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Par

M. Qiang Sun

A contribution to the theory of (signed) graph
homomorphism bound and Hamiltonicity

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*“When things had been classified in organic categories,
knowledge moved toward fulfillment.”*

Confucius (551-479 B.C.), The Great Learning

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Abstract

In this thesis, we study two main problems in graph theory: homomorphism problem of planar (signed) graphs and Hamiltonian cycle problems.

As an extension of the Four-Color Theorem, it is conjectured ([80], [41]) that every planar consistent signed graph of unbalanced-girth $d+1$ ($d \geq 2$) admits a homomorphism to signed projective cube $SPC(d)$ of dimension d . It is naturally asked that:

Is $SPC(d)$ an optimal bound of unbalanced-girth $d+1$ for all planar consistent signed graphs of unbalanced-girth $d+1$? (*)

In Chapter 2, we prove that: if (B, Ω) is a consistent signed graph of unbalanced-girth d which bounds the class of planar consistent signed graphs of unbalanced-girth d , then $|B| \geq 2^{d-1}$. Furthermore, if no subgraph of (B, Ω) bounds the same class, $\delta(B) \geq d$, and therefore, $|E(B)| \geq d \cdot 2^{d-2}$. Our results showed that if the conjecture ([80], [41]) holds, then $SPC(d)$ is an optimal bound both in terms of number of vertices and number of edges.

When $d = 2k$, the problem (*) is equivalent to the homomorphisms of graphs: is $PC(2k)$ an optimal bound of odd-girth $2k+1$ for \mathcal{P}_{2k+1} (the class of all planar graphs of odd-girth at least $2k+1$)? Note that K_4 -minor free graphs are planar graphs, is $PC(2k)$ also an optimal bound of odd-girth $2k+1$ for all K_4 -minor free graphs of odd-girth $2k+1$? The answer is negative. In [6], a family of graphs of order $\mathcal{O}(k^2)$ bounding the K_4 -minor free graphs of odd-girth $2k+1$ were given. Is this an optimal bound? In Chapter 3, we proved that: if B is a graph of odd-girth $2k+1$ which bounds all the K_4 -minor free graphs of odd-girth $2k+1$, then $|B| \geq \frac{(k+1)(k+2)}{2}$. Our result together with the result in [6] shows that order $\mathcal{O}(k^2)$ is optimal.

Furthermore, if $PC(2k)$ bounds \mathcal{P}_{2k+1} , then $PC(2k)$ also bounds \mathcal{P}_{2r+1} ($r > k$). However, in this case we believe that a proper subgraph of $PC(2k)$ would suffice to bound \mathcal{P}_{2r+1} , then what's the optimal subgraph of $PC(2k)$ that bounds \mathcal{P}_{2r+1} ? The first case of this problem which is not studied is $k = 3$ and $r = 5$. For this case, Naserasr [81] conjectured that the Coxeter graph bounds \mathcal{P}_{11} . Supporting this conjecture, in Chapter 4, we prove that the Coxeter graph bounds \mathcal{P}_{17} .

In Chapters 5, 6, we study the Hamiltonian cycle problems. Dirac showed in 1952 that every graph of order n is Hamiltonian if any vertex is of degree at least $\frac{n}{2}$. This result started a new approach to develop sufficient conditions on degrees for a graph to be Hamiltonian. Many results have been obtained in generalization of Dirac's theorem. In the results which strengthen Dirac's theorem, there is an interesting research area: to

control the placement of a set of vertices on a Hamiltonian cycle such that these vertices have some certain distances among them on the Hamiltonian cycle.

In this thesis, we consider two related conjectures. One conjecture is given by Enomoto: if G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $dist_C(x, y) = \lfloor \frac{n}{2} \rfloor$. Under the same condition of this conjecture, it was proved in [32] that a pair of vertices are located at distances no more than $\frac{n}{6}$ on a Hamiltonian cycle. In [33], Faudree and Li studied the case $\delta(G) \geq \frac{n+k}{2}$, $2 \leq k \leq \frac{n}{2}$. They proved that any pair of vertices can be located at any given distance from 2 to k on a Hamiltonian cycle. Moreover, Faudree and Li proposed a more general conjecture: if G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = k$.

Using Regularity Lemma and Blow-up Lemma, we gave a proof of Enomoto's conjecture for graphs of sufficiently large order in Chapter 5, and gave a proof of Faudree-Li conjecture for graphs of sufficiently large order in Chapter 6.

Keywords: signed graphs, projective cubes, homomorphism, walk-power, Hamiltonian cycle, Regularity Lemma.

Résumé

Dans cette thèse, nous étudions deux principaux problèmes de la théorie des graphes: le problème d'homomorphisme des graphes planaires (signé) et celui du cycle Hamiltonien.

Généralisant le théorème des quatre couleurs, il est conjecturé ([80], [41]) que tout graphe signé cohérent planaire de maille-déséquilibré $d + 1$ ($d \geq 2$) est homomorphe au cube projectif signé $SPC(d)$ de dimension d . On se demande naturellement:

$SPC(d)$ est-elle une borne optimale de maille-déséquilibré $d+1$ pour tous les graphes signé cohérent planaires de maille-déséquilibré $d + 1$?

Au Chapitre 2, nous prouvons que si (B, Ω) est un graphe signé cohérent de maille-déséquilibré d qui borne la classe des graphes signés cohérents planaires de maille-déséquilibré $d + 1$, alors $|B| \geq 2^{d-1}$. Par ailleurs, si aucun sous-graphe de (B, Ω) ne borne la même classe, alors le degré minimum de B est au moins d , et par conséquent, $|E(B)| \geq d \cdot 2^{d-2}$. Notre résultat montre que si la conjecture ci-dessus est vérifiée, alors le cube $SPC(d)$ est une borne optimale à la fois en termes des nombre de sommets et de nombre des arêtes.

Lorsque le $d = 2k$, le problème est équivalent aux homomorphismes de graphe: $PC(2k)$ est-elle une borne optimale de maille-impair $2k + 1$ pour \mathcal{P}_{2k+2} (la classe de tous graphes planaires de maille-impair au moins $2k + 1$)? Observant que les graphes K_4 -mineur libres sont les graphes planaires, $PC(2k)$ est-elle aussi une borne optimale de maille-impair $2k + 1$ pour tous les graphes K_4 -mineur libres de maille-impair $2k + 1$? La réponse est négative, dans [6], une famille de graphes d'ordre $\mathcal{O}(k^2)$ qui borne les graphes K_4 -mineur libres de maille-impair $2k + 1$ est donnée. La borne est-elle optimale? Au Chapitre 3, nous prouvons que si B est un graphe de maille-impair $2k + 1$ qui borne tous les graphes K_4 -mineur libres de maille-impair $2k + 1$, alors $|B| \geq \frac{(k+1)(k+2)}{2}$. Notre résultat, avec que le résultat de [6] montre que l'ordre $\mathcal{O}(k^2)$ est optimal.

En outre, si $PC(2k)$ borne \mathcal{P}_{2k+1} , alors $PC(2k)$ borne également \mathcal{P}_{2r+1} ($r > k$). Cependant, dans ce cas, nous croyons qu'un sous-graphe propre de $PC(2k)$ suffiront à borne \mathcal{P}_{2r+1} . Alors quel est le sous-graphe optimal de $PC(2k)$ qui borne \mathcal{P}_{2r+1} ? Le premier cas de ce problème qui n'est pas étudiée est $k = 3$ et $r = 5$. Dans ce cas, Naseras [81] conjecturé que le graphe Coxeter borne \mathcal{P}_{11} . Soutenir cette conjecture, au Chapitre 4, nous prouvons que le graphe Coxeter borne \mathcal{P}_{17} .

Au Chapitres 5, 6, nous étudions les problèmes du cycle hamiltonien. Dirac a montré en 1952 que chaque graphe d'ordre n est Hamiltonien si tout sommet est de degré au moins $\frac{n}{2}$. Ce résultat a commencé une nouvelle approche pour développer

des conditions suffisantes sur degrés pour caractériser les graphes hamiltoniens. De nombreux résultats ont été obtenus généralisant le théorème de Dirac. Parmi eux, il y a une zone de recherche intéressant: autour de la mise en place d'un ensemble de sommets sur un cycle hamiltonien tel que ces sommets aient une certaine distance entre eux sur ce cycle.

Dans cette thèse, nous considérons deux conjectures connexes, une proposée par Enomoto: si G est un graphe d'ordre $n \geq 3$ et $\delta(G) \geq \frac{n}{2} + 1$. Alors pour toute paire de sommets x, y dans G , il y a un cycle hamiltonien C de G tel que $dist_C(x, y) = \lfloor \frac{n}{2} \rfloor$. Motivé par cette conjecture, il a été prouvé, dans [32], qu'une paire de sommet ne peut être séparée par une distance supérieure à $\frac{n}{6}$ sur un cycle hamiltonien. Dans [33], les cas $\delta(G) \geq \frac{n+k}{2}$, $2 \leq k \leq \frac{n}{2}$, sont considérés, et il est prouvé qu'une paire de sommets à distance 2 à k peut être posé sur un cycle hamiltonien. En outre, Faudree et Li ont proposé une conjecture plus générale: si G est un graphe d'ordre $n \geq 3$ et $\delta(G) \geq \frac{n}{2} + 1$, alors pour toute paire de sommets x, y dans G et tout entier $2 \leq k \leq \frac{n}{2}$, il y a un hamiltonien cycle C de G tel que $dist_C(x, y) = k$.

Utilisant le Lemme de Régularité et le Blow-up Lemma, dans le chapitre 5, nous donnons une preuve de Enomoto conjecture pour les graphes d'ordre suffisant, et dans le chapitre 6, nous donnons une preuve de la conjecture de Faudree et Li pour les graphes d'ordre suffisant.

Mots-clés : graphes signé, cubes projectifs, homomorphisme, walk-power, cycle hamiltonien, Regularity Lemma.

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Symbols

$\omega(G)$	clique number of G
$\chi(G)$	chromatic number of G
$\alpha(G)$	independence number of G
$c(G)$	circumference of G
$\delta(G)$	minimum degree of G
$\Delta(G)$	maximum degree of G
$(G)^{(r)}$	r -th walk-power of G
$G \rightarrow H$	G has a homomorphism to H
$\mathcal{C} \prec H$	H bounds \mathcal{C}
$PC(n)$	projective cube of dimension n
$SPC(n)$	signed projective cube of dimension n
$Forb_m(\mathcal{H})$	the set of \mathcal{H} -minor free graphs

Chapter 1

Introduction

In 1736, Leonhard Euler gave a nice proof of a negative solution of well-known Seven Bridges of Königsberg problem. In his solution, he replaced each mass land with, in modern term, an abstract vertex, and each bridge with an abstract connection, in modern term, an edge. The resulting mathematical structure is called a graph now. Thus Leonhard Euler laid the foundation of graph theory and his solution of Seven Bridges of Königsberg problem is considered as the first theorem of graph theory.

In this thesis, we will work on two topics in the graph theory: “Homomorphisms of planar (signed) graphs to (signed) projective cubes”, which will be shown in Part I (Chapter 2, 3, 4), and “Locating any two vertices on Hamiltonian cycles”, which will be shown in Part II (Chapters 5, 6).

1.1 Some background

Around 1890, P. G. Tait concerned about the relationship between the existence of Hamiltonian cycles and the Four-Colour Problem. The relationship is simply that if a planar graph has a Hamiltonian cycle, then its faces can be 4-colored. Tait proved that the Four-Colour Problem is equivalent to the problem of finding 3-edge-coloring of bridgeless cubic planar graphs. Then Tait confined the attention on the cubic planar graphs and conjectured that: Every bridgeless cubic planar graph has a Hamilton cycle. Though this conjecture was disproved by W. T. Tutte in 1946, finding the extensions of Four-Colour Problem and finding the Hamiltonian cycles in a given graph motivate researchers a lot, these are the two main parts of this thesis.

1.1.1 Background of Part I

The Four-Colour Theorem is one of the most notable theorem in graph theory, searching a proof of it has motivated the development of graph theory a lot. The problem was first proposed in 1852 by Francis Guthrie. The intuitive statement of the Four-Colour Theorem was not in exact mathematical form, which is “that given any separation of a plane into contiguous regions, called a map, the regions can be colored using at most four colors so that no two adjacent regions have the same color”. To put everything in exact mathematical form, the set of regions can be represented abstractly as a set of vertices, two vertices are connected by an edge if the two regions be represented are adjacent. Thus the Four-Colour Theorem states, in graph-theoretic term, that “every planar graph is 4-colorable”. In 1890, using Kempe chain method, Percy Heawood[50] proved that “every planar graph is 5-colorable”. In 1976, Kenneth Appel and Wolfgang Haken [3] proved the Four-Colour Theorem. In their proof, computer and program are used.

The study of the Four-Colour Theorem led to the theory of vertex-coloring. Naturally, many other kinds of graph colorings have been defined and studied, such as edge-coloring, fractional coloring, acyclic coloring, etc.. Graph coloring came out to be a fruitful branch of graph theory, even a notable branch of mathematics. During the approach to The Four-Colour Theorem, it is generalized in many different ways: fractional coloring, circular coloring, graph minors and, in particular, the theory of graph homomorphisms, which is one of the main topics of this thesis.

In the language of graph minors, the Four-Colour Theorem states that: “Any graph G which does not contain complete graph K_5 or complete bipartite graph $K_{3,3}$ as a minor is 4-colorable”. As one of the most well-known conjectures that extend the Four-Colour Theorem, Hadwiger’s conjecture was proposed by H. Hadwiger in 1943.

Conjecture 1.1 (Hadwiger). *Any graph G which does not contain K_n as a minor is $(n - 1)$ -colorable.*

In the language of graph homomorphisms, the Four-Colour Theorem states that: “Every planar graph admits a homomorphism to the complete graph K_4 ”. If the planar graph is bipartite, it admits a homomorphism to complete graph K_2 . Thus we only need to consider the non-bipartite planar graphs, i.e., planar graphs of odd-girth at least 3. Note that K_4 is isomorphic to $PC(2)$ (projective cube of dimension 2, defined in Section 1.2), in 2007, R. Naserasr proposed a conjecture which extends the Four-Colour Theorem:

Conjecture 1.2 ([80]). *Given an integer $k \geq 1$, every planar graph of odd-girth at least $2k + 1$ admits a homomorphism to $PC(2k)$.*

Using the notation of signed graphs and $SPC(k)$ (signed projective cubes of dimension k , defined in Section 1.2), the above conjecture was extended to include the planar bipartite graphs, as introduced in [80] and [41]:

Conjecture 1.3. *Given an integer $k \geq 2$, every planar consistent signed graph of unbalanced-girth $k + 1$ admits a homomorphism to $SPC(k)$.*

A lot of work has been done with respect to these two conjectures while a lot of problems are left to be solved. The Part I of this thesis will focus on some related problems.

1.1.2 Background of Part II

The Hamiltonian paths and Hamiltonian cycles are named after Sir William Rowan Hamilton who invented the Icosian Game. In 1856, Hamilton invented a mathematical game, the Icosian Game, which consists of a dodecahedron. Each vertex of the dodecahedron is labeled with the name of a city and the game's object is finding a (Hamiltonian) cycle along the edges of the dodecahedron such that every vertex is visited a single time, and the ending point is the same as the starting point. Since then, the Hamiltonian problem, determining when a graph contains a Hamiltonian cycle, has been fundamental in graph theory. In fact, as a generalization of Hamiltonian cycles, circumferences, dominating cycles, pancyclic, cyclability etc. are well studied, and a huge number of results have been produced by researchers.

Note that it is NP-complete to determine whether there exists a Hamiltonian cycle in a graph, finding necessary or sufficient conditions for hamiltonicity become an interesting topic in graph theory. Every complete graph on at least three vertices is evidently Hamiltonian, indeed, to get a Hamiltonian cycle in a complete graph, we can start from any vertex, and choose the vertices one by one in an arbitrary order. If a graph does not have so many edges, how large of a minimum degree can guarantee the existence of a Hamiltonian cycle? In 1952, G. A. Dirac[23] answered this question by showing that if a simple graph has order at least 3 and each vertex has the degree at least half of the order, then the graph is Hamiltonian. This original result started a new approach to develop sufficient conditions on degrees for a graph to contain a Hamiltonian cycle.

There are plenty of results to strengthen Dirac's theorem. One of the interesting research areas is to control the placement of a set of vertices on the Hamiltonian cycle such that these vertices have some certain distances among them on the Hamiltonian cycle. Enomoto proposed the following conjecture of exact placement for a pair of vertices at a precise distance (half of the graph order) on a Hamiltonian cycle.

Conjecture 1.4 ([39]). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

In 2012, Faudree and Li proposed a more general conjecture.

Conjecture 1.5 ([33]). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

The Part II of this thesis will focus on these two conjectures.

1.2 Basic terminology and notation

In this section we provide some basic terminology and notations for the rest of the thesis. The definitions not given here will be mentioned in the beginning of the respective chapters.

First, we give some basic terminology and notations of graph.

1.2.1 Terminology and notations of graphs

A *graph* G is an ordered pair $(V(G), E(G))$ with a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G . Given an edge e , if $\psi_G(e) = \{u, v\}$, then e is said to *join* u and v ; u and v are called the *ends* of e ; moreover, u and v are said to be *adjacent*. In this thesis, we write $e = uv$ instead of $\psi_G(e) = \{u, v\}$. A *loop* is an edge with identical ends. Two edges e_1 and e_2 (which are not loops) are said to be *parallel* if they have the same pair of ends. A graph is *simple* if it has neither loops nor parallel edges. A graph with parallel edges and without loops is called a *multigraph*.

The *order* of a graph is the cardinality of its set of vertices and the *size* of a graph is the cardinality of its set of edges. The order of a graph G is denoted by $|V(G)|$ or $|G|$.

Subgraphs

A *subgraph* $H = (V(H), E(H))$ of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$ if H is a subgraph of G .

Given a nonempty subset V' of $V(G)$, the subgraph with vertex set V' and edge set $\{uv \in E(G) \mid u, v \in V'\}$ is called the subgraph of G *induced* by V' , denoted $G[V']$. We say that $G[V']$ is an *induced subgraph* of G .

Let F be a set of graphs. A graph is said to be *F-free* if it does not contain any graph from the set F as a subgraph.

Disjoint union of graphs

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *disjoint union* of G_1 and G_2 , denoted $G_1 \cup G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

Complete join of graphs

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *complete join* of G_1 and G_2 , denoted $G_1 + G_2$, is the graph obtained by starting with $G_1 \cup G_2$ and adding edges joining every vertex of G_1 to every vertex of G_2 .

Walk, path and cycle

A *walk* in a graph G is a finite non-null sequence $W := v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$, whose terms are alternately vertices and edges of G (not necessarily distinct), such that the ends of e_i ($1 \leq i \leq k$) are v_{i-1} and v_i . We say that v_0 and v_k are connected by W . The vertices v_0 and v_k are called the ends of W , v_0 being its *initial vertex* and v_k being its *terminal vertex*; the vertices v_1, \dots, v_{k-1} are its *internal vertices*. A v_0 -walk is a walk with initial vertex v_0 . The length of a walk is the number of its edge. A walk of length k is also called a *k-walk*. If $v_0 = v_k$, we call W a *closed walk*.

If the vertices v_0, v_1, \dots, v_k of W are distinct, W is called a *path* or v_0 - v_k *path*.

If the vertices v_0, v_1, \dots, v_{k-1} of W are distinct and $v_0 = v_k$, W is called a *cycle*.

The *length* of a path or a cycle is the number of its edges. A path or a cycle of length k is called a *k-path* or *k-cycle*, respectively; the path or cycle is *odd* or *even* according to the parity of its length.

Hamiltonian cycle

In a graph G , a *Hamiltonian cycle* is a cycle that visits each vertex of G exactly once. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

Distance, diameter and neighbors

The *distance* $d_G(x, y)$ of two vertices x, y in G is the length of a shortest x - y path in G . Whenever the underlying graph is clear from the context, we will write $d(x, y)$ instead of $d_G(x, y)$.

The *diameter* of a connected graph G is the greatest distance between any two vertices in G .

Given a positive integer i and a vertex x of G , $N_i(x)$ denotes the set of i -th neighbors of x , i.e., the set of vertices at distance exactly i from x . When $i = 1$, we simply write $N(x)$. For $U \subseteq V(G)$ we write $N_i(U) = \bigcup_{x \in U} N_i(x)$.

Girth and circumference

The *girth* of a graph is the length of a shortest cycle contained in the graph. The *odd-girth* of a graph is the length of a shortest odd-cycle contained in the graph. The *circumference* of a graph G is the length of a longest cycle contained in G , denoted $c(G)$. If a graph does not contain any cycle, its girth and circumference are defined to be infinity.

Complete graphs and cliques

A *complete* graph is a simple graph such that any two vertices are connected by an edge. If a complete graph is of order n , we denote it by K_n .

A *clique* of a graph G is a complete graph contained in G as a subgraph. The *clique number* $\omega(G)$ of a graph G is the order of a maximum clique in G .

Bipartite graphs

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that every edge has one end in V_1 and the other end in V_2 . Equivalently, a graph is bipartite if it does not contain any odd-cycle.

Planar graphs

A graph is *planar* if it can be drawn on the plan such that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph. Given a planar embedding of a planar graph, it divides the plan into a set of connected regions, including an outer unbounded connected region. Each of these regions is called a *face* of the planar graph. The *boundary* of a face is the cycle of the graph containing it. A planar graph with a given planar embedding is called a *plane* graph. We denote the class of planar graph of odd-girth at least $2k + 1$ by \mathcal{P}_{2k+1} .

Degree and regularity

In a simple graph G , for any vertex v of G , the *degree* of v is the number of vertices adjacent to v in G . We will use $d_G(v)$ (or $d(v)$ when there is no chance of confusion) to denote the degree of v in Chapter 2,3,4, while in Chapter 5, 6 we use $\deg_G(v)$ or $\deg(v)$. A d -regular graph is a graph in which every vertex has degree d . A 3-regular graph is also known as a *cubic* graph.

Cayley graphs

Let Γ be a group, S be a set of elements of Γ not including the identity element. Suppose, furthermore, that the inverse of every element of S also belongs to S . The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$.

Hypercubes

The *hypercube* of dimension n , denoted $H(n)$, is the graph whose vertex set is the set all n -tuples of 0's and 1's, where two n -tuples are adjacent if and only if they differ in precisely one coordinate. It can be checked that, $H(n)$ is a Cayley graph $(\mathbb{Z}_2^n, \{e_1, e_2, \dots, e_n\})$ where e_i 's are the standard basis of \mathbb{Z}_2^n . $H(n)$ is also called *n -cube*.

Projective cubes

The *Projective cube* of dimension n , denoted $PC(n)$, is the graph obtained by identifying the antipodal vertices of the hypercube $H(n+1)$, or equivalently, by adding edges between pairs of antipodal vertices of the hypercube $H(n)$. $PC(n)$ can be represented as a Cayley graph, that is, $PC(n) = (\mathbb{Z}_2^n, \{e_1, e_2, \dots, e_n, J\})$ where e_i 's are the standard basis of \mathbb{Z}_2^n and J is the all 1 vector of relevant length (n here). Projective cubes are also known as *folded cubes*.

Kneser graph $K(n, k)$

Given positive integers n and k such that $n \geq 2k$, the Kneser graph $K(n, k)$ is defined to be the graph whose vertices correspond to the k -element subsets of a set of n elements, where two vertices are adjacent if the two corresponding sets are disjoint.

Minors

In a graph G , an *edge contraction* is an operation which removes an edge and identify the vertices of the edge. A graph H is called a *minor* of the graph G if H can be obtained from G by a series of deleting edges, vertices and contracting edges. Given a graph H , a graph is said to be *H -minor free* if it does not contain H as a minor. Let \mathcal{H} be a set of graphs. A graph is said to be *\mathcal{H} -minor free* if it does not contain any graph from \mathcal{H} as a minor. Moreover, we use $Forb_m(\mathcal{H})$ to denote the class of all graphs that have no member of \mathcal{H} as a minor, that means the set of \mathcal{H} -minor free graphs.

Connectivity

A graph is *connected* if any pair of vertices is connected by a path. A connected graph G is said to be *k -vertex-connected* (or *k -connected*) if it has more than k vertices and remains connected whenever fewer than k vertices are removed. Similarly, a connected graph G is said to be *k -edge-connected* if it remains connected whenever fewer than k edges are removed. The *vertex-connectivity* (or just *connectivity*) (or *edge-connectivity*, respectively), of a graph is the largest k for which the graph is k -vertex-connected (or k -edge-connected, respectively). A *bridge*, or *cut-edge*, is an edge of a graph whose deletion increases its number of components. A graph is *bridgeless* if it contains no bridges, that means each component of it is 2-edge-connected.

k -tough graph

Let $t(G)$ denote the number of components of a graph G . A graph G is *k -tough* if $kt(G - S) \leq |S|$ for every subset S of the vertex set $V(G)$ with $t(G - S) > 1$. The *toughness* of G , denoted $\tau(G)$, is the maximum value of k for which G is k -tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$).

Pancyclic and bipancyclic graphs

A graph G is called *pancyclic* if it contains cycles of all length k for $3 \leq k \leq |V(G)|$. Analogously, a bipartite graph G is called *bipancyclic* if it contains cycles of all even lengths from 4 to $|V(G)|$.

Independent set

An *independent set* of a graph G is a subset of the vertices such that no two vertices in the subset induce an edge of G . The cardinality of a maximum independent set in a graph G is called the independence number of G , denoted $\alpha(G)$.

Homomorphisms

Let G and H be two graphs. A *homomorphism* of G to H is a mapping $\varphi : V(G) \rightarrow V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$. If G admits a homomorphism to H , we write $G \rightarrow H$. We say that two graphs G and H are *hom-equivalent* if $G \rightarrow H$ and $H \rightarrow G$.

The image of G under φ is called a *homomorphic image* of G . Given a class \mathcal{C} of graphs and a graph H , if every graph in \mathcal{C} admits a homomorphism to H we write $\mathcal{C} \preceq H$ and we say H *bounds* \mathcal{C} . Given a finite set \mathcal{H} of connected graphs, we use $Forb_{\mathcal{H}}$ to denote the class of all graphs which do not admit a homomorphism from any member of \mathcal{H} .

Isomorphisms

Let G and H be two graphs. An *isomorphism* between G and H is a bijection $\varphi : V(G) \rightarrow V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ if and only if $uv \in E(G)$. Two graphs are *isomorphic* if there exists an isomorphism between them.

Embedding

An *embedding* of a graph H into a graph G is an isomorphism between H and a subgraph of G . We say H is *embeddable* into G if there exists an embedding of H into G .

Vertex-coloring and edge-coloring

A *k-vertex-coloring* (or *k-edge-coloring*, respectively) of a graph G is a mapping: $c : V(G) \rightarrow S$ (or $c : E(G) \rightarrow S$, respectively), where S is a set of k colors, usually the set S of colors is taken to be $\{1, 2, \dots, k\}$. Thus a *k-vertex-coloring* (or *k-edge-coloring*, respectively) is an assignment of k colors to the vertices (or edges, respectively) of G . A *vertex-coloring* (or *edge-coloring*, respectively) c is *proper* if no two adjacent vertices (or incident edges, respectively) are assigned a same color. A graph is *k-vertex-colorable* (or *k-edge-colorable*, respectively) if it has a *k-vertex-coloring* (or *k-edge-coloring*, respectively). The minimum k for which a graph G is *k-vertex-colorable* (or *k-edge-colorable*, respectively) is called its *chromatic number* (or *chromatic index*, respectively).

Fractional coloring

Let $\mathcal{I}(G)$ denote the set of all independent vertex sets of a graph G , and let $\mathcal{I}(G, u)$ denote the independent vertex sets of G that contain the vertex u . A *fractional coloring* of G is defined as a nonnegative real function f on $\mathcal{I}(G)$ such that for any vertex u of G , $\sum_{S \in \mathcal{I}(G, u)} f(S) \geq 1$. The sum of values of f is called its *weight*, and the minimum possible weight of a fractional coloring is called the *fractional chromatic number* of G .

Circular coloring

Given a graph G and positive integers p and q and a color set $C = \{0, 1, \dots, p-1\}$, if there is a mapping $c : V(G) \rightarrow C$ such that: for each edge $uv \in E(G)$, $q \leq |c(u) - c(v)| \leq p - q$, then we say G has a *circular- $\frac{p}{q}$ -coloring*, or G is *circular- $\frac{p}{q}$ -colorable*.

Walk-powers of graphs

Given a graph G and a positive integer k , we define the *k-th walk-power* of G , denoted $G^{(k)}$, to be the graph whose vertex set is also $V(G)$ with two vertices x and y being adjacent if there is a walk of length k connecting x and y in G .

In the following, we give some basic terminology and notations of signed graphs.

1.2.2 Terminology and notations of signed graphs

Signed graphs

Given a graph G , we assign a sign “+” or “−” to each edge of G . The edges labeled “+” are called *positive* edges while the ones labeled “−” are called *negative* edges. We can see this assignment as a mapping of the edges of G to the set $\{+, -\}$. Such a mapping is called a *signature* of G . We normally denote the set of negative edges by Σ . Note that, a signature of G is given if and only if the set of negative edges is given, thus the set of edges Σ will be referred to as the signature of G , and (G, Σ) is called a *signed* graph.

Resigning

Given a signed graph (G, Σ) and a vertex $v \in V(G)$, a *resigning* at v is to change the sign of each edge incident to v . Two signatures Σ_1, Σ_2 on a graph G are said to be *equivalent* if one can be obtained from the other by a sequence of resignings, moreover, (G, Σ_1) and (G, Σ_2) are also said to be *equivalent*. Thus resigning defines an equivalence relations on the set of all signed graphs over a graph. Given a signed graph (G, Σ) , denote $[G, \Sigma]$ the set of all signed graphs equivalent to (G, Σ) .

Unbalanced-girth

In a signed graph (G, Σ) , a cycle with an odd (or even, respectively) number of negative edges is called *unbalanced* (or *balanced*, respectively). Note that resignings do not change the balance of a cycle. Recall that an odd-cycle is a cycle of odd length while a cycle of $(G, E(G))$ is unbalanced if and only if it is an odd-cycle of G , the notation of unbalanced cycle is, in some sense, an extension of the notation of an odd-cycle. Similar to the definition of odd-girth of G , we define the *unbalanced-girth* of (G, Σ) as the shortest length of an unbalanced cycle of (G, Σ) . A signed graph is *balanced* if all its cycles are balanced.

Consistent signed graphs

A *consistent signed* graph is a signed graph in which every balanced cycle is of even length and all unbalanced cycles are of the same parity. Thus there are two types of consistent signed graphs:

- i. when all unbalanced cycles are of odd length (it can be shown that this is the case if and only if $\Sigma \equiv E(G)$), such a signed graph will be called an *odd signed* graph;

- ii. when all unbalanced cycles are of even length (which will be the case if and only if G is bipartite), such a signed graph thus will be referred to as a *signed bipartite* graph.

Signed projective cubes

The *signed projective cube* of dimension n , denoted $SPC(n)$, is the signed graph $(PC(n), \Sigma)$, where Σ is the set of edges corresponding to J according to the definition of $PC(n)$. On the other hand, Σ can be viewed as the set of edges added between pairs of antipodal vertices of hypercube H_n to get $PC(n)$.

Homomorphisms of signed graphs

Given two signed graphs (G_1, Σ_1) and (G_2, Σ_2) , we say that there is a *homomorphism* of (G_1, Σ_1) to (G_2, Σ_2) if there exist a signed graph (G, Σ'_1) equivalent to (G_1, Σ_1) , a signed graph (G_2, Σ'_2) equivalent to (G_2, Σ_2) and a mapping $\varphi : V(G_1) \rightarrow V(G_2)$ such that: $\varphi(x)\varphi(y) \in E(G_2)$ whenever $xy \in E(G_1)$ and $xy \in \Sigma'_1$ if and only if $\varphi(x)\varphi(y) \in \Sigma'_2$. When there exists a homomorphism of (G_1, Σ_1) to (G_2, Σ_2) , we write $(G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$. Given a class \mathcal{C} of signed graphs, we say a signed graph (H, Σ) *bounds* \mathcal{C} if every member of \mathcal{C} admits a homomorphism to (H, Σ) .

Minors of signed graphs

A (signed) *minor* of a signed graph (G_1, Σ_1) is a signed graph (G_2, Σ_2) obtained from (G_1, Σ_1) by a sequence of the following operations: (i) delete an edge (and remove it from the signature if it is present), (ii) contract a positive edge (that means it is not in the signature), (iii) resign at any vertex. These operations can be taken in any order. We say that (G_1, Σ_1) is (G_2, Σ_2) -*minor free* if it does not contain (G_2, Σ_2) as a minor.

Cover and pack

Given a signed graph (G, Σ) and a set B of edges of (G, Σ) , we call B an (unbalanced cycle) *cover* if every unbalanced cycle of (G, Σ) contains at least one edge of B . Moreover, denote by $\tau(G, \Sigma)$ the unbalanced-girth of (G, Σ) and denote by $\nu(G, \Sigma)$ the maximum number of pairwise disjoint covers of (G, Σ) . Since every unbalanced cycle intersect every cover, $\tau(G, \Sigma) \geq \nu(G, \Sigma)$. If $\tau(G, \Sigma) = \nu(G, \Sigma)$, we say that (G, Σ) *packs*.

1.3 Motivations and overview

1.3.1 Motivations and overview of Part I

The existence of a homomorphism from a class of graphs to a projective cube is of special importance. Generally, Conjecture 1.2 and Conjecture 1.3 capture a certain packing problem and edge-coloring problem.

A packing problem of signed graphs was introduced by Guenin [41]. In this paper, Guenin proposed a conjecture which he called *main conjecture*:

Conjecture 1.6. [41] *Consistent signed graphs which are $(K_5, E(K_5))$ -minor free, pack.*

Guenin pointed out that Conjecture 1.6 is a special case of a conjecture on binary clutters in [93]. Using the Proposition 1.7 below, Guenin showed that the Four-Colour Theorem is a special case of the Conjecture 1.6. Also it is proved that Conjecture 1.6 implies other conjectures, we will introduce this later.

