamma} &= \mu\partial_x^2 + \partial_z^2 & \text{if } d = 1 \\ \nabla^{\mu,\gamma} &= {}^t(\sqrt{\mu}\partial_x, \gamma\sqrt{\mu}\partial_y, \partial_z) & \text{if } d = 2 & \text{ and} & \nabla^{\mu,\gamma} &= {}^t(\sqrt{\mu}\partial_x, \partial_z) & \text{if } d = 1. \end{aligned}$$

Remark 4.1.3 *All the results proved in this paper do not need the assumption that the typical wave lengths are the same in both directions, i.e. $\gamma = 1$. However, if one is not interested in the dependence of γ , it is possible to take $\gamma = 1$ in all the following proofs. A typical situation where $\gamma \neq 1$ is for weakly transverse waves for which $\gamma = \sqrt{\mu}$; this leads to weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili equation (see [40]).*

For all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we write

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$$

and

$$|\alpha| = \alpha_1 + \dots + \alpha_d.$$

We denote for all $a, b \in \mathbb{R}$:

$$a \vee b = \max(a, b).$$

We denote, for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, the Fourier transform of φ by $\mathcal{F}(\varphi)$ or more simply $\hat{\varphi}$.

We use the classical Fourier multiplier

$$\Lambda^s = (1 - \Delta)^{s/2} \text{ on } \mathbb{R}^d$$

defined by its Fourier transform as

$$\mathcal{F}(\Lambda^s u)(\xi) = (1 + |\xi|^2)^{s/2} (\mathcal{F}u)(\xi)$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$. The operator \mathfrak{P} is defined as

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} \quad (4.1.9)$$

where

$$\mathcal{F}(f(D)u)(\xi) = f(\xi)\mathcal{F}(u)(\xi)$$

is defined for any smooth function of polynomial growth f and $u \in \mathcal{S}'(\mathbb{R}^d)$. The pseudo-differential operator \mathfrak{P} acts as the square root of the Dirichlet Neumann operator, since $\mathfrak{P} \sim \sqrt{\mathcal{G}_0}$ (recall the definition of \mathcal{G}_0 given by (4.1.7)) where the implicit constant does not depend on μ .

We denote as before by $\mathcal{G}_{\mu,\gamma}$ the Dirichlet-Neumann operator, which is defined as followed in the scaled variables:

$$\mathcal{G}_{\mu,\gamma}\psi = \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi = \sqrt{1 + \varepsilon^2|\nabla^\gamma\zeta|^2}\partial_n\Phi|_{z=\varepsilon\zeta} = (\partial_z\Phi - \mu\nabla^\gamma(\varepsilon\zeta) \cdot \nabla^\gamma\Phi)|_{z=\varepsilon\zeta},$$

where Φ solves the Laplace equation

$$\begin{cases} \Delta^{\gamma,\mu}\Phi = 0 \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_n\Phi|_{z=-1+\beta b} = 0. \end{cases}$$

For the sake of simplicity, we use the notation $\mathcal{G}[\varepsilon\zeta, \beta b]\psi$ or even $\mathcal{G}\psi$ when no ambiguity is possible.

The Dirichlet-Neumann problem

In order to study the Dirichlet-Neumann problem (4.1.5), we need to map the domain occupied by the water Ω_t into a fixed domain (and not on a moving subset). For this purpose, we define:

$$\zeta^\delta(\cdot, z) = \chi(\delta z |D^\gamma|)\zeta, \quad b^\delta(\cdot, z) = \chi(\delta(z+1)|D^\gamma|)b$$

where $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is a compactly supported smooth function equals to one in the neighbourhood of the origin, and $\delta > 0$. We now introduce the following fixed strip:

$$\mathcal{S} = \mathbb{R}^d \times (-1; 0)$$

and the diffeomorphism

$$\Sigma : \begin{array}{l} \mathcal{S} \rightarrow \Omega_t \\ (X, z) \mapsto (X, (1 + \varepsilon\zeta^\delta(X) - \beta^\delta b(X))z + \varepsilon\zeta^\delta(X)) \end{array} \quad (4.1.10)$$

It is quite easy to check that Φ is the variational solution of (4.1.5) if and only if $\phi = \Phi \circ \Sigma$ is the variational solution of the following problem:

$$\begin{cases} \nabla^{\mu, \gamma} \cdot P(\Sigma) \nabla^{\mu, \gamma} \phi = 0 \\ \phi_{z=0} = \psi, \quad \partial_n \phi_{z=-1} = 0, \end{cases} \quad (4.1.11)$$

and where

$$P(\Sigma) = |\det J_\Sigma| J_\Sigma^{-1} {}^t(J_\Sigma^{-1}),$$

where J_Σ is the Jacobian matrix of the diffeomorphism Σ .

Remark 4.1.4 *By smoothing the functions ζ and b in the choice of the diffeomorphism Σ , we ensure a better estimate for the solutions of (4.1.11).*

For a complete statement of the result, and a proof of existence and uniqueness of solutions to these problems, see later Section 4.3.2 and also [37] Chapter 2.

We introduce here the notations for the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\Gamma = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \quad \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$\mathcal{G}[\varepsilon \cdot, \beta \cdot] : \begin{array}{l} \Gamma \longrightarrow H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) \longmapsto \mathcal{G}[\varepsilon\zeta, \beta b]\psi. \end{array}$$

We can prove the differentiability of this mapping. See Appendix 4.B for more details. We denote $d^j \mathcal{G}(h, k)\psi$ the j -th derivative of the mapping at (ζ, b) in the direction (h, k) . When we only differentiate in one direction, and when no ambiguity is possible, we simply denote $d^j \mathcal{G}(h)\psi$ or $d^j \mathcal{G}(k)\psi$.

Functional spaces

The standard scalar product on $L^2(\mathbb{R}^d)$ is denoted by $(\cdot, \cdot)_2$ and the associated norm $|\cdot|_2$. We will denote the norm of the Sobolev spaces $H^s(\mathbb{R}^d)$ by $|\cdot|_{H^s}$. We denote the norms of $W^{k, \infty}(\mathbb{R}^d)$ by $|\cdot|_{W^{k, \infty}}$ or simply $|\cdot|_\infty = |\cdot|_{W^{0, \infty}}$ when no ambiguity is possible.

We introduce the following functional Sobolev-type spaces, or Beppo-Levi spaces:

Definition 4.1.5 *We denote $\dot{H}^{s+1}(\mathbb{R}^d)$ the topological vector space*

$$\dot{H}^{s+1}(\mathbb{R}^d) = \{u \in L^2_{loc}(\mathbb{R}^d), \quad \nabla u \in H^s(\mathbb{R}^d)\}$$

endowed with the (semi) norm $|u|_{\dot{H}^{s+1}(\mathbb{R}^d)} = |\nabla u|_{H^s(\mathbb{R}^d)}$.

Just remark that $\dot{H}^{s+1}(\mathbb{R}^d)/\mathbb{R}^d$ is a Banach space (see for instance [27]).

The space variables $z \in \mathbb{R}$ and $X \in \mathbb{R}^d$ play different roles in the equations since the Euler formulation (4.1.1) is posed for $(X, z) \in \Omega_t$. Therefore, X lives in the whole space \mathbb{R}^d (which allows to take fractional Sobolev type norms in space), while z is actually bounded. For this reason, we denote the L^2 norm on \mathcal{S} by $\|\cdot\|$, and we introduce the following Banach spaces:

Definition 4.1.6 *The Banach space $(H^{s,k}((-1, 0) \times \mathbb{R}^d), \|\cdot\|_{H^{s,k}})$ is defined by*

$$H^{s,k}((-1, 0) \times \mathbb{R}^d) = \bigcap_{j=0}^k H^j((-1, 0); H^{s-j}(\mathbb{R}^d)), \quad \|u\|_{H^{s,k}} = \sum_{j=0}^k \|\Lambda^{s-j} \partial_z^j u\|_2.$$

We will denote $\|\cdot\|_{H^s} = \|\cdot\|_{H^{s,0}}$ when no ambiguity is possible. To sum up, $|\cdot|$ will denote a norm on \mathbb{R}^d while $\|\cdot\|$ will denote a norm on the flat strip \mathcal{S} .

4.2 A dispersive estimate for the linear Water-Waves equations in dimension 1

We prove in this section the dispersive estimate of Theorem 4.1.1 and Theorem 4.1.2. We first introduce some classical results on the oscillatory integrals, and some technical results on the Littlewood-Paley decomposition.

4.2.1 Technical tools

Littlewood-Paley decomposition

We briefly recall the Littlewood-Paley decomposition. Let $\psi \in C_0^\infty(\mathbb{R})$ be such that

$$\text{supp } \psi \subset (-1; 1), \quad \psi(\xi) = 1 \text{ for } |\xi| \leq 1/2.$$

We now define for all $k \in \mathbb{Z}$, a function ψ_k by:

$$\forall \xi \in \mathbb{R}, \quad \psi_k(\xi) = \psi(\xi/2^k) - \psi(\xi/2^{k-1}) \tag{4.2.12}$$

which is compactly supported in $2^{k-1} \leq |\xi| \leq 2^{k+1}$. We then define the operators P_k for all $k \in \mathbb{Z}$ by:

$$\forall \xi \in \mathbb{R}, \quad \widehat{P_k f}(\xi) = \psi_k(\xi) \widehat{f}(\xi) \tag{4.2.13}$$

for all $f \in \mathcal{S}'(\mathbb{R})$. We recall here Bernstein's Lemma:

Lemma 4.2.1 *Let $k \in \mathbb{Z}$ and P_k defined by (4.2.13). For every $1 \leq p \leq q \leq \infty$ and all $s \geq 0$, one has:*

$$|P_k g|_{L^q} \leq C 2^{k(1/p-1/q)} |P_k g|_{L^p}$$

and

$$|P_k g|_{L^p} \leq C 2^{-sk} |(-\Delta)^{s/2} P_k g|_{L^p}$$

for all $g \in \mathcal{S}(\mathbb{R})$, where C does not depend on k, s, p, q .

We also give the two following technical results (see for instance [15] for a complete proof):

Lemma 4.2.2 *Let $k \in \mathbb{Z}$ and P_k defined by (4.2.13). One has:*

$$|\partial_\xi \widehat{P_k \varphi}|_2 \leq C 2^{-k} (|\varphi|_2 + |x \partial_x \varphi|_2),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, where C does not depend on k .

Lemma 4.2.3 *Let $k \in \mathbb{Z}$ and P_k defined by (4.2.13). For all $s > 1/2$ one has*

$$|\widehat{P_k \varphi}|_\infty \leq C 2^{-sk} (|\varphi|_{H^s} + |x \partial_x \varphi|_2),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$, where C does not depend on k, s .

Some results on oscillatory integrals

We invoke later in this paper the following Van der Corput Lemma, which is a refinement of the stationary phase lemma:

Lemma 4.2.4 *Let $\Phi \in C^k(\mathbb{R})$, and $a < b$ be such that, either:*

- (1) $\forall x \in [a; b], \quad |\Phi^{(k)}(x)| \geq 1$ if $k > 1$;
- (2) $\forall x \in [a; b], \quad |\Phi'(x)| \geq 1$ and Φ' is monotonic.

Then, there exists $C > 0$ which only depends on k such that

$$\forall t > 0, \quad \left| \int_a^b e^{it\Phi(\xi)} d\xi \right| \leq \frac{C}{t^{1/k}}.$$

Note that in the above Lemma, C does not depend on a nor b .

4.2.2 Proof of the main result

We prove in this section the following dispersive estimate for the linearized Water-Waves equations in dimension 1:

Theorem 4.2.5 *Let*

$$\omega : \begin{cases} \mathbb{R} \rightarrow \mathbb{R} \\ \xi \mapsto \sqrt{\frac{|\xi| \tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}}} \end{cases}.$$

Then, there exists $C > 0$ independent on μ such that, for all $\mu > 0$:

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C \left(\frac{1}{\mu^{1/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/8}} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}} \right) (|\varphi|_{H^1} + |x\partial_x\varphi|_2).$$

Note that the dependence of μ in the dispersive estimate has been precisely mentioned. This is crucial in view of using a decay estimate for the Water-Waves equations, since the properties of the solutions, and even the dispersive properties of the problem may completely vary with respect to the size of the shallowness parameter μ , as one should see by studying for example the asymptotic regimes when μ goes to zero. See for instance the Chapter 5 on shallow water models in [37]. Notice that, if one sets $\lambda = \frac{t}{\sqrt{\mu}}$ in the statement of Theorem 4.2.5, one recovers the result by Aynur Bulut (4.1.8) when μ goes to $+\infty$: indeed, one writes $e^{it\omega(D)} = e^{i\frac{t}{\sqrt{\mu}}g(\sqrt{\mu}D)}$ with $g(x) \xrightarrow{x \rightarrow +\infty} \sqrt{|x|}$. Moreover, this result does not need any assumption on the size of μ (while $\mu \leq \mu_0$ is a common assumption in the Water-Waves results).

In [45], a similar decay as one given by Theorem 4.2.5 is proved but only for functions φ such that $\widehat{\varphi}(0) = 0$ and with L^1 and H^2 weighted spaces, which are less convenient for practical use than H^1 space and L^2 -weighted space. However, a short adaptation of the proof of [45] shows that a decay of order $\frac{1}{t^{1/3}}$ can be obtained if $\widehat{\varphi}(0) \neq 0$, which is a better decay than one of Theorem 4.2.5. We can however improve such result by adapting the proof of Theorem 4.2.5, and we prove also in this paper the following decay results:

Theorem 4.2.6 *With the notations of Theorem 4.2.5, the following estimates hold:*

- 1. *There exists $C > 0$ independent on μ such that, for all $\mu > 0$:*

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C \left(\frac{1}{\mu^{3/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/3}} |\varphi|_{L^1} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}} (|\varphi|_{H^1} + |x\partial_x\varphi|_2) \right).$$

2. There exists $C > 0$ independent on μ such that, for all $\mu > 0$:

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_\infty \leq C \left(\frac{1}{\mu^{3/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/3}} |x\varphi|_{L^2} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}} (|\varphi|_{H^1} + |x\partial_x\varphi|_2) \right).$$

The Theorem 4.2.6 gives a better decay than Theorem 4.2.5 for the linear operator of the Water-Waves, however its use in view of long time results for the full Water-Waves equation (4.1.4) may require to prove that solutions are bounded in L^1 norm, or in $|x \cdot |_{L^2}$ norm, which is difficult. Indeed, the proof of local existence for this equation in weighted spaces requires the control of the commutator $[\mathcal{G}, x]$, which is difficult to get (while the one for $[\mathcal{G}, x]\partial_x$ is less difficult to get, see later Section 4.3.2).

The proof is based on a stationary phase result and a use of the Littlewood decomposition. More precisely, the control of oscillatory integrals of the form

$$\int_a^b e^{it\Phi(\xi)} d\xi$$

consists in the precise study of where the phase Φ may be stationary. Here, the derivative of the phase

$$\Phi(\xi) = t(\omega(\xi) + x\xi/t) \tag{4.2.14}$$

may vanish and we are therefore led to study the behaviour of the second derivative Φ'' . However, $|\Phi''(\xi)| \underset{\xi \rightarrow \infty}{\sim} \frac{C}{|\xi|^{3/2}}$ and therefore Φ'' cannot be bounded from below by a constant and one cannot apply Van der Corput's Lemma 4.2.4. We therefore need to compensate the bad bound of Φ'' with good weighted estimates on $\widehat{\varphi}$.

Proof The result is easy to get for $|t/\sqrt{\mu}| \leq 1$ by using the continuous injection $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, and therefore we assume that $t > \sqrt{\mu}$ (the case $t < -\sqrt{\mu}$ is similar). Let fix $x \in \mathbb{R}$ with $x \neq 0$. In all this proof, we will denote by C any constant which does not depend on μ, x, t, k . As explained above, the derivative of the phase, Φ' , may vanish and therefore one needs a close study of the second derivative $\Phi'' = \omega''$. Note that $\omega(\xi) = \frac{1}{\sqrt{\mu}}g(\sqrt{\mu}\xi)$ with $g(\xi) = \sqrt{|\xi| \tanh(|\xi|)}$. It is easy to show that

$$|g''(\xi)| \underset{\xi \rightarrow 0}{\sim} |\xi|, \quad |g''(\xi)| \underset{\xi \rightarrow +\infty}{\sim} |\xi|^{-3/2}. \tag{4.2.15}$$

As suggested by the behaviour of ω'' , we split the study of the linear operator $e^{it\omega(D)}$ into low and high frequencies cases. To this purpose, one can define χ a smooth compactly supported function equals to 1 in the neighbourhood of the origin, and write $\varphi = \chi(D)\varphi + (1 - \chi(D))\varphi$, with as usual $\chi(D)\varphi(\xi) = \chi(\xi)\widehat{\varphi}(\xi)$ for all $\varphi \in \mathcal{S}(\mathbb{R})$. Note that the estimate of Theorem 4.2.5 stay true if we prove it for $\chi(D)(\sqrt{\mu}|D^\gamma|)\varphi$ or $(1 - \chi(\sqrt{\mu}|D^\gamma|))\varphi$ instead of φ .

We write:

$$\begin{aligned} e^{it\omega(D)}\varphi &= \int_{\mathbb{R}} e^{i(t\omega(\xi)+x\xi)} \widehat{\varphi}(\xi) d\xi \\ &= \int_{|\xi| \leq y_0/\sqrt{\mu}} e^{i(t\omega(\xi)+x\xi)} \widehat{\varphi}(\xi) d\xi + \int_{|\xi| > y_0/\sqrt{\mu}} e^{i(t\omega(\xi)+x\xi)} \widehat{\varphi}(\xi) d\xi. \end{aligned} \tag{4.2.16}$$

Taking the last remark into account, we first assume that $\widehat{\varphi}$ is compactly supported in some $[0; \frac{y_0}{\sqrt{\mu}}[$ for some $y_0 > 0$. In this case, we only need to control the first term of (4.2.16). One sets $\delta > 0$ and splits the integral into two parts (recall that the phase Φ is defined by (4.2.14)):

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq \left| \int_{|\xi| \leq \delta} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| + \left| \int_{\delta \leq |\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right|. \tag{4.2.17}$$

We now use Cauchy-Schwarz inequality to control the first integral of the right hand side of (4.2.17):

$$\left| \int_{|\xi| \leq \delta} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq \sqrt{2\delta} |\varphi|_2. \tag{4.2.18}$$

For the second integral of the right hand side of (4.2.17), we only consider the integral over $[\delta; y_0/\sqrt{\mu}]$ where Φ is smooth (the integral over $[-y_0/\sqrt{\mu}; -\delta]$ is controlled by the exact same technique using the symmetry of Φ''). We integrate by parts in the second integral of the right hand side of (4.2.17) (remember that $\widehat{\varphi}$ is compactly supported in $[0; \frac{y_0}{\sqrt{\mu}}]$):

$$\left| \int_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq \left| - \int_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} \int_{\delta}^{\xi} e^{it\Phi(s)} ds \frac{d}{d\xi} \widehat{\varphi}(\xi) d\xi \right|.$$

Using Cauchy-Schwarz inequality, one gets:

$$\begin{aligned} \left| \int_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| &\leq \frac{\sqrt{y_0}}{\mu^{1/4}} \sup_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} \left| \int_{\delta}^{\xi} e^{it\Phi(s)} ds \right| \times |1_{\delta \leq |\xi|} \frac{1}{\xi} (\xi \frac{d}{d\xi} \widehat{\varphi})(\xi)|_2 \\ &\leq \frac{C}{\delta \mu^{1/4}} \sup_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} \left| \int_{\delta}^{\xi} e^{it\Phi(s)} ds \right| (|\varphi|_2 + |x \partial_x \varphi|_2). \end{aligned} \quad (4.2.19)$$

We now give a control of the oscillatory integral of (4.2.19). Recall that $\omega = \frac{1}{\sqrt{\mu}} g(\sqrt{\mu} \xi)$ with (4.2.15). Therefore, there exists $C > 0$ independent on μ such that $|\Phi''(s)| \geq C\mu|\xi|$ on $[0; y_0/\sqrt{\mu}]$ (see also Figure 4.1). Therefore, one has:

$$\forall \delta \leq s \leq \frac{y_0}{\sqrt{\mu}}, \quad |\Phi''(s)| \geq \mu s \geq \mu \delta.$$

Using Van der Corput's Lemma 4.2.4, one gets the control:

$$\sup_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} \left| \int_{\delta}^{\xi} e^{it\Phi(s)} ds \right| \leq \frac{C}{\sqrt{\mu \delta t}}. \quad (4.2.20)$$

Putting together (4.2.20) into (4.2.19), one gets:

$$\left| \int_{\delta \leq \xi \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq \frac{C}{\delta \mu^{1/4}} \frac{1}{\sqrt{\mu \delta t}} (|\varphi|_2 + |x \partial_x \varphi|_2). \quad (4.2.21)$$

Combining the estimates of (4.2.18) and (4.2.21), one obtains:

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq (\sqrt{2\delta} + \frac{C}{\delta \mu^{1/4}} \frac{1}{\sqrt{\mu \delta t}}) (|\varphi|_2 + |x \partial_x \varphi|_2).$$

The above quantity is minimal for $\delta = \frac{1}{\mu^{3/8} t^{1/4}}$ and therefore, we finally get:

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi \right| \leq \frac{1}{\mu^{3/16}} t^{1/8}. \quad (4.2.22)$$

We now assume that $\widehat{\varphi}$ has its support in $[\frac{y_0}{\sqrt{\mu}}; +\infty[$. In this case, we only need to focus on the second term of the right hand side of (4.2.16). We are led to control in L^∞ norm the quantity $\int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{\varphi}(\xi) d\xi$ where $\text{supp } \widehat{\varphi} \subset [\frac{y_0}{\sqrt{\mu}}; +\infty[$. The following lines are an adaptation of [15]. Using the Littlewood-Paley decomposition, we split $e^{it\omega(D)} \varphi$ into

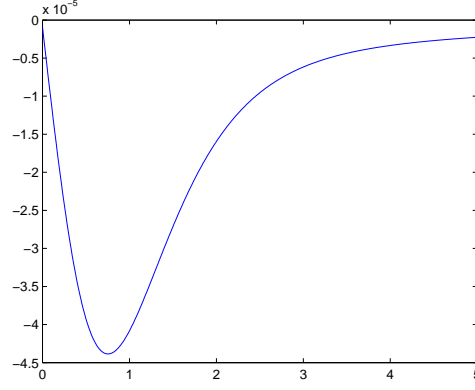
$$e^{it\omega(D)} \varphi = \sum_{\substack{k \in \mathbb{Z} \\ 2^k \leq \lambda(t)}} P_k e^{it\omega(D)} \varphi + \sum_{\lambda(t) \leq 2^k \leq \Lambda(t)} P_k e^{it\omega(D)} \varphi + \sum_{2^k \geq \Lambda(t)} P_k e^{it\omega(D)} \varphi$$

where

$$\lambda(t) = C_1 (1 + |t/\sqrt{\mu}|)^{-1}, \quad \Lambda(t) = \frac{1}{C_1} (1 + |t/\sqrt{\mu}|),$$

with $C_1 > 1$. We therefore have

$$|(e^{it\omega(D)} \varphi)(x)| \leq S_1 + S_2 + S_3,$$


 Figure 4.1: Graph of Φ''

with:

$$\begin{aligned} S_1 &= \sum_{2^k \leq \lambda(t)} |P_k e^{it\omega(D)} \varphi(x)|, \\ S_2 &= \sum_{\lambda(t) \leq 2^k \leq \Lambda(t)} |P_k e^{it\omega(D)} \varphi(x)|, \\ S_3 &= \sum_{2^k \geq \Lambda(t)} |P_k e^{it\omega(D)} \varphi(x)|. \end{aligned}$$

The term S_1 is controlled by using Bernstein's Lemma 4.2.1:

$$\begin{aligned} S_1 &\leq \sum_{2^k \leq \lambda(t)} |P_k e^{it\omega(D)} \varphi(x)|_\infty \\ &\leq C \sum_{2^k \leq \lambda(t)} 2^{k/2} |P_k e^{it\omega(D)} \varphi(x)|_2 \\ &\leq C \lambda(t)^{1/2} |\varphi(x)|_2 \\ &\leq C(1 + |t/\sqrt{\mu}|)^{-1/2} |\varphi|_2. \end{aligned} \tag{4.2.23}$$

Using again Bernstein's Lemma 4.2.1, one gets the control of S_3 :

$$\begin{aligned} S_3 &\leq \sum_{2^k \geq \Lambda(t)} |P_k e^{it\omega(D)} \varphi(x)|_\infty \\ &\leq \sum_{2^k \geq \Lambda(t)} 2^{k/2} |P_k e^{it\omega(D)} \varphi(x)|_2 \\ &\leq C \sum_{2^k \geq \Lambda(t)} 2^{-k/2} |P_k e^{it\omega(D)} \varphi(x)|_{H^1} \\ &\leq C(1 + |t/\sqrt{\mu}|)^{-1/2} |\varphi|_{H^1}. \end{aligned} \tag{4.2.24}$$

For the control of S_2 , we need a close study of oscillatory integrals of the form $\int_{\mathbb{R}} e^{it(x/t\xi + \omega(\xi))} \widehat{P_k \varphi}(\xi) d\xi$. Therefore, we need precise bounds for the oscillatory phase $\xi \mapsto (x/t)\xi + \omega(\xi)$ and its derivatives. We are led to split the summation set of S_2 into three parts:

$$\begin{aligned} I_1 &= \{k \in \mathbb{Z}, \lambda(t) \leq 2^k \leq \Lambda(t), \quad 2^{k/2} \leq |t/x|/C_2\}, \\ I_2 &= \{k \in \mathbb{Z}, \lambda(t) \leq 2^k \leq \Lambda(t), \quad |t/x|/C_2 \leq 2^{k/2} \leq C_2|t/x|\}, \\ I_3 &= \{k \in \mathbb{Z}, \lambda(t) \leq 2^k \leq \Lambda(t), \quad 2^{k/2} \geq C_2|t/x|\}, \end{aligned}$$

where C_2 has to be set. We therefore set

$$S_{2j} = \sum_{I_j} \left| \int_{\mathbb{R}} e^{i(x\xi + t\omega(\xi))} \widehat{P_k \varphi}(\xi) d\xi \right|, \quad j = 1, 2, 3.$$

The contributions of I_1 and I_3 are the most easy to get. One writes, for all $k \in I_1$:

$$\begin{aligned} \int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi &= \int_{\mathbb{R}} \frac{d}{d\xi} \left(\int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \right) \widehat{P_k \varphi}(\xi) d\xi \\ &= - \int_{\mathbb{R}} \int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \frac{d}{d\xi} \widehat{P_k \varphi}(\xi) d\xi \end{aligned}$$

by integrating by parts, and recalling that $\text{supp} \widehat{P_k \varphi} \subset \text{supp} \psi_k \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. We now use Cauchy-Schwarz inequality to get:

$$\left| \int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| \leq C 2^{k/2} \sup_{\xi \in \text{supp} \psi_k \cap \text{supp} \widehat{\varphi}} \left| \int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \right| \times \left| \frac{d}{d\xi} \widehat{P_k \varphi} \right|_2. \quad (4.2.25)$$

We are therefore led to control the oscillatory integral $\int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds$ with $\xi \in \text{supp} \psi_k \cap \text{supp} \widehat{\varphi}$. We put this integral under the form

$$\int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds = \int_{2^{k-1}}^{\xi} e^{it((x/t)s + \omega(s))} ds$$

and we use the following lower bound for the derivative of the phase:

$$|x/t + \omega'(s)| \geq |\omega'(s)| - |x/t|.$$

Now, recall that $k \in I_1$ and therefore $|x/t| \leq 2^{-k/2} C_2$. Moreover, recall that $\omega(\xi) = \frac{1}{\sqrt{\mu}} g(\sqrt{\mu} \xi)$ with $|g'(\xi)| \underset{\xi \rightarrow +\infty}{\sim} |\xi|^{-1/2}$, and that $\text{supp} \widehat{\varphi} \subset [y_0/\sqrt{\mu}; +\infty[$. Therefore, the derivative of ω is bounded from below on $\text{supp} \widehat{\varphi}$ by $\frac{C_3}{\mu^{1/4}} |s|^{-1/2}$ where C_3 is independent on μ . We therefore get:

$$|x/t + \omega'(s)| \geq \frac{C_3}{\mu^{1/4}} |s|^{-1/2} - 2^{-k/2} C_2.$$

Now, since $s \in \text{supp} \psi_k$, one has

$$\begin{aligned} |x/t + \omega'(s)| &\geq \frac{C_3}{\mu^{1/4}} \frac{1}{\sqrt{2}} 2^{-k/2} - 2^{-k/2} C_2 \\ &\geq \frac{C_3}{\mu^{1/4}} 2^{-k/2} \end{aligned}$$

provided

$$\frac{C_3}{\mu^{1/4} \sqrt{2}} - C_2 \geq C \mu^{1/4}$$

with C independent on k, μ, x . We therefore set

$$C_2 = \frac{C_3}{2\sqrt{2}\mu^{1/4}}. \quad (4.2.26)$$

Now, one can apply the Van der Corput Lemma 4.2.4 and get:

$$\left| \int_{2^{k-1}}^{\xi} e^{it((x/t)s + \omega(s))} ds \right| \leq C |t|^{-1} 2^{k/2} \mu^{1/4}. \quad (4.2.27)$$

Using Lemma 4.2.2, one has:

$$\left| \frac{d}{d\xi} \widehat{P_k \varphi} \right|_2 \leq 2^{-k} (|\varphi|_2 + |x \partial_x \varphi|_2). \quad (4.2.28)$$

Putting together (4.2.25), (4.2.27) and (4.2.28), one finally gets, for all $k \in I_1$:

$$\int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \leq \mu^{1/4} \frac{C}{|t|} (|\varphi|_2 + |x \partial_x \varphi|_2).$$

We now sum over $k \in I_1$. Since the set has a $O(\log(|t|))$ number of elements (recall that it is included in $\{\lambda(t) \leq 2^k \leq \Lambda(t)\}$), we get:

$$\begin{aligned} S_{21} &\leq \mu^{1/4} \frac{C}{|t|} \sum_{k \in I_1} (|\varphi|_2 + |x \partial_x \varphi|_2) \\ &\leq \frac{C \mu^{1/4}}{|t|} \log(|t|) (|\varphi|_2 + |x \partial_x \varphi|_2) \\ &\leq C \mu^{1/4} |t|^{-1/2} (|\varphi|_2 + |x \partial_x \varphi|_2). \end{aligned} \quad (4.2.29)$$

The control for S_{23} is similar and therefore we omit it and focus on the most difficult term which is S_{22} . One starts to notice that there is a finite number of terms which is of the form $C \log(\mu)$ with C independent on t, x, μ in the set I_2 . Indeed, if $k \in I_2$ then one has

$$-\log(C_2) + \log(|t/x|) \leq k \leq \log(C_2) + \log(|t/x|),$$

and C_2 has been set in (4.2.26). Therefore, it suffices to control the integrals $\int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi$ for $k \in I_2$ by a term of the form $\frac{C}{t^{1/2}}$ with C independent on x, t, k . For $k \in I_2$, the derivative of the phase Φ may vanishes, and one needs to control the second derivative and use a Van der Corput type result. Let c be the minimum of $|\Phi'|$ on $[2^{k-1}; 2^{k+1}]$.

