



Elastodynamic homogenization of periodic media

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École Doctorale Sciences de l'Ingénieur et de l'Environnement

Thèse présentée pour l'obtention du titre de

Docteur en Mécanique

Elastodynamic homogenization of periodic media

par

Hussein Nassar

à soutenir le 1^{er} octobre 2015 devant le jury composé de

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<i>Rapporteur</i>	Jean-Jacques Marigo	Professeur École Polytechnique
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Abstract

The recent issue of metamaterials design has renewed the interest in homogenization theories under dynamic loadings. In particular, the elastodynamic homogenization theory initiated by J.R. Willis has gained special attention while studying elastic cloaking. The present thesis reformulates Willis theory for periodic media, investigates its outcome and assesses its physical suitability in the sense of a few suggested “homogenizability conditions”. Based on the results of this first part, approximate asymptotic expansions of Willis theory are explored in connection with strain-gradient media. A necessary convergence condition then shows that all optical dispersion branches are lost when long-wavelength low-frequency Taylor asymptotic expansions are carried out. Finally, a new homogenization theory is proposed to generalize Willis theory and improve it at finite frequencies in such a way that selected optical branches, formerly lost, are recovered. It is also proven that the outcome of the new theory is an effective homogeneous generalized continuum satisfying a generalized elastodynamic version of Hill-Mandel lemma.

Résumé

La problématique récente de la conception de métamatériaux a renouvelé l'intérêt dans les théories de l'homogénéisation en régime dynamique. En particulier, la théorie de l'homogénéisation élastodynamique initiée par J.R. Willis a reçu une attention particulière suite à des travaux sur l'invisibilité élastique. La présente thèse reformule la théorie de Willis dans le cas des milieux périodiques, examine ses implications et évalue sa pertinence physique au sens de quelques “conditions d'homogénéisabilité” qui sont suggérées. En se basant sur les résultats de cette première partie, des développements asymptotiques approximatifs de la théorie de Willis sont explorés en relation avec les théories à gradient. Une condition nécessaire de convergence montre alors que toutes les branches optiques de la courbe de dispersion sont omises quand des développements asymptotiques de Taylor de basse fréquence et de longue longueur d'onde sont déployés. Enfin, une nouvelle théorie de l'homogénéisation est proposée. On montre qu'elle généralise la théorie de Willis et qu'elle l'améliore en moyenne fréquence de sorte qu'on retrouve certaines branches optiques omises auparavant. On montre également que le milieu homogène effectif défini par la nouvelle théorie est un milieu généralisé dont les champs satisfont une version élastodynamique généralisée du lemme de Hill-Mandel.

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*These things which cannot be bought:
The childhood memories between your brother and yourself,
Your sudden sense of manhood,
This shy sentiment which holds back the longing when you embrace him,
The silence – smiling – to your mother’s scolding
As though you were still children!*

Amal Donqol, *Do not reconcile*

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Champs-sur-Marne, October 11, 2015.

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Revenge is never a straight line. It's a forest, and like a forest it's easy to lose your way.

Hattori Hanzo, *Kill Bill: Volume 1*

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General introduction

All right, everybody be cool, this is a robbery!

Pumpkin, *Pulp Fiction*

1 Metamaterials

Metamaterials are man-made materials purposely tailored to have an unusual behavior be it optical, electromagnetic, acoustic, elastic or any other physical behavior. Metamaterials can have a negative permittivity, permeability, shear and bulk moduli, a negative or an anisotropic mass, a negative refraction index or any combination thereof, for instance. By featuring these non-standard properties, metamaterials allow for transcendent applications such as “superlensing”, i.e., lensing beyond the diffraction limit, and cloaking. Other applications include wave attenuation, wave channeling and, more generally, any specific and purposeful form of wave control.

On the microscopic scale, a metamaterial appears as an arrangement of multiple constituents whose shapes, short- and long-range orders convey to the metamaterial its exquisite properties on the macroscopic scale. In contrast, taken individually, each constituent has a standard behavior. Therefore, metamaterials are only “meta” on the macroscopic scale, i.e., when the wavelength of the wave, whether light, sound or any other disturbance, traveling in the metamaterial is large enough with respect to the typical size of the constituents so as to interact with them collectively instead of individually. Furthermore, metamaterials behavior is only observable at specific frequencies which trigger internal resonant mechanisms. For example, a metamaterial’s mass will be negative over a more or less narrow band centered around some particular frequency.

For “negative” electromagnetic and elastic materials, the interested reader is referred to the pioneering papers by [Smith et al. \(2000\)](#), [Pendry \(2000\)](#) and by [Liu et al. \(2000\)](#). It is worth mentioning here that the potential of “negative materials” was known long before ([Veselago, 1968](#)) and so were the unusual macroscopic properties of mixtures ([Willis, 1985](#); [Auriault and Bonnet, 1985](#)). However, it is only recently that the complete chain, from microstructure to application, has been fully recognized. For a recent exhaustive review of metamaterials, see the introductory chapter of the book edited by [Craster and Guenneau \(2012\)](#).

2 Cloaking

Perhaps the most exciting application of metamaterials is the possibility for rendering objects invisible to probing waves of a given nature, light or sound waves for instance, by placing a well conceived metamaterial, a cloak, around it. By suitably varying the index of refraction within the metamaterial, rays can be guided around a region of space towards the paths they would have followed in the absence of the cloak. An object is then cloaked by simply putting it in the circumvented region.

Going further into details, there exists a finite transformation which maps the straight paths of rays, in the absence of the cloak, to the desired curved paths. Then, this finite transformation seen as a curvilinear change of coordinates can be applied to the governing wave equation whose coefficients are constants. The coefficients of the transformed wave equation, interpreted in Cartesian coordinates as varying coefficients, define then the constitutive properties of the cloak. This idea is at the basis of what became known as transformation optics and allows for conceiving more general devices for wave control than cloaks. The founding papers were written by [Leonhardt \(2006\)](#) and [Pendry et al. \(2006\)](#) in the context of electromagnetic waves. For more references and insightful remarks see the papers by [Kohn et al. \(2008\)](#) and by [Norris \(2008\)](#).

The transformation method described above will work as long as the governing equations are invariant under curvilinear changes of coordinates. Otherwise, the transformed equation will include new terms unaccounted for by the physics. Hence, the implementation for electromagnetic and acoustic waves is straightforward. In contrast, the standard full elasticity equations do not retain their form and, in particular, when a curvilinear change of coordinates is applied, third-order tensors coupling stress to velocity, on one hand, and strain to momentum, on the other hand, appear. Interestingly, alternative non-standard elasticity equations somehow derived by [Willis \(1997\)](#) do feature the mentioned coupling tensors and therefore are form invariant. This was demonstrated by [Milton et al. \(2006\)](#). See also the paper by [Norris and Shuvalov \(2011\)](#) for further references.

Next, we shall describe the considerations that lead to the mentioned “Willis equations”. Accordingly, a few words on micromechanics and homogenization are in order.

3 Micromechanics and homogenization

[Hill \(1965\)](#) estimated the mechanical properties of an “aggregate of elastoplastic crystals”, “as a whole”, assuming known the constitutive law of the individual crystals. He described his theory as being micromechanical. More generally, micromechanics, as put by [Zaoui \(2001\)](#), is an attempt to invert the “inductive” or “phenomenological” approach to the characterization of materials behavior into a more “deductive” one. We speak of a scale transition from the laws governing “the parts” at the microscopic scale to the laws governing “the whole” at the macroscopic scale. This transition is called homogenization.

A material, or a metamaterial as described earlier, when resolved, appears indeed as an aggregate of different phases or constituents and we are presented with two options when studying its behavior. One is to model it as an inhomogeneous one on the microscale and to deal directly with the different constituents. The other is to model it as an effectively homogeneous one on the macroscale. Undoubtedly, how the constituents behave individually and how they are geometrically and statistically distributed over space are decisive to the macroscopic behavior of the material under consideration. Therefore, ideally, a deductive micromechanical approach permits to solve inverse problems as in finding a microstructure whose macroscopic behavior is sought and desired.

Scale transition, under one form or another, appears in multiple branches of physics. In some circumstances, Keller (1977) notes, “the equations governing the gross behavior are of a different form from those which govern the small scale behavior”. For instance, sound traveling at the same speed in all directions in each phase does not imply it traveling at the same speed in all directions in the mixture. The same as before holds if “in all directions” is substituted by “at all frequencies”. More fundamental examples are provided in the context of statistical physics where time reversal symmetry is broken on the macroscale or in the context of fluid mechanics where Darcy’s law is derived from the Navier-Stokes equations.

As for the elasticity equations in statics, it is known that a mixture of elastic phases behave as an elastic, possibly anisotropic, solid. In dynamics however, several homogenization theories with different outcomes have been suggested depending on the involved length scales, frequencies and phases. Most remarkably, the homogenization theory initiated by Willis (1997) features, among other peculiarities, the formerly mentioned third-order coupling tensors mandatory for elastic cloaking applications.

After the particular framework of the present manuscript has been laid down, a brief overview of some homogenization theories for elastodynamics, including Willis theory in particular, is presented.

4 Framework

The present manuscript, hopefully, contributes to the effort of understanding how the macroscopic properties of (periodic) materials in general, and metamaterials in particular, relate to their microscopic counterparts for linearly elastic solids. In connection to the mentioned observation of Keller, *we shall focus our attention on the form taken by the macroscopic behavior, its nature, consistency and physical suitability, rather than on the actual, numerical, calculation of the parameters involved in said behavior.* In the remainder of this section, we comment on the choices we have made in terms of the microstructures to be homogenized, the considered frequency regimes and the adopted homogenization approaches.

4.1 On random and periodic microstructures

An incident elastic wave interacts with the heterogeneities of a given random medium: it is scattered by inclusions, reflected, refracted and transmitted across interfaces. When the wavelength is infinitely larger than the typical “radius” of heterogeneities, the incident wave is little sensitive to the exact features of the microstructure and only feels its overall, averaged or smoothed, effect. The inhomogeneous medium behaves then, at the scale of the wave, as an effective homogeneous medium. Conversely, the incident wave is weakly scattered. However, even weak scattering from a disordered, random, distribution of scatterers can perturb drastically the incident wave across large travel distances. Most of the carried energy gets shifted from the “coherent” mean wave towards the scattered waves. The effective homogeneous medium approximation will then only be valid locally for travel distances of the order of several wavelengths. Beyond this limit, other approximations, of a diffusive nature for instance, are called for.

In contrast, for a periodic medium, due to the particular distribution of scatterers, the scattered waves can only have specific, discrete, directions of propagation¹. As a consequence, even when the incident wave is strongly scattered, i.e., even for high frequencies and small wavelengths, the change in the traveling wave across large distances is entirely accounted for by simple phase shifts. The energetic content of the scattered waves will of course depend heavily on frequencies and wavelengths so that the notion of the effective homogeneous medium will need adaptation but will remain pertinent for a wider domain of small and finite frequencies as we shall see.

The introductory chapter of the book by [Sheng \(2006\)](#) and the overview by [Papanicolaou \(1998\)](#) seem to be good starting points for the reader willing to tackle diffusion and other wave phenomena in random media. When dealing with the propagation of (infinitely-) high-frequency waves, scientists speak of ray theory and of geometrical and physical optics. The relevant chapter of the book by [Bender and Orszag \(1999\)](#) might be of assistance for readers. In the remainder of the present work, we shall be only interested in the propagation of waves in periodic media at low and high but finite frequencies. Note that periodic materials are sufficiently rich in that they offer a wide array of potential macroscopic properties, including those of metamaterials. Their manufacturing is also easy and direct.

4.2 Three kinds of homogenization approaches

A microstructured medium can be modeled, as depicted earlier, either microscopically as an inhomogeneous one or macroscopically as an effective homogeneous one. There exist at least three points of view regarding how the scale transition should be made.

If one carries out tensile tests on multiple samples of the same (inhomogeneous) material, and given that these samples are sufficiently representative, the plots of the obtained stress-strain curves are expected to be identical up to a small statistical

¹This fact will later be named “Floquet-Bloch theorem”.

fluctuation and all of them characterize then the macroscopic constitutive stress-strain relationship. In a structure made of this material, each representative volume element corresponds to a macroscopic material point whose constitutive behavior is given by the results of the performed mechanical tests. Such is the first type of approaches. Its major advantage is the immediate link it establishes between the macroscopic constitutive behavior and its straightforward experimental characterization. All homogenization textbooks discuss the notion of representative volume elements and the suitable tensile tests to be carried over them. These tests are defined through specific boundary conditions in relation to a constraint on virtual work known as Hill-Mandel lemma. For more information, we refer to the book edited by [Bornert et al. \(2001\)](#), to the book by [Forest \(2006\)](#) and to references therein.

Alternatively, homogenization can be seen as a limit process during which a scale separation parameter, namely the ratio of the typical size of heterogeneities to the size of the loading wavelength, tends to zero. The wave solution to the microscopic motion equation has then increasingly finer and smaller fluctuations and converges toward a macroscopic solution of another motion equation interpreted as the macroscopic motion equation. Several mathematical theories define and deal with suitable convergence notions of motion equations such as G -, Γ -, H - and two-scale convergence. The interested reader might start with the book dedicated to “beginners” by [Braides \(2002\)](#). See also the paper by [Allaire \(1992\)](#) and references therein.

Finally, instead of filtering microstructure-related fluctuations by passing to the limit, one can smooth the microscopic wave by carrying averages of some kind. When smoothed, the microscopic wave becomes a macroscopic wave and the motion equation whose solutions are the macroscopic waves is taken to be the macroscopic motion equation. These approaches are discussed subsequently in more detail.

All three kinds of approaches seem to yield the same effective homogeneous behavior for infinitely large wavelengths and infinitely low frequencies. However, for finite wavelengths and wavenumbers, it is unclear how the standard quasi-static theory should be improved. We shall be interested in Willis and similar theories and in approaches of the third kind more generally. These are rigorous methods in the sense that they transform a microscopic motion into a macroscopic one but lack the convergence results of the methods of the second kind. It is freeing to some extent to escape strict mathematical rigor. Nonetheless, this comes at the cost of some inconsistencies that will be encountered.

Concluding our comparison, approaches of the first kind can be described as local as they deal with representative volume elements regardless of the whole structure or medium whereas approaches of the second and third kinds can be said to be global since they are concerned with the structure as a whole. In particular, the latter viewpoint does not presume the existence of a representative volume element nor the locality of the effective constitutive behavior. This being said, the three paradigms need not to be opposed but to complete one another.

5 A brief overview of homogenization theories

One can divide the homogenization approaches of the third kind, described above, into two major categories. The first one includes theories which make no assumptions on the involved frequencies and wavelengths. These are mostly variations or particular cases of the smoothing method by Keller such as Willis theory. Methods of the second category are based on a scaling of frequencies, wavelengths and other physical parameters. It is obvious in what follows that the theory initiated by Willis holds a special place. Indeed, it was our starting point and its assessment motivated several developments in connection with other homogenization approaches.

Bibliographical elements are provided within each chapter. Hereafter is a brief overview which helps integrate the references to come in a relatively coherent body. Many, if not all, approximate variational, self-consistent and numerical, among other, methods have been left out. These can be found in the book by [Bornert et al. \(2001\)](#) or in the surveys by [Zaoui \(2002\)](#) and by [Charalambakis \(2010\)](#).

5.1 The smoothing method

A microscopic displacement field is expected to vary on two different scales. It will have large scale variations induced by the macroscopically applied loading, or the long-wavelength incident wave, and will fluctuates on the scale of the heterogeneities. The macroscopic displacement field is then thought of as an approximation to the microscopic field where the rapid small scale fluctuations have been smoothed. Smoothing can be done in different ways. One popular approach devised by [Keller \(1960, 1977, 1979\)](#) and others is to define the macroscopic wave as the average of the microscopic one over an ensemble of realizations of the microstructure for a random medium. In other words, the macroscopic wave is the expectancy of the microscopic one when the microstructure is randomized. The macroscopic wave is then what is referred to as the coherent wave.

Keller only spoke in terms of motion equations and did not investigate the repercussions of the smoothing method on the constitutive behavior. Furthermore, only low contrast media were handled due to the complexity of the required calculations. A few years later, it is Willis who suggested systematic variational methods for approximating the effective behavior of composites of arbitrary microstructures. As for the corresponding effective constitutive law, it first appeared in 1997.

One should underline the fact that the formalism based on projectors that [Keller \(1977\)](#) came up with is highly insightful. Indeed, speaking in terms of projections clarifies the meaning of the approximation made in the course of the upscaling process. More importantly, known properties of projections regarding the conservation of scalar products entail an energy equivalency property between the microscale and the macroscale usually referred to as Hill-Mandel lemma. Such is the main driving idea of chapter 4 for instance.

5.2 Willis theory

In an attempt to describe the scattering of an incident wave by a single inclusion, Willis (1980a) transformed the differential motion equation into a fixed point integral one. The integral formulation is based on the use of a fictional homogeneous comparison medium. The idea was not new and undoubtedly reminds of the Eshelby problem in statics. For instance, such a scattering problem was earlier investigated by Gubernatis et al. (1977) by means of integral equations as well. The novelty of Willis' approach was to substitute the displacement integral equation with two other equations written in terms of two auxiliary unknown fields: the formerly known stress polarization field and the newly introduced momentum polarization field.

In the second part of his paper, Willis (1980b) adapted his polarization approach in order to deal with scattering by multiple inclusions. Randomness arises hence necessarily when describing the distribution of scatterers in space. Then, speaking of the solution to the integral equations, and in the spirit of the smoothing method, Willis notes that “the solution of these equations, even if it could be obtained, would contain an excess of detail and a reasonable objective would be to extract from it some “locally-averaged” information”. Nonetheless, unlike periodic media as shall be seen, defining suitable local space averages over a random medium is far from being obvious. Regretfully, Willis seeks for the expectancy value of the displacement field, i.e., the coherent wave, instead of its local space averages.

In a more general fashion than for matrix-inclusion composites, an approach by polarizations allow to construct new variational principles of the Hashin-Shtrikman type describing the motion of the coherent wave in an inhomogeneous medium. These are stationarity principles in general and hence do not produce any bounds except in the Laplace domain for a real Laplace variable. Willis (1981) explored these possibilities without speaking of any effective constitutive law the first mention of which came 4 years later while investigating the overall effective mass density of composites and the possibility for nonlocal and anisotropic effects (Willis, 1985). Incidentally, in that paper, in order to focus attention on inertial considerations, Willis assumed stiffnesses to be homogeneous in which case the effective constitutive law takes a much simpler form. In fact, back then, the existence of a coupling between stresses and velocities or between momenta and strains was not suspected at all.

The chapter “Dynamics of composites” written by Willis (1997) is the first instance where the outcome of the smoothing method of Keller is written in terms of an effective constitutive behavior. It turns out that, at the macroscopic level, the existence of some coupling terms allow for velocity-generated stresses and for strain-generated momenta. In addition, the behavior is nonlocal in both space and time and, as hinted earlier, the effective mass density is possibly anisotropic.

Finally, Willis theory was generalized to describe at the macroscopic scale some weighted expectancy of the displacement field by Milton and Willis (2007) and by Willis (2011). Some examples are treated in (Willis, 2009, 2012a) and a remake of the paper from 1981 on variational methods in light of recent developments can be found in (Willis, 2012b).

The renewed interest in Willis theory in recent years is caused by two aspects both related to the very trending metamaterials. On one hand, the outcome of Willis theory is sufficiently rich to account for all kinds of unusual effects: anisotropic masses, negative masses and stiffnesses, nonlocal effects, non-propagating frequencies, ... On the other hand, the Willis elasticity equations retain their form under curvilinear changes of coordinates as demonstrated by [Milton et al. \(2006\)](#) so that the macroscopic effective behavior outcome of Willis theory can be used to realize elastic cloaks.

When the distribution of scatterers is periodic, the wave takes a simple form which separates microstructure related fluctuations from loading related variations. The coherent wave can then be obtained by space averaging some periodic amplitude over a unit cell thanks to what is known as Floquet-Bloch expansions¹. This idea was first pursued independently from Willis theory by [Amirkhizi and Nemat-Nasser \(2008\)](#) for electromagnetic waves and then by others for elastic waves ([Nemat-Nasser and Srivastava, 2011, 2013](#); [Nemat-Nasser et al., 2011](#); [Norris et al., 2012](#); [Shuvalov et al., 2011](#); [Srivastava and Nemat-Nasser, 2011](#)).

5.3 Asymptotic theories

For infinitely long wavelengths and low frequencies, there exists a well-established more or less standard theory of homogenization based on two-scale asymptotic expansions and dealing with static and quasi-static behaviors. This theory is usually traced back to the work of [Bensoussan et al. \(1978\)](#) and [Sanchez-Palencia \(1980\)](#). It proves, by convergence analysis, that the macroscopic behavior is homogeneous and of the same nature as the microscopic one with possible elastic anisotropy, the effective mass density being simply the arithmetic mean of the phases mass densities. Unfortunately, the macroscopic behavior thus obtained has its limitations. Most remarkably, all information related to internal length scales are lost and no dispersion phenomena, i.e., phenomena related to the frequency dependence of the speed of sound, are captured.

In the language of Taylor expansions, the above theory is of the lowest order in frequency. One would want to improve the accuracy of the above described effective behavior by relaxing the corresponding assumptions and allowing for relatively higher frequencies. Two ways for doing this can be found in the literature. The first one simply pushes farther in the Taylor expansion and takes into account higher order terms (see², e.g., [Boutin and Auriault, 1993](#); [Boutin, 1996](#); [Smyshlyaev and Cherednichenko, 2000](#); [Andrianov et al., 2008](#); [Kalamkarov et al., 2009](#); [Andrianov et al., 2011](#)). These macroscopic models capture well some weak dispersive effects but miss stronger ones related to internal oscillation modes³. Note that these approaches, except to lowest order, lack convergence results.

¹These, and related references, are discussed in detail in the next chapter.

²Not to mention all “continualization” schemes such as those of [Askes et al. \(2002\)](#), [Askes and Aifantis \(2011\)](#) and [Polyzos and Fotiadis \(2012\)](#) and references therein.

³Think of the difference between standard and spinning roller coasters ...

Another way of looking at things is to say that, strictly speaking, infinite wavelengths correspond to uniform translatory rigid body motions. The macroscopic waves described by the standard homogenization theory can then be seen as slow or large scale modulations of rigid body motions. A particularity of rigid body motions is that the medium can sustain them in the absence of external loadings. They are a particular case, and the only static kind, of what is called a free wave. Therefore, a natural generalization of the quasi-static theory is to consider large scale modulations of other high frequency free waves. In terms of Taylor expansions, instead of increasing the order of the expansion, the neighborhood of the expansions, which was centered around the null frequency, is now modified and centered around another non-null eigenfrequency of the medium. This theory was first suggested by [Craster et al. \(2010\)](#) and further developed by [Nolde et al. \(2011\)](#), [Antonakakis et al. \(2014\)](#), [Colquitt et al. \(2014\)](#) and by [Boutin et al. \(2014\)](#). The advantage of the outlined high-frequency homogenization theory is that it captures some high-frequency internal oscillation modes (in the same manner in which the quasi-static theory captures the trivial oscillation mode: rigid body motion) and the related dispersive effects. Its major inconvenience is that it is a local theory: in the vicinity of each eigenfrequency, a motion equation describing the modulations of the corresponding free wave is needed.

5.4 High contrasts

Say a matrix-inclusion composite is to be homogenized and assume that the matrix is infinitely stiffer than the inclusions. When the matrix experiences a low-frequency motion, the inclusions, being softer, would still oscillate in a high-frequency manner. Taking this fact into account when carrying Taylor expansions leads to a high-contrast low-frequency long-wavelength asymptotic homogenization theory capable of capturing the large scale modulations of rigid body transformations in addition to some localized states induced by the internal oscillations of inclusions. The earliest¹ investigations of this idea in the context of elastodynamics were carried by [Auriault and Bonnet \(1985\)](#)² and later pursued by others. We refer to ([Smyshlyaev, 2009](#)), to ([Auriault and Boutin, 2012](#)) and to references therein for more details.

The described high-contrast theory has two (expected) inconveniences. First, it only approximates the microscopic displacement field in the stiff matrix. Second, the presence of an infinite number of internal oscillation modes makes that the effective behavior is nonlocal. Consequently, the effective behavior, just like for Willis theory, can turn out to be highly inextricable.

¹Earliest to my knowledge that is ...

²It is worth mentioning that the cited paper is hard to find, written in French and poorly recognized.

6 An outline

We start by investigating Willis theory due to its close connection to modern research on metamaterials and cloaking. It turns out that the mathematical rigor of this theory cannot avoid many physical inconsistencies. Chapter 2 shall reformulate and assess said theory without going as far as applications however. Our conclusions will call for two kinds of developments. One pushes toward simplifying the Willis effective behavior by the use of asymptotics (chapter 3). The results are discussed in connection with the so-called higher-order strain-gradient media ([Mindlin, 1964](#); [Askes and Aifantis, 2011](#)). The other invites us to give up the coherent wave in favor of other more significant components of the microscopic wave and will give rise to macroscopic behaviors resembling those of a class of generalized continua, i.e., continua with an extended set of degrees of freedom ([Maugin, 2010](#)). Chapter 1, on the other hand, is a technical introduction to the elastodynamics of periodic media and should recall, and make the reader familiar with, some basic useful tools of a geometrical nature, such as direct and reciprocal lattices, unit cells and Brillouin zones, and others of an analytical nature, such as Fourier and Floquet-Bloch transforms.

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Chapter 1

Preliminaries

[Subject appears well-rested and enthusiastic. Apparently, ignorance is bliss.] All right, let us begin.

Sheldon Cooper, *The Big Bang Theory*

Before speaking of homogenization, hereafter is a brief introduction to the study of wave propagation in infinite periodically inhomogeneous media. The necessary mathematical tools, both analytical and geometrical, are introduced: direct and reciprocal lattices, the first Brillouin zone, virtual work theorem, Fourier transform and, most importantly, Floquet-Bloch transform. The latter allows in particular to simplify the motion equation by reducing the domain of definition of the fields. Two “educational” examples are treated and commented last. Much of the (scientific) materials presented here are reformulations of what can be found in classic literature on the subject. The books by [Graff \(2012\)](#), by [Achenbach \(2012\)](#) and by [Brillouin \(1953\)](#) were most helpful.

1 Elastodynamics of infinite homogeneous media

In this first section, we briefly recall some facts about wave propagation in infinite homogeneous media. This is an opportunity to introduce the infinite elastic body Ω and its constitutive law and motion equation.

1.1 Ω , the infinite medium

Let Ω be the d -dimensional Euclidean *point space* and $\mathcal{E}$ be the associated *vector space*. Vectors $\mathbf{r} \in \mathcal{E}$ act as translations on points $\mathbf{x} \in \Omega$. These two spaces can be identified with one another according to

$$\begin{aligned}\mathcal{E} &\longrightarrow \Omega \\ \mathbf{r} &\longmapsto \mathbf{x}_o + \mathbf{r},\end{aligned}$$

once an arbitrary origin $\mathbf{x}_o \in \Omega$ has been chosen. Note that distinguishing Ω and $\mathcal{E}$ will turn out to be occasionally of use and that, otherwise, they can be thought of as equal.

We assume that the motion of Ω is completely described by a displacement field $\mathbf{u}$ function of time $t \in \mathbb{R}$ and position $\mathbf{x} \in \Omega$, in contrast with generalized continua where other kinematical degrees of freedom come into play (see [Mindlin, 1964](#), for instance). The infinitesimal strain $\boldsymbol{\varepsilon}$ and the velocity $\mathbf{v}$ are given by

$$\boldsymbol{\varepsilon} \equiv \nabla \otimes^s \mathbf{u}, \quad \mathbf{v} \equiv \dot{\mathbf{u}},$$

where $\equiv$ stands for equality by definition, ∇ is the nabla operator, $\otimes$ designates tensor product, the superscript s implies symmetrization and a superscripted dot denotes differentiation with respect to time. The stress $\boldsymbol{\sigma}$ and the linear momentum density $\mathbf{p}$ measures are related to strains and velocities through the constitutive law of Ω :

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \quad \mathbf{p} = \rho \mathbf{v},$$

with $\mathbf{C}$ and ρ being the $\mathbf{x}$ -dependent t -independent fourth-order elastic stiffness tensor and scalar mass density, respectively, and the colon $(:)$ standing for double contraction.

The motion equation reads

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \dot{\mathbf{p}},$$

where the symbol $(\cdot)$ stands for the dot product and $\mathbf{f}$ is a field of body forces¹. Finally, we will mostly be interested in harmonic motion for various frequencies $\omega \in \mathbb{R}$. Correspondingly, time dependency is taken to be of the form $e^{i\omega t}$ (with $i^2 = -1$) and is replaced by an implicit dependency over ω while time differentiation is systematically substituted by $i\omega$ -multiplication. Consequently, the motion equation in terms of $\mathbf{u}$ becomes the Helmholtz equation

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \mathbf{u})] + \mathbf{f} = -\omega^2 \rho \mathbf{u}. \quad (1.1)$$

So far, we have assumed that Ω is infinite, made of local linearly elastic materials and that it undergoes an infinitesimal transformation². For the purposes of the present section, we make one additional assumption: $\mathbf{C}$ and ρ are $\mathbf{x}$ -independent.

¹One can account for initial displacement and velocity conditions by changing the unknown displacement field according to

$$\mathbf{u}(\mathbf{x}, t) \mapsto H(t)\mathbf{u}(\mathbf{x}, t),$$

where $H(t)$ is the Heaviside function, and correspondingly altering the body force following

$$\mathbf{f}(\mathbf{x}, t) \mapsto \mathbf{f}(\mathbf{x}, t) + \delta(t)\rho(\mathbf{x})\dot{\mathbf{u}}(\mathbf{x}, t=0) + \dot{\delta}(t)\rho(\mathbf{x})\mathbf{u}(\mathbf{x}, t=0),$$

where $\delta(t)$ is the Dirac function. Initial conditions can then be thought of as a particular loading applied at $t = 0$.

²For the derivation of the linearized theory, see the book by [Achenbach \(2012\)](#) and references therein.

1.2 Forced waves

Let $\mathcal{E}^*$ be the space dual to $\mathcal{E}$. A *wavenumber* $\mathbf{k} \in \mathcal{E}^*$ acts on a translation $\mathbf{r} \in \mathcal{E}$ to produce a phase shift $\mathbf{k} \cdot \mathbf{r}$. By identifying Ω and $\mathcal{E}$, we abuse notations and write $\mathbf{k} \cdot \mathbf{x}$ instead of $\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_o)$. Then, a scalar or tensorial field h is called a *plane wave* if it admits the form

$$h(\mathbf{x}) = \tilde{h}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$$

for some $\mathbf{x}$ -independent amplitude $\tilde{h}$ and wavenumber $\mathbf{k}$. Its *wavelength* is then $2\pi/\|\mathbf{k}\|$ where $\|\mathbf{k}\|$ is the modulus of $\mathbf{k}$. By Fourier analysis, we know that all fields h can be expanded into a sum of plane waves such that

$$h(\mathbf{x}) = \int_{\mathcal{E}^*} \tilde{h}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k},$$

with the amplitudes $\tilde{h}$ given by

$$\tilde{h}_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{\Omega} h(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}.$$

Assuming that body force is a plane wave, one can look for a solution in the form of a plane wave as well:

$$\mathbf{f}(\mathbf{x}) = \tilde{\mathbf{f}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{u}(\mathbf{x}) = \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}.$$

In other words, we apply Fourier transform to the motion equation which becomes

$$i\mathbf{k} \cdot [\mathbf{C} : (i\mathbf{k} \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}})] + \tilde{\mathbf{f}}_{\mathbf{k}} = -\omega^2 \rho \tilde{\mathbf{u}}_{\mathbf{k}}$$

thanks to the formal rule $\nabla \mapsto i\mathbf{k}$. Consequently, $\mathbf{I}$ being the second-order identity tensor, it comes that

$$\mathbf{Z}_{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{\mathbf{k}} \equiv (\mathbf{k} \cdot \mathbf{C} \cdot \mathbf{k} - \omega^2 \rho \mathbf{I}) \cdot \tilde{\mathbf{u}}_{\mathbf{k}} = \tilde{\mathbf{f}}_{\mathbf{k}},$$

where we have defined the *impedance*¹ $\mathbf{Z}$. Inverting the above equation yields

$$\tilde{\mathbf{u}}_{\mathbf{k}} = \mathbf{Z}_{\mathbf{k}}^{-1} \cdot \tilde{\mathbf{f}}_{\mathbf{k}} \equiv \mathbf{G}_{\mathbf{k}} \cdot \tilde{\mathbf{f}}_{\mathbf{k}},$$

where $\mathbf{G}$ denotes the inverse of the impedance $\mathbf{Z}$ and is the Green's tensor of Ω . Last, the displacement solution for an arbitrary body force $\mathbf{f}$ can be obtained by summation with respect to $\mathbf{k} \in \mathcal{E}^*$ according to

$$\mathbf{u}(\mathbf{x}) = \int_{\mathcal{E}^*} \mathbf{G}_{\mathbf{k}} \cdot \tilde{\mathbf{f}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} = \frac{1}{(2\pi)^d} \int_{\Omega} \mathbf{G}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}.$$

Note that both $\mathbf{G}$ and $\mathbf{Z}$ depend on ω and that the transient field $\mathbf{u}(\mathbf{x}, t)$ can similarly be obtained by integration with respect to $\omega \in \mathbb{R}$.

¹This is not standard terminology. An impedance is usually defined as the ratio of a force to a velocity (Graff, 2012) but here is defined as the ratio of a volume force density to a displacement.

1.3 Free waves

The body Ω can sustain waves, called *free waves*, even when body forces vanish. Free waves of wavenumber $\mathbf{k}$ and frequency ω are therefore *non-null* solutions of the equation

$$i\mathbf{k} \cdot [\mathbf{C} : (i\mathbf{k} \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}})] + \mathbf{0} = -\omega^2 \rho \tilde{\mathbf{u}}_{\mathbf{k}}.$$

Their existence is subject to the condition

$$\det \mathbf{Z}_{\mathbf{k}} \equiv \det (\mathbf{k} \cdot \mathbf{C} \cdot \mathbf{k} - \omega^2 \rho \mathbf{I}) = 0,$$

called *dispersion relation*, on the couple $(\mathbf{k}, \omega)$. The set of all couples $(\mathbf{k}, \omega \geq 0)$ satisfying the dispersion relation, hereafter referred to as *eigenmodes*, defines the dispersion curve¹ $\mathcal{C}$.

Given a wavenumber $\mathbf{k} \in \mathcal{E}^*$, the second-order tensor $\mathbf{k} \cdot \mathbf{C} \cdot \mathbf{k} / \rho$ is positive and symmetric and has therefore d positive eigenvalues $\omega_n^2 = \omega_n^2(\mathbf{k})$ to which are associated d linearly independent eigenvectors or free waves $\tilde{\psi}_{\mathbf{k},n}$, for $n = 1, \dots, d$. Hence, curve $\mathcal{C}$ is constituted of d *branches*. Since ω_n is taken to be positive by definition, it is easy to see that ω_n is a positively homogeneous function of degree 1 in $\mathbf{k}$, namely²

$$\forall a \in \mathbb{R}, \quad \omega_n(a\mathbf{k}) = |a| \omega_n(\mathbf{k}),$$

because the impedance $\mathbf{Z}$, involved in the dispersion relation, itself is a homogeneous polynomial of degree 2 in $\mathbf{k}$ and ω . Consequently, the *phase velocity* $\omega_n(\mathbf{k}) / \|\mathbf{k}\|$ may depend on the selected branch n and on the direction of $\mathbf{k}$ but not on its amplitude $\|\mathbf{k}\|$. In addition, for $\mathbf{k} = \mathbf{0}$ all branches pass by $\omega = 0$.

The infinite homogeneous *Cauchy continuum*³ has therefore two characteristics or two limitations, depending on the point of view.

1. All waves of a given branch propagating in a given direction travel at the same velocity. The medium Ω is then said to be *non-dispersive*. Intuitively, dispersion cannot arise when the medium does not provide any characteristic internal length in which case the medium is the same at all scales and shortening wavelengths by a factor a simply amounts to increasing the frequency by the same factor.
2. Other than rigid body translations, there are no uniform oscillation modes. All branches pass by $(\mathbf{0}, 0)$ and are said to be *acoustic branches* in contrast to *optical branches* which yield a finite frequency for a null wavenumber.

Of course, dispersion phenomena and optical branches can still occur for Cauchy media for instance either by allowing for an inhomogeneous constitutive behavior or by

¹In (Graff, 2012), this curve is called the “frequency spectrum”, while the term “dispersion curve” is reserved to the plot of phase velocity against wavenumber.

² $|a| \equiv \max(a, -a)$.

³I.e., a continuum with kinematics fully described by one displacement field and whose constitutive behavior is the previously introduced classical “first order” one.

considering geometries with finite cross sections. In the latter case, we speak of wave propagation in waveguides¹ and one can refer to *exact* rod, beam or plate theories for examples (Achenbach, 2012; Graff, 2012).

Now, on a different note, by symmetry, we have²

$$\omega_n^2 \rho \tilde{\psi}_{\mathbf{k},m} \cdot \tilde{\psi}_{\mathbf{k},n} = \tilde{\psi}_{\mathbf{k},m} \cdot (\mathbf{k} \cdot \mathbf{C} \cdot \mathbf{k}) \cdot \tilde{\psi}_{\mathbf{k},n} = \omega_m^2 \rho \tilde{\psi}_{\mathbf{k},n} \cdot \tilde{\psi}_{\mathbf{k},m},$$

which entails

$$(\omega_n^2 - \omega_m^2) \rho \tilde{\psi}_{\mathbf{k},m} \cdot \tilde{\psi}_{\mathbf{k},n} = 0.$$

Therefore, there exists an orthonormal basis of the space of amplitudes composed of eigenvectors satisfying

$$\rho \tilde{\psi}_{\mathbf{k},n} \cdot \tilde{\psi}_{\mathbf{k},m} = \delta_{nm}$$

where δ_{nm} is the Kronecker delta. By carrying the corresponding diagonalization of $\mathbf{Z}$, we obtain the expression

$$\mathbf{Z}_{\mathbf{k}} = \sum_{n=1}^d (\omega_n^2 - \omega^2) \rho^2 \tilde{\psi}_{\mathbf{k},n} \otimes \tilde{\psi}_{\mathbf{k},n}$$

valid for *all* couples $(\mathbf{k}, \omega)$. We finally deduce the following representation of the Green's tensor

$$\mathbf{G}_{\mathbf{k}} = \sum_{n=1}^d \frac{1}{\omega_n^2 - \omega^2} \tilde{\psi}_{\mathbf{k},n} \otimes \tilde{\psi}_{\mathbf{k},n}.$$

The above representations show that the distinction between the study of forced and free motions is, to some extent, artificial. We conclude however with the following remarks:

1. the dispersion curve on its own holds information on propagation velocities in all directions and related potential anisotropy or dispersion phenomena,
2. but it needs to be completed by attributing a *polarization*, or an eigenvector, to each eigenmode.
3. Finally, only $\mathbf{C}/\rho$ can be characterized this way and some genuine forced solutions need to be obtained in order to access $\mathbf{C}$ and ρ separately.

For more details on dispersion curves, wave velocities and the Green's function in the isotropic case, we refer to the textbooks by Achenbach (2012) and Graff (2012). Anisotropic materials seem to be much harder to deal with. A good starting point might be the reference (Burridge, 1996).

¹It seems that “waveguide” is a “clever” word meaning “cylinder”.

²This is basically a direct consequence of the virtual work theorem.

2 Elastodynamics of infinite periodic media

Next, periodically inhomogeneous media are introduced. Some geometrical and analytical elements useful to the study of periodic microstructures are recalled. Most importantly, Floquet-Bloch transform and theorem are exploited to significantly simplify the elastodynamic problem at hand.

2.1 Periodicity

We say that Ω “is periodically inhomogeneous”, “has a periodic microstructure”, or simply “is periodic” when the elastic stiffness tensor $\mathbf{C}$ and the scalar mass density ρ are invariant with respect to a family $\{\mathbf{b}_j\}_{j=1\dots d} \subset \mathcal{E}$ of d linearly independent translations which we assume henceforth. It follows immediately that $\mathbf{C}$ and ρ are invariant with respect to the set $\mathcal{R} \subset \mathcal{E}$ called a *lattice* and defined by

$$\mathcal{R} = \sum_{j=1}^d \mathbb{Z} \mathbf{b}_j$$

as the set of all integer combinations of the $\mathbf{b}_j$. More generally, we say that a scalar or tensor field h is $\mathcal{R}$ -periodic whenever h satisfies the invariance property

$$\forall \mathbf{x} \in \Omega, \quad \forall \mathbf{r} \in \mathcal{R}, \quad h(\mathbf{x} + \mathbf{r}) = h(\mathbf{x}).$$

It is clear that such functions, $\mathbf{C}$ and ρ in particular, are characterized by the values they take over the *unit cell*

$$T = \mathbf{x}_o + \sum_{j=1}^d \left[-\frac{1}{2}, \frac{1}{2} \right] \mathbf{b}_j \subset \Omega,$$

of center $\mathbf{x}_o$. An example in 2D ($d = 2$) of $\{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{R}$ and T is given in figure 1.1. Note that given a notion of periodicity defined by a lattice $\mathcal{R}$, many choices for the $\mathbf{b}_j$, for $\mathbf{x}_o$ and for T are possible¹.