Proposition 1.7. [41] *Let (G, Σ) be a signed graph and let k be a positive integer. Suppose k is even and G is bipartite or k is odd and $\Sigma = E(G)$. Then the following statements are equivalent,*

- (1) *there exist k disjoint covers of (G, Σ) ,*
- (2) *(G, Σ) is homomorphic to $SPC(k - 1)$.*

In [82], Naserasr, Rollová and Sopena independently Proposition 1.7. This proposition shows that the problem of finding a mapping of a consistent signed graph to a signed projective cube is equivalent to a packing problem.

The study of edge-coloring has a long history in graph theory, it has a close link to the Four-Colour Theorem. Edge-coloring of graphs were first considered in two short papers by Tait [97] published in the same proceeding between 1878 and 1880. Tait proved a theorem relating face-coloring and edge-coloring of planar graphs. Tait's theorem says that if G is a bridgeless cubic planar (simple) graph, then G admits a 3-edge-coloring if and only if the faces of G can be colored with four colors such that the adjacent faces receive different colors. Thus Tait conjectured that every bridgeless 3-regular planar graph admits a 3-edge-coloring. Since the conjecture is equivalent to the Four-Color Theorem, we state Tait's conjecture as a theorem.

Theorem 1.8. (Tait's Conjecture [100]) *Every bridgeless 3-regular planar graph is 3-edge-colorable.*

In 1981, P.D. Seymour proposed a generalization to Tait's conjecture, which use the following notations.

Given a graph G , let X, Y be a partition of $V(G)$ and let $[X, Y]$ denote the set of all edges with one end in X and the other end in Y . Then $[X, Y]$ is said to be a *cut* of G . Moreover, if $|X|$ or $|Y|$ is odd, we call $[X, Y]$ an *odd cut* of G . The *size* of a cut $[X, Y]$ is $|[X, Y]|$. An *r -graph* is an r -regular graph with no odd cut of size less than r .

Conjecture 1.9. ([95]) *Every planar r -graph is r -edge-colorable.*

It is proved that Conjecture 1.9 can be implied by Conjecture 1.6. The cases $3 \leq r \leq 8$ of this conjecture are proved, as shown in the following table. In [80] Naserasr

Cases	Proved by
$r = 3$	Tait [100]
$r = 4, 5$	Guenin [42]
$r = 6$	Dvořák, Kawarabayashi, Král [24]
$r = 7$	Chudnovsky, Edwards, Kawarabayashi, Seymour [19]
$r = 8$	Chudnovsky, Edwards and Seymour [20]
$r \geq 9$	open

TABLE 1.1: Cases of Conjecture 1.9

proved that for $r = 2k + 1$, Conjecture 1.9 is equivalent to Conjecture 1.2; in [82] Naserasr, Rollová and Sopena proved that for $r = 2k$, Conjecture 1.9 is equivalent to Conjecture 1.3. Thus Conjecture 1.3 is verified up to $k \leq 8$.

Besides capturing a certain packing problem and edge-coloring problem, Conjecture 1.2 and Conjecture 1.3 are in a relation to a famous work of P. Ossona de Mendez and J. Nešetřil. Indeed, Conjecture 1.2 was introduced in [80] in relation to a question of J. Nešetřil who asked if there is a triangle-free graph to which every triangle-free planar graph admits a homomorphism. This question was answered in a larger frame work by P. Ossona de Mendez and J. Nešetřil.

Theorem 1.10. [88] *Given a finite set \mathcal{M} of graphs and a finite set \mathcal{H} of connected graphs, there is a graph in $Forb_h(\mathcal{H})$ to which every graph in $Forb_h(\mathcal{H}) \cap Forb_m(\mathcal{M})$ admits a homomorphism.*

The bound (graphs) that are build using known proofs of the Theorem 1.10 are of super exponential orders. To find an optimal bound in this theorem, in general, is a very difficult question.

Indeed, if we take $\mathcal{M} = \mathcal{H} = \{K_n\}$, then $Forb_m(\{K_n\})$ is the class of all K_n -minor free graphs and $Forb_h(\{K_n\})$ is the class of all graphs do not admit a homomorphism from K_n . Note that if K_n admits a homomorphism to a graph G , then G must have a

subgraph isomorphic to K_n . Therefore, $Forb_h(\{K_n\})$ is the class of all graphs without a subgraph isomorphic to K_n . The graph in $Forb_h(\{K_n\})$ may have K_n as a minor, thus $Forb_h(\{K_n\}) \supseteq Forb_m(\{K_n\})$ and $Forb_h(\{K_n\}) \cap Forb_m(\{K_n\})$ is the class of all K_n -minor free graphs. Using Theorem 1.10, we can see that there is a graph G without a subgraph isomorphic to K_n such that every K_n -minor free graph admits a homomorphism to G . Note that K_{n-1} is a K_n -minor free graph, thus G contains a subgraph isomorphic to K_{n-1} . If K_{n-1} is the optimal bound in this case, then every K_n -minor free graph is $(n-1)$ -colorable, which implies Hadwiger's conjecture.

Furthermore, if we take $\mathcal{M} = \{K_5, K_{3,3}\}$ and $\mathcal{H} = \{C_{2k-1}\}$, then $Forb_h(\{C_{2k-1}\})$ is a class of all graphs of odd-girth at least $2k+1$, and $Forb_h(\{K_5, K_{3,3}\}) \cap Forb_m(\{C_{2k+1}\})$ is the class of all planar graphs of odd-girth $2k+1$. In this case, for $k=1$, (C_1 being a loop), since $PC(2) = K_4$ is a planar graph, it is the optimal answer by the Four-Color Theorem. For $k=2$, it is proved in [81] that $PC(4)$, known as the Clebsch graph, is the optimal bound. For other cases, we have the natural question:

Problem 1.11. *For $k \geq 3$, is $PC(2k)$ an optimal bound of odd-girth $2k+1$ for planar graphs of odd-girth $2k+1$?*

To answer this question, in the remarks and open problems section of [80], Naserasr gave us a direction of proving the chromatic number of the walk-power of planar graphs. Given a graph G with at least one edge and an integer k , according to the definition of $G^{(2k-1)}$, $G^{(2k-1)}$ is loopless if and only if G is of odd-girth at least $2k+1$. If φ is a homomorphism of G to H , then φ is also a homomorphism of $G^{(k)}$ to $H^{(k)}$. To see this, let u, v be any two vertices in $G^{(k)}$, if $uv \in E(G^{(k)})$, then there is a walk of length k which connects u and v in G , denote this walk by $uu_1 \dots u_{k-1}v$. Since φ is a homomorphism of G to H , $\varphi(u)\varphi(u_1) \dots \varphi(u_{k-1})\varphi(v)$ is a walk of length k which connects $\varphi(u)$ and $\varphi(v)$ in H , and thus $\varphi(u)\varphi(v) \in E(H^{(k)})$. Naserasr proposed a relaxation of Conjecture 1.2 in following:

Conjecture 1.12. [80] *For every planar graph G of odd-girth $2k+1$, we have $\chi(G^{(2k-1)}) \leq 2^{2k}$.*

Note that $|PC(2k)| = 2^{2k}$. Is the bound of Conjecture 1.12 tight? If so, then there is a big chance that $PC(2k)$ is the optimal bound of planar graph G of odd-girth $2k+1$. Thus a problem was asked by Naserasr:

Problem 1.13. [80] *Is there a planar graph G of odd-girth $2k+1$ with $\chi(G^{(2k-1)}) \geq 2^{2k}$?*

Actually, our first work answers this question. We construct a planar graph G of odd-girth $2k+1$ with $\omega(G^{(2k-1)}) \geq 2^{2k}$. This result shows that, if Conjecture 1.2 holds,

$PC(2k)$ is an optimal bound. The development of the notion of homomorphisms for signed graphs has began very recently and, therefore, it is not yet known if an analogue of Theorem 1.10 would hold for the class of signed bipartite graphs. We believe that it would be the case. By similar methods, we prove that if the signed bipartite case of Conjecture 1.3 holds, $SPC(2k+1)$ is an optimal bound for the signed bipartite planar graphs of unbalanced-girth $2k+1$. In a uniform language, we prove that if Conjecture 1.3 holds, $SPC(k)$ is an optimal bound for the planar consistent signed graph of unbalanced-girth $k+1$.

As a continue of our first work, if we replace $\mathcal{M} = \{K_5, K_{3,3}\}$ by $\mathcal{M} = \{K_4\}$ and keep $\mathcal{H} = \{C_{2k-1}\}$, that means we replace the condition of “planar” by “ K_4 -minor free”, what will be the optimal bound of Theorem 1.10? Our second work gives some partial result of this. We construct a K_4 -minor free graph G of odd-girth $2k+1$ with $\omega(G^{(2k-1)}) \geq \frac{(k+1)(k+2)}{2}$. And we prove the tightness of the bound for $k=2$.

If Conjecture 1.2 holds, then $PC(2k)$ bounds \mathcal{P}_{2k+1} . For $r > k$, since \mathcal{P}_{2r+1} is included in \mathcal{P}_{2k+1} , $PC(2k)$ also bounds \mathcal{P}_{2r+1} . However, in this case we believe that a proper subgraph of $PC(2k)$ would suffice to bound \mathcal{P}_{2r+1} . Then, what are the minimal subgraphs of $PC(2k)$ that suffice to bound \mathcal{P}_{2r+1} ? This question was first asked by Naserasr:

Problem 1.14. [81] *Given integers $l \geq k \geq 1$, what are the minimal subgraphs of $PC(2k)$ to which every planar graph of odd-girth $2l+1$ admits a homomorphism?*

In [81], Naserasr conjectured that $K(2k+1, k)$, as a subgraph of $PC(2k)$, is an answer for the case $l = k+1$:

Conjecture 1.15. *For $l = k+1$, the smallest subgraph of $PC(2k)$ to which every planar graph of odd-girth $2l+1$ admits a homomorphism is the Kneser graph $K(2k+1, k)$.*

The Conjecture 1.15 is related to the study of the fractional chromatic number of planar graphs of a given odd-girth. Since in a manuscript [79], Naserasr conjectured that the fractional chromatic number of planar graphs of odd-girth $2l+1$ is bounded by $2 + \frac{1}{l-1}$ and he showed that if the conjecture holds, the bound is the best possible. Note that the fractional chromatic number of $K(2k+1, k)$ is $\frac{2k+1}{k} = 2 + \frac{1}{k}$, if the Conjecture 1.15 holds, then it would determine the fractional chromatic number of the planar graph of odd-girth $2l+1$. In this regards, when $l = 2$ and $k = 1$, the conjecture is implied by Grötzsch’s theorem which states that:

Theorem 1.16. (Grötzsch’s theorem[40]) *Every loop-free and triangle-free planar graph is 3-colorable.*

For the case $l = 3$ and $k = 2$, the conjecture states that every planar graph of odd-girth 7 is bounded by $K(5, 2)$. Note that $K(5, 2)$ is the well-known Petersen graph. The best result for this case is given by Dvořák, Škrekovski and Valla in [25]: every planar graph of odd-girth 9 is bounded by $K(5, 2)$.

For the cases $l \geq 2k$, the smallest subgraph of the projective cube $PC(2k)$, which is not bipartite, is C_{2k+1} . It is a classic result that when l is much larger than k , then C_{2k+1} is the answer to Problem 1.14. This can be implied by the following theorems proved by Zhang.

Theorem 1.17. [58] *There is a function $f(\epsilon)$ for each $\epsilon > 0$ such that, if the odd-girth of a planar graph G is at least $f(\epsilon)$, then G is circular- $(2+\epsilon)$ -colorable.*

The function $f(\epsilon)$ in Theorem 1.17 was presented as graph homomorphism result as follows.

Theorem 1.18. [58] *Every planar graph with odd-girth at least $10k - 3$ has a homomorphism to the cycle of length $2k + 1$.*

Actually, Zhang conjectured that C_{2k+1} is the answer to the Problem 1.14 as soon as $l \geq 2k$. This is related to the theory of flows.

Conjecture 1.19. (Zhang [107], Jaeger [54], also see [55], or Conjecture 9.1.5 in [106]) *Let k be a positive integer. Every graph with edge connectivity at least $4k + 1$ admits a nowhere-zero circular- $(2 + \frac{1}{k})$ -flow.*

This case would determine the circular chromatic number of planar graph of odd-girth $4k + 1$. For general k , Zhu [108] proved that every planar graph of odd-girth $8k - 3$ is bounded by C_{2k+1} . And the best result in this case is that: every planar graph of odd-girth $6k + 1$ is bounded by C_{2k+1} , which is implied by Corollary 4.14 of [76].

The first case of Problem 1.14 which is not covered by any of these theorems and conjectures is $k = 3$ and $r = 5$. For this case, we conjecture that the Coxeter graph, which is a subgraph of $K(7, 3)$, bounds the planar graph of odd-girth at least 11.

Conjecture 1.20. [46] *Every planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph.*

Supporting this conjecture, we prove in Chapter 4 that:

Theorem 1.21. *Every planar graph of odd-girth at least 17 admits a homomorphism to the Coxeter graph.*

The organization of Part I is as follows. In Chapter 2, we discuss some results related to Problem 1.11 and Problem 1.13. In Chapter 3, we continue the work in Chapter 3 by replacing the condition of “planar graph” to “ K_4 -minor free graph”. In Chapter 4, we discuss some results related to Problem 1.14 and Conjecture 1.20.

1.3.2 Motivations and overview of Part II

On the Hamiltonian problems, one may find many well-known theorems in any textbook of graph theory, thus it is not necessary and also impossible to give a detailed survey in this thesis. For an excellent general introduction to the Hamiltonian problem, the reader can see the article by J. C. Bermond [8]. For the Hamiltonian problem on Cayley graphs, we refer to the survey paper by Witte and Gallian [102]. For the toughness of graphs and the Hamiltonian problem, we refer to the survey paper by D. Beauer et al. [5]. For the Hamiltonian problem on digraphs, we refer to the survey papers by J. C. Bermond and C. Thomassen [9] and by J. A. Bondy [11]. For pancyclic and bypancyclic graphs, we refer to the survey paper by J. Mitchem and E. Schmeichel [77]. For claw-free graphs, we refer to the survey paper by R. Faudree et al. [30]. Moreover, R. J. Coudé gave three nice surveys in [37–39] which contain many problems on generalizations of Hamiltonian problem.

In this thesis, we will work on the generalizations of Dirac’s theorem in Hamiltonian graph theory. Dirac’s theorem states that:

Theorem 1.22 (Dirac’s theorem [23]). *If G is a 2-connected graph with $n \geq 3$ vertices and minimum degree $\delta(G)$, then the circumference $c(G) \geq \min\{n, 2\delta(G)\}$. Thus, if $\delta(G) \geq \frac{n}{2}$, G is Hamiltonian.*

There are a lot of results that generalize or strengthen Dirac’s theorems, some of them are based on degrees and neighborhood, some results generalize the hamiltonicity to the circumferences of graphs, and some results hunt for more edge-disjoint Hamiltonian cycles in the graphs satisfying the Dirac’s degree conditions or Ore’s degree conditions. Moreover, some results try to control the placement of a set of vertices on a Hamiltonian cycle so that certain distances are maintained between these vertices, which is one of the main topics of this thesis. We will introduce some results related to the aspects mentioned above. Obviously, there are some other results. For some results based on the conditions involving independence number and connectivity, see [14, 21, 45]; for some results on pancyclic, see [29, 34, 51]; for some results on regular graphs, see [53, 68, 75]. For more details, we refer to the survey paper by Li [72].

Degrees and neighborhood

Now, we introduce the generalizations of Dirac's theorem based on the degree and neighborhood. First, we introduce some notations that will be used.

For a subset (or a subgraph) S of $V(G)$ (or G), denote by $\alpha(S) = \alpha(G[S])$ the maximum number of vertices in S which are independent in the graph G . For any integer $k \geq 1$, when $\alpha(S) \geq k$, define $\sigma_k(S) = \min\{\sum_{i=1}^k d(x_i) : x_1, x_2, \dots, x_k\}$, where x_1, x_2, \dots, x_k are vertices in S and are pair-wisely nonadjacent (i.e. independent) in G ; $\bar{\sigma}_k(S) = \min\{\sum_{i=1}^k d(x_i) - |\cap_{i=1}^k N(x_i)| : x_1, x_2, \dots, x_k\}$, where x_1, x_2, \dots, x_k are vertices in S and are pair-wisely nonadjacent (i.e. independent) in G . When S does not have k vertices that are independent in G , we define $\sigma_k(S) = \bar{\sigma}_k(S) = \infty$.

Note that, using the notation of $\sigma_k(S)$, Dirac's theorem says that if $\sigma_1(G) \geq \frac{n}{2}$, then G is Hamiltonian.

The first important generalization of Dirac's theorem was given by Ore in 1960.

Theorem 1.23 (Ore's theorem [90]). *Let G be a graph of order n . If $\sigma_2(G) \geq n$ (i.e. $d(x) + d(y) \geq n$ for any pair of nonadjacent vertices x and y in G), then G is Hamiltonian.*

Let G be a graph of order n , if $\sigma_1(G) \geq \frac{n}{2}$, we say that G satisfies Dirac's degree condition; if $\sigma_2(G) \geq n$, we say that G satisfies Ore's degree condition

To search weaker conditions than Ore's theorem, one way is to relax Ore's degree condition, some results were given in [1, 56, 78, 92]; another way is that when a graph G satisfies Ore's degree condition, delete a given set of edges of G such that the remaining graph is still Hamiltonian, some results were given by Hu and Li [51], Li et al. [73].

If we consider k -connected graphs, Bondy gave a sufficient condition of hamiltonicity which relates to σ_k .

Theorem 1.24. [12] *Let G be a k -connected graph of order $n \geq 3$. If $\sigma_{k+1}(G) \geq \frac{(k+1)(n-1)}{2}$, then G is Hamiltonian.*

Now we introduce some results based on the neighborhoods, which was given by Flandrin et al [35].

Theorem 1.25. [35] *If G is a 2-concerned graph of order n such that $\bar{\sigma}_3(G) \geq n$, then G is Hamiltonian.*

The following corollaries are also given in [35].

Corollary 1.26. [35] *If G is a 2-concerned graph of order n such that $|N(u) \cup N(v)| \geq n - \max\{d(u), d(v)\}$ for any pair of nonadjacent vertices u and v , then G is Hamiltonian.*

Corollary 1.27. [35] *If G is a 2-concerned graph of order n such that $3|N(u) \cup N(v)| + \max\{2, |N(u) \cap N(v)|\} \geq 2n - 1$ for any pair of nonadjacent vertices u and v , then G is Hamiltonian.*

Corollary 1.26 was improved in [17], but the bound in Corollary 1.27 is sharp in the sense that $\max\{2, |N(u) \cap N(v)|\}$ cannot be replaced by $\max\{3, |N(u) \cap N(v)|\}$. We can see this from the example $K_2 + 3K_m, m \geq 1$.

On circumferences of graphs

Another generalization of Dirac's theorem is from the parameter of circumferences of graphs. If a graph satisfies Dirac's degree condition or Ore's degree condition, it is Hamiltonian, thus the circumference of the graph is its order. But if a graph satisfies a weaker Dirac type condition or Ore condition, what the lower bound can be given for the circumference?

In 1979, Bigalke and Jung [10] proved that:

Theorem 1.28. [10] *If G is a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{3}$, then every longest cycle C is a dominating cycle, i.e., the vertices of $V(G) - V(C)$ form an independent set.*

Based on the result of Theorem 1.28, Bauer et al. [4], in 1989, proposed the following conjecture:

Conjecture 1.29. [4] *If G is a 1-tough graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{3}$, then $c(G) \geq \min\{n, \frac{3n+1}{4}\} + \frac{\delta(G)}{2} \geq \frac{11n+3}{12}$.*

The best result so far is a result of Li [70] from 1995, which uses a variation of Woodall's Hopping Lemma (see [103]).

Theorem 1.30. [70] *Let G be a 1-tough graph of order $n \geq 3$. Then the circumference $c(G) \geq \min\{n, \frac{2n+1+2\delta(G)}{3}, \frac{3n+2\delta(G)-2}{4}\} \geq \min\{\frac{8n+3}{9}, \frac{11n-6}{12}\}$.*

The above results are based on the minimum degree of the graphs, if we don't use the minimum degree condition and consider the graphs with enough vertices of large degree, what the lower bound can be given for the circumference? As a possible improvement of Dirac's theorem, Woodall proposed the following conjecture in 1975, which was one of the 50 unsolved problems in graph theory in Bondy and Murty's book [13].

Conjecture 1.31. [104] *If G is a 2-connected graph of order n with at least $\frac{n}{2} + k$ vertices of degree at least k , then $c(G) \geq \min\{n, 2k\}$.*

Note that when $k = \frac{n}{2}$, Conjecture 1.31 implies Dirac's theorem.

Supporting to Conjecture 1.31, Häggkvist and Jackson obtained some partial results in 1985.

Theorem 1.32. [43] (1) Let G be a 2-connected graph of order $n \leq 3k - 2$. If G has at least $2k$ vertices of degree at least k , then $c(G) \geq 2k$.

(2) Let G be a 2-connected graph of order $n \geq 3k - 2$. If G has at least $n - \frac{k-1}{2}$ vertices of degree at least k , then $c(G) \geq 2k$.

Li essentially verified Woodall's conjecture in 2002 by showing the followings:

Theorem 1.33. [71] If G is a 2-connected graph of order n with at least $\frac{n}{2} + k$ vertices of degree at least k , then $c(G) \geq \min\{n, 2k - 13\}$.

Theorem 1.34. [66] Let $k \geq 683$. If G is a 2-connected graph of order n with at least $\frac{n}{2} + k$ vertices of degree at least k , then $c(G) \geq \min\{n, 2k\}$.

Edge-disjoint Hamiltonian cycles

There are plenty of results which strengthen Dirac's theorem. One of the most interesting research area is to find more than one Hamiltonian cycle in the graphs satisfying the Dirac's degree condition or Ore's degree condition. One of the fundamental results is given by Nash-Williams [85], which shows that Dirac's degree condition, despite being best possible, even guarantee the existence of many edge-disjoint Hamilton cycles.

Theorem 1.35. [85] Every graph on n vertices of minimum degree at least $\frac{n}{2}$ contains at least $\lfloor \frac{5n}{224} \rfloor$ edge-disjoint Hamiltonian cycles.

Nash-Williams asked whether the number of edge-disjoint Hamiltonian cycles can be improved. It is natural to see that, we could not to expect this number to be larger than $\lfloor \frac{n+1}{4} \rfloor$. In [86], Nash-Williams conjectured that $\lfloor \frac{n+1}{4} \rfloor$ is achieved, unfortunately, Babai (see [86]) pointed out that this conjecture is false, according to his idea, Nash-Williams [86] gave an example of a graph on $n = 4m$ vertices with minimum degree $2m$ having at most $\lfloor \frac{n+4}{8} \rfloor$ edge-disjoint Hamilton cycles.

Christofides et al. gave a similar example in [18]: Let A be an empty graph on $2m$ vertices, B a graph consisting of $m + 1$ disjoint edges and let G be the graph obtained from the disjoint union of A and B by adding all possible edges between A and B . So G is a graph on $4m + 2$ vertices with minimum degree $2m + 1$. Observe that any Hamilton cycle of G must use at least 2 edges from B and thus G has at most $\lfloor \frac{m+1}{2} \rfloor$ edge-disjoint Hamilton cycles. It is shown by Christofides et al. [18] that this example is asymptotically best possible.

Theorem 1.36. [18] *For every $\alpha > 0$, there is an integer n_0 so that every graph on $n \geq n_0$ vertices of minimum degree at least $(\frac{1}{2} + \alpha)n$ contains at least $\frac{n}{8}$ edge-disjoint Hamiltonian cycles.*

Noting that the construction given above depends on the graph being non-regular, Nash-Williams [87] conjectured that:

Conjecture 1.37. [87] *Let G be a d -regular graph on at most $2d$ vertices. Then G contains $\lfloor \frac{d}{2} \rfloor$ edge-disjoint Hamiltonian cycles.*

The conjecture was also raised independently by Jackson [52], where he proved the following theorem.

Theorem 1.38. [52] *Let G be a d -regular graph on $14 \leq n \leq 2d + 1$ vertices. Then G contains $\lfloor \frac{3d-n+1}{6} \rfloor$ edge-disjoint Hamiltonian cycles.*

In [18], the following approximate version of the Conjecture 1.37 was shown.

Theorem 1.39. [18] *For every $\alpha > 0$, there is an integer n_0 so that every d -graph on $n \geq n_0$ vertices with $d \geq (\frac{1}{2} + \alpha)n$ contains at least $\frac{d-\alpha n}{2}$ edge-disjoint Hamiltonian cycles.*

The proofs use the Regularity Lemma, so the order of the graph is accordingly large.

It is also interesting to see if Ore's degree condition may ensure multiple edge-disjoint Hamiltonian cycles. The first results about this were obtained by Faudree, Rousseau and Schelp [28] in 1985, but they required $n + 2k - 2$ instead of n in Ore's degree condition.

Theorem 1.40. [28] *Let G be a graph of order n and k a positive integer. If $\sigma_2(G) \geq n + 2k - 2$, then for n sufficiently large ($n \geq 60k^2$ will suffice), G has k edge-disjoint Hamiltonian cycles.*

In [74], Li and Zhu proved that Ore's degree condition can ensure two edge-disjoint Hamiltonian cycles for most graphs.

Theorem 1.41. [74] *Let G be a graph of order $n \geq 20$. If $\delta(G) \geq 5$ and $\sigma_2(G) \geq n$, then G has at least two edge-disjoint Hamiltonian cycles.*

Furthermore, Li [67] proved the following theorem:

Theorem 1.42. [67] *If G is a graph of order n such that $\sigma_2(G) \geq n$ and either $\delta(G) < \frac{n}{2} - 2$ or $\Delta(G) \geq \frac{n}{2} - 6$, then for any $3 \leq l_1 \leq l_2 \leq n$, G has two edge-disjoint cycles with lengths l_1 and l_2 , respectively.*

In 1986, Faudree and Schelp conjectured that if n is sufficiently larger than $\delta(G)$ and $\sigma_2(G) \geq n$, then the graph of order n has $\lfloor \frac{\delta(G)-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles. Their conjecture was confirmed in 1989 by Li [69].

Theorem 1.43. [69] *Let G be a graph on n vertices and k a positive integer. If $2k+1 \leq \delta(G) \leq 2k+2$, $n \leq 8k^2 - 5$ and $\sigma_2(G) \geq n$, for any $3 \leq l_1 \leq l_2 \leq \dots \leq l_k \leq n$ with $k = \lfloor \frac{\delta(G)-1}{2} \rfloor$, G contains k edge-disjoint cycles with lengths l_1, l_2, \dots, l_k , respectively.*

Corollary 1.44. [69] *Let G be a graph on n vertices. If $n \geq 2(\delta(G))^2$ and $\sigma_2(G) \geq n$, then G has at least $\lfloor \frac{\delta(G)-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles.*

Distributing vertices on the Hamiltonian cycle

Another research area that strengthens Dirac's theorem is to control the placement of a set of vertices on a Hamiltonian cycle such that these vertices have some certain distances among them on the Hamiltonian cycle. In 2001, Kaneko and Yoshimoto [57] showed that in a graph satisfies Dirac's degree condition, given any sufficiently small subset S of vertices, there exists a Hamiltonian cycle C such that the distances on C between successive pairs of vertices of S have a uniform lower bound.

Theorem 1.45. [57] *Let G be a graph of order n with $\delta(G) \geq \frac{n}{2}$, and let d be a positive integer such that $d \leq \frac{n}{4}$. Then, for any vertex subset S with $|S| \leq \frac{n}{2d}$, there is a Hamiltonian cycle C such that $\text{dist}_C(u, v) \geq d$ for any $u, v \in S$.*

The result in this theorem is sharp, we can see this from the graph $2K_{\frac{n}{2}-1} + K_2$, if we place the vertices of A in one of the $K_{\frac{n}{2}-1}$, then the bound is clear.

In 2008, Sárközy and Selkow [91] showed that almost all of the distances between successive pairs of vertices of S can be specified almost exactly.

Theorem 1.46. [91] *There are $\omega, n_0 > 0$ such that if G is a graph with $\delta(G) \geq \frac{n}{2}$ on $n \geq n_0$ vertices, d is an arbitrary integer with $3 \leq d \leq \frac{\omega n}{2}$ and S is an arbitrary subset of $V(G)$ with $2 \leq |S| = k \leq \frac{\omega n}{2}$, then for every sequence of integers with $3 \leq d_i \leq d$, and $1 \leq i \leq k-1$, there is a Hamiltonian cycle C of G and an ordering of the vertices of S , a_1, a_2, \dots, a_k , such that the vertices of S are encountered in this order on C and we have $|\text{dist}_C(a_i, a_{i+1}) - d_i| \leq 1$, for all but one $1 \leq i \leq k-1$.*

In [91], the authors believe that Theorem 1.46 remains true for greater values of d as well. In a personal communication, Enomoto proposed the following conjecture of exact placement for a pair of vertices at a precise distance (half of the graph order) on a Hamiltonian cycle.

Conjecture 1.47. [39] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

The degree condition of Enomoto's conjecture is sharp. First, we consider the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$. For any Hamiltonian cycle of $K_{\frac{n}{2}, \frac{n}{2}}$, any pair of vertices in the same part will be at an even distance on this cycle and any pair of vertices in different parts will be at an odd distance on this cycle. Since $\delta(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n}{2}$, the minimum degree $\delta(G) \geq \frac{n}{2}$ is not sufficient to imply the existence of a Hamiltonian cycle with a fixed pair of vertices at distance $\lfloor \frac{n}{2} \rfloor$. Second, we consider the graph $(K_{\frac{n-3}{2}} \cup K_{\frac{n-3}{2}}) + K_3$. If x, y are both in one of the copies of $K_{\frac{n-3}{2}}$, then we cannot find a Hamiltonian cycle C of $(K_{\frac{n-3}{2}} \cup K_{\frac{n-3}{2}}) + K_3$ such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$. Since $\delta((K_{\frac{n-3}{2}} \cup K_{\frac{n-3}{2}}) + K_3) = \frac{n+1}{2}$, the minimum degree $\delta(G) \geq \frac{n+1}{2}$ is not sufficient to imply the existence of the desired Hamiltonian cycle.

Motivated by Enomoto's conjecture, Faudree et al. [32] deal with locating a pair of vertices at precise distances on a Hamiltonian cycle.

Theorem 1.48. [32] *Let $k \geq 2$ be a fixed positive integer. If G is a graph of order $n \geq 6k$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

This theorem was generalized in [31].

Theorem 1.49. [31] *Given a set of $k - 1$ integers $\{p_1, p_2, \dots, p_{k-1}\}$ and a fixed set of k vertices $\{x_1, x_2, \dots, x_k\}$ in a graph G of sufficiently large order n with $\delta(G) \geq \frac{n+2k-2}{2}$, then there is a Hamiltonian cycle C such that $\text{dist}_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k - 1$.*

Furthermore, Faudree and Li [33] obtained the following theorem.

Theorem 1.50. [33] *If k is a positive integer with $2 \leq k \leq \frac{n}{2}$ and G is a graph of order n with $\delta(G) \geq \frac{n+k}{2}$, then for any pair of vertices x and y in G and for any $2 \leq p \leq k$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = p$.*

Moreover, Faudree and Li [33] proposed a more general conjecture.

Conjecture 1.51. [33] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

In Chapter 5, we prove Conjecture 1.47 for graphs of sufficiently large order. Our main result is the following:

Theorem 1.52. [48] *There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair x, y of vertices, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

To prove Theorem 1.52, we define two extremal cases of G as follows.

Extremal Case 1: If there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$.

Extremal Case 2: If there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$.

Here, a *balanced partition* of $V(G)$ into V_1 and V_2 is a partition of $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2|$, and α is a parameter we fix before the proof.

The proof of Theorem 1.52 will be divided into three parts: the non-extremal case, the Extremal case 1 and Extremal case 2. Obviously, the non-extremal cases part is the main part of the proof, we use Regularity Lemma and Blow-up Lemma to prove it. For the Extremal case 1, apart from a small number of vertices, the rest forms a super-regular pair, we will use Blow-up Lemma to construct the Hamiltonian cycles desired.

In Chapter 6, as an extension of Theorem 1.52, we prove Conjecture 1.51 for graphs of sufficiently large order. Our main result is the following:

Theorem 1.53. [47] *There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair x, y of vertices and any integer k , $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

The main idea to prove this theorem is same to the proof the Theorem 1.52, we use some tricks to change the length of the path connecting x and y on the Hamiltonian cycle.

The tools we use

Note that, to prove Theorem 1.52 and Theorem 1.53, we use Regularity Lemma and Blow-up Lemma, which are very powerful tools of graph theory. To introduce these two lemmas, we need some more notations.

We denote the degree of a vertex v in G by $\text{deg}_G(v)$. Given a graph G , let X and Y be two disjoint sets of vertices of G . We define the *density*, $d(X, Y)$, of pair (X, Y) as the ratio

$$d(X, Y) := \frac{e_G(X, Y)}{|X||Y|},$$

here $e_G(X, Y)$ is defined to be the number of edges in G with one end vertex in X and the other in Y , if no ambiguity arises, we write $e(X, Y)$ instead of $e_G(X, Y)$.

Let $\epsilon > 0$. Given two disjoint vertex sets $X \subseteq V(G), Y \subseteq V(G)$ we say the pair (X, Y) is ϵ -regular if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| > \epsilon|X|$ and $|B| > \epsilon|Y|$ we have

$$|d(A, B) - d(X, Y)| < \epsilon.$$

Given a graph G and disjoint vertex sets $X, Y \subseteq V(G)$ let $\epsilon, \delta > 0$, the pair (X, Y) is (ϵ, δ) -super-regular if it is ϵ -regular, and $\deg_Y(x) > \delta|Y|$ for all $x \in X$ and $\deg_X(y) > \delta|X|$ for all $y \in Y$.

Now we introduce the degree form of Regularity Lemma and Bipartite Version of Blow-up Lemma.

