Quantification of the model risk in finance and related problems

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"Essentially, all models are wrong, but some are useful."

George E. P. Box

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Introduction

The actors in any financial market are exposed to a myriad of risks. Derivative contracts allow the transfer of a specific risk from a party that wishes to cover that risk to a party willing to be exposed to it. For example, a call option covers its buyer from the rise of the underlying price. The rapid growth of the derivatives market during the last century generated an extensive interest in several associated financial risks, e.g. the market risk, the counterparty credit risk, the liquidity risk etc.

Historically, the market risk was the first which was taken into account. It denotes the risk that the price variations of the underlying financial assets impact the party's portfolio composed of multiple positions in derivatives contracts. The seminal work of Merton [1973] and Black and Scholes [1973] postulated the geometric Brownian motion as a stochastic model for the underlying price dynamics and constructed pricing and hedging strategies for basic derivative contracts. Since then, the (derivative-related) market risk has been managed by proposing a long stream of models (stochastic volatility, local volatility, jump-diffusion etc.), which were supposed to correctly describe the dynamics of the option underlying, such as interest rates, equities, inflation, currencies etc.

In addition to market risk, derivatives contracts carry other financial risks, as *counterparty credit risk* and *liquidity risk*. The recent subprime crisis stressed the importance of the counterparty credit risk when managing derivatives. This constitutes the risk that a counterparty may not be able to fulfill its engagements in a derivative contract, for example in the case of bankruptcy. This risk is generally mitigated through the clearing houses for the exchange traded derivatives or through tailored collateral arrangements for over-the-counter products. Typical instruments for risk reduction are credit derivatives as CDS. Additionally to the uncertainty on the counterparty credit worthiness, the liquidity risk is also a critical risk for derivatives dealers. It represents the risk that the strategies that a company decides to implement may become inapplicable because of a lack of liquidity of the underlying. For instance, a hedging strategy of an exotic option may become too expensive and inefficient, because the trading in the underlying asset and/or the vanilla instruments may suffer from an unexpected reduced market liquidity, producing a bid/ask spread widening.

The quantification of all the three financial risks described above requires the postulation of a complex dynamic model, which for instance, involves the volatility modeling for the market risk or the default probability modeling for the counterparty credit risk. Assuming the validity of that model, the specification of a pricing and hedging strategy ensures, theoretically, that the market risk is managed and quantified. However, as pointed out by Hobson [2011], "although market risk (the known unknown) is eliminated, model risk (the unknown unknown) remains". Indeed, the initial and subjective choice of the model and its practical use may become a source of ambiguity and give rise to a new risk concept, i.e. the model uncertainty. Quoting the European Directive (2013/36/EU), "model risk means the potential loss an institution

may incur, as a consequence of decisions that could be principally based on the output of internal models, due to errors in the development, implementation or use of such models". The new regulations demand the assessment of the model risk for the derivatives market participants. For instance, the European Regulation No 575/2013 states that, "with regard to complex products, [...] institutions shall explicitly assess the need for valuation adjustments to reflect the model risk associated with using a possibly incorrect valuation methodology and the model risk associated with using unobservable (and possibly incorrect) calibration parameters in the valuation model."

Model risk has being the subject of early interest, starting with Merton et al. [1978, 1982] which quantifies the performance of the celebrated Black-Scholes model. As mentioned by Hénaff and Martini [2011], the model risk can be highlighted by two means. The first one is the ambiguity entailed by the pricing of a fixed exotic option, using various models that fit the market standard option data. The second aspect is the hedging quality of the exotic option, using dynamic (and possibly static) strategies.

In regard to the first aspect, i.e. the pricing ambiguity, Schoutens et al. [2003] (c.f. also Hirsa et al. [2003]) produce a thorough empirical comparison of several models that match almost identically a given implied volatility surface. In spite of the fact that all those models fit accurately the market standard options prices, they give a large window of prices for the exotic option. This highlights the model uncertainty that arises when pricing an exotic option, using only the vanilla options prices information. We mention also the work of Eberlein and Madan [2009] which concentrates on the variance option with a long maturity. In many models, the average realized variance, that defines the payoff of this option, converges to a constant when the maturity goes to infinity. Consequently, if one chooses such a model, the out-of-the-money option price would be negligible. The authors show that the underlying models based on the so-called Sato processes overcome this drawback and generate a value for long dated out-of-the-money option on variance.

From a hedging point of view (which concerns the second aspect) many authors used the simulation framework to quantify the impact of model misspecification on the hedging strategies, c.f. for example Bakshi et al. [1997], Hull and Suo [2002], An and Suo [2009], Poulsen et al. [2009], Branger et al. [2012], Schroter et al. [2012] and references therein. The underlying methodology consists in choosing a rich model which takes into account most of the features justified by empirical observations, for instance stochastic volatility, jumps and stochastic interest rates. This will constitute the benchmark generated market for the analysis, i.e. a *true model*, from which the market data are simulated. The idea consists in comparing the hedging strategies inherent to that market to the ones resulting for models including less features. Two important questions are then raised and formulated as follows.

1. What do we gain from the involvement of each feature from a pricing and replication point of view ?

2. Does the possible benefit of including a given feature balances the additional

complexity or implementational costs?

According to many studies, while models including many features allow for a better fit to the standard market data, models with less complexity yield better hedging performances. For example, the quantitative results of Bakshi et al. [1997] conclude that incorporating stochastic volatility and jumps allows for a better pricing and internal consistency. On the other hand, modeling stochastic volatility alone yields the best performance for the hedging of an exotic option. The considerations above about the pricing and hedging ambiguity lead naturally to the problem of the quantification of the range of possible prices for an exotic option and the specification of hedging strategies, which are robust towards model risk.

In that direction, the **Uncertain Volatility Model** (UVM) introduced by Lyons [1995] and Avellaneda et al. [1995] goes beyond the plain empirical studies and solves the theoretical problem of finding the no-arbitrage bound option prices, for a European payoff h paid at a maturity date T, when the volatility is uncertain and bounded by two constants $\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}$. We consider a fixed probability $\mathbb{Q}$ and W a $\mathbb{Q}$-Brownian motion. The no-arbitrage UVM bound option prices appear as follows.

$$\begin{aligned} \overline{P}(\sigma_{\min}, \sigma_{\max}) &= \sup_{\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}} \mathbb{E}^{\mathbb{Q}}[h(S_T^\sigma)], \\ \underline{P}(\sigma_{\min}, \sigma_{\max}) &= \inf_{\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}} \mathbb{E}^{\mathbb{Q}}[h(S_T^\sigma)], \end{aligned}$$

where

$$dS_t^\sigma = \sigma_t S_t^\sigma dW_t.$$

The motivation behind the restriction of the volatility range $\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}$ is to avoid trivial bounds for the option price. Indeed, Frey and Sin [1999] (and similarly Cvitanić et al. [1999]) give conditions, so that the option supremum (resp. infimum) value over a set of equivalent martingale measures is given by trivial super-hedging (resp. sub-hedging) strategy. In many popular stochastic volatility models (as for instance Heston), the super-hedging strategy for the seller of a call option consists in buying the stock and holding it until the maturity. So the concept of super- or sub-replication, without the volatility bounds constraints, is often of little practical use for the pricing and hedging of derivative securities. Lyons [1995] and Avellaneda et al. [1995] characterize the upper and lower prices and their corresponding hedging strategies through a nonlinear PDE, called the Black-Scholes-Barenblatt equation, yielding hedging strategies that are robust to the uncertainty on the volatility level. For instance, if the upper bound price $\overline{P}(\sigma_{\min}, \sigma_{\max})$ and the associated hedging strategy are used by an option seller, then he is fully covered with respect to the model risk induced by the uncertainty on the volatility level. In some sense, the distance $\overline{P}(\sigma_{\min}, \sigma_{\max}) - \underline{P}(\sigma_{\min}, \sigma_{\max})$ between the upper and lower bounds constitutes a reasonable measure of the model risk for the considered option. Another model risk measure, in a similar spirit, was proposed by Cont [2006]. We stress that all the model risk measures in the literature are strongly associated with the given exotic option h, as it is also discussed in the numerical experiments of Cont [2006].

Coming back to the UVM model, when h is a convex payoff, the work of El Karoui et al. [1998] implies that the upper (resp. lower) bound for the option price is attained for $\sigma_t \equiv \sigma_{\max}$ (resp. $\sigma_t \equiv \sigma_{\min}$) and the extreme prices are the Black-Scholes prices related to $\sigma_{\max}$ and $\sigma_{\min}$. The authors consider the case when a stock S follows a stochastic volatility model with an unknown (*true*) volatility process σ_t, and an investor uses a misspecified model with local volatility $(\gamma(t, S_t))$, with γ being some real function, to hedge the payoff h. The key result of El Karoui et al. [1998] is a quantification of the hedging error (between the misspecified and the true model), in terms of the difference of the squared misspecified volatility and the true volatility. Concerning the hedging strategies, they focus a significant monotonicity property: when the misspecified volatility $(\gamma(t, S_t))$ is larger (resp. lower) than the true volatility (σ_t), the final value of the hedging portfolio implied by the misspecified model is larger (resp. lower) than the payoff h. Moreover, the initial price in the misspecified model is larger (resp. lower) than the true option price. Consequently, when the payoff is convex and $\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}$, the possible prices lie between the Black-Scholes price with volatility $\sigma_{\min}$ and the Black-Scholes price with volatility $\sigma_{\max}$. Note that this result does not hold in general for non-convex payoffs.

UVM model has been a subject of extensive interest. For instance, Romagnoli and Vargiolu [2000] considered UVM in a multidimensional setting where the interval value for the volatility $[\sigma_{\min}, \sigma_{\max}]$ is replaced by a compact subset where the volatility matrix takes values. They also treated the case of known volatility and uncertain correlation and found the optimal bounds for exchange options and geometrical mean options. Mykland [2000] defines the model uncertainty through bounds on integrals of the squared volatility. The UVM was further developed in Denis and Martini [2006], who also considered path-dependent contingent claims. More recently, Fouque and Ren [2014] have studied the asymptotic behavior of the option bounds prices when the model ambiguity vanishes, i.e. as the volatility interval $[\sigma_{\min}, \sigma_{\max}]$ degenerates to a single point. They have provided an approximation that works well even for non small uncertainty intervals.

The main criticism to UVM concerns the fact that the computation of the price bounds requires a difficult numerical resolution of a fully non linear PDE (c.f. Pooley et al. [2003]) and the obtained values may be prohibitively too extreme to be used for trading purposes, due to the prudent formulation of the problem. This is essentially due to the small number of parameters, i.e. the two bounds $\sigma_{\min}$, $\sigma_{\max}$. As we have already mentioned, the growing derivatives market made available the prices of a large set of vanilla instruments. The inclusion of this market information into the model constraints, rather than a simple bounds on the volatility process, is then likely to give tighter and more concrete meaningful option price bounds. The knowledge of call/put market prices has a direct consequence on the underlying price dynamics, more particularly on its marginals. Indeed, the original work of Breeden and Litzenberger [1978] states that if a risk-neutral model $\mathbb{Q}$ on a asset S, fits the market call prices $\mathcal{C}(T, K)$, for a fixed maturity T and all strikes K, then the (marginal) distribution of S_T is given by $\mathbb{Q}(S_T > K) = e^{rT} \left| \frac{\partial}{\partial K} \mathcal{C}(T, K) \right|$. Moreover, Carr and Madan

[2001] showed that

$$g(S) = g(F_0)+(S-F_0)g'(F_0)+\int_0^{F_0} g''(K)(K-S)^+dK+\int_{F_0}^{\infty} g''(K)(S-K)^+dK, \forall S \geq 0, \tag{0.1}$$

where g is a twice differentiable function and F_0 is an arbitrary positive constant. So the knowledge of the call and put prices (which are essentially expectations under a risk-neutral probability) for all strikes K determines the price of the contingent claim $g(S_T)$.

In the limiting case where the call/put prices for all maturities and strikes are known, all the marginals distribution of the process S are known. The set of non-arbitrage models that match the market data is then the set of probability measures that make the discounted price process a martingale *and* that have fixed prescribed marginals. Those considerations lead to consider an uncertainty model, whose range of option prices is related to all the underlying models with given marginals, instead of prices corresponding to volatility bounds. Interestingly, if one adds some supplementary assumptions, then the model risk can be even eliminated: for instance, the pioneer work of Dupire [1994] shows that under the additional assumption that the set of acceptable models is restrained to diffusion processes, the market prices of vanillas options specify completely (under some regularity conditions) the process distribution, so that the price of any exotic option is uniquely determined (inside this restricted class of diffusion processes). This result is related to the work of Krylov [1985] and Gyöngy [1986], and more particularly, it states that there exists a unique diffusion of the form

$$dS_t = rS_t dt + S_t\sigma(t, S_t)dW_t,$$

such that

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = \mathcal{C}(T, K),\ \forall T, K > 0.$$

In particular, σ verifies

$$\frac{1}{2}K^2\sigma^2(T, S)\partial_{KK}\mathcal{C}(T, K) = rK\partial_K\mathcal{C}(T, K) + \partial_T\mathcal{C}(T, K).$$

This local volatility model seems to solve the modeling problem, since it matches exactly the law marginals, as induced by the vanilla options prices. In this case, the price of an exotic option is known without ambiguity. Nevertheless, it was documented (see e.g. Dumas et al. [1998] and Hagan et al. [2002]) that the volatility surface dynamics predicted by the local volatility model is inconsistent with some market observed ones. This entails that if, supposedly, the market dynamics results from some model, the local volatility model does not reproduce some of its features even though it matches exactly the given marginals. One should then look into the set of all models $\mathcal{M}$ that make the discounted price process a martingale and that have the same (market implied) marginals; the exotic option price will vary between the supremum and infimum of the option prices over the set $\mathcal{M}$.

This idea gave place to the notion of **model-independent hedging** (or equivalently **robust hedging**), within a class of models having fixed marginal laws. Logically the robustness of the corresponding hedging strategy is with respect to all non-arbitrage models that are compliant with the market vanilla prices. In other words, the model-independent (super or sub)-hedging strategy, if it exists, is supposed to (super or sub)-hedge the contingent claim for all such models. Since the late 90thies, the problem of model-independent hedging has been formulated in different settings, and its relation to the optimal bound prices has been established. For instance, consider the case where S represents the stock price, whose only marginal law at the maturity time $T = 1$ is known and denoted μ. The problem of model-independent super-hedging of an exotic option paying $\Phi(S)$ at maturity can be written as

$$\underline{D} := \inf_{g,\gamma} \Big\{ \mathbb{E}_\mu[g(S_1)] \mid \exists\, \gamma \text{ such that } g(S_1) + \int_0^1 \gamma(S)_u dS_u \geq \Phi(S), \\ \text{for every continuous function } S : [0,1] \to \mathbb{R}_+^\star \Big\}, \tag{0.2}$$

where $g(S_1)$ represents a vanilla option and $\gamma(S)$ defines a time-continuous (bounded variation) dynamic strategy with respect to the stock price path S. For a rigorous statement of the problem, see for example Dolinsky and Soner [2014]. Note that, while the robust hedging in the setup (0.2) is stated *path-wise*, other formulations using *quasi-sure* definition of the super-replication exist, c.f. Galichon et al. [2014] for example. Finally, we mention that Dolinsky and Soner [2014] proved the equivalence of the two latter formulations.

The model-independent super-hedging problem (0.2) is closely linked to the upper bound price of the option Φ, over the set $\mathcal{M}(\mu)$ of all martingale measures with the specified marginal μ. Indeed, [Dolinsky and Soner, 2014, Theorem 2.7] proved that

$$\underline{D} = \overline{P}, \quad \text{where} \quad \overline{P} := \sup_{\mathbb{Q} \in \mathcal{M}(\mu)} \mathbb{E}_{\mathbb{Q}}\big[\Phi(S)\big]. \tag{0.3}$$

Similarly, one defines the lower bound $\underline{P}$ (which corresponds to the sub-hedging cost $\overline{D}$), and the price range $\overline{P} - \underline{P}$ can be interpreted as the model risk associated with the option Φ.

The problem of computing the option bounds $\overline{P}$ and $\underline{P}$, when one marginal μ is known, i.e. when the call prices are known for all strikes K and only one maturity T, was firstly approached using the *Skorokhod embedding problem* (SEP) results. Given a probability measure μ, the SEP consists in looking for a Brownian motion B and a stopping time τ such that $B_\tau \sim \mu$ and the stopped process $(B_{\tau \wedge t})_t$ is uniformly integrable. This problem has been extensively studied and many solutions were found, see for example Obłój [2004] for a survey. The SEP can be linked to the problem of constructing a martingale M such that $M_T \sim \mu$, via the Dambis, Dubins-Schwarz theorem, see [Revuz and Yor, 1991, Chapter V, Theorem 1.6]. This relation was exploited for obtaining the price optimal bounds in the seminal work of Hobson [1998]. The

author supposes the availability of the market prices of call options (equivalent to the prices of call options, see Equation (0.1)) and characterizes the upper optimal bound among the prices of an exotic option, in particular a Lookback option, whose payoff is the maximum of the underlying prices during the time interval $[0, T]$. The mentioned upper bound price for this option was expressed through the Azema-Yor solution of the SEP (c.f. Azéma and Yor [1979]). The SEP approach was also applied by Brown et al. [2001] to get optimal lower and upper bounds for Digital and Barrier options, and Cox and Obłój [2011b,a] made use of it for obtaining optimal upper and lower bounds for Double-Touch and Double-No-Touch barrier options and the corresponding super and sub-hedging strategies. Hobson and Neuberger [2012] used the SEP approach to study the form of the optimal martingale measure that realizes the upper bound price for the forward start straddle.

A recent breakthrough in the study of the model-independent pricing and hedging is the use of the *optimal transport theory* for the computation of the option bounds under marginals constraints. The original optimal transport problem, as formulated by Monge [1784], is the minimization of the cost, denoted c, of transporting a mass from an initial distribution μ to a target distribution ν. This can be stated as the following minimization problem

$$\inf_{T\in\mathcal{T}(\mu,\nu)} \int_{\mathbb{R}^n} c(x, T(x))d\nu(x), \tag{0.4}$$

where

$$\mathcal{T}(\mu,\nu) = \Big\{T : \mathbb{R}^n \to \mathbb{R}^n, \text{ such that } \int \varphi(T(x))d\nu(x) = \int \varphi(x)d\mu(x)\Big\}.$$

The maps T of this kind are classically called transport (or transference) plans. We observe that Problem (0.4) is equivalent to

$$\inf_{\mathbb{P}\in\Pi^T(\mu,\nu)} \mathbb{E}_{\mathbb{P}}\Big[c(X, Y)\Big], \tag{0.5}$$

where $\Pi^T(\mu,\nu)$ is the set of probabilities $\mathbb{P}$ on $\mathbb{R}^n\times\mathbb{R}^n$ of the type $\mathbb{P}(dx, dy) = \mu(dx)\delta_{T(x)}(dy)$, for some $T \in \mathcal{T}(\mu,\nu)$. We remark that the marginal laws of any $\mathbb{P} \in \Pi^T(\mu,\nu)$ are μ and ν.

Problem (0.5) was relaxed by Kantorovich [1942, 1948] by replacing the set $\Pi^T(\mu,\nu)$ of transport plans by the set $\Pi(\mu,\nu)$ of probability measures on $\mathbb{R}^n\times\mathbb{R}^n$ which have the specified marginals μ and ν. The so-called Monge-Kantorovich (MK) problem writes

$$P_{MK} := \inf_{\mathbb{P}\in\Pi(\mu,\nu)} \mathbb{E}_{\mathbb{P}}[c(X, Y)], \tag{0.6}$$

where

$$\Pi(\mu,\nu) = \{\mathbb{P}\text{: probability measure on } \mathbb{R}^n \times \mathbb{R}^n \text{ with marginals } \mu \text{ and } \nu\}. \tag{0.7}$$

The dual formulation of Problem (0.6) is

$$D_{MK} := \sup_{(\varphi,\psi)\in\mathcal{D}(\mu,\nu)} \int \varphi(x)\mu(dx) + \int \psi(y)\nu(dy),$$

where

$$\mathcal{D}(\mu,\nu) = \quad \{ \quad (\varphi,\psi) : \varphi \text{ is } \mu - \text{integrable}, \psi \text{ is } \nu - \text{integrable and } \varphi(x) + \psi(y) \leq c(x,y)\}.$$

The optimal transport theory has been the subject of extensive studies, we refer e.g. to Villani [2003, 2009] and to the references therein. For instance, for a large set of functions c that verify the so-called Spence-Mirrlees condition $c_{xy} := \frac{\partial^2 c}{\partial x \partial y}(x,y) > 0$, the Monge problem (0.5) and the Monge-Kantorovich problem (0.6) are shown to be equivalent. Indeed, for such cost functions, the primal problem (0.6) is attained by the so-called Fréchet-Hoeffding probability measure $\mathbb{P}^\star(dx,dy) := \mu(dx)\delta_{T^\star(x)}(dy) \in \Pi^T(\mu,\nu)$, where

$$T^\star(x) := F_\nu^{-1}(F_\mu(x)), \tag{0.8}$$

and F_μ (resp. F_ν) is the cumulative distribution function (c.d.f) of the probability measure μ (resp. ν) and F_ν^{-1} denotes the right-continuous inverse of F_ν. Moreover, there is no duality gap, i.e. $P_{MK} = D_{MK}$. Of course, analogous results exist for the profit maximization problem

$$\sup_{\mathbb{P}\in\Pi(\mu,\nu)} \mathbb{E}_{\mathbb{P}}[c(X,Y)], \tag{0.9}$$

which is symmetric with respect to the cost minimization problem (0.6).

The maximization problem (0.9) (resp. minimization problem (0.6)) is closely related to the financial problem of computing the no-arbitrage upper (resp. lower) price bound. In this context, X (resp. Y) represents the price S_1 (resp. S_2) of a stock at time 1 (resp. 2) and the function c denotes the payoff of an exotic option that depends on the stock price at dates 1 and 2. The price bounds for this option can be expressed as the optimal values of the problems

$$\underline{P}(\mu,\nu,c) = \inf_{\mathbb{Q}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{Q}}\Big[c(S_1,S_2)\Big] \text{ and } \overline{P}(\mu,\nu,c) = \sup_{\mathbb{Q}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{Q}}\Big[c(S_1,S_2)\Big], \tag{0.10}$$

where $\mathcal{M}(\mu,\nu)$ is the subset of $\Pi(\mu,\nu)$ of measures $\mathbb{Q}$ verifying the martingale property $\mathbb{E}_{\mathbb{Q}}[S_2|S_1] = S_1$, $\mathbb{E}_{\mathbb{Q}}[S_1|S_0] = S_0$. The maximization problem in (0.10) is similar to the one in (0.9) with a restricted class of probability measures. Generally (0.6) and (0.9) are called **(Monge-Kantorovich) optimal transport problems** and (0.10) is called **optimal martingale transport problem**.

Subsequently, a new stream of works has considered the *martingale* optimal transport and its applications to model-independent pricing and hedging in various settings (discrete, continuous time, one ore more marginals etc.). Indeed, Galichon et al. [2014] has shown that the dual formulation of the time-continuous martingale optimal transport problem (0.3) yields the super-hedging problem (0.2). Using this result,

they recover the optimality of the Azéma-Yor solution for the Lookback option, already established using the SEP techniques by Hobson [1998]. On the other hand, Beiglböck et al. [2013] consider, in a pure time-discrete setup, a finite set of dates $0 < t_1 < t_2 < \cdots < t_n$, assuming as Hobson [1998] the availability of all call options maturing at those dates. In Beiglböck et al. [2013], the admissible strategies allow to dynamically trade the stock (buy/sell the quantity $\Delta(S_1, \cdots, S_i)$ at date t_i, where $S_1, \cdots, S_n$ are the stock price) at the given set of dates and to take static positions on vanilla options (of payoffs $u_1(S_1), \cdots, u_n(S_n)$) at the initial date 0. Similarly as for (0.2), the model-independent super-hedging of an option paying $\Phi(S_1, \cdots, S_n)$ is then written as

$$\begin{aligned} \underline{D} := \inf_{u,\Delta} \Big\{ \sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i(S_i)] \mid \quad & \exists \Delta = (\Delta_1, \cdots \Delta_n) \text{ such that} \\ & \sum_{i=1}^{n} u_i(S_i) + \sum_{i=1}^{n-1} \Delta_i(S_1, \cdots, S_i)(S_{i+1} - S_i) \geq \Phi(S_1, \cdots, S_n) \Big\}. \end{aligned} \tag{0.11}$$

We recall again, taking into account (0.1), that any vanilla option is equivalent to a portfolio of call options of same maturity. As we have mentioned for the case of one maturity, the availability of vanilla option prices having maturities $t_1 < t_2 < \cdots < t_n$, specifies the stock marginals at these dates, denoted $\mu_1, \cdots, \mu_n$. In particular the expressions $\mathbb{E}_{\mu_i}[u_i(S_i)]$, which represents the initial price of the vanilla option $u_i(S_i)$, are completely determined by the marginals μ_i.

Beiglböck et al. [2013] show that the model-independent super-hedging problem (0.11) is a dual formulation of the following primal problem, which consists in looking for the upper bound of the option price

$$\overline{P} := \sup_{\mathbb{Q} \in \mathcal{M}(\mu_1, \cdots, \mu_n)} \mathbb{E}_{\mathbb{Q}}\big[\Phi\big], \tag{0.12}$$

where $\mathcal{M}(\mu_1, \cdots, \mu_n)$ is the set of martingale measures with specified marginals $\mu_1, \cdots, \mu_n$.

Similarly as for the continuous time case (0.3), Beiglböck et al. [2013] show that, under suitable conditions, there is no duality gap, i.e. $\overline{P} = \underline{D}$. Moreover, the primal value $\overline{P}$ is attained, i.e. there exists a martingale measure $\mathbb{Q}^\star \in \mathcal{M}(\mu_1, \cdots, \mu_n)$ such that

$$\overline{P} = \mathbb{E}_{\mathbb{Q}^\star}\big[\Phi\big].$$

Close attention was devoted to the two marginals optimal martingale transport problem (0.10). For instance, Hobson and Neuberger [2012] and Hobson and Klimmek [2015] studied the model-free lower and upper bound price for an option paying $C^1_{II}(S_1, S_2) := |S_2 - S_1|$. This type of option is called, in the literature, the **type II forward start straddle** with the strike $\alpha = 1$, or at-the-money (ATM). Other type II forward start straddle with coefficient $\alpha > 0$ are defined via the payoff

$$C^\alpha_{II}(x, y) = |y - \alpha x|, \quad \forall x, y > 0. \tag{0.13}$$

On the other hand, the payoff of the so-called **type I forward start straddle** is given by

$$C_I^{\alpha}(x,y) = \left|\frac{y}{x} - \alpha\right|, \quad \forall x, y > 0. \tag{0.14}$$

Indeed, Hobson and Neuberger [2012] give explicit information on the optimal martingale measure realizing the upper bound price of the ATM type II forward starting straddle C_{II}^1 and show the existence of a super-replication strategy that realizes the infimum $\underline{D}$. As a byproduct, they have established the same no-duality gap result, $\overline{P} = \underline{D}$. Moreover, Hobson and Klimmek [2015] derive an explicit expression for an optimal martingale measure (also called **coupling**), denoted by $\mathbb{Q}_{HK}$, that realizes the lower bound price of C_{II}^1 and they determine the form of the dual sub-hedging strategy. The probability $\mathbb{Q}_{HK}$ is indeed concentrated on a three points transition $\{p(x), x, q(x)\}$, in the following sense

$$\begin{aligned}\mathbb{Q}_{HK}(dx,dy) = \mu(dx)\Big[&\delta_x(dy)\mathbb{1}_{x\le a} + \delta_x(dy)\mathbb{1}_{x\ge b} \\ &+ \big(l(x)\delta_{p(x)}(dy) + u(x)\delta_{q(x)}(dy) + (1-l(x)-u(x))\delta_x(dy)\big)\,\mathbb{1}_{a<x<b}\Big],\end{aligned} \tag{0.15}$$

where p and q are two decreasing functions, $a < b$ are two positive reals and $0 \le u, l \le 1$ are two functions.

Elsewhere, Beiglböck and Juillet [2012] used the optimal transport theory for the study of the primal problem (0.12), for a large class of payoffs $\Phi(S_1, S_2)$. Inspired by the results about the Fréchet-Hoeffding probability measure (0.8) in the classical optimal transport theory, they introduce the notion of left-monotone martingale measure, and show that it constitutes a solution for (0.12). Its form is given by

$$\mathbb{Q}_L(\mu,\nu)(dx,dy) = \mu(dx)\left[\delta_x(dy)\mathbb{1}_{x\le a} + (q(x)\delta_{L_u(x)}(dy) + (1-q(x))\delta_{L_d(x)}(dy))\mathbb{1}_{x\ge a}\right], \tag{0.16}$$

where $a \in \mathbb{R}$ and L_u (resp. L_d) is an increasing (resp. decreasing) function such that $L_d(x) \le x \le L_u(x)$ and $q(x) := \frac{x - L_d(x)}{L_u(x) - L_d(x)}$. Henry-Labordère and Touzi [2013] obtained explicit expressions for the two maps L_u, L_d and the transition probability q; they also show that $\mathbb{Q}_L(\mu,\nu)$ realizes the upper bound $\overline{P}$ for a set of payoff Φ verifying the *generalized* Spence-Mirrlees type condition $\Phi_{xyy} := \frac{\partial^3 \Phi}{\partial x \partial^2 y} > 0$, which includes the ones discussed in Beiglböck and Juillet [2012]. Similarly Henry-Labordère and Touzi [2013] characterized the so-called right-monotone martingale measure, denoted $\mathbb{Q}_R(\mu,\nu)$, which realizes the upper bound price for payoffs verifying $\Phi_{xyy} < 0$. This was done via a *mirror* change of variable of the type $x \mapsto -x$. Moreover, Henry-Labordère and Touzi [2013] give the explicit form of the corresponding super and sub-hedging strategies. Finally, we mention that Henry-Labordère et al. [2014] have studied the continuous time analogue of the problem treated of Henry-Labordère and Touzi [2013].

In this thesis, we are interested in the model risk problem from the empirical

and theoretical point of views. Chapter 1 continues the martingale optimal transport approach as described above, for computing the model-free prices of a given path-dependent contingent claim Φ in a two-periods model. We are interested in the two-marginals maximization martingale problem (0.10), i.e.

$$\overline{P}(\mu,\nu,\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{Q}}\Big[\Phi(S_1,S_2)\Big], \tag{0.17}$$

where we recall that $\mathcal{M}(\mu,\nu)$ is the set of the laws of all two-period martingales with marginals μ,ν. By assumption we suppose that the marginals μ and ν are supported by the positive real half-line: in that case the corresponding processes are positive martingales as in the usual context of finance. We also suppose that the difference of c.d.f. $F_\mu - F_\nu$ admits a single maximizer. Under these assumptions, which do not restrict too much the scope of the financial applications, it has been possible to give a simple and instructive construction of the optimal martingale measures (0.15) and (0.16).

Our expression of (0.16) and (0.15) reveals to be more suitable for studying the effect of the change of numeraire transformation, which associates to a (strictly) positive martingale M the process $\frac{1}{M}$. We consider the operator $\mathbb{S}$ that, to every $\mathbb{Q}\in\mathcal{M}(\mu,\nu)$, associates a measure $\mathbb{S}(\mathbb{Q})$ defined by

$$E^{\mathbb{S}(\mathbb{Q})}[f(S_1,S_2)] = E^{\mathbb{Q}}\left[S_2 f\left(\frac{1}{S_1},\frac{1}{S_2}\right)\right], \text{ for every bounded measurable function } f.$$

We remark that $\mathbb{S}(\mathbb{Q})$ belongs to $\mathcal{M}(\mathcal{S}(\mu),\mathcal{S}(\nu))$, where $\mathcal{S}$ is an operator which associates to μ with density p_μ a measure $\mathcal{S}(\mu)$ with density $p_{\mathcal{S}(\mu)}$ defined by

$$p_{\mathcal{S}(\mu)}(x) = \frac{p_\mu(\frac{1}{x})}{x^3}, \quad x>0.$$

We observe that the set $\mathcal{M}(\mu,\nu)$ is non-empty if and only if $\mathcal{M}(\mathcal{S}(\mu),\mathcal{S}(\nu))$ is non-empty. If we define the payoff $\mathbb{S}^*(\Phi)(x,y) := y\Phi(\frac{1}{x},\frac{1}{y})$ for $x,y>0$, then, by Proposition 1.3.3 of Chapter 1, the symmetry property

$$\overline{P}(\mathcal{S}(\mu),\mathcal{S}(\nu),\mathbb{S}^*(\Phi)) = \overline{P}(\mu,\nu,\Phi), \tag{0.18}$$

holds. We remind that $\overline{P}$ has been defined in (0.17). Using the definition of $\mathbb{S}^*(\Phi)$, it is clear that the generalized Spence-Mirrlees condition $\Phi_{xyy}>0$ holds true if and only if $\mathbb{S}^*(\Phi)_{xyy}<0$. This elementary remark allows to find the upper bound price for payoffs verifying $\Phi_{xyy}<0$, passing by the symmetry $\mathbb{S}$, which defines the so called *mirror transference plan*. This approach gives similar results to the ones related to the mirror coupling $x\mapsto -x$ in [Henry-Labordère and Touzi, 2013, Remark 3.14], where the martingale measures marginals have support on $\mathbb{R}$. The symmetry operator $\mathbb{S}$ permits, similarly to the mirror coupling above, to reproduce analogue properties for the case of measures whose marginals are supported by $\mathbb{R}_+^*$, and allows to construct, from the left-monotone martingale measure $\mathbb{Q}_L(\mu,\nu)$, the right-monotone martingale

measure $\mathbb{Q}_R(\mu,\nu)$, which again realizes the upper bound price for payoffs Φ verifying $\Phi_{xyy} < 0$. We apply the same analysis in the case of positive martingales to retrieve the three points martingale measure $\mathbb{Q}_{HK}(\mu,\nu)$ introduced in Hobson and Klimmek [2015], see (0.15).

Furthermore, we show that the martingale measures $\mathbb{Q}_L(\mu,\nu)$, $\mathbb{Q}_R(\mu,\nu)$ and $\mathbb{Q}_{HK}(\mu,\nu)$ are extremal points of the convex set $\mathcal{M}(\mu,\nu)$ and that they verify the symmetry relations

$$\begin{aligned}\mathbb{S}\big(\mathbb{Q}_L(\mathcal{S}(\mu),\mathcal{S}(\nu))\big) &= \mathbb{Q}_R(\mu,\nu)\\ \mathbb{S}\big(\mathbb{Q}_{HK}(\mathcal{S}(\mu),\mathcal{S}(\nu))\big) &= \mathbb{Q}_{HK}(\mu,\nu).\end{aligned}$$

Interestingly, there is a symmetry relation between type I and type II forward start straddles, which is given by

$$\mathbb{S}^*(C_{II}^{\alpha}) = \alpha C_I^{\frac{1}{\alpha}}, \tag{0.19}$$

which implies, taking into account (0.18), that the lower bound price of the type I forward start C_I^1 is also attained by the 3-point martingale measure $\mathbb{Q}_{HK}$.

Finally, in Section 1.6 of Chapter 1, we introduce a new martingale measure which is also extremal in the convex set $\mathcal{M}(\mu,\nu)$ and we characterize some of the payoff functions Φ for which this measure is optimal.

In Chapter 2, we approach the model risk problem from an empirical point of view. We conduct a quantitative study of the impact of the model risk, in the context of the commodities market. More precisely, we are interested in the valuation and hedging of a natural gas storage and the related model risk impact. The owner of a natural gas storage has the possibility, at each date t_i, to make a decision u_i: either inject, withdraw gas or take no action. Each decision yields a profit/cost $\phi_{u_i}(S_{t_i})$ resulting from selling/buying the gas at the Spot price S_{t_i}. In parallel with this Spot strategy, one can buy/sell Futures contracts, that allows to reduce the variance of the final profit. All these decisions have to be made under many operational constraints, such as maximal and minimal volume of the storage and limited injection and withdrawal rates. For a fixed probability parametrized by a vector θ, the gas storage value is given by the profit expectation maximization

$$J(\theta) = \max_{(u_i)_{i=0\dots n-1}} \mathbb{E}^{\theta}\Big[\sum_{i=0}^{n-1}\phi_{u_i}(S_{t_i})\Big]. \tag{0.20}$$

In the first step of our work, we highlight important stylized facts about natural gas markets, which are the price seasonality and the presence of spikes. These features are the two main sources of value for a gas storage unit, since the ownership of a storage facility enables one to take advantage from seasonality and price spikes. Then, we propose a model that unifies the dynamics of the Futures curve and Spot price, and involves the seasonality and the presence of price spikes.

The second aspect of Chapter 2 is related to the quantification of model uncertainty

related to the Spot dynamics. In order to quantify the stability of the storage valuation with respect to model uncertainty, we define two model risk measures, inspired by the work of Cont [2006]. Our context is however different from Cont's one, in the sense that our models are estimated on historical data, and not on derivative market data. The first model risk measure is defined by

$$\pi_1 = \max_{\theta_i \in \Gamma} J^\star(\theta_i) - \min_{\theta_i \in \Gamma} J^\star(\theta_i), \tag{0.21}$$

where Γ is a set of statistically acceptable models, with respect to historical data. The second model risk measure π_2 has been defined in a similar spirit, but using the historically realized profits instead of profit expectations, see (2.7.3) of Chapter 2. Using those risk measures, we observe the great sensitivity of gas storage value to the modeling assumptions. Indeed the model uncertainty, as measured by the size of price range (0.21), represents a large proportion of the storage value. This puts into perspective the concentration of all the effort on the numerical resolution of problem (0.20) described in the literature. We conclude that much more attention should probably be devoted to the discussion of modeling assumptions.

In Chapter 2, we remind that the optimal trading strategy is based on the natural gas Spot physical prices, while the variance reduction hedging strategy uses financial Futures contracts. The two prices, Spot (physical) and Future (financial) are not perfectly correlated. Consequently, the use of these Futures, as hedging instruments of the Spot strategy profit, gives rise to an additional risk, commonly called *basis risk*. In Chapter 3, we consider a pair of processes (X, S) where X is a non traded or illiquid, but observable asset and S is a traded asset, correlated to X. Our starting point is the problem of the hedging of a contingent claim of the type $h := g(X_T, S_T)$, using only a trading strategy on the available asset S.

At least two approaches were used to define a hedging strategy in the presence of basis risk. The first one uses the utility function as a risk aversion criterion, see for example Davis [2006], Henderson and Hobson [2002], Monoyios [2004], Monoyios [2007], Ceci and Gerardi [2009, 2011]. Another approach is based on the quadratic hedging error criterion: it follows the idea of the seminal work of Föllmer and Schweizer [1991] that introduces the theoretical bases of quadratic hedging in incomplete markets. In particular, they show the close relation between the quadratic hedging problem with a special semimartingale decomposition, known as the **Föllmer-Schweizer (F-S) decomposition**. A triplet (h_0, Z^h, O^h) is said to be an F-S decomposition of the random variable $h = g(X_T, S_T)$ if

$$g(X_T, S_T) = h_0 + \int_0^T Z_s^h dS_s + O_T^h, \tag{0.22}$$

where O^h is a martingale which is strongly orthogonal to the martingale part of the hedging asset process S. This problem was studied by Hulley and McWalter [2008] in the simple two-dimensional Black-Scholes model for the non-traded (but observable

asset) X and the hedging asset S, described by

$$\begin{aligned} dX_t &= \mu_X X_t dt + \sigma_X X_t dW_t^X, \\ dS_t &= \mu_S S_t dt + \sigma_S S_t dW_t^S, \end{aligned} \tag{0.23}$$

where (W^X, W^S) is a standard correlated two-dimensional Brownian motion. The authors characterize the F-S decomposition (0.22) through a PDE terminal-value problem. Extensions of those results to the case of stochastic correlation between the two assets X and S have been performed by Ankirchner and Heyne [2012].

Note that the F-S decomposition (0.22) can be seen as a special case of the well-known backward stochastic differential equations (BSDEs), where we look for a triplet of processes (Y, Z, O) being solution of an equation of the form

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_{s-}, Z_s) dV_s^S - \int_t^T Z_s dM_s^S - (O_T - O_t), \tag{0.24}$$

where M^S (resp. V^S) is the local martingale (resp. the bounded variation process) appearing in the semimartingale decomposition of S, O is a strongly orthogonal martingale to M^S and $\hat{f}(\omega, s, y, z) = -z$. It is well-known that the forward-backward SDEs driven by Brownian motion are closely related with semilinear parabolic PDEs. The first motivation of this Chapter is to introduce a formalism which extends this analytical tool to to the case of forward-backward SDEs driven by a càdlàg martingale.

In this Chapter we consider a forward-backward SDE, issued from (0.24), where the forward process solves a sort of martingale problem, instead of the usual stochastic differential equation (0.23) appearing in the Brownian case. More particularly we suppose the existence of an operator $\mathfrak{a} : \mathfrak{D}(\mathfrak{a}) \subset C([0,T] \times \mathbb{R}^2) \to \mathcal{L}$, where $\mathcal{L}$ is a suitable space of functions $[0,T] \times \mathbb{R}^2 \to \mathbb{C}^2$, such that (X, S) verifies the following:

$$\forall y \in \mathfrak{D}(\mathfrak{a}), \quad \left(y(t, X_t, S_t) - \int_0^t \mathfrak{a}(y)(u, X_{u-}, S_{u-}) dA_u \right)_{0 \le t \le T} \text{ is an } \mathcal{F}_t\text{-local martingale,}$$

and A is some fixed predictable bounded variation process. With $\mathfrak{a}$ we associate the operator $\widetilde{\mathfrak{a}}$ defined by

$$\widetilde{\mathfrak{a}}(y) := \mathfrak{a}(\widetilde{y}) - y\mathfrak{a}(id) - id\mathfrak{a}(y),$$

where $id(t,x,s) = s, \widetilde{y} = y \times id$. In the elementary case when $X \equiv 0$ and S is a Brownian motion, $A_t = t$, $\mathfrak{a}(y) = \partial_t + \frac{1}{2}\partial_{ss}^2 y$ and $\widetilde{\mathfrak{a}}(y) = \partial_s y$. This justifies that the operator $\widetilde{\mathfrak{a}}$ can be considered as a *generalized derivative*.

In the forward-backward SDE we are interested in, the driver $\hat{f}$ verifies

$$\mathfrak{a}(id)(t, X_{t-}(\omega), S_{t-}(\omega)) \hat{f}(\omega, t, y, z) = f(t, X_{t-}(\omega), S_{t-}(\omega), y, z), \ (t, y, z) \in [0,T] \times \mathbb{C}^2, \omega \in \Omega, \tag{0.25}$$

for some $f : [0,T] \times \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{C}$.

The object of this chapter is threefold.

1) As already mentioned, to provide a general methodology for solving forward-backward SDEs (0.24) driven by a càdlàg martingale, via the solution of a deterministic problem generalizing the classical partial differential problem appearing in the case of Brownian martingales.

2) To give applications to the hedging problem in the case of basis risk via the F-S decomposition. In particular we revisit the case when (X, S) is a diffusion process whose particular case of Black-Scholes was treated by Hulley and McWalter [2008], discussing some analysis related to a corresponding PDE.

3) To furnish a quasi-explicit solution when the pair of processes (X, S) is an exponential of additive processes, which constitutes a generalization of the results of Goutte et al. [2014] and Hubalek et al. [2006], established in the absence of basis risk. This yields a characterization of the hedging strategy in terms of Fourier-Laplace transform and the moment generating function.

We formulate the deterministic problem which consists in looking for a pair of functions (y, z) which solves

$$\begin{aligned} \mathfrak{a}(y)(t,x,s) &= -f(t,x,s,y(t,x,s),z(t,x,s)), \\ \widetilde{\mathfrak{a}}(y)(t,x,s) &= z(t,x,s)\widetilde{\mathfrak{a}}(id)(t,x,s), \end{aligned} \tag{0.26}$$

for all $t \in [0,T]$ and $(x,s) \in \mathbb{R}^2$, with the terminal condition $y(T,.,.) = g(.,.)$. This is related to the BSDE (0.24) with final condition $h = g(X_T, S_T)$, under the condition (0.25). Indeed, Theorem 3.3.2 shows that any solution to the deterministic problem (0.26) will provide a solution (Y, Z, O) to the BSDE mentioned above, setting

$$Y_t = y(t, X_t, S_t), Z_t = z(t, X_{t-}, S_{t-}).$$

As we have pointed out before, a significant application concerns the hedging problem under basis risk of a contingent claim $g(X_T, S_T)$ via the F-S decomposition. According to Corollary 3.4.7 and Remark 3.4.8, if a couple of functions (y, z) (fulfilling some integrability conditions) solves the problem

$$\begin{aligned} \mathfrak{a}(y)(t,x,s) &= \mathfrak{a}(id)(t,x,s)z(t,x,s), \\ \widetilde{\mathfrak{a}}(y)(t,x,s) &= \widetilde{\mathfrak{a}}(id)(t,x,s)z(t,x,s), \end{aligned}$$

with terminal condition $y(T,.,.) = g(.,.)$, then the triplet (Y_0, Z, O), where

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_{t-}, S_{t-}),\ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

is an F-S decomposition of the random variable $g(X_T, S_T)$.

Special interest was devoted to the case when (X, S) is a couple of additive processes. In this context, Theorem 3.4.15 provides the quasi-explicit F-S decomposition

of a random variable of the form

$$g(X_T, S_T) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_T^{z_1} S_T^{z_2},$$

where Π is a finite complex Borel measure on $\mathbb{C}^2$, in terms of the moment generating function. This yields as a byproduct a characterization of the variance optimal hedging strategy. This extends the results of Goutte et al. [2014] and Hubalek et al. [2006], established in the absence of basis risk.

1

Change of numeraire in the two-marginals martingale transport problem.

This chapter is the object of Campi et al. [2014].

Abstract

In this paper we consider the optimal transport approach for computing the model-free prices of a given path-dependent contingent claim in a two periods model. More precisely, we first specialize the optimal transport plan introduced in Beiglböck and Juillet [2012], following the construction of Henry-Labordère and Touzi [2013], as well as the one in Hobson and Klimmek [2015], to the case of positive martingales and a single maximizer for the difference between the c.d.f.'s of the two marginals. These characterizations allow us to study the effect of the change of numeraire on the corresponding super and subhedging model-free prices. It turns out that, for Henry-Labordère and Touzi [2013]'s construction, the change of numeraire can be viewed as a mirror coupling for positive martingales, while for Hobson and Klimmek [2015] it exchanges forward start straddles of type I and type II giving also that the optimal transport plan in the subhedging problems is the same for both types of options. Some numerical applications are provided.

1.1 Introduction

Let μ and ν be two probability measures on $\mathbb{R}_+^*$ such that $\mu \preccurlyeq \nu$ in the sense of convex ordering, that is $\int f d\mu \leq \int f d\nu$ for all convex functions $f : \mathbb{R}_+^* \to \mathbb{R}$. In particular, μ and ν have the same mean. The classical theorem by Strassen [1965] proves the existence of a discrete martingale $\{M_i : i = 0, 1, 2\}$ with $M_0 = 1$ such that, if $X := M_1$ and $Y = M_2$, then $X \sim \mu$ and $Y \sim \nu$.

Let $\mathcal{M}(\mu, \nu)$ be the set of the laws of all such discrete martingales with marginals μ, ν. For functions $C : (\mathbb{R}_+^*)^2 \to \mathbb{R}$ with linear growth, Beiglböck and Juillet [2012] study the two-marginals martingale problem:

$$\overline{P}(\mu, \nu, C) = \sup_{Q \in \mathcal{M}(\mu,\nu)} E^Q C(X, Y). \tag{1.1}$$

Similarly, the inf-problem can be defined by:

$$\underline{P}(\mu,\nu,C) = \inf_{Q\in\mathcal{M}(\mu,\nu)} E^Q C(X,Y). \tag{1.2}$$

Using these two bounds, we can define a range of price measure, we denote its width by R

$$R(\mu,\nu,C) = \overline{P}(\mu,\nu,C) - \underline{P}(\mu,\nu,C). \tag{1.3}$$

The *primal* problems (1.1) and (1.2) have the following dual formulation

$$\overline{D}(\mu,\nu,C) = \inf_{(\varphi,\psi,h)\in\overline{\mathcal{H}}} \mu(\varphi)+\nu(\psi), \tag{1.4}$$

$$\underline{D}(\mu,\nu,C) = \sup_{(\varphi,\psi,h)\in\underline{\mathcal{H}}} \mu(\varphi)+\nu(\psi), \tag{1.5}$$

with

$$\overline{\mathcal{H}} = \Big\{(\varphi,\psi,h)\in\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu)\times\mathbb{L}^0 : \varphi(x)+\psi(y)+h(x)(y-x)\geq C(x,y),\ \forall x,y\in\mathbb{R}_+^*\Big\},$$

$$\underline{\mathcal{H}} = \Big\{(\varphi,\psi,h)\in\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu)\times\mathbb{L}^0 : \varphi(x)+\psi(y)+h(x)(y-x)\leq C(x,y),\ \forall x,y\in\mathbb{R}_+^*\Big\}.$$

Under suitable conditions, Beiglböck et al. [2013] show that there is no *duality gap*, i.e.

$$\begin{aligned}\overline{P}(\mu,\nu,C) &= \overline{D}(\mu,\nu,C),\\ \underline{P}(\mu,\nu,C) &= \underline{D}(\mu,\nu,C).\end{aligned}$$

and that the *primal problems* (1.1) and (1.2) are attained. Beiglböck and Juillet [2012] prove that these optimal probabilities are of special type, called the left-monotone and right-monotone transference plans which realize the extremum in (1.1) and (1.2), for a certain class of payoffs. On the other hand, Henry-Labordère and Touzi [2013] provide an explicit construction of the optimal transference plan for a more general class of payoffs C that satisfy the so-called *generalized Spence-Mirrlees* condition:

$$C_{xyy} > 0. \tag{1.6}$$

The construction is relatively easy when the difference of the cumulative distribution functions $\delta F := F_\nu - F_\mu$ has a single maximizer, and much trickier otherwise. Finally, Hobson and Klimmek [2015] construct an optimal transference plan giving a model-free sub-replicating price of a forward start straddle of type II, whose payoff $|X-Y|$ does not satisfy the generalized Spence-Mirrlees condition above.

In this work, we want to study the effect of a change of numeraire on those optimal transference plans. To do so, we start with revisiting the construction of Henry-Labordère and Touzi [2013] and Hobson and Klimmek [2015] in a way which is more suited to our study. Unlike these authors, we consider the case of *positive* martingales. Our motivation is to give a simple and instructive construction of the optimal transference plan, assuming additional properties on the marginals which considerably sim-

plify the proofs without restricting too much the scope of the financial applications. In particular, we specialize both constructions to the case of a single maximizer for the difference between the two cumulative distributions functions δF.[1]

We restate a characterization of the optimal two-point conditional distributions which reveals to be more suitable for studying the effect of a change of numeraire transformation, which associates to a positive martingale M the martingale $\frac{1}{M}$ under the change of probability with density M_2. In particular, it turns out that, for Beiglböck and Juillet [2012] and Henry-Labordère and Touzi [2013] optimal transport plan the change of numeraire can be viewed as a mirror coupling for positive martingales, while, for Hobson and Klimmek [2015], it exchanges forward start straddles of type I and type II giving also that the optimal transport plan in the subhedging problems is the same for two types of options.

The paper is organised as follows. In Section 1.2, we provide a self-contained explicit construction of the left-monotone transference plan for positive martingales in the single maximizer case. We define and study in Section 1.3 the change of numeraire and the transformation of the two-marginals problem by change of numeraire. In Section 1.4 we characterize the right-monotone transference plan and show that the change of numeraire operates like a mirror-coupling for positive martingales. We study in Section 1.5 the transference plan introduced by Hobson and Klimmek [2015], we characterize its existence and uniqueness and give some symmetry properties. In the last Section 1.7, we study the symmetric case where μ and ν are invariant by change of numeraire. This covers the case of the Black-Scholes model and of the stochastic volatility models with no correlation between the volatility and the spot (c.f. Renault and Touzi [1996]).

Notations.

1. Let $\mathcal{P}_1 = \mathcal{P}(\mathbb{R}_+^*)$ denote the set of probability measures μ on $\mathbb{R}_+^*$ with a positive density p_μ with respect to the Lebesgue measure, such that
$$\int_0^\infty x p_\mu(x) dx = 1. \tag{1.7}$$
2. If $\mu, \nu \in \mathcal{P}_1$, then p_μ, p_ν will denote the densities of μ, ν, and F_μ, F_ν their cumulative distribution functions. We also introduce the function δF defined by
$$\delta F = F_\nu - F_\mu.$$
3. id will denote the identity function.

We will work under the following assumption in the rest of the paper.

Assumption 1.1.1. *We suppose that the two measures μ and ν do not agree on any interval.*

[1] The case of a single maximizer for the construction in Henry-Labordère and Touzi [2013] is also treated in forthcoming Lecture Notes on Martingale Optimal Transport by Nizar Touzi, with no emphasis on the symmetry yet. To keep this article self-contained and to emphasize the role of symmetry we decided to keep this part.

This assumption is fulfilled in most classical diffusions and stochastic volatility models. Note however that this excludes the case of marginals with bounded support.

1.2 Basic left-monotone transference plan: existence and uniqueness

In this section, we consider two measures $\mu, \nu \in \mathcal{P}_1$ such that $\mu \preccurlyeq \nu$. The maximization problem (1.1) is strongly related to the concept of *left* and *right monotone transference plans*. The latter was introduced in Beiglböck and Juillet [2012], who show its existence and uniqueness for convex ordered marginals and that it realizes the optimum in the problem (1.1) for a specific set of payoffs. On the other hand, Henry-Labordère and Touzi [2013] extended these results to a wider set of payoffs, and more importantly give an *explicit* construction of the left-monotone transference plan. In this paper, we will use a more convenient notion that we call a *basic left-monotone transference plan*, motivated by the form of the optimal transference plan found by Henry-Labordère and Touzi [2013], and we study its properties.

Definition 1.2.1 (Basic left-monotone transference plan)**.** *A basic left-monotone transference plan is a triplet $(x_\star, L_d, L_u)$, where $x_\star \in \mathbb{R}_+^*$ and L_d, L_u are positive continuous functions on $]0, \infty[$, such that:*

i) $L_d(x) = L_u(x) = x$, *for* $x \leq x_\star$;

ii) $L_d(x) < x < L_u(x)$, *for* $x > x_\star$;

iii) on the interval $]x_\star, \infty[$, L_d *is decreasing,* L_u *is increasing;*

iv) $\mathcal{L}\mu = \nu$ *where the transition kernel* $\mathcal{L}$ *is defined by*

$$\mathcal{L}(x, dy) = \delta_x \mathbb{1}_{x \leq x\star} + (q_L(x)\delta_{L_u(x)} + (1 - q_L(x))\delta_{L_d(x)})\mathbb{1}_{x > x\star}$$

where $q_L(x) := \frac{x - L_d(x)}{L_u(x) - L_d(x)}$.

For the rest of the article, we will work under the following assumption:

Assumption 1.2.2. *δF has a single maximizer m.*

In order to prove the existence of the basic left-monotone transference plan, we first look for necessary conditions on $(x_\star, L_d, L_u)$ which ensure that the resulting law is ν i.e. property (iv)). For convenience we shall denote $1 - q_L$ by $\overline{q}_L$.

1.2.1 Necessary conditions

In this section, we give some necessary conditions for a given triplet $(x_\star, L_d, L_u)$ to be a left-monotone transference plan. First, we show that it verifies an ODE.