Let $\boldsymbol{\xi} \in \mathcal{E}^*$ be a wavenumber, then the plane wave $e^{i\boldsymbol{\xi} \cdot \mathbf{x}}$ is $\mathcal{R}$ -periodic if and only if

$$\forall \mathbf{r} \in \mathcal{R}, \quad \boldsymbol{\xi} \cdot \mathbf{r} \in 2\pi\mathbb{Z}.$$

The set of all such wavenumbers is called the *reciprocal lattice* $\mathcal{R}^*$. It can be constructed as

$$\mathcal{R}^* = \sum_{j=1}^d \mathbb{Z} \mathbf{b}_j^*$$

where the wavenumbers $\mathbf{b}_j^*$ satisfy

$$\mathbf{b}_i^* \cdot \mathbf{b}_j = 2\pi \delta_{ij}.$$

¹In other words, $\mathcal{R}$ is intrinsic, T is conventional.

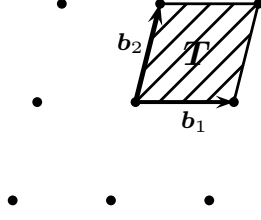


Figure 1.1. A 2D lattice. Each point represents a vector $\mathbf{r}$ of the lattice $\mathcal{R}$. Many choices for $\mathbf{b}_1$, $\mathbf{b}_2$ and T are possible.

The reciprocal lattice $\mathcal{R}^*$ plays the role of the support of all $\mathcal{R}$ -periodic functions. Precisely, if h is $\mathcal{R}$ -periodic then its Fourier transform $\tilde{h}$ is zero everywhere except over $\mathcal{R}^*$. The plane-wave expansion reads then

$$h(\mathbf{x}) = \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \tilde{h}_{\boldsymbol{\xi}} e^{i\boldsymbol{\xi} \cdot \mathbf{x}},$$

with^{1,2}

$$\tilde{h}_{\boldsymbol{\xi}} = \frac{1}{|T|} \int_T h(\mathbf{x}) e^{-i\boldsymbol{\xi} \cdot \mathbf{x}} d\mathbf{x},$$

where $|T|$ is the volume of T .

For later purposes, define the first Brillouin zone T^* as the set of wavenumbers closer to the null wavenumber than to any other wavenumber of $\mathcal{R}^*$, i.e.,

$$T^* = \{\mathbf{k} \in \mathcal{E}^* \mid \forall \boldsymbol{\xi} \in \mathcal{R}^* - \{\mathbf{0}\}, \quad \|\mathbf{k}\| < \|\mathbf{k} - \boldsymbol{\xi}\| \} \subset \mathcal{E}^*.$$

This zone helps segregate short and long wavelengths. As a matter of fact, wavenumbers $\boldsymbol{\xi}$ of $\mathcal{R}^* - \{\mathbf{0}\}$ correspond to $\mathcal{R}$ -periodic plane waves oscillating at least as fast as $\mathbf{C}$ and ρ . By proximity, all $\mathbf{k}$ close to such $\boldsymbol{\xi}$ define “rapidly” oscillating fields and all $\mathbf{k}$ close to $\mathbf{0}$, i.e., $\mathbf{k} \in T^*$, define “slowly” oscillating fields. In addition, note that if a wavenumber $\mathbf{k}$ does belong to T^* then $\mathbf{k} + \boldsymbol{\xi}$ does not, whatever is $\boldsymbol{\xi} \in \mathcal{R}^* - \{\mathbf{0}\}$. In other words, T^* is isomorphic to the quotient $\mathcal{E}^*/\mathcal{R}^*$ implying that the partition $\mathcal{E}^* = T^* + \mathcal{R}^*$ holds³.

In 1D, if $\mathcal{R} = |T|\mathbb{Z}$ then $\mathcal{R}^* = (2\pi/|T|)\mathbb{Z}$ and $T^* =]-\pi/|T|, \pi/|T|[$. In 2D, examples of $\{\mathbf{b}_1^*, \mathbf{b}_2^*\}$, $\mathcal{R}^*$ and T^* are depicted in figure 1.2.

¹All choices of T are equivalent by $\mathcal{R}$ -periodicity.

²Using the same notation for Fourier transform and series is consistent as far as we are concerned.

³In fact, for this to be rigorously true, one has to add to T^* “half” of its boundary, which we assume done henceforth.

¹Vectors $\mathbf{b}_i$ and $\mathbf{b}_i^*$ have different dimensions (respectively, a length and the inverse of a length) and their magnitudes, therefore, cannot be compared.

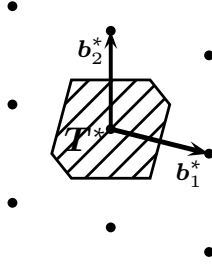


Figure 1.2. The reciprocal lattice¹ $\mathcal{R}^*$ of the direct lattice $\mathcal{R}$ depicted in figure 1.1 and its first Brillouin zone T^* . Note that T^* is uniquely defined and is independent of the choice of T .

2.2 Fourier Transform

Now that $\mathbf{C}$ and ρ are not uniform anymore, applying Fourier transform to the motion equation (1.1) is poorly motivated. As a matter of fact, for general stiffnesses and mass densities, both writings of the motion equation, as a differential one over Ω or as an integral one over $\mathcal{E}^*$, are equally complicated. Periodicity, we hope, will imply some simplification.

In any case, by the convolution theorem, the Fourier transform of (1.1) reads²

$$\forall \mathbf{k} \in \mathcal{E}^*, \quad i\mathbf{k} \cdot \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \mathbf{C}_{\boldsymbol{\xi}} : [(\mathbf{k} - \mathbf{i}\boldsymbol{\xi}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}-\boldsymbol{\xi}}] + \tilde{\mathbf{f}}_{\mathbf{k}} = -\omega^2 \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \rho_{\boldsymbol{\xi}} \tilde{\mathbf{u}}_{\mathbf{k}-\boldsymbol{\xi}},$$

where each equation is indexed by $\mathbf{k}$. Then, it is easy to see that the *sub-system* of equations indexed by $\mathbf{k} + \boldsymbol{\xi}$, for $\boldsymbol{\xi} \in \mathcal{R}^*$, is uncoupled from all other equations indexed by $\mathbf{k}'$ whenever $\mathbf{k} - \mathbf{k}' \notin \mathcal{R}^*$. Having $\mathcal{E}^* = T^* + \mathcal{R}^*$ as mentioned earlier, and upon the corresponding change of variable $\mathbf{k} \rightarrow \mathbf{k} + \boldsymbol{\zeta}$, we re-write the above motion equation as

$$\begin{aligned} \forall \mathbf{k} \in T^*, \\ \forall \boldsymbol{\zeta} \in \mathcal{R}^*, \quad (i\mathbf{k} + \mathbf{i}\boldsymbol{\zeta}) \cdot \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \mathbf{C}_{\boldsymbol{\xi}} : [(\mathbf{k} + \mathbf{i}\boldsymbol{\zeta} - \mathbf{i}\boldsymbol{\xi}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\zeta}-\boldsymbol{\xi}}] + \tilde{\mathbf{f}}_{\mathbf{k}+\boldsymbol{\zeta}} = -\omega^2 \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \rho_{\boldsymbol{\xi}} \tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\zeta}-\boldsymbol{\xi}}, \end{aligned} \quad (1.2)$$

where now the sub-system indexed by $\mathbf{k} \in T^*$ is *uncoupled* from all others and involves only and exclusively the components $\{\tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\xi}}\}_{\boldsymbol{\xi} \in \mathcal{R}^*}$ of the displacement field.

2.3 Floquet-Bloch transform

The above decomposition of the motion equation into uncoupled sub-systems motivates a similar decomposition of the displacement field into uncoupled components according

²The fields $\{\mathbf{u}, \mathbf{f}, \mathbf{v}, \mathbf{p}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}\}$ are “tilded” when written in Fourier domain. Other quantities, such as $\mathbf{G}$, $\mathbf{Z}$, $\mathbf{C}$ and ρ are given the same name in both Fourier and real domains. Context shall help clear any ambiguities.

to the *Floquet-Bloch-wave expansion*

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \int_{\mathcal{E}^*} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} = \int_{T^* + \mathcal{R}^*} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \\ &= \int_{T^*} \sum_{\xi \in \mathcal{R}^*} \tilde{\mathbf{u}}_{\mathbf{k}+\xi} e^{i\xi \cdot \mathbf{x}} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \equiv \int_{T^*} \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \end{aligned}$$

where we have introduced the *Floquet-Bloch transform*¹ of the displacement field:

$$\forall \mathbf{k} \in T^*, \quad \forall \mathbf{x} \in \Omega, \quad \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}) = \sum_{\xi \in \mathcal{R}^*} \tilde{\mathbf{u}}_{\mathbf{k}+\xi} e^{i\xi \cdot \mathbf{x}}.$$

Note that a Floquet-Bloch (FB) component, or amplitude, $\tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})$ is $\mathcal{R}$ -periodic with respect to $\mathbf{x}$ due to its definition as a Fourier series.

Three preemptive remarks are in order.

1. It is always possible to expand thusly any field h whether it is a solution or not to a motion equation. The motion equation only motivated the use of this particular type of expansions.
2. This expansion has two parameters. The first one is the lattice $\mathcal{R}$ which is chosen to be the periodicity lattice of $\mathbf{C}$ and ρ for convenience. The second one is T^* which can be replaced by any other domain isomorphic to $\mathcal{E}^*/\mathcal{R}^*$. We make use of T^* , however, as it generally has nice symmetry properties and, most importantly, it is the support of slowly oscillating fields.
3. Regarding notations: $\tilde{h}_{\mathbf{k}}(\mathbf{x})$ designates a Fourier component of h whenever it is $\mathbf{x}$ -independent and an FB component otherwise.

In order to apply FB transform to the kinematical, constitutive and motion equations, one needs to study its behavior with respect to two fundamental operations.

Differentiation.

$$\mathbf{b} = \nabla a \implies \tilde{\mathbf{b}}_{\mathbf{k}} = (i\mathbf{k} + \nabla) \tilde{a}_{\mathbf{k}}.$$

Indeed,

$$\begin{aligned} \tilde{\mathbf{b}}_{\mathbf{k}}(\mathbf{x}) &= \sum_{\xi \in \mathcal{R}^*} (i\mathbf{k} + i\xi) a_{\mathbf{k}+\xi} e^{i\xi \cdot \mathbf{x}} \\ &= i\mathbf{k} \sum_{\xi \in \mathcal{R}^*} a_{\mathbf{k}+\xi} e^{i\xi \cdot \mathbf{x}} + \nabla \sum_{\xi \in \mathcal{R}^*} a_{\mathbf{k}+\xi} e^{i\xi \cdot \mathbf{x}} = (i\mathbf{k} + \nabla) \tilde{a}_{\mathbf{k}}(\mathbf{x}). \end{aligned}$$

In other words, the operator ∇ is not transformed into $i\mathbf{k}$ anymore as by Fourier transform but into $\nabla + i\mathbf{k}$.

¹Named after Gaston Floquet and Felix Bloch.

Multiplication. Let c be an $\mathcal{R}$ -periodic function. Then,

$$b = ca \implies \tilde{b}_{\mathbf{k}} = c\tilde{a}_{\mathbf{k}}.$$

As a matter of fact, we have

$$\tilde{b}_{\mathbf{k}+\boldsymbol{\xi}} = \sum_{\boldsymbol{\zeta} \in \mathcal{R}^*} \tilde{c}_{\boldsymbol{\zeta}} \tilde{a}_{\mathbf{k}+\boldsymbol{\xi}-\boldsymbol{\zeta}},$$

so that

$$\begin{aligned} \tilde{b}_{\mathbf{k}}(\mathbf{x}) &= \sum_{\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathcal{R}^*} \tilde{c}_{\boldsymbol{\zeta}} \tilde{a}_{\mathbf{k}+\boldsymbol{\xi}-\boldsymbol{\zeta}} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} = \sum_{\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathcal{R}^*} \tilde{c}_{\boldsymbol{\zeta}} e^{i\boldsymbol{\zeta} \cdot \mathbf{x}} \tilde{a}_{\mathbf{k}+\boldsymbol{\xi}-\boldsymbol{\zeta}} e^{(i\boldsymbol{\xi}-i\boldsymbol{\zeta}) \cdot \mathbf{x}} \\ &= \sum_{\boldsymbol{\zeta} \in \mathcal{R}^*} \tilde{c}_{\boldsymbol{\zeta}} e^{i\boldsymbol{\zeta} \cdot \mathbf{x}} \sum_{\boldsymbol{\xi} \in \mathcal{R}^*} \tilde{a}_{\mathbf{k}+\boldsymbol{\xi}} e^{i\boldsymbol{\xi} \cdot \mathbf{x}} = c(\mathbf{x}) \tilde{a}_{\mathbf{k}}(\mathbf{x}), \end{aligned}$$

where the change of variable $\boldsymbol{\xi} \mapsto \boldsymbol{\xi} + \boldsymbol{\zeta}$ has been exploited.

In conclusion, using the above first property of the FB transform, we can re-write the kinematical relations and the motion equation in terms of the FB components as, respectively,

$$\tilde{\boldsymbol{\varepsilon}}_{\mathbf{k}} = (\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}}, \quad \tilde{\mathbf{v}}_{\mathbf{k}} = i\omega \tilde{\mathbf{u}}_{\mathbf{k}}, \quad (1.3)$$

and

$$(\nabla + i\mathbf{k}) \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}} = i\omega \tilde{\mathbf{p}}_{\mathbf{k}}, \quad (1.4)$$

whereas the second property permits to transform the constitutive relations into

$$\tilde{\boldsymbol{\sigma}}_{\mathbf{k}} = \mathbf{C} : \tilde{\boldsymbol{\varepsilon}}_{\mathbf{k}}, \quad \tilde{\mathbf{p}}_{\mathbf{k}} = \rho \tilde{\mathbf{v}}_{\mathbf{k}}.$$

2.4 The central equation

In terms of $\tilde{\mathbf{u}}_{\mathbf{k}}$, the motion equation reads

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})] \} + \tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}), \quad (1.5)$$

which is equivalent to the previously derived equation (1.2) in Fourier domain. All fields involved above being $\mathcal{R}$ -periodic implies that it is enough to solve this equation over a unit cell T under periodic boundary conditions¹.

For a homogeneous medium, each Fourier component of the displacement field satisfies an algebraic equation independently of all other Fourier components. For a periodic medium, each FB component satisfy a differential equation independently of all other FB components. The benefit of using FB transform in the latter case resides in the reduction of the domain of the unknowns from Ω to T by periodicity.

¹There seems to be an implicit agreement on the meaning of the term “periodic boundary conditions” in the literature. It is really the continuity of displacements and tractions which matters the most.

In both homogeneous and periodic settings, one needs to carry an integration over wavenumbers to recover the total displacement field.

More on FB transform can be found in the lecture notes by [Muthukumar \(2014\)](#). As for the FB theorem properly speaking, it will be presented in a following subsection on free waves. Meanwhile, we present the virtual work theorem and formal solutions to the above central equation in forced regime using Green's tensor.

2.5 Virtual work theorem

Whenever a and $\mathbf{b}$ are regular enough¹, the divergence theorem dictates that

$$\int_T (\nabla \cdot \mathbf{b}) a \, dV = \int_{\partial T} (\mathbf{b} \cdot \mathbf{n}) a \, dS - \int_T \mathbf{b} \cdot (\nabla a) \, dV$$

where $\mathbf{n}$ is the unitary outward vector normal to the boundary ∂T of T and where dV and dS designate volume and surface measures, respectively. If, in addition, a and $\mathbf{b}$ are $\mathcal{R}$ -periodic, and a and $\mathbf{b} \cdot \mathbf{n}$ are continuous across ∂T , then the boundary term vanishes and we have the algebraic identity

$$\int_T (\nabla \cdot \mathbf{b}) a = - \int_T \mathbf{b} \cdot (\nabla a).$$

It goes without saying that the same holds for tensors of all orders.

Now let $(\boldsymbol{\varepsilon}, \mathbf{v})$ be a kinematically admissible strain-velocity couple of fields deriving from one displacement field $\mathbf{u}$ and $(\boldsymbol{\sigma}', \mathbf{p}')$ be a stress-momentum couple of fields dynamically equilibrated by a body force field $\mathbf{f}'$. Their respective FB components are mutually related through (1.3) and (1.4). Hence,

$$\begin{aligned} \int_T \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* &= \int_T [-(\nabla + i\mathbf{k}) \cdot \tilde{\boldsymbol{\sigma}}'_k + i\omega \tilde{\mathbf{p}}'_k] \cdot \tilde{\mathbf{u}}_k^* \\ &= \int_T \tilde{\boldsymbol{\sigma}}'_k : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_k]^* - \tilde{\mathbf{p}}'_k \cdot (i\omega \tilde{\mathbf{u}}_k)^* = \int_T \tilde{\boldsymbol{\sigma}}'_k : \tilde{\boldsymbol{\varepsilon}}_k^* - \tilde{\mathbf{p}}'_k \cdot \tilde{\mathbf{v}}_k^*, \end{aligned}$$

where a superscripted $*$ denotes complex conjugation for fields. By summing with respect to $\mathbf{k}$ over T^* , we obtain

$$\int_\Omega \mathbf{f}' \cdot \mathbf{u}^* = \int_\Omega \boldsymbol{\sigma}' : \boldsymbol{\varepsilon}^* - \mathbf{p}' \cdot \mathbf{v}^*.$$

The last step needs perhaps some explanation. It is not surprising that FB transform inherits from Fourier transform a Plancherel equality stated as

$$\int_\Omega \mathbf{f}' \cdot \mathbf{u}^* = \frac{(2\pi)^d}{|T|} \int_{T^*} \int_T \tilde{\mathbf{f}}'_k(\mathbf{x}) \cdot \tilde{\mathbf{u}}_k^*(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{k},$$

¹Assuming piecewise regularity, the continuity of a and $\mathbf{b} \cdot \mathbf{n}$ “across pieces” is the important property for our purposes.

valid for all couples $(\mathbf{u}, \mathbf{f}')$. Indeed,

$$\begin{aligned} \int_{\Omega} \mathbf{f}' \cdot \mathbf{u}^* &= (2\pi)^d \int_{\mathcal{E}^*} \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* d\mathbf{k}, & (\text{Plancherel's, Fourier transform}) \\ &= (2\pi)^d \int_{T^*} \sum_{\xi \in \mathcal{R}^*} \tilde{\mathbf{f}}'_{k+\xi} \cdot \tilde{\mathbf{u}}_{k+\xi}^* d\mathbf{k}, & (\text{by } \mathcal{E}^* = T^* + \mathcal{R}^*) \\ &= \frac{(2\pi)^d}{|T|} \int_{T^*} \int_T \tilde{\mathbf{f}}'_k(\mathbf{x}) \cdot \tilde{\mathbf{u}}_k^*(\mathbf{x}) d\mathbf{x} d\mathbf{k}. & (\text{Parseval's, Fourier series}) \end{aligned}$$

Again, same holds for tensors of all orders.

Therefore, we have given the virtual work theorem two forms.

1. The first one,

$$\int_{\Omega} \mathbf{f}' \cdot \mathbf{u}^* = \int_{\Omega} \boldsymbol{\sigma}' : \boldsymbol{\varepsilon}^* - \mathbf{p}' \cdot \mathbf{v}^*,$$

is fully written in the real domain. It only concerns the whole body Ω and does not generally hold for sub-domains, nor does it hold for unit cells.

2. The second one,

$$\forall \mathbf{k} \in T^*, \quad \int_T \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* = \int_T \tilde{\boldsymbol{\sigma}}'_k : \tilde{\boldsymbol{\varepsilon}}_k^* - \tilde{\mathbf{p}}'_k \cdot \tilde{\mathbf{v}}_k^*,$$

thanks to Parseval's identity, is equivalent to the first one except that it is written in terms of FB components. Therein, even though the integration domain is a unit cell, one should recall that the definition of a FB component involves contributions from all of the medium Ω . In other words, this second form is *not* a restriction of the first one to a unit cell T , *but* an expansion based on FB waves¹.

Finally, when $(\mathbf{u}, \mathbf{f})$ and $(\mathbf{u}', \mathbf{f}')$ are two solutions to the motion equation *for the same frequency* ω , $\mathbf{C}$ and ρ being symmetric and real valued, we have

$$\boldsymbol{\sigma}' : \boldsymbol{\varepsilon}^* = \boldsymbol{\sigma}^* : \boldsymbol{\varepsilon}', \quad \mathbf{p}' \cdot \mathbf{v}^* = \mathbf{p}^* \cdot \mathbf{v}',$$

which immediately implies the reciprocity law²

$$\int_{\Omega} \mathbf{f}' \cdot \mathbf{u}^* = \int_{\Omega} \mathbf{f}^* \cdot \mathbf{u}',$$

or, using the FB components,

$$\forall \mathbf{k} \in T^*, \quad \int_T \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* = \int_T \tilde{\mathbf{f}}_k^* \cdot \tilde{\mathbf{u}}'_k.$$

¹Taking the risk of over-insisting: using Fourier transform, we know that

$$\forall \mathbf{k} \in \mathcal{E}^*, \quad \tilde{\mathbf{f}}_k \cdot \tilde{\mathbf{u}}_k^* = \tilde{\boldsymbol{\sigma}}_k : \tilde{\boldsymbol{\varepsilon}}_k^* - \tilde{\mathbf{p}}_k \cdot \tilde{\mathbf{v}}_k^*.$$

However, $\mathbf{f} \cdot \mathbf{u}^* = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^* - \mathbf{p} \cdot \mathbf{v}^*$ does not hold pointwisely. Furthermore, it does not hold when integrated over sub-domains of Ω .

²Also known as “Betti reciprocal theorem”, also known as “dynamic reciprocal identity”, also known as “Betti-Rayleigh theorem”, also known as “Maxwell-Betti theorem”, ...

2.6 Forced waves

We denote by $\mathbf{g}_k(\cdot, \mathbf{y})$ the second-order Green's tensor solution, under periodic boundary conditions imposed on T , to the equation

$$(\nabla + i\mathbf{k}) \cdot \{\mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \mathbf{g}_k(\cdot, \mathbf{y})]\} + |T| \delta_{\mathbf{y}} \mathbf{I} = -\omega^2 \rho \mathbf{g}_k(\cdot, \mathbf{y}) \quad (1.6)$$

where $\mathbf{y}$ is a parameter and $\delta_{\mathbf{y}}$ is the Dirac function centered at $\mathbf{y}$. The component $[\mathbf{g}_k]_{ij}(\mathbf{x}, \mathbf{y})$ is then interpreted as the displacement observed at position $\mathbf{x}$ in the direction i when a concentrated load, of intensity $|T|$, is applied at $\mathbf{y}$ in the direction j .

Letting $\tilde{\mathbf{u}}_k$ be the solution to equation (1.5), the reciprocity law dictates

$$\int_T |T| \delta_{\mathbf{y}}(\mathbf{x}) \tilde{\mathbf{u}}_k^*(\mathbf{x}) \, d\mathbf{x} = \int_T \tilde{\mathbf{f}}_k^*(\mathbf{x}) \cdot \mathbf{g}_k(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}$$

entailing

$$\tilde{\mathbf{u}}_k(\mathbf{y}) = \frac{1}{|T|} \int_T \tilde{\mathbf{f}}_k(\mathbf{x}) \cdot \mathbf{g}_k^*(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}.$$

In particular, substituting for $\tilde{\mathbf{f}}_k$ a concentrated load, of intensity $|T|$, applied at $\mathbf{y}'$ in the direction j gives rise to the following symmetry property of the Green's tensor:

$$[\mathbf{g}_k]_{ij}(\mathbf{y}, \mathbf{y}') = [\mathbf{g}_k]_{ji}^*(\mathbf{y}', \mathbf{y}).$$

Hence,

$$\tilde{\mathbf{u}}_k(\mathbf{y}) = \frac{1}{|T|} \int_T \mathbf{g}_k(\mathbf{y}, \mathbf{x}) \cdot \tilde{\mathbf{f}}_k(\mathbf{x}) \, d\mathbf{x},$$

and

$$\mathbf{u}(\mathbf{y}) = \frac{1}{|T|} \int_{T^*} \int_T \mathbf{g}_k(\mathbf{y}, \mathbf{x}) \cdot \tilde{\mathbf{f}}_k(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{y}} \, d\mathbf{x} \, d\mathbf{k}.$$

Recall that $\mathbf{g}$ depends implicitly on the frequency ω . Finally, unlike for homogeneous media, the above localization equation cannot be written as a convolution one, the inhomogeneous medium lacking translation invariance.

2.7 Free waves: Floquet-Bloch theorem

A free wave $\mathbf{u}$, when it exists, is a non-null solution to the motion equation

$$\nabla \cdot \mathbf{C} : (\nabla \otimes^s \mathbf{u}) = -\omega^2 \rho \mathbf{u}. \quad (1.7)$$

We then say that ω is an eigenfrequency. Just like before, periodicity of $\mathbf{C}$ and ρ does not imply that of $\mathbf{u}$. Therefore, in order to reduce the domain of resolution of the above differential equation, we apply FB transform and obtain the central equation particularized to $\mathbf{f} = \mathbf{0}$, namely¹

$$(\nabla + i\mathbf{k}) \cdot \mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_k] = -\omega^2 \rho \tilde{\mathbf{u}}_k. \quad (1.8)$$

¹This is the so-called “shifted cell problem” in mathematical literature.

A free wave being non-null, the consequences are such that: ω is an eigenfrequency if and only if there exist a wavenumber $\mathbf{k} \in T^*$ and a *non-null* $\mathcal{R}$ -periodic field $\tilde{\mathbf{u}}_{\mathbf{k}}$ satisfying (1.8). The free wave associated with the eigenmode $(\mathbf{k}, \omega)$ takes then the form

$$\mathbf{u}_{\mathbf{k}}(\mathbf{x}) = \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The above theorem is originally due to Floquet (1883) who proved it for ordinary differential equations of arbitrary orders with periodic coefficients. He demonstrated the existence of one particular “fundamental system of solutions” composed of “periodic functions of the second kind” satisfying $h(\cdot + r) = \lambda h(\cdot)$ for all (1D) lattice translations r and where λ is named “multiplier” of the solution h . The presented proof is somehow direct and hence a bit lengthy. Another proof was presented by Bloch (1928) while trying to describe the eigenstates of an electron in a crystal, i.e., a periodic potential. The relevant equation is now the Schrödinger equation. Once the group of lattice translations acting on fields was represented as a matrix group, the problem of finding said fundamental system became a matrix diagonalization problem. It is worth mentioning that Bloch made use of the Born-von Karman boundary conditions¹. For a rather detailed survey of proofs, one can refer to the paper by Gazalet et al. (2013). The Bloch proof can be found in virtually any book on solid state physics. The lecture notes by Quéré (1988) were particularly helpful.

Whatever the proof is, the key point resides in the fact that if $\mathbf{u}$ is a free wave then so is $\mathbf{u}(\cdot + \mathbf{r})$, for $\mathbf{r} \in \mathcal{R}$, due to the $\mathcal{R}$ -periodicity of $\mathbf{C}$ and ρ . Here, using FB transform, this fact was hidden in the *commutativity property* we have presented earlier between FB transform and multiplication by $\mathcal{R}$ -periodic functions: $\tilde{c}\tilde{a} = \tilde{c}\tilde{a}$ for all $\mathcal{R}$ -periodic c .

Back to the statement of the theorem, we now have a way of indexing the eigenfrequencies and the corresponding free waves. Letting $\mathbf{k} \in T^*$, we already know² that equation (1.8) admits a countable set³ of eigenfrequencies $\{\omega_n(\mathbf{k})\}_{n \geq 1}$ and eigenvector solutions $\{\tilde{\psi}_{\mathbf{k},n}\}_{n \geq 1}$. Then, by the FB theorem, the set of all eigenfrequencies of (1.7) can be obtained as

$$\mathcal{S} \equiv \{\omega_n(\mathbf{k}), \mathbf{k} \in T^*\}_{n \geq 1} = \bigcup_{n \geq 1} [\min_{\mathbf{k} \in T^*} \omega_n(\mathbf{k}), \max_{\mathbf{k} \in T^*} \omega_n(\mathbf{k})], \quad (1.9)$$

the corresponding free waves being the

$$\psi_{\mathbf{k},n}(\mathbf{x}) = \tilde{\psi}_{\mathbf{k},n}(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Up to multiplicity, a free wave of the above fundamental system of solutions is uniquely designated by two indices, $\mathbf{k} \in T^*$ and $n \in \mathbb{N} - \{0\}$.

¹Unfortunately, I do not speak German. The equations and a dictionary are enough however to manage through selected parts of the original paper.

²by spectral theory, or vibration theory.

³For discrete media, this set is finite and has $d \times N$ elements where N is the number of particles per unit cell.

1. The first one relates to the eigenvalue of the free wave seen as an eigenvector of all of the translation operators

$$t_{\mathbf{r}} : h(\cdot) \mapsto h(\cdot + \mathbf{r}),$$

for $\mathbf{r} \in \mathcal{R}$. Indeed,

$$\psi_{\mathbf{k},n}(\cdot + \mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \psi_{\mathbf{k},n}(\cdot).$$

However, the set of eigenvalues $\{e^{i\mathbf{k} \cdot \mathbf{r}}\}_{\mathbf{r} \in \mathcal{R}}$ can be characterized with the d values $\{e^{i\mathbf{k} \cdot \mathbf{b}_j}\}_{j=1\dots d}$ eigenvalues of the operators $\{t_{\mathbf{b}_j}\}_{j=1\dots d}$. Moreover, these d values only determine $\mathbf{k}$ modulo $\mathcal{R}^*$ and uniquely over T^* and similar domains isomorphic to $\mathcal{E}^*/\mathcal{R}^*$. In any case, it is possible, and sometimes convenient, to extend the definition of $\psi_{\mathbf{k},n}$ and of $\omega_{\mathbf{k},n}$ as functions of $\mathbf{k}$ from T^* to $\mathcal{E}^*$ by $\mathcal{R}^*$ -periodicity. As for its significance, $\mathbf{k}$ informs on how the periodic amplitude $\tilde{\psi}_{\mathbf{k},n}$ propagates and on the phase shifts it undergoes across large distances.

2. The dispersion curve $\mathcal{C}$, set of all eigenmodes $\{(\mathbf{k}, \omega_n(\mathbf{k})) \in T^* \times \mathbb{R}^+\}_{n \geq 1}$, is now composed of an infinite, but countable, number of branches in general. We know that the eigenmode $(\mathbf{0}, 0)$ corresponds to exactly d free waves: the d rigid body translations. Hence, only the first d branches, $n = 1, \dots, d$, pass through $(\mathbf{0}, 0)$ and are called acoustic branches. All other branches yield a non-null finite frequency for $\mathbf{k} = \mathbf{0}$ and are called optical branches. Now, the second index relates to the eigenvalue of $\psi_{\mathbf{k},n}$ seen as an eigenvector of the motion equation, i.e., its frequency. Thus, n , the branch number, dictates the shape of $\tilde{\psi}_{\mathbf{k},n}$ as a mode of vibration of the unit cell T . For the acoustic branches, we expect $\tilde{\psi}_{\mathbf{k},n}$ to have small fluctuations and the particles of T to have an in-phase overall motion whereas for optical branches, energy gets shifted toward shorter wavelengths and out-of-phase motion will eventually take over. Illustrations of this phenomenon, and much more, can be found in the book by Brillouin (1953).

Last, the spectrum $\mathcal{S}$ has a band structure described by equation (1.9). Unlike for homogeneous media, it is unclear whether $\mathcal{S}$ is equal to $\mathbb{R}^+$. In general, it is not. The gaps, $\mathbb{R}^+ - \mathcal{S}$, are called *band gaps*, or *stop bands*, and correspond to frequencies at which no free waves can propagate. More details are given subsequently.

2.8 Green's tensor revisited

To conclude this section, we re-write the already introduced Green's tensor in terms of the eigenvectors $\tilde{\psi}_{\mathbf{k},n}$. In fact, for each $\mathbf{k}$, the set $\{\tilde{\psi}_{\mathbf{k},n}\}_{n \geq 1}$ is an orthonormal basis for $\mathcal{R}$ -periodic vector fields. We can easily check orthogonality while normality is a matter of choice. It is understood henceforth that the eigenfrequencies depend implicitly on $\mathbf{k}$.

Let $m \geq 1$ and $n \geq 1$ be two integers, then by definition

$$\begin{aligned} (\nabla + i\mathbf{k}) \cdot \mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\psi}_{\mathbf{k},n}] &= -\omega_n^2 \rho \tilde{\psi}_{\mathbf{k},n}, \\ (\nabla + i\mathbf{k}) \cdot \mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\psi}_{\mathbf{k},m}] &= -\omega_m^2 \rho \tilde{\psi}_{\mathbf{k},m}. \end{aligned}$$

The latter equation can be slightly altered to give

$$(\nabla + i\mathbf{k}) \cdot \mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\psi}_{\mathbf{k},m}] + (\omega_m^2 - \omega_n^2) \rho \tilde{\psi}_{\mathbf{k},m} = -\omega_n^2 \rho \tilde{\psi}_{\mathbf{k},m},$$

so that both eigenvectors, number m and n , now satisfy two motion equations of the same frequency. Application of the reciprocity law yields

$$(\omega_m^2 - \omega_n^2) \int_T \rho \tilde{\psi}_{\mathbf{k},m} \cdot \tilde{\psi}_{\mathbf{k},n}^* = 0.$$

Accordingly, whenever $\omega_m^2(\mathbf{k}) \neq \omega_n^2(\mathbf{k})$, the eigenvectors are automatically orthogonal with respect to the *weighted* scalar product with weight ρ . If $\omega_m^2(\mathbf{k}) = \omega_n^2(\mathbf{k})$, the eigenvectors belong to the same eigenspace and can always be chosen orthogonal. It is as good a time¹ as any to introduce the averaging operator $\langle \cdot \rangle$ which acts, *exclusively*, on $\mathcal{R}$ -periodic fields and yields their average over a unit cell, say T :

$$\langle \cdot \rangle : h \mapsto \langle h \rangle \equiv \frac{1}{|T|} \int_T h.$$

Then, if normalized properly, the $\tilde{\psi}_{\mathbf{k},n}$ satisfy

$$\forall \mathbf{k} \in T^*, \quad \forall m, n \geq 1, \quad \langle \rho \tilde{\psi}_{\mathbf{k},m} \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle = \delta_{mn},$$

and any $\mathcal{R}$ -periodic vector field $\mathbf{h}$ can be expanded into the series

$$\mathbf{h} = \sum_{n \geq 1} \langle \rho \mathbf{h} \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle \tilde{\psi}_{\mathbf{k},n}.$$

If now $\tilde{\mathbf{u}}_{\mathbf{k}}$ is a solution to the forced motion equation (1.5) with a forcing term $\tilde{\mathbf{f}}_{\mathbf{k}}$ then the same kind of manipulation combined with the reciprocity law ensures that

$$\langle [(\omega^2 - \omega_n^2) \rho \tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}] \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle = 0,$$

or

$$\langle \rho \tilde{\mathbf{u}}_{\mathbf{k}} \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle = \frac{\langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle}{\omega_n^2 - \omega^2}.$$

Therefore, the expansion of $\tilde{\mathbf{u}}_{\mathbf{k}}$ can be written in terms of the work done by $\tilde{\mathbf{f}}_{\mathbf{k}}$ in the course of the virtual fields $\tilde{\psi}_{\mathbf{k},n}$. Precisely, we have

$$\tilde{\mathbf{u}}_{\mathbf{k}} = \sum_{n \geq 1} \frac{\langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot \tilde{\psi}_{\mathbf{k},n}^* \rangle}{\omega_n^2 - \omega^2} \tilde{\psi}_{\mathbf{k},n}.$$

Since

$$\tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{y}) = \langle \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \cdot) \cdot \tilde{\mathbf{f}}_{\mathbf{k}}(\cdot) \rangle,$$

¹I have been reluctant to introduce averages. Their use is purely conventional here and has no profound meaning. I would go farther and say that their use in homogenization is also purely conventional but I would be getting ahead of myself.

we identify Green's tensor as

$$\mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) = \sum_{n \geq 1} \frac{\tilde{\psi}_{\mathbf{k},n}(\mathbf{y}) \otimes \tilde{\psi}_{\mathbf{k},n}^*(\mathbf{x})}{\omega_n^2 - \omega^2}.$$

Therein, the eigenvectors $\tilde{\psi}_{\mathbf{k},n}$ and eigenfrequencies ω_n depend on $\mathbf{k}$ but not on ω . The ω -dependency was made completely explicit.

The total displacement response to a harmonic point load centered at $\mathbf{x}$ can now be obtained by summing with respect to $\mathbf{k}$ over T^* :

$$\mathbf{g}(\mathbf{y}, \mathbf{x}) = \int_{T^*} \sum_{n \geq 1} \frac{\psi_{\mathbf{k},n}(\mathbf{y}) \otimes \psi_{\mathbf{k},n}^*(\mathbf{x})}{\omega_n^2(\mathbf{k}) - \omega^2} d\mathbf{k} = \int_{T^*} \sum_{n \geq 1} \frac{\tilde{\psi}_{\mathbf{k},n}(\mathbf{y}) \otimes \tilde{\psi}_{\mathbf{k},n}^*(\mathbf{x})}{\omega_n^2(\mathbf{k}) - \omega^2} e^{i\mathbf{k} \cdot (\mathbf{y} - \mathbf{x})} d\mathbf{k}.$$

Depending on the frequency of the harmonic load, we can examine two cases of figure.

1. When ω belongs to $\mathcal{S}$, there exist modes n and wavenumbers $\mathbf{k}$ such that $\omega_n(\mathbf{k}) = \omega$. The corresponding terms in the expression of $\mathbf{g}$ will have a zero denominator and their contributions will be dominant. A combination of the relevant free waves will then propagate to infinity while decaying slowly for higher than 1 dimensions and without decay in 1D.
2. In contrast, if ω is chosen in the band gap $\mathcal{S} - \mathbb{R}^+$, there are no real wavenumbers $\mathbf{k}$ satisfying $\omega_n(\mathbf{k}) = \omega$ and no free waves will propagate to infinity. Instead, for increasing distances $\|\mathbf{y} - \mathbf{x}\|$, the radiated field will decay *exponentially*¹ in all dimensions. The speed of the decay will then be given by the imaginary part β of the *complex* wavenumbers $\mathbf{k}$ solution to $\omega_n(\mathbf{k}) = \omega$. The inverse of a component of β in a given direction is homogeneous to a distance and estimates the *penetration depth* of the decaying wave in that particular direction before it approximately ceases to exist.

Note that the motion remains harmonic and that the described “decay”, in both cases of passing and stopping bands, happens in space over distances and not over time. No viscosity or damping of any kind are involved.

3 Two examples

Many aspects of the above, more or less, postulated phenomena (optical branches, band gaps, exponential decay) can be understood by inspecting simple examples. Now when it comes to simplicity, one can hardly compete with the work of Brillouin (1953).

1. The following first example is barely a generalization of what is referred to, therein, as “Born’s model for sodium chloride”. The studied structure has the minimal requirements to qualify as a periodic medium. The (educational) benefit is that closed form expressions for many quantities can be easily obtained.

¹In a slightly different context, a fairly simple proof based on Cauchy’s integral theorem can be found in (De Boor et al., 1993, page 82). See also the paper by Martin (2006).

2. The second example deals with a homogeneous medium whose stiffness and mass density are weakly and periodically perturbed. This is a class of standard problems widely and exhaustively investigated in solid state physics literature (see [Quéré, 1988](#), for instance). [Brillouin \(1953\)](#), among many others, studied these problems formulated for the 1, 2 and 3D *wave equation*¹. Here, we slightly generalize the study to the elastodynamics of 1D media². We will focus our attention on the dispersion curve and, in particular, on the mechanism by which band gaps and optical branches appear.

3.1 A 1D 2-periodic mass-spring lattice

For the purposes of this example, Ω is a discrete medium embedded in a 1D space, and correspondingly $\mathcal{R}$ reduces to a 1D lattice defined by $\{a\mathbf{b}, a \in \mathbb{Z}\}$ with $\mathbf{b}$ being the only basis vector. A unit cell T is formed of two different masses and two springs of distinct stiffnesses but of the same length as shown in figure 1.3. In the situation

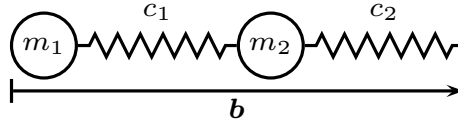


Figure 1.3. The model unit cell T .

under consideration, the space variable x is discrete. Given the characteristic length $\ell = |T|/2 = \|\mathbf{b}\|/2$ of the structure, all positions can be specified by $x = n\ell$ with $n \in \mathbb{Z}$. With no loss of generality, we take ℓ to be the unit length, and refer to position x with the integer n . Masses m and stiffnesses c are functions of $n \in \mathbb{Z}$. The periodicity of the system dictates that for every position n in $\mathbb{Z}$, $m_{n+2} = m_n$ and $c_{n+2} = c_n$. The displacement u , velocity v and momentum p are defined pointwisely only for the masses while the strain ε and stress σ make sense only for the springs. Concerning these quantities, we have

$$\varepsilon_n = u_{n+1} - u_n, \quad p_n = m_n v_n, \quad \sigma_n = c_n \varepsilon_n.$$

The motion equation reads

$$\sigma_n - \sigma_{n-1} + f_n = i\omega p_n$$

where f_n is a concentrated force applied to mass number n .

¹This is: $\nabla \cdot (\nabla \otimes \mathbf{u}) + \omega^2/V^2 \mathbf{u} = 0$ where $\mathbf{u}$ is some potential and V is a periodic function.

²To the best of my knowledge, weak periodic perturbations have never been applied to the full equations of elasticity where a coupling between many types of waves, shear and pressure for instance, can arise. Our first intention was to treat a 2D example for the mentioned reason but calculations turned out to be a bit lengthy and necessitated an undesirable digression.