Lemma 1.54 (Regularity Lemma-Degree Form). *For every $\epsilon > 0$ and every integer m_0 there is an $M_0 = M_0(\epsilon, m_0)$ such that if $G = (V, E)$ is any graph on at least M_0 vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $l + 1$ clusters V_0, V_1, \dots, V_l , and there is a subgraph $G' = (V, E')$ with the following properties:*

- (1) $m_0 \leq l \leq M_0$;
- (2) $|V_0| \leq \epsilon|V|$ for $0 \leq i \leq l$, and $|V_1| = |V_2| = \dots = |V_l| = L$;
- (3) $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$ for all $v \in V$;
- (4) $G'[V_i] = \emptyset$ (i.e. V_i is an independent set in G') for all i ;
- (5) each pair (V_i, V_j) , $1 \leq i < j \leq l$, is ϵ -regular, each with a density either 0 or at least d .

Lemma 1.55 (Blow-up Lemma-Bipartite Version [59]). *For every $\delta, \Delta, c > 0$, there exists an $\epsilon = \epsilon(\delta, \Delta, c) > 0$ and $\alpha = \alpha(\delta, \Delta, c) > 0$ such that the following holds. Let (X, Y) be an (ϵ, δ) -super-regular pair with $|X| = |Y| = N$. If a bipartite graph H with $\Delta(H) \leq \Delta$ can be embedded in $K_{N, N}$ by a function ϕ , then H can be embedded in (X, Y) . Moreover, in each $\phi^{-1}(X)$ and $\phi^{-1}(Y)$, fix at most αN special vertices z , each of which is equipped with a subset S_z of X or Y of size at least cN . The embedding of H into (X, Y) exists even if we restrict the image of z to be S_z for all special vertices z .*

More details about Regularity Lemma will be presented in Section 5.1. Recently, there are many beautiful results about Hamiltonian problems obtained by using Regularity Lemma, e. g. Theorem 1.36 and Theorem 1.39. Besides these, Chen et al.[16] prove that:

Theorem 1.56. [16] *There exists $N > 0$ such that for all even integers $n \geq N$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 92$, then G contains an ESHC.*

Here *ESHC* is short for *Even Squared Hamiltonian Cycle*, that is a Hamiltonian cycle $C = v_1v_2 \dots v_nv_1$ of a graph with chords v_iv_{i+3} for all $1 \leq i \leq n$.

For more progress on F -packing, Hamiltonian problems and tree embedding, see [62]. For the progress on Hamiltonian cycles in directed graphs, oriented graphs and tournaments, see [63]

The organization of Part II is as follows. In Chapter 5, we prove Conjecture 1.47 for graphs of sufficiently large order. In Chapter 6, we prove Conjecture 1.51 for graphs of sufficiently large order.

Chapter 2

Cliques in walk-powers of planar graphs

In this chapter, we will discuss some results related to Conjecture 1.3. We show that if Conjecture 1.3 holds, then the proposed projective cube is an optimal bound of the given unbalanced-girth both in terms of number of vertices and number of edges. More precisely we prove the following.

Theorem 2.1. [84] *If (B, Ω) is a consistent signed graph of unbalanced-girth d which bounds the class of consistent signed planar graphs of unbalanced-girth d , then B has at least 2^{d-1} vertices. Furthermore, if no subgraph of (B, Ω) bounds the same class, then minimum degree of B is at least d , and therefore, B has at least $d \cdot 2^{d-2}$ edges.*

The first part of this theorem will follow from the following theorems (to be proved in the Section 2.2 and Section 2.3 respectively) and Lemmas 2.18, 2.19, 2.20 and 2.21.

Theorem 2.2. [84] *There exists a planar graph G of odd-girth $2k+1$ with $\omega(G^{(2k-1)}) \geq 2^{2k}$.*

Theorem 2.3. [84] *There exists a planar signed bipartite graph (G, Σ) of unbalanced-girth $2k$ for which there are two cliques of order 2^{2k-2} in $(G, \Sigma)^{(2k-2)}$ (defined later), one for each part (induced by the bipartition) of G .*

Our proof of both theorems are constructive and we provide a concrete construction.

We start with some preliminaries on planar graphs, signed graphs and (signed) projective cubes.

2.1 Preliminaries

First, we recall two characterizations of planar graphs in terms of forbidden subgraphs with subdivision and minors.

2.1.1 Planar graphs

We introduce the characterization of planar graphs from Kuratowski, which uses the following definitions. In a graph G , a *subdivision* of an edge $e = uv$, is to delete e , add a new vertex w and join w to u and v . Any graph derived from a graph G by a sequence of edge subdivision is called a *subdivision* of G . Two examples of subdivisions of K_5 and $K_{3,3}$ are presented in Figure 2.1.

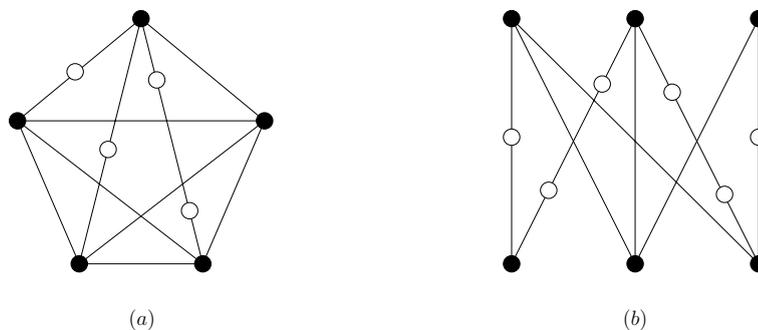


FIGURE 2.1: (a) A subdivision of K_5 , (b) a subdivision of $K_{3,3}$

Theorem 2.4. (*Kuratowski's theorem*)[\[64\]](#) *A finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

A closely related characterization is given by Wagner, using the concept of forbidden minors, which is proved to be equivalent to Theorem 2.4 in [\[13\]](#)

Theorem 2.5. (*Wagner's theorem*)[\[99\]](#) *A finite graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

2.1.2 Signed graphs

The term of signed graphs was first introduced by F. Harary [\[44\]](#) in 1955 to handle a problem in social psychology. In 1982, T. Zaslavsky [\[105\]](#) introduced the matroids of signed graph, there some notations were given and be used until now. Due to the notation of homomorphisms of signed graphs, the term signed graph was used for the equivalence class by Naserasr, Rollová and Sopena in [\[82\]](#) and [\[83\]](#).

One of the first theorems about signed graph was given by Harary [\[44\]](#), which develops a characterization of balance for a signed graph.

Theorem 2.6. [44] *A signed graph (G, Σ) is balanced if and only if its vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 (either of which may be empty), such that each positive edge of (G, Σ) joins two vertices of the same subset and each negative edge joins two vertices of different subsets.*

Harary developed the characterization of balanced signed graph. Zaslavsky [105] extended his work to develop the characterization of equivalence of two signed graphs, which says that the set of equivalent signed graphs is determined uniquely by the set of unbalanced cycles.

Theorem 2.7. [105] *Two signed graphs (G, Σ_1) and (G, Σ_2) are equivalent if and only if they have the same set of unbalanced cycles.*

From Theorem 2.6 and Theorem 2.7 we can see that: if (G, Σ_1) and (G, Σ_2) are equivalent, they have the same set of unbalanced cycles and the symmetric difference of Σ_1 and Σ_2 is an edge-cut. We give an example in Figure 2.2. The signed K_5 in (a) is equivalent to the signed K_5 in (b), where the negative edges are colored red. The symmetric difference of the two signatures is an edge-cut, as shown in (b) by a dashed line.

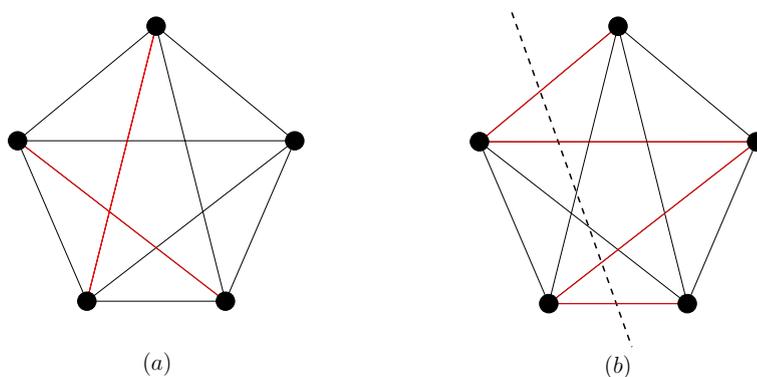


FIGURE 2.2: An example of two equivalent signed graphs

The consistent signed graphs are the main concern of our work. Recall that a consistent signed graph is a signed graph in which every balanced cycle is of even length and all unbalanced cycles are of the same parity. Respect to the parity of the unbalanced cycles, there are two types of consistent signed graphs: odd signed graphs and signed bipartite graphs.

If a signed graph (G, Σ) is an odd signed graphs, then all unbalanced cycles of (G, Σ) are of odd length and all balanced cycles of (G, Σ) are of even length. Since all unbalanced cycles of $(G, E(G))$ are of odd length and all balanced cycles of $(G, E(G))$ are of even length, by Theorem 2.7, (G, Σ) and $(G, E(G))$ are equivalent.

If in a signed graph (G, Σ) the lengths of all balanced cycles and unbalanced cycles are even, then all cycles of G are even, thus G must be a bipartite graph.

2.1.3 (Signed) projective cubes

In Conjecture 1.3, signed projective cube, $SPC(k)$, is proposed as a bound of planar consistent signed graphs of unbalanced-girth $k + 1$. We give some properties of signed projective cubes in this section.

In [82], Naserasr, Rollová and Sopena proved that $SPC(n)$ is a consistent signed graph and determined its unbalanced-girth.

Theorem 2.8. [82] *All balanced cycles of $SPC(n)$ are of even length, all unbalanced cycles of $SPC(n)$ are of the same parity, and the unbalanced-girth of $SPC(n)$ is $n + 1$. Furthermore, for each unbalanced cycle UC of $SPC(n)$ and for each $x \in \{e_1, e_2, \dots, e_n\} \cup \{J\}$, there is an odd number of edges of UC labeled by x .*

As a corollary to Theorem 2.8, one can easily check that the signed projective cube $SPC(2k)$ is equivalent to $(PC(2k), E(PC(2k)))$ and $SPC(2k + 1)$ is a signed bipartite graph.

On the one hand, if a signed graph (G, Σ) admits a homomorphism to $SPC(2k + 1)$, then G admits a homomorphism to the underlying graph of $SPC(2k + 1)$, note that $SPC(2k + 1)$ is bipartite, G must be bipartite, which implies that (G, Σ) is a signed bipartite graph.

On the other hand, if a signed graph (G, Σ) admits a homomorphism to $SPC(2k)$, which is equivalent to $(PC(2k), E(PC(2k)))$, then (G, Σ) must be equivalent to $(G, E(G))$, which implies that (G, Σ) is an odd signed graph.

Thus if a signed graph (G, Σ) admits a homomorphism to a signed projective cube, then it must be a consistent signed graph. Moreover, if (G, Σ) is an odd signed graph and $(G, \Sigma) \rightarrow SPC(2k)$, then equivalently $(G, E(G)) \rightarrow (PC(2k), E(PC(2k)))$, furthermore, this homomorphism can be seen as a homomorphism of G to $PC(2k)$. Thus part (i) of Conjecture 1.3 claims that every planar graph of odd-girth at least $2k + 1$ admits a homomorphism to $PC(2k)$.

The projective cubes, known by other names such as folded cube, are well studied. By definitions, the graphs $PC(1), PC(2), PC(3)$ are isomorphic to $K_2, K_4, K_{4,4}$ respectively. It is easy to check that $PC(n)$ is $(n + 1)$ -regular for $n \geq 2$. In general, every projective cube of odd order, $PC(2n + 1)$, is a bipartite graph. Every projective cube of even order, $PC(2n)$, is of odd-girth $2n + 1$.

Lemma 2.9. [80] *The graph $PC(2n)$ has following properties:*

- (a) *It is $(2n + 1)$ -regular.*
- (b) *It has edge-chromatic number equal to $2n + 1$.*
- (c) *It is of odd-girth $2n + 1$.*

The following lemma is a folklore fact about projective cubes also known as folded cubes (e.g. see A. E. Brouwer, A. M. Cohen and A. Neumaier [15], page 231).

Lemma 2.10. *The graph $PC(n)$ is vertex transitive. It is, furthermore, distance transitive, i.e., for any two pairs $\{x, y\}$ and $\{u, v\}$ of vertices, if $d(x, y) = d(u, v)$, then there is an automorphism σ of $PC(n)$ such that $\sigma(u) = x$ and $\sigma(v) = y$.*

Since part (i) of Conjecture 1.3 focuses on mapping planar graphs into the projective cubes of even dimensions, it is necessary to understand the basic properties on homomorphisms of the projective cubes of even dimensions.

First, Naserasr [81] showed that all the shortest odd-cycles in $PC(2k)$ are isomorphic.

Lemma 2.11. [81] *For every pair C and C' of $(2l + 1)$ -cycles in $PC(2k)$, there is an automorphism that takes vertices of C into vertices of C' .*

Then, Naserasr [81] showed that every the projective cube of even order admits a homomorphism to a projective cube of smaller even order. Precisely,

Lemma 2.12. [81] *We have $PC(2k + 2) \rightarrow PC(2k)$. Furthermore, $PC(2k)$ is 4-chromatic.*

Furthermore, in [7] Beaudou, Naserasr and Tardif conjectured that all these kinds of homomorphisms are onto.

Conjecture 2.13. [7] *Given $r \geq k$, any mapping of $PC(2r)$ to $PC(2k)$ must be onto.*

Supporting Conjecture 2.13, Beaudou et al. proved that the case $r = 3, k = 2$ of the conjecture holds.

Theorem 2.14. [7] *Any homomorphism of $PC(6)$ into $PC(4)$ must be onto.*

Note that every projective cube of odd order, $PC(2k + 1)$, is a bipartite graph, the ordinary homomorphism problem on $PC(2k + 1)$ can be easily solved.

For the homomorphism relation between signed projective cubes, we should mention this result, which is an extension of Lemma 2.12.

Theorem 2.15. [82] *There is a homomorphism of $SPC(k)$ to $SPC(l)$ if and only if $k \geq l$ and $k \equiv l \pmod{2}$.*

As an analogue of Conjecture 2.13, together with M. Chen and R. Naserasr we conjectured that:

Conjecture 2.16. *Given $r \geq k$, any mapping of $SPC(2r + 1)$ to $SPC(2k + 1)$ must be onto.*

Supporting this conjecture, we proved that the case $r = 2, k = 1$ of the conjecture holds.

Theorem 2.17. *Any homomorphism of $SPC(5)$ into $SPC(3)$ must be onto.*

2.1.4 Walk-powers

We will use two notions of graph powers, one for each type of consistent signed graphs. Since the homomorphism of odd signed graphs are reduced to graph homomorphism problems, we use the terminology of graphs for this case.

Recall that given a graph G and a positive integer k , the k -th walk-power of G , denoted $G^{(k)}$, is the graph whose vertex set is also $V(G)$ with two vertices x and y being adjacent if there is a walk of length k connecting x and y in G . Assuming G has at least one edge, $G^{(k)}$ is loopless if and only if k is odd and G has odd-girth at least $k + 2$. As an example we have:

Lemma 2.18. *We have $(PC(2d))^{(2d-1)} \cong K_{2^{2d}}$.*

Proof. We will prove this by showing that each pair of vertices of $PC(2d)$ belong to a cycle of length $2d + 1$. Recall that $PC(2d) = (\mathbb{Z}_2^{2d}, \{e_1, e_2, \dots, e_{2d}, J\})$, $|PC(2d)| = 2^{2d}$. For a given pair x, y of vertices of $PC(2d)$, denote $x - y = e_{i_1} + e_{i_2} + \dots + e_{i_j}$, where $1 \leq j \leq 2d$. Denote $\{e_1, e_2, \dots, e_{2d}, J\} \setminus \{e_{i_1}, e_{i_2}, \dots, e_{i_j}\} = \{e_{i_{j+1}}, \dots, e_{i_{2d}}, J\}$. Let $x_1 = x + e_{i_1}$ and $x_t = x_{t-1} + e_{i_t}$, $2 \leq t \leq 2d$. Note that $y = x_j$ and $x - x_{2d} = J$. Then $x, x_1, \dots, y = x_j, \dots, x_{2d}$ form a cycle of length $2d + 1$. \square

A property of walk-power, which is important for our work, is that:

Lemma 2.19. *If ϕ is a homomorphism of a graph G to a graph H , then ϕ is also a homomorphism of $G^{(r)}$ to $H^{(r)}$ for any positive integer r .*

Proof. We only need to show that for any two vertices u and v , if $uv \in E(G^{(r)})$, then $\phi(u)\phi(v) \in E(H^{(r)})$. Note that if $uv \in E(G^{(r)})$, by definition of walk-power, there is a walk of length r connecting u and v in G , denote this walk by $uu_1u_2 \dots u_{r-1}v$. Since ϕ is homomorphism of G to H , $\phi(u)\phi(u_1) \dots \phi(v)$ is a walk of length r in H . Thus $\phi(u)\phi(v) \in E(H^{(r)})$. \square

Note that for odd values of r if we consider a pair x, y of adjacent vertices in $G^{(r)}$ and identify them in G , then there will be a cycle of odd-length at most r in the resulting graph. This is a key tool for us and we define power of signed bipartite graph to have an analogous property. For this case we shall use the notion of unbalanced cycles instead of odd-cycles.

Given a signed bipartite graph (G, Σ) and an even integer $r \geq 2$ we define $(G, \Sigma)^{(r)}$ to be a graph (not signed) on vertex set $V(G)$ where vertices x and y are adjacent if the following two conditions satisfy:

- x and y are in a same part of bipartite graph G ,
- if x and y are identified in (G, Σ) , then there will be a (new) unbalanced cycle of (even) length at most r .

Note that second condition is equivalent to saying that there are x, y -paths P_1 and P_2 (connecting x and y), each of length at most r , such that one has an odd number of negative edges and the other has an even number of negative edges.

By showing that each pair of vertices from the same part in $SPC(2d+1)$ belong to an unbalanced cycle of length $2d+2$ we have the following bipartite analogue of Lemma 2.18.

Lemma 2.20. *We have $(SPC(2d+1))^{(2d)} \cong 2K_{2^{2d}}$, here $2K_{2^{2d}}$ means two disjoint copies of $K_{2^{2d}}$.*

Proof. We will prove this by showing the that each pair of vertices of $SPC(2d+1)$ belong to an unbalanced cycle of length $2d+2$. Recall that $SPC(2d+1) = (\mathbb{Z}_2^{2d+1}, \{e_1, e_2, \dots, e_{2d+1}, J\})$, with the set of edges corresponding to the J vector assigned by $-$. For any two vertices x, y of $SPC(2d+1)$, denote $x - y = e_{i_1} + e_{i_2} + \dots + e_{i_j}$, where $1 \leq j \leq 2d+1$. Denote $\{e_1, e_2, \dots, e_{2d+1}, J\} \setminus \{e_{i_1}, e_{i_2}, \dots, e_{i_j}\} = \{e_{i_{j+1}}, \dots, e_{i_{2d+1}}, J\}$. Let $x_1 = x + e_{i_1}$ and $x_t = x_{t-1} + e_{i_t}$, $2 \leq t \leq 2d+1$. Note that $y = x_j$ and $x - x_{2d+1} = J$. Then $x, x_1, \dots, y = x_j, \dots, x_{2d+1}$ form an unbalanced cycle of length $2d+2$, since there is only one edge xx_{2d+1} assigned $-$. Note that each part of $SPC(2d+1)$ has 2^{2d} vertices, thus $(SPC(2d+1))^{(2d)} \cong 2K_{2^{2d}}$. \square

The homomorphism property also holds the same:

Lemma 2.21. *Given a positive integer r , if ϕ is a homomorphism of a signed bipartite graph (G, Σ) to a signed bipartite graph (H, Π) , then ϕ is also a homomorphism of the graph $(G, \Sigma)^{(2r)}$ to the graph $(H, \Pi)^{(2r)}$.*

Proof. We only need to show that for any two vertices u and v , if $uv \in E((G, \Sigma)^{(2r)})$, then $\phi(u)\phi(v) \in E((H, \Pi)^{(2r)})$. Note that if $uv \in E((G, \Sigma)^{(2r)})$, by definition of walk-power of signed bipartite graph, u and v are in the same part and there are two u, v -paths P_1 and P_2 (connecting u and v in (G, Σ)), each of length at most $2r$, with one, say P_1 , has an odd number of negative edges and the other, say P_2 , has an even number of negative edges.

Since ϕ is homomorphism of (G, Σ) to (H, Π) , there is signature Σ' equivalent to Σ such that ϕ preserves both adjacency and signs of edges (with respect to Σ' and Π). Note that the parity of the number of negative edges is changed if and only if there is a resigning at one of its end vertices. Since P_1 and P_2 have the same end vertices u and v , the parities of the numbers of negative edges of them are switched if there is a resigning at u or v . Thus P_1 and P_2 also have different parities of the numbers of negative edges with respect to Σ' . Since ϕ preserves both adjacency and signs of edges (with respect to Σ' and Π), the images of P_1 and P_2 in (H, Π) also have different parities of the numbers of negative edges in (H, Π) , if we identify $\phi(u)$ and $\phi(v)$ in (H, Π) , there must be a (new) unbalanced cycle of length at most $2r$. Thus $\phi(u)\phi(v) \in E((H, \Pi)^{(2r)})$. \square

2.2 Optimal bound for planar odd signed graphs

In this section we prove Theorem 2.2. Since this is for odd signed graphs, the homomorphism problem is equivalent to the homomorphisms of graphs. Thus, we will use the terminology of graphs rather than signed graph in this section.

As mentioned, our proof is constructive and we will build an example of a planar graph G of odd-girth $2k+1$ for which we have $\omega(G^{(2k)}) \geq 2^{2k}$. The construction is based on the following local construction.

Lemma 2.22. *Let G be the graph obtained by subdividing edges of K_4 such that in a planar embedding of G each of the four faces is a cycle of length $2k+1$. Then $G^{(2k-1)}$ is isomorphic to K_{4k} .*

Proof. Let a, b, c and d be the original vertices of the K_4 from which G is constructed. For $x, y \in \{a, b, c, d\}$ let P_{xy} be the subdivision of xy , and let t_{xy} be the length of this

path. For an internal vertex w of P_{xy} , let P_{xw} (or P_{wx}) be the part of P_{xy} connecting w to x . Let t_{xw} be the length of P_{xw} .

We have

$$\begin{aligned}
 t_{ab} + t_{bc} + t_{ca} &= t_{ab} + t_{bd} + t_{da} \\
 &= t_{ac} + t_{cd} + t_{da} \\
 &= t_{bc} + t_{cd} + t_{db} \\
 &= 2k + 1.
 \end{aligned} \tag{2.1}$$

From Equation (2.1) we have

$$t_{xy} = t_{wz} \text{ for } \{x, y, w, z\} = \{a, b, c, d\}, \tag{2.2}$$

that is to say that if all four faces have the same length, then any pair of disjoint edges of K_4 are subdivided the same number of times (the parity of the length of the faces is not important here and we will use this fact to prove Lemma 2.25 in Section 2.3).

First, we show that $|V(G)| = 4k$. Note that G has four faces, each of length $2k + 1$. In each face of G , there are $2k - 2$ added vertices which are not the original vertices of the K_4 from which G is constructed. Since each added vertex is in two faces, we have $|V(G)| = 4 + \frac{4(2k-2)}{2} = 4k$.

Now we show that for every pair of vertices u, v of G there is a walk of length $2k - 1$ between them. If u and v are both vertices of a facial cycle of G , then there is a walk of length $2k - 1$ connecting them since each facial cycle is of length $2k + 1$. If there is no facial cycle of G containing both u and v , then they are internal vertices (after subdivision) of two distinct parallel edges of K_4 , thus we may assume, without loss of generality, that u is a vertex of the path P_{ab} and v is a vertex of the path P_{cd} .

Note that by Equation (2.2) we have

$$\begin{aligned}
 t_{au} + t_{bu} &= t_{cv} + t_{dv} \\
 &= t_{ab} = t_{cd}.
 \end{aligned} \tag{2.3}$$

If $t_{ab} = t_{cd}$ is even (odd respectively), then t_{au} and t_{bu} have the same parity (different parities respectively) and t_{cv} and t_{dv} have the same parity (different parities respectively). Moreover, since t_{cd} is even (odd respectively) and $t_{ac} + t_{cd} + t_{da} = 2k + 1$, t_{ac} and t_{ad} have different parities (same parity respectively).

Now one of the paths connecting u, v , say $P_{ua} \cup P_{ac} \cup P_{cv}$, is of length $t_{au} + t_{ac} + t_{cv}$, and another path, say $P_{ub} \cup P_{bd} \cup P_{dv}$, is of length $t_{bu} + t_{bd} + t_{dv}$. By (2.3) we have $(t_{bu} + t_{bd} + t_{dv}) + (t_{au} + t_{ac} + t_{cv}) = 2(t_{ab} + t_{bd})$, hence $t_{bu} + t_{bd} + t_{dv}$ and $t_{au} + t_{ac} + t_{cv}$ have the same parity. Furthermore, since $P_{ab} \cup P_{ad} \cup P_{bd}$ forms a facial cycle we have $t_{ab} + t_{ad} + t_{bd} = 2k + 1$, thus $2(t_{ab} + t_{bd}) = 4k + 2 - 2t_{ad} \leq 4k$.

Hence we have $\min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\} \leq 2k$. Similarly, we can show that $\min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\} \leq 2k$.

But note that $\min\{(t_{au} + t_{ac} + t_{cv}), (t_{bu} + t_{bd} + t_{dv})\}$ and $\min\{(t_{au} + t_{ad} + t_{dv}), (t_{bu} + t_{bc} + t_{cv})\}$ have different parities irrespective of the parity of $t_{ab} = t_{cd}$. Therefore, there is a walk of length $2k - 1$ from u to v . \square

The subdivided K_4 where two parallel edges are subdivided $2k - 1$ times will be the base of our construction. Next we will use two operations to enlarge this construction.

Operation copy threads: Let G be a graph and $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a set of threads of G . For each thread $P_i = xv_1v_2 \dots v_r y$ in \mathcal{P} , add a new thread $P'_i = xv'_1v'_2 \dots v'_r y$ where all the internal vertices are new and distinct. Denote the new graph by $CT(G)$. Let W be a clique in $G^{(l)}$ on vertex set $V(W)$. Consider a set U of vertices of $CT(G)$ which consists of $V(W)$ and copy vertices v' for each vertex v of $V(W)$ that has degree 2 in G . It is now easy to check that the subgraph W' of $CT(G)^{(l)}$ induced by U is a complete graph minus a matching. To be precise, the missing matching matches pairs v, v' where v is a degree 2 vertex of G and v' is its copy.

Next we want to introduce an operation which will complete W' into a complete graph.

Operation shorten threads: Let G be a graph. Consider a collection \mathcal{P} of threads of G , in which every thread is of length at most $2k - 1$. Let $CT(G)$ be the graph obtained after applying the operation copy thread with respect to \mathcal{P} . For $P \in \mathcal{P}$, let P' be its copy in $CT(G)$. Suppose P is of length $r + 1$ with x and y its end vertices and with v_1, v_2, \dots, v_r its internal vertices. Let v'_1, v'_2, \dots, v'_r be the internal vertices of P' . Add a new path to $CT(G)$ of length $2k - r$ which connects v_1 and v'_r (all internal vertices are new and distinct). The new graph obtained after repeating the process for all paths in \mathcal{P} will be denoted by $ST(G)$. Note that the operation $ST(G)$ creates two

shortened threads; $v_1v_2\cdots v_r y$, a shortening of P and $xv'_1v'_2\cdots v'_r$, a shortening of P' . The length of each of the shortened threads is one less than the length of P .

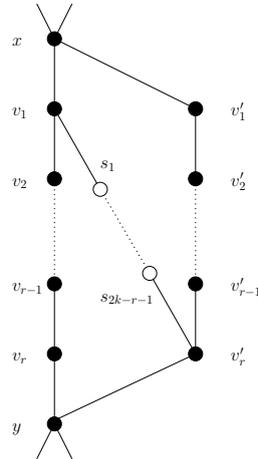


FIGURE 2.3: Copy and shortening of a thread.

See Figure 2.3 for presentation of the two operations.

Observation 2.23. *If G is a planar graph, then $ST(G)$ is also planar with respect to any choice of \mathcal{P} . Furthermore, if a thread P is in a cycle of length $2k + 1$ in G , then both of its corresponding shorten threads are each in a cycle of length $2k + 1$ in $ST(G)$.*

The next lemma is the key property of this operation.

Lemma 2.24. *Let G be of odd-girth $2k + 1$ and let \mathcal{P} be a collection of threads of G , each of them contained in a cycle of length $2k + 1$. Let $ST(G)$ be the graph obtained after applying operations copy threads and shorten threads with respect to \mathcal{P} . Then $ST(G)$ is also of odd-girth $2k + 1$.*

Proof. We may assume \mathcal{P} consists of one thread only (say P). We may apply the proof repeatedly if \mathcal{P} has more threads. Note that a new cycle C' in $CT(G)$ must contain P' , the copy of P . If C' contains both P and P' , then C' is formed of the union of the two and is of even length. Otherwise, by replacing P' with P we obtain a cycle C of G which has the same length as C' . As G has odd-girth at least $2k + 1$, the cycle C' must either be of even length or have odd length at least $2k + 1$. Thus, $CT(G)$ is also of odd-girth $2k + 1$.

Suppose that C is an odd-cycle of length $2l + 1$ ($l \leq k - 1$) in $ST(G)$ that contains the path $v_1s_1\cdots s_{2k-r-1}v'_r$ which connects $P = xv_1\cdots v_r y$ and $P' = xv'_1\cdots v'_r y$. The cycle C must contain at least one of x or y . If it contains only one, say x , then it is $xv'_1\cdots v'_r s_{2k-r-1}v_1$ but this is of length $2k + 1$ which is a contradiction.

If C contains both x and y , then either it contains the path $xv'_1 \cdots v'_r s_{2k-r-1} \cdots s_1 v_1 v_2 \cdots v_r y$ or the path $Q = xv_1 s_1 \cdots s_{2k-r-1} v'_r y$. The former path is already of length $2k + r$, contradicting with length of C . Thus the latter must be the case. In such a case, the path obtained by deleting the edges of Q from the cycle C is a path of length $2l + 1 - (2k - r + 2)$ in G that connects x and y , call it Q'' . On the other hand, since, by our assumption, P is part of a $(2k + 1)$ -cycle, there exists a path Q' of length $2k - r$ connecting x and y . These two paths, Q' and Q'' , together induce a closed walk of length $2l - 1$ in G , contradicting the fact that G has odd-girth $2k + 1$. \square

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let G_0 be the graph obtained from K_4 by subdividing two parallel edges $2k - 2$ times each. Note that the result is a graph of odd-girth $2k + 1$ in which each face is of length $2k + 1$. Thus, by Lemma 2.22, the $(2k - 1)$ -th walk power of G_0 is a clique of order $4k$. Let \mathcal{P}_0 be set of the two threads of G_0 (each of length $2k - 1$), note that each of the threads belongs to a facial cycle of length $2k + 1$. Thus when we apply operation shorten thread on G_0 with respect to \mathcal{P}_0 , both Observation 2.23 and Lemma 2.24 applies.

Starting from G_0 and \mathcal{P}_0 , we will build a graph inductively in $2k - 2$ steps as follows: given G_i and \mathcal{P}_i , we define G_{i+1} to be $ST(G_i)$ with respect to \mathcal{P}_i . We then define \mathcal{P}_{i+1} to be the collection of shortened threads and their copies. Thus \mathcal{P}_{i+1} has twice as many elements as \mathcal{P}_i . Each thread in \mathcal{P}_{i+1} has $2k - i$ vertices. And furthermore Observation 2.23 and Lemma 2.24 applies at each step. Therefore at each step we have a graph G_i of odd-girth $2k + 1$.

At final step, i.e. G_{2k-2} , we have;

$$\begin{aligned}
\omega(G_{2k-2}^{(2k-1)}) &\geq 4k + \sum_{j=1}^{2k-2} 2^j (2k - j - 1) \\
&= 4k + (2k - 1) \sum_{j=1}^{2k-2} 2^j - 2 \sum_{j=1}^{2k-2} j 2^{j-1} \\
&= 4k + [(2k - 1)(2^{2k-1} - 2)] - \\
&\quad 2[(1 - 2^{2k-1}) - (-1)(2k - 1)2^{2k-2}] \\
&= 4k + (k2^{2k} - 4k - 2^{2k-1} + 2) - \\
&\quad (2 - 2^{2k} + k2^{2k} - 2^{2k-1}) \\
&= 2^{2k}.
\end{aligned}$$

□

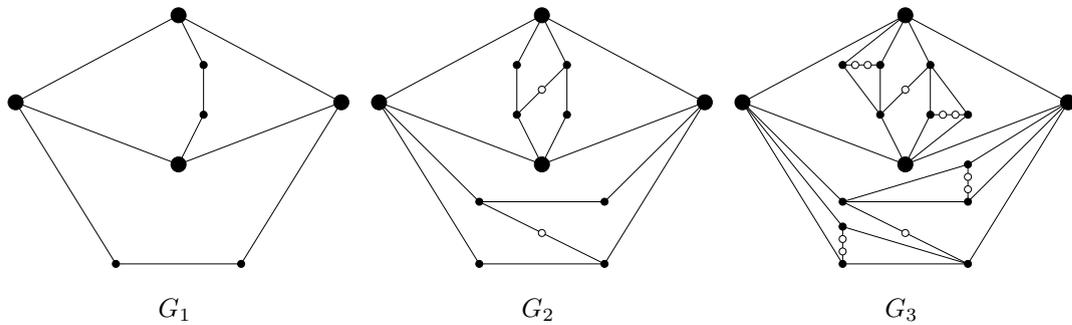


FIGURE 2.4: Example of building a planar graph G_3 of odd-girth 5 with $\omega(G_3^{(3)}) \geq 2^4$.

In Figure 2.4 we present our construction for the case $k = 2$. The result is a graph on 26 vertices. The black vertices correspond to the clique of order 2^4 in $G_3^{(3)}$. Note that the construction of [81] has more than 600 vertices.