Lemma 1.2.3. *If $(x_\star, L_d, L_u)$ is a basic left-monotone transference plan, then the pair (L_d, L_u) verifies the following ODE:*

$$p_\nu(L_u)dL_u = q_L p_\mu dy, \quad \forall y > x_\star, \tag{2.8}$$

$$(p_\nu(L_d) - p_\mu(L_d))dL_d = -\overline{q}_L p_\mu dy, \quad \forall y > x_\star. \tag{2.9}$$

Proof. C.f. Appendix 1.A. □

Using these ODEs, we get the following necessary conditions on the triplet $(x_\star, L_d, L_u)$.

Proposition 1.2.4. *If $(x_\star, L_d, L_u)$ is a basic left-monotone transference plan, then we have the following properties:*

1. *$x_\star$ must be the unique maximizer of δF;*
2. *L_d verifies the equation*

$$F_\nu^{-1}(F_\mu(x) + \delta F(L_d(x))) = G_\nu^{-1}(G_\mu(x) + \delta G(L_d(x))), \ \forall x > x_\star, \tag{2.10}$$

   where $G_\mu(x) = \int_0^x y p_\mu(y)dy$, $G_\nu(x) = \int_0^x y p_\nu(y)dy$ and $\delta G = G_\nu - G_\mu$.

3. *L_u is related to L_d throughout the equation*

$$F_\nu(L_u(x)) = F_\mu(x) + \delta F(L_d(x)), \ \forall x > x_\star. \tag{2.11}$$

Proof. We postpone a detailed proof of these three properties to Appendix 1.B □

Remark 1.2.5. *Equation (2.10) defines $L_d(x)$, at least formally (the uniqueness will be proven in Lemma 1.2.6), for $x > x_\star$, and $L_u(x)$ follows from Equation (2.11). This equation is well defined if it has a unique solution $L_d(x)$ for all $x > x_\star$. This will be proved in the next section.*

1.2.2 Sufficient conditions

Observe first that the single maximizer Assumption 1.2.2 implies, by item 1 in Proposition 1.2.4, that if $(x_\star, L_d, L_u)$ is a basic left transference plan, one necessarily has $x_\star = m$. Now, we want to prove that equation (2.10) has a unique solution $L_d(x) < m$ for any $x > m$.
We can state the following preliminary lemma:

Lemma 1.2.6. *Given $x > m$, let $t_F(x)$ be defined by:*

1. *$t_F(x) = m$ if $F_\mu(x) + \delta F(m) < 1$;*
2. *$t_F(x)$ solves $F_\mu(x) + \delta F(t_F(x)) = 1$ in $]0, m[$ otherwise.*

Then, Equation (2.10) has a unique solution, denoted $L_d(x)$, taking value in the interval $]0, t_F(x)[$. Moreover, $L_u(x)$, given by Equation (2.11), is well defined.

Proof. C.f. Appendix 1.C. □

Remark 1.2.7. *Note that if $F_\mu(x) + \delta F(m) \geq 1$, then the equation $F_\mu(x) + \delta F(t_F(x)) = 1$ in $]0, m[$ has a unique solution. This is due to the fact that $\mu \preccurlyeq \nu$ and to Assumption 1.1.1.*

To complete the construction, we will prove that the graphs of L_d and L_u, defined as solutions of (2.10) and (2.11) have the following properties:

Proposition 1.2.8 (Properties of L_d and L_u)**.** *On (m, ∞), we have:*

1. *L_d and L_u are C^1 functions,*
2. *$L_u(x) > x > L_d(x)$,*
3. *L_d is decreasing and L_u is increasing.*

Proof.

1. It follows from the implicit function theorem.
2. Let $x > m$. By definition, we have $L_d(x) < x$, since $L_d(x) < m$. Moreover, $L_u(x) > L_d(x)$ is equivalent to $F_\mu(x) > F_\mu(L_d(x))$, which follows from $L_d(x) < m < x$. Now, we prove that $L_u(x) > x$. Using the definition of L_u, one has

$$L_u(x) > x \Leftrightarrow \delta F(L_d(x)) > \delta F(x).$$

Now, note that because of the convex ordering of μ and ν we know that δF has at least one zero, otherwise δF would have a constant sign, which contradicts the convex ordering. If it had one more zero, this last property would imply that δF has at least two local maximizers. We denote this unique zero by $z_{\delta F}$. Let us distinguish two cases. If $x \geq z_{\delta F}$, then $\delta F(x) \leq 0$, and since $\delta F(L_d(x)) > 0$, $L_u(x) > x$ follows. On the other hand, if $x \leq z_{\delta F}$, then by continuity, there exists $\overline{x} < m$ such that $\delta F(x) = \delta F(\overline{x})$. Let us introduce the function

$$Z_x(t) := G_\mu(x) + \delta G(t) - G_\nu\Big[F_\nu^{-1}(F_\mu(x) + \delta F(t))\Big].$$

We have

$$\begin{aligned} Z_x(\overline{x}) &= G_\mu(x) + \delta G(\overline{x}) - G_\nu(x) \\ &= -\int_{\overline{x}}^{x} y\delta F'(y)dy \\ &= -\delta F(x)(\overline{x} - x) + \int_{\overline{x}}^{x} \delta F(y)dy \\ &= \int_{\overline{x}}^{x} (\delta F(y) - \delta F(x))dy. \end{aligned}$$

Finally, since $\delta F(y) > \delta F(x)$ for all $y \in (\overline{x}, x)$, we get $Z_x(\overline{x}) > 0$.
By Lemma 1.2.6, $L_d(x)$ is the unique zero of Z_x in the interval $]0, t_F(x)[$. Therefore $L_d(x)$ will be in $(\overline{x}, x)$, and we get our desired result $\delta F(L_d(x)) > \delta F(x)$.

3. If we differentiate the equation (2.10) with respect to x, we get $\delta F'(L_d(x))L_d'(x) = -\overline{q}_L(x)p_\mu(x)$ where $\overline{q}_L(x) = \frac{L_u(x)-x}{L_u(x)-L_d(x)}$ is well defined and positive, and $\delta F'(L_d(x))$ is also positive since the function δF is increasing on $(0, m)$. This implies that L_d is decreasing.
To prove that L_u is increasing, recall that L_u is defined via $F_\nu(L_u(x)) = F_\mu(x) + \delta F(L_d(x))$. Differentiating this equation with respect to x gives

$$L_u'(x)p_\nu(L_u(x)) = p_\mu(x) + L_d'(x)\delta F'(L_d(x)) = q_L(x)p_\mu(x)$$

where the function $q_L(x) = \frac{x-L_d(x)}{L_u(x)-L_d(x)}$ is well defined and positive.

□

Now, we can give the main result of this section, stating the existence and uniqueness of the basic left-monotone transference plan.

Theorem 1.2.9. *Assume that δF has a single maximizer m. Then there is a unique basic left-monotone transference plan (m, L_d, L_u) where L_d is the unique solution of (2.10) and L_u is given by the relation (2.11).*

Having proved the existence and uniqueness of the basic left-monotone transference plan, we can introduce similarly the basic right-monotone transference plan. Before that, let us introduce the change of numeraire transformation and study some symmetry properties of the maximization problem (1.1).

1.3 Change of numeraire

Let X be a positive random variable with law $\mu \in \mathcal{P}_1$. The change of numeraire with respect to X amounts to define a new probability measure $Xd\mu$ and to look at the law of $\frac{1}{X}$ under $Xd\mu$. Its density q satisfies for any measurable bounded function f:

$$\int f(y)q(y)dy = E\left[Xf\left(\frac{1}{X}\right)\right] = \int xf\left(\frac{1}{x}\right)p(x)dx = \int f(y)\frac{p(\frac{1}{y})}{y^3}dy.$$

This motivates the introduction of a symmetry operator, denoted S.

1.3.1 The symmetry operator S

We associate to $\mu \in \mathcal{P}_1$ with density p_μ a measure $S(\mu)$ with density $p_{S(\mu)}$ defined by:

$$p_{S(\mu)}(x) = \frac{p_\mu(\frac{1}{x})}{x^3}, \quad x > 0. \tag{3.12}$$

It is straightforward to verify that the function $p_{S(\mu)}$ is indeed a density defining a measure in $\mathcal{P}_1$. Let us also observe that S is an involution, i.e. $S \circ S = id$. Indeed, we have

$$\frac{p_{S(\mu)}(\frac{1}{x})}{x^3} = \frac{x^3 p_\mu(x)}{x^3} = p_\mu(x).$$

We summarize our findings in the following lemma.

Lemma 1.3.1. *$p_{S(\mu)}$ is the density of a measure $S(\mu) \in \mathcal{P}_1$. Moreover the operator S is an involution, preserving the convex order in the set of measures $\mathcal{P}_1$, i.e. if $\mu, \nu \in \mathcal{P}_1$ satisfy $\mu \preccurlyeq \nu$, then $S(\mu) \preccurlyeq S(\nu)$.*

Proof. Let $\mu, \nu \in \mathcal{P}_1$ such that for any convex function f, $\int f d\mu \leq \int f d\nu$. Since $S(\mu)$ and $S(\nu)$ have the same (unit) mass and first moment, it is enough to show that for any positive constant K, L we have

$$\int (Kx - L)_+ dS(\mu) \leq \int (Kx - L)_+ dS(\nu).$$

Now $\int (Kx-L)_+ dS(\mu) = \int (Kx-L)_+ \frac{p(\frac{1}{x})}{x^3} dx = -\int (K-\frac{L}{x})_+ p(\frac{1}{x}) d\frac{1}{x} = \int (K-Ly)_+ d\mu$, the same for ν. Since $y \mapsto (K-Ly)_+$ is a convex function, the result follows. □

It is easy to show the following properties for the image of μ by the operator S. Its proof is therefore omitted.

Proposition 1.3.2. *If $\mu \in \mathcal{P}_1$, then for all $y > 0$ we have*

$$F_{S(\mu)}(y) = 1 - G_\mu(1/y) \quad \textit{and} \quad G_{S(\mu)}(y) = 1 - F_\mu(1/y).$$

1.3.2 The symmetric two-marginals martingale problem

For μ, ν in $\mathcal{P}_1$, with $\mu \preccurlyeq \nu$, we recall that $\mathcal{M}(\mu,\nu)$ denotes the set of all discrete martingales with marginals μ, ν. By the classical theorem of Strassen [1965], we know that there exists a discrete martingale $\{M_i : i = 0,1,2\}$ with $M_0 = 1$ such that, if $X := M_1$ and $Y = M_2$, then $X \sim \mu$ and $Y \sim \nu$. We also recall the problem (1.1), for functions $C(x,y)$ with linear growth:

$$C \mapsto \overline{P}(\mu,\nu,C) = \sup_{Q \in \mathcal{M}(\mu,\nu)} E^Q[C(X,Y)].$$

Then, we introduce the symmetric two-marginals martingale problem, defined as $C \mapsto \overline{P}(S(\mu), S(\nu), C)$. We start a study of its properties with the following proposition:

Proposition 1.3.3. *Let $\mathbb{S}$ be the operator that to every $Q \in \mathcal{M}(\mu,\nu)$ associates a measure $\mathbb{S}(Q)$ defined by*

$$E^{\mathbb{S}(Q)}[f(X,Y)] = E^Q\left[Y f\left(\frac{1}{X}, \frac{1}{Y}\right)\right], \textit{ for every bounded measurable function } f.$$

Then, we have the following properties:

1. *$\mathbb{S}(Q)$ defines a probability in $\mathcal{M}(S(\mu), S(\nu))$, and the symmetry $\mathbb{S}$ is an involution, i.e. $\mathbb{S} \circ \mathbb{S} = id$.*
2. *$\mathbb{S}\left(\mathcal{M}(\mu,\nu)\right) = \mathcal{M}(S(\mu), S(\nu))$.*
3. *Let us define the payoff $\mathbb{S}^*(C)(x,y) := yC(\frac{1}{x}, \frac{1}{y})$ for $x, y \geq 0$. Then*

$$\overline{P}(S(\mu), S(\nu), \mathbb{S}^*(C)) = \overline{P}(\mu,\nu,C). \tag{3.13}$$

Proof. 1. First, let us prove that $\mathbb{S}(Q) \in \mathcal{M}(S(\mu), S(\nu))$ for $Q \in \mathcal{M}(\mu,\nu)$. The fact that Y has law $S(\nu)$ under $\mathbb{S}(Q)$ amounts to the definition of S on $\mathcal{P}_1$. Regarding X, by the martingale property under Q, for functions f that depend only on the x-variable:

$$E^{\mathbb{S}(Q)}[f(X)] = E^Q\left[Y f\left(\frac{1}{X}\right)\right] = E^Q\left[X f\left(\frac{1}{X}\right)\right],$$

and we conclude since X has law μ under Q. It remains to show the martingale property:

$$E^{\mathbb{S}(Q)}[Y f(X)] = E^Q\left[Y\frac{1}{Y}f\left(\frac{1}{X}\right)\right] = E^Q\left[f\left(\frac{1}{X}\right)\right] = E^Q\left[X\frac{1}{X}f\left(\frac{1}{X}\right)\right].$$

Now by the martingale property under Q this is also $E^Q[Y\frac{1}{X}f(\frac{1}{X})] = E^{\mathbb{S}(Q)}[Xf(X)]$, which implies $E^{\mathbb{S}(Q)}[Y|X] = X$.

2. In order to prove that $\mathbb{S}(\mathcal{M}(\mu,\nu)) = \mathcal{M}(S(\mu),S(\nu))$, we note that one inclusion is implied by the property 1. in this proposition. The other inclusion is a consequence of the fact that the symmetry operator S is an involution.

3. It is an easy consequence of the previous property 2.

□

In the rest of this section, we will study the effect of change of numeraire on the generalized Spence-Mirrless condition and how the symmetry operator $\mathbb{S}^*$ introduced in Proposition 1.3.3 acts on the space of hedgeable claims.

1.3.3 Relation to the generalized Spence-Mirrlees condition

The model-free bounds $\overline{P}$ and $\underline{P}$ are linked to the left and right monotone transference plans, under a generalized Spence-Mirrlees type condition on C: $C_{xyy} > 0$ (or $C_{xyy} < 0$). In fact, Henry-Labordère and Touzi [2013] show that for payoffs C verifying this condition, the optimal problem (1.1) is attained by the left-monotone transference plan, extending the results of Beiglböck and Juillet [2012].

Using the definition of $\mathbb{S}^*(C)$, it is clear that

$$\mathbb{S}^*(C)_{xyy}(x,y) = -\frac{1}{x^2y^3}C_{xyy}\left(\frac{1}{x},\frac{1}{y}\right),\ \forall x,y>0. \tag{3.14}$$

Hence, we have that $C_{xyy} > 0$ holds true if and only if $\mathbb{S}^*(C)_{xyy} < 0$. This elementary remark allows to find the bound price for payoffs verifying $C_{xyy} < 0$, passing by the symmetry S, which defines the mirror transference plan R_d, R_u. This is similar to the change of variables of the mirror coupling in Henry-Labordère and Touzi [2013], Remark 3.14, where the martingale measures have support on $\mathbb{R}$. The symmetry operator S permits to handle this case for $\mathbb{R}_+^*$-supported measures.

Definition 1.3.4. *We say that a payoff function C is symmetric if it satisfies $\mathbb{S}^*(C) = C$.*

If the payoff C is symmetric and verifies the generalized Spence-Mirrlees condition i.e. $\mathbb{S}^*(C) = C$ and $C_{xyy} \geq 0$, then using (3.14), we have $C_{xyy}(x,y) = -\frac{1}{x^2y^3}C_{xyy}(\frac{1}{x},\frac{1}{y})$, hence $C_{xyy} = 0$. Integrating with respect to y twice and with respect to x, we see that C is necessarily of the form $C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x)$, for some functions φ, ψ and h.

Remark 1.3.5. *Since a symmetric payoff verifies $C(x,y) = yC(1/x,1/y),\ \forall x,y>0$, a way of constructing it could go as follows: it suffices to choose its value on $[0,1]\times\mathbb{R}_+^*$, then for*

$(x,y) \in (1,\infty) \times \mathbb{R}_+^*$, define the payoff value by $C(x,y) = yC(1/x,1/y)$, since $(1/x,1/y) \in [0,1] \times \mathbb{R}_+^*$. One may easily check that C satisfies the symmetry relation $\mathbb{S}^*(C) = C$.

1.3.4 Symmetry and model risk

The quantity $R(\mu,\nu,C) = \overline{P}(\mu,\nu,C) - \underline{P}(\mu,\nu,C)$ is a natural indicator of the model risk associated with a given payoff C. Obviously, model-risk free payoffs include payoffs which can be written like $C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x)$, since $R(C) = 0$ in this case. What about the converse?

Proposition 1.3.6. *Let C be a payoff such that $R(C) = 0$ and such that the dual problem $\overline{D}$ in (1.5) is attained and that there is no dual gap. Then, there exist functions $\varphi \in \mathbb{L}^1(\mu),\ \psi \in \mathbb{L}^1(\nu),\ h \in \mathbb{L}^0$ such that*

$$C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x),\ Q-a.e. \quad \forall Q \in \mathcal{M}(\mu,\nu). \tag{3.15}$$

Proof. Let C be a payoff such that $R(C) = 0$ and the dual problem $\overline{D}$ (c.f. the equation (1.5)) of $\overline{P}(\mu,\nu,C)$ is attained. The first condition, $R(C) = 0$, implies that $E^Q[C(X,Y)] = \overline{P}(\mu,\nu,C),\ \forall Q \in \mathcal{M}(\mu,\nu)$. The second condition means that there exist dual functions φ, ψ, h such that

$$\mu(\varphi) + \nu(\psi) = \overline{P}(\mu,\nu,C)$$

and

$$C(x,y) \geq \varphi(x) + \psi(y) + h(x)(y-x),\ \forall x,y > 0.$$

Since all $Q \in \mathcal{M}(\mu,\nu)$ have marginals μ and ν and verify the martingale property, we have

$$\overline{P}(\mu,\nu,C) = E^Q[C(X,Y)] = E^Q[\varphi(X) + \psi(Y) + h(X)(Y-X)],\ \forall Q \in \mathcal{M}(\mu,\nu).$$

Consequently, we have the two equations

$$\begin{aligned} &C(x,y) - \varphi(x) - \psi(y) - h(x)(y-x) \geq 0, \quad \forall x,y > 0, \\ &E^Q\left[C(X,Y) - \varphi(X) - \psi(Y) - h(X)(Y-X)\right] = 0,\ \forall Q \in \mathcal{M}(\mu,\nu). \end{aligned}$$

This gives $C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x)$, Q-a.e. for all $Q \in \mathcal{M}(\mu,\nu)$. □

We denote $\mathcal{H}(\mu,\nu)$ the set of payoffs that can be represented as in (3.15), i.e.

$$\begin{aligned} \mathcal{H}(\mu,\nu) = \Big\{ \quad & C : (R_+^*)^2 \to \mathbb{R} | \text{ there exist functions } \varphi \in \mathbb{L}^1(\mu),\ \psi \in \mathbb{L}^1(\nu),\ h \in \mathbb{L}^0 \\ & C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x)\ Q-a.e. \quad \forall Q \in \mathcal{M}(\mu,\nu) \Big\} \end{aligned}$$

This set contains all the payoffs that can be replicated by investing in the stock and in European options. An interesting property of this set is that it is invariant by the symmetry operator $\mathbb{S}^*$.

Proposition 1.3.7. *The set $\mathcal{H}(\mu,\nu)$ is invariant by $\mathbb{S}^*$, i.e. $\mathbb{S}^*(\mathcal{H}(\mu,\nu)) = \mathcal{H}(S(\mu),S(\nu))$.*

Proof. Let $C \in \mathcal{H}(\mu,\nu)$, i.e. there exist functions $\varphi \in \mathbb{L}^1(\mu),\ \psi \in \mathbb{L}^1(\nu),\ h \in \mathbb{L}^0$ such that

$$C(x,y) = \varphi(x) + \psi(y) + h(x)(y-x) \quad Q-a.e. \quad \forall Q \in \mathcal{M}(\mu,\nu).$$

Let $x_0, y_0 > 0$ such that $C(x_0,y_0) = \varphi(x_0) + \psi(y_0) + h(x_0)(y_0 - x_0)$, then

$$\begin{aligned}\mathbb{S}^*(C)(x_0,y_0) &:= y_0 C(1/x_0, 1/y_0)\\ &= y_0\varphi(1/x_0) + y_0\psi(1/y_0) + y_0 h(1/x_0)(1/y_0 - 1/x_0)\\ &= \tilde{\varphi}(x_0) + \tilde{\psi}(y_0) + \tilde{h}(x_0)(y_0 - x_0),\end{aligned}$$

where we define the functions $\tilde{\varphi}$, $\tilde{\psi}$ and $\tilde{h}$ by

$$\tilde{\varphi}(x) = x\varphi(1/x),\ \tilde{\psi}(y) = y\psi(1/y),\ \tilde{h}(x) = \left(\varphi(1/x) - 1/x h(1/x)\right),\ \forall x,y > 0.$$

These functions verify $\tilde{\varphi} \in \mathbb{L}^1(S(\mu)),\ \tilde{\psi} \in \mathbb{L}^1(S(\nu)),\ \tilde{h} \in \mathbb{L}^0$. We have the following equivalences, since $\mathbb{S}(\mathcal{M}(\mu,\nu)) = \mathcal{M}(S(\mu), S(\nu))$

$$\begin{aligned}\forall Q \in \mathcal{M}(\mu,\nu),\ \mathbb{E}^Q\left[|C(X,Y) - \varphi(X) - \psi(Y) - h(X)(Y-X)|\right] &= 0\\ \Leftrightarrow \forall Q \in \mathcal{M}(\mu,\nu),\ \mathbb{E}^{\mathbb{S}(Q)}\left[\left|\mathbb{S}^*(C)(X,Y) - \tilde{\varphi}(X) - \tilde{\psi}(Y) - \tilde{h}(X)(Y-X)\right|\right] &= 0\\ \Leftrightarrow \forall Q \in \mathcal{M}(S(\mu),S(\nu)),\ \mathbb{E}^Q\left[\left|\mathbb{S}^*(C)(X,Y) - \tilde{\varphi}(X) - \tilde{\psi}(Y) - \tilde{h}(X)(Y-X)\right|\right] &= 0.\end{aligned}$$

So,

$$\mathbb{S}^*(C)(x,y) = \tilde{\varphi}(x) + \tilde{\psi}(y) + \tilde{h}(x)(y-x),\ Q-a.e.\ \forall Q \in \mathcal{M}(S(\mu),S(\nu)),$$

i.e. $\mathbb{S}^*(C) \in \mathcal{H}(S(\mu),S(\nu))$. □

1.4 Construction of the basic right-monotone transference map via change of numeraire

The goal of this section is to use the symmetry operator and our previous results on the basic left-monotone transport plan to provide a simple construction of the basic right-monotone transport plan. We suppose given two measures μ and ν verifying the same conditions as in the previous section, i.e. $\mu,\nu \in \mathcal{P}_1$ such that μ,ν are convex ordered ($\mu \preccurlyeq \nu$).

Definition 1.4.1 (Basic right-monotone transference plan)**.** *A basic right-monotone transference plan is a triplet $(x^\star, R_d, R_u)$, where R_d, R_u are positive continuous functions on $]0,\infty[$, such that:*

i) $R_d(x) = R_u(x) = x$, for $x \geq x_\star$;

ii) $R_d(x) < x < R_u(x)$, for $x < x_\star$;

iii) On the interval $]0, x_\star[$, R_d is increasing, R_u is decreasing,

iv) $\mathcal{L}\mu = \nu$ *where the transition kernel* $\mathcal{L}$ *is defined by*

$$\mathcal{L}(x, dy) = \delta_x \mathbb{1}_{x \leq x\star} + (q_R(x)\delta_{R_u(x)} + (1 - q_R(x))\delta_{R_d(x)})\mathbb{1}_{x > x\star}$$

where $q_R(x) := \frac{x - R_d(x)}{R_u(x) - R_d(x)}$.

As we have proved in the previous section, if $\mu, \nu \in \mathcal{P}_1$ satisfy $\mu \preccurlyeq \nu$, then their images by the symmetry operator S verify the same conditions, i.e : $S(\mu), S(\nu) \in \mathcal{P}_1$ such that $S(\mu) \leq S(\nu)$. We know from Theorem 1.2.9 that there exists a basic left-monotone transference plan $(x^S_\star, L^S_d, L^S_u)$ for $(S(\mu), S(\nu))$, under the condition that δF_S admits a single maximizer. Note that

$$F_{S(\mu)}(y) = \int_0^y \frac{p_\mu(\frac{1}{x})}{x^3} dx = 1 - \int_0^{1/y} x p_\mu(x) dx,$$

so that

$$\delta F_S(y) = -\int_0^{1/y} x \partial_x(\delta F)(x) dx.$$

Hence, δF_S has a single maximizer $x^S_\star$ if and only if δF has a single minimizer $x^\star$, such that $x^\star = \frac{1}{x^S_\star}$.

Let us now proceed to the construction of a right-monotone transport plan based on the symmetric left-monotone transport plan $(x^S_\star, L^S_d, L^S_u)$. Denote

$$R_d(x) := \frac{1}{L^S_u(1/x)}, \quad R_u(x) := \frac{1}{L^S_d(1/x)}, \quad x^\star := 1/x^S_\star. \tag{4.16}$$

Then, by definition of the left-monotone transference plan, we have

1. $0 < R_d(x) < x < R_u(x)$, for all $x < x^\star$,

2. R_d is increasing and R_u is decreasing.

At this point, it suffices to prove that (R_d, R_u) transports μ to ν. To this end, we let $g : \mathbb{R}^*_+ \to \mathbb{R}$ be any measurable bounded function. By definition of $(x^S_\star, L^S_d, L^S_u)$, we have

$$\begin{aligned}\int_0^\infty g(y) p_{S(\nu)}(y) dy = &\int_0^{x^S_\star} g(y) p_{S(\mu)}(y) dy + \int_{x^S_\star}^\infty g(L^S_u(x)) q^S_L(x) p_{S(\mu)}(x) dx \\ &+ \int_{x^S_\star}^\infty g(L^S_d(x))(1 - q^S_L(x)) p_{S(\mu)}(x) dx,\end{aligned}$$

where $q_L^S(x) = \dfrac{x - L_d^S(x)}{L_u^S(x) - L_d^S(x)}$. Performing a change of variable, we get

$$\begin{aligned}\int_0^\infty g(1/y)p_{S(\nu)}(1/y)\frac{dy}{y^2} &= \int_{x^\star}^\infty g(1/y)p_{S(\mu)}(1/y)\frac{dy}{y^2} + \int_0^{x^\star} g(L_u^S(1/x))q_L^S(1/x)p_{S(\mu)}(1/x)\frac{dx}{x^2}\\ &\quad + \int_0^{x^\star} g(L_d^S(1/x))(1-q_L^S(1/x))p_{S(\mu)}(1/x)\frac{dx}{x^2},\\ \int_0^\infty \widehat{g}(y)p_\nu(y)dy &= \int_{x^\star}^\infty \widehat{g}(y)p_\mu(y)dy + \int_0^{x^\star} \widehat{g}(R_u(x))q_R(x)p_\mu(x)dx\\ &\quad + \int_0^{x^\star} \widehat{g}(R_d(x))(1-q_R(x))p_\mu(x)dx,\end{aligned}$$

where $\widehat{g}(y) = yg(1/y)$, $p_{S(\mu)}(x) = p_\mu(1/x)/x^3$, $p_{S(\nu)}(x) = p_\nu(1/x)/x^3$ and $q_R(x) = \frac{x-R_d(x)}{R_u(x)-R_d(x)}$. Hence, $(x^\star, R_d, R_u)$ transports μ to ν. We have just proved the following

Theorem 1.4.2. *Assume that δF has a single maximizer. Then there exists a basic right-monotone transference plan, which is defined in (4.16).*

The left-monotone transference plan $(x_\star^S, L_d^S, L_u^S)$ verifies the equations

$$\begin{aligned}F_{S(\nu)}^{-1}\Big(F_{S(\mu)}(x) + \delta F_S(L_d^S(x))\Big) &= G_{S(\nu)}^{-1}\Big(G_{S(\mu)}(x) + \delta G_S(L_d^S(x))\Big)\\ F_{S(\nu)}(L_u^S(x)) &= F_{S(\mu)}(x) + \delta F_S(L_d^S(x)).\end{aligned}$$

We substitute the following expressions in the two equations above:

$$\begin{aligned}L_d^S(x) =& \frac{1}{R_u(1/x)},\ L_u^S(x) = \frac{1}{R_d(1/x)},\\ F_{S(\mu)}(y) =& 1 - G_\mu(1/y),\ F_{S(\nu)}(y) = 1 - G_\nu(1/y),\ \delta F_S(y) = -\delta G(1/y),\\ G_{S(\mu)}(y) =& 1 - F_\nu(1/y),\ G_{S(\nu)}(y) = 1 - F_\nu(1/y),\ \delta G_S(y) = -\delta F(1/y),\end{aligned}$$

and we get the following proposition.

Proposition 1.4.3. *The basic right-monotone transference plan $(x^\star, R_d, R_u)$ is characterized by the fact that $x^\star$ is the unique minimizer of δF, and by the two equations.*

$$F_\nu^{-1}\Big(F_\mu(x) + \delta F(R_u(x))\Big) = G_\nu^{-1}\Big(G_\mu(x) + \delta G(R_u(x))\Big) \tag{4.17}$$

$$G_\nu(R_d) = G_\mu(x) + (G_\nu(R_u(x)) - G_\mu(R_u(x))).$$

Moreover, the transition probabilities corresponding to the left and right transference plans are related by

$$q_L(x) = 1 - q_R^S(1/x)\frac{x}{L_d(x)}, \quad q_R(x) = \frac{x}{R_u(x)}(1 - q_L^S(1/x)).$$

Note that the equations (2.10) and (4.17) defining L_d and R_u are actually the same equation, but with different domains. This equation, where the unknown is denoted z, can be written as

$$F_\nu^{-1}(F_\mu(x) + (F_\nu(z) - F_\mu(z))) = G_\nu^{-1}(G_\mu(x) + (G_\nu(z) - G_\mu(z))), \text{ with } x > 0. \tag{4.18}$$

Denote by m and $\tilde{m}$, respectively, the maximizer and the minimizer of δF, with $m < \tilde{m}$. Thus we have

1. for $x > m$, $L_d(x)$ is the unique solution of (4.18) on the interval $(0, m)$;
2. for $x < \tilde{m}$, $R_u(x)$ is the unique solution of (4.18) on the interval $(\tilde{m}, \infty)$.

Hence, Equation (4.18) has three solutions in the interval $(m, \tilde{m})$, obviously x, $L_d(x)$ in the interval $(0, m)$ and $R_u(x)$ in the interval $(\tilde{m}, \infty)$. This may be important when going to numerically solving Equation (4.18) that gives the basic left and right-monotone transference plans.

Remark 1.4.4. *Note that by construction of the basic right-monotone transference plan, we have the symmetry relation*

$$\mathbb{S}\big(\mathbb{Q}_L(S(\mu), S(\nu))\big) = \mathbb{Q}_R(\mu, \nu), \tag{4.19}$$

i.e. the basic right-monotone transference plan is the symmetric of the left-monotone transference plan related to the symmetric of the marginals. One could use this equality to prove the optimality results in Henry-Labordère and Touzi [2013] when the marginals μ and ν have supports in $\mathbb{R}_+^*$. Indeed, recall first that using Lagrangian techniques, Henry-Labordère and Touzi [2013] show that $\mathbb{Q}_L(\mu, \nu)$ attains the upper bound (1.1) for payoffs verifying the generalized Spence-Mirrlees condition (1.6) $C_{xyy} > 0$. Now, assume that the payoff satisfies $C_{xyy} < 0$ instead. By item 3. of Proposition 1.3.3 and that $C_{xyy} < 0$ if and only if $\mathbb{S}^*(C)_{xyy} > 0$, we have

$$\begin{aligned} \overline{P}(\mu, \nu, C) &= \overline{P}(S(\mu), S(\nu), \mathbb{S}^*(C)) \\ &= \mathbb{E}^{\mathbb{Q}_L(S(\mu), S(\nu))}\left[\mathbb{S}^*(C)(X, Y)\right] \\ &= \mathbb{E}^{\mathbb{S}(\mathbb{Q}_L(S(\mu), S(\nu)))}\left[C(X, Y)\right]. \end{aligned}$$

Hence, $\overline{P}(\mu, \nu, C)$ is attained by $\mathbb{S}\left(\mathbb{Q}_L(S(\mu), S(\nu))\right)$, which is equal to $\mathbb{Q}_R(\mu, \nu)$, by the symmetry equation (4.19). One can prove in a similar way that if $C_{xyy} > 0$ (resp. $C_{xyy} < 0$), the lower bound (1.2) is attained by $\mathbb{Q}_R(\mu, \nu)$ (resp. $\mathbb{Q}_L(\mu, \nu)$).

Now, we end this part with the interesting remark that the measures induced by the left and right transference plans (L_d, L_u) and (R_d, R_u) are extremal points of $\mathcal{M}(\mu, \nu)$.

Proposition 1.4.5. *We denote by $\mathbb{Q}_L(\mu, \nu)$ and $\mathbb{Q}_R(\mu, \nu)$ the martingale measures in $\mathcal{M}(\mu, \nu)$ entailed by the two transference plans (L_d, L_u) and (R_d, R_u).*

The probability measures $\mathbb{Q}_L(\mu, \nu)$ and $\mathbb{Q}_R(\mu, \nu)$ are extremal points of $\mathcal{M}(\mu, \nu)$.

Proof. Suppose that there exist two probabilities Q_1 and Q_2 and a real number $0 \leq \alpha \leq 1$ such that $\mathbb{Q}_L(\mu, \nu) = \alpha Q_1 + (1 - \alpha) Q_2$. Since $\mathbb{Q}_L(\mu, \nu)\Big(Y = L_d(X) \text{ or } Y = L_u(X)\Big) = 1$, then $Q_i(Y = L_d(X) \text{ or } Y = L_u(X)) = 1$ for $i = 1, 2$. Hence, Q_1 and Q_2 are concentrated on the two graphs L_u, L_d. Since, the two measures $(Q_i)_{i=1,2}$ preserve the marginals μ and ν, they are characterized by their transition probabilities

$q_i(X) = Q_i\Big(Y = L_u(X)\big|X\Big)$. On the other hand, $(Q_i)_{i=1,2}$ are martingale measures, i.e. $\mathbb{E}^{Q_i}[Y|X] = X$, which implies that

$$q_i(x)L_u(x) + (1 - q_i(x))L_d(x) = x, \ \forall x > m.$$

Hence, for $x > m$, $q_i(x) = \frac{x - L_d(x)}{L_u(x) - L_d(x)}$, which is equal to $q_L(x)$, the transition probability of $\mathbb{Q}_L$. In conclusion, Q_1 and Q_2 are equal to $\mathbb{Q}_L$. We get similarly that $\mathbb{Q}_R$ is an extremal point of $\mathcal{M}(\mu, \nu)$. □

1.5 Symmetry: Hobson and Klimmek [2015] revisited

Another important result in the optimal bound price literature is the work of Hobson and Klimmek [2015] on the model-free lower bound price for an option paying $|Y - X|$. This type of option is called, in the literature, the Type II forward start straddle, with the strike $\alpha > 0$ and the payoff

$$C^{\alpha}_{II}(x, y) = |y - \alpha x|, \quad \forall x, y > 0, \tag{5.20}$$

while the Type I forward start straddle is given by

$$C^{\alpha}_{I}(x, y) = \left|\frac{y}{x} - \alpha\right|, \quad \forall x, y > 0, \tag{5.21}$$

c.f. Lucic [2003] and Jacquier and Roome [2012]. Hobson and Klimmek [2015] derive explicit expressions for the coupling which minimizes the price of the at-the-money (ATM) Type II forward starting straddle C^1_{II} and for the form of the dual strategy. Note that this payoff does not satisfy the generalized Spence-Mirrlees condition (1.6). Their main result is that an optimal martingale coupling for the forward starting straddle is concentrated on a three point transition $\{p(x), x, q(x)\}$ where p and q are two decreasing functions.

This result was obtained under a *dispersion assumption* [Hobson and Klimmek, 2015, Assumption 2.1] on the supports of the marginal laws: assume that the support of $(\mu - \nu)^+$ is contained in a finite interval E and the support of $(\nu - \mu)^+$ is contained in E^c. One important preliminary remark is the equivalence of the dispersion assumption 2.1 in Hobson and Klimmek [2015] and the Assumption 1.2.2 of a single maximizer of δF.

Lemma 1.5.1. *Let $\mu, \nu \in \mathcal{P}_1$ with $\mu \preccurlyeq \nu$. Then the Assumption 2.1 in Hobson and Klimmek [2015] is equivalent to the single maximizer Assumption 1.2.2.*

Proof. Let $\mu, \nu \in \mathcal{P}_1$ with $\mu \preccurlyeq \nu$. Observe that $\text{Supp}(\mu - \nu)^+ = \{x \geq 0, \ p_\mu(x) - p_\nu(x) > 0\}$.

Suppose that Assumption 2.1 in Hobson and Klimmek [2015] holds, i.e. there exist $a, b \geq 0$ such that

$$\forall x \in (a, b), \ p_\mu(x) - p_\nu(x) \geq 0$$
$$\forall x \in (a, b)^c, \ p_\mu(x) - p_\nu(x) \leq 0$$

Consequently, δF is decreasing on $[a,b]$ and increasing on $(0,a)$ and (b,∞). Hence it admits a global maximizer on a and a global minimizer on b. So the single maximizer Assumption 1.2.2.

Conversely, suppose that the single maximizer Assumption 1.2.2 holds. Then δF admits a global maximum in $m > 0$. Moreover, by the convex order of μ and ν, δF admits a global minimum at $\tilde{m} > m$. Hence, $\forall x \in (m,\tilde{m}),\ p_\mu(x) - p_\nu(x) > 0$. and $\forall x \in (m,\tilde{m})^c,\ p_\mu(x) - p_\nu(x) \leq 0$, and finally the dispersion Assumption 2.1 in Hobson and Klimmek [2015] is fulfilled. □

Starting from the optimal coupling found in Hobson and Klimmek [2015], we introduce the following transference plan definition. Note that contrary to the left and right-monotone transference plans which are concentrated on a two point band, the following definition have a 3-points structure.

Definition 1.5.2 (Basic 3-points band transference plan)**.** *A basic 3-points band transference plan is a tuple (a,b,p,q,l,u), where $0 < a < b$ are real numbers and*

i) $p : [a,b] \to [0,a]$ *and* $q : [a,b] \to [b,\infty]$ *are continuous decreasing functions.*

ii) $p(x) < x < q(x)$*, for all* $x \in (a,b)$*.*

iii) $l, u : (a,b) \to [0,1]$ *satisfy* $0 \leq l(x) + u(x) \leq 1$*.*

iv) $\mathcal{L}\mu = \nu$ *where the transition kernel* $\mathcal{L}$ *is defined by*

$$\mathcal{L}(x,dy) = \delta_x \mathbb{1}_{x\leq a} + (l(x)\delta_{p(x)} + u(x)\delta_{q(x)} + (1-l(x)-u(x))\delta_x)\mathbb{1}_{a<x<b} + \delta_x \mathbb{1}_{x\geq b}. \tag{5.22}$$

Using the same ideas as in Section 1.2, we write down necessary conditions for such a tuple to exist. Suppose there exists a tuple (a,b,p,q,l,u) as in Definition 1.5.2, with the additional conditions $p(a) = a, p(b) = 0$ and $\lim_{a+} q(x) = \infty, q(b) = b$.

The condition that $\mathcal{L}$ transports μ into ν implies that for any bounded measurable function g we have

$$\begin{aligned}
\int_0^a g(y)p_\nu(y)dy &= \int_a^b g(p(x))l(x)p_\mu(x)dx + \int_0^a g(y)p_\mu(y)dy,\\
\int_b^\infty g(y)p_\nu(y)dy &= \int_a^b g(q(x))u(x)p_\mu(x)dx + \int_b^\infty g(y)p_\mu(y)dy,\\
\int_a^b g(y)p_\nu(y)dy &= \int_a^b g(x)(1-l(x)-u(x))p_\mu(x)dx.
\end{aligned} \tag{5.23}$$

Suppose moreover that p and q are differentiable, then a change of variables gives

$$\begin{aligned}
\int_a^b g(p(x))p_\nu(p(x))p'(x)dx &= -\int_a^b g(p(x))l(x)p_\mu(x)dx + \int_a^b g(p(x))p_\mu(p(x))p'(x)dx,\\
\int_a^b g(q(x))p_\nu(q(x))q'(x)dx &= -\int_a^b g(q(x))u(x)p_\mu(x)dx + \int_a^b g(q(x))p_\mu(q(x))q'(x)dx,\\
\int_a^b g(x)p_\nu(x)dx &= \int_a^b g(x)(1-l(x)-u(x))p_\mu(x)dx.
\end{aligned}$$

On the other hand, the martingale condition gives

$$l(x)p(x) + u(x)q(x) + (1 - l(x) - u(x))x \quad = \quad x, \quad \forall x \in (a, b).$$

Consequently, we get for $x \in (a, b)$

$$\begin{aligned} p_\nu(p(x))p'(x) &= -l(x)p_\mu(x) + p_\mu(p(x))p'(x), \\ p_\nu(q(x))q'(x) &= -u(x)p_\mu(x) + p_\mu(q(x))q'(x), \\ l(x) + u(x) &= \frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)}, \\ l(x)(x - p(x)) &= u(x)(q(x) - x). \end{aligned} \tag{5.24}$$

Remark 1.5.3. *Interestingly, a (resp. b) is necessarily a global maximum (resp. minimum) of δF. Indeed, the condition $0 \leq l(x) + u(x) \leq 1$ implies that $\frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)} \geq 0$, hence $\delta F'(x) = p_\nu(x) - p_\mu(x) \leq 0$ for all $x \in (a, b)$. Consequently, δF is decreasing in (a,b). Moreover*

$$p'(x)\left(p_\mu(p(x)) - p_\nu(p(x))\right) = l(x)p_\mu(x)$$

and since p is decreasing, then $p_\mu(p(x)) - p_\nu(p(x)) \leq 0$. So $\delta F'(y) = p_\nu(y) - p_\mu(y) \geq 0$ for all $y \in (0, a)$. Similarly, we show that $\delta F'(y) = p_\nu(y) - p_\mu(y) \geq 0$ for all $y \in (b, \infty)$. In conclusion, δF is increasing on $(0, a)$, then decreases on (a, b) and finally increases on (b, ∞). As a corollary, a (resp. b) is a global maximum (resp. minimum) of δF.

Rearranging the terms in the equations (5.24), we recover the equations (3.3) and (3.4) in Hobson and Klimmek [2015] , $\forall x \in (a, b)$

$$\begin{aligned} p'(x) &= \frac{q(x) - x}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(p(x)) - p_\nu(p(x))}, \\ q'(x) &= \frac{x - p(x)}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(q(x)) - p_\nu(q(x))}, \\ u(x) &= \frac{x - p(x)}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)}, \\ l(x) &= \frac{q(x) - x}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)}. \end{aligned} \tag{5.25}$$

Combining the first two equations above, we get for all $x \in (a, b)$

$$\begin{aligned} p'(x)(p_\mu(p(x)) - p_\nu(p(x))) + q'(x)(p_\mu(q(x)) - p_\nu(q(x))) \quad &= \quad p_\mu(x) - p_\nu(x) \\ p(x)p'(x)(p_\mu(p(x)) - p_\nu(p(x))) + q(x)q'(x)(p_\mu(q(x)) - p_\nu(q(x))) \quad &= \quad x(p_\mu(x) - p_\nu(x)). \end{aligned}$$

Integrating gives us the following relation between p and q, for all $x \in (a, b)$,

$$\begin{aligned} \delta F(q(x)) + \delta F(p(x)) &= \delta F(x), \\ \delta G(q(x)) + \delta G(p(x)) &= \delta G(x). \end{aligned} \tag{5.26}$$

This equations correspond to (6.1) to (6.2) in Hobson and Klimmek [2015].

1.5.1 Sufficient conditions

Now, we prove the well-posedness of equations (5.26), under the single maximizer Assumption 1.2.2. Recall that Lemma 1.5.1 states that this is equivalent to Assumption 2.1 in Hobson and Klimmek [2015].

Lemma 1.5.4. *Suppose that the single maximizer Assumption 1.2.2 is verified, so δF has a single maximizer a and a single minimizer $b > a$. For a given $x \in (a, b)$, let $t_F^1(x)$ $t_F^2(x)$ be defined by:*

1. *$t_F^1(x) = 0$ if $\delta F(x) \leq 0$;*
2. *$t_F^1(x)$ solves $\delta F(t_F^1(x)) = \delta F(x)$ in $(0, a)$ otherwise,*

and

1. *$t_F^2(x) = a$ if $\delta F(x) - \delta F(b) \geq \delta F(a)$;*
2. *$t_F^2(x)$ solves $\delta F(y) = \delta F(x) - \delta F(b)$ in $(0, a)$ otherwise.*

Then the equations (5.26) admit a unique solution $p(x), q(x)$, such that $p(x) \in [t_F^1(x), t_F^2(x)]$, $p(x) \in [0, a]$ and $q(x) \in [b, \infty)$.

Proof. C.f. the proof in Appendix 1.D. □

Remark 1.5.5.
1. *Note that $t_F^1(x)$ and $t_F^2(x)$ are well defined, by the intermediate values theorem and the continuity of δF.*
2. *Recall that, by the convex ordering of μ and ν, the single maximizer Assumption 1.2.2 implies that δF has a single minimizer b, such that $a < b$ and also, $\delta F(a) > 0, \delta F(b) < 0$ and δF is increasing on $(0, a)$, decreasing on (a, b) then increasing (b, ∞).*

Now, we prove that this unique pair (p, q) has the following properties:

Proposition 1.5.6 (Properties of p and q)**.** *Let (p, q) be the unique pair in Lemma 1.5.4, then the following properties hold:*

1. *p and q are C^1 functions;*
2. *$p(x) < x < q(x)$, for all $x \in (a, b)$;*
3. *they are both decreasing;*
4. *They satisfies the following boundary conditions:*

$$\lim_{x \to a^+} p(x) = a, \quad \lim_{x \to b^-} p(x) = 0$$
$$\lim_{x \to a^+} q(x) = \infty, \quad \lim_{x \to b^-} q(x) = b.$$

Proof.
1. It follows from the implicit function theorem.
2. Obvious since for $x \in (a, b)$, $p(x) < a$ and $q(x) > b$.

3. Let $x \in (a,b)$, then $p(x)$ is the unique solution of

$$\delta G\Big[\delta F^{-1}\Big(\delta F(x) - \delta F(p(x))\Big)\Big] + \delta G(p(x)) - \delta G(x) = 0.$$

Differentiating this equation gives

$$p'(x)\delta F'(p(x))\left(p(x) - \delta F^{-1}\left(\delta F(x) - \delta F(p(x))\right)\right) = \delta F'(x)\left(x - \delta F^{-1}\left(\delta F(x) - \delta F(p(x))\right)\right). \tag{5.27}$$

Since $x \in (a,b)$ and $p(x) \in (0,a)$, then

$$\begin{aligned} \delta F'(p(x)) &> 0 \\ p(x) &< a < b < \delta F^{-1}\left(\delta F(x) - \delta F(p(x))\right) \\ \delta F'(x) &< 0 \\ x &< b < \delta F^{-1}\left(\delta F(x) - \delta F(p(x))\right). \end{aligned}$$

Hence, $p'(x) < 0$, so that p is decreasing.

On the other hand $q(x)$ is given by $q(x) = \delta F^{-1}\Big(\delta F(x) - \delta F(p(x))\Big)$. Hence

$$q'(x) = \frac{\delta F'(x) - p'(x)\delta F'(p(x))}{\delta F'(q(x))}.$$

Rewriting equation (5.27) gives $p'(x)\delta F'(p(x)) = \delta F'(x)\frac{x-q(x)}{p(x)-q(x)}$, consequently

$$q'(x) = \frac{\delta F'(x)}{\delta F'(q(x))}\frac{x-p(x)}{q(x)-p(x)}.$$

Since $x \in (a,b)$, $p(x) \in (0,a)$ and $q(x) \in (b,\infty)$, then

$$\delta F'(q(x)) > 0, \quad \delta F'(x) < 0, \quad \frac{x-p(x)}{q(x)-p(x)} > 0.$$

In conclusion $q'(x) < 0$ and q is also decreasing.

4. We have for $x \in (a,b)$

$$t_F^1(x) \le p(x) \le t_F^2(x).$$

Using the definition of $t_F^1(x)$ and $t_F^2(x)$ we obtain

$$\begin{aligned} \lim_{x\to a^+} t_F^1(x) = a, \ \lim_{x\to b^-} t_F^1(x) = a, \\ \lim_{x\to a^+} t_F^2(x) = 0, \ \lim_{x\to b^-} t_F^2(x) = 0. \end{aligned}$$

Consequently

$$\lim_{x\to a^+} p(x) = a, \ \lim_{x\to b^-} p(x) = 0.$$

On the other hand, q is defined by $q(x) = \delta F^{-1}\left(\delta F(x) - \delta F(p(x))\right)$, for $x \in (a,b)$.

Using the limits of p, we get

$$\lim_{x\to a^+} q(x) = \infty, \quad \lim_{x\to b^-} q(x) = b.$$

□

A direct consequence of Lemma 1.5.4 and Proposition 1.5.6 is the following existence and uniqueness theorem:

Theorem 1.5.7. *Assume that δF has a single maximizer m. Then there is a unique basic 3-points band transference plan (a, b, p, q, l, u) where a (resp. b) is the global maximizer (resp. minimizer) of δF, (p, q) are given by Equations (5.26) and the transition probabilities (l, u) are given by Equations (5.25).*

We conclude this section with a discussion on the extremality and symmetry properties of the Hobson-Klimmek measure denoted by $\mathbb{Q}_{HK}(\mu,\nu)$, which is the martingale measure in $\mathcal{M}(\mu,\nu)$ entailed by the pair (p, q) (c.f. the transition equation (5.22)). In particular, using change of numeraire techniques we will show that $\mathbb{Q}_{HK}(\mu,\nu)$ attains the lower bound price for the type I forward start straddle C_I^1. This result complement the result in Hobson and Klimmek [2015] about type II forward start straddle C_{II}^1.

Proposition 1.5.8. *The measure $\mathbb{Q}_{HK}(\mu,\nu)$ is an extremal point of $\mathcal{M}(\mu,\nu)$.*

Proof. Suppose that there exist two probabilities Q_1 and Q_2 and a real number $0 \leq \alpha \leq 1$ such that $\mathbb{Q}_{HK}(\mu,\nu) = \alpha Q_1 + (1-\alpha)Q_2$. Since $\mathbb{Q}_{HK}(\mu,\nu)\Big(Y = p(X),\ Y = q(X) \text{ or } Y = X\Big) = 1$, then $Q_i\Big(Y = p(X),\ Y = q(X) \text{ or } Y = X\Big) = 1$ for $i = 1, 2$. Hence, Q_1 and Q_2 are concentrated on the three band graph $\{p(x), x, q(x)\}$. Since, the two measures $(Q_i)_{i=1,2}$ preserve the marginals μ and ν, they are characterized by their transition probabilities l_i and u_i:

$$u_i(X) = Q_i\Big(Y = q(X)\big|X\Big),\ l_i(X) = Q_i\Big(Y = p(X)\big|X\Big).$$

The fact that $(Q_i)_{i=1,2}$ are martingale measures and has marginals μ and ν implies that, for any bounded measurable function g

$$\begin{aligned} \int_a^b g(x)(1 - l_i(x) - u_i(x))p_\mu(x)dx &= \int_a^b g(x)p_\nu(x)dx \\ l_i(x)p(x) + u_i(x)q(x) + (1 - l_i(x) - u_i(x))x &= x, \quad \forall x \in (a, b). \end{aligned}$$

This has a unique solution given by

$$\begin{aligned} u_i(x) &= \frac{x - p(x)}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)} \\ l_i(x) &= \frac{q(x) - x}{q(x) - p(x)} \frac{p_\mu(x) - p_\nu(x)}{p_\mu(x)}, \quad \forall x \in (a, b), \end{aligned}$$

which are equal to the transition probabilities of $\mathbb{Q}_{HK}(\mu,\nu)$. In conclusion, Q_1 and Q_2 are equal to $\mathbb{Q}_{HK}(\mu,\nu)$, which is then an extremal point of $\mathcal{M}(\mu,\nu)$. □

Proposition 1.5.9. *The martingale measure $\mathbb{Q}_{HK}(\mu,\nu)$ verifies the symmetry relation*

$$\mathbb{S}\big(\mathbb{Q}_{HK}(S(\mu),S(\nu))\big)=\mathbb{Q}_{HK}(\mu,\nu)$$

where the symmetry operator $\mathbb{S}$ has been defined in Proposition 1.3.3.

Proof. Let the pair (p^S,q^S) define the measure $\mathbb{Q}_{HK}(S(\mu),S(\nu))$.

Then, a simple computation shows that the symmetric of $\mathbb{Q}_{HK}(S(\mu),S(\nu))$ is concentrated on $\{1/p^S(1/x),1/x,1/q^S(1/x)\}$. Let use write the equations satisfied by this three-band graph.

First, recall the symmetry relations

$$\delta F_S(y)=-\delta G(1/y),\quad \delta G_S(y)=-\delta F(1/y).$$

By definition, (p^S,q^S) is characterized by the two equations

$$\begin{aligned}\delta F^S(q^S(x))+\delta F^S(p^S(x)) &= \delta F^S(x),\\ \delta G^S(q^S(x))+\delta G^S(p^S(x)) &= \delta G^S(x).\end{aligned}$$

Hence

$$\begin{aligned}\delta F(1/q^S(1/x))+\delta F(1/p^S(1/x)) &= \delta F(x),\\ \delta G(1/q^S(1/x))+\delta G(1/p^S(1/x)) &= \delta G(x).\end{aligned}$$

These two equations are the same as the ones that characterize the pair (p,q), and by uniqueness we get our desired result. □

Hobson and Klimmek [2015] prove that lower bound price of the type II forward start straddle paying $C^1_{II}(x,y)=|y-x|$ is attained by $\mathbb{Q}_{HK}(\mu,\nu)$, i.e.

$$\underline{P}(\mu,\nu,C^1_{II}):=\inf_{Q\in\mathcal{M}(\mu,\nu)}\mathbb{E}^Q\left[|Y-X|\right]=\mathbb{E}^{\mathbb{Q}_{HK}(\mu,\nu)}\Big[|Y-X|\Big] \tag{5.28}$$

Interestingly, there is a symmetry relation between Type I and Type II forward start straddles, which is given by

$$\mathbb{S}^*(C^\alpha_{II})(X,Y)=Y\left|\frac{1}{Y}-\frac{\alpha}{X}\right|=\alpha\left|\frac{Y}{X}-\frac{1}{\alpha}\right|=\alpha C_I^{\frac{1}{\alpha}}. \tag{5.29}$$

In particular, the ATM straddles are related by $\mathbb{S}^*(C^1_{II})(X,Y)=C^1_I(X,Y)$. This relation can be exploited to obtain the following proposition, that concludes this section. Its proof is straightforward.

Proposition 1.5.10. *The lower bound price of the Type I forward start is also attained by a basic 3-points band transference plan, i.e.*

$$\underline{P}(\mu,\nu,C^1_I):=\inf_{Q\in\mathcal{M}(\mu,\nu)}\mathbb{E}^Q\left[\left|\frac{Y}{X}-1\right|\right]=E^{\mathbb{Q}_{HK}(\mu,\nu)}\left[C^1_I\right]. \tag{5.30}$$

Proof.