In order to obtain the central equation (1.5), we postulate that u and f are both FB waves:

$$u_n = \tilde{u}_n e^{ikn}, \quad f_n = \tilde{f}_n e^{ikn},$$

where $\tilde{u}$ and $\tilde{f}$ are 2-periodic. The above kinematical and constitutive equations become

$$\tilde{\varepsilon}_n = \tilde{u}_{n+1} - \tilde{u}_n + (e^{ik} - 1)\tilde{u}_{n+1}, \quad \tilde{p}_n = m_n \tilde{v}_n, \quad \tilde{\sigma}_n = c_n \tilde{\varepsilon}_n,$$

while the motion equation transforms into

$$\tilde{\sigma}_n - \tilde{\sigma}_{n-1} + (1 - e^{-ik})\tilde{\sigma}_{n-1} + \tilde{f}_n = i\omega \tilde{p}_n.$$

Note that the modified differential operator does not include ik anymore, as in $\nabla + i\mathbf{k}$, but some discrete version $(e^{ik} - 1)$ or $(1 - e^{-ik})$ depending on whether the operator is a gradient or a divergence one. It is of course enough to consider $n = 1$ then $n = 2$ by 2-periodicity. In matrix form, this leads to

$$\begin{bmatrix} -\omega^2 m_1 + c_1 + c_2 & -c_1 e^{ik} - c_2 e^{-ik} \\ -c_2 e^{ik} - c_1 e^{-ik} & -\omega^2 m_2 + c_1 + c_2 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}.$$

The inversion of the above system is straightforward and entails

$$\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \frac{1}{D_{k,\omega}} \begin{bmatrix} -\omega^2 m_2 + c_1 + c_2 & c_1 e^{ik} + c_2 e^{-ik} \\ c_2 e^{ik} + c_1 e^{-ik} & -\omega^2 m_1 + c_1 + c_2 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix},$$

where D is the determinant of the stiffness matrix and reads

$$D_{k,\omega} = (-\omega^2 m_1 + c_1 + c_2)(-\omega^2 m_2 + c_1 + c_2) - (-c_1 e^{ik} - c_2 e^{-ik})(-c_2 e^{ik} - c_1 e^{-ik}).$$

The dispersion relation is evidently

$$D_{k,\omega} = 0.$$

After elementary manipulations, it takes the form

$$\frac{m_1 m_2}{4c_1 c_2} \omega^4 - \frac{c_1 + c_2}{2c_1 c_2} \frac{m_1 + m_2}{2} \omega^2 + \sin^2(k) = 0.$$

It can be insightful to define a dimensionless frequency ν by

$$\nu^2 = \frac{c_1 + c_2}{2c_1 c_2} \frac{m_1 + m_2}{2} \omega^2,$$

and to re-write the dispersion relation thusly

$$\delta \nu^4 / 4 - \nu^2 + \sin^2(k) = 0, \tag{1.10}$$

with

$$\delta = \delta_c \delta_m, \quad \delta_c = \frac{2c_1 c_2}{c_1 + c_2} \frac{2}{c_1 + c_2}, \quad \delta_m = \frac{2m_1 m_2}{m_1 + m_2} \frac{2}{m_1 + m_2}.$$

These dimensionless parameters can be interpreted as measuring the heterogeneity of the structure or the contrast between the phases. They are all comprised between 0 and 1 by the harmonic-arithmetic means inequality and are equal to 1 for a homogeneous structure. When one spring is infinitely stiffer than the other, δ_c tends to 0 and when one mass is infinitely lighter than the other, δ_m tends to 0 and in either cases δ vanishes. It is remarkable that up to a change in units, δ alone characterizes the dispersion curves of all similar 1D 2-periodic lattices. Some of these curves are plotted in figure 1.4. Solving the dispersion equation, we see that the spectrum of the motion equation is

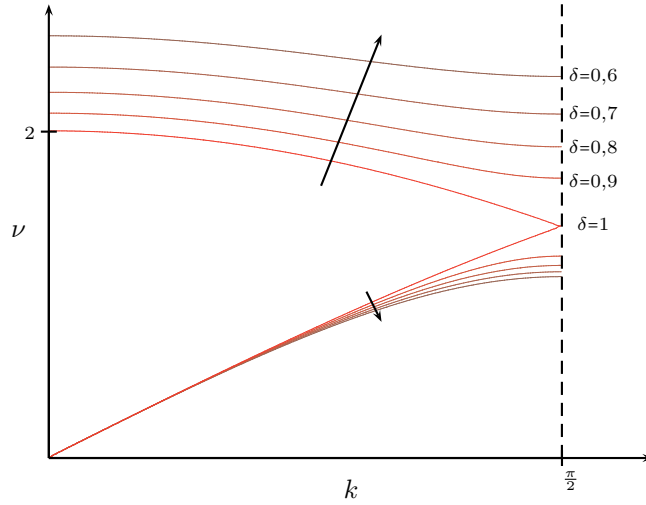


Figure 1.4. Dispersion curves family (equation (1.10)) indexed by δ (decreasing with the arrows). Since the homogeneous medium ($\delta = 1$) is treated as a 2-periodic medium, its dispersion curve is folded into the first Brillouin zone of a 2-periodic medium and has only one branch. All other curves have two branches (of the same shade of red): an acoustic one of low frequency ν , and an optical one of higher frequency ν .

composed of two passing bands, namely

$$\mathcal{S} = \left[0, \sqrt{2(1 - \sqrt{1 - \delta})/\delta} \right] \cup \left[\sqrt{2(1 + \sqrt{1 - \delta})/\delta}, 2/\sqrt{\delta} \right],$$

separated by a band gap of width

$$\nu_2(\pi/2) - \nu_1(\pi/2) = \sqrt{2(1 + \sqrt{1 - \delta})/\delta} - \sqrt{2(1 - \sqrt{1 - \delta})/\delta}.$$

The frequency $2/\sqrt{\delta}$ is the cut-off frequency and no waves can propagate beyond it. We underline the fact that the band gap “closes” for a homogeneous medium ($\delta = 1$) and is increasingly wide for high contrasts ($\delta \rightarrow 0$). Moreover, we see that, for higher contrasts, the band gap appears at lower frequencies. Finally, there is a drastic change

in the oscillation modes when comparing the in-phase motion at acoustic frequencies and the out-of-phase motion at optical ones. Unsurprisingly, for more details and illustrations, we refer to the book of Brillouin (1953, figure 15.3, for instance).

3.2 Weak periodic perturbations

This second example shall help to understand the mechanism by which the acoustic branches of a homogeneous medium break into pieces and give birth to new optical ones. In other words, using perturbation theory, we gradually transform a homogeneous medium into a periodic one and capture the moment when new degrees of freedom start manifesting themselves.

3.2.1 Setup

Let Ω be a weakly and periodically perturbed homogeneous medium in the sense that its stiffness and inertia can be written as

$$\mathbf{C} = \mathbf{C}^0 + \mathbf{C}', \quad \rho = \rho^0 + \rho',$$

where $\mathbf{C}^0$ and ρ^0 are constant and $\mathbf{C}'$ and ρ' are *small* $\mathcal{R}$ -periodic perturbations. For simplicity, we make two assumptions. The medium Ω is taken to be 1D. Tensors hence become scalars. Also, we take $\mathbf{C}'$ and ρ' to have zero average so that¹

$$C^0 = \langle C \rangle = C_0, \quad \rho^0 = \langle \rho \rangle = \rho_0.$$

Our purpose is to calculate the dispersion curve, up to the first order in the perturbation magnitude. Given $k \in \mathcal{E}^*$, we denote by u_k some free wave and by $\nu = \omega^2$ its frequency (squared). The weak perturbation assumption suggests we look for solutions in the form

$$\tilde{u}_k = \tilde{u}_k^0 + \tilde{u}_k' + \dots, \quad \nu = \nu^0 + \nu' + \dots,$$

where $\tilde{u}_k^0$ and ν^0 are the zero order terms, $\tilde{u}_k'$ and ν' are the first order corrections and higher order corrections have been omitted.

3.2.2 A discussion

Equation (1.8) remains the relevant motion equation. To the lowest order, it reads

$$(\nabla + ik)C^0 [(\nabla + ik)\tilde{u}_k^0] = -\nu^0 \rho^0 \tilde{u}_k^0.$$

The solutions are the free waves of the homogeneous medium satisfying the periodicity constraint: $\tilde{u}_k^0(x) = \tilde{u}_{k+\eta}^0 e^{i\eta x}$ where $\tilde{u}_{k+\eta}^0$ is a constant amplitude and η belongs to $\mathcal{R}^*$.

¹The superscript 0 relates to the order of magnitude whereas the subscript 0 designates a Fourier component.

However, since we have allowed for arbitrary k , k and $k + \eta$ span the same set so that taking $\eta = 0$ will be enough. Then, to the lowest order, the eigenfrequency is given by

$$\nu^0 = c^0 k^2$$

where c^0 is the speed (squared) of waves traveling in the unperturbed homogeneous medium, i.e., C^0/ρ^0 .

At the first order, the motion equation is

$$\begin{aligned} (\nabla + ik)C^0 [(\nabla + ik)\tilde{u}'_k] + (\nabla + ik)C' [(\nabla + ik)\tilde{u}^0_k] \\ = -\nu^0 \rho^0 \tilde{u}'_k - \nu^0 \rho' \tilde{u}^0_k - \nu' \rho^0 \tilde{u}^0_k. \end{aligned} \quad (1.11)$$

The above equation admits a solution if and only if the forcing term, driven by $\tilde{u}^0_k$, is orthogonal to *all* the oscillation modes of the unperturbed medium at frequency ν^0 :

$$\forall \xi \in \mathcal{R}^*, \quad \nu^0 = c^0(\xi + k)^2 \implies [(i\xi + ik)C'_\xi ik + (\nu^0 \rho'_\xi + \nu' \rho^0 \delta_0(\xi))] \cdot \tilde{u}^0_k = 0. \quad (1.12)$$

The wavenumber $\xi = 0$ always satisfies the if-clause of (1.12). The existence of other wavenumbers $\xi \neq 0$ solutions to said if-clause entails

$$k^2 = \nu^0/c^0 = (\xi + k)^2,$$

which means that k is on the boundary of a Brillouin zone¹.

We distinguish in what follows two types of points. Regular points are those k which do not fall on a Brillouin zone boundary and jump points are those which do. The latter points form the subset $(1/2)\mathcal{R}^* - \{0\}$ of $\mathcal{E}^*$.

3.2.3 Regular points

For these points, all we need is to check (1.12) for $\xi = 0$. In fact, it immediately yields

$$\nu' = 0,$$

so that the dispersion relation of the unperturbed medium holds at the first order as well:

$$\nu = c^0 k^2.$$

Finally, the first order correction to the free wave $\tilde{u}^0_k$ can be obtained from (1.11). Its Fourier series components are such, for all $\xi \in \mathcal{R}^*$,

$$\tilde{u}'_{k+\xi} = \frac{-(\xi + k)C'_\xi k + \nu^0 \rho'_\xi}{(\xi + k)C^0(\xi + k) - \nu^0 \rho^0} \tilde{u}^0_k. \quad (1.13)$$

Under these circumstances, Ω can be said to have only one degree of freedom, $\tilde{u}^0_k$. The other components of the displacement field are obtained as the result of the diffraction of the “incident wave” $\tilde{u}^0_k e^{ikx}$ by the “weak” heterogeneities.

¹Not necessarily the first one T^* , but a similar one. See (Brillouin, 1953, figure 31.5).

3.2.4 Jump points

Let $k = -\zeta/2$ for some $\zeta \in \mathcal{R}^* - \{0\}$ so that $k^2 = (k + \eta)^2$ has over $\mathcal{R}^*$ the two solutions $\eta = 0$ and $\eta = \zeta \neq 0$. The former solution injected in the existence condition entails $\nu' = 0$ just like before whereas the latter requires

$$(i\zeta + ik)C'_\zeta ik + (\nu^0 \rho'_\zeta + \nu' \rho^0 \delta_0(\zeta)) = 0,$$

or simply

$$C'_\zeta + \rho'_\zeta c^0 = 0.$$

The above condition has no particular reason to hold. When it does, a second free wave solution can be added at first order but when it does not, said solution must be added at order zero. As a matter of fact, a violation of the existence condition can be understood as a divergence problem in formula (1.13) where one of the Fourier components of the correction becomes infinitely large. One should therefore “move” said component to a lower order in the asymptotic series. Correspondingly, we change the zero order solution into

$$\tilde{u}_k^0 + \tilde{u}_{k+\zeta}^0 e^{i\zeta x},$$

where $\tilde{u}_k^0$ and $\tilde{u}_{k+\zeta}^0$ are two constant amplitudes. We underline the fact that this is possible since $\nu^0 = c^0 k^2 = c^0 (k + \zeta)^2$. We now look for ν' by writing the existence condition of the first order motion equation. We immediately have

$$\begin{aligned} [(i\zeta + ik)C'_\zeta ik + \nu^0 \rho'_\zeta] \tilde{u}_k^0 + \nu' \rho^0 \tilde{u}_{k+\zeta}^0 &= 0, \\ \nu' \rho^0 \tilde{u}_k^0 + [ikC'_{-\zeta}(i\zeta + ik) + \nu^0 \rho'_{-\zeta}] \tilde{u}_{k+\zeta}^0 &= 0, \end{aligned}$$

which amounts to no more than

$$\det \begin{bmatrix} \nu' \rho^0 & \frac{\pi^2}{\ell^2} C'_{-\zeta} + \nu^0 C'_{-\zeta} \\ \frac{\pi^2}{\ell^2} C'_\zeta + \nu^0 \rho'_\zeta & \nu' \rho^0 \end{bmatrix} = 0,$$

where ℓ is the direct lattice parameter. Now since C and ρ are real functions of x , their Fourier transforms satisfy

$$C_{-\xi} = C_\xi^*, \quad \rho_{-\xi} = \rho_\xi^*.$$

In conclusion, the first order correction to the acoustic branch at jump points is given by

$$\nu' = \pm \frac{\pi^2}{\ell^2} \left| \frac{\rho'_\zeta}{\rho^0} \right| \left| \frac{C'_\zeta}{\rho'_\zeta} + \frac{C^0}{\rho^0} \right|.$$

3.2.5 Comments

We have seen that the dispersion curve is not perturbed, at first order, at regular points and experiences jumps over the discrete set $(1/2)\mathcal{R}^* - \{0\}$. It is when k crosses jump points that the zero order displacement field needs to be enriched and that the degrees

of freedom of the inhomogeneous medium start manifesting themselves. The new component turns out to have a well-defined wavenumber $k + \zeta$ of the same magnitude as k .

It should be noted that the calculations in 1D setting are misleadingly simple, especially for large wavenumbers for which, in higher dimensions, the boundaries of Brillouin zones get increasingly closer to one another (Brillouin, 1953, figure 31.5). Nonetheless, as far as physical interpretation goes, things are better understood in a d dimensional setting. So let be given a wavenumber $\mathbf{k}$. Jump points are now those for which there exists at least one $\boldsymbol{\xi} \in \mathcal{R}^* - \{\mathbf{0}\}$ such that

$$c_I^0 \|\mathbf{k}\|^2 = \nu^0 = c_R^0 \|\mathbf{k} + \boldsymbol{\xi}\|^2, \quad (1.14)$$

where c_I^0 is the speed (squared) of the original component of the zero order displacement field and c_R^0 is that of the new component. This equality can be understood as the “time part” of some *reflexion law*. Indeed, let $\mathbf{k}_I = \mathbf{k}$ be the wavenumber of an “incident” wave of speed c_I^0 . Reflexion will occur on planes of heterogeneities and will produce a wave of wavenumber $\mathbf{k}_R$ and speed c_R^0 and of the same frequency (squared) ν^0 . Hence,

$$c_I^0 \|\mathbf{k}_I\|^2 = \nu^0 = c_R^0 \|\mathbf{k}_R\|^2.$$

For the reflected amplitude to be significant, the reflexion must be systematic in the sense that it must take place over a family of identical parallel planes of heterogeneities of common normal $\boldsymbol{\xi} \in \mathcal{R}^*$. Accordingly, the “spatial part” of the reflexion law will state that

$$\mathbf{k}_I - \mathbf{k}_R \parallel \boldsymbol{\xi},$$

or that $\mathbf{k}_I - \mathbf{k}_R = \alpha \boldsymbol{\xi}$ for some real number α . Finally, for the interference between all reflected waves to be constructive, α must be an integer. Hence, $\alpha \boldsymbol{\xi}$ is also a reciprocal lattice wavenumber. Renaming variables, we recover $\mathbf{k}_R = \mathbf{k}_I + \boldsymbol{\xi}$.

The sketched reflection formalism is a brief summary, and perhaps a light generalization to elasticity, of what is known as *Bragg’s reflexion*. Equation (1.14) would then be the generalized Bragg law. We refer as usual to the book by Brillouin (1953) for a more thorough study and insightful illustrations. Note that passing from the wave equation to full elasticity requires dealing with a coupling between many types of waves and hence the boundaries of new zones will come into play. The corresponding generalization of the first Brillouin zone would be

$$T_{IR}^* = \{\mathbf{k} \in \mathcal{E}^* \mid \forall \boldsymbol{\xi} \in \mathcal{R}^* - \{\mathbf{0}\}, \quad c_I \|\mathbf{k}\|^2 < c_R \|\mathbf{k} - \boldsymbol{\xi}\|^2\},$$

from which T^* is recovered when the incident and reflected waves have the same speed.

Finally, as for energy considerations, when a jump occurs, the dominant part of the displacement field keeps the same polarizations and wavenumbers. What changes is the energy repartition among components. Indeed, we expect the carried energy to get shifted towards larger wavenumbers, i.e., shorter wavelengths.

3.3 A remark on contrast

Twice so far we have seen that contrast plays a major role in determining the position and the width of band gaps. The second example, in particular, showed that the frequency at which the first band gap appears for low contrast media is of the order of $(\pi/\ell)\sqrt{\langle C \rangle / \langle \rho \rangle}$. Hence, strong dispersion and band-gap-related effects are only visible at similar and higher frequencies. For some soundproofing applications, a material is designed to have a band gap around an audible frequency, say 1kHz. The structure, the wall for instance, should then have a thickness comparable to $L = 5\ell$, i.e., 5 unit cells, at least so that the transmitted wave decays sufficiently. Using the stiffness and density of a concrete- or wood-like material, we obtain $L \geq 7\text{m}$. In conclusion, high contrast composites seem then to be inevitable in the design of the so-called “metamaterials” (Liu et al., 2000, for instance).

In the literature, periodically inhomogeneous materials are referred to either as metamaterials or as phononic crystals. Orders of magnitude seem to be one way of segregating them. Metamaterials are only “meta” if their weird effects are observable at low frequencies, otherwise they are simply inhomogeneous materials solicited at high frequencies. Physically speaking, the mechanism which triggers dispersion and band gaps is Bragg’s reflexion for low contrast composites as discussed earlier. For high contrast composites, the resonance of some phase freely oscillating within another is key (Liu et al., 2000, again).

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Chapter 2

Willis elastodynamic homogenization theory revisited

Do not try and bend the spoon – that’s impossible. Instead, only try to realize the truth. *There is no spoon.* Then you will see that it is not the spoon that bends, it is only yourself.

Spoon boy, *The Matrix*

The theory of elastodynamic homogenization initiated by J.R. Willis is revisited for periodically inhomogeneous media through a careful scrutiny of the main aspects of that theory in the 3D continuum context and by applying it to the thorough treatment of a simple 1D discrete periodic system. Floquet-Bloch transform appears to be central to appropriately defining and interpreting effective fields. Based on some physical arguments, three necessary conditions are derived for the transition from the microscopic description to the macroscopic description of periodic media. The parameters involved in the Willis effective constitutive relation are expressed in terms of two localization tensors and specified with the help of the corresponding Green’s function in the spirit of micromechanics. These results are illustrated and discussed for the 1D discrete periodic system considered. In particular, inspired by Brillouin’s study, the dependency of the effective constitutive parameters on the frequency is physically interpreted in terms of oscillation modes of the underlying microstructure. The contents of this chapter, as well as the present abstract, are a somehow matured version of the paper by Nassar, He and Auffray (2015).

1 Introduction

The beginnings of the elastodynamic homogenization theory of J.R. Willis can be traced back to the relevant papers he published during the first half of the 1980s (Willis, 1980a,b, 1981, 1985). The main body of this theory was, in a rather complete manner, presented more than 10 years later in a chapter of a book edited after a course

dedicated to continuum micromechanics (Willis, 1997). Recently, increasing interest in acoustic metamaterials and cloaking (see, e.g., papers by Chen and Chan 2010; Lee et al. 2012; Liu et al. 2000, 2012; Milton et al. 2006; Norris 2008; Norris and Shuvalov 2011; Simovski 2007) has, in particular, given an impetus to the development and application of the elastodynamic homogenization theory of Willis (Milton and Willis, 2007, 2010; Nemat-Nasser and Srivastava, 2011, 2013; Nemat-Nasser et al., 2011; Norris et al., 2012; Shuvalov et al., 2011; Srivastava and Nemat-Nasser, 2011; Willis, 2009, 2011, 2012).

The elastodynamic homogenization theory of Willis exhibits the following salient features: (i) in the microscopic-to-macroscopic upscaling process, no approximation hypotheses are made, so that, in this sense, the resulting theory can be considered as exact; (ii) the effects of material microscopic inhomogeneities are, after homogenization, all incorporated only in the resulting non-classical effective constitutive law, so that the macroscopic (or effective) motion equation takes the same classical form as the one at the microscopic level; (iii) for a composite formed of elastic phases whose constitutive laws are local in time and space, the effective constitutive law obtained by homogenization becomes generally nonlocal both in time and space; (iv) the effective mass density is, in general, no longer a scalar but a second-order tensor quantity¹; (v) a non-classical coupling between the effective stress tensor and the effective velocity, and another one between the effective momentum and the effective strain tensor, occur generally in the effective constitutive law; (vi) the parameters involved in the effective constitutive law are non-unique but can be rendered unique by prescribing, for example, an additional eigenstrain field. Note that the features (iii) through (vi) make that the effective constitutive law derived in the homogenized elastodynamic theory of Willis is very different from the constitutive law involved in the classical elastodynamic theory and that its explicit determination in terms of the phase properties is a quite tough task and in general necessitates using a numerical method.

The present work consists in revisiting the elastodynamic homogenization theory of Willis for periodic composites so as to reach the following threefold objective. First, it aims to derive, on the basis of some physically sound arguments, a few necessary conditions for the application of that theory to be physically meaningful. Second, it has the purpose of expressing the effective constitutive parameters of the effective constitutive law in terms of some appropriate localization tensors in the spirit of micromechanics, so that a general numerical method, such as the finite element method, can be directly used to numerically compute said parameters. Thirdly, it aims to gain physical insights into the general theory by applying it to thoroughly and analytically study a simple one-dimensional (1D) periodic discrete system. By achieving these three objectives, the present work contributes not only to getting a better understanding of but also developing the elastodynamic homogenization theory initiated by Willis.

The next sections of this chapter are structured and summarized as follows. In section 2, the elastodynamic homogenization theory of Willis is carefully reformulated

¹More accurately, the effective mass density is no longer an isotropic second-order tensor but an anisotropic one, in general.

mathematically and examined physically for periodic composites. Floquet-Bloch (FB) expansions are shown to play a central role in properly defining and interpreting the effective (or macroscopic) fields. Three necessary conditions are then proposed for the elastodynamic homogenization theory of Willis applied to a periodic composite to lead to a physically meaningful effective behavior. The first necessary condition corresponds to the requirement that the microscopic virtual work be equal to the effective (or macroscopic) virtual work, which is reminiscent of the well-known Hill-Mandel relation in micromechanics. The second necessary condition concerns wavenumbers and demands that the effective (or macroscopic) fields capture the long-wavelength parts of the relevant microscopic fields. The third necessary condition is relative to frequencies and comes from the requirement that the effective elastodynamic behavior of a composite be a good approximation of its microscopic one. In section 3, the original ensemble-average-based approach of Willis is presented and commented. Finally, in section 4, inspired by the work of Brillouin (1953), a simple 1D periodic discrete system is analytically and exhaustively studied to illustrate and discuss the main results of section 2.

2 Elements of an elastodynamic homogenization theory for periodic media

Considering the periodic medium Ω of chapter 1, we recall the kinematical relations,

$$\forall \mathbf{k} \in T^*, \quad \tilde{\boldsymbol{\varepsilon}}_{\mathbf{k}} = (\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}}, \quad \tilde{\mathbf{v}}_{\mathbf{k}} = i\omega \tilde{\mathbf{u}}_{\mathbf{k}}, \quad (2.1)$$

the motion equation,

$$\forall \mathbf{k} \in T^*, \quad (\nabla + i\mathbf{k}) \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}} = i\omega \tilde{\mathbf{p}}_{\mathbf{k}}, \quad (2.2)$$

and constitutive law,

$$\forall \mathbf{k} \in T^*, \quad \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} = \mathbf{C} : \tilde{\boldsymbol{\varepsilon}}_{\mathbf{k}}, \quad \tilde{\mathbf{p}}_{\mathbf{k}} = \rho \tilde{\mathbf{v}}_{\mathbf{k}}, \quad (2.3)$$

written in terms of FB components.

The problem to be treated in what follows is the homogenization of Ω . Precisely, that periodically heterogeneous medium will be replaced by a *homogeneous* medium over which appropriate effective fields are defined and whose elastodynamic behavior is determined in such a way that the most important elastodynamic features of the initial periodic medium are captured. To achieve this objective, we shall work with transformed fields instead of dealing directly with fields of space and time. We have seen that FB transform and the superposition principle allow to considerably simplify the motion equation and the way in which the fields depend on the space variables. Furthermore, FB transform will help introduce and interpret the effective fields in elementary terms by means of space averages. The original point of view of Willis based on *expectancies* is postponed until the next section.

2.1 Effective fields

The fields describing Ω exactly, as it is, are referred to as “microscopic” or “exact” whereas the fields describing Ω macroscopically or “effectively” are referred to as “effective”. We define¹ the effective FB component $\tilde{H}_{\mathbf{k}}$ of a microscopic FB component $\tilde{h}_{\mathbf{k}}(\mathbf{x})$ as being the $\mathbf{x}$ -independent amplitude

$$\tilde{H}_{\mathbf{k}} = \frac{1}{|T|} \int_T \tilde{h}_{\mathbf{k}}(\mathbf{y}) \, d\mathbf{y} \equiv \langle \tilde{h}_{\mathbf{k}} \rangle, \quad (2.4)$$

for all $\mathbf{k} \in T^*$. We insist on the fact that only $\mathcal{R}$ -periodic fields can be averaged without any ambiguity as all choices of T are equivalent. A *non* $\mathcal{R}$ -periodic field will have different averages over different unit cells. Furthermore, given $\mathbf{k} \in T^*$, $\tilde{H}_{\mathbf{k}}$, being a uniform amplitude, can be seen as both a *Fourier component* and a *FB component*. Accordingly, we can extend the definition of the FB transform of H with zeros outside of T^* and speak indifferently of Fourier or FB transforms. Of course, this does not hold for the microscopic field h which generally has non-null Fourier components outside of T^* .

For the moment, we argue that the effective field definition (2.4) allows to satisfy exactly the usual motion equation written as a conservation law. First, we recast the central equation (2.2) into the equivalent form

$$\nabla \cdot \tilde{\sigma}_{\mathbf{k}} + i\mathbf{k} \cdot \tilde{\sigma}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}} = i\omega \tilde{p}_{\mathbf{k}}.$$

Then, by the divergence theorem and the periodic boundary conditions, averaging over T yields

$$i\mathbf{k} \cdot \langle \tilde{\sigma}_{\mathbf{k}} \rangle + \langle \tilde{f}_{\mathbf{k}} \rangle = i\omega \langle \tilde{p}_{\mathbf{k}} \rangle.$$

Taking the inverse Fourier transform, the above equation becomes the effective motion equation

$$\nabla \cdot \Sigma + \mathbf{F} = i\omega \mathbf{P}. \quad (2.5)$$

By the same logic, the kinematical relations are kept exact and we have

$$\tilde{\mathbf{E}}_{\mathbf{k}} = i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_{\mathbf{k}}, \quad \tilde{\mathbf{V}}_{\mathbf{k}} = i\omega \tilde{\mathbf{U}}_{\mathbf{k}},$$

or, back in the real domain,

$$\mathbf{E} = \nabla \otimes^s \mathbf{U}, \quad \mathbf{V} = i\omega \mathbf{U}. \quad (2.6)$$

¹The suggested form of the effective fields is different than that of [Amirkhizi and Nemat-Nasser \(2008\)](#) and [Willis \(2011\)](#). It remains however directly inspired from the mentioned papers. We shall present our point of view first.

2.2 Necessary conditions for homogenization

Having the effective motion equation and the effective kinematical relations, all that is left to do in order to have a complete macroscopic description of Ω is to see what becomes of the constitutive behavior. However, even when said behavior is still unknown, it is possible to draw a priori a validity domain for the present approach. In fact, for the effective fields defined in (2.4) to act as physically meaningful macroscopic fields, frequencies and wavenumbers have to satisfy some conditions, hereafter called *homogenizability conditions*. Below, we derive three of them. The first one amounts to a generalized version of the Hill-Mandel relation. The second one is based on the physically sound requirement that the effective field H defined for a microscopic field h be a slowly space-varying approximation of h . The third one comes from the demand that H be a good approximation of h .

2.2.1 First condition: Hill-Mandel relation

It is desirable that the macroscopic and the microscopic descriptions of Ω be energetically equivalent in the sense that the work of microscopic internal forces be equal to that of macroscopic internal forces, namely that the equality

$$\int_{\Omega} \boldsymbol{\sigma}' : \boldsymbol{\varepsilon}^* - \mathbf{p}' \cdot \mathbf{v}^* = \int_{\Omega} \boldsymbol{\Sigma}' : \mathbf{E}^* - \mathbf{P}' \cdot \mathbf{V}^* \quad (2.7)$$

holds for *any* combination of microscopic and macroscopic fields (h, H) and (h', H') related through (2.1), (2.2), (2.5) and (2.6). We refer to such relation as *Hill-Mandel relation*. Thanks to Parseval's identity for FB transform, the above is equivalent to

$$\forall \mathbf{k} \in T^*, \quad \langle \tilde{\boldsymbol{\sigma}}'_k : \tilde{\boldsymbol{\varepsilon}}_k^* - \tilde{\mathbf{p}}'_k \cdot \tilde{\mathbf{v}}_k^* \rangle = \langle \tilde{\boldsymbol{\Sigma}}'_k : \tilde{\mathbf{E}}_k^* - \tilde{\mathbf{P}}'_k \cdot \tilde{\mathbf{V}}_k^* \rangle,$$

the averaging sign on the right-hand side being unnecessary, the FB components of the effective fields being uniform in space. Now, by the virtual work theorem, it comes that

$$\begin{aligned} \langle \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* \rangle &= \langle \tilde{\boldsymbol{\sigma}}'_k : \tilde{\boldsymbol{\varepsilon}}_k^* - \tilde{\mathbf{p}}'_k \cdot \tilde{\mathbf{v}}_k^* \rangle, \\ \langle \tilde{\mathbf{F}}'_k \cdot \tilde{\mathbf{U}}_k^* \rangle &= \langle \tilde{\boldsymbol{\Sigma}}'_k : \tilde{\mathbf{E}}_k^* - \tilde{\mathbf{P}}'_k \cdot \tilde{\mathbf{V}}_k^* \rangle. \end{aligned}$$

Hence, Hill-Mandel relation holds whenever one has

$$\forall \mathbf{k} \in T^*, \quad \frac{1}{|T|} \int_T \tilde{\mathbf{f}}'_k \cdot \tilde{\mathbf{u}}_k^* = \tilde{\mathbf{F}}'_k \cdot \tilde{\mathbf{U}}_k^*.$$

Such requirement is unbearable except when T reduces to a single point and T^* extends to $\mathcal{E}^*$, i.e., when Ω is homogeneous. In general, the foregoing relation can only hold for a particular set of microscopic body forces which we qualify as *admissible*. Under these circumstances, it is straightforward to see that the unique maximal¹ set of admissible body forces is the one defined by

$$\forall \mathbf{k} \in T^*, \quad \tilde{\mathbf{F}}_k = \langle \tilde{\mathbf{f}}_k \rangle = \tilde{\mathbf{f}}_k.$$

¹In the sense that it cannot be enriched.

Finally, recall that the choice of T^* as a basis for FB expansions is, so far, conventional and that T^* can be substituted by any other zone isomorphic to the quotient $\mathcal{E}^*/\mathcal{R}^*$. What is common to all of these choices is the property

$$\forall \mathbf{k} \in \mathcal{E}^*, \quad \forall \boldsymbol{\xi} \in \mathcal{R}^*, \quad \mathbf{k} \in \mathcal{E}^*/\mathcal{R}^* \implies \mathbf{k} + \boldsymbol{\xi} \notin \mathcal{E}^*/\mathcal{R}^*.$$

Our first homogenizability condition can therefore be stated as follows. The wavenumbers of body forces under which homogenization is carried cannot be arbitrarily rich and must constitute a subset of some zone isomorphic to $\mathcal{E}^*/\mathcal{R}^*$ in which case the Hill-Mandel relation is valid and *body forces are the same on both microscopic and macroscopic scales*.

Two remarks are in order.

1. Hill-Mandel relation being valid for a specific set of loadings is not new. In statics already, Hill-Mandel lemma is proven for specific rather than arbitrary boundary conditions.
2. Other than its energetic significance, once (2.7) is satisfied, the symmetry of the left-hand side, resulting from the symmetry of the microscopic constitutive behavior, is transmitted to the right-hand side implying, a priori, the symmetry of the to-be-found effective constitutive behavior. We then know in particular that the effective motion equation admits a variational formulation.

2.2.2 Second condition: effective vs macroscopic

Let $\mathbf{k} \in T^*$ and let $\tilde{h}_{\mathbf{k}}(\mathbf{x})$ be some microscopic FB component, $\tilde{H}_{\mathbf{k}} = \langle \tilde{h}_{\mathbf{k}} \rangle$ being the corresponding effective component. The microscopic wave has the wavenumbers $\{\mathbf{k} + \boldsymbol{\xi}\}_{\boldsymbol{\xi} \in \mathcal{R}^*}$ whereas the effective one has one wavenumber, $\mathbf{k}$. One can then think of the effective wave as a filtered wave where the microscopic component is rid of all wavenumbers, i.e., all wavelengths, but one. However, for H to be the outcome of a *long-wavelength filtering* of h , the involved wavenumbers must satisfy

$$\forall \boldsymbol{\xi} \in \mathcal{R}^* - \{\mathbf{0}\}, \quad \|\mathbf{k}\| < \|\mathbf{k} + \boldsymbol{\xi}\|,$$

which holds automatically having $\mathbf{k} \in T^*$.

If FB expansions are carried over a different zone than T^* , the effective fields can no longer be seen as long-wavelength approximations to the microscopic ones. When the first homogenizability condition requires that the set of body force wavenumbers be isomorphic to $\mathcal{E}^*/\mathcal{R}^*$, the present second homogenizability condition states that said wavenumbers must be in the first Brillouin zone T^* .

For a 1D medium, if $|T|$ is the length of the unit cell T , the second homogenizability condition reduces simply to

$$-\frac{\pi}{|T|} < k < \frac{\pi}{|T|}.$$

Therefore, the effective field wavelength $\lambda = |2\pi/k|$ must be at least twice as large as T (see figure 2.1).

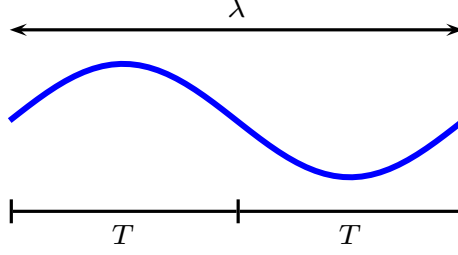


Figure 2.1. An illustration of the second homogenizability condition showing the shortest allowed wavelength.

2.2.3 Third condition: on free waves

We now assess the quality of the effective description as an approximation to the microscopic one. More precisely, we are interested in knowing which of the Fourier components $\tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\xi}}$ of a FB free wave $\tilde{\mathbf{u}}_{\mathbf{k}}$ is the dominant component in the elastic energy sense for instance. We assume that such dominant component exists and that it has a wavenumber $\mathbf{k}' = \mathbf{k} + \boldsymbol{\xi}$ for some reciprocal lattice wavenumber $\boldsymbol{\xi}$. By $\mathcal{R}^*$ -periodicity of free waves, we know that $\tilde{\mathbf{u}}'_k(\mathbf{x}) = \tilde{\mathbf{u}}_k(\mathbf{x})e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}$ is also a free wave. Furthermore, dominance of $\tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\xi}} = \langle \tilde{\mathbf{u}}'_k \rangle$ is formulated as

$$\langle [C : (\nabla \otimes^s \tilde{\mathbf{u}}'_k)] : (\nabla \otimes^s \tilde{\mathbf{u}}'^{*}_k) \rangle \ll \langle [C : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'^{*}_k) \rangle, \quad (2.8)$$

where the relation $\ll$ means “is negligible with respect to”. As a matter of fact, the component $\tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\xi}}$ contributes to the right-hand side exclusively¹ while the sum of the strains $i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k$ and $\nabla \otimes^s \tilde{\mathbf{u}}'_k$ yields the strain of $\tilde{\mathbf{u}}'_k$. The virtual work theorem for free waves implies

$$\omega^2 \langle \rho \tilde{\mathbf{u}}'_k \cdot \tilde{\mathbf{u}}'^{*}_k \rangle = \left\langle \left\{ C : [(\nabla + i\mathbf{k}') \otimes^s \tilde{\mathbf{u}}'_k] \right\} : [(\nabla + i\mathbf{k}') \otimes^s \tilde{\mathbf{u}}'_k]^* \right\rangle,$$

so that the triangle inequality gives rise to

$$\omega^2 \langle \rho \tilde{\mathbf{u}}'_k \cdot \tilde{\mathbf{u}}'^{*}_k \rangle \leq \left(\sqrt{\langle [C : (\nabla \otimes^s \tilde{\mathbf{u}}'_k)] : (\nabla \otimes^s \tilde{\mathbf{u}}'_k)^* \rangle} + \sqrt{\langle [C : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \rangle} \right)^2.$$

Hence, the condition (2.8) yields

$$\omega^2 \langle \rho \tilde{\mathbf{u}}'_k \cdot \tilde{\mathbf{u}}'^{*}_k \rangle \lesssim \langle [C : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \rangle$$

¹The component $\tilde{\mathbf{u}}_{\mathbf{k}+\boldsymbol{\xi}}$ is the average of $\tilde{\mathbf{u}}'_k$. The gradient of the latter is hence independent of the former.

where the relation $\lesssim$ is defined as “smaller than or almost equal to”¹. Consequently, we have

$$\omega^2 \lesssim \frac{\langle [C : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \rangle}{\langle \rho \tilde{\mathbf{u}}'_k \cdot \tilde{\mathbf{u}}'^*_{\mathbf{k}} \rangle}.$$

Next, we compare the $\mathbf{C}$ -norm and the ρ -norm. In this regard, we have

$$\begin{aligned} \langle [C : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \rangle &= \left\langle \rho \left[\frac{C}{\rho} : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k) \right] : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \right\rangle \\ &\leq \left\langle \rho (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k) : (i\mathbf{k}' \otimes^s \tilde{\mathbf{u}}'_k)^* \right\rangle \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \leq (\mathbf{k}' \cdot \mathbf{k}') \langle \rho \tilde{\mathbf{u}}'_k \cdot \tilde{\mathbf{u}}'^*_{\mathbf{k}} \rangle \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \end{aligned}$$

where $d' = d(d+1)/2$ and the $(c_I)_{I=1\dots d'}$ are the eigenvalues of the elasticity tensor $\mathbf{C}$. It can then be concluded that

$$\omega^2 \lesssim \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \|\mathbf{k}'\|^2 = \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \|\mathbf{k} + \boldsymbol{\xi}\|^2. \quad (2.9)$$

This condition says that the higher the frequency is, the shorter the wavelength $2\pi/\|\mathbf{k} + \boldsymbol{\xi}\|$ of the dominant component is. Elastic energy gets shifted towards short wavelengths for high frequencies in agreement with the analysis of chapter 1 based on Bragg’s reflexion.

The third homogenizability condition demands that the macroscopic field, for which $\mathbf{k}' = \mathbf{k} \in T^*$, be the dominant component of the microscopic displacement and requires *necessarily*

$$\omega^2(\mathbf{k}) \lesssim \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \|\mathbf{k}\|^2, \quad (2.10)$$

where we have reminded that $\omega = \omega(\mathbf{k})$ depends on $\mathbf{k}$ through the dispersion relation.

Let us consider two examples.

1. In the quasi-static limit $\omega \rightarrow 0$, we know by the Voigt bound that $\omega(\mathbf{k})^2 \langle \rho \rangle \mathbf{I} \leq \mathbf{k} \cdot \langle \mathbf{C} \rangle \cdot \mathbf{k}$ in the sense of quadratic forms. It is easy then to see that these frequencies, and the whole acoustic branches most likely, are homogenizable according to the above third condition.
2. Let $\tilde{\mathbf{u}}_{\mathbf{k}}$ be an *optical* eigenmode $\tilde{\boldsymbol{\psi}}_{\mathbf{k},n}$ for $\mathbf{k} = \mathbf{0}$, $n \geq d+1$. Then, if the macroscopic amplitude is a good approximation to the microscopic field $\tilde{\mathbf{u}}_{\mathbf{k}}$, we would have $0 \neq \omega^2 \lesssim 0$. Hence, the considered eigenmode is not homogenizable. Even worse, when ρ is uniform, the orthogonality of eigenmodes for $\mathbf{k} = \mathbf{0}$ implies that $\langle \tilde{\boldsymbol{\psi}}_{\mathbf{k},n} \rangle$ is identically null. Therefore, when the above condition is violated, not only the effective description is a poor approximation to the microscopic one but some microscopic free waves may simply vanish on the macroscopic scale.