Proof of Theorem 2.1 (for even values of d). Let $G = G_{2k-2}$ be the graph built in the previous proof. Since G is of odd-girth $2k + 1$, by the assumption, it maps to B . Since B is also of odd-girth $2k + 1$, both $B^{(2k-1)}$ and $G^{(2k-1)}$ are simple graphs and $G^{(2k-1)}$ admits a homomorphism to $B^{(2k-1)}$. Hence $K_{2^{2k}} \subset B^{(2k-1)}$ which, in particular, implies $|V(B)| \geq 2^{2k}$.

To prove the lower bound on the minimum degree, we first introduce the following graph: let $P = x_1, x_2, \dots, x_{2k+1}$ be a path of length $2k$. Now subdivide each edge $x_i x_{i+1}$ of P by replacing it with the path $x_i y_1^i y_2^i \dots y_{2k-2}^i x_{i+1}$. Note that now x_i is at distance $2k - 1$ from x_{i+1} . Then, we obtain a new graph P' by adding some shortcut edges $x_1 y_1^2, y_1^2 y_2^3, y_2^3 y_3^4, \dots, y_{2k-2}^{2k} x_{2k+1}$ so that the shortest odd walk between each x_i and x_j becomes of length $2k - 1$. Now, given a vertex u , the graph P_u is the graph obtained from a disjoint copy of P' by adding the edges $u x_i$ for all $i \in \{1, 2, \dots, 2k + 1\}$. Note that the graph P_u is of odd-girth $2k + 1$ and that in $P_u^{(2k-1)}$ the vertices of P (i.e., the x_i 's) induce a $(2k + 1)$ -clique.

Now, since B is minimal, there exists a planar graph G_B of odd-girth $2k + 1$ whose mappings to B are always onto. Let G_B^* be a new graph obtained from G_B by adding a copy of P_u for each vertex u of G_B . This new graph is also of odd-girth $2k + 1$, thus, by the choice of B , it maps to B . Let ϕ be such a mapping of G_B^* to B . This mapping induces a mapping of G_B to B . Thus, by the choice of G_B , each vertex v of B is the image of a vertex u of G_B . But in the mapping G_B^* to B , all x_i 's of P_u must map to distinct vertices all of which are neighbors of $\phi(u) = v$. □

Note that since $\mathcal{PC}(2k)$ is a $(2k + 1)$ -regular graph on 2^{2k} vertices, if Conjecture 1.3 holds, then $\mathcal{PC}(2k)$ is an optimal homomorphism bound

2.3 Optimal bound for planar signed bipartite graphs

The development of the notion of homomorphisms for signed graphs has began very recently and, therefore, it is not yet known if an analogue of Theorem 1.10 would hold for the class of signed bipartite graphs. While we believe that it would be the case, here we prove that $SPC(d)$ is the optimal homomorphism bound for the signed bipartite case of Conjecture 1.3 if the conjecture holds.

Note that if both graphs are of unbalanced-girth at least $r + 2$, then $(G, \Sigma)^r$ and $(H, \Pi)^r$ are both loopless, and, therefore, the existence of a homomorphism $\phi : (G, \Sigma) \rightarrow (H, \Pi)$ would imply $\omega((G, \Sigma)^r) \leq \omega((H, \Pi)^r)$. Furthermore, assuming that G and H are both connected, since ϕ is also a homomorphism of G to H , it would preserve bipartition. Thus in what follows we will built a signed bipartite planar graph (G, Σ) of unbalanced-girth $2k$ such that each part of G contains a clique of size 2^{k-2} in $(G, \Sigma)^{2k-2}$.

To this end we start with the following lemma which is the signed bipartite analogue of Lemma 2.22.

Lemma 2.25. *Let (G, Σ) be a planar signed graph which is obtained by assigning a signature to a subdivision of K_4 in such a way that each of the four facial cycles is an unbalanced cycle of length $2k$. Then $(G, \Sigma)^{(2k-2)}$ is isomorphic to two disjoint copies of $K_{2^{k-1}}$ induced by the two parts of G .*

Proof. We consider a fixed signature Σ of (G, Σ) . We will use the same notations (P_{xy} , t_{xy} , etc.) as in Lemma 2.22. Thus as proved in that lemma, parallel edges of K_4 are subdivided the same number of times. Furthermore, repeating the same argument modulo 2, we can conclude that the number of negative edges in P_{xy} and the number of negative edges in P_{wz} have the same parity for all $\{x, y, w, z\} = \{a, b, c, d\}$.

Let u and v be two vertices from the same part of G (thus any path connecting u and v has even length). We would like to prove that they are adjacent in $(G, \Sigma)^{(2k-2)}$. If they both belong to a facial cycle, then the two paths connecting these two vertices in that (unbalanced) cycle satisfy the conditions and we are done. Hence, assume without loss of generality that $u \in P_{ab}$ and $v \in P_{cd}$.

Removing the edges of the parallel paths P_{ad} and P_{bc} will result in a cycle of length $4k - 2t_{ad}$ containing u, v . This implies:

$$(t_{ua} + t_{ac} + t_{cv}) + (t_{ub} + t_{bd} + t_{dv}) \leq 4k - 2,$$

and thus

$$\min\{(t_{ua} + t_{ac} + t_{cv}), (t_{ub} + t_{bd} + t_{dv})\} \leq 2k - 2. \quad (2.4)$$

Similarly by removing P_{ac} and P_{bd} we get

$$\min\{(t_{ua} + t_{ad} + t_{dv}), (t_{ub} + t_{bc} + t_{cv})\} \leq 2k - 2. \quad (2.5)$$

It remains to show that the two paths of Equations (2.4) and (2.5) have different numbers of negative edges modulo 2. To see this note that the union of any of the two paths from (2.4) with a path from (2.5) covers a facial cycle exactly once and one of P_{ab} or P_{cd} twice. Since each facial cycle is unbalanced, our claim is proved. \square

Next we will use two operations, similar to the ones done the previous section, to enlarge this construction.

Operation copy threads: Let (G, Σ) be a signed bipartite graph and $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a set of threads of (G, Σ) . For each thread $P_i = xv_1v_2 \cdots v_r y$ in \mathcal{P} , add a new thread $P'_i = xv'_1v'_2 \cdots v'_r y$ where all the internal vertices are new and distinct. Assign signs to the new edges in such a way that the edges xv'_1 and $v_r y$ have the same sign, the edges $v'_r y$ and xv_1 have the same sign and the edges $v'_i v'_{i+1}$ and $v_{r-i+1} v_{r-i}$ have the same sign. Denote the new signed graph by $(CT(G), CT(\Sigma))$.

Let W be a clique in $(G, \Sigma)^{(l)}$ on vertex set $V(W)$. Consider a set U of vertices of $(CT(G), CT(\Sigma))$ which consists of $V(W)$ and copy vertices v' for each vertex v of $V(W)$ that has degree 2 in G . It is now easy to check that the subgraph W' of $(CT(G), CT(\Sigma))^{(l)}$ induced by U is a complete graph minus a matching. To be precise, the missing matching is between pairs v, v' where v is a degree 2 vertex of (G, Σ) and v' is its copy.

Next we want to introduce an operation which will complete W' into a complete graph.

Operation shorten threads: Let (G, Σ) be a signed bipartite graph. Consider a collection \mathcal{P} of threads of length at most $2k - 2$ of (G, Σ) and let $(CT(G), CT(\Sigma))$

be the graph obtained after applying the operation copy thread with respect to \mathcal{P} . For $P \in \mathcal{P}$ let P' be its copy in $(CT(G), CT(\Sigma))$. Suppose P is of length $r + 1$ with x and y its end vertices and with $v_1, v_2 \dots, v_r$ as its internal vertices. Let $v'_1, v'_2 \dots v'_r$ be the internal vertices of P' . Add a new path N of length $2k - r$ to $(CT(G), CT(\Sigma))$ which connects v_1 and v'_r (all internal vertices are new and distinct). Moreover, we assign signs to the edges of the new path N in such a way that the cycles induced by $V(N) \cup (V(P) \setminus \{x\})$ and $V(N) \cup (V(P') \setminus \{y\})$ are both unbalanced. The new graph obtained after repeating the process for all paths in \mathcal{P} will be denoted by $(ST(G), ST(\Sigma))$. Note that the operation $(ST(G), ST(\Sigma))$ creates two shortened threads; $v_1 v_2 \dots v_r y$ a shortening of P and $x v'_1 v'_2 \dots v'_r$ a shortening of P' . The length of each of the shortened threads is one less than the length of P .

Observation 2.26. *If (G, Σ) is a planar signed bipartite graph, then $(ST(G), ST(\Sigma))$ is also a planar signed bipartite graph with respect to any choice of \mathcal{P} . Furthermore, if a thread P is in an unbalanced cycle of length $2k$ in (G, Σ) , then both of its corresponding shorten threads are each in an unbalanced cycle of length $2k$ in $(ST(G), ST(\Sigma))$.*

The next lemma is the key property of this operation.

Lemma 2.27. *Let (G, Σ) be a signed bipartite graph of unbalanced-girth $2k$ and let \mathcal{P} be a collection of threads of (G, Σ) , each of which is contained in an unbalanced cycle of length $2k$. Let $(ST(G), ST(\Sigma))$ be the signed graph obtained after applying operations copy threads and shorten threads with respect to \mathcal{P} . Then $(ST(G), ST(\Sigma))$ is also a signed bipartite graph of unbalanced-girth $2k$.*

The proof of this lemma is analogous to the proof of Lemma 2.24 and is omitted here. We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let (G_0, Σ_0) be the signed graph obtained from K_4 by subdividing two parallel edges and by assigning a signature in such a way that each of the four facial cycles is an unbalanced cycle of length $2k$. Note that the result is a planar bipartite signed graph of unbalanced-girth $2k$ in which each face is of length $2k$. Thus, by Lemma 2.25, the $(2k - 2)$ -th walk power of (G_0, Σ_0) is a disjoint union of two cliques, each of order $2k - 1$. Let \mathcal{P}_0 be the set of two threads of (G_0, Σ_0) (each of length $2k - 2$), note that each thread in \mathcal{P}_0 is contained in an unbalanced facial cycle of length $2k$. Thus when we apply operation shorten threads on (G_0, Σ_0) with respect to \mathcal{P}_0 , both Observation 2.26 and Lemma 2.27 applies.

Starting from (G_0, Σ_0) and \mathcal{P}_0 , we will build a signed graph inductively in $2k - 2$ steps as follows: given (G_i, Σ_i) and \mathcal{P}_i we define (G_{i+1}, Σ_{i+1}) to be $(ST(G_i), ST(\Sigma_i))$

with respect to \mathcal{P}_i . We then define \mathcal{P}_{i+1} to be the collection of shortened threads and their copies. Thus, \mathcal{P}_{i+1} has twice as many elements as \mathcal{P}_i . Also, note that each thread in \mathcal{P}_{i+1} has $2k - 2 - i$ vertices. Furthermore Observation 2.26 and Lemma 2.27 applies at each step. Therefore at each step we have a signed graph (G_i, Σ_i) of unbalanced-girth $2k$.

At final step we have a planar bipartite signed graph $(G_{2k-2}, \Sigma_{2k-2})$ of unbalanced-girth $2k$. The size of each of the two cliques induced on two parts of $(G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}$ is calculated as follows:

$$\begin{aligned}
\omega((G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}) &= \\
&= 2k - 1 + \sum_{j=1}^{2k-2} 2^{j-1}(2k - j - 2) = 2k - 1 + (k - 1) \sum_{j=1}^{2k-2} 2^j - \sum_{j=1}^{2k-2} j2^{j-1} \\
&= 2k - 1 + [(k - 1)(2^{2k-1} - 2)] - [(1 - 2^{2k-1}) - (-1)(2k - 1)2^{2k-2}] \\
&= 2k - 1 + [k2^{2k-1} - 2k - 2^{2k-1} + 2] - [1 - 2^{2k-1} + k2^{2k-1} - 2^{2k-2}] \\
&= 2^{2k-2}.
\end{aligned}$$

□

Now we are ready to conclude the proof of Theorem 2.1.

Proof of Theorem 2.1 (for odd values of d). The proof is similar to the proof for even values of d . The only thing we need to do is to provide a gadget graph similar to P_u . We will use the graph $(G_{2k-2}, \Sigma_{2k-2})$ as the gadget graph P_u where the role of u is played by one of the original vertices of the K_4 from which the graph was built.

More formally, let x be one of the original vertices of the K_4 from which the signed graph $(G_{2k-2}, \Sigma_{2k-2})$ was built in the proof of Theorem 2.3. Note that x has exactly $2k$ neighbors in $(G_{2k-2}, \Sigma_{2k-2})$, each of which is part of a clique in $(G_{2k-2}, \Sigma_{2k-2})^{(2k-2)}$.

Now, since B is minimal, there exists a planar bipartite signed graph (G_B, Σ_B) of unbalanced-girth $2k$ whose mappings to B are always onto. Let (G_B^*, Σ_B^*) be a new graph obtained from (G_B, Σ_B) by gluing a copy of $(G_{2k-2}, \Sigma_{2k-2})$ to each vertex u of (G_B, Σ_B) by identifying the vertex x of $(G_{2k-2}, \Sigma_{2k-2})$ with the vertex u of (G_B, Σ_B) . This new graph (G_B^*, Σ_B^*) , clearly, is a planar bipartite signed graph of unbalanced-girth $2k$. The rest of the proof is similar to the proof for even values of d . □

2.4 Concluding remarks and further work

P. Seymour has conjectured in [94] that the edge-chromatic number of a planar multi-graph is equal to its fractional edge-chromatic number. It turns out that the restriction of this conjecture for k -regular multigraphs can be proved if and only if Conjecture 1.3 is proved for $d = k + 1$ [81, 82]. This special case of Seymour's conjecture is proved for $k \leq 8$ in a series of works using induction and the Four-Color Theorem in [42] ($k = 4, 5$), [24] ($k = 6$), [26] ($k = 7$) and [20] ($k = 8$). Thus Conjecture 1.3 is verified for $d \leq 7$. Hence we have the following corollary.

Corollary 2.28. *For $d \leq 7$ the signed graph $SPC(d)$ is the smallest consistent graph (both in terms of number of vertices and edges) of unbalanced-girth $d + 1$ which is a homomorphism bound for all consistent planar signed graphs of unbalanced-girth exactly $d + 1$.*

B. Guenin has proposed a strengthening of Conjecture 1.3 by replacing the condition of planarity by the condition of having no $(K_5, E(K_5))$ -minor [41].

For further generalization one can consider the following general question:

Problem 2.29. *Given d and r , $d \geq r$ and $d \equiv r \pmod{2}$, what is the optimal homomorphism bound having unbalanced girth r for all consistent signed graphs of unbalanced-girth d with no $(K_n, E(K_n))$ -minor?*

We do not know yet whether such a homomorphism bound exists in general. For $n = 3$, consistent signed graphs with no $(K_n, E(K_n))$ -minor are bipartite graphs with all edges positive, and, therefore, have K_2 as their homomorphism bound. For $n = 5$ if the input and target graphs are both of unbalanced-girth $d + 1$, then our work and Geunin's extension of Conjecture 1.3 propose projective cubes as the optimal solutions. For $d = r = 3$, the answer would be K_{n-1} if the Odd Hadwiger Conjecture is true. For the case $n = 4$ some partial answers are given by Beadou, Foucaud and Naserasr [6]. For all other cases there is not even a conjecture yet.

Chapter 3

Cliques in walk-powers of K_4 -minor free graphs

In Chapter 2, to show that the proposed projective cube is an optimal bound if Conjecture 1.3 holds, we introduce the notation of the walk-power, what we considered are planar graphs of given odd-girth. If we restrict the planar graphs to K_4 -minor free graphs, is $PC(2k)$ also an optimal bound? In a manuscript [6], Naserasr et al. show that $PC(2k)$ is far from being optimal (with respect to the order), they give a family of graphs of order $\mathcal{O}(k^2)$ bounding the K_4 -minor free graphs of odd-girth $2k + 1$. Is this an optimal bound?

In this chapter, we consider the clique number in the walk-powers of K_4 -minor free graphs. More precisely, we conjecture that:

Conjecture 3.1. [49] *Let G be a K_4 -minor free graph of odd-girth $2k + 1$ ($k \in \mathbb{Z}^+$), then $\omega(G^{(2k-1)}) \leq \frac{(k+1)(k+2)}{2}$.*

Respecting to this conjecture, first we show that, if this conjecture holds, then the bound $\frac{(k+1)(k+2)}{2}$ is optimal, we can see this from the following theorem.

Theorem 3.2. [49] *Given integer $k \geq 1$, there exists a K_4 -minor free graph G of odd-girth $2k + 1$ such that $\omega(G^{(2k-1)}) \geq \frac{(k+1)(k+2)}{2}$.*

Our result shows that order $\mathcal{O}(k^2)$ is optimal for the graphs bounding all K_4 -minor free graphs of odd-girth $2k + 1$.

Claim of Conjecture 3.1 for $k = 1$ is immediate. To support our conjecture we provide a proof for $k = 2$.

Theorem 3.3. [49] *For any K_4 -minor free graph G with no triangle, we have $\omega(G^{(3)}) \leq 6$.*

We start with some preliminaries on K_4 -minor free graphs.

3.1 K_4 -minor free graphs of odd-girth $2k + 1$

The class of K_4 -minor free graphs are graphs without a K_4 as a minor, they are also known as the series-parallel graphs. The series-parallel graph can be defined in the following way. Given a graph G with two distinguished vertices, say s and t , we call G a *2-terminal graph* with terminal pair (s, t) . Let G_1 and G_2 be (vertex disjoint) 2-terminal graphs with terminal pairs (s_1, t_1) and (s_2, t_2) , respectively. A *series combination* of G_1 and G_2 is a 2-terminal graph obtained from $G_1 \cup G_2$ by identifying t_1 with s_2 , and choosing (s_1, t_2) as the new terminal pair. A *parallel combination* of G_1 and G_2 is a 2-terminal graph obtained from $G_1 \cup G_2$ by identifying s_1 with s_2 into a new vertex s , identifying t_1 with t_2 into a new vertex t , and choosing (s, t) as the new terminal pair. A *series-parallel graph* is a graph that can be obtained from copies of K_2 by iterated series and parallel combinations.

In [89], Nešetřil and Nigussie give a lemma showing the configuration of a K_4 -minor free graph which is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth. To recall the lemma, we need the following definitions.

Let G be a graph and let G^s denote the multigraph obtained by two following operations.

- Contract each thread of G to an edge;
- If cycle C is a component of G or a block connected to the rest at u , then contract C to parallel edges e and e' , where in the latter case u is a vertex of e and e' .

For each edge e of G^s , if e is obtained as in first item, we say that the thread is represented by e in G^s , and denote it by P_e ; if e is an original edge of G , we also denote it by P_e . Moreover, the end vertices of e are of degree at least three both in G and G^s ; if e , together with e' , is obtained as in second item, the end vertices of e divide C into two edge-disjoint paths, denoted P_e and $P_{e'}$, which are represented by e and e' in G^s , respectively. In all cases, let l_e denote the length of P_e . Note that if $d_G(v) > 2$, then $d_G(v) = d_{G^s}(v)$.

It was proved by Nešetřil and Nigussie that,

Lemma 3.4. [89] *Let G be a K_4 -minor free graph of odd-girth $2k + 1$ and let e, e' be parallel edges in G^s with common end vertices x and y . If G is not homomorphic to a strictly smaller graph of the same odd-girth, then $l_e + l_{e'} = 2k + 1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k + 1$ containing both x and y .*

Denote by G^* the graph obtained from G^s by identifying its parallel edges. Lemma 3.4 is strengthened by the following:

Lemma 3.5. [89] *Let G be a K_4 -minor free graph of odd-girth $2k + 1$ such that G is not hom-equivalent with C_{2k+1} and G is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth. Then for any $y \in V(G^*)$, if $d_{G^*}(y) = 2$, then $d_{G^s}(y) = d_G(y) = 4$. Moreover, if such y exists then G has a configuration of Figure 3.1, where $P_{e_1} \cup P_{e_2}$, $P_{e_3} \cup P_{e_4}$ and $P_{e_5} \cup P$ are pairwise edge-disjoint cycles of length $2k + 1$, such that $l_{e_i} \geq 2$, for each i , $1 \leq i \leq 5$.*

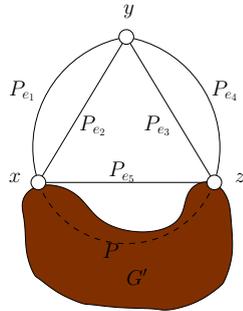


FIGURE 3.1: Configuration 1

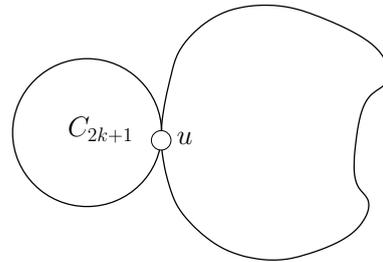


FIGURE 3.2: Configuration 2

From Lemma 3.4 and Lemma 3.5 we get following corollary.

Corollary 3.6. *Let G be a K_4 -minor free graph of odd-girth $2k + 1$ such that G is not hom-equivalent with C_{2k+1} and G is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth. Then G has a configuration of either Figure 3.1 or Figure 3.2.*

Proof. By the definition, G^s and G^* are both K_4 -minor free graphs. Thus there exists a vertex v such that $d_{G^*}(v) \leq 2$.

If $d_{G^*}(v) = 2$, by Lemma 3.5, G has a configuration of Figure 3.1.

If $d_{G^*}(v) = 1$, denote by e the only edge incident to v in G^* . Denote by u the other end vertex of e . If u and v are joined only by e in G^s , that means $d_{G^s}(v) = 1$ and e is an original edge in G , then $d_G(v) = 1$. Thus v can be mapped to any other neighbor of u , this is contrary to that G is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth.

If u and v are joined by another edge e' parallel to e , by Lemma 3.4, $P_e \cup P_{e'}$ is the unique cycle of length $2k + 1$ containing both u and v in G . Moreover, u is the only vertex on this cycle of degree at least three in G . We claim that $P_e \cup P_{e'}$ is the unique cycle of length $2k + 1$ containing u in G . Suppose to the contrary that C is another cycle of length $2k + 1$ containing u . Note that no vertex of $(P_e \cup P_{e'}) \setminus u$ is on C , otherwise all vertices of $P_e \cup P_{e'}$ are on C , which implies that $(P_e \cup P_{e'}) = C$. Thus $V(C) \cap V(P_e \cup P_{e'}) = u$. Since all vertices of $(P_e \cup P_{e'}) \setminus u$ are of degree two in G , $P_e \cup P_{e'}$ can be mapped to C , this is contrary to that G is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth. Thus G has a configuration of Figure 3.2. \square

3.2 Proof of Theorem 3.2

In this section, we will prove Theorem 3.2.

First, we construct a graph, then show that it satisfies the theorem.

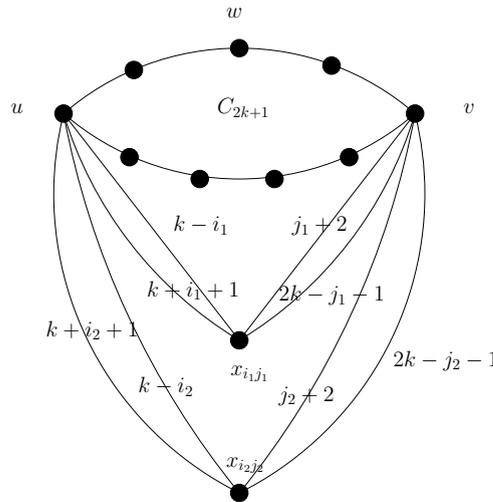


FIGURE 3.3: Constructing a K_4 -minor free graph G of odd-girth $2k + 1$ with $\omega(G^{(2k-1)}) \geq \frac{(k+1)(k+2)}{2}$.

As shown in the Figure 3.3, we construct the graph G from a cycle C of length $2k + 1$. It is easy to check that these $2k + 1$ vertices on C will be in a clique of $G^{(2k-1)}$. Denote these vertices by a set X . Take two vertices of distance k on the cycle C , denote them by u and v . Then we add new vertices step by step to G and X , for convenience, we always call the new graph G and the new set X . For $0 \leq i \leq k - 2$, $i \leq j \leq k - 2$, we add a new vertex x together with two parallel paths connecting u and x , of length $k - i$ and $k + i + 1$ respectively; also together with two parallel paths connecting v and x , of length $j + 2$ and $2k - j - 1$ respectively. All vertices on these four new paths are unused except u and v . Also we add x to X .

First, we show that at each step the graph G is K_4 -minor free and of odd-girth $2k + 1$. Note that at each step, we only use series and parallel connections of copies of K_2 , it is easy to check that G is also K_4 -minor free. Now we show that G is of odd-girth $2k + 1$. Note that, each time we add a vertex, say $x_{i_1j_1}$, together with four paths, say P_1, P_2, P_3, P_4 , of lengths $k - i_1, k + i_1 + 1, j_1 + 2, 2k - j_1 - 1$ ($i_1 \leq j_1 \leq k - 2$ by assumption), respectively, to G . Then in the subgraph formed by P_1, P_2, P_3, P_4 together with C , except three cycles of length $2k + 1$, all the other cycles each must contain $u, v, x_{i_1j_1}$, thus each of them is of length at least $k + k - i_1 + j_1 + 2 \geq 2k + 2$. We are done for this case. If there exists a vertex, say $x_{i_2j_2}$, added before $x_{i_1j_1}$, together with four paths, say P'_1, P'_2, P'_3, P'_4 , of lengths $k - i_2, k + i_2 + 1, j_2 + 2, 2k - j_2 - 1$ ($i_2 \leq j_2 \leq k - 2$ by assumption), respectively, then in the subgraph formed by P'_1, P'_2, P'_3, P'_4 together with P_1, P_2, P_3, P_4 , except four cycles of lengths $2k + 1$, all the other cycles each must contain $u, v, x_{i_1j_1}, x_{i_2j_2}$, thus each of them is of length at least $k - i_2 + j_2 + 2 + k - i_1 + j_1 + 2 = 2k + 4 + (j_2 - i_2) + (j_1 - i_1) \geq 2k + 4$. We are done.

Second, we show that at each step, the new vertex added to X is connected by an odd path of length at most $2k - 1$ to each vertex on C . To see this, we take a vertex, say w , of C . Since u and x are on a cycle of length $2k + 1$, u and x are surely connected by an odd path of length at most $2k - 1$, it is the same for v and x . Thus we may assume that w is different from u and v . Denote the distance of u and w by l_1 and the distance of v and w by l_2 . Then $l_1 + l_2 = k$ or $k + 1$ and $l_1 \geq 1, l_2 \geq 1$. There are two paths connecting w and x which contain u , of distances $k + l_1 - i$ and $k + l_1 + i + 1$ respectively. Also there are two paths connecting w and x which contain v , of distances $j + l_2 + 2$ and $2k + l_2 - j - 1$ respectively. We need to show that one of $k + l_1 - i, k + l_1 + i + 1, j + l_2 + 2$ and $2k + l_2 - j - 1$ is odd and no more than $2k - 1$. Note that $k + l_1 - i$ and $k + l_1 + i + 1$ are of different parities, $j + l_2 + 2$ and $2k + l_2 - j - 1$ are of different parities, also both $k + l_1 - i$ and $j + l_2 + 2$ are no more than $2k$. If one of $k + l_1 - i$ and $j + l_2 + 2$ is odd, we are done. Now assume that both $k + l_1 - i$ and $j + l_2 + 2$ are even, then both $k + l_1 + i + 1$ and $2k + l_2 - j - 1$ are odd. Since $k + l_1 + i + 1 + 2k + l_2 - j - 1 = 3k + (l_1 + l_2) - (j - i) - 2 \leq 4k$, one of $k + l_1 + i - 1$ and $2k + l_2 - j - 1$ is no more than $2k - 1$, we are done.

Finally we show that each pair of vertices added to X is connected by an odd path of length no more than $2k - 1$. Take two vertices $x_{i_tj_t}$'s ($t = 1, 2$) such that $x_{i_tj_t}$ is connected by two paths to u of lengths $k - i_t$ and $k + i_t + 1$ respectively, and $x_{i_tj_t}$ is connected by two paths to v of lengths $j_t + 2$ and $2k - j_t - 1$ respectively. There are four paths connecting $x_{i_1j_1}$ and $x_{i_2j_2}$ which contain u of lengths $2k - i_1 - i_2, 2k + i_1 + i_2 + 2, 2k + i_1 - i_2 + 1$ and $2k + i_2 - i_1 + 1$ respectively. If i_1 and i_2 are of different parities, then $2k - i_1 - i_2$ is odd and no more than $2k - 1$, we are done. We assume that i_1 and i_2 are of the same parity. If $i_1 \neq i_2$, then one of $2k + i_1 - i_2 + 1$ and $2k + i_2 - i_1 + 1$ is odd

and no more than $2k - 1$, we are done. Thus we assume that $i_1 = i_2$. Since $x_{i_1j_1}$ and $x_{i_2j_2}$ are distinct, $j_1 \neq j_2$. Then there are four paths connecting $x_{i_1j_1}$ and $x_{i_2j_2}$ which contain v of lengths $j_1 + j_2 + 4$, $4k - j_1 - j_2 - 2$, $2k + j_1 - j_2 + 1$ and $2k + j_2 - j_1 + 1$ respectively. If j_1 and j_2 are of different parities, then $j_1 + j_2 + 4$ is odd and no more than $2k - 1$, we are done. If j_1 and j_2 are of the same parity, then one of $2k + j_1 - j_2 + 1$ and $2k + j_2 - j_1 + 1$ is odd and no more than $2k - 1$, we are done.

Now we calculate the size of X . At beginning, there are $2k + 1$ vertices in X . Each pair (i, j) gives a new vertex, where $0 \leq i \leq k - 2$ and $i \leq j \leq k - 2$. Thus there are $(k - 2 + 1) + (k - 2) + \dots + 1$ new vertices. Totally, we get $|X| = 1 + 2 + \dots + k - 1 + 2k + 1 = \frac{(k+1)(k+2)}{2}$.

The proof of Theorem 3.2 is completed.

3.3 Preliminaries for the proof of Theorem 3.3

In this section, we give some Lemmas for the proof of Theorem 3.3.

Lemma 3.7. *Let G be a graph and C be a cycle of length at least 3 in G . If there exist 3 walks, say W_1, W_2, W_3 , starting from x in $G \setminus C$ such that the first vertex where W_i meets C is x_i ($i = 1, 2, 3$), where x_1, x_2 and x_3 are mutually distinct, then G has K_4 as a minor.*

Proof. Denote by T the component containing x in $G \setminus C$. It is easy to check that x_i ($i = 1, 2, 3$) is adjacent to a vertex in H . Contract T to be a new vertex x' , then x' is adjacent to all of $\{x_1, x_2, x_3\}$. Contract some edges of C such that x_1, x_2 and x_3 are mutually adjacent, then we get a K_4 as a minor. \square

Lemma 3.8. *Let G be a K_4 -minor free graph of odd-girth 5. Let X be a set of vertices of a clique in $G^{(3)}$. Then for any vertex w , $|N_1(w) \cap X| \leq 2$.*

Proof. Suppose to the contrary that there exists a vertex w adjacent to 3 vertices x_1, x_2, x_3 in X . Since x_1 and x_2 are in X , there exists a 3-walk W connecting them in G . Then W together with x_2wx_1 walk forms a closed walk of length 5. Thus it should contain an odd-cycle of length at most 5. Since G is triangle free, this closed walk should be a 5-cycle, denote it by $C_{x_1x_2}$. We claim that x_3 is not on $C_{x_1x_2}$. To see this, note that two neighbors of w in $C_{x_1x_2}$ are x_1 and x_2 . If x_3 is also one of vertices of $C_{x_1x_2}$, then since x_3 is adjacent to w , we will have a smaller odd-cycle.

Similarly, since x_3 and x_1 are in X , there exists a 3-walk W_1 connecting them in G . Then W_1 together with x_1wx_3 walk forms a 5-cycle, and x_2 is not on it. Denote

by w_1 the vertex where W_1 meets W . Note that $w_1 \neq x_2$. Since W is of length 3, the lengths of w_1x_1 walk and w_1x_2 walk in W have different parities. Thus x_3w_1 walk in W_1 together with w_1x_2 walk in W is of even length, and W_1 together with x_1w_2 walk is of length 5. We need another 3-walk W_2 connecting x_3 and x_2 in G . Similarly, W_2 together with x_2w_3 walk forms a 5-cycle, and x_1 is not on it. Denote by w_2 the vertex where W_2 meets W . Note that $w_2 \neq x_1$.

We claim that $w_1 \neq w_2$. To the contrary, let $w_1 = w_2$. Note that $x_2 \neq w_1, x_1 \neq w_2$. Then W_1 together with x_1w_2 walk and W_2 is a closed walk of length 8, which contains $C_{x_1x_2}$ of length 5. Thus x_3w_1 walk in W_1 together with x_3w_2 walk in W_2 is a closed walk of length 3, a contradiction.

Since x_3 is adjacent to w , W_1 and W_2 meet $C_{x_1x_2}$ at another two different vertices w_1 and w_2 , by Lemma 3.7, we can get a K_4 as a minor. A contradiction. \square

In this Lemma we show that, at most 2 vertices in $N(w)$ of any w can form a clique in $G^{(3)}$, in the following, we show that at most 3 vertices in $N_2(w)$ of any w can form a clique in $G^{(3)}$.

Lemma 3.9. *Let G be a K_4 -minor free graph of odd-girth 5. Let X be a set of vertices of a clique in $G^{(3)}$. Then for any vertex w , $|N_2(w) \cap X| \leq 3$.*

Proof. Suppose to the contrary that there exist a vertex w and four vertices x_1, x_2, x_3, x_4 such that $\{x_1, x_2, x_3, x_4\} \subseteq N_2(w) \cap X$. Since x_1 and x_2 are in X , there exists a 3-walk W connecting them in G . Respect to the existence of common neighbors of x_1 and x_2 in $N_1(w)$, we continue our proof in two cases.

Case 1. $N_1(x_1) \cap N_1(x_2) \cap N_1(w) \neq \emptyset$. Denote one of the common neighbors of w and x_i 's ($i = 1, 2$) by w_1 . Since $\{x_1, x_2\} \subseteq X$, there exists a 3-walk connecting x_1 and x_2 , denote it by $x_1y_1y_2x_2$. Note that G is triangle free, it is to check that $\{y_1, y_2\} \cap \{x_1, x_2, w_1, w\} = \emptyset$.