Using point 3. of Proposition 1.3.3 and Equation 5.29, we have

$$\begin{aligned}\underline{P}(\mu,\nu,C_I^1) &= \underline{P}(S(\mu),S(\nu),\mathbb{S}^*(C_I^1)) \\ &= \underline{P}(S(\mu),S(\nu),C_{II}^1) \\ &= E^{\mathbb{Q}_{HK}(S(\mu),S(\nu))}\left[C_{II}^1\right] \\ &= E^{\mathbb{Q}_{HK}(\mu,\nu)}\left[C_I^1\right].\end{aligned}$$

□

1.6 Two new transference plans

In this Section, we characterize two new transference plans and some of their properties. For the rest of the article, we will work under the classical assumption that δF has a single maximizer m. Under this assumption, we show the following proposition.

Proposition 1.6.1. *Suppose that $\mu \preccurlyeq \nu$ and that δF has a single maximizer m. Then*

1. *δF admits a single zero, denoted $z_F^\star$, δF is positive on $(0, z_F^\star)$ and negative on $(z_F^\star, \infty)$.*
2. *For all $x > 0$, we have $0 < F_\nu(z_F^\star) - \delta F(x) < 1$.*
3. *δG admits a single zero, denoted $z_G^\star$, δG is positive on $(0, z_G^\star)$ and negative on $(z_G^\star, \infty)$.*
4. *For all $x > 0$, we have $0 < G_\nu(z_G^\star) - \delta G(x) < 1$.*

Proof. Item 1) was proved in the previous Sections.

For Item 2), let $x > 0$. If $x > z_F^\star$, then, by Item 1), $F_\mu(x) - F_\nu(x) > 0$; So $F_\mu(x) - F_\nu(x) + F_\nu(z_F^\star) > 0$. Moreover, since F_ν is increasing, $F_\nu(z_F^\star) < F_\nu(x)$. So $F_\mu(x) - F_\nu(x) + F_\nu(z_F^\star) < F_\mu(x) < 1$.

Otherwise, if $x < z_F^\star$, then $F_\nu(x) < F_\nu(z_F^\star)$ so that again $F_\mu(x) - F_\nu(x) + F_\nu(z_F^\star) > 0$. Moreover, by Item 1), $F_\mu(x) - F_\nu(x) < 0$. So $F_\mu(x) - F_\nu(x) + F_\nu(z_F^\star) < F_\nu(z_F^\star) < 1$.

Items 3) and 4) can be proved similarly, because δG verifies the same properties as δF, i.e. δG is increasing and bounded by 0 and 1. □

We introduce the following two special transference plans.

Definition 1.6.2 (F-Increasing transference plan)**.** *We say that a pair of functions (f, g) is a F-increasing transference plan if the following conditions are fulfilled*

1. *f and g are increasing.*
2. *$f(x) < x < g(x)$ for all $x > 0$.*
3. *$f(0) = 0$, $\lim_\infty f(x) = z_F^\star$ and $g(0) = z_F^\star$.*
4. *$\mathcal{L}\mu = \nu$, where the transition kernel $\mathcal{L}$ is defined by*

$$\mathcal{L}(x, dy) = q_F(x)\delta_{f(x)} + (1 - q_F(x))\delta_{g(x)},$$

where $q_F(x) := \frac{g(x)-x}{g(x)-f(x)}$.

Definition 1.6.3 (G-Increasing transference plan). *We say that a pair of functions $(\tilde{f}, \tilde{g})$ is a G-increasing transference plan if the following conditions are fulfilled*

1. *$\tilde{f}$ and $\tilde{g}$ are increasing.*
2. *$\tilde{f}(x) < x < \tilde{g}(x)$ for all $x > 0$.*
3. *$\tilde{f}(0) = 0$, $\lim_\infty \tilde{f}(x) = z_G^\star$ and $\tilde{g}(0) = z_G^\star$.*
4. *$\mathcal{L}\mu = \nu$, where the transition kernel $\mathcal{L}$ is defined by*

$$\mathcal{L}(x, dy) = q_G(x)\delta_{\tilde{f}(x)} + (1 - q_G(x))\delta_{\tilde{g}(x)},$$

where $q_G(x) := \frac{\tilde{g}(x)-x}{\tilde{g}(x)-\tilde{f}(x)}$.

In the following, we will show the existence and uniqueness of the F-increasing transference plan. The same results for the G-increasing transference plan follow similarly. Note that condition 1) and 3) of the two definitions imply that

1) $f(x) < z_F^\star$, $g(x) > z_F^\star$, $\tilde{f}(x) < z_G^\star$, $\tilde{g}(x) > z_G^\star$ for all $x > 0$.

2) $\lim_\infty g(x) = \infty$, $\lim_\infty \tilde{g}(x) = \infty$.

Following the same steps as in Section 1.2, we start by writing the necessary conditions for the existence of such a pair (f, g). The fact that $\mathcal{L}$ transfers μ to ν gives that, for every bounded measurable function h

$$\begin{aligned}
\int_0^{z^\star} h(y)p_\nu(y)dy &= \int_0^\infty h(f(x))q_F(x)p_\mu(x)dx \\
\int_{z^\star}^\infty h(y)p_\nu(y)dy &= \int_0^\infty h(g(x))(1 - q_F(x))p_\mu(x)dx
\end{aligned}$$

Assume moreover that f and g are differentiable, then the two equations above translate into

$$\begin{aligned}
p_\nu(f(x))f'(x) &= q_F(x)p_\mu(x) \\
p_\nu(g(x))g'(x) &= (1 - q_F(x))p_\mu(x), \quad \forall x > 0.
\end{aligned}$$

We get then the following necessary conditions

Proposition 1.6.4. *Let (f, g) be an F-increasing transference plan and suppose moreover that f and g are differentiable. Then, $\forall x > 0$*

$$\begin{aligned}
F_\nu(g(x)) + F_\nu(f(x)) - F_\nu(z_F^\star) &= F_\mu(x) \\
G_\nu(g(x)) + G_\nu(f(x)) - G_\nu(z_F^\star) &= G_\mu(x)
\end{aligned}$$

Having obtained these equations, we show now their well-posedness for every $x > 0$. We introduce the bounds $t_F^1(x), t_F^2(x)$ for every $x > 0$ by

1) If $F_\nu(z_F^\star) + F_\mu(x) \leq 1$, then $t_F^1(x) = 0$..

2) Else $t_F^1(x)$ is the solution of $F_\nu(t_F^1(x)) = F_\mu(x) + F_\nu(z_F^\star) - 1$ on $(0, z_F^\star)$, i.e. $t_F^1(x) = F_\nu^{-1}(F_\mu(x) + F_\nu(z_F^\star) - 1)$.

and

1) If $x \geq z_F^\star$, then $t_F^2(x) = z_F^\star$.

2) Else, $t_F^2(x)$ is the solution of $F_\nu(t_F^2(x)) = F_\mu(x)$ on $(0, x)$, i.e. $t_F^2(x) = F_\nu^{-1}(F_\mu(x))$.

Remark 1.6.5.
1) If $x < z_F^\star$, then $\delta F(x) = F_\nu(x) - F_\mu(x) > 0$. So $F_\nu(x) > F_\mu(x)$ and $F_\nu(0) = 0$, so the intermediate values theorem ensure the existence of $t_F^2(x)$. Moreover, since F_ν is increasing, we have also

$$\forall y \in (0, t_F^2(x)), \quad F_\nu(y) < F_\mu(x).$$

2) By definition of the interval $(t_F^1(x), t_F^2(x))$, we have

$$F_\nu(z_F^\star) < F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star) < 1, \quad \forall y \in (t_F^1(x), t_F^2(x)), \tag{6.31}$$

so that $F_\nu^{-1}\left(F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star)\right)$ is well-defined and is greater that $z_F^\star$.
3) We have $t_F^2(x) \leq x$ for all $x > 0$.

Proposition 1.6.6. *For every $x > 0$, the equation*

$$G_\nu\left[F_\nu^{-1}\left(F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star)\right)\right] + G_\nu(y) - G_\nu(z_F^\star) = G_\mu(x) \tag{6.32}$$

admits a unique solution on the interval $(t_F^1(x), t_F^2(x))$.

Proof. For every $x > 0$, we introduce the function ϕ, defined on $(t_F^1(x), t_F^2(x))$ by

$$\phi_x(y) = G_\nu\left[F_\nu^{-1}\left(F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star)\right)\right] + G_\nu(y) - G_\nu(z_F^\star) - G_\mu(x), \quad \forall y \in (t_F^1(x), t_F^2(x)).$$

We show that ϕ_x has a unique zero on $(t_F^1(x), t_F^2(x))$. For this goal we will prove that

1. ϕ_x is decreasing.
2. $\phi_x(t_F^1(x)) > 0$.
3. $\phi_x(t_F^2(x)) < 0$.

We have, for $x > 0$ and $y \in (t_F^1(x), t_F^2(x)) \subset (0, z_F^\star)$.

$$\phi_x'(y) = F_\nu'(y)\left[y - F_\nu^{-1}\left(F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star)\right)\right].$$

By Remark 1.6.5, we have $F_\nu(y) - F_\mu(x) + F_\nu(y) - F_\nu(z_F^\star) < F_\nu(y) - F_\nu(z_F^\star)$. The right-hand side is negative, since $y < z_F^\star$. Hence $F_\nu(y) - F_\mu(x) + F_\nu(y) - F_\nu(z_F^\star) < 0$, which implies that $y - F_\nu^{-1}\left(F_\mu(x) - F_\nu(y) + F_\nu(z_F^\star)\right) < 0$. Consequently, $\phi_x'(y) < 0$,

i.e. ϕ_x is decreasing on $(t_F^1(x), t_F^2(x))$, so Item 1) is established.

We denote by φ_1 the function defined by $\varphi_1(x) = \phi_x(t_F^1(x)), \forall x > 0$. Let $x > 0$. If $x \in (0, F_\nu^{-1}(1 - F_\nu(z_F^\star)))$, then $F_\nu(z_F^\star) + F_\mu(x) \leq 1$. So, $t_F^1(x) = 0$ and $\varphi_1(x) = \phi_x(0) = G_\nu \circ F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star)\right) - G_\mu(x) - G_\nu(z_F^\star)$. We have

$$\varphi_1'(x) = F_\mu'(x)\left[F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star)\right) - x\right].$$

By item 2) of Proposition 1.6.1, we have $F_\mu(x)+F_\nu(z_F^\star) > F_\nu(x)$, so $F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star)\right) > x$. Hence $\varphi_1'(x) > 0$ i.e. φ_1 is increasing on the interval $(0, F_\nu^{-1}(1-F_\nu(z_F^\star)))$. Moreover, $\varphi_1(0) = 0$, so that $\varphi_1 > 0$ on $(0, F_\nu^{-1}(1 - F_\nu(z_F^\star)))$.

Otherwise, if $x > F_\nu^{-1}(1-F_\nu(z_F^\star))$, i.e. $F_\nu(z_F^\star)+F_\mu(x) > 1$, then $t_F^1(x) = F_\nu^{-1}(F_\nu(z_F^\star)+ F_\mu(x) - 1)$ and $\varphi_1(x) = 1 - G_\nu(z_F^\star) - G_\mu(x) + G_\nu \circ F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star) - 1\right)$.

We have

$$\varphi_1'(x) = F_\mu'(x)\left[F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star) - 1\right) - x\right].$$

By item 2) of Proposition 1.6.1, we have $F_\mu(x)+F_\nu(z_F^\star)-1 < F_\nu(x)$, so $F_\nu^{-1}\left(F_\mu(x) + F_\nu(z_F^\star) - 1\right) < x$. Hence $\varphi_1'(x) < 0$ i.e. φ_1 is decreasing on the interval $(F_\nu^{-1}(1 - F_\nu(z_F^\star)), \infty)$. Moreover, $\lim_\infty \varphi_1(x) = 0$, so that $\varphi_1 > 0$ on $(F_\nu^{-1}(1 - F_\nu(z_F^\star)), \infty)$. Thus $\varphi_1(x)$ is positive for all $x > 0$, which yields Item 2): $\phi_x(t_F^1(x)) < 0$.

Finally, we establish Item 3). A simple computation yields that for all $x > 0$, $\phi_x(t_F^2(x)) = G_\nu \circ F_\nu^{-1}\left(F_\mu(x)\right) - G_\mu(x)$. We denote this function by φ_2, i.e. $\varphi_2(x) = G_\nu \circ F_\nu^{-1}\left(F_\mu(x)\right) - G_\mu(x), \quad \forall x > 0$. Then

$$\varphi_2'(x) = F_\mu'(x)\left[F_\nu^{-1}\left(F_\mu(x)\right) - x\right], \quad \forall x > 0.$$

Hence, using Proposition 1.6.1, we get that φ_2 is decreasing on $(0, z_F^\star)$ and increasing on $(z_F^\star, \infty)$. Moreover, $\varphi_2(0) = 0$ and $\lim_\infty \varphi_2(x) = 0$. Thus $\varphi_2(x)$ is negative for all $x > 0$, which yields point 3): $\phi_x(t_F^2(x)) < 0$.

Conclusion: we have that ϕ_x is decreasing, $\phi_x(t_F^1(x)) > 0$ and $\phi_x(t_F^2(x)) < 0$. Consequently, for every $x > 0$, Equation 6.32 admits a unique solution on the interval $(t_F^1(x), t_F^2(x))$.

□

Now, we prove the existence of the F-increasing transference plan in the following proposition.

Proposition 1.6.7. *We denote by $f : (0, \infty) \to \mathbb{R}$ the function such that, for every $x > 0$, $f(x)$ is the unique solution of Equation 6.32 given in Proposition 1.6.6.*

We define the function $g : (0, \infty) \to \mathbb{R}$ by $g(x) = F_\nu^{-1}\left(F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star)\right), \quad \forall x > 0$.

Then the pair (f, g) is an F-increasing transference plan. Moreover, f and g are differentiable.

Proof. First, note that by Inequality 6.31, g is well-defined. By implicit theorem, we have that f is differentiable, and so g is also differentiable. By definition, f and g

verify the system of equations

$$\begin{aligned} F_\nu(g(x)) + F_\nu(f(x)) - F_\nu(z_F^\star) &= F_\mu(x), \\ G_\nu(g(x)) + G_\nu(f(x)) - G_\nu(z_F^\star) &= G_\mu(x). \end{aligned}$$

This ensures that Condition 4) of the F-increasing transference plan definition 1.6.2 is fulfilled. It remains to show items 1,2,3) of this definition. First, we show that $f(x) < x < g(x), \quad \forall x > 0$.

Let $x > 0$. By definition, we have $f(x) < t_F^2(x)$, and item 3) of Remark 1.6.5 states that $t_F^2(x) \leq x$, so that $f(x) < x$. If $x \leq z_F^\star$, then by Inequality 6.31, we have $F_\nu^{-1}\left(F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star)\right) \geq z_F^\star > x$. Hence $g(x) > x$. Now, if $x > z_F^\star$, then by Item 1) of Proposition 1.6.1, $F_\mu(x) > F\nu(x)$. Moreover, $F_\nu(z_F^\star) - F_\nu(f(x)) > 0$ because $f(x) < z_F^\star$. So $F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star) > F_\nu(x)$, i.e. $g(x) > x$. Consequently, Item 2) of Definition 1.6.2 is verified.

By definition of the two bounds t_F^1 and t_F^2 we get directly $t_F^1(0) = t_F^2(0) = 0$ and $\lim_\infty t_F^1(x) = \lim_\infty t_F^2(x) = z_F^\star$. Hence, $f(0) = 0$, $\lim_\infty f(x) = z_F^\star$, and $g(0) = 0$.

It remains to show that f and g are increasing. We know that f verify the equation, for all $x > 0$

$$G_\nu\left[F_\nu^{-1}\left(F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star)\right)\right] + G_\nu(f(x)) - G_\nu(z_F^\star) = G_\mu(x).$$

Differentiating this equation, we get

$$\begin{aligned} F_\mu'(x)\{F_\nu^{-1}\left[F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star)\right] - x\} &= f'(x)F_\nu'(f(x)) \times \\ &\quad \{F_\nu^{-1}\left[F_\mu(x) - F_\nu(f(x)) + F_\nu(z_F^\star)\right] - f(x)\} \end{aligned}$$

which is equivalent to $F_\mu'(x)\{g(x) - x\} = f'(x)F_\nu'(f(x))\{g(x) - f(x)\}$. Since, $f(x) < x < g(x)$, then $f'(x) > 0$ i.e. f is increasing. Similarly, we get $F_\mu'(x)\{x - f(x)\} = g'(x)F_\nu'(g(x))\{g(x) - f(x)\}$ which implies $g'(x) > 0$ i.e. g is increasing. □

Remark 1.6.8. *Using the same methodology we prove the existence and uniqueness of the G-increasing transference plan.*

We denote by $\mathbb{Q}_F(\mu,\nu)$ (resp. $\mathbb{Q}_G(\mu,\nu)$) the martingale measure entailed by the F-increasing (resp. G-increasing) transference plan given by the marginals (μ,ν). More precisely,

$$\mathbb{Q}_F(\mu,\nu)(dx,dy) = \mu(dx)\left[q_F(x)\delta_{f(x)}(dy) + (1 - q_F(x))\delta_{g(x)}(dy)\right], \tag{6.33}$$

where we recall that $q_F(x) := \frac{g(x)-x}{g(x)-f(x)}$.

$$\mathbb{Q}_G(\mu,\nu)(dx,dy) = \mu(dx)\left[q_G(x)\delta_{\tilde{f}(x)}(dy) + (1 - q_G(x))\delta_{\tilde{g}(x)}(dy)\right], \tag{6.34}$$

where we recall that $q_G(x) := \frac{\tilde{g}(x)-x}{\tilde{g}(x)-\tilde{f}(x)}$.

We have the following symmetry property

Proposition 1.6.9. *$\mathbb{Q}_F(\mu,\nu)$ and $\mathbb{Q}_G(\mu,\nu)$ verify*

$$\mathbb{S}(\mathbb{Q}_F(\mu,\nu)) = \mathbb{Q}_G(S(\mu),S(\nu))$$

Proof. Let ϕ be a bounded measurable function on $\mathbb{R}_+^\star$. We have

$$\begin{aligned}\mathbb{E}^{\mathbb{S}(\mathbb{Q}_F(\mu,\nu))}\left[\phi(X,Y)\right] &= \mathbb{E}^{\mathbb{Q}_F(\mu,\nu)}\left[Y\phi(1/X,1/Y)\right]\\ &= \mathbb{E}^{\mu}\left[\frac{g(X)-X}{g(X)-f(X)}f(X)\phi(\frac{1}{X},\frac{1}{f(X)}) + \frac{X-f(X)}{g(X)-g(X)}f(X)\phi(\frac{1}{X},\frac{1}{g(X)})\right]\\ &= \mathbb{E}^{S(\mu)}\left[\frac{X-\tilde{f}(X)}{\tilde{g}(X)-\tilde{f}(X)}\phi(X,\tilde{g}(X)) + \frac{\tilde{g}(X)-X}{\tilde{g}(X)-\tilde{f}(X)}\phi(X,\tilde{f}(X))\right],\end{aligned}$$

where $\tilde{g}$ and $\tilde{f}$ are defined by $\tilde{g}(x) = 1/f(1/x)$ and $\tilde{f}(x) = 1/g(1/x)$.

By definition, f and g verify, for $x>0$

$$\begin{aligned}F_\nu(g(x)) + F_\nu(f(x)) - F_\nu(z_F^\star) &= F_\mu(x)\\ G_\nu(g(x)) + G_\nu(f(x)) - G_\nu(z_F^\star) &= G_\mu(x)\end{aligned}$$

Using the fact that, for every $\tau\in\mathcal{P}_1$, $F_{S(\tau)}(x) = 1 - G_\tau(1/x)$, we get

$$\begin{aligned}F_{S(\nu)}(\tilde{f}(x)) + F_{S(\nu)}(\tilde{g}(x)) - F_{S(\nu)}(1/z_F^\star) &= F_{S(\mu)}(x)\\ G_{S(\nu)}(\tilde{f}(x)) + G_{S(\nu)}(\tilde{g}(x)) - G_{S(\nu)}(1/z_F^\star) &= G_{S(\mu)}(x)\end{aligned}$$

and, obviously

1. $\tilde{g}$ and $\tilde{f}$ are increasing.
2. $\tilde{f}(x) < x < \tilde{g}(x)$ for all $x>0$.
3. $\tilde{f}(0) = 0$, $\lim_\infty \tilde{f}(x) = \frac{1}{z_F^\star}$ and $\tilde{g}(0) = \frac{1}{z_F^\star}$.

Note that since $\delta F(z_F^\star) = 0$ and $\delta G_S(x) = \delta F(1/x)$, then $z_G^\star := 1/z_F^\star$ is a zero of δG_S. Hence $(\tilde{f},\tilde{g})$ is the G-increasing transference plan corresponding to the marginals $(S(\mu),S(\nu))$, so that

$$\mathbb{E}^{\mathbb{S}(\mathbb{Q}_F(\mu,\nu))}\left[\phi(X,Y)\right] = \mathbb{E}^{\mathbb{Q}_G(S(\mu),S(\nu))}\left[\phi(X,Y)\right],$$

i.e. $\mathbb{S}(\mathbb{Q}_F(\mu,\nu)) = \mathbb{Q}_G(S(\mu),S(\nu))$. □

1.6.1 What are the payoffs for which this transference plan is optimal ?

Let u be a C^2 function, we look for conditions such that the F-increasing transference plan realizes its maximum. For this goal, we look for a triplet of functions (α,β,θ) such that the Lagrangian function $L^{\alpha,\beta,\theta}$ defined by

$$L^{\alpha,\beta,\theta}(x,y) = u(x,y) - \alpha(y) - \beta(x) - \theta(x)(x-y), \quad \forall x,y>0$$

verifies the equations $L^{\alpha,\beta,\theta}(x,f(x)) = 0$, $L^{\alpha,\beta,\theta}(x,g(x)) = 0$ and $L^{\alpha,\beta,\theta}(x,y)\leq 0$ for all $x,y>0$. Hence, we get the four equations

$$L^{\alpha,\beta,\theta}(x,f(x)) = L^{\alpha,\beta,\theta}(x,g(x)) = 0$$
$$L_y^{\alpha,\beta,\theta}(x,f(x)) = L_y^{\alpha,\beta,\theta}(x,g(x)) = 0.$$

They lead to the following system

$$\begin{aligned} u(x,f(x)) - \alpha(f(x)) - \beta(x) - \theta(x)(x-f(x)) &= 0 \\ u(x,g(x)) - \alpha(g(x)) - \beta(x) - \theta(x)(x-g(x)) &= 0 \\ u_y(x,f(x)) - \alpha'(f(x)) + \theta(x) &= 0 \\ u_y(x,g(x)) - \alpha'(g(x)) + \theta(x) &= 0. \end{aligned}$$

Differentiating the first two equations, and combining them with the two others, we get, for all $x > 0$

$$\begin{aligned} \theta'(x) &= -\frac{u_x(x,g(x)) - u_x(x,f(x))}{g(x)-f(x)}, \quad \forall x > 0 \\ \alpha'(y) &= u_y(f^{-1}(y),y) + \theta(f^{-1}(y)), \quad \forall y \in (0,z^\star) \\ \alpha'(y) &= u_y(g^{-1}(y),y) + \theta(g^{-1}(y)), \quad \forall y \in (z^\star,\infty) \\ \beta(x) &= u(x,f(x)) - \alpha(f(x)) - \theta(x)(x-f(x)) = u(x,g(x)) - \alpha(g(x)) - \theta(x)(x-g(x)), \quad \forall x > 0. \end{aligned}$$

Hence

$$\begin{aligned} L^{\alpha,\beta,\theta}(x,y) &= [u(x,y) - u(x,f(x))] - [\alpha(y) - \alpha(f(x))] - \theta(x)(f(x)-y) \\ &= [u(x,y) - u(x,g(x))] - [\alpha(y) - \alpha(g(x))] - \theta(x)(g(x)-y), \quad \forall x,y > 0 \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} L^{\alpha,\beta,\theta}(x,y) &= \int_y^{f(x)} \left[u_y(f^{-1}(t),t) - u_y(x,t) + \gamma(f^{-1}(t))(y-t)f^{-1}(t)'\right] dt, \ \forall y < f(x) \\ L^{\alpha,\beta,\theta}(x,y) &= -\int_{f(x)}^{y} \left[u_y(f^{-1}(t),t) - u_y(x,t) + \gamma(f^{-1}(t))(y-t)f^{-1}(t)'\right] dt, \ \forall f(x) < y < z^\star \\ L^{\alpha,\beta,\theta}(x,y) &= \int_y^{g(x)} \left[u_y(g^{-1}(t),t) - u_y(x,t) + \gamma(g^{-1}(t))(y-t)g^{-1}(t)'\right] dt, \ \forall z^\star < y < g(x) \\ L^{\alpha,\beta,\theta}(x,y) &= -\int_{g(x)}^{y} \left[u_y(g^{-1}(t),t) - u_y(x,t) + \gamma(g^{-1}(t))(y-t)g^{-1}(t)'\right] dt, \ \forall y > g(x). \end{aligned}$$

where $\gamma(x) = -\dfrac{u_x(x,g(x)) - u_x(x,f(x))}{g(x)-f(x)}$.

Proposition 1.6.10. *Let u be a C^2 function such that $L^{\alpha,\beta,\theta}(x,y) \leq 0, \forall x,y > 0$, where $L^{\alpha,\beta,\theta}$ is defined above, then $\mathbb{Q}_F(\mu,\nu)$ is optimal for u.*

1.7 Applications

The symmetric case denotes the fact that $S(\mu) = \mu$ and $S(\nu) = \nu$, we will say then that μ and ν are symmetric. Note that the use of the word 'symmetry' in this context comes from the fact that the corresponding volatility smiles at each maturity are symmetric in log-forward moneyness. Symmetric models have been further studied e.g. by Carr and Lee [2009] and Tehranchi [2009]. In Carr and Lee [2009], this concept is called put-call symmetry (PCS). They give many examples of symmetric models, c.f. [Carr and Lee, 2009, Section 3 and 4].

The stochastic volatility models with zero correlation between the volatility and the spot is a classical example of symmetric model. If μ and ν are induced by a stochastic volatility model of the type

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t^1,\ S_0 = 1 \\ dV_t &= \alpha(t,V_t)dt + \beta(t,V_t)dW_t^2 \end{aligned}$$

where W^1 and W^2 are two independent Brownian motions, then a simple application of Girsanov's theorem yields $S(\mu) = \mu$ and $S(\nu) = \nu$ (c.f. [Renault and Touzi, 1996, Proposition 3.1]). This includes as a special case the Black-Scholes model.

Remark 1.7.1. *The stochastic volatility model above verifies actually a seemingly "stronger" version of symmetry. By Girsanov's theorem, we know that there exists a martingale probability Q, which is symmetric, i.e.*

$$\mathbb{S}(Q) = Q.$$

Such a "stronger" notion of symmetry is actually equivalent to the weaker symmetry of the marginals μ and ν. To see this, let us consider an element P in $\mathcal{M}(\mu,\nu)$ and let us define the probability measure

$$Q := \frac{P + \mathbb{S}(P)}{2}.$$

Then, Q is clearly an element of $\mathcal{M}(\mu,\nu)$ and it is a symmetric measure, i.e. $\mathbb{S}(Q) = Q$.

Assumption 1.7.2. *We suppose in the rest of this section that the marginals μ and ν are symmetric.*

The symmetric models have some additional properties. For example:

Proposition 1.7.3. *If δF has a single maximizer m, then its unique minimizer $\tilde{m}$ satisfies $\tilde{m} > m$ and is given by $\tilde{m} = \frac{1}{m}$. As a consequence $m < 1$.*

Proof. Let m be the single maximizer of $\delta F_{\mu,\nu}$ and $\tilde{m}$ its minimizer, the existence of which is ensured by the convex order of μ and ν.

We know from the previous section (c.f. the equation (4.16)) that the minimizer $\tilde{m}_S$ of $\delta F_{S(\mu),S(\nu)}$ verifies the relation $m = \frac{1}{\tilde{m}_S}$. Since μ and ν are symmetric, then $m = 1/\tilde{m}$. Since $\mu \preccurlyeq \nu$, we know that $m < \tilde{m}$, and consequently we get $m < 1$. □

1.7.1 Symmetrized payoffs have a lower model risk

We show in this subsection how the symmetry property can be used to reduce the model risk. By Proposition 1.3.3, we have for any payoff C (with linear growth) and

any symmetric model that

$$\overline{P}(\mu,\nu,C)=\overline{P}(\mu,\nu,\mathbb{S}^*(C)),\quad \underline{P}(\mu,\nu,C)=\underline{P}(\mu,\nu,\mathbb{S}^*(C)),$$

implying

$$R(\mu,\nu,C)=R(\mu,\nu,\mathbb{S}^*(C)).$$

We define the family of payoffs $C_\alpha=\alpha C+(1-\alpha)\mathbb{S}^*(C)$, for $\alpha\in[0,1]$. Then $R(C_\alpha)\leq R(C)$.

In financial terms, this means that the new payoff C_α reduces the model risk. Note that $R(C_0)=R(C_1)=R(C)$. Moreover, we have $R(\mathbb{S}^*(C_\alpha))=R(C_\alpha)$, and since S is an involution, we get $R(C_{1-\alpha})=R(C_\alpha)$.

On the other hand, $C_{1/2}=(C+\mathbb{S}^*(C))/2=(C_\alpha+C_{1-\alpha})/2$, and because of the symmetry of $R(C_\alpha)$ around $1/2$ we get

$$\begin{aligned}R(C_{1/2})=R\left(\frac{C_\alpha+C_{1-\alpha}}{2}\right) &\leq \frac{1}{2}R(C_\alpha)+\frac{1}{2}R(C_{1-\alpha})\\ &= \frac{1}{2}R(C_\alpha)+\frac{1}{2}R(C_\alpha)=R(C_\alpha).\end{aligned}$$

Hence, $\alpha=\frac{1}{2}$ realizes the minimum model risk for the portfolio C_α.

1.7.2 Example: the symmetric log normal case

We give an example of symmetric model, where the laws μ and ν are log-normal

$$\mu\sim\ln\mathcal{N}(-\frac{\sigma_\mu^2}{2},\sigma_\mu^2),\ \nu\sim\ln\mathcal{N}(-\frac{\sigma_\nu^2}{2},\sigma_\nu^2)\quad\text{with }\sigma_\mu<\sigma_\nu.$$

Their probability densities and cumulative distribution functions are given by

$$p_i(x)=\frac{1}{x\sqrt{2\pi}\sigma_i}\exp\left[-\frac{(\ln(x)+\frac{1}{2}\sigma_i^2)^2}{2\sigma_i^2}\right]\text{ and }F_i(x)=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\ln(x)+\frac{1}{2}\sigma_i^2}{\sqrt{2}\sigma_i}\right)\right],$$

where $i=\mu,\nu$ and erf is the error function defined by $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\frac{t^2}{2}}dt,\ \forall x\in\mathbb{R}$.

In this case, the maximum m and minimum $\tilde{m}$ of $\delta F:=F_\nu-F_\mu$ can be computed explicitly. They are solutions of:

$$\ln(y)^2=2\frac{\sigma_\mu^2\sigma_\nu^2}{\sigma_\nu^2-\sigma_\mu^2}\ln\left(\frac{\sigma_\nu}{\sigma_\mu}\right)+\frac{\sigma_\mu^2\sigma_\nu^2}{4},$$

so that we have

$$m=\exp\left\{-\left(2\frac{\sigma_\mu^2\sigma_\nu^2}{\sigma_\nu^2-\sigma_\mu^2}\ln(\frac{\sigma_\nu}{\sigma_\mu})+\frac{\sigma_\mu^2\sigma_\nu^2}{4}\right)^{1/2}\right\}\quad\text{and }\tilde{m}=\frac{1}{m}.$$

Note that $m<1<\tilde{m}$.

Figure 1.1: δF

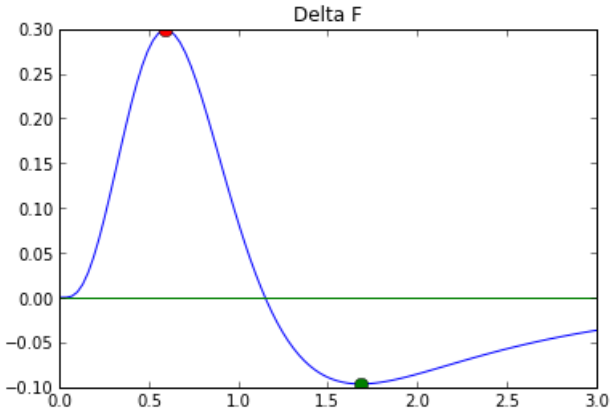
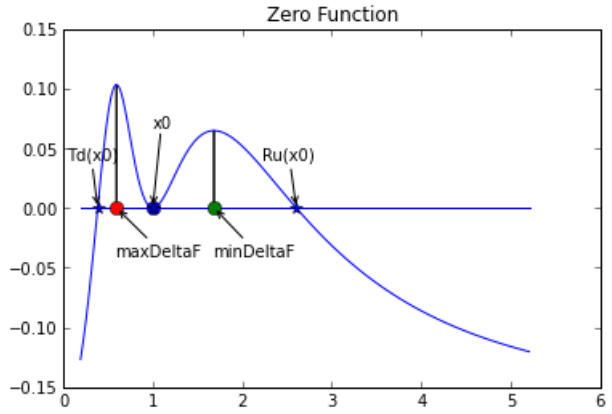


Figure 1.2: Z_x



We recall that the left and the right-monotone transference plan components $L_d(x)$ and $R_u(x)$ are defined as zeros of the function $Z(.;x)$ (c.f. the proof of Lemma 1.2.6)

$$y \mapsto Z(y;x) := G_\nu \left[F_\nu^{-1}\left(F_\mu(x) + \delta F(y)\right)\right] - G_\nu(x) - \delta G(y).$$

If $F_\mu(x) + \delta F(m) \geq 1$, then there exist $t_F^1(x) < m$ such that $F_\mu(x) + \delta F(t_F^1(x)) = 1$ and $t_F^2(x) > m$ such that $F_\mu(x) + \delta F(t_F^2(x)) = 1$. Similarly, if $F_\mu(x) + \delta F(\tilde{m}) \leq 0$ then there exist $\tilde{t}_F^1(x) < \tilde{m}$ such that $F_\mu(x) + \delta F(\tilde{t}_F^1(x)) = 0$ and $\tilde{t}_F^2(x) > \tilde{m}$ such that $F_\mu(x) + \delta F(\tilde{t}_F^2(x)) = 0$. Hence, the function $Z(y;x)$ is defined on the set $\mathbb{R}_*^+ \setminus ((t_F^1(x), t_F^2(x)) \cup (\tilde{t}_F^1(x), \tilde{t}_F^2(x)))$.

Moreover, for $y \in I$, we have $\frac{\partial Z}{\partial y}(y;x) = \delta F'(y)\left(F_\nu^{-1}\left(F_\mu(x) + \delta F(y)\right) - y\right)$, which implies the following:

- $L_d(x)$ is defined, for $x > m$, as the zero of $Z(.;x)$ on the interval $(0, m)$;
- $R_u(x)$ is defined, for $x < \tilde{m}$, as the zero of $Z(.;x)$ on the interval $(\tilde{m}, \infty)$.

Figure 1.3: Left and right-monotone transference plan

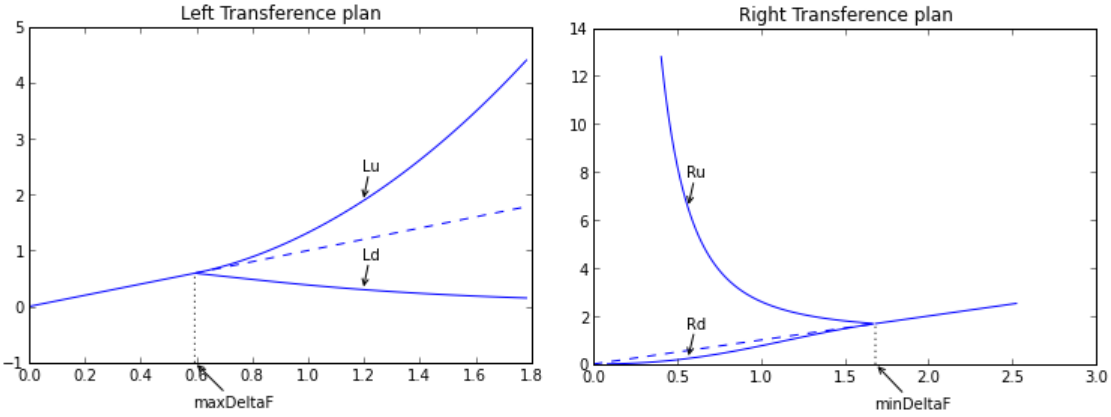
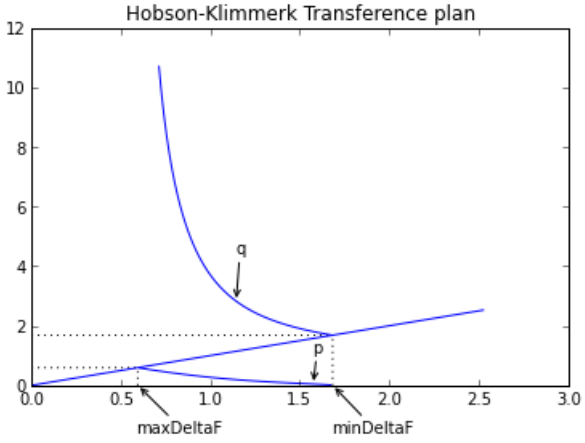


Figure 1.4: 3-band transference plan



Moreover one can show that $Z(.;x)$, for $x \in (m, \tilde{m})$, has three zeros x, $L_d(x) \in (0, m)$ and $R_u(x) \in (\tilde{m}, \infty)$.

We give a numerical example below, where we plot in Figure 1.1 the function δF with a special mention of the maximum and minimum m and $\tilde{m}$. Figure 1.2 represents the function $Z(.;x)$ whose zeros yield the left and right-monotone transference plans.

The two figures 1.3 and 1.4 represent the basic left and right-monotone transference plans (L_d, L_u) and (R_d, L_u), and the basic H-K decreasing transference plan.

1.8 Conclusion

In this work we revisit the explicit construction of Henry-Labordère and Touzi [2013] of the optimal transference plan in the two-marginals martingale problem introduced

by Beiglböck and Juillet [2012], in the particular but important case of positive martingales and of a single maximizer for the difference between the two cumulative distribution functions. We show that the change of numeraire transformation exchanges the left- and the right-monotone transference plans, so that the change of numeraire may be viewed has a mirror coupling acting on positive martingales with pre-specified marginals. We repeat our analysis for another important transference plan, which has been introduced by Hobson and Klimmek [2015]. We study some of its symmetry properties and we show in particular that the change of numeraire exchanges type I with type II forward start straddle, so that the lower bound prices are attained for both options by the same probability measure, the one associated with the Hobson-Klimmek transference plan. Moreover, we show the extremality of these transference plans. We conclude this paper with some numerical illustrations. We leave the multi-maximizer case for further research.

Appendix 1.A Proof of Lemma 1.2.3

First of all, notice that the third condition in the definition of a basic left-monotone transference plan is equivalent to the following equation, for any bounded measurable function g:

$$\int_0^\infty g(y)p_\nu(y)dy = \int_0^{x_\star} g(y)p_\mu(y)dy + \int_{x_\star}^\infty g(L_u(x))q_L(x)p_\mu(x)dx + \int_{x_\star}^\infty g(L_d(x))\overline{q}_L(x)p_\mu(x)dx.$$

A simple change of variable gives

$$\begin{aligned}\int_0^\infty g(y)p_\nu(y)dy = &\int_0^{x_\star} g(y)p_\mu(y)dy + \int_{x_\star}^\infty g(y)q_L(L_u^{-1}(y))p_\mu(L_u^{-1}(y))dL_u^{-1}(y)\\ &- \int_0^{x_\star} g(y)\overline{q}_L(L_d^{-1}(y))p_\mu(L_d^{-1}(y))dL_d^{-1}(y),\end{aligned} \tag{1.A.1}$$

where we have used the fact that $L_u(x) \to \infty$ as $x \to \infty$, which follows from $L_u(x) > x$, and $L_d(x) \to 0$ as $x \to \infty$. This latter is due the assumption that μ and ν do not agree on any interval. Indeed, if $L_d(x) \to d_\star > 0$, then μ and ν would necessarily agree on $]0, d_\star[$, by definition of L_d ($u \in]0, d_\star[$ can only be reached by the points where $L_u = L_d$).

Using (1.A.1), we get that:

- On $y > x_\star$, $p_\nu(y)dy = q_L(L_u^{-1}(y))p_\mu(L_u^{-1}(y))dL_u^{-1}(y)$,
- On $y < x_\star$, $p_\nu(y)dy = p_\mu(y)dy - \overline{q}_L(L_d^{-1}(y))p_\mu(L_d^{-1}(y))dL_d^{-1}(y)$.

The two ODEs (2.8) and (2.9) follows by a change of variable $x = L_u^{-1}(y)$ in the two ODEs before stating the present Lemma, together with the observation that, L_d being decreasing, the inequality $y < x_\star$ in the second ODE turns to $y > x_\star$.

Appendix 1.B Proof of Proposition 1.2.4

First, we show Equation (2.11). Subtracting the two relations (2.8) and (2.9), we get:

$$p_\nu(L_u)dL_u - (p_\nu(L_d) - p_\mu(L_d))dL_d = p_\mu dy. \tag{1.B.1}$$

Integrating between $x_\star$ and some $x \geq x_\star$ we get

$$F_\nu(L_u) - F_\nu(x_\star) - F_\nu(L_d) + F_\nu(x_\star) + F_\mu(L_d) - F_\mu(x_\star) = F_\mu(x) - F_\mu(x_\star),$$

which gives Equation (2.11).

Now we prove that $x_\star$ must be the unique maximizer of δF. Note that the equation (2.9) and the fact that L_d is decreasing entail that $p_\nu - p_\mu > 0$ on the support of the image of L_d, which is equal to $]0, x_\star[$, since we have $L_d(x) \to 0$ as $x \to \infty$. This means that $\delta F' = p_\nu - p_\mu > 0$ on $]0, x_\star[$. Hence, δF is increasing on $]0, x_\star[$. In other words, if

$x_{\star\star}$ denotes any maximizer of δF, we have

$$x_\star \leq x_{\star\star}.$$

On the other hand, using the equation (2.11) that relates L_u and L_d, one can show that the point $x_\star$ is necessarily a maximizer of δF. In fact, we have by definition, $\forall x > x_\star,\ L_d(x) < x < L_u(x)$, which is equivalent to $\forall x > x_\star,\ F_\nu(L_u(x)) > F_\nu(x)$. Hence, the equation (2.11) implies that

$$\forall x > x_\star,\ F_\nu(L_d(x)) - F_\mu(L_d(x)) > F_\nu(x) - F_\mu(x). \tag{1.B.2}$$

So, if $x_{\star\star}$ denotes any maximizer of δF, we have that

$$x_{\star\star} \leq x_\star.$$

Otherwise, if $x_{\star\star} > x_\star$, then by (1.B.2), $\delta F(L_d(x_{\star\star})) > \delta F(x_{\star\star})$, which contradicts the fact that $x_{\star\star}$ is a maximizer of δF. In conclusion, we get $x_\star = x_{\star\star}$.

Finally, we show that L_d verifies Equation (2.10). For that, let us rewrite equation (2.8) above by replacing the explicit expression for q_L:

$$(L_u - L_d)p_\nu(L_u)dL_u = (y - L_d)p_\mu dy$$

or equivalently

$$L_u p_\nu(L_u)dL_u - L_d(p_\nu(L_u)dL_u - p_\mu dy) = y p_\mu dy,$$

so that eventually, after using equation (1.B.1), we obtain

$$L_u p_\nu(L_u)dL_u - L_d(p_\nu(L_d) - p_\mu(L_d))dL_d = y p_\mu dy. \tag{1.B.3}$$

Let us introduce the cumulated expectations G_μ, G_ν where $G(b) = \int_0^b ap(a)da$ for $p \in \{p_\mu, p_\nu\}$. Integrating (1.B.3) between $x_\star$ and some $x > x_\star$ we get

$$G_\nu(L_u(x)) - G_\nu(x_\star) - G_\nu(L_d(x)) + G_\nu(x_\star) + G_\mu(L_d(x)) - G_\mu(x_\star) = G_\mu(x) - G_\mu(x_\star),$$

giving

$$G_\nu(L_u(x)) - G_\nu(L_d(x)) + G_\mu(L_d(x)) = G_\mu(x),$$

so that, using (2.11), we have that $L_d(x)$ is solution to

$$G_\nu(F_\nu^{-1}(F_\mu(x) + (F_\nu(L_d(x)) - F_\mu(L_d(x))))) - G_\nu(L_d(x)) + G_\mu(L_d(x)) = G_\mu(x).$$

Appendix 1.C Proof of Lemma 1.2.6

Let $f(t) = F_\nu^{-1}(F_\mu(x) + \delta F(t))$ and $g(t) = G_\nu^{-1}(G_\mu(x) + \delta G(t))$. We already know from the previous section that $L_d(x)$ is solution to equation (2.10). Thus, it remains to show the uniqueness of the solution for the equation $f(t) = g(t)$. Since G_ν is strictly

increasing and continuous, this equation is equivalent to $G_\nu(g(t)) - G_\nu(f(t)) = 0$. So let us introduce

$$Z_x(t) := G_\nu(g(t)) - G_\nu(f(t)) = G_\mu(x) + \delta G(t) - G_\nu(f(t)),$$

which is defined on $]0, t_F(x)[$. We want to prove that Z_x has a unique zero on the interval $]0, t_F(x)[$. We split the rest of the proof into three steps.

Step 1. First we prove that Z_x is decreasing. Let $t < t_F(x)$. We have $Z_x'(t) = \delta G'(t) - (G_\nu F_\nu^{-1})'(F_\mu(x) + \delta F(t))\delta F'(t)$. Observe now that:

i) $\delta G'(t) = t\delta F'(t)$,

ii) $G_\mu'(t) = tF_\mu'(t) = tp_\mu(t)$ and $(G_\mu F_\mu^{-1})'(t) = F_\mu^{-1}(t)$. The same equalities hold for the measure ν.

so that:

$$Z_x'(t) = (t - f(t))\delta F'(t).$$

Now $t < f(t)$ if and only if $x > t$, which holds on $]0, m[$ since we assumed $x > m$.

Step 2. Now we prove that $Z_x(0+) > 0$. Indeed, let:

$$z(x) := Z_x(0+) = G_\mu(x) - G_\nu(F_\nu^{-1})(F_\mu(x)).$$

Then $z(0+) = 0$ and $z'(x) = (x - F_\nu^{-1}(F_\mu(x)))p_\mu(x)$ is positive on the set $\{\delta F > 0\}$. Therefore $z(x) = z(0+) + \int_0^x z'(y)dy$ is positive on the set $\{\delta F > 0\}$. Now thanks to our assumption $p_\mu > 0$, we see that the extrema of z are the zeros of δF. Because of the convex ordering, we know that δF has at least one zero, otherwise δF would have a constant sign, which contradicts the convex ordering. If it had one more zero, this last property would imply that δF has at least two local maximizers.

Step 3. To end the proof, we show that $Z_x(t_F(x)) < 0$. This can be done by looking at possible values of $t_F(x)$. We distinguish between two cases.

i) $Z_x(m) < 0$ (case $t_F(x) = m$).

Let us denote $y(x) = Z_x(m)$. Then $y(m) = G_\nu(m) - G_\nu(m) = 0$ and $y'(x) = p_\mu(x)(x - f(m))$. Now $x < f(m)$ if and only if $\delta F(x) < \delta F(m)$, which is always true by definition of m. Therefore y is decreasing and $Z_x(m) = y(x) < 0$.

ii) $Z_x(t_F(x)) < 0$ (case $t_F(x) < m$).

Let $u(x) = Z_x(t_F(x)) = G_\mu(x) + \delta G(t_F(x)) - 1$. Note that $t_F(x) \to 0$ as $x \to \infty$, so that $u(x) \to 1 + 0 - 1 = 0$. Therefore it suffices to show that u is increasing. Now $u'(x) = xp_\mu(x) + t_F'(x)t_F(x)(p_\nu(t_F(x)) - p_\mu(t_F(x)))$ and by the equation defining $t_F(x)$, $p_\mu(x) + t_F'(x)(p_\nu(t_F(x)) - p_\mu(t_F(x))) = 0$. Therefore $u'(x) = p_\mu(x)(x - t_F(x))$ with $x > m > t_F(x)$.

Appendix 1.D Proof of Lemma 1.5.4

Proof. First, we denote by δF^{-1} the function mapping $y \in (\delta F(b), 0)$ to the unique $z \in (b, \infty)$ solving $\delta F(z) = y$.

For each $x \in (a,b)$, we introduce the following continuous function Z_x given by

$$Z_x(y) = \delta G\Big[\delta F^{-1}\Big(\delta F(x) - \delta F(y)\Big)\Big] + \delta G(y) - \delta G(x).$$

Such a function is well defined on the interval $[t_F^1(x), t_F^2(x)]$, since for any y in that interval we have

$$\delta F(b) \leq \delta F(x) - \delta F(y) \leq 0.$$

Note that $(p(x), q(x))$ such that $p(x) \in [0,a]$, $q(x) \in [b,\infty)$ is a solution of (5.26) if and only if $p(x) \in [t_F^1(x), t_F^2(x)]$, $Z_x(p(x)) = 0$ and $q(x) = \delta F^{-1}\Big(\delta F(x) - \delta F(p(x))\Big)$.

We show now that for any $x \in (a,b)$, Z_x admits a unique zero on $(t_F^1(x), t_F^2(x))$. This will be done in three steps: first we show that Z_x is decreasing in $(t_F^1(x), t_F^2(x))$, then prove that $Z_x(t_F^1(x)) > 0$ and finally $Z_x(t_F^2(x)) < 0$.

First, recall that for $z \geq 0$, $\delta G'(z) = z\delta F'(z)$, hence

$$\frac{d\delta G\Big(\delta F^{-1}(z)\Big)}{dz} = \frac{1}{\delta F'(\delta F^{-1}(z))}\delta F^{-1}(z)\delta F'(\delta F^{-1}(z))) = \delta F^{-1}(z),\ \forall z \in (\delta F(b), 0).$$

Then, for $x \in (a,b)$ and $y \in [t_F^1(x), t_F^2(x)]$

$$\begin{aligned} Z_x'(y) &= -\delta F'(y)\delta F^{-1}(\delta F(x) - \delta F(y)) + y\delta F'(y) \\ &= \delta F'(y)\Big[y - \delta F^{-1}(\delta F(x) - \delta F(y))\Big]. \end{aligned} \tag{1.D.1}$$

Since δF is increasing on $(0,a)$ then $\delta F'(y) \geq 0$ for all $y \in (t_F^1(x), t_F^2(x)) \subset (0,a)$. Also, by definition of δF^{-1}, $\delta F^{-1}(\delta F(x) - \delta F(y)) \geq b$, so that $y \leq \delta F^{-1}(\delta F(x) - \delta F(y))$ for all $y \in (t_F^1(x), t_F^2(x))$. Consequently, Z_x is decreasing. In order to conclude, we need to show that $Z_x(t_F^1(x)) > 0$ and $Z_x(t_F^2(x)) < 0$.

Let $x \in (a,b)$, we compute $Z_x(t_F^1(x))$.

1. If $\delta F(x) \leq 0$ then $t_F^1(x) = 0$. Let $\eta(x) = \delta F^{-1}(\delta F(x)) \geq b$, i.e. $\delta F(\eta(x)) = \delta F(x)$. Thus

$$\begin{aligned} Z_x(0) &= \delta G\Big[\delta F^{-1}\Big(\delta F(x)\Big)\Big] - \delta G(x) \\ &= \delta G(\eta(x)) - \delta G(x) \\ &= \eta(x)\delta F(\eta(x)) - \int_0^{\eta(x)} \delta F(y)dy - x\delta F(x) + \int_0^x \delta F(y)dy \\ &= (\eta(x) - x)\delta F(x) - \int_x^{\eta(x)} \delta F(y)dy \\ &= \int_x^{\eta(x)} (\delta F(x) - \delta F(y))dy. \end{aligned}$$

   Since δF is decreasing on (x,b) and increasing on $(b, \eta(x))$ we have $\delta F(x) \geq \delta F(y)\ \forall y \in (x, \eta(x))$. Consequently, $Z_x(0) > 0$.

2. If $\delta F(x) > 0$, $t_F^1(x)$ is the solution of $\delta F(z) = \delta F(x)$ on $(0, a)$. Then

$$\begin{aligned} Z_x(t_F^1(x)+) &= \delta G(t_F^1(x)) - \delta G(x) \\ &= \int_{t_F^1(x)}^{x} (\delta F(y) - \delta F(x))dy. \end{aligned}$$

Since δF is increasing on $(t_F^1(x), a)$ and decreasing on (a, x) then $\delta F(x) \leq \delta F(y)$ $\forall y \in (t_F^1(x), x)$. Consequently, $Z_x(t_F^1(x)+) > 0$.

Finally, we show that $Z_x(t_F^2(x)) < 0$. Let us denote by $f : (a, b) \to \mathbb{R}$ defined by $f(y) = \delta F(y) - \delta F(a) - \delta F(b)$. We have $f'(y) = \delta F'(y) < 0,\ \forall y \in (a, b)$, $f(a) = -\delta F(b) > 0$ and $f(b) = -\delta F(a) < 0$. Hence, there exist a unique $z^* \in (a, b)$ such that $f(z^*) = 0$.

Hence $\forall y \in (a, z^*]$, $\delta F(y) - \delta F(a) \geq \delta F(b)$ and $\forall y \in (z^*, b)$, $\delta F(y) - \delta F(a) < \delta F(b)$.

1. If $x \in (a, z^*]$, then $\delta F(x) - \delta F(b) \geq \delta F(a)$ and $t_F^2(x) = a$. We introduce the function z defined on (a, z^*).

$$z(y) := Z_y(a) = \delta G\Big[\delta F^{-1}\Big(\delta F(y) - \delta F(a)\Big)\Big] + \delta G(a) - \delta G(y),\ \forall y \in (a, z^*).$$

Let $y \in (a, z^*)$, then $z'(y) = \delta F'(y)\Big[\delta F^{-1}\Big(\delta F(y) - \delta F(a)\Big) - y\Big]$. Since $y \in (a, b)$ and $\delta F(y) - \delta F(b) \geq \delta F(a)$, we have $\delta F'(y) < 0$ and $\delta F^{-1}\Big(\delta F(y) - \delta F(a)\Big) \geq b > y$. Hence z is decreasing on (a, z^*). Moreover, $z(a+) = \lim\limits_{u \to \infty} \delta G(u) = 1 - 1 = 0$, consequently, $z(y) < z(a+) = 0,\ \forall y \in (a, z^*)$. In particular, $Z_x(a) = z(x) < 0$.