¹More precisely, $a \lesssim b$ means that $a \leq (1 + \epsilon)b$ with $\epsilon \ll 1$. Here, ϵ is the ratio between the two components of the elastic energy.

Requiring the three homogenizability conditions yields an upper bound for the set of homogenizable frequencies ω :

$$\omega^2 \lesssim \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \max_{\mathbf{k} \in T^*} \|\mathbf{k}\|^2 \equiv \max_{\substack{\mathbf{x} \in T \\ I=1\dots d'}} \left(\frac{c_I(\mathbf{x})}{\rho(\mathbf{x})} \right) \frac{\pi^2}{4\ell^2}, \quad (2.11)$$

where ℓ is a microstructural length depending only on the geometry of $\mathcal{R}$. The homogenizable frequency domain can contain the acoustic branches of the dispersion curve and, depending on the actual microstructure of Ω , parts of some optical branches. This result is in agreement with and generalizes that of [Srivastava and Nemat-Nasser \(2014\)](#). The cited authors proved that, for particular layered composites, the acoustic branch was homogenizable, in another energy-based sense, and so is the first optical branch for composites with internal resonances.

Finally, condition (2.11) can be seen as a relaxed scale separation condition in comparison with the following classical one

$$\forall m, \quad \omega^2 \ll \min_{I=1\dots d'} \left(\frac{c_I^m}{\rho^m} \right) \frac{\pi^2}{4\ell_m^2},$$

where ℓ_m means the “radius” of phase m .

2.2.4 Two remarks

The first two homogenizability conditions describe geometric conditions under which some internal consistency of the averaging scheme can be guaranteed. They do not make use of any of the features of Ω . In particular, they are valid whether Ω is periodic or not. In the latter case, it is the utility of the averaging scheme in defining a homogeneous effective behavior that will be at stake but not its discussed consistency.

The third and last homogenizability condition estimates the magnitude of the error committed in describing Ω as an effectively homogeneous medium. If closely inspected, it can be seen that it actually suggests that the zone over which FB expansions are carried must be frequency dependent. The zone T^* of small wavenumbers is appropriate for low frequencies and some other zones are more suitable at higher frequencies. This is not the same as defining ω -dependent effective constitutive parameters but rather similar to changing the kinematics of the effective medium based on the frequency band under consideration. This is not without resemblance to how [Craster et al. \(2010\)](#) proceeds.

2.3 Formal solutions and homogenization

In this subsection, the methodology developed by [Willis \(2011\)](#) to derive effective constitutive laws is revisited in such a way that the main aspects for periodic media are specified and clarified. In fact, we present a purely spatial¹ counterpart of said

¹This is: without referring to random descriptions or expectancies.

methodology. It is clear from equations (2.1), (2.2) and (2.3) that FB components can be investigated independently. Also, we drop the explicit $\mathbf{k}$ dependency hereafter in order to simplify notations.

2.3.1 Preliminary discussion

In deriving the effective constitutive relation, the displacement solution of (2.2) needs to be expressed first in terms of the applied loadings (representation step) and then in terms of the effective strain and velocity (localization step). Formally, in the localization step, we write

$$\tilde{\mathbf{u}} = \langle \tilde{\mathbf{u}} \rangle + \mathbf{A} : \langle \tilde{\boldsymbol{\varepsilon}} \rangle + \mathbf{B} \cdot \langle \tilde{\mathbf{v}} \rangle$$

where $\mathbf{A}$ and $\mathbf{B}$ are two localization tensor fields to be determined. However, $\langle \tilde{\boldsymbol{\varepsilon}} \rangle$ and $\langle \tilde{\mathbf{v}} \rangle$ are not algebraically independent since

$$i\omega \langle \tilde{\boldsymbol{\varepsilon}} \rangle = i\omega (\langle \nabla \otimes^s \tilde{\mathbf{u}} \rangle + i\mathbf{k} \otimes^s \langle \tilde{\mathbf{u}} \rangle) = i\mathbf{k} \otimes^s \langle \tilde{\mathbf{v}} \rangle,$$

where $\nabla \otimes^s \tilde{\mathbf{u}}$ has zero average given the periodic boundary conditions. This will result in an ambiguity in the definition of tensors $\mathbf{A}$ and $\mathbf{B}$.

The foregoing fact that velocity and strain are dependent is unusual. Indeed, for a local medium, the strain tensor $\boldsymbol{\varepsilon}(\mathbf{y})$ and the velocity vector $\mathbf{v}(\mathbf{y})$ at a given point $\mathbf{y}$ are independent of each other and they are also independent of the displacement $\mathbf{u}(\mathbf{y})$ at the same point. However, for a nonlocal medium, the variables are some fields over the whole medium. For example, the strain and velocity fields are *not* independent since the time derivative of the former is equal to the symmetric part of the space gradient of the latter. The effective behavior of the periodic medium once homogenized is, *a priori*, nonlocal and characterized by a relation between some $(\mathbf{x}, t)$ -dependent effective fields over Ω . In contrast, some numerical homogenization schemes seek, at isolated macroscopic points, for an effective behavior which links total averages of the microscopic fields. Such schemes do not seem to have to deal with $\mathbf{E}$ - $\mathbf{V}$ dependency because they explicitly presume locality through the use of spatial Taylor expansions and, thus, leave no place for a nonlocal behavior (see Pham et al., 2013).

One way to solve the ambiguity in the definition of tensors $\mathbf{A}$ and $\mathbf{B}$ is to prohibit “abusive” time integration and derivation, i.e., division and multiplication by $i\omega$. To this end, let $\boldsymbol{\gamma}$ be an eigenstrain field over Ω so that the stress-strain relationship becomes

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\gamma}).$$

Now, since $\boldsymbol{\gamma}$ do not derive from the displacement field $\mathbf{u}$ and is therefore independent of $\mathbf{v}$, by keeping track of $(\boldsymbol{\varepsilon} - \boldsymbol{\gamma})$ as a whole, no confusion between the strain and velocity components is possible. Correspondingly, the motion equation (2.2) takes the form

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C} : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}} - \tilde{\boldsymbol{\gamma}}] \} + \tilde{\mathbf{f}} = -\omega^2 \rho \tilde{\mathbf{u}} \quad (2.12)$$

where $\tilde{\boldsymbol{\gamma}}$ can be viewed as a generalized loading. The use of eigenstrain in (Willis, 2011) was inspired by the numerical scheme of Fietz and Shvets (2010).

We underline the fact that non-uniqueness is *not* due to a particular choice of boundary conditions, even though the dependency takes a simple algebraic form in the present periodic case. This is mentioned explicitly in the more general study (Willis, 2011). Incidentally, eigenstrain γ can be seen to play the dual role of body force $\mathbf{f}$. If $\mathbf{f}$ was not taken into account, stress Σ and momentum $\mathbf{P}$ would have been dependent through the effective motion equation

$$i\mathbf{k} \cdot \tilde{\Sigma} = i\omega \tilde{\mathbf{P}},$$

which is equivalent to restraining our attention to free wave solutions. The bottom line is that different loadings will allow to investigate more general aspects of the effective behavior. We also refer to the discussion presented in (Willis, 2012) for other interesting aspects of non-uniqueness.

Here, we have adopted the choice of localizing $\mathbf{u}$ with respect to $\mathbf{E} - \gamma$ and $\mathbf{V}$. Other legitimate choices exist. For example, we could have renounced the use of γ and localized $\mathbf{u}$ directly with respect to $\mathbf{U}$ (as sketched in Willis, 2009). Such choices will lead to other forms of the constitutive equations, which are, however, compatible in the sense that they give rise to the same motion equation in terms of $\mathbf{U}$.

Note that in order to derive the effective constitutive law, we take $\tilde{\mathbf{f}}$ to be uniform in space in agreement with the first homogenizability condition guaranteeing Hill-Mandel relation. Also, it is straightforward to check that including a uniform eigenstrain $\tilde{\gamma}$ does not perturb said relation.

2.3.2 Representation equation

Given the periodic Green's tensor $\mathbf{g}$ of the inhomogeneous medium Ω , we can write the microscopic displacement solution as a sum of two contributions from body force and eigenstrain:

$$[T] \tilde{\mathbf{u}}(\mathbf{y}) = \int_T \mathbf{g}(\mathbf{y}, \mathbf{x}) \cdot \tilde{\mathbf{f}} \, d\mathbf{x} + \int_T [\mathbf{g}(\mathbf{y}, \mathbf{x}) \otimes^s (\nabla_{\mathbf{x}} - i\mathbf{k})] : [\mathbf{C}(\mathbf{x}) : \tilde{\gamma}] \, d\mathbf{x}, \quad (2.13)$$

where $\nabla_{\mathbf{x}}$ is the usual differential operator applied to the $\mathbf{x}$ variable. The operator $\nabla_{\mathbf{y}}$ will be similarly understood. In addition, use will be made of the volume average operators $\langle \cdot \rangle_{\mathbf{x}}$, $\langle \cdot \rangle_{\mathbf{y}}$ and $\langle \langle \cdot \rangle \rangle$ with respect to $\mathbf{x}$, $\mathbf{y}$ and both $\mathbf{x}$ and $\mathbf{y}$, respectively. However, no distinction is indicated when confusion is impossible, i.e., for fields of one variable. Unless otherwise specified, the symbol $\mathbf{g}$ stands for $\mathbf{g}(\mathbf{y}, \mathbf{x})$ hereafter. A last notation convention is that all fields written on the right of the operator $\mathbf{g}$ are $\mathbf{x}$ -dependent while all those on the left of $\mathbf{g}$ are $\mathbf{y}$ -dependent. Finally, equation (2.13) can be written in the compact form¹

$$\tilde{\mathbf{u}} = \langle \mathbf{g} \rangle_{\mathbf{x}} \cdot \tilde{\mathbf{f}} + \langle [\mathbf{g} \otimes^s (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : \tilde{\gamma} \quad (2.14)$$

¹This is an equality between vector fields of the variable $\mathbf{y}$. The operator $\mathbf{g}$ is to be seen as a second order tensor.

where $\tilde{\mathbf{f}}$ and $\tilde{\gamma}$ are uniform in space.

For later purposes, we recall that $\mathbf{g}$ satisfies the equation

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C} : [(\nabla + i\mathbf{k})^{\otimes s} \mathbf{g}(\cdot, \mathbf{y})] \} + |T| \delta_{\mathbf{y}} \mathbf{I} = -\omega^2 \rho \mathbf{g}(\cdot, \mathbf{y})$$

whose volume average over T , by transposition, can be put in the form

$$\langle [\mathbf{g}(\mathbf{y}, \mathbf{x})^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C}(\mathbf{x}) \rangle_{\mathbf{x}} \cdot i\mathbf{k} = \mathbf{I} + \omega^2 \langle \mathbf{g}(\mathbf{y}, \mathbf{x}) \rho(\mathbf{x}) \rangle_{\mathbf{x}}. \quad (2.15)$$

2.3.3 Localization

Aiming to localize $\tilde{\mathbf{u}}$ with respect to $\langle \tilde{\mathbf{v}} \rangle$ and to $\langle \tilde{\varepsilon} - \tilde{\gamma} \rangle$, we can re-write $\tilde{\gamma}$ as $-\langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + \langle \tilde{\varepsilon} \rangle$ in equation (2.14) which becomes

$$\begin{aligned} \tilde{\mathbf{u}} = \langle \mathbf{g} \rangle_{\mathbf{x}} \cdot \tilde{\mathbf{f}} - \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle \\ + \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : \langle \tilde{\varepsilon} \rangle. \end{aligned} \quad (2.16)$$

The third term in the right-side member of this equation can be further written as

$$\begin{aligned} \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : \langle \tilde{\varepsilon} \rangle &= \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : (i\mathbf{k}^{\otimes s} \langle \tilde{\mathbf{u}} \rangle) \\ &= \{ \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} \cdot i\mathbf{k} \} \cdot \langle \tilde{\mathbf{u}} \rangle \\ &= \langle \tilde{\mathbf{u}} \rangle + \omega^2 \langle \mathbf{g} \rho \rangle_{\mathbf{x}} \cdot \langle \tilde{\mathbf{u}} \rangle, \end{aligned} \quad (2.17)$$

where the last equality stems from (2.15).

Combining (2.16) and (2.17) delivers

$$\tilde{\mathbf{u}} - \langle \tilde{\mathbf{u}} \rangle = \langle \mathbf{g} \rangle_{\mathbf{x}} \cdot \tilde{\mathbf{f}} - \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}} : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + \omega^2 \langle \mathbf{g} \rho \rangle_{\mathbf{x}} \cdot \langle \tilde{\mathbf{u}} \rangle. \quad (2.18)$$

The volume average of this equation with respect to $\mathbf{y}$ results in an *algebraic* equation

$$\mathbf{0} = \langle \langle \mathbf{g} \rangle \rangle \cdot \tilde{\mathbf{f}} - \langle \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle \rangle : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + \omega^2 \langle \langle \mathbf{g} \rho \rangle \rangle \cdot \langle \tilde{\mathbf{u}} \rangle$$

whose solution for $\tilde{\mathbf{f}}$ is given by

$$\tilde{\mathbf{f}} = \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle \rangle : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle - \omega^2 \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle \mathbf{g} \rho \rangle \rangle \cdot \langle \tilde{\mathbf{u}} \rangle \quad (2.19)$$

provided $\det \langle \langle \mathbf{g} \rangle \rangle \neq 0$.

Introducing (2.19) into (2.18), we finally obtain

$$\tilde{\mathbf{u}}(\mathbf{y}) - \langle \tilde{\mathbf{u}} \rangle = \mathbf{A}(\mathbf{y}) : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + \mathbf{B}(\mathbf{y}) \cdot \langle \tilde{\mathbf{v}} \rangle,$$

with

$$\begin{aligned} \mathbf{A} &= \langle \mathbf{g} \rangle_{\mathbf{x}} \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle \rangle - \langle [\mathbf{g}^{\otimes s} (\nabla_{\mathbf{x}} - i\mathbf{k})] : \mathbf{C} \rangle_{\mathbf{x}}, \\ \mathbf{B} &= i\omega \langle \mathbf{g} \rangle_{\mathbf{x}} \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle \mathbf{g} \rho \rangle \rangle - i\omega \langle \mathbf{g} \rho \rangle_{\mathbf{x}}. \end{aligned} \quad (2.20)$$

Then, it is immediate that

$$\begin{aligned} \tilde{\boldsymbol{\sigma}} &= \{ \mathbf{C} + \mathbf{C} : [(\nabla_{\mathbf{y}} + i\mathbf{k})^{\otimes s} \mathbf{A}] \} : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + \{ \mathbf{C} : [(\nabla_{\mathbf{y}} + i\mathbf{k})^{\otimes s} \mathbf{B}] \} \cdot \langle \tilde{\mathbf{v}} \rangle, \\ \tilde{\boldsymbol{\rho}} &= i\omega \rho \mathbf{A} : \langle \tilde{\varepsilon} - \tilde{\gamma} \rangle + (\rho \mathbf{I} + i\omega \rho \mathbf{B}) \cdot \langle \tilde{\mathbf{v}} \rangle. \end{aligned} \quad (2.21)$$

2.3.4 Homogenization step

With the expressions for $\tilde{\sigma}$ and $\tilde{p}$, from equation (2.21), the homogenization step can be accomplished easily by volume averaging. Indeed, using the definition (2.4) for $\tilde{\Sigma}$ and $\tilde{P}$, the effective constitutive law of a periodic medium is specified by

$$\begin{bmatrix} \tilde{\Sigma} \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} C^e & S^1 \\ S^2 & \rho^e \end{bmatrix}_{k,\omega} \begin{bmatrix} \tilde{E} - \tilde{\gamma} \\ \tilde{V} \end{bmatrix} \quad (2.22)$$

where the fourth-order effective stiffness tensor C^e , the second-order effective mass tensor ρ^e and the third-order coupling tensors S^1 and S^2 depend on $(\mathbf{k}, \omega)$ and have the expressions

$$\begin{aligned} C^e &= \langle C \rangle + \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{A}] \rangle, \\ \rho^e &= \langle \rho \rangle \mathbf{I} + i\omega \langle \rho \mathbf{B} \rangle, \\ S^1 &= \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{B}] \rangle, \\ S^2 &= i\omega \langle \rho \mathbf{A} \rangle. \end{aligned} \quad (2.23)$$

Accounting for the formulae (2.20) for $\mathbf{A}$ and $\mathbf{B}$, the constitutive tensors can further be specified as follows (an equivalent form of equation (3.19) of Willis, 2011):

$$\begin{aligned} C^e &= \langle C \rangle + \langle \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{g}] \rangle \rangle \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle [\mathbf{g} \otimes^s (\nabla_x - i\mathbf{k})] : C \rangle \rangle \\ &\quad - \langle \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{g} \otimes^s (\nabla_x - i\mathbf{k})] : C \rangle \rangle, \\ \rho^e &= \langle \rho \rangle \mathbf{I} - \omega^2 \langle \langle \rho \mathbf{g} \rangle \rangle \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle \rho \mathbf{g} \rangle \rangle + \omega^2 \langle \langle \rho \mathbf{g} \rho \rangle \rangle, \\ S^1 &= i\omega \langle \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{g}] \rangle \rangle \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle \rho \mathbf{g} \rangle \rangle - i\omega \langle \langle C : [(\nabla_y + i\mathbf{k}) \otimes^s \mathbf{g}] \rho \rangle \rangle, \\ S^2 &= i\omega \langle \langle \rho \mathbf{g} \rangle \rangle \cdot \langle \langle \mathbf{g} \rangle \rangle^{-1} \cdot \langle \langle [\mathbf{g} \otimes^s (\nabla_x - i\mathbf{k})] : C \rangle \rangle - i\omega \langle \langle \rho [\mathbf{g} \otimes^s (\nabla_x - i\mathbf{k})] : C \rangle \rangle. \end{aligned} \quad (2.24)$$

2.4 Comments on the effective behavior

Non-locality. All the effective parameters involved in (2.22) depend on the wavenumber and the frequency of the effective fields. When the effective constitutive law (2.22) is written back in the real domain, by means of inverse Fourier transform, it gives rise to a law which is nonlocal in both space and time, namely

$$\begin{aligned} \Sigma(\mathbf{x}, t) &= \frac{1}{(2\pi)^{d+1}} \int_{\Omega \times \mathbb{R}} \left[C^e(\mathbf{x} - \mathbf{x}', t - t') : (\mathbf{E}(\mathbf{x}', t') - \gamma(\mathbf{x}', t')) \right. \\ &\quad \left. + S^1(\mathbf{x} - \mathbf{x}', t - t') \cdot \mathbf{V}(\mathbf{x}', t') \right] d\mathbf{x}' dt', \\ P(\mathbf{x}, t) &= \frac{1}{(2\pi)^{d+1}} \int_{\Omega \times \mathbb{R}} \left[S^2(\mathbf{x} - \mathbf{x}', t - t') : (\mathbf{E}(\mathbf{x}', t') - \gamma(\mathbf{x}', t')) \right. \\ &\quad \left. + \rho^e(\mathbf{x} - \mathbf{x}', t - t') \cdot \mathbf{V}(\mathbf{x}', t') \right] d\mathbf{x}' dt', \end{aligned} \quad (2.25)$$

where γ is a field of inelastic strain. Hence, strains and velocities from all positions $\mathbf{x}'$ and all instants of time t' contribute to the stress and momentum density at a given position $\mathbf{x}$ at a given instant t . Once more, strains and velocities being derived from the same displacement field, contributions of $\mathbf{E}$ and $\mathbf{V}$ can be mixed up. The inelastic strain γ helps segregate said contributions.

Homogeneity. It is unclear why the outcome of an averaging scheme of any kind would be a homogeneous rather than an inhomogeneous effective behavior. From equations (2.1), (2.2) and (2.3), it appears that the FB components are uncoupled. The averaging procedure (2.4) then transforms these FB components into uncoupled Fourier components. Finally, the uncoupling of Fourier components is, for infinite media at least, equivalent to homogeneity. Homogeneity can also be directly checked from (2.25) where translating strains and velocities amounts to translating stresses and momenta.

Linearity. It is legitimate to wonder whether the effort we have made, following Willis (2011), in order to obtain explicit expressions for the effective parameters in terms of the Green's tensor was necessary and in any way useful. Indeed, linearity directly implies the existence of the effective constitutive law (2.22). The calculation allows simply to check in an elementary fashion at least three facts about the effective behavior: Galilean invariance¹, self-adjointness and the need for coupling. These are investigated subsequently.

Invariance. By Galilean invariance, *uniform* velocities should not produce any stresses. The Galilean invariance on the effective scale is naturally inherited from that on the microscopic scale. In explicit terms, $\mathbf{S}^1$ vanishes when ω vanishes (see equation (2.24)). It is then more fitting to recast (2.22) into

$$\begin{bmatrix} \tilde{\Sigma} \\ \tilde{P} \end{bmatrix} = \begin{bmatrix} C^e & \mathbf{0} \\ \mathbf{0} & \rho^e \end{bmatrix}_{\mathbf{k},\omega} \begin{bmatrix} \tilde{\mathbf{E}} - \tilde{\gamma} \\ \tilde{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \hat{\mathbf{S}}^1 \\ \hat{\mathbf{S}}^2 & \mathbf{0} \end{bmatrix}_{\mathbf{k},\omega} \begin{bmatrix} i\omega(\tilde{\mathbf{E}} - \tilde{\gamma}) \\ i\omega\tilde{\mathbf{V}} \end{bmatrix},$$

with $\mathbf{S}_{\mathbf{k},\omega}^{1,2} = i\omega\hat{\mathbf{S}}_{\mathbf{k},\omega}^{1,2}$, and to say that stresses are coupled with accelerations, not velocities, and momenta with strain rates, and not strains.

Adjointness. The Hill-Mandel relation and the self-adjointness of the microscopic constitutive behavior bestow a self-adjointness property on the effective constitutive behavior. It reads

$$\begin{aligned} C_{\mathbf{k},\omega}^{e\top} &= C_{\mathbf{k},\omega}^{e*}, \\ \rho_{\mathbf{k},\omega}^{e\top} &= \rho_{\mathbf{k},\omega}^{e*}, \\ \mathbf{S}_{\mathbf{k},\omega}^{1\top} &= -\mathbf{S}_{\mathbf{k},\omega}^{2*}, \\ \mathbf{S}_{\mathbf{k},\omega}^{2\top} &= -\mathbf{S}_{\mathbf{k},\omega}^{1*}. \end{aligned}$$

Note that this is *different* from Hermitian symmetry^{2,3}. The above is also evident if (2.24) is considered. Furthermore *and independently*, the real-valuedness of the

¹We mean that a Galilean transformation does not generate any stresses.

²Not unless the coupling vanishes.

³The minus sign appearing between $\mathbf{S}^1$ and $\mathbf{S}^2$ is basically the minus of the difference $\mathcal{T} - \mathcal{K}$ in the expression of the Lagrangian.

effective parameters translates into

$$\begin{aligned} \mathbf{C}_{-\mathbf{k},-\omega}^e &= \mathbf{C}_{\mathbf{k},\omega}^{e*}, \\ \boldsymbol{\rho}_{-\mathbf{k},-\omega}^e &= \boldsymbol{\rho}_{\mathbf{k},\omega}^{e*}, \\ \mathbf{S}_{-\mathbf{k},-\omega}^1 &= \mathbf{S}_{\mathbf{k},\omega}^{1*}, \\ \mathbf{S}_{-\mathbf{k},-\omega}^2 &= \mathbf{S}_{\mathbf{k},\omega}^{2*}. \end{aligned}$$

Finally, the microscopic motion equation only depends on ω through ω^2 so that we have

$$\begin{aligned} \mathbf{C}_{\mathbf{k},-\omega}^e &= \mathbf{C}_{\mathbf{k},\omega}^e, \\ \boldsymbol{\rho}_{\mathbf{k},-\omega}^e &= \boldsymbol{\rho}_{\mathbf{k},\omega}^e, \\ \mathbf{S}_{\mathbf{k},-\omega}^1 &= -\mathbf{S}_{\mathbf{k},\omega}^1, \\ \mathbf{S}_{\mathbf{k},-\omega}^2 &= -\mathbf{S}_{\mathbf{k},\omega}^2. \end{aligned}$$

Coupling. On the microscopic scale, an acceleration can generate stresses through the microscopic motion equation. Similarly, strains generate momenta. The coupling terms $\mathbf{S}^1$ and $\mathbf{S}^2$ capture partially this mechanism through the macroscopic constitutive behavior. Such couplings appear naturally in (2.23) and will later be proven inevitable using a simple example. When either $\mathbf{C}$ or $\boldsymbol{\rho}$ are uniform, or when $\omega = 0$, coupling vanishes.

Symmetry. The *material* symmetry of the microstructure, if any, implies the existence of some relations between the components of the effective parameters. What should be noted is that, due to nonlocality, material symmetries must transform wavenumbers as well as microstructure. Precisely, if $\mathbf{R}$ is a material symmetry, then¹

$$\begin{aligned} [\mathbf{C}_{\mathbf{k},\omega}^e]_{pqrs} &= [\mathbf{C}_{\mathbf{k}',\omega}^e]_{abcd} R_{ap} R_{bq} R_{cr} R_{ds}, \\ [\boldsymbol{\rho}_{\mathbf{k},\omega}^e]_{pq} &= [\boldsymbol{\rho}_{\mathbf{k}',\omega}^e]_{ab} R_{ap} R_{bq}, \\ [\mathbf{S}_{\mathbf{k},\omega}^1]_{pqr} &= [\mathbf{S}_{\mathbf{k}',\omega}^1]_{abc} R_{ap} R_{bq} R_{cr}, \\ [\mathbf{S}_{\mathbf{k},\omega}^2]_{pqr} &= [\mathbf{S}_{\mathbf{k}',\omega}^2]_{abc} R_{ap} R_{bq} R_{cr}, \end{aligned}$$

with $\mathbf{k}' = \mathbf{R}^\top \cdot \mathbf{k}$. We would like simply to underline the fact that, given a material symmetry, the shape of the effective parameters for a nonlocal behavior is different than that for a local one. In particular, the odd order coupling tensors $\mathbf{S}^1$ and $\mathbf{S}^2$ do *not* generally vanish for centrosymmetric microstructures.

¹To my poorly educated self, the expression of this transformation is counterintuitive. When a unit cell is rotated in one direction, $\mathbf{k}$ seems to go in the opposite direction.

Elastostatics. Homogenization under quasi-static loadings can be obtained as a particular case of the present theory by specifying $\omega = 0$. Coupling vanishes as noted earlier, the effective mass density ρ^e becomes the isotropic average mass density $\langle \rho \rangle \mathbf{I}$ and time nonlocality naturally disappears. The effective behavior remains nonlocal in space however. When the long-wavelength limit is considered, the effective behavior becomes local in space and we recover the standard quasi-static effective behavior. More details are given in chapter 3.

2.5 The effective motion equation

When all strains are elastic, i.e., $\gamma = \mathbf{0}$, using the effective constitutive parameters, the effective motion equation (2.5) becomes

$$i\mathbf{k} \cdot (\mathbf{C}_k^e : \tilde{\mathbf{E}}_k + \mathbf{S}_k^1 \cdot \tilde{\mathbf{V}}_k) + \tilde{\mathbf{f}}_k = i\omega(\mathbf{S}_k^2 : \tilde{\mathbf{E}}_k + \rho_k^e \cdot \tilde{\mathbf{V}}_k)$$

or, in terms of the macroscopic displacement,

$$\mathbf{Z}_k \cdot \tilde{\mathbf{U}}_k \equiv [\mathbf{k} \cdot \mathbf{C}_k^e \cdot \mathbf{k} + \omega(\mathbf{k} \cdot \mathbf{S}_k^1 - \mathbf{S}_k^2 \cdot \mathbf{k}) - \omega^2 \rho_k^e] \cdot \tilde{\mathbf{U}}_k = \tilde{\mathbf{f}}_k.$$

Note that as dictated by the first homogenizability condition, macroscopic body forces are the same as on the microscopic scale. The operator $\mathbf{Z}$ is referred to as *the effective impedance*. It depends on both $\mathbf{k}$ and ω . For a homogeneous Cauchy medium, $\mathbf{Z}$ is a second order differential operator, i.e., a second order polynomial in $\mathbf{k}$ and ω . For the effective medium obtained by homogenization, it is a general integro-differential operator. The knowledge of $\mathbf{Z}$ is enough to characterize displacement solutions but is not sufficient to describe stresses and momenta. Regardless of this shortcoming, in later chapters, we will mostly focus our attention on $\mathbf{Z}$ rather than on the effective constitutive parameters for reasons of simplicity and coherence, especially when speaking of asymptotic approximations.

In terms of the Green's function, $\mathbf{Z}$ admits a simple expression. Indeed, from (2.14), given a zero eigenstrain, it is immediate to see that

$$\tilde{\mathbf{U}}_k = \langle \tilde{\mathbf{u}}_k \rangle = \langle \langle \mathbf{g}_k \rangle \rangle \cdot \tilde{\mathbf{f}}_k \equiv \mathbf{G}_k \cdot \tilde{\mathbf{f}}_k,$$

where $\mathbf{G}$ is the effective Green's tensor of the effective medium. Consequently, we have

$$\mathbf{Z}_k = \mathbf{G}_k^{-1} = \langle \langle \mathbf{g}_k \rangle \rangle^{-1}.$$

It is in fact insightful to write a series representation of the effective Green's tensor. From the results of chapter 1, the microscopic Green's tensor admits the form

$$\mathbf{g}_k(\mathbf{y}, \mathbf{x}) = \sum_{n \geq 1} \frac{\tilde{\psi}_{k,n}(\mathbf{y}) \otimes \tilde{\psi}_{k,n}^*(\mathbf{x})}{\omega_n^2(\mathbf{k}) - \omega^2},$$

so that the macroscopic one can be written as

$$\mathbf{G}_k = \sum_{n \geq 1} \frac{\langle \tilde{\psi}_{k,n} \rangle \otimes \langle \tilde{\psi}_{k,n} \rangle^*}{\omega_n^2(\mathbf{k}) - \omega^2}.$$

All that Willis theory reduces to is to substitute an eigenmode by its volume average. The theory is then appropriate for describing eigenmodes that have dominant averages. Such was the subject of the third homogenizability condition.

We have seen that the macroscopic fields automatically satisfy a macroscopic motion equation. As a matter of fact, Willis theory transforms a microscopic solution into a macroscopic one. In particular, it transforms a free wave into a free wave of the same frequency and of the same wavenumber. This last statement is obvious from the above representation of the effective Green's tensor where the eigenfrequencies are not affected by the scale transition. Only the associated polarizations are modified. Hence, a microscopic eigenfrequency is also a macroscopic one except when $\langle \tilde{\psi}_{\mathbf{k},n} \rangle$ happens to be null:

$$\mathcal{C}^e = \bigcup_{n \geq 1} \{(\mathbf{k}, \omega_n(\mathbf{k})) \in T^* \times \mathbb{R}^+, \quad \langle \tilde{\psi} \rangle_{\mathbf{k},n} \neq \mathbf{0}\} \subset \mathcal{C},$$

where $\mathcal{C}^e$ is the macroscopic dispersion curve and $\mathcal{C}$ is the microscopic one. Note that when the subtracted set of zero average eigenmodes is of empty interior, $\mathcal{C}$ can be recovered from $\mathcal{C}^e$ by continuous extrapolation. We shall then speak indifferently of $\mathcal{C}$ and $\mathcal{C}^e$ henceforth.

3 Willis theory à la Willis

In what follows, we present the Willis homogenization theory as introduced by Willis (1997, 2011). This point of view based on a random description of the microstructure and on ensemble averages, i.e., expectancies, rather than on volume averages is discussed and commented in the light of the previously presented homogenizability conditions.

3.1 Random and randomized microstructures

The microstructure of Ω characterized by the stiffnesses $\mathbf{C}$ and inertia ρ is now assumed to be random and not necessarily periodic¹. In other words, in addition to their spatial dependence, $\mathbf{C}$ and ρ now depend on some random variable X of probability distribution $\mu(X)$ and they are denoted, respectively, by $\mathbf{C}^X$ and ρ^X . Given a *deterministic* body force field $\mathbf{f}$, the motion equation reads

$$\nabla \cdot [\mathbf{C}^X : (\nabla \otimes^s \mathbf{u}^X)] + \mathbf{f} = -\omega^2 \rho^X \mathbf{u}^X,$$

where $\mathbf{u}^X$ is the displacement field in the *realization* X . The effective fields are then defined at each position $\mathbf{x}$ as the *expected value* with respect to X at that same position, namely

$$H(\mathbf{x}) = \int h^X(\mathbf{x}) \mathrm{d}\mu(X).$$

¹Periodic media will shortly be seen as particular random media.

The above average is the so-called “ensemble average”. In order to compute H , one has to carry the same experiment, defined by $\mathbf{f}$, for different realizations $\mathbf{X}$ and then average the $h^{\mathbf{X}}$ outcomes at each position.

In order to apply the foregoing definition to periodic media, one needs to describe a periodic microstructure in terms of a random one. This is done as follows. Let $\mathbf{C}$ and ρ be some *deterministic* $\mathcal{R}$ -periodic fields of elastic stiffness and mass density. Let $\mathbf{X} \in \mathcal{E}$ be a random translation and define the random fields $\mathbf{C}^{\mathbf{X}} \equiv \mathbf{C}(\cdot + \mathbf{X})$ and $\rho^{\mathbf{X}} \equiv \rho(\cdot + \mathbf{X})$. Therefore, the only random feature of the microstructure described by $\mathbf{C}^{\mathbf{X}}$ and $\rho^{\mathbf{X}}$ is the absolute position of said microstructure in space. In other words, the interior of a unit cell of the considered composite is deterministically known, however, its absolute position in space, say in the lab frame of reference, is subject to uncertainties. This uncertainty takes its full meaning when absolute sure body forces are taken into account. The motion equation, in terms of FB components, reads

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x} + \mathbf{X}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x})] \} + \tilde{\mathbf{f}} = -\omega^2 \rho(\mathbf{x} + \mathbf{X}) \tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x}).$$

Translating the above equation by $-\mathbf{X}$, it comes that

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x} - \mathbf{X})] \} + \tilde{\mathbf{f}} = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x} - \mathbf{X}).$$

Therefore $\tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x} - \mathbf{X})$, whatever the realization $\mathbf{X}$ is, is a solution to the equation

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}(\mathbf{x})] \} + \tilde{\mathbf{f}} = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}(\mathbf{x})$$

governing the motion at $\mathbf{X} = \mathbf{0}$. Hence, assuming the uniqueness of the solution,

$$\tilde{\mathbf{u}}^{\mathbf{X}}(\mathbf{x}) = \tilde{\mathbf{u}}(\mathbf{x} + \mathbf{X}).$$

The last ingredient that needs to be specified before defining $\tilde{\mathbf{U}}$ is the probability distribution μ . Said distribution is taken to be uniform over one unit cell T , i.e., all positions are equiprobable. Correspondingly, the effective FB components of H are given by

$$\tilde{H}(\mathbf{x}) = \int \tilde{h}(\mathbf{x} + \mathbf{X}) d\mu(\mathbf{X}) = \frac{1}{|T|} \int_T \tilde{h}(\mathbf{x} + \mathbf{X}) d\mathbf{X} = \langle \tilde{h} \rangle \quad (2.26)$$

and *turn out* to be independent of $\mathbf{x}$ and the same as the ones previously introduced in equation (2.4), up to one detail that is discussed next.

3.2 Domain of definition

In (2.4), the FB components of the effective fields are given in terms of the FB components of the microscopic one, for all $\mathbf{k} \in T^*$. Then, we have assumed that microscopic and macroscopic body forces are the same in agreement with the requirement of the Hill-Mandel relation. Finally, for each $\mathbf{k} \in T^*$, we have calculated expressions for some effective parameters. Consequently, the effective constitutive behavior is only defined for $\mathbf{k} \in T^*$ and has no meaning so far outside of T^* .

Nonetheless, the above methodology can be “naively” followed whatever $\mathbf{k} \in \mathcal{E}^*$ is. First, let $\mathbf{k}$ be an arbitrary wavenumber and $\tilde{\mathbf{f}}_{\mathbf{k}}$ be a uniform body force amplitude. Second, solve the motion equation for $\tilde{\mathbf{u}}_{\mathbf{k}}$. Finally, define $\tilde{\mathbf{U}}_{\mathbf{k}} = \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle$ and obtain $\mathbf{Z}_{\mathbf{k}}$ as the mapping between $\tilde{\mathbf{f}}_{\mathbf{k}}$ and $\tilde{\mathbf{U}}_{\mathbf{k}}$. Similarly, the effective constitutive parameters can be extrapolated to the whole space $\mathcal{E}^*$. In doing so, the effective field $\mathbf{U}$ is no longer defined one FB component at a time as in (2.4), but one response to a plane wave body force at a time as in (2.26). Furthermore, note that we were able to speak of (2.4) independently of any motion equation whereas (2.26) was the consequence of some forced motion equation. The point of view of Willis summarized in (2.26) has two shortcomings which motivated the alternative formulation (2.4).

1. Willis defines the effective fields driven by a sure body force. When body forces vanish, the effective fields are ill-defined. One might argue that one only needs to define the effective fields in order to calculate the effective constitutive parameters. Once said parameters are known in forced regimes, they can be extrapolated to unforced regimes, which brings us to the second issue.
2. Let us assume that the effective behavior is given a meaning over the whole space $\mathcal{E}^*$ and let $\mathbf{k}$ and $\mathbf{k} + \boldsymbol{\xi}$, for some $\boldsymbol{\xi} \in \mathcal{R}^*$, be two different wavenumbers. If $\omega_n(\mathbf{k})$ is an eigenfrequency then so is $\omega_n(\mathbf{k} + \boldsymbol{\xi})$ by $\mathcal{R}^*$ -periodicity of the dispersion curve. On the microscopic scale, the free waves associated with $\omega_n(\mathbf{k})$ and with $\omega_n(\mathbf{k} + \boldsymbol{\xi})$ are the same. However, on the macroscopic scale, these two free waves become distinct plane waves of wavenumbers $\mathbf{k}$ and $\mathbf{k} + \boldsymbol{\xi}$ respectively. Hence, each microscopic free wave gets split into an infinity, one for each $\boldsymbol{\xi} \in \mathcal{R}^*$, of macroscopic free waves the majority of which are non-physical solutions and merely artefacts of the Willis averaging scheme.

In fact, the Willis scheme transforms some components *coupled* on the microscale into some *uncoupled* components on the macroscale. Forcing said uncoupling entails the mentioned inconsistencies. These issues disappear once the effective description is restricted to T^* (or to any other region isomorphic to $\mathcal{E}^*/\mathcal{R}^*$) as suggested by the first homogenizability condition. As a consequence, the effective medium has reduced kinematics ($\tilde{\mathbf{U}}_{\mathbf{k}}$ is uniform, for all $\mathbf{k} \in T^*$) in comparison to the original medium ($\tilde{\mathbf{u}}_{\mathbf{k}}$ is $\mathcal{R}$ -periodic, for all $\mathbf{k} \in T^*$).

Incidentally, Willis (1997) presented a Hill-Mandel relation valid whatever the sure body forces are. However, his version is only concerned with the *expectancies* of microscopic and macroscopic works.

4 An application: 1D discrete systems

Hereafter, we reconsider Born’s model for sodium chloride which was thoroughly investigated in chapter 1. This model is studied here as an example to illustrate and to gain insight into the general elastodynamic homogenization theory presented in the foregoing section for periodic media.

4.1 Some closed form expressions

If $\tilde{h}_n$ is a (2-periodic) microscopic FB component, definition (2.4) is interpreted as

$$\tilde{H} = \frac{\tilde{h}_1 + \tilde{h}_2}{2}.$$

Given the above and the results of chapter 1, it is straightforward to obtain¹

$$\begin{bmatrix} \Sigma \\ P \end{bmatrix} = \begin{bmatrix} C^e & S^1 \\ S^2 & \rho^e \end{bmatrix}_{k,\omega} \begin{bmatrix} E \\ V \end{bmatrix}$$

where

$$\begin{aligned} C_{k,\omega}^e &= \frac{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \left\langle \frac{\langle m \rangle}{c} \right\rangle}{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \frac{\langle m \rangle}{\langle c \rangle}} \left\langle \frac{1}{c} \right\rangle^{-1}, \\ \rho_{k,\omega}^e &= \frac{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \left\langle \frac{\langle c \rangle}{m} \right\rangle^{-1}}{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \frac{\langle m \rangle}{\langle c \rangle}} \langle m \rangle, \\ S_{k,\omega}^1 &= \frac{i\omega \Delta m \Delta c \cos \left(\frac{k}{2} \right) e^{ik/2}}{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \frac{\langle m \rangle}{\langle c \rangle}} \frac{1}{2 \langle c \rangle}, \\ S_{k,\omega}^2 &= \frac{i\omega \Delta m \Delta c \cos \left(\frac{k}{2} \right) e^{-ik/2}}{4 \cos^2 \left(\frac{k}{2} \right) - \omega^2 \frac{\langle m \rangle}{\langle c \rangle}} \frac{1}{2 \langle c \rangle}, \end{aligned}$$

with $\Delta m = m_1 - m_2$ and $\Delta c = c_1 - c_2$.

As ω goes toward zero, the effective stiffness C^e approaches its static limit $1/\langle 1/c \rangle$ and the effective mass approaches the static effective mass $\langle m \rangle$. At the same time, the coupling terms S^1 and S^2 vanish for $\omega = 0$ as expected. This also guarantees that no stress is generated from a change of the Galilean frame of reference. It is therefore more correct to say that stresses are coupled with accelerations and not velocities. In the same manner, momenta are coupled with strain rates and not strains. Furthermore, if the masses m or the stiffnesses c were uniform, the coupling terms would vanish as well. If both masses and stiffnesses are uniform, the initial classical constitutive relation is recovered.