Subcase 1. $|\{x_3, x_4\} \cap \{y_1, y_2\}| = 2$. Note that G is triangle free, w_1 is not adjacent to y_1 or y_2 , moreover, y_1 and y_2 has no common neighbors. Since $w_1x_1y_1y_2x_2w_1$ forms a cycle of length 5, denoted C , and there exist 3 walks starting from w in $G \setminus C$ such that the first vertices they meet C are w_1, y_1 and y_2 respectively. By Lemma 3.7, G has a K_4 as a minor, a contradiction.

Subcase 2. $|\{x_3, x_4\} \cap \{y_1, y_2\}| = 1$. Assume w.l.o.g. that $x_3 = y_1$. Since G is triangle free, y_1 is not adjacent to w_1 . Note that $y_1 = x_3 \in N_2(w)$, y_1 and w must have a common neighbor. We claim that this common neighbor could be not one of $\{w_1, x_1, x_2\}$, otherwise there will be a triangle, a contradiction.

First, we assume that y_2 is one of the common neighbors of w and y_1 . By Lemma 3.8, x_4 is not adjacent to w_1 . Note that $x_4 \in N_2(w)$.

If x_4 is adjacent to y_2 , the $\{x_2, x_3, x_4\} \subseteq X \cap N(y_2)$, a contradiction by Lemma 3.8.

If x_4 is not adjacent to y_2 , denote by y_3 the common neighbor of w and x_4 , then $y_3 \notin \{w_1, x_1, x_2, y_1, y_2\}$. There are 3-walks connecting x_4 and x_1, x_2, y_1 respectively. Considering the distances between w and x_i 's, none of these 3-walks contains w . Similarly to the proof in Lemma 3.8, the number of the first vertices where these 3-walks meet the cycle $w_1x_1y_1y_2x_2w_1$ must be at least 2. In this case, there are walks starting from x_4 which meet one of the cycles $ww_1x_2y_2$ and $ww_1x_1y_1y_2$ at three distinct vertices (first meet vertices). Then by Lemma 3.7 we get a contradiction.

Second, we assume that y_2 is not the common neighbor of w and y_1 . This case is very similar to case that the common neighbor of y_1 and w is y_2 , we can use Lemma 3.7 to get a contradiction.

Subcase 3. $|\{x_3, x_4\} \cap \{y_1, y_2\}| = 0$. By Lemma 3.8, both x_3 and x_4 are not adjacent to w_1 . Note that $w_1x_1y_1y_2x_2w_1$ is a cycle of length 5, denote it by C , to get two 3-walks connecting x_i ($i = 3, 4$) and x_1 or x_2 , there are two ways: one is connecting x_3 to w_1 by a 2-walk, the other one is connecting x_3 by two walks which meet C at two different vertices. It is the same for x_4 . We only give one case here: x_3 is connected to w_1 by a 2-walk, and x_4 is connected by two walks to C which meet C at two different vertices except w_1 . In this case, since x_3 and x_4 is connected by 3-walk and x_3 is connected to w_1 by a 2-walk, it is easy to check that there is a walk connecting x_4 and w_1 such that w_1 is the first vertex where the walk meets C . Thus, by Lemma 3.7 we get a contradiction. The other cases are similar.

Case 2. $N_1(x_1) \cap N_1(x_2) \cap N_1(w) = \emptyset$. Moreover, we can assume $N_1(x_i) \cap N_1(x_j) \cap N_1(w) = \emptyset$ ($i \neq j, \{i, j\} \subseteq \{1, 2, 3, 4\}$), otherwise it is the same as **Case 1**. Denote one of the common neighbors of w and x_i by y_i ($i = 1, 2, 3, 4$). Note that y_i 's are distinct.

First, we claim that no pair of x_i and x_j are adjacent. To see this, suppose to the contrary we assume that x_1 and x_2 are adjacent. Then $wy_1x_1x_2y_2w$ is a cycle of length 5, denoted it C . Assume that W_1 and W_2 are two 3-walks connecting x_3 to x_1 and x_2 , then W_1 and W_2 do not contain w . Moreover, the first vertices W_1 and W_2 meet C must be the same, otherwise by Lemma 3.7 we get a contradiction. Thus the only way is that one of $\{W_1, W_2\}$ is of length 2, the other is of length 3, W_1 and W_2 meet at x_1 or x_2 , say x_2 . Note that x_3 could not be adjacent to x_1 or x_2 . In this case there must exist a cycle containing w, x_1, x_2, x_3 , denote it by C' . Now consider the 3-walks, say W_3, W_4, W_5 , connecting x_4 to x_i 's ($i = 1, 2, 3$). Since w and x_i ($i = 1, 2, 3, 4$) are of distance 2, W_3, W_4, W_5 do not contain w . Furthermore, since x_2 is of different distance

to x_1 and x_3 on C' , W_3, W_4, W_5 must meet C' at at least 2 vertices, then together with x_4y_4w , there are 3 walks starting from x_4 which meet C' at three distinct vertices, we get a contradiction by Lemma 3.7.

Second, we claim that no pair of x_i and x_j are connected by a 3-walk without using vertices in $\{y_1, y_2, y_3, y_4\}$. To see this, suppose to the contrary we assume that x_1 and x_2 are connected by a 3-walk $x_1y_5y_6x_2$. Thus $wy_1x_1y_5y_6x_2y_2w$ form a cycle of length 7, denoted by C . Once again, let W_1 and W_2 be two 3-walks connecting x_3 to x_1 and x_2 , respectively. Considering the distance between x_1 and x_2 , W_1 and W_2 must meet C at two distinct vertices. Including the walk x_3y_3w , there are three walks starting from x_4 which meet C at three distinct vertices, by Lemma 3.7 a contradiction.

Furthermore, we claim that no pair of x_i and x_j are connected by a 3-walk using two vertices in $\{y_1, y_2, y_3, y_4\}$, otherwise a triangle arises.

Thus each pair of x_i and x_j are connected by a 3-walk using exactly one vertex in $\{y_1, y_2, y_3, y_4\}$ and one vertex not in $\{w, x_i, y_i\}$ ($i = 1, 2, 3, 4$). Since there are $C_4^2 = 6$ pairs of x_i and x_j , and there are only 4 y_i 's, on these six 3-walks, there are two cases: one of y_i 's is used 3 times and the others each is used once; two of y_i 's are used twice and the others each is used once. We give the proof of the first case here, the second case is similar. Assume w.l.o.g that y_1 is used 3 times in the 3-walks connecting x_1 to x_2, x_3 or x_4 . On these three 3-walks, y_1 can be adjacent to 1, 2 or 3 new vertices. We give the proof for the case y_1 adjacent to one new vertex, say y_1^* , the other cases are similar. These three 3-walks are $x_1y_1y_1^*x_i$ ($i = 2, 3, 4$). Moreover, assume x_2 and x_3 are connected by a 3-walk $x_2y_2y_2^*x_3$. Note that $y_2^* \neq y_1^*$, otherwise a triangle arises. Now $wy_2y_2^*x_3y_3w$ form a cycle of length 5, denote it by C . Then there are three walks starting from x_3 which meet C at three distinct vertices y_2, x_3 and w , by Lemma 3.7 we get a contradiction.

This completes the proof. □

Lemma 3.10. *Let G be a K_4 -minor free graph of odd-girth 5. Let u and v be two vertices of G connected by a thread of length at least 2. For any two vertices x, y in $V(G) \setminus \{u, v\}$, if $d(x, u) = d(y, u) = 2$ and $d(x, v) = d(y, v) = 1$, then x and y are not adjacent in $G^{(3)}$.*

Proof. Suppose to the contrary that x and y are adjacent in $G^{(3)}$, that means there exists a 3-walk W connecting them in G , considering the distances between x, y and u, v , x, y are not on this thread. Since $d(x, u) = d(y, u) = 2$, u and x have a common neighbor, denote it by w_1 . And u and y also have a common neighbor, denote it by w_2 .

Note that the vxw_1u walk together with uv -thread forms a cycle, denoted C , of length at least 5.

We claim that u is not on W , otherwise, since W is of length 3, u is adjacent to either x or y , which is contrary to $d(x, u) = d(y, u) = 2$. Note that W together with xvy walk forms a closed walk of length 5, thus it should contain an odd-cycle of length at most 5. Since G is triangle free, this closed walk should be a 5-cycle, that means v is not on W .

If $w_1 = w_2$, similarly to v , we can prove that w_1 is not on W . Now there exist 3 walks starting from y which meet C at 3 different vertices w_1, v and x . By Lemma 3.7, G has a K_4 as a minor, a contradiction.

If $w_1 \neq w_2$, then one of $\{w_1, w_2\}$ is not on W , otherwise w_1 and w_2 are adjacent and a triangle uw_1w_2 arises, a contradiction. If one of $\{w_1, w_2\}$, assume w.l.o.g. w_1 , is on W , then there exist 3 walks starting from y which meet C at 3 different vertices u, v and x . By Lemma 3.7, G has a K_4 as a minor, a contradiction. If neither w_1 nor w_2 is on W , then there exist 3 walks starting from y which meet C at 3 different vertices u, v and x . By Lemma 3.7, G has a K_4 as a minor, a contradiction.

This completes the proof. □

3.4 Proof of Theorem 3.3

Now we prove the Theorem 3.3.

Given two graphs G and H , if $G \rightarrow H$, then any clique in G will be mapped to a clique of same size in H . Thus $\omega(G) \leq \omega(H)$. If φ is a homomorphism of G to H , then by Lemma 2.19 φ is also a homomorphism of $G^{(k)}$ to $H^{(k)}$. Define a class \mathcal{G} of graphs such that every member of \mathcal{G} is K_4 -minor free graph of odd-girth 5 and is not homomorphic to a strictly smaller K_4 -minor free graph of the same odd-girth. Note that every K_4 -minor free graph of odd-girth 5 is homomorphic to a member of \mathcal{G} . To prove the Theorem 3.3, we only need to prove that every graph $G \in \mathcal{G}$, $\omega(G^{(3)}) \leq 6$. Suppose to the contrary that there exists counterexamples in \mathcal{G} , take one of minimum order, denote it by G . Note that $\omega(G^{(3)}) \geq 7$, since $\omega(C_5^{(3)}) = 5$, $G \not\rightarrow C_5$. Thus by Corollary 3.6 G has a configuration either in Figure 3.4 or in Figure 3.5. Let X be a set of vertices of a maximum clique in $G^{(3)}$, then $|X| \geq 7$. We will get contradictions according to the two configurations of G .

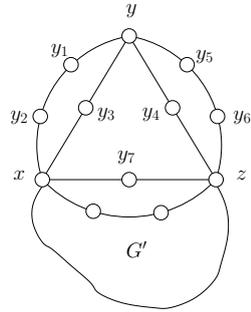


FIGURE 3.4: Configuration 3

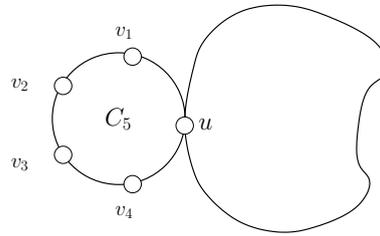


FIGURE 3.5: Configuration 4

3.4.1 G has a configuration in Figure 3.5

Firstly, we claim that $\{v_1, v_2, v_3, v_4\} \cap X \neq \emptyset$. To see this, assume that $\{v_1, v_2, v_3, v_4\} \cap X = \emptyset$, then $X \subseteq V(G) \setminus \{v_1, v_2, v_3, v_4\}$. For any two vertices in X connected by a 3-walk W , if one of $\{v_1, v_2, v_3, v_4\}$ is on W , then all of them are on W , contrary to that W is a 3-walk. Thus none of $\{v_1, v_2, v_3, v_4\}$ is on a 3-walk connecting a pair of vertices in X . Deleting the vertices in $\{v_1, v_2, v_3, v_4\}$, we get a smaller counterexample, a contradiction.

Secondly, we claim that $\{v_1, v_4\} \cap X \neq \emptyset$ and $\{v_2, v_3\} \cap X = \emptyset$. To see this, suppose that $\{v_2, v_3\} \cap X \neq \emptyset$. Assume w.l.o.g. that $v_2 \in X$. If $v_1 \in X$, since $|X| \geq 7$, there are at least two vertices w_1, w_2 in $X \cap V(G \setminus C_5)$. Note that $d(v_2, u) = 2$ and there exists a 3-walk connecting v_2 and each of w_1, w_2 , we get that w_1 and w_2 must be adjacent to u . Now three neighbors of u are in X , by Lemma 3.8, we get a contradiction. If $v_1 \notin X$, since $|X| \geq 7$, there are at least 3 vertices in $X \cap V(G \setminus C_5)$ which must be adjacent to u , a contradiction by Lemma 3.8. Therefore, $\{v_2, v_3\} \cap X = \emptyset$ and $\{v_1, v_4\} \cap X \neq \emptyset$.

Since $\{v_1, v_4\} \cap X \neq \emptyset$, assume w.l.o.g. that $v_1 \in X$. Note that $|X| \geq 7$, there are at least 4 vertices of X in $X \cap V(G \setminus C_5)$, which are in $N_2(u)$. By Lemma 3.9, we get a contradiction.

3.4.2 G has a configuration in Figure 3.4

We first claim that at most one of $\{y_1, y_5\}$ is in X . To see this, note that any walk W connecting y_1 and y_5 , W contains either y or both z and x . If W contains y , then it is of even length or of length at least 5. If W contains both z and x , then it is of length at least 6. Thus there is no walk of length 3 connecting y_1 and y_5 , at most one of $\{y_1, y_5\}$ could be in X . Similarly we can prove that at most one of $\{y_2, y_6\}$, one of $\{y_1, y_4\}$, one of $\{y_6, y_7\}$, one of $\{y_3, y_4, y_7\}$ could be in X . Note that at most 3 of $\{y_1, y_2, \dots, y_7\}$ could be in X . Considering y is in X or not, there are two cases.

Case 1. Vertex $y \in X$. Note that 3 of $\{y_1, y_2, \dots, y_7\}$ together with x, y, z , we now have at most 6 vertices in X . Since $|X| \geq 7$, there exists at least one vertex w_1 in G' such that $w_1 \in X$, $w_1 \notin \{z, x\}$. Thus there exists a 3-walk connecting w_1 to y . Since $d(y, x) = d(y, z) = 2$, w_1 must be adjacent to x or z . Assume w.l.o.g. that w_1 is adjacent to x . We show that $y_2 \notin X$. To see this, if there exists a 3-walk W connecting y_2 to w_1 , the y_2xw_1 walk together W forms a 5-walk. Note that y_2 is on the unique 5-cycle, this 5-walk must contain a cycle of length 3, a contradiction. Similarly, we can show that $y_3 \notin X$.

If $y_1 \in X$, then $y_4 \notin X, y_5 \notin X$. Since at most one of $\{y_6, y_7\}$ could be in X , together with x, y, z, w_1, y_1 , now we have at most 6 vertices in X . Since $|X| \geq 7$, there exists another vertex w_2 in G' such that $w_2 \in X$, $w_2 \notin \{z, x\}$. Since $d(y_1, x) = 2$ and $d(y_1, z) = 3$, to connect y_1 with w_2 by a 3-walk, w_2 must be adjacent to x . Moreover, for any vertex w in G' such that $w \in X$, $w \notin \{z, x\}$, w must be adjacent to x . By Lemma 3.8, at most two neighbors of x could be in X . Thus one of $\{y_6, y_7\}$ must be in X . Since y_7 is adjacent to x , $y_7 \notin X$. Therefore, $y_6 \in X$. Since $d(y_6, x) = 3$, $d(y_6, z) = 1$, y_6 and $w_i (i = 1, 2)$ are connected by a 3-walk, we can get that $d(w_1, z) = d(w_2, z) = 2$. Together with $d(w_1, x) = d(w_2, x) = 1$ and by Lemma 3.10, w_1 and w_2 could not be adjacent in $G^{(3)}$, a contradiction.

If $y_1 \notin X$. Now y_5 could be in X . If $y_5 \in X$, since $d(y_5, z) = 2$ and $d(y_5, x) = 3$, for any vertex w in G' such that $w \in X$, $w \notin \{z, x\}$, w must be adjacent to z . By Lemma 3.8, at most two neighbors of z could be in X , together with z, x, y, y_5 , we have at most 6 vertices in X , a contradiction. If $y_5 \notin X$. By Lemma 3.8, at most 3 of $\{z\} \cup N(z)$ can be in X , together with y, z, w_1 , we now have at most 6 vertices in X . Since $|X| \geq 7$, there exists another vertex w_2 in G' such that $w_2 \in X$, $w_2 \notin \{z, x\}$, moreover $w_2 \notin N(z)$ and w_2 is connected to y by a 3-walk, thus w_2 is adjacent to x . In this case, if one of $\{y_4, y_6, y_7\}$ is in X , then $d(w_1, z) = d(w_2, z) = 2$. together with $d(w_1, x) = d(w_2, x) = 1$, and by Lemma 3.10, w_1 and w_2 could not be adjacent in $G^{(3)}$, a contradiction. If none of $\{y_4, y_6, y_7\}$ is in X , now we have 5 vertices y, z, x, w_1, w_2 in X , Since $|X| \geq 7$, there exists another two vertex w_3, w_4 in G' such that $w_3 \in X$, $w_4 \in X$, $w_3 \notin \{z, x\}$, $w_4 \notin \{z, x\}$. Note that w_3 and w_4 are adjacent to z or x , by Lemma 3.8, both of them are adjacent to z . In this case, $\{w_1, w_2, w_3, w_4\} \subseteq N_2(y_7)$, by Lemma 3.9 we get a contradiction.

Case2. Vertex $y \notin X$. It is easy to check that y can be connected by a 3-walk to each of $\{x, z, y_1, y_2, \dots, y_7\}$. If none of $\{y_1, y_2, \dots, y_6\}$ is in X , note that no pair of vertices in X are connected by a 3-walk containing a vertex in $\{y, y_1, y_2, \dots, y_6\}$, then deleting vertices in $\{y, y_1, y_2, \dots, y_6\}$, we get a smaller counterexample, a contradiction. Thus at least one of $\{y_1, y_2, \dots, y_6\}$ is in X . Note that at most 3 of $\{y_1, y_2, \dots, y_7\}$ could

be in X . Together with x, z , we now have at most 5 vertices in X , thus at least two vertices in X which are in $G' \setminus \{z, x\}$.

If one of $\{y_1, y_5\}$ is in X , then all vertices in X which are in $G' \setminus \{z, x\}$ are adjacent to x or z , thus y is connected to each of them by a 3-walk, we get a bigger clique with vertex set $X \cup \{y\}$, a contradiction.

If $y_1 \notin X, y_5 \notin X$. We claim that at least one of $\{y_4, y_6\}$ is in X , otherwise, note that now $X \subseteq V(G') \cup \{y_2, y_3\}$, none of $\{y_4, y_5, y_6\}$ is necessarily on a 3-walk connecting a pair of vertices in X , deleting them we get a smaller counterexample, a contradiction. It is same for $\{y_2, y_3\}$. Since at most one of $\{y_2, y_6\}$ could be in X , at most one of $\{y_3, y_4\}$ could be in X , assume w.l.o.g. that y_3 and y_6 are in X . Now none of $\{y_1, y_2\}$ is on a 3-walk connecting a pair of vertices in X , deleting them we get a smaller counterexample, a contradiction.

The proof of Theorem 3.3 is completed.

3.5 Other cases of Conjecture 3.1

Even though we have not proved the other cases of Conjecture 3.1, we get some partial results.

Claim 3.11. *Given a graph G of odd-girth at least $2k+1$ and a u, v -thread P of length l_p in G , denote the set of internal vertices of P by S . For any two vertices x, y in $V(G \setminus P)$, assume that in the graph induced by $V(G) \setminus S$, W is a walk connecting x and y of odd length l , P_1, P_2, P_3, P_4 , are shortest paths connecting x and v , y and v , x and u , y and u respectively, with P_i being of length of l_i , $1 \leq i \leq 4$, satisfying: $V(P_1) \cap V(P_2) = v$, $V(P_3) \cap V(P_4) = u$, $l_1 \equiv l_2 \pmod{2}$, $l_3 \equiv l_4 \pmod{2}$, and $l + l_i \not\leq 2k + 1$ for $1 \leq i \leq 4$. Then G has K_4 as a minor.*

Proof. First, we claim that none of the vertices of P is on W . To see this, suppose that $V(P) \cap V(W) \neq \emptyset$, thus there are two cases.

Case 1. $|V(W) \cap V(P)| = 1$. Assume w.l.o.g. that $V(W) \cap V(P) = \{v\}$. Then W together with P_1 and P_2 forms a closed walk W' . Since $l_1 \equiv l_2 \pmod{2}$, l is odd, W' is of odd length. Thus W' contains an odd-cycle C which is formed by part of P_1 , or part of P_2 , together with W . Since $l + l_i \not\leq 2k + 1$ for $1 \leq i \leq 4$, $|C| \leq l + \max\{l_1, l_2\} \leq 2k - 1$, which is contrary to that the odd-girth of G is at least $2k + 1$.

Case 2. $V(W) \cap V(P) = V(P)$. Then W must be formed by a x, v -walk, say W_1 , together with v, u -thread and a u, y -walk, say W_2 . Then W together with P_4 and P_3

forms a closed walk W' with length $l + l_3 + l_4$. If W_1 together with v, u -thread and P_3 forms a closed walk of odd length, then it contains an odd-cycle of length smaller than $l + l_3 \not\leq 2k + 1$, a contradiction. Thus W' contains an odd-cycle formed by W and part of P_4 , which is of length smaller than $l + l_4 \not\leq 2k + 1$, a contradiction.

Now, we prove that G contains a K_4 as a minor. Note that P_1 together with P and P_3 forms a cycle of length at least 3, denote it by C' . Let W start from y , denote the first vertex where W meets C' by w . Since $V(P) \cap V(W) = \emptyset$, w, u, v are distinct. By Lemma 3.7, G has a K_4 as a minor. \square

Next we generalize the Lemma 3.8 to general odd-girth.

Claim 3.12. *Let G be a K_4 -minor free graph of odd-girth $2k + 1$, $k \geq 1$. Let X be a set of vertices of a clique in $G^{(2k-1)}$. Then for any vertex w , $|N(w) \cap X| \leq 2$.*

Proof. The proof of this Lemma is very similar to Lemma 3.8. Suppose to the contrary that there exists a vertex w adjacent to 3 vertices x_1, x_2, x_3 in X . Since x_1 and x_2 are in X , there exists a $(2k - 1)$ -walk W connecting them in G . Then W together with x_2wx_1 walk forms a closed walk of length $2k + 1$. Thus it should contain an odd-cycle of length at most $2k + 1$. Since G is of odd-girth $2k + 1$, this closed walk should be a $(2k + 1)$ -cycle, denote it by $C_{x_1x_2}$. We claim that x_3 is not on $C_{x_1x_2}$. To see this, note that two neighbors of w in $C_{x_1x_2}$ are x_1 and x_2 . If x_3 is also one of vertices of $C_{x_1x_2}$, since x_3 is adjacent to w , we will have a smaller odd-cycle, a contradiction.

Similarly, since x_3 and x_1 are in X , there exists a $(2k - 1)$ -walk W_1 connecting them in G . Then W_1 together with x_1wx_3 walk forms a $(2k + 1)$ -cycle, denoted $C_{x_1x_3}$, and x_2 is not on $C_{x_1x_3}$. Denote by w_1 the first vertex where W_1 meets $C_{x_1x_2}$. Note that $w_1 \neq x_2, w$. Since W is of length $2k - 1$, the w_1x_1 -walk and w_1x_2 -walk in W have different parities. Without of losing generality, we assume that w_1x_1 -walk in W is of even length l_1 and w_1x_2 -walk in W is of odd length l_2 such that $l_1 + l_2 = 2k - 1$. In the one hand, since x_3w_1 -walk in W' together with w_1x_1 walk in W , which actually forms W_1 , is of an odd length $2k - 1$, x_3w_1 -walk in W_1 together with w_1x_2 walk in W forms a even walk. On the other hand, W_1 together with x_1wx_2 -walk forms a walk of length $2k + 1$ which is greater than $2k - 1$. We need another walk, say W_2 , of length $(2k - 1)$, connecting x_3 and x_2 in G . Similarly, W_2 together with x_2wx_3 -walk forms a $(2k + 1)$ -cycle, denoted $C_{x_2x_3}$, and x_1 is not on $C_{x_2x_3}$. Denote by w_2 the vertex where W_2 meets $C_{x_2x_3}$. Note that $w_2 \neq x_1, w$.

We claim that $w_1 \neq w_2$. To the contrary, let $w_1 = w_2$. Note that $x_2 \neq w_1 = w_2 \neq x_1$. Then W_1 together with x_1wx_2 walk and W_2 forms a closed walk of length $2k$, which

contains $C_{x_1x_2}$ of length $2k + 1$, thus there must exist an odd-cycle of at most $2k - 1$, a contradiction.

This completes the proof. □

3.6 Concluding remarks and further work

In this chapter, we consider the clique number of the $(2k - 1)$ -walk-powers of K_4 -minor free graphs of odd girth $2k + 1$. We conjectured that the upper bound is $\frac{(k+1)(k+2)}{2}$.

First, we constructed a graph such that this upper bound is achieved. Our result together with the result from [6] implies that order $\mathcal{O}(k^2)$ is optimal for the graphs of odd-girth that bounding all K_4 -minor free graphs of odd-girth $2k + 1$.

Then, we show that for $k = 2$, our conjecture holds. We hope to prove the general cases of the conjecture.

Chapter 4

Homomorphism and planar graphs

In this chapter, we will discuss some results related to Problem 1.14 and Conjecture 1.20. To make it easier to read, we state again the problems here.

Problem 4.1. [81] *Given integers $l \geq k \geq 1$, what are the minimal subgraphs of $PC(2k)$ to which every planar graph of odd-girth $2l + 1$ admits a homomorphism?*

In [81], Naserasr conjectured that $K(2k + 1, k)$, as a subgraph of $PC(2k)$, is an answer for the case $r = k + 1$. The first case of Problem 4.1 which is not being studied is $k = 3$ and $r = 5$. For this case, we conjecture that the Coxeter graph, which is a subgraph of $K(7, 3)$, bounds the planar graph of odd-girth at least 11. We state again the Conjecture 1.20 here.

Conjecture 4.2. [46] *Every planar graph of odd-girth at least 11 admits a homomorphism to the Coxeter graph.*

Supporting this conjecture, we prove Theorem 1.21. We state again this theorem here.

Theorem 4.3. [46] *Every planar graph of odd-girth at least 17 admits a homomorphism to the Coxeter graph.*

We will prove this theorem by contradiction and discharging technique. In section 4.1 we will present a list properties of Kneser graphs and Coxeter graph. In section 4.2, we present the folding lemma which is used to characterize the faces a plane graph and Euler formula which is used to give an initial charge. In section 4.3, we assume there exists a counterexample and give a list of ten reducible configurations. In section 4.4, we will use discharging technique to obtain a contradiction.

4.1 Kneser graphs and Coxeter graph

Recall that the Kneser graph $K(n, k)$, $n \geq 2k$, is defined to be the graph whose vertices corresponding to the k -element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Kneser graphs are named after Martin Kneser, who first investigated them in 1955. Kneser graph $K(2k-1, k-1)$ is the well-known *odd graph*, usually denoted $O(k)$.

It is well-known that Kneser Graph $K(n, k)$ is both vertex-transitive and edge-transitive. For the Kneser graph $K(n, k)$, if $n = 2k$, then $K(n, k)$ is a matching; if $k = 1$, then $K(n, k)$ is isomorphic to K_n . $K(5, 2)$ is isomorphic to the well-known Petersen graph and $O(3)$. $K(7, 3)$ is isomorphic to $O(4)$.

It is shown that $K(2k+1, k)$ is isomorphic to an induced subgraph of $PC(2k)$ (see [81]). Here we show this fact by giving an isomorphic mapping from $K(2k+1, k)$ to an induced subgraph of $PC(2k)$.

Proposition 4.4. $K(2k+1, k)$ is isomorphic to an induced subgraph of $PC(2k)$.

Proof. In the binary representation of $PC(2k)$, vertices are the elements of \mathbb{Z}_2^{2k} , $PC(2k) = (\mathbb{Z}_2^{2k}, \{e_1, e_2, \dots, e_{2k}, J\})$ where e_i 's are the standard basis of \mathbb{Z}_2^{2k} and J is the all 1 vector of length $2k$. Denote the set of all vertices with k 1's by U and the set of all vertices with $k-1$ 1's by V . Note that the subgraph induced by $U \cup V$ in $PC(2k)$ is of order $\binom{2k}{k} + \binom{2k}{k-1} = \binom{2k+1}{k}$. Moreover, in this induced subgraph, $x - y = J$ only if $x, y \in U$, $x - y = e_i$ only if one of $\{x, y\}$ is in U and the other is in V .

For any vertex u in U , we extend u to an element, denoted u' , of \mathbb{Z}_2^{2k+1} by the following operations: first, take a complement of u , which is $u + J$; second, add a $(2k+1)$ -th coordination with 0 to $u + J$. For any vertex v in V , we extend v to an element, denoted v' , of \mathbb{Z}_2^{2k+1} by adding a $(2k+1)$ -th coordination with 1 to v , denoted $u + e_{2k+1}$. Note that this is an one-to-one extension. Denote the set of u' by U' , the set of v' by V' .

Now we present the vertices of $K(2k+1, k)$ as the elements of \mathbb{Z}_2^{2k+1} . Denote the set of $2k+1$ elements by $\{1, 2, \dots, 2k+1\}$. Given a k -set, $\{i_1, i_2, \dots, i_k\}$, take the element of \mathbb{Z}_2^{2k+1} with i_j 'th coordination 1, $1 \leq j \leq k$, to correspond to it. It is easy to check that, the vertices of $K(2k+1, k)$ can be presented as vertices in $U' \cup V'$. Note that two vertices, say x and y , of $K(2k+1, k)$ are adjacent if and only if two corresponding sets are disjoint, if we see x, y as vertices in $U' \cup V'$, then x and y are adjacent if and only if $x - y = J' - e'_i$, here e'_i is one of the standard basis of \mathbb{Z}_2^{2k+1} and J' is the all 1 vector of length $2k+1$.

Now we give an isomorphic mapping, φ , from $U \cup V$ to $U' \cup V' = V(K(2k+1, k))$ as following:

$$\varphi(x) = x', \forall x \in U \cup V.$$

We need to show that for any two vertices x, y of $U \cup V$, $xy \in E(PC(2k))$ if and only if $x'y' \in U' \cup V'$. To see this, $x'y' \in U' \cup V'$ if and only if $x' - y' = J' - e'_i$.

$e'_i = e_{2k+1}$ if and only if x and y are in U and $x - y = J$. Thus $x' - y' = J' - e_{2k+1} \Leftrightarrow x - y = J$.

$e_i \neq e_{2k+1}$ if and only if one of x, y , say x , is in U and the other, y , is in V . Thus $x' - y' = J' - e'_i \Leftrightarrow (x + J - y) + e_{2k+1} = J' - e'_i \Leftrightarrow x + J - y = J - e_i \Leftrightarrow x - y = e_i$.

In total, we get that $xy \in E(PC(2k))$ if and only if $x'y' \in U' \cup V'$.

This completes the proof. \square

Next, following [36], we will give a definition of the Coxeter graph based on the Fano plane.

Given a set U of size 7, a *Fano plane* is a set of seven 3-subsets of U such that each pair of elements from U appears exactly in one 3-subset. It can be checked that there is a unique such collection up to isomorphism. This collection then satisfies the axioms of finite geometry and triples would be called lines. Throughout this chapter we will use the labeling of Figure 4.1 to denote the Fano plane.

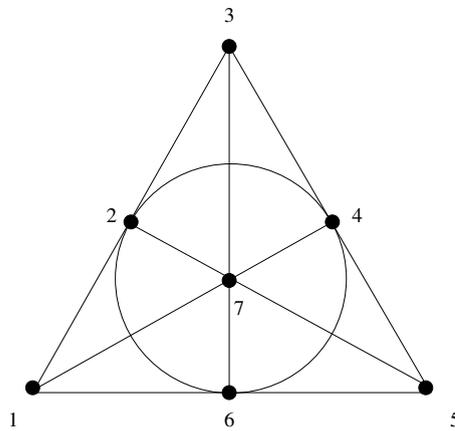


FIGURE 4.1: Fano plane

The Coxeter graph, denoted Cox , is a subgraph of $K(7, 3)$ obtained by deleting the vertices corresponding to the lines of the Fano plane. By Proposition 4.4, $K(7, 3)$ is an induced subgraph of $PC(6)$, therefore, Cox is an induced subgraph of $PC(6)$. Hence, we propose that Cox is an answer for the case $k = 3$ and $r = 5$ of Problem 4.1.

The Coxeter graph is well-known for its highly symmetric structure. There are many symmetric representations of it, but we will use the representation of Figure 4.2. Note that the labeling in Figure 4.2 is based on the labeling of the Fano plane given in

Figure 4.1. The main properties of this graph we will need are collected in the following lemma.