- **1st case** : $\Phi'(c) = 0$ We split the integral into three terms:

$$\int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi = \int_{2^{k-1}}^{c-\delta} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi + \int_{c-\delta}^{c+\delta} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi + \int_{c+\delta}^{2^{k+1}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi. \quad (4.2.30)$$

The second integral of the right hand side of (4.2.30) is estimated as follows:

$$\begin{aligned} \left| \int_{c-\delta}^{c+\delta} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| &\leq 2\delta |\widehat{P_k \varphi}(\xi)|_{\infty} \\ &\leq 2\delta 2^{-sk} (|\varphi|_{H^s} + |x \partial_x \varphi|_2) \end{aligned} \quad (4.2.31)$$

where we used Lemma 4.2.3 to derive the last inequality, with $s > 1/2$ to be set. We now focus on the control of the first integral of the right hand side of (4.2.30) (the last integral is controlled by using the same technique). Integrating by parts, one gets:

$$\begin{aligned} \left| \int_{2^{k-1}}^{c-\delta} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| &= \left| - \int_{2^{k-1}}^{c-\delta} \int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \frac{d}{d\xi} (\widehat{P_k \varphi})(\xi) d\xi \right| \\ &\leq 2^{k/2} \sup_{\xi \in \text{supp} \widehat{\varphi} \cap [2^{k-1}; c-\delta]} \left| \int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \right| \frac{d}{d\xi} |\widehat{P_k \varphi}|_2. \end{aligned} \quad (4.2.32)$$

One now estimates $\left| \int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \right|$ for $\xi \in [2^{k-1}; c-\delta] \cap \text{supp} \widehat{\varphi}$. Recall that $\omega(\xi) = \frac{1}{\sqrt{\mu}} g(\sqrt{\mu} \xi)$ with (4.2.15), and that $\text{supp} \widehat{\varphi} \subset [y_0/\sqrt{\mu}; +\infty[$. Therefore, for $\xi \in [2^{k-1}; c-\delta] \cap \text{supp} \widehat{\varphi}$, there exists $C > 0$ such that $|\Phi''(\xi)| \geq \frac{C}{\mu^{1/4} |\xi|^{3/2}}$ and therefore, one has:

$$\begin{aligned} |\Phi'(s)| &= \int_c^s |\Phi''(s)| ds + \Phi'(c) \\ &\geq \int_c^s \frac{C}{\mu^{1/4} \xi^{3/2}} \\ &\geq C \mu^{-1/4} \left(\frac{1}{\sqrt{c}} - \frac{1}{\sqrt{s}} \right) \\ &\geq C \mu^{-1/4} \frac{c-s}{\sqrt{c} \sqrt{s} (\sqrt{c} + \sqrt{s})} \geq C \mu^{-1/4} \frac{\delta}{2^{3k/2}} \end{aligned}$$

where we used the fact that $|c-s| \geq \delta$ for $s \in [2^{k-1}; \xi]$ and $\xi \in [2^{k-1}; c-\delta]$. Therefore, using Van der Corput's Lemma 4.2.4, one gets

$$\int_{2^{k-1}}^{\xi} e^{it\Phi(s)} ds \leq C \frac{\mu^{1/4} 2^{3k/2}}{t\delta}. \quad (4.2.33)$$

Using Lemma 4.2.2, one has

$$\left| \frac{d}{d\xi} \widehat{P_k \varphi} \right|_2 \leq 2^{-k} (|\varphi|_2 + |x \partial_x \varphi|_2) \quad (4.2.34)$$

and putting together (4.2.33) and (4.2.34) in (4.2.32), one obtains:

$$\left| \int_{2^{k-1}}^{c-\delta} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| \leq C \frac{\mu^{1/4} 2^{3k/2}}{t\delta} 2^{k/2} 2^{-k} (|\varphi|_2 + |x \partial_x \varphi|_2). \quad (4.2.35)$$

Putting together (4.2.35) and (4.2.31), one finally obtains:

$$\left| \int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| \leq C \left(\frac{\mu^{1/4} 2^k}{t\delta} + \delta 2^{-sk} \right) (|\varphi|_{H^1} + |x \partial_x \varphi|_2). \quad (4.2.36)$$

The above right hand side is minimal with respect to δ if $\delta = 2^{sk/2} \mu^{1/8} \frac{2^{k/2}}{t^{1/2}}$. We therefore set $s = 1$ and get the control:

$$\left| \int_{\mathbb{R}} e^{it\Phi(\xi)} \widehat{P_k \varphi}(\xi) d\xi \right| \leq \mu^{1/8} \frac{C}{t^{1/2}}.$$

At last, one gets by summation on $k \in I_2$ (recall that there is a $O(\log(\mu))$ number of terms in I_2):

$$S_{22} \leq \frac{\mu^{1/4}}{t^{1/2}} (|\varphi|_{H^1} + |x \partial_x \varphi|_2). \quad (4.2.37)$$

-2nd case : $\Phi'(c) \neq 0$ It is the same technique as above, noticing that in this case Φ' is monotonic and does not vanish, and therefore c is one of the bounds of the integral.

Conclusion : Putting together (4.2.22), (4.2.23), (4.2.24), (4.2.29) and (4.2.37), and taking the supremum over all $x \in \mathbb{R}^*$ (note that all these estimates are independent on x), one gets:

$$|e^{it\omega(D)} \varphi|_{\infty} \leq C \left(\frac{1}{\mu^{1/4}} \frac{1}{(1+t/\sqrt{\mu})^{1/8}} + \frac{1}{(1+t/\sqrt{\mu})^{1/2}} \right) (|\varphi|_{H^1} + |x \partial_x \varphi|_2).$$

□

We now give a short proof for a better decay estimate, but with other norms of control:

Proof of Theorem 4.2.6 We use the same notations as ones of the proof of Theorem 4.2.5. One can control the first integral of the right hand side of (4.2.16) differently:

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{i(t\omega(\xi)+x\xi)} \varphi(\xi) d\xi \right| \leq \left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{i(t\omega(\xi)+x\xi)} d\xi \right|_{\infty} |\varphi|_{L^1}.$$

We now control the integral $\int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} d\xi$ in L^∞ norm. One has $\Phi'''(\xi) = \omega'''(\xi) = \mu g'''(\sqrt{\mu}\xi)$ with the notations of the proof of Theorem 4.2.5. One can check by computation that $g'''(\xi) \underset{\xi \rightarrow 0}{\sim} -1$ and therefore, using Van der Corput's Lemma 4.2.4, one gets:

$$\int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{it\Phi(\xi)} d\xi \leq \frac{C}{(\mu t)^{1/3}}.$$

One gets the first estimate of Theorem 4.2.6.

Another way to control the first integral of the right hand side of (4.2.16) is by integrating by parts:

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{i(t\omega(\xi)+x\xi)} \varphi(\xi) d\xi \right| = \left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} \int_0^\xi e^{i(t\omega(s)+xs)} ds \frac{d}{d\xi} \widehat{\varphi}(\xi) d\xi \right|$$

and thus, using Cauchy-Schwarz inequality, one gets:

$$\left| \int_{|\xi| \leq \frac{y_0}{\sqrt{\mu}}} e^{i(t\omega(\xi)+x\xi)} \varphi(\xi) d\xi \right| \leq \frac{\sqrt{y_0}}{\mu^{1/4}} \sup_{\xi \in [0, \frac{y_0}{\sqrt{\mu}}]} \left| \int_0^\xi e^{i(t\omega(s)+xs)} ds \right| \frac{d}{d\xi} \widehat{\varphi}|_2.$$

One estimates as above

$$\sup_{\xi \in [0; \frac{20}{\sqrt{\mu}}]} \left| \int_0^\xi e^{i(t\omega(s)+xs)} ds \right| \leq \frac{C}{(\mu t)^{1/3}}$$

and one gets the second estimate of Theorem 4.2.6. \square

Remark 4.2.7 *It should be possible to prove a dispersive decay in dimension 2, using the previous results, since the phase Φ introduced in the proof of Theorem 4.2.5 has a radial symmetry.*

4.3 Local existence for the Water-Waves equations in weighted spaces in dimension $d = 1, 2$

In view of practical use of Theorem 4.2.5 to prove some long time (or even global time) results in the case of the full Water-Waves equations (4.1.4), one may need to control $x\partial_x\varphi$ in L^2 norm, for φ a solution of the equations. We prove in this section a local existence result for the Water-Waves equations in weighted spaces. To this purpose, we briefly give some recalls about the Water-Waves equations, and we state the local existence result proved by [6]. We then give a commutator estimate which is the key point for local existence in weighted spaces. Note that in this Section we do not make any assumption on the dimension unlike in the previous one, and that all the results proved in this section stand in dimensions $d = 1, 2$. The local existence result in weighted spaces is used for instance in [49], where the default of compactness in the rigid lid limit for the Water-Waves equations is investigated.

4.3.1 The Water-Waves equations

We briefly give some recalls about the Water-Waves equations and its local existence (see [37] Chapter 4 for a complete study). Let $t_0 > d/2$ and $N \geq t_0 + t_0 \vee 2 + 3/2$ (where $a \vee b = \max(a, b)$). The energy for the Water-Waves equations is the following (see 4.1.3 for the notations):

$$\mathcal{E}^N(U) = |\mathfrak{P}\psi|_{H^{t_0+3/2}} + \sum_{|\alpha| \leq N} |\zeta_{(\alpha)}|_2 + |\psi_{(\alpha)}|_2 \tag{4.3.38}$$

where $\zeta_{(\alpha)}, \psi_{(\alpha)}$ are the so called Alinhac's good unknowns:

$$\forall \alpha \in \mathbb{N}^d, \zeta_{(\alpha)} = \partial^\alpha \zeta, \quad \psi_{(\alpha)} = \partial^\alpha \psi - \varepsilon \underline{w} \partial^\alpha \zeta$$

with

$$\underline{w} = \frac{\mathcal{G}\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}.$$

We consider solutions $U = (\zeta, \psi)$ of the Water Waves equations in the following space:

$$E_T^N = \{U \in C([0, T]; H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d)), \mathcal{E}^N(U(\cdot)) \in L^\infty([0, T])\}.$$

The following quantity, called the Rayleigh-Taylor coefficient plays an important role in the Water-Waves problem:

$$\underline{a}(\zeta, \psi) = 1 + \varepsilon(\partial_t + \varepsilon \underline{V} \cdot \nabla^\gamma) \underline{w} = -\varepsilon \frac{P_0}{\rho a g} (\partial_z P)|_{z=\varepsilon \zeta}$$

where

$$\underline{V} = \nabla^\gamma \psi - \varepsilon \underline{w} \nabla^\gamma \zeta.$$

As suggested by the notations, \underline{V} and \underline{w} are respectively the horizontal and vertical components of the velocity evaluated at the surface. We recall that the notation $a \vee b$ stands for $\max(a, b)$. We can now state the local existence result by Alvarez-Samaniego Lannes (see [6] and [37] Chapter 3 for reference):

Theorem 4.3.1 *Let $t_0 > d/2, N \geq t_0 + t_0 \vee 2 + 3/2$. Let $U^0 = (\zeta^0, \psi^0) \in E_0^N, b \in H^{N+1 \vee t_0+1}(\mathbb{R}^d)$. Let $\varepsilon, \beta, \gamma, \mu$ be such that*

$$0 \leq \varepsilon, \beta, \gamma \leq 1, \quad 0 \leq \mu \leq \mu_{\max}$$

with $\mu_{\max} > 0$ and moreover assume that:

$$\exists h_{\min} > 0, \exists a_0 > 0, \quad 1 + \varepsilon \zeta^0 - \beta b \geq h_{\min} \quad \text{and} \quad \underline{\mathbf{a}}(U^0) \geq a_0.$$

Then, there exists $T > 0$ and a unique solution $U^\varepsilon \in E_{\varepsilon \sqrt{\beta}}^N$ to (4.1.4) with initial data U^0 . Moreover,

$$\frac{1}{T} = C_1, \quad \text{and} \quad \sup_{t \in [0; \frac{T}{\varepsilon \sqrt{\beta}}]} \mathcal{E}^N(U^\varepsilon(t)) = C_2$$

with $C_i = C(\mathcal{E}^N(U^0), \frac{1}{h_{\min}}, \frac{1}{a_0})$ for $i = 1, 2$.

4.3.2 A commutator estimate

The key point of the local existence result we prove in this Section is the commutator result of Proposition 4.3.7 below. We first need to introduce some technical results about the resolution of the Dirichlet-Neumann problem (4.1.5). We use here the notations of Section 4.3.1. We recall the introduction of the diffeomorphism Σ defined by (4.1.10) which maps Ω into $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$. We also recall that Φ is a solution of (4.1.5) if and only if $\phi = \Phi \circ \Sigma$ is a solution of the following problem:

$$\begin{cases} \nabla^{\mu, \gamma} \cdot P(\Sigma) \nabla^{\mu, \gamma} \phi = 0 \\ \phi_{z=0} = \psi, \quad \partial_n \phi_{z=-1} = 0, \end{cases} \quad (4.3.39)$$

where

$$P(\Sigma) = |\det J_\Sigma| J_\Sigma^{-1} {}^t(J_\Sigma^{-1}), \quad (4.3.40)$$

where J_Σ is the Jacobian matrix of the diffeomorphism Σ . For the sake of clarity in the proof of the main result of this section, we introduce the following notations, for all ζ, b, t_0 satisfying the hypothesis of Theorem 4.3.1:

$$\begin{aligned} M_0 &= C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}\right), \\ M &= C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+2}}, |b|_{H^{t_0+2}}\right), \\ M(s) &= M_0 = C(M_0, |\zeta|_{H^s}, |b|_{H^s}), \end{aligned} \quad (4.3.41)$$

where C denotes a non decreasing function of its arguments. One can prove (see [37] Chapter 2 and equation (2.26)) that

$$P(\Sigma) = I_d + Q(\Sigma) \quad (4.3.42)$$

with

$$\|Q(\Sigma)\|_{H^{s,1}} \leq M_0 |(\varepsilon \zeta, \beta b)|_{H^{s+1/2}}. \quad (4.3.43)$$

One can also prove the coercivity of $P(\Sigma)$:

$$\forall \Theta \in \mathbb{R}^d, \quad P(\Sigma)\Theta \cdot \Theta \geq k(\Sigma)\Theta^2 \quad (4.3.44)$$

where

$$\frac{1}{k(\Sigma)} \leq M_0.$$

Technical results about the boundary problem (4.3.39)

As usual for an elliptic problem of the form (4.3.39) with a Dirichlet condition, we are looking for solutions in the space $\psi + H_{0, \text{surf}}^1(\mathcal{S})$ where $H_{0, \text{surf}}^1(\mathcal{S})$ is the set of functions of $H^1(\mathcal{S})$ with a vanishing trace at $z = 0$ (recall that \mathcal{S} is the flat strip $\mathbb{R}^d \times (-1; 0)$). More precisely, we have:

Definition 4.3.2 We define $H_{0, \text{surf}}^1$ as the completion of $D(\mathbb{R}^d \times [-1; 0])$ endowed with the H^1 norm of \mathcal{S} .

We now define the variational solutions to the elliptic equation (4.3.39). To this purpose, we introduce for all $\psi \in \mathcal{S}'(\mathbb{R}^d)$ the smoothed distribution

$$\psi^\dagger(\cdot, z) = \chi(\sqrt{\mu}z|D^\gamma|)\psi \tag{4.3.45}$$

where χ is a smooth compactly supported function equals to 1 in the neighbourhood of the origin.

Definition 4.3.3 For all $\psi \in \dot{H}^{1/2}$ a variational solution to (4.3.39) is $\phi = \tilde{\phi} + \psi^\dagger$ such that

$$\int_{\mathcal{S}} \nabla^{\mu, \gamma} \tilde{\phi} \cdot P(\Sigma) \nabla^{\mu, \gamma} \varphi = - \int_{\mathcal{S}} \nabla^{\mu, \gamma} \psi^\dagger \cdot P(\Sigma) \nabla^{\mu, \gamma} \varphi$$

for all $\varphi \in H_{0, surf}^1(\mathcal{S})$.

Remark 4.3.4 As expected for an elliptic problem of the form (4.3.39), if $\psi \in H^s(\mathbb{R}^d)$, the solution Φ should be in $H^{s+1/2}(\mathcal{S})$ (as ψ is the trace of Φ on $z = 0$). Therefore, if one uses ψ instead of ψ^\dagger in Definition 4.3.3, then the formulation provides the same regularity for Φ as for ψ . Instead of brutally considering ψ (which is a function defined on \mathbb{R}^d) as a function of \mathcal{S} , we introduce ψ^\dagger which is indeed $1/2$ more regular than ψ and is defined on all \mathcal{S} .

We are now able to give the existence result for the problem (4.3.39) (recall the notations of (4.3.41) for the constants M and $M(s)$, and see [37] for reference):

Proposition 4.3.5 Let $t_0 > d/2$ and $s \geq 0$. Let $\zeta, b \in H^{s+1/2} \cap H^{t_0+1}(\mathbb{R}^d)$ be such that

$$\exists h_{\min} > 0, \quad \forall X \in \mathbb{R}^d, \quad 1 + \varepsilon\zeta(X) - \beta b(X) \geq h_{\min}.$$

Then, for all $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, there exists a unique variational solution ϕ to (4.3.39). Moreover, this solution satisfies:

$$\begin{aligned} \forall 0 \leq s \leq t_0 + 3/2, \quad & \|\Lambda^s \nabla^{\mu, \gamma} \phi\|_2 \leq \sqrt{\mu} M(s + 1/2) |\mathfrak{P}\psi|_{H^s}, \\ \forall t_0 + 3/2 \leq s, \quad & \|\Lambda^s \nabla^{\mu, \gamma} \phi\|_2 \leq \sqrt{\mu} M(|\mathfrak{P}\psi|_{H^s} + |(\varepsilon\zeta, \beta b)|_{H^{s+1/2}}) |\mathfrak{P}\psi|_{H^{t_0+3/2}}. \end{aligned}$$

Moreover, if $s \geq -t_0 + 1$, the same estimates hold on $\|\nabla^{\mu, \gamma} \phi\|_{H^{s,1}}$.

Main result

We prove in this section an estimate for the commutator $[\frac{1}{\mu}\mathcal{G}, x]\partial_x$ in H^s norm, where x is one of the variable of \mathbb{R}^d (we denote x instead of x_j for the sake of clarity). In the case of a flat bottom and a flat surface in dimension 1, one has for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $\xi > 0$:

$$[\frac{1}{\mu}\widehat{\mathcal{G}}, x]\partial_x \varphi = \frac{d}{d\xi} \left(\frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}} \xi \right) \xi \widehat{\varphi}(\xi) = \left(\frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}} + (1 - \tanh(\sqrt{\mu}\xi)^2) \xi \right) \xi \widehat{\varphi}(\xi),$$

and thus one should expect a control of the form

$$|[\frac{1}{\mu}G, x]\partial_x \varphi|_{H^{s+1/2}} \leq C |\mathfrak{P}\varphi|_{H^{s+3/2}},$$

with C independent on μ , where we recall that \mathfrak{P} acts like the square root of the Dirichlet-Neumann operator and is defined by

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}}.$$

Remark 4.3.6 – The Dirichlet-Neumann operator has to be seen as a $3/2$ order operator instead of an one order operator if one needs a uniform bound with respect to μ . Indeed, the brutal bound $\frac{\tanh(\sqrt{\mu}\xi)}{\sqrt{\mu}} \leq \frac{1}{\mu^{1/2}}$ is singular in μ . We still gain one derivative in the commutator $[\frac{1}{\mu}\mathcal{G}, x]$ and have controls uniforms with respect to μ .

– There are two cases depending on s in the statement of Proposition 4.3.5. It is to have tame estimates with respect to the H^s norms of the unknowns, for high values of s .

The following Proposition shows that the result stands true with non flat bottom and surface, and in all dimensions. We denote $\langle a \rangle_{cond} = a$ if $cond$ is satisfied, and else 0.

Proposition 4.3.7 *Let $t_0 > d/2$, $s \geq -1/2$, and $\zeta, b \in H^{t_0+2}(\mathbb{R}^d)$ be such that*

$$\exists h_{\min} > 0, \quad \forall X \in \mathbb{R}^d, \quad 1 + \varepsilon\zeta(X) - \beta b(X) \geq h_{\min}.$$

We denote x one of the variables of \mathbb{R}^d . Then, one has for all $\varphi \in \dot{H}^{s+2}(\mathbb{R}^d)$:

$$|[\frac{1}{\mu}\mathcal{G}, x]\partial_x\varphi|_{H^{s+1/2}} \leq \mu M(s+1)|\mathfrak{P}\varphi|_{H^{s+3/2}} + \langle |\mathfrak{P}\varphi|_{H^{t_0+2}} |(\varepsilon\zeta, \beta b)|_{H^{s+1}} \rangle_{s+1/2 > t_0}.$$

Moreover, if $\nabla f \in H^{t_0+1/2} \cap H^{s+1/2}$, one has for all $\varphi \in \dot{H}^{s+2}(\mathbb{R}^d)$:

$$\begin{aligned} |[\frac{1}{\mu}\mathcal{G}, x]f\partial_x\varphi|_{H^{s+1/2}} &\leq \mu M(s+1)|\nabla^\gamma f|_{H^{t_0+1/2}}|\mathfrak{P}\varphi|_{H^{s+3/2}} \\ &\quad + \langle |\mathfrak{P}\varphi|_{H^{t_0+2}} |(\varepsilon\zeta, \beta b)|_{H^{s+1}} |\nabla^\gamma f|_{H^{s+1/2}} \rangle_{s+1/2 > t_0}. \end{aligned}$$

Remark 4.3.8 *The fact that the commutator is applied to $\partial_x\varphi$ instead of φ is crucial in this result. Remark that the second point of the Proposition implies the first one (just take $f = 1$), but its proof requires to use the first point as one shall see during the proof below.*

Proof The proof is an adaptation of the commutator estimate $[\Lambda^s, \frac{1}{\mu}G]$ which is proved in [37], using a duality argument. We set

$$v = \partial_x\varphi.$$

For all $v \in \dot{H}^{1/2}$, we will denote $v^{\mathfrak{h}}$ the solution of the Dirichlet-Neumann problem (4.3.39) with boundary condition $v^{\mathfrak{h}}_{z=0} = v$. This notation stands for "harmonic extension of v ". We recall the notation ψ^\dagger for all $\psi \in \mathcal{S}'(\mathbb{R}^d)$ given by (4.3.45). We now write, for all $u \in \mathcal{S}(\mathbb{R}^d)$:

$$(\Lambda^{s+1/2}u, [\mathcal{G}, x]v)_2 = (\Lambda^{s+1/2}u, \mathcal{G}xv)_2 - (x\Lambda^{s+1/2}u, \mathcal{G}v)_2.$$

Since $(\Lambda^{s+1/2}xu)^\dagger_{z=0} = \Lambda^{s+1/2}xu$, we get using Green's identity that

$$\begin{aligned} (u, [\mathcal{G}, x]v) &= \int_{\mathcal{S}} \nabla^{\mu, \gamma} \Lambda^{s+1/2} u^\dagger \cdot P(\Sigma) \nabla^{\mu, \gamma} (xv)^{\mathfrak{h}} - \int_{\mathcal{S}} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \cdot \nabla^{\mu, \gamma} (x\Lambda^{s+1/2} u^\dagger) \\ &= \int_{\mathcal{S}} \nabla^{\mu, \gamma} \Lambda^{s+1/2} u^\dagger \cdot P(\Sigma) \nabla^{\mu, \gamma} ((xv)^{\mathfrak{h}} - xv^{\mathfrak{h}}) + \int_{\mathcal{S}} \nabla^{\mu, \gamma} \Lambda^{s+1/2} u^\dagger \cdot P(\Sigma) (\nabla^{\mu, \gamma} x)v^{\mathfrak{h}} \\ &\quad - \int_{\mathcal{S}} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \cdot (\nabla^{\mu, \gamma} x)\Lambda^{s+1/2} u^\dagger. \end{aligned} \tag{4.3.46}$$

We start to control the easiest term of (4.3.46), using Cauchy-Schwarz inequality (recall that $|\cdot|$ stands for norms on \mathbb{R}^d while $\|\cdot\|$ stands for norms on the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1; 0)$):

$$|\int_{\mathcal{S}} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \cdot (\nabla^{\mu, \gamma} x)\Lambda^{s+1/2} u^\dagger| \leq \sqrt{\mu} \|u^\dagger\|_2 \left\| \Lambda^{s+1/2} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_2$$

where the $\sqrt{\mu}$ factor comes from the definition of $\nabla^{\mu, \gamma} x = {}^t(\sqrt{\mu}\partial_x, \gamma\sqrt{\mu}\partial_y, \partial_z)x$ (see Section 4.1.3) and the fact that $\partial_z x = 0$. Using the definition of u^\dagger , one has easily

$$|u^\dagger|_2 \leq |u|_2. \tag{4.3.47}$$

We now use the product estimate of Proposition 4.A.1 and the decomposition $P(\Sigma) = I_d + Q$ of (4.3.42) to write:

$$\begin{aligned} \forall 0 \leq s+1/2 \leq t_0, \quad \left\| \Lambda^{s+1/2} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_2 &\leq C(1 + \|Q\|_{L^\infty H^{t_0}}) \left\| \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_{H^{s+1/2, 0}} \\ \forall t_0 + 3/2 < s+1/2, \quad \left\| \Lambda^{s+1/2} P(\Sigma) \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_2 &\leq C(1 + \|Q\|_{L^\infty H^{t_0}}) \left\| \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_{H^{s+1/2, 0}} \\ &\quad + \left\| \Lambda^{s+1/2} Q \right\|_2 \left\| \nabla^{\mu, \gamma} v^{\mathfrak{h}} \right\|_{L^\infty H^{t_0}}. \end{aligned} \tag{4.3.48}$$

Remark 4.3.9 *We don't treat the case $t_0 < s \leq t_0 + 3/2$, since we will obtain it by interpolation of the two cases above. One has to combine the difference in the product estimate of Proposition 4.A.1 between the cases $0 \leq s+1 \leq t_0$ and $t_0 < s$, and the difference in Lemma 4.3.10 between the cases $0 \leq s+1 \leq t_0 + 3/2$ and $t_0 + 3/2 < s$. For this reason, we split the proof in only two cases, and get the third one by interpolation.*

One has, using (4.3.43) and the embedding $H^{s+1/2,1}$ in $L^\infty H^s(\mathbb{R}^d)$ given by Proposition 4.A.2:

$$\|Q\|_{L^\infty H^{t_0}} \leq M_0, \quad \|\Lambda^{s+1/2}Q\|_2 \leq M_0|(\varepsilon\zeta, \beta b)|_{H^{s+1}}. \quad (4.3.49)$$

We use Proposition 4.3.5 and Proposition 4.A.2 to write:

$$\begin{aligned} \forall 0 \leq s + 1/2 \leq t_0, \quad & \|\nabla^{\mu,\gamma}v^{\flat}\|_{H^{s+1/2,0}} \leq \sqrt{\mu}M(s+1)|\mathfrak{P}v|_{H^{s+1}} \\ \forall t_0 + 3/2 < s + 1/2, \quad & \|\nabla^{\mu,\gamma}v^{\flat}\|_{H^{s+1/2,0}} \leq \sqrt{\mu}M(|\mathfrak{P}v|_{H^{s+1/2}} + |(\varepsilon\zeta, \beta b)|_{H^{s+1}}|\mathfrak{P}v|_{H^{t_0+3/2}}) \\ & \|\nabla^{\mu,\gamma}v^{\flat}\|_{L^\infty H^{t_0}} \leq \sqrt{\mu}M_0|\mathfrak{P}\psi|_{H^{t_0+1/2}}. \end{aligned} \quad (4.3.50)$$

Combining (4.3.49) and (4.3.50) in (4.3.48), one finally gets:

$$\begin{aligned} \forall 0 \leq s + 1/2 \leq t_0, \quad & \|\Lambda^{s+1/2}P(\Sigma)\nabla^{\mu,\gamma}v^{\flat}\|_2 \leq \sqrt{\mu}M(s+1)|\mathfrak{P}v|_{H^{s+1/2}} \\ \forall t_0 + 3/2 < s + 1/2, \quad & \|\Lambda^{s+1/2}P(\Sigma)\nabla^{\mu,\gamma}v^{\flat}\|_2 \leq \sqrt{\mu}M(|\mathfrak{P}v|_{H^{s+1/2}} + |\mathfrak{P}v|_{H^{t_0+1}}|(\varepsilon\zeta, \beta b)|_{H^{s+1}}). \end{aligned} \quad (4.3.51)$$

Combining (4.3.51) with (4.3.47), one finally gets:

$$|\int_{\mathcal{S}} P(\Sigma)\nabla^{\mu,\gamma}v^{\flat} \cdot (\nabla^{\mu,\gamma}x)\Lambda^{s+1/2}u^\dagger| \leq \mu M(s+1)|\mathfrak{P}v|_{H^{s+1/2}} + \langle |\mathfrak{P}v|_{H^{t_0+1}}|(\varepsilon\zeta, \beta b)|_{H^{s+1}} \rangle_{s+1/2 > t_0} |u|_2 \quad (4.3.52)$$

where we recall the notation $\langle a \rangle_{cond} = a$ if $cond$ is satisfied, and else 0. Note that we got the result for $t_0 < s < t_0 + 3/2$ by interpolation. Remembering that $v = \partial_x \varphi$, one gets the control of Proposition 4.3.7 for this term.