It is remarkable that the effective medium, of our mass-spring lattice, is also *discrete*². In fact, the effective motion equation is

$$\Sigma_n - \Sigma_{n-1} + f_n = i\omega P_n.$$

¹We underline the fact that here, unlike the general continuous case, Σ is homogeneous to a force, P to a momentum and not to a momentum density, E to a length and V to a velocity. The dimensions of the effective parameters change therefore accordingly.

²Homogenization is not continualization.

In addition, the effective constitutive parameters are 2π -periodic functions in the wavenumber k . The corresponding convolution kernels are inverse Fourier transforms with respect to k and have therefore a discrete support of step 1 (i.e., the set of integers $\mathbb{Z}$).

4.2 Effective impedance, nonlocality and coupling

The determination of the effective displacement response U to a solicitation f does not entail knowing the whole effective properties. Indeed, a direct relation between $\tilde{U}$ and $\tilde{f}$ can be found and is characterized by the effective impedance

$$Z_{k,\omega} = |e^{ik} - 1|^2 C^e + i\omega(e^{-ik} - 1)S^1 + i\omega(e^{ik} - 1)S^2 - \omega^2 \rho^e$$

which can be further specified as

$$Z_{k,\omega} = \frac{2m_1 m_2 \omega^4 - 2(h_1 + h_2)(m_1 + m_2)\omega^2 + 8h_1 h_2 \sin^2(k)}{4(h_1 + h_2) \cos^2\left(\frac{k}{2}\right) - (m_1 + m_2)\omega^2}.$$

Note that the condition $Z_{k,\omega} = 0$ yields the exact dispersion curve up to the singularities that $Z_{k,\omega}$ may have due to its poles. These poles corresponds to the zero average eigenmodes previously discussed.

It is known and easy to see that, at low frequencies, the effective mass is the arithmetic mean of masses since point masses are oscillating in phase. Let us then investigate, through direct analysis, the high frequency behavior of the effective mass through two examples $\omega = \omega_2(0)$, i.e, the cut-off frequency, and $\omega \rightarrow \infty$.

1. The microscopic solution for the eigenmode ($k = 0, \omega = \omega_2(0)$) is $u_n = \tilde{u}_n$. Since $\tilde{u}$ is 2-periodic, the eigenmode ($0, \omega_2(0)$) decouples the medium into two rigid subsystems, even indexed masses and odd indexed masses, connected by two springs and oscillating in two opposite directions, so as not to violate Newton's first law (see figure 2.2). It is easy then to see that $E = 0$ since $k = 0$, that $P = 0$ since $u_1/u_2 = -m_2/m_1$ and that $V \neq 0$ for $m_1 \neq m_2$. Therefore, $\rho_{k=0,\omega=\omega_2(0)}^e = 0$.
2. For even higher frequencies $\omega \rightarrow \infty$, no free waves can propagate meaning that, given a force field $\tilde{f}$, masses will oscillate as if independent of each other. We then have $-\omega^2 m_1 \tilde{u}_1 = -\omega^2 m_2 \tilde{u}_2 = \tilde{f}$ so that $i\omega \langle \tilde{u} \rangle = \langle 1/m \rangle \tilde{f} / i\omega$ and $\rho_{k,\omega \rightarrow \infty}^e = 1 / \langle 1/m \rangle$.

These two examples, along with the quasi-static case, show that the nonlocality in time which is caused by the ω -dependency of the effective mass, among others, can be interpreted as the effect, over effective mass, of changing modes of oscillation (in/out of phase for example) of internal degrees of freedom.

The need for coupling terms can as well be demonstrated even in this very simple context. Again, let k be null so that *the macroscopic strain is null* and let ω be the cut-off frequency $\omega_2(0)$ as in the example treated above. Then, the macroscopic stress is

$$\tilde{\Sigma} = \frac{c_1(\tilde{u}_2 - \tilde{u}_1) + c_2(\tilde{u}_1 - \tilde{u}_2)}{2} = \frac{(c_1 - c_2)(\tilde{u}_1 - \tilde{u}_2)}{2}$$

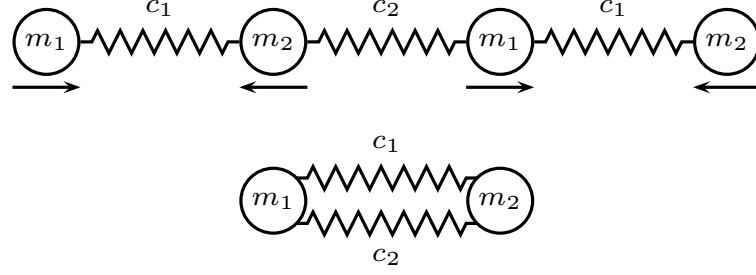


Figure 2.2. The mode ($k = 0, \omega = \omega_2(0)$) (top) and the two-body problem that it reduces to (bottom). The arrows illustrate the direction of the velocity at a given time.

and is *non-null* as long as $c_1 \neq c_2$ and $m_1 \neq m_2$, since otherwise $\tilde{u}_1 = \tilde{u}_2$. The effective stresses are then inevitably coupled with something other than strains.

4.3 Homogenizability conditions illustrated

For the 1D discrete medium under consideration, the second homogenizability condition ($\mathbf{k} \in T^*$) is simply

$$-\frac{\pi}{2} < k < \frac{\pi}{2}.$$

As the gradient operator is replaced by its discrete version, the third homogenizability condition becomes

$$\omega^2(k) \lesssim 4 \max\left(\frac{c}{m}\right) \sin^2\left(\frac{k}{2}\right).$$

Solving the dispersion relation for $\omega = \omega(k)$, it is seen that the acoustic branch is always entirely homogenizable whereas most of the optical branch violates the above condition. Borrowing Brillouin's interpretation, the combined homogenizability conditions state that *consecutive* masses should not oscillate in an obvious out-of-phase fashion.

Figure 2.3 shows an example of an exact high-frequency free wave solution where a macroscopic description appears to be out of context. Therein, the effective wavelength satisfies the second homogenizability condition. The large difference between the amplitudes of the effective and microscopic waves indicates that the dominant component is the short-wavelength one. This is due to the chosen high frequency that violates the third homogenizability condition. In figure 2.4 is given an example of an exact low-frequency free wave solution where the macroscopic behavior is rather obvious. The dominant component is clearly the effective long-wavelength one due to the low frequency setting in agreement with the third homogenizability condition. For these examples, details (masses, stiffnesses, time, ...) which are not important from the physical standpoint are not specified.

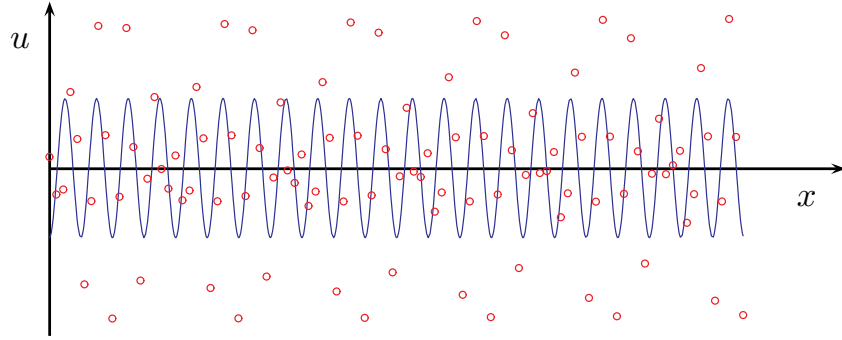


Figure 2.3. In small circles, an exact high-frequency free wave solution: the (real) displacement of 100 consecutive masses. The continuous line draws the corresponding effective plane wave.

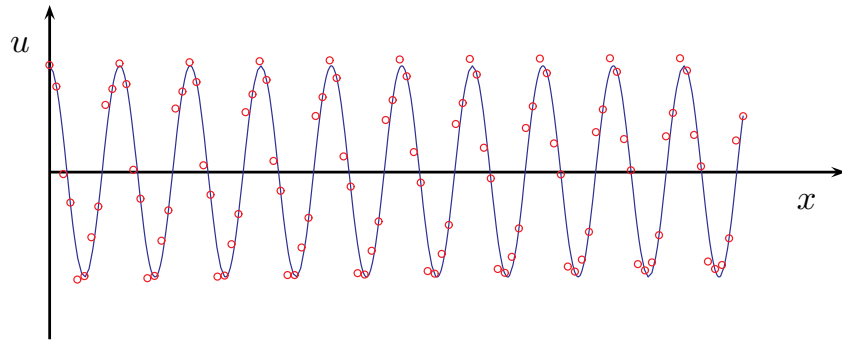


Figure 2.4. In small circles, an exact low-frequency free wave solution: the (real) displacement of 100 consecutive masses. The continuous line draws the corresponding effective plane wave.

5 Concluding remarks

The elastodynamic homogenization theory of composites, initiated by J.R. Willis more than thirty years ago, escaping due attention during a long period, and developed recently by him and other researchers, has been carefully and systematically revisited in the present work for periodic media. The theory of Willis is general and exact, leading to an effective elastodynamic constitutive law which is nonlocal in time and space and takes a form quite different from and much more complicated than the usual constitutive laws in classical elastodynamics. Now, in spite of its mathematical generality and exactness, the corresponding physical validity (or suitability) domain

remains far from being clarified. Aiming mainly at giving an answer to this important question, the present work has established three necessary conditions for the application of that theory to be physically sound. The first condition guaranteed the existence of a Hill-Mandel relation at the cost of reducing the kinematics of the effective medium. Then, the third condition imposed upper bounds on the frequencies for which the effective behavior is a good approximation to the microscopic one. In later chapters, we explore two directions.

1. The Willis constitutive behavior is highly complicated. In addition, it is likely that it is mostly valid for relatively low frequencies. It is of interest then to investigate the asymptotics of said behavior as ω approaches 0. More generally, connections between Willis theory and asymptotic homogenization methods need to be clarified.
2. In order to extend the frequency validity domain, the kinematics of the effective medium need to be enriched. Enriching said kinematics while keeping valid a Hill-Mandel relation will receive our attention later on.

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Chapter 3

On some asymptotic elastodynamic homogenization approaches

Il serait aisé d'aller plus loin, mais ce serait un vain jeu de l'esprit ; on n'imaginerait que des symboles sans application possible, et personne ne s'en avisera. Le continu du troisième ordre auquel conduit la considération des divers ordres d'infiniment petits est lui-même trop peu utile pour avoir conquis droit de cité, et les géomètres ne le regardent que comme une simple curiosité. L'esprit n'utilise de sa faculté créatrice que quand l'expérience lui en impose la nécessité.

[...]

Enfin M. Veronese et M. Hilbert ont imaginé de nouvelles géométries plus étranges encore, qu'ils appellent non-archimédiennes. Ils les construisent en rejetant l'axiome d'Archimède en vertu duquel toute longueur donnée, multipliée par un entier suffisamment grand, finira par surpasser toute autre longueur donnée si grande qu'elle soit. Sur une droite non archimédienne, les points de notre géométrie ordinaire existent tous, mais il y en a une infinité d'autres qui viennent s'intercaler entre eux, de telle sorte qu'entre deux segments, que les géomètres de la vieille école auraient regardés comme contigus, on puisse caser une infinité de points nouveaux. En un mot, l'espace non archimédien n'est plus un continu du second ordre, pour employer le langage du chapitre précédent, mais un continu du troisième ordre.

Henri Poincaré, *La Science et l'Hypothèse*

Two long-wavelength and low-frequency asymptotic approximations to Willis theory are carefully analyzed and compared in connection with higher-order strain-gradient media. In particular, these approaches are proven to be unable to capture, at least in the one-dimensional setting, the optical branches of the dispersion curve. As an example, a two-phase string is thoroughly studied. Finally, a fairly large family of average-based

asymptotic homogenization methods is shown to be derivable from Willis theory under appropriate approximation assumptions about, for example, frequencies, wavelengths and phase contrast.

1 Willis theory summarized

In the context of periodic media, the elastodynamic homogenization theory of Willis can be completely established through a purely spatial formulation in which the Floquet-Bloch (FB) transform plays a key role. Such a spatial formulation is adopted in what follows.

Consider a periodic medium Ω and let be given a pair of wavenumber $\mathbf{k}$ and frequency ω denoted as $(\mathbf{k}, \omega)$. We prescribe over Ω a harmonic plane wave body force

$$\mathbf{f}(\mathbf{x}, t) = \tilde{\mathbf{f}} e^{i(\mathbf{k} \cdot \mathbf{x} + \omega t)} \quad (3.1)$$

where $\tilde{\mathbf{f}}$ is a constant force vector amplitude. Then, the resulting displacement field $\mathbf{u}$ in Ω takes the form of a FB wave

$$\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(\mathbf{x}) e^{i(\mathbf{k} \cdot \mathbf{x} + \omega t)} \quad (3.2)$$

where $\tilde{\mathbf{u}}(\mathbf{x})$ is a time-independent and spatially $\mathcal{R}$ -periodic displacement amplitude with $\mathcal{R}$ representing the periodicity lattice associated to Ω .

In what follows, the time dependence will be dropped when there is no risk of confusion. In terms of $(\mathbf{u}, \mathbf{f})$, the harmonic motion equation over Ω can be written as

$$\nabla \cdot [\mathbf{C}(\mathbf{x}) : (\nabla \otimes^s \mathbf{u}(\mathbf{x}))] + \mathbf{f}(\mathbf{x}) = -\omega^2 \rho(\mathbf{x}) \mathbf{u}(\mathbf{x}). \quad (3.3)$$

In this equation, the fourth-order elastic stiffness tensor $\mathbf{C}$ and the scalar mass density ρ are periodic functions of the material point position vector $\mathbf{x}$. Accounting for (3.1) and (3.2) in (3.3), we obtain the reduced but equivalent motion equation in terms of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{f}}$:

$$(\nabla + i\mathbf{k}) \cdot \{\mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}(\mathbf{x})]\} + \tilde{\mathbf{f}} = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}(\mathbf{x}) \quad (3.4)$$

which holds over any unit cell T of the periodic medium in question and is supplemented with the appropriate boundary conditions implied by $\mathcal{R}$ -periodicity and by the continuity of displacements and tractions. Below, the solution to equation (3.3) is noted as $\mathbf{u}_{\mathbf{k}, \omega}$ while the one to equation (3.4) is denoted by $\tilde{\mathbf{u}}_{\mathbf{k}, \omega}$. We pass from the latter to the former by adding the phase exponential factor $e^{i\mathbf{k} \cdot \mathbf{x}}$.

The effective displacement field corresponds to the *plane wave* obtained through averaging the periodic FB amplitude of the microscopic field:

$$\mathbf{U}_{\mathbf{k}, \omega}(\mathbf{x}) \equiv \tilde{\mathbf{U}}_{\mathbf{k}, \omega} e^{i\mathbf{k} \cdot \mathbf{x}} \equiv \frac{1}{|T|} \left(\int_T \tilde{\mathbf{u}}_{\mathbf{k}, \omega}(\mathbf{x}') d\mathbf{x}' \right) e^{i\mathbf{k} \cdot \mathbf{x}} \equiv \langle \tilde{\mathbf{u}}_{\mathbf{k}, \omega} \rangle e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (3.5)$$

It has been shown in chapter 2 that the effective motion equation takes the form

$$\mathbf{Z}_{\mathbf{k}, \omega} \cdot \tilde{\mathbf{U}}_{\mathbf{k}, \omega} = \tilde{\mathbf{f}} \quad (3.6)$$

where $\mathbf{Z}_{\mathbf{k},\omega}$ is the second order effective impedance tensor. It has the following expression (Willis, 1997, equation (3.28)):

$$\mathbf{Z}_{\mathbf{k},\omega} = i\omega \boldsymbol{\rho}_{\mathbf{k},\omega}^e i\omega + i\omega \mathbf{S}_{\mathbf{k},\omega} \cdot i\mathbf{k} - i\mathbf{k} \cdot \mathbf{C}_{\mathbf{k},\omega}^e \cdot i\mathbf{k}, \quad (3.7)$$

where $\boldsymbol{\rho}^e$ is the effective mass density tensor of order 2, $\mathbf{S}$ is a tensor of order 3 combining¹ the stress-velocity and momentum-strain coupling terms and $\mathbf{C}^e$ is the effective elastic stiffness tensor of order 4. Except in simple cases, these tensors are found numerically.

The dependence of the impedance tensor $\mathbf{Z}_{\mathbf{k},\omega}$ on $\mathbf{k}$ and ω is in general non-polynomial and far from being trivial. The determination of $\mathbf{Z}_{\mathbf{k},\omega}$ necessitates solving equation (3.4) for each pair $(\mathbf{k}, \omega)$ and for d independent loadings $\mathbf{f}$ in a d -dimensional space. It goes without saying that such task is hard and costly. Most importantly, the knowledge of $\mathbf{Z}_{\mathbf{k},\omega}$ at distinct points $(\mathbf{k}, \omega)$ does not reveal the nature of the effective behavior and is rather unsatisfying. An approach which is less accurate but more uniform over (regions of) the $(\mathbf{k}, \omega)$ -space is preferable.

The main purpose of the present chapter is to discuss different asymptotic expansions, Long-Wavelength (LW) Low-Frequency (LF) asymptotics in particular, of the effective impedance $\mathbf{Z}_{\mathbf{k},\omega}$. The exact effective behavior described by $\mathbf{Z}_{\mathbf{k},\omega}$ will then be replaced by a simpler approximate asymptotic one $\mathbf{Z}^e$ truncated at some order of accuracy R .

2 LW-LF asymptotics

As noted above, effective impedances $\mathbf{Z}$, whether exact or approximate, depend on the Fourier variables $\mathbf{k}$ and ω . Therefore, back in the real space-time domain, $\mathbf{Z}$ corresponds to an integro-differential operator which acts on the effective displacement field $\mathbf{U}(\mathbf{x}, t)$ and yields $\mathbf{f}(\mathbf{x}, t)$. The nature of this operator is determined by the way in which $\mathbf{Z}$ depends on $\mathbf{k}$ and ω . LW-LF asymptotics give birth to an approximate effective impedance which is a polynomial in $\mathbf{k}$ and ω . Accordingly, the LW-LF approximate effective behavior is *local* in both space and time and corresponds, to the lowest order, to the usual wave equation as will be seen. Other asymptotic schemes yield other types of approximate effective behaviors, the nature of which, whether local or not for instance, depends strongly on the underlying microstructure and the hypotheses made regarding wavelengths, frequencies and phase contrast. In this section, we will restrict our attention to selected topics in LW-LF asymptotics.

2.1 Scaled motion equation

Our purpose is to define and calculate the terms of a Taylor expansion of $\mathbf{Z}_{\mathbf{k},\omega}$ as $\mathbf{k} \rightarrow \mathbf{0}$ and $\omega \rightarrow 0$, say at the same rate. We then introduce a “small” parameter ϵ in

¹Namely, $\mathbf{S} = \mathbf{S}^2 - \mathbf{S}^{1\top}$.

the expression of $\mathbf{Z}_{\mathbf{k},\omega}$ and consider the “scaled” effective impedance

$$\mathbf{Z}^\epsilon = \frac{1}{\epsilon^2} \mathbf{Z}_{\epsilon\mathbf{k},\epsilon\omega} = i\omega \boldsymbol{\rho}_{\epsilon\mathbf{k},\epsilon\omega}^e i\omega + i\omega \mathbf{S}_{\epsilon\mathbf{k},\epsilon\omega} \cdot i\mathbf{k} - i\mathbf{k} \cdot \mathbf{C}_{\epsilon\mathbf{k},\epsilon\omega}^e \cdot i\mathbf{k}. \quad (3.8)$$

As $\epsilon \rightarrow 0$, $\|\mathbf{k}\|$ and ω decay toward 0 at the same speed as needed. The parameter ϵ hence allows to summarize the hypotheses we made on $\mathbf{k}$ and on ω on one hand, and to carry one-variable Taylor expansions in what follows on the other hand. The scaled effective impedance describes a scaled effective motion equation which reads

$$\mathbf{Z}^\epsilon \cdot \tilde{\mathbf{U}}^\epsilon = \frac{1}{\epsilon^2} \mathbf{Z}_{\epsilon\mathbf{k},\epsilon\omega} \cdot \tilde{\mathbf{U}}^\epsilon = \tilde{\mathbf{f}},$$

or

$$\mathbf{Z}_{\epsilon\mathbf{k},\epsilon\omega} \cdot \tilde{\mathbf{U}}^\epsilon = \epsilon^2 \tilde{\mathbf{f}},$$

which is, by the definition of the effective impedance, the counterpart of the microscopic motion equation

$$(\nabla + i\epsilon\mathbf{k}) \cdot \{\mathbf{C} : [(\nabla + i\epsilon\mathbf{k}) \otimes^s \tilde{\mathbf{u}}^\epsilon]\} + \epsilon^2 \tilde{\mathbf{f}} = -(\epsilon\omega)^2 \rho \tilde{\mathbf{u}}^\epsilon, \quad (3.9)$$

with $\tilde{\mathbf{U}}^\epsilon$ being the volume average of $\tilde{\mathbf{u}}^\epsilon$.

Injecting the Taylor expansion

$$\tilde{\mathbf{u}}^\epsilon = \sum_r \epsilon^r \tilde{\mathbf{u}}^r$$

into equation (3.9) and solving for $\tilde{\mathbf{u}}^r$ at each order in ϵ , we obtain the terms of the expansions

$$\tilde{\mathbf{U}}^\epsilon = \sum_r \epsilon^r \tilde{\mathbf{U}}^r, \quad \mathbf{Z}^\epsilon = \sum_r \epsilon^r \mathbf{Z}^r.$$

2.2 Hierarchical motion equations

We start by writing the first three motion equations of orders ϵ^{-2} , ϵ^{-1} and $\epsilon^0 = 1$. The lowest order motion equation is

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_{\mathbf{k},\omega}^0)] = \mathbf{0},$$

which yields

$$\tilde{\mathbf{u}}_{\mathbf{k},\omega}^0(\mathbf{x}) = \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0.$$

The first order one takes the form

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_{\mathbf{k},\omega}^1 + i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0)] = \mathbf{0},$$

and leads to

$$\tilde{\mathbf{u}}_{\mathbf{k},\omega}^1(\mathbf{x}) = \tilde{\mathbf{U}}_{\mathbf{k},\omega}^1 + \mathbf{A}(\mathbf{x}) : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0),$$

where $\mathbf{A}$ is a microstructure dependent, $(\mathbf{k}, \omega)$ -independent, zero-average localization operator. Finally, the second order motion equation, averaged with respect to $\mathbf{x} \in T$, gives the classical quasi-static homogenized motion equation

$$i\mathbf{k} \cdot [\langle \mathbf{C} + \mathbf{C} : \nabla \mathbf{A} \rangle : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0)] \cdot \tilde{\mathbf{f}} = -\omega^2 \langle \rho \rangle \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0.$$

Such a procedure is more or less a standard one and was described in great detail, although in the absence of body forces, by [Boutin and Auriault \(1993\)](#).

In any case, the lowest order term $\mathbf{Z}_{\mathbf{k},\omega}^0$ in the LW-LF Taylor expansion of $\mathbf{Z}$ is equal to

$$\mathbf{Z}_{\mathbf{k},\omega}^0 = \mathbf{k} \cdot \langle \mathbf{C} + \mathbf{C} : (\nabla \otimes^s \mathbf{A}) \rangle \cdot \mathbf{k} - \omega^2 \langle \rho \rangle \mathbf{I}. \quad (3.10)$$

Writing and solving higher-order motion equations allow to calculate all other terms $\mathbf{Z}^r$. Next, we present two methods of defining the approximate effective impedance of a given order R denoted by $\mathbf{Z}^{(R)}$ for the first method and by $\mathbf{Z}^{[R]}$ for the second one.

2.3 Approximate effective behavior: first approach

This first approach is an elastodynamic extension of the method used by [Boutin \(1996\)](#) to describe microstructural effects in elastostatics.

Having calculated the expansion of $\mathbf{Z}^\epsilon$ up to order R , the approximate effective behavior can be defined through the approximate effective impedance

$$\mathbf{Z}_{\mathbf{k},\omega}^{(R)} = \sum_{r \leq R} \mathbf{Z}_{\mathbf{k},\omega}^r.$$

Therein, ϵ was substituted by 1 so as to recover an approximation of the original, unscaled, effective impedance. The approximate effective motion equation accordingly reads

$$\mathbf{Z}_{\mathbf{k},\omega}^{(R)} \cdot \tilde{\mathbf{U}}_{\mathbf{k},\omega} = \tilde{\mathbf{f}}. \quad (3.11)$$

The lowest order approximate effective behavior corresponds therefore to $\mathbf{Z}^{(0)} = \mathbf{Z}^0$ given in equation (3.10).

Going in the other direction, i.e., assuming given the Willis effective constitutive parameters, a general formula for the expansion of $\mathbf{Z}^\epsilon$ is

$$\begin{aligned} \mathbf{Z}^\epsilon = \sum_r \epsilon^r \sum_{\alpha+\beta=r} \frac{(i\omega)^\beta}{\alpha! \beta!} \frac{\partial^r}{\partial (i\mathbf{k}')^\alpha \partial (i\omega')^\beta} [i\omega \boldsymbol{\rho}_{\mathbf{k}',\omega'}^e i\omega \\ + i\omega \mathbf{S}_{\mathbf{k}',\omega'} \cdot i\mathbf{k} - i\mathbf{k} \cdot \mathbf{C}_{\mathbf{k}',\omega'}^e \cdot i\mathbf{k}]|_{\mathbf{k}'=0,\omega'=0} \odot^\alpha [\otimes^\alpha i\mathbf{k}] \end{aligned} \quad (3.12)$$

where $\odot^\alpha$ stands for contraction over α indices and $\otimes^\alpha i\mathbf{k}$ represents the α^{th} tensor power of $i\mathbf{k}$. Thus, we have

$$\begin{aligned} \mathbf{Z}_{\mathbf{k},\omega}^{(R)} = \sum_{\alpha+\beta \leq R} \frac{(i\omega)^\beta}{\alpha! \beta!} \frac{\partial^{\alpha+\beta}}{\partial (i\mathbf{k}')^\alpha \partial (i\omega')^\beta} [i\omega \boldsymbol{\rho}_{\mathbf{k}',\omega'}^e i\omega \\ + i\omega \mathbf{S}_{\mathbf{k}',\omega'} \cdot i\mathbf{k} - i\mathbf{k} \cdot \mathbf{C}_{\mathbf{k}',\omega'}^e \cdot i\mathbf{k}]|_{\mathbf{k}'=0,\omega'=0} \odot^\alpha [\otimes^\alpha i\mathbf{k}]. \end{aligned} \quad (3.13)$$

In the above expansions, we can restrict the summation to *even* powers of ω since all motion equations depend on ω uniquely through ω^2 (this would not be the case for a linear visco-elastic behavior for example). In particular, for $R = 0$, the above sum reduces to the sole term

$$\mathbf{Z}_{\mathbf{k},\omega}^{(0)} = \mathbf{Z}_{\mathbf{k},\omega}^0 = i\omega \rho_{0,0}^e i\omega + i\omega \mathbf{S}_{0,0} \cdot i\mathbf{k} - i\mathbf{k} \cdot \mathbf{C}_{0,0}^e \cdot i\mathbf{k},$$

meaning that the Willis effective constitutive parameters, to the lowest order, thanks to equation (3.10), are given by:

$$\begin{aligned} \rho_{0,0}^e &= \langle \rho \rangle \mathbf{I}, \\ \mathbf{S}_{0,0} &= \mathbf{0}, \\ \mathbf{C}_{0,0}^e &= \langle \mathbf{C} + \mathbf{C} : \nabla \otimes^s \mathbf{A} \rangle. \end{aligned}$$

The expression of $\mathbf{Z}^{(R)}$ for $R \geq 1$ requires pushing further in the hierarchical motion equations as noted earlier and will allow to characterize¹ the higher order terms in the expansion of the Willis effective constitutive parameters. Finally, to obtain the corresponding approximate effective motion equation in the real space-time domain, we simply use the mappings $i\omega \mapsto \partial/\partial t$ and $i\mathbf{k} \mapsto \nabla$ in the expression of $\mathbf{Z}^{(R)}$. For $R = 2$, the approximate effective behavior is that of a strain-gradient medium with micro-inertia, i.e., new terms with time derivatives, whose motion equation involves the following derivatives of $\mathbf{U}$:

$$\frac{\partial^{\alpha+2\beta}}{\partial \mathbf{x}^\alpha \partial t^{2\beta}} \mathbf{U} \text{ for } \alpha + 2\beta \leq 4. \quad (3.14)$$

An explicit example of such approximate effective motion equation is given in section 3.

2.4 Approximate effective behavior: second approach

This second approach is based on two ingredients: a localization operator and a variational formulation. This approach, introduced by [Smyshlyaev and Cherednichenko \(2000\)](#) for elastostatics, is generalized here to elastodynamics.

As for the localization operator, one can summarize the results of the first two motion equations of the above hierarchy by writing

$$\begin{aligned} \tilde{\mathbf{u}}^\epsilon(\mathbf{x}) &= \tilde{\mathbf{u}}_{\mathbf{k},\omega}^0(\mathbf{x}) + \epsilon \tilde{\mathbf{u}}_{\mathbf{k},\omega}^1(\mathbf{x}) + O(\epsilon^2) \\ &= \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0 + \epsilon \tilde{\mathbf{U}}_{\mathbf{k},\omega}^1 + \epsilon \mathbf{A}(\mathbf{x}) : i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0 + O(\epsilon^2) \\ &= \tilde{\mathbf{U}}^\epsilon + \epsilon \mathbf{A}(\mathbf{x}) : i\mathbf{k} \otimes^s \tilde{\mathbf{U}}^\epsilon + O(\epsilon^2), \end{aligned} \quad (3.15)$$

where $\mathbf{A}$ appears as the first order term of some localization operator $\mathbf{L}^\epsilon$ defined by

$$\tilde{\mathbf{u}}^\epsilon(\mathbf{x}) = \mathbf{L}^\epsilon(\mathbf{x}) \cdot \tilde{\mathbf{U}}^\epsilon. \quad (3.16)$$

¹Non-uniquely however.

As for the variational formulation, starting from (3.9), it can be seen that $\tilde{\mathbf{u}}^\epsilon$ is the stationary point of the action defined by

$$\begin{aligned} & \frac{1}{2} \left\langle \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \tilde{\mathbf{u}}^\epsilon \right]^{*\top} : \mathbf{C} : \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \tilde{\mathbf{u}}^\epsilon \right] - \omega^2 \rho \tilde{\mathbf{u}}^{\epsilon* \top} \cdot \tilde{\mathbf{u}}^\epsilon \right\rangle - \Re(\langle \tilde{\mathbf{u}}^{\epsilon*} \cdot \tilde{\mathbf{f}} \rangle) \\ &= \frac{1}{2} \left\langle \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \tilde{\mathbf{u}}^\epsilon \right]^{*\top} : \mathbf{C} : \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \tilde{\mathbf{u}}^\epsilon \right] - \omega^2 \rho \tilde{\mathbf{u}}^{\epsilon* \top} \cdot \tilde{\mathbf{u}}^\epsilon \right\rangle - \Re(\tilde{\mathbf{U}}^{\epsilon*} \cdot \tilde{\mathbf{f}}), \end{aligned}$$

over the set of $\mathcal{R}$ -periodic fields, where $\Re$ symbolizes the real part of a complex number. However, we know that such stationary point can be written as in equation (3.16). Upon the corresponding change of unknown field, the action becomes

$$\begin{aligned} & \frac{1}{2} \tilde{\mathbf{U}}^{\epsilon*} \cdot \left\langle \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \mathbf{L}^\epsilon \right]^{*\top} : \mathbf{C} : \left[\left(\frac{1}{\epsilon} \nabla + i\mathbf{k} \right) \otimes^s \mathbf{L}^\epsilon \right] - \omega^2 \rho \mathbf{L}^{\epsilon* \top} \cdot \mathbf{L}^\epsilon \right\rangle \cdot \tilde{\mathbf{U}}^\epsilon \\ & - \Re(\tilde{\mathbf{U}}^{\epsilon*} \cdot \tilde{\mathbf{f}}) = \frac{1}{2} \tilde{\mathbf{U}}^{\epsilon*} \cdot \mathbf{Z}^\epsilon \cdot \tilde{\mathbf{U}}^\epsilon - \Re(\tilde{\mathbf{U}}^{\epsilon*} \cdot \tilde{\mathbf{f}}). \quad (3.17) \end{aligned}$$

Finally, instead of truncating the expansion of $\mathbf{Z}^\epsilon$, it is the expansion of $\mathbf{L}^\epsilon$ that is truncated at order R so as to yield an approximate localization operator $\mathbf{L}^{(R)}$ which is then injected into (3.17). To sum up, we define the approximate effective impedance by

$$\mathbf{Z}_{\mathbf{k},\omega}^{[2R]} = \left\langle \left[(\nabla + i\mathbf{k}) \otimes^s \mathbf{L}_{\mathbf{k},\omega}^{(R)} \right]^{*\top} : \mathbf{C} : \left[(\nabla + i\mathbf{k}) \otimes^s \mathbf{L}_{\mathbf{k},\omega}^{(R)} \right] - \omega^2 \rho \mathbf{L}_{\mathbf{k},\omega}^{(R)* \top} \cdot \mathbf{L}_{\mathbf{k},\omega}^{(R)} \right\rangle. \quad (3.18)$$

The approximate effective motion equation is accordingly

$$\mathbf{Z}_{\mathbf{k},\omega}^{[2R]} \cdot \tilde{\mathbf{U}}_{\mathbf{k},\omega} = \tilde{\mathbf{f}}.$$

Impedance $\mathbf{Z}^{[2R]}$ is *not* a partial sum of any Taylor series and is different from $\mathbf{Z}^{(2R)}$. In particular, $\mathbf{Z}^{[0]}$ is different from the first term in the expansion: $\mathbf{Z}^0$. In fact, since $\mathbf{L}^{(0)} = \mathbf{I}$ as can be seen from equation (3.15), we have:

$$\mathbf{Z}_{\mathbf{k},\omega}^{[0]} = \mathbf{k} \cdot \langle \mathbf{C} \rangle \cdot \mathbf{k} - \omega^2 \langle \rho \rangle \mathbf{I}.$$

Using again equation (3.15), we see that for $R = 1$, the approximate effective behavior corresponds to a strain-gradient medium with micro-inertia whose motion equation involves the same derivatives as in (3.14) *except for* $\partial^4 \mathbf{U} / \partial t^4$. An explicit example of such approximate effective motion equation is given in section 3.

2.5 Discussion

It is of interest to discuss some of the differences distinguishing the above two approaches. Comparing, in particular, $\mathbf{Z}^{(2)}$ and $\mathbf{Z}^{[2]}$ will be enough.

First, both approximate impedances yield a medium of order 4 in the sense that the highest order derivative of $\mathbf{U}$ appearing in the motion equation is of order 4.

However, $\mathbf{Z}^{(2)}$ involves a 4th-order time derivative while $\mathbf{Z}^{[2]}$ does not and is hence formally identical to the impedance of the strain-gradient medium derived by Mindlin (1964).

Second, the most important advantage of the second approach is that $\mathbf{Z}^{[2]}$ guarantees a positive definite elastic energy. As a matter of fact, truncating $\mathbf{L}^\epsilon$ according to (3.15) amounts to changing the set of admissible fields of the action (3.17) and not the action itself. Since $\mathbf{C}$ is positive definite, so is the elastic energy defined by $\mathbf{Z}^{[2]}$. The first approach on the other hand does not necessarily yield a positive definite elastic energy. However, if one were to interpret the motion equation (3.11) asymptotically and solve for the $\tilde{\mathbf{U}}^r$ instead of $\tilde{\mathbf{U}}$, then equation (3.11) will be replaced by the following hierarchy:

$$\begin{aligned} \mathbf{Z}_{\mathbf{k},\omega}^0 \cdot \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0 &= \tilde{\mathbf{f}}, \\ \mathbf{Z}_{\mathbf{k},\omega}^0 \cdot \tilde{\mathbf{U}}_{\mathbf{k},\omega}^1 + \mathbf{Z}_{\mathbf{k},\omega}^1 \cdot \tilde{\mathbf{U}}_{\mathbf{k},\omega}^0 &= \mathbf{0} \dots \end{aligned}$$

The elastic energy at each stage remains therefore positive definite since $\mathbf{Z}^0$ defines a positive definite effective elastic stiffness tensor as is well-known.

Third, only the first approach defines a “good” limit process in the sense that $\mathbf{Z}^{(0)}$ is equal to the correct homogenization limit while $\mathbf{Z}^{[0]}$ is some kind of a low-contrast approximation of said limit. At the second order, the expression of the classical quasi-static effective stiffness tensor is recovered by $\mathbf{Z}^{[2]}$. Generally speaking, for $R \geq 1$, $\mathbf{Z}^{(2R)}$ and $\mathbf{Z}^{[2R]}$ agree over all terms of order lower than $R - 1$ (these are the coefficients of all derivatives in the approximate effective motion equation up to order $R + 1$).

Fourth, in order to define the effective 4th order medium, the first approach uses *more information* since it requires to solve for $\tilde{\mathbf{u}}^r$ up to $r = 3$ whereas to define $\mathbf{Z}^{[2]}$ only the solutions of order $r = 0$ and $r = 1$ are needed. It is indeed surprising that a 4th order behavior can be defined with only a first order approximation of the microscopic solution. Smyshlyaev and Cherednichenko (2000) assessed however the quality of the approximation in a variational sense. What should be stressed is that said variational sense is different from the usual asymptotic one. We refer to the cited paper for details.

Finally, there are other approaches in the literature seeking to define a strain-gradient behavior from microstructures and crystal structures in particular. We mention the works of DiVincenzo (1986) and Maranganti and Sharma (2007). They identified the coefficients of an enriched Lagrangian thanks to a computation of the dispersion curve. Their method has however one important shortcoming since it is completely based on free waves propagation. The implications are that the effective mass density is *postulated* to be the mass per unit volume and that micro-inertia terms can never appear. In fact, their approach is fundamentally different and cannot be obtained through the present formalism since they have not clearly defined what the effective displacement field must be.

2.6 The dispersion relation: a theorem

It has been argued in chapter 2 that the microscopic dispersion curve can be recovered from that of the Willis effective medium. A LW-LF approximation to the latter is hence also an approximation of the former.

2.6.1 On dispersion

The approximate dispersion relation is naturally defined through

$$\{(\mathbf{k}, \omega) \in \mathcal{E}^* \times \mathbb{R}^+, \quad \det \mathbf{Z}_{\mathbf{k}, \omega}^{(R)} = 0\}. \quad (3.19)$$

The lowest order approximation of the dispersion relation is based on $\mathbf{Z}^0$ from (3.10) and given by

$$\det(-\omega^2 \langle \rho \rangle \mathbf{I} + \mathbf{k} \cdot \mathbf{C}_{0,0}^e \cdot \mathbf{k}) = 0.$$

Such curve has d branches $\omega = \omega_n^{(0)}(\mathbf{k})$ with $n \in \{1, \dots, d\}$, and all are acoustic. We recall the fact that the $\omega_n^{(0)}$ are positively homogeneous functions of degree 1. This implies that all waves of branch number n propagating in direction $\mathbf{k}/\|\mathbf{k}\|$ have the same speed regardless of their wavelengths. In other words, the approximate effective behavior described by $\mathbf{Z}^0$ is *non-dispersive*, as is well known. The classical homogenization limit has therefore two limitations: no dispersive effects and no optical branches can be captured. Dispersive effects can be however modeled by taking into account higher order approximations: $\mathbf{Z}^{(R)}$, $R > 0$, which yield non-homogeneous functions $\omega = \omega_n^{(R)}(\mathbf{k})$, $n \in \{1, \dots, d\}$, and, correspondingly, a weakly nonlocal behavior. Nonetheless, LW-LF asymptotics of order R would still be unable to approximate the optical branches whatever R is. This intrinsic limitation of LW-LF asymptotics to acoustic branches is proven by the following theorem.

2.6.2 On the loss of the optical branches

Generally speaking, if the LW-LF assumption is not strictly satisfied, the approximate effective behavior will not have the expected accuracy. To improve accuracy, we can increase the truncation order R or make better suited assumptions. Increasing the order R augments the accuracy of the approximate effective behavior over some neighborhood, or *convergence domain*, of the $(\mathbf{k}, \omega)$ -space. However, increasing R does not change this domain. Here, we prove a theorem, for arbitrary 1D media, stating that the convergence domain of any Taylor expansions in $\mathbf{k}$ and ω are limited to a neighborhood of one branch, at most, of the dispersion curve. Further, we demonstrate that LW-LF asymptotics systematically *miss all* optical branches¹. The keystone is, unsurprisingly, Cauchy's theorem.

¹One can argue that it is obvious that strain-gradient media are unable to capture optical branches as they do not describe the corresponding kinematical “degrees of freedom”. To that I respond by saying that the Willis medium has only one displacement field and produces the microscopic dispersion curve in its entirety. To which one may retaliate by noting that the

Let $\mathbf{k}$ be a given wavenumber. From the results of chapter 2, we have a series representation of the effective Green's function $\mathbf{G}$:

$$\mathbf{G}_{\mathbf{k},\omega} \equiv \mathbf{Z}_{\mathbf{k},\omega}^{-1} = \sum_{n \geq 1} \frac{\langle \tilde{\psi}_{\mathbf{k},n} \rangle \otimes \langle \tilde{\psi}_{\mathbf{k},n} \rangle^*}{\omega_n^2(\mathbf{k}) - \omega^2}. \quad (3.20)$$

Note that therein, all $\tilde{\psi}_{\mathbf{k},n}$ and $\omega_n(\mathbf{k})$ are $\mathbf{k}$ -dependent and ω -independent. A corresponding expression for $\mathbf{Z}_{\mathbf{k},\omega}$ can be derived using Cramer's rule for instance. This is not done here since we shall consider $d = 1$ next.