2. The second case is $x \in (z^*, b)$. This implies that $\delta F(x) - \delta F(b) < \delta F(a)$ and $t_F^2(x)$ is solution of the equation $\delta F(z) = \delta F(x) - \delta F(b)$ on $(0, a)$. We evaluate $Z_x(t_F^2(x))$.

$$\begin{aligned} Z_x(t_F^2(x)) &= \delta G\Big[\delta F^{-1}\Big(\delta F(x) - \delta F(t_F^2(x))\Big)\Big] + \delta G(t_F^2(x)) - \delta G(x) \\ &= \delta G(b) + \delta G(t_F^2(x)) - \delta G(x). \end{aligned}$$

We consider the function z defined on (z^*, b) as

$$z(y) := Z_y(t_F^2(y)) = \delta G(b) + \delta G(t_F^2(y)) - \delta G(y),\ \forall y \in (z^*, b).$$

Similarly to the computation of the derivative in (1.D.1), we get for all $y \in (z^*, b)$

$$z'(y) := \delta F'(y)\big[t_F^2(y) - y\big] < 0,$$

since δF is decreasing in (a, b) and by definition $t_F^2(y) < a < y$. On the other hand, $t_F^2(b) = 0$, so that $z(b) = \delta G(b) + \delta G(0) - \delta G(b) = 0$. Consequently, $z(y) < 0,\ \forall y \in (z^*, b)$ and in particular

$$z(x) = Z_x(t_F^2(x)) < 0.$$

□

2

Gas storage valuation and hedging. A quantification of model risk.

This chapter is the object of Hénaff et al. [2013].

Abstract

This paper focuses on the valuation and hedging of gas storage facilities, using a spot-based valuation framework coupled with a financial hedging strategy implemented with futures contracts. The contributions of this paper are two-fold. Firstly, we propose a model that unifies the dynamics of the futures curve and spot price, and accounts for the main stylized facts of the US natural gas market such as seasonality and the presence of price spikes. Secondly, we evaluate the associated model risk, and show that the valuation is strongly dependent upon the dynamics of the spot price.

2.1 Introduction

Natural gas storage units are used to reconcile the variable seasonal demand for gas with the more constant rate of natural gas production. These gas storage facilities are mainly owned by distribution companies which use them for system supply regulation, and for reducing the risk of shortages. In fact, regulation requires that local distribution companies own storage units, in order to secure their gas supply and to be able to meet any sudden increase in demand or any disruption in the pipeline transportation system.

Several techniques may be used to value a gas storage facility. They can be broadly classified in two categories: the *intrinsic* and the *extrinsic* valuation methods.

Traditionally, the demand for natural gas is seasonal, with winter peaks and summer lows. This motivates the *intrinsic* valuation methodology, which exploits the seasonal shape of the natural gas futures curve. Following this strategy, the storage manager observes the futures curve at the beginning of the storage contract and buys/sell futures contracts, thereby determining once and for all the complete schedule of injection and withdrawals. In order to determine the optimal futures positions, a linear optimization problem is solved, with constraints defined by the physical and financial particulars of the storage contract (see Annexe 2.B which refers to Eydeland and Krzysztof [2002]). We emphasize that the storage manager keeps the optimal futures

positions for the entire duration of the storage contract. This strategy does not take advantage of possible profitable movements of the futures curve.

This static methodology is extended by Gray and Khandelwal [2004] to the *rolling intrinsic valuation*, to take advantage of the changing dynamics of the futures curve. According to this variant, optimal futures positions are chosen at the beginning of the storage contract, but when the futures curve moves away from its initial shape, new optimal futures positions are recalculated, and the portfolio is rebalanced if this is found to be profitable.

These two futures-based approaches capture the predictable seasonal pattern of natural gas prices: they lead to buying cheap summer futures and selling expensive winter futures. The corresponding storage value greatly depends on the summer-winter spread. The intrinsic valuation methodology has been popular in the storage industry, especially during periods where seasonal patterns were very pronounced. During the last few years, the seasonal spreads have been reduced, and this puts into question the futures-based methodology. The *2011 State of the Markets* report, issued by the US Federal Energy Regulatory Commission (FERC) FERC [2012], noticed the following: "We have also seen a decline in the seasonal difference between winter and summer natural gas prices. Falling seasonal spreads reflect increased production and storage capacity, as well as greater year-round use of natural gas by power generators. This decline has developed over the past several years and we expect the trend to continue."

The narrowing winter/summer spread, mentioned earlier, is mainly due to two factors that put a downward pressure on winter gas prices and an upward pressure on summer prices. The first factor is the recent surge in non conventional shale gas supply, with geographical locations that are closer to gas consumption areas. The main result of this new abundant source of gas is a downward pressure on winter prices. The second factor is related to power consumption by cooling systems during summer periods and the growing use of natural gas as a fuel for electricity generation. This puts an upward pressure on summer gas prices.

The combination of these two factors has the logical consequence of narrowing the seasonal spreads between winter and summer prices, diminishing the intrinsic value of gas storage units. Static strategies based on futures contracts are no longer adequate to monetize the value of gas storage facilities; they sometimes fail to recover the operating expenses. This motivates the interest in dynamic strategies that take advantage of the real options embedded in gas storage facilities. This so-called *extrinsic valuation* method still takes advantage of the remaining price seasonality, but, more importantly, monetizes the high volatility of natural gas spot price.

The first contribution of this paper is to present a new modeling framework that unifies the dynamics of the futures curve and spot price, and is consistent with the two stylized facts that are essential to the gas storage valuation problem: price seasonality and spot price spikes. The second aspect of the paper is related to the quantification of model uncertainty related to the spot dynamics. We highlight in particular the significant sensitivity of gas storage value to the specification and estimation of

the spot model. This result puts into perspective the extensive literature on gas storage valuation, and calls for a more careful assessment of the model risk inherent to these valuations. We believe that it is crucial to devote more attention to the choice of the spot-futures modeling framework, rather than to concentrate all the efforts on the specification of an optimal trading strategy.

The rest of the paper is organized as follows. In Section 2.2 we describe the important stylized facts related to the natural gas market. In Section 2.3 we present the characteristics of typical gas storage unit and the valuation method, using an optimal spot strategy and a futures-based hedging scheme. Section 2.4 is devoted to a review of the modeling approaches in the gas storage literature. In Section 2.5 we introduce the modeling framework combining futures and spot dynamics and in Section 2.6 we perform several numerical tests. In Section 2.7 we introduce two natural model risk measures to quantify the sensitivity of a class of models with respect to the parameters; those risk measures are computed in several test cases.

2.2 Natural gas stylized facts

In this section, we highlight important stylized facts about natural gas markets that influence the value of a storage unit. These properties are related to the demand and use of natural gas. In fact, the demand for natural gas for heating in cold periods of the year produces a seasonal price pattern, while unpredictable changes in weather can cause sudden shifts in gas prices. These facts are the two main sources of value for a gas storage unit, since the ownership of a storage facility enables one to take advantage from seasonality and price spikes.

As for all other commodities, the price of natural gas (NG) is influenced by its point of delivery. In this study, we will be interested in the United States market, specifically in a storage location near Henry Hub (Louisiana), which justifies the use of gas daily spot prices and the Nymex natural gas futures as hedge instruments. One can buy natural gas in the spot market for next-day delivery, or in the futures market for rated delivery over a future period of one calendar month. The NYMEX futures market provides quotes for the next 72 monthly futures contracts, but only the first 24 or so are liquidly traded.

In what follows, S_t will denote the spot price of natural gas at date t, and $(F(t,T_i))_i$ represent the futures contracts price at t, for a set $\{T_i\}$ of maturities. We consider monthly spaced maturities, so every futures contract is related to a delivery month. Also, we denote by P_t the price of prompt contract, i.e. the futures contract with the closest maturity to current time t. Natural gas prices are quoted in U.S. dollars per million British thermal units (MMBtu).

As mentioned above, the first main feature of natural gas prices is constituted by the presence of a seasonal component. We plot the NG futures curve for several dates in Figure 2.1 and observe a periodic winter increase in price, which are clearly due to the demand for heating during cold periods of the year. In addition to this traditional seasonal feature, the use of natural gas for electricity generation has created a

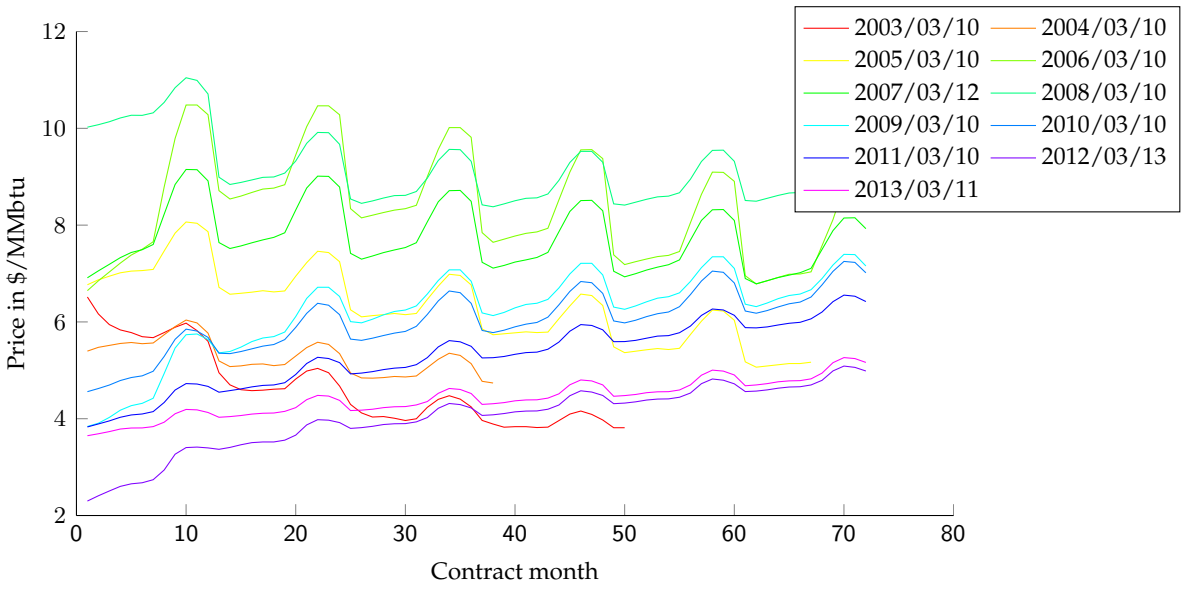


Figure 2.1: Futures curve for Nymex NG at different observation dates

second smaller increase during the summer period, related to the increasing demand for cooling. These expected patterns in natural gas prices are the first source of value for a storage unit. In the futures market, buy gas for summer delivery and simultaneously sell gas for winter delivery. Store it in between, and you have locked a certain profit which is the summer-winter spread less the storage cost. This is the essence of the so-called "intrinsic strategy," which is based only on futures contracts and exploits the calendar spreads in the futures curve (see Appendix 2.B for more details about the intrinsic value of a storage unit.)

The second important aspect of natural gas prices, is the presence of sudden moves due to unexpected imbalances between supply and demand, caused by such factors as unpredicted weather changes, disruptions in the supply chain, or poor anticipations of the global amount of gas in storage. Such events are almost instantaneously reflected in the spot dynamics, giving rise to large shifts in prices, rapidly absorbed by the storage capacities available in the market. These large and quickly absorbed jumps, commonly called spikes, can be viewed in Figure 2.2, which shows many sudden dislocations between spot and prompt prices. [1] For example, we can notice a large spike in the spot price during late February 2003, when the natural gas price jumped by almost 78.00% and 54.26% in two successive days, then went down by -43.34% and -19.58% during the two following days. As noted by the US Federal Energy Regulatory Commission (FERC) by FERC [2003], this spike in gas price was due to "physical market conditions leading to low supply and high demand for a short time." FERC [2003] also observed that "similar natural gas price spikes are possible when episodes of cold weather occur at times when storage inventories are limited."

[1] We use a 1997-2013 historical data of spot and prompt price, published by the U.S. Energy Information Administration. cf http://www.eia.gov/dnav/ng/ng_pri_fut_s1_d.htm

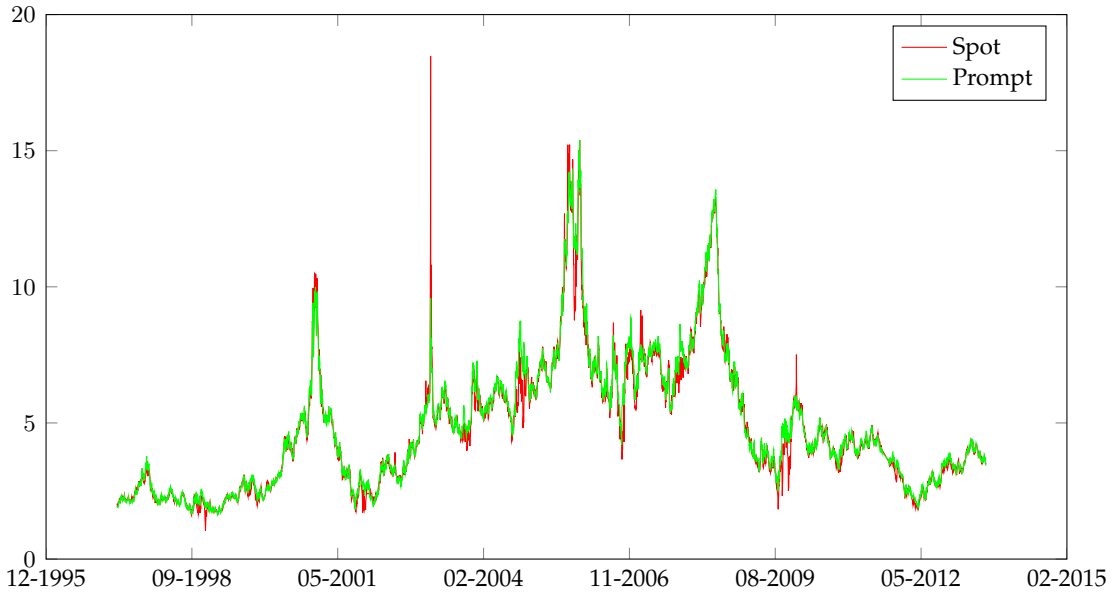


Figure 2.2: Spot and prompt historical prices

The appearance of the spikes is related to the spot and prompt prices spread. While the prompt contract is a good proxy for the spot price, it does not suffer from sudden shocks of the same amplitude as spot prices, because of time-to-maturity factor. The spikes provoke 'unusual' gaps between the two contracts

In our study, we detect spikes by identifying the outliers from the time series (x_t) of the spread between the spot price S_t and the prompt price P_t given by $x_t := \frac{S_t - P_t}{P_t}$; we study separately the positive and negative spikes, since they reflect different market conditions. Positive spikes are often caused by unpredicted weather changes, such as a cold front or a heat wave. On the other hand, negative spikes are generally due to a poor anticipation of market-wide gas storage levels. In Figure 2.3 we plot the number of occurrences of negative and positive spikes during each month. We remark that the distribution of spikes is clearly dependent on their sign: most of the positive spikes happen during the winter months of January and February and the summer month of June, which can be explained by the occurrence of an unpredicted cold front or heat wave. On the other hand, negative spikes appear during the Fall. One plausible explanation is given by Mastrangelo [2007], which states: "October is the last month of the refill season. There may be increased competition from storage facilities looking to meet end-of-season refill goals as well as increased anticipation regarding the upcoming heating season."

In order to take into account the stylized facts mentioned above, our futures model incorporates seasonality in the futures curve, and the spot model describes the existence of spikes and takes into account the correlation between spot and futures prices, through the prompt contract. To the best of our knowledge, these two facts have not

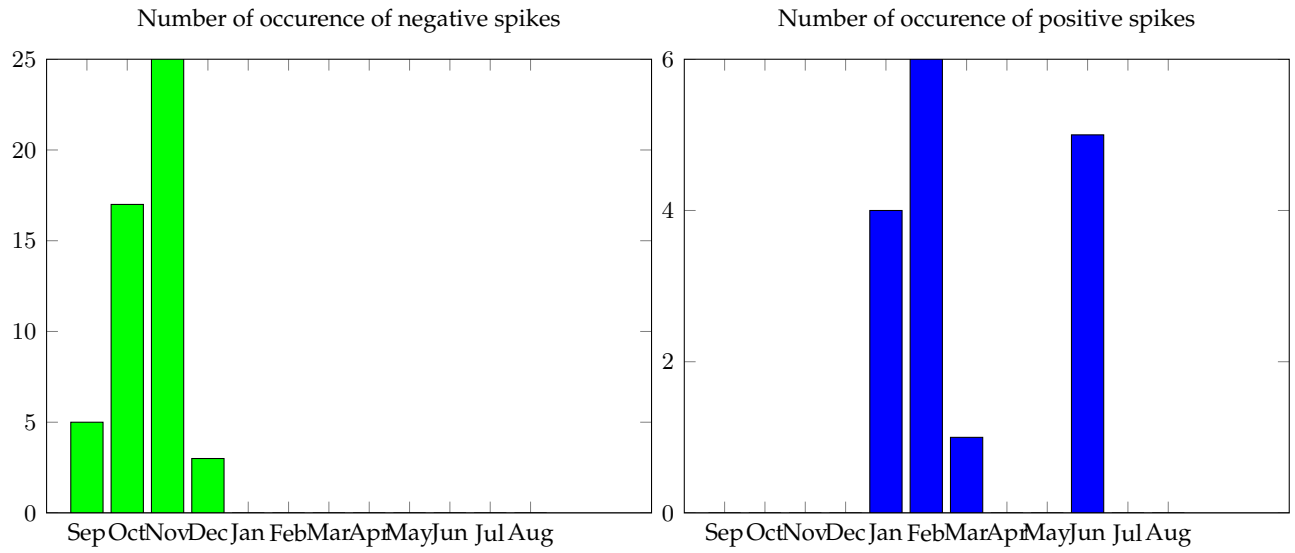


Figure 2.3: Occurrences of spikes

yet been taken into account in the literature related to gas storage valuation, although, in our opinion, they constitute the two main sources of storage value.

Before discussing modeling issues, we formalize the gas storage valuation and hedging problem, and recall numerical simulation algorithms.

2.3 Valuation and hedging of a gas storage utility

The problem of valuing gas storage units has been discussed from many angles in the literature, yielding different approaches and numerical methods. Leasing a gas storage unit is equivalent to paying for the right, but not the obligation, to inject or withdraw gas from the unit. Hence the goal of the owner is to optimize the use of the gas storage facility, by injecting or withdrawing gas from the unit and, at the same time, trading gas on the spot and/or futures market. All these decisions have to be made under many operational constraints, such as maximal and minimal volume of the storage, and limited injection and withdrawal rates. This forces the resolution of a constrained stochastic control problem.

The gas spot price is modeled by a process, denoted by S. We suppose that this process is given as a function of a Markov process X in term of which the optimal control problem will be expressed. For example, in the framework (2.4.1), used by Boogert and De Jong [2008], the spot process is a Markov process, so we take obviously $X = S$.

Warin [2012] takes the futures curve as the underlying process, such that

$$F(t,T) = F(0,T) \exp\left[-\frac{1}{2}V(t,T) + \sum_{i=1}^{n} e^{-a_i(T-t)} W_t^i\right], \tag{2.3.1}$$

with $W_t^i = \int_0^t \sigma_i(u)e^{-a_i(T-u)}dZ_u^i$, Z^i being standard Wiener processes, and $V(t,T) = var(\sum_{i=1}^{n} e^{-a_i(T-t)}W_t^i)$. Indeed, the Markov process X can be chosen to be equal to the random sources $X = (W^1, ..., W^n)$.

In fact the futures prices modeled in (2.3.1) are martingales. In particular Warin [2012] supposes that the underlying probability is risk-neutral probability, and not necessarily unique.

Remark 2.3.1. *For simplicity, in the rest of this article, we suppose the discount interest rate to vanish, and consider the problem specification in a time discrete setting.*

We next present the specification of the gas storage valuation problem, using the notations of Warin [2012].

2.3.1 Gas storage specification

We consider a gas storage facility with technical constraints (either physical or regulatory) on the volume of stored gas, V_{min} and V_{max} i.e. at all time, the volume of stored gas V should verify $V_{min} \leq V \leq V_{max}$.

We assume a discrete set of dates $t_i = i\Delta t$ for $i = 0, ..., n-1$ with $\Delta t = T/n$. At each date t_i and starting from a volume V_{t_i}, the user has the possibility to make one of three decisions: either inject gas at rate of a_{inj}, or withdraw gas at rate of a_{with} or take no action. We denote by u_i the decision at time t_i, and write $u_i = inj$ (resp. $with$, no) if the decision is injecting gas at rate a_{inj} (resp. withdrawing gas at rate a_{with}, no action).

If the user follows a strategy $(u_i)_{i=0...n-1}$, then the volume of gas in storage $(V_{t_i})_i$ is given by the iteration

$$V_0 = v, \tag{2.3.2}$$

$$V_{t_{i+1}}(u) = \begin{cases} \min(V_{t_i}(u) + a_{inj}\Delta t, V_{max}) & \text{if } u_i = ing \\ \max(V_{t_i}(u) - a_{with}\Delta t, V_{min}) & \text{if } u_i = with \\ V_{t_i}(u) & \text{if } u_i = no, \end{cases} \tag{2.3.3}$$

for $i = 0, ..., n-2$. The generated cash flow (positive when withdrawing gas, negative when injecting) is given by

$$\phi_{u_i}(S_{t_i}) := S_{t_i}(V_{t_{i+1}}(u) - V_{t_i}(u)).$$

In general the maximum injection and withdrawal rates (a_{inj} and a_{with}) are functions of the amount of gas in storage. However, without loss of generality, we assume for simplicity that these rates are constant. In Table 2.1 we summarize the possible decisions and their consequences on gas volume and generated cash flow. We recall that a_{inj} indicates the injection rate per time unit Δt and a_{with} the withdrawal rate per

Decision u	**Next volume**	**Cash flow**
Injection: $u_i = inj$	$V_{t_{i+1}}(u) = \min(V_{max}, V_{t_i}(u) + a_{inj}\Delta t)$	$\phi_{inj} = S_{t_i}(V_{t_i}(u) - V_{t_{i+1}}(u))$
Withdrawal: $u_i = with$	$V_{t_{i+1}}(u) = \max(V_{min}, V_{t_i}(u) - a_{with}\Delta t)$	$\phi_{with} = S_{t_i}(V_{t_i}(u) - V_{t_{i+1}}(u))$
No Action: $u_i = no$	$V_{t_{i+1}}(u) = V_{t_i}(u)$	$\phi_{no} = 0$

Table 2.1: Possible decisions

time unit Δt. Consequently, the wealth generated by following a strategy u is given by

$$\text{Wealth}_{\text{spot}}(u) = \sum_{i=0}^{n-1} \phi_{u_i}(S_{t_i}). \tag{2.3.4}$$

Finally, we are interested in the expectation of this cumulative cash flows, which we denote by J. More precisely we set

$$\begin{aligned} J(t_0, x_0, v_0; u) &:= \mathbb{E}\left[\text{Wealth}_{\text{spot}}\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{n-1} \phi_{u_i}(S_{t_i})\right], \end{aligned} \tag{2.3.5}$$

J is a function that depends on the initial time t_0, the value of the Markov process $X_0 = x_0$, the initial volume in storage v_0 and the strategy u.

The goal of the storage operator is to find a strategy u maximizing the expected cumulative cash flows. We denote this optimal value by $J^\star$. So the problem to solve is the following:

$$\begin{aligned} J^\star(t_0, x_0, v_0) &= \max_{(u_i)_{i=0\ldots n-1}} J(t_0, x_0, v_0; u) = \max_{(u_i)_{i=0\ldots n-1}} \mathbb{E}\left[\sum_{i=0}^{n-1} \phi_{u_i}(S_{t_i})\right] \\ &= J(t_0, x_0, v_0; u^\star). \end{aligned} \tag{2.3.6}$$

A priori, the underlying probability is the historical probability measure, at least if the futures do not intervene in the spot model. This quantity constitutes an objective for the manager. However, it is not a "fair" price in the sense of "absence of arbitrage," since the spot is not traded as a financial asset; in that case the price would be an expectation with respect to a risk-neutral probability. On the other hand, $J^\star$ constitutes a price indicator; practitioners lease gas storage units at a proportion of this price.

In our proposed framework (see Section 2.5) the spot and the futures are jointly modeled on a product space $\Omega = (\Omega_s, \Omega_f)$ equipped with a probability $\mathbb{Q}$. The futures are first directly described as martingales on Ω_f with respect to their corresponding

risk-neutral probability $\mathbb{P}^*$ and they are extended trivially to Ω. Formally $\mathbb{Q}$ is defined by $\mathbb{Q}(d\omega_s, d\omega_f) = Q^{\omega_f}(d\omega_s)P^*(d\omega_f)$, where $Q^{\omega_f}(d\omega_s)$ is a random probability kernel (historically considered), describing the random behavior of S for each realization ω_f of the futures asset. The expectation of the optimal cumulative cash flows with respect to Q will then be a price indicator, compatible with classic financial principles, as far as futures assets are concerned. Indeed, we will also estimate the volatility parameters for the diffusion describing the futures assets F using historical data, that is under some historical probability $\mathbb{P}$ and not $\mathbb{P}^*$ as we would need. However, the probability $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and this justifies the coherence of the estimation.

2.3.2 Dynamic programming equation

From Table 2.1, we recall that $V_{t_{i+1}}(u)$ only depends on $V_{t_i}(u)$ and u_i. To emphasize this fact, if $V_{t_i} = v$, we also express $V_{t_{i+1}}(u)$ by $\widehat{V}_{u_i}(v)$.

At time t, for $X_t = x$ and with current volume level v, the (optimal) value for gas storage will be of course denoted by $J^\star(t, x, v)$. The dynamic programming principle implies

$$J^\star(t_i, x, v) = \max_{u_i \in \{inj,\, no,\, with\}} \left\{ \phi_{u_i} + \mathbb{E}\left[J^\star(t_{i+1}, X_{t_{i+1}}, \widehat{V}_{u_i}(v)) | X_{t_i} = x, V_{t_i} = v \right] \right\} \quad (2.3.7)$$

The classic way to solve this problem numerically is to use Monte Carlo simulations, combined with the Longstaff and Schwartz [2001] algorithm, which approximates the above conditional expectation, using a regression technique. This backward algorithm yields an estimate of the optimal strategy. As noted by Boogert and De Jong [2008], the main difficulty comes from the fact that the value function also depends on volume level, which in turn depends on the optimal strategy.

To circumvent this difficulty, Boogert and De Jong [2008] suggest discretizing the volume into a finite grid, $v_l = V_{min} + l\delta, \quad l = 0, ..., L = (V_{max} - V_{min})/\delta$ where L is the number of volume subintervals. However, the fact that the time grid is discrete and that at each time the storage unit manager has only three possible actions implies that the number of attainable volumes for any strategy is finite. In fact, at each time t_i, the set $\mathcal{V}(i)$ of possible volumes is given by

$$\mathcal{V}(i) = \{V_i = v_0 + k a_{inj}\Delta t + l a_{with}\Delta t \text{ , such that } V_{min} \leq V_i \leq V_{max} \text{ and } k, l \in \mathbb{N} \text{ , } k+l \leq i\};$$

consequently, it is possible to solve the dynamic programming equation (2.3.7) for all volumes in $\mathcal{V}(i)$. The only motivation to use a restricted volume grid would be the reduction of computation time.

We then get the following equation at time t_i, for each path simulation X^m, where $m = 1, \ldots, M$, M being the total number of paths, and each volume level $v_l \in \mathcal{V}(i)$:

$$J^{\star}(t_i, X^m_{t_i}, v_l) = \max_{u_i \in \{inj,\ no,\ with\}} \left\{ \phi_{u_i} + \mathbb{E}\left[J^{\star}(t_{i+1}, X_{t_{i+1}}, \widehat{V}_{u_i}(v_l)) | X_{t_i} = X^m_{t_i}, V_{t_i} = v_l \right] \right\}. \tag{2.3.8}$$

The conditional expectation above is estimated using the Longstaff-Schwartz regression algorithm, at each volume grid point v_l.

This will give us an estimation of the optimal strategy, denoted $u^{\star}$, and the initial value of the gas storage unit $J^{\star}(t_0, X_{t_0}, v_0) = J(t_0, X_{t_0}, v_0; u^{\star})$. The numerical resolution of problem (2.3.7) is done in two phases. The first stage consists in estimating the optimal strategy $u^{\star}$, by performing the backward iterations of equation (2.3.8). The second phase consists in estimating the value function $J^{\star}$ through the forward iterations of (2.3.8), along a new set of simulated paths, where we apply the optimal strategy given by the backward algorithm.

One important remark about problem (2.3.7) is that the maximization is carried out for the expected wealth generated by the spot-trading strategy. Consequently, a manager who follows the optimal strategy on a single path is not assured to recover the initial storage value $J^{\star}$. There will certainly be a discrepancy between the realized cumulative cash flows on a given path and the expected value $J^{\star}$. Hence, it is crucial for the storage manager to reduce the variance of the cumulative cash flows, which is a random variable. As we will explain in the next section, this will be achieved by conducting a financial hedging strategy, based on futures contracts on natural gas.

2.3.3 Financial hedging strategy

After estimating the optimal strategy, the storage unit manager will follow these optimal decisions on the sample path revealed by the market. But one should keep in mind that, if one follows this optimal strategy $u^{\star}$, the cumulative wealth is only the realization of a random variable whose expectation equals the initial price $J^{\star}$ of the gas storage unit. This motivates the interest in hedging strategies that can reduce the variance of this random variable. This can be done by combining the optimal operating strategy with additional financial trades, so that the expectation of the related cumulative wealth generated by both physical and financial operations is still $J^{\star}$, but its variance (or some other risk criterion) is reduced. Analogously to Bjerksund et al. [2011], who treats the intrinsic value case, this additional financial hedging strategy plays a role analogous to control variates in the variance reduction of Monte Carlo simulations, as it preserves the expected cumulative cash flows and reduces its variance. In order to reduce the variance of a Monte Carlo estimator of a r.v. Y, one adds to it a mean zero control variate, which is highly (negatively) correlated to Y. Since futures contracts are the most liquid assets in the natural gas market, and are strongly correlated to the spot price, they form an ideal hedging instrument. In fact, although a futures contract price $F(t,T)$ does not converge to the spot price, when the time to maturity $T-t$ goes to zero, the correlation between the prompt contract (for example) and the spot price is very high, and often the two contracts move in the same direction. The basic idea of a financial hedging strategy is to add to the physical spot trading,

a strategy of buying and selling, at a trading date t_i, a quantity $\Delta(t_i, T_j)$ of futures contracts $F(., T_j)$ for $1 \leq j \leq m$. Logically, those quantities will depend on the spot and futures prices, S and $\{F(., T_j)\}_{1 \leq j \leq m}$, but also on the current volume level.

If the gas storage manager follows such a hedging strategy, in addition to the spot physical trading, then the cumulative cash flows of the combined strategies is:

$$\text{Wealth}_{\text{spot+futures}} = \sum_{i=0}^{n-1} \phi_{u_i^\star}(S_{t_i}) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} \Delta(t_i, T_j)(F(t_{i+1}, T_j) - F(t_i, T_j)). \quad (2.3.9)$$

Because the futures contract $F(., T_j)$ stops trading after its expiration date T_j, we use the convention $\Delta(t, T_j) = 0$, for $t \geq T_j$.

Since the futures price process is a martingale under the risk neutral probability, we have $\mathbb{E}_{t_i}\big[F(t_{i+1}, T_j)\big] = F(t_i, T_j)$. Hence, the expectation of this hedging strategy is null i.e.

$$\mathbb{E}\left[\sum_{i=0}^{n-1} \sum_{j=1}^{m} \Delta(t_i, T_j)(F(t_{i+1}, T_j) - F(t_i, T_j))\right] = 0.$$

Consequently, following the optimal spot strategy in parallel with a futures hedging portfolio gives the same cash flows in expectation, but very likely with lower variance.

$$\mathbb{E}\left[\text{Wealth}_{\text{spot+futures}}\right] = \mathbb{E}\left[\text{Wealth}_{\text{spot}}\right] = \mathbb{E}\left[\sum_{i=0}^{n-1} \phi_{u_i^\star}(S_{t_i})\right] = J^\star.$$

The specification of such a hedging strategy will of course depend on the nature of the relation between the spot price and the futures curve.

A heuristic strategy that is widely used in the industry is to take the quantity $\Delta_1(t_i, T_j)$ of futures $F(., T_j)$ to be equal to the conditional expectation of volume to be exercised during the delivery period of the futures contract, conditional on the information at t_i. More precisely, the heuristic delta is equal to the t_i-conditional expectation

$$\Delta_1(t_i, T_j) = \mathbb{E}_{t_i}[\sum_{T_{j-1} \leq t_l < T_j} V_{l+1}(u^\star) - V_l(u^\star)]. \quad (2.3.10)$$

We also propose a modification of this heuristic delta, where we use the concept of tangent process (Warin [2012]). If we assume that the prompt converges towards the spot, then we can write

$$\begin{aligned}\text{Wealth}_{\text{spot}} &= \sum_{i=0}^{n-1} \left(V_{i+1}(u^\star) - V_i(u^\star)\right) S_{t_i} \\ &\simeq \sum_{i=0}^{n-1} \left(V_{i+1}(u^\star) - V_i(u^\star)\right) P_{t_i} \\ &= \sum_{j} \sum_{T_{j-1} \le t_l < T_j} \left(V_{l+1}(u^\star) - V_l(u^\star)\right) F(t_l, T_j).\end{aligned}$$

which yields another heuristic delta Δ_2:

$$\Delta_2(t_i, T_j) = \mathbb{E}_{t_i}[\sum_{T_{j-1} \le t_l < T_j} \left(V_{l+1}(u^\star) - V_l(u^\star)\right) \frac{F(t_l, T_j)}{F(t_i, T_j)}] \tag{2.3.11}$$

We emphasize that the definition of these two hedging strategies is based on heuristic reasoning. Therefore, the hedging will not be perfect and a residual risk will still remain.

As we will see in the numerical experiments, this financial hedging strategy yields a significant reduction in the cash flows uncertainty of the spot trading strategy. This will be illustrated later by an out-of-sample test applied over a 10 years price history.

In the next section, we present several modeling approaches for the spot and futures prices processes, and study the consequences of various model choices.

2.4 Literature on price processes

Generally, the problem of gas storage unit valuation has been studied from the angle of numerical methods, and not much interest has been paid to the modeling itself and its effects on the final outcome of the calculation. Two modeling approaches may be found in he literature. The first approach consists in modeling the spot price by itself, with classical mean-reverting models, as proposed by Boogert and De Jong [2008]. We refer to Lautier [2005] for a review on the commodities spot models.

This approach does not offer the possibility of a hedging strategy based on futures and does not take into account the price spikes. The second approach is based on a model of the futures curve by multi-factor log-normal dynamics, and assumes that the spot price is the limit of the futures price, when time to maturity goes to zero. We note, however, that this assumption does not conform to reality in NG market. This assumption, however, enables the definition of a delta hedging strategy based on futures contracts.

We now describe in some details these two approaches.

2.4.1 Spot price processes

A very common framework consists in modeling the spot price as a mean-reverting process. For instance, Boogert and De Jong [2008] developed a Monte Carlo method for storage valuation, using the Least Square Monte Carlo method, as proposed by

Longstaff and Schwartz [2001] for American options. They consider one factor model for the spot price, which is calibrated to the initial futures curve. The price process S is given by

$$\frac{dS_t}{S_t} = \kappa[\mu(t) - \log(S_t)]dt + \sigma dW_t, \tag{2.4.1}$$

where W is a standard Brownian motion, μ is a time-dependent parameter, calibrated to the initial futures curve $(F(0,T))_{T\geq 0}$, provided by the market; the mean reversion parameter κ and the volatility σ are two positive constants.

As pointed out by Bjerksund et al. [2011], this framework has several drawbacks with respect to the goal of capturing the value of the gas storage. The calibration of the time-varying function $\mu(t)$ is, as expected, quite unstable and gives unrealistic sensitivity of the spot dynamics, and hence of the gas storage value with respect to the initial futures curve. More importantly, this spot modeling does not take into account the futures market and the possibility of trading strategies on futures contracts. The futures curve is only used as an initial input to calibrate the parameter μ, but no dynamics for the futures curve is assumed. Modeling the futures curve is indispensable in order to formulate the hedging strategies based on futures contracts. Finally, (2.4.1) does not account for price spikes, which is an important source of storage value.

An enhancement of this model is proposed by Parsons [2013], who considers the following two-factor mean-reverting model:

$$\frac{dS_t}{S_t} = a[\mu(t) + \log(L_t) - \log(S_t)]dt + \sigma_{S,t}dW_t, \tag{2.4.2}$$

$$\frac{dL_t}{L_t} = b[\log(\mathcal{L}) - \log(L_t)]dt + \sigma_{L,t}dZ_t, \tag{2.4.3}$$

where the spot price S follows a mean-reverting process, with a long-run mean which is itself a stochastic process reverting to a deterministic value $\mathcal{L}$.

While this model is more realistic than the one factor model, it still suffers from the instability of the deterministic function μ, and still does not include the possibility of spikes in the spot price. The author defines the futures contract price as the expectation of the spot price at maturity date T. We emphasize that this definition implies that natural gas is delivered at the futures expiration T; in reality, however, the delivery period spans an entire calendar month.

A second way to model the spot process is to consider it as the limit of the futures contract price as time to maturity goes to zero; in particular we have $S_t = \lim_{T\downarrow t} F(t,T)$. This approach was adopted by Warin [2012]; the author considers a n-factor log-normal dynamics for the futures curve:

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(t) e^{-a_i(T-t)} dZ_t^i, \tag{2.4.4}$$

and by continuity the spot is given by $S_t = F(t,t)$. In this framework, the author

presented a similar algorithm to Boogert and De Jong [2008] to estimate the optimal strategy for the spot; moreover he gives formulae for the sensitivities of storage value with respect to futures contracts. This provides a hedging strategy based on futures in parallel to the spot optimal trading strategy, aimed at reducing the uncertainty on the realized cash flows. This futures-based hedging strategy presents a big advantage, compared to the first approach, since it increases the manager's chances of recovering the storage value and consequently the price paid to rent the storage unit.

To summarize, the available research on storage valuation is based on models for the price processes that either do not capture important features of the spot process, or assume a convergence of the futures to the spot price that does not conform to reality. In contrast, we present next a joint, multi-factor model for the futures curve and the spot process. It captures the seasonality of the natural gas futures prices and the correlation between the spot and prompt prices. It also accounts for the presence of spikes in the spot price.

2.5 Our modeling framework

In Section 2.2 we discussed the main stylized facts of natural gas prices, which are seasonality and spikes. We believe that the incorporation of these two features is essential in order to monetize these two sources of value. Also, we emphasize that it is crucial to use a modeling framework that combines spot and futures curve dynamics, and that accounts for the presence of a basis between the spot and prompt prices.

In Section 2.5.1, we introduce a two-factors model for the futures curve, with a seasonal component for instantaneous volatility. This parsimonious model has easy-to-interpret parameters and can be efficiently calibrated using futures curve historical data.

In Section 2.5.2, we discuss the spot price model: we consider two formulations, with a clear relation to the prompt contract. We also include spikes by means of a fast-reverting jump process, similar to a model by Hambly et al. [2009], which was applied to the electricity market.

2.5.1 Modeling the futures curve

Early models of commodities futures prices $F(t,T)$ were obtained through conditional expectations of S_T with respect to the current information at time t, where S is the spot price process. c.f. for example the classical approaches by Gibson and Schwartz [1990] and Schwartz [1997] and. This process was linked to futures prices through additional, possibly stochastic, quantities such as convenience yield and interest rates.

This approach has several drawbacks such as the difficulty of observing or estimating those quantities and the problem of fitting the initial curve $F(0,T)$.

Hence a second generation of models was proposed to directly describe the futures curve, using multi-factor log-normal dynamics. For instance, Clewlow and Strickland [1999b] proposes a one-factor model for the futures curve; this was then extended by Clewlow and Strickland [1999a] to a multi-factor setting. A two-factor version of this

model can be expressed as

$$\frac{dF(t,T)}{F(t,T)} = e^{-\lambda(T-t)}\sigma_{ST}dW_t^S + \sigma_{LT}dW_t^L,$$

where λ, σ_{ST} and σ_{LT} are positive constants, and W^S and W^L are two correlated Brownian motions. This model has the advantage of exactly fitting the initial futures curve, and the dependence of the volatility on the maturity parameter, i.e. it is of term-structure type; however it does not take into account the essential seasonality feature. Note that this model is an adaptation of the well-known Gabillon [1991] model, originally proposed for spot prices. Our framework slightly modifies previous models, adding a seasonality component and introducing parameters that have an economical significance.

We will call it the Seasonal Gabillon two-factor model. It is formulated as

$$\frac{dF(t,T)}{F(t,T)} = e^{-\lambda(T-t)}\phi(t)\sigma_S dW_t^S + (1-e^{-\lambda(T-t)})\sigma_L dW_t^L, \tag{2.5.1}$$

where W^S and W^L are two correlated Brownian motions, with $d\langle W^L, W^S\rangle_t = \rho dt$. The letters L and S stand respectively for Long term and Short term; λ, σ_S and σ_L are positive constants. The function $\phi(t) = 1 + \mu_1 cos(2\pi(t-t_1)) + \mu_2 cos(4\pi(t-t_2))$ weights instantaneous volatility with a periodic behaviour. It takes into account the winter seasonal peaks (resp. the secondary summer peak) by taking for example t_1 equal to January (resp. t_2 equal to August). There exist alternative ways to model price seasonality, e.g. in Nowotarski et al. [2013], in the context of electricity markets. The coefficients μ_1 and μ_2 quantify the winter and summer seasonal contribution to volatility: we expect the winter parameter μ_1 to be larger, in absolute value, than the summer parameter μ_2.

This model constitutes an efficient framework, whose parameters are economically meaningful. Indeed, the parameters σ_L and σ_S can be interpreted as 'long-term' and 'short term' volatility. Note that even if the model is expressed with a continuous set of maturities, in the real world we only have access to a finite number of maturities, for example, monthly spaced futures contracts.

In the next section we give more details about the meaning of each parameter and their estimation, using historical data of futures prices.

Model estimation

Many of the model parameters are almost observable, if we have sufficient historical data of futures curves at hand. In fact, σ_S and σ_L could be approximated by the volatility of short and long-dated continuous futures contracts, and ρ by their empirical correlation.

For $T \to \infty$, we can formally write $\frac{dF(t,T)}{F(t,T)} \simeq \sigma_L dW_t^L$, so a good approximation for the long-term volatility is

$$\sigma_L^2 \simeq \frac{1}{m-1}\sum_{i=1}^{m}(\frac{z_{t_i}^L}{\sqrt{\Delta t_i}} - \bar{\mu}^L)^2,$$

where z_t^L is the log-return of a constant maturity long-dated contract, four years for example, and $\bar{\mu}^L = \frac{1}{m}\sum_{i=1}^{m}\frac{z_{t_i}^L}{\sqrt{\Delta t_i}}$.

Similarly, for small times to maturity, i.e. $T - t \to 0$, we can ignore the long-term noise effect, and write $\frac{dF(t,T)}{F(t,T)} \simeq \sigma_S dW_t^S$, so that a good proxy for the spot volatility is the volatility of the rolling prompt contract, i.e. the contract with the nearest maturity

$$\sigma_S^2 \simeq \frac{1}{m-1}\sum_{i=1}^{m}(\frac{z_{t_i}^P}{\sqrt{\Delta t_i}} - \bar{\mu}^P)^2,$$

where z_t^P is the log-return of a prompt futures contracts and $\bar{\mu}^P = \frac{1}{m}\sum_{i=1}^{m}\frac{z_{t_i}^P}{\sqrt{\Delta t_i}}$.

We can also give an initial estimate for the correlation parameter ρ as

$$\rho \simeq \frac{1}{m-1}\frac{\sum_{i=1}^{m}(\frac{z_{t_i}^P}{\sqrt{\Delta t_i}} - \bar{\mu}^P)(\frac{z_{t_i}^L}{\sqrt{\Delta t_i}} - \bar{\mu}^L)}{\sigma_S \sigma_L}.$$

These rough estimates could be used directly, or as input parameters for a more rigorous statistical estimation procedure. For example, we can use the maximum likelihood method. For that, suppose we have a time series over dates $t_1, \ldots, t_m$ of futures prices maturing at $T_1, ..., T_n$. We denote $z_t, t = t_i, i \in \{0, \ldots, t_{m-1}\}$ the vector of price returns, Δt being the corresponding step $t_{i+1} - t_i$ and θ is the model parameters vector $\theta = (\lambda, \mu_1, \mu_2, \sigma_S, \sigma_L, \rho)$, we have

$$z_t = \begin{pmatrix} \frac{\Delta F(t,T_1)}{F(t,T_1)} \\ . \\ . \\ . \\ \frac{\Delta F(t,T_n)}{F(t,T_n)} \end{pmatrix}, \quad H_t = \sqrt{\Delta t}\begin{pmatrix} e^{-\lambda(T_1-t)}\phi(t)\sigma_S, & (1-e^{-\lambda(T_1-t)})\sigma_L \\ . & . \\ . & . \\ . & . \\ e^{-\lambda(T_n-t)}\phi(t)\sigma_S, & (1-e^{-\lambda(T_n-t)})\sigma_L \end{pmatrix},$$

where $\Delta F(t,T_1) = F(t+\Delta t, T_1) - F(t,T_1)$. Then an Euler discretization of the SDE (2.5.1) gives the equation

$$z_t = H_t x_t, \ t \in \{t_1, \ldots, t_m\},$$

where (x_{t_i}) are independents Gaussian 2-d vectors such that

$$x_{t_i} \sim \mathcal{N}(0, \Sigma),\ 1 \leq i \leq m,$$

where

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

The likelihood maximization could then be written as the minimization of the function

$$L(x_{t_1}, x_{t_2}, ..., x_{t_m}|\theta) = \frac{1}{m}\sum_{i=1}^{m} log(det(\Sigma)) + x_{t_i}^T \Sigma^{-1} x_{t_i},$$

and the $x_t, t \in \{t_1, \ldots, t_m\}$ are given by $z_t = H_t x_t$, i.e.

$$x_t = (H_t^T H_t)^{-1} H_t^T z_t.$$

So, the model estimation procedure is equivalent to the following minimization problem

$$\begin{cases} \min\ L(x_{t_1}, x_{t_2}, ..., x_{t_m}|\theta) = log(det(\Sigma)) + \frac{1}{m}\sum_{i=1}^{m} x_{t_i}^T \Sigma^{-1} x_{t_i} \\ \theta = (\lambda, \mu_1, \mu_2, \sigma_S, \sigma_L, \rho). \end{cases} \tag{2.5.2}$$

To illustrate, we apply this estimation procedure, using daily futures curves from 1997 to 2007. As mentioned, the estimation problem (2.5.2) is solved using an optimization algorithm, with the rough estimates of σ_S, σ_L and ρ as initial point for the algorithm. We report in Table 2.2 the estimated parameters of the futures curve model.

Parameter	Value	Confidence interval
σ_S	0.4580	[0.4462,0.4698]
σ_L	0.1655	[0.1617,0.1694]
λ	0.7896	[0.7518,0.8274]
μ_1	0.0246	[-0.0015,0.0507]
μ_2	0.0038	[-0.0218,0.0294]
ρ	0.4113	[0.3737,0.4488]

Table 2.2: Estimated parameters using 1997-2007 futures curves history.

As expected, the short-term volatility is larger than the long-term volatility, which is a common feature in energy futures curve dynamics, and the winter contribution μ_1 in the seasonality component is larger than summer contribution μ_2.

2.5.2 Modeling spot price

We have argued in Section 2.4 that the spot price should be considered as a separate stochastic process, correlated to the prompt price, but which not the limit of this

prompt price when time to maturity tends to 0:

$$S_t \neq \lim_{T \to t} F(t,T).$$

A model in that sense was proposed by Gray and Palamarchuk [2010], where the logarithm of the spot is a mean reverting process, whose mean-reversion level is a stochastic process equal to the prompt price. For a family of maturities $(T_i)_i$, the futures contract $F(t,T_i)$ is a log-normal process fulfilling

$$\frac{dF(t,T_i)}{F(t,T_i)} = \sigma(t,T_i)dW_t$$

and the spot price S_t evolves according to

$$d\log(S_t) = (\theta_t + a\log(P_t) - a\log(S_t))dt + \sigma_t^S dB_t, \tag{2.5.3}$$

where B and W are two correlated Brownian motions, and for the current date t, P_t denotes the prompt price, i.e.

$$P_t = F(t,T_i) \quad \text{for} \quad T_{i-1} \leq t < T_i.$$

In our opinion it is crucial to incorporate futures curve dynamics into the modeling of the spot prices, for instance a dynamics relating the spot and prompt futures price. Indeed, as shown by the historical paths of spot and prompt prices in Figure 2.2, the two processes are closely related. In fact they seem to move very often in the same direction, with some occasional dislocations of spot and prompt prices.

In what follows we will study two spot models, connected to our futures curve model. They will be stated in discrete time.

Spot model 1

Our first spot model is similar to (2.5.3), which was introduced by Gray and Palamarchuk [2010]. Its dynamics, based on the spot log-return $y_t = \log(S_t/S_{t-1})$, is given by

$$\log(S_t/S_{t-1}) = a_1 + a_2\log(P_{t-1}/S_{t-1}) + a_3\log(P_t/P_{t-1}) + \epsilon_t \tag{2.5.4}$$

where (ϵ_t) is a GARCH(p,q) process and again P is the prompt price.

Recall that a GARCH(p, q) process ϵ verifies an autoregressive moving-average equation for its conditional variance σ:

$$\begin{aligned} \epsilon_t &= \sigma_t z_t \text{ , where} \\ \sigma_t^2 &= \kappa + \sum_{i=1}^{p} \gamma_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 \end{aligned} \tag{2.5.5}$$

where z_t is a white noise.

This model captures both the heteroscedasticity of the natural gas spot price and

the correlation between the spot price and the prompt futures price. Similarly to (2.5.3), the spot price dynamics described by (2.5.4) is mean reverting around a stochastic level equal to the prompt price. In addition, the prompt log return is a supplementary explanatory variable of the spot log return. Recall that our futures model (2.5.1) incorporates seasonality in the futures curve dynamics; this implies that the spot dynamics itself follows a seasonal pattern, transmitted by the prompt price.

Spot model 2

As an alternative, we model the spot process by modeling the return of the spot to prompt spread: $y_t = \frac{S_t - P_t}{P_t}$, using the so-called front-back spread as independent variable.

This alternate model is:

$$\frac{S_t - P_t}{P_t} = a_1 + a_2 \frac{S_{t-1} - P_{t-1}}{P_{t-1}} + a_3 \frac{P_{t-1} - B_{t-1}}{B_{t-1}} + \epsilon_t, \tag{2.5.6}$$

where B_t is the price of the second nearby futures (also known as the back contract) and ϵ is a GARCH(p, q) process.

(2.5.6) has the advantage of directly handling the spread between the spot and the prompt price, which is a key variable in gas storage management. Intuitively, a large positive spread value will generally induce the decision to withdraw gas, while the reverse is likely to motivate a gas injection. Also, as we pointed out in the introduction, the narrowing of the seasonal spread in the futures curve during last years has diminished the intrinsic value of gas storage units. Consequently, almost all the storage value is now concentrated in the extrinsic value, which is heavily dependent on the spot-prompt spread.

Spikes modeling

In Section 2.2, we showed that natural gas prices have two distinct characteristics: seasonality and presence of spikes. The first feature (seasonality), is captured by the seasonal factor in the futures curve dyamics (2.5.1). This seasonal pattern is transferred to the spot process by means of models that include the prompt and/or the back contract price as explanatory variables. There is no need to include a separate seasonal element in the spot dynamics.

Price spikes are another matter. These large and rapidly absorbed jumps are an essential feature of the spot process, since they can be source of value for gas storage and they can be monetized if injection/withdrawal rates are high enough.

They are mostly observed in the spot market, and we account for them by including a fast mean-reverting jump process in the spot model, in the same spirit as the electricity price model of Hambly et al. [2009].

These authors propose a spot model for the power price that incorporates spikes via a process Y, which is the solution of the equation

$$dY_t = -\beta Y_{t-}dt + dZ_t, \quad Y_0 = 0, \tag{2.5.7}$$

where Z is a compound Poisson process of the type $Z_t = \sum_{i=1}^{N_t} J_i$, (N_t) is a Poisson process with intensity λ and $(J_i)_{i\in\mathbb{N}}$ is a family of independent identically distributed (iid) variables representing the jump size. Furthermore (N_t) and (J_i) are supposed to be mutually independent. The process Y can be written explicitly as

$$Y_t = Y_0 e^{-\beta t} + \sum_{i=1}^{N_t} e^{-\beta(t-\tau_i)} J_i. \tag{2.5.8}$$

We recall that the spot model is directly expressed as a discrete time process, indexed on the grid (t_i) introduced in Section 2.3. For that reason Y will be restricted to the same time grid.

Choosing a high value for the mean-reversion parameter β forces the jump process Y to revert very quickly to zero after the jump times τ_i, which constitutes a desired feature for natural gas spikes. In practice, the jumps in natural gas spot prices are rapidly absorbed, precisely thanks to the existence of storage facilities.

Models (2.5.4) and (2.5.6) alone do not take into account the possibility of sudden spikes in the spot price. In order to add a jump component, the dynamics in (2.5.4) and (2.5.6) are multiplied by the process $\exp(Y_t)$:

$$\tilde{S}_t = \exp(Y_t) S_t.$$

As noted in Section 2.2, the natural gas spikes are clearly distinguished by their signs. Positive spikes, due to unpredicted weather changes, occur exclusively during the winter and summer months. Conversely, negative spikes, generally caused by a poor market anticipation of the storage situation, happen mostly during "shoulder months" such as October and November. This motivates a separate modeling for these two categories of spikes. We will consider two processes Y^+ and Y^- for positive and negative spikes, each one verifying a slightly modified version of equation (2.5.8):

$$Y_t^+ = \sum_{i=1}^{N_t} e^{-\beta(t-\tau_i)} J_i \mathbb{1}_{\tau_i \in I^+}, \tag{2.5.9}$$

where I^+ (resp. I^-) represents the time period where positive (resp. negative) spikes are observed, i.e. winter and summer (resp. shoulder months), as we observed in Section 2.2.

Putting the pieces together, the spot process that we consider for our gas storage valuation is

$$\tilde{S}_t = \exp(Y_t^+ + Y_t^-) S_t. \tag{2.5.10}$$

Finally (2.5.10) possesses all the desired properties: it includes seasonality in both

futures and spot prices, and it features positive and negative spikes in the spot process, each one generated by a separate jump process Y^+ and Y^-.

Model estimation

As for the futures model, we estimate the spot models with historical data for spot and futures prices. The parameters estimation for the two spot dynamics (2.5.4), (2.5.6) is based on regression techniques and the classic estimation procedure for GARCH processes. Following Hambly et al. [2009], we use the likelihood method to estimate the spike process parameters, after filtering the underlying time series to extract the jumps. Note that the coefficient β is heuristically fixed.

Regression parameters	Value	C.I.
a_1	-0.0054	[-0.0082, -0.0026]
a_2	0.2937	[0.2542, 0.3333]
a_3	0.4606	[0.3920, 0.5293]
Garch(1,1) parameters	Value	C.I.
κ	2.5936e-5	[1.5121e-5, 3.6751e-5]
γ_1	0.8458	[0.8225, 0.8691]
α_1	0.1452	[0.1180 , 0.1724]
Spike process Y^+	Value	C.I.
β	300	
λ	0.249905	[0.11691, 0.3829]
Jump Law	$\mathcal{N}(0.2499, 0.1169)$	[0.1824, 0.3174]
		[0.0848, 0.1884]
Spike process Y^-	Value	C.I.
β	300	
λ	1.2131	[0.70655, 1.71965]
Jump Law	$\mathcal{N}(-0.2295, 0.1124)$	[-0.3010, -0.1581]
		[0.0797, 0.1909]

Table 2.3: Spot model 1 parameters using 1997-2007 data

An analysis of the spot and futures historical data shows that a GARCH(1, 1) process is adequate. As mentioned before, we use a large value for the spike reversion parameter β.