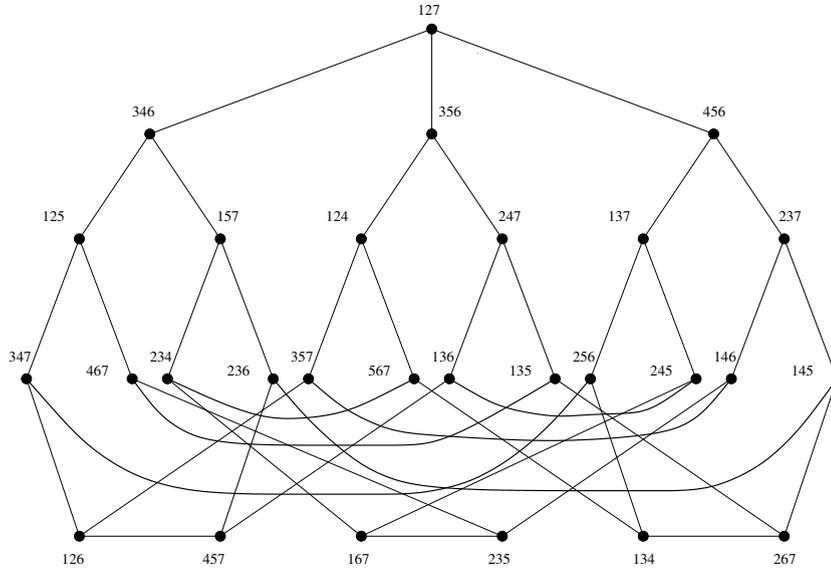


FIGURE 4.2: A representation of the Coxeter graph

Lemma 4.5. *The Coxeter graph satisfies the following:*

- (i) *It is distance-transitive.*
- (ii) *It is of diameter four.*
- (iii) *Its girth is seven.*
- (iv) *Given a vertex A , we have $|N(A)| = 3$, $|N_2(A)| = 6$, $|N_3(A)| = 12$ and $|N_4(A)| = 6$.*
- (v) *The independence number of Cox is 12.*
- (vi) *Let A and B be a pair of vertices in Cox. If $d(A, B) \leq 3$, then there exists a 7-cycle passing through A and B . If $d(A, B) = 4$, then there exists a 9-cycle passing through A and B .*
- (vii) *No homomorphic image of Cox is a proper subgraph of Cox.*
- (viii) *Given an edge A_1A_2 , there exist exactly two vertices B_1 and B_2 such that $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. Furthermore, B_1 and B_2 are adjacent vertices of Cox.*
- (ix) *Let \mathcal{A} and \mathcal{B} be two (not necessarily distinct) subsets of $V(\text{Cox})$ each of size at least 14. Then there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $AB \in E(\text{Cox})$.*
- (x) *For any two distinct vertices A and B of Cox we have $|N(A) \cap N(B)| \leq 1$.*

- (xi) For any pair A and C of vertices of Cox we have $|N_2(A) \cap N(C)| \leq 2$ with equality only when $A \sim C$.
- (xii) For any pair A and C of vertices of Cox we have $|N_3(A) \cap N_3(C)| \geq 4$. Furthermore, when equality holds, there does not exist a vertex B in $N_2(A)$ and a vertex D in $N_2(C)$ such that $N_3(A) \cap N_3(C) \subseteq N(B) \cup N(D)$.

Proof. The properties (i) through (v) are well known. We comment on the remaining seven.

- (vi) Since Cox is distance-transitive, assume w.o.l.g. that A is the vertex 127, B is one of 346, 125, 347, 126 of distance 1, 2, 3, 4 to A , respectively. Our claim holds following the 7-cycle $127 - 346 - 125 - 347 - 256 - 137 - 456 - 127$ and the 9-cycle $127 - 346 - 125 - 347 - 126 - 457 - 136 - 247 - 356 - 127$.
- (vii) For contradiction, let ϕ be a homomorphism of Cox to a proper subgraph of itself. Then ϕ must identify at least two vertices, say A_1 and A_2 . From (vi) we can see that, if $d(A_1, A_2) \leq 3$, then there exists a 7-cycle passing through A_1 and A_2 . Thus there exists an A_1 - A_2 path P of odd length at most 5. If $d(A_1, A_2) = 4$, then there exists a 9-cycle passing through A_1 and A_2 . Thus there exists an A_1 - A_2 path P of odd length 5. Hence, the image of P under ϕ contains a closed odd walk of length at most 7, contradicting (iii).
- (viii) Since Cox is edge-transitive, without loss of generality, we may assume that $A_1 = 127$ and $A_2 = 346$. It is then implied that $\{B_1, B_2\} = \{134, 267\}$.
- (ix) Suppose some subsets \mathcal{A} and \mathcal{B} provide a counter-example, and let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$. We may assume each of \mathcal{A} and \mathcal{B} is of size 14. Note that by connectivity of Cox, \mathcal{C} is not empty. Let $\mathcal{C}' = (\mathcal{A} \cup \mathcal{B})^c$. By our assumption, \mathcal{C} is an independent set of Cox, thus $|\mathcal{C}| \leq 12$. Furthermore, for each vertex C in \mathcal{C} all three neighbors of C are in \mathcal{C}' . Since Cox is 3-regular and $|\mathcal{C}'| = 28 - |\mathcal{A}| - |\mathcal{B}| + |\mathcal{C}| = |\mathcal{C}|$, for each C' of \mathcal{C}' , all three neighbors of C' are in \mathcal{C} , thus $\mathcal{C} \cup \mathcal{C}'$ induces a proper 3-regular subgraph of Cox, contradicting the connectivity of Cox.
- (x) For otherwise, a 4-cycle would appear in Cox.
- (xi) If A is not adjacent to C , then existence of two elements in $N_2(A) \cap N(C)$ would result in a cycle of length at most 6 which is a contradiction. If A is adjacent to C , then $N_2(A) \cap N(C) = N(C) \setminus \{A\}$.
- (xii) Using the distance-transitivity of Cox, this is proved by considering the five possibilities for $d(A, C)$. If $d(A, C) = 0$ then $|N_3(A) \cap N_3(C)| = |N_3(A)| = 12$. Since

for any B and D , $|N(B) \cup N(D)| \leq 6$, the last part of the statement holds. If $d(A, C) = 1$, we may assume $A = 127$ and $C = 346$. Then $N_3(A) \cap N_3(C) = \{567, 135, 256, 145\}$ and it is readily checked that each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement. If $d(A, C) = 2$, we may assume $A = 127$ and $C = 125$. Then $N_3(A) \cap N_3(C) = \{234, 236, 357, 146\}$ and it is readily checked that the vertex 146 is respectively at distances 1, 3, 3 from the vertices 357, 236, 234. This clearly implies the last part of the statement. If $d(A, C) = 3$, we may assume $A = 127$ and $C = 347$. Then $N_3(A) \cap N_3(C) = \{236, 567, 136, 135, 245, 146\}$. Finally, if $d(A, C) = 4$, we may assume $A = 127$ and $C = 126$. Then $N_3(A) \cap N_3(C) = \{467, 567, 245, 145\}$ and each of the second-neighbors of A has at most one neighbor among these four vertices, implying the last part of the statement.

□

4.2 Folding lemma and Euler formula

In this section, we present Folding lemma and Euler formula which will be used in the proof of Theorem 4.3.

In [58], Klostermeyer and Zhang proved a very useful lemma, namely Folding lemma.

Lemma 4.6 (Folding lemma). [58] *Let G be a plane graph of odd-girth $2k + 1$. If $C = v_0 v_1 \dots v_{r-1} v_0$ is a facial cycle of G with $r \neq 2k + 1$, then there exists an $i \in \{0, 1, \dots, r - 1\}$ such that the planar graph G' obtained from G by identifying v_{i-1} and $v_{i+1} \pmod{r}$ is of odd-girth $2k + 1$.*

Here is a direct corollary of Lemma 4.6.

Corollary 4.7. *Given a 2-connected planar graph G of odd-girth at least $2k + 1$, there is a homomorphic image G' of G such that G' is a plane graph of odd-girth $2k + 1$, and moreover every face of G' is a $(2k + 1)$ -cycle.*

The well-known Euler formula is one of the oldest mathematic formulas related to plane graphs, it was first established for polyhedral graphs by Euler in 1752.

Theorem 4.8 (Euler formula). [13] *If a finite, connected plane graph has V vertices, E edges and F faces, then*

$$V - E + F = 2.$$

4.3 Reducible configurations

We may refer to a mapping of a graph H to the Coxeter graph as a Cox-coloring of H . A *partial Cox-coloring* of H is a mapping from a subset of vertices of H to vertices of the Coxeter graph which preserves adjacency among the mapped vertices. Let H be a graph, ϕ be a partial Cox-coloring of H and u be a vertex of H not colored yet. We define $\text{ad}_{H,\phi}(u)$ to be the set of *admissible* colors for u , i.e., the set of distinct choices $A \in V(\text{Cox})$ such that the assignment $\phi(u) = A$ is extendable to a Cox-coloring of H . When H and ϕ are clear from the context, we will simply write $\text{ad}(u)$.

Our proof of Theorem 4.3 is based on the contradiction and the discharging technique. Suppose to the contrary that Theorem 4.3 is false and there is a planar graph of odd-girth 17 not mapping to Cox, we choose X to be such a graph with the smallest value of $|V(X)| + |E(X)|$. Hence, X is simple and no proper homomorphic image of X is in \mathcal{P}_{17} . Since Cox is a vertex-transitive graph, X is 2-connected. Hence, Corollary 4.7 implies that X has a plane embedding whose faces are all 17-cycles. We fix such an embedding and denote it also by X .

Given a subgraph T of X , let *boundary* of T , denoted $\text{Bdr}(T)$, be the set of vertices of T which have at least one neighbor in $X - T$. Let the *interior* of T be $\text{Int}(T) = T - \text{Bdr}(T)$.

Let $X_T = X - \text{Int}(T)$ be a subgraph of X induced by vertices not in $\text{Int}(T)$. If at least one Cox-coloring of X_T can be extended to a Cox-coloring of X , then $(T, \text{Bdr}(T))$ is called a *reducible* configuration. Each reducible configuration we will consider in this paper is a tree having all its leaf vertices as its boundary. Thus, we will simply use T to denote $(T, \text{Bdr}(T))$.

Note that by the minimality, X cannot contain any reducible configuration.

In this section, we provide a list of ten reducible configurations, all of which are trees of small order. Sometimes to prove that a configuration is reducible, we will consider smaller configurations and prove that most of the local Cox-colorings on the boundary are extendable.

Our first lemma is about paths. Given a u - v path P of length at most five we characterize all possible Cox-colorings of $\{u, v\}$ which are extendable to P .

Lemma 4.9. *Let P be a u - v path of length l , $2 \leq l \leq 5$. Consider a partial Cox-coloring ϕ given by $\phi(u) = A$ and $\phi(v) = B$. Then, ϕ is extendable to P if and only if:*

- (i) $l = 2$ and $d(A, B) \in \{0, 2\}$, or

(ii) $l = 3$ and $d(A, B) \in \{1, 3\}$, or

(iii) $l = 4$ and $d(A, B) \neq 1$, or

(iv) $l = 5$ and $A \neq B$.

Proof. Note that extending ϕ to P is actually finding a l -walk with two ends A and B in Cox. By Lemma 4.5, Cox is of girth seven, thus:

When $l = 2$, there is a 2-walk connecting A and B if and only if $d(A, B) \in \{0, 2\}$.

When $l = 3$, there is a 3-walk connecting A and B if and only if $d(A, B) \in \{1, 3\}$.

When $l = 4$, if $d(A, B) \in \{0, 2, 4\}$, it is easy to find a 4-walk connecting A and B ; if $d(A, B) = 3$, by (vi) of Lemma 4.5, there is a 7-cycle passing through A and B , thus there is a 4-walk connecting A and B ; if $d(A, B) = 1$, there is no 4-walk connecting A and B , otherwise there will be a closed walk of length 5 in Cox, a contradiction by (iii) of Lemma 4.5.

When $l = 5$, if $d(A, B) \in \{1, 2, 3\}$, by (vi) of Lemma 4.5, there is a 7-cycle passing through A and B , thus there is a 5-walk connecting A and B ; if $d(A, B) = 4$, then by (vi) of Lemma 4.5, there is a 9-cycle passing through A and B , thus there is a 5-walk connecting A and B ; if $d(A, B) = 0$, there is no 5-walk connecting A and B , otherwise there will be a closed walk of length 5 in Cox, a contradiction by (iii) of Lemma 4.5.

□

Before proceeding further, we give more notations. Given a graph G , a vertex of degree d is called a d -vertex. Analogously, a d^+ -vertex is a vertex whose degree is d or more. A *thread* in X is a path $P = ux_1x_2 \dots x_nv$ where all the internal vertices x_1, x_2, \dots, x_n are 2-vertices of X . We will also say that P is a u - v thread. The length of a thread is the number of its edges. Distinct vertices x and y are said to be *weakly adjacent* if there exists a thread in X containing both of them. Given a 3^+ -vertex x , the number of 2-vertices weakly adjacent to x is denoted by $d_{\text{weak}}(x)$.

Our first reducible configuration is the thread of length 6.

Proposition 4.10. *Any thread of length 6 is a reducible configuration.*

Proof. Let P be a thread of length 6 with the two end vertices u and v . We need to show that any Cox-coloring of $X - \text{Int}(P)$ can be extended. Let ϕ be a Cox-coloring of $X - \text{Int}(P)$, with $\phi(u) = A$ and $\phi(v) = B$. Choose a neighbor C of B distinct from A , this is possible because Cox is 3-regular. Let v' be the neighbor of v in P . Extend ϕ to v' by setting $\phi(v') = C$. Then by Lemma 4.9 (iv), ϕ is extendable to a Cox-coloring of P . □

By Proposition 4.10, we can see that the maximum length of a thread in X is at most 5. It follows immediately that:

Corollary 4.11. *Given a vertex v of X we have $d_{\text{weak}}(v) \leq 4d(v)$.*

Observe that, as a consequence of the fact that X is 2-connected, any 2-vertex x in X has exactly two weakly adjacent 3^+ -vertices. Thus, there exists a unique maximal thread having x as an internal vertex.

Proposition 4.12. *Given distinct 3^+ -vertices u and v of X , there exists at most one u - v thread.*

Proof. Suppose to the contrary that there are two such threads, say P and P' , of lengths l and l' , respectively. Since the length of each thread is at most 5, l and l' must have the same parity, otherwise there would be an odd-cycle of length less than 17 in $P \cup P'$. Without loss of generality, we may assume that $l \geq l'$. But then there exists a homomorphism $P \rightarrow P'$ that leaves u and v fixed, and hence there exists a homomorphism $X \rightarrow X - E(P)$, contradicting the fact that no proper homomorphic image of X is in \mathcal{P}_{17} . \square

For a path which is not a thread in X , it is not easy for us prove that whether it is a reducible configuration. Actually, what we need to know is that when we give a Cox-coloring of its two end vertices, how many ways can we extend this coloring to the interior vertices. Precisely, let ϕ be a partial Cox-coloring of a path P , with the two end vertices colored, for a interior vertex, say u , which is not colored, what is the number of $ad_{P,\phi}(u)$? This is given in the next two lemmas. We only consider the paths of length 5 or 6.

Lemma 4.13. *Let $P = xv_1v_2v_3v_4y$ be a 5-path. Let $\phi(x) = A$ and $\phi(y) = B$, with $B \neq A$, be a partial Cox-coloring. If $d(A, B) = 2$, then $|ad(v_1)| = |ad(v_2)| = 2$ with the two possible choices for v_2 being at distance three in Cox. Otherwise, $|ad(v_1)| = 3$ and $|ad(v_2)| \geq 4$.*

Proof. Since Cox is distance-transitive, the statement can be proven by considering the four possibilities for $d(A, B)$ and applying Lemma 4.9. If $d(A, B) = 1$, we may assume $B = 127$ and $A = 346$. Then $ad(v_1) = N(A)$ and $ad(v_2) = \{346, 356, 456, 347, 467, 234, 236\}$, hence $|ad(v_1)| = 3$ and $|ad(v_2)| = 7$. In case $d(A, B) = 2$, we may assume $B = 127$ and $A = 125$. Then $ad(v_1) = \{347, 467\}$ and $ad(v_2) = \{135, 256\}$, hence $|ad(v_1)| = 2$ and $|ad(v_2)| = 2$, with the two admissible colors for v_2 being at distance three in Cox. If $d(A, B) = 3$, we may assume $B = 127$ and $A = 347$. Then $ad(v_1) = N(A)$

and $\text{ad}(v_2) = \{346, 347, 467, 357\}$, hence $|\text{ad}(v_1)| = 3$ and $|\text{ad}(v_2)| = 4$. Finally, if $d(A, B) = 4$, we may assume $B = 127$ and $A = 126$. Then $\text{ad}(v_1) = N(A)$ and $\text{ad}(v_2) = \{236, 136, 256, 146\}$, hence $|\text{ad}(v_1)| = 3$ and $|\text{ad}(v_2)| = 4$. \square

Lemma 4.14. *Let $P = xv_1v_2v_3v_4v_5y$ be a 6-path. Let $\phi(x) = A$ and $\phi(y) = B$ be a partial Cox-coloring. If $d(A, B) = 1$, then $|\text{ad}(v_3)| = 4$, furthermore these four colors constitute the neighbors of an edge of Cox. If $d(A, B) \neq 1$, then $|\text{ad}(v_3)| \geq 8$.*

Proof. We again apply Lemma 4.9. First we consider the case of $d(A, B) = 1$. Since Cox is distance-transitive, we may assume without loss of generality that $A = 127$ and $B = 346$. In this case $\text{ad}(v_3) = \{567, 135, 256, 145\}$. Note that these are the neighbors of the edge $A'B'$, where $A' = 134$ and $B' = 267$. We note that each of A' and B' is at distance 4 from both A and B . This property uniquely determines the edge $A'B'$.

If $A = B$, then each vertex in $N(A) \cup N_3(A)$ is an admissible color for v_3 , and we have $|\text{ad}(v_3)| = 15$. If $d(A, B) = 2$, since Cox is distance-transitive, we may assume $A = 127$ and $B = 125$. In this case $\text{ad}(v_3) = \{346, 356, 456, 347, 467, 234, 236, 357, 146\}$. For the case of $d(A, B) = 3$ we may assume $A = 127$ and $B = 347$, thus $\text{ad}(v_3) = \{456, 256, 135, 245, 567, 146, 236, 136\}$. Finally, if $d(A, B) = 4$ we may assume $A = 127$ and $B = 126$. In this case we have $\text{ad}(v_3) = \{346, 356, 347, 357, 467, 567, 145, 245\}$. \square

To give some other reducible configurations, we need the following notations. We define $T_{k_1k_2\dots k_r}$ with $0 \leq k_1 \leq k_2 \leq \dots \leq k_r$ to be a graph obtained from $K_{1,r}$ by subdividing the ut_i -edge k_i times, where t_i 's are the leaf vertices and u is the central vertex of $K_{1,r}$. Given an r -vertex u , with $r \geq 3$, we will denote by $T(u)$ the union of all the threads in X which have u as an end-vertex. A direct consequence of Proposition 4.12 is that $T(u)$ is a $T_{k_1k_2\dots k_r}$ with $k_r \leq 4$. The next few lemmas are about the possibilities for $T(u)$ when u is of degree 3 or 4.

Lemma 4.15. *Let $T = T_{222}$. Then the partial Cox-coloring $\phi(t_i) = A_i$, $i = 1, 2, 3$ is extendable to T unless $\{A_1, A_2, A_3\}$ induces a 2-path in Cox.*

Proof. Consider the t_1 - t_2 path P in T . Let v be the middle vertex of this path and let \mathcal{A} be the set of colors whose assignment to v is extendable to P . We use the proof of Lemma 4.14 for the different values of $d(A_1, A_2)$. In three of these possibilities, to be precise, when $d(A_1, A_2) \neq 1, 2$, we have

$$N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox}).$$

Thus, in these cases any choice of A_3 is extendable.

If $d(A_1, A_2) = 1$, then

$$N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox}) \setminus N(\{A_1, A_2\}) \cup \{A_1, A_2\}.$$

Thus, in this case a choice of A_3 is extendable unless either $A_3 \sim A_1$ and $A_3 \neq A_2$ or $A_3 \sim A_2$ and $A_3 \neq A_1$.

Finally if $d(A_1, A_2) = 2$, then

$$N(\mathcal{A}) \cup N_3(\mathcal{A}) = V(\text{Cox}) \setminus \{B\},$$

where B is the common neighbor of A_1 and A_2 . Thus, in this case a choice of A_3 is extendable unless either $A_3 = B$.

□

Proposition 4.16. *The configurations T_{123} and T_{034} are reducible.*

Proof. We give a proof for T_{123} , the proof for T_{034} is similar. Denote the unique 3-vertex of T_{123} by u . Let X' be the subgraph of X obtained by deleting the interior of the u - t_3 thread. By the minimality of X , there is a Cox-coloring ϕ of X' . By Lemma 4.9, $\phi(t_1) \neq \phi(t_2)$. Now consider the Cox-coloring $\phi' = \phi|_{X - \text{Int}(T_{123})}$ of $X - \text{Int}(T_{123})$. By Lemma 4.13, there is an extension of ϕ' to the t_1 - t_2 path of T_{123} such that

- (i) there are two choices for $\phi'(u)$, say A_1, A_2 , with $d(A_1, A_2) = 3$ or
- (ii) there are at least four choices for $\phi'(u)$.

In case (i), we have $A_1 \approx \phi'(t_3)$ or $A_2 \approx \phi'(t_3)$, otherwise the $A_1 A_2$ -path of length 3 together with the path $A_1 \phi'(t_3) A_2$ forms a closed walk of length 5 in Cox, a contradiction. In case (ii), since Cox is 3-regular, there is a choice for $\phi'(u)$ such that $\phi'(u) \approx \phi'(t_3)$. Anyway, we can find a $\phi'(u) \approx \phi'(t_3)$. By Lemma 4.9, this ϕ' can be extended to the rest of T_{123} , which implies that T_{123} is reducible. □

Proposition 4.16 yields the following corollary.

Corollary 4.17. *If v is a 3-vertex in X , then $d_{\text{weak}}(v) \leq 6$. Furthermore, if $d_{\text{weak}}(v) = 6$, then $T(v)$ is one of the following trees: $T_{024}, T_{033}, T_{114}, T_{222}$.*

Proof. Denote the central vertex of $T_{k_1 k_2 k_3}$ by v .

If $d_{\text{weak}}(v) \geq 7$, then $k_3 \geq 3$. By Proposition 4.10, $k_3 \leq 4$. Thus $T_{k_1 k_2 k_3}$ must contain the reducible configurations T_{034} or T_{123} , a contradiction.

If $d_{\text{weak}}(v) = 6$, note that $k_3 \leq 4$, there are 5 possible configurations: T_{024} , T_{033} , T_{123} , T_{114} , T_{222} . Note that T_{123} is reducible, $T_{k_1 k_2 k_3} \neq T_{123}$.

□

Proposition 4.18. *The configurations T_{1334} , T_{2234} , T_{2333} are reducible.*

Proof. Let T be one of the three configurations, and denote its central vertex by v . Let ϕ be a Cox-coloring of $X - \text{Int}(T)$ with $\phi(t_1) = A_1, \phi(t_2) = A_2, \phi(t_3) = A_3, \phi(t_4) = A_4$.

First we assume $T = T_{1334}$. Using Lemma 4.9, we have

$$\text{ad}(v) = (\{A_1\} \cup N_2(A_1)) \setminus (N(A_2) \cup N(A_3) \cup \{A_4\}).$$

By Lemma 4.5 (xi), we have

$$|N_2(A_1) \cap N(A_2)| \leq 2, |N_2(A_1) \cap N(A_3)| \leq 2.$$

Thus $|\text{ad}(v)| \geq 2$, and $T = T_{1334}$ is reducible.

For the case of $T = T_{2234}$, using Lemma 4.9, we have

$$\text{ad}(v) = ((N(A_1) \cup N_3(A_1)) \cap (N(A_2) \cup N_3(A_2))) \setminus (N(A_3) \cup \{A_4\}).$$

By Lemma 4.14, if $d(A_1, A_2) \neq 1$, then we have

$$|(N(A_1) \cup N_3(A_1)) \cap (N(A_2) \cup N_3(A_2))| \geq 8,$$

thus $|\text{ad}(v)| \geq 8 - 4 = 4$. If $d(A_1, A_2) = 1$, then

$$|(N(A_1) \cup N_3(A_1)) \cap (N(A_2) \cup N_3(A_2))| = 4,$$

moreover, these four vertices constitute the neighbors of an edge of Cox. By Lemma 4.5 (xi), we have

$$|(N(A_1) \cup N_3(A_1)) \cap (N(A_2) \cup N_3(A_2)) \cap N(A_3)| \leq 2,$$

then $|\text{ad}(v)| \geq 2 - 1 = 1$. Thus $T = T_{2234}$ is reducible.

For the last case, i.e., $T = T_{2333}$, using Lemma 4.9, we have

$$\text{ad}(v) = (N(A_1) \cup N_3(A_1)) \setminus (N(A_2) \cup N(A_3) \cup N(A_4)).$$

Note that

$$|N(A_1) \cup N_3(A_1)| = 15, |N(A_2) \cup N(A_3) \cup N(A_4)| \leq 9,$$

we have $|\text{ad}(v)| \geq 6$. Thus $T = T_{2333}$ is reducible. \square

Corollary 4.19. *If v is a 4-vertex in X , then $d_{\text{weak}}(v) \leq 12$. Furthermore, if $d_{\text{weak}}(v) = 12$, then $T(v)$ is T_{0444} . Otherwise, $d_{\text{weak}}(v) \leq 11$.*

Proof. Denote $T(v) = T_{k_1 k_2 k_3 k_4}$.

If $d_{\text{weak}}(v) \geq 13$, then $k_4 \geq 4$. By Proposition 4.10, $k_4 \leq 4$, thus $k_4 = 4$. Then $T_{k_1 k_2 k_3 k_4}$ have 7 possible configurations: $T_{1444}, T_{2344}, T_{2444}, T_{3334}, T_{3344}, T_{3444}, T_{4444}$. By Proposition 4.18, $T_{1334}, T_{2234}, T_{2333}$ are reducible. Note that T_{1444} contains T_{1334} , we get that T_{1444} is reducible. Moreover, the other 6 configurations contains T_{2234} , thus they are all reducible. In total, we have $d_{\text{weak}}(v) \leq 12$.

If $d_{\text{weak}}(v) = 12$, Then $T_{k_1 k_2 k_3 k_4}$ have 5 possible configurations: $T_{0444}, T_{1344}, T_{2244}, T_{2334}, T_{3333}$. Note that T_{1344} contains T_{1334} and each of $T_{2244}, T_{2334}, T_{3333}$ contains T_{2234} , then T_{0444} is the only configuration in this case. \square

Let u and v be weakly adjacent 3-vertices. We now would like to investigate $T(u) \cup T(v)$ (see Figures 4.3 and 4.4 where the black vertices have degrees as depicted in the figures, whereas the white vertices have arbitrary degrees greater than 2).

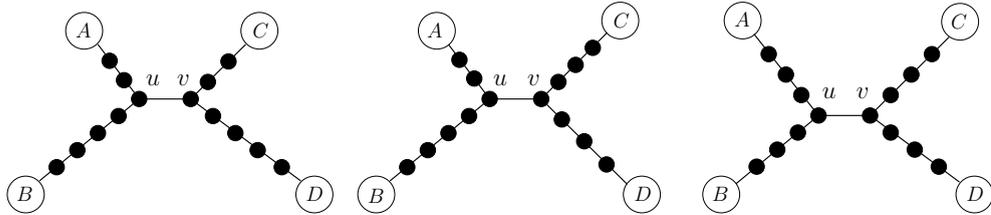


FIGURE 4.3: Reducible configurations of adjacent 3-vertices with a Cox-coloring of the boundary.

Proposition 4.20. *The three trees in Figure 4.3 are reducible.*

Proof. Denote by X' the graph obtained by deleting the edge uv in X . By the minimality, there is a Cox-coloring ϕ' of $X' - \text{Int}(T(u) \cup T(v))$, with the four vertices on the boundary, say x_1, x_2, x_3, x_4 , being colored A, B, C and D , respectively. We will extend this partial coloring to u and v , then to X .

For the first configuration, by Lemma 4.9,

$$\text{ad}(u) = (N(A) \cup N_3(A)) \setminus \{B\}, \text{ad}(v) = (N(C) \cup N_3(C)) \setminus \{D\}.$$

Thus

$$|\text{ad}(u)| \geq 14, \text{ad}(v) \geq 14.$$

By (ix) of Lemma 4.5, we can find a color in $\text{ad}(u)$ and a color in $\text{ad}(v)$ such that they are adjacent in Cox, thus ϕ' can be extended to X and $T(u) \cup T(v)$ is reducible.

For the second configuration, by Lemma 4.9,

$$\text{ad}(u) = (N(A) \cup N_3(A)) \setminus \{B\}, \text{ad}(v) = V(\text{Cox}) \setminus (N(C) \cup N(D)).$$

Thus

$$|\text{ad}(u)| \geq 14, \text{ad}(v) \geq 22.$$

For the third configuration, by Lemma 4.9,

$$\text{ad}(u) = V(\text{Cox}) \setminus (N(A) \cup N(B)), \text{ad}(v) = V(\text{Cox}) \setminus (N(C) \cup N(D)).$$

Thus

$$|\text{ad}(u)| \geq 22, \text{ad}(v) \geq 22.$$

Using the same argument, we can show that these two configurations are reducible. \square

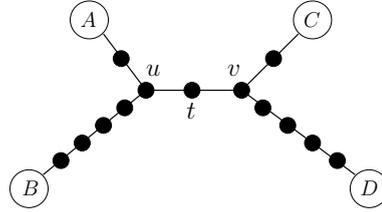


FIGURE 4.4: Reducible configuration with a Cox-coloring of the boundary.

Proposition 4.21. *The tree in Figure 4.4 is reducible.*

Proof. We can see this configuration as $T(u) \cup (T(v) \setminus \{u, v\})$. Consider a partial Cox-coloring, ϕ , of its leaf vertices with A, B, C, D , respectively. Now we calculate minimum number of $|\text{ad}_{\phi, T(u)}(v)|$, and $|\text{ad}_{\phi, (T(v) \setminus \{u, v\})}(v)|$.

First we calculate $|\text{ad}_{\phi, T(u)}(v)|$.

By Lemma 4.9, we have

$$\text{ad}_{\phi, T(u)}(u) = (\{A\} \cup N_2(A)) \setminus \{B\}.$$

If $B \notin (\{A\} \cup N_2(A))$, then

$$\text{ad}_{\phi, T(u)}(v) = (\{A\} \cup N_2(A)) \cup N_2((\{A\} \cup N_2(A))) = V(\text{Cox}) \setminus N(A), |\text{ad}_{\phi, T(u)}(v)| = 25.$$

If $B \in \{A\}$, then

$$\text{ad}_{\phi, T(u)}(v) = N_2(A) \cup N_2(N_2(A)) = V(\text{Cox}) \setminus N(A), |\text{ad}_{\phi, T(u)}(v)| = 25.$$

If $B \in N_2(A)$, by the vertex-transitivity of Cox, it is readily checked that

$$\text{ad}_{\phi, T(u)}(v) = V(\text{Cox}) \setminus (N(A) \cup (N_2(B) \cap N_3(A))), |N_2(B) \cap N_3(A)| = 2.$$

Thus we have $|\text{ad}_{\phi, T(u)}(v)| = 23$.

In sum, we have $|\text{ad}_{\phi, T(u)}(v)| \geq 23$.

Now, we calculate $|\text{ad}_{\phi, (T(v) \setminus \{u, v\})}(v)|$.

By Lemma 4.9, we have that

$$\text{ad}_{\phi, (T(v) \setminus \{u, v\})}(v) = (\{C\} \cup N_2(C)) \setminus \{D\},$$

thus

$$|\text{ad}_{\phi, (T(v) \setminus \{u, v\})}(v)| \geq 6.$$

Note that

$$|\text{ad}_{\phi, T(u)}(v)| + |\text{ad}_{\phi, (T(v) \setminus \{u, v\})}(v)| \geq 23 + 6 = 29 > |V(\text{Cox})|,$$

the partial coloring can be extended to v , and the configuration in Figure 4.4 is reducible. □

The next two configurations we consider are not reducible. But we show that, up to isomorphism, there is a unique Cox-coloring of the boundary which is not extendable to the interior. This implies, in particular, that if there exists a second choice for a color of one of the vertices on the boundary, then the coloring is extendable.

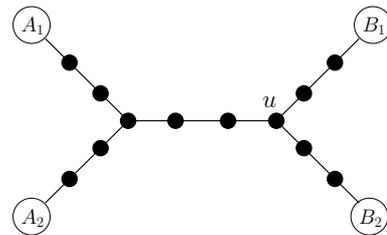


FIGURE 4.5: Configuration F_1 with a Cox-coloring of the boundary.

Proposition 4.22. *The partial Cox-coloring of the configuration F_1 given in Figure 4.5 is extendable to the whole configuration unless $d(A_1, A_2) = d(B_1, B_2) = 1$ and $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$.*

Proof. Consider the T_{222} configuration whose boundary consists of the vertex u and the two vertices colored with A_1 and A_2 , respectively. If $d(A_1, A_2) \neq 1$, then by Lemma 4.15, there are at least 27 choices of color for u which is extendable to the interior of this T_{222} . On the other hand, by Lemma 4.14, there are at least 4 choices of color for u that is extendable to a Cox-coloring of the partially colored 6-path connecting B_1 and B_2 . Thus, there are at least three common choices of color for u which is extendable on the whole configuration.

If $d(A_1, A_2) = 1$, then, again by Lemma 4.15, there are exactly four non-extendable choices of color for u for the considered T_{222} configuration. These particular four choices are the neighbors of A_1 and A_2 distinct from A_1 and A_2 . If any of the other 24 choices is extendable on the B_1 - B_2 path, then the coloring is extendable to the whole configuration. Otherwise, by the proof of Lemma 4.14, we have $d(B_1, B_2) = 1$ and, furthermore, $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$. \square

Corollary 4.23. *For the configuration F_1 of Figure 4.5, if the given partial Cox-coloring is not extendable to the whole configuration, then A_1 is uniquely determined by A_2 , B_1 and B_2 .*

Proof. Note that given an edge A_1A_2 , the property $d(A_i, B_j) = 4$ for $i, j \in \{1, 2\}$ determines a unique edge in Cox, as shown in Lemma 4.5 (viii). \square

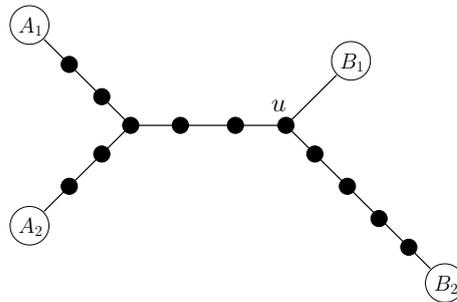


FIGURE 4.6: Configuration F_2 with a Cox-coloring of the boundary.

Proposition 4.24. *The partial Cox-coloring of the configuration F_2 given in Figure 4.6 is extendable to the whole configuration unless $d(A_1, A_2) = 1$ and $\{B_1, B_2\} = \{A_1, A_2\}$.*

Proof. The proof is similar to that of the previous proposition. Again we consider the possibilities on u . If $d(A_1, A_2) \neq 1$, then, by Lemma 4.15, there are at least 27 choices of color for u that would be extendable on the part connecting to A_1 and A_2 . Of these 27 vertices in Cox, at least two are neighbors of B_1 and of these two, one is distinct from B_2 . This color is an extendable choice.