We now focus on the most difficult term of (4.3.46) (the last term of (4.3.46) is estimated by a similar technique). Using Cauchy-Schwarz inequality, one gets:

$$|\int_{\mathcal{S}} \nabla^{\mu,\gamma}\Lambda^{s+1/2}u^\dagger \cdot P(\Sigma) \cdot \nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})| \leq \|\Lambda^{s+1}P(\Sigma)\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \|\Lambda^{-1/2}\nabla^{\mu,\gamma}u^\dagger\|_2.$$

Using the definition of u^\dagger given by (4.3.45), one has easily

$$\|\Lambda^{-1/2}\nabla^{\mu,\gamma}u^\dagger\|_2 \leq C\mu^{1/4}|u|_2. \quad (4.3.53)$$

The product estimate of Proposition 4.A.1 shows that

$$\begin{aligned} \forall 0 \leq s + 1 \leq t_0, \quad & \|\Lambda^{s+1}P(\Sigma)\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \leq C(1 + \|Q\|_{L^\infty H^{t_0}})\|\Lambda^{s+1}\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \\ \forall t_0 + 3/2 \leq s + 1, \quad & \|\Lambda^{s+1}P(\Sigma)\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \leq C(1 + \|Q\|_{L^\infty H^{t_0}})\|\Lambda^{s+1}\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \\ & + \|\Lambda^{s+1}Q\|_2 \|\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_{L^\infty H^{t_0}}. \end{aligned}$$

One has $\|Q\|_{L^\infty H^{t_0}} \leq M_0$ and $\|\Lambda^{s+1}Q\|_2 \leq M_0|(\varepsilon\zeta, \beta b)|_{H^{s+3/2}}$. The proof of Proposition 4.3.7 is completed if one can prove:

$$\begin{aligned} \forall s \geq -1/2, \quad & \|\Lambda^{s+1}\nabla^{\mu,\gamma}((xv)^{\flat} - xv^{\flat})\|_2 \leq \mu M(s+1/2)|\mathfrak{P}\varphi|_{H^{s+1}} \\ & + \langle |\mathfrak{P}\varphi|_{H^{t_0+2}}|(\varepsilon\zeta, \beta b)|_{H^{s+1}} \rangle_{s+1 > t_0}. \end{aligned} \quad (4.3.54)$$

Note that the case $t_0 \leq s \leq t_0 + 3/2$ is obtained by interpolation. We now prove the estimate (4.3.54) for $s+1 \leq t_0$. The case $s+1 > t_0 + 3/2$ is estimated by the same technique, so we omit it for the sake of clarity. The case $t_0 \leq s \leq t_0 + 3/2$ is obtained by interpolation. The quantity $w = (xv)^{\flat} - xv^{\flat}$ satisfies the following elliptic equation:

$$\begin{cases} \nabla^{\mu,\gamma} \cdot (P(\Sigma)\nabla^{\mu,\gamma}w) = -\nabla^{\mu,\gamma} \cdot (P(\Sigma)\nabla^{\mu,\gamma}xv^{\flat}) - P(\Sigma)\nabla^{\mu,\gamma}x \cdot \nabla^{\mu,\gamma}v^{\flat} \\ w_{z=0} = 0, \quad \partial_n w_{w=-1} = 0. \end{cases} \quad (4.3.55)$$

We now prove the following elliptic regularity type result:

Lemma 4.3.10 *Let $t_0 > d/2$, $s \geq 0$ and Σ be the diffeomorphism from Ω to \mathcal{S} defined by (4.1.10). Let $\zeta, b \in H^{t_0+1} \cap H^{s+1/2}(\mathbb{R}^d)$ be such that*

$$\exists h_{\min} > 0, \quad \forall X \in \mathbb{R}^d, \quad 1 + \varepsilon\zeta(X) - \beta b(X) \geq h_{\min}.$$

We consider the following elliptic problem:

$$\begin{cases} \nabla^{\mu,\gamma} \cdot (P(\Sigma)\nabla^{\mu,\gamma}w) = -\nabla^{\mu,\gamma} \cdot g + f \\ w_{z=0} = 0, \quad \partial_n w_{w=-1} = 0. \end{cases} \quad (4.3.56)$$

Then, there exists a unique variational solution $w \in H_{0,surf}^1(\mathcal{S})$ to the boundary value problem (4.3.56). Moreover, one has :

$$\begin{aligned} \forall 0 \leq s \leq t_0 + 3/2, & \quad \|\Lambda^s \nabla^{\mu,\gamma} w\|_2 \leq M(s+1/2)(\|g\|_{H^{s,1}} + \|f\|_{H^{s-1,0}}) \\ \forall t_0 + 3/2 \leq s, & \quad \|\Lambda^s \nabla^{\mu,\gamma} w\|_2 \leq M(\|g\|_{H^{s,1}} + \|f\|_{H^{s-1,0}} \\ & \quad + |(\varepsilon\zeta, \beta b)|_{H^{s+1/2}}(\|g\|_{H^{t_0+1/2,1}} + \|f\|_{H^{t_0-1/2,0}})). \end{aligned}$$

Proof By definition, w is a variational solution to (4.3.56) if for all $\theta \in H_{0,surf}^1(\mathcal{S})$, one has

$$\begin{aligned} \int_{\mathcal{S}} \nabla^{\mu,\gamma} w \cdot P \nabla^{\mu,\gamma} \theta &= - \int_{\mathcal{S}} (\nabla^{\mu,\gamma} \cdot g) \theta + \int_{\mathcal{S}} f \theta \\ &= \int_{\mathcal{S}} g \cdot \nabla^{\mu,\gamma} \theta + \int_{\mathcal{S}} f \theta - \int_{z=-1} g \theta. \end{aligned} \quad (4.3.57)$$

The existence and uniqueness follows from the coercivity of P given by (4.3.44) and the Lax-Milgram Theorem. We now introduce $\Lambda_\delta^s = \Lambda^s \chi(\delta\Lambda)$ for $\delta > 0$ and χ a smooth and compactly supported function, equals to 1 in a neighbourhood of zero. If $w \in H_{0,surf}^1(\mathcal{S})$ is the variational solution of (4.3.56), then $(\Lambda_\delta^s)^2 w$ is also in $H_{0,surf}^1(\mathcal{S})$, and thus, taking $\theta = (\Lambda_\delta^s)^2 w$ in (4.3.57) (recall that $P = I + Q$):

$$\begin{aligned} \int_{\mathcal{S}} \nabla^{\mu,\gamma} \Lambda_\delta^s w \cdot P \nabla^{\mu,\gamma} \Lambda_\delta^s \theta &= \int_{\mathcal{S}} \Lambda_\delta^s g \cdot \nabla^{\mu,\gamma} \Lambda_\delta^s w + \int_{\mathcal{S}} \Lambda_\delta^s f \Lambda_\delta^s w - \int_{z=-1} \Lambda_\delta^s g \Lambda_\delta^s w \\ & \quad + \int_{\mathcal{S}} [Q, \Lambda_\delta^s] \nabla^{\mu,\gamma} w \cdot \nabla^{\mu,\gamma} \Lambda_\delta^s w. \end{aligned}$$

We now use Cauchy-Schwarz inequality and the coercivity of P (see (4.3.44)) to get:

$$\begin{aligned} k(\Sigma) \|\nabla^{\mu,\gamma} \Lambda_\delta^s w\|_2^2 &\leq \|\Lambda_\delta^s g\|_2 \|\nabla^{\mu,\gamma} \Lambda_\delta^s w\|_2 + \|\Lambda_\delta^{s-1} f\|_2 \|\Lambda_\delta^{s+1} w\|_2 + |\Lambda_\delta^{s+1/2} w(\cdot, -1)|_2 |\Lambda_\delta^{s-1/2} g|_2 \\ & \quad + \|[Q, \Lambda_\delta^s] \nabla^{\mu,\gamma} w\|_2 \|\nabla^{\mu,\gamma} \Lambda_\delta^s w\|_2. \end{aligned} \quad (4.3.58)$$

Since $\Lambda_\delta^{s+1} w \in H_{0,surf}^1(\mathcal{S})$, one has using Poincaré's inequality (recall that $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$ and $H_{0,surf}^1(\mathcal{S})$ is the set of H^1 functions of \mathcal{S} with vanishing trace at the surface):

$$\begin{aligned} \|\Lambda_\delta^{s+1} w\|_2 &\leq \|\Lambda_\delta^s w\|_{H^{1,0}} \\ &\leq \|\partial_z \Lambda_\delta^s w\|_2 \\ &\leq \|\Lambda_\delta^s \nabla^{\mu,\gamma} w\|_{L^2}. \end{aligned} \quad (4.3.59)$$

Moreover, one has for all $s \in \mathbb{R}$, using Proposition 4.A.2:

$$|u|_{H^{s-1/2}} \leq \|u\|_{H^{s,1}}$$

and thus one can write:

$$|\Lambda_\delta^{s-1/2} g(\cdot, -1)|_2 \leq \|g\|_{H^{s,1}}. \quad (4.3.60)$$

From now, the idea of the proof is to show a commutator estimate of the form

$$\begin{aligned} \forall 0 \leq s \leq t_0 + 3/2, & \quad \|[\Lambda_\delta^s, Q] (\nabla^{\mu,\gamma} w)\|_2 \leq M(s+1/2) \|\Lambda_\delta^{s-\varepsilon} \nabla^{\mu,\gamma} w\|_2 \\ \forall t_0 + 3/2 \leq s, & \quad \|[\Lambda_\delta^s, Q] (\nabla^{\mu,\gamma} w)\|_2 \leq M(|(\varepsilon\zeta, \beta b)|_{H^{s+1/2}} (\|g\|_{H^{t_0+1/2,1}} \\ & \quad + \|f\|_{H^{t_0-1/2,0}}) \|\Lambda_\delta^{s-\alpha} \nabla^{\mu,\gamma} w\|_2, \end{aligned} \quad (4.3.61)$$

for some $\alpha > 0$.

Putting together (4.3.59), (4.3.60) and (4.3.61) into (4.3.58), letting δ goes to zero and using a finite induction on s , one gets the result of Lemma 4.3.10. However the commutator estimate (4.3.61) is technical to obtain, and therefore we omit the proof for the sake of clarity (see [37] Lemma 2.38 for details). \square

We now go back to the proof of (4.3.54). For $0 \leq s + 1 \leq t_0$, one has, using Lemma 4.3.10:

$$\begin{aligned} \|\Lambda^{s+1} \nabla^{\mu, \gamma} ((xv)^{\flat} - xv^{\flat})\|_2 &\leq M(s + 1/2) \|P(\Sigma) (\nabla^{\mu, \gamma} x) v^{\flat}\|_{H^{s+1,1}} + \|P(\Sigma) \nabla^{\mu, \gamma} x \cdot \nabla^{\mu, \gamma} v^{\flat}\|_{H^{s,0}} \\ &\leq M(s + 1/2) (1 + \|Q\|_{H^{t_0+1,1}}) (\|v^{\flat}\|_{H^{s+1,1}} + \|\nabla^{\mu, \gamma} v^{\flat}\|_{H^{s,0}}) \end{aligned}$$

where we used the product estimate of Proposition 4.A.1 to derive the last inequality. Using Lemma 4.3.5, we get the bound

$$\|\nabla^{\mu, \gamma} v^{\flat}\|_{H^{s,0}} \leq \sqrt{\mu} |\mathfrak{P}v|_{H^s}. \quad (4.3.62)$$

To control v^{\flat} in $H^{s+1,1}$ norm, we recall that $v = \partial_x \varphi$ and we notice that

$$(\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat} \in H_{0, surf}^1(\mathcal{S})$$

and we write

$$\|v^{\flat}\|_{H^{s+1,1}} \leq \|(\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat}\|_{H^{s+1,1}} + \|\partial_x (\varphi)^{\flat}\|_{H^{s+1,1}}. \quad (4.3.63)$$

To control the first term of the right hand side of (4.3.63), we use the Poincaré's inequality on the flat strip \mathcal{S} :

$$\|(\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat}\|_{H^{s+1,1}} \leq \|\nabla^{\mu, \gamma} ((\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat})\|_{H^{s,0}}.$$

Now, if one defines

$$w = (\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat}$$

then w satisfies the following boundary problem:

$$\begin{cases} \nabla^{\mu, \gamma} \cdot (P(\Sigma) \nabla^{\mu, \gamma} w) = -\nabla^{\mu, \gamma} \cdot g \\ w_{z=0} = 0, \quad \partial_n w_{z=-1} = -g \cdot e_z \end{cases} \quad (4.3.64)$$

with $g = [P(\Sigma), \partial_x] \nabla^{\mu, \gamma} \varphi^{\flat}$, and e_z the unit normal vector in the vertical direction. Adapting the proof of Lemma 4.3.10 (see also Lemma 2.38 in [37]), one can prove

$$\|\nabla^{\mu, \gamma} w\|_{H^{s,0}} \leq M(s + 1/2) \|\nabla^{\mu, \gamma} \varphi^{\flat}\|_{H^s}$$

and using Proposition 4.3.5, one finally gets

$$\|(\partial_x \varphi)^{\flat} - \partial_x (\varphi)^{\flat}\|_{H^{s+1,1}} \leq M(s + 1/2) |\mathfrak{P}\varphi|_{H^s}. \quad (4.3.65)$$

To control the second term of the rhs of (4.3.63), one uses Proposition 4.3.5 again:

$$\begin{aligned} \|\partial_x (\varphi)^{\flat}\|_{H^{s+1,1}} &\leq \|\nabla^{\mu, \gamma} \varphi^{\flat}\|_{H^{s+1,1}} \\ &\leq M(s + 1/2) |\mathfrak{P}\varphi|_{H^{s+1}}. \end{aligned} \quad (4.3.66)$$

Putting together (4.3.65) and (4.3.66) into (4.3.63), one gets

$$\|v^{\flat}\|_{H^{s+1,1}} \leq M(s + 1/2) |\mathfrak{P}\varphi|_{H^{s+1}}. \quad (4.3.67)$$

Putting together (4.3.62) and (4.3.67), we proved:

$$\|\Lambda^{s+1} \nabla^{\mu, \gamma} ((xv)^{\flat} - xv^{\flat})\|_2 \leq M(s + 1/2) |\mathfrak{P}\varphi|_{H^{s+1}}$$

which is the desired result (4.3.54).

The proof of the second point of the Proposition only requires a small adaptation of the proof above. The only technical change is the control of $\|v^{\flat}\|_{H^{s+1,1}}$. We write, with $v = f \partial_x \varphi$:

$$\|v^{\flat}\|_{H^{s+1,1}} \leq \|(f \partial_x \varphi)^{\flat} - f(\partial_x \varphi)^{\flat}\|_{H^{s+1,1}} + \|f(\partial_x \varphi)^{\flat}\|_{H^{s+1,1}} \quad (4.3.68)$$

The second term of the right hand side of (4.3.68) is controlled using Proposition 4.A.1, and the control of $(\partial_x \varphi)^{\flat}$ proved above. To control the first term of the right hand side of (4.3.68), one remarks that $w = (f \partial_x \varphi)^{\flat} - f(\partial_x \varphi)^{\flat} \in H_{0, surf}^1(\mathcal{S})$ solves the following boundary problem:

$$\begin{cases} \nabla^{\mu, \gamma} \cdot (P(\Sigma) \nabla^{\mu, \gamma} w) = -\nabla^{\mu, \gamma} f P(\Sigma) \nabla^{\mu, \gamma} (\partial_x \varphi)^{\flat} - \nabla^{\mu, \gamma} \cdot (P(\Sigma) (\partial_x \varphi)^{\flat} \nabla^{\mu, \gamma} f) \\ w_{z=0} = 0, \quad \partial_n w_{z=-1} = -P(\Sigma) (\partial_x \varphi)^{\flat} \nabla^{\mu, \gamma} \zeta \cdot e_z \end{cases} \quad (4.3.69)$$

and we use the Poincaré's inequality on the flat strip \mathcal{S} to control $\|(f \partial_x \varphi)^{\flat} - f(\partial_x \varphi)^{\flat}\|_{H^{s+1,1}}$ by $\|\nabla^{\mu, \gamma} w\|_{H^{s,0}}$, and adapt the proof of Lemma 4.3.10 above to get the control of this latter term.

□

4.3.3 Local existence in weighted Sobolev Spaces

We prove here an existence result for the Water-Waves equation in weighted Sobolev spaces. We recall that x denotes the identity of \mathbb{R}^d , and we define, for all $N \geq 2$ the energy \mathcal{E}_x^N by

$$\mathcal{E}_x^N = \mathcal{E}^N(\zeta, \psi) + \sum_{\alpha \in \mathbb{N}^d, 1 \leq |\alpha| \leq N-2} |x\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}x\psi_{(\alpha)}|_2^2$$

where \mathcal{E}^N is the standard energy for the Water-Waves equations given by (4.3.38).

Theorem 4.3.11 *Let us consider the assumptions of Theorem 4.3.1, and then consider $T > 0$ and (ζ, ψ) the unique solution provided by the theorem on $[0; \frac{T}{\varepsilon\sqrt{\beta}}$ of the Water-Waves equation (4.1.4). If $(\zeta^0, \psi^0) \in \mathcal{E}_x^N$, then one has*

$$(\zeta, \psi) \in L^\infty([0; \frac{T}{\varepsilon\sqrt{\beta}}], \mathcal{E}_x^N),$$

with

$$(\zeta, \psi)_{L^\infty([0; \frac{T}{\varepsilon\sqrt{\beta}}], \mathcal{E}_x^N)} \leq C_2,$$

where C_2 is a constant of the form $C_2 = C(\mathcal{E}^N(U^0), \frac{1}{h_{\min}}, \frac{1}{a_0})$ with C a non decreasing continuous function of its arguments.

Remark 4.3.12 – *Note that there are less space derivatives for the weighted norms $|x\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}x\psi_{(\alpha)}|_2^2$ than for the "Sobolev" norms \mathcal{E}^N . This is due to the presence of commutators of the form $[\mathcal{G}, x]\psi_{(\alpha)}$ in the evolution equation for $\psi_{(\alpha)}$, which are of order 1 (at least) in $\psi_{(\alpha)}$.*

– *Note also that we control $\mathfrak{P}\psi_{(\alpha)}, \zeta_{(\alpha)}$ only for $|\alpha| \geq 1$. This is due to the fact that we only control terms of the form $x\partial_x\varphi$.*

Proof The proof is an adaptation of the proof of the Theorem 4.3.1 (see [37] Chapter 4 for a full proof). Therefore, we only give the main ideas and insist on the specificity of using weights. Considering the result given by Theorem 4.3.1, we only need to recover estimates for weighted norms (estimates for the "classical Sobolev" norms of \mathcal{E}^N are done in the proof of the local existence result of [37]). We recall (see for instance [37] Chapter 3 for reference) that one has:

$$(\psi, \frac{1}{\mu}G\psi)_2 \leq M_0|\mathfrak{P}\psi|_2^2 \quad \text{and} \quad |\mathfrak{P}\psi|_2^2 \leq M_0(\psi, \frac{1}{\mu}G\psi)_2$$

Therefore, we set, for all $0 \leq |\alpha| \leq N-2$:

$$E^\alpha = \frac{1}{2\mu}(\mathcal{G}x\psi_{(\alpha)}, x\psi_{(\alpha)})_2 + \frac{1}{2}(x\zeta_{(\alpha)}, x\zeta_{(\alpha)})$$

and look for a control of E^α . We now differentiate E^α with respect to time and get, using the symmetry of \mathcal{G} :

$$\frac{d}{dt}E^\alpha = (\mathcal{G}x\psi_{(\alpha)}, \partial_t x\psi_{(\alpha)})_2 + (d\mathcal{G}(\varepsilon\partial_t\zeta)x\psi_{(\alpha)}, x\psi_{(\alpha)})_2 + (\partial_t x\zeta_{(\alpha)}, x\zeta_{(\alpha)})_2 + \frac{1}{2}(x\zeta_{(\alpha)}, (\partial_t \mathbf{a})x\zeta_{(\alpha)})_2.$$

We now need an equation in terms of $\zeta_{(\alpha)}, \psi_{(\alpha)}$. To this purpose, one computes ∂^α of the equations (4.1.4). One gets in the first equation a term of the form

$$\partial^\alpha \mathcal{G}[\varepsilon\zeta, \beta b]\psi = \mathcal{G}[\varepsilon\zeta, \beta b]\partial^\alpha\psi + \sum_{\nu < \alpha, \delta_1 + \dots + \delta_m + l_1 + \dots + l_n + \nu = \alpha} d\mathcal{G}(\partial^{\delta_1}\varepsilon\zeta, \dots, \partial^{\delta_m}\varepsilon\zeta, \partial^{l_1}\beta b, \dots, \partial^{l_n}\beta b)\partial^\nu\psi$$

where $d\mathcal{G}$ denotes the shape derivative of $\mathcal{G}[\varepsilon\zeta, \beta b]$ with respect to the bottom b and the surface ζ . We therefore obtain, after computations, a system of the form (see [37] Chapter 4 for details):

$$\begin{cases} \partial_t \zeta_{(\alpha)} + \varepsilon \bar{V} \cdot \nabla^\gamma \zeta_{(\alpha)} - \frac{1}{\mu} \mathcal{G}\psi_{(\alpha)} = R^\alpha \\ \partial_t \psi_{(\alpha)} + \mathbf{a}\zeta_{(\alpha)} + \varepsilon \bar{V} \cdot \nabla^\gamma \psi_{(\alpha)} = S^\alpha \end{cases} \quad (4.3.70)$$

with

$$|xR^\alpha|_2 + |\mathfrak{P}xS^\alpha|_2 \leq C(\mathcal{E}_x^N) \quad (4.3.71)$$

with C a continuous function of its arguments. In order to get a control of the form (4.3.71), one can adapt the proof of the control for the shape derivatives of \mathcal{G} given in Proposition 3.28 of [37] (we do not detail this proof here). We therefore have, replacing $\partial_t(\zeta_{(\alpha)}, \psi_{(\alpha)})$ by their expression given by (4.3.70):

$$\begin{aligned} \frac{d}{dt} E^\alpha &= \frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, \underline{\mathbf{a}}x\zeta_{(\alpha)})_2 - \frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, \underline{\mathbf{a}}x\zeta_{(\alpha)})_2 \\ &\quad + \varepsilon (x\underline{\mathbf{a}}\zeta_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \zeta_{(\alpha)})_2 + \frac{\varepsilon}{\mu} (\mathcal{G}x\psi_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \psi_{(\alpha)})_2 \\ &\quad + \frac{1}{\mu} (d\mathcal{G}(\varepsilon\partial_t\zeta)x\psi_{(\alpha)}, x\psi_{(\alpha)})_2 + \frac{1}{\mu} ([\mathcal{G}, x]\psi_{(\alpha)}, x\underline{\mathbf{a}}\zeta_{(\alpha)})_2. \end{aligned} \quad (4.3.72)$$

The first two terms of (4.3.72) are the one of order 1 with respect to the unknowns $x\zeta_{(\alpha)}, x\psi_{(\alpha)}$ but cancel one another, thanks to the symmetry of the equation.

The two terms of the second line of (4.3.72) are of contributions of order 0 to the energy estimate, with respect to the unknowns, thanks to the symmetry. More precisely, one computes, integrating by parts:

$$\begin{aligned} (x\underline{\mathbf{a}}\zeta_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \zeta_{(\alpha)})_2 &= \sum_{j=1}^d ((x\underline{\mathbf{a}}\zeta_{(\alpha)}, x\bar{\mathbf{V}}_j \partial_j \zeta_{(\alpha)})_2 \\ &= - \sum_{j=1}^d (x\bar{\mathbf{V}}_j \partial_j \zeta_{(\alpha)}, x\underline{\mathbf{a}}\zeta_{(\alpha)})_2 - (\partial_j (\bar{\mathbf{V}}_j \underline{\mathbf{a}}) x\zeta_{(\alpha)}, x\zeta_{(\alpha)})_2 - ((\partial_j x) \bar{\mathbf{V}}_j \underline{\mathbf{a}}\zeta_{(\alpha)}, x\zeta_{(\alpha)})_2 \end{aligned}$$

and therefore one has

$$(x\underline{\mathbf{a}}\zeta_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \zeta_{(\alpha)})_2 = -\frac{1}{2} (\partial_j (\bar{\mathbf{V}}_j \underline{\mathbf{a}}) x\zeta_{(\alpha)}, x\zeta_{(\alpha)})_2 - \frac{1}{2} ((\partial_j x) \bar{\mathbf{V}}_j \underline{\mathbf{a}}\zeta_{(\alpha)}, x\zeta_{(\alpha)})_2. \quad (4.3.73)$$

Using Proposition 4.B.1, it is possible to prove that $|\bar{\mathbf{V}}_j|_{W^{1,\infty}} + |\underline{\mathbf{a}}|_{W^{1,\infty}} \leq \mathcal{E}^N$ and therefore one gets from (4.3.73) the control:

$$\begin{aligned} |(x\underline{\mathbf{a}}\zeta_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \zeta_{(\alpha)})_2| &\leq C(\mathcal{E}^N) (|x\zeta_{(\alpha)}|_2^2 + |\zeta_{(\alpha)}|_2 |x\zeta_{(\alpha)}|_2) \\ &\leq C(\mathcal{E}^N) E^\alpha \end{aligned} \quad (4.3.74)$$

where C is continuous and non decreasing. For the control of the second term of the second line of (4.3.72), one writes:

$$\frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, x\bar{\mathbf{V}} \cdot \nabla^\gamma \psi_{(\alpha)})_2 = \frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, \bar{\mathbf{V}} \cdot (x\psi_{(\alpha)}))_2 - \frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, (\bar{\mathbf{V}} \cdot \nabla^\gamma x)\psi_{(\alpha)})_2.$$

We use Proposition 4.B.5 to write (recall the notations of M given by (4.3.41)):

$$|\frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, \bar{\mathbf{V}} \cdot (x\psi_{(\alpha)}))_2| \leq M |\underline{\mathbf{V}}|_{W^{1,\infty}} |\mathfrak{P}x\psi|_2^2$$

and again, using the Proposition (4.B.1) one can control the $W^{1,\infty}$ norm of $\underline{\mathbf{V}}$ by the energy and get

$$|\frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, \bar{\mathbf{V}} \cdot (x\psi_{(\alpha)}))_2| \leq C(\mathcal{E}^N) E^\alpha. \quad (4.3.75)$$

We now use Proposition 4.B.3 with $s = 1$ to compute:

$$|\frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, (\bar{\mathbf{V}} \cdot \nabla^\gamma x)\psi_{(\alpha)})_2| \leq \mu M |\mathfrak{P}x\psi_{(\alpha)}|_2 |(\bar{\mathbf{V}} \cdot \nabla^\gamma x)\psi_{(\alpha)}|_2$$

and one can prove, using the definition of \mathfrak{P} and standard Sobolev estimates:

$$|(\bar{\mathbf{V}} \cdot \nabla^\gamma x)\psi_{(\alpha)}|_2 \leq |\bar{\mathbf{V}}|_{H^{t_0}} |\mathfrak{P}\psi_{(\alpha)}|_2$$

and therefore, using Proposition (4.B.1) again to control $|\bar{\mathbf{V}}|_{H^{t_0}}$ by the energy, one finally gets

$$|\frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, (\bar{\mathbf{V}} \cdot \nabla^\gamma x)\psi_{(\alpha)})_2| \leq C(\mathcal{E}^N) E^\alpha. \quad (4.3.76)$$

Putting together (4.3.75) and (4.3.76), one proved

$$\left| \frac{1}{\mu} (\mathcal{G}x\psi_{(\alpha)}, x\bar{V} \cdot \nabla^\gamma \psi_{(\alpha)})_2 \right| \leq C(\mathcal{E}^N)E^\alpha. \quad (4.3.77)$$

The first term of the third line of (4.3.72) is estimated by using Proposition 4.B.3. The only non trivial remaining term to control in (4.3.72) is the last one, which is the commutator $\frac{1}{\mu}([\mathcal{G}, x]\psi_{(\alpha)}, x\underline{a}\zeta_{(\alpha)})_2$. Recall that $|\alpha| > 1$ and that:

$$\psi_{(\alpha)} = \partial^\alpha \zeta - \varepsilon \underline{w} \partial^\alpha \psi$$

and one gets therefore, using Proposition 4.3.7, one can control both of these terms by $C(\mathcal{E}^N)E^\alpha$. One can obtain by summing on all $\alpha, 1 \leq |\alpha| \leq N-1$ the following energy estimate:

$$\frac{d}{dt} \mathcal{E}_x^N \leq C(\mathcal{E}^N) \mathcal{E}_x^N$$

with C a continuous function of its arguments. Using a Gronwall's Lemma, one can conclude and end the proof of the Theorem. \square .