To illustrate, the estimation of spot model 1, using a GARCH(1,1) process and a historical data from 1997 to 2007, yields the parameters summarized in Table 2.3.

2.6 Numerical results

In this section we use our futures-spot models to value various storage contracts, and compare our results to the intrinsic value of storage units. We consider two types of storage units, characterized by their maximum injection/withdrawal rates. A fast gas storage can be filled in, say, one month, but it often has limited capacity: salt caverns

are a common example of high deliverability storage units. Depleted oil/gas fields, or aquifers can also be used as storage facilities. They have very large capacities, but they suffer from low injection/withdrawal rates (see Appendix 2.A for details).

We will consider fast and slow storage units whose characteristics are described in Table 2.4. For simplicity, all the quantities are expressed in 10^6 MMBtu[2], while the storage values are expressed in million of US dollars. This means that the fast storage unit can be filled in 25 days, and emptied in 17 days. The slow storage unit needs 125 days to be completely filled and 83 days to be completely emptied. In all calculations, we ignore transaction costs.

	Fast storage	Slow storage
Total capacity	100	100
Injection rate	4 per day	0.8 per day
Withdrawal rate	6 per day	1.2 per day
Initial gas volume	0	0
Final gas volume	0	0
Lease duration	1 year	1 year

Table 2.4: Gas storage characteristics (fast and slow units)

The experiments were run using the Matlab software. We use 5000 simulations for the Monte Carlo method, with independent paths for the backward and forward phases of the Longstaff & Schwartz algorithm (see Section 2.3.2). First, we simulate a set of spot and futures paths, then we apply the dynamic programming algorithm (2.3.8) to estimate the optimal spot strategy; in parallel we evaluate the hedging strategy, based on futures contracts, according either to (2.3.10) or (2.3.11). We then re-simulate a new set of spot and futures paths, independent from the paths used in the preceding backward phase, and we apply the estimated optimal spot strategy, combined with the futures hedging strategy, to the new trajectories. We store the cumulative cash flows $\text{Wealth}_{\text{spot+futures}}(u^\star)$ resulting from these physical and financial operations for each sample path, and we compute the empirical mean and standard deviations of those cash flows. The mean of the cumulative wealth gives an estimate of the extrinsic value $J^\star$ of the gas storage unit, given in (2.3.7), while the standard deviation is an indicator of the dispersion of the realized cash flows around the extrinsic value. We emphasize that the empirical mean estimates the cash flow generated by the optimal strategy, while the empirical standard deviation gives an indicator of the variance reduction obtained through the financial hedging strategy. A lower standard deviation means that the manager will face less uncertainty on a single realization of the spot and futures prices. Numerical results confirm that the hedging strategy provides a significant reduction in the variance of the cumulative cash flows. Sample outputs from this valuation procedure are presented in Figure 2.4 (fast storage) and 2.5 (slow storage). In these figures, different colors correspond to different simulated spot trajectories.

Note also that the analysis described above depends on the choice of the model,

[2]This energy unit can be naturally converted into a volume, under standard conditions for temperature and pressure.

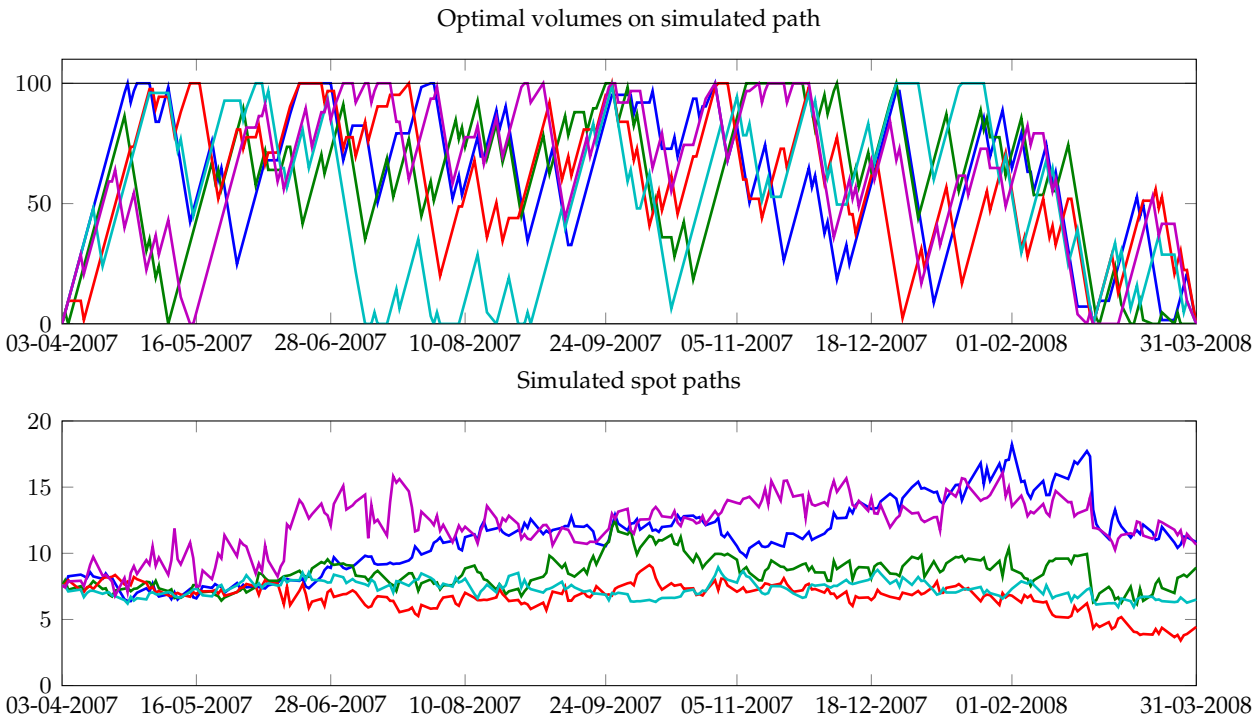


Figure 2.4: Valuation of a fast storage unit. For each simulated path (bottom panel), we display (top panel) the storage level corresponding to the optimal policy. Because of the fast injection/withdrawal speed, the storage level reacts quickly to changing market conditions.

because the backward and forward phases are executed on the sample paths generated by the model itself. In order to make the comparison less model-dependent, we calculate the cumulative cash flows of the estimated optimal strategy, based on spot and futures *historical paths*. For this reason, we will consider a series of spot and futures curve data from 2003 to 2012, and split it into periods of one year: the storage lease contracts specified in Table 2.4 start in April each year, for a one-year period. We run the optimal strategy obtained in the backward phase (for the corresponding storage duration) on the spot and futures historical paths for the related period. This constitutes a real case test for the optimal strategy and corroborates the relevance of the spot modeling, since it provides the profit that would have been accumulated by the storage manager in a realized path.

Figures 2.6 and 2.7 represent the historical spot path realized during the contract period (for both slow and fast units) from April 2007 to April 2008, and the natural gas volumes resulting from the optimal strategy computed on simulated paths (see Figures 2.4 and 2.5 for examples of these simulated paths).

We summarize the results of the valuation algorithm for each period in Tables 2.5 and 2.6, for the fast and slow storage units, when the spot paths are generated according to spot model 2 (2.5.6). The tables report, for each period, the intrinsic value (IV) (see (O_t) in Appendix 2.B), the estimated extrinsic value (EV) (computed on simulated

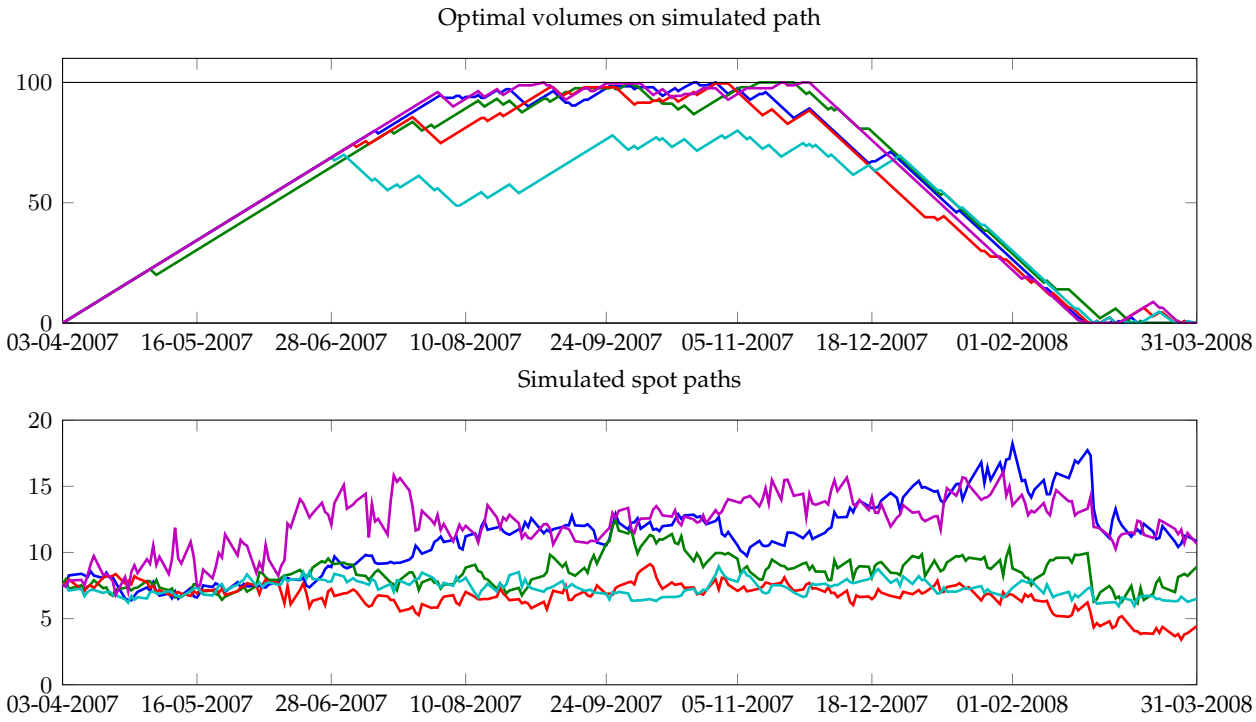


Figure 2.5: Valuation of a slow storage unit. Because of the slow injection/withdrawal rate, there is only one storage cycle per year, and the storage value is essentially function of the summer-winter spread.

paths), and finally the actual cumulative cash flow obtained by applying the optimal strategy on the historical path. The last two columns show the standard deviation of the simulated cash flows under the optimal strategy.

We expect that the extrinsic spot-based strategy will give a larger value than the intrinsic physical futures-based strategy, while our financial hedging strategy is supposed to reduce the uncertainty of gas storage cash flows. For example, the fast storage contract starting in April 2007 has an intrinsic value of $\$222.9689\ 10^6$ while the spot-based strategy gives an extrinsic value of $\$697.0003\ 10^6$. As expected, the extrinsic strategy allows better financial exploitation of the rights (without obligation) of injection/withdrawal natural gas compared to the conservative intrinsic strategy. In other words, the extrinsic strategy allows better extraction of the optionality of storage. We also note that the hedging strategy yields a significant empirical variance reduction of the cumulative cash flows from $\$340.2193\ 10^6$ to $\$190.8546\ 10^6$. On the other hand, the intrinsic value of slow storage is equal to $\$195.5517\ 10^6$, while the spot-based strategy captures a larger optionality value of $\$251.0064\ 10^6$. Similarly to fast storage, the financial hedging strategy allows an important reduction in variance, from $\$232.7825\ 10^6$ to $\$28.0414\ 10^6$.

Previous observations about year 2007 remain valid for the other test periods; indeed the extrinsic spot-based strategy always out-performs the intrinsic futures-based strategy, along both simulated and historical paths. The historical backtesting over the period 2003-2012 shows that the extrinsic strategy allows for better extraction of storage unit optionality, with a ratio of extrinsic value to intrinsic value as high as 500% for a fast storage unit. This performance of the extrinsic strategy is less significant in the case of slow storage unit, with a ratio up to 100%. This is due to limitations in the deliverability of slow storage. The optimal strategy is not able to fully benefit from gas price volatility, and cannot respond rapidly to favorable price movements. In all cases, hedging with financial instruments provides a significant reduction in the cumulative cash flows uncertainty. The last two columns of Tables 2.5 and 2.6 show a standard deviation reduction factor of up to 10, with better performance for slow storage. This gives the storage manager more insurance to recover a large percentage of the value of the storage contract.

Remark 2.6.1. 1. *In Section 2.3.3, we presented two heuristic hedging strategies,* (2.3.10) *and* (2.3.11)*, based on financial futures contracts. The numerical tests that we have conducted show that the hedging strategy defined by* (2.3.11) *gives better results in the variance reduction of the simulated cash flows under the optimal strategy; in addition, in the historical backtesting,* (2.3.11) *yields a better cumulative wealth performance than* (2.3.10)*. We emphasize that we have only reported about the better performing hedging strategy* (2.3.11)*.*

2. *We also note that the historical intrinsic value of the gas storage attains a peak in 2006, and shows a clear decline afterwards. This can be intuitively explained by observing the*

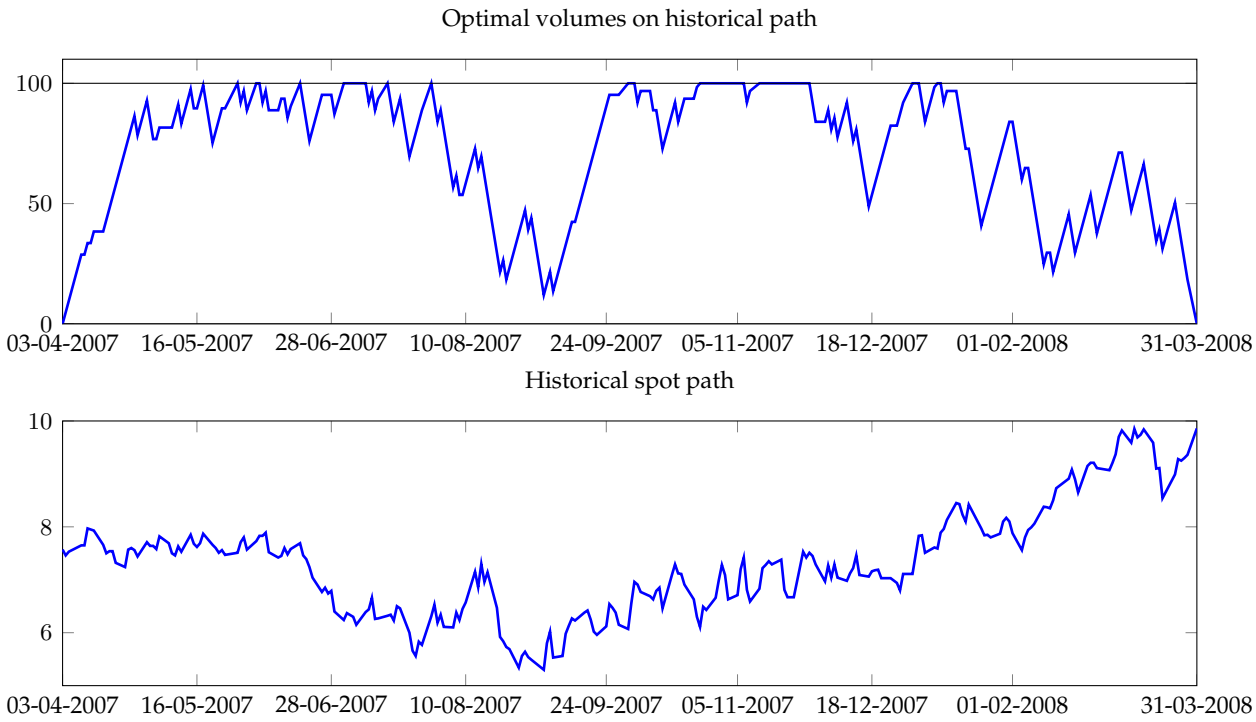


Figure 2.6: Historical spot path and optimal volumes (fast storage)

futures curve samples in Figure 2.1: in 2006, the seasonal spreads were very pronounced, but have been shrinking steadily ever since.

	Simulated paths test		Historical path test		Standard deviation	
Starting Date	IV	EV	IV	EV	Without hedge	With hedge
2003-Apr	39.9542	337.7276	42.6441	184.1178	189.5820	119.3606
2004-Apr	63.0335	395.6198	63.6000	347.2736	213.1796	126.3763
2005-Apr	115.2008	592.0854	112.0473	528.6510	306.3232	179.4792
2006-Apr	371.1724	860.9714	416.3992	616.2357	390.0864	194.9693
2007-Apr	222.9689	697.0003	241.8000	399.7347	340.2193	190.8546
2008-Apr	119.5200	674.6745	129.6000	427.9652	359.7650	210.6817
2009-Apr	204.6539	459.5531	205.9000	302.6958	203.2847	100.9753
2010-Apr	144.1958	420.2250	153.7000	259.1776	202.1802	112.6989
2011-Apr	86.5488	352.7785	92.9000	134.7794	190.0749	102.9312
2012-Apr	125.8968	272.5376	130.2000	215.9591	118.0606	55.4645

Table 2.5: Fast gas storage valuation (under spot model 2 (2.5.6))

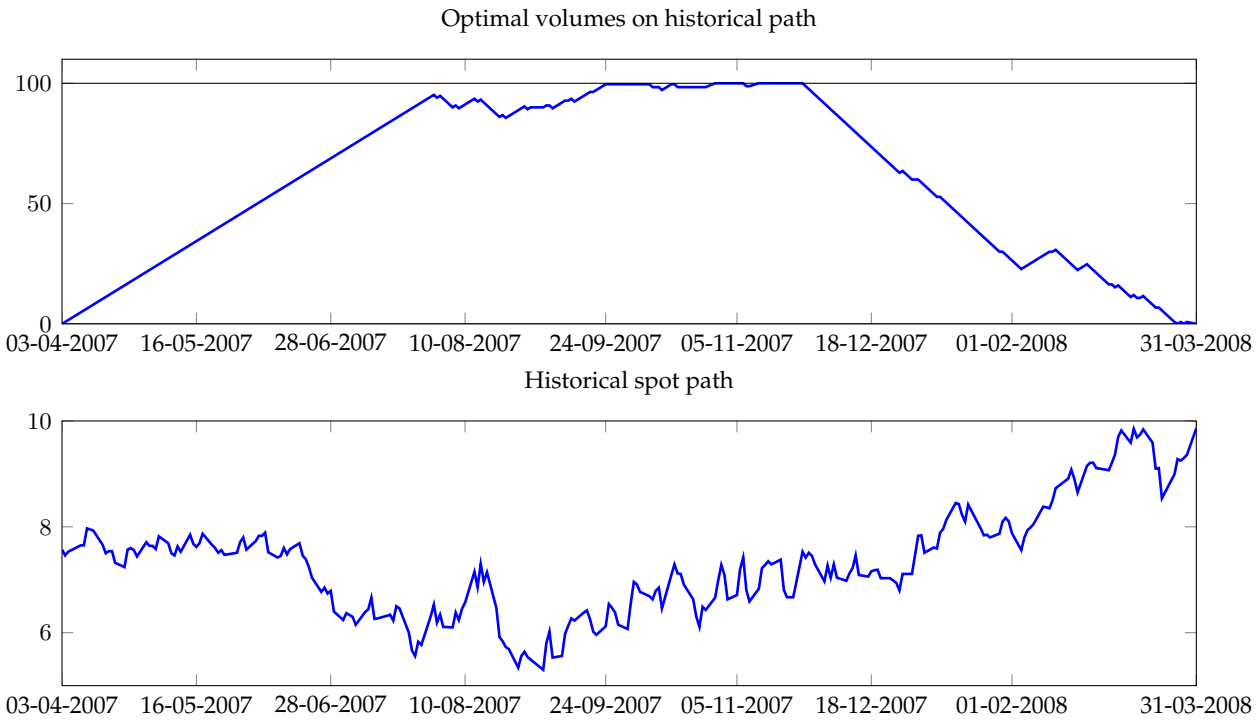


Figure 2.7: Historical spot path and optimal volumes (slow storage)

	Simulated paths test		Historical paths test		Standard deviation	
Starting Date	IV	EV	IV	EV	Without hedge	With hedge
2003-Apr	24.6556	67.5795	26.0563	16.7890	83.9382	18.5218
2004-Apr	45.2183	91.1053	44.6833	53.8389	119.8064	20.1098
2005-Apr	93.6136	157.9486	92.2304	146.3097	189.6219	28.0376
2006-Apr	333.1988	386.4656	333.1972	356.0564	282.2993	29.6749
2007-Apr	195.5517	251.0064	195.3024	221.4466	232.7825	28.0414
2008-Apr	96.8824	169.6740	98.5936	141.5038	216.1477	32.6633
2009-Apr	180.5010	206.6439	180.4980	210.8117	148.3145	14.6222
2010-Apr	122.4013	152.9924	122.3784	128.1330	140.5389	16.5936
2011-Apr	68.5264	104.2509	68.1356	72.3083	118.2672	18.8294
2012-Apr	107.4493	122.5703	107.3928	110.0214	86.2897	8.4167

Table 2.6: Slow gas storage valuation (under spot model 2 (2.5.6))

We conclude from the numerical results presented above that the joint modeling of the natural gas spot price and futures curve is a pertinent framework for the gas storage valuation and hedging problem. It allows the unit manager to better exploit storage optionality by monetizing the spot price volatility and seasonality. Indeed, the historical backtesting shows that the extrinsic value under this modeling always outperforms the classical intrinsic value, even in the case of slow storage. A joint model for the futures curve with its own risk factors is a more realistic framework for spot and futures markets, since it takes into account the seasonality of the futures curve and the non-convergence of the futures price to the spot price, an unrealistic

hypothesis that is often made in the literature. This also allows for a more relevant hedging strategy based on futures contracts, and better tracking of the extrinsic value of gas storage in real market conditions.

2.7 Model risk

As we showed in the introduction, seasonal spreads have become narrower these last years, which leads to a concentration of almost all the value of gas storage in the extrinsic part, based on spot trading. Hence it is very important to look closely into the spot modeling and its effect on storage valuation and hedging. We believe that the uncertainty surrounding storage value is due more to the uncertainty of the spot modeling than of the futures modeling, since only spot evolution determines the optimal strategy even though the futures contract prices intervene in spot modeling, see (2.5.4) and (2.5.6). In fact, the futures model mostly affects the quality of the hedge or variance reduction, not the expected value of the storage unit. This section is divided in two parts: we first compare the performances of the two spot models proposed in Section 2.5.2, using historical data. We focus on the effect of various modeling hypotheses, and on the sensitivity of the storage estimated value with respect to the model parameters. We next define a model risk measure to quantify these uncertainties, following Cont [2006].

2.7.1 Spot modeling

In Section 2.5.2, we proposed two discrete models for the spot price dynamics. The first model, defined in (2.5.4), is a discrete version of a mean-reverting model, with a stochastic mean-reversion level equal to the prompt price. The second model (2.5.6), directly captures the spread between the spot and the prompt prices, which is a key variable in the optimal management of a storage unit: one tends to buy and store gas when the spot-prompt spread is negative and withdraw it in the opposite case. Since the seasonality of gas prices has been getting weaker in recent years, the principal source of value for the storage unit is the spot-prompt spread rather than the winter-summer spread, so we expect the second model (2.5.6) to give good results in recent years.

We run the valuation procedure explained in Section 2.6 for the two spot models, during the testing periods between 2003 and 2012, and compute the performance of both models through historical spot and futures paths: in particular, we report in Figures 2.8 and 2.9 the cumulative cash flows using the optimal spot strategy for historical spot and futures trajectories. In the fast storage case, Figure 2.8 shows that spot-prompt spread model 2 yields slightly better results than spot model 1 in all the test cases, except for the year 2004. In the slow storage case, see Figure 2.9, the two spot models give comparable results for all periods. In the fast storage case, other tests, not reported in this article, show that with spot model 2, the cumulative cash flows has a lower standard deviation than with spot model 1, which reinforces the observation that spot model 2 is globally better suited for our purpose.

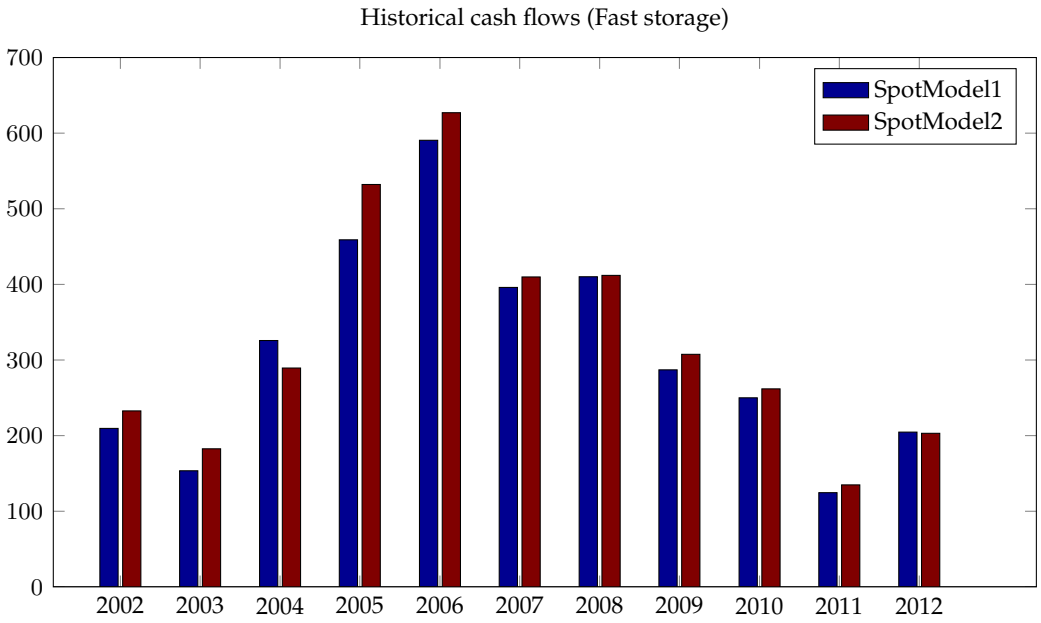


Figure 2.8: Historical cash flows for spot models 1 and 2 (fast storage)

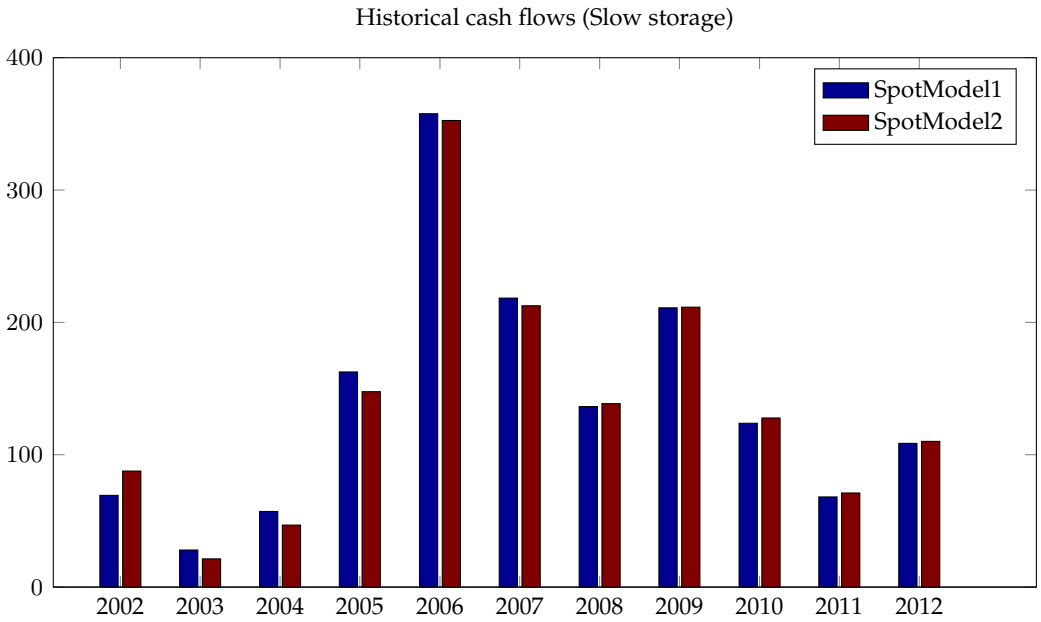


Figure 2.9: Historical cash flows for spot models 1 and 2 (slow storage)

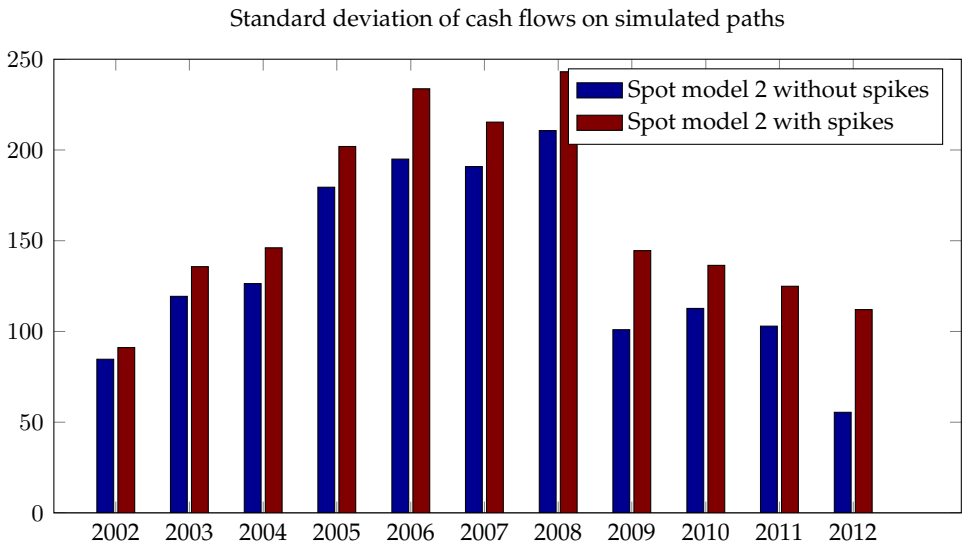


Figure 2.10: Standard deviation of cash flows on simulated paths

Effect of spikes modeling

The presence of spikes in natural gas prices is an essential feature of the dynamics of spot prices. As we noted in Section 2.2, these jumps are sudden dislocations between the cash and futures markets due to unexpected imbalances between supply and demand, caused by such factors as unpredicted weather changes, disruptions in the supply chain, or poor anticipations of the global amount of gas in storage.

These spikes can be a source of value for the storage manager, since a large gap between spot and prompt prices can be monetized by buying gas during a negative spike, and selling gas during a positive spike. Since these are rapidly absorbed by the market, the value can only be captured by fast storage units.

Figure 2.11 represents the expected cumulative cash flows of a fast storage unit, on simulated paths under spot model (2.5.6). All the test periods show that modeling the spikes in the spot dynamics gives a larger extrinsic value for the storage unit, but at the same time it introduces a larger standard deviation for the cumulative cash flows, as illustrated in Figure 2.10.

A final test of the effect of the spikes modeling is performed on historical spot paths for each test period, and results are shown in Figure 2.12.

The graph shows that modeling the spikes does not make a significant contribution to realized optimal value. This accords with the fact that the models with spikes produce a large standard deviation. In conclusion, surprisingly, this historical back test does not support the need for incorporating spikes in the spot model.

2.7.2 Model risk measure

In order to quantify the modeling uncertainty, we use an approach introduced by Cont [2006] for measuring the model risk inherent in the pricing of exotic derivative products. The approach may be summarized as follows: Given a set of benchmark quotes for vanilla options (or bid/ask intervals), model uncertainty for an exotic payoff H, is

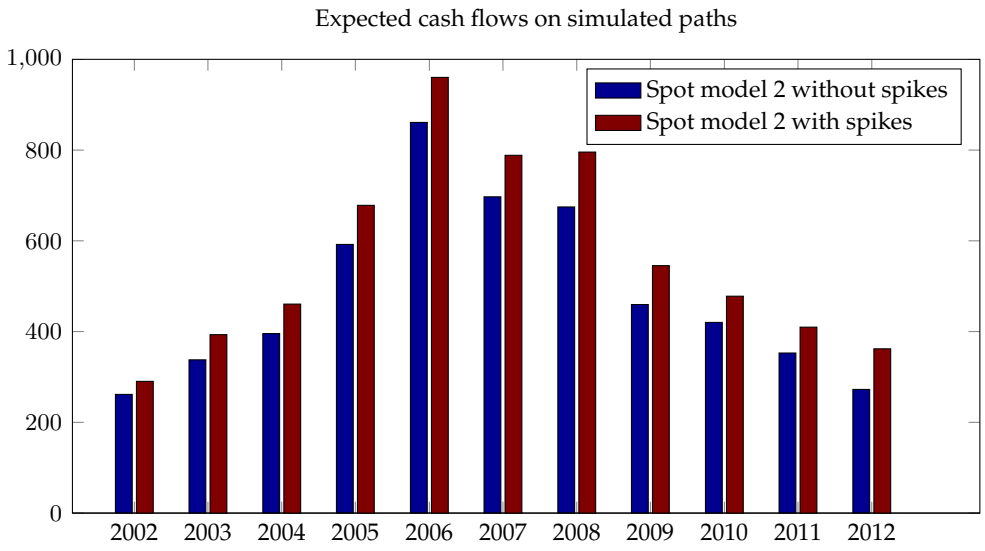


Figure 2.11: Expected cash flows on simulated paths

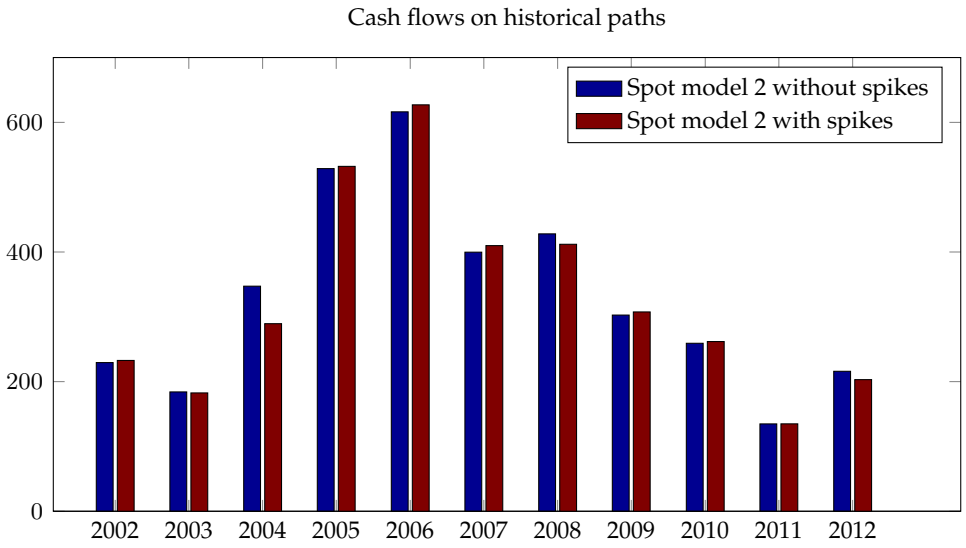


Figure 2.12: Cash flows on historical paths

quantified by computing the range of prices of this exotic product, using a set of risk neutral models Γ calibrated to the benchmark vanilla prices, i.e.

$$\pi(H) = \max_{Q\in\Gamma} \mathbb{E}^Q[H] - \min_{Q\in\Gamma} \mathbb{E}^Q[H]. \tag{2.7.1}$$

For our gas storage valuation problem, we will adapt this risk measure by using as "calibration" data the historical prices of the futures and spot contracts. The constraint of calibration on vanilla prices is replaced by the success of suitable statistical tests and closeness to the optimal likelihood objective function value of the model.

The family Γ consists of a set of spot models, (2.5.4) or (2.5.6), which pass the statistical tests imposed by the modeling hypothesis for the noise (ϵ_t), which is assumed to be GARCH(1,1). Moreover, the family Γ is restricted to the models that have a likelihood function value close to the optimal one found during the model estimation.

This methodology for the generating the set Γ is broadly similar to the one proposed by Dumont and Lunven [2006], and applied to multi-asset options. In this study, the authors calibrate a multi-assets model to single-asset vanilla options, then build the set Γ by perturbation of the correlation matrix. This yields a family of models that price the benchmark vanilla options perfectly, but differ by their correlation matrix.

In our case, the statistical estimation of the spot model parameters, in (2.5.4) or (2.5.6), is obtained by classical maximum likelihood methods. The estimation procedure solves:

$$\max_{\theta=\{a_1,a_2,a_3,\kappa,\gamma_1,\alpha_1\}} L(\theta),$$

where $L(\theta)$ is the likelihood function associated with the spot model (2.5.4) or (2.5.6). This maximization yields an optimal parameters vector $\theta^\star = \{a_1^\star, a_2^\star, a_3^\star, \kappa^\star, \gamma_1^\star, \alpha_1^\star\}$, an optimal likelihood function value $L(\theta^\star)$, and an empirical variance-covariance matrix $\Sigma^\star$ of the parameter estimates, from which confidence intervals may be computed.

In order to generate the family of spot models, we perturb the optimal parameters $\theta^\star$ by adding a Gaussian noise with the specified covariance matrix $\Sigma^\star$ to $\theta^\star$. This yields a set of perturbed parameters $\{\theta_i\}_{i\in I}$, from which we only retain those that satisfy two constraints: first, the inferred GARCH white noise $z(\theta_i)$ in (2.5.5) must pass a statistical test for normality [3]; second, the corresponding likelihood function value $L(\theta_i)$ has to be close to the optimal value $L(\theta^\star)$: $L(\theta_i) > (1-\epsilon)L(\theta^\star)$, where ϵ is a small constant.

In the following discussion, Γ will be the set $\{\theta_i, i \in I\}$, fulfilling the two conditions above.

We can now define the associated model risk. The analogue risk measure to (2.7.1) can be expressed using (2.3.7), with the value function now written$J^\star(\theta)$ to emphasize the dependence of this value function on the parameters θ. The normalized risk

[3] We use a Kolmogorov-Smirnov test for the normality test of the inferred noise z.

measure is given by:

$$\pi_1 = \frac{\max_{\theta_i \in \Gamma} J^\star(\theta_i) - \min_{\theta_i \in \Gamma} J^\star(\theta_i)}{J^\star(\theta^\star)}. \tag{2.7.2}$$

In this risk measure evaluation, each $J^\star(\theta_i)$, is calculated using spot and futures paths simulated under the perturbed model θ_i.

Moreover, we propose a second model risk measure based on the performance on realized historical spot and futures paths. For this we define

$$\pi_2 = \frac{\max_{\theta_i \in \Gamma} \text{Wealth}_{\text{spot+futures}}(\theta_i) - \min_{\theta_i \in \Gamma} \text{Wealth}_{\text{spot+futures}}(\theta_i)}{\text{Wealth}_{\text{spot+futures}}(\theta^\star)}, \tag{2.7.3}$$

where $\text{Wealth}_{\text{spot+futures}}$ represents the cumulative cash flows, computed on the historical path, as defined in (2.3.9).

The two risk measures π_1 and π_2 are computed for each of the test periods from 2003 to 2012, under the two spot models 1 and 2, using a set of 30 perturbed models. The results reported in Table 2.7 again show a better performance for spot model 2. In fact, this model seems to be less subject to model risk, since it gives a smaller range of prices, compared to spot model 1.

	Risk measure π_1		Risk measure π_2	
Starting date	Spot model 1	Spot model 2	Spot model 1	Spot model 2
2003-Apr	51.33 %	44.8085 %	70.8465 %	39.3852 %
2004-Apr	25.4987 %	23.6942 %	26.5597 %	22.3195 %
2005-Apr	26.0388 %	27.0318 %	50.7306 %	38.352 %
2006-Apr	14.9666 %	15.9873 %	10.6853 %	6.6954 %
2007-Apr	93.8336 %	14.7645 %	29.4626 %	18.6143 %
2008-Apr	37.9839 %	13.8195 %	16.6811 %	8.6166 %
2009-Apr	20.7969 %	10.1216 %	15.1415 %	8.1936 %
2010-Apr	26.7845 %	12.8976 %	33.0669 %	7.5285 %
2011-Apr	25.9442 %	12.3857 %	35.8704 %	30.9282 %
2012-Apr	16.7783 %	9.1489 %	13.1014 %	7.1694 %

Table 2.7: Model risk measure for spot models 1 and 2.

One observation that follows clearly from Table 2.7 is that the range of prices induced by the model uncertainty and measured by π_1 and π_2 represents a large proportion of the storage value. This shows that the dependence of gas storage valuation on spot modeling is quite significant. While the literature has concentrated its efforts until now on the specification of an optimal valuation strategy, we believe that one should pay more attention to the choice of the spot-futures modeling framework. Referring again to Table 2.7, model 2 appears to be less sensitive to the change of parameters and is therefore more robust. Fortunately, this is in concordance with the better performance of spot model 2 already observed in Section 2.7.1. Table 2.7 shows that the spot-futures valuation framework is subject to a large model risk (average: 25%). For

comparison, the model risk for a basket option has been evaluated to 3% (see Dumont and Lunven [2006]).

2.8 Conclusion

In this paper we consider the problem of gas storage valuation. After restating the main stylized facts of natural gas prices, specifically seasonality and spikes, we present a joint modeling framework for the futures curve and the spot price, and introduce two different models for the spot process. Using a Monte Carlo simulation method, we determine the optimal spot trading strategy; in addition, for the purpose of variance reduction of the cumulative cash flow, we set up a financial hedging strategy. We also conduct extensive back testing using historical data of futures and spot prices over a period of 10 years. This confirms the superior performance of the extrinsic strategy compared to the classic intrinsic futures-based strategy.

More importantly, we study the model uncertainty associated with this valuation method, concentrating on the risk associated with the spot model. After a quantitative comparison of the two spot models, we conclude that the model based on the spot-prompt spread performs better.

In order to quantify the stability of our valuation estimates with respect to model uncertainty, we next define two model risk measures, inspired by the work of Cont [2006]. Our context is however different from Cont's, in the sense that our models are estimated on historical data, and not on market data. This motivates a redefinition of the notion of "benchmark data".

Using those risk measures, we observe the great sensitivity of gas storage value to the modeling assumptions. In fact the model uncertainty, as measured by the size of price range, represents a large proportion of the storage value. This puts into perspective the concentration of effort in the literature on the specification of an optimal valuation strategy. Much more attention should probably be devoted to the discussion of modeling assumptions.

Appendix 2.A Different types of gas storage facilities

Natural gas storage units are underground facilities; as a result, their characteristics essentially depend on the geological properties of the storage area. There are three types of gas storage units: depleted gas/oil fields, aquifers and salt caverns. The main characteristics that distinguish these gas storage units are their injection/withdrawal rates, the total capacity and the so-called cushion volume and working volume. The cushion volume is the quantity of gas that must remain in the storage unit to provide the required pressurization, and the working volume is the volume of gas that can be extracted. Using the notations in this article, the cushion volume corresponds to the minimum volume V_{min}, the total capacity corresponds to V_{max}, and the working volume is represented by the actual volume minus the cushion volume, i.e. $V_t - V_{min}$.

These characteristics distinguish two different types of gas storage: base-load and peak-load. Base-load units are used to meet seasonal demand (a more or less predictable phenomenon). In fact the demand for gas is highly concentrated in the winter season, so in order to ensure sufficient supply, gas is bought and stored in the summer season then withdrawn and sold in winter. The main characteristics of base-load units are their large volume capacity and low deliverability rates.

On the other hand, peak-load units are used to mitigate the risk of unpredictable increases in the gas demand, generally caused by weather changes or technical problems in the pipeline system. Hence, they have to be very reactive and have high deliverability rates, higher injection/withdrawal rates, and in general they contain less gas than base-load units. While the injection/withdrawal cycle of a base-load is in general one year, peak-loads can have a turn-over period of a few weeks.

The depleted gas/oil fields and aquifers are of the base-load type, while salt caverns are peak-load facility. We summarize here their main characteristics.

- **Depleted gas and oil fields** are the most commonly used underground storage sites because of their wide availability. Besides their large capacities, they benefit from the already available wells and injection/withdrawal equipments, pipelines etc. Their main drawbacks are low deliverability rate and the large *cushion* gas percentage (although part of this non-usable gas already exists in the geological formation). Therefore, these depleted fields naturally belong to the base-load category.

- **Aquifers** are underground, porous and permeable rock formations that act as natural water reservoirs.

  They are flexible units with small volume, but more expensive than depleted fields since everything has to be built from scratch (wells, extraction equipments, pipelines, etc). The construction of the required infrastructure can take up to four years, which is more than twice the time needed to transform depleted reservoirs into storage facilities.

  Aquifers, however, require a greater percentage of *cushion* gas (up to 80% of the total gas volume) than depleted reservoirs.

Similar to depleted fields, aquifers operate on a single annual cycle, so they still belong to the base-load category.

- **Salt caverns** are the third common choice for gas storage. They are created by dissolving and extracting a certain amount of salt from the geological formation; this process then leaves a cavern that can be used for natural gas storage.

  A salt cavern offers storage with high deliverability, with low *cushion* gas requirements (30% of *cushion* gas), but with lower capacities than depleted fields and aquifers. They cannot be used to meet base-load storage requirements, but they are well suited to rapid actions, which are distinctive features of the peak-load category.

Table 2.8, compiled by the Federal Energy Regulatory Commission (FERC),[4] summarizes the three types of storage and their characteristics.

Type	**Cushion to working gas ratio**	**Injection period (days)**	**Withdrawal period (days)**
Aquifer	50% to 80%	200 to 250	100 to 150
Depleted oil/gas reservoir	50%	200 to 250	100 to 150
Salt cavern	20% to 30%	20 to 40	10 to 20

Table 2.8: Types of natural gas storage.

Appendix 2.B Futures-based valuation methods

Futures-based valuation is still very commonly used in natural gas storage management. This is mainly due to its simplicity and its low risk profile. It is based on trading natural gas futures and using the storage facility to handle physical delivery.[5] The optimal trading schedule is determined once and for all at the beginning of the lease, based on the futures prices, thereby securing a certain profit.

In order to determine the optimal futures positions, a linear optimization problem is solved, with constraints imposed by the specifics of the storage unit Eydeland and Krzysztof [2002]. Assume that N futures contracts, expiring at times $T_J, j = 1, \ldots, N$, are available for trading, and let $F(t, T_j)$ be the price at t of a futures contract expiring at time T_j. The optimization problem amounts to finding the number of futures contracts $\alpha_j(t) \equiv \alpha_j(t_0)$ to buy or sell at the start of the lease, and can be expressed as follows:

[4] *Current State of and Issues Concerning Underground Natural Gas Storage*, Federal Energy Regulatory Commission (FERC), Staff Report, September 30, 2004. cf: http://www.ferc.gov/EventCalendar/Files/20041020081349-final-gs-report.pdf

[5] For example, the Nymex NG futures have monthly spaced maturities, and the delivery period extends over the calendar month following each maturity date.

$$
\begin{aligned}
IV(t) := \max_{(\alpha_j(t))_{j=1,\dots,N}} & -\sum_j \alpha_j(t) F(t,T_j) \\
-a_{with} \leq \alpha_j(t) \leq a_{inj}, & \text{ for } j = 1, ..., N \\
V_{min} \leq V(t) + \sum_{j=1}^{n} \alpha_j(t) \leq V_{max}, & \text{ for } n = 1, ..., N,
\end{aligned}
\tag{O_t}
$$

where $V(t)$ is the gas in storage at time t, $V_{min/max}$ are bounds on storage volume and $a_{inj/with}$ are the maximum volumes that can be injected or withdrawn.

In its basic formulation, the storage manager solves (O_t) once and for all at time $t = t_0$, and $IV(t_0)$ represents a certain optimal profit, based on the information available at that time.

It is called *intrinsic value* because it does not take advantage of the rebalancing option available to the storage manager.

This static methodology was extended by Gray and Khandelwal [2004] to the *rolling intrinsic valuation*, to take advantage of shifts in the futures curve. Consider a set of trading dates $t_0 < t_1 < ... < t_{n-1} < t_n$, where t_n is the end of the storage lease, and set $\Delta t = t_{i+1} - t_i, i = 1, \dots, n-1$.

As before, the storage manager solves O_t at the beginning of the storage lease, and builds his initial portfolio of futures contract. He also repeats this calculation at each trading date t_i, and rebalances his portfolio if this is profitable.

More precisely, suppose that at date t, the manager owns a futures portfolio $\alpha^\star(t)$; then, at date $t + \Delta t$, the manager solves $O_{t+\Delta t}$, calculating an optimal portfolio $\alpha^\star(t + \Delta t)$ and $IV(t + \Delta t)$. The value of rebalancing the futures portfolio from $\alpha(t)^\star$ to $\alpha^\star(t + \Delta t)$ is equal to

$$
C(t, \Delta t) := \sum_j [\alpha_j^\star(t) - \alpha_j^\star(t + \Delta t)] F(t + \Delta t, T_j),
$$

and the manager alters his portfolio if this rebalancing value is positive. We denote by $RI(t)$ the so-called rolling intrinsic value at time t, with, by definition, $RI(t_0) = IV(t_0)$. For each trading date, define recursively the cumulative profit generated by this enhanced strategy by:

$$
RI(t + \Delta t) := RI(t) + \max(C(t, \Delta t), 0)
$$

Obviously, at each rebalancing date t, the rolling intrinsic value $RI(t)$ is always greater or equal to the intrinsic value $IV(t_0)$.

Remark 2.B.1. *The intrinsic and rolling intrinsic methodologies capture the seasonal pattern of natural gas prices: they lead to buying cheap summer futures and selling expensive winter futures. The corresponding storage value greatly depends on the seasonal spread. With the recent tightening of the spread (cf Section 2.2) the (rolling) intrinsic strategy has becomes less attractive to practitioners.*

3

BSDEs, càdlàg martingale problems and mean-variance hedging under basis risk.

This chapter is the object of Laachir and Russo [2014].

Abstract

The aim of this paper is to introduce a new formalism for the deterministic analysis associated with backward stochastic differential equations driven by general càdlàg martingales. When the martingale is a standard Brownian motion, the natural deterministic analysis is provided by the solution of a semilinear PDE of parabolic type. A significant application concerns the hedging problem under basis risk of a contingent claim $g(X_T, S_T)$, where S (resp. X) is an underlying price of a traded (resp. non-traded but observable) asset, via the celebrated Föllmer-Schweizer decomposition. We revisit the case when the couple of price processes (X, S) is a diffusion and we provide explicit expressions when (X, S) is an exponential of additive processes.

3.1 Introduction

The motivation of this work comes from the hedging problem in the presence of basis risk. When a derivative product is based on a non traded or illiquid underlying, the specification of a hedging strategy becomes problematic. In practice one frequent methodology consists in constituting a portfolio based on a (traded and liquid) additional asset which is correlated with the original one. The use of a non perfectly correlated asset induces a residual risk, often called **basis risk**, that makes the market incomplete. A common example is the hedging of a basket (or index based) option, only using a subset of the assets composing the contract. Commodity markets also present many situations where basis risk plays an essential role, since many goods (as kerosene) do not have liquid future markets. For instance, kerosene consumers as airline companies, who want to hedge their exposure to the fuel use alternative future contracts, as crude oil or heating oil. The latter two commodities are strongly correlated to kerosene and their corresponding future market is liquid. Weather derivatives constitute an example of contract written on a non-traded underlying, since they are based on heating temperature; natural gas or electricity are in general used to hedge

these contracts.

In this work, we consider a maturity $T > 0$, a pair of processes (X, S) and a contingent claim of the type $h := g(X_T)$ or even $h := g(X_T, S_T)$. X is a non traded or illiquid, but observable asset and S is a traded asset, correlated to X. In order to hedge this derivative, in general the practitioners use the *proxy* asset S as a hedging instrument, but since the two assets are not perfectly correlated, a basis risk exists. Because of the incompleteness of this market, one should define a risk aversion criterion. One possibility is to use the utility function based approach to define the hedging strategy, see for example Davis [2006], Henderson and Hobson [2002], Monoyios [2004], Monoyios [2007], Ceci and Gerardi [2009, 2011], Ankirchner et al. [2010]. We mention also Ankirchner et al. [2013] who consider the case when an investor has tow possibilities, either hedge with an illiquid instrument, which implies liquidity costs, or hedge using a liquid correlated asset, which entails basis risk. Another approach is based on the quadratic hedging error criterion: it follows the idea of the seminal work of Föllmer and Schweizer [1991] that introduces the theoretical bases of the quadratic hedging in incomplete markets. In particular, they show the close relation of the quadratic hedging problem with a special semimartingale decomposition, known as the Föllmer-Schweizer (F-S) decomposition. The reader can consult Schweizer [1994, 2001] for basic information on F-S decomposition, which provides the so called *local risk minimizing hedging strategy* and it is a significant tool for solving the *mean variance hedging problem* in an incomplete market.

Hulley and McWalter [2008] applied this general framework to the quadratic hedging under basis risk in a simple log-normal model. They consider for instance the two-dimensional Black-Scholes model for the non traded (but observable) X and the hedging asset S, described by

$$\begin{aligned} dX_t &= \mu_X X_t dt + \sigma_X X_t dW_t^X, \\ dS_t &= \mu_S S_t dt + \sigma_S S_t dW_t^S, \end{aligned}$$

where (W^X, W^S) is a standard correlated two-dimensional Brownian motion. They derive the F-S decomposition of a European payoff $h = g(X_T)$, i.e.

$$g(X_T) = h_0 + \int_0^T Z_s^h dS_s + L_T^h, \tag{3.1.1}$$

where L^h is a martingale which is strongly orthogonal to the martingale part of the hedging asset process S. Using the Feynman-Kac theorem, they relate the decomposition components h_0 and Z^h to a PDE terminal-value problem. This yields, as byproduct, the price and hedging portfolio of the European option h. These quantities can be expressed in closed formulae in the case of call-put options. Extensions of those results to the case of stochastic correlation between the two assets X and S, have been performed by Ankirchner and Heyne [2012].

Coming back to the general case, the F-S decomposition of h with respect to the $\mathcal{F}_t$-

semimartingale S can be seen as a special case of the well-known backward stochastic differential equations (BSDEs). We look for a triplet of processes (Y, Z, O) being solution of an equation of the form

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_{s-}, Z_s) dV_s^S - \int_t^T Z_s dM_s^S - (O_T - O_t), \tag{3.1.2}$$

where M^S (resp. V^S) is the local martingale (resp. the bounded variation process) appearing in the semimartingale decomposition of S, O is a strongly orthogonal martingale to M^S, and $\hat{f}(\omega, s, y, z) = -z$.

BSDEs were first studied in the Brownian framework by Pardoux and Peng [1990] with an early paper of Bismut [1973]. Pardoux and Peng [1990] showed existence and uniqueness of the solutions when the coefficient $\hat{f}$ is globally Lipschitz with respect to (y, z) and h being square integrable. It was followed by a long series of contributions, see for example El Karoui et al. [2008] for a survey on Brownian based BSDEs and applications to finance. For example, the Lipschitz condition was essential in z and only a monotonicity condition is required for y. Many other generalizations were considered. We also drive the attention on the recent monograph by Pardoux and Rascanu [2014].