For 1D media, the effective impedance and Green's tensors become scalars. In particular, we have

$$G_{k,\omega} = \sum_{n \geq 1} \frac{|\langle \tilde{\psi}_n(k) \rangle|^2}{\omega_n^2 - \omega^2}. \quad (3.21)$$

Therein and hereafter, $\tilde{\psi}_n(k) \equiv \tilde{\psi}_{k,n}$. We now assume that the eigenvalues ω_n are simple, for clarity purposes. Consider the limits of $G_{k,\omega}$ as $\omega \rightarrow \omega_n^+$ and $\omega \rightarrow \omega_{n+1}^-$ for two consecutive eigenfrequencies ω_n and ω_{n+1} with $\omega_n < \omega_{n+1}$. It can be seen that

$$G_{k,\omega_n^+} = -\infty \quad \text{and} \quad G_{k,\omega_{n+1}^-} = +\infty, \quad (3.22)$$

given $|\langle \tilde{\psi}_n(k) \rangle|^2 \neq 0$ and $|\langle \tilde{\psi}_{n+1}(k) \rangle|^2 \neq 0$.

In the LW-LF setting, one needs to consider the Taylor expansion of $Z_{\epsilon k, \epsilon \omega}$. More generally, let $\epsilon \mapsto p^\epsilon = (k^\epsilon, \omega^\epsilon)$ be a “path” connecting one point from the n^{th} branch, denoted by $(k^{\epsilon_n}, \omega_n(k^{\epsilon_n}))$ and attained for $\epsilon = \epsilon_n > 0$, to another from branch $n + 1$, called $(k^{\epsilon_{n+1}}, \omega_{n+1}(k^{\epsilon_{n+1}}))$ and reached for $\epsilon = \epsilon_{n+1} > \epsilon_n$. Without loss of generality, we assume that p^ϵ remains between the two branches n and $n + 1$. Recalling that $Z_{k,\omega} = 1/G_{k,\omega}$, the question is whether a Taylor expansion of $Z^\epsilon = Z_{k^\epsilon, \omega^\epsilon}$ can converge at, and recover, both eigenmodes $(k^{\epsilon_n}, \omega_n(k^{\epsilon_n}))$ and $(k^{\epsilon_{n+1}}, \omega_{n+1}(k^{\epsilon_{n+1}}))$.

First of all, if either $|\langle \tilde{\psi}_n(k = k^{\epsilon_n}) \rangle|^2$ or $|\langle \tilde{\psi}_{n+1}(k = k^{\epsilon_{n+1}}) \rangle|^2$ is null then the answer is negative since $Z_{k,\omega}$ is, already itself, unable to recover the corresponding eigenmode. We can assume henceforth that both $|\langle \tilde{\psi}_n(k = k^{\epsilon_n}) \rangle|^2$ and $|\langle \tilde{\psi}_{n+1}(k = k^{\epsilon_{n+1}}) \rangle|^2$ are non-null. Then, thanks to (3.22), $G^\epsilon = G_{k^\epsilon, \omega^\epsilon}$ changes sign when ϵ passes from ϵ_n to ϵ_{n+1} and attains, by continuity, zero for some value of ϵ in $]\epsilon_n, \epsilon_{n+1}[$. The existence of a zero value for G^ϵ implies that Z^ϵ “explodes” at some point on the path p^ϵ between ϵ_n and ϵ_{n+1} . This singularity of Z^ϵ , by Cauchy's theorem, forbids convergence of Taylor expansions of Z^ϵ near $\epsilon = 0 \leq \epsilon_n$ at $\epsilon = \epsilon_{n+1}$.

2.6.3 A few remarks

In conclusion, and in particular, LW-LF Taylor asymptotics can converge only near the acoustic branch. Optical branches are systematically *lost*. LW finite-frequency Taylor asymptotics converge near one optical branch at one time.

Willis medium is strongly nonlocal. I then ask: is a nonlocal medium somehow kinematically enriched?

For higher dimensions, singularities seem also to limit the convergence domain although not in the nice one-branch-at-a-time manner and a (much) weaker version of the above theorem holds but is not presented here.

In order to have a Taylor expansion valid for multiple branches, one needs a path $p = p^\epsilon$ which avoids the singularities of Z . Needless to say that such a path will have to get out of the (k, ω) -space and surf other dimensions. High-contrast asymptotics provide, sometimes, depending on the underlying microstructure, such paths. A LW-LF-high-contrast scaling can hence extend the convergence domain to englobing multiple acoustic and optical branches. We refer to papers by [Auriault and Bonnet \(1985\)](#) and by [Auriault and Boutin \(2012\)](#) for examples of such approximations.

Here, it is the definition of the approximate effective impedance in the first approach that is adopted: $Z^{(R)}$. Similar results for $Z^{[R]}$ are thought to hold. However, a rigorous proof has not been found yet.

Last, the loss of optical branches means that there could exist solutions to the approximate dispersion equation with complex ω . Such solutions are unstable and non-physical. A “tuning” technique, due to [Pichugin et al. \(2008\)](#), introduces artificial optical branches so that the approximate motion equation becomes stable. However, the optical modes thus introduced still have no physical meaning. In addition, note that a tuned motion equation is an implicit motion equation in the sense that derivatives of f appear when writing the tuned motion equation in forced regime. This explains how tuned approximate motion equations are not unique while the partial sums $Z^{(R)}$, and accordingly equation (3.11), are unique by the uniqueness of Taylor expansions.

3 Example: a 1D string

Next, the exact effective impedance Z of a two-phase 1D string is calculated, scaled and expanded in order to exemplify what has been presented in the previous section.

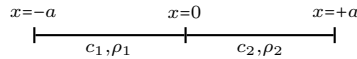


Figure 3.1. A unit cell of a 1D 2-phase string.

3.1 Setup and Willis effective behavior

The periodic 1D string to be studied has two phases of stiffness c_j and mass density ρ_j with $j \in \{1, 2\}$. A unit cell of length $2a$ is illustrated by figure 3.1. The waves propagating along the string are either transverse or longitudinal waves but not both, the example being genuinely 1D.

The motion equation (3.3) reduces, over each phase, to

$$c_j u'' + \tilde{f} e^{ikx} = -\omega^2 \rho_j u,$$

where the superscript $(')$ stands for d/dx and $\tilde{f}$ is the amplitude of body forces and is constant. The displacement u and traction Cu' are required to be continuous across phases and to satisfy the boundary conditions given by the periodicity of $u(x)e^{-ikx} = \tilde{u}(x)$ and of $[Cu'](x)e^{-ikx} = C(x)(\tilde{u}'(x) + ik\tilde{u}(x))$.

Once u is calculated, the effective impedance Z can be defined as $Z = \tilde{f}/\langle \tilde{u} \rangle$ and is a function, exclusively, of the period half-length a , stiffnesses c_1 and c_2 , mass densities ρ_1 and ρ_2 , frequency ω and wavenumber k . The general form of the solution u for phase j is known to be

$$u(x) = B_j^+ \exp\left(i\omega\sqrt{\rho_j/c_j}x\right) + B_j^- \exp\left(-i\omega\sqrt{\rho_j/c_j}x\right) + \frac{\tilde{f}}{c_j k^2 - \rho_j \omega^2} \exp(ikx),$$

where the $B_j^{+, -}$ are constants to be identified by the continuity and boundary conditions. Note that the above equation is valid as long as pairs (k, ω) satisfying $c_j k^2 - \rho_j \omega^2 \neq 0$ are excluded.

The identification of the constants presents no particular interest for our purpose and is skipped here. We directly give a closed form analytical expression for Z as

$$Z = P/Q, \quad (3.23)$$

where

$$\begin{aligned} P = a(c_1 k^2 - \rho_1 \omega^2)^2 (c_2 k^2 - \rho_2 \omega^2)^2 \{ & 4\sqrt{c_1 \rho_1 c_2 \rho_2} \cos(2ka) \\ & + (\sqrt{c_1 \rho_1} - \sqrt{c_2 \rho_2})^2 \cos\left[\omega(\sqrt{\rho_1/c_1} - \sqrt{\rho_2/c_2})a\right] \\ & - (\sqrt{c_1 \rho_1} + \sqrt{c_2 \rho_2})^2 \cos\left[\omega(\sqrt{\rho_1/c_1} + \sqrt{\rho_2/c_2})a\right] \}, \end{aligned}$$

and where Q is specified in 3.2.5. Note that Z can be continuously extrapolated over the previously excluded values of k and ω . The dependence of Z on k and ω is complex and cannot be easily interpreted. What is certain is that Z describes a nonlocal effective behavior in both space and time since it is *neither* a polynomial in k nor in ω .

The dispersion relation, as can be found in (Andrianov et al., 2008) and references therein, is given by

$$\begin{aligned} \cos(2ka) = & \frac{(\sqrt{c_1 \rho_1} + \sqrt{c_2 \rho_2})^2}{4\sqrt{c_1 \rho_1 c_2 \rho_2}} \cos\left[\omega(\sqrt{\rho_1/c_1} + \sqrt{\rho_2/c_2})a\right] \\ & - \frac{(\sqrt{c_1 \rho_1} - \sqrt{c_2 \rho_2})^2}{4\sqrt{c_1 \rho_1 c_2 \rho_2}} \cos\left[\omega(\sqrt{\rho_1/c_1} - \sqrt{\rho_2/c_2})a\right]. \end{aligned} \quad (3.24)$$

This is the same as $P = 0$ up to the $(c_j k^2 - \rho_j \omega^2)^2$ factors. In fact, the dispersion relation can be obtained from $Z = \det(Z) = 0$ and not $P = 0$ since zeros of P can get “canceled” by zeros of Q . In particular, whenever $(c_j k^2 - \rho_j \omega^2)^2$ is null, Q is null. This is not obvious and can be checked through calculations. Note that the dispersion curve is even, periodic¹ in k and of period π/a .

¹Incidentally, the dispersion curve is also periodic in the frequency ω if and only if the expression $\frac{\sqrt{\rho_1/c_1} - \sqrt{\rho_2/c_2}}{\sqrt{\rho_1/c_1} + \sqrt{\rho_2/c_2}}$ is a rational number.

A typical dispersion curve is plotted in figure 3.2. It shows the first three branches of the dispersion curve $Z = 0$: $\omega_n(k)$ with $n \in \{1, 2, 3\}$; the first two branches of the curve $1/Z = 0$ are also displayed: $\varpi_n(k)$ with $n \in \{1, 2\}$. We can see, as implied by our theorem, that $\omega_1 \leq \varpi_1 \leq \omega_2 \leq \varpi_2 \leq \omega_3$ and so on. In the plot, wavenumbers have been normalized by a factor of $1/a$ and frequencies ω by a factor $\omega_o = \frac{\pi}{2a\sqrt{\langle 1/C \rangle \langle \rho \rangle}}$. The

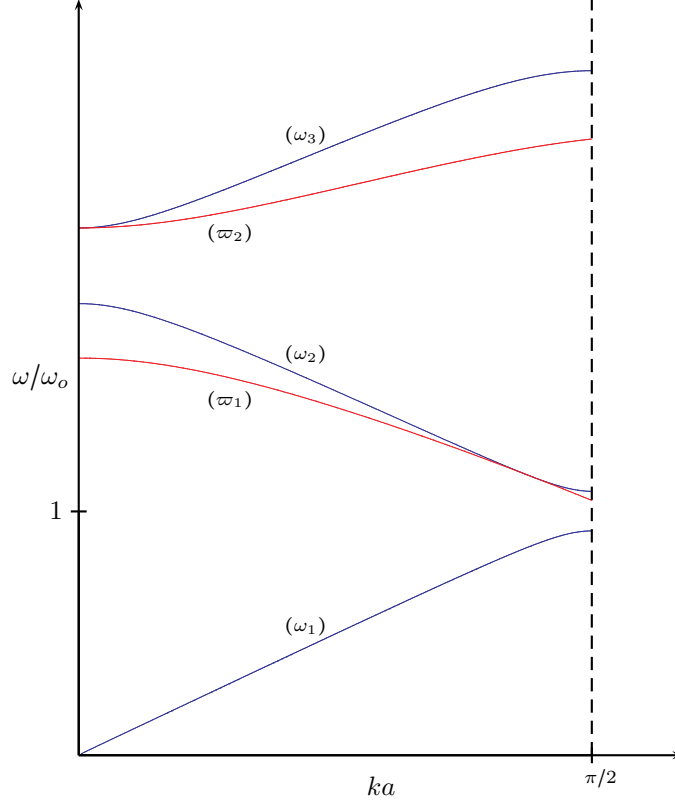


Figure 3.2. A typical dispersion curve. The first three branches, (ω_1) , (ω_2) and (ω_3) , are plotted against the first two branches of singularities of the effective impedance Z , ϖ_1 and ϖ_2 . The dashed line marks off the first Brillouin zone boundary.

numerical values used in the plots are:

$$c_1 = 10^{10} \text{N}, c_2 = 4.10^{12} \text{N}, \rho_1 = 4900 \text{kg/m}, \rho_2 = 900 \text{kg/m}, a = 5.10^{-3} \text{m}.$$

3.2 LW-LF asymptotics

We next explore second order expansions of the Willis impedance Z according to the two approaches discussed earlier and make a brief comparison with Mindlin's 1D second-order strain-gradient medium.

3.2.1 First approach

Define

$$Z^\epsilon = \frac{1}{\epsilon^2} Z_{\epsilon k, \epsilon \omega}.$$

The lowest order approximation to Z is therefore

$$Z^{(0)} = Z^0 = \lim_{\epsilon \rightarrow 0} Z^\epsilon = 2 \frac{c_1 c_2}{c_1 + c_2} k^2 - \frac{\rho_1 + \rho_2}{2} \omega^2,$$

as expected. An approximation of order 2 has the form

$$\begin{aligned} Z^{(2)} = & 2 \frac{c_1 c_2}{c_1 + c_2} k^2 - \frac{\rho_1 + \rho_2}{2} \omega^2 \\ & - \frac{a^2 (c_1 - c_2) [2(\rho_1 c_1 - \rho_2 c_2) + (c_1 + c_2)(\rho_1 - \rho_2)]}{24 (c_1 + c_2)^2} \omega^2 k^2 \\ & - \frac{a^2}{96} (\rho_1 - \rho_2)^2 \frac{(c_1 + c_2)}{c_1 c_2} \omega^4. \end{aligned}$$

The corresponding approximate effective motion equation, in the real space-time domain, has the expression

$$\begin{aligned} -2 \frac{c_1 c_2}{c_1 + c_2} u''(x, t) + \frac{\rho_1 + \rho_2}{2} \ddot{u}(x, t) \\ - \frac{a^2 (c_1 - c_2) [2(\rho_1 c_1 - \rho_2 c_2) + (c_1 + c_2)(\rho_1 - \rho_2)]}{24 (c_1 + c_2)^2} \ddot{u}''(x, t) \\ - \frac{a^2}{96} (\rho_1 - \rho_2)^2 \frac{(c_1 + c_2)}{c_1 c_2} \ddot{u}(x, t) = f(x, t), \end{aligned}$$

where the superscripted dot stands for d/dt . Note that there is no term in u'''' . This means that there are no purely spatial higher order strain-gradient effects. In 1D, this is *not* a particular feature of the approximation order. In fact, for all similar 1D systems, we have in statics:

$$Z_{\omega=0} = 2 \frac{c_1 c_2}{c_1 + c_2} k^2. \quad (3.25)$$

Higher order strain gradient effects, and general strong nonlocal effects, caused by heterogeneities appear only under dynamic loadings or when homogenizing in higher dimensions (2D and 3D layered composites for instance), at least according to Willis theory. Strain gradient effects do appear in LW asymptotics when *continualizing* discrete 1D lattices (see for example [Pichugin et al., 2008](#)). At this stage, *continualization* of discrete lattices and *homogenization* of already continuous media need to be clearly distinguished.

3.2.2 Second approach

It is easy to see that the first order localization tensor is given by

$$A(y) = \left(\frac{1}{C(y)} - \left\langle \frac{1}{C} \right\rangle \right) \frac{y}{\langle 1/C \rangle} - \frac{a}{2} \frac{c_1 - c_2}{c_1 + c_2}.$$

When this expression is injected into (3.18), we get

$$Z^{[2]} = 2 \frac{c_1 c_2}{c_1 + c_2} k^2 - \frac{\rho_1 + \rho_2}{2} \omega^2 - \frac{a^2}{24} \frac{(c_1 - c_2)^2 (\rho_1 + \rho_2)}{(c_1 + c_2)^2} \omega^2 k^2 + \frac{a^2}{24} \frac{(c_1 - c_2)^2}{c_1 + c_2} k^4.$$

As expected, no ω^4 terms appear. However, the k^4 coefficient is non-null contrarily to what happens in the first approach. Therefore, for higher order approximations, we expect the coefficient of k^4 to be corrected and eventually to be set to zero (the same goes for all k^n with $n > 2$, according to (3.25)).

3.2.3 Signature

In order to compare both approximate effective models with Mindlin's theory, it is convenient to define a signature which is the series of signs of the coefficients appearing in the expression of the effective impedance. The results are as follows:

	k^2	ω^2	$k^2 \omega^2$	k^4	ω^4
Mindlin (1964)	+	-	-	+	0
first approach	+	-	+ or -	0	-
second approach	+	-	-	+	0

It is seen that the second approach yields the same signature as in Mindlin's theory (for 1D media) whereas the first approach has generally a different signature.

3.2.4 Approximate dispersion curve

The approximate dispersion curves, $\omega_1^{(0)}(k)$ and $\omega_1^{(2)}(k)$, are calculated, according to (3.19) and have the equations

$$Z^{(0)} = 0 \quad \text{and} \quad Z^{(2)} = 0.$$

They are plotted in figure 3.3 and are almost indistinguishable. Note that these approximations have only acoustic branches as expected. As can be seen, they yield a good approximation for low frequencies (near the acoustic branch) and for almost all low wavelengths up to, and except near, the frontier of the first Brillouin zone.

The same goes for the second approach which defines an approximation to the dispersion curve, $\omega^{[2]}(k)$, of equation:

$$Z^{[2]} = 0.$$

Curve $\omega^{[2]}(k)$ is a poor approximation except in the quasi-static range near $\omega = 0$, $k = 0$ (see figure 3.3). As said earlier, the second approach defines an approximate effective impedance in a variational sense and is based on the asymptotics of the localization operator. The first approach on the other hand relies on the Taylor series of Z directly. Having $Z = \det(\mathbf{Z})$ in 1D, it is understandable that the first approach yields a better approximation of the dispersion curve. In higher dimensions, similar observations are expected to hold even though a truncated Taylor series of $\mathbf{Z}$ does not produce a truncated Taylor series of $\det(\mathbf{Z})$. This is because the error committed therein is expected to remain smaller than the one committed by approximating $\det(\mathbf{Z})$ based on a truncated series of the localization operator.

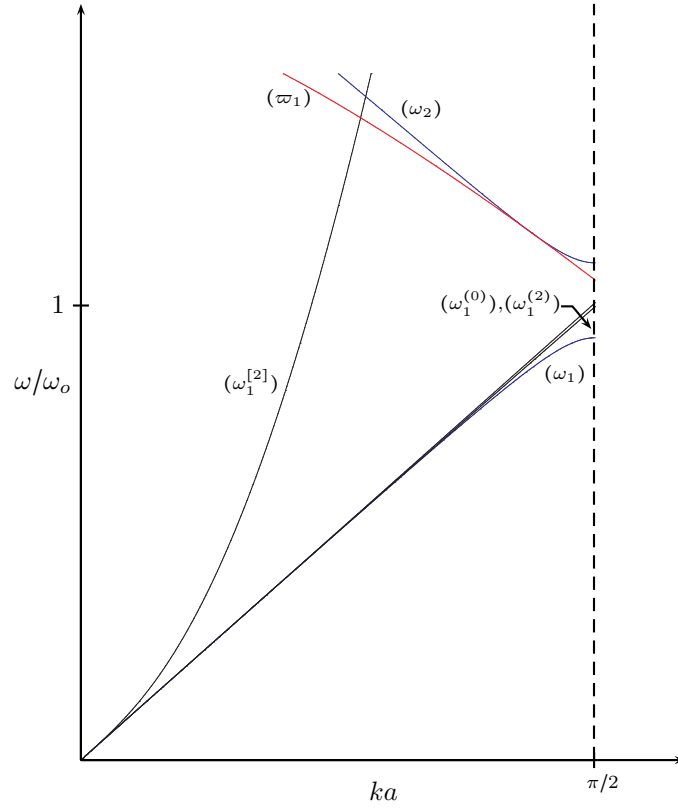


Figure 3.3. LW-LF approximations to the dispersion curve.

3.2.5 The expression of Q

On the off chance that it is of interest, the denominator Q of Z from equation (3.23) is given by

$$Q = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6,$$

with

$$\begin{aligned} Q_1 = & 2a\sqrt{c_1\rho_1c_2\rho_2} \times \\ & (c_2k^2 - \rho_2\omega^2)(c_1k^2 - \rho_1\omega^2) \left[(c_1 + c_2)k^2 - (\rho_1 + \rho_2)\omega^2 \right] \cos(2ka) \\ & - 4\sqrt{c_1\rho_1c_2\rho_2}(c_1\rho_2 - c_2\rho_1) \left[(c_1 - c_2)k^2 - (\rho_1 - \rho_2)\omega^2 \right] k\omega^2 \sin(2ka), \end{aligned}$$

$$\begin{aligned} Q_2 = & \sqrt{\rho_2c_2} \left[\sqrt{\rho_1}(c_1 - c_2)k - \sqrt{c_1}(\rho_1 - \rho_2)\omega \right]^2 \omega \\ & \left\{ (\sqrt{c_1}k + \sqrt{\rho_1}\omega)^2 \sin \left[a(k + \sqrt{\rho_1/c_1}\omega) \right] \right. \\ & \left. - (\sqrt{c_1}k - \sqrt{\rho_1}\omega)^2 \sin \left[a(k - \sqrt{\rho_1/c_1}\omega) \right] \right\}, \end{aligned}$$

$$\begin{aligned} Q_3 = & \sqrt{c_1\rho_1} \left[\sqrt{\rho_2}(c_1 - c_2)k - \sqrt{c_2}(\rho_1 - \rho_2)\omega \right]^2 \omega \\ & \left\{ (\sqrt{c_2}k + \sqrt{\rho_2}\omega)^2 \sin \left[a(k + \sqrt{\rho_2/c_2}\omega) \right] \right. \\ & \left. - (\sqrt{c_2}k - \sqrt{\rho_2}\omega)^2 \sin \left[a(k - \sqrt{\rho_2/c_2}\omega) \right] \right\}, \end{aligned}$$

$$\begin{aligned} Q_4 = & \frac{a}{2}(c_1k^2 - \rho_1\omega^2)(c_2k^2 - \rho_2\omega^2) \left[(c_1 + c_2)k^2 - (\rho_1 + \rho_2)\omega^2 \right] \\ & \left\{ (\sqrt{c_1\rho_1} - \sqrt{c_2\rho_2})^2 \cos \left[\omega a(\sqrt{\rho_2/c_2} - \sqrt{\rho_1/c_1}) \right] \right. \\ & \left. - (\sqrt{c_1\rho_1} + \sqrt{c_2\rho_2})^2 \cos \left[\omega a(\sqrt{\rho_2/c_2} + \sqrt{\rho_1/c_1}) \right] \right\}, \end{aligned}$$

$$\begin{aligned} Q_5 = & -\sqrt{c_1\rho_1c_2\rho_2} \left[(c_1 - c_2)k^2 - (\rho_1 - \rho_2)\omega^2 \right]^2 \omega \\ & \left\{ (\sqrt{c_1\rho_1} + \sqrt{c_2\rho_2}) \sin \left[\omega a(\sqrt{\rho_2/c_2} + \sqrt{\rho_1/c_1}) \right] \right. \\ & \left. - (\sqrt{c_1\rho_1} - \sqrt{c_2\rho_2}) \sin \left[\omega a(\sqrt{\rho_2/c_2} - \sqrt{\rho_1/c_1}) \right] \right\}, \end{aligned}$$

and at last

$$\begin{aligned} Q_6 = & -(c_1\rho_2 - c_2\rho_1)^2 \omega^3 k^2 \\ & \left\{ (\sqrt{c_1\rho_1} + \sqrt{c_2\rho_2}) \sin \left[\omega a(\sqrt{\rho_2/c_2} + \sqrt{\rho_1/c_1}) \right] \right. \\ & \left. + (\sqrt{c_1\rho_1} - \sqrt{c_2\rho_2}) \sin \left[\omega a(\sqrt{\rho_2/c_2} - \sqrt{\rho_1/c_1}) \right] \right\}. \end{aligned}$$

4 A panel of asymptotic approaches

Next, we briefly comment on a number of asymptotic homogenization approaches and establish a few connections with Willis theory. In particular, we show that many of said homogenization approaches make use of the same notion of effective behavior while scaling and approximating it over different neighborhoods, i.e., under different assumptions on orders of magnitude.

Asymptotic homogenization methods proceed systematically in two steps: two-scale representations and imbeddings.

4.1 Two scales

The microscopic displacement field $\mathbf{u}(\mathbf{x})$ depends on $\mathbf{x}$ due to two sources of “inhomogeneities”. The first one is microstructure as modeled by $\mathbf{C}(\mathbf{x})$ and $\rho(\mathbf{x})$. The second one is loading and stems from $\mathbf{f}(\mathbf{x})$. For the sake of argument, we assume that we have a closed-form expression of $\mathbf{u}(\mathbf{x})$ where $\mathbf{x}$ appears multiple times. In principle, to each appearance, we can assign one of the two mentioned inhomogeneity sources and we replace $\mathbf{x}$ by an auxiliary variable, $\mathbf{x}'$, each time the appearance is microstructure-related. The obtained expression, which depends on both $\mathbf{x}$ and $\mathbf{x}'$, corresponds to a two-scale representation of the displacement field, $\mathbf{u}(\mathbf{x}, \mathbf{x}')$, where loading and microstructure influences have been segregated.

The variable $\mathbf{x}'$ is generally understood to be a “fast” variable whereas $\mathbf{x}$ is seen as a “slow” variable. Whether it is really the case or not is governed by the assumptions we will make next on the orders of magnitude of wavenumbers. For now however, the field $\mathbf{u}(\mathbf{x})$ being a priori unknown, its two-scale representation is implicitly defined as the solution to the two-scale motion equation

$$(\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}'}) \cdot \{ \mathbf{C}(\mathbf{x}') : [(\nabla_{\mathbf{x}} + \nabla_{\mathbf{x}'}) \otimes^s \mathbf{u}(\mathbf{x}, \mathbf{x}')] \} + \mathbf{f}(\mathbf{x}) = -\omega^2 \rho(\mathbf{x}') \mathbf{u}(\mathbf{x}, \mathbf{x}'),$$

where we have applied the above procedure. Therein, the gradient operators $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{x}'}$ act with respect to $\mathbf{x}$ and $\mathbf{x}'$ respectively. Note that all fields are $\mathcal{R}$ -periodic in $\mathbf{x}'$ and that putting $\mathbf{x}' = \mathbf{x}$ yields back the original fields and motion equation.

Let us prescribe a plane wave body force $\mathbf{f}(\mathbf{x}) = \tilde{\mathbf{f}} e^{i\mathbf{k} \cdot \mathbf{x}}$. The above equation having *coefficients homogeneous in $\mathbf{x}$* , its solution $\mathbf{u}_{\mathbf{k}, \omega}(\mathbf{x}, \mathbf{x}')$ can be looked for in the form

$$\mathbf{u}_{\mathbf{k}, \omega}(\mathbf{x}, \mathbf{x}') = \tilde{\mathbf{u}}_{\mathbf{k}, \omega}(\mathbf{x}') e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (3.26)$$

The two-scale motion equation becomes

$$(\nabla_{\mathbf{x}'} + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}') : [(\nabla_{\mathbf{x}'} + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}, \omega}(\mathbf{x}')] \} + \tilde{\mathbf{f}} = -\omega^2 \rho(\mathbf{x}') \tilde{\mathbf{u}}_{\mathbf{k}, \omega}(\mathbf{x}')$$

and needs to be solved under $\mathcal{R}$ -periodic boundary conditions and homogeneous body force amplitude $\tilde{\mathbf{f}}$. The foregoing equation is identical to equation (3.4) of the summarized Willis theory and have therefore the same displacement solution $\mathbf{u}_{\mathbf{k}, \omega}$.

Consequently, the two-scale representation of the FB waves used in Willis theory is given by (3.26). FB waves appear then to naturally separate the fast and the

slow variables. The latter can be seen as $\mathbf{k} \cdot \mathbf{x}$ whereas the former is $\mathbf{x} \pmod{\mathcal{R}}$ by $\mathcal{R}$ -periodicity. By the same logic, all methods based on an “averaging-over- $\mathbf{x}$ ” scheme are equivalent to one another and are equivalent to Willis theory in the sense that they all make use of the same notion of effective displacement

$$\langle \tilde{\mathbf{u}}_{\mathbf{k},\omega}(\mathbf{x}') \rangle_{\mathbf{x}'} e^{i\mathbf{k} \cdot \mathbf{x}} = \langle \tilde{\mathbf{u}}_{\mathbf{k},\omega} \rangle e^{i\mathbf{k} \cdot \mathbf{x}}$$

and, accordingly, of the same notion of effective behavior incarnated by $\mathbf{Z}_{\mathbf{k},\omega}$. Differences rise next as different asymptotic approximations are derived for the *same* quantity $\mathbf{Z}_{\mathbf{k},\omega}$.

4.2 Imbeddings

In order to approximate the microscopic displacement solution, assumptions are made on the order of magnitude of several of the parameters of the elastodynamic problem at hand. Such parameters include the loading frequency ω or wavenumber $\mathbf{k}$, the size of an inclusion in a matrix-inclusion composite, the size of a unit-cell, the mean stiffness, the mean mass density and any phase contrast parameter, for instance.

The basic idea is to make Taylor expansions with respect to one or several of the above parameters. The assumptions concerning these parameters are summarized by the introduction of a “small” scaling variable ϵ such that said assumptions can be recovered from the limit $\epsilon \rightarrow 0$. Taylor expansions are then carried with respect to ϵ in the vicinity of $\epsilon = 0$. Once calculations of the unknown fields are done, ϵ can be substituted by 1 so as to recover the original, unscaled, fields.

Borrowing the words of [Bensoussan et al. \(1978\)](#), introducing ϵ amounts to “imbedding” the motion equation (3.4) into “a family of problems (parametrized by ϵ)”:

$$(\nabla + i\mathbf{k}^\epsilon) \cdot \{ \mathbf{C}^\epsilon(\mathbf{x}) : [(\nabla + i\mathbf{k}^\epsilon) \otimes^s \tilde{\mathbf{u}}^\epsilon(\mathbf{x})] \} + \tilde{\mathbf{f}} = -(\omega^\epsilon)^2 \rho^\epsilon(\mathbf{x}) \tilde{\mathbf{u}}^\epsilon(\mathbf{x}). \quad (3.27)$$

While the scaling of $\mathbf{C}$, ρ , $\mathbf{k}$ and ω can be chosen arbitrarily according to our assumptions, the scaling, or the expansion, of the solution $\tilde{\mathbf{u}}$ is to be deduced so as to fulfill the above equation. Body force scaling is of no importance since $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{u}}$ are in a linear relationship. Examples of what these scalings might be are given, with non-exhaustive references, in table 3.1. Finally, it is assumed that all the fields h^ϵ have Taylor expansions with respect to ϵ of the form

$$h^\epsilon = \sum_r \epsilon^r h^r.$$

Thus, from (3.27), a hierarchy of motion equations, one for each order r of ϵ , is recovered and solved up to some desired degree of accuracy. A corresponding approximation for the effective impedance is obtained in a similar manner to the LW-LF setting detailed above.

Label	Scaling	Assumptions
LW-LF (Boutin and Auriault, 1993)	$\mathbf{k}^\epsilon = \epsilon \mathbf{k},$ $\omega^\epsilon = \epsilon \omega$	$\ \mathbf{k}\ \ll 2\pi/\ell,$ $\omega \ll \omega_o$
LW-FF (Craster et al., 2010)	$\mathbf{k}^\epsilon = \epsilon \mathbf{k},$ $\omega^\epsilon = \omega^0 + \epsilon \omega^1 + \dots$	$\ \mathbf{k}\ \ll 2\pi/\ell,$ $\omega \approx \omega^0 \neq 0$
High-contrast (LW-LF) (Auriault and Bonnet, 1985)	$\mathbf{k}^\epsilon = \epsilon \mathbf{k},$ $\omega^\epsilon = \epsilon \omega,$ $\mathbf{C}^\epsilon = \mathbf{C}_2 + \chi_1(\epsilon^2 \mathbf{C}_1 - \mathbf{C}_2)$	$\ \mathbf{k}\ \ll 2\pi/\ell,$ $\omega \ll \omega_o,$ $\ \mathbf{C}_1\ \ll \ \mathbf{C}_2\ $
Low-contrast (Willis, 1985, 1997)	$\mathbf{C}^\epsilon = \langle \mathbf{C} \rangle + \epsilon \mathbf{C}',$ $\rho^\epsilon = \langle \rho \rangle + \epsilon \rho'$	$\ \mathbf{C}'\ \ll \ \langle \mathbf{C} \rangle\ ,$ $\rho' \ll \langle \rho \rangle$

Table 3.1. Examples of expansions from the literature. Symbol $\approx$ means “of the same order of magnitude”, symbol $\ll$ means “very small with respect to”, ℓ and ω_o are, respectively, a characteristic length and frequency of the microstructure, $\mathbf{C}_1$ and $\mathbf{C}_2$ are the stiffness tensors of a two-phase composite, χ_1 is the indicator function of phase 1 in that composite, $\mathbf{C}' = \mathbf{C} - \langle \mathbf{C} \rangle$, $\rho' = \rho - \langle \rho \rangle$ and $\|\mathbf{C}\|$ is the norm of $\mathbf{C}$.

4.3 Concluding remarks on the suitability of averaging

In chapter 2, we have seen that averaging the FB amplitude, i.e., averaging over the fast variable, is only well justified under a homogenizability condition bounding frequencies to relatively low ones. In high-frequency regimes, averaging was shown to yield some negligible components of the displacement field.

For asymptotic methods, to the lowest order, it appears that the displacement field $\mathbf{u}^0(\mathbf{x}, \mathbf{x}')$ usually admits the scale-separated form

$$\mathbf{u}^0(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x}') \mathbf{D}^0(\mathbf{x}).$$

The inconvenience of Willis theory is that it describes

$$\mathbf{U}^0(\mathbf{x}) = \langle \phi \rangle \mathbf{D}^0(\mathbf{x}),$$

where $\langle \phi \rangle$ can be small making $\mathbf{U}^0$ irrelevant to the undergoing physics. It is in fact better to consider $\mathbf{D}^0$ as being the relevant macroscopic displacement field as it gives the correct asymptotics of the microscopic field to the lowest order given a set of assumptions. Now $\mathbf{U}^0$ and $\mathbf{D}^0$ will be governed by the same kind of motion equations and will yield the same approximation to the dispersion curve, however, the associated oscillation modes will be better captured by $\mathbf{D}^0$. One can refer to the high-contrast setting investigated by Auriault and Bonnet (1985), by Smyshlyaev (2009), by Auriault and Boutin (2012) and by others, and to the finite frequency one studied by Craster et al. (2010), Nolde et al. (2011), Antonakakis et al. (2014), Colquitt et al. (2014) and

Boutin et al. (2014). These approaches also have some limited validity domain in the same manner Willis theory does. They have the compared merit of being simpler.

It is worth mentioning that a weighted form of Willis theory first introduced by Milton and Willis (2007) defines $\mathbf{U}^0$ as a weighted average of $\mathbf{u}^0$:

$$\mathbf{U}^0(\mathbf{x}) = \langle w\phi \rangle \mathbf{D}^0(\mathbf{x}),$$

where w is some weight. By choosing w adequately, i.e., $w = \phi / \langle \phi^2 \rangle$, Willis theory succeeds in recovering $\mathbf{U}^0 = \mathbf{D}^0$. The bottom line is that Willis theory, in its most general form, appears to be the non-asymptotic counterpart of a fairly large family of asymptotic homogenization methods, each of which is valid under some assumptions over the orders of magnitude of some physical parameters.

The purpose of the next chapter is to present a theory where we try to extend the validity domain of these approaches by enriching the kinematics of the effective medium.

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Chapter 4

A generalized elastodynamic homogenization theory

Mais en Physique, il n'en est pas de même : si les phénomènes physiques sont dus à des mouvements, c'est aux mouvements de molécules que nous ne voyons pas. Si alors l'accélération d'un des corps que nous voyons nous paraît dépendre d'autre chose que des positions ou des vitesses des autres corps visibles ou des molécules invisibles dont nous avons été amenés antérieurement à admettre l'existence, rien ne nous empêchera de supposer que cette autre chose est la position ou la vitesse d'autres molécules dont nous n'avions pas jusque-là soupçonné la présence. La loi se trouvera sauvegardée.

Qu'on me permette d'employer un instant le langage mathématique pour exprimer la même pensée sous une autre forme. Je suppose que nous observions n molécules et que nous constatons que leurs $3n$ coordonnées satisfont à un système de $3n$ équations différentielles du quatrième ordre (et non du deuxième ordre, comme l'exigerait la loi d'inertie). Nous savons qu'en introduisant $3n$ variables auxiliaires, un système de $3n$ équations du quatrième ordre peut être ramené à un système de $6n$ équations de deuxième ordre. Si alors nous supposons que ces $3n$ variables auxiliaires représentent les coordonnées de n molécules invisibles, le résultat est de nouveau conforme à la loi d'inertie.

Henri Poincaré, *La Science et l'Hypothèse*

For periodically inhomogeneous media, a generalized theory of elastodynamic homogenization is proposed so that even the long-wavelength and low-frequency asymptotic expansions of the resulting effective (or macroscopic) motion equation can, approximately but simultaneously, capture all the acoustic and some of the optical branches of the microscopic dispersion curve. This is in sharp contrast with the presently available elastodynamic homogenization theories. The key to constructing the generalized theory resides in incorporating rapidly oscillating body forces as microscopic and macroscopic loadings and in postulating an energy equivalency principle. By this principle, an

effective displacement field is naturally defined as the projection of a microscopic one onto the dual to the space of body forces. By varying the space of body forces, the generalized theory leads to a family of elastodynamic homogenization schemes with effective kinematic and dynamic quantities of different degrees of complexity. To illustrate these results, a two-phase string is studied in detail.

1 Introduction

The elastodynamic homogenization approaches reported up to now in the literature are observed to run into difficulties when being used to model dynamical effects over a wide frequency range.

1. The first-order Long-Wavelength (LW) Low-Frequency (LF) homogenization approaches as described by [Bensoussan et al. \(1978\)](#) and [Sanchez-Palencia \(1980\)](#) yield a homogeneous substitution Cauchy medium which misses all dispersive effects and all internal resonances, i.e., all optical oscillation modes.
2. The higher-order LW-LF asymptotic homogenization approaches ([Boutin and Auriault, 1993](#); [Andrianov et al., 2008](#)) lead to effective strain-gradient media which can model well dispersive behaviors and size effects but are valid only near the acoustic branches independently of the order of the asymptotic approximations used.
3. The high-frequency asymptotic approaches ([Craster et al., 2010](#); [Nolde et al., 2011](#); [Antonakakis et al., 2014](#); [Colquitt et al., 2014](#); [Boutin et al., 2014](#)) are successful in capturing high-frequency optical modes but still valid only in the vicinity of some finite frequency.
4. The high-contrast asymptotic approaches of [Auriault and Bonnet \(1985\)](#), [Smyshlyaev \(2009\)](#) and [Auriault and Boutin \(2012\)](#), among others, have a wide frequency validity domain englobing an infinite number of optical branches. However, the corresponding effective behavior is complex and nonlocal in time.
5. The non-asymptotic theory of [Willis \(1997, 2011\)](#) yields exactly the whole dispersion curve. Nonetheless, the described effective fields are only relevant for low frequencies, not to mention the inextricability of the effective behavior.

The main purpose of the present chapter is to construct a generalized theory of elastodynamic homogenization for periodic media which improves the quality of the Willis effective behavior as an approximation to the microscopic behavior in a way that LW-LF asymptotic expansions become able to capture, approximately but simultaneously, all the acoustic and some of the optical branches of the microscopic dispersion curve. To achieve this purpose, a balance between the microscopic and macroscopic works, hereafter called Energy Equivalency Principle (EEP), is postulated. Once a set of expected body force loadings is specified, the EEP dictates what the

effective, or macroscopic, displacement field for a given microscopic displacement field should be. Then, by incorporating various rapidly oscillating body forces on the microscale and on the macroscale, the error committed during the upscaling process is reduced. With respect to Willis theory, we underline two major differences. First, the expected loadings are much richer than those employed by Willis (1997, 2011). This has the consequence of providing an extended frequency validity domain. Second, the EEP concerns works and not their expectancies. From the physical standpoint, this leads to a clear distinction between the macroscale and the microscale in terms of wavelengths. Nevertheless, it should be pointed out that the generalized theory presented here is by construction limited to periodically inhomogeneous media while Willis theory is formally valid both for periodically and randomly inhomogeneous media.