4.A Estimates on the flat strip \mathcal{S}

We recall the notation $a \vee b = \max(a, b)$ and we define $L^\infty H^s = L^\infty((-1; 0); H^s(\mathbb{R}^d))$ and use the notation $\langle a \rangle_{s>t_0} = a$ if $s > t_0$ and else 0.

Proposition 4.A.1 *Let $t_0 > d/2$. If $s \geq -t_0$, $f, g \in L^\infty H^{t_0} \cap H^{s,0}$, one has $fg \in H^{s,0}$ and*

$$\|fg\|_{H^{s,0}} \leq C \|f\|_{L^\infty H^{t_0}} \|g\|_{H^{s,0}} + \langle \|f\|_{H^{s,0}} \|g\|_{L^\infty H^{t_0}} \rangle_{s>t_0}.$$

The following Proposition states a L^∞ embedding result for the Beppo-Levi spaces:

Proposition 4.A.2 *For all $s \in \mathbb{R}$:*

- (1) *The mapping $u \mapsto u|_{z=0}$ extends continuously from $H^{s+1,1}$ to $H^{s+1/2}(\mathbb{R}^d)$.*
- (2) *The space $H^{s+1/2,1}$ is continuously embedded in $L^\infty H^s$.*

4.B The Dirichlet Neumann Operator

Here are for the sake of convenience some technical results about the Dirichlet Neumann operator, and its estimates in Sobolev norms. See [37] Chapter 3 for complete proofs. The first two propositions give a control of the Dirichlet-Neumann operator.

Proposition 4.B.1 *Let $t_0 > d/2$, $0 \leq s \leq t_0 + 3/2$ and $(\zeta, \beta) \in H^{t_0+1} \cap H^{s+1/2}(\mathbb{R}^d)$ such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0.$$

- (1) *The operator G maps continuously $\dot{H}^{s+1/2}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has*

$$|G\psi|_{H^{s-1/2}} \leq \mu^{3/4} M(s+1/2) |\mathfrak{P}\psi|_{H^s},$$

where $M(s+1/2)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1/2}}, |b|_{H^{s+1/2}})$.

- (2) *The operator G maps continuously $\dot{H}^{s+1}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has*

$$|G\psi|_{H^{s-1/2}} \leq \mu M(s+1) |\mathfrak{P}\psi|_{H^{s+1/2}},$$

where $M(s+1)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1}}, |b|_{H^{s+1}})$.

Moreover, it is possible to replace $G\psi$ by \underline{w} in the previous result, where $\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}$ (vertical component of the velocity $U = \nabla_{X,z} \Phi$ at the surface).

Remark 4.B.2 *In all this paper, we consider the Water-Waves problem in finite depth. This is crucial for all these regularity results on G . For instance, in the linear case $\zeta = b = 0$, the Dirichlet-Neumann operator is $|D^\gamma| \tanh(|D^\gamma|)$ in finite depth, while it is $|D^\gamma|$ in infinite depth. The low frequencies are therefore affected differently.*

Proposition 4.B.3 *Let $t_0 > d/2$, and $0 \leq s \leq t_0 + 1/2$. Let also $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, 1 + \varepsilon\zeta(X) - \beta b(X) \geq h_0.$$

Then, for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, we have

$$(\Lambda^s G\psi_1, \Lambda^s \psi_2)_2 \leq \mu M_0 |\mathfrak{P}\psi_1|_{H^s} |\mathfrak{P}\psi_2|_{H^s},$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

The second result gives a control of the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\Gamma = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$G[\varepsilon \cdot, \beta \cdot] : \begin{array}{ccc} \Gamma & \longrightarrow & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \longmapsto & G[\varepsilon\zeta, \beta b]\psi. \end{array} \quad (4.B.78)$$

We can prove the differentiability of this mapping. The following Proposition gives estimates of the shape derivatives of \mathcal{G} .

Proposition 4.B.4 *Let $t_0 > d/2$ and $(\zeta, b) \in H^{t_0+1}$ be such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) - \beta b(X) + 1 \geq h_0.$$

Then, for all $0 \leq s \leq t_0 + 1/2$,

$$|d^j G(h, k)\psi|_{H^{s-1/2}} \leq M_0 \mu^{3/4} \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^s}.$$

The following commutator estimate is useful (see [37] Proposition 3.30):

Proposition 4.B.5 *Let $t_0 > d/2$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^d)$ such that:*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) - \beta b(X) + 1 \geq h_0.$$

For all $\underline{V} \in H^{t_0+1}(\mathbb{R}^d)^2$ and $u \in H^{1/2}(\mathbb{R}^d)$, one has

$$((\underline{V} \cdot \nabla^\gamma u), \frac{1}{\mu} Gu) \leq M |\underline{V}|_{W^{1,\infty}} |\mathfrak{P}u|_2^2,$$

where M is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+2}}, |b|_{H^{t_0+2}})$.

Chapter 5

The rigid lid limit for the water waves equations

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Abstract

The Water-Waves equations constitute an equivalent model to the free surface Euler equations. The rescaled Water-Waves equations depend strongly on the ratio ε between the amplitude of the wave and the depth of the water. We investigate in this paper the convergence as ε goes to zero of the free surface Euler equations to the so called rigid lid model. We first prove that the only solutions of this model are zero. Due to the conservation of the Hamiltonian, the solutions of the free surface Euler equations converge weakly to zero, but not strongly in the general case, as ε goes to zero. We then study this default of convergence. More precisely, we show a strong convergence result of the solutions of the water waves equations in the Zakharov-Craig-Sulem formulation to the solutions of the linear water-waves equations. It is then easy to observe these latter converge weakly to zero. The simple structure of this system also allows us to explain the mechanisms of the weak convergence to zero. Finally, we show that this convergence to the rigid lid model also holds for the solutions of the Euler equations. To this end we give a new proof of the equivalence of the free surface Euler equations and of the Zakharov-Craig-Sulem equation by building an extension of the velocity and pressure fields.

5.1 Introduction

We recall here classical formulations of the Water Waves problem. We then shortly introduce the meaningful dimensionless parameters of this problem, and introduce the so called rigid lid equations. We then present the main result of this paper.

5.1.1 Formulations of the Water Waves problem

The Water Waves problem puts the motion of a fluid with a free surface into equations. We recall here two equivalent formulations of the Water Waves equations for an incompressible and irrotational fluid.

Free surface d -dimensional Euler equations

The motion, for an incompressible, inviscid and irrotational fluid occupying a domain Ω_t delimited below by a fixed bottom and above by a free surface is described by the following quantities:

- the velocity of the fluid $U = (V, w)$, where V and w are respectively the horizontal and vertical components;
- the free top surface profile ζ ;
- the pressure P .

All these functions depend on the time and space variables t and $(X, z) \in \Omega_t$. There exists a function $b : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the domain of the fluid at the time t is given by

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -H_0 + b(X) < z < \zeta(t, X)\},$$

where H_0 is the typical depth of the water. The unknowns (U, ζ, P) are governed by the Euler equations:

$$\begin{cases} \partial_t U + U \cdot \nabla_{X,z} U = -\frac{1}{\rho} \nabla P - g e_z & \text{in } \Omega_t \\ \operatorname{div}(U) = 0 & \text{in } \Omega_t \\ \operatorname{curl}(U) = 0 & \text{in } \Omega_t. \end{cases} \quad (5.1.1)$$

We denote here $-g e_z$ the acceleration of gravity, where e_z is the unit vector in the vertical direction, and ρ the density of the fluid. Here, $\nabla_{X,z}$ denotes the $d + 1$ dimensional gradient with respect to both variables X and z .

These equations are completed by boundary conditions:

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla \zeta - w = 0 \\ U \cdot n = 0 & \text{on } \{z = -H_0 + b(X)\} \\ P = P_{atm} & \text{on } \{z = \zeta(t, X)\}. \end{cases} \quad (5.1.2)$$

In these equations, \underline{V} and \underline{w} are the horizontal and vertical components of the velocity evaluated at the surface. The vector n in the last equation stands for the normal upward vector at the bottom ($X, z = -H_0 + b(X)$). We denote P_{atm} the constant pressure of the atmosphere at the surface of the fluid. The first equation of (5.1.2) states the assumption that the fluid particles do not cross the surface, while the last equation of (5.1.2) states the assumption that they do not cross the bottom. The equations (5.1.1) with boundary conditions (5.1.2) are commonly referred to as the free surface Euler equations.

Craig-Sulem-Zakharov formulation

Since the fluid is by hypothesis irrotational, it derives from a scalar potential:

$$U = \nabla_{X,z} \Phi.$$

Zakharov remarked in [65] that the free surface profile ζ and the potential at the surface $\psi = \Phi|_{z=\zeta}$ fully determine the motion of the fluid, and gave an Hamiltonian formulation of the problem. Later, Craig-Sulem, and Sulem ([25] and [26]) gave a formulation of the Water Waves equation involving the Dirichlet-Neumann operator. The following Hamiltonian system is equivalent (see [37] and [3] for more details) to the free surface Euler equations (5.1.1) and (5.1.2):

$$\begin{cases} \partial_t \zeta - G\psi = 0 \\ \partial_t \psi + g\zeta + \frac{1}{2} |\nabla \psi|^2 - \frac{(G\psi + \nabla \zeta \cdot \nabla \psi)^2}{2(1 + |\nabla \zeta|^2)} = 0, \end{cases} \quad (5.1.3)$$

where the unknowns are ζ (free surface profile) and ψ (velocity potential at the surface) with t as time variable and $X \in \mathbb{R}^d$ as space variable. The fixed bottom profile is b , and G stands for the Dirichlet-Neumann operator, that is

$$G\psi = G[\zeta, b]\psi = \sqrt{1 + |\nabla \zeta|^2} \partial_n \Phi|_{z=\zeta},$$

where Φ stands for the potential, and solves Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions:

$$\begin{cases} \Delta_{X,z} \Phi = 0 & \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -H_0 + b(X) < z < \zeta(X)\} \\ \phi|_{z=\zeta} = \psi, & \partial_n \Phi|_{z=-H_0+b} = 0, \end{cases} \quad (5.1.4)$$

with the notation, for the normal derivative

$$\partial_n \Phi|_{z=-H_0+b(X)} = \nabla_{X,z} \Phi(X, -H_0 + b(X)) \cdot n$$

where n stands for the normal upward vector at the bottom ($X, -H_0 + b(X)$). See also [37] for more details.

Dimensionless equations

Since the properties of the solutions depend strongly on the characteristics of the fluid, it is more convenient to non-dimensionalize the equations by introducing some characteristic lengths of the wave motion:

- (1) The characteristic water depth H_0 ;
- (2) The characteristic horizontal scale L_x in the longitudinal direction;
- (3) The characteristic horizontal scale L_y in the transverse direction (when $d = 2$);
- (4) The size of the free surface amplitude a_{surf} ;
- (5) The size of bottom topography a_{bott} .

Let us then introduce the dimensionless variables:

$$x' = \frac{x}{L_x}, \quad y' = \frac{y}{L_y}, \quad \zeta' = \frac{\zeta}{a_{surf}}, \quad z' = \frac{z}{H_0}, \quad b' = \frac{b}{a_{bott}},$$

and the dimensionless variables:

$$t' = \frac{t}{t_0}, \quad \Phi' = \frac{\Phi}{\Phi_0},$$

where

$$t_0 = \frac{L_x}{\sqrt{gH_0}}, \quad \Phi_0 = \frac{a_{surf}}{H_0} L_x \sqrt{gH_0}.$$

After rescaling, several dimensionless parameters appear in the equation. They are

$$\frac{a_{surf}}{H_0} = \varepsilon, \quad \frac{H_0^2}{L_x^2} = \mu, \quad \frac{a_{bott}}{H_0} = \beta, \quad \frac{L_x}{L_y} = \gamma,$$

where $\varepsilon, \mu, \beta, \gamma$ are commonly referred to respectively as "nonlinearity", "shallowness", "topography" and "transversality" parameters.

For instance, the Zakharov-Craig-Sulem system (5.1.3) becomes (see [37] for more details) in dimensionless variables (we omit the "primes" for the sake of clarity):

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0, \end{cases} \quad (5.1.5)$$

where $G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi$ stands for the dimensionless Dirichlet-Neumann operator,

$$G_{\mu, \gamma}[\varepsilon \zeta, \beta b] \psi = \sqrt{1 + \varepsilon^2 |\nabla^\gamma \zeta|^2} \partial_n \Phi|_{z=\varepsilon \zeta} = (\partial_z \Phi - \mu \nabla^\gamma(\varepsilon \zeta) \cdot \nabla^\gamma \Phi)|_{z=\varepsilon \zeta},$$

where Φ solves the Laplace equation with Neumann (at the bottom) and Dirichlet (at the surface) boundary conditions

$$\begin{cases} \Delta^{\mu, \gamma} \Phi = 0 & \text{in } \{(X, z) \in \mathbb{R}^d \times \mathbb{R} - 1 + \beta b(X) < z < \varepsilon \zeta(X)\} \\ \phi|_{z=\varepsilon \zeta} = \psi, \quad \partial_n \Phi|_{z=-1+\beta b} = 0. \end{cases} \quad (5.1.6)$$

We used the following notations:

$$\begin{aligned} \nabla^\gamma &= {}^t(\partial_x, \gamma \partial_y) & \text{if } d = 2 & \text{ and } & \nabla^\gamma &= \partial_x & \text{if } d = 1 \\ \Delta^{\mu, \gamma} &= \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{if } d = 2 & \text{ and } & \Delta^{\mu, \gamma} &= \mu \partial_x^2 + \partial_z^2 & \text{if } d = 1 \end{aligned}$$

and

$$\partial_n \Phi|_{z=-1+\beta b} = \frac{1}{\sqrt{1 + \beta^2 |\nabla^\gamma b|^2}} (\partial_z \Phi - \mu \nabla^\gamma(\beta b) \cdot \nabla^\gamma \Phi)|_{z=-1+\beta b}.$$

5.1.2 The rigid lid model

The Euler system (5.1.1), (5.1.2), can be written in the dimensionless variables:

$$\begin{cases} \partial_t V + \varepsilon (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\nabla^\gamma P & \text{in } \Omega_t \\ \partial_t w + \varepsilon (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -(\partial_z P) & \text{in } \Omega_t \\ \partial_t \zeta + \varepsilon \underline{V} \cdot \nabla^\gamma \zeta - \frac{1}{\mu} \underline{w} = 0 \\ \nabla^{\mu, \gamma} \cdot U = 0 & \text{in } \Omega_t \\ \text{curl}^{\mu, \gamma}(U) = 0 & \text{in } \Omega_t \\ U \cdot n = 0 & \text{for } z = -1 + \beta b \\ P = \varepsilon \zeta & \text{for } z = \varepsilon \zeta. \end{cases} \quad (5.1.7)$$

We changed the pressure into hydrodynamic pressure $P = P - (z - \varepsilon \zeta)$ and set $P_{atm} = 0$. The rigid lid model models the motion of the fluid as if the top surface was fixed at $z = 0$ (see for instance [16]). But to derive this model, we do not brutally take $\varepsilon = 0$ in the Euler system (5.1.7) but rather start by a change of time scale, and a change of scale for the pressure:

$$t' = \varepsilon t$$

and

$$P' = \frac{P}{\varepsilon}.$$

Formally, if one takes the limit ε goes to zero in the newly scaled Euler equations, one finds the following so called rigid lid equations (we omit the "primes" for the sake of clarity):

$$\begin{cases} \partial_t V + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\nabla^\gamma P & \text{in } \Omega \\ \partial_t w + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -(\partial_z P) & \text{in } \Omega \\ \underline{w} = 0 \\ \nabla^{\mu, \gamma} \cdot U = 0 & \text{in } \Omega \\ \text{curl}^{\mu, \gamma}(U) = 0 & \text{in } \Omega \\ U \cdot n = 0 & \text{for } z = -1 + \beta b \\ P = \zeta & \text{for } z = 0 \end{cases} \quad (5.1.8)$$

where Ω is now the fixed domain

$$\Omega = \{(X, z) \in \mathbb{R}^{d+1}, -1 + \beta b(X) \leq z \leq 0\}.$$

5.1.3 Reminder on the local existence for the Water-Waves equations with flat bottom

We briefly give some recalls about the Water-Waves equations and their local existence (see [37] Chapter 4 for a complete study). Taking $\beta = 0$ (flat bottom), and after scaling

$$t' = \varepsilon t$$

as for the rigid lid equation, one gets from (5.1.5) the equations:

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} G[\varepsilon \zeta, 0] \psi = 0 \\ \varepsilon \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G[\varepsilon \zeta, 0] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0. \end{cases} \quad (5.1.9)$$

As explained above, we prove later that the solutions of the full Water-Waves equations (5.1.9) converge strongly to the solutions of the linearized equations (5.2.22). The strategy is to treat the non-linear terms of the Water-Waves equations as a perturbation of the linearized equation.

Let $t_0 > d/2$ and $N \geq t_0 + t_0 \vee 2 + 3/2$ (where $a \vee b = \sup(a, b)$). The energy for the Water-Waves equations is the following (see Section 5.1.5 for the notations):

$$\mathcal{E}^N(U) = |\mathfrak{R}\psi|_{H^{t_0+3/2}} + \sum_{|\alpha| \leq N} |\zeta_{(\alpha)}|_2 + |\psi_{(\alpha)}|_2$$

where $\zeta_{(\alpha)}, \psi_{(\alpha)}$ are the so called Alinhac's good unknowns:

$$\forall \alpha \in \mathbb{N}^d, \zeta_{(\alpha)} = \partial^\alpha \zeta, \quad \psi_{(\alpha)} = \partial^\alpha \psi - \varepsilon \underline{w} \partial^\alpha \zeta.$$

We consider solutions $U = (\zeta, \psi)$ of the Water-Waves equations in the following space:

$$E_T^N = \{U \in C([0, T]; H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d)), \mathcal{E}^N(U(\cdot)) \in L^\infty([0, T])\}.$$

The following quantity, called the Rayleigh-Taylor coefficient plays an important role in the Water-Waves problem:

$$\underline{\mathfrak{a}}(\zeta, \psi) = 1 + \varepsilon(\varepsilon \partial_t + \varepsilon \underline{V} \cdot \nabla^\gamma) \underline{w} = -\varepsilon \frac{P_0}{\rho a g} (\partial_z P)|_{z=\varepsilon \zeta}.$$

We can now state the local existence result by Alvarez-Samaniego Lannes (see [6] and [37] Chapter 3):

Theorem 5.1.1 *Let $t_0 > d/2, N \geq t_0 + t_0 \vee 2 + 3/2$. Let $U^0 = (\zeta^0, \psi^0) \in E_0^N$. Let ε, γ be such that*

$$0 \leq \varepsilon, \gamma \leq 1,$$

and moreover assume that:

$$\exists h_{min} > 0, \exists a_0 > 0, \quad 1 + \varepsilon \zeta^0 \geq h_{min} \quad \text{and} \quad \underline{a}(U^0) \geq a_0.$$

Then, there exists $T > 0$ and a unique solution $U^\varepsilon \in E_T^N$ to (5.1.9) with initial data U^0 . Moreover,

$$\frac{1}{T} = C_1, \quad \text{and} \quad \sup_{t \in [0; T]} \mathcal{E}^N(U^\varepsilon(t)) = C_2$$

with $C_i = C(\mathcal{E}^N(U^0), \frac{1}{h_{min}}, \frac{1}{a_0})$ for $i = 1, 2$.

Remark 5.1.2 *Note that the time existence provided by Theorem 5.1.1 does not depend on ε . Actually, the full Theorem in non flat bottom (see [37]) states that the time of existence for the equations in the original scaling (5.1.5) is of size $\frac{1}{\varepsilon \vee \beta}$ where β is the size of the topography. In our case, $\beta = 0$ and therefore one gets a time of existence of size $\frac{1}{\varepsilon}$ for (5.1.5), and of size 1 for the rescaled equations (5.1.9). In the case of a non flat bottom, this long time result stands true in presence of surface tension (see [50]).*

5.1.4 Main result

We investigate in this paper the rigorous limit ε goes to zero in the rescaled Euler equation (5.1.7) in view of the mathematical justification of the derivation of the rigid lid model (5.1.8). However, as one shall see in Section 5.2.1, the only solutions of the rigid lid equations are trivially null. Though the rigid lid model does not then seem of much interest, it implies that the solutions of the rescaled Euler equations (5.1.7) converge weakly to zero as ε goes to zero. In [13], the same limit in the rigid lid regime is investigated by Bresch and Métivier for the Shallow-Water equations with large bathymetry, which consist in an asymptotic model for the Water-Waves problem in the shallow water regime (μ small). They prove that the rescaled equations are well-posed on a time interval independent on ε (which is equivalent to a large time of existence of size $\frac{1}{\varepsilon}$ for the system written in the original variables) and they rigorously pass to the limit as ε goes to zero, and prove the weak convergence and the strong local convergence in space of the solutions.

In [50], a similar long time existence result is proved for the Water-Waves equations with large bathymetry and with surface tension. If we restrict the problem to the flat bottom case, then the local existence result (5.1.1) can be read as (we don't give a precise statement):

Theorem 5.1.3 *Let N be large enough, and let $(\zeta_0, \psi_0) \in H^N(\mathbb{R}^d) \times H^{N+1/2}(\mathbb{R}^d)$. Then, there exists $T > 0$, and a unique solution $(\zeta, \psi) \in C^1([0; \frac{T}{\varepsilon}; H^N \times H^{N+1/2}(\mathbb{R}^d))$ to the Water-Waves equations (5.1.5). Moreover, one has:*

$$\frac{1}{T} = C_1(\zeta_0, \psi_0), \quad |(\zeta, \psi)|_{C^1([0; \frac{T}{\varepsilon}; H^N \times H^{N+1/2}(\mathbb{R}^d))} \leq C_2(\zeta_0, \psi_0), \quad (5.1.10)$$

where C_i are continuous functions of their arguments, and are independent on μ, ε .

The bound on the solutions given by (5.1.10) is uniform with respect to ε , which allows by a compactness argument to extract a convergent sub-sequence of solutions as ε goes to zero. The limit should be zero (at least for ψ) as we discussed above. The question is to understand if the convergence is strong in $C([0; T]; H^{N-1} \times H^{N-1/2}(\mathbb{R}^d))$, i.e. "globally in time and space". In this paper, we complete the study of the rigid lid limit for the Water-Waves problem, and give an answer to this question.

Since the equations (5.1.5) have the structure of a Hamiltonian equation, the following quantity is conserved:

$$\frac{1}{2\mu}(\mathcal{G}\psi, \psi)_2 + (\zeta, \zeta)_2. \quad (5.1.11)$$

We recall that \mathcal{G} is symmetric and positive for the L^2 scalar product. Therefore, even after the time change of scale

$$t' = \varepsilon t$$

and the limit ε goes to zero, the Hamiltonian is conserved and the strong convergence in L^2 of the unknowns cannot be (except in particular cases). In Section 5.2.2, we precisely try to highlight the default of compactness that prevents the strong convergence in dimension 1, in presence of a flat bottom. As one shall see, the default comes from the linear operator of the Water-Waves equations which has quite a similar behavior as the linear operator of the wave equation. In Section 5.3, we prove the equivalence between the Water-Waves equations and the Euler equations, which completes the study of the weak but not strong convergence to zero in Sobolev spaces for the solutions of the Euler equation (5.1.7) in the rigid lid regime. To sum up, we give here the plan of this article:

- In Section 5.2.2, we prove that the solutions of the linearized equation does not converge strongly in $L^2(\mathbb{R}^d)$ at a fixed time, in the rigid lid scaling ($t' = \varepsilon t$) for $d = 1, 2$.
- In Section 5.2.3, we prove the strong convergence in $L^\infty([0; T[; L^2 \times H^{1/2}(\mathbb{R}^d))$ of the solutions of the full Water-Waves equation in rigid lid scaling to the solutions of the linearized equations, for $d = 1$. It proves that the solutions of the full Water-Waves equations does not converge strongly in $L^\infty([0; T[; L^2 \times H^{1/2}(\mathbb{R}^d))$ to zero, for $d = 1$.
- In Section 5.3, we prove the equivalence between the free-surface Euler equations (5.1.7) and the Water-Waves equations (5.1.5) for $d = 1, 2$. We show that the solutions of Euler equations converge weakly as ε goes to zero, to the solutions of the rigid lid equation (5.1.8). The convergence is therefore not strong, at least for $d = 1$, according to the preceding points.

5.1.5 Notations

We introduce here all the notations used in this paper.

Operators and quantities

Because of the use of dimensionless variables (see before the "dimensionless equations" paragraph), we use the following twisted partial operators:

$$\begin{array}{llll} \nabla^\gamma = {}^t(\partial_x, \gamma \partial_y) & \text{if } d = 2 & \text{and} & \nabla^\gamma = \partial_x \quad \text{if } d = 1 \\ \Delta^{\mu, \gamma} = \mu \partial_x^2 + \gamma^2 \mu \partial_y^2 + \partial_z^2 & \text{if } d = 2 & \text{and} & \Delta^{\mu, \gamma} = \mu \partial_x^2 + \partial_z^2 \quad \text{if } d = 1 \\ \nabla^{\mu, \gamma} = {}^t(\sqrt{\mu} \partial_x, \gamma \sqrt{\mu} \partial_y, \partial_z) & \text{if } d = 2 & \text{and} & {}^t(\sqrt{\mu} \partial_x, \partial_z) \quad \text{if } d = 1 \\ \nabla^{\mu, \gamma} = \sqrt{\mu} \partial_x + \gamma \sqrt{\mu} \partial_y + \partial_z & \text{if } d = 2 & \text{and} & \sqrt{\mu} \partial_x + \partial_z \quad \text{if } d = 1 \\ \text{curl}^{\mu, \gamma} = {}^t(\sqrt{\mu} \gamma \partial_y - \partial_z, \partial_z - \sqrt{\mu} \partial_x, \partial_x - \gamma \partial_y) & \text{if } d = 2 & & \end{array}$$

Remark 5.1.4 *All the results proved in this paper do not need the assumption that the typical wave lengths are the same in both directions, ie $\gamma = 1$. However, if one is not interested in the dependence of γ , it is possible to take $\gamma = 1$ in all the following proofs. A typical situation where $\gamma \neq 1$ is for weakly transverse waves for which $\gamma = \sqrt{\mu}$; this leads to weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili equation (see [40]).*

For all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we write

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$$

and

$$|\alpha| = \alpha_1 + \dots + \alpha_d.$$

We use the classical Fourier multiplier

$$\Lambda^s = (1 - \Delta)^{s/2} \text{ on } \mathbb{R}^d$$

defined by its Fourier transform as

$$\mathcal{F}(\Lambda^s u)(\xi) = (1 + |\xi|^2)^{s/2} \mathcal{F}u(\xi)$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$. The operator \mathfrak{P} is defined as

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} \quad (5.1.12)$$

where

$$\mathcal{F}(f(D)u)(\xi) = f(\xi)\mathcal{F}(u)(\xi)$$

is defined for any smooth function f of polynomial growth and $u \in \mathcal{S}'(\mathbb{R}^d)$. The pseudo-differential operator \mathfrak{P} acts as the square root of the Dirichlet-Neumann operator.

We denote as before by $G_{\mu,\gamma}$ the Dirichlet-Neumann operator, which is defined as followed in the scaled variables:

$$G_{\mu,\gamma}\psi = G_{\mu,\gamma}[\varepsilon\zeta, \beta b]\psi = \sqrt{1 + \varepsilon^2|\nabla^\gamma\zeta|^2}\partial_n\Phi|_{z=\varepsilon\zeta} = (\partial_z\Phi - \mu\nabla^\gamma(\varepsilon\zeta) \cdot \nabla^\gamma\Phi)|_{z=\varepsilon\zeta},$$

where Φ solves the Laplace equation

$$\begin{cases} \Delta^{\gamma,\mu}\Phi = 0 \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_n\Phi|_{z=-1+\beta b} = 0. \end{cases}$$

For the sake of simplicity, we use the notation $G[\varepsilon\zeta, \beta b]\psi$ or even $G\psi$ when no ambiguity is possible.

The Dirichlet-Neumann problem

In order to study the Dirichlet-Neumann problem (5.1.4), we need to map Ω_t into a fixed domain (and not on a moving subset). For this purpose, we introduce the following fixed strip:

$$\mathcal{S} = \mathbb{R}^d \times (-1; 0)$$

and the diffeomorphism

$$\Sigma_t^\varepsilon : \begin{array}{ccc} \mathcal{S} & \rightarrow & \Omega_t \\ (X, z) & \mapsto & (1 + \varepsilon\zeta(X) - \beta b(X))z + \varepsilon\zeta(X). \end{array} \quad (5.1.13)$$

It is quite easy to check that Φ is the variational solution of (5.1.4) if and only if $\phi = \Phi \circ \Sigma_t^\varepsilon$ is the variational solution of the following problem:

$$\begin{cases} \nabla^{\mu,\gamma} \cdot P(\Sigma_t^\varepsilon)\nabla^{\mu,\gamma}\phi = 0 \\ \phi_{z=0} = \psi, \quad \partial_n\phi_{z=-1} = 0, \end{cases} \quad (5.1.14)$$

and where

$$P(\Sigma_t^\varepsilon) = |\det J_{\Sigma_t^\varepsilon}| J_{\Sigma_t^\varepsilon}^{-1} {}^t(J_{\Sigma_t^\varepsilon}^{-1}),$$

where $J_{\Sigma_t^\varepsilon}$ is the jacobian matrix of the diffeomorphism Σ_t^ε . For a complete statement of the result, and a proof of existence and uniqueness of solutions to these problems, see [37] Chapter 2.