If $d(A_1, A_2) = 1$, then the four neighbors of A_1 and A_2 , distinct from A_1 and A_2 , are the only choices for u that would make the coloring non-extendable on the left side. If $B_1 \notin \{A_1, A_2\}$, then there are at least two neighbors of B_1 whose assignments to u are extendable on the left side of u , and at least one of these two is different from B_2 . Thus, we may assume without loss of generality that $B_1 = A_1$. Then A_2 is an extendable choice for u unless $B_2 = A_2$. \square

Corollary 4.25. *If the partial Cox-coloring of the configuration F_2 given in Figure 4.6 is not extendable to the whole configuration, then A_1 is uniquely determined by A_2 , B_1 and B_2 .*

4.4 Discharging and further reducible configurations

In this section, we will discharging technique to get a contradiction. Recall that X , our minimal counterexample, is a 2-connected plane graph whose faces are all 17-cycles.

Since X is a plane graph, by Euler formula we have:

$$V - E + F = 2.$$

Here V, E, F are the number of vertices, edges and faces of X , respectively. Denote by $l(f)$ the length of face f . Then we have $\sum_f l(f) = 2E = \sum_v d(v)$.

Thus

$$E - (V + F) = -2 \Rightarrow 6E - 6(V + F) = -12 \Rightarrow 3 \cdot 2E - \left(\sum_v 6 + \sum_f 6 \right) = -12.$$

If we choose two real numbers α and β satisfying $\alpha + \beta = 3$, then

$$(\alpha + \beta) \cdot 2E - \left(\sum_v 6 + \sum_f 6 \right) = -12.$$

Furthermore, we have

$$\sum_v (\alpha \cdot d(v) - 6) + \sum_f (\beta \cdot l(f) - 6) = -12, \quad (4.1)$$

Note that all the faces of X are 17-cycles, thus $l(f) = 17$ for all f 's. If we set $\beta = \frac{6}{17}$, then $\beta \cdot l(f) - 6 = 0$, and identity 4.1 reduces to

$$\sum_v (45 \cdot d(v) - 102) + \sum_f 0 = -204. \quad (4.2)$$

We give each vertex v of X the following initial charge:

$$w_0(v) = 45 \cdot d(v) - 102.$$

Note that the total sum of the initial charge $\sum_{v \in V(X)} w_0(v) = -204 < 0$. Since X is 2-connected, there is no 1-vertex, and it is easy to check that each 2-vertex has initial charge -12 , each 3-vertex has initial charge 33 , each 4-vertex has initial charge 78 . To prove Theorem 4.3, we will redistribute the charges on the vertices so that at the final step, the charge on each vertex is non-negative. This contradicts that the total charge is negative and would disprove the existence of X . We will accomplish this through two phases of discharging.

In the first phase, we will take care of vertices of degree 2.

Then, in the second phase, we design a discharging rule that would take care of all negatively charged vertices after the first phase. We will then show that each configuration which may lead to a vertex of negative charge is reducible. This would complete our proof.

4.4.1 First phase of discharging

Here we use the following discharging rule:

- (R1) For each pair x, y of weakly adjacent vertices in X with $d(x) = 2$ and $d(y) \geq 3$, y sends charge of 6 to x .

Let $w_1(v)$ denote the new charge at each vertex v .

If $d(v) = 2$ then v receives a total charge of 12 (6 from each of its weakly adjacent 3^+ -vertices), hence $w_1(v) = 0$.

If $d(v) = 3$, then by Corollary 4.17, we have $w_1(v) \geq 33 - 6 \cdot 6 = -3$. Furthermore, if $d_{\text{weak}}(v) \neq 6$, then $w_1(v) \geq 33 - 5 \cdot 6 = 3$.

If $d(v) = 4$, we have $w_1(v) \geq 78 - 6 \cdot 12 = 6$ by Corollary 4.19.

If $d(v) \geq 5$, we have $w_1(v) \geq 45d(v) - 102 - 6 \cdot 4d(v) = 21d(v) - 102 \geq 3$ by Corollaries 4.11.

A vertex v of X is called *poor* if $w_1(v) < 0$. As a consequence of Corollary 4.17, we have the following characterization of poor vertices.

Proposition 4.26. *A vertex v of X is poor if and only if $d(v) = 3$ and $d_{\text{weak}}(v) = 6$.*

Corollary 4.17 also implies that for each poor vertex v , $T(v)$ is one of the following trees: T_{024} , T_{033} , T_{114} , T_{222} . Our aim is to seek charge for v from its closest leaf vertices of $T(v)$. Given a 3^+ -vertex $x \in V(X)$, we say x *supports* v if:

- (i) $w_1(x) > 0$, and
- (ii) x is a leaf vertex of $T(v)$ on a shortest thread of $T(v)$.

Note that each such thread has length at most 3. Furthermore, observe that v may have more than one supporting vertex x .

4.4.2 Second phase of discharging

In this phase we try to increase the charge of all poor vertices. The discharging rule is as follows:

- (R2) Whenever y supports a poor vertex x , then y gives charge of 3 to x if $d(x, y) = 1$, and charge of 1.5 to x if $d(x, y) \neq 1$.

Let $w_2(v)$ be the charge of an arbitrary vertex v after this phase. We will show that $w_2(v) \geq 0$, for every vertex v of X .

Observe that the charge of each 2-vertex v remains the same, i.e. $w_2(v) = 0$.

If v is a 5^+ -vertex, then by Corollary 4.11 we have

$$w_1(v) \geq w_0(v) - 24d(v) = 21d(v) - 102 \geq 3.$$

Furthermore, if v is a support for a vertex u , then the number of 2-vertices on the v - u thread is at most two. Thus, if v supports r vertices then

$$w_1(v) \geq w_0(v) - 6 \cdot 4 \cdot (d(v) - r) - 6 \cdot 2r \geq 21d(v) - 102 + 12r \geq 3 + 12r.$$

This implies that

$$w_2(v) \geq 3 + 12r - 3r \geq 3.$$

Now, assume v is a 4-vertex. By Corollary 4.19, unless $T(v) = T_{0444}$, we have $w_1(v) \geq 78 - 6 \cdot 11 = 12$ and this clearly gives

$$w_2(v) \geq w_1(v) - 4 \cdot 3 \geq 0.$$

If $T(v) = T_{0444}$, then v supports at most one vertex, and therefore

$$w_2(v) \geq 78 - 6 \cdot 12 - 3 = 3.$$

We are left to consider the case of a 3-vertex v , which supports at most 3 vertices.

If $d_{\text{weak}}(v) \leq 4$, then $w_2(v) \geq w_1(v) - 3 \cdot 3 \geq 33 - 6 \cdot 4 - 9 = 0$.

If $d_{\text{weak}}(v) = 5$ and v support only one poor vertex, then $w_2(v) \geq w_1(v) - 3 \geq 33 - 6 \cdot 5 - 3 = 0$.

The remaining two possibilities for v are as follows: either

- (1) $d_{\text{weak}}(v) = 6$, that means v is a poor vertex, or
- (2) $d_{\text{weak}}(v) = 5$ and v supports at least two poor vertices.

We will complete our proof by showing that:

- (i) If v is a poor vertex, then either v has an adjacent supporting vertex or it has at least two supporting vertices.
- (ii) If $d(v) = 3$ and $d_{\text{weak}}(v) = 5$, then when applying (R2) v sends total charge of at most 3.

To prove (i), note that by Corollary 4.17 and Proposition 4.26, $T(v)$ is one of the following trees: T_{024} , T_{033} , T_{114} , T_{222} .

If $T(v)$ is T_{024} or T_{033} , then v is adjacent to a 3^+ -vertex, say x . We claim that x is the adjacent supporting vertex of v . To see this, suppose by contradiction x is poor. Then, the union $T(x) \cup T(v)$ must be one of the configurations of Figure 4.3. But these are reducible configurations as shown in Proposition 4.20.

If $T(v)$ is T_{114} , there are two 3^+ -vertices at distance 2 from v . It remains to prove that neither of these two vertices is poor. By contradiction, suppose one of these two 3^+ -vertices, say x , is a poor vertex. Then $T(x)$ must be one of T_{024} , T_{033} , T_{114} , T_{222} . Since x and v are of distance 2, $T(x)$ must be T_{114} . Since each face of X is a 17-cycle, the union $T(x) \cup T(v)$ must be the configuration of Figure 4.4, which is shown to be reducible in Proposition 4.21.

If $T(v)$ is T_{222} , we prove that at most one vertex in $N_3(v)$ is poor. If $x \in N_3(v)$ is a poor vertex, then $T(x)$ must be either T_{222} or T_{024} . Then, the union $F = T(v) \cup T(x)$ is, respectively, the configuration of Figure 4.5 or the configuration of Figure 4.6. Let y be another vertex in $N_3(v)$. If y is also a poor vertex, then in $X - \text{Int}(F)$ the vertex y

is an internal vertex of an induced 5-path P . Thus, by Lemma 4.13, there are at least two choices for extending a Cox-coloring of $(X - \text{Int}(F)) - \text{Int}(P)$ to y , one of which is extendable to a Cox-coloring of X by Corollary 4.23 or Corollary 4.25.

To prove (ii), we begin by observing that $T(v)$ must be one of the configurations: $T_{014}, T_{023}, T_{113}, T_{122}$.

If $T(v)$ is T_{014} , we need to prove that v supports at most 1 poor vertex. Suppose to the contrary that v supports 2 poor vertices, say x_1 and x_2 , then x_1 and x_2 must be at distance 1 and 2 to v , respectively. Assume without loss of generality that $d(v, x_1) = 1$. Since x_1 is a poor vertex, $T(x_1)$ must be in $\{T_{024}, T_{033}\}$. Then $T(v) \cup T(x_1)$ has only two such possible configurations, shown in Figure 4.7. In this figure, the vertex in square is the poor vertex x_1 . We claim that each configuration of Figure 4.7 is reducible. To prove this, we need to show that any partial Cox-coloring, ϕ , of the leaf vertices A, B, C, D can be extended to the interior of $T(v) \cup T(x_1)$.

For the first configuration, note that it can be seen as $T(x_1) \cup (T(v) \setminus x_1)$, v is a leaf vertex of $T(x_1)$ and an internal vertex of $T(v) \setminus x_1$. Now we calculate the minimum number of $|\text{ad}_{\phi, T(x_1)}(v)|$ and $|\text{ad}_{\phi, T(v) \setminus x_1}(v)|$. By Lemma 4.9 we have that

$$\text{ad}_{\phi, T(x_1)}(x_1) = (N(A) \cup N_3(A)) \setminus \{B\}, \text{ad}_{\phi, T(x_1)}(v) = N((N(A) \cup N_3(A)) \setminus \{B\}).$$

Using the vertex-transitivity of Cox, it is readily observed that

$$N((N(A) \cup N_3(A))) = \{A\} \cup N_2(A) \cup N_3(A) \cup N_4(A).$$

Note that if $B \notin (N(A) \cup N_3(A))$, then

$$\text{ad}_{\phi, T(x_1)}(v) = \{A\} \cup N_2(A) \cup N_3(A) \cup N_4(A), |\text{ad}_{\phi, T(x_1)}(v)| = 25.$$

If $B \in N(A)$, then

$$\text{ad}_{\phi, T(x_1)}(v) = N((N(A) \cup N_3(A)) \setminus \{B\}) = N((N(A) \cup N_3(A))), |\text{ad}_{\phi, T(x_1)}(v)| = 25.$$

If $B \in N_3(A)$, then

$$\text{ad}_{\phi, T(x_1)}(v) = N((N(A) \cup N_3(A)) \setminus \{B\}) = N((N(A) \cup N_3(A)) \setminus \{C : C \in N_3(A) \cap N(B)\}).$$

Note that $|N_3(A) \cap N(B)| = 1$, then $|\text{ad}_{\phi, T(x_1)}(v)| = 24$.

In sum, $|\text{ad}_{\phi, T(x_1)}(v)| \geq 24$.

By Lemma 4.9, we have

$$\text{ad}_{\phi, T(v) \setminus x_1}(v) = (\{C\} \cup N_2(C)) \setminus \{D\},$$

thus $|\text{ad}_{\phi, T(v) \setminus x_1}(v)| \geq 6$.

Note that

$$|\text{ad}_{\phi, T(x_1)}(v)| + |\text{ad}_{\phi, T(v) \setminus x_1}(v)| \geq 30 > |V(\text{Cox})|,$$

there is a good common choice for coloring v , which implies that ϕ can be extended to the interior of $T(v) \cup T(x_1)$ and the first configuration is reducible.

For the second configuration, by the same method and notations and by Lemma 4.9, we have that

$$\text{ad}_{\phi, T(x_1)}(x_1) = V(\text{Cox}) \setminus \{N(A) \cup N(B)\}.$$

Note that,

$$\text{ad}_{\phi, T(x_1)}(v) = N(\text{ad}_{\phi, T(x_1)}(x_1)),$$

a vertex C is in $\text{ad}_{\phi, T(x_1)}(v)$ if and only if $N(C) \not\subseteq \{N(A) \cup N(B)\}$. By (x) of Lemma 4.5, if $C \notin \{A, B\}$, then $|N(C) \cap N(A)| \leq 1$ and $|N(C) \cap N(B)| \leq 1$, thus

$$|N(C) \cap (N(A) \cup N(B))| \leq 2,$$

which implies that $C \in \text{ad}_{\phi, T(x_1)}(v)$. We have $\text{ad}_{\phi, T(x_1)}(v) = V(\text{Cox}) \setminus \{A, B\}$ and $|\text{ad}_{\phi, T(x_1)}(v)| \geq 26$. Note that $|\text{ad}_{\phi, T(v) \setminus x_1}(v)| \geq 6$, we have

$$|\text{ad}_{\phi, T(x_1)}(v)| + |\text{ad}_{\phi, T(v) \setminus x_1}(v)| \geq 32 > |V(\text{Cox})|.$$

This implies that the second configuration is reducible.

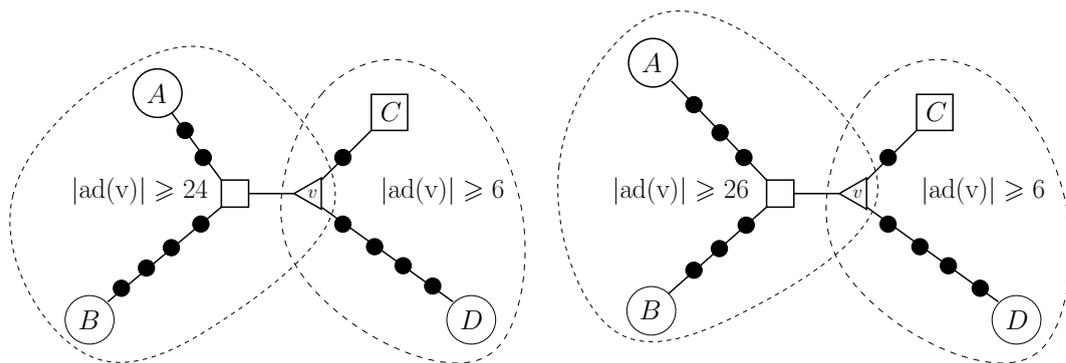
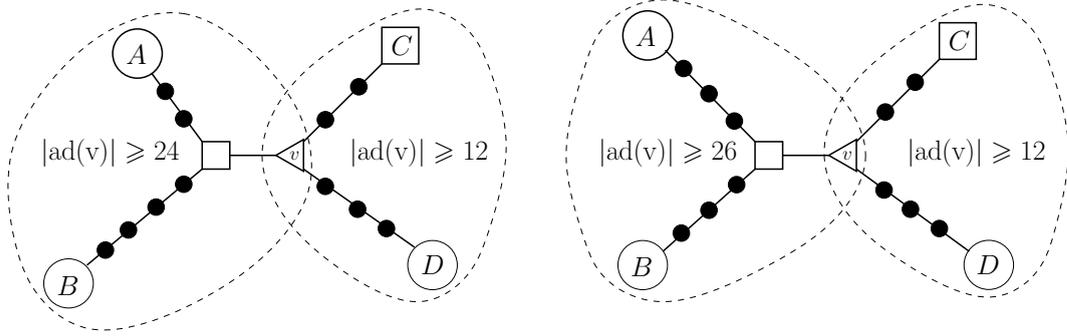


FIGURE 4.7: Local configurations of a center of T_{014} supporting two poor vertices.

If $T(v)$ is T_{023} , using the Figure 4.8, a similar argument is applied.

If $T(v)$ is T_{113} , then v is a support of at most two poor vertices (leave vertices at distance 2 to v) to each of which it may send charge of 1.5. Hence, in the second phase v gives as support at most charge of +3.

FIGURE 4.8: Local configurations of a center of T_{023} supporting two poor vertices.

Finally, assume that $T(v)$ is T_{122} . By (R_2) , v gives each of its supported vertex of charge 1.5. Note that $w_1(v) = 3$, if $w_2(v) < 0$, v supports 3 poor vertices, namely three leaf vertices of $T(v)$, denoted x_1, x_2, x_3 . Assume without loss of generality that $d(v, x_1) = 2$, $d(v, x_2) = d(v, x_3) = 3$. Note that $T(x_i) \in \{T_{024}, T_{033}, T_{114}, T_{222}\}$, $T(x_1)$ must be T_{114} , $T(x_2)$ and $T(x_3)$ must be T_{222} . In this case, we have a unique local configuration, given in Figure 4.9, we can see it by $T(x_1) \cup T(x_2) \cup P_{vx_3}$, here P_{vx_3} is the thread connecting v and x_3 . We claim that $T(x_1) \cup T(x_2) \cup P_{vx_3}$ is reducible. To see this, consider a partial Cox-coloring, ϕ , of its leaf vertices with A, B, C, D, F , respectively. Now we calculate minimum number of $|\text{ad}_{\phi, T(x_1)}(v)|$, $|\text{ad}_{\phi, T(x_2)}(v)|$ and $|\text{ad}_{\phi, P_{vx_3}}(v)|$. First we calculate $|\text{ad}_{\phi, T(x_1)}(v)|$.

By Lemma 4.9, we have

$$\text{ad}_{\phi, T(x_1)}(x_1) = (\{A\} \cup N_2(A)) \setminus \{B\}.$$

If $B \notin (\{A\} \cup N_2(A))$, then

$$\text{ad}_{\phi, T(x_1)}(v) = (\{A\} \cup N_2(A)) \cup N_2((\{A\} \cup N_2(A))) = V(\text{Cox}) \setminus N(A), |\text{ad}_{\phi, T(x_1)}(v)| = 25.$$

If $B \in \{A\}$, then

$$\text{ad}_{\phi, T(x_1)}(v) = N_2(A) \cup N_2(N_2(A)) = V(\text{Cox}) \setminus N(A), |\text{ad}_{\phi, T(x_1)}(v)| = 25.$$

If $B \in N_2(A)$, by the vertex-transitivity of Cox, it is readily checked that

$$\text{ad}_{\phi, T(x_1)}(v) = V(\text{Cox}) \setminus (N(A) \cup (N_2(B) \cap N_3(A))),$$

and $|N_2(B) \cap N_3(A)| = 2$. We have $|\text{ad}_{\phi, T(x_1)}(v)| = 23$.

In sum, we have $|\text{ad}_{\phi, T(x_1)}(v)| \geq 23$.

By Lemma 4.15, $|\text{ad}_{\phi, T(x_2)}(v)| \geq 28 - 4 = 24$.

By Lemma 4.9, $|\text{ad}_{\phi, P_{vx_3}}(v)| \geq 15$. It is easy to check that

$$|\text{ad}_{\phi, T(x_1)}(v) \cap \text{ad}_{\phi, T(x_2)}(v) \cap \text{ad}_{\phi, P_{vx_3}}(v)| \geq 6.$$

Thus there is a common choice for v .

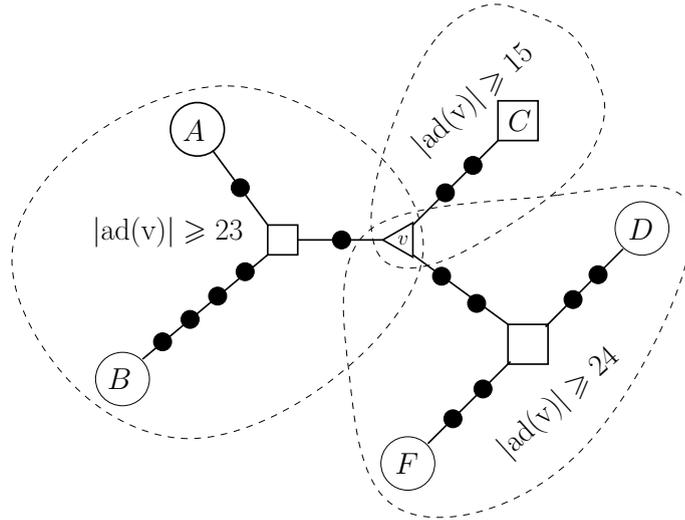


FIGURE 4.9: Local configuration of a center of T_{122} supporting three poor vertices.

4.5 Concluding remarks and further work

We have shown in this paper that one may use the existence of a combinatorial design to propose answer for special cases of the Problem 4.1. Our primary concern in this paper was the case $r = 5$ and $k = 3$ of this question and we proposed an answer using the Fano plane. At a 2011 summer workshop in Prague, Peter Cameron has proposed a similar conjecture for the case of $r = 7$ and $k = 5$ based on the existence of a unique Steiner quintuple system of order 11.

The condition of odd-girth 17 was used only when applying Euler formula, indeed each of the 15 reducible configurations we used in our proof is a tree. Thus if X is a minimal graph which admits no homomorphism to Cox (i.e., every proper subgraph admits a homomorphism to Cox), then X does not contain any of these reducible configurations. We believe that with a larger set of reducible trees and together with cumbersome discharging steps we can improve the result for odd-girth 15. However, it seems that to prove the conjecture using the discharging technique, if possible at all, one has to consider reducible configurations that involve cycles.

Chapter 5

Locating any two vertices on Hamiltonian cycles

In this chapter we give a proof of Conjecture 1.47 (Enomoto's conjecture) for graphs of sufficiently large order. We recall Enomoto's conjecture here.

Conjecture 5.1. [39] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

Our main result is the Theorem 1.52. We state again this theorem here.

Theorem 5.2. [48] *There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

The main tools of our proof are Regularity Lemma of Szemerédi and Blow-up Lemma of Komlós et al. In section 5.1, we will introduce these lemmas and some applications of them.

In this chapter, we use a new notation of degree of a vertex. For any vertex v of G and a subset X of $V(G)$, we denote the degree of v in G by $\text{deg}_G(v)$ and the degree of v in X by $\text{deg}_G(v, X)$ (if no ambiguity arises, we denote them by $\text{deg}(v)$ and $\text{deg}(v, X)$ respectively).

5.1 Regularity Lemma and Blow-up Lemma

In this section we introduce Regularity Lemma and Blow-up Lemma.

The Regularity Lemma, Szemerédi's Regularity Lemma [96], is a powerful tool of graph theory. It was invented as an auxiliary lemma in the proof of the famous conjecture of Erdős and Turán [27] which states that sequences of integers of positive upper density must always contain long arithmetic progressions.

To introduce Regularity Lemma and Blow-up Lemma, we first give some definitions about the regular pairs and some related results.

5.1.1 Regular pairs and related properties

Given a graph G , let X and Y be two disjoint sets of vertices of G . In particular, G could be a bipartite graph with vertex classes X and Y . We define the *density*, $d(X, Y)$, of the pair (X, Y) as the ratio

$$d(X, Y) := \frac{e_G(X, Y)}{|X||Y|},$$

here $e_G(X, Y)$ is defined to be the number of edges in G with one end vertex in X and the other in Y , if no ambiguity arises, we write $e(X, Y)$ instead of $e_G(X, Y)$.

Let $\epsilon > 0$. Given two disjoint vertex sets $X \subseteq V(G), Y \subseteq V(G)$ we say the pair (X, Y) is ϵ -regular if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| > \epsilon|X|$ and $|B| > \epsilon|Y|$ we have

$$|d(A, B) - d(X, Y)| < \epsilon.$$

From the definition we can see that the edges between an ϵ -regular pair are distributed fairly uniformly. Moreover, most of vertices of one part have a fairly large number of neighbors in the other part, we can see this from the following lemma.

Lemma 5.3. [61] *Let (A, B) be an ϵ -regular pair of density d and $Y \subseteq B$ such that $|Y| > \epsilon|B|$. Then all but at most $\epsilon|A|$ vertices in A have more than $(d - \epsilon)|Y|$ neighbors in Y .*

The following lemma says that subgraphs of regular pairs with reasonable size are also regular.

Lemma 5.4 (Slicing Lemma). [61] *Let $\alpha > \epsilon > 0$ and $\epsilon' := \max\{\frac{\epsilon}{\alpha}, 2\epsilon\}$. Let (A, B) be an ϵ -regular pair with density d . Suppose $A' \subseteq A$ such that $|A'| \geq \alpha|A|$, and $B' \subseteq B$ such that $|B'| \geq \alpha|B|$. Then (A', B') is an ϵ' -regular pair with density d' such that $|d' - d| < \epsilon$.*

The Slicing Lemma tells us that not too small subgraphs of an ϵ -regular pair are also regular with density close to that of the original pair. Sometimes, in some situations,

we only consider some of the vertices in an ϵ -regular pair, we hope that as many as possible the properties of the original pair do not just disappear. For this, we consider an application of the Slicing Lemma which links the notation of regularity to that of super-regularity.

Given a graph G and disjoint vertex sets $X, Y \subseteq V(G)$ let $\epsilon, \delta > 0$, the pair (X, Y) is (ϵ, δ) -super-regular if it is ϵ -regular, and $\deg_Y(x) > \delta|Y|$ for all $x \in X$ and $\deg_X(y) > \delta|X|$ for all $y \in Y$.

From next lemma we can see that: given a regular pair we can approximate it by a super-regular pair.

Lemma 5.5. [98] *If (X, Y) is an ϵ -regular pair with density d in a graph G (where $0 < \epsilon < \frac{1}{3}$), then there exists $A' \subseteq A, B' \subseteq B$ with $|A'| \geq (1 - \epsilon)|A|$ and $|B'| \geq (1 - \epsilon)|B|$, such that (A', B') is a $(2\epsilon, d - 3\epsilon)$ -super-regular pair.*

For a bipartite graph $G = X \cup Y$, let $\delta(X, Y) := \min_{x \in X} \{\deg_G(x, Y)\}$. We can say that a bipartite graph with very large minimum degree has a super-regular pair.

Lemma 5.6. [16] *Given $0 < \rho < 1$, let $G = X \cup Y$ be a bipartite graph such that $\delta(X, Y) \geq (1 - \rho)|Y|$ and $\delta(Y, X) \geq (1 - \rho)|X|$. Then (X, Y) is $(\sqrt{\rho}, 1 - \rho)$ -super-regular.*

5.1.2 Regularity Lemma and Blow-up Lemma

Now we introduce Szemerédi's Regularity Lemma.

First we state the original version of the Regularity Lemma.

Theorem 5.7. [96] *For every $\epsilon > 0$ and every $m \in \mathbb{N}$, there exist two integers $M(\epsilon, m)$ and $N(\epsilon, m)$ such that every graph G of order $n \geq N(\epsilon, m)$ admits a partition $\{V_0, V_1, \dots, V_k\}$ of $V(G)$ such that:*

- (i) $m \leq k \leq M$,
- (ii) $0 \leq |V_0| \leq \epsilon|G|$,
- (iii) $|V_1| = |V_2| = \dots = |V_k|$,
- (iv) all but at most ϵk^2 of the pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ϵ -regular.

The classes V_i 's are usually called *clusters* and the partition described is called an ϵ -regular partition. Note that V_0 may be empty, we call it an exceptional set because its role is purely technical: to make possible that all other clusters have exactly the same cardinality. The role of m is to make the clusters V_i 's sufficiently small, so that the number of edges inside those clusters are negligible.

In the applications of the original Regularity Lemma, one usually, in the first step, apply the lemma to create a regular partition, then in the second step, get rid of all edges except the edges between regular pairs of high enough densities, the leftover graphs still contain most of the original edges, and much easier to handle. This process leads the degree form of the Regularity Lemma, which is more applicable.

Lemma 5.8 (Regularity Lemma-Degree Form). *For every $\epsilon > 0$ and every integer m_0 there is an $M_0 = M_0(\epsilon, m_0)$ such that if $G = (V, E)$ is any graph on at least M_0 vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $l + 1$ clusters V_0, V_1, \dots, V_l , and there is a subgraph $G' = (V, E')$ with the following properties:*

- (1) $m_0 \leq l \leq M_0$;
- (2) $|V_0| \leq \epsilon|V|$ for $0 \leq i \leq l$, and $|V_1| = |V_2| = \dots = |V_l| = L$;
- (3) $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$ for all $v \in V$;
- (4) $G'[V_i] = \emptyset$ (i.e. V_i is an independent set in G') for all i ;
- (5) each pair (V_i, V_j) , $1 \leq i < j \leq l$, is ϵ -regular, each with a density either 0 or at least d .

We refer to [61] for more versions of the Regularity Lemma.

How to apply the Regularity Lemma? In the earlier time, many applications of the Regularity Lemma are concerned with embedding and packing problem, there are basically three steps to apply the lemma (suppose we try to embed a graph H into G).

Step 1. Apply Theorem 5.8 to G with suitable parameters ϵ and d , and construct a reduced graph which is defined as following:

Let G be a graph and V_1, V_2, \dots, V_r a partition of $V(G)$. Given two parameters $\epsilon > 0$ and $d \in [0, 1)$, we define the *reduced graph* R of G as follows: its vertices are the clusters V_1, V_2, \dots, V_r and there exists an edge between V_i and V_j precisely when (V_i, V_j) is ϵ -regular with density more than d . Let $R(t)$ be a graph obtained from R by replacing the edges of R by copies of $K_{t,t}$.

Step 2. Find a graph containing H in $R(t)$.

Step 3. Apply the Key Lemma to imply that $H \subseteq G$ as required. Key Lemma is given in [61].

Theorem 5.9 (Key Lemma). [61] *Given $d > \epsilon > 0$, a graph R , and a positive integer m , let us construct a graph G by replacing every vertex of R by m vertices, and replacing the edges of R with ϵ -regular pairs of density at least d . Let $R(t)$ be a graph obtained from R by replacing the edges of R by copies of $K_{t,t}$. Let H be a subgraph of $R(t)$ with h vertices and maximum degree $\Delta > 0$, and let $\delta := d - \epsilon$ and $\epsilon_0 := \frac{\delta \Delta}{2 + \Delta}$. If $\epsilon \leq \epsilon_0$ and $t - 1 \leq \epsilon_0 m$, then $H \subseteq G$.*

Remark: The order of R plays no role here. The order of H is smaller than the order of G .

An application of the Regularity Lemma is about embedding spanning graphs into dense graphs. Some of the proofs use a powerful tool — Blow-up Lemma. Basically, it sees the super-regular pairs as complete bipartite graphs.

Theorem 5.10 (Blow-up Lemma [59]). *Given a graph R of order r and positive parameters δ, Δ , there exists an $\epsilon > 0$ such that the following holds.*

Let n_1, n_2, \dots, n_r be arbitrarily positive integers and let us replace the vertices of R with pairwise disjoint sets V_1, V_2, \dots, V_r of sizes n_1, n_2, \dots, n_r (respectively, blowing up). We construct two graphs on the same vertex set $V = \cup_{i=1}^r V_i$. The first graph \mathbf{R} is obtained by replacing each edge $v_i v_j$ of R with the complete bipartite graphs between the corresponding vertex sets V_i and V_j . A sparser graph G is constructed by replacing each edge $v_i v_j$ of R with an (ϵ, δ) -super-regular pair between the corresponding vertex sets V_i and V_j . If a graph H with $\Delta(H) \leq \Delta$ is embeddable into \mathbf{R} , then it is already embedded into G .

Note that for embedding spanning subgraphs, one needs all degrees of the host graphs are large enough, that's why we use super-regular pairs while not the regular pairs when we construct the second graph G in the above theorem. In total, while Key Lemma plays a role in embedding smaller graphs H into G , Blow-up Lemma plays the role in embedding spanning graphs H into G .

The following special case of the Blow-up Lemma, Bipartite Version of Blow-up Lemma, which also restricts the mappings of a smaller number of vertices.

Lemma 5.11 (Blow-up Lemma-Bipartite Version [59]). *For every $\delta, \Delta, c > 0$, there exists an $\epsilon = \epsilon(\delta, \Delta, c) > 0$ and $\alpha = \alpha(\delta, \Delta, c) > 0$ such that the following holds. Let (X, Y) be an (ϵ, δ) -super-regular pair with $|X| = |Y| = N$. If a bipartite graph H with $\Delta(H) \leq \Delta$ can be embedded in $K_{N,N}$ by a function ϕ , then H can be embedded in (X, Y) . Moreover, in each $\phi^{-1}(X)$ and $\phi^{-1}(Y)$, fix at most αN special vertices z , each of which is equipped with a subset S_z of X or Y of size at least cN . The embedding of H into (X, Y) exists even if we restrict the image of z to be S_z for all special vertices z .*

Actually, the following special case of the Blow-up Lemma is frequently used in this thesis.

Lemma 5.12. *For every $\delta > 0$ there are $\epsilon_{BL} = \epsilon_{BL}(\delta)$, $n_{BL} = n_{BL}(\delta) > 0$ such that if $\epsilon \leq \epsilon_{BL}$ and $N \geq n_{BL}$, $G = (X, Y)$ is an (ϵ, δ) -super-regular pair with $|X| = |Y| = N$, $x_1, x_2 \in X$ ($x_1 \neq x_2$), $y_1, y_2 \in Y$ ($y_1 \neq y_2$) and l^i is an even integer with $4 \leq l^i \leq 2N - 4$*

($i = 1, 2$), $l^1 + l^2 = 2N$, then there are two vertex-disjoint paths P_1 and P_2 in G such that the end vertices of P_i are x_i, y_i and $|V(P_i)| = l^i$ ($i = 1, 2$).