The chapter is organized as follows. In section 2, the EEP is first postulated; the space of admissible body forces is then defined as the set of macroscopically applied loadings; the effective displacement field associated to a microscopic displacement is obtained by the EEP; the effective motion equation is finally derived in a formal way. In section 3, the generalized kinematics of microdilatation-, Cosserat-, microstretch- and micromorphic-like media are discussed as examples of the generalized theory of section 2. As an application, in section 4, we study the LW-LF asymptotic expansions of the effective motion equation so as to obtain the simplest generalized effective substitution medium. An analytical expression is provided for the approximate effective motion equation of a 1D two-phase string. Exact and approximate dispersion curves are plotted and compared to show how the resulting asymptotic model can simultaneously capture acoustic and optical branches while conserving a low-order local motion equation. Finally, for arbitrary d -dimensional media, we prove that the present theory is able to produce simultaneously, in the LW-LF limit, hints of any combination of dispersion branches.

2 A general theory

Let Ω be the infinite periodically inhomogeneous medium of chapter 1. The homogenization of Ω amounts to finding the motion equation, hereafter called “effective motion equation”, of a homogeneous medium substituting the initial inhomogeneous one, under an energy equivalency constraint to be specified. Said constraint is based on an equality between two external work quantities and is referred to hereafter as the Energy Equivalency Principle (EEP). Next, the EEP is postulated, given a simple form and exploited to define the effective displacement field which is subsequently thoroughly investigated and interpreted. A formal derivation of the effective motion equation is then presented. The effective constitutive behavior is nonlocal in both space and time which raises questions about its uniqueness (Fietz and Shvets, 2010; Willis, 2011). In order to avoid this difficulty, we will be interested only in the effective motion equation which is unique. Nonetheless, we will derive expressions for the generalized stress, momentum, velocity and strain measures which are, in particular, needed for determining an effective constitutive law. Finally, in terms of these measures,

a generalized Hill-Mandel relation is proven to hold.

2.1 Energy equivalency¹

In classical static or quasi-static homogenization, an energy equivalency relation, known as Hill-Mandel lemma, is proven for a family of boundary conditions prescribed on a representative volume element as macroscopic loadings. Once the boundary conditions have been specified, Hill-Mandel lemma can be used to define, by duality, the macroscopic stress in a strain-based approach or the macroscopic strain in a stress-based approach. In the present formulation, admissible body forces applied globally to Ω instead of locally prescribed boundary conditions are taken to be the macroscopic loading. Then, an EEP is postulated so as to dualize body forces and displacements. This duality will allow us to define the macroscopic displacement field, called $\mathbf{D}$, in terms of the microscopic one $\mathbf{u}$, once admissible body forces have been imposed.

Let $\mathcal{F}$ be the space of FB amplitudes $\tilde{\mathbf{f}}_{\mathbf{k}}$ of *admissible* body forces $\mathbf{f}$. In other words, having the FB expansion

$$\mathbf{f}(\mathbf{x}) = \int_{T^*} \tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) e^{i\mathbf{k} \cdot \mathbf{x}},$$

the loading $\mathbf{f}$ is admissible if and only if $\tilde{\mathbf{f}}_{\mathbf{k}}$ belongs to $\mathcal{F}$ for all the wavenumbers $\mathbf{k}$ of T^* . The elements of $\mathcal{F}$ are seen as external loadings likely to be applied to Ω . Note that *they will remain the same* after the scale transition. The space $\mathcal{F}$ acts as a parameter of the approach to be elaborated and needs to be chosen adequately.

Next, let $\mathcal{F}^*$, the space dual to $\mathcal{F}$, be the space of FB amplitudes $\tilde{\mathbf{D}}_{\mathbf{k}}$ of admissible effective displacement fields² $\mathbf{D}$. For a given microscopic displacement field $\mathbf{u}$, the corresponding effective (or macroscopic) displacement field is defined as the *unique* admissible displacement such that

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{D}^* = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}^* \quad (4.1)$$

for all admissible virtual body forces $\mathbf{f}$. Physically, the EEP (4.1) can be interpreted as requiring that the effective displacement field $\mathbf{D}$ associated to a given microscopic field $\mathbf{u}$ be such that the work done by every admissible virtual body force $\mathbf{f}$ in the course of $\mathbf{D}$ is equal to the one done by $\mathbf{f}$ in the course of $\mathbf{u}$. Geometrically, the EEP (4.1) simply means that $\mathbf{D}$ is the projection of $\mathbf{u}$ onto the space of admissible displacements (figure 4.1). On the basis of the EEP (4.1), a generalized Hill-Mandel lemma will be proven in subsection 2.4.

Using FB transform, combined with Parseval's identity, the EEP (4.1) can be equivalently written in terms of FB amplitudes as

$$\forall \mathbf{k} \in T^*, \quad \forall \tilde{\mathbf{f}}_{\mathbf{k}} \in \mathcal{F}, \quad \langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot \tilde{\mathbf{D}}_{\mathbf{k}}^* \rangle = \langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{\mathbf{k}}^* \rangle. \quad (4.2)$$

¹It was pointed out to me that the formalism used hereafter bears resemblance to the one developed by [Nayroles \(2011\)](#). Duality experts are kindly invited to comment.

²Identifying $\mathcal{F}^*$ to a set of displacements is not straightforward and is discussed later on.

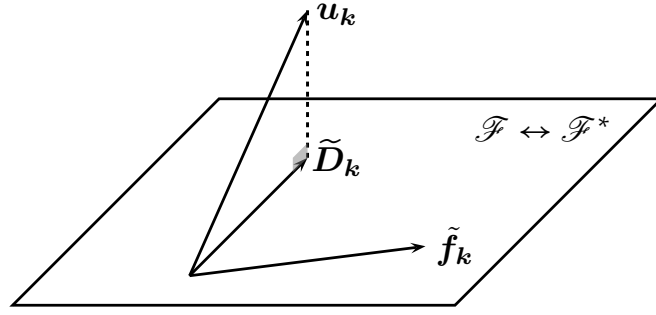


Figure 4.1. The effective displacement field $\mathbf{D}$ associated to a given microscopic one $\mathbf{u}$ is geometrically interpreted as the projection of the latter onto the space of admissible displacements. Spaces $\mathcal{F}$ and $\mathcal{F}^*$ are isomorphic and, here, are taken to be equal up to a change in units.

2.2 Effective displacement field

Bearing in mind the EEP (4.1), choosing the space $\mathcal{F}$ of admissible body forces becomes a key step toward elaborating a generalized theory. The choice of $\mathcal{F}$ depends ultimately on the degree of accuracy with which $\mathbf{D}$ is required to approximate $\mathbf{u}$. The bigger $\mathcal{F}$ is, the closer $\mathbf{D}$ is to $\mathbf{u}$. When all body forces are considered as admissible, the relation (4.1) implies $\mathbf{D} = \mathbf{u}$ and the effective medium is trivially the original one. In what follows, we study the rather general case of practical importance where $\mathcal{F}$ is finite-dimensional and show how $\mathbf{D}$ derives from (4.1) correspondingly.

2.2.1 Admissible body forces

Given N linearly independent, $\mathbf{k}$ - and ω -independent, $\mathcal{R}$ -periodic *vector* fields ϕ_α with $\alpha = 1 \dots N$, a body force field $\mathbf{f}$ is admissible if and only if it has ($\mathcal{R}$ -periodic) FB amplitudes $\tilde{\mathbf{f}}_{\mathbf{k}}$ of the form

$$\tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) = \sum_{\alpha=1}^N \tilde{f}_{\mathbf{k}}^\alpha \phi_\alpha(\mathbf{x}), \quad (4.3)$$

where the $\tilde{f}_{\mathbf{k}}^\alpha$ are constants. The space $\mathcal{F}$ is therefore of dimension N .

With no loss of generality, let the *subset* $(\phi_\alpha)_{\alpha=1 \dots d}$, where d is the dimension of Ω , be formed of *constant vectors* and constitute a basis for $\mathcal{E}$. We call $\tilde{\mathbf{F}}_{\mathbf{k}}$ the constant

component of $\tilde{\mathbf{f}}_{\mathbf{k}}$ and write

$$\tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) = \sum_{\alpha=1}^d \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha} + \sum_{\alpha=d+1}^N \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha}(\mathbf{x}) \equiv \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha}(\mathbf{x}). \quad (4.4)$$

Above and from now on, the repeated Greek letters are understood to be summed over from $d+1$ to N . Integrating with respect to $\mathbf{k}$ over T^* , we obtain the generic form of admissible body forces:

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + f^{\alpha}(\mathbf{x}) \phi_{\alpha}(\mathbf{x}).$$

Of most importance is the fact that fields $\mathbf{F}$ and f^{α} have their supports contained in T^* . As such, they have wavelengths at least twice as large as the characteristic length of a unit cell. Consequently, the DOFs $\mathbf{F}$ and f^{α} of admissible body forces are said to be “macroscopic”. These DOFs are carried by $\mathcal{R}$ -periodic *shape functions*, the ϕ_{α} , describing the ways in which $\mathbf{f}$ can vary on the microscale. For example, taking $N = d$, we have $\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x})$ implying that body forces are not allowed to vary on the microscale. As another example, setting $N = d+1$ and $\phi_{d+1}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{e}$, where $\mathbf{e}$ is a vertically oriented vector, we have $\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + f^{d+1}(\mathbf{x})\rho(\mathbf{x})\mathbf{e}$ so that the admissible variations of body forces on the microscale are gravitational.

2.2.2 Effective displacement field by the EEP

From now on, we assume that the ϕ_{α} form an orthonormal¹ basis of $\mathcal{F}$ so that

$$\forall \alpha, \beta \in \{1, \dots, N\}, \quad \langle \phi_{\alpha} \cdot \phi_{\beta}^* \rangle = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. For $\beta \in \{1, \dots, d\}$, ϕ_{β} being constant entails

$$\forall \alpha \in \{d+1, \dots, N\}, \quad \langle \phi_{\alpha} \rangle = 0,$$

meaning that being orthogonal to a constant is equivalent to having a zero average.

Injecting (4.4) in the expression of the virtual work, we obtain

$$\begin{aligned} \langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{\mathbf{k}}^* \rangle &= \langle \tilde{\mathbf{F}}_{\mathbf{k}} \cdot \tilde{\mathbf{u}}_{\mathbf{k}}^* \rangle + \langle \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha} \cdot \tilde{\mathbf{u}}_{\mathbf{k}}^* \rangle && \text{(by orthogonality)} \\ &= \tilde{\mathbf{F}}_{\mathbf{k}} \cdot \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle^* + \tilde{f}_{\mathbf{k}}^{\alpha} \langle \phi_{\alpha}^* \cdot \tilde{\mathbf{u}}_{\mathbf{k}} \rangle^* && \text{(by constancy)} \\ &= \tilde{\mathbf{F}}_{\mathbf{k}} \cdot \tilde{\mathbf{U}}_{\mathbf{k}}^* + \tilde{f}_{\mathbf{k}}^{\alpha} \tilde{u}_{\mathbf{k}}^{\alpha*} && \text{(by definition (4.5))} \\ &= \langle \tilde{\mathbf{F}}_{\mathbf{k}} \cdot \tilde{\mathbf{U}}_{\mathbf{k}}^* \rangle + \langle \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha} \cdot (\tilde{u}_{\mathbf{k}}^{\beta} \phi_{\beta})^* \rangle && \text{(by orthogonality)} \\ &= \langle \tilde{\mathbf{f}}_{\mathbf{k}} \cdot (\tilde{\mathbf{U}}_{\mathbf{k}} + \tilde{u}_{\mathbf{k}}^{\beta} \phi_{\beta})^* \rangle, && \text{(by orthogonality)} \end{aligned}$$

with

$$\tilde{\mathbf{U}}_{\mathbf{k}} \equiv \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle; \quad \forall \beta \in \{d+1, \dots, N\}, \quad \tilde{u}_{\mathbf{k}}^{\beta} \equiv \langle \phi_{\beta}^* \cdot \tilde{\mathbf{u}}_{\mathbf{k}} \rangle. \quad (4.5)$$

¹Just like the identification of $\mathcal{F}^*$, orthonormality needs to be clarified.

Then, it follows from (4.2) that

$$\tilde{\mathbf{D}}_k(\mathbf{x}) = \tilde{\mathbf{U}}_k + \tilde{u}_k^\beta \phi_\beta(\mathbf{x}).$$

Finally, summing over T^* , it comes that

$$\mathbf{D}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) + u^\beta(\mathbf{x}) \phi_\beta(\mathbf{x}).$$

The above expression of the effective displacement field results from the EEP combined with a particular choice of admissible body forces. It contains the classical translational displacement vector $\mathbf{U}$ and additional generalized “displacements” u^β carried by the shape functions ϕ_β . Once more, the shape functions define the way in which $\mathbf{D}$ varies on the microscale whereas the slowly varying DOFs $\mathbf{U}$ and u^β describe how $\mathbf{D}$ varies on the macroscale.

Willis (2011) proposed a homogenization theory in which shape functions are taken to be $\phi_\alpha(\mathbf{x}) = w(\mathbf{x})\mathbf{e}_\alpha$, for $\alpha \in \{1, \dots, d\}$, where $w(\mathbf{x})$ is a fixed $\mathcal{R}$ -periodic function and the $\mathbf{e}_\alpha$ form a basis for $\mathcal{E}$. Taking $w \equiv 1$ yields the unweighted theory of Willis (1997) and amounts to taking $\mathbf{f} = \mathbf{F}$ and $\mathbf{D} = \mathbf{U}$. Here, we combine both the weighted and unweighted Willis theories and use even more general shape functions. As a consequence, $\mathbf{D}$ is a better approximation of $\mathbf{u}$ than $\mathbf{U}$ as will be seen in more detail.

2.2.3 Effective displacement field through error minimization

First of all, rewriting (4.2) in the equivalent form

$$\forall \tilde{\mathbf{f}}_k \in \mathcal{F}, \quad \langle \tilde{\mathbf{f}}_k \cdot (\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k)^* \rangle = 0,$$

it is clear that $\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k$ is orthogonal to $\mathcal{F}$ and $\tilde{\mathbf{D}}_k$ acts as the orthogonal projection of $\tilde{\mathbf{u}}_k$ onto $\mathcal{F}^*$ (figure 4.1). Using the Pythagorean theorem, it is easy to see that for any $\mathcal{R}$ -periodic field $\mathbf{h} \in \mathcal{F}^*$,

$$\begin{aligned} \langle (\tilde{\mathbf{u}}_k - \mathbf{h}) \cdot (\tilde{\mathbf{u}}_k - \mathbf{h})^* \rangle &= \langle (\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k) \cdot (\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k)^* \rangle + \langle (\tilde{\mathbf{D}}_k - \mathbf{h}) \cdot (\tilde{\mathbf{D}}_k - \mathbf{h})^* \rangle \\ &\geq \langle (\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k) \cdot (\tilde{\mathbf{u}}_k - \tilde{\mathbf{D}}_k)^* \rangle. \end{aligned}$$

Thus,

$$\tilde{\mathbf{D}}_k = \arg \min_{\mathbf{h} \in \mathcal{F}^*} \langle (\tilde{\mathbf{u}}_k - \mathbf{h}) \cdot (\tilde{\mathbf{u}}_k - \mathbf{h})^* \rangle. \quad (4.6)$$

This shows that the effective displacement FB amplitude is the best admissible approximation to the microscopic one. Consequently, the effective displacement field $\mathbf{D}$, associated to a microscopic displacement field $\mathbf{u}$, can be seen as the best admissible approximation to $\mathbf{u}$. Note that this *global* optimal argument definition is different from the *local* one introduced by Forest and Sab (1998) and Forest (2006) in elastostatics (see subsection 3.2.5), despite an apparent resemblance.

The preceding definition of the effective displacement field concretizes the intuition that the richer the DOFs of the generalized substitution medium are, the closer $\mathbf{D}$ is to $\mathbf{u}$:

$$\text{if } \mathcal{F}_1 \subset \mathcal{F}_2 \text{ then } \min_{\mathbf{h} \in \mathcal{F}_2^*} \langle (\tilde{\mathbf{u}}_k - \mathbf{h}) \cdot (\tilde{\mathbf{u}}_k - \mathbf{h})^* \rangle \leq \min_{\mathbf{h} \in \mathcal{F}_1^*} \langle (\tilde{\mathbf{u}}_k - \mathbf{h}) \cdot (\tilde{\mathbf{u}}_k - \mathbf{h})^* \rangle.$$

In this sense, the generalized substitution medium to be obtained by our theory is more realistic than the Willis substitution medium in the above minimal error sense, at the cost of an additional kinematical complexity.

A remark is now in order. We have used a scalar product on the space of body forces twice up till now: once to identify $\mathcal{F}$ and $\mathcal{F}^*$ and once to orthonormalize the set of shape functions. This scalar product is not unique and can be modified by adding a weighting function for example. Note that such choice has influence neither on the definition of the DOFs u^α , nor on the effective motion equation to be found. It simply changes the above quadratic error function and determines the mapping $(u^\alpha)_{\alpha=1\dots N} \mapsto \mathbf{D}$. In other words, the DOFs $(u^\alpha)_{\alpha=1\dots N}$ are intrinsic whereas the field $\mathbf{D}$ is conventional and only optimal with respect to some error function. For instance, when dealing with eigenmodes, as in subsection 4.6, the natural choice appears to be indeed a scalar product weighted with mass density.

2.2.4 Effective displacement field under infinite scale separation

It is of interest to examine what $\mathbf{D}$ becomes under the hypothesis of infinite scale separation, namely, when $\mathbf{k} \rightarrow \mathbf{0}$ and $\omega \rightarrow 0$. It is known that in this case, to the lowest order, the displacement field depends only on the “slow variable”. In terms of FB amplitudes, this means that $\tilde{\mathbf{u}}_{\mathbf{k}}$ is constant. Consequently,

$$\begin{aligned} \tilde{\mathbf{U}}_{\mathbf{k}} &= \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle = \tilde{\mathbf{u}}_{\mathbf{k}}, \\ \forall \alpha \in \{d, \dots, N-1\}, \quad \tilde{u}_{\mathbf{k}}^\alpha &= \langle \phi_\alpha^* \cdot \tilde{\mathbf{u}}_{\mathbf{k}} \rangle = \langle \phi_\alpha^* \rangle \cdot \tilde{\mathbf{u}}_{\mathbf{k}} = 0. \end{aligned}$$

Therefore, the translational DOF $\mathbf{U}$ is the only non-null component of $\mathbf{D}$, to the lowest order. The use of a generalized kinematics is hence justified only under weak scale separation when microscopic deformation modes become significant. Otherwise, it is enough to keep track of $\mathbf{U}$ exclusively as in the unweighted Willis theory. As a matter of fact, it has been observed that a periodic medium was “homogenizable” in the Willis sense over the acoustic and the first optical branches only (Srivastava and Nemat-Nasser, 2014; Nassar et al., 2015). For higher frequencies, one needs to use non-uniform shape functions.

Finally, the classical macroscopic displacement field $\mathbf{U}$, as defined by (4.5), is the combination of all long-wavelength ($\mathbf{k} \in T^*$) components of $\mathbf{u}$, which is *not* a moving average of $\mathbf{u}$. It is important to remark that a moving average does not correspond to a projection and, hence, does not define any acceptable effective displacement in the above EEP sense, except asymptotically, to the first order when $\mathbf{k} \rightarrow \mathbf{0}$.

2.3 Effective motion equation

Having specified body forces, the central motion equation reads

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})] \} + \tilde{\mathbf{F}}_{\mathbf{k}} + f^\alpha(\mathbf{x}) \phi_\alpha(\mathbf{x}) = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}), \quad (4.7)$$

which needs to be solved over Ω . Since $\tilde{\mathbf{u}}_{\mathbf{k}}$ is $\mathcal{R}$ -periodic, it is enough to solve the above equation over a unit cell T under periodic boundary conditions. Let $\mathbf{g}_{\mathbf{k}}$ be the corresponding periodic second order Green's operator. Then,

$$\begin{aligned}\tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{y}) &= \frac{1}{T} \int_T \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \cdot \tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) \, d\mathbf{x} \\ &= \left(\frac{1}{T} \int_T \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \, d\mathbf{x} \right) \cdot \tilde{\mathbf{F}}_{\mathbf{k}} + \left(\frac{1}{T} \int_T \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \cdot \tilde{\phi}_{\alpha} \, d\mathbf{x} \right) \tilde{f}_{\mathbf{k}}^{\alpha}\end{aligned}\tag{4.8}$$

which, combined with (4.5), delivers the following expressions for the components of the macroscopic dispicement field:

$$\begin{aligned}\tilde{\mathbf{U}}_{\mathbf{k}} &= \langle \langle \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \rangle \rangle \cdot \tilde{\mathbf{F}}_{\mathbf{k}} + \langle \langle \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \cdot \phi_{\alpha}(\mathbf{x}) \rangle \rangle \tilde{f}_{\mathbf{k}}^{\alpha}, \\ \tilde{u}_{\mathbf{k}}^{\beta} &= \langle \langle \phi_{\beta}^*(\mathbf{y}) \cdot \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \rangle \rangle \cdot \tilde{\mathbf{F}}_{\mathbf{k}} + \langle \langle \phi_{\beta}^*(\mathbf{y}) \cdot \mathbf{g}_{\mathbf{k}}(\mathbf{y}, \mathbf{x}) \cdot \phi_{\alpha}(\mathbf{x}) \rangle \rangle \tilde{f}_{\mathbf{k}}^{\alpha},\end{aligned}\tag{4.9}$$

where $\langle \langle \rangle \rangle$ means averaging with respect to both $\mathbf{x}$ and $\mathbf{y}$.

The Green's operator of the effective medium $G_{\mathbf{k}}$ is given by the last two equalities which can be written concisely as

$$\forall \beta \in \{1, \dots, N\}, \quad \tilde{u}_{\mathbf{k}}^{\beta} = \sum_{\alpha=1}^N G_{\mathbf{k}}^{\beta\alpha} \tilde{f}_{\mathbf{k}}^{\alpha},$$

where no distinction is made between the classical and generalized DOFs (recall that $\tilde{\mathbf{U}}_{\mathbf{k}} = \sum_{\beta=1}^d \tilde{u}_{\mathbf{k}}^{\beta} \phi_{\beta}$). Inverting the preceding equation delivers the effective motion equation in Fourier domain:

$$\forall \alpha \in \{1, \dots, N\}, \quad \sum_{\beta=1}^N Z_{\mathbf{k}}^{\alpha\beta} \tilde{u}_{\mathbf{k}}^{\beta} = \tilde{f}_{\mathbf{k}}^{\alpha},\tag{4.10}$$

where $Z_{\mathbf{k}}$, the inverse of $G_{\mathbf{k}}$, is called the effective impedance. It depends implicitly on the frequency ω . By summing over $\mathbf{k} \in T^*$ and over ω , we obtain the effective motion equation in $\mathbf{x}$ and t as

$$\forall \alpha \in \{1, \dots, N\}, \quad \sum_{\beta=1}^N Z^{\alpha\beta}(\mathbf{x}, t) * u^{\beta}(\mathbf{x}, t) = f^{\alpha}(\mathbf{x}, t),$$

where $Z(\mathbf{x}, t)$ is an integro-differential operator and $*$ denotes convolution product with respect to space and time. The effective motion equation is hence nonlocal in both space and time and involves long wavelengths ($\mathbf{k} \in T^*$) only. This generalizes equation (3.28) of (Willis, 1997) for periodic media.

2.4 Internal work

As mentioned earlier, it is not essential for achieving the main purpose of the present work to derive an explicit expression for the underlying effective constitutive law which

is not unique as in the theory of Willis (2011). However, it is of interest to specify the macroscopic stress, momentum, strain and velocity measures that an effective constitutive law involves. In addition, these generalized macroscopic measures will be shown to be related to their microscopic counterparts through an extended Hill-Mandel relation.

2.4.1 Generalized stress and momentum measures

Said measures are taken to be the ones involved in the effective motion equation written as a conservation law equivalent to (4.10). Starting with the microscopic motion equation

$$(\nabla + i\mathbf{k}) \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} + \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha} = i\omega \tilde{\mathbf{p}}_{\mathbf{k}}, \quad (4.11)$$

where $\tilde{\boldsymbol{\sigma}}_{\mathbf{k}}$ and $\tilde{\mathbf{p}}_{\mathbf{k}}$ are the FB amplitudes of stress and momentum, we take its volume average over a unit cell to obtain, with the help of the divergence theorem,

$$i\mathbf{k} \cdot \tilde{\boldsymbol{\Sigma}}_{\mathbf{k}} + \tilde{\mathbf{F}}_{\mathbf{k}} = i\omega \tilde{\mathbf{P}}_{\mathbf{k}}. \quad (4.12)$$

This is the first effective motion equation involving the classical macroscopic stress and momentum measures:

$$\tilde{\boldsymbol{\Sigma}}_{\mathbf{k}} \equiv \langle \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle, \quad \tilde{\mathbf{P}}_{\mathbf{k}} \equiv \langle \tilde{\mathbf{p}}_{\mathbf{k}} \rangle.$$

Further, projecting equation (4.11) onto the space spanned by the other shape functions ϕ^{β} gives rise to

$$\forall \beta \in \{d+1, \dots, N\}, \quad i\mathbf{k} \cdot \langle \phi^{\beta*} \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle - \langle (\nabla \otimes^s \phi^{\beta*}) : \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle + \tilde{f}_{\mathbf{k}}^{\beta} = i\omega \langle \phi^{\beta*} \cdot \tilde{\mathbf{p}}_{\mathbf{k}} \rangle,$$

where, for simplicity, we have assumed the continuity of ϕ^{β} so that the boundary term vanishes. The generalized stress and momentum measures can be identified as

$$\forall \beta \in \{d+1, \dots, N\}, \quad \tilde{\boldsymbol{\sigma}}_{\mathbf{k}}^{\beta} \equiv \langle \phi^{\beta*} \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle, \quad \tilde{s}_{\mathbf{k}}^{\beta} \equiv -\langle (\nabla \otimes^s \phi^{\beta*}) : \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle, \quad \tilde{\mathbf{p}}_{\mathbf{k}}^{\beta} \equiv \langle \phi^{\beta*} \cdot \tilde{\mathbf{p}}_{\mathbf{k}} \rangle.$$

The additional motion equation becomes then simply

$$\forall \beta \in \{d+1, \dots, N\}, \quad \tilde{s}_{\mathbf{k}}^{\beta} + i\mathbf{k} \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}}^{\beta} + \tilde{f}_{\mathbf{k}}^{\beta} = i\omega \tilde{\mathbf{p}}_{\mathbf{k}}^{\beta}. \quad (4.13)$$

Note that equations (4.12) and (4.13) on one hand, and (4.10) on the other, are related to one another through a non-unique effective constitutive law whose characterization is beyond the purpose of the present work (see the discussion by Willis, 2011, 2012).

In summary, the motion equations in the space domain are given by

$$\begin{aligned} \nabla \cdot \boldsymbol{\Sigma} + \mathbf{F} &= i\omega \mathbf{P}, \\ \forall \beta \in \{d+1, \dots, N\}, \quad s^{\beta} + \nabla \cdot \boldsymbol{\sigma}^{\beta} + f^{\beta} &= i\omega p^{\beta}. \end{aligned} \quad (4.14)$$

These equations are the “dynamic version” of the “equations of equilibrium” phenomenologically derived by Germain (1973, equation (45)). There are however some differences; see subsection 3.2.5 for details.

2.4.2 Generalized strain and velocity measures

Said measures are obtained by duality. The virtual work theorem combined with the EEP yields

$$\langle \tilde{\sigma}_k : \tilde{\varepsilon}_k^* - \tilde{p}_k \cdot \tilde{v}_k^* \rangle = \langle \tilde{f}_k \cdot \tilde{u}_k^* \rangle = \tilde{F}_k \cdot \tilde{U}_k^* + \tilde{f}_k^\alpha \tilde{u}_k^{\alpha*},$$

where $\tilde{\varepsilon}_k$ and $\tilde{v}_k$ are the FB amplitudes of the strain and velocity fields, respectively, given by

$$\tilde{\varepsilon}_k = (\nabla + i\mathbf{k})^{\otimes s} \tilde{u}_k, \quad \tilde{v}_k = i\omega \tilde{u}_k.$$

Substituting body forces by the corresponding stress and momentum measures according to (4.13) delivers

$$\begin{aligned} \langle \tilde{\sigma}_k : \tilde{\varepsilon}_k^* - \tilde{p}_k \cdot \tilde{v}_k^* \rangle &= \tilde{\Sigma}_k : (i\mathbf{k} \otimes^s \tilde{U}_k)^* - \tilde{P}_k \cdot (i\omega \tilde{U}_k)^* \\ &\quad + \tilde{\sigma}_k^\alpha \cdot (i\mathbf{k} \tilde{u}_k^\alpha)^* - \tilde{p}_k^\alpha (i\omega \tilde{u}_k^\alpha)^* - \tilde{s}_k^\alpha \tilde{u}_k^{\alpha*}. \end{aligned}$$

Summing over $\mathbf{k}$, and using Parseval's identity, we obtain a generalized version of the Hill-Mandel lemma:

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^* - \mathbf{p} \cdot \mathbf{v}^* \\ = \int_{\Omega} \boldsymbol{\Sigma} : (\nabla \otimes^s \mathbf{U})^* - \mathbf{P} \cdot (i\omega \mathbf{U})^* + \boldsymbol{\sigma}^\alpha \cdot (\nabla u^\alpha)^* - p^\alpha (i\omega u^\alpha)^* - s^\alpha u^{\alpha*}. \end{aligned} \quad (4.15)$$

This result is valid for all virtual couples $(\boldsymbol{\sigma}, \mathbf{p})$ equilibrated by an *admissible* body force field and for all couples $(\boldsymbol{\varepsilon}, \mathbf{v})$ derived from an *arbitrary* displacement field $\mathbf{u}$. The classical macroscopic measures of strain and velocity are $\nabla \otimes^s \mathbf{U}$ and $i\omega \mathbf{U}$ while the generalized ones are ∇u^α , u^α and $i\omega u^\alpha$. It is clear that all of the above measures can be specified once particular shape functions have been chosen. This will be done in the next section.

3 A hierarchy of standard generalized substitution media

In the present section, we choose particular admissible body forces so as to produce generalized kinematics having clear geometric interpretations.

3.1 Local coordinates

Given a unit cell T of center $\mathbf{x}_o$, the position vector $\mathbf{x}$ of a point of Ω can be uniquely decomposed as $\mathbf{x} = \mathbf{x}_o + \mathbf{r} + \tilde{\mathbf{x}}$, with $\mathbf{r} \in \mathcal{R}$ and $\tilde{\mathbf{x}} \in T - \mathbf{x}_o$ (see figure 4.2). The local coordinates of this point refer to the components of $\tilde{\mathbf{x}}$. In particular, for $\mathbf{x} \in T$, $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_o$. Note that the local coordinates are $\mathcal{R}$ -periodic and depend on the shape of T and on its center $\mathbf{x}_o$ modulo $\mathcal{R}$. The choice of T will influence the set of admissible body forces and, consequently, the effective properties of the substitution medium, and

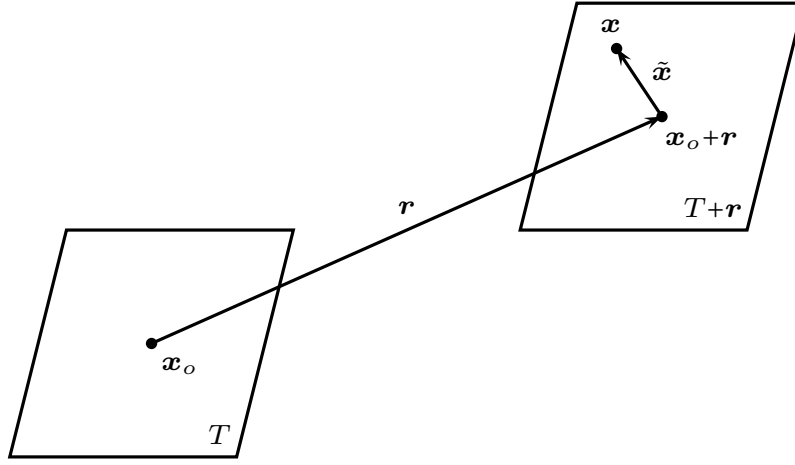


Figure 4.2. An illustration of the decomposition $\mathbf{x} = \mathbf{x}_o + \mathbf{r} + \tilde{\mathbf{x}}$ in 2D.

should be made with regard to the actual microstructure of Ω . Last, the previous local coordinates have zero average and the important property that $\langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle$ is a symmetric positive definite, and therefore invertible, tensor.

3.2 Four shades

In what follows, by choosing appropriate sets of admissible body forces, we obtain the generalized kinematics for a hierarchy of generalized substitution media. In particular, interest will be in the interpretation of admissible body forces and the corresponding generalized DOFs. Derivation of the effective motion equation remains unchanged with respect to the general theory presented above.

Retracing the history of generalized continuum mechanics is out of the scope of the present chapter, and one can refer, for example, to the texts by [Maugin \(2010\)](#) and by [Askes and Aifantis \(2011\)](#) for exhaustive reviews. We would like simply to point out the fact that enriching the kinematics of standard continua has a well established potential in producing dispersive effects and internal oscillation modes. For instance, [Mindlin \(1964\)](#) derived the equations of a generalized “micromorphic” medium and described its possible oscillation modes, both acoustic and optical. Though he provided a careful geometrical interpretation of the additional DOFs, he did not describe how these DOFs can or should be obtained from the standard kinematics of a hidden underlying microstructure. [Germain \(1973\)](#) made a clearer, yet only qualitative, connection between the DOFs of the generalized continuum and the microscopic displacement field. According to him, at the microscopic level, a particle M appears as a continuum of small extent $P(M)$ and the DOFs at M define an approximation to the displacement field over $P(M)$. Our purpose here is to concretize this intuition and write the generalized

DOFs explicitly in terms of the microscopic displacement field as in equation (4.5). It is worth noting that the kinematics obtained next are not rigorously the same as their phenomenological counterparts. Similarities and differences in comparison to the approach of Germain, and to that of [Forest and Sab \(1998\)](#) and [Forest \(2006\)](#), are commented at the end of the present subsection.

3.2.1 Cauchy kinematics and strain-gradient media

Taking slowly varying body forces as the only admissible body forces requires no more than the usual Cauchy kinematics:

$$\mathbf{f} = \mathbf{F}, \quad \mathbf{D} = \mathbf{U},$$

whose support is T^* . Shape functions are constants and $\mathbf{D}$ is the long-wavelength part of $\mathbf{u}$ given by

$$\forall \mathbf{k} \in T^*, \quad \tilde{\mathbf{D}}_{\mathbf{k}} = \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle.$$

A macroscopic particle can translate *and deform* as much as allowed by the long-wavelength restriction ($\mathbf{k} \in T^*$). This medium corresponds to Willis' substitution medium and has a nonlocal behavior in both space and time. Taking an asymptotic expansion of the effective motion equation in the vicinity of ($\mathbf{k} \rightarrow \mathbf{0}, \omega \rightarrow 0$) yields a strain-gradient-media-type motion equation with micro-inertia. We refer to the papers by [Boutin \(1996\)](#) and by [Smyshlyaev and Cherednichenko \(2000\)](#) and to chapter 3 for details.

3.2.2 Microdilatation-like media

Admissible body forces can be enriched according to

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + q(\mathbf{x})\tilde{\mathbf{x}}$$

where $\tilde{\mathbf{x}}$ plays the role of a shape function and q is an additional slowly varying scalar function carried by the rapidly oscillating local coordinates. It models a hydrostatic pressure on a cellular level. Body forces $\mathbf{F}$ can also produce such effects but on a wholly different scale, that of Ω . The key to this distinction is again the restriction to T^* of the supports of $\mathbf{F}$ and q . Same goes for the effective displacement which admits a similar form

$$\mathbf{D}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) + \chi(\mathbf{x})\tilde{\mathbf{x}}$$

where χ is dual to q and reflects the “microdilatational” effects.

With respect to the microscopic displacement, the new DOF can be obtained according to either (4.5) or (4.6). For diversity, we adopt the latter. DOFs $\mathbf{U}$ and χ span two orthogonal spaces so that it is possible to proceed with the minimization in two independent steps over $\mathbf{U}$ and χ . The first step entails the already known

expression of $\tilde{U}_k$. The second amounts to minimizing

$$\begin{aligned} \langle (\tilde{u}_k - \tilde{\chi}_k \tilde{x}) \cdot (\tilde{u}_k - \tilde{\chi}_k \tilde{x})^* \rangle &= \langle \tilde{u}_k \cdot \tilde{u}_k^* \rangle - \tilde{\chi}_k \langle \tilde{x} \cdot \tilde{u}_k^* \rangle - \tilde{\chi}_k^* \langle \tilde{x} \cdot \tilde{u}_k \rangle + \tilde{\chi}_k \tilde{\chi}_k^* \langle \tilde{x} \cdot \tilde{x} \rangle \\ &= \langle \tilde{u}_k \cdot \tilde{u}_k^* \rangle + \langle \tilde{x} \cdot \tilde{x} \rangle \left| \tilde{\chi}_k - \frac{\langle \tilde{x} \cdot \tilde{u}_k \rangle}{\langle \tilde{x} \cdot \tilde{x} \rangle} \right|^2 - \frac{\langle \tilde{x} \cdot \tilde{u}_k \rangle \langle \tilde{x} \cdot \tilde{u}_k \rangle^*}{\langle \tilde{x} \cdot \tilde{x} \rangle}, \end{aligned}$$

where $\|\cdot\|$ represents the modulus of a complex number and has zero minimum. Therefore,

$$\forall k \in T^*, \quad \tilde{\chi}_k = \frac{\langle \tilde{x} \cdot \tilde{u}_k \rangle}{\langle \tilde{x} \cdot \tilde{x} \rangle}.$$

3.2.3 Microstretch-like media

For $d \geq 2$, admissible body forces can be enriched according to

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + q(\mathbf{x})\tilde{\mathbf{x}} + \mathbf{M}(\mathbf{x}) \cdot \tilde{\mathbf{x}},$$

where $\mathbf{M}$ is an additional second-order antisymmetric tensor. The term $\mathbf{M}(\mathbf{x}) \cdot \tilde{\mathbf{x}}$ characterizes force couples at the microscopic scale. In order to produce couples, body forces need to oscillate rapidly at the microscale. This is prohibited by classical homogenization schemes. Here, couples are modeled with the additional admissible body forces. However, couples themselves are restricted to having slow variations.

The effective displacement field takes the dual form

$$\mathbf{D}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) + \chi(\mathbf{x})\tilde{\mathbf{x}} + \mathbf{R}(\mathbf{x}) \cdot \tilde{\mathbf{x}},$$

where $\mathbf{R}$ is an additional kinematical second-order antisymmetric tensor dual to $\mathbf{M}$. It is interpreted as an infinitesimal rotation at the microscale different from the macroscale infinitesimal rotation of the antisymmetric gradient of $\mathbf{U}$. Note that $\mathbf{R}$ corresponds to a “quasi-rigid” transformation since it varies slowly within a unit cell.

The spaces spanned by $\mathbf{U}$, χ and $\mathbf{R}$ are orthogonal to each other, so that the previous expressions of $\mathbf{U}$ and χ hold and that the expression of $\mathbf{R}$ results from the EEP. Precisely, for all antisymmetric tensors $\tilde{\mathbf{M}}_k$,

$$\tilde{\mathbf{M}}_k : \langle \tilde{u}_k^* \otimes \tilde{x} \rangle = \tilde{\mathbf{M}}_k : (\tilde{\mathbf{R}}_k^* \cdot \langle \tilde{x} \otimes \tilde{x} \rangle).$$

Hence, the tensor

$$\langle \tilde{u}_k^* \otimes \tilde{x} \rangle - \tilde{\mathbf{R}}_k^* \cdot \langle \tilde{x} \otimes \tilde{x} \rangle$$

needs to be symmetric. Consequently, $\tilde{\mathbf{R}}_k$ is the *unique* antisymmetric solution to the equation

$$2 \langle \tilde{u}_k \wedge \tilde{x} \rangle = \tilde{\mathbf{R}}_k \cdot \langle \tilde{x} \otimes \tilde{x} \rangle + \langle \tilde{x} \otimes \tilde{x} \rangle \cdot \tilde{\mathbf{R}}_k,$$

where $\wedge$ stands for antisymmetric tensor product. When T is a cube of edge $2a$, the rotation admits the expression

$$\forall k \in T^*, \quad \tilde{\mathbf{R}}_k = \frac{6}{a^2} \langle \tilde{u}_k \wedge \tilde{x} \rangle.$$

In general, we have the formula¹

$$\forall \mathbf{k} \in T^*, \quad \tilde{\mathbf{R}}_{\mathbf{k}} = 2 \int_0^{+\infty} e^{-\eta \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle} \cdot \langle \tilde{\mathbf{u}}_{\mathbf{k}} \wedge \tilde{\mathbf{x}} \rangle \cdot e^{-\eta \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle} d\eta.$$

Cosserat-like media can be obtained by simply eliminating q and χ .

3.2.4 Micromorphic-like media

Admissible body forces can also be enriched so that they englobe all affine transformations of the local coordinates:

$$\mathbf{f}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \boldsymbol{\Psi}(\mathbf{x}) \cdot \tilde{\mathbf{x}},$$

where $\boldsymbol{\Psi}$ is an arbitrary second-order tensor. The effective displacement field admits the similar representation

$$\mathbf{D}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x}) \cdot \tilde{\mathbf{x}}.$$

The new tensorial DOF $\boldsymbol{\chi}$ describes a general linear transformation at the microscale similar to the one described by the gradient of $\mathbf{U}$ at the macroscale. In addition, $\boldsymbol{\chi}$ derives from the microscopic displacement according to

$$\forall \mathbf{k} \in T^*, \quad \tilde{\boldsymbol{\chi}}_{\mathbf{k}} = \langle \tilde{\mathbf{u}}_{\mathbf{k}} \otimes \tilde{\mathbf{x}} \rangle \cdot \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle^{-1}.$$

We underline the fact that the symmetric and antisymmetric parts of $\boldsymbol{\chi}$ span *non-orthogonal* spaces in the sense of virtual work so that the rotation of a microstretch medium *is not* the antisymmetric part of $\boldsymbol{\chi}$. For a cubical T , however, $\mathbf{R}$ is indeed the antisymmetric part of $\boldsymbol{\chi}$.