When the driving martingale in the BSDE is a Brownian motion, $h = g(S_T)$, and S is a Markov diffusion, a solution of a BSDE constitutes a probabilistic representation of a semilinear parabolic PDE. In particular if u is a solution of the mentioned PDE, then, roughly speaking setting $Y_t = u(t, S_t), Z = \partial_s u(t, S_t), O \equiv 0$, the triplet (Y, Z, O) is a solution to (3.1.2). That PDE is a deterministic problem naturally related to the BSDE. When M^S is a general càdlàg martingale, the link between a BSDE (3.1.2) and a deterministic problem is less obvious.

As far as backward stochastic differential equations driven by a martingale, the first paper seems to be Buckdahn [1993]. Later, several authors have contributed to that subject, for instance Briand et al. [2002] and El Karoui and Huang [1997]. More recently [Carbone et al., 2007, Theorem 3.1] give sufficient conditions for existence and uniqueness for BSDEs of the form (3.1.2). BSDEs with partial information driven by càdlàg martingales were investigated by Ceci et al. [2014a,b].

In this paper we consider a forward-backward SDE, issued from (3.1.2), where the forward process solves a sort of martingale problem (in the strong probability sense, i.e. where the underlying filtration is fixed) instead of the usual stochastic differential equation appearing in the Brownian case. More particularly we suppose the existence of an operator $\mathfrak{a} : \mathfrak{D}(\mathfrak{a}) \subset C([0,T] \times \mathbb{R}^2) \to \mathcal{L}$, where $\mathcal{L}$ is a suitable space of functions $[0,T] \times \mathbb{R}^2 \to \mathbb{C}^2$ (see (3.2.2)), such that (X, S) verifies the following:

$$\forall y \in \mathfrak{D}(\mathfrak{a}), \quad \left(y(t, X_t, S_t) - \int_0^t \mathfrak{a}(y)(u, X_{u-}, S_{u-}) dA_u \right)_{0 \le t \le T} \text{ is an } \mathcal{F}_t\text{-local martingale,}$$

and A is some fixed predictable bounded variation process. With $\mathfrak{a}$ we associate the

operator $\widetilde{\mathfrak{a}}$ defined by

$$\widetilde{\mathfrak{a}}(y) := \mathfrak{a}(\widetilde{y}) - y\mathfrak{a}(id) - id\mathfrak{a}(y),$$

where $id(t,x,s) = s, \widetilde{y} = y \times id$.
In the forward-backward BSDE we are interested in, the driver $\hat{f}$ verifies

$$\mathfrak{a}(id)(t, X_{t-}(\omega), S_{t-}(\omega))\hat{f}(\omega,t,y,z) = f(t, X_{t-}(\omega), S_{t-}(\omega), y, z),\ (t,y,z) \in [0,T]\times\mathbb{C}^2, \omega \in \Omega, \tag{3.1.3}$$

for some $f : [0,T] \times \mathbb{R}^2 \times \mathbb{C}^2 \to \mathbb{C}$. The main idea is to settle a deterministic problem which is naturally associated with the forward-backward SDE (3.1.2).

The deterministic problem consists in looking for a pair of functions (y,z) which solves

$$\begin{aligned} \mathfrak{a}(y)(t,x,s) &= -f(t,x,s,y(t,x,s),z(t,x,s)) \\ \widetilde{\mathfrak{a}}(y)(t,x,s) &= z(t,x,s)\widetilde{\mathfrak{a}}(id)(t,x,s), \end{aligned} \tag{3.1.4}$$

for all $t \in [0,T]$ and $(x,s) \in \mathbb{R}^2$, with the terminal condition $y(T,.,.) = g(.,.)$.

Any solution to the deterministic problem (3.1.4) will provide a solution (Y,Z,O) to the corresponding BSDE, setting

$$Y_t = y(t, X_t, S_t), Z_t = z(t, X_{t-}, S_{t-}).$$

For illustration, let us consider the elementary case when S is a diffusion process fulfilling $dS_t = \sigma_S(t,S_t)dW_t + b_S(t,S_t)dt$, and $X \equiv 0$. Then $A_t \equiv t$, $\langle M^S \rangle = \int_0^{\cdot}(\sigma^S)^2(r,S_r)dr$, $V^S = \int_0^{\cdot} b(r,S_r)dr = \int_0^{\cdot} \mathfrak{a}(id)(r,S_r)dr$; $\mathfrak{a}$ is the parabolic generator of S, $\mathfrak{D}(\mathfrak{a}) = C^{1,2}([0,T]\times\mathbb{R}^2 \to \mathbb{C}$. In that case (3.1.4) becomes

$$\begin{aligned} \partial_t y(t,x,s) + (b_s\partial_s y + \frac{1}{2}\sigma_S^2\partial_{ss}y)(t,x,s) &= -f(t,x,s,y(t,x,s),z(t,x,s)) \\ z &= \partial_s y \end{aligned} \tag{3.1.5}$$

In that situation $\widetilde{\mathfrak{a}}$ is closely related to the classical derivation operator. When S models the price of a traded asset and $f(t,x,s,y,z) = -b_S(t,s)z$, the resolution of (3.1.5) emerging from the BSDE (3.1.2) with (3.1.3), allows to solve the usual (complete market Black-Scholes type) hedging problem with underlying S. Consequently, in the general case, $\widetilde{\mathfrak{a}}$ appears to be naturally associated with a sort of "generalized derivation map". A first link between the hedging problem in incomplete markets and generalized derivation operators was observed for instance in Goutte et al. [2013].

The aim of our paper is threefold.

1) To provide a general methodology for solving forward-backward SDEs driven by a càdlàg martingale, via the solution of a deterministic problem generalizing the classical partial differential problem appearing in the case of Brownian martingales.

2) To give applications to the hedging problem in the case of basis risk via the Föllmer-Schweizer decomposition. In particular we revisit the case when (X,S)

is a diffusion process whose particular case of Black-Scholes was treated by Hulley and McWalter [2008], discussing some analysis related to a corresponding PDE.

3) To furnish a quasi-explicit solution when the pair of processes (X, S) is an exponential of additive processes, which constitutes a generalization of the results of Goutte et al. [2014] and Hubalek et al. [2006], established in the absence of basis risk. This yields a characterization of the hedging strategy in terms of Fourier-Laplace transform and the moment generating function.

The paper is organized as follows. In Section 3.2, we state the strong inhomogeneous martingale problem, and we give several examples, as Markov flows and the exponential of additive processes. In Section 3.3, we state the general form of a BSDE driven by a martingale and we associate a deterministic problem with it. We show in particular that a solution for this deterministic problem yields a solution for the BSDE. In Section 3.4, we apply previous methodology to the F-S decomposition problem under basis risk. In the case of exponential of additives processes, we obtain a quasi-explicit decomposition of the mentioned F-S decomposition.

3.2 Strong inhomogeneous martingale problem

3.2.1 General considerations

In this paper T will be a strictly positive number. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, fulfilling the usual conditions. By default, all the processes will be indexed by $[0, T]$. Let (X, S) a couple of $\mathcal{F}_t$-adapted processes. We will often mention concepts as *martingale, semimartingale, adapted, predictable* without mentioning the underlying filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Given a bounded variation function $\phi : [0, T] \to \mathbb{R}$, we will denote by $t \mapsto \|\phi\|_t$ the associated total variation function.

We introduce a notion of martingale type problem related to (X, S), which is a generalization of a stochastic differential equation. We emphasize that the present notion looks similar to the classical notion of Stroock and Varadhan [2006] but here the notion is probabilistically strong and relies on a fixed filtered probability space. In the context of Stroock and Varadhan, however, a solution is a probability measure and the underlying process is the canonical process on some canonical space. Here a filtered probability space is given at the beginning. A similar notion was introduced in Russo and Trutnau [2007] Definition 5.1. A priori, we will not suppose that our strong martingale problem is well-posed (existence and uniqueness).

Definition 3.2.1. *Let $\mathcal{O}$ be an open set of $\mathbb{R}^2$. Let (A_t) be an $\mathcal{F}_t$-adapted bounded variation continuous process, such that, a.s.*

$$dA_t \ll d\rho_t,$$

for some bounded variation function ρ, and $\mathfrak{a}$ a map

$$\mathfrak{a} : \mathfrak{D}(\mathfrak{a}) \subset \mathcal{C}([0,T] \times \mathcal{O}, \mathbb{C}) \longrightarrow \mathcal{L}, \tag{3.2.1}$$

where

$$\begin{aligned} \mathcal{L} = \{ f :&[0,T] \times \mathcal{O} \to \mathbb{C}, \text{such that for every compact } K \text{ of } \mathcal{O} \\ &\|f\|_K (t) := \sup_{(x,y)\in K} |f(t,x,y)| < \infty \quad d\rho_t \ a.e.\}. \end{aligned} \tag{3.2.2}$$

We say that a couple of càdlàg processes (X, S) is a solution of the **strong martingale problem** *related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, if for any $g \in \mathfrak{D}(\mathfrak{a})$, $(g(t, X_t, S_t))_t$ is a semimartingale with continuous bounded variation component such that*

$$\int_0^t |\mathfrak{a}(g)(u, X_{u-}, S_{u-})| d \|A\|_u < \infty \ a.s. \tag{3.2.3}$$

and

$$t \longmapsto g(t, X_t, S_t) - \int_0^t \mathfrak{a}(g)(u, X_{u-}, S_{u-}) dA_u \tag{3.2.4}$$

is an $\mathcal{F}_t$- local martingale.

We start introducing some significant notations.

Notation 3.2.2.

1) $id : (t, x, s) \longmapsto s$.

2) For any $y \in \mathcal{C}([0,T] \times \mathcal{O})$, we denote by $\widetilde{y}$ the function $\widetilde{y} := y \times id$, i.e.

$$(y \times id)(t, x, s) = sy(t, x, s). \tag{3.2.5}$$

3) Suppose that $id \in \mathfrak{D}(\mathfrak{a})$. For $y \in \mathfrak{D}(\mathfrak{a})$ such that $\widetilde{y} \in \mathfrak{D}(\mathfrak{a})$, we set

$$\widetilde{\mathfrak{a}}(y) := \mathfrak{a}(\widetilde{y}) - y\mathfrak{a}(id) - id\mathfrak{a}(y). \tag{3.2.6}$$

As we have mentioned in the introduction, the map $\widetilde{\mathfrak{a}}$ will play the role of a generalized derivative. We state first a preliminary lemma.

Lemma 3.2.3. *Let $y \in \mathfrak{D}(\mathfrak{a})$. Suppose that $y, id, y \times id \in \mathfrak{D}(\mathfrak{a})$. We set $Y_t = y(t, S_t, X_t)$ and M^Y be its martingale component given in (3.2.4). Then*

$$\langle M^Y, M^S \rangle_t = \int_0^t \widetilde{\mathfrak{a}}(y)(u, X_{u-}, S_{u-}) dA_u.$$

Proof. In order to compute the angle bracket $\langle M^Y, M^S \rangle$, we start by expressing the square bracket of M^Y and M^S. First, note that, since $y, id \in \mathfrak{D}(\mathfrak{a})$ and A is a continuous process, then the bounded variation parts of the semimartingales $(S_t)_t$ and

$(y(t,S_t,X_t))_t$ are continuous. We have

$$\begin{aligned}[M^Y,M^S]_t &= [Y,S]_t \\ &= (SY)_t - \int_0^t Y_{s-}dS_s - \int_0^t S_{s-}dY_s,\end{aligned}$$

where the first equality is justified by the fact that the square bracket of any process with a continuous bounded variation process vanishes. Using moreover the fact that $y \times id \in \mathfrak{D}(\mathfrak{a})$, the process

$$[M^Y,M^S] - \int_0^\cdot \mathfrak{a}(y\times id)(r,X_{r-},S_{r-})dA_r + \int_0^\cdot y(r,X_{r-},S_{r-})\mathfrak{a}(id)(r,X_{r-},S_{r-})dA_r + \int_0^\cdot S_{r-}\mathfrak{a}(y)(r,X_{r-},S_{r-})dA_r$$

is an $\mathcal{F}_t$-local martingale.

Consequently, $[M^Y,M^S]$ is a special $\mathcal{F}_t$-semimartingale. Since $\langle M^Y,M^S\rangle - [M^Y,M^S]$ is a local martingale, the result follows by uniqueness of the decomposition of a special semimartingale. □

In the sequel, we will make the following assumption.

Assumption 3.2.4. *$(\mathfrak{D}(\mathfrak{a}),\mathfrak{a},A)$ verifies the following axioms.*

i) $id \in \mathfrak{D}(\mathfrak{a})$.

ii) $(t,x,s) \mapsto s^2 \in \mathfrak{D}(\mathfrak{a})$.

Corollary 3.2.5. *Let (X,S) be a solution of the strong martingale problem introduced in Definition 3.2.1 then, under Assumption 3.2.4, S is a special semimartingale with decomposition $M^S + V^S$ given below.*

i) $V_t^S = \int_0^t \mathfrak{a}(id)(u,X_{u-},S_{u-})dA_u$.

ii) $\langle M^S\rangle_t = \int_0^t \widetilde{\mathfrak{a}}(id)(u,X_{u-},S_{u-})dA_u$.

Proof. i) is obvious since $id \in \mathfrak{D}(\mathfrak{a})$ and ii) is a consequence of Lemma 3.2.3 and the fact that $(t,x,s) \mapsto s^2$ belongs to $\mathfrak{D}(\mathfrak{a})$. □

In many situations, the operator $\mathfrak{a}$ is related to the classical infinitesimal generator, when it exists. We will make this relation explicit in the below example of Markov processes.

3.2.2 The case of Markov semigroup

In this section $\mathcal{O}$ will be for simplicity $\mathbb{R}^2$. In this example, for illustration, we only consider a single process S instead of a couple (X,S). For this reason, it is more comfortable to re-express Definition 3.2.1 into the following simplified version. In the present case we will always have $A_t \equiv t$.

Definition 3.2.6. *We say that S is a solution of the* **strong martingale problem** *related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, if there is a map*

$$\mathfrak{a} : \mathfrak{D}(\mathfrak{a}) \subset \mathcal{C}([0,T] \times \mathbb{R}) \longrightarrow \mathcal{L}, \tag{3.2.7}$$

where

$$\begin{aligned} \mathcal{L} = \{ f :[0,T] \times \mathbb{R} \to \mathbb{C}, &\text{ such that for every compact } K \text{ of } \mathbb{R} \\ \|f\|_K (t) := \sup_{x \in K} |f(t,x)| < \infty \quad &dt \ a.e.\}, \end{aligned} \tag{3.2.8}$$

such that for any $g \in \mathfrak{D}(\mathfrak{a})$, $(g(t, S_t))_t$ is a (special) semimartingale with continuous bounded variation component verifying

$$\int_0^t |\mathfrak{a}(g)(u, S_{u-})| du < \infty \ a.s. \tag{3.2.9}$$

and

$$t \longmapsto g(t, S_t) - \int_0^t \mathfrak{a}(g)(u, S_{u-}) du \tag{3.2.10}$$

is a $\mathcal{F}_t^S$- local martingale, where $\mathcal{F}_t^S$ is the canonical filtration associated with S.

Let $(X_t^{s,x})_{t \geq s, x \in \mathbb{R}}$ be a time-homogeneous Markovian flow. In particular, if $S = X^{\ell,x}$ and f is a bounded Borel function, then

$$\mathbb{E}\left[f(S_t)|\mathcal{F}_s^S\right] = \Psi(t-s, S_s), \tag{3.2.11}$$

where $\ell \leq s \leq t \leq T$ and

$$\Psi(r,y) = \mathbb{E}\left[f(X_r^{0,y})\right] = \mathbb{E}\left[f(X_{s+r}^{s,y})\right], \tag{3.2.12}$$

for any $r, s \geq 0$ and $\mathcal{F}^S$ is the canonical filtration for S. We suppose moreover that $X_t^{\ell,x}$ is square integrable for any $0 \leq \ell \leq t \leq T$ and $x \in \mathbb{R}$. We denote by E the linear space of functions such that

$$E = \left\{ f \in C \text{ such that } \widetilde{f} := x \mapsto \frac{f(x)}{1+x^2} \text{ is uniformly continuous and bounded } \right\}, \tag{3.2.13}$$

equipped with the norm

$$\|f\|_E := \sup_x \frac{|f(x)|}{1+x^2} < \infty.$$

The set E can easily be shown to be a Banach space equipped with the norm $\|.\|_E$. Indeed E is a suitable space for Markov processes which are square integrable. In particular, (3.2.11) and (3.2.12) remain valid if $f \in E$. From now on we consider the family of linear operators $(P_t, t \geq 0)$ defined on the space E by

$$P_t f(x) = \mathbb{E}\left[f(X_t^{0,x})\right] \text{ , for } t \in [0,T], x \in \mathbb{R}, \quad \forall f \in E. \tag{3.2.14}$$

We formulate now a fundamental assumption.

Assumption 3.2.7.

i) $P_t E \subset E$ *for all* $t \in [0,T]$.

ii) The linear operator P_t *is bounded, for all* $t \in [0,T]$.

iii) (P_t) *is* **strongly continuous**, *i.e.* $\lim_{t\to 0} P_t f = f$ *in the* E *topology.*

Using the Markov flow property (3.2.11), it is easy to see that the family of continuous operators (P_t) defined above has the semigroup property. In particular, under Assumption 3.2.7, the family (P_t) is strongly continuous semigroup on E.

Assumption 3.2.7 is fulfilled in many common cases, as mentioned in Proposition 3.2.8 and Remarks 3.2.9 and 3.2.10.

The proposition below concerns the validity of items i) and ii).

Proposition 3.2.8. *Let* $t \in [0,T]$. *Suppose that* $x \mapsto X_t^{0,x}$ *is differentiable in* $L^2(\Omega)$ *such that*

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[|\partial_x X_t^{0,x}|^2\right] < \infty. \tag{3.2.15}$$

Then $P_t f \in E$ *for all* $f \in E$ *and* P_t *is a bounded operator.*

The proof of this proposition is reported in Appendix 3.A.

Remark 3.2.9. *Condition* (3.2.15) *of Proposition 3.2.8 is fulfilled in the following two cases.*

1) If (L_t) *is a Lévy process, the Markov flow* $X^{0,x} = x + L$ *verifies* $\partial_x X^{0,x} = 1$.

2) If $(X_t^{0,x})$ *is a diffusion process verifying*

$$X_t^{0,x} = x + \int_0^t b(X_s^{0,x})ds + \int_0^t \sigma(X_s^{0,x})dW_s,$$

where b *and* σ *are* C_b^1 *functions.*

Remark 3.2.10. *Item iii) of Assumption 3.2.7 is verified in the case of square integrable Lévy processes, c.f. Proposition 3.B.1 in Appendix 3.B.*

For the rest of this subsection we work under Assumption 3.2.7.

Item iii) of Assumption 3.2.7 permits to introduce the definition of the generator of (P_t) as follows.

Definition 3.2.11. *The* **generator** L *of* (P_t) *in* E *is defined on the domain* $D(L)$ *which is the subspace of* E *defined by*

$$D(L) = \left\{ f \in E \text{ such that } \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists in } E \right\}. \tag{3.2.16}$$

We denote by Lf *the limit above. We refer to [Jacob, 2001, Chapter 4], for more details.*

Remark 3.2.12. *If $f \in E$ such that there is $g \in E$ such that*

$$(P_t f)(x) - f(x) - \int_0^t P_s g(x) ds = 0, \ \forall t \geq 0, \ x \in E,$$

then $f \in D(L)$ and $g = Lf$.
Previous integral is always defined as E-valued Bochner integral. Indeed, since (P_t) is strongly continuous, then by [Jacob, 2001, Lemma 4.1.7], we have

$$\|P_t\| \leq M_w e^{wt}, \tag{3.2.17}$$

for some real w and related constant M_w. $\|\cdot\|$ denotes here the operator norm.

A useful result which allows to deal with time-dependent functions is given below.

Lemma 3.2.13. *Let $f : [0,T] \to D(L) \subset E$. We suppose the following.*

i) f is continuous as a $D(L)$-valued function, where $D(L)$ is equipped with the graph norm.

ii) $f : [0,T] \to E$ is of class C^1.

Then, the below E-valued equality holds:

$$P_t f(t,.) = f(0,.) + \int_0^t P_s(Lf(s,.))ds + \int_0^t P_s(\frac{\partial f}{\partial s}(s,.))ds, \ \forall t \in [0,T]. \tag{3.2.18}$$

Remark 3.2.14. *We observe that the two integrals above can be considered as E-valued Bochner integrals because, by hypothesis, $s \mapsto Lf(s,\cdot)$ and $s \mapsto \frac{\partial f}{\partial s}(s,.)$ are continuous with values in E, and so we can apply again (3.2.17) in Remark 3.2.12.*

Proof. It will be enough to show that

$$\frac{d}{dt} P_t f(t,.) = P_t(Lf(t,.)) + P_t\left(\frac{\partial f}{\partial t}(t,.)\right), \ \forall t \in [0,T]. \tag{3.2.19}$$

In fact, even if Banach space valued, a differentiable function at each point is also absolutely continuous.

Since the right-hand side of (3.2.19) is continuous it is enough to show that the right-derivative of $t \mapsto P_t f(t,\cdot)$ coincides with the right-hand side of (3.2.19). Let $h > 0$. We evaluate

$$P_{t+h} f(t+h,.) - P_t f(t,.) = I_1(t,h) + I_2(t,h),$$

where

$$I_1(t,h) = P_{t+h} f(t+h,.) - P_t f(t+h,.), I_2(t,h) = P_t f(t+h,.) - P_t f(t,.).$$

Now by [Jacob, 2001, Lemma 4.1.14], we get

$$I_1(t,h) := P_{t+h}f(t+h,.) - P_tf(t+h,.) = \int_t^{t+h} P_sLf(t+h,.)ds.$$

We divide by h and we get

$$\begin{aligned}\left\|\frac{1}{h}\int_t^{t+h}(P_sLf(t+h,.) - P_sLf(t,.))ds\right\|_E &\leq \frac{1}{h}\int_t^{t+h} ds\,\|P_s\{Lf(t+h,.) - Lf(t,.)\}\|_E \\ &\leq \|Lf(t+h,.) - Lf(t,.)\|_E \frac{1}{h}\int_t^{t+h}\|P_s\|\,ds \\ &\leq \|f(t+h,.) - f(t,.)\|_{D(L)} \frac{1}{h}\int_t^{t+h}\|P_s\|\,ds,\end{aligned}$$

where $\|.\|_{D(L)}$ is the graph norm of L. This converges to zero (note that $\|P_s\|$ is bounded by (3.2.17)), and we get that

$$\frac{1}{h}I_1(t,h) \xrightarrow{h\to 0} P_t(Lf(t,.)).$$

We estimate now $I_2(t,h)$.

$$\left\|\frac{P_tf(t+h,.) - P_tf(t,.)}{h} - P_t(\frac{\partial f}{\partial t}(t,.))\right\|_E \leq \|P_t\|\left\|\frac{f(t+h,.) - f(t,.)}{h} - \frac{\partial f}{\partial t}(t,.)\right\|_E.$$

This goes to zero as h goes to zero, by Assumption ii).

This concludes the proof of Lemma 3.2.13.

□

We can now discuss the fact that a process $S = X^{0,x}$, where $X_t^{s,x}$ is a Markovian flow (as precised at the beginning of Section 3.2.2) is a solution to our (time inhomogeneous) strong martingale problem (3.2.6).

Theorem 3.2.15. *We denote*

$$\mathfrak{D}(\mathfrak{a}) = \{g : [0,T] \to D(L) \text{ such that assumptions i) and ii) of Lemma 3.2.13 are fulfilled}\}$$

and for $g \in \mathfrak{D}(\mathfrak{a})$

$$\mathfrak{a}(g)(t,x) = \frac{\partial g}{\partial t}(t,x) + Lg(t,\cdot)(x),\ \forall t \in [0,T], x \in \mathbb{R}.$$

Then S is a solution of the strong martingale problem introduced in Definition 3.2.6.

Remark 3.2.16. *Let $g \in \mathfrak{D}(\mathfrak{a})$. Since for each $t \in [0,T]$, by assumptions i) and ii) of Lemma 3.2.13, $\mathfrak{a}(g)(t,\cdot) \in E$, then, obviously $\mathfrak{a}(g) \in \mathcal{L}$. Moreover, the same assumptions imply that $t \mapsto \frac{\partial g}{\partial t}(t,\cdot)$ and $t \mapsto Lg(t,\cdot)$ are continuous on $[0,T]$ and hence are bounded, i.e.*

$$\sup_{t\in[0,T]}\left\|\frac{\partial g}{\partial t}(t,\cdot)\right\|_E < \infty, \quad \sup_{t\in[0,T]}\|Lg(t,\cdot)\|_E < \infty.$$

This yields in particular that Condition (3.2.9) is fulfilled.

Proof of Theorem 3.2.15.

It remains to show the martingale property (3.2.10). We fix $0 \leq s < t \leq T$ and a bounded random variable $\mathcal{F}_s^S$-measurable G. It will be sufficient to show that

$$\mathbb{E}\left[A(s,t)\right] = 0, \tag{3.2.20}$$

where

$$A(s,t) = G\left(g(t,S_t) - g(s,S_s) - \int_s^t \partial_r g(r,S_r)dr - \int_s^t Lg(r,.)(S_r)dr\right).$$

By taking the conditional expectation of $A(s,t)$ with respect to $\mathcal{F}_s^S$, using (3.2.11) and Fubini's theorem, we get

$$\mathbb{E}\left[A(s,t)|\mathcal{F}_s^S\right] = G\phi(S_s),$$

where

$$\phi(x) = \left(P_{t-s}g(t,.)(x) - g(s,x) - \int_s^t (P_{r-s}\partial_r g(r,.))(x)dr - \int_s^t (P_{r-s}Lg(r,.))(x)dr\right),\ \forall x \in \mathbb{R}.$$

We define $f : [0, T-s] \times \mathbb{R} \to \mathbb{R}$ by $f(\tau,\cdot) = g(\tau+s,\cdot)$. f fulfills the assumptions of Lemma 3.2.13 with T being replaced by $T-s$. By the change of variable $u = r-s$, setting $\tau = t-s$, the equality above becomes

$$\phi(x) = \left(P_\tau f(\tau,.)(x) - f(0,x) - \int_0^\tau (P_u \partial_r f(u,.))(x)du - \int_0^\tau (P_u Lf(u,.))(x)dr\right).$$

Now by Lemma 3.2.13 we get that $\phi(x) = 0,\ \forall x \in \mathbb{R}$. Consequently $\mathbb{E}\left[A(s,t)|\mathcal{F}_s^S\right] = 0$ and (3.2.20) is fulfilled. □

Remark 3.2.17. *We introduce the following subspace E_0^2 of C^2.*

$$E_0^2 = \{f \in C^2 \text{ such that } f'' \text{ vanishes at infinity}\}. \tag{3.2.21}$$

Note that only the second derivative is supposed to vanish at infinity.

E_0^2 is included in E. Indeed, if $f \in E_0^2$, then the Taylor expansion $f(x) = f(0) + xf'(0) + x^2\int_0^1 (1-\alpha)f''(x\alpha)d\alpha$ implies that $\widetilde{f}$ is bounded. On the other hand, by straightforward calculus we see that the first derivative $\frac{d\widetilde{f}}{dx}$ is bounded. This implies that $\widetilde{f}$ is uniformly continuous.

In several examples it is easy to identify E_0^2 as a significant subspace of $D(L)$, see for instance the example of Lévy processes which is described below.

A significant particular case: Lévy processes

As anticipated above, an insightful example for Markov flows is the case of Lévy processes. We specify below the corresponding infinitesimal generator.

Let (X_t) be a square integrable Lévy process with characteristic triplet (A, ν, γ), such that $X_0 = 0$. We refer to, e.g., [Cont and Tankov, 2004, Chapter 3] for more details.

We suppose that (X_t) is of pure jump, i.e. $A = 0$ and $\gamma = 0$. Since X is square integrable, then (c.f. [Cont and Tankov, 2004, Proposition 3.13])

$$\int_{\mathbb{R}} |x|^2 \nu(dx) < \infty \tag{3.2.22}$$

and

$$c_1 := \frac{\mathbb{E}[X_t]}{t} = \int_{|x|>1} x\nu(dx) < \infty, \quad c_2 := \frac{\mathrm{Var}[X_t]}{t} = \int_{\mathbb{R}} |x|^2 \nu(dx) < \infty. \tag{3.2.23}$$

Clearly the corresponding Markov flow is given by $X_t^{0,x} = x + X_t, t \geq 0, x \in \mathbb{R}$.

The classical theory of semigroup for Lévy processes is for instance developed in Section 6.31 of Sato [2013]. There one defines the semigroup P on the set C_0 of continuous functions vanishing at infinity, equipped with the sup-norm $\|u\|_\infty = \sup_x |u(x)|$. By [Sato, 2013, Theorem 31.5], the semigroup P is strongly continuous on C_0, with norm $\|P\| = 1$, and its generator L_0 is given by

$$L_0 f(x) = \int \left(f(x+y) - f(x) - y f'(x) \mathbb{1}_{|y|<1} \right) \nu(dy). \tag{3.2.24}$$

Moreover, the set C_0^2 of functions $f \in C^2$ such that f, f' and f'' vanish at infinity, is included in $D(L_0)$. We remark that the domain $D(L)$ includes the classical domain $D(L_0)$. In fact, we have

$$\|g\|_E \leq \|g\|_{C_0}, \ \forall g \in C_0.$$

Consequently, if $f \in D(L_0) \subset C_0$, then for $t > 0$

$$\left\| \frac{P_t f - f}{t} - L_0 f \right\|_E \leq \left\| \frac{P_t f - f}{t} - L_0 f \right\|_{C_0}.$$

So $f \in D(L)$ and $Lf = L_0 f$.

Assumption 3.2.7 is verified because of Proposition 3.2.8, item 1) of Remark 3.2.9 and Remark 3.2.10.

The theorem below shows that the space E_0^2, defined in Remark 3.2.17, is a subset of $D(L)$.

Theorem 3.2.18. *Let L be the infinitesimal generator of the semigroup (P_t). Then $E_0^2 \subset D(L)$ and*

$$Lf(x) = \int \left(f(x+y) - f(x) - y f'(x) \mathbb{1}_{|y|<1} \right) \nu(dy), \ f \in E_0^2. \tag{3.2.25}$$

A proof of this result, using arguments in Figueroa-López [2008], is developed in Appendix 3.B.

In conclusion, C^2 functions whose second derivative vanishes at infinity belong to $D(L)$. For instance, $id : x \mapsto x \in D(L)$. On the other hand the function $x^2 : x \mapsto x^2$

also belongs to $D(L)$.

In fact, for every $x \in \mathbb{R}, t \geq 0$ we have

$$P_t f(x) - f(x) = \frac{\mathbb{E}\left[(x + X_t)^2\right] - x^2}{t} = \frac{2xc_1 t + c_2 t + c_1^2 t^2}{t}.$$

Obviously, this converges to the function $x \mapsto 2xc_1 + c_2$ according to the E-norm. Finally it follows that $L(x^2) = 2xc_1 + c_2$.

Corollary 3.2.19. *We have the following inclusion:*

$$E_0^2 \oplus x^2 \subset D(L)$$

3.2.3 Diffusion processes

Here we will suppose again $\mathcal{O} = \mathbb{R} \times E$, where $E = \mathbb{R}$ or $]0, \infty[$. A function $f : [0, T] \times \mathcal{O}$ will be said to be **globally Lipschitz** if it is Lipschitz with respect to the second and third variable uniformly with respect to the first.

We consider here the case of a diffusion process (X, S) whose dynamics is described as follows:

$$\begin{aligned} dX_t &= b_X(t, X_t, S_t)dt + \sum_{i=1}^{2} \sigma_{X,i}(t, X_t, S_t)dW_t^i \\ dS_t &= b_S(t, X_t, S_t)dt + \sum_{i=1}^{d} \sigma_{S,i}(t, X_t, S_t)dW_t^i, \end{aligned} \tag{3.2.26}$$

where $W = (W^1, W^2)$ is a standard 2-dimensional Brownian motion with canonical filtration $(\mathcal{F}_t)$, b_X, b_S, $\sigma_{X,i}$, and $\sigma_{S,i}$, for $i = 1, 2$, $b, \sigma : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions which are globally Lipschitz.

We also suppose (X_0, S_0) to have all moments and that S takes value in E. For instance a geometric Brownian motion takes value in $E =]0, \infty[$, if it starts from a positive point.

Remark 3.2.20. *Let $p \geq 1$. It is well-known, that there is a constant $C(p)$, only depending on p, such that*

$$\mathbb{E}\left[\sup_{t \leq T}(|X_t|^p + |S_t|^p)\right] \leq C(p)(|X_0|^p + |S_0|^p).$$

By Itô formula, for $f \in \mathcal{C}^{1,2}([0, T[\times\mathcal{O})$, we have

$$\begin{aligned} df(t, X_t, S_t) &= \partial_t f(t, X_t, S_t)dt + \partial_s f(t, X_t, S_t)dS_t + \partial_x f(t, X_t, S_t)dX_t \\ &+ \frac{1}{2}\left\{\partial_{ss} f(t, X_t, S_t)d\langle S\rangle_t + \partial_{xx} f(t, S_t, X_t)d\langle X\rangle_t + 2\partial_{sx} f(t, X_t, S_t)d\langle S, X\rangle_t\right\}. \end{aligned}$$

We denote $|\sigma_S|^2 = \sum_{i=1}^{2} \sigma_{S,i}^2$, $|\sigma_X|^2 = \sum_{i=1}^{2} \sigma_{X,i}^2$ and $\langle \sigma_S, \sigma_X \rangle = \sum_{i=1}^{2} \sigma_{S,i}\sigma_{X,i}$.

Hence, the operator $\mathfrak{a}$ can be defined as

$$\begin{aligned}\mathfrak{a}(f) &= \partial_t f + b_S \partial_s f + b_X \partial_x f \\ &+ \frac{1}{2}\left\{|\sigma_S|^2 \partial_{ss} f + |\sigma_X|^2 \partial_{xx} f + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} f\right\},\end{aligned}$$

associated with $A_t \equiv t$ and a domain $\mathfrak{D}(\mathfrak{a}) = \mathcal{C}^{1,2}([0,T[\times\mathcal{O}) \cap \mathcal{C}^0([0,T]\times\mathcal{O})$.
Note that Assumption 3.2.4 is verified since id and $id \times id$ belong to $\mathfrak{D}(\mathfrak{a})$. Moreover, a straightforward calculation gives

$$\widetilde{\mathfrak{a}}(f) = |\sigma_S|^2 \partial_s f(t,x,s) + \langle \sigma_S, \sigma_X \rangle \partial_x f(t,x,s)$$

In particular,

$$\widetilde{\mathfrak{a}}(id) = |\sigma_S|^2.$$

Remark 3.2.21. *By Itô formula, for $0 \le u \le T$, we obviously have*

$$\begin{aligned}f(u, X_u, S_u) - \int_0^u \mathfrak{a}(f)(r, X_r, S_r) dr &= \int_0^u \partial_x f(r, X_r, S_r) \left(\sigma_{X,1}(r, X_r, S_r) dW_r^1 + \sigma_{X,2}(r, X_r, S_r) dW_r^2\right) \\ &+ \int_0^u \partial_s f(r, X_r, S_r) \left(\sigma_{S,1}(r, X_r, S_r) dW_r^1 + \sigma_{S,2}(r, X_r, S_r) dW_r^2\right).\end{aligned}$$

3.2.4 Variant of diffusion processes

Let (W_t) be an $\mathcal{F}_t$-standard Brownian motion and S be a solution of the SDE

$$dS_t = \sigma(t, S_t) dW_t + b_1(t, S_t) da_t + b_2(t, S_t) dt, \tag{3.2.27}$$

where $b_1, b_2, \sigma : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions which are globally Lipschitz, and $a : [0,T] \to \mathbb{R}$ is an increasing function such that da is singular with respect to Lebesgue measure. We set $A_t = a_t + t$.

The equation (3.2.27) can be written as

$$dS_t = \sigma(t, S_t) dW_t + \left(b_1(t, S_t) \frac{d\rho_t}{dA_t} + b_2(t, S_t) \frac{dt}{dA_t}\right) dA_t.$$

A solution S of (3.2.27) verifies the strong martingale problem with respect to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, in the sense where $\mathfrak{D}(\mathfrak{a}) = \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and for $f \in \mathfrak{D}(\mathfrak{a})$,

$$\mathfrak{a}(f)(t,s) = \left(\partial_t f(r,s) \frac{dr}{dA_r} + \partial_s f(r,s) \widetilde{b}(r,s) + \frac{1}{2} \partial_{ss} f(r,s) \widetilde{\sigma}^2(r,s)\right),$$

where $\widetilde{b}(t,s) = b_1(t,s) \frac{d\rho_t}{dA_t}(t) + b_2(t,s) \frac{dt}{dA_t}(t)$ and $\widetilde{\sigma}^2(t,s) = \sigma^2(t,s) \frac{dt}{dA_t}(t)$.

Indeed, by Itô formula, the process

$$t \mapsto f(t, S_t) - \int_0^t \mathfrak{a}(f)(r, S_r) dA_r$$

is a local martingale.

3.2.5 Exponential of additive processes

A càdlàg process (Z^1, Z^2) is said to be an **additive process** if $(Z^1, Z^2)_0 = 0$, (Z^1, Z^2) is continuous in probability and it has independent increments, i.e. $(Z^1_t - Z^1_s, Z^2_t - Z^2_s)$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t \leq T$ and $(\mathcal{F}_s)$ is the canonical filtration associated with (Z^1, Z^2).

In this section we restrict ourselves to the case of exponential of additive processes which are semimartingales (shortly semimartingale additive processes) and we specify a corresponding martingale problem $(\mathfrak{a}, \mathfrak{D}(\mathfrak{a}), A)$ for this process. This will be based on Fourier-Laplace transform techniques. The couple of processes (X, S) is defined by

$$\begin{aligned} X &= \exp(Z^1) \\ S &= \exp(Z^2), \end{aligned}$$

where (Z^1, Z^2) is an (two-dimensional) semimartingale additive process.
We denote by D the set

$$D := \{z = (z_1, z_2) \in \mathbb{C}^2 | \quad \mathbb{E}\left[|X_T^{\mathrm{Re}(z_1)} S_T^{\mathrm{Re}(z_2)}|\right] < \infty\}.$$

We convene that $\mathbb{C}^2 = \mathbb{R}^2 + i\mathbb{R}^2$, associating the couple (z_1, z_2) with $(\mathrm{Re}z_1, \mathrm{Re}z_2) + i(\mathrm{Im}z_1, \mathrm{Im}z_2)$. Clearly we have $D = (D \cap \mathbb{R}^2) + i\mathbb{R}^2$. We also introduce the notation

$$D/2 := \{z \in \mathbb{C}^2 | \quad 2z \in D\} \subset D.$$

Remark 3.2.22. *By Cauchy-Schwarz inequality, $z, y \in D/2$ implies that $z + y \in D$.*

We denote by $\kappa : D \to \mathbb{C}$, the generating function of (Z^1, Z^2), see for instance [Goutte et al., 2014, Definition 2.1]. In particular κ verifies

$$\exp(\kappa_t(z_1, z_2)) = \mathbb{E}\left[\exp(z_1 Z^1_t + z_2 Z^2_t)\right] = \mathbb{E}\left[X_t^{z_1} S_t^{z_2}\right].$$

We will adopt similar notations and assumptions as in Goutte et al. [2014], which treated the problem of variance optimal hedging for a one-dimensional exponential of additive process. We introduce a function ρ, defined, for each $t \in [0, T]$, as follows:

$$\begin{aligned} \rho_t(z_1, z_2, y_1, y_2) &:= \kappa_t(z_1 + y_1, z_2 + y_2) - \kappa_t(z_1, z_2) - \kappa_t(y_1, y_2), \quad \text{for } (z_1, z_2), (y_1, y_2) \in D/2, \\ \rho_t(y_1, y_2) &:= \rho_t(z_1, z_2, \bar{z_1}, \bar{z_2}), \quad \text{for } (z_1, z_2) \in D/2, \\ \rho^S_t &:= \rho_t(0, 1) = \kappa_t(0, 2) - 2\kappa_t(0, 1), \text{ if } (0, 1) \in D/2. \end{aligned} \tag{3.2.28}$$

We remark that for $(z_1, z_2) \in D/2$, $t \mapsto \rho_t(z_1, z_2)$ is a real function. These functions appear naturally in the expression of the angle brackets of (M^X, M^S) where M^X (resp. M^S) is the martingale part of X (resp. S).

From now on, in this section, the assumption below will be in force.

Assumption 3.2.23.

1) ρ^S *is strictly increasing.*

2) $(0,2) \in D$. *This is equivalent to the existence of the second order moment of* S.

Note that, by Cauchy-Schwarz, the second item implies that, $D/2 + (0,1) \subset D$.

We remind that previous assumption implies that Z^2 has no deterministic increments, see [Goutte et al., 2014, Lemma 3.9].

Similarly as in [Goutte et al., 2014, Propositions 3.4 and 3.15], one can prove the following.

Proposition 3.2.24.

1) For every $(z_1, z_2) \in D$, $\left(X_t^{z_1} S_t^{z_2} e^{-\kappa_t(z_1,z_2)}\right)$ *is a martingale.*

2) $t \mapsto \kappa_t(z_1, z_2)$ *is a bounded variation continuous function, for every* $(z_1, z_2) \in D$. *In particular,* $t \mapsto \rho_t(z_1, z_2)$ *is also a bounded variation continuous function, for every* $(z_1, z_2) \in D/2$.

3) Let I *be a compact real set included in* D. *Then*

$$\sup_{(x,y)\in I} \sup_{t\leq T} \mathbb{E}\left[X_t^x S_t^y\right] = \sup_{(x,y)\in I} \sup_{t\leq T} e^{\kappa_t(x,y)} < \infty.$$

4) $\forall (z_1, z_2) \in D/2$, $t \mapsto \rho_t(z_1, z_2)$ *is non-decreasing.*

5) $\kappa_{dt}(z_1, z_2) \ll \rho_{dt}^S$ *, for every* $z \in D$.

6) $\rho_{dt}(z_1, z_2, y_1, y_2) \ll \rho_{dt}^S$ *, for every* $(z_1, z_2), (y_1, y_2) \in D/2$.

Remark 3.2.25. *Note that, for any* $(z_1, z_2) \in D$, $X^{z_1} S^{z_2}$ *is a special semimartingale. Indeed, by Proposition 3.2.24,* $X_t^{z_1} S_t^{z_2} = N_t e^{\kappa_t(z_1,z_2)}$ *where* $\kappa(z_1, z_2)$ *is a bounded variation continuous function and* N *is a martingale. Hence, integration by parts implies that* $X^{z_1} S^{z_2}$ *is a special semimartingale whose decomposition is given by*

$$X^{z_1} S^{z_2} = M(z_1, z_2) + V(z_1, z_2), \tag{3.2.29}$$

where $M_t(z_1, z_2) = X_0^{z_1} S_0^{z_2} + \int_0^t e^{\kappa_u(z_1,z_2)} dN_u$ *and* $V_t(z_1, z_2) = \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \kappa_{du}(z_1, z_2)$, $\forall t \in [0, T]$.

The following proposition shows that the local martingale part of the decomposition above is a square integrable martingale if $(z_1, z_2) \in D/2$ and gives its angle bracket in terms of the generating function.

Proposition 3.2.26. *Let* $z = (z_1, z_2), y = (y_1, y_2) \in D/2$. *Then* $X^{z_1} S^{z_2}$ *is a special semimartingale, whose decomposition* $X^{z_1} S^{z_2} = M(z_1, z_2) + V(z_1, z_2)$ *satisfies, for* $t \in [0, T]$,

$$\begin{aligned} V(z_1, z_2)_t &= \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \kappa_{du}(z_1, z_2) \\ \langle M(z_1, z_2), M(y_1, y_2)\rangle_t &= \int_0^t X_{u-}^{z_1+y_1} S_{u-}^{z_2+y_2} \rho_{du}(z_1, z_2, y_1, y_2). \end{aligned}$$

In particular,

$$\langle M(z_1,z_2)\rangle_t := \langle M(z_1,z_2), \overline{M(z_1,z_2)}\rangle_t = \int_0^t X_{u-}^{2\mathrm{Re}(z_1)} S_{u-}^{2\mathrm{Re}(z_2)} \rho_{du}(z_1,z_2).$$

Moreover, $M(z_1,z_2)$ is a square integrable martingale.

Proof. This can be done adapting the techniques of [Hubalek et al., 2006, Lemma 3.2] and its generalization to one-dimensional additive processes, i.e. [Goutte et al., 2014, Proposition 3.17 and Lemma 13.19]. □

The measure $d\rho^S$, called **reference variance measure** in Goutte et al. [2014], plays a central role in the expression of the canonical decomposition of special semimartingales depending on the couple (X,S).

Corollary 3.2.27. *The semimartingale decomposition of S is given by $S = M^S + V^S$, where, for $t \in [0,T]$*

$$\begin{aligned} V_t^S &= \int_0^t S_{u-}\kappa_{du}(0,1) \\ \langle M^S\rangle_t &= \int_0^t S_{u-}^2 \rho_{du}^S. \end{aligned}$$

Proof. It follows from Proposition 3.2.26 setting $z_1 = 0, z_2 = 1$. □

Now we state some useful estimates.

Lemma 3.2.28. *Let $(a,b) \in D \cap \mathbb{R}^2$. Then*

$$\mathbb{E}\left[\sup_{t\le T} X_t^a S_t^b\right] < \infty.$$

Proof. Let $(a,b) \in D \cap \mathbb{R}^2$, then $(a/2, b/2) \in D/2$. By Proposition 3.2.26, we have

$$X_t^{a/2} S_t^{b/2} = M_t(a/2,b/2) + \int_0^t X_{u-}^{a/2} S_{u-}^{b/2} \kappa_{du}(a/2,b/2),\ \forall t \in [0,T]$$

and $M(a/2,b/2)$ is a square integrable martingale. Hence, by Doob inequality, we have

$$\mathbb{E}\left[\sup_{t\le T} |M_t(a/2,b/2)|^2\right] \le 4\mathbb{E}\left[|M_T(a/2,b/2)|^2\right] < \infty.$$

On the other hand, using Cauchy-Schwarz inequality and Fubini theorem, we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{t\le T}\left|\int_0^t X_{u-}^{a/2} S_{u-}^{b/2} \kappa_{du}(a/2,b/2)\right|^2\right] &\le \|\kappa(a/2,b/2)\|_T \int_0^T \mathbb{E}\left[X_{u-}^a S_{u-}^b\right] \|\kappa(a/2,b/2)\|_{du} \\ &= \|\kappa(a/2,b/2)\|_T \int_0^T e^{\kappa_u(a,b)} \|\kappa(a/2,b/2)\|_{du} \\ &\le e^{\|\kappa(a,b)\|_T} \|\kappa(a/2,b/2)\|_T^2 < \infty. \end{aligned}$$

Finally

$$\mathbb{E}\left[\sup_{t\leq T} X_t^a S_t^b\right] \quad = \quad \mathbb{E}\left[\sup_{t\leq T}\left|X_t^{a/2} S_t^{b/2}\right|^2\right] < \infty.$$

□

In the general case, when $(z_1, z_2) \in D$, the local martingale part of the special semimartingale $X^{z_1} S^{z_2}$ is a true (not necessarily square integrable) martingale.

Proposition 3.2.29. *Let $(z_1, z_2) \in D$, then, $M(z_1, z_2)$, the local martingale part of $X^{z_1} S^{z_2}$, is a true martingale such that*

$$\mathbb{E}\left[\sup_{t\leq T} |M_t(z_1, z_2)|\right] < \infty.$$

Proof. Let $(z_1, z_2) \in D$. Adopting the notations of (3.2.29), we remind that, by Proposition 3.2.26, $\forall t \in [0, T]$, $M_t(z_1, z_2) = X_t^{z_1} S_t^{z_2} - \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \kappa_{du}(z_1, z_2)$. For this local martingale we can write

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq T} |M_t(z_1, z_2)|\right] &\leq \mathbb{E}\left[\sup_{t\leq T} |X_t^{z_1} S_t^{z_2}|\right] + \mathbb{E}\left[\int_0^T \left|X_{t-}^{z_1} S_{t-}^{z_2}\right| \|\kappa(z_1, z_2)\|_{dt}\right] \\
&\leq \mathbb{E}\left[\sup_{t\leq T}\left|X_t^{\mathrm{Re}(z_1)} S_t^{\mathrm{Re}(z_2)}\right|\right] \left(1 + \|\kappa(z_1, z_2)\|_T\right).
\end{aligned}$$

Since $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ belongs to D, by Lemma 3.2.28, the right-hand side is finite. Consequently the local martingale $M(z_1, z_2)$ is indeed a true martingale. □

The goal of this section is to show that (X, S) is a solution of a strong martingale problem, with related triplet $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, which will be specified below. For this purpose, we determine the semimartingale decomposition of $(f(t, X_t, S_t))_t$ for functions $f : [0, T] \times \mathcal{O} \to \mathbb{C}$, where $\mathcal{O} =]0, \infty[^2$, of the form

$$f(t, x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \lambda(t, z_1, z_2), \ \forall t \in [0, T], x, y > 0, \tag{3.2.30}$$

where Π is finite complex Borel measure on $\mathbb{C}^2$ and $\lambda : [0, T] \times \mathbb{C}^2 \longrightarrow \mathbb{C}$. The family of those functions will include the set $\mathfrak{D}(\mathfrak{a})$ defined later.

Proposition 3.2.29 and item 5) of Proposition 3.2.24 say that, for $z = (z_1, z_2) \in D$,

$$t \mapsto X_t^{z_1} S_t^{z_2} - \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \kappa_{du}(z_1, z_2) = X_t^{z_1} S_t^{z_2} - \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \frac{d\kappa_u(z_1, z_2)}{d\rho_u^S} \rho_{du}^S$$

is a martingale. This provides the semimartingale decomposition of the basic functions $(t, x, s) \mapsto x^{z_1} z^{z_2}$ for $z_1, z_2 \in D$, applied to (X, S). Those functions are expected to be elements of $\mathfrak{D}(\mathfrak{a})$ and one candidate for the bounded variation process A is ρ^S. It remains to precisely define $\mathfrak{D}(\mathfrak{a})$ and the operator $\mathfrak{a}$.

A first step in this direction is to consider a Borel function $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ such that, for any $(z_1, z_2) \in D$, $t \in [0,T] \mapsto \lambda(t, z_1, z_2)$ is absolutely continuous with respect to ρ^S.

Lemma 3.2.30. *Let $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ such that, for any $(z_1, z_2) \in D$, $t \in [0,T] \mapsto \lambda(t, z_1, z_2)$ is absolutely continuous with respect to ρ^S. Then for any $(z_1, z_2) \in D$,*

$$t \mapsto M_t^{\lambda}(z_1, z_2) := S_t^{z_1} X_t^{z_2} \lambda(t, z_1, z_2) - \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \left\{ \frac{d\lambda(u, z_1, z_2)}{d\rho_u^S} + \lambda(u, z_1, z_2) \frac{d\kappa_u(z_1, z_2)}{d\rho_u^S} \right\} \rho_{du}^S, \tag{3.2.31}$$

is a martingale. Moreover, if $(z_1, z_2) \in D/2$ then $M^{\lambda}(z_1, z_2)$ is a square integrable martingale and

$$\mathbb{E}\left[|M_t^{\lambda}(z_1, z_2)|^2 \right] = \int_0^t e^{\kappa_u(2\mathrm{Re}(z_1), 2\mathrm{Re}(z_2))} |\lambda(u, z_1, z_2)|^2 \rho_{du}(z_1, z_2).$$

Proof. Let $(z_1, z_2) \in D$, $M(z_1, z_2)$ and $V(z_1, z_2)$ be the random fields introduced in Remark 3.2.25. Since $\lambda(dt, z_1, z_2) \ll \rho_{dt}^S$, then $t \mapsto \lambda(t, z_1, z_2)$ is a bounded continuous function on $[0,T]$. By item 5) of Proposition 3.2.24 $M^{\lambda}(z_1, z_2)$ is well-defined. Integrating by parts and taking into account Remark 3.2.25 allows to show

$$M_t^{\lambda}(z_1, z_2) = \lambda(0, z_1, z_2) M_0(z_1, z_2) + \int_0^t \lambda(u, z_1, z_2) dM_u(z_1, z_2), \ \forall t \in [0,T]. \tag{3.2.32}$$

Obviously $M^{\lambda}(z_1, z_2)$ is a local martingale. In order to prove that it is a true martingale, we establish that

$$\mathbb{E}\left[\sup_{t \le T} \left| M_t^{\lambda}(z_1, z_2) \right| \right] < \infty.$$

Indeed, by integration by parts in (3.2.32), for $t \in [0,T]$ we have

$$M_t^{\lambda}(z_1, z_2) = \lambda(t, z_1, z_2) M_t(z_1, z_2) - \int_0^t M_{u-}(z_1, z_2) \lambda(du, z_1, z_2).$$

Hence, as in the proof of Lemma 3.2.28,

$$\begin{aligned} \mathbb{E}\left[\sup_{t \le T} \left| M_t^{\lambda}(z_1, z_2) \right| \right] \le & \mathbb{E}\left[\sup_{t \le T} |\lambda(t, z_1, z_2) M_t(z_1, z_2)| \right] + \mathbb{E}\left[\int_0^T |M_{u-}(z_1, z_2)| \, \|\lambda(., z_1, z_2)\|_{dt} \right] \\ \le & 2\mathbb{E}\left[\sup_{t \le T} |M_t(z_1, z_2)| \right] \|\lambda(., z_1, z_2)\|_T . \end{aligned} \tag{3.2.33}$$

Thanks to Proposition 3.2.29, the right-hand side of (3.2.33) is finite and finally $M^{\lambda}(z_1, z_2)$ is shown to be a martingale so that the first part of Lemma 3.2.30 is proved.

Now, suppose that $(z_1, z_2) \in D/2$. By (3.2.32) and Proposition 3.2.26, we have

$$\begin{aligned}
\mathbb{E}\left[\langle M^{\lambda}(z_1,z_2)\rangle_T\right] &= \mathbb{E}\left[\int_0^T |\lambda(t,z_1,z_2)|^2 \langle M(z_1,z_2)\rangle_{dt}\right] \\
&= \mathbb{E}\left[\int_0^T X_{u-}^{2\mathrm{Re}(z_1)} S_{u-}^{2Re(z_2)} |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2)\right] \\
&= \int_0^T e^{\kappa_u(2Re(z_1),2Re(z_2))} |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2) \\
&\leq \sup_{u\leq T} e^{\kappa_u(2Re(z_1),2Re(z_2))} \int_0^T |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2) < \infty.
\end{aligned} \tag{3.2.34}$$

The latter term is finite by point 3) of Proposition 3.2.24 and by the fact that $\lambda(.,z_1,z_2)$ is bounded on $[0,T]$. Consequently, $M^{\lambda}(z_1,z_2)$ is a square integrable martingale and since $|M^{\lambda}(z_1,z_2)|^2 - \langle M^{\lambda}(z_1,z_2)\rangle$ is a martingale, then

$$\mathbb{E}\left[|M_t^{\lambda}(z_1,z_2)|^2\right] = \int_0^t e^{\kappa_u(2Re(z_1),2Re(z_2))} |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2),$$

because of (3.2.34). □

Now, let Π be a finite Borel measure on $\mathbb{C}^2$ and let us make the following assumption on it.

Assumption 3.2.31. *We set* $I_0 := \mathrm{Re}(supp\ \Pi)$.