We introduce here the notations for the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\mathbf{\Gamma} \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\mathbf{\Gamma} = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \quad \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$G[\varepsilon \cdot, \beta \cdot] : \begin{array}{ccc} \mathbf{\Gamma} & \longrightarrow & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \longmapsto & G[\varepsilon\zeta, \beta b]\psi. \end{array}$$

We can prove the differentiability of this mapping. See Appendix 5.A for more details. We denote $d^j G(h, k)\psi$ the j -th derivative of the mapping at (ζ, b) in the direction (h, k) . When we only differentiate in one direction, and no ambiguity is possible, we simply denote $d^j G(h)\psi$ or $d^j G(k)\psi$.

Functional spaces

The standard scalar product on $L^2(\mathbb{R}^d)$ is denoted by $(\cdot, \cdot)_2$ and the associated norm $|\cdot|_2$. We will denote the norm of the Sobolev spaces $H^s(\mathbb{R}^d)$ by $|\cdot|_{H^s}$.

We introduce the following functional Sobolev-type spaces, or Beppo-Levi spaces:

Definition 5.1.5 *We denote $\dot{H}^{s+1}(\mathbb{R}^d)$ the topological vector space*

$$\dot{H}^{s+1}(\mathbb{R}^d) = \{u \in L^2_{loc}(\mathbb{R}^d), \quad \nabla u \in H^s(\mathbb{R}^d)\}$$

endowed with the (semi) norm $|u|_{\dot{H}^{s+1}(\mathbb{R}^d)} = |\nabla u|_{H^s(\mathbb{R}^d)}$.

Just remark that $\dot{H}^{s+1}(\mathbb{R}^d)/\mathbb{R}^d$ is a Banach space (see for instance [27]).

The space variables $z \in \mathbb{R}$ and $X \in \mathbb{R}^d$ play different roles in the equations since the Euler formulation (5.1.1) is posed for $(X, z) \in \Omega_t$. Therefore, X lives in the whole space \mathbb{R}^d (which allows to take fractional Sobolev type norms in space), while z is actually bounded. For this reason, we need to introduce the following Banach spaces:

Definition 5.1.6 *The Banach space $(H^{s,k}((-1, 0) \times \mathbb{R}^d), |\cdot|_{H^{s,k}})$ is defined by*

$$H^{s,k}((-1, 0) \times \mathbb{R}^d) = \bigcap_{j=0}^k H^j((-1, 0); H^{s-j}(\mathbb{R}^d)), \quad |u|_{H^{s,k}} = \sum_{j=0}^k |\Lambda^{s-j} \partial_z^j u|_2.$$

5.2 The rigid lid limit for the Water-Waves equations

In all this Section, we set the bottom to zero:

$$\beta = 0.$$

We prove that the solutions of the linearized Water-Waves equations in flat bottom converge weakly but not strongly in L^2 as ε goes to zero, in the rigid lid regime. We then prove the strong convergence in $L_t^\infty L_x^2$ of the solutions of the full Water-Waves equations to the solutions of the linearized equations, in the same regime, in dimension 1.

Remark 5.2.1 *The hypothesis " $d = 1$ " can be removed, if one can prove a dispersive estimate for the linear operator of the Water-Waves equation of the form of Theorem (5.2.9), in dimension $d = 2$. The flat bottom hypothesis seems however less easy to remove, due to technical reasons, such as the complexity of the asymptotic expansion of \mathcal{G} with respect to the surface when the bottom is non flat.*

5.2.1 Solutions of the rigid lid equations

We recall the formulation of the rigid lid equations with a flat bottom (see also [16] for reference):

$$\begin{cases} \partial_t V + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\nabla^\gamma P \text{ in } \Omega \\ \partial_t w + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -(\partial_z P) \text{ in } \Omega \\ \underline{w} = 0 \\ \nabla^{\mu, \gamma} \cdot U = 0 \text{ in } \Omega \\ \text{curl}^{\mu, \gamma}(U) = 0 \text{ in } \Omega \\ U \cdot e_z = 0 \text{ for } z = -1 \\ P = \zeta \text{ for } z = 0 \end{cases} \quad (5.2.15)$$

where Ω is the fixed domain

$$\Omega = \{(X, z) \in \mathbb{R}^{d+1}, -1 \leq z \leq 0\}.$$

We prove here that in fact, there is only one trivial solution to the system (5.2.15):

Lemma 5.2.2 *Let us consider the following div-curl problem:*

$$\begin{cases} \nabla^{\mu,\gamma} \cdot U = 0 & \text{in } \Omega \\ \text{curl}^{\mu,\gamma}(U) = 0 & \text{in } \Omega \\ U \cdot e_z = 0 & \text{for } z = -1 \\ \underline{w} = 0 \end{cases} \quad (5.2.16)$$

with the above notations. Then, the unique solution in $\dot{H}^1(\Omega)$ to the system (5.2.16) is $U = 0$.

Proof The vector field $U = 0$ is of course a solution to the system (5.2.16). Let us now consider a solution U to this problem. The curl free condition over U states that there exists a potential Φ such that $U = \nabla^{\mu,\gamma}\Phi$ in Ω . The divergence free condition provides $\Delta^{\mu,\gamma}\Phi = 0$ in Ω . With the boundary conditions (recall that \underline{w} is the vertical component of the velocity at the surface), the potential Φ satisfies the following Laplace equation:

$$\begin{cases} \Delta^{\mu,\gamma}\Phi = 0 & \text{in } \Omega \\ \nabla^{\mu,\gamma}\Phi|_{z=0} \cdot n = 0 \\ \nabla^{\mu,\gamma}\Phi|_{z=-1} \cdot n = 0, \end{cases}$$

where n denotes the upward normal vector in the vertical direction. It is well known (see for instance [37] Chapter 2 and Appendix A) that the unique solution to this system in $\dot{H}^1(\Omega)$ satisfies $\nabla^{\mu,\gamma}\Phi = 0$.¹ \square

Let us explain this result on a physical point of view. The rigid-lid equations are obtained after scaling the time

$$t' = \varepsilon t$$

in the free surface Euler equations, and passing to the limit as ε goes to zero. Therefore, in the original time variable, $t = \frac{t'}{\varepsilon}$, the rigid lid limit consists in letting the amplitude of the waves goes to zero AND moving forward in time at a rate $\frac{1}{\varepsilon}$. At the limit, after an infinite time, all the interesting components of the waves moved to $\pm\infty$ and there only remains a static fluid ($U = 0$) with a flat surface. It is possible that some vortices remain, but are not seen by the model ($\text{curl}(U) = 0$).

5.2.2 The linearized equation around $\zeta = 0$ in the rigid lid regime

We prove here that the solutions to the Water-Waves equations does not converge strongly in L^2 as ε goes to zero. To this purpose, we study the lack of compactness induced by the linear operator of the Water-Waves equations. We recall that in the rigid lid regime, we perform a time scaling

$$t' = \varepsilon t.$$

The Water-Waves equations can be written in dimensionless form, and in the rigid lid scaling:

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} G[\varepsilon \zeta, 0] \psi = 0 \\ \varepsilon \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G[\varepsilon \zeta, 0] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0. \end{cases} \quad (5.2.17)$$

Since we are interested in the convergence as ε goes to zero, we write this system under the form:

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = \varepsilon f \\ \varepsilon \partial_t \psi + \zeta = \varepsilon g. \end{cases} \quad (5.2.18)$$

We will treat the non-linear terms f, g present in (5.2.18) as a perturbation of the linearized equations. To this purpose, we start to study the solutions of the linearized system:

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = 0 \\ \varepsilon \partial_t \psi + \zeta = 0, \end{cases} \quad (5.2.19)$$

¹In fact the result stands even if the bottom is non flat.

where $\mathcal{G}_0 = \mathcal{G}[0, 0]$ is the Dirichlet Nemann operator in flat bottom and flat surface, and is given by (see for instance [37] Chapter 1):

$$\mathcal{G}_0\psi = \sqrt{\mu}|D^\gamma| \tanh(\sqrt{\mu}|D^\gamma|)\psi \quad (5.2.20)$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. It is easy to check that the solution ζ of (5.2.19) satisfies the following equation:

$$\varepsilon^2 \partial_t \zeta + \frac{1}{\mu} \mathcal{G}_0 \zeta = 0. \quad (5.2.21)$$

The equation (5.2.21) looks like a wave equation, except that \mathcal{G}_0 is not exactly an order two operator (since it acts like an order one operator for high frequencies). In the case of the wave equation on \mathbb{R}

$$\begin{cases} \varepsilon^2 \partial_t u - \partial_x^2 u = 0 \\ u(x, 0) = u_0 \\ (\varepsilon \partial_t u)(x, 0) = u_1, \end{cases}$$

with u_0, u_1 smooth, the solutions have components of the form

$$u_0(x - \frac{1}{\varepsilon}t) + u_0(x + \frac{1}{\varepsilon}t)$$

(plus other terms we do not detail). Each of these two components converges pointwise to 0 as ε goes to zero (since $u_0(t, x) \xrightarrow{|x| \rightarrow +\infty} 0$) and does not converge strongly in $L^2(\mathbb{R})$, since for instance its

$L^2(\mathbb{R})$ norm does not depend on ε . The same behavior, combined with some dispersive effects stands for (5.2.21). We start to give an existence result for the linearized equations (5.2.19). Note that it is not difficult to prove formally that if (ζ, ψ) is the solution of (5.2.19), then the following quantity, called "Hamiltonian" is conserved through time:

$$\frac{1}{2\mu} (\mathcal{G}_0 \psi, \psi)_2 + \frac{1}{2} |\zeta|_2^2.$$

Therefore, we introduce the following operator:

$$\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}}.$$

Note that \mathfrak{P} has the same behavior as the square root of \mathcal{G}_0 defined by (5.2.20).

Proposition 5.2.3 *Let $s \geq 1$. Let also $\zeta_0 \in H^s(\mathbb{R}^d)$ and ψ_0 be such that $\mathfrak{P}\psi_0 \in H^s(\mathbb{R}^d)$. Then, there exists a unique solution (ζ, ψ) to the equation:*

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = 0 \\ \varepsilon \partial_t \psi + \zeta = 0 \\ (\zeta(0, X), \psi(0, X)) = (\zeta^0(X), \psi^0(X)) \end{cases} \quad (5.2.22)$$

such that $(\zeta, \mathfrak{P}\psi) \in C(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^d))$. Moreover, one has:

$$\forall t \in \mathbb{R}, \quad \begin{pmatrix} \zeta \\ \psi \end{pmatrix} (t) = e^{-\frac{t}{\varepsilon}L} \begin{pmatrix} \zeta_0 \\ \psi_0 \end{pmatrix}$$

where

$$e^{tL} = \frac{1}{2} \begin{pmatrix} 1 & -i\omega(D) \\ -\frac{1}{i\omega(D)} & 1 \end{pmatrix} e^{-i\omega(D)t} + \frac{1}{2} \begin{pmatrix} 1 & i\omega(D) \\ \frac{1}{i\omega(D)} & 1 \end{pmatrix} e^{i\omega(D)t} \quad (5.2.23)$$

and

$$\omega(D) = \sqrt{\frac{|D^\gamma| \tanh(\sqrt{\mu}|D^\gamma|)}{\sqrt{\mu}}}. \quad (5.2.24)$$

Remark 5.2.4 *The notation e^{tL} refers to the linear operator of the Water-Waves, defined by*

$$L = \begin{pmatrix} 0 & -\frac{1}{\mu} \mathcal{G}_0 \\ 1 & 0 \end{pmatrix}.$$

Proof Let us fix $s \geq 1$, and $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d)^2$. If (ζ, ψ) is a solution to (5.2.22), then ζ satisfies the following equation:

$$\begin{cases} \varepsilon^2 \partial_t^2 \zeta + \frac{1}{\mu} \mathcal{G}_0 \zeta = 0 \\ \zeta(x, 0) = \zeta_0 \\ (\varepsilon \partial_t \zeta)(x, 0) = \zeta_1, \end{cases} \quad (5.2.25)$$

in the space $C(\mathbb{R}, H^s(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^d))$, with $\zeta_1 = \frac{1}{\mu} \mathcal{G}_0 \psi_0$. As usual, we take the Fourier transform in space of the equation (5.2.25) and we denote $\widehat{\zeta}(t, \xi)$ the Fourier transform of ζ with respect to the variable X for a fixed t . The distribution $\widehat{\partial_t \zeta}$ of $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ is equal to the distribution $\partial_t \widehat{\zeta}$ using the Banach-Steinhaus theorem. We therefore get the following equation in the distributional sense of $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$:

$$\begin{cases} \varepsilon^2 \partial_t^2 \widehat{\zeta} + \frac{1}{\sqrt{\mu}} |\xi^\gamma| \tanh(\sqrt{\mu} |\xi^\gamma|) \widehat{\zeta} = 0 \\ \widehat{\zeta}(\xi, 0) = \widehat{\zeta}_0(\xi) \\ (\varepsilon \partial_t \widehat{\zeta})(\xi, 0) = \widehat{\zeta}_1(\xi). \end{cases}$$

It is then easy to check that the unique solution to this equation is given by

$$\widehat{\zeta}(\xi, t) = \left(\frac{\widehat{\zeta}_0(\xi) - i\omega(\xi) \widehat{\psi}_0(\xi)}{2} \right) \exp^{i\omega(\xi) \frac{t}{\varepsilon}} + \left(\frac{\widehat{\zeta}_0(\xi) + i\omega(\xi) \widehat{\psi}_0(\xi)}{2} \right) \exp^{-i\omega(\xi) \frac{t}{\varepsilon}} \quad (5.2.26)$$

with

$$\omega(\xi) = \sqrt{\frac{|\xi^\gamma| \tanh(\sqrt{\mu} |\xi^\gamma|)}{\sqrt{\mu}}}$$

such that $\frac{1}{\mu} \mathcal{G}_0 = \omega(D)^2$. The desired regularity is easy to get by noticing that

$$\begin{aligned} \omega(\xi) \widehat{\psi}_0(\xi) &= \frac{|\xi^\gamma|}{(1 + \sqrt{\mu} |\xi^\gamma|)^{1/2}} \widehat{\psi}_0 \sqrt{\frac{\tanh(\sqrt{\mu} |\xi^\gamma|)(1 + \sqrt{\mu} |\xi^\gamma|)}{\sqrt{\mu} |\xi^\gamma|}} \\ &= \mathfrak{P} \widehat{\psi}_0 \sqrt{\frac{\tanh(\sqrt{\mu} |\xi^\gamma|)(1 + \sqrt{\mu} |\xi^\gamma|)}{\sqrt{\mu} |\xi^\gamma|}}. \end{aligned}$$

with $\mathfrak{P} \widehat{\psi}_0 \in H^s(\mathbb{R}^d)$ and that there exists $C_1, C_2 > 0$ independent of μ, ε such that:

$$\forall \xi \in \mathbb{R}^d, \quad C_1 \leq \sqrt{\frac{\tanh(\sqrt{\mu} |\xi^\gamma|)(1 + \sqrt{\mu} |\xi^\gamma|)}{\sqrt{\mu} |\xi^\gamma|}} \leq C_2.$$

The same method applies for ψ and one gets that if (ζ, ψ) satisfies (5.2.22), then necessarily:

$$\widehat{\psi}(t, \xi) = \widehat{\psi}_0 \cos(\omega(\xi) \frac{t}{\varepsilon}) - \frac{\sin(\omega(\xi) \frac{t}{\varepsilon})}{\omega(\xi)} \widehat{\zeta}_0. \quad (5.2.27)$$

It is easy to check that (ζ, ψ) given by (5.2.26), (5.2.27) is a solution to the system (5.2.19). \square

The oscillating component of the solution of the linearized Water-Waves equations (5.2.22) appears in the explicit formula (5.2.26), but we need to prove an oscillating phase method type result in order to conclude to the weak convergence to zero and strong non-convergence. The phase ω is actually not differentiable at $\xi = 0$, which prevent the direct use of standards results on stationary phase methods.

Proposition 5.2.5 *Let $t > 0$ and $u \in C_0^1(\mathbb{R}^d)$. Then, one has*

$$\int_{\mathbb{R}^d} \exp^{i \frac{t}{\varepsilon} \omega(\xi)} u(\xi) d\xi \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof One can compute:

$$\forall \xi \neq 0, \quad (\nabla \omega)(\xi) = \frac{\tanh(|\xi^\gamma|) + |\xi^\gamma|(1 - \tanh^2(|\xi^\gamma|))}{2\sqrt{|\xi^\gamma| \tanh(|\xi^\gamma|)}} \frac{\xi}{|\xi^\gamma|}. \quad (5.2.28)$$

It is easy to see that $\nabla \omega$ does not vanish for $\xi \neq 0$. It is also easy to show that ω is not differentiable in $\xi = 0$. But the derivative $\nabla \omega$ stay bounded as ξ goes to zero. We treat separately the cases $d = 1$ and $d > 1$.

- *Case $d = 1$* One writes

$$\int_{\mathbb{R}} \exp^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi = \int_{-\infty}^0 \exp^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi + \int_0^{+\infty} \exp^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi. \quad (5.2.29)$$

Since ω is right and left differentiable in $\xi = 0$, according to the expression (5.2.28), one can use standard methods of stationary phase. Indeed, one can write:

$$\begin{aligned} \int_{-\infty}^0 \exp^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi &= \frac{\varepsilon}{it} \int_{-\infty}^0 \frac{1}{\omega'(\xi)} \frac{d}{d\xi} (\exp^{i\frac{\pm}{\varepsilon}\omega(\xi)}) u(\xi) d\xi \\ &= \frac{\varepsilon}{it} \frac{u(0)}{\omega'^-(0)} e^{i\frac{\pm}{\varepsilon}\omega(0)} - \frac{\varepsilon}{it} \int_{-\infty}^0 e^{i\frac{\pm}{\varepsilon}\omega(\xi)} \frac{d}{d\xi} \left(\frac{u(\xi)}{\omega'^-(\xi)} \right) d\xi \end{aligned}$$

where $\omega'^+(0)$ and $\omega'^-(0)$ are respectively the right and left derivatives of ω at $\xi = 0$. We then get the desired result. The same goes for the second integral of (5.2.29).

- *Case $d > 1$* We use the standard integrating by part method for the truncated integral:

$$\begin{aligned} \int_{|\xi| \geq \delta} e^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi &= \int_{|\xi| \geq \delta} \frac{\varepsilon}{it} \frac{1}{|\omega'(\xi)|^2} \sum_{j=1}^d \frac{\partial}{\partial \xi_j} \omega(\xi) \frac{\partial}{\partial \xi_j} (e^{i\frac{\pm}{\varepsilon}\omega(\xi)}) u(\xi) d\xi \\ &= \frac{\varepsilon}{it} \int_{|\xi| = \delta} \frac{1}{|\omega'(\xi)|^2} \sum_{j=1}^d \frac{\partial}{\partial \xi_j} \omega(\xi) e^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) n_j d\sigma(\xi) \\ &\quad - \frac{\varepsilon}{it} \int_{|\xi| \geq \delta} \sum_{j=1}^d \frac{\partial}{\partial \xi_j} \left(\frac{1}{|\omega'(\xi)|^2} \frac{\partial}{\partial \xi_j} \omega(\xi) u(\xi) \right) e^{i\frac{\pm}{\varepsilon}\omega(\xi)} d\xi \end{aligned} \quad (5.2.30)$$

where n_j stands for the j -th component of the normal external vector at the surface $|\xi| = \delta$. Now, one can check that:

$$\frac{1}{|\omega'(\xi)|^2} \frac{\partial}{\partial \xi_j} \omega(\xi) = \frac{2\xi_j \sqrt{|\xi^\gamma| \tanh(|\xi^\gamma|)}}{|\xi^\gamma| \tanh(|\xi^\gamma|) + (1 - \tanh^2(|\xi^\gamma|)) |\xi^\gamma|^2}$$

and thus this function is bounded as ξ goes to zero. Moreover, one can check by computation that

$$\left| \frac{\partial}{\partial \xi_j} \left(\frac{1}{|\omega'(\xi)|^2} \frac{\partial}{\partial \xi_j} \omega(\xi) \right) \right| \leq \frac{C}{|\xi|}$$

as ξ goes to zero, and consequently this function is integrable at $\xi = 0$ (remember that $d > 1$ here). Therefore, one can pass the limit as δ goes to zero in the formula (5.2.30) and get

$$\int_{\mathbb{R}^d} e^{i\frac{\pm}{\varepsilon}\omega(\xi)} u(\xi) d\xi = -\frac{\varepsilon}{it} \int_{\mathbb{R}^d} \sum_{j=1}^d \frac{\partial}{\partial \xi_j} \left(\frac{1}{|\omega'(\xi)|^2} \frac{\partial}{\partial \xi_j} \omega(\xi) u(\xi) \right) e^{i\frac{\pm}{\varepsilon}\omega(\xi)} d\xi$$

and we get the desired result.

□

Now it is easy to prove the following result:

Theorem 5.2.6 *Let $s \geq 1$. Let (ζ_0, ψ_0) be such that $(\zeta_0, \mathfrak{P}\psi_0) \in H^s(\mathbb{R}^d)$. Then, for all $t \in \mathbb{R}$, the solution $(\zeta, \psi)(t)$ of the system (5.2.19) converges weakly to zero, and does not converge strongly, in $L^2(\mathbb{R}^d)$.*

Proof We prove it for ζ . We first assume that $\widehat{\zeta}, \widehat{\psi} \in C_0^1(\mathbb{R}^d)$. The weak convergence to zero is a consequence of the explicit formulation of the solution given by (5.2.26) and the Proposition 5.2.5. To prove the strong non convergence, one computes:

$$\begin{aligned} |\zeta(t, \cdot)|_2^2 &= \frac{1}{(2\pi)^d} |\widehat{\zeta}(t, \cdot)|_2^2 \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{2} \left((1 + \cos(2\omega(\xi) \frac{t}{\varepsilon})) |\widehat{\zeta}_0(\xi)|^2 + \omega(\xi)^2 (1 - \cos(2\omega(\xi) \frac{t}{\varepsilon})) |\widehat{\psi}_0(\xi)|^2 \right. \\ &\quad \left. + \operatorname{Re}(\widehat{\zeta}_0 \overline{\widehat{\psi}_0})(\xi) \sin(2\omega(\xi) \frac{t}{\varepsilon}) \right) d\xi \end{aligned}$$

where we expanded the square modulus of the explicit formulation of $\widehat{\zeta}$ given by (5.2.26). We can then conclude, using Proposition 5.2.5 to the following convergence:

$$|\zeta|_2^2 \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} (|\zeta_0|_2^2 + |\omega(D)\psi_0|_2^2)$$

and thus $\zeta(t, \cdot)$ does not converge strongly to zero.

For the general case, one proceeds by density of $C_0^1(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. \square

Remark 5.2.7 *As explained above, if ζ, ψ is a solution to (5.2.22), then the Hamiltonian*

$$\frac{1}{2\mu} (\mathcal{G}_0\psi, \psi)_2 + \frac{1}{2} |\zeta|_2^2$$

is conserved through time. Since $\frac{1}{\mu} (\mathcal{G}_0\psi, \psi)_2 \sim |\mathfrak{P}\psi|_2^2$, the strong convergence of both ζ and ψ to zero cannot happen (except for zero initial conditions). However, it would have been possible that some transfers of energy occur between ζ and ψ , with one of the unknown strongly converging to zero. The Theorem 5.2.6 states that such behavior does not happen.

5.2.3 Lack of strong convergence for the full Water-Waves equations in dimension 1

In all this section, we work with a flat bottom and in dimension 1:

$$d = 1, \quad b = 0.$$

The study in dimension 2 should however not be hard to do, if one gets a dispersive estimate of the form of Theorem 5.2.9 for the Water-Waves operator in dimension 2. In the previous section, we proved that the solutions of the linearized Water-Waves equations (5.2.19) weakly converge as ε goes to zero in $L^2(\mathbb{R})$, and do not converge strongly. We now establish a similar result for the solutions of the full nonlinear Water-Waves system (5.1.9). To this purpose, we write this system under the form:

$$\begin{cases} \varepsilon \partial_t \zeta - \frac{1}{\mu} \mathcal{G}_0 \psi = \varepsilon f \\ \varepsilon \partial_t \psi + \zeta = \varepsilon g \end{cases}$$

with

$$f = \frac{1}{\mu \varepsilon} (G[\varepsilon \zeta, 0] \psi - \mathcal{G}_0 \psi)$$

and

$$g = -\left(\frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} \frac{(G[\varepsilon \zeta, 0] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} \right).$$

Remark 5.2.8 *As suggested by the notations, the quantity f is of size $O(1)$ with respect to ε (although it has a $\frac{1}{\varepsilon}$ factor), as given by the asymptotic extension of $G[\varepsilon \zeta, 0]$ given later in Proposition 5.2.13. We are going to treat the non-linear terms f, g as perturbations of the linear equation as ε goes to zero.*

For this proof, we need to use a dispersive estimate for the linear Water-Waves equations with flat bottom, proved in [48]:

Theorem 5.2.9 *Let*

$$\omega : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ \xi \longmapsto \sqrt{\frac{|\xi| \tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}}}. \end{cases}$$

Then, there exists $C > 0$ independent on μ such that, for all $\mu > 0$:

$$\forall t > 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad |e^{it\omega(D)}\varphi|_{\infty} \leq C \left(\frac{1}{\mu^{1/4}} \frac{1}{(t/\sqrt{\mu})^{1/8}} + \frac{1}{(t/\sqrt{\mu})^{1/2}} \right) (|\varphi|_{H^1} + |x\partial_x\varphi|_2).$$

Remark 5.2.10 *The dispersion in $\frac{1}{t^{1/8}}$ stated by this Theorem is absolutely not optimal. Actually, one should expect a decay of order $\frac{1}{t^{1/3}}$ in dimension 1 (see for instance [48] for variants of Theorem 5.2.9). However, the $\frac{1}{t^{1/8}}$ decay suffices to prove the main result of this section.*

In view of use of Theorem 5.2.9, we also need a local existence result in weighted Sobolev spaces, proved also in [48]. For all $N \in \mathbb{N}$, we define \mathcal{E}_x^N by

$$\mathcal{E}_x^N = \mathcal{E}^N(\zeta, \psi) + \sum_{1 \leq |\alpha| \leq N-2} |x\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}x\psi_{(\alpha)}|_2^2.$$

Theorem 5.2.11 *Let us consider the assumption of Theorem 5.1.1, and then consider (ζ, ψ) the unique solution provided by the Theorem 5.1.1, of the Water-Waves equation (5.1.9). If $(\zeta^0, \psi^0) \in \mathcal{E}_x^N$, then one has*

$$(\zeta, \psi) \in L^\infty([0; T], \mathcal{E}_x^N),$$

with

$$(\zeta, \psi)_{L^\infty([0; T], \mathcal{E}_x^N)} \leq C_2.$$

We now state the main result of this section:

Theorem 5.2.12 *Let (ζ_0, ψ_0) be such that $(\zeta_0, \psi_0) \in \mathcal{E}_x^N$. Let denote $T > 0$, (ζ^{WW}, ψ^{WW}) the solution of the Water-Waves equations (5.1.9) in $(\zeta, \psi) \in L^\infty([0; T], \mathcal{E}_x^N)$ given by Theorem 5.1.1 on $[0; T]$ and (ζ^L, ψ^L) the solution of the linearized equation (5.2.19) given by Proposition 5.2.3, both with initial condition (ζ_0, ψ_0) . Then, one has:*

$$|(\zeta^L, \mathfrak{P}\psi^L) - (\zeta^{WW}, \mathfrak{P}\psi^{WW})|_{L^\infty([0; T]; L^2(\mathbb{R}))} \leq \left(\frac{\varepsilon^{1/8}}{\mu^{3/16}} + \varepsilon^{1/2}\mu^{1/4} \right) C_2 \quad (5.2.31)$$

where C_2 is given by Theorem 5.1.1.

Proof In all this proof, we will denote by C^N any constant of the form

$$C^N = C(\mathcal{E}_x^N(\zeta^0, \psi^0), \frac{1}{h_{\min}}, \frac{1}{a_0}) \quad (5.2.32)$$

where C is a non decreasing function of its arguments. Let us define $(\zeta, \psi) = (\zeta^{WW}, \mathfrak{P}\psi^{WW}) - (\zeta^L, \mathfrak{P}\psi^L)$ which is defined on $[0; T]$. We use the evolution operator $e^{\frac{t}{\varepsilon}L}$ defined by (5.2.23) to write:

$$\partial_t(e^{\frac{t}{\varepsilon}L} \begin{pmatrix} \zeta \\ \psi \end{pmatrix}) = e^{\frac{t}{\varepsilon}L} F \left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix} \right) (t)$$

where

$$F \left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{\mu\varepsilon} (\mathcal{G}(\varepsilon\zeta, 0)\psi - \mathcal{G}_0\psi) \\ -\frac{1}{2} |\nabla^\gamma \psi|_2^2 - \frac{1}{\mu} \frac{(\mathcal{G}(\varepsilon\zeta, 0)\psi + \varepsilon\mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2\mu |\nabla^\gamma \zeta|^2)} \end{pmatrix} \quad (5.2.33)$$

and thus one has

$$\forall t \in [0; T], \quad \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \int_0^t e^{\frac{s-t}{\varepsilon}L} F \left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix} \right) (s) ds. \quad (5.2.34)$$

We set $A = \begin{pmatrix} I & 0 \\ 0 & \mathfrak{P} \end{pmatrix}$ and look for an estimate of the L^2 norm of $A^t(\zeta, \psi)$. The proof consists in using the decay estimate of the linear operator e^{tL} of the Water-Waves equation of Theorem 5.2.9. More precisely, the operator F is "almost" bilinear (at least up to a $O(\varepsilon)$ order term), which allows us to control the L^2 norm of the integral (5.2.34) by writing estimates of the form

$$\left| \int_0^t A e^{\frac{s-t}{\varepsilon}L} F\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right)(s) ds \right|_2 \leq \left| \int_0^t A e^{\frac{s-t}{\varepsilon}L} B\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}, \begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right)(s) ds \right|_2 + \varepsilon C$$

where C does not depend on ε , and with B a bilinear operator. The proof then consists in using a Strichartz type of estimate, using the dispersive nature of e^{itL} , given by Theorem 5.2.9 to get a control of the remaining integral of the form

$$\left| \int_0^t e^{\frac{s-t}{\varepsilon}L} B\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}, \begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right)(s) ds \right|_2 \leq (\varepsilon)^{1/8} t^\alpha$$

with $\alpha > 0$.