The generalized stress measures are obtained as discussed in the general case (subsection 2.4). Here, we have three generalized stress tensor measures of second, third and second orders, respectively, defined by

$$\begin{aligned} \forall \mathbf{k} \in T^*, \quad \tilde{\boldsymbol{\Sigma}}_{\mathbf{k}} &= \langle \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \rangle, \\ \tilde{\boldsymbol{\sigma}}_{\mathbf{k}}^{\chi} &= \langle \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \otimes \tilde{\mathbf{x}} \rangle \cdot \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle^{-1}, \\ \tilde{\mathbf{s}}_{\mathbf{k}}^{\chi} &= \left(\frac{1}{|T|} \int_{\partial T} \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}}_{\mathbf{k}} \otimes \tilde{\mathbf{x}} dS(\mathbf{x}) - \tilde{\boldsymbol{\Sigma}}_{\mathbf{k}} \right) \cdot \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle^{-1}, \end{aligned}$$

where $\mathbf{n}$ is the unitary outward vector normal to the boundary ∂T of T and dS is a surface measure. Note that $\mathbf{s}^{\chi}$ quantifies the difference between the volume average of $\boldsymbol{\sigma}$ and its average surface flux on the boundary of a unit cell. Finally, these measures relate to the momentum measures given by:

$$\forall \mathbf{k} \in T^*, \quad \tilde{\mathbf{P}}_{\mathbf{k}} = \langle \tilde{\mathbf{p}}_{\mathbf{k}} \rangle, \quad \tilde{\mathbf{p}}_{\mathbf{k}}^{\chi} = \langle \tilde{\mathbf{p}}_{\mathbf{k}} \otimes \tilde{\mathbf{x}} \rangle \cdot \langle \tilde{\mathbf{x}} \otimes \tilde{\mathbf{x}} \rangle^{-1},$$

¹This solution was encountered during my last “colle”, almost 6 years ago. Professor Pascal Guelfi from the lycée Henri Poincaré is to be thanked. Sadly, I was unable to find a corresponding geometrical interpretation.

through the motion equations

$$\begin{aligned}\nabla \cdot \boldsymbol{\Sigma} + \mathbf{F} &= \mathbf{P}, \\ \mathbf{s}^\chi + \nabla \cdot \boldsymbol{\sigma}^\chi + \boldsymbol{\Psi} &= \mathbf{p}^\chi.\end{aligned}$$

3.2.5 Remarks

The equations of motion of the above medium resemble the ones derived by Germain (1973) for his first order micromorphic medium. They are not identical however. For instance, a difference can be spotted by closely checking the stress measure $\boldsymbol{\Sigma}$ appearing in the first equation: it is symmetric here but not in the work of Germain. Symmetry, or lack thereof, of stress measures is dictated by the choice of a set of “distributors”, i.e., rigid body motions, which must yield zero internal work. The expressions of rigid body motions, *postulated* by Germain (1973), affect both the classical and the generalized DOFs. His choice turns out not to require any particular symmetry on behalf of $\boldsymbol{\Sigma}$ (therein called $\boldsymbol{\tau}$). In the present micromechanical approach, rigid body motions *are not postulated* but deduced by answering the question: what are the expressions of $\mathbf{U}$ and $\boldsymbol{\chi}$ when $\mathbf{u}$ is a rigid body motion? As a rigid body motion $\mathbf{u}$ has only infinitely-long-wavelength components, we have $\mathbf{U} = \mathbf{u}$ and $\boldsymbol{\chi} = \mathbf{0}$. The superposition of a rigid body motion hence only affects $\mathbf{U}$ which is consistent with $\boldsymbol{\Sigma}$ being symmetric.

Forest and Sab (1998) and Forest (2006) have defined generalized kinematics by minimizing a quadratic error similar to (4.6). Their minimization procedure is local: it defines the best admissible approximation to the microscopic displacement field over a representative volume element. In doing so, they obtain similar kinematics to those of Germain, at least regarding rigid body motions. The corresponding effective displacement field, combination of the generalized DOFs carried by shape functions, has a meaning only in the vicinity of each material point. In contrast, our approach is global: it defines the best admissible approximation to the microscopic displacement field over Ω . We can summarize the difference by saying that Germain’s kinematics is based on a couple of global-local position variables while our’s is based on a couple of slow-fast position variables. Let us underline then the fact that the term “two-scale” is ambiguous. Our approach is a two-scale approach in the sense of two-scale asymptotics for instance but not in the sense of Forest and Sab (1998) or Germain. The major advantage of the present approach resides in the fact that *a microscopic displacement field solution is transformed into the best effective displacement field solution of the same FB wavenumber and of the same frequency*. Particularized to free waves, this entails that the dispersion curve is unharmed by the scale transition.

On a different note, due to their definitions, the macroscopic DOFs are expected to be continuous functions of $\mathbf{x}$. However, the shape functions used here, i.e., the local coordinates, are discontinuous and consequently $\mathbf{D}$ is expected to be discontinuous as well. Discontinuity means that gaps and/or overlappings, however small, may occur when Ω undergoes the macroscopic transformation prescribed by $\mathbf{D}$. One can be tolerant toward discontinuities and say that $\mathbf{D}$ is only an approximation to the real microscopic transformation of Ω . One can also avoid discontinuities altogether

by regularizing the local coordinates across unit cell boundaries. The latter attitude is much more interesting as it would correspond to a situation where a unit cell is composed of a hard core, “the brick”, and a soft boundary, “the mortar”. When the hard cores undergo different rigid-like transformations for instance, the soft phase will regularize the differences across unit cells. A regularization of $\tilde{\mathbf{x}}$ can be simply obtained by truncating its Fourier series expansion. In the example treated in subsection 4.5, we replace $\tilde{\mathbf{x}}$ by its first Fourier series component.

4 LW-LF asymptotics

At this point, we have generalized Willis theory by using enriched kinematics to improve the quality of approximation of a microscopic displacement $\mathbf{u}$ by a macroscopic one $\mathbf{D}$. The cost is however the increasing complexity of the resulting effective motion equation. A numerical procedure dedicated to the implementation of Willis’ theory or our previous one is quite heavy, the effective behavior being nonlocal in both space and time with infinite radii of influence in general.

Taylor asymptotic expansions provide an efficient way to approximate the nonlocal behavior with a local one under appropriate assumptions on $\mathbf{k}$ and ω . LW-LF expansions have the main advantage of only requiring the solution of static problems but present the disadvantage of being limited to the LF behavior as was proven in chapter 3. The purpose of this section is to show how the simplest enriched kinematics makes it possible to extend the validity domain of LW-LF expansions to high-frequency behavior. In particular, the microdilation-like generalized substitution medium will be proven able to capture, even in the LF limit, approximately but simultaneously, all of the acoustic branches and selected optical ones.

4.1 Setup

In the spirit of microdilatation media, let admissible body forces FB amplitudes be of the form

$$\tilde{\mathbf{f}}_{\mathbf{k}}(\mathbf{x}) = \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{q}_{\mathbf{k}}\phi(\mathbf{x}),$$

where ϕ is a fixed $\mathcal{R}$ -periodic shape function, $\tilde{\mathbf{F}}_{\mathbf{k}}$ and $\tilde{q}_{\mathbf{k}}$ are constants. Orthonormality requires $\langle \phi \rangle = \mathbf{0}$ and $\langle \phi \cdot \phi^* \rangle = 1$ which are assumed subsequently. It is then known that the effective displacement field has the expression

$$\tilde{\mathbf{D}}_{\mathbf{k}}(\mathbf{x}) = \tilde{\mathbf{U}}_{\mathbf{k}} + \tilde{\chi}_{\mathbf{k}}\phi(\mathbf{x}),$$

where $\tilde{\mathbf{U}}_{\mathbf{k}}$ and $\tilde{\chi}_{\mathbf{k}}$ are the DOFs of the generalized substitution medium. They derive from the microscopic displacement according to (4.5):

$$\tilde{\mathbf{U}}_{\mathbf{k}} = \langle \tilde{\mathbf{u}}_{\mathbf{k}} \rangle, \quad \tilde{\chi}_{\mathbf{k}} = \langle \phi^* \cdot \tilde{\mathbf{u}}_{\mathbf{k}} \rangle.$$

In order to find the effective motion equation, the microscopic one (4.7) which reads

$$(\nabla + i\mathbf{k}) \cdot \{ \mathbf{C}(\mathbf{x}) : [(\nabla + i\mathbf{k}) \otimes^s \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x})] \} + \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{q}_{\mathbf{k}}\phi(\mathbf{x}) = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}_{\mathbf{k}}(\mathbf{x}), \quad (4.16)$$

when applied to the case under consideration, needs to be solved.

4.2 Expansions

As discussed earlier, we will be only interested in an LW-LF asymptotic expansion of the effective motion equation and the corresponding asymptotic expansion of the effective impedance Z from (4.10). This requires the asymptotic expansion of the solution to (4.16). Let us then start by imbedding the foregoing equation into the family of equations

$$\left(\frac{1}{\epsilon}\nabla + i\mathbf{k}\right) \cdot \left\{ \mathbf{C}(\mathbf{x}) : \left[\left(\frac{1}{\epsilon}\nabla + i\mathbf{k}\right) \otimes^s \tilde{\mathbf{u}}_k^\epsilon(\mathbf{x}) \right] \right\} + \tilde{\mathbf{F}}_k^\epsilon + \tilde{q}_k^\epsilon \phi(\mathbf{x}) = -\omega^2 \rho \tilde{\mathbf{u}}_k^\epsilon(\mathbf{x}) \quad (4.17)$$

scaled with the “small parameter” ϵ . Putting $\epsilon = 1$ generates back the original microscopic motion equation. However, before doing so, expansions will be carried in the vicinity of $\epsilon = 0$, the limit $\epsilon \rightarrow 0$ corresponding to vanishing $\mathbf{k}$ and ω and infinite scale separation. Displacements and body forces are expanded into Taylor series according to

$$\tilde{\mathbf{u}}_k^\epsilon = \sum_{n \in \mathbb{N}} \epsilon^n \tilde{\mathbf{u}}_k^n, \quad \tilde{\mathbf{F}}_k^\epsilon = \sum_{n \in \mathbb{N}} \epsilon^n \tilde{\mathbf{F}}_k^n, \quad \tilde{q}_k^\epsilon = \sum_{n \geq -1} \epsilon^n \tilde{q}_k^n. \quad (4.18)$$

Body forces of all orders are introduced for ease of tractability. Otherwise, scaling of body forces is of no importance as it does not influence the expansion of Z .

4.3 Hierarchical microscopic motion equations

Injecting (4.18) into (4.17) yields a hierarchy of microscopic motion equations, one for each order in ϵ .

4.3.1 Orders -2 and -1

At the lowest order ϵ^{-2} we have

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_k^0)] = \mathbf{0}.$$

Therefore

$$\tilde{\mathbf{u}}_k^0 = \tilde{\mathbf{U}}_k^0, \quad \tilde{\chi}_k^0 = 0.$$

As expected the classical homogenization limit produces no DOFs other than $\tilde{\mathbf{U}}^0$. This does not hold at higher orders.

At the order ϵ^{-1} , we have

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_k^1 + i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0)] + \tilde{q}_k^{-1} \phi = 0.$$

Note that the presence of $\tilde{q}_k^{-1}$ does not obstruct solution existence since $\langle \phi \rangle = \mathbf{0}$. The solution can be written as

$$\tilde{\mathbf{u}}_k^1(\mathbf{x}) = \tilde{\mathbf{U}}_k^1 + \mathbf{R}^{1,0}(\mathbf{x}) : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0) + \mathbf{P}(\mathbf{x}) \tilde{q}_k^{-1},$$

where $\mathbf{R}^{1,0}$ and $\mathbf{P}$ are two zero-average $(\mathbf{k}, \omega)$ -independent localization tensors. Therefore, the new DOF, to first order, is given by

$$\tilde{\chi}_k^1(\mathbf{x}) = \langle \phi^* \cdot \mathbf{R}^{1,0} \rangle : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0) + \langle \phi^* \cdot \mathbf{P} \rangle \tilde{q}_k^{-1},$$

or equivalently by

$$\langle \phi^* \cdot \mathbf{P} \rangle^{-1} \tilde{\chi}_k^1(\mathbf{x}) = \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \langle \phi^* \cdot \mathbf{R}^{1,0} \rangle : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0) + \tilde{q}_k^{-1}. \quad (4.19)$$

Accordingly, the first order solution can be localized with respect to the macroscopic DOFs by

$$\tilde{\mathbf{u}}_k^1(\mathbf{x}) = \tilde{\mathbf{U}}_k^1 + \mathbf{X}^{1,0}(\mathbf{x}) : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0) + \mathbf{Y}^{0,0}(\mathbf{x}) \tilde{\chi}_k^1$$

with

$$\begin{aligned} \mathbf{X}^{1,0}(\mathbf{x}) &= \mathbf{R}^{1,0} - \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \mathbf{P}(\mathbf{x}) \otimes \langle \phi^* \cdot \mathbf{R}^{1,0} \rangle, \\ \mathbf{Y}^{0,0}(\mathbf{x}) &= \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \mathbf{P}(\mathbf{x}). \end{aligned}$$

4.3.2 Order 0 and higher orders

At the order $\epsilon^0 = 1$, we have

$$\nabla \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_k^2 + i\mathbf{k} \otimes^s \tilde{\mathbf{u}}_k^1)] + i\mathbf{k} \cdot [\mathbf{C} : (\nabla \otimes^s \tilde{\mathbf{u}}_k^1 + i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0)] + \tilde{\mathbf{F}}_k^0 + \tilde{q}_k^0 \phi = -\omega^2 \rho \tilde{\mathbf{U}}_k^0, \quad (4.20)$$

whose volume average is no longer trivially satisfied and yields, thanks to the periodic boundary conditions,

$$i\mathbf{k} \cdot \langle \mathbf{C} + \mathbf{C} : \nabla \otimes^s \mathbf{X}^{1,0} \rangle : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0) + i\mathbf{k} \cdot \langle \mathbf{C} : \nabla \otimes^s \mathbf{Y}^{0,0} \rangle \tilde{\chi}_k^1 + \tilde{\mathbf{F}}_k^0 = -\omega^2 \langle \rho \rangle \tilde{\mathbf{U}}_k^0. \quad (4.21)$$

Compared to the classical homogenization limit, the lowest order effective motion equation here involves an additional DOF and needs to be completed with equation (4.19).

Subtracting (4.21) from (4.20) and solving for $\tilde{\mathbf{u}}^2$, we similarly get

$$\begin{aligned} \tilde{\mathbf{u}}_k^2 &= \tilde{\mathbf{U}}_k^2 + \mathbf{R}^{1,0} : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^1) + \mathbf{R}^{2,0} \cdot : [i\mathbf{k} \otimes (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0)] \\ &\quad + \mathbf{R}^{0,1} \cdot \omega^2 \mathbf{U}_k^0 + \mathbf{S}^{1,0} \cdot (i\mathbf{k} \tilde{\chi}_k^1) + \mathbf{P} \tilde{q}_k^0, \end{aligned}$$

where $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{P}$ are zero-average $(\mathbf{k}, \omega)$ -independent localization tensors and the symbol $\cdot :$ stands for triple contraction. In order to localize the microscopic displacements with respect to the effective DOFs, we average the above equation weighted with ϕ^* . We then arrive at

$$\begin{aligned} \tilde{\mathbf{u}}_k^2 &= \tilde{\mathbf{U}}_k^2 + \mathbf{X}^{1,0} : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^1) + \mathbf{X}^{2,0} \cdot : [i\mathbf{k} \otimes (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k^0)] \\ &\quad + \mathbf{X}^{0,1} \cdot \omega^2 \mathbf{U}_k^0 + \mathbf{Y}^{1,0} \cdot (i\mathbf{k} \tilde{\chi}_k^1) + \mathbf{Y}^{0,0} \tilde{\chi}_k^2, \end{aligned}$$

with

$$\begin{aligned} \mathbf{X}^{2,0} &= \mathbf{R}^{2,0} - \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \mathbf{P} \otimes \langle \phi^* \cdot \mathbf{R}^{2,0} \rangle, \\ \mathbf{X}^{0,1} &= \mathbf{R}^{0,1} - \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \mathbf{P} \otimes \langle \phi^* \cdot \mathbf{R}^{0,1} \rangle, \\ \mathbf{Y}^{1,0} &= \mathbf{S}^{1,0} - \langle \phi^* \cdot \mathbf{P} \rangle^{-1} \mathbf{P} \otimes \langle \phi^* \cdot \mathbf{S}^{1,0} \rangle. \end{aligned}$$

Finally, following the above methodology, we obtain a series of localization operators $\mathbf{X}^{m,n}$ and $\mathbf{Y}^{m,n}$ yielding the field u^r , for all orders $r \geq 0$, with respect to the macroscopic DOFs. As for notations, $\mathbf{X}$ is associated with $\tilde{\mathbf{U}}$, $\mathbf{Y}$ with $\tilde{\chi}$, m with tensorial powers of $i\mathbf{k}$ and n with powers of ω^2 .

4.4 Approximate effective motion equation

We have formally obtained a localization operator (L_k^U, L_k^χ) satisfying

$$\tilde{\mathbf{u}}_k = L_k^U \cdot \tilde{\mathbf{U}}_k + L_k^\chi \tilde{\chi}_k. \quad (4.22)$$

Its expression, to the second order, requires solving the above hierarchy up to order 1 and is given by

$$\begin{aligned} \tilde{\mathbf{u}}_k = & \tilde{\mathbf{U}}_k + \mathbf{Y}^{0,0} \tilde{\chi}_k + \mathbf{X}^{1,0} : (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k) + \mathbf{Y}^{1,0} \cdot (i\mathbf{k} \tilde{\chi}_k) \\ & + \mathbf{X}^{2,0} \cdot [i\mathbf{k} \otimes (i\mathbf{k} \otimes^s \tilde{\mathbf{U}}_k)] + \mathbf{Y}^{2,0} : (i\mathbf{k} \otimes i\mathbf{k} \tilde{\chi}_k) + \omega^2 \mathbf{X}^{0,1} \cdot \tilde{\mathbf{U}}_k + \omega^2 \mathbf{Y}^{0,1} \tilde{\chi}_k + \dots \end{aligned} \quad (4.23)$$

Orders of magnitude can be traced thanks to $\mathbf{k}$ and ω upon mapping ϵ to 1 so as to recover the unscaled microscopic displacement field.

The action describing the microscopic motion is known to be

$$\mathcal{A} = \frac{1}{2} \langle \tilde{\epsilon}_k^* : \mathbf{C} : \tilde{\epsilon}_k - \rho \tilde{\mathbf{v}}_k^* \cdot \tilde{\mathbf{v}}_k \rangle - \Re \langle \tilde{\mathbf{u}}_k^* \cdot \tilde{\mathbf{f}}_k \rangle,$$

where $\Re$ gives the real part of a complex number. Its stationarity condition over arbitrary $\mathcal{R}$ -periodic fields $\tilde{\mathbf{u}}_k$ yields the microscopic motion equation (4.7) whereas its stationarity condition over fields of the form (4.22) gives rise to the macroscopic motion equation (4.10). Finally, its stationarity condition over fields of the form (4.23), truncated at order 2 in $\mathbf{k}$ and ω , results in the second order approximate effective motion equation

$$\begin{aligned} \mathbf{Z}_k^{UU} \cdot \tilde{\mathbf{U}}_k + \mathbf{Z}_k^{U\chi} \tilde{\chi}_k &= \tilde{\mathbf{F}}_k, \\ \mathbf{Z}_k^{\chi U} \cdot \tilde{\mathbf{U}}_k + \mathbf{Z}_k^{\chi\chi} \tilde{\chi}_k &= \tilde{\mathbf{q}}_k, \end{aligned} \quad (4.24)$$

where the second order tensor $\mathbf{Z}_k^{UU}$, the first order tensors $\mathbf{Z}_k^{U\chi}$ and $\mathbf{Z}_k^{\chi U}$ and the scalar $\mathbf{Z}_k^{\chi\chi}$ are second degree polynomials in $\mathbf{k}$ and ω . Their expressions are specified next for a 1D example.

4.5 Example: one string, two phases

4.5.1 Setting

Consider the periodically inhomogeneous string whose unit cell is depicted in figure 4.3, and define

$$\phi(x) = \sqrt{2} \sin(\pi x/a).$$

The form of ϕ is inspired by a low-contrast analysis carried out in the context of solid state physics by [Quéré \(1988\)](#). The resulting 1D generalized substitution medium can be seen as a microstretch medium whose local coordinate $\tilde{x}$ has been replaced by its first Fourier component.

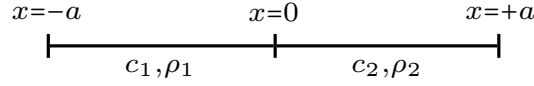


Figure 4.3. Unit cell.

4.5.2 Expression of the approximate effective motion equation

It is straightforward to solve the family of motion equations of subsection 4.3 up to order 1 and to get the corresponding localization operators. Calling c_i and ρ_i the stiffness and mass density of phase i for $i \in \{1, 2\}$, and a the half-length of a unit cell, the approximate effective impedance Z is given by

$$\begin{aligned} Z^{UU} &= 2 \frac{c_1 c_2}{c_1 + c_2} k^2 - \frac{\rho_1 + \rho_2}{2} \omega^2, \\ Z^{\chi U} = Z^{U\chi} &= \frac{4\sqrt{2}}{\pi} \frac{c_1 c_2 (c_1 - c_2)}{(c_1 + c_2)^2} k^2 + \frac{\sqrt{2}}{\pi} (\rho_1 - \rho_2) \omega^2, \\ Z^{\chi\chi} &= \frac{2\pi^2}{a^2} \frac{c_1 c_2}{c_1 + c_2} \\ &\quad - \frac{2}{\pi^2} \frac{[(\pi^2 - 6)\rho_2 + 2\rho_1]c_1^2 + 4(\rho_1 + \rho_2)c_1 c_2 + [(\pi^2 - 6)\rho_1 + 2\rho_2]c_2^2}{(c_1 + c_2)^2} \omega^2 \\ &\quad - \frac{2}{\pi^2} \frac{c_1 c_2 [(3\pi^2 - 8)c_1^2 + 2(3\pi^2 + 8)c_1 c_2 + (3\pi^2 - 8)c_2^2]}{(c_1 + c_2)^3} k^2. \end{aligned}$$

Recall that this impedance is obtained through LW-LF asymptotics and that neither the approximate impedance nor the exact one have meaning outside $T^* =] - \pi/2a, \pi/2a[$. The approximate effective motion equation takes the form

$$\begin{aligned} &-2 \frac{c_1 c_2}{c_1 + c_2} U'' + \frac{\rho_1 + \rho_2}{2} \ddot{U} - \frac{4\sqrt{2}}{\pi} \frac{c_1 c_2 (c_1 - c_2)}{(c_1 + c_2)^2} \chi'' - \frac{\sqrt{2}}{\pi} (\rho_1 - \rho_2) \ddot{\chi} = F, \\ &- \frac{4\sqrt{2}}{\pi} \frac{c_1 c_2 (c_1 - c_2)}{(c_1 + c_2)^2} U'' - \frac{\sqrt{2}}{\pi} (\rho_1 - \rho_2) \ddot{U} + \frac{2\pi^2}{a^2} \frac{c_1 c_2}{c_1 + c_2} \chi \\ &+ \frac{2}{\pi^2} \frac{[(\pi^2 - 6)\rho_2 + 2\rho_1]c_1^2 + 4(\rho_1 + \rho_2)c_1 c_2 + [(\pi^2 - 6)\rho_1 + 2\rho_2]c_2^2}{(c_1 + c_2)^2} \ddot{\chi} \\ &+ \frac{2}{\pi^2} \frac{c_1 c_2 [(3\pi^2 - 8)c_1^2 + 2(3\pi^2 + 8)c_1 c_2 + (3\pi^2 - 8)c_2^2]}{(c_1 + c_2)^3} \chi'' = f^w, \end{aligned}$$

where a superscripted dot denotes $\partial/\partial t$ and the prime symbol means $\partial/\partial x$.

4.5.3 Dispersion curves: exact vs approximate

The expression of the exact dispersion curve is known and was derived elsewhere (chapter 3). It reads:

$$\begin{aligned} \cos(2ka) = & \frac{(\sqrt{c_1\rho_1} + \sqrt{c_2\rho_2})^2}{4\sqrt{c_1\rho_1c_2\rho_2}} \cos\left[\omega(\sqrt{\rho_1/c_1} + \sqrt{\rho_2/c_2})a\right] \\ & - \frac{(\sqrt{c_1\rho_1} - \sqrt{c_2\rho_2})^2}{4\sqrt{c_1\rho_1c_2\rho_2}} \cos\left[\omega(\sqrt{\rho_1/c_1} - \sqrt{\rho_2/c_2})a\right]. \end{aligned} \quad (4.25)$$

The approximate dispersion curve is derived from the approximate effective impedance according to

$$Z^{UU} Z^{\chi\chi} - Z^{\chi U} Z^{U\chi} = 0.$$

The first branches of the exact and approximate dispersion curves are drawn in figure 4.4 for the following numerical values of the string parameters

$$c_1 = 10^{10} \text{N}, \quad c_2 = 10^{12} \text{N}, \quad \rho_1 = 100 \text{kg/m}, \quad \rho_2 = 200 \text{kg/m}, \quad a = 5 \cdot 10^{-3} \text{m}.$$

On figure 4.4, with respect to the classical quasi-static approximation (Q), enriching the kinematics not only allows capturing the first optical branch but also improves the approximation of the acoustical one over the entire first Brillouin zone. Why is it possible that low-frequency Taylor expansions lead to a correct estimate of some optical modes? Physically speaking, for high frequencies, inertial forces become important and shift the energy carried by displacements toward shorter wavelengths. Correspondingly, including rapidly oscillating body forces have two benefits. First, they simulate the effects of inertial forces. Second, and most importantly, they oblige the macroscopic displacement field to include some short-wavelength components, necessary for approximating the high-frequency behavior. Mathematically speaking, a proof of the accuracy of the above asymptotics is yet to be found.

Craster et al. (2010) utilized high-frequency asymptotics to calculate optical modes. They had to numerically solve the unit cell problem at the lowest order and to use two initialization points for each optical branch. In comparison, the present formulation is capable of capturing not only the acoustic mode but also, simultaneously, the *entire* first optical branch thanks to the enriched kinematics while maintaining a simple motion equation of order 2. Note that we only had to solve *static* cell problems. This being said, based on the results of Craster et al. (2010), and on the next theorem, we are inclined to conjecture that the best shape functions to be supplied to the present theory are actual periodic and anti-periodic microscopic solutions.

4.6 A theorem: on the recovery of optical branches

Inspired by the approach of Craster et al. (2010), we now choose the shape functions ϕ to be actual $\mathcal{R}$ -periodic optical eigenmodes associated to the null wavenumber.

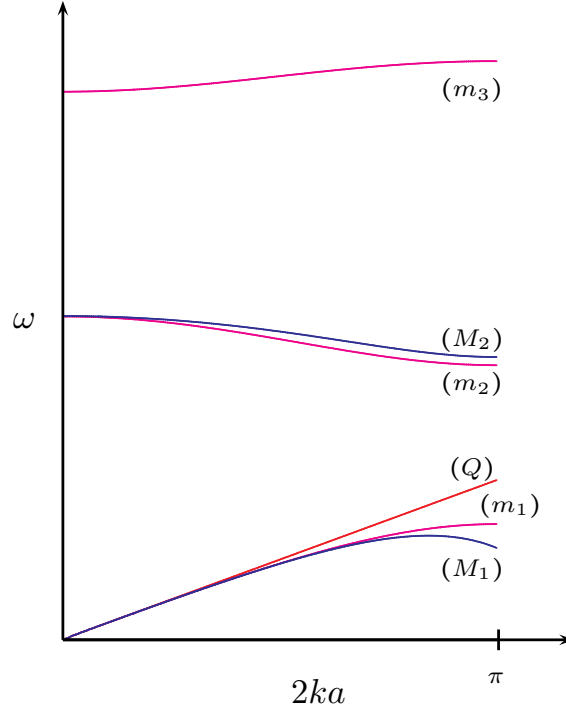


Figure 4.4. Exact dispersion curve (three branches $(m_{1,2,3})$) compared to its classical quasi-static approximation (one branch (Q)) and to its approximation by the present theory (two branches $(M_{1,2})$).

Precisely¹,

$$\tilde{\mathbf{f}}_{\mathbf{k}} = \frac{\rho}{\langle \rho \rangle} \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}}^{\alpha} \phi_{\alpha} = \rho \tilde{\mathbf{F}}_{\mathbf{k}} + \tilde{f}_{\mathbf{k}}^{\alpha} \rho \tilde{\psi}_{\mathbf{0},\alpha}.$$

Note that the above shape functions do not satisfy the orthonormality² condition but that this shall not interfere hereafter. Also, note that $\tilde{\mathbf{F}}_{\mathbf{k}}$ has now the weight $\rho/\langle \rho \rangle$.

Our purpose here is to prove that *the substitution medium corresponding to the above choice yields, in the LF limit, exactly and simultaneously, the values of the optical eigenfrequencies $\omega_{\alpha}(\mathbf{0})$, for $\alpha = d+1, \dots, N$, and of the acoustic ones $\omega_1(\mathbf{0}) = \dots = \omega_d(\mathbf{0}) = 0$.*

Since we are only interested here in the behavior at $\mathbf{k} = \mathbf{0}$, instead of solving the hierarchy of motion equations, it is easier to consider the macroscopic Green's function. Let us then first derive a series representation for the macroscopic Green's tensor for

¹The $\tilde{\psi}$ have been introduced in chapter 1.

²Of course, one can always orthonormalize the set of shape functions while keeping invariant the spanned space. A better attitude would be to change the scalar product. See the remark at the end of 2.2.3.

general shape functions. Starting with the now known representation

$$\mathbf{g}_k(\mathbf{y}, \mathbf{x}) = \sum_{n \geq 1} \frac{\tilde{\psi}_{k,n}(\mathbf{y}) \otimes \tilde{\psi}_{k,n}^*(\mathbf{x})}{\omega_n^2(\mathbf{k}) - \omega^2},$$

and using (4.9), we obtain

$$\begin{aligned} \tilde{U}_k &= \sum_{n \geq 1} \frac{1}{\langle \rho \rangle^2} \frac{\langle \rho \tilde{\psi}_{k,n} \rangle \otimes \langle \tilde{\psi}_{k,n}^* \rho \rangle}{\omega_n^2(\mathbf{k}) - \omega^2} \cdot \tilde{\mathbf{F}}_k + \frac{1}{\langle \rho \rangle} \frac{\langle \rho \tilde{\psi}_{k,n} \rangle \langle \tilde{\psi}_{k,n}^* \cdot \phi_\alpha \rangle}{\omega_n^2(\mathbf{k}) - \omega^2} \tilde{f}_k^\alpha, \\ \tilde{u}_k^\beta &= \sum_{n \geq 1} \frac{1}{\langle \rho \rangle} \frac{\langle \phi_\beta^* \cdot \tilde{\psi}_{k,n} \rangle \langle \tilde{\psi}_{k,n}^* \rho \rangle}{\omega_n^2(\mathbf{k}) - \omega^2} \cdot \tilde{\mathbf{F}}_k + \frac{\langle \phi_\beta^* \cdot \tilde{\psi}_{k,n} \rangle \langle \tilde{\psi}_{k,n}^* \cdot \phi_\alpha \rangle}{\omega_n^2(\mathbf{k}) - \omega^2} \tilde{f}_k^\alpha. \end{aligned}$$

For $\mathbf{k} = \mathbf{0}$, the above expressions become

$$\begin{aligned} \tilde{U}_0 &= -\frac{1}{\langle \rho \rangle \omega^2} \tilde{\mathbf{F}}_0 - \frac{1}{\langle \rho \rangle \omega^2} \langle \phi_\alpha \rangle \tilde{f}_0^\alpha, \\ \tilde{u}_0^\beta &= -\frac{1}{\langle \rho \rangle \omega^2} \langle \phi_\beta^* \rangle \cdot \tilde{\mathbf{F}}_0 + \sum_{n \geq 1} \frac{\langle \phi_\beta^* \cdot \tilde{\psi}_{0,n} \rangle \langle \tilde{\psi}_{0,n}^* \cdot \phi_\alpha \rangle}{\omega_n^2(\mathbf{0}) - \omega^2} \tilde{f}_0^\alpha. \end{aligned}$$

Therein, we have used the orthonormality of eigenmodes:

$$\forall n, r \geq 1, \quad \langle \rho \tilde{\psi}_{k,n}^* \cdot \tilde{\psi}_{k,r} \rangle = \delta_{nr},$$

and, in particular,

$$\forall n \geq d+1, \quad \langle \rho \tilde{\psi}_{0,n}^* \rangle = \mathbf{0},$$

since the acoustic eigenmodes, $r = 1, \dots, d$, are uniform in space being rigid body translations. For our particular choice of shape functions, the foregoing expression of the macroscopic Green's function is even further simplified into

$$\begin{aligned} \tilde{U}_0 &= -\frac{1}{\langle \rho \rangle \omega^2} \tilde{\mathbf{F}}_0, \\ \tilde{u}_0^\beta &= \sum_{n \geq 1} \frac{\delta_{\beta n} \delta_{n\alpha}}{\omega_n^2(\mathbf{0}) - \omega^2} \tilde{f}_0^\alpha = \frac{1}{\omega_\beta^2(\mathbf{0}) - \omega^2} \tilde{f}_0^\beta. \end{aligned}$$

Therefore, for $\mathbf{k} = \mathbf{0}$, the effective motion equation is

$$\begin{aligned} -\langle \rho \rangle \omega^2 \tilde{U}_0 &= \tilde{\mathbf{F}}_0, \\ (\omega_\beta^2(\mathbf{0}) - \omega^2) \tilde{u}_0^\beta &= \tilde{f}_0^\beta, \end{aligned}$$

with $\beta = d+1, \dots, N$. This effective motion equation is already of the second order so that LF asymptotics will leave it as it is. Its dispersion relation reads

$$(-\langle \rho \rangle \omega^2)^d \prod_{\beta=d+1}^N (\omega_\beta^2(\mathbf{0}) - \omega^2) = 0$$

which ends the proof of the stated theorem.

In conclusion, an appropriate choice of shape functions is capable of producing the “exact germs”, i.e., at $\mathbf{k} = \mathbf{0}$, of any combination of opto-acoustical branches even in the LF limit. What happens for small but non-null $\mathbf{k}$ requires more involved calculations. This result demonstrates the potential of the presented theory. However, it is not directly useful unless the concerned eigenmodes at $\mathbf{k} = \mathbf{0}$ were known, or calculable, beforehand. Otherwise, one will have to guess alternative appropriate shape functions as was done for the example of the 1D string.

5 Concluding remarks

The present chapter has proposed an elastodynamic homogenization theory generalizing the one of Willis in the case of periodic media. The effective motion equation resulting from the generalized theory was then used as the basis for deriving asymptotic elastodynamic homogenization models of different degrees of complexity according to the richness of body forces involved. In particular, it has been shown that the LW-LF asymptotic expansion of the effective motion equation of the generalized theory is capable of simultaneously capturing all the acoustic and some of the optical branches of the microscopic dispersion curve.

Two problems remain open and need solving. The first concerns the effective elastodynamic constitutive law produced by the generalized theory proposed. Here, to avoid the difficulty related to its non-uniqueness, the effective motion equation has been directly treated and exploited. However, in numerous situations, it is useful and important to explicitly know the effective elastodynamic constitutive law. The second problem regards the optimal choice of shape functions for body forces acting as microscopic and macroscopic loading. For the LW-LF behavior, constant shape functions have extensively been employed in the literature on asymptotic elastodynamic homogenization methods. For more involved shape functions, it seems possible to distinguish two types. The first one is based on actual particular solutions of the motion equation such as some periodic oscillation modes proposed by [Craster et al. \(2010\)](#) and [Boutin et al. \(2014\)](#). The other one relies on the geometry and contrast of the microstructure, resulting in a hierarchy of standard generalized media such as the one from microdilatation-like to micromorphic-like media. In any case, criteria for guiding the optimal choice of shape functions are still lacking.

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Conclusion and openings

You and I have unfinished business.

The Bride, Kill Bill: Volume 2

Improving the predictions of the standard quasi-static homogenization theory is essential to the macroscopic modeling of the elastodynamic behavior of composites, in general, and metamaterials, in particular. These improvements should permit to capture dispersion effects, optical oscillation modes and band gaps. Accordingly, the strict hypotheses made on frequencies and wavelengths must be loosened or, in any case, modified. Willis theory is a radical attempt to overlook all such scale separation assumptions and while it succeeds in upscaling correctly all the information related to the dispersion curve, it fails to reproduce faithfully the associated free waves beyond the homogenizability limit of weak scattering where the coherent wave is dominant.

Higher-order strain-gradient theories offer simple and efficient approximations valid exclusively at acoustical frequencies. These theories model well the dispersive effects but miss all optical modes due to some inevitable localization singularities. Then, seeking for a theory valid over an extended, however finite, frequency band englobing simultaneously acoustic and optical modes, we have seen that a family of generalized media is up for the task.

Hereafter, in order to conclude, we freely comment on, criticize and discuss extensions and applications of the foregoing ideas.

1 On the use of body forces

We argue that body forces must be taken into account by any homogenization scheme for, at least, two reasons.

1. Even if the effective parameters do not depend on body forces, it is indispensable to know how body forces should be incorporated in the effective motion equation. For instance, gravitational forces are irrelevant to the results of compression tests carried on concrete samples. Nonetheless, the fact that the microscopic forces $\rho g \mathbf{e}_3$ become $\langle \rho \rangle g \mathbf{e}_3$ on the macroscale is not trivial a priori and needs to be justified.

2. When enriching the degrees of freedom of the effective homogeneous substitution medium at finite frequencies, i.e., beyond the quasi-static limit, the effective parameters become dependent on the space of admissible body forces. Being aware of this dependency, including body forces allows to preserve energetic consistency in the sense of Hill-Mandel relation.

2 On averaging and contrast

What is striking¹ in the present work is the abundance of definitions. All these definitions involve some kind of averaging which, as intuitive as one might think it is, is poorly justified and is mainly a simple way to force the smoothing of microscopic fields where smoothing should be a natural outcome of the governing physics. Another bizarre feature of said definitions is the absence of the role of microstructure. All the suggested homogenization approaches seem to be indifferently applicable to all kinds of periodic microstructures and can yield strain-gradient, Cosserat-like or micromorphic-like effective media. It is desirable however that the rise of dispersive effects or that of new degrees of freedom be also a natural outcome of the governing physics and of the underlying microstructure.

The nature of the effective medium, be it a strain-gradient, a non-local and/or a generalized one, can be justified by three means;

1. ideally, through a convergence theorem,
2. or, otherwise, by some formal asymptotic expansion,
3. or, as a last resort, by some homogenizability conditions which should help control the error committed by the averaging scheme.

Paradoxically, the most known convergence theorem of homogenization theory is also microstructure-independent implying that the effective medium is a Cauchy medium and that dispersion and enriched kinematics are only relevant as first order corrections. However, in some physical situations where contrasts are high enough and phases are connected in a particular fashion, said first order corrections become dominant. A natural question then comes to mind: under what conditions on contrast and connectedness can a strain-gradient medium, a Cosserat-like medium or even a Willis medium be obtained as homogenization limits rather than by an averaging scheme?

Effective media with a nonlocal behavior in time have been obtained as homogenization limits of a two-phase high-contrast stiff-matrix soft-inclusion composite. I speculate that Cosserat-like media, i.e., with additional rotational degrees of freedom, can be obtained by inverting the contrast in the aforementioned composite. As for Willis media, the question remains largely open to my knowledge.

¹In a negative sense.

3 Back to cloaking

The realization of some elastic cloaks requires a material whose effective behavior couples stresses and velocities as well as momenta and strains as in the one defined by Willis theory. A project aiming to conceive such cloaks would encounter two difficulties.

1. For the metamaterials used in the cloak to behave macroscopically as dictated by Willis theory, said theory must be valid which invokes the same above questions on homogenizability and convergence.
2. Assuming the validity of Willis theory, the effective behavior is nonlocal in space whereas the behavior of the cloak is local. An appropriate local long-wavelength asymptotic approximation of Willis theory preserving the coupling terms needs then to be found.

4 Other behaviors and media

The (formal) extension of the presented approaches to other microscopic linear behaviors seems straightforward. The physical implications on the characteristics of the effective behavior can be however important. For instance, including linearly viscoelastic components would localize time nonlocality in some recent past as small as viscous effects are important. This can be analytically checked on the simple discrete 1D example treated in chapter 2. As a matter of fact, the therein derived formulae remain valid if any spring is replaced by any combination of springs and linear dashpots.

It seems to me that the methods presented here are profoundly linear as they are based on different kinds of expansions such as Floquet-Bloch and Fourier expansions. Non-linear behaviors are much more challenging.

Other problems of theoretical and practical importance arise when dealing with finite media as boundaries, interfaces and associated reflexion, refraction and transmission phenomena come into play. Non-periodic media, such as quasi-crystals, perturbed periodic media or plain random media, are also undoubtedly worthy of attention.