1. *I_0 is bounded.*
2. *$I_0 \subset D$.*

Note that this assumption implies that $supp\ \Pi \subset D$.

Theorem 3.2.32. *Suppose that Assumptions 3.2.23 and 3.2.31 are verified. Let $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ be a function such that*

$$\forall (z_1,z_2) \in supp\ \Pi,\ \lambda(dt,z_1,z_2) \ll \rho_{dt}^S, \tag{3.2.35}$$

$$\forall t \in [0,T],\ \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|\lambda(t,z_1,z_2)|^2 < \infty, \tag{3.2.36}$$

$$\int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\left|\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right| < \infty. \tag{3.2.37}$$

Then the function f defined by

$$f(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \lambda(t,z_1,z_2),\ \forall t \in [0,T], x,s > 0. \tag{3.2.38}$$

is continuous. Moreover

$$t \mapsto M_t^{\lambda} := f(t,X_t,S_t) - \int_0^t \rho_{du}^S \int_{\mathbb{C}^2} d\Pi(z_1,z_2) X_{u-}^{z_1} S_{u-}^{z_2} \left\{\frac{d\lambda(u,z_1,z_2)}{d\rho_u^S} + \lambda(u,z_1,z_2)\frac{d\kappa_u(z_1,z_2)}{d\rho_u^S}\right\} \tag{3.2.39}$$

is a martingale.

Remark 3.2.33. *In (3.2.37) and (3.2.39), part of the integrand with respect to Π is only defined for $(z_1, z_2) \in D$. By convention the integrand will be set to zero for $(z_1, z_2) \notin D$. In the sequel we will adopt the same rule.*

Proof. Let $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ verifying the hypotheses of the theorem.

The function f is well-defined. Indeed, for $t \in [0,T], x, y > 0$,

$$|f(t,x,s)| \leq \sup_{(a,b)\in I_0} x^a s^b \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|\lambda(t,z_1,z_2)|,$$

which is finite because of Condition (3.2.36) and Assumption 3.2.31, taking into account Cauchy-Schwarz inequality.

Moreover, by Fubini theorem and (3.2.38), we get

$$\begin{aligned}\mathbb{E}\left[|f(t,X_t,S_t)|\right] &\leq \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\mathbb{E}\left[X_t^{\mathrm{Re}(z_1)} S_t^{\mathrm{Re}(z_2)}\right]|\lambda(t,z_1,z_2)| \\ &\leq \sup_{u\in[0,T],(a,b)\in I_0} \mathbb{E}\left[X_u^a S_u^b\right] \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|\lambda(t,z_1,z_2)|. \quad (3.2.40)\end{aligned}$$

The right-hand side is finite by item 3) of Proposition 3.2.24 and Condition (3.2.36).

We observe that $t \mapsto \lambda(t,z_1,z_2)$ is a continuous function since it is absolutely continuous with respect to ρ^S for fixed $(z_1,z_2) \in \mathbb{C}^2$. On the other hand, Condition (3.2.36) implies that the family $(\lambda(t,z_1,z_2), t \in [0,T])$ is $|\Pi|$ -uniformly integrable. These properties, together with Lebesgue dominated convergence theorem imply that f defined in (3.2.38) is continuous with respect to all the variables.

We show now that the process $t \mapsto M_t^\lambda$ is well-defined. This holds because

$$\mathbb{E}\left[\int_0^t \rho^S_{du} \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|X^{z_1}_{u-}S^{z_2}_{u-}|\left|\frac{d\lambda(u,z_1,z_2)}{d\rho^S_u} + \lambda(u,z_1,z_2)\frac{d\kappa_u(z_1,z_2)}{d\rho^S_u}\right|\right] \quad (3.2.41)$$

$$\leq \sup_{u\in[0,T],(a,b)\in I_0} \mathbb{E}\left[X_u^a S_u^b\right] \int_0^t \rho^S_{du} \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\left|\frac{d\lambda(u,z_1,z_2)}{d\rho^S_u} + \lambda(u,z_1,z_2)\frac{d\kappa_u(z_1,z_2)}{d\rho^S_u}\right|,$$

which is finite by point 3) of Proposition 3.2.24 and Condition (3.2.37). Inequality (3.2.41) allows to apply Fubini theorem to the integral term in (3.2.39), so that we get

$$\begin{aligned}M_t^\lambda &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\left(X_t^{z_1}S_t^{z_2}\lambda(t,z_1,z_2) - \int_0^t X^{z_1}_{u-}S^{z_2}_{u-}\left\{\frac{d\lambda(u,z_1,z_2)}{d\rho^S_u} + \lambda(u,z_1,z_2)\frac{d\kappa_u(z_1,z_2)}{d\rho^S_u}\right\}\rho^S_{du}\right) \\ &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2) M_t^\lambda(z_1,z_2), \quad (3.2.42)\end{aligned}$$

where $M^\lambda(z_1,z_2)$ is defined in (3.2.31) for any $(z_1,z_2) \in D$. We observe that

$$\mathbb{E}\left[\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\left|M_t^\lambda(z_1,z_2)\right|\right] < \infty, \quad (3.2.43)$$

taking into account (3.2.40) and (3.2.41). It remains to show that M^λ is a martingale. Let $0 \leq s \leq t \leq T$ and a bounded, $\mathcal{F}_s$-measurable random variable G. By Fubini theorem and Lemma 3.2.30 it follows

$$\begin{aligned}
\mathbb{E}\left[M_t^\lambda G\right] &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\mathbb{E}\left[M_t^\lambda(z_1,z_2)G\right] \\
&= \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\mathbb{E}\left[M_s^\lambda(z_1,z_2)G\right] \\
&= \mathbb{E}\left[\int_{\mathbb{C}^2} d\Pi(z_1,z_2)M_s^\lambda(z_1,z_2)G\right] \\
&= \mathbb{E}\left[M_s^\lambda G\right],
\end{aligned}$$

which implies the desired result. □

We proceed now to the definition of the domain $\mathfrak{D}(\mathfrak{a})$ in view of the specification of the corresponding martingale problem. We set

$$\begin{aligned}
\mathfrak{D}(\mathfrak{a}) = \Big\{ & f : (t,x,s) \mapsto \int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}\lambda(t,z_1,z_2), \forall t \in [0,T], x,y > 0, \\
& \text{where } \Pi \text{ is a finite Borel measure on } \mathbb{C}^2 \text{ verifying Assumption 3.2.31,} \\
& \text{with } \lambda : [0,T]\times\mathbb{C}^2 \to \mathbb{C} \text{ Borel such that conditions (3.2.35), (3.2.36)} \\
& \text{and (3.2.37) are fulfilled} \Big\}.
\end{aligned} \tag{3.2.44}$$

Corollary 3.2.34. *Suppose that Assumptions 3.2.23 and 3.2.31 are verified. Then (X,S) is a solution of the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}),\mathfrak{a},\rho^S)$ where, for $f \in \mathfrak{D}(\mathfrak{a})$ of the type (3.2.38), $\mathfrak{a}(f)$ is defined by*

$$\mathfrak{a}(f)(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}\left\{\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right\}, \tag{3.2.45}$$

for all $t \in [0,T], x,s > 0$.

Proof. By Theorem 3.2.32 note that $f \in \mathfrak{D}(\mathfrak{a})$ defined in (3.2.38) is continuous, which implies that (3.2.1) is fulfilled. By (3.2.37), $\mathfrak{a}(f)$ belongs to $\mathcal{L}$ defined in (3.2.2) and Condition (3.2.3) is fulfilled. Finally (3.2.4) is a consequence of (3.2.39) in Theorem 3.2.32.

□

Under additional conditions, one can say more about the martingale decomposition given by the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}),\mathfrak{a},\rho^S)$.

Proposition 3.2.35. *Let $f \in \mathfrak{D}(\mathfrak{a})$ as defined in (3.2.38). Suppose the following.*

a) $I_0 := \text{Re}(supp\ \Pi) \subset D/2$,

b) $\displaystyle\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\int_0^T |\lambda(u,z_1,z_2)|^2\rho_{du}(z_1,z_2) < \infty.$

Then, the martingale $t \mapsto M_t^\lambda = f(t,X_t,S_t) - \displaystyle\int_0^t \mathfrak{a}(f)(u,X_{u-},S_{u-})\rho_{du}^S$ is square integrable.

Proof. Let $t \in [0,T]$ and M^λ as defined in (3.2.39), which is a martingale by Theorem 3.2.32. By (3.2.42) we have

$$M_t^\lambda = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) M_t^\lambda(z_1,z_2), \tag{3.2.46}$$

where $M^\lambda(z_1,z_2)$ was defined in (3.2.31). By Lemma 3.2.30, for every $(z_1,z_2) \in D/2$, we have

$$\mathbb{E}\left[|M_t^\lambda(z_1,z_2)|^2\right] = \int_0^t e^{\kappa_u(2Re(z_1),2Re(z_2))} |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2). \tag{3.2.47}$$

By Fubini theorem, integrating both sides of (3.2.47) with respect to $|\Pi|$, gives

$$\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|M_t^\lambda(z_1,z_2)|^2\right] &= \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\mathbb{E}\left[|M_t^\lambda(z_1,z_2)|^2\right] \\
&= \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \int_0^t e^{\kappa_u(2Re(z_1),2Re(z_2))} |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2) \\
&\leq \sup_{u\in[0,T],(a,b)\in I_0} e^{\kappa_u(a,b)} \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \int_0^t |\lambda(u,z_1,z_2)|^2 \rho_{du}(z_1,z_2).
\end{aligned}$$

Now, by point 3) of Proposition 3.2.24 and condition b), the right-hand side is finite. This together with (3.2.46) and Cauchy-Schwarz show that M^λ is square integrable. □

Proposition 3.2.36. *We suppose the validity of Assumptions 3.2.23.*

1) *Assumption 3.2.4 is verified. More precisely*

   i) $id : (t,x,s) \longmapsto s \in \mathfrak{D}(\mathfrak{a})$ *and*

$$\mathfrak{a}(id)(t,x,s) = s\frac{d\kappa_t(0,1)}{d\rho_t^S}, \quad \forall t \in [0,T], x,s > 0. \tag{3.2.48}$$

   ii) $(t,x,s) \longmapsto s^2 \in \mathfrak{D}(\mathfrak{a})$ *and*

$$\widetilde{\mathfrak{a}}(id)(t,x,s) = s^2, \quad \forall t \in [0,T], x,s > 0. \tag{3.2.49}$$

2) *Let Π be a finite signed Borel measure on $\mathbb{C}^2$ verifying Assumption 3.2.31. Let $f \in \mathfrak{D}(\mathfrak{a})$ of the form (3.2.44), such that $\widetilde{f} = f \times id \in \mathfrak{D}(\mathfrak{a})$. Then $\widetilde{\mathfrak{a}}$, defined in (3.2.6), is given by, $\forall t \in [0,T], x,s > 0$,*

$$\widetilde{\mathfrak{a}}(f)(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\lambda(t,z_1,z_2) x^{z_1} s^{z_2+1} \frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S}. \tag{3.2.50}$$

Proof.

We first address item 1).

i) Let $\Pi_1(z_1,z_2) = \delta_{\{z_1=0,z_2=1\}}$ and $\lambda \equiv 1$. Since by Assumption 3.2.23 $(0,1) \in D$, Π_1 fulfills Assumption 3.2.31. The other conditions (3.2.35), (3.2.36), (3.2.37)

defining $\mathfrak{D}(\mathfrak{a})$ in (3.2.44) are trivially satisfied. Consequently $id \in \mathfrak{D}(\mathfrak{a})$ and (3.2.48) follows from (3.2.45).

ii) Let $\Pi_2(z_1, z_2) = \delta_{\{z_1=0, z_2=2\}}$ and $\lambda \equiv 1$. Again, by Assumption 3.2.23 $(0, 2) \in D$, and Π_2 fulfills Assumption 3.2.31. Similar arguments as for i) allow to show that $(t, x, s) \longmapsto s^2 \in \mathfrak{D}(\mathfrak{a})$.

Formula (3.2.50) constitutes a direct application of (3.2.45), taking into account (3.2.44), which establishes item 2). In particular (3.2.49) holds. □

3.3 The basic BSDE and the deterministic problem

3.3.1 General framework

We consider two $\mathcal{F}_t$-adapted processes (X, S) fulfilling the martingale problem related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$ stated in Definition 3.2.1 under Assumption 3.2.4. We denote by M^S the martingale part of S.

Let $\hat{f} : \Omega \times [0, T] \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a predictable random field (i.e. such that for every y, z, $s \mapsto \hat{f}(\cdot, s, y, z)$ is predictable) and h be an $\mathcal{F}_T$-measurable, complex valued, random variable. As we have mentioned in the introduction, the object of our interest is a BSDE of the type (3.1.2). We focus on a deterministic natural problem associated with it, which plays the role of the semilinear PDE of the Brownian case.

Definition 3.3.1. *A triplet (Y, Z, O) of processes is called* **solution** *of (3.1.2) if the following holds.*

1) (Y_t) is $\mathcal{F}_t$-adapted;

2) (Z_t) is $\mathcal{F}_t$-predictable and

(a) $\int_0^T |Z_s|^2 d\langle M^S\rangle_s < \infty$ a.s.

(b) $\int_0^T |\hat{f}(\omega, s, Y_{s-}, Z_s)| d \left\|V^S\right\|_s < \infty$ a.s.

3) Equality (3.1.2) holds and (O_t) is an $\mathcal{F}_t$-local martingale such that $\langle O, M^S\rangle = 0$ and $O_0 = 0$ a.s.

In this section we are more particularly interested in the BSDE (3.1.2) when $\hat{f}$ is given by (3.1.3).

3.3.2 The forward-backward case and the deterministic problem

As we have already mentioned in the Introduction, the BSDE on which we will focus on, arises when the *driver coefficient* $\hat{f}$ is associated with a locally bounded function $f : [0, T] \times \mathcal{O} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ and with the $\mathcal{F}_t$-special semimartingale (X, S) which solves the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$. In conformity with (3.1.3), we suppose the form of $\hat{f}$ and of the target r.v. h as follows.

$$\begin{aligned} \mathfrak{a}(id)(t, X_{t-}(\omega), S_{t-}(\omega))\hat{f}(\omega, t, y, z) &= f(t, X_{t-}(\omega), S_{t-}(\omega), y, z), \\ h &= g(X_T, S_T), \end{aligned}$$

for some continuous function $g : \mathcal{O} \to \mathbb{C}$.

Therefore, our subject of study is the particular BSDE given below.

$$Y_t = h + \int_t^T f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r - \int_t^T Z_r dM_r^S - (O_T - O_t), \quad t \in [0, T]. \tag{3.3.1}$$

As we remarked in the introduction, when M^S is a Brownian motion and $(\mathcal{F}_t)$ is its canonical filtration, (3.3.1) can be linked to a semilinear partial differential equation. We will formulate a deterministic problem, generalizing that "classical" semilinear PDE. In particular we look for solutions (Y, Z, O) for which there is a function $y \in \mathfrak{D}(\mathfrak{a})$ such that $\widetilde{y} = y \times id \in \mathfrak{D}(\mathfrak{a})$ and a locally bounded Borel function $z : [0, T] \times \mathcal{O} \longrightarrow \mathbb{C}$, such that

$$Y_t = y(t, X_t, S_t), \tag{3.3.2}$$

$$Z_t = z(t, X_{t-}, S_{t-}), \quad \forall t \in [0, T], \tag{3.3.3}$$

and

$$\begin{aligned} \int_0^t |Z_s|^2 d\langle M^S \rangle_s &< \infty \text{ a.s.} \\ \int_0^t |f(s, X_{s-}, S_{s-}, Y_{s-}, Z_s)| d \left\| A \right\|_s &< \infty \text{ a.s.} \end{aligned} \tag{3.3.4}$$

By (3.3.2) and (3.3.4), Conditions 1) and 2) of Definition 3.3.1 are obviously fulfilled. Consequently the triplet (Y, Z, O) where

$$O_t := Y_t - Y_0 - \int_0^t Z_r dM_r^S + \int_0^t f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r, \tag{3.3.5}$$

is a solution of (3.1.2) provided that

1. (O_t) is an $\mathcal{F}_t$-local martingale, (3.3.6)
2. $\langle O, M^S \rangle = 0$, (3.3.7)
3. $Y_T = g(X_T, S_T)$. (3.3.8)

Since (X, S) solves the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, replacing (3.3.2) in expression (3.3.5), Condition (3.3.6) can be rewritten saying that

$$\int_0^t \mathfrak{a}(y)(r, X_{r-}, S_{r-}) dA_r + \int_0^t f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r$$

is an $\mathcal{F}_t$-local martingale. This implies that

$$\int_0^t \mathfrak{a}(y)(r, X_{r-}, S_{r-}) dA_r + \int_0^t f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r = 0. \tag{3.3.9}$$

On the other hand, Condition (3.3.7) implies

$$\langle M^Y, M^S\rangle_t = \int_0^t Z_s d\langle M^S\rangle_s, \tag{3.3.10}$$

where M^Y denotes the martingale part of Y. By Lemma 3.2.3 and item ii) of Corollary 3.2.5, we have

$$\begin{aligned}\langle M^Y, M^S\rangle_t &= \int_0^t \widetilde{\mathfrak{a}}(y)(r, X_{r-}, S_{r-})dA_r \\ \langle M^S\rangle_t &= \int_0^t \widetilde{\mathfrak{a}}(id)(r, X_{r-}, S_{r-})dA_r.\end{aligned}$$

Consequently, Condition (3.3.10) can be re-expressed as

$$\int_0^t \widetilde{\mathfrak{a}}(y)(r, X_{r-}, S_{r-})dA_r = \int_0^t z(r, X_{r-}, S_{r-})\widetilde{\mathfrak{a}}(id)(r, X_{r-}, S_{r-})dA_r. \tag{3.3.11}$$

Condition (3.3.8) requires $y(T,\cdot,\cdot) = g$.

This allows to state the following representation theorem.

Theorem 3.3.2. *Suppose the existence of a function y, such that $y, \widetilde{y} := y \times id$ belong to $\mathfrak{D}(\mathfrak{a})$, and a Borel locally bounded function z, solving the system*

$$\mathfrak{a}(y)(t,x,s) = -f(t,x,s,y(t,x,s),z(t,x,s)) \tag{3.3.12}$$

$$\widetilde{\mathfrak{a}}(y)(t,x,s) = z(t,x,s)\widetilde{\mathfrak{a}}(id)(t,x,s), \tag{3.3.13}$$

for $t \in [0,T]$ and $(x,s) \in \mathcal{O}$, where the equalities hold in $\mathcal{L}$, with the terminal condition $y(T,.,.) = g(.,.)$.

Then the triplet (Y,Z,O) defined by

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_{t-}, S_{t-}) \tag{3.3.14}$$

and (O_t) given by (3.3.5), is a solution to the BSDE (3.3.1).

Proof. The triplet (Y,Z,O) fulfills the three conditions of Definition 3.3.1 provided that (3.3.4) is verified. Indeed, since $y \in \mathfrak{D}(\mathfrak{a})$ then the integral $\int_0^t |f(s, X_{s-}, S_{s-}, Y_{s-}, Z_s)|d\,\|A\|_s$, is finite taking into account (3.2.3).
Since z is locally bounded, then $\int_0^T |Z_s|^2 d\langle M\rangle_s$ is also finite. This concludes the proof of the theorem. □

Remark 3.3.3.

1. *The statement of Theorem 3.3.2 can be generalized relaxing the assumption on z to be locally bounded. We replace this with the condition*

$$\int_0^T z^2(r, X_{r-}, S_{r-})\widetilde{\mathfrak{a}}(id)(r, X_{r-}, S_{r-})dA_r < \infty \text{ a.s.} \tag{3.3.15}$$

This is equivalent to $\int_0^T |Z_s|^2 d\langle M\rangle_s < \infty$ a.s.

2. *In particular, if z is locally bounded a.s., then* (3.3.15) *is fulfilled.*

Remark 3.3.4. *Theorem 3.3.2 constitutes also an existence theorem for particular BSDEs. If M^S is a square integrable martingale and the function $\hat{f}$ associated with f, fulfills some Lipschitz type conditions then the solution (Y, Z, O) provided by* (3.3.14) *is unique in the class of processes introduced in [Carbone et al., 2007, Theorem 3.1].*

The presence of the local martingale O is closely related to the classical martingale representation property. In fact, if $(\Omega, \mathcal{F}, \mathbb{P})$ verifies the local martingale representation property with respect to M^S, then O vanishes.

Proposition 3.3.5. *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ fulfills the local martingale representation property with respect to M. Then, if (Y, Z, O) is a solution to* (3.3.1)*, then, necessarily $O_t = 0,\ \forall t \in [0, T]$.*

Proof. Since (O_t) is an $\mathcal{F}_t$-local martingale, there is a predictable process (Z_t) such that

$$O_t = O_0 + \int_0^t Z_s dM_s^S,\ \forall t \in [0, T].$$

So the condition $\langle O, M^S \rangle \equiv 0$ implies

$$\int_0^{\cdot} Z_s d\langle M^S \rangle = 0.$$

Consequently,

$$Z \equiv 0\ d\mathbb{P} \otimes d\langle M^S \rangle \text{ a.e.},$$

and so $O_t = O_0 = 0\ \forall t \in [0, T]$.

□

3.3.3 Illustration 1: the Markov semigroup case

Let us consider the case of Section 3.2.2 with related notations. Let $S = X^{0,x}$ be a solution of the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$, see Definition 3.2.6. Let (P_t) be the semigroup introduced in (3.2.14), fulfilling Assumption 3.2.7 with generator L defined in Definition 3.2.11. Let $f : [0, T] \times \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C}$ be a locally bounded function and a continuous function $g : \mathbb{R} \longrightarrow \mathbb{C}$.
Here we have of course $S = M^S + V^S$ where $V^S = \int_0^{\cdot} \mathfrak{a}(id)(r, S_{r-})dr$ and $id(s) \equiv s$.

Theorem 3.3.2 gives the following.

Proposition 3.3.6. *Suppose the existence of a function $y : [0, T] \times \mathbb{R} \to \mathbb{C}$ and a Borel locally bounded function $z : [0, T] \times \mathbb{R} \to \mathbb{C}$ verifying the following.*

i) $t \mapsto y(t, \cdot)$ (resp. $\tilde{y}(t, \cdot)$) takes value in $D(L)$ and it is continuous with respect to the graph norm.

ii) $t \mapsto y(t, \cdot)$ (resp. $\tilde{y}(t, \cdot)$) is of class C^1 with values in E.

iii) For $(t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} \partial_t y(t,x) + Ly(t,\cdot)(x) &= -f(t,x,y(t,x),z(t,x)), \\ z(t,\cdot)\widetilde{L}(id) &= \widetilde{L}y(t,\cdot), \\ y(T,.) &= g, \end{aligned}$$

where $\widetilde{L}\varphi = L\tilde{\varphi} - \varphi Lid - idL\varphi$.

Then the triplet (Y,Z,O)*, where*

$$Y_t := y(t,S_t),\ Z_t := z(t,S_{t-}),$$

$$O_t := Y_t - Y_0 - \int_0^t Z_r dM_r^S + \int_0^t \hat{f}(r,\omega,Y_{r-},Z_r)dV_r^S,\ t \in [0,T],$$

is a solution of the BSDE

$$Y_t = g(S_T) + \int_t^T \hat{f}(r,\omega,Y_{r-},Z_r)dV_r^S - \int_t^T Z_r dM_r^S - (O_T - O_t), \quad t \in [0,T],$$

in the sense of Definition 3.3.1, where

$$\mathfrak{a}(id)(r,S_{r-}(\omega))\hat{f}(r,\omega,y,z) = f(r,S_{r-}(\omega),y,z).$$

Remark 3.3.7. *If* $S = \sigma W$ *with* $\sigma > 0$ *and* $\varphi : [0,T] \times \mathbb{R} \to \mathbb{C}$ *is of class* $C^{1,2}$ *then* $\mathfrak{a}(\varphi) = \partial_t\varphi + \frac{\sigma^2}{2}\partial_{ss}\varphi$ *and* $\widetilde{\mathfrak{a}}(\varphi) = \sigma^2\partial_s\varphi = \widetilde{\mathfrak{a}}(id)\partial_s\varphi$.
In the case where L *is a generic generator, the formal quotient* $\frac{\widetilde{\mathfrak{a}}(\varphi)}{\widetilde{\mathfrak{a}}(id)}$ *can be considered as a sort of generalized derivative.*

3.3.4 Illustration 2: the diffusion case

Consider the case of where (X,S) is diffusion process as given in equations (3.2.26). We remind that in that case, the operator $\mathfrak{a}$, for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^2)$, is given by

$$\begin{aligned} \mathfrak{a}(\varphi) &= \partial_t\varphi + b_S\partial_s\varphi + b_X\partial_x\varphi \\ &+ \frac{1}{2}\left\{|\sigma_S|^2\partial_{ss}\varphi + |\sigma_X|^2\partial_{xx}\varphi + 2\langle\sigma_S,\sigma_X\rangle\partial_{sx}\varphi\right\}. \end{aligned}$$

Corollary 3.3.8. *Let* (y,z) *be a solution of the PDE*

$$\mathfrak{a}(y)(t,x,s) = -f(t,x,s,y(t,x,s),z(t,x,s)) \tag{3.3.16}$$

$$|\sigma_S|^2 z(t,x,s) = |\sigma_S|^2\partial_s y(t,x,s) + \langle\sigma_S,\sigma_X\rangle\partial_x y(t,x,s), \tag{3.3.17}$$

with terminal condition $y(T,.,.) = g(.,.)$*. Then the triplet* (Y,Z,O)*, where*

$$Y_t = y(t,X_t,S_t),\ Z_t = z(t,X_t,S_t),$$

and (O_t) *is given by* (3.3.5) *is a solution to the BSDE* (3.3.1).

3.4 Explicit solution for Föllmer-Schweizer decomposition in the basis risk context

3.4.1 General considerations

We will discuss in this section the important Föllmer-Schweizer decomposition, denoted shortly F-S decomposition. It is a generalization of the well-known Galtchouk-Kunita-Watanabe decomposition for martingales, to the more general case of semimartingales. Our task will consist in providing explicit expressions for the F-S decomposition in several situations. Let S be a special semimartingale with canonical decomposition $S = M^S + V^S$. In the sequel we will convene that the space $L^2(M^S)$ consists of the predictable processes $(Z_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T |Z_s|^2 d\langle M^S\rangle_s\right] < \infty$ and $L^2(V^S)$ will denote the set of all predictable processes $(Z_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\left(\int_0^T |Z_s| d\|V^S\|_s\right)^2\right] < \infty$. The intersection of these two spaces is denoted

$$\Theta := L^2(M^S) \cap L^2(V^S). \tag{3.4.1}$$

The Föllmer-Schweizer decomposition is defined as follows.

Definition 3.4.1. *Let h be a (possibly complex valued) square integrable $\mathcal{F}_T$-measurable random variable. We say that h admits an F-S decomposition with respect to S if it can be written as*

$$h = h_0 + \int_0^T Z_s dS_s + O_T, \mathbb{P} - a.s., \tag{3.4.2}$$

where h_0 is an $\mathcal{F}_0$-measurable r.v., $Z \in \Theta$ and $O = (O_t)_{t\in[0,T]}$ is a square integrable martingale, strongly orthogonal to M^S.

Remark 3.4.2.

1) *The notion of weak and strong orthogonality is discussed for instance in [Protter, 2005, Section 4.3] and [Jacod and Shiryaev, 2003, Section 1.4b]. Let L and N be two $\mathcal{F}_t$-local martingales, with null initial value. L and N are said to be strongly orthogonal if LN is a local martingale. If L and N are locally square integrable, then they are strongly orthogonal if and only if $\langle L, N\rangle = 0$. The definition of locally square integrable martingale is given for instance just before [Protter, 2005, Theorem 49 in Chapter 1].*

2) *The F-S decomposition makes also sense for complex valued square integrable random variable h. In that case the triplet (h_0, Z, O) is generally complex.*

3) *If h admits an F-S decomposition (3.4.2) then the complex conjugate $\bar{h}$ admits an F-S decomposition given by*

$$\bar{h} = \bar{h}_0 + \int_0^T \bar{Z}_s dS_s + \bar{O}_T, \mathbb{P} - a.s. \tag{3.4.3}$$

The F-S decomposition has been extensively studied in the literature: sufficient conditions on the process S were given so that every square integrable random vari-

able has such a decomposition. A well-known condition ensuring the existence of such a decomposition is the so called **structure condition** (SC).

Definition 3.4.3. *We say that a special semimartingale $S = V^S + M^S$ satisfies the **structure condition** (SC) if there exists a predictable process α such that*

1. $V_t^S = \int_0^t \alpha_s d\langle M^S\rangle_s$,
2. $\int_0^T \alpha_s^2 d\langle M^S\rangle_s < \infty$ *a.s.*

The latter quantity plays a central role in the F-S decomposition. The associated process

$$K_t := \int_0^t \alpha_s^2 d\langle M^S\rangle_s \text{ for } t \in [0,T], \tag{3.4.4}$$

is called **mean variance trade-off** process.

Remark 3.4.4. *Monat and Stricker [1995] proved that, under (SC) and the additional condition that the process K is uniformly bounded, the F-S decomposition of any real valued square integrable random variable exists and it is unique. More recent papers about the subject are Schweizer [2001], Černý and Kallsen [2007] and references therein.*

This general decomposition refers to the process S as underlying and it will be applied in the context of mean-variance hedging under basis risk, where X is an observable price process of a non-traded asset.
As in previous sections, we consider a couple (X, S) verifying the martingale problem (3.2.7), and we suppose Assumption 3.2.4 to be fulfilled. In the sequel we do not necessarily assume (SC) for S.

Definition 3.4.5. *Let h be a square integrable $\mathcal{F}_T$-measurable random variable. We say that h admits a* **weak F-S decomposition** *with respect to S if it can be written as*

$$h = h_0 + \int_0^T Z_s dS_s + O_T, \mathbb{P}-\text{a.s.}, \tag{3.4.5}$$

where h_0 is an $\mathcal{F}_0$-measurable r.v., Z is a predictable process such that $\int_0^T |Z_s|^2 d\langle M^S\rangle_s < \infty$ a.s., $\int_0^T |Z_s| d\|V^S\|_s < \infty$ a.s. and O is a local martingale such that $\langle O, M^S\rangle = 0$ with $O_0 = 0$.

Finding a weak F-S decomposition (3.4.5) (h_0, Z, O) for some r.v. h is equivalent to finding a solution (Y, Z, O) of the BSDE

$$Y_t = h - \int_t^T Z_s dS_s - (O_T - O_t). \tag{3.4.6}$$

The link is given by $Y_0 = h_0$. The latter equation (3.4.6) can be seen as a special case of BSDE (3.3.1), where the driver f is linear in z, of the form

$$f(t,x,s,y,z) = -\mathfrak{a}(id)(t,x,s)z. \tag{3.4.7}$$

This point of view was taken for instance by Schweizer [1994].

Remark 3.4.6. *Let (Y, Z, O) be a solution of (3.4.6) with $Z \in \Theta$, where Θ has been defined in (3.4.1) and O is a square integrable martingale. Then h admits an F-S decomposition (3.4.2) with $Y_0 = h_0$.*

We consider the case of the final value $h = g(X_T, S_T)$ for some continuous function g. Theorem 3.3.2 can be applied to obtain the result below.

Corollary 3.4.7. *Let y (resp. z): $[0,T] \times \mathcal{O} \to \mathbb{C}$. We suppose the following.*

1) *$y, \widetilde{y} := y \times id$ belong to $\mathfrak{D}(\mathfrak{a})$.*

2) *z verifies (3.3.15) of Remark 3.3.3.*

3) *(y, z) solve the problem*

$$\mathfrak{a}(y)(t,x,s) = \mathfrak{a}(id)(t,x,s)z(t,x,s), \tag{3.4.8}$$
$$\widetilde{\mathfrak{a}}(y)(t,x,s) = \widetilde{\mathfrak{a}}(id)(t,x,s)z(t,x,s), \tag{3.4.9}$$

where the equalities hold in $\mathcal{L}$, with the terminal condition $y(T,.,.) = g(.,.)$.

Then the triplet (Y, Z, O), where

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_{t-}, S_{t-}),\ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

is a solution to the linear BSDE (3.4.6) linked to the weak F-S decomposition.

Remark 3.4.8. *We remind that, setting $h_0 = y(0, X_0, S_0)$, the triplet (h_0, Z, O) is a candidate for a true F-S decomposition, see Definition 3.4.1. Sufficient conditions for this are the following.*

a) $h = g(X_T, S_T) \in L^2(\Omega)$.

b) $(z(t, X_{t-}, S_{t-}))_t \in \Theta$ *i.e.*

- $\mathbb{E}\left[\int_0^T |z(t, X_{t-}, S_{t-})|^2 \widetilde{\mathfrak{a}}(id)(t, X_{t-}, S_{t-}) dA_t\right] < \infty$.
- $\mathbb{E}\left[\left(\int_0^T |z(t, X_{t-}, S_{t-})| \, \|\mathfrak{a}(id)(t, X_{t-}, S_{t-}) dA\|_t\right)^2\right] < \infty$.

c) $\left(y(t, X_t, S_t) - \int_0^t \mathfrak{a}(y)(u, X_{u-}, S_{u-}) dA_u\right)_t$ *is an $\mathcal{F}_t$-square integrable martingale.*

We remark that b) and c) imply by additivity that O is a square integrable martingale. In fact

$$O_t = y(t, X_t, S_t) - \int_0^t \mathfrak{a}(y)(u, X_{u-}, S_{u-}) dA_u - \int_0^t z(u, X_{u-}, S_{u-}) dM_u^S,\ \forall t \in [0,T]. \tag{3.4.10}$$

3.4.2 Application: exponential of additive processes

We will investigate in this section a significant context where the equations in Corollary 3.4.7 can be solved, yielding the weak F-S decomposition and we can give sufficient conditions so that the true F-S decomposition is fulfilled. We focus on exponential of additive processes. Another example will be given in Section 3.4.3.

Let (X, S) be a couple of exponential of semimartingale additive processes, as introduced in Section 3.2.5.

Proposition 3.4.9. *Under Assumption 3.2.23, S verifies the (SC) condition given in Definition 3.4.3 if and only if*

$$\int_0^T \left(\frac{d\kappa_t(0,1)}{d\rho_t^S}\right)^2 d\rho_t^S < \infty \tag{3.4.11}$$

In this case, the mean variance trade-off process K is deterministic and given by

$$K_t = \int_0^t \left(\frac{d\kappa_u(0,1)}{d\rho_u^S}\right)^2 d\rho_u^S < \infty, \ \forall t \in [0,T]. \tag{3.4.12}$$

Proof. It follows from Corollary 3.2.27 and item 5) of Proposition 3.2.24. □

We look for the F-S decomposition of an $\mathcal{F}_T$-measurable random variable h of the form $h := g(X_T, S_T)$ for a function g such that

$$g(x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2}, \tag{3.4.13}$$

where Π is finite Borel complex measure.

In Section 3.2.5, Corollary 3.2.34 states that (X, S) fulfills the martingale problem with respect to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, \rho^S)$ where the objects $\mathfrak{D}(\mathfrak{a}), \mathfrak{a}$ and ρ^S were introduced respectively in (3.2.44), (3.2.45), (3.2.28). In order to determine the F-S decomposition (in its weak form given in (3.4.5)) we make use of Corollary 3.4.7. We look for a function y (resp. z): $[0,T] \times \mathbb{R}^2 \to \mathbb{C}$ such that Hypotheses 1), 2) and 3) are fulfilled. In agreement with definition of $\mathfrak{D}(\mathfrak{a})$ given in (3.2.44) we select y of the form

$$y(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \lambda(t,z_1,z_2), \tag{3.4.14}$$

where Π being the same finite complex measure as in (3.4.13) and $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$. We will start by writing "necessary" conditions for a couple (y, z), such that y has the form (3.4.14), to be solutions of (3.4.8) and (3.4.9).

Suppose that the couple (y, z) fulfills (3.4.8) and (3.4.9) of Corollary 3.4.7. We consider the expressions of $\mathfrak{a}(id), \widetilde{\mathfrak{a}}(id)$ given by (3.2.48), (3.2.49), and $\mathfrak{a}(y), \widetilde{\mathfrak{a}}(y)$ given by (3.2.45) and (3.2.50), for $f = y$. We replace them in the two above mentioned conditions (3.4.8) and (3.4.9) to obtain the following equations for λ ($d\rho_t^S$ a.e.).

$$\begin{aligned}\int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}\left\{\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S}+\lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right\} &= s\frac{d\kappa_t(0,1)}{d\rho_t^S}z(t,x,s)\\ \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\lambda(t,z_1,z_2)x^{z_1}s^{z_2+1}\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S} &= s^2 z(t,x,s).\end{aligned} \tag{3.4.15}$$

The final condition $y(T,\cdot,\cdot)=g$ produces

$$\int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}\lambda(T,z_1,z_2)=\int_{\mathbb{C}^2} d\Pi(z_1,z_2)x^{z_1}s^{z_2}. \tag{3.4.16}$$

Replacing z from the second line of (3.4.15) in the first line, by identification of the inverse Fourier-Laplace transform, it follows that λ verifies

$$\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} = \lambda(t,z_1,z_2)\left\{\frac{d\kappa_t(0,1)}{d\rho_t^S}\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S}-\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right\} \tag{3.4.17}$$

$$\lambda(T,z_1,z_2) = 1, \tag{3.4.18}$$

for all $(z_1,z_2)\in supp\ \Pi$. Without restriction of generality we can clearly set $\lambda(\cdot,z_1,z_2)=0$ for (z_1,z_2) outside the support of Π. We observe that for fixed z_1,z_2, (3.4.17) constitutes an ordinary differential equation (in the Lebesgue-Stieltjes sense) in time t.

We solve now the linear differential equation (3.4.17). Provided that

$$u\mapsto \frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\frac{d\kappa_u(0,1)}{d\rho_u^S}\in L^1([0,T],d\rho_s), \tag{3.4.19}$$

the (unique) solution of (3.4.17), is given by

$$\begin{aligned}\lambda(t,z_1,z_2) &= \exp\left(\int_t^T\left[\frac{d\kappa_u(z_1,z_2)}{d\rho_u^S}-\frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\frac{d\kappa_u(0,1)}{d\rho_u^S}\right]d\rho_u^S\right)\\ &= \exp\left(\int_t^T \kappa_{du}(z_1,z_2)-\frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\kappa_{du}(0,1)\right)\\ &= \exp\left(\int_t^T \eta(z_1,z_2,du)\right),\end{aligned} \tag{3.4.20}$$

where

$$\eta(z_1,z_2,t):=\kappa_t(z_1,z_2)-\int_0^t\frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\kappa_{du}(0,1), \tag{3.4.21}$$

which is clearly absolutely continuous with respect to $d\rho^S$.
At this point, we have an explicit form of λ defining the function y intervening in the weak F-S decomposition. In the sequel we will show that such a choice of λ will constitute a sufficient condition so that (y,z) where y is defined by (3.4.14) and z determined by the second line of (3.4.15), is a solution of the deterministic problem given by (3.4.8) and (3.4.9).

In order to check (3.4.19) and the validity of (3.4.15) and (3.4.16), we formulate the following assumption reinforcing Assumptions 3.2.23 and 3.2.31.

Assumption 3.4.10. *Recall* $I_0 := \mathrm{Re}(supp\ \Pi)(\subset \mathbb{R}^2)$, *where we convene that* $\mathrm{Re}(z_1, z_2) = (\mathrm{Re}(z_1), \mathrm{Re}(z_2))$. *We denote* $I := 2I_0 \cup \{(0,1)\}$ *and* $\mathcal{D}$ *the set*

$$\mathcal{D} = \Big\{ z \in D,\ \int_0^T \left| \frac{d\kappa_u(z_1, z_2)}{d\rho_u^S} \right|^2 d\rho_u^S < \infty \Big\}. \tag{3.4.22}$$

We assume the validity of the properties below.

1) ρ^S *is strictly increasing.*

2) I_0 *is bounded.*

3) $\forall z \in supp\ \Pi,\ z, z + (0,1) \in \mathcal{D}$.

4) $\sup_{x \in I} \left\| \frac{d(\kappa_t(x))}{d\rho_t^S} \right\|_\infty < \infty$.

Remark 3.4.11.

1) *Assumptions 3.2.23 and 3.2.31 are consequences of Assumption 3.4.10.*

2) *Taking into account Remark 3.2.33, we emphasize that, for the rest of this section, the statements would not change if we consider that the quantities integrated with respect to the measure* Π *are null outside its support.*

3) $I \subset \mathcal{D}$, *in particular* $(0,1) \in \mathcal{D}$ *because of item 4) of Assumption 3.4.10.*

4) *By previous item and Proposition 3.4.9,* S *verifies the (SC) condition and the mean variance trade-off process* K *given by (3.4.12) is deterministic.*

5) $I_0 \subset D/2$ *(i.e.* $supp\ \Pi \subset D/2$*). This follows again by item 4) of Assumption 3.4.10.*

In the sequel we will introduce the following notation.

$$\gamma_t(z_1, z_2) := \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S},\ \forall (z_1, z_2) \in D/2, t \in [0, T]. \tag{3.4.23}$$

Similarly to [Goutte et al., 2014, Lemma 3.28], we can show the upper bounds below.

Lemma 3.4.12. *Under Assumption 3.4.10, we have the following.*

1) *Condition (3.4.19) is verified for* $t \in [0,T], (z_1, z_2) \in supp\ \Pi$.

2) *There is a positive constant* c_1, *such that* $d\rho_s^S$ *a.e.* $\sup_{(z_1,z_2) \in I_0 + i\mathbb{R}^2} \frac{d\mathrm{Re}(\eta(z_1, z_2, t))}{d\rho_t^S} \leq c_1$.

3) *There are positive constants* c_2, c_3 *such that,* $d\rho_s$ *a.e. the following holds.*
For any $(z_1, z_2) \in I_0 + i\mathbb{R}^2$, $\left| \gamma_t(z_1, z_2) \right|^2 \leq \frac{d\rho_t(z_1, z_2)}{d\rho_t^S} \leq c_2 - c_3 \frac{d\mathrm{Re}(\eta(z_1, z_2, t))}{d\rho_t^S}$.

4) $\sup_{(z_1,z_2) \in I_0 + i\mathbb{R}^2} - \int_0^T 2\mathrm{Re}(\eta(z_1, z_2, dt)) \exp\left(\int_t^T 2\mathrm{Re}(\eta(z_1, z_2, ds)) \right) < \infty$.

Proof. For illustration we prove item 1), the other points can be shown by similar techniques as in [Goutte et al., 2014, Lemma 3.28].
Let $t \in [0,T], (z_1,z_2) \in supp\ \Pi$. Condition (3.4.19) is valid since $(0,1) \in \mathcal{D}$, $z, z+(0,1) \in \mathcal{D}$ and

$$\left(\int_0^t \left|\frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\frac{d\kappa_u(0,1)}{d\rho_u^S}\right| \rho_{du}^S\right)^2 \leq \int_0^t \left|\frac{d\rho_u(z_1,z_2,0,1)}{d\rho_u^S}\right|^2 \rho_{du}^S \int_0^t \left|\frac{d\kappa_u(0,1)}{d\rho_u^S}\right|^2 \rho_{du}^S.$$

□

Now, we can state a proposition that gives indeed the weak F-S decomposition of a random variable $h = g(X_T, S_T)$.

Proposition 3.4.13. *We suppose the validity of Assumption 3.4.10. Let λ be defined as*

$$\lambda(t,z_1,z_2) = \exp\left(\int_t^T \eta(z_1,z_2,du)\right), \forall (z_1,z_2) \in D/2, \tag{3.4.24}$$

where η has been defined at (3.4.21). Then (Y,Z,O) is a solution of the BSDE (3.4.6), where

$$\begin{aligned} Y_t &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2) X_t^{z_1} S_t^{z_2} \lambda(t,z_1,z_2) \\ Z_t &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2) X_{t-}^{z_1} S_{t-}^{z_2-1} \lambda(t,z_1,z_2)\gamma_t(z_1,z_2) \\ O_t &= Y_t - Y_0 - \int_0^t Z_s dS_s, \end{aligned}$$

recalling that γ has been defined in (3.4.23).

Proof. The result will follow from Corollary 3.4.7 for which we need to check the assumptions.
First we prove that the function y defined by

$$y(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \lambda(t,z_1,z_2),$$

where λ is defined in (3.4.24), is indeed an element of $\mathfrak{D}(\mathfrak{a})$. Secondly, we prove that the associated $\widetilde{y}$ also belongs to $\mathfrak{D}(\mathfrak{a})$. Third, we check the condition (3.3.15) for z. Finally we need to check the validity of the system of equations (3.4.8) and (3.4.9).
Concerning y, the function $\lambda(\cdot, z_1, z_2)$ is well-defined for $(z_1,z_2) \in supp\ \Pi$, thanks to point 1) of Lemma 3.4.12 and by definition we have $\lambda(dt,z_1,z_2) \ll \rho_{dt}^S, \quad \forall (z_1,z_2) \in D$, which is Condition (3.2.35).
In order to prove that $y \in \mathfrak{D}(\mathfrak{a})$, which was defined in (3.2.44), it remains to prove the two conditions below which constitute conditions (3.2.36) and (3.2.37) of Theorem 3.2.32.

$$\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\,|\lambda(t,z_1,z_2)|^2 < \infty, \quad \forall t \in [0,T]; \tag{3.4.25}$$

$$\int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left|\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right| < \infty. \tag{3.4.26}$$

Let $t \in [0,T],\ (z_1,z_2) \in D/2$. By (3.4.24), we have

$$|\lambda(t,z_1,z_2)| = \exp\left(\int_t^T \frac{dRe(\eta(z_1,z_2,u))}{d\rho_u^S}\rho_{du}^S\right),$$

which implies, by item 2) of Lemma 3.4.12, that

$$|\lambda(t,z_1,z_2)| \leq \exp\left(c_1\rho_T^S\right), \tag{3.4.27}$$

which gives in particular (3.4.25): in fact $\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)|\lambda(t,z_1,z_2)|^2 \leq e^{2c_1\rho_T^S}|\Pi|(\mathbb{C}^2) < \infty$.

Finally, to conclude that $y \in \mathfrak{D}(\mathfrak{a})$, we need to show (3.4.26). By construction, λ verifies equation (3.4.17). Hence, by (3.4.17) and Cauchy-Schwarz we get

$$\begin{aligned}
&\left(\int_0^T d\rho_t^S \left|\frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S}\right|\right)^2 \\
&\qquad = \left(\int_0^T d\rho_t^S\, |\lambda(t,z_1,z_2)| \left|\frac{d\kappa_t(0,1)}{d\rho_t^S}\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S}\right|\right)^2 \\
&\qquad \leq \int_0^T |\lambda(t,z_1,z_2)|^2\, |\gamma_t(z_1,z_2)|^2\, d\rho_t^S \int_0^T \left|\frac{d\kappa_t(0,1)}{d\rho_t^S}\right|^2 d\rho_t^S \\
&\qquad \leq (I_1(z_1,z_2) + I_2(z_1,z_2)) \int_0^T \left|\frac{d\kappa_t(0,1)}{d\rho_t^S}\right|^2 d\rho_t^S,
\end{aligned} \tag{3.4.28}$$

with

$$\begin{aligned}
I_1(z_1,z_2) &:= c_2 \int_0^T |\lambda(t,z_1,z_2)|^2\, d\rho_t^S, \\
I_2(z_1,z_2) &:= -c_3 \int_0^T |\lambda(t,z_1,z_2)|^2 \frac{d\mathrm{Re}(\eta(z_1,z_2,t))}{d\rho_t} d\rho_t^S,
\end{aligned} \tag{3.4.29}$$

where we have used item 3) of Lemma 3.4.12. Since λ is uniformly bounded, see (3.4.27), we have

$$I_1(z_1,z_2) \leq c_2\rho_T^S \exp\left(2c_1\rho_T^S\right). \tag{3.4.30}$$

On the other hand,

$$\begin{aligned}
I_2(z_1,z_2) &= -c_3 \int_0^T \mathrm{Re}(\eta(z_1,z_2,dt)) \exp\left(\int_t^T 2\mathrm{Re}(\eta(z_1,z_2,ds))\right) \\
&\leq c_3 \sup_{y\in I_0+i\mathbb{R}^2} -\int_0^T \mathrm{Re}(\eta(y_1,y_2,dt)) \exp\left(\int_t^T 2\mathrm{Re}(\eta(y_1,y_2,ds))\right),
\end{aligned} \tag{3.4.31}$$

which is finite by item 4) of Lemma 3.4.12. Integrating (3.4.28) with respect to $|\Pi|$, taking into account the two uniform bounds in (z_1,z_2), i.e. (3.4.30) and (3.4.31), we can conclude to the validity of (3.4.26), so that $y \in \mathfrak{D}(\mathfrak{a})$.

We show similarly that $\widetilde{y} := y \times id \in \mathfrak{D}(\mathfrak{a})$. In fact, for $t \in [0,T]$ and $x, y > 0$, we have

$$\begin{aligned}\widetilde{y}(t,x,s) &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2+1} \lambda(t,z_1,z_2) \\ &= \int_{\mathbb{C}^2} d\widetilde{\Pi}(z_1,z_2) x^{z_1} s^{z_2} \widetilde{\lambda}(t,z_1,z_2),\end{aligned}$$

where $\widetilde{\lambda}(t,z_1,z_2) = \lambda(t,z_1,z_2-1)$ and $\widetilde{\Pi}$ is the Borel complex measure defined by

$$\int_{\mathbb{C}^2} d\widetilde{\Pi}(z_1,z_2)\varphi(z_1,z_2) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2)\varphi(z_1,z_2+1),$$

for every bounded measurable function φ. Hence, $supp\ \widetilde{\Pi} = supp\ \Pi + (0,1)$. By Remark 3.4.11 1) and 5), we have $(0,1) \in D/2$ and $supp\ \Pi \subset D/2$. Then, by Remark 3.2.22, $supp\ \widetilde{\Pi} \subset D$, so that Assumption 3.2.31 is verified for $\widetilde{\Pi}$. Moreover, by definition of $\widetilde{\Pi}$, the conditions (3.2.35) and (3.2.36) are fulfilled replacing Π and λ with $\widetilde{\Pi}$ and $\widetilde{\lambda}$. In order to conclude that $\widetilde{y} \in \mathfrak{D}(\mathfrak{a})$, we need to show

$$A := \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\widetilde{\Pi}|(z_1,z_2) \left| \frac{d\lambda(t,z_1,z_2-1)}{d\rho_t^S} + \lambda(t,z_1,z_2-1)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S} \right| < \infty, \tag{3.4.32}$$

which corresponds to Condition (3.2.37) for Π and λ replaced by $\widetilde{\Pi}$ and $\widetilde{\lambda}$. Note that

$$\begin{aligned}A &= \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2+1)}{d\rho_t^S} \right| \\ &= \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\left(\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S} + \frac{d\kappa_t(z_1,z_2)}{d\rho_t^S} + \frac{d\kappa_t(0,1)}{d\rho_t^S} \right) \right| \\ &\leq A_1 + A_2 + A_3,\end{aligned}$$

where

$$\begin{aligned}A_1 &:= \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2)\frac{d\kappa_t(z_1,z_2)}{d\rho_t^S} \right|, \\ A_2 &:= \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \lambda(t,z_1,z_2)\frac{d\kappa_t(0,1)}{d\rho_t^S} \right|, \\ A_3 &:= \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \lambda(t,z_1,z_2)\frac{d\rho_t(z_1,z_2,0,1)}{d\rho_t^S} \right|.\end{aligned}$$

The first term A_1 is finite, since we already proved that $y \in \mathfrak{D}(\mathfrak{a})$ and so condition (3.4.26) is fulfilled. Moreover

$$A_2 \leq \left\| \frac{d\kappa_t(0,1)}{d\rho_t^S} \right\|_\infty \int_0^T d\rho_t^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \left| \lambda(t,z_1,z_2) \right|.$$

The right-hand side is finite, thanks to point 4) of Assumption 3.4.10 and the fact that λ is uniformly bounded.

Finally, by Cauchy-Schwarz and item 3) of Lemma 3.4.12, taking into account Notation (3.4.23), by similar arguments as (3.4.28), we have

$$\begin{aligned}(A_3)^2 &\leq |\Pi|(\mathbb{C}^2)\rho_T^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2) \int_0^T d\rho_t^S \, |\lambda(t,z_1,z_2)|^2 \, |\gamma_t(z_1,z_2)|^2 \\ &\leq |\Pi|(\mathbb{C}^2)\rho_T^S \int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)(I_1(z_1,z_2)+I_2(z_1,z_2)),\end{aligned}$$

where $I_1(z_1,z_2)$ and $I_2(z_1,z_2)$ have been defined in (3.4.29). We have already shown in (3.4.30) and (3.4.31) that I_1 and I_2 are bounded on $supp\ \Pi$, hence $A_3 < \infty$. In conclusion, it follows indeed that $\widetilde{y} \in \mathfrak{D}(\mathfrak{a})$ and Hypothesis 1) of Corollary 3.4.7 is verified.
We define $(t,x,s) \mapsto z(t,x,s)$ so that $s^2 z(t,x,s) = \widetilde{\mathfrak{a}}(y)(t,x,s)$. This gives

$$z(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2-1} \lambda(t,z_1,z_2)\gamma_t(z_1,z_2), \ \forall t \in [0,T], \ x,y>0, \quad (3.4.33)$$

Lemma 3.4.14 below shows that (3.3.15) is fulfilled and so Hypothesis 2) of Corollary 3.4.7 is verified.

We go on verifying Hypothesis 3) of Corollary 3.4.7, i.e. the validity of (3.4.15) and (3.4.16). Condition (3.4.16) is straightforward since $\lambda(T,\cdot,\cdot) = 1$. The second equality in (3.4.15) takes place by definition of z. The first equality holds true integrating (3.4.17) thanks to (3.4.26). This proves 3) of Corollary 3.4.7.

Finally Corollary 3.4.7 implies that (Y,Z,O), is a solution of the BSDE (3.4.6) provided we establish the following. □

Lemma 3.4.14. *Let z be as in (3.4.33), where λ, γ have been respectively defined in (3.4.24) and (3.4.23). We have*

$$\mathbb{E}\left[\int_0^T |z(r,X_{r-},S_{r-})|^2 \, S_{r-}^2 \rho_{dr}^S\right] < \infty.$$

In particular (3.3.15) is fulfilled.

Proof. First, let us show that

$$\int_{\mathbb{C}^2} d\Pi(z_1,z_2) \int_0^T |\lambda(t,z_1,z_2)|^2 \rho_{dt}(z_1,z_2) < \infty. \quad (3.4.34)$$

For this, we use points 3) and 4) of Lemma 3.4.12, (3.4.27) and (3.4.20) we get

$$\begin{aligned}\int_0^T |\lambda(t,z_1,z_2)|^2 \rho_{dt}(z_1,z_2) &= \int_0^T |\lambda(t,z_1,z_2)|^2 \frac{d\rho_t(z_1,z_2)}{d\rho_t^S}\rho_{dt}^S \\ &\leq \int_0^T |\lambda(t,z_1,z_2)|^2 \left(c_2 - c_3\frac{d\mathrm{Re}(\eta(y_1,y_2,t))}{d\rho_t^S}\right)\rho_{dt}^S \\ &\leq c_2 e^{2c_1\rho_T^S}\rho_T^S - c_3\int_0^T \mathrm{Re}(\eta(z_1,z_2,dt))\exp\left(\int_t^T 2\mathrm{Re}(\eta(z_1,z_2,ds))\right) \\ &\leq c_2 e^{2c_1\rho_T^S}\rho_T^S + \\ &\quad c_3 \sup_{(\xi_1,\xi_2)\in I_0+i\mathbb{R}^2} -\int_0^T \mathrm{Re}(\eta(\xi_1,\xi_2,dt))\exp\left(\int_t^T 2\mathrm{Re}(\eta(\xi_1,\xi_2,ds))\right).\end{aligned}$$

Hence (3.4.34) is fulfilled.