We start to write F under the form $F = B + \varepsilon R$ where B is bilinear, and R is a least of size $O(1)$ with respect to ε . To this purpose, recalling that F is given by (5.2.33), we get inspired by the following Proposition, which gives an asymptotic extension of $\mathcal{G}[\varepsilon\zeta, 0]$ (see [37] Proposition 3.44) with respect to ε :

Proposition 5.2.13 *Let $t_0 > d/2, s \geq 0$ and $k = 0, 1$. Let $\zeta \in H^{s+(k+1)/2} \cap H^{t_0+2}(\mathbb{R}^d)$ be such that:*

$$\exists h_{\min} > 0, \forall X \in \mathbb{R}^d, 1 + \varepsilon\zeta(X) \geq h_{\min}$$

and $\psi \in H^{s+k/2}(\mathbb{R}^d)$. We get:

$$|\mathcal{G}\psi - \mathcal{G}_0\psi - \varepsilon\mathcal{G}_1|_{H^{s-1/2}} \leq \varepsilon^2 \mu^{\frac{3+k}{4}} C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}}, |\zeta|_{H^{s+(k+1)/2}}\right) |\mathfrak{P}\psi|_{H^{s+k/2}},$$

where $\mathcal{G}_1 = -\mathcal{G}_0(\zeta(\mathcal{G}_0 \cdot)) - \mu \nabla^\gamma \cdot (\zeta \nabla^\gamma \cdot)$.

We therefore write

$$\frac{1}{\mu\varepsilon} (\mathcal{G}(\varepsilon\zeta, 0) - \mathcal{G}_0)\psi = \frac{1}{\mu}\mathcal{G}_1 - \frac{1}{\mu\varepsilon} (\mathcal{G}_1 + \mathcal{G}_0 - \mathcal{G}(\varepsilon\zeta, 0))\psi \quad (5.2.35)$$

where the second term of the right hand side satisfies the following estimate, using Proposition 5.2.13 with $k = 1$ and $s = 1/2$:

$$\left| \frac{1}{\mu\varepsilon} (\mathcal{G}_1 + \mathcal{G}_0 - \mathcal{G}(\varepsilon\zeta, 0)) \right|_2 \leq C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}}, |\zeta|_{H^{1/2+1}}\right) |\mathfrak{P}\psi|_{H^1}. \quad (5.2.36)$$

The second component of F (recall that it is given by (5.2.33)) is easier to decompose, and using Proposition 5.A.1 one gets

$$-\frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} \frac{(\mathcal{G}(\varepsilon\zeta, 0)\psi + \varepsilon\mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = -\frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} (\mathcal{G}(\varepsilon\zeta, 0)\psi)^2 + \varepsilon R(\psi) \quad (5.2.37)$$

with

$$|R(\psi)|_L^2 \leq C^N, \quad (5.2.38)$$

where C^N is given by (5.2.32). Using (5.2.35) with the control (5.2.36), and (5.2.37) with the control (5.2.38), one gets:

$$F\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right) = B\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}, \begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right) + \varepsilon R\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right) \quad (5.2.39)$$

with

$$B\left(\begin{pmatrix} \zeta \\ \psi \end{pmatrix}, \begin{pmatrix} \zeta \\ \psi \end{pmatrix}\right) = \left(-\frac{1}{2} |\nabla^\gamma \psi|^2 - \frac{1}{\mu} \mathcal{G}(\varepsilon\zeta, 0)\psi^2 \right) \quad (5.2.40)$$

and

$$|R\left(\begin{smallmatrix} \zeta \\ \psi \end{smallmatrix}\right)|_2 \leq C^N. \quad (5.2.41)$$

We therefore get, using (5.2.39) and (5.2.41) (recall that C^N is a constant of the form (5.2.32)):

$$\left| \int_0^t A e^{\frac{s-t}{\varepsilon}L} F\left(\begin{smallmatrix} \zeta \\ \psi \end{smallmatrix}\right)(s) ds \right|_2 \leq \left| \int_0^t e^{\frac{s-t}{\varepsilon}L} AB(s) ds \right|_2 + \varepsilon C^N.$$

We denote in the following lines $(\cdot, \cdot)_{L_x^2}$ the L^2 scalar product with respect to the space variable. One writes:

$$\left| \int_0^t e^{\frac{s-t}{\varepsilon}L} AB(s) ds \right|_2^2 = \left| \int_0^t \int_0^t (e^{\frac{s-t}{\varepsilon}L} AB(s), e^{\frac{u-t}{\varepsilon}L} AB(u))_{L_x^2} duds \right|.$$

Now, note that $A e^{itL}$ is sum of terms of the form

$$\begin{pmatrix} 1 & i\omega(D) \\ +\frac{\mathfrak{P}}{i\omega(D)} & \mathfrak{P} \end{pmatrix} e^{it\omega(D)}$$

and therefore, we are led to estimate terms of the form

$$\int_0^t \int_0^t (e^{\frac{s-t}{\varepsilon}L} f_2(s), e^{\frac{u-t}{\varepsilon}L} f_1(u))_{L_x^2} duds, \quad (5.2.42)$$

where f_1, f_2 are terms of the form

$$\frac{1}{\mu} \mathcal{G}_1, \quad \omega(D) \frac{1}{\mu} \mathcal{G}(\varepsilon \zeta, 0)^2, \quad \omega(D) |\nabla^\gamma \psi|^2, \quad \frac{\mathfrak{P}}{\omega(D)} \frac{1}{\mu} \mathcal{G}_1, \quad \mathfrak{P} \frac{1}{\mu} \mathcal{G}(\varepsilon \zeta, 0)^2, \quad \mathfrak{P} |\nabla^\gamma \psi|_2^2. \quad (5.2.43)$$

One has to notice that the control related to the latter three terms can be deduced from the control of the first three terms since \mathfrak{P} acts like the square root $\omega(D)$ of \mathcal{G}_0 . Indeed, one has

$$\begin{aligned} \omega(D) &= \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}} \sqrt{\frac{\tanh(\sqrt{\mu}|D^\gamma|)(1 + \sqrt{\mu}|D^\gamma|)}{\sqrt{\mu}|D^\gamma|}} \\ &= \mathfrak{P} \sqrt{\frac{\tanh(\sqrt{\mu}|D^\gamma|)(1 + \sqrt{\mu}|D^\gamma|)}{\sqrt{\mu}|D^\gamma|}} \end{aligned}$$

and there exists $C_1, C_2 > 0$ independent of μ, ε such that:

$$\forall \xi \in \mathbb{R}^d, \quad C_1 \leq \sqrt{\frac{\tanh(\sqrt{\mu}|\xi^\gamma|)(1 + \sqrt{\mu}|\xi^\gamma|)}{\sqrt{\mu}|\xi^\gamma|}} \leq C_2. \quad (5.2.44)$$

We now estimate terms of the form (5.2.42) using a similar technique as for Strichartz estimates for dispersive PDE's. One computes, using the symmetry of e^{itL} :

$$\int_0^t \int_0^t (e^{\frac{s-t}{\varepsilon}L} f_2(s), e^{\frac{u-t}{\varepsilon}L} f_1(u))_{L_x^2} duds = \int_0^t \int_0^t (f_2(s), e^{\frac{u+s-2t}{\varepsilon}L} f_1(u))_{L_x^2} duds.$$

Now recall that f_i are of the form (5.2.43). We are not treating all the possible cases, but only the most difficult one (the others are treated by the same technique), which is $f_1 = f_2 = \frac{1}{\mu} \mathcal{G}_1 \psi$. Using the definition of \mathcal{G}_1 given in Proposition 5.2.13, one computes, integrating by parts:

$$\begin{aligned} & \int_0^t \int_0^t (f_2(s), e^{\frac{u+s-2t}{\varepsilon}L} f_1(u))_{L_x^2} duds \\ &= \int_0^t \int_0^t \left(-\frac{1}{\mu} \mathcal{G}_0(\zeta \mathcal{G}_0(\psi))(s) - \nabla^\gamma \cdot (\zeta \nabla^\gamma \psi)(s), e^{\frac{u+s-2t}{\varepsilon}L} \frac{1}{\mu} \mathcal{G}_1 \psi(u) \right)_{L_x^2} duds \\ &= \int_0^t \int_0^t \left(-\frac{1}{\mu} \zeta \mathcal{G}_0(\psi)(s), e^{\frac{u+s-2t}{\varepsilon}L} \mathcal{G}_0 \frac{1}{\mu} \mathcal{G}_1(u) \right)_{L_x^2} + (\zeta \nabla^\gamma \psi(s), e^{\frac{u+s-2t}{\varepsilon}L} \nabla^\gamma \frac{1}{\mu} \mathcal{G}_1 \psi(u))_{L_x^2} duds. \end{aligned} \quad (5.2.45)$$

We only control the first integral of the right hand side of (5.2.45) (the other one is estimated by a similar technique). We set

$$\tilde{f}_2 = -\frac{1}{\mu}\zeta\mathcal{G}_0\psi(s), \quad \tilde{f}_1 = \mathcal{G}_0\frac{1}{\mu}\mathcal{G}_1\psi(u) \quad (5.2.46)$$

and we now use the dispersive estimate of Theorem 5.2.9:

$$\begin{aligned} \left| \int_0^t \int_0^t (\tilde{f}_2(s), e^{\frac{u+s-2t}{\varepsilon}L}(\tilde{f}_1)_{L_x^2} duds) \right| &\leq \int_0^t \int_0^t |\tilde{f}_2|_{L_x^1} |e^{\frac{u+s-2t}{\varepsilon}L}(\tilde{f}_1)|_{L_x^\infty} \\ &\leq C^N \int_0^t \int_0^t \frac{\varepsilon^{1/8}}{\mu^{3/16}} \frac{1}{(2t-u-s)^{1/8}} \\ &\quad + \frac{\varepsilon^{1/2}\mu^{1/4}}{(2t-u-s)^{1/2}} (|\tilde{f}_1(u)|_{H_x^1} + |x\partial_x\tilde{f}_1(u)|_{L_x^2}) duds \\ &\leq \left(\frac{\varepsilon^{1/8}}{\mu^{3/16}} t^{7/8} + \varepsilon^{1/2}\mu^{1/4} t^{3/2} \right) C^N \end{aligned}$$

if one can prove the following controls:

$$\sup_{u \in [0; T]} |\tilde{f}_2(u)|_{L_x^1} + |\tilde{f}_1|_{H_x^1} + |x\partial_x\tilde{f}_1(u)|_{L_x^2} \leq C^N, \quad i = 1, 2. \quad (5.2.47)$$

One has, using Cauchy-Schwarz inequality:

$$\begin{aligned} |\tilde{f}_2(u)|_{L_x^1} &\leq |\zeta|_2 \left| \frac{1}{\mu}\mathcal{G}_0\psi \right|_2 \\ &\leq |\zeta|_2 |\mathfrak{P}\psi|_{H^1} \\ &\leq C^N \end{aligned}$$

where we used Proposition 5.A.1 with $\zeta = b = 0$ to control $\frac{1}{\mu}\mathcal{G}_0\psi$. We now focus on the most difficult term of (5.2.47) which is $|x\partial_x\mathcal{G}_0\frac{1}{\mu}\mathcal{G}_1\psi|_{L_x^2}$. We use again the definition of \mathcal{G}_1 given by Proposition 5.2.13, and we control $\frac{1}{\mu}|x\partial_x\mathcal{G}_0\mathcal{G}_0(\zeta\mathcal{G}_0(\psi))|_2$ (the other one is similar). One computes:

$$\frac{1}{\mu}|x\partial_x\mathcal{G}_0\mathcal{G}_0(\zeta\mathcal{G}_0(\psi))|_2 \leq \frac{1}{\mu}|x\mathcal{G}_0^2(\partial_x\zeta)\mathcal{G}_0\psi|_2 + \frac{1}{\mu}|x\mathcal{G}_0^2(\zeta\mathcal{G}_0(\partial_x\psi))|_2. \quad (5.2.48)$$

We only control the first term of the right hand side of (5.2.48) (the other one is similar):

$$|x\mathcal{G}_0^2(\partial_x\zeta)\mathcal{G}_0\psi|_2 \leq |\mathcal{G}_0^2x(\partial_x\zeta)\mathcal{G}_0\psi|_2 + |[x, \mathcal{G}_0^2](\partial_x\zeta)\mathcal{G}_0\psi|_2.$$

Now, note that for all $f \in \mathcal{S}(\mathbb{R})$, one has:

$$\begin{aligned} |[x, \mathcal{G}_0^2]f|_2 &= \left| \frac{d}{d\xi}(\sqrt{\mu}|\xi| \tanh(\sqrt{\mu}|\xi|))^2 \hat{f} \right|_2 \\ &\leq \mu^2 |f|_{H^3} \end{aligned}$$

using the definition of \mathcal{G}_0 given by (5.2.20). Therefore, one has:

$$\begin{aligned} \frac{1}{\mu}|x\mathcal{G}_0^2(\partial_x\zeta)\mathcal{G}_0\psi|_2 &\leq \mu|x(\partial_x\zeta)\mathcal{G}_0\psi|_{H^4} + \mu|(\partial_x\zeta)\mathcal{G}_0\psi|_{H^3} \\ &\leq \mu^2|x(\partial_x\zeta)|_{H^4}|\mathcal{G}_0\psi|_{H^4}, \end{aligned}$$

where we used the gross estimate $|\mathcal{G}_0^2u| \leq |u|_{H^4}$. One can conclude using Proposition 5.A.1 with $\zeta = b = 0$, and Theorem 5.2.11.

This achieves the proof of the controls (5.2.47).

Conclusion: We proved

$$|(\zeta, \mathfrak{P}\psi)(t)|_2 \leq \left(\frac{\varepsilon^{1/8}}{\mu^{3/16}} t^{7/8} + \varepsilon^{1/2}\mu^{1/4} t^{3/2} \right) C^N.$$

One can then take the supremum over $t \in [0; T]$ and get the desired result. \square .

It is very important to note that the strong convergence stated by Theorem 5.2.12 does not stand anymore if μ is related to ε . For instance, if $\varepsilon = \mu^{3/2}$, the result is not true anymore. In this case, the dispersive effects would be at the same size as the non-linear effects, and one should expect a behavior of solutions similar to ones of the Korteweg de Vries's equation. In particular, it is no longer possible to treat the non-linearities f, g as perturbations of the linearized system. If $\varepsilon \ll \mu^{3/2}$, then the non-linear effects are dominants on the dispersive effects, and one should expect breaking waves. This is the main interest of having a dispersive result of the form of Theorem 5.2.9 which depends on the small parameters.

It is now easy to prove the following Corollary:

Corollary 5.2.14 *Let (ζ_0, ψ_0) be such that $(\zeta_0, \psi_0) \in \mathcal{E}_x^N$. Let denote $T > 0$, (ζ, ψ) the solution of the Water-Waves equations defined on $[0; T]$ of (5.1.9). Then, $(\zeta, \mathfrak{P}\psi)$ converge weakly to zero in $L_t^\infty L_x^2$ and does not converge strongly in this space.*

Proof Let us denote (ζ^L, ψ^L) the solution of the linearized equation (5.2.19) with initial condition (ζ_0, ψ_0) given by Theorem 5.2.11 on $[0; T]$ (just take $\varepsilon = 0$ in the equation (5.1.9) to get the solutions). Using Proposition 5.2.3, one has

$$(\zeta^L, \psi^L)(t) = e^{-\frac{t}{\varepsilon}L}(\zeta^0, \psi^0).$$

Using the definition of e^{tL} given by (5.2.23), $\zeta, \mathfrak{P}\psi$, are sum of terms of the form $e^{\frac{it\omega(D)}{\varepsilon}}(\zeta^0, \psi^0)$ and therefore one has, using the dispersive estimate of Theorem 5.2.9:

$$\forall t \in [0; T], \quad |(\zeta^L(t), \mathfrak{P}\psi^L(t))| \leq \left(\frac{\varepsilon^{1/8}}{\mu^{3/16}} t^{7/8} + \varepsilon^{1/2} \mu^{1/4} t^{3/2} \right) (|(\zeta^0, \mathfrak{P}\psi^0)|_{H^1} + |x\partial_x \zeta^0, \mathfrak{P}x\partial_x \psi^0|_{L^2})$$

and therefore one gets the weak convergence of $(\zeta, \mathfrak{P}\psi)$ in $L^\infty((0; T); L_x^2(\mathbb{R}))$ as ε goes to zero. The convergence is not strong in $L^\infty((0; T); L_x^2(\mathbb{R}))$ since the quantity

$$\frac{1}{2\mu}(\mathcal{G}_0\psi, \psi)_2 + \frac{1}{2}|\zeta|_2^2 \sim \frac{1}{2}(|\mathfrak{P}\psi|_2^2 + |\zeta|_2^2)$$

is conserved through time.

Now, according to Theorem 5.2.12, $(\zeta, \mathfrak{P}\psi) - (\zeta^L, \psi^L)$ converges strongly to zero as ε goes to zero in $L^\infty((0; T); L_x^2(\mathbb{R}))$. Therefore, (ζ^L, ψ^L) converges weakly, but not strongly as ε goes to zero, in this space. \square

5.3 Equivalence between free surface Euler and Water-Wave equation

In all this Section, we do not make any assumption on the dimension, and we set the bottom to zero:

$$d = 1, 2, \quad b = 0.$$

We consider the standard dimensionless version of the Water-Waves equations with flat bottom:

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} G[\varepsilon \zeta, 0] \psi = 0 \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla^\gamma \psi|^2 - \frac{\varepsilon}{\mu} \frac{(G[\varepsilon \zeta, 0] \psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2)} = 0. \end{cases} \quad (5.3.49)$$

In Section 5.2, we studied the rigid lid limit for the Water-Waves equations (5.2.17). In order to complete the study of the rigid lid equations (5.2.15), we study in this Section the rigid lid limit for the Euler equations (5.1.7). To this purpose, we prove that one can build rigorously solutions to the free surface Euler equations from the solutions of the Water-Waves equations. The first problem is to define rigorously a solution to the free surface Euler equations. Because the free surface Euler equations (5.1.7) are posed on a domain Ω_t^ε which depends on ε and t , we have to be careful with the functional spaces used. The idea is that, by hypothesis, the height $1 + \varepsilon \zeta(t, X)$ of the water is

bounded with respect to $X \in \mathbb{R}^d$ at a fixed time t . By continuity in time, we can include Ω_t^ε in a fixed strip \mathcal{S}^* , which does not depend either on t nor ε .

We can then define rigorously what is a solution to the Euler equation (5.1.7): it is (U, P) living in a functional space like "continuous in time with value to a Sobolev-type space in X on the fixed strip \mathcal{S}^* " such that there exists $T > 0$ such that, for all $0 \leq t \leq T$, the derivative (in the sense of the distributions of $\mathcal{D}'_{X,z,t}([0, T] \times \mathcal{S}^*)$) satisfies the equalities of (5.1.7).

5.3.1 Main result

In order to extract a convergent sub-sequence of solutions to the Water-Waves equations (5.3.49) as ε goes to zero, we need some compactness results. Indeed, a space such as $C([0; T]; H^N(\mathbb{R}^d))$ is not compactly embedded into $L^\infty([0; T]; H^N(\mathbb{R}^d))$. We need a control for time derivatives of the unknowns. For this reason, the stated local existence Theorem 5.1.1 is not sufficient. We recall here a slightly different version, including time derivatives in the energy space. The following framework is used for instance to prove the local existence for the Water-Waves equations with surface tension in [37] Chapter 9, and in [50] for a long time existence result. We first introduce the following energy:

Definition 5.3.1 *Let $N \geq 1$ and $t_0 > d/2$. We define*

$$\mathcal{E}_1^N = |\mathfrak{P}\psi|_{H^{t_0+3/2}}^2 + \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leq N} |\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}\psi_{(\alpha)}|_2^2 \tag{5.3.50}$$

with

$$\zeta_{(\alpha)} = \partial^\alpha \zeta, \quad \psi_{(\alpha)} = \partial^\alpha \psi - \varepsilon \underline{w} \partial^\alpha \zeta$$

and

$$\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}$$

with the notation:

$$\forall \alpha = (\alpha_1, \dots, \alpha_d, k) \in \mathbb{N}^{d+1}, \partial^\alpha = \partial_{X_1}^{\alpha_1} \dots \partial_{X_d}^{\alpha_d} \partial_t^k.$$

Remark 5.3.2 – *It is very important to notice that we consider here also time derivatives in the energy.*

– *We recall that \underline{w} coincides as suggested with the horizontal component of the velocity evaluated at the surface.*

We now give a slightly different version of the local existence Theorem for the Water-Waves equations (5.3.49) proved by Alvarez-Samaniego and Lannes (see also [37] Chapter 4 and Chapter 9):

Theorem 5.3.3 (Alvarez Samaniego, Lannes) *Let $t_0 > d/2$, $N > t_0 + t_0 \vee 2 + 3/2$, and $(\varepsilon, \mu, \gamma)$ be such that*

$$0 \leq \varepsilon, \mu, \gamma \leq 1.$$

Let (ζ_0, ψ_0) with $\mathcal{E}_1^N(\zeta_0, \psi_0) < \infty$, where \mathcal{E}_1^N is defined by (5.3.50). We assume that:

$$\exists h_{\min} > 0, \exists a_0 > 0, \quad 1 + \varepsilon \zeta_0 \geq h_{\min} \text{ and } \underline{a}(\zeta_0, \psi_0) \geq a_0.$$

Then, there exists $T > 0$ and a unique solution $(\zeta, \psi) \in C([0; T]; H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d))$ to the Water-Waves equation (5.3.49) such that $\mathcal{E}_1^N(\zeta, \psi) \in L^\infty([0; \frac{T}{\varepsilon}])$. Moreover, one has

$$\frac{1}{T} = C_1 \quad \text{and} \quad \sup_{t \in [0; \frac{T}{\varepsilon}]} \mathcal{E}_1^N(t) = C_2$$

with C_i a constant of the form $C_i = C(\mathcal{E}_1^N(\zeta_0, \psi_0), \frac{1}{h_{\min}}, \frac{1}{a_0})$ for $i = 1, 2$.

We use from now on the notations $H^{s,k}$ for the Beppo-Levi spaces, and Σ_t^ε for the diffeomorphism from the flat strip to the water domain (see Section 5.1.5 for notations). We also recall that Ω_t^ε denote the domain occupied by the water at time t . We now state the main result of this Section:

Theorem 5.3.4 *Under the hypothesis of Theorem 5.3.3, let $T > 0$ be given by Theorem 5.3.3. Then, for all $\varepsilon > 0$, there exists a flat strip $\mathcal{S}^* = \mathbb{R}^d \times (-(k+1); k)$ with $k \geq 1$ and $(U^\varepsilon, \zeta^\varepsilon, P^\varepsilon)$ where $U^\varepsilon = (V^\varepsilon, w^\varepsilon)$ in the following spaces:*

$$\begin{cases} \forall k \leq N, & \partial_t^k U^\varepsilon \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}^*)^{d+1}) \\ \forall k \leq N, & \partial_t^k \zeta^\varepsilon \in L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k}(\mathbb{R}^d)) \\ \forall k \leq N, & \partial_t^k P^\varepsilon \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}^*)) \end{cases} \quad (5.3.51)$$

with bounds uniform in ε in these spaces, such that for all $t \in [0; \frac{T}{\varepsilon}]$, for all $\varepsilon > 0$, the domain Ω_t^ε is included in \mathcal{S}^* . Moreover, $(V^\varepsilon, w^\varepsilon, \zeta^\varepsilon, P^\varepsilon)$ satisfy in the sense of the distributions of $\mathcal{D}'_{X,z,t}([0, \frac{T}{\varepsilon}] \times \mathcal{S}^*)$ the Euler equations (5.1.7).

The proof follows these steps:

- We claim some regularity for the velocity potential ϕ solving (5.1.14) on the fixed strip $\mathcal{S} = \mathbb{R}^d \times (-1; 0)$.
- We extend ϕ to a larger strip than \mathcal{S} .
- We recover regularity for the velocity potential Φ defined originally on the water domain by $\phi = \Phi \circ \Sigma_t^\varepsilon$, after defining it on a large strip containing all the fluid domain Ω_t^ε for all t, ε .

5.3.2 Regularity of the solutions of the Water Waves problem

According to Theorem 5.3.3, for all ε , there exists a unique solution $(\zeta^\varepsilon, \psi^\varepsilon)$ to (5.3.49) on a time interval $[0, \frac{T}{\varepsilon}]$, with T only depending on the initial energy (and not on ε), with the following control:

$$\forall t \in [0; \frac{T}{\varepsilon}], \quad \mathcal{E}_1^N(U^\varepsilon)(t) \leq C_2 \quad (5.3.52)$$

with C_2 a constant of the form $C(\mathcal{E}_1^N(\zeta_0, \psi_0), \frac{1}{h_{\min}}, \frac{1}{a_0})$, where \mathcal{E}_1^N is defined by (5.3.50). We can then prove the following regularity result:

Proposition 5.3.5 *Let $(\zeta^\varepsilon, \psi^\varepsilon)$ be the solution of 5.3.49 given by Theorem 5.3.3, and $T > 0$ such that (5.3.52) is satisfied. Then, the following regularity results stand:*

- 1) For all $0 \leq k \leq N$, one has:

$$\partial_t^k \zeta^\varepsilon \in L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k}(\mathbb{R}^d)) \quad \text{and} \quad |\partial_t^k \zeta^\varepsilon|_{L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k}(\mathbb{R}^d))} \leq C_2;$$

- 2) For all $0 \leq k \leq N-1$, one has:

$$\partial_t^k \mathfrak{P}\psi^\varepsilon \in L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k-1/2}(\mathbb{R}^d)) \quad \text{and} \quad |\partial_t^k \mathfrak{P}\psi^\varepsilon|_{L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k-1/2}(\mathbb{R}^d))} \leq C_2,$$

with C_2 a constant of the form $C(\mathcal{E}^N(\zeta_0, \psi_0), \frac{1}{h_{\min}}, \frac{1}{a_0})$.

Proof Recall that the energy is:

$$\mathcal{E}^N(U) = |\mathfrak{P}\psi|_{H^{t_0+3/2}}^2 + \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leq N} |\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}\psi_{(\alpha)}|_2^2.$$

1) Since the time derivatives of the unknowns appear in the energy, the control given by (5.3.52) implies that for all $0 \leq k \leq N$, $\partial_t^k \zeta^\varepsilon$ is in $L_t^\infty(H^{N-k}(\mathbb{R}^d))$, with the desired estimate.

2) Let us fix k such that $0 \leq k \leq N-1$. We have to get a control of $\mathfrak{P}\partial_t^k \partial^\alpha \psi$ by $\mathfrak{P}\psi_{(\alpha, k)}$. Adapting a proof from Lemma 4.6 in [37], one computes:

$$\begin{aligned} |\mathfrak{P}\partial_t^k \psi|_{H^{N-k-1/2}} &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1-k} |\mathfrak{P}\partial_t^k \partial^\beta \psi|_{H^{1/2}} \\ &\leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1-k} (|\mathfrak{P}\psi_{(\beta, k)}|_{H^{1/2}} + \varepsilon |\mathfrak{P}(\underline{w}\partial_t^k \partial^\beta \zeta)|_{H^{1/2}}) \end{aligned}$$

where we used the definition of $\psi_{(\beta,k)}$ given by Definition 5.3.1. We now use the identity $|\mathfrak{P}f|_{H^{1/2}} \leq \max\{1, \mu^{-1/4}\} |\nabla f|_2$ to conclude:

$$|\mathfrak{P}\partial_t^k \psi|_{H^{N-k-1/2}} \leq \sum_{\beta \in \mathbb{N}^d, |\beta| \leq N-1-k} (|\mathfrak{P}\psi_{(\beta,k)}|_{H^1} + \varepsilon \max\{1, \mu^{-1/4}\} |\underline{w}|_{H^{t_0+1}} |\partial_t^k \zeta|_{H^{N-k}}).$$

One already has by the first point that $|\partial_t^k \zeta|_{H^{N-k}} \leq C_2$. Using Proposition 5.A.1, we have $|\underline{w}|_{H^{t_0+1}} \leq \mu^{3/4} M |\mathfrak{P}\psi|_{H^{t_0+3/2}} \leq C_2$. We now control $\psi_{(\beta,k)}$ in H^1 norm, for all $|\beta| \leq N-1-k$. One computes:

$$\begin{aligned} |\mathfrak{P}\psi_{(\beta,k)}|_{H^1} &\leq |\mathfrak{P}\psi_{(\beta,k)}|_2 + \sum_{\delta \in \mathbb{N}^d, |\delta|=1} |\partial^\beta \mathfrak{P}\psi_{(\beta,k)}|_2 \\ &\leq |\mathfrak{P}\psi_{(\beta,k)}|_2 + \sum_{\delta \in \mathbb{N}^d, |\delta|=1} (|\mathfrak{P}\psi_{(\beta+\delta,k)}|_2 + \varepsilon |\mathfrak{P}(\partial^\delta \underline{w} \partial_t^k \partial^\beta \zeta)|_2). \end{aligned}$$

Since $|\beta + \delta| + k \leq N$, the first component of the right hand side from above is controlled by a constant of the form C_2 . For the second component, we use the identity:

$$\exists C > 0, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \quad |\mathfrak{P}f| \leq C \mu^{-1/4} |f|_{H^{1/2}}$$

where C does not depend on μ . Therefore,

$$|\mathfrak{P}(\partial^\delta \underline{w} \partial^\beta \zeta)|_2 \leq C \mu^{-1/4} |\partial^\delta \underline{w} \partial^\beta \zeta|_{H^{1/2}} \leq C \mu^{-1/4} |\underline{w}|_{H^{t_0+1}} |\partial_t^k \zeta|_{H^{N-k}}.$$

We now use as before Proposition 5.A.1 to control $\mu^{-1/4} |\underline{w}|_{H^{t_0+1}}$ by C_2 and we remark that $|\mathfrak{P}\psi_{(\beta+\delta,k)}|_2 \leq C_2$ for all $|\beta + \delta| + k \leq N$. We obtained the control:

$$|\mathfrak{P}\partial_t^k \psi|_{H^{N-k-1/2}} \leq C_2$$

with C_2 independent on the time. Taking the supremum on time over $[0; \frac{T}{\varepsilon}]$, one gets the desired result.

□

In order to define a velocity $U = \nabla^{\mu,\gamma} \Phi$ on the fluid domain Ω_t^ε , where Φ solves the Dirichlet-Neumann problem (5.1.6) on the fluid domain, we first study the regularity of ϕ where ϕ is the solution of the Dirichlet-Neumann problem (5.1.14) on the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1; 0)$. We recall the relation $\phi = \Phi \circ \Sigma_t^\varepsilon$, where Σ_t^ε is the diffeomorphism defined by (5.1.13).

Proposition 5.3.6 *Let ϕ be the unique variational solution to (5.1.14). We have, for all $0 \leq k \leq N-1$:*

$$\partial_t^k \nabla^{\mu,\gamma} \phi \in L^\infty((0, \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}))$$

with

$$|\partial_t^k \nabla^{\mu,\gamma} \phi|_{L^\infty((0, \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}))} \leq C_2$$

where C_2 is a constant of the form $C(\mathcal{E}^N(\zeta_0, \psi_0), \frac{1}{h_{\min}}, \frac{1}{a_0})$.

This notation is a bit heavy, but we have to deal with the fact that we only use integer derivatives in z .

Proof Using the notations of the Proposition 5.A.8, we recall that:

$$\phi = \mathfrak{A}_\psi(\zeta, 0).$$

By taking the derivative in time, $\partial_t^k \phi$ is sum of terms of the form:

$$d^j \mathfrak{A}_{\partial_t^{k-j} \psi}(\zeta, 0)(\partial_t^{l_1} \zeta, \dots, \partial_t^{l_j} \zeta)$$

with $0 \leq j \leq k$ and $l_1 + \dots + l_j = j$. Let $l_{\max} = \max_{1 \leq m \leq j} l_m$. We assume $l_{\max} = l_1$ by symmetry. We then have two cases:

1) Case $l_1 \leq N - t_0 - 1$. By using Proposition 5.A.8, with $t'_0 = N - l_1 - 1$ (note that $t'_0 \geq t_0 > d/2$) and $s \leq N - k - 1/2$ we have:

$$\begin{aligned} |\Lambda^s \nabla^{\mu, \gamma} d^j \mathfrak{A}_{\partial_t^{k-j} \psi}(\zeta, 0)(\partial_t^{l_1} \zeta, \dots, \partial_t^{l_j} \zeta)|_2 &\leq \sqrt{\mu} M_0 \prod_{m=1}^j |\varepsilon \partial_t^{l_m} \zeta|_{H^{N-l_1}} |\mathfrak{P} \partial_t^{k-j} \psi|_{H^s} \\ &\leq C_2 \end{aligned}$$

where we used the $H^{N-k-1/2}$ regularity of $\mathfrak{P} \partial_t^{k-j} \psi$ given by Proposition 5.3.5.

2) Case $l_1 \geq N - t_0 - 1$. Let us use Proposition 5.A.9, with $t'_0 = N - 1 - (k - l_1)$ (just note that $t'_0 \geq 3/2 > d/2$ for $d = 1, 2$). We have, for all $s \leq N - k - 1/2$ (we have $N - k - 1/2 \geq t'_0$):

$$\begin{aligned} |\Lambda^s \nabla^{\mu, \gamma} d^j \mathfrak{A}_{\partial_t^{k-j} \psi}(\zeta, 0)(\partial_t^{l_1} \zeta, \dots, \partial_t^{l_j} \zeta)|_2 &\leq \sqrt{\mu} M_0 |\varepsilon \partial_t^{l_1} \zeta|_{H^{s+1/2}} \prod_{m>1} |\varepsilon \partial_t^{l_m} \zeta|_{H^{t'_0+1}} |\mathfrak{P} \partial_t^{k-j} \psi|_{H^{t'_0+1/2}} \\ &\leq C_2 \end{aligned}$$

because $t'_0 + 1/2 + k - j \leq N - 1/2$, and where we used again the Proposition 5.3.5 for the $H^{N-k-1/2}$ regularity of $\mathfrak{P} \partial_t^{k-j} \psi$.

We proved that $\partial_t^k \nabla^{\mu, \gamma} \phi \in L^\infty((0, \frac{T}{\varepsilon}; H^{N-k-1/2})$. Using the last part of Proposition 5.A.8 and Proposition 5.A.9, we can adapt the previous proof to get

$$\partial_t^k \nabla^{\mu, \gamma} \phi \in L^\infty((0, \frac{T}{\varepsilon}; H^{N-k-1/2, N-k-1}(\mathcal{S})))$$

with the desired control for the norm. \square

5.3.3 Extension of the solution ϕ

We want to extend the distribution ϕ defined on the strip $\mathcal{S} = \mathbb{R}^d \times (-1; 0)$ to a larger strip. This is the point of the following result:

Theorem 5.3.7 *Let $s \in \mathbb{R}^+$ and $k \in \mathbb{N}$. Let us denote $\mathcal{S}_j = (-(j+1), j) \times \mathbb{R}^d$ for all $j \in \mathbb{N}$. Then, there exists an extension*

$$P : \begin{array}{ccc} H^{s,k}(\mathcal{S}_0) & \rightarrow & H^{s,k}(\mathcal{S}_j) \\ u & \mapsto & Pu \end{array}$$

such that:

$$\forall u \in H^{s,k}(\mathcal{S}_0), \quad |u|_{H^{s,k}(\mathcal{S}_j)} \leq C(k, j) |u|_{H^{s,k}(\mathcal{S}_0)} \quad (5.3.53)$$

where $C(k, j)$ only depends on k and j .

The proof is postponed to Appendix 5.B. We now have the full artillery to extend the solution Φ defined in the moving domain Ω_t^ε into a function (with the same regularity) defined on a fixed strip \mathcal{S}^* .

Let Σ_t^ε be the following diffeomorphism, mapping the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$ into Ω_t^ε :

$$\begin{aligned} \Sigma_t^\varepsilon : \mathbb{R}^{d+1} &\rightarrow \mathbb{R}^{d+1} \\ (X, z) &\mapsto (X, (1 + \varepsilon \zeta(t, X))z + \varepsilon \zeta(t, X)). \end{aligned}$$

It is easy to check that Σ_t^ε is a homeomorphism that maps \mathcal{S} exactly into Ω_t^ε . We now include Ω_t^ε into a fixed strip \mathcal{S}^* :

Proposition 5.3.8 *With the notations of Theorem 5.B.1, there exists a strip \mathcal{S}_k , which does not depend either in ε or t such that:*

$$\forall 0 \leq t \leq \frac{T}{\varepsilon}, \quad \Omega_t^\varepsilon \subset \mathcal{S}_k.$$

Proof Proposition 5.3.5 gives that:

$$\begin{aligned} [0; \frac{T}{\varepsilon}] \times \mathbb{R}^d &\rightarrow \mathbb{R} \\ (t, X) &\mapsto 1 + \varepsilon \zeta(t, X) \end{aligned}$$

is (in particular) in $C([0; \frac{T}{\varepsilon}], H^2(\mathbb{R}^d))$ with a bound uniform in ε . The continuous embedding $H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $d \leq 2$ gives the desired result. \square

Now, with the uniform (with respect to t, X, ε) bound of $\zeta(t, X)$, it is easy to check that all strip S_j is mapped by Σ_t^ε into a strip S_k :

$$\forall j \in \mathbb{N}^* \quad \exists k \in \mathbb{N}^*, \quad \forall t \in [0, \frac{T}{\varepsilon}], \quad \forall \varepsilon \leq 1, \Sigma_t^\varepsilon(S_j) \subset S_k.$$

Definition 5.3.9 Let k, l, j be such that:

$$\begin{aligned} S_l &\subset S_j \\ \Omega_t^\varepsilon &\subset S_k \text{ for all } t, \varepsilon \\ S_l &\subset \Sigma_t^\varepsilon(S_j) \text{ for all } t, \varepsilon \\ S_k &\subset \Sigma_t^\varepsilon(S_l) \text{ for all } t, \varepsilon \end{aligned}$$

We denote $S_k = S^*$, $\tilde{\phi}$ the extension of ϕ provided by theorem 5.B.1 into S_l , $\Phi = \phi \circ (\Sigma_t^{\varepsilon^{-1}})$ and $\tilde{\Phi} = \tilde{\phi} \circ (\Sigma_t^{\varepsilon^{-1}})$.

Φ is the potential on the domain Ω_t^ε , while $\tilde{\Phi}$ is an extension into the fixed strip S^* .

Remark 5.3.10 The purpose of this definition is to have a distribution $\tilde{\Phi}$ extending Φ and defined on a flat strip S_k . We therefore extend ϕ to S_l , and $\tilde{\Phi} = \tilde{\phi} \circ (\Sigma_t^{\varepsilon^{-1}})$ is then defined on S_k (since it contains $\Sigma_t^\varepsilon(S_l)$) which also contains Ω_t^ε . The fact that we need to use three strips instead of two is purely technical (see later the proof of Proposition 5.3.11).

5.3.4 Regularity after diffeomorphism

We now need to recover regularity for Φ .

Proposition 5.3.11 Using the notations of Definition 5.3.9, let ϕ be a distribution on $[0; \frac{T}{\varepsilon}] \times \mathcal{S}$ with the following regularity:

$$\forall 0 \leq k \leq N-1, \quad \partial_t^k \nabla^{\mu, \gamma} \phi \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}))$$

with a C_2 bound. Let $\tilde{\phi}$ be an extension of ϕ on the strip S_l , and $\tilde{\Phi} = \tilde{\phi} \circ \Sigma_t^{\varepsilon^{-1}}$. Then, we have, for all $k \leq N-1$:

$$\partial_t^k \nabla^{\mu, \gamma} \tilde{\Phi} \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}^*))$$

with a C_2 bound.

The proof is postponed in Section 5.B for the sake of clarity.

5.3.5 Craig-Sulem-Zakharov formulation to Euler formulation

We now build solutions to the free surface Euler equation (5.1.7) from the potential $\tilde{\Phi}$ introduced before. We first prove that $\tilde{\Phi}$ satisfies the Bernoulli equations on the domain Ω_t^ε . One considers the following distribution of $\mathcal{D}'((0; \frac{T}{\varepsilon}); \mathcal{S}^*)$:

$$\partial_t \tilde{\Phi} + \frac{\varepsilon}{2} |\nabla^{\mu, \gamma} \tilde{\Phi}|^2 + \frac{1}{\varepsilon} z,$$

which exists in $L^\infty((0; \frac{T}{\varepsilon}); H^{N-1/2, N-1}(\mathcal{S}^*))$.

Let us consider a measurable function F of the variable t , such that for almost every $t \in (0; \frac{T}{\varepsilon})$, F is equal to this quantity. For almost every t , F is an element of $H^{N-1/2, N-1}(\mathcal{S}^*)$ which trace

on the boundary $z = \varepsilon\zeta$ of Ω_t^ε is equal to zero (this statement implies the second equation of the Craig-Sulem-Zakharov formulation (5.3.49)). For a complete and rigorous proof of this statement, see [3]. In particular, for almost every t , F is equal to an element of

$$H_{0,surf}^{N-1/2, N-1}(\mathcal{S}^*) = \overline{\mathcal{S}^* \setminus \{z = \varepsilon\zeta\}}_{\|\cdot\|_{H^{N-1/2, N-1}(\mathcal{S}^*)}}.$$

We denote this element $-\frac{1}{\varepsilon}P$. Of course, P corresponds physically to the pressure. We can rewrite this as the Bernoulli equation:

$$\partial_t \tilde{\Phi} + \frac{\varepsilon}{2} |\nabla^{\mu, \gamma} \tilde{\Phi}|^2 + \frac{1}{\varepsilon} z = -\frac{1}{\varepsilon} P \quad (5.3.54)$$

in $\mathcal{D}'((0; \frac{T}{\varepsilon}); \Omega_t^\varepsilon)$, where $((0; \frac{T}{\varepsilon}); \Omega_t^\varepsilon) = \{(t, X, z) \in (0; \frac{T}{\varepsilon}) \times \mathcal{S}^*, (X, z) \in \Omega_t^\varepsilon\}$ (note that this is an open set of $(0; \frac{T}{\varepsilon}) \times \mathcal{S}^*$).

Remark 5.3.12 *It is easy to deduce from the Bernoulli equation (5.3.54) that for all $k \leq N_0$:*

$$\partial_t^k P \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}^*))$$

with a bound uniform in ε .

Now, defining $U = \nabla^{\mu, \gamma} \tilde{\Phi}$, we have immediately

$$\begin{cases} \partial_t V + \varepsilon(V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\frac{1}{\varepsilon} \nabla^\gamma P \text{ in } \mathcal{D}'((0; \frac{T}{\varepsilon}); \Omega_t^\varepsilon) \\ \partial_t w + \varepsilon(V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -\frac{1}{\varepsilon} (\partial_z P + 1) \text{ in } \mathcal{D}'((0; \frac{T}{\varepsilon}); \Omega_t^\varepsilon) \\ \partial_t \zeta + \varepsilon \underline{V} \cdot \nabla^\gamma \zeta - \frac{1}{\mu} w = 0 \text{ in } \mathcal{D}'((0; \frac{T}{\varepsilon}); \mathbb{R}^d) \\ \nabla^{\mu, \gamma} \cdot (U) = 0 \text{ in } \mathcal{D}'((0; \frac{T}{\varepsilon}); \Omega_t^\varepsilon) \\ \text{curl}^{\mu, \gamma} (U) = 0 \text{ in } \mathcal{S}^* \\ U \cdot n = 0 \text{ on } z = -1 + \beta b \end{cases} \quad (5.3.55)$$

Remark 5.3.13 - *The third equation of the Euler system (5.3.55) is just the first equation of the Zakharov-Craig-Sulem equation.*

- *Note that the regularity of $U = \nabla^{\mu, \gamma} \tilde{\Phi}$ is given by Proposition 5.3.11*

5.4 The non strong convergence from Euler to rigid-lid

We now have a proper set of solutions $(U^\varepsilon, \zeta^\varepsilon, P^\varepsilon)$ to the free surface Euler equations (5.1.7), defined in the following spaces:

$$\begin{cases} \forall k \leq N-1, & \partial_t^k U^\varepsilon \in L^\infty((0; \frac{T}{\varepsilon}); H^{N-k-1/2, N-k-1}(\mathcal{S}^*)) \\ \forall k \leq N-1, & \partial_t^k \zeta^\varepsilon \in L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k}(\mathbb{R}^d)) \\ \forall k \leq N-1, & \partial_t^k P^\varepsilon \in L^\infty([0; \frac{T}{\varepsilon}]; H^{N-k-1/2, N-k-1}(\mathcal{S}^*)) \end{cases} \quad (5.4.56)$$

with bounds uniform in ε of the form $C(\mathcal{E}^N(\zeta_0, \psi_0), \frac{1}{h_{\min}}, \frac{1}{a_0})$. In order to prove the weak convergence to the solutions of the rigid lid model, we need to prove some compactness results in the Beppo-Levi spaces in order to extract a convergent sub-sequence of solutions.

5.4.1 Compactness result

We recall that $A \subset\subset B$ means that there exists a compact K such that $A \subset K \subset B$.

Lemma 5.4.1 *Let $k \geq 1$. We have the following compact embedding:*

$$H_{loc}^{s, k}(\mathcal{S}^*) \subset H_{loc}^{s-1, k-1}(\mathcal{S}^*) \quad \forall 1 \leq k \leq s$$

Proof The proof remains the same as for the Rellich-Kondrachov theorem, in the case of Sobolev spaces (see [14]).

Let $\omega \subset\subset \mathcal{S}^*$. 1) We first show the compact embedding of $H^{s,1}(\omega)$ into $H^{s-1,0}(\omega)$. We write, for $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^d$, $|h| < d(\omega, (\mathcal{S}^*)^c)$ and $u \in C^\infty(\mathcal{S}^*)$:

$$\begin{aligned} & \int_{\omega} |(\Lambda^{s-1}u)(X - h_1, z - h_2) - (\Lambda^{s-1}u)(X, z)|^2 dX dz \\ & \leq \int_{\omega} \left| \int_0^1 (\nabla_{X,z} \Lambda^s u)(X - th_1, z - th_2) \cdot (h_1, h_2) dt \right|^2 dX dz. \end{aligned}$$

We now use Jensen's inequality to write:

$$\begin{aligned} & \int_{\omega} |(\Lambda^{s-1}u)(X - h_1, z - h_2) - (\Lambda^{s-1}u)(X, z)|^2 dX dz \\ & \leq \int_{\mathcal{S}^*} \int_0^1 |\nabla_{X,z} \Lambda^{s-1}u(X, z)|^2 |h|^2 dt dX dz \\ & \leq |u|_{H^{s,1}(\mathcal{S}^*)}^2 |h|^2 \end{aligned}$$

and the result stands true for all $u \in H^{s,1}(\mathcal{S}^*)$ by density of $C^\infty(\mathcal{S}^*)$ in this space.

It follows from the theorem of Riesz-Frechet-Kolmogorov that if $(u_n)_n$ is bounded in $H^{s,1}(\mathcal{S}^*)$, then $(\Lambda^{s-1}u_n|_{\omega})_n$ is precompact in $L^2(\omega)$. Such result is true for all $\omega \subset\subset \mathcal{S}^*$ and therefore, by a diagonal sub-sequence argument, one proves the compact embedding $H_{loc}^{s,1}(\mathcal{S}^*)$ into $H_{loc}^{s-1,0}(\mathcal{S}^*)$.

2) For the general case, let $k \in \mathbb{N}^*$ and $s \geq k$, let $(u_n)_n$ be bounded in $H^{s,k+1}(\mathcal{S}^*)$ with $s \geq k+1$. By the definition of Beppo-Levi spaces (recall Section 5.1.5), to prove the convergence of a sub-sequence in $H_{loc}^{s-1,k}(\mathcal{S}^*)$, one must prove the existence of a sub-sequence $(u_{\varphi(n)})_n$ such that for all $0 \leq l \leq k$, the sequence $(\partial_z^l u_{\varphi(n)})_n$ converges in $L_{loc}^2((-k+1, k), H_{loc}^{s-1-l}(\mathbb{R}^d))$ (recall that $\mathcal{S}^* = (-(k+1), k)$). From the case $k = 1$, there exists $u \in L^2(\mathcal{S}^*)$ and a sub-sequence $(u_{\varphi(n)})_n$ convergent to u in $L_{loc}^2(\mathcal{S}^*)$. In particular, $(\partial_z^l u_{\varphi(n)})_n$ is convergent to $\partial_z^l u$ in $\mathcal{D}'(\mathcal{S}^*)$. But one has also $(\partial_z^l u_{\varphi(n)})_n \in H^{s-l, k+1-l}(\mathcal{S}^*)$ and therefore this sequence converges, up to a sub-sequence, in $L_{loc}^2((-k+1, k), H_{loc}^{s-1-l}(\mathbb{R}^d))$, since $k+1-l \geq 1$. By uniqueness of the limit in $\mathcal{D}'(\mathcal{S}^*)$, the limit is $\partial_z^l u$.

5.4.2 Extraction of a convergent sub-sequence for U^ε

We now perform the rigid lid time scaling:

$$t' = \varepsilon t$$

and the change of unknown $P' = \frac{1}{\varepsilon^2}(P + z)$. We now have

$$U^\varepsilon \in L^\infty((0; T); H^{N-1/2, N-1}(\mathcal{S}^*)^{d+1}), \zeta^\varepsilon \in L^\infty((0; T); H^N(\mathbb{R}^d))$$

and the following equations are satisfied:

$$\begin{cases} \partial_t V^\varepsilon + (V^\varepsilon \cdot \nabla^\gamma + \frac{1}{\mu} w^\varepsilon \partial_z) V^\varepsilon = -\frac{1}{\varepsilon} \nabla^\gamma P^\varepsilon & \text{in } \mathcal{D}'((0; T); \Omega_t^\varepsilon) \\ \partial_t w^\varepsilon + (V^\varepsilon \cdot \nabla^\gamma + \frac{1}{\mu} w^\varepsilon \partial_z) w^\varepsilon = -\frac{1}{\varepsilon} (\partial_z P^\varepsilon) & \text{in } \mathcal{D}'((0; T); \Omega_t^\varepsilon) \\ \partial_t \zeta^\varepsilon + \underline{V}^\varepsilon \cdot \nabla^\gamma \zeta^\varepsilon - \frac{1}{\mu \varepsilon^2} \underline{w}^\varepsilon = 0 & \text{in } \mathcal{D}'((0; T); \mathbb{R}^d) \\ \nabla^{\mu, \gamma} \cdot (U^\varepsilon) = 0 & \text{in } \mathcal{D}'((0; T); \Omega_t^\varepsilon) \\ \text{curl}^{\mu, \gamma} (U^\varepsilon) = 0 & \text{in } \mathcal{S}^* \\ U^\varepsilon \cdot n = 0 & \text{on } z = -1 + \beta b \end{cases} \quad (5.4.57)$$

We now prove the following Theorem:

Theorem 5.4.2 *There exists U, P, ζ distributions of $D'((0; T); \Omega)$ such that*

$$U \in H^{N-3/2, N-2}(\mathcal{S}^*), U \in H^{N-5/2, N-3}(\mathcal{S}^*), \zeta \in H^{N-1}(\mathbb{R}^d)$$

are solutions in the distributional sense of $D'((0; T); \Omega)$ of the rigid lid equations (5.1.8). Moreover, the solutions of (5.4.57) converge in the distributional sense of $D'((0; T); \Omega_t^\varepsilon)$ to these solutions of the rigid-lid equation 5.1.8. The convergence of $(U^\varepsilon)_\varepsilon$ is not strong in $L^\infty((0; T); L^2(\Omega))$.

By Lemma 5.4.1, there exists $U^\varepsilon \in H^{N-3/2, N-2}(\mathcal{S}^*)$ such that $(U^\varepsilon)_\varepsilon$ converges up to a sub-sequence to U in $H_{loc}^{N-3/2, N-2}(\mathcal{S}^*)$. Moreover, since $(U^\varepsilon)_\varepsilon \in L^\infty((0; T); H^{N-1/2, N-1}(\mathcal{S}^*)^{d+1})$, one has also $U \in L^\infty((0; T); H^{N-3/2, N-2}(\mathcal{S}^*)^{d+1})$. Therefore, $(U^\varepsilon)_\varepsilon$ converges up to a sub-sequence to U in $D'((0; T); \mathcal{S}^*)$. Therefore, $(\partial_t U^\varepsilon)_\varepsilon$ converges (up to a sub-sequence) to $\partial_t U$ in $D'((0; T); \mathcal{S}^*)$.

In view of taking the limit as ε goes to zero in the first equation of (5.4.57), one must prove the convergence of the non-linear terms. We do it for the term $V^\varepsilon \cdot \nabla^\gamma V^\varepsilon$. One uses the following estimate, which proof can be found in [37] Corollary B.5.

$$|fg|_{H^{N-3/2, N-2}} \leq C |f|_{H^{N-1/2, N-1}} |g|_{H^{N-3/2, N-2}}$$

for all $N-2 \geq d/2$ (which is true for our choice of N , since $N-2 \geq t_0 + 3/2 > d/2$). Such estimate proves that $V^\varepsilon \cdot \nabla^\gamma V^\varepsilon \in L^\infty H^{N-3/2, N-2}$, and therefore one has the convergence of this term (up to a sub-sequence) as ε goes to zero, in $D'((0; T); \mathcal{S}^*)$. It proves, using the equation (5.4.57) that $(\nabla^\gamma P^\varepsilon)_\varepsilon$ converges in $D'((0; T); \mathcal{S}^*)$ to an element G . By De Rham's Theorem, there exists $P \in D'((0; T); \mathcal{S}^*)$ such that $G = \nabla^\gamma P$. The same study can be done for the second equation of 5.4.57 and one obtains, passing to the limit as ε goes to zero:

$$\begin{aligned} \partial_t V + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V &= -\nabla^\gamma P \text{ in } \mathcal{D}'((0; T); \Omega) \\ \partial_t w + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w &= -\frac{1}{\varepsilon} (\partial_z P) \text{ in } \mathcal{D}'((0; T); \Omega) \end{aligned}$$

with $U = (V, w) \in L^\infty((0; T); H^{N-3/2, N-2}(\mathcal{S}^*)^{d+1})$, $\nabla^{\mu, \gamma} P \in L^\infty((0; T); H^{N-5/2, N-3}(\mathcal{S}^*)^{d+1})$.

We now pass to the limit in the third equation (on ζ^ε) of (5.4.57). The same reasoning as before proves that $(\zeta^\varepsilon)_\varepsilon$ converges up to a sub-sequence to a $\zeta \in H^{N-1}(\mathbb{R}^d)$ in $H_{loc}^{N-1}(\mathbb{R}^d)$. Therefore, the same convergence occurs in $D'((0; T); \mathcal{S}^*)$, and $(\partial_t \zeta)_\varepsilon$ converges up to a sub-sequence to $\partial_t \zeta$ in $D'((0; T); \mathcal{S}^*)$. We prove as before the convergence of the bi-linear term $(\underline{V}^\varepsilon \cdot \nabla^\gamma \zeta^\varepsilon)_\varepsilon$ in $D'((0; T); \mathcal{S}^*)$. Multiplying the third equation of 5.4.57 by ε^2 , one finds at the limit the following equation:

$$\underline{w} = 0 \text{ in } \mathcal{D}'((0; T); \Omega_t^\varepsilon).$$

Finally, one obtains at the limit ε goes to zero, the following equations:

$$\begin{cases} \partial_t V + (V \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) V = -\nabla^\gamma P \text{ in } \Omega \\ \partial_t w + (V^\varepsilon \cdot \nabla^\gamma + \frac{1}{\mu} w \partial_z) w = -(\partial_z P) \text{ in } \Omega \\ \underline{w} = 0 \\ \nabla^{\mu, \gamma} \cdot U = 0 \text{ in } \Omega \\ \text{curl}^{\mu, \gamma}(U) = 0 \text{ in } \Omega \\ U \cdot n = 0 \text{ for } z = -1 + \beta b \end{cases}$$

and the study above prove the following regularity for the unknowns: $U \in H^{N-3/2, N-2}(\mathcal{S}^*)$, $\nabla^{\mu, \gamma} P \in U \in H^{N-5/2, N-3}(\mathcal{S}^*)$, $\zeta \in H^{N-1}(\mathbb{R}^d)$. The sequence $(U^\varepsilon)_\varepsilon$ does not converge in $L^\infty((0; T); L^2(\Omega))$. Indeed, it would imply the strong convergence of $\mathfrak{P}\psi$ in $L^2(\mathbb{R}^d)$ which is not satisfied, according to Theorem 5.2.6. \square

5.A The Dirichlet Neumann Operator

Here are for the sake of convenience some technical results about the Dirichlet Neumann operator, and its estimates in Sobolev norms. See [37] Chapter 3 for complete proofs. The first two propositions give a control of the Dirichlet-Neumann operator.

Proposition 5.A.1 *Let $t_0 > d/2$, $0 \leq s \leq t_0 + 3/2$ and $(\zeta, \beta) \in H^{t_0+1} \cap H^{s+1/2}(\mathbb{R}^d)$ such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0$$

(1) *The operator G maps continuously $\dot{H}^{s+1/2}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has*

$$|G\psi|_{H^{s-1/2}} \leq \mu^{3/4} M(s+1/2) |\mathfrak{P}\psi|_{H^s},$$

where $M(s+1/2)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1/2}}, |b|_{H^{s+1/2}})$.

(2) *The operator G maps continuously $\dot{H}^{s+1}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ and one has*

$$|G\psi|_{H^{s-1/2}} \leq \mu M(s+1) |\mathfrak{P}\psi|_{H^{s+1/2}},$$

where $M(s+1)$ is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}, |\zeta|_{H^{s+1}}, |b|_{H^{s+1}})$.

Moreover, it is possible to replace G by \underline{w} in the previous result, where $\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}$ (vertical component of the velocity $U = \nabla_{X,z} \Phi$ at the surface).