Using Cauchy-Schwarz inequality, Fubini theorem and point 3) of Lemma 3.4.12, we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T |z(r,X_{r-},S_{r-})|^2 S_{r-}^2 d\rho_r^S\right] = \mathbb{E}\left[\int_0^T \left|\int_{\mathbb{C}^2} d\Pi(z_1,z_2) X_{t-}^{z_1} S_{t-}^{z_2}\lambda(t,z_1,z_2)\gamma_t(z_1,z_2)\right|^2 \rho_{ds}^S\right] \\ \leq\ & |\Pi|(\mathbb{C}^2)\sup_{t\in[0,T],(a,b)\in I_0}\mathbb{E}\left[X_t^{2a}S_t^{2b}\right]\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\int_0^t |\lambda(t,z_1,z_2)\gamma_t(z_1,z_2)|^2\rho_{ds}^S \\ \leq\ & |\Pi|(\mathbb{C}^2)\sup_{t\in[0,T],(a,b)\in I_0}\mathbb{E}\left[X_t^{2a}S_t^{2b}\right]\int_{\mathbb{C}^2} d|\Pi|(z_1,z_2)\int_0^t |\lambda(t,z_1,z_2)|^2\rho_{dt}(z_1,z_2).\end{aligned}$$

The right-hand side is finite, thanks to (3.4.34).

□

One can prove that the **weak** F-S decomposition in Proposition 3.4.13 is actually a strong F-S decomposition in the sense of Definition 3.4.1.

Theorem 3.4.15. *Under Assumption 3.4.10, the random variable*

$$h = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) X_T^{z_1} S_T^{z_2}$$

admits an F-S decomposition (3.4.2) where $h_0 = Y_0$ and (Y,Z,O) is given in Proposition 3.4.13.

Moreover, if h is real-valued then the decomposition (Y,Z,O) is real-valued and it is therefore the unique F-S decomposition.

Remark 3.4.16. *This statement is a generalization of the results of Goutte et al. [2014] (and Hubalek et al. [2006]) to the case of hedging under basis risk. This yields a characterization of the hedging strategy in terms of Fourier-Laplace transform and the moment generating function.*

Proof. Since Π is a finite measure, then h is square integrable. Indeed by Cauchy-

Schwarz

$$\begin{aligned}\mathbb{E}\left[h^2\right] &\leq |\Pi|(\mathbb{C}^2)\int_{\mathbb{C}^2}\mathbb{E}\left[|X_T|^{2\mathrm{Re}(z_1)}|S_T|^{2\mathrm{Re}(z_2)}\right]d\Pi(z_1,z_2)\\ &\leq \left(|\Pi|(\mathbb{C}^2)\right)^2\sup_{(a,b)\in I}\mathbb{E}\left[|X_T|^a|S_T|^b\right],\end{aligned} \tag{3.4.35}$$

where I is a bounded subset of $\mathbb{R}^2$ defined in Assumption 3.4.10. By item 2) of Assumption 3.4.10 and item 3) of Proposition 3.2.24, previous quantity is finite.

By item 4) of Remark 3.4.11 and by Remark 3.4.4, the real-valued F-S decomposition of any real valued square integrable $\mathcal{F}_T$-measurable random variable is unique.

As a consequence, if h is real-valued then its F-S decomposition is also real-valued. In fact, if (Y_0, Z, O) is an F-S decomposition of h, then $(\overline{Y}_0, \overline{Z}, \overline{O})$ is also an F-S of $\overline{h}$ by item 3) of Remark 3.4.2. Thus, by subtraction, $(\mathrm{Im}(Y_0), \mathrm{Im}(Z), \mathrm{Im}(O))$ is an F-S decomposition with real-valued triplet of the real-valued r.v. $\mathrm{Im}(h) = 0$. By uniqueness $\mathrm{Im}(Y_0)$, $\mathrm{Im}(Z)$ and $\mathrm{Im}(O)$ are null and the decomposition (Y_0, Z, O) is real valued.

Now, let (Y, Z, O) defined in Proposition 3.4.13. It remains to prove that (Y_0, Z, O) is a strong (possibly complex) F-S decomposition in the sense of Definition 3.4.1. For this we need to show items a),b),c) of Remark 3.4.8. Item a) has been the object of (3.4.35).

We show below item b) i.e. $\mathbb{E}\left[\int_0^T |Z_s|^2 d\langle M^S\rangle_s\right] < \infty$ and $\mathbb{E}\left[\left(\int_0^T |Z_s| d\|V^S\|_s\right)^2\right] < \infty$. The first inequality is stated in Lemma 3.4.14. In order to prove the second one, we remind that, by Corollary 3.2.27,

$$dV_t^S = S_{t-}\kappa_{dt}(0,1) = S_{t-}\frac{d\kappa_t(0,1)}{d\rho_t^S}\rho_{dt}^S.$$

Consequently

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |Z_s| d\|V^S\|_s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T |Z_u|\left|\frac{d\kappa_u(0,1)}{d\rho_u^S}\right| S_{u-}\rho_{du}^S\right)^2\right]\\ &\leq \int_0^T \left|\frac{d\kappa_u(0,1)}{d\rho_u^S}\right|^2 \rho_{du}^S \mathbb{E}\left[\int_0^T |Z_u|^2 S_{u-}^2 \rho_{du}^S\right],\end{aligned}$$

which is finite since, by item 3) of Remark 3.4.11 which says that $(0,1) \in \mathcal{D}$, taking into account Lemma 3.4.14.

To end this proof, we need to show item c) of Remark 3.4.8. For this we use Proposition 3.2.35 for which we need to check conditions a) and b). By item 5) of Remark 3.4.11 we have $I_0 \subset D/2$ which constitutes item a). Item b) is verified by condition (3.4.34) is verified. Hence Proposition 3.2.35 implies that

$$t \mapsto y(t, X_t, S_t) - \int_0^t \mathfrak{a}(y)(u, X_{u-}, S_{u-})\rho_{dt}^S$$

is a square integrable martingale. □

3.4.3 Diffusion processes

We set $\mathcal{O} = \mathbb{R} \times E$, where $E = \mathbb{R}$ or $]0, \infty[$. In this Section we apply Corollary 3.4.7 to the diffusion processes (X, S) modeled in Section 3.2.3 whose dynamics is given by (3.2.26). We are interested in the F-S decomposition of $h = g(X_T, S_T)$. We recall the assumption in that context.

Assumption 3.4.17.

- *b_X, b_S, σ_X and σ_S are continuous and globally Lipschitz.*
- *$g : \mathcal{O} \to \mathbb{R}$ is continuous.*

We remind that (X, S) solve the strong martingale problem related to $(\mathfrak{D}(\mathfrak{a}), \mathfrak{a}, A)$ where $A_t = t$, $\mathfrak{D}(\mathfrak{a}) = \mathcal{C}^{1,2}([0,T[\times\mathcal{O}) \cap \mathcal{C}^1([0,T]\times\mathcal{O})$. For a function $y \in \mathfrak{D}(\mathfrak{a})$, obviously $\tilde{y} \in \mathfrak{D}(\mathfrak{a})$ and the operators $\mathfrak{a}$ and $\widetilde{\mathfrak{a}}$ are given by

$$\begin{aligned} \mathfrak{a}(y) &= \partial_t y + b_S \partial_s y + b_X \partial_x y \\ &+ \frac{1}{2}\left\{|\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} y\right\}, \\ \widetilde{\mathfrak{a}}(y) &= |\sigma_S|^2 \partial_s y + \langle \sigma_S, \sigma_X \rangle \partial_x y. \end{aligned}$$

Conditions 3) of that Corollary 3.4.7 translates into

$$\begin{aligned} b_S z &= \partial_t y + b_S \partial_s y + b_X \partial_x y + \frac{1}{2}\left\{|\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} y\right\}, \\ y(T, ., .) &= g(., .), \\ |\sigma_S|^2 z &= |\sigma_S|^2 \partial_s y + \langle \sigma_S, \sigma_X \rangle \partial_x y. \end{aligned} \tag{3.4.36}$$

If, moreover, $\frac{1}{|\sigma_S|}$ is locally bounded, then we have the following:

$$\begin{cases} \partial_t y + B \partial_x y + \frac{1}{2}\left(|\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} y\right) = 0, \\ y(T, ., .) = g(., .), \end{cases} \tag{3.4.37}$$

and

$$z = \partial_s y + \frac{\langle \sigma_S, \sigma_X \rangle}{|\sigma_S|^2} \partial_x y, \tag{3.4.38}$$

where

$$B = b_X - b_S \frac{\langle \sigma_S, \sigma_X \rangle}{|\sigma_S|^2}. \tag{3.4.39}$$

z is then locally bounded since σ_S, σ_X and $\frac{1}{|\sigma_S|}$ are locally bounded and because $y \in \mathfrak{D}(\mathfrak{a})$.

Proposition 3.4.18. *We suppose the validity of Assumption 3.4.17 and that $|\sigma_S|$ is always strictly positive.*
If (y, z) is a solution of the system (3.4.37) and (3.4.38), such that $y \in \mathfrak{D}(\mathfrak{a})$, then (Y, Z, O) is

a solution of the BSDE (3.4.6), *where*

$$\begin{aligned} Y_t &= y(t, X_t, S_t), \\ Z_t &= z(t, X_t, S_t), \\ O_t &= Y_t - Y_0 - \int_0^t Z_s dS_s. \end{aligned}$$

Proof. It follows from Corollary 3.4.7 for which we need to check the conditions 1), 2) and 3). Indeed, since $y, \tilde{y} \in \mathfrak{D}(\mathfrak{a})$, Condition 1) holds; since z is locally bounded, by item 2. of Remark 3.3.3, Condition 2) is fulfilled. Condition 3) has been the object of the considerations above the statement of the Proposition. □

The result above yields the weak F-S decomposition for h. In order to show that (Y_0, Z, O) constitutes a true F-S decomposition, we need to make use of Remark 3.4.8. First we introduce the following assumption.

Assumption 3.4.19. *Suppose that the process (X, S) takes values in $\mathcal{O}$ and the following.*

i) $g \in C^1$ *such that* g, $\partial_x g$ *and* $\partial_s g$ *have polynomial growth.*

ii) B *is globally Lipschitz.*

iii) $\partial_x B$, $\partial_s B$, $\partial_x \sigma_X$, $\partial_s \sigma_X$, $\partial_x \sigma_S$ *and* $\partial_s \sigma_S$ *exist, are continuous and have polynomial growth.*

iv) σ_S *never vanishes.*

We formulate the following.

Theorem 3.4.20. *Suppose that Assumptions 3.4.17 and 3.4.19 are fulfilled, and suppose the existence of a function $y : [0, T] \times \mathcal{O} \to \mathbb{R}$ such that*

$$y \in C^0([0, T] \times \mathcal{O}) \cap C^{1,2}([0, T[\times \mathcal{O}) \text{ verifies the PDE (3.4.37) and has polynomial growth.} \tag{3.4.40}$$

Then the F-S decomposition (3.4.2) *of $h = g(X_T, S_T)$ is provided by (h_0, Z, O) where, $h_0 = Y_0$ and*

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_t, S_t), O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

and $z : [0, T] \times \mathcal{O} \to \mathbb{R}$ is given by (3.4.38).

Proof. Let $y : [0, T] \times \mathcal{O} \to \mathbb{R}$ verifying (3.4.40) and z defined by (3.4.38). In order to show that the triplet given in Proposition 3.4.18 yields a true F-S decomposition, we need to show items a), b), c) of Remark 3.4.8.

First note that the random variable $g(X_T, S_T)$ is square integrable, because g has polynomial growth and X and S admit all moments, see Remark 3.2.20. So a) is verified.

In view of verifying item b) of Remark 3.4.8 we remind that

$$\mathfrak{a}(id) = b_S, \widetilde{\mathfrak{a}}(id) = |\sigma_S|^2, A_t \equiv t \text{ and } \ z = \partial_s y + \frac{\langle \sigma_S, \sigma_X \rangle}{|\sigma_S|^2} \partial_x y.$$

Indeed, since y has polynomial growth, it is forced to be unique since [Karatzas and Shreve, 1991, Theorem 7.6, chapter 5] implies that

$$y(t,x,s) = \mathbb{E}\left[g(X_T^{t,x,s}, S_T^{t,x,s}) \right], \tag{3.4.41}$$

where $(\widetilde{X} = X^{t,x,s}, \widetilde{S} = S^{t,x,s})$ is a solution of

$$d\begin{pmatrix} \widetilde{X}_r \\ \widetilde{S}_r \end{pmatrix} = \Sigma(r, \widetilde{X}_r, \widetilde{S}_r) d\widetilde{W}_r + \begin{pmatrix} B(r, \widetilde{X}_r, \widetilde{S}_r) \\ 0 \end{pmatrix} dr,$$

with $\widetilde{X}_t = x, \ \widetilde{S}_t = s$, where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2)$ is a standard two-dimensional Brownian motion, and

$$\Sigma = \begin{pmatrix} \sigma_{X,1} & \sigma_{X,2} \\ \sigma_{S,1} & \sigma_{S,2} \end{pmatrix}.$$

We remind that B has been defined in (3.4.39).

By (3.4.41), a straightforward adaptation of [Friedman, 1975, Theorem 5.5] yields that the partial derivatives $\partial_x y$ and $\partial_s y$ exist and are continuous on $[0,T] \times \mathcal{O}$ and they have polynomial growth.

Using (3.4.38), we have

$$z b_S = b_S \partial_s y + b_X \partial_x y - B \partial_x y.$$

Now, since $\partial_x y$ and $\partial_s y$ have polynomial growth, and by assumption b_S, b_X and B have linear growth, we get that $z b_S$ has polynomial growth. This gives, by Remark 3.2.20,

$$\mathbb{E}\left[\left(\int_0^T |z b_S| \, (t, X_t, S_t) dt \right)^2 \right] < \infty.$$

On the other hand, using (3.4.38) and Cauchy-Schwarz, we have

$$\begin{aligned} |z\sigma_S| &= ||\sigma_S| \partial_s y + \frac{\langle \sigma_X, \sigma_S \rangle}{|\sigma_S|} \partial_x y| \\ &\leq |\sigma_S| |\partial_s y| + |\sigma_X| |\partial_x y|. \end{aligned}$$

Since σ_X, σ_S have linear growth and $\partial_x y$ and $\partial_s y$ have polynomial growth, we get that $z\sigma_S$ has polynomial growth, which implies, by Remark 3.2.20, that

$$\mathbb{E}\left[\int_0^T |z\sigma_S|^2 \, (t, X_t, S_t) dt \right] < \infty.$$

Consequently, item b) of Remark 3.4.8 is fulfilled.

In order to show the last item c), taking into account Remark 3.2.21, we need to

prove that

$$\begin{aligned} u \mapsto M_u^Y &= \int_0^u \partial_x y(r, X_r, S_r) \left(\sigma_{X,1}(r, X_r, S_r) dW_r^1 + \sigma_{X,2}(r, X_r, S_r) dW_r^2\right) \\ &+ \int_0^u \partial_s y(r, X_r, S_r) \left(\sigma_{S,1}(r, X_r, S_r) dW_r^1 + \sigma_{S,2}(r, X_r, S_r) dW_r^2\right) \end{aligned}$$

is a square integrable martingale. This is due to the fact that $\partial_x y$ and $\partial_s y$ have polynomial growth, and that σ_X and σ_S have linear growth, and Remark 3.2.20, which implies that

$$\mathbb{E}\left[\int_0^T \{(\partial_x y(r, X_r, S_r))^2 |\sigma_X(r, X_r, S_r)|^2 + (\partial_s y(r, X_r, S_r))^2 |\sigma_S(r, X_r, S_r)|^2\} du\right] < \infty.$$

This concludes the proof of Theorem 3.4.20. □

Below we show that, under Assumptions 3.4.17 and 3.4.19, Condition (3.4.40) is not really restrictive.

Proposition 3.4.21. *We assume the validity of Assumptions 3.4.17 and 3.4.19.*
Moreover we suppose the validity of one of the three items below.

1) *We set $\mathcal{O} = \mathbb{R}^2$. Suppose that the second (partial, with respect to (x, s)) derivatives of B, σ_X, σ_S and g exist, are continuous and have polynomial growth.*

2) *We set $\mathcal{O} = \mathbb{R}^2$. We suppose B, σ_X, σ_S to be bounded and there exist $\lambda_1, \lambda_2 > 0$ such that*
$$\lambda_1 |\xi|^2 \leq (\xi_1, \xi_2) C(t, x, s) (\xi_1, \xi_2)^T \leq \lambda_2 |\xi|^2, \ \forall \xi = (\xi_1, \xi_2) \in \mathcal{O},$$
where $C(t, x, s) = \begin{pmatrix} |\sigma_X|^2(t,x,s) & \langle \sigma_X, \sigma_S \rangle(t,x,s) \\ \langle \sigma_X, \sigma_S \rangle(t,x,s) & |\sigma_S|^2(t,x,s) \end{pmatrix}$.

3) *(Black-Scholes case.) We suppose $\mathcal{O} =]0, +\infty[^2$.*

$$\begin{aligned} b_S(t, x, s) &= s\hat{b}_S, \\ b_X(t, x, s) &= x\hat{b}_X, \\ \sigma_S(t, x, s) &= (s\hat{\sigma}_{S,1}, \ s\hat{\sigma}_{S,2}), \\ \sigma_X(t, x, s) &= (x\hat{\sigma}_{X,1}, \ x\hat{\sigma}_{X,2}), \end{aligned}$$

where $\hat{b}_S$, $\hat{b}_X$, $\hat{\sigma}_{S,1}$, $\hat{\sigma}_{S,2}$, $\hat{\sigma}_{X,1}$ and $\hat{\sigma}_{X,2}$ are constants, such that $\langle \hat{\sigma}_X, \hat{\sigma}_S \rangle < |\hat{\sigma}_X||\hat{\sigma}_S|$.

We have the following.

i) There is a (unique) strict solution y of (3.4.37) in the class $C^{1,2}([0, T[\times\mathcal{O}) \cap C^0([0, T] \times \mathcal{O})$ with polynomial growth.

ii) The F-S decomposition (3.4.2) of $h = g(X_T, S_T)$ is provided by (h_0, Z, O) where (Y, Z, O) fulfill

$$Y_t = y(t, X_t, S_t), \ Z_t = z(t, X_t, S_t) \text{ and } O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

where z is given by (3.4.38).

Remark 3.4.22. *We will show below that under the hypotheses above, conclusion i) holds, i.e. there is a function y fulfilling* (3.4.40). *We observe that, by the proof of Theorem 3.4.20, if such a y exists then it admits the probabilistic representation* (3.4.41) *and so it is necessarily the unique $C^{1,2}([0,T[\times\mathcal{O})\cap C^0([0,T]\times\mathcal{O})$, with polynomial growth, solution of* (3.4.37).

Proof. We proceed to discussing the existence of y mentioned in Remark 3.4.22. So we distinguish now the mentioned three cases.

Suppose first item 1). The function y defined by (3.4.41) is a continuous function by the fact that the flow $(\widetilde{X},\widetilde{S})$ is continuous in all variables and Remark 3.2.20, taking into account Lebesgue dominated convergence theorem. [Friedman, 1975, Theorem 6.1], states that y belongs to $C^{1,2}([0,T]\times\mathcal{O})$, and it verifies the PDE (3.4.37). [Friedman, 1975, Theorem 5.5] says in particular that y has polynomial growth. In that case conclusion i) is established.

Under the assumption described in item 2), the conclusion i) can be obtained by simply adapting the proof of [Friedman, 1964, Theorem 12, p.25]. Indeed, according to [Friedman, 1964, Theorem 8, p.19] there is a fundamental solution $\Gamma:\{(t_1,t_2),0\leq t_1<t_2\leq T\}\times\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\Gamma(t_1,t_2;\gamma,\xi)\leq\frac{1}{a_1(t_2-t_1)}\exp\left(-\frac{-|\gamma-\xi|^2}{a_1(t_2-t_1)}\right), \tag{3.4.42}$$

where a_1 is a positive constant.

Now, by [Friedman, 1964, Theorem 12, p.25], the function y defined by

$$y(t,x,s)=\int_{\mathbb{R}^2}\Gamma(t,T;(x,s),(\xi_1,\xi_2))g(\xi_1,\xi_2)d\xi_1d\xi_2, \tag{3.4.43}$$

is a strict solution of (3.4.37), in particular it belongs to $\mathcal{C}^{1,2}([0,T[\times\mathbb{R}^2)\cap\mathcal{C}^0([0,T]\times\mathbb{R}^2)$.

Since g has polynomial growth then there exist $a_2>0$, $p>1$ such that, $\forall x,s\in\mathbb{R}$,

$$|g(x,s)|\leq a_2(1+|x|^p+|s|^p). \tag{3.4.44}$$

Thus, by (3.4.43), (3.4.42) and (3.4.44), for $x,s\in\mathbb{R}$ and $0\leq t\leq T$, we have

$$|y(t,x,s)|\leq\frac{a_2}{a_1(T-t)}\int_{\mathbb{R}^2}(1+|\xi_1|^p+|\xi_2|^p)\exp\left(-\frac{|x-\xi_1|^2+|s-\xi_2|^2}{a_1(T-t)}\right)d\xi_1d\xi_2.$$

So there is a constant $C_1(p,T)>0$ such that

$$|y(t,x,s)|\leq C_1(p,T)\left(1+\mathbb{E}\left[|x+G_1|^p+|x+G_2|^p\right]\right), \tag{3.4.45}$$

where $G=(G_1,G_2)$ is a two dimensional centered Gaussian vector with covariance matrix equal to $\frac{a_1(T-t)}{2}$ times the identity matrix. Since $p>1$, then there is a constant

$C_2(p,T)$ such that

$$\begin{aligned} |y(t,x,s)| &\leq C_2(p,T)\left(1+|x|^p+|s|^p+\mathbb{E}\left[|G_1|^p+|G_2|^p\right]\right) \\ &\leq C_3(p,T)\left(1+|x|^p+|s|^p\right), \end{aligned}$$

where $C_3(p,T)$ is another positive constant. In conclusion the solution y given by (3.4.43) has polynomial growth.

We discuss now the Black-Scholes case 3) showing that, also in that case, there is y such that (3.4.40) is fulfilled. First note that the uniform ellipticity condition in 2) is not fulfilled for this dynamics, so we consider a logarithmic change of variable. For a function $y \in \mathfrak{D}(\mathfrak{a})$, we introduce the function $\hat{y} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\hat{y}(t,x,s) = y(t,\log(x),\log(s)),\ \forall t \in [0,T], x, s > 0.$$

By inspection we can show that y is a solution of (3.4.37) if and only if $\hat{y}$ fulfills

$$\begin{aligned} 0 &= \partial_t \hat{y} + \left(\hat{b}_X - \hat{b}_S \frac{\langle \hat{\sigma}_S, \hat{\sigma}_X \rangle}{|\hat{\sigma}_S|^2} - \frac{1}{2}|\hat{\sigma}_X|^2\right)\partial_x \hat{y} - \frac{1}{2}|\hat{\sigma}_S|^2 \partial_s \hat{y} + \\ &\quad + \frac{1}{2}\left(|\hat{\sigma}_S|^2 \partial_{ss}\hat{y} + |\hat{\sigma}_X|^2 \partial_{xx}\hat{y} + 2\langle \hat{\sigma}_S, \hat{\sigma}_X \rangle \partial_{sx}\hat{y}\right), \qquad (3.4.46) \\ \hat{y}(T,.,.) &= \hat{g}(.,.), \end{aligned}$$

where $\hat{g}(x,s) = g(e^x, e^s),\ \forall x, s \in \mathbb{R}$. Note that the PDE problem (3.4.46) has constant coefficients and it verifies the uniform ellipticity condition in 2).

Moreover, since g has polynomial growth, then there exist $c > 0, p > 1$ such that $g(x,s) \leq c(1+x^p+s^p),\ \forall x, s > 0$ again. Hence $\hat{g}(x,s) \leq c(1+e^{px}+e^{ps}),\ \forall x, s \in \mathbb{R}$.

Again, by simple adaptation of the proof of [Friedman, 1964, Theorem 12, p.25], we observe that equation (3.4.46) admits a solution $\hat{y}$ in $C^{1,2}([0,T[\times\mathbb{R}^2) \cap C^0([0,T] \times \mathbb{R}^2)$, such that

$$\hat{y}(t,x,s) \leq K(1+e^{px}+e^{ps}),\ \forall x, s \in \mathbb{R},$$

where $K > 0$. This yields that y has polynomial growth, since $y(t,x,s) = \hat{y}(t,\log(x),\log(s)),\ \forall t \in [0,T], x, s > 0$, so

$$y(t,x,s) \leq K(1+x^p+s^p),\ \forall t \in [0,T], x, s > 0.$$

This concludes the proof of conclusion i).

Conclusion ii) is now a direct consequence of Theorem 3.4.20 together with condition i). □

Remark 3.4.23. *The last item of Proposition 3.4.21 permits to recover the results already*

found in Hulley and McWalter [2008], by replacing

$$\begin{aligned}\hat{b}_S &= (\mu_S - r),\\ \hat{b}_X &= (\mu_U - r),\\ \hat{\sigma}_S &= (\sigma_S,\ 0),\\ \hat{\sigma}_X &= (\rho\sigma_U,\ \sqrt{1-\rho^2}\sigma_U),\end{aligned}$$

where μ_S, μ_U, r, σ_S and σ_U are constants.

Appendix 3.A Proof of Proposition 3.2.8

Proof. Let $f \in E$ and set $\widetilde{f}(x) = \frac{f(x)}{1+x^2}$, $\forall x \in \mathbb{R}$. Condition (3.2.15) implies, by mean value theorem, that there exists a constant $c(t)$ such that

$$\mathbb{E}\left[\left|X_t^{0,x} - X_t^{0,y}\right|^2\right] \leq c(t)\,|x-y|^2\,,\ \forall x, y \in \mathbb{R}.$$

Then, by the Garsia-Rodemich-Rumsey criterion, see for instance [Barlow and Yor, 1982, Section 3], there exists a r.v. Γ_t such that $\mathbb{E}\left[\Gamma_t^2\right] < \infty$ and $\forall x, y \in \mathbb{R}$

$$\left|X_t^{0,x} - X_t^{0,y}\right| \leq \Gamma_t\,|x-y|^\alpha\,,\ \text{for } 0 < \alpha < \frac{1}{2}, \tag{3.A.1}$$

possibly up to a modified version of the flow.

This implies in particular that for $x \in \mathbb{R}$

$$\begin{aligned}
\frac{|X_t^{0,x}|^2}{1+x^2} &\leq \frac{2}{1+x^2}\left(|X_t^{0,0}|^2 + |X_t^{0,x} - X_t^{0,0}|^2\right)\\
&\leq \frac{2}{1+x^2}\left(|X_t^{0,0}|^2 + |\Gamma_t|^2\,|x|^{2\alpha}\right)\\
&\leq 2\left(|X_t^{0,0}|^2 + |\Gamma_t|^2\right).
\end{aligned}$$

Hence

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[\frac{|X_t^{0,x}|^2}{1+x^2}\right] < \infty. \tag{3.A.2}$$

Consequently, for $x \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{|P_t f(x)|}{1+x^2} &= \frac{\left|\mathbb{E}\left[f(X_t^{0,x})\right]\right|}{1+x^2}\\
&\leq \|f\|_E\,\frac{1+\mathbb{E}\left[|X_t^{0,x}|^2\right]}{1+x^2}\\
&\leq \|f\|_E \sup_{\xi\in\mathbb{R}} \frac{1+\mathbb{E}\left[|X_t^{0,\xi}|^2\right]}{1+\xi^2}.
\end{aligned}$$

The right-hand side is finite, thanks to (3.A.2), so that

$$\|P_t f\|_E \leq \|f\|_E \sup_{\xi\in\mathbb{R}} \frac{1+\mathbb{E}\left[|X_t^{0,\xi}|^2\right]}{1+\xi^2}. \tag{3.A.3}$$

After we will have shown that $\widetilde{P_t f}$ is also uniformly continuous, (3.A.3) will also imply that $P_t f \in E$ and that P_t is a bounded linear operator.

Therefore it remains to show that $\widetilde{P_t f}$ is uniformly continuous. For this, let $x, y \in$

$\mathbb{R}$. We have

$$\begin{aligned}\frac{P_t f(x)}{1+x^2} - \frac{P_t f(y)}{1+y^2} &= \mathbb{E}\left[\frac{f(X_t^{0,x})}{1+x^2} - \frac{f(X_t^{0,y})}{1+y^2}\right] \\ &= \mathbb{E}\left[I_1 + I_2\right],\end{aligned} \tag{3.A.4}$$

where

$$\begin{aligned}I_1 &= \left(\widetilde{f}(X_t^{0,x}) - \widetilde{f}(X_t^{0,y})\right)\frac{1+(X_t^{0,x})^2}{1+x^2} \\ I_2 &= \widetilde{f}(X_t^{0,y})\left(\frac{1+(X_t^{0,x})^2}{1+x^2} - \frac{1+(X_t^{0,y})^2}{1+y^2}\right).\end{aligned}$$

Let $\epsilon > 0$. By uniform continuity of $\widetilde{f}$, there exists $\delta_1 > 0$ such that

$$\forall a, b \in \mathbb{R},\ |a-b| \leq \delta_1 \Rightarrow \left|\widetilde{f}(a) - \widetilde{f}(b)\right| < \epsilon. \tag{3.A.5}$$

Since $\lim_{M\to\infty} \mathbb{E}\left[|I_1|\mathbb{1}_{|\Gamma_t|\geq M}\right] = 0$, there exists $M_1 > 0$ such that

$$\mathbb{E}\left[|I_1|\mathbb{1}_{|\Gamma_t|\geq M_1}\right] < \epsilon. \tag{3.A.6}$$

We fix $0 < \alpha < \frac{1}{2}$ and we choose $\delta_2 = \left(\frac{\delta_1}{M_1}\right)^{1/\alpha}$. Taking into account (3.A.1) and (3.A.5), for $|x-y| < \delta_2$ we have

$$\begin{aligned}\mathbb{E}\left[|I_1|\mathbb{1}_{|\Gamma_t|<M_1}\right] &\leq \mathbb{E}\left[\frac{1+(X_t^{0,x})^2}{1+x^2}\left(\widetilde{f}(X_t^{0,x}) - \widetilde{f}(X_t^{0,y})\right)\mathbb{1}_{|X_t^{0,x}-X_t^{0,y}|<\delta_1}\right] \\ &< \sup_{\xi\in\mathbb{R}} \mathbb{E}\left[\frac{1+|X_t^{0,\xi}|^2}{1+\xi^2}\right]\epsilon.\end{aligned}$$

The right-hand side is finite thanks to (3.A.2). Consequently, if $|x-y| < \delta_2$, then (3.A.6) implies that

$$\mathbb{E}\left[|I_1|\right] < A_1\epsilon, \tag{3.A.7}$$

where $A_1 = 1 + \sup_{\xi\in\mathbb{R}} \mathbb{E}\left[1 + \frac{|X_t^{0,\xi}|^2}{1+\xi^2}\right]$.

Concerning I_2, we define

$$F(\omega, z) = \frac{1+|X_t^{0,z}(\omega)|^2}{1+z^2}, \omega \in \Omega, z \in \mathbb{R}. \tag{3.A.8}$$

Since $z \mapsto F(\cdot, z)$ is differentiable in $L^2(\Omega)$, by mean value theorem we get

$$\begin{aligned}\mathbb{E}\left[|I_2|\right] &= |x-y|\mathbb{E}\left[\left|\widetilde{f}(X_t^{0,y})\int_0^1 \partial_z F(\cdot, ax+(1-a)y)da\right|\right] \\ &\leq |x-y|\,\|f\|_E \sup_z \mathbb{E}\left[|\partial_z F(\cdot,z)|\right].\end{aligned}$$

It remains to estimate the previous supremum. We have for $z \in \mathbb{R}$

$$\partial_z F(\cdot,z) = 2\frac{X_t^{0,z}\partial_z X_t^{0,z}}{1+z^2} - 2z\frac{1+|X_t^{0,z}|^2}{(1+z^2)^2}.$$

So by Cauchy-Schwarz we get

$$\begin{aligned}\mathbb{E}\left[|\partial_z F(\cdot,z)|\right] &\leq 2\left(\frac{\mathbb{E}\left[|X_t^{0,z}|^2\right]}{1+z^2}\frac{\mathbb{E}\left[|\partial_z X_t^{0,z}|^2\right]}{1+z^2}\right)^{1/2} + 2\frac{|z|}{1+z^2}\frac{1+\mathbb{E}\left[|X_t^{0,z}|^2\right]}{1+z^2} \\ &\leq A_2,\end{aligned}$$

where

$$A_2 = 2\left(\sup_z \frac{\mathbb{E}\left[|X_t^{0,z}|^2\right]}{1+z^2}\sup_z \mathbb{E}\left[|\partial_z X_t^{0,z}|^2\right]\right)^{1/2} + \left(1+\sup_z \frac{\mathbb{E}\left[|X_t^{0,z}|^2\right]}{1+z^2}\right).$$

By (3.2.15) and (3.A.2) A_2 is finite and we get

$$\mathbb{E}\left[|I_2|\right] \leq A_2\,\|f\|_E\,|x-y|. \tag{3.A.9}$$

Combining inequalities (3.A.7) and (3.A.9), (3.A.4) gives the existence of $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow \left|\frac{P_t f(x)}{1+x^2} - \frac{P_t f(y)}{1+y^2}\right| < \epsilon,$$

so that the function $x \mapsto \frac{P_t f(x)}{1+x^2}$ is uniformly continuous.

In conclusion we have proved that $P_t f \in E$. P_t is a bounded linear operator follows as a consequence of (3.A.3). □

Appendix 3.B Proof of Theorem 3.2.18

We recall that the semigroup P is here given by $P_t f(x) = \mathbb{E}\left[f(x+X_t)\right], x \in \mathbb{R}, t \geq 0$ and X is a square integrable Lévy process vanishing at zero. The classical theory of semigroup for Lévy processes defines the semigroup P on the set C_0 of continuous functions vanishing at infinity, equipped with the sup-norm $\|u\|_\infty = \sup_x |u(x)|$, c.f. for example [Sato, 2013, Theorem 31.5]. On C_0, the semigroup P is strongly continuous, with norm $\|P\| = 1$, and its generator L_0 is given by

$$L_0 f(x) = \int \left(f(x+y) - f(x) - yf'(x)\mathbb{1}_{|y|<1}\right)\nu(dy),\ f \in C_0. \tag{3.B.1}$$

Moreover, [Sato, 2013, Theorem 31.5] shows that $C_0^2 \subset D(L_0)$, where C_0^2 is the set of functions $f \in C^2$ such that f, f' and f'' vanish at infinity.

To prove Theorem 3.2.18 which concerns the infinitesimal generator of the semigroup P defined on the set E (c.f. (3.2.13)) related to a square integrable pure jump Lévy process, we adapt the classical theory. Since we consider a space $(E, \|.\|_E)$, different from the classical one, i.e. $(C_0, \|.\|_\infty)$, we need to show that (P_t) is still a strongly continuous semigroup.

Proposition 3.B.1. *Let X be a square integrable Lévy process, then the semigroup $(P_t) : E \to E$ is strongly continuous.*

Proof. The idea of the proof is an adaptation of the proof in [Sato, 2013, Theorem 31.5].

Let $f \in E$ and $\widetilde{f}$ defined by $\widetilde{f}(x) = \frac{f(x)}{1+x^2}$, $\forall x \in \mathbb{R}$. We evaluate, for $t > 0$, $x \in \mathbb{R}$

$$\frac{P_t f(x) - f(x)}{1+x^2} = \mathbb{E}\left[\widetilde{f}(x+X_t) - \widetilde{f}(x)\right] + \mathbb{E}\left[\widetilde{f}(x+X_t)\frac{X_t^2 + 2xX_t}{1+x^2}\right].$$

So

$$\|P_t f - f\|_E \leq \sup_{x\in\mathbb{R}} \left|\mathbb{E}\left[\widetilde{f}(x+X_t) - \widetilde{f}(x)\right]\right| + \sup_{x\in\mathbb{R}} \left|\mathbb{E}\left[\widetilde{f}(x+X_t)\frac{X_t^2 + 2xX_t}{1+x^2}\right]\right|. \quad \text{(3.B.2)}$$

First, note that

$$\begin{aligned}\left|\mathbb{E}\left[\widetilde{f}(x+X_t)\frac{X_t^2 + 2xX_t}{1+x^2}\right]\right| &\leq \|f\|_E \,\mathbb{E}\left[\frac{X_t^2 + 2|xX_t|}{1+x^2}\right] \\ &\leq \|f\|_E \left(\mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[|X_t|\right]\right),\end{aligned}$$

hence

$$\sup_{x\in\mathbb{R}} \left|\mathbb{E}\left[\widetilde{f}(x+X_t)\frac{X_t^2 + 2xX_t}{1+x^2}\right]\right| \leq \|f\|_E \left(\mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[|X_t|\right]\right).$$

Since X is a square integrable Lévy process, $\mathbb{E}\left[X_t^2\right] = c_2 t + c_1^2 t^2$ where c_1, c_2 were defined in (3.2.23). Hence, the right-hand side of the inequality above goes to zero as t goes to zero.

Now we prove that the first term $\sup_{x\in\mathbb{R}} \left|\mathbb{E}\left[\widetilde{f}(x+X_t) - \widetilde{f}(x)\right]\right|$ in the right-hand side of (3.B.2) goes to zero as well. Let $\epsilon > 0$ be a fixed positive real. Since $\widetilde{f}$ is uniformly continuous, then there is $\delta > 0$ such that

$$\forall x, y \; |x-y| < \delta \;\Rightarrow\; |\widetilde{f}(x) - \widetilde{f}(y)| < \frac{\epsilon}{2}.$$

Moreover, since X is continuous in probability

$$\exists t_0 > 0, \text{ such that } \forall t < t_0, \; \mathbb{P}(|X_t| > \delta) < \frac{\epsilon}{4\|f\|_E}.$$

For all $x \in \mathbb{R}$, $t < t_0$ we have

$$\begin{aligned}\left|\mathbb{E}\left[\widetilde{f}(x+X_t)-\widetilde{f}(x)\right]\right| &\leq \mathbb{E}\left[\left|\widetilde{f}(x+X_t)-\widetilde{f}(x)\right| \mathbb{1}_{\{|X_t|\leq\delta\}}\right] + \mathbb{E}\left[\left|\widetilde{f}(x+X_t)-\widetilde{f}(x)\right| \mathbb{1}_{\{|X_t|>\delta\}}\right] \\ &\leq \frac{\epsilon}{2} + 2\left\|f\right\|_E \mathbb{P}(|X_t|>\delta) \\ &\leq \epsilon.\end{aligned}$$

Since the inequality above is valid for every $x \in \mathbb{R}$, then

$$\sup_{x\in\mathbb{R}}\left|\mathbb{E}\left[\widetilde{f}(x+X_t)-\widetilde{f}(x)\right]\right| \xrightarrow{t\to 0} 0.$$

This concludes the proof that P is a strongly continuous semigroup. □

Remark 3.B.2. *Note that the semigroup (P_t) is not a contraction. In fact, if $f \in E, t > 0$, then*

$$\left\|P_t f\right\|_E = \sup_{x\in\mathbb{R}}\left|\mathbb{E}\left[\frac{f(x+X_t)}{1+x^2}\right]\right|. \tag{3.B.3}$$

Let $f_0(x) = 1 + x^2$ and denote again $c_1 = \mathbb{E}[X_1]$ and $c_2 = \mathrm{Var}(X_1)$. Obviously $f_0 \in E$, $\left\|f_0\right\|_E = 1$ and

$$\begin{aligned}\left\|P_t f_0\right\|_E &= \sup_{x\in\mathbb{R}}\mathbb{E}\left[\frac{1+(x+X_t)^2}{1+x^2}\right] \\ &= 1+\sup_{x\geq 0}\frac{2x|c_1|t+c_2 t+c_1^2 t^2}{1+x^2} \\ &= 1+|c_1|t+c_1^2 t^2.\end{aligned} \tag{3.B.4}$$

Hence (P_t) cannot be not a contraction since

$$\left\|P_t\right\| \geq \left\|P_t f_0\right\|_E > 1.$$

On the other hand, for $f \in E$, (3.B.3) gives

$$\left\|P_t f\right\|_E \leq \left\|f\right\|_E \left\|P_t f_0\right\|_E.$$

By (3.B.4) this implies that

$$\left\|P_t\right\| \leq 1+(|c_1|+c_2)t+c_1^2 t^2.$$

So, there exists a positive real $\omega > 0$ such that

$$\left\|P_t\right\| \leq e^{\omega t}.$$

Semigroups verifying the latter inequality are called **quasi-contractions**, see Pazy [1983]. For instance, [Pazy, 1983, Corollary 3.8] implies that

$$\forall \lambda > \omega,\ \lambda I - L \text{ is invertible.} \tag{3.B.5}$$

At this point we show that the space E_0^2, defined in (3.2.21), is a subset of $D(L)$ and that formula (3.B.1) remains valid in E_0^2. This will be done adapting a technique described in [Sato, 2013, Theorem 31.5], where it is stated that C_0^2 is included in $D(L_0)$. The main tool used for the proof of [Sato, 2013, Theorem 31.5] is the small time asymptotics

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[g(X_t)\right] = \int g(x)\nu(dx), \tag{3.B.6}$$

which holds for bounded continuous function g vanishing on a neighborhood of the origin, see [Sato, 2013, Corollary 8.9]. This result has been extended to a class of unbounded functions by [Figueroa-López, 2008, Theorem 1.1]. (3.B.6) is used in [Figueroa-López, 2008, Proposition 2.3] to prove that the quantity $\lim_{t\to 0}\dfrac{P_t g - g}{t}(x)$ converges point-wise, under some suitable conditions on the function g.

We state a similar lemma below.

Lemma 3.B.3. *Let $f \in E_0^2$. For all $x \in \mathbb{R}$, the quantity*

$$\lim_{t\to 0}\frac{P_t f - f}{t}(x) \tag{3.B.7}$$

exists and equals the right-hand side of (3.B.1).

Remark 3.B.4.

1) *To be self-contained, we give below a simple proof of Lemma 3.B.3, in the case when X is a square integrable pure jump process.*

2) *Later we will need to show that the point-wise convergence* (3.B.7) *holds according to the norm of E.*

Proof. Let $f \in E_0^2$. First, we verify that the integral

$$\int \left(f(x+y) - f(x) - yf'(x)\mathbb{1}_{|y|<1}\right)\nu(dy) \tag{3.B.8}$$

is well-defined for all $x \in \mathbb{R}$, taking into account $\int y^2\nu(dy) < \infty$ by (3.2.22).

In fact, by Taylor expansion and since $f \in E_0^2$, then for every $x \in \mathbb{R}$ there exist $a, b \geq 0$ such that, for all $y \in \mathbb{R}$

$$\begin{aligned} |f(x+y) - f(x)|\mathbb{1}_{|y|\geq 1} &\leq a(y^2+1)\mathbb{1}_{|y|\geq 1}, \\ |f(x+y) - f(x) - f'(x)y|\mathbb{1}_{|y|<1} &\leq by^2\mathbb{1}_{|y|<1}. \end{aligned}$$

Let $t > 0, x \in \mathbb{R}$. By Taylor expansion and Fubini theorem, recalling that $P_t f(x) = \mathbb{E}\left[f(x+X_t)\right]$ we have

$$\frac{P_t f - f}{t}(x) = c_1 f'(x) + \int_0^1 (1-a)\frac{1}{t}\mathbb{E}\left[f''(aX_t + x)X_t^2\right]da.$$

By abuse of notation, we denote by $L_0 f(x)$ the integral (3.B.8). Taking into account

(3.2.23) we have

$$\begin{aligned} L_0 f(x) &= c_1 f'(x) + \int \big(f(x+y) - f(x) - y f'(x)\big)\,\nu(dy) \\ &= c_1 f'(x) + \int_0^1 (1-a) \int_{\mathbb{R}} y^2 f''(ay+x)\nu(dy)da. \end{aligned} \tag{3.B.9}$$

Hence, it remains to show that (x being fixed)

$$\frac{P_t f - f}{t}(x) - L_0 f(x) = \int_0^1 (1-a)\Big(\frac{1}{t}\mathbb{E}\left[X_t^2 f''(aX_t + x)\right] - \int_{\mathbb{R}} y^2 f''(ay+x)\nu(dy)\Big)da \xrightarrow[t\to 0]{} 0. \tag{3.B.10}$$

For $a \in [0,1]$, we denote $g(y) = y^2 f''(ay+x)$. We have $g(y) \underset{y\to 0}{\sim} y^2 f''(x)$. If $f''(x) \neq 0$, then [Figueroa-López, 2008, Theorem 1.1] (ii) implies that

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[g(X_t)\right] = \int_{\mathbb{R}} g(y)\nu(dy). \tag{3.B.11}$$

If $f''(x) = 0$, then $g(y) = o(y^2)$ and (3.B.11) is still valid by [Figueroa-López, 2008, Theorem 1.1] (i). We conclude to the validity of (3.B.10) by Lebesgue dominated convergence theorem taking into account that f'' is bounded. □

As observed in a similar case in [Figueroa-López, 2008, Remark 2.4], we will prove that the point-wise convergence proved in Lemma 3.B.7 holds in the strong sense.

For this purpose, we introduce the linear subspace

$$\widetilde{E} = \Big\{ f \in C \text{ such that } \widetilde{f} := x \mapsto \frac{f(x)}{1+x^2} \text{ is vanishing at infinity } \Big\}$$

of E. It it is easy to show that $\widetilde{E}$ is closed in E so that it is a Banach subspace of E.

Lemma 3.B.5. *Let $f,\ g \in \widetilde{E}$, such that*

$$\lim_{t\to 0} \frac{P_t f - f}{t}(x) = g(x),\ \forall x \in \mathbb{R}. \tag{3.B.12}$$

Then $f \in D(L)$ and $Lf = g$.

Proof. We first introduce a restriction $\widetilde{P}$ of the semigroup P to the linear subspace $\widetilde{E}$. By Lebesgue dominated convergence theorem and the fact that $\frac{1+(X_t+x)^2}{1+x^2} \leq 2(|X_t|^2 + 1)$, one can show that $P_t f \in \widetilde{E}$ for any $f \in \widetilde{E}$, $t \geq 0$. Hence $(\widetilde{P}_t)$ is a semigroup on $\widetilde{E}$; we denote by $\widetilde{L}$ its infinitesimal generator.

As in [Sato, 2013, Lemma 31.7], we denote by $L^{\#} f = g$, the operator defined by the equation (3.B.12) for $f, g \in \widetilde{E}$ and by $D(L^{\#})$ its domain, i.e. the set of functions f for which (3.B.12) exists. Then $L^{\#}$ is an extension of $\widetilde{L}$.

Fix $q > |c_1| + c_2$. We prove first that

$$\forall f \in D(L^{\#}) \quad (qI - L^{\#})f = 0 \Rightarrow f = 0. \tag{3.B.13}$$

Let $f \in D(L^{\#})$ such that $(qI - L^{\#})f = 0$. We denote $f^- = -(f \wedge 0)$ and $f^+ = f \vee 0$. Suppose that $f^+ \neq 0$. Since f^+ is continuous and vanishing at infinity, there exists x_1 such that $\frac{f^+(x_1)}{1+x_1^2} = \max_x \frac{f^+(x)}{1+x^2} > 0$. Moreover $f(x_1) = f^+(x_1)$. Then

$$\frac{\mathbb{E}\left[f(x_1 + X_t) - f(x_1)\right]}{t} \leq \frac{1}{t}\left(f(x_1)\frac{\mathbb{E}\left[1 + (x_1 + X_t)^2\right]}{1 + x_1^2} - f(x_1)\right).$$

Passing to the limit when $t \to 0$ it follows

$$L^{\#}f(x_1) \leq f(x_1)(|c_1| + c_2).$$

Then

$$(q - |c_1| - c_2)f(x_1) \leq 0,$$

which contradicts the fact that $f(x_1) > 0$. Hence, $f^+ = 0$. With similar arguments, we can show that $f^- = 0$ and so $f = 0$, which proves (3.B.13).

By restriction, $(\widetilde{P}_t)$ fulfills $\|\widetilde{P}_t\| \leq e^{\omega t}$, in particular it is a quasi-contraction semigroup, so by (3.B.5), we can certainly choose $q > \max(|c_1| + c_2, \omega)$, so that $qI - \widetilde{L}$ is invertible and $R(qI - \widetilde{L}) = \widetilde{E}$.
We observe that $D(\widetilde{L}) \subset D(L^{\#})$. Let now $f \in D(L^{\#})$; then $(qI - L^{\#})f \in \widetilde{E} = R(qI - \widetilde{L})$. Consequently, there is $v \in D(\widetilde{L})$ such that $(qI - L^{\#})f = (qI - \widetilde{L})v$. So, $(qI - L^{\#})(f - v) = 0$.

By (3.B.13), $(qI - L^{\#})$ is injective, so $f = v$ and $f \in D(\widetilde{L})$. Consequently $\widetilde{L}f$ is given by g defined in (3.B.12). Finally, the fact that $D(\widetilde{L}) \subset D(L)$ and $\widetilde{L}$ is a restriction of L allow to conclude the proof of Lemma 3.B.5.

□

We continue the proof of Theorem 3.2.18 making use of Lemmas 3.B.3 and Lemma 3.B.5.
First, let us prove that $E_0^2 \subset \widetilde{E}$. Indeed by Taylor expansion, we have, for $f \in E_0^2$

$$\frac{f(x)}{1+x^2} = \frac{f(0)}{1+x^2} + \frac{x}{1+x^2}f'(0) + \frac{x^2}{1+x^2}\int_0^1 (1-\alpha)f''(x\alpha)d\alpha.$$

Since $\lim_{x\to\infty} f''(x\alpha) = 0$ for all $\alpha \in]0,1[$, then by Lebesgue theorem, we have that $\lim_{x\to\infty} \frac{f(x)}{1+x^2} = 0$, so $f \in \widetilde{E}$.

By Lemma 3.B.3, it follows

$$\lim_{t\to 0} \frac{P_t f - f}{t}(x) = L_0 f(x), \ \forall x \in \mathbb{R},$$

where L_0 is given in (3.B.8). In order to apply Lemma 3.B.5, it remains to show that

$L_0 f \in \widetilde{E}$. Using relation (3.B.9), for $x \in \mathbb{R}$ we get

$$\frac{L_0 f(x)}{1+x^2} = c_1 \frac{f'(x)}{1+x^2} + \int_0^1 (1-a) \int_{\mathbb{R}} y^2 \frac{f''(ay+x)}{1+x^2} \nu(dy) da.$$

Since $f \in E_0^2$, then f'' is bounded and f' has linear growth. So, the fact that $\int_{\mathbb{R}} y^2 \nu(dy) < \infty$ implies indeed $\lim_{x\to\infty} \frac{L_0 f(x)}{1+x^2} = 0$ and $L_0 f \in \widetilde{E}$.

Finally, Lemma 3.B.5 implies that $E_0^2 \subset D(L)$ and for $f \in E_0^2$, Lf is given by (3.2.25).

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Résumé

L'objectif central de la thèse est d'étudier diverses mesures du risque de modèle, exprimées en terme monétaire, qui puissent être appliquées de façon cohérente à une collection hétérogène de produits financiers. Les deux premiers chapitres traitent cette problématique, premièrement d'un point de vue théorique, ensuite en menant un étude empirique centrée sur le marché du gaz naturel. Le troisième chapitre se concentre sur une étude théorique du risque dit *de base* (en anglais *basis risk*). Dans le premier chapitre, nous nous sommes intéressés à l'évaluation de produits financiers complexes, qui prend en compte le risque de modèle et la disponibilité dans le marché de produits dérivés basiques, appelés aussi *vanille*. Nous avons en particulier poursuivi l'approche du transport optimal (connue dans la littérature) pour le calcul des bornes de prix et des stratégies de sur (sous)-couverture robustes au risque de modèle. Nous reprenons en particulier une construction de probabilités martingales sous lesquelles le prix d'une option exotique atteint les dites bornes de prix, en se concentrant sur le cas des martingales positives. Nous mettons aussi en évidence des propriétés significatives de symétrie dans l'étude de ce problème. Dans le deuxième chapitre, nous approchons le problème du risque de modèle d'un point de vue empirique, en étudiant la gestion optimale d'une unité de gaz naturel et en quantifiant l'effet de ce risque sur sa valeur optimale. Lors de cette étude, l'évaluation de l'unité de stockage est basée sur le prix spot, alors que sa couverture est réalisée avec des contrats à termes. Comme mentionné auparavant, le troisième chapitre met l'accent sur le *risque de base*, qui intervient lorsque l'on veut couvrir un actif conditionnel basé sur un actif non-traité (par exemple la température) en se servant d'un portefeuille constitué d'actifs traités sur le marché. Un critère de couverture dans ce contexte est celui de la minimisation de la variance qui est étroitement lié à la décomposition dite de *Föllmer-Schweizer*. Cette décomposition peut être déduite de la résolution d'une certaine équation différentielle stochastique rétrograde (EDSR) dirigée par une martingale éventuellement à sauts. Lorsque cette martingale est un mouvement brownien standard, les EDSR sont fortement associées aux EDP paraboliques semilinéaires. Dans le cas général nous formulons un problème déterministe qui étend les EDPs mentionnées. Nous appliquons cette démarche à l'important cas particulier de la décomposition de Föllmer-Schweizer, dont nous donnons des expressions explicites de la décomposition du payoff d'une option lorsque les sous-jacents sont exponentielles de processus additifs.

Abstract

The main objective of this thesis is the study of the model risk and its quantification through monetary measures. On the other hand we expect it to fit a large set of complex (exotic) financial products. The first two chapters treat the model risk problem both from the empirical and the theoretical point of view, while the third chapter concentrates on a theoretical study of another financial risk called *basis risk*. In the first chapter of this thesis, we are interested in the model-independent pricing and hedging of complex financial products, when a set of standard (*vanilla*) products are available in the market. We follow the optimal transport approach for the computation of the option bounds and the super (sub)-hedging strategies. We characterize the optimal martingale probability measures, under which the exotic option price attains the model-free bounds; we devote special interest to the case when the martingales are positive. We stress in particular on the symmetry relations that arise when studying the option bounds. In the second chapter, we approach the model risk problem from an empirical point of view. We study the optimal management of a natural gas storage and we quantify the impact of that risk on the gas storage value. As already mentioned, the last chapter concentrates on the basis risk, which is the risk that arises when one hedges a contingent claim written on a non-tradable but observable asset (e.g. the temperature) using a portfolio of correlated tradable assets. One hedging criterion is the mean-variance minimization, which is closely related to the celebrated *Föllmer-Schweizer* decomposition. That decomposition can be deduced from the resolution of a special Backward Stochastic Differential Equations (BSDEs) driven by a càdlàg martingale. When this martingale is a standard Brownian motion, the related BSDEs are strongly related to semi-linear parabolic PDEs. In that chapter, we formulate a deterministic problem generalizing those PDEs to the general context of martingales and we apply this methodology to discuss some properties of the Föllmer-Schweizer decomposition. We also give an explicit expression of such decomposition of the option payoff when the underlying prices are exponential of additives processes.

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