Proposition 5.A.2 *Let $t_0 > d/2$, and $0 \leq s \leq t_0 + 1/2$. Let also $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, 1 + \varepsilon \zeta(X) - \beta b(X) \geq h_0$$

Then, for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, we have

$$(\Lambda^s G\psi_1, \Lambda^s \psi_2)_2 \leq \mu M_0 |\mathfrak{P}\psi_1|_{H^s} |\mathfrak{P}\psi_2|_{H^s},$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

The second result gives a control of the shape derivatives of the Dirichlet-Neumann operator. More precisely, we define the open set $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ as:

$$\Gamma = \{\Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) + 1 - \beta b(X) \geq h_0\}$$

and, given a $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping:

$$G[\varepsilon \cdot, \beta \cdot] : \begin{array}{ccc} \Gamma & \longrightarrow & H^{s-1/2}(\mathbb{R}^d) \\ \Gamma = (\zeta, b) & \longmapsto & G[\varepsilon \zeta, \beta b] \psi. \end{array} \quad (5.A.58)$$

We can prove the differentiability of this mapping. The following Theorem gives a very important explicit formula for the first-order partial derivative of G with respect to ζ :

Theorem 5.A.3 *Let $t_0 > d/2$. Let $\Gamma = (\zeta, b) \in \Gamma$ and $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$. Then, for all $h \in H^{t_0+1}(\mathbb{R}^d)$, one has*

$$dG(h)\psi = -\varepsilon G(h\underline{w}) - \varepsilon \mu \nabla^\gamma \cdot (h\underline{V}),$$

with

$$\underline{w} = \frac{G\psi + \varepsilon \mu \nabla^\gamma \zeta \cdot \nabla^\gamma \psi}{1 + \varepsilon^2 \mu |\nabla^\gamma \zeta|^2}, \quad \text{and} \quad \underline{V} = \nabla^\gamma \psi - \varepsilon \underline{w} \nabla^\gamma \zeta.$$

The following result gives estimates of the derivatives of the mapping (5.A.58).

Proposition 5.A.4 *Let $t_0 > d/2$, $0 \leq s \leq t_0 + 1/2$ and $(\zeta, \beta) \in H^{t_0+1}(\mathbb{R}^d)$ such that:*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon \zeta(X) - \beta b(X) + 1 \geq h_0$$

Then, for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, for all $(h, k) \in H^{t_0+1}(\mathbb{R}^d)$ one has

$$|(\Lambda^s d^j G(h, k)\psi_1, \Lambda^s \psi_2)| \leq \mu M_0 \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi_1|_s |\mathfrak{P}\psi_2|_s,$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

The following Proposition gives the same type of estimate that the previous one:

Proposition 5.A.5 *Let $t_0 > d/2$ and $(\zeta, b) \in H^{t_0+1}$ be such that*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) - \beta b(X) + 1 \geq h_0.$$

Then, for all $0 \leq s \leq t_0 + 1/2$,

$$|d^j G(h, k)\psi|_{H^{s-1/2}} \leq M_0 \mu^{3/4} \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^s}$$

We need the following commutator estimate:

Proposition 5.A.6 *Let $t_0 > d/2$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^d)$ such that:*

$$\exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) - \beta b(X) + 1 \geq h_0$$

For all $\underline{V} \in H^{t_0+1}(\mathbb{R}^d)^2$ and $u \in H^{1/2}(\mathbb{R}^d)$, one has

$$((\underline{V} \cdot \nabla^\gamma u), \frac{1}{\mu} Gu) \leq M |\underline{V}|_{W^{1,\infty}} |\mathfrak{P}u|_2^2,$$

where M is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+2}}, |b|_{H^{t_0+2}})$.

In order to have regularity estimates of the potential Φ (recall that it solves the Dirichlet-Neumann problem (5.1.4)), we need to introduce the following mapping:

Definition 5.A.7 *Let $t_0 > d/2$.*

(1) We denote by $\Gamma \subset H^{t_0+1}(\mathbb{R}^d)^2$ the open set:

$$\Gamma = \{ \Gamma = (\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)^2, \exists h_0 > 0, \forall X \in \mathbb{R}^d, \varepsilon\zeta(X) + 1 - \beta b(X) \geq h_0 \}$$

(2) One can define, for all $0 \leq s \leq t_0 + 1/2$ and $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$ a mapping \mathfrak{A}_ψ as

$$\mathfrak{A}_\psi : \begin{array}{l} \Gamma \rightarrow \dot{H}^{s+1}\mathcal{S} \\ \Gamma \mapsto \Phi_\Gamma \end{array}$$

where Φ_Γ is the unique variational solution to (5.1.14).

One can prove the analyticity of \mathfrak{A}_ψ , which means that the solution ϕ_Γ of (5.1.14) is analytic with respect to the boundaries. The following result then gives some estimates about the derivatives of the mapping \mathfrak{A}_ψ

Proposition 5.A.8 *Let $t_0 > d/2$ and $0 \leq s \leq t_0 + 1/2$, $\psi \in H^{s+1/2}(\mathbb{R}^d)$, and $\Gamma = (\zeta, b) \in \Gamma$. For all $j \in \mathbb{N}$ and $(h, k) = (h_1, \dots, h_j, k_1, \dots, k_j) \in H^{t_0+1}(\mathbb{R}^d)^2$, one has*

$$|\Lambda^s \nabla^{\mu, \gamma} d^j \mathfrak{A}_\psi(\Gamma)(h, k)|_2 \leq \sqrt{\mu} M_0 \prod_{m=1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^s},$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

If $s = t_0 + 1/2$, then the same estimate holds on $\|\nabla^{\mu, \gamma} d^j \mathfrak{A}_\psi(\Gamma)(h, k)\|_{H^{t_0+1/2, k}(\mathcal{S})}$ for $k \leq t_0 + 1/2$.

In Proposition 5.A.8 below, we need at least the $H^{t_0+1/2}$ -norm of the component (h, k) , even if $s = 0$. The following proposition allows to relax this constraint, by using only the $H^{s+1/2}$ norm of the first component of (h, k) .

Proposition 5.A.9 *Let $t_0 > d/2$ and $0 \leq s \leq t_0$, $\psi \in H^{t_0+1/2}(\mathbb{R}^d)$, and $\Gamma = (\zeta, b) \in \Gamma$. For all $j \in \mathbb{N}$ and $(h, k) = (h_1, \dots, h_j, k_1, \dots, k_j) \in H^{t_0+1}(\mathbb{R}^d)^2$, one has*

$$|\Lambda^s \nabla^{\mu, \gamma} d^j \mathfrak{A}_\psi(\Gamma)(h, k)|_2 \leq \sqrt{\mu} M_0 |(\varepsilon h_1, \beta k_1)|_{H^{s+1/2}} \prod_{m>1}^j |(\varepsilon h_m, \beta k_m)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^{t_0+1/2}},$$

where M_0 is a constant of the form $C(\frac{1}{h_0}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}})$.

5.B Extensions on Beppo-Levi Spaces

This section contains proofs of the existence and regularity of extensions used in Section 5.3.

Theorem 5.B.1 *Let $s \in \mathbb{R}^+$ and $k \in \mathbb{N}$. Let us denote $\mathcal{S}_j = (-(j+1), j) \times \mathbb{R}^d$ for all $j \in \mathbb{N}$. Then, there exists an extension*

$$P : \begin{array}{ccc} H^{s,k}(\mathcal{S}_0) & \rightarrow & H^{s,k}(\mathcal{S}_j) \\ u & \mapsto & Pu \end{array}$$

such that:

$$\forall u \in H^{s,k}(\mathcal{S}_0), \quad |u|_{H^{s,k}(\mathcal{S}_j)} \leq C(k, j) |u|_{H^{s,k}(\mathcal{S}_0)} \quad (5.B.59)$$

where $C(k, j)$ only depends on k and j .

Proof We first construct an extension to $H^{s,k}((-1, 1) \times \mathbb{R}^d)$.

The proof requires a small adaptation from the case $k = 1$ which is proved in [14]. When $k = 1$, u is extended by reflection:

$$\tilde{u}(X, z) = \begin{cases} u(X, z) & \forall z \in (-1, 0) \\ u(X, -z) & \forall z \in (0, 1). \end{cases}$$

But such extension \tilde{u} is indeed in $H^{s,1}(S_1)$ but is not in general in $H^{s,2}(S_1)$ since derivatives in z of \tilde{u} differs for $z = 0$:

$$\partial_z \tilde{u}(X, 0^+) \neq -\partial_z \tilde{u}(X, 0^-)$$

We should therefore define an extension which derivatives of order $i \leq k-1$ have the same value in 0^+ and in 0^- . We are looking for an extension under the form:

$$Pu(X, z) = \begin{cases} u(X, z) & \forall z \in (-1, 0) \\ \sum_{i=0}^{k-1} c_i u(X, -\alpha_i z) & \forall z \in (0, 1). \end{cases}$$

The condition over the α_i and c_i , $0 \leq i \leq k-1$ is:

$$\forall 0 \leq j \leq k-1, \quad \sum_{i=0}^{k-1} c_i (-\alpha_i)^j \partial_z^j u(X, 0^+) = \partial_z^j u(X, 0^-)$$

i.e. $\sum_{i=0}^k c_i (-\alpha_i)^j = 1$. We should therefore find (c_0, \dots, c_{k-1}) and $(\alpha_0, \dots, \alpha_{k-1})$ such that:

$$\begin{pmatrix} 1 & \cdots & 1 \\ -\alpha_1 & \cdots & -\alpha_{k-1} \\ (-\alpha_1)^2 & \cdots & (-\alpha_{k-1})^2 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ \vdots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \quad (5.B.60)$$

which is a Vandermonde system. It suffices to take the α_i distincts, non zero, and taken in $(0, 1)$ in order for $u(X, -\alpha_i z)$ to make sense. Then the c_i are defined as solutions of the Vandermonde system (5.B.60).

Now, let us prove that such defined extension Pu maps continuously $H^{s,1}(S_1)$ into $H^{s,1}((-1, 1) \times \mathbb{R}^d)$.

1) Case $u \in C^\infty(\overline{\mathcal{S}_0})$

It is clear that Pu is measurable. Let us check that it is $L^2((-1, 1), H^s(\mathbb{R}^d))$:

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{R}^d} \Lambda^{2s} |Pu(X, z)|^2 dX dz &= \int_{-1}^0 \int_{\mathbb{R}^d} \Lambda^{2s} |u(X, z)|^2 dX dz + \int_0^1 \int_{\mathbb{R}^d} \Lambda^{2s} \left| \sum_{i=0}^{k-1} c_i u(X, -\alpha_i z) \right|^2 dX dz \\ &\leq |u|_{H^{s,k}((-1,0) \times \mathbb{R}^d)}^2 + \sum_{i=0}^{k-1} \frac{c_i^2}{\alpha_i} \int_{-\alpha_i}^0 \int_{\mathbb{R}^d} \Lambda^{2s} |u(X, z)|^2 dX dz \\ &\leq C_k |u|_{H^{s,k}((-1,0) \times \mathbb{R}^d)}^2. \end{aligned}$$

The first inequality is Cauchy-Schwarz's inequality. The constant C_k only depends on k . Now, let us check that Pu is $H^{k-j}((-1, 1); H^{s-k+j}(\mathbb{R}^d))$ for all $0 \leq j \leq k$. Let $\varphi \in C_0^\infty((-1, 1) \times \mathbb{R}^d)$. One set j such that $0 \leq j \leq k$, and computes $\partial_z^j(Pu)$ in the distributional sense of $D'((-1, 1) \times \mathbb{R}^d)$:

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{R}^d} u(X, z) \partial_z^j \varphi(X, z) dX dz &= \int_{-1}^0 \int_{\mathbb{R}^d} u(X, z) \partial_z^j \varphi(X, z) dX dz \\ &+ \int_0^1 \int_{\mathbb{R}^d} \sum_{i=0}^{k-1} c_i u(X, -\alpha_i z) \partial_z^j \varphi(X, z) dX dz \\ &= \int_{-1}^0 \int_{\mathbb{R}^d} (-1)^j \partial_z^j u(X, z) \varphi(X, z) dX dz \\ &+ \sum_{l=0}^{j-1} (-1)^{j-1-l} \int_{\mathbb{R}^d} \partial_z^{j-l-1} u(X, 0) \partial_z^l \varphi(X, 0) dX dz \\ &+ \int_0^1 \int_{\mathbb{R}^d} (-1)^j \sum_{i=0}^{k-1} c_i (-\alpha_i)^j (\partial_z^j u)(X, -\alpha_i z) \varphi(X, z) dX dz \\ &+ \sum_{l=0}^{j-1} (-1)^{j-l} \int_{\mathbb{R}^d} c_i (-\alpha_i)^{j-l-1} \partial_z^{j-l-1} u(X, 0) \partial_z^l \varphi(X, 0) dX dz \end{aligned}$$

by integrating by parts, (recall that $u \in C^\infty(\overline{\mathcal{S}_1})$). Note that these calculus does not make sense for $u \in H^{s,k}(\mathcal{S}_0)$ since φ is not in $C_0^\infty(\mathcal{S}_0)$.

Since we have the identity

$$\forall 0 \leq j \leq k-1, \quad \sum_{i=0}^{k-1} c_i (-\alpha_i)^j \partial_z^j u(X, 0) = \partial_z^j u(X, 0)$$

the two integrals over \mathbb{R}^d cancel one another. Therefore, we have:

$$\begin{aligned} \int_{-1}^1 \int_{\mathbb{R}^d} u(X, z) \partial_z^j \varphi(X, z) dX dz &= \int_{-1}^0 \int_{\mathbb{R}^d} (-1)^j \partial_z^j u(X, z) \varphi(X, z) dX dz \\ &+ \int_0^1 \int_{\mathbb{R}^d} \sum_{i=0}^{k-1} c_i (\alpha_i)^j (\partial_z^j u)(X, -\alpha_i z) \varphi(X, z) dX dz \end{aligned} \quad (5.B.61)$$

which prove that

$$\partial_z^j(Pu) = (1 - \operatorname{sgn}(z)) \partial_z^j u(X, z) + (1 + \operatorname{sgn}(z)) \sum_{i=0}^{k-1} c_i (-\alpha_i)^j (\partial_z^j u)(X, -\alpha_i z)$$

on $\mathcal{D}'((-1, 1) \times \mathbb{R}^d)$, with the notation $\operatorname{sgn}(z) = 1$ if $z \geq 0$ and $\operatorname{sgn}(z) = -1$ if $z < 0$. It is then quite easy to check that $\partial_z^j(Pu) \in H^{k-j}((-1, 1); H^{s-k+j}(\mathbb{R}^d))$ (proceed as for case $j = 0$) with

$$|Pu|_{H^{k-j}((-1, 1); H^{s-k+j}(\mathbb{R}^d))} \leq C_k |u|_{H^{k-j}((-1, 0); H^{s-k+j}(\mathbb{R}^d))} \quad (5.B.62)$$

2) Case $u \in H^{s,k}(\mathcal{S}_0)$

By density of $C^\infty(\overline{\mathcal{S}_0})$ in $H^{s,k}(\mathcal{S}_0)$, it is easy to check that (5.B.61) stands true in $H^{s,k}(\mathcal{S}_0)$, and the result is proved, with the control (5.B.62).

We can construct by exactly the same way an extension of $u \in H^{s,k}(\mathcal{S}_0)$ into $H^{s,k}((-2, 0) \times \mathbb{R}^d)$ and then combine the two extensions to have an extension to $H^{s,k}(\mathcal{S}_1)$. By the same way it is easy to construct the extension into $H^{s,k}(\mathcal{S}_k)$. Finally, the main of the theorem is proved, with the control (5.B.59). \square

Proposition 5.B.2 *Using the notations of Definition 5.3.9, let ϕ be a distribution on $[0; T_0] \times \mathcal{S}$ with the following regularity:*

$$\forall 0 \leq k \leq N, \quad \partial_t^k \nabla^{\mu, \gamma} \phi \in L^\infty((0; T_0); H^{N-k-1/2, N-k-1}(\mathcal{S}))$$

Let $\tilde{\phi}$ be an extension of ϕ on the strip \mathcal{S}_l , and $\tilde{\Phi} = \tilde{\phi} \circ \Sigma_t^{\varepsilon-1}$. Then, we have, for all $k \leq N$:

$$\partial_t^k \nabla^{\mu, \gamma} \tilde{\Phi} \in L^\infty((0; T_0); H^{N-k-1/2, N-k-1}(\mathcal{S}^*))$$

with a bound uniform in t and ε .

Proof The proof consists in proving that if Θ is a function defined on $[0; T_0] \times \mathcal{S}^*$, with $\theta = \Theta \circ \Sigma_t^\varepsilon$ such that:

$$\forall k \leq N, \partial_t^k \theta \in L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^j))$$

then we have the same regularity for Θ :

$$\forall k \leq N, \partial_t^k \Theta \in L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^*))$$

with a control of the norm:

$$|\partial_t^k \Theta|_{L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^*))} \leq C |\partial_t^k \theta|_{L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^j))} \quad (5.B.63)$$

where C does not depend on ε .

Let us do it by induction on k . The induction hypothesis is the following:

"For all $N \in \mathbb{N}$ and $\theta \in L^\infty(0; T_0; H^{N+1/2, N}(\mathcal{S}^j))$ if $\Theta = \theta \circ \Sigma_t^{\varepsilon-1}$, then we have:

$$\partial_t^k \Theta \in L^\infty(0; T_0; H^{N-k+1/2, N-k}(\mathcal{S}^*))$$

with a control

$$|\partial_t^k \Theta|_{L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^*))} \leq C |\partial_t^k \theta|_{L^\infty((0; T_0); H^{N-k+1/2, N-k}(\mathcal{S}^j))} \quad (5.B.64)$$

where C does not depend on ε "

1) $k = 0$

Let us prove that $\Theta \in L^\infty(0; T_0, H^{N+1/2, N}(\mathcal{S}^*))$. We are using the following characterization of $H^{N+1/2}(\mathbb{R}^d)$:

$$|u|_{H^{N+1/2}(\mathbb{R}^d)}^2 \sim |u|_{H^N(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D^N u(x) - D^N u(y)|^2}{|x - y|^{2(1/2+d/2)}} dx dy$$

Let first prove that $\Theta \in L^\infty((0; T_0); L^2(-(k+1); k), H^{N+1/2}(\mathbb{R}^d))$ with the control (5.B.64) by induction on N . Let the induction hypothesis be:

"For for all $\theta \in L^\infty((0; T_0); L^2(-(l+1); l), H^{N+1/2}(\mathbb{R}^d))$, if $\Theta = \theta \circ \Sigma_t^{\varepsilon-1}$ then we have:

$$\Theta \in L^\infty((0; T_0); L^2(-(k+1); k), H^{N+1/2}(\mathbb{R}^d))$$

with a norm control such as (5.B.64)"

For $N = 0$, we need the previous characterization of $H^{1/2}(\mathbb{R}^d)$:

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{-(k+1)}^k \frac{|\Theta(X, z) - \Theta(Y, z)|^2}{|X - Y|^{2(1/2+d/2)}} dX dY dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{-(k+1)}^k \frac{|\theta \circ \Sigma_t^{\varepsilon-1}(X, z) - \theta \circ \Sigma_t^{\varepsilon-1}(Y, z)|^2}{|X - Y|^{2(1/2+d/2)}} dX dY dz \\ &= \int_{\Sigma_t^{\varepsilon-1}(-(k+1))_z}^{\Sigma_t^{\varepsilon-1}(k)_z} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\theta(X, u) - \theta(Y, u)|^2}{|X - Y|^{2(1/2+d/2)}} |J_{\Sigma_t^{\varepsilon-1}}(X)| dX dY du \\ &\leq \int_{-(l+1)}^l \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\theta(X, u) - \theta(Y, u)|^2}{|X - Y|^{2(1/2+d/2)}} |J_{\Sigma_t^{\varepsilon-1}}(X)| dX dY du \\ &= \int_{\Sigma_t^{\varepsilon-1}(-(l+1))_z}^{\Sigma_t^{\varepsilon-1}(l)_z} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\theta(X, v) - \theta(Y, v)|^2}{|X - Y|^{2(1/2+d/2)}} |J_{\Sigma_t^{\varepsilon-1}}(X)| |J_{\Sigma_t^{\varepsilon-1}}(Y)| dX dY dv \\ &\leq \int_{-(j+1)}^j \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\theta(X, v) - \theta(Y, v)|^2}{|X - Y|^{2(1/2+d/2)}} |J_{\Sigma_t^{\varepsilon-1}}(X)| |J_{\Sigma_t^{\varepsilon-1}}(Y)| dX dY dv \\ &\leq C |\theta|_{L^\infty((0; T_0); H^{1/2, 0}(\mathcal{S}^j))} \end{aligned}$$

with C uniform in ε . We used the change of variable $(X, z) = \Sigma_t^\varepsilon(X, u)$ and $(Y, u) = \Sigma_t^\varepsilon(Y, v)$ in the integrals. The first inequality comes from $\mathcal{S}^k \subset \Sigma_t^\varepsilon(\mathcal{S}^l)$. The last comes from $\mathcal{S}^l \subset \Sigma_t^\varepsilon(\mathcal{S}^j)$.

Suppose the result true for $N - 1$ with $N \geq 1$. Let prove it for N . We have:

$$\begin{aligned} \nabla_{X, z} \Theta &= \nabla_{X, z} \left(\theta \left(X, \frac{z - \varepsilon \zeta}{h_B} \right) \right) \\ &= (\nabla_{X, z} \theta) \left(X, \frac{z - \varepsilon \zeta}{h_B} \right) + \nabla_{X, z} \left(\frac{z - \varepsilon \zeta}{h_B} \right) (\partial_z \theta) \left(X, \frac{z - \varepsilon \zeta}{h_B} \right) \end{aligned}$$

We have $(\nabla_{X,z}\theta) \in L^\infty((0; T_0); L^2(-(k+1); k), H^{N-1+1/2}(\mathbb{R}^d))$, so by induction hypothesis, this term is controlled. For the latter one, just remark that $\nabla_{X,z}(\frac{z-\varepsilon\zeta}{h_B}) \in H^{t_0}(\mathbb{R}^d)$, so with classical product estimates:

$$\begin{aligned}
& \int_{-(k+1)}^k |\nabla_{X,z}(\frac{z-\varepsilon\zeta}{h_B})(\partial_z\theta)(X, \frac{z-\varepsilon\zeta}{h_B})|_{H^{N-1+1/2}(\mathbb{R}^d)}^2 dz \\
& \leq \int_{-(l+1)}^l |\nabla_{X,z}(\frac{z-\varepsilon\zeta}{h_B})|_{H^{t_0}}^2 |(\partial_z\theta)(X, \frac{z-\varepsilon\zeta}{h_B})|_{H^{N-1+1/2}(\mathbb{R}^d)}^2 dz \\
& \leq C \int_{-(k+1)}^k |(\partial_z\theta)(X, \frac{z-\varepsilon\zeta}{h_B})|_{H^{N-1+1/2}(\mathbb{R}^d)}^2 dz \\
& = C |(\partial_z\theta) \circ \Sigma_t^{\varepsilon-1}|_{L^2(-(k+1); k); H^{N-1+1/2}(\mathbb{R}^d)} \\
& \leq C |\partial_z\theta|_{L^2(-(l+1); l); H^{N-1+1/2}(\mathbb{R}^d)} \\
& \leq C |\partial_z\theta|_{L^2(-(j+1); j); H^{N-1+1/2}(\mathbb{R}^d)}
\end{aligned}$$

using the induction hypothesis and $\mathcal{S}_l \subset \mathcal{S}_j$. The constant C does not depends either on t or ε , so the result is proved.

2) Now, suppose this result is proved for $k-1$, with $k \geq 1$. Let prove it for k . We write:

$$\begin{aligned}
\partial_t\Theta &= \partial_t\theta(t, X, \frac{z-\varepsilon\zeta}{h_B}) \\
&= (\partial_t\theta)(t, X, \frac{z-\varepsilon\zeta}{h_B}) + \partial_t(\frac{z-\varepsilon\zeta}{h_B})(\partial_z\theta)(t, X, \frac{z-\varepsilon\zeta}{h_B})
\end{aligned}$$

Since $\partial_t\theta$ is $L^\infty(0; t_0); H^{N-1+1/2, N-1}(\mathbb{R}^d)$, by induction hypothesis the $(k-1)$ -th time derivative of the first term of the r.h.s. is controlled as in (5.B.64).

The same argument stands for $(\partial_z\theta)(t, X, \frac{z-\varepsilon\zeta}{h_B})$. The term $\partial_t(\frac{z-\varepsilon\zeta}{h_B})$ is harmless, since it is in $L^\infty((0; T_0); H^{N-1+1/2, N-1})$ with $N-1 \geq t_0$, so standard product estimates in Sobolev spaces will finally give the control of the second term of the $(k-1)$ -th time derivative of the r.h.s. \square

Index des notations

Notations générales

- d désigne la dimension de la surface (on a $d = 1$ ou 2).
- $|\alpha| = \sum_{j=1}^d \alpha_j$ pour tout vecteur $\alpha = (\alpha_j)_{j=1..d} \in \mathbb{N}^d$.
- $\langle A \rangle_{s>r}$ vaut A si $s > r$ et 0 sinon, pour tous réels A, s, r .
- $a \vee b = \max(a, b)$ pour tous réels a, b .
- Ω_t désigne le domaine occupé par le fluide à un instant $t \in \mathbb{R}^+$.
- \mathcal{S} désigne en général la bande $\mathcal{S} = \{(X, z) \in \mathbb{R}^d \times \mathbb{R}, -1 < z < 0\}$.
- $\Gamma \subset H^{t_0+2}(\mathbb{R}^d)^2$ désigne l'ensemble des couples (ζ, b) tels que $\inf_{X \in \mathbb{R}^d} (1 + \varepsilon\zeta - \beta b)(X) > 0$.
- ψ^\dagger est une extension régularisée de ψ (voir (4.3.45)).
- ψ^h désigne l'extension harmonique de ψ (voir section 4.3.2).

Variables et opérateurs

- $X \in \mathbb{R}^d$ désigne la variable horizontale.
- z désigne la variable verticale.
- $\nabla = {}^t(\partial_x, \partial_y)$ si $d = 2$ et $\nabla = \partial_x$ si $d = 1$.
- $\Delta = \partial_x^2 + \partial_y^2$ si $d = 2$ et $\Delta = \partial_x^2$ si $d = 1$.
- $\nabla_{X,z}$ désigne le gradient en dimension $d + 1$ par rapport aux variables X et z .
- curl désigne le rotationnel en dimension $d + 1$.
- div désigne la divergence en dimension d ou $d + 1$.
- $\nabla \cdot$ désigne la divergence en dimension d ou $d + 1$.
- $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ pour tout vecteur $\alpha = (\alpha_i)_{i=1..d} \in \mathbb{N}^d$.
- \mathcal{F} désigne la transformée de Fourier.
- \hat{f} désigne la transformée de Fourier de f pour toute distribution tempérée f .
- $D = \frac{1}{i} \nabla$ et pour toute fonction f à croissance polynomiale, $f(D)$ est l'opérateur différentiel

$$\forall u \in \mathcal{S}'(\mathbb{R}^d), \forall \xi \in \mathbb{R}^d, \quad \widehat{f(D)u} = f(\xi)\hat{u}(\xi).$$

- $\Lambda = (1 + |D^\gamma|^2)^{1/2}$ est une dérivée fractionnaire.

Quantités dépendant de paramètres

- $\mu, \varepsilon, \gamma, \beta, B_0$: voir introduction du Chapitre 2, paragraphe "Dimensionless equations"
- $\nabla^\gamma, \nabla^{\mu,\gamma}, \Delta^{\mu,\gamma}, \text{curl}^{\mu,\gamma}$: voir Section 2.1.3.
- $\mathfrak{P} = \frac{|D^\gamma|}{(1 + \sqrt{\mu}|D^\gamma|)^{1/2}}$.
- $G, \mathcal{G}, \mathcal{G}_{\mu,\gamma}$: opérateur de Dirichlet-Neumann (voir aussi Section 2.1.3).
- $\underline{w}, \underline{V}$: composantes respectivement horizontales et verticales de la vitesse évaluée à la surface.
- $\zeta_{(\alpha)}, \psi_{(\alpha)}$: bonnes inconnues d'Alinhac (voir (2.2.14)).
- $M_0, M, M(s)$: voir 4.3.41.

Espaces fonctionnels

- $D'(\Omega)$ désigne l'espace des distributions sur Ω , pour tout $\Omega \subset \mathbb{R}^d$.
- $\mathcal{S}'(\mathbb{R}^d)$ désigne l'espace des distributions tempérées sur \mathbb{R}^d .
- $L^2(\Omega)$ désigne l'espace de Lebesgue sur Ω associé à la norme

$$\|f\|_2 = \left(\int_{\mathbb{R}^d} |f(t)|^2 dt \right)^{1/2}.$$

- $L^\infty(\Omega)$ désigne l'espace de Lebesgue sur Ω de la classe des fonctions vérifiant $\|f\|_\infty = \sup_{\mathbb{R}^d} |f| < \infty$.
- $W^{k,\infty}(\mathbb{R}^d) = \{f \in L^\infty(\mathbb{R}^d), \|f\|_{W^{k,\infty}} < \infty\}$ où

$$\|f\|_{W^{k,\infty}} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_\infty.$$

- $H^s(\mathbb{R}^d)$ est l'espace de Sobolev sur \mathbb{R}^d associé à la norme

$$\|f\|_{H^s} = \|\Lambda^s f\|_{L^2}.$$

- $\dot{H}(\mathbb{R}^d)$ est l'espace de Beppo-Levi sur \mathbb{R}^d , voir Section 2.1.3.
- $H^{s,k}(\mathbb{R}^d)$: voir Section 2.1.3.

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