Proof. Let $X^* = X - \{x_1, x_2\}$, $Y^* = Y - \{y_1, y_2\}$ and $H = H_1 \cup H_2$ be the union of two vertex-disjoint paths H_1, H_2 satisfied $|V(H_i)| = l^i - 2$ ($i = 1, 2$). It is not hard to see that H can be embedded in $K_{N-2, N-2}$. By Slicing Lemma, we know that (X^*, Y^*) is also a super-regular pair. Fix the end vertices of H_1 and H_2 to be the special vertices. For H_i , one of its end vertices is equipped with the neighbor set of x_i and the other end vertex is equipped with the neighbor set of y_i ($i = 1, 2$). By Lemma 5.11, H can be embedded in (X^*, Y^*) satisfying the restrictions of the special vertices. Since one of the end vertices of H_i is a neighbor of x_i and the other end vertex of H_i is a neighbor of y_i , we can extend H_i to a path P_i with end vertices x_i and y_i ($i = 1, 2$). Then $P_1 \cup P_2$ is a spanning subgraph of G and $|V(P_i)| = l^i$ ($i = 1, 2$). \square

5.1.3 Some applications of Regularity Lemma

Early applications of Regularity Lemma include Ramsey-Turán theory, generalized random graphs, building packing with small graphs, et. al.

In 1992, Alon and Yuster [2] proved the following theorem:

Theorem 5.13. [2] *For any $\alpha > 0$ and graph H , there is an n_0 such that*

$$n \geq n_0, \quad \delta(G_n) > \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$$

imply that there are $\frac{(1-\alpha)n}{|V(H)|}$ vertex-disjoint subgraphs, each being isomorphic to H , in G_n .

In 1998, Komlós, Sárközy and Szemerédi [60] proved Pósa-Seymour conjecture for graphs of large enough orders.

Theorem 5.14. [60] *For any $\epsilon > 0$ and any positive integer k , there is an n_0 such that if G has order $n \geq n_0$ and minimum degree at least $\left(1 - \frac{1}{k+1} + \epsilon\right) n$, then G contains the k -th power of a Hamiltonian cycle.*

Note that in Theorem 5.14, the k -th power of a Hamiltonian graph H is the graph obtained from H by joining every pair of vertices with distance at most k in H .

For some other early applications of Regularity Lemma we refer to [61].

Recently many long-standing conjectures about Hamiltonian problems are proved or partially proved by using the Regularity Lemma.

In 1971, Nash-Williams [87] conjectured that:

Conjecture 5.15. [87] *Let G be a d -regular graph on at most $2d$ vertices. Then G contains $\lfloor \frac{d}{2} \rfloor$ edge-disjoint Hamiltonian cycles.*

In 2012, Christofides, Kühn and Osthus [18] proved an approximate version of Conjecture 5.15.

Theorem 5.16. [18] *For every $\alpha > 0$ there is an integer n_0 so that every d -regular graph on $n \geq n_0$ vertices with $d \geq (\frac{1}{2} + \alpha)n$ contains at least $\frac{d-\alpha n}{2}$ edge-disjoint Hamiltonian cycles.*

In [22], Csaba et al. proved the following theorems.

Theorem 5.17 (1-factorization conjecture). [22] *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil \frac{n}{4} \rceil - 1$. Then every D -regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

Here a 1-factorization of a graph G consists of a set of edge-disjoint perfect matchings covering all edges of G .

Theorem 5.18 (Hamilton decomposition conjecture). [22] *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor \frac{n}{2} \rfloor$. Then every D -regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

For more progress on F -packing, Hamiltonian problems and tree embedding, we refer to [62]. For the progress on Hamiltonian cycles in directed graphs, oriented graphs and tournaments, we refer to [63]

5.2 Overview of the proof of Theorem 5.2

In our proof for Theorem 5.2 we will use the Regularity Lemma and Blow-up Lemma as many other studies (see [16], [91] for some similar ideas).

For proving Theorem 5.2, we only need to consider the graphs of even order. For a graph G of odd order n , we choose one vertex v in G which is not either of the two vertices x, y , then the minimum degree of graph $G^* = G - \{v\}$ is at least $\lceil \frac{n}{2} \rceil = \frac{n-1}{2} + 1$. Since G^* is a graph of even order, by assumption we can locate x, y with distance $\frac{n-1}{2}$ on a Hamiltonian cycle C^* of G^* . By the degree condition of v in G , there exist two

consecutive vertices u_1, u_2 on C^* which are adjacent to v . Replacing the edge u_1u_2 on C^* by the 2-path u_1vu_2 , we obtain a Hamiltonian cycle of G in which x, y have distance $\frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$.

Now let us consider a graph G of even order n with

$$\delta(G) \geq \frac{n}{2} + 1. \quad (5.1)$$

We assume that n is sufficiently large and we fix the following sequence of parameters,

$$0 < \epsilon \ll d \ll \alpha \ll 1. \quad (5.2)$$

Here $a \ll b$ means a is sufficiently small compared to b . For simplicity, we don't specify their dependencies in the proof, although we could.

A *balanced partition* of $V(G)$ into V_1 and V_2 is a partition of $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = \frac{n}{2}$. We define two extremal cases as follows.

Extremal Case 1: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$.

Extremal Case 2: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$.

The proof of Theorem 5.2 will be divided into two parts: the non-extremal case part in Section 5.3 and the extremal cases part in Section 5.4.

For the non-extremal case, the proof consists of the following four steps.

Step 1: We apply the Regularity Lemma to the graph G and find a Hamiltonian cycle in the reduced graph.

Step 2: By the Hamiltonian cycle in the reduced graph, we obtain a perfect matching of the reduced graph and we define the pairs of clusters according to the matching. We construct some paths to connect the clusters in different pairs.

Step 3: We use all the vertices in V_0 to extend those paths constructed in step 2.

Step 4: We apply Lemma 5.12 in each pair of clusters to finally construct a Hamiltonian cycle in G . And we make sure that x, y have distance $\lfloor \frac{n}{2} \rfloor$ on this cycle.

Indeed, due to the parity of $\frac{n}{2}$, our proof will have some cases discussions.

For two extremal cases, we will prove a structure lemma for each case to make the partition of $V(G)$ more helpful for the proof (Lemma 5.27 and Lemma 5.29). By Lemma

5.27, a graph in extremal case 1 is like a balanced bipartite graph except some special vertices, then we will use Lemma 5.12 to construct our desired Hamiltonian cycle. And by Lemma 5.29, a graph in extremal case 2 is like a union of two high minimum degree subgraphs except some special vertices. All these two extremal cases will have some sub-case discussions based on the parity of $\frac{n}{2}$.

5.3 Non-extremal case

5.3.1 Applying the Regularity Lemma

Let G be a graph which is not either of the extremal cases and the vertices x, y have been chosen. We apply the Regularity Lemma (in this thesis we always refer to Theorem 5.8) to G with parameter ϵ and d as in equation 5.2. We get a partition of $V(G)$ into $l + 1$ clusters $V_0, V_1, V_2, \dots, V_l$. Assume that l is even, if not, we move the vertices of one of the clusters into V_0 (also denote it by V_0) to make l be an even number. Now, by (2) of Theorem 5.8, we have $|V_0| \leq 2\epsilon n$ and

$$(1 - 2\epsilon)n \leq lN \leq n. \quad (5.3)$$

Let $k := \frac{l}{2}$.

We define the reduced graph R based on the clusters V_1, \dots, V_l (a partition of $V(G) \setminus V_0$): the vertices of R are r_1, r_2, \dots, r_l , and there is an edge between r_i and r_j if the pair (V_i, V_j) is ϵ -regular in G' with density at least d . If no ambiguity arises, we won't distinguish the cluster and its corresponding vertex in R .

The following claim shows that R inherits the minimum degree condition.

Claim 5.19. $\delta(R) \geq (\frac{1}{2} - 2d)l$.

Proof. For any cluster V_i ($1 \leq i \leq l$), the neighbors of $v \in V_i$ in G' can only be in V_0 and in the clusters which are neighbors of V_i in R . Consider the sum of degrees of vertices in V_i , in one hand,

$$\left(\frac{n}{2} + 1 - (d + \epsilon)n\right)L \leq \sum_{v \in V_i} \deg_{G'}(v);$$

in the other hand,

$$\sum_{v \in V_i} \deg_{G'}(v) \leq 2\epsilon nL + \deg_R(r_i)L^2.$$

Thus $\deg_R(r_i) \geq (\frac{1}{2} - d - 3\epsilon)\frac{n}{L} > (\frac{1}{2} - 2d)l$ provided $3\epsilon < d$. □

The assumption that G is not either of the extremal cases leads the following claim.

Claim 5.20. G is a graph which is not either of the extremal cases, then

- (1) the independence number of R is less than $(\frac{1}{2} - 8d)l$,
- (2) R contains no two disjoint subsets R_1, R_2 of size at least $(\frac{1}{2} - 6d)l$ such that $e_R(R_1, R_2) = 0$.

Proof. (1) Suppose to the contrary that R contains an independent set R_1 of size $(\frac{1}{2} - 8d)l$. We will show that G is in the Extremal Case 1 with parameter α . Let $A = \cup_{r_i \in R_1} V_i$ and $B = V(G) - A$. By equation 5.3,

$$\left(\frac{1}{2} - 9d\right)n \leq \left(\frac{1}{2} - 8d\right)lL = |R_1|L = |A| < \left(\frac{1}{2} - 2d\right)n.$$

For each $x \in A$, by (3) of Theorem 5.8,

$$\deg_G(x, A) \leq \deg_{G'}(x, A) + (d + \epsilon)n < 2dn,$$

then we have

$$\deg_G(x, B) > \frac{n}{2} - 2dn = \left(\frac{1}{2} - 2d\right)n.$$

Hence,

$$e_G(A, B) > \left(\frac{1}{2} - 9d\right)n \cdot \left(\frac{1}{2} - 2d\right)n > \left(\frac{1}{4} - \frac{11}{2}d\right)n^2.$$

Now move at most $9dn$ vertices from B to A such that A and B are of size $\frac{n}{2}$. We still have

$$e_G(A, B) > \left(\frac{1}{4} - \frac{11}{2}d\right)n^2 - 9dn \cdot \frac{n}{2} = \left(\frac{1}{4} - 10d\right)n^2 = (1 - 40d)\left(\frac{n}{2}\right)^2.$$

By specializing $40d \leq \alpha$, we get that G is in the Extremal Case with parameter α .

(2) Suppose to the contrary that R contains two disjoint subsets R_1, R_2 of size $(\frac{1}{2} - 6d)l$ such that $e_R(R_1, R_2) = 0$. We will show that G is in the Extremal Case 2 with parameter α . Let $A = \cup_{r_i \in R_1} V_i$ and $B = \cup_{r_i \in R_2} V_i$. Since $e_R(R_1, R_2) = 0$, we have $e_{G'}(A, B) = 0$. By (3) of Theorem 5.8, we have $|E(G)| \leq |E(G')| + dn^2$, we have

$$e_G(A, B) \leq e_{G'}(A, B) + dn^2 = dn^2.$$

Note that

$$|A| = |R_1|L = \left(\frac{1}{2} - 6d\right)lL > \left(\frac{1}{2} - 7d\right)n,$$

and

$$|B| = |R_2|L = \left(\frac{1}{2} - 6d\right)lL > \left(\frac{1}{2} - 7d\right)n.$$

By adding at most $7dn$ vertices to each of A and B , we obtain two subsets of size $\frac{n}{2}$ and still name them as A and B , respectively. Then,

$$e_G(A, B) \leq dn^2 + 2 \cdot (7dn) \binom{n}{2} = 8dn^2,$$

which in turn shows that the density

$$d_G(A, B) = \frac{e_G(A, B)}{\binom{n}{2}^2} \leq 32d.$$

Since $\alpha > 32d$, we obtain that G is in the Extremal Case 2 with parameter α . \square

Note that Claim 5.20 shows that there exists an upper bound of the independence number of R . The proof of this lemma is similar to the proof of Claim 4.5 in [16],

By Claim 5.20, we can say that R is Hamiltonian. To prove this, we need a theorem of Nash-Williams.

Theorem 5.21. [85] *Let G be a 2-connected graph of order n . If minimum degree $\delta(G) \geq \max\{\frac{n+2}{3}, \alpha(G)\}$, here $\alpha(G)$ is the independence number of G , then G contains a Hamiltonian cycle.*

Claim 5.22. *R is a Hamiltonian graph.*

Proof. By $\delta(R) \geq (\frac{1}{2} - 2d)l$ and Lemma 5.20, we can say that $\delta(R) \geq \max\{\frac{l+2}{3}, \alpha(R)\}$. Now we only need to prove that R is a 2-connected graph.

Actually, we can show that R is dl -connected. To the contrary suppose that there exists a vertex cut S in R with $|S| < dl$. Let X and Y be two components in $R - S$. Since $\delta(R) \geq (\frac{1}{2} - 2d)l$, the size of X and Y should be more than $(\frac{1}{2} - 3d)l$. And $e_G(X, Y) = 0$, which is impossible by Lemma 5.20. So R is dl -connected.

Since $n \leq Ll + 2\epsilon n \leq (l + 2)\epsilon n$, $l \geq \frac{1}{\epsilon} - 2$. Suppose $5\epsilon \leq d$, we can get $l \geq \frac{3}{d}$. So $dl \geq 3$ which leads to the end of the proof. \square

There are also some similar arguments in [16].

5.3.2 Constructing paths to connect clusters

We say that a vertex v is *friendly* to a cluster X , denoted $v \sim X$, if $\deg_G(v, X) \geq (d - \epsilon)|X|$. Moreover, given an ϵ -regular pair (X, Y) of clusters and a subset $Y' \subseteq Y$, we say that a vertex $v \in X$ is *friendly* to Y' , if $\deg(v, Y') \geq (d - \epsilon)|Y'|$. Actually by Lemma 5.3, at most $\epsilon|X|$ vertices of X are not friendly to Y' whenever $|Y'| > \epsilon|Y|$.

Claim 5.23. *Every vertex $v \in V(G)$ is friendly to at least $(\frac{1}{2} - 2d)l$ clusters.*

Proof. Assume for a contradiction that there are less than $(\frac{1}{2} - 2d)l$ friendly clusters for v . Then

$$\deg_G(v) \leq (\frac{1}{2} - 2d)lL + (d - \epsilon)Ll + 2\epsilon n \leq (\frac{1}{2} - d + 2\epsilon)n < \frac{n}{2},$$

provided that $2\epsilon < d$, which is a contradiction to equation 5.1. \square

If two clusters X and Y are a regular pair, we denote this relation by $X \sim Y$. Given two vertices $u, v \in V(G)$, a u, v -chain of length $2s$ with distinct clusters $A_1, B_1, \dots, A_s, B_s$ is $u \sim A_1 \sim B_1 \sim \dots \sim A_s \sim B_s \sim v$ and $\{A_j, B_j\} = \{X_i, Y_i\}$ for some $1 \leq i \leq k$. We have the following claim. There are also some similar discussions in [16], but we give the claim with a different bound on the number of the chains.

Claim 5.24. *Let L be a list of at most $2\epsilon n$ pairs of vertices of G . For each pair $\{u, v\} \in L$, we can find u, v -chains of length two or four such that every cluster is used in at most $\frac{d}{10}L$ chains.*

Proof. We deal with one pair in L at each step. Suppose we have found the desired chains for $s < 2\epsilon n$ pairs such that no cluster is used in more than $\frac{d}{10}L$ chains. Let \mathcal{O} be the set of clusters which are used $\frac{d}{10}L$ times.

We have a bound on the cardinality of \mathcal{O} ,

$$\frac{d}{10}L|\mathcal{O}| \leq 4s \leq 8\epsilon n \leq 8\epsilon \frac{2kL}{1 - 2\epsilon}.$$

So $|\mathcal{O}| \leq \frac{160\epsilon k}{(1 - 2\epsilon)d} \leq \frac{160\epsilon l}{d} \leq dl$, provided $d^2 \geq 160\epsilon$.

Now consider a new pair $\{u, v\}$ in L , we try to find a u, v -chain of length at most four such that every cluster is used in at most $\frac{d}{10}L$ chains. Let \mathcal{U} be the set of clusters which are friendly to u and not in \mathcal{O} . Similarly, let \mathcal{V} be the set of clusters which are friendly to v and not in \mathcal{O} . Let $P(\mathcal{U})$ and $P(\mathcal{V})$ be the set of partners of clusters in \mathcal{U} and \mathcal{V} , respectively. It is easy to see that $|\mathcal{U}| = |P(\mathcal{U})|$ and $|\mathcal{V}| = |P(\mathcal{V})|$. Moreover, since $|\mathcal{O}| \leq dl$, by Claim 5.23, we know that $|\mathcal{U}| = |P(\mathcal{U})| \geq (\frac{1}{2} - 3d)l$ and $|\mathcal{V}| = |P(\mathcal{V})| \geq (\frac{1}{2} - 3d)l$. Denote by R_1 the set of vertices in R which are corresponding to the clusters in $P(\mathcal{U})$, and by R_2 the set of vertices in R which are corresponding to the clusters in $P(\mathcal{V})$.

First, we claim that $e_R(R_1, R_2) \neq 0$. Suppose to the contrary that $e_R(R_1, R_2) = 0$. If $R_1 \cap R_2 = \emptyset$, Then R_1 and R_2 are disjoint and $e_R(R_1, R_2) = 0$. Since $|R_1| = |P(\mathcal{U})| \geq (\frac{1}{2} - 3d)l$ and $|R_2| = |P(\mathcal{V})| \geq (\frac{1}{2} - 3d)l$, we get a contradiction to Claim 5.20.

If there exists a vertex $x \in R_1 \cap R_2$, then $\deg_R(x) \geq (\frac{1}{2} - 2d)l$. Since $e_R(R_1, R_2) = 0$, x is not adjacent to any vertex in $R_1 \cup R_2$. Thus $|R_1 \cup R_2| \leq (\frac{1}{2} + 2d)l$. Since $|R_1| \geq (\frac{1}{2} - 3d)l$ and $|R_2| \geq (\frac{1}{2} - 3d)l$, $|R_1 \cap R_2| \geq (\frac{1}{2} - 8d)l$. $R_1 \cap R_2$ is an independent set in R , which is a contradiction to Claim 5.20.

Now $e_R(R_1, R_2) \neq 0$, that means there exist a cluster in $P(\mathcal{U})$, say A , and a cluster in $P(\mathcal{V})$, say B , such that $A \sim B$. Denote the partner of A in \mathcal{U} by A' and the partner of B in \mathcal{V} by B' . If A, A', B, B' are mutually distinct, then we get a u, v -chain $u \sim A' \sim A \sim B \sim B' \sim v$ of length four. If $A' = B$ or $A = B'$, we get a u, v -chain $u \sim B \sim B' \sim v$ or $u \sim A' \sim A \sim v$ of length two, respectively.

Continue this process until there is no pair in L is left. □

Let C_R be a Hamiltonian cycle in R . We choose two distinct clusters X, Y which are as close as possible on C_R such that x is friendly to Y and y is friendly to X .

Claim 5.25. *We can choose distinct clusters X and Y such that x is friendly to Y , y is friendly to X and $\text{dist}_{C_R}(X, Y) \leq 3dl$.*

Proof. Let \mathcal{X} be the the family of friendly clusters for x and \mathcal{Y} be the the family of friendly clusters for y . By Claim 5.23, $|\mathcal{X}|, |\mathcal{Y}| \geq (\frac{1}{2} - 2d)l$. We won't distinguish a cluster and its corresponding vertex on C_R .

We call a segment on C_R an \mathcal{X} -segment if it is a maximal segment(or we can say a maximal path) on C_R with both end vertices in \mathcal{X} such that it contains no clusters in \mathcal{Y} . Similarly, a \mathcal{Y} -segment is a maximal segment on C_R with both end vertices in \mathcal{Y} such that it contains no clusters in \mathcal{X} . Each cluster in $\mathcal{X} \cap \mathcal{Y}$ forms an \mathcal{X} -segment(a \mathcal{Y} -segment) with one vertex on C_R . Now C_R is divided by all these segments. We choose $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ such that X and Y are two closest end vertices in two continuous segments on C_R .

If $\mathcal{X} \cap \mathcal{Y}$ is equal to \mathcal{X} or \mathcal{Y} , then $|\mathcal{X} \cap \mathcal{Y}| \geq (\frac{1}{2} - 2d)l$. The distance between X and Y should have

$$\text{dist}_{C_R}(X, Y) \leq \frac{l - |\mathcal{X} \cap \mathcal{Y}|}{|\mathcal{X} \cap \mathcal{Y}|} + 1 \leq \frac{l - (\frac{1}{2} - 2d)l}{(\frac{1}{2} - 2d)l} + 1 \leq \frac{8d}{1 - 4d} + 2 \leq 4$$

provided $d \leq \frac{1}{8}$. Since the distance between two vertices in a path is the number of internal vertices plus one, we have a “+1” in the above calculation.

If $\mathcal{X} \cap \mathcal{Y}$ is not equal to either of \mathcal{X} or \mathcal{Y} , then the number of segments should be no less than $|\mathcal{X} \cap \mathcal{Y}| + 2$.

$$\text{dist}_{C_R}(X, Y) \leq \frac{l - (|\mathcal{X}| + |\mathcal{Y}| - |\mathcal{X} \cap \mathcal{Y}|)}{|\mathcal{X} \cap \mathcal{Y}| + 2} + 1 \leq \frac{4dl + |\mathcal{X} \cap \mathcal{Y}|}{|\mathcal{X} \cap \mathcal{Y}| + 2} + 1 \leq 2dl + 2.$$

Since $n = Ll + |V_0| \leq (l + 2)\epsilon n$, we have $l \geq \frac{1}{\epsilon} - 2 \geq \frac{2}{d}$, provided $\epsilon \leq \frac{d}{4}$. Then $dl \geq 2$, $\text{dist}_{C_R}(X, Y) \leq 3dl$. \square

By Claim 5.25 we choose such two clusters X and Y . We give a new notation for all clusters except V_0 . We choose a direction of C_R , which is along the longer path from Y to X on C_R (there are two paths from Y to X on C_R , and we choose the longer one), then starting from Y , we denote the clusters by $Y_1, X_2, Y_2, X_3, Y_3, \dots, X_k, Y_k, X_1$ along this direction (recall that $k = \frac{l}{2}$). Y is denoted by Y_1 and X is denoted by a X_i or a Y_i . We call X_i, Y_i *partners* of each other ($1 \leq i \leq k$) and write $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

We need to mention that the parity of $\frac{n}{2}$ and the new notation of X would affect our following discussions. In the following arguments, we assume that $\frac{n}{2}$ is even and X is denoted by Y_t . We call this the non-extremal case 1. For the other cases ($\frac{n}{2}$ is odd or X is denoted by some X_i), we will discuss them in Subsection 5.3.5.

We know that $t \neq 1$. By Claim 5.25, $\text{dist}_{C_R}(Y_1, Y_t) = l - 2t + 2 \leq 3dl$. So

$$t - 1 \geq \frac{1 - 3d}{2}l. \quad (5.4)$$

This will be used in Subsection 5.3.4.

Now we construct some paths to connect Y_i and X_{i+1} ($1 \leq i \leq k$). We always say $X_{k+1} = X_1$.

Since x is friendly to Y_1 , we can choose two neighbors of x in Y_1 , denoted w_x and y_1^1 , such that w_x is friendly to X_2 and y_1^1 is friendly to X_1 . This is possible because x has at least $(d - \epsilon)L$ neighbors in Y_1 and $(X_1, Y_1), (Y_1, X_2)$ are both regular pairs, by Lemma 5.3 at least $(d - \epsilon)L - \epsilon L$ vertices of Y_1 can be chosen as w_x and y_1^1 . Choose a neighbor of w_x in X_2 , denoted x_2^1 , such that x_2^1 is friendly to Y_2 . We know that at least $(d - \epsilon)L - \epsilon L$ vertices of X_2 can be chosen as x_2^1 . And we have a path from y_1^1 to x_2^1 , precisely $P_1 := y_1^1 x w_x x_2^1$. We call this procedure *joining* x to Y_1 . Similarly we can construct a path $P_t = y_t^1 y w_y x_{t+1}^1$, where $y_t^1 \in Y_t$ is friendly to X_t , $x_{t+1}^1 \in X_{t+1}$ is friendly to Y_{t+1} and $w_y \in Y_t$. We call this procedure *joining* y to Y_t . For $1 \leq i \leq k$ and $i \neq 1, t$, we choose two adjacent vertices y_i^1 and x_{i+1}^1 such that $y_i^1 \in Y_i$ is friendly to X_i

and $x_{i+1}^1 \in X_{i+1}$ is friendly to Y_{i+1} (we always use x_1^1 to denote x_{k+1}^1). It is possible because by Lemma 5.3 at least $(d - \epsilon)L$ vertices of Y_i can be chosen as y_i^1 and at least $(d - \epsilon)L - \epsilon L$ vertices can be chosen as x_{i+1}^1 . Let P_i ($1 \leq i \leq k$ and $i \neq 1, t$) be the path $y_i^1 x_{i+1}^1$, which connects Y_i and X_{i+1} . We always call the vertices in P_i ($1 \leq i \leq k$) *used* vertices.

We need some other vertex-disjoint paths to connect Y_i and X_{i+1} ($1 \leq i \leq k$). By the same method, we choose two adjacent unused vertices y_i^2 and x_{i+1}^2 such that $y_i^2 \in Y_i$ is friendly to X_i and $x_{i+1}^2 \in X_{i+1}$ is friendly to Y_{i+1} (we always use x_1^2 to denote x_{k+1}^2). Let Q_i ($1 \leq i \leq k$) be the path $y_i^2 x_{i+1}^2$, which connects Y_i and X_{i+1} . By Lemma 5.3, it is possible to find these unused vertices.

In summary, we have constructed paths P_i and Q_i ($1 \leq i \leq k$), which are vertex-disjoint and connect Y_i and X_{i+1} (see Figure 5.1). x is on P_1 and y is on P_t . Every end vertex of these paths is friendly to its cluster's partner. We use INT to denote the vertex set of all internal vertices on all P_i 's and Q_i 's. Since, except for P_1 and P_t , all P_i 's and Q_i 's are edges, now we have $INT = \{x, y, w_x, w_y\}$.

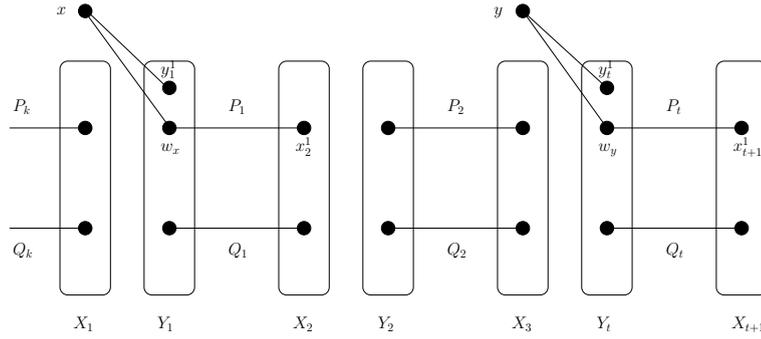


FIGURE 5.1: Construction of P_i 's and Q_i 's.

For every $1 \leq i \leq k$, let

$$X'_i := \{u \in X_i : \deg(u, Y_i) \geq (d - \epsilon)L\}, Y'_i := \{v \in Y_i : \deg(v, X_i) \geq (d - \epsilon)L\}.$$

Since (X_i, Y_i) is ϵ -regular, by Lemma 5.3 have $|X'_i|, |Y'_i| \geq (1 - \epsilon)L$. We move all the vertices in $X_i - X'_i$ and $Y_i - Y'_i$ to V_0 . Meanwhile, we need to make sure (X'_i, Y'_i) is balanced, so by Lemma 5.3 we may move at most ϵLl vertices in (X'_i, Y'_i) to guarantee that. We also remove all the vertices in INT out of V_0, X'_i and Y'_i . This may cause that some (X'_i, Y'_i) is not balanced. For example, if $w_x \in Y'_1$, and we remove it from Y'_1 , then it causes (X'_1, Y'_1) be not balanced. In this example, to make sure (X'_1, Y'_1) is balanced, we move a vertex in X'_1 to V_0 . We do the same operations for all the vertices in INT . Since $|INT| = 4$ and n is sufficiently large, at most $\epsilon Ll + 4 \leq 2\epsilon n$ vertices are moved to V_0 . We derive that $|V_0| \leq 4\epsilon n$ and $|X'_i| = |Y'_i| \geq (1 - \epsilon)L - 1$ ($1 \leq i \leq k$) in this step.

Since

$$\epsilon L \geq \epsilon \frac{(1-2\epsilon)n}{l} \geq \frac{\epsilon(1-2\epsilon)}{M_0} n$$

and n is sufficiently large, we say $\epsilon L \geq 1$. Thus $|X'_i| = |Y'_i| \geq (1-2\epsilon)L$ ($1 \leq i \leq k$). The minimum degree in each pair is at least $(d-\epsilon)L - \epsilon L - 1 \geq (d-3\epsilon)L$.

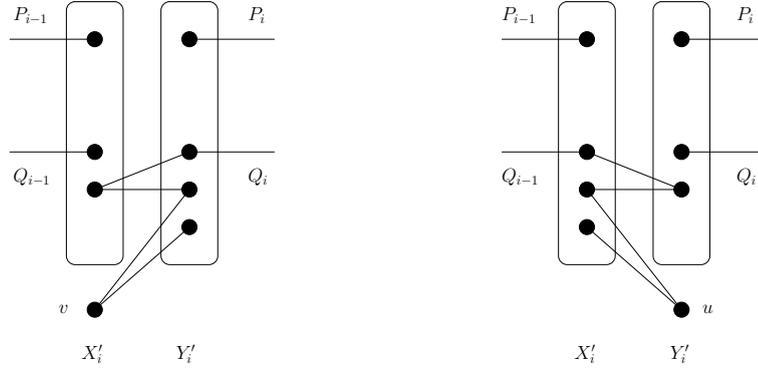
5.3.3 Handling of all the vertices of V_0

In this step, we extend those paths Q_i 's ($1 \leq i \leq k$) by adding all the vertices of V_0 to them. Note that, $|V_0|$ is even because $|X'_i| = |Y'_i|$ for all i and $|V_0| \leq 4\epsilon n$. We arbitrarily partition V_0 into at most $2\epsilon n$ pairs. Applying Claim 5.24, we have chains of length at most four for each pair such that every cluster is used in at most $\frac{d}{10}L$ chains.

Now we extend those paths Q_i 's by using vertices of V_0 . Recall that the end vertices of Q_i are $y_i^2 \in Y_i$ and $x_{i+1}^2 \in X_{i+1}$. We deal with the vertices of V_0 pair by pair. Assume that we deal with the pair (u, v) now.

If there is a chain of length two between u and v , assume that this chain is $u \sim X_i \sim Y_i \sim v$, for some $1 \leq i \leq k$. We choose two adjacent vertices $w_1 \in X'_i$ and $w_2 \in Y'_i$ such that w_1 is a neighbor of y_i^2 and w_2 is a neighbor of v . Since y_i^2 is friendly to X_i and v is friendly to Y_i , the size of the neighbor sets of w_1 and w_2 are at least $(d-3\epsilon)L$. By Lemma 5.3, it is possible to choose w_1 and w_2 . We choose another neighbor of v in Y'_i , denoted w_3 . Then we extend Q_i to $Q_i \cup \{w_3v, vw_2, w_2w_1, w_1y_i^2\}$. We still denote this new path by Q_i and call w_3 the new y_i^2 to make sure that the end vertices of the new Q_i are denoted by $y_i^2 \in Y_i$ and $x_{i+1}^2 \in X_{i+1}$. Similarly, for u , we can choose $w_5, w_6 \in X'_i$ and $w_4 \in Y'_i$ to extend Q_{i-1} to $Q_{i-1} \cup \{x_{i-1}^2w_4, w_4w_5, w_5u, uw_6\}$. We still denote this new path by Q_{i-1} and call w_6 the new x_i^2 to make sure that the end vertices of P_{i-1} are $y_{i-1}^2 \in Y_{i-1}$ and $x_i^2 \in X_i$. And we update the set INT . Indeed, three vertices of X'_i are added to INT (also for Y'_i) and totally eight vertices are added to INT including u, v . Since v behaves like a vertex in X'_i and u behaves like a vertex in Y'_i , we call this procedure *inserting* v into X'_i to extend Q_i and *inserting* u into Y'_i to extend Q_{i-1} (see Figure 5.2).

Now we consider that the u, v -chain has length four. Without loss of generality, we assume that the chain is $u \sim X_i \sim Y_i \sim X_j \sim Y_j \sim v$, for some i, j . We extend the path Q_{i-1} by inserting u into Y'_i . We choose a vertex of Y'_i which is friendly to X_j and insert it into Y'_j to extend Q_{j-1} . At last we extend the path Q_j by inserting v into X'_j . Meanwhile, we update the set INT . Indeed, two vertices of X'_i are added to INT (also for Y'_i) and three vertices of X'_j are added to INT (also for Y'_j). So totally twelve vertices are added into INT including u, v .


 FIGURE 5.2: Extending Q_{i-1} and Q_i when u, v have a chain of length two.

We continue this process till there is no vertex left in V_0 . Denote $X_i^* = X'_i - INT$ and $Y_i^* = Y'_i - INT$. It is not hard to see that the pair (X_i^*, Y_i^*) is still balanced. For inserting each pair of vertices, at most three vertices of a cluster in the chain are used. So

$$|X_i^*| = |Y_i^*| \geq (1 - 2\epsilon)L - 3\frac{d}{10}L \geq (1 - \frac{d}{2})L$$

provided $\epsilon < \frac{d}{10}$.

For each vertex $u \in X_i^*$, we have

$$\deg(u, Y_i^*) \geq (d - 3\epsilon)L - 3\frac{d}{10}L \geq \frac{d}{2}L$$

provided $\epsilon < \frac{d}{15}$. And it is the same for the degree of any vertex in Y_i^* .

Thus by Slicing Lemma, we can say the pair (X_i^*, Y_i^*) is $(2\epsilon, \frac{d}{2})$ -super-regular ($1 \leq i \leq k$).

5.3.4 Constructing the desired Hamiltonian cycle

In this step, first we use Lemma 5.12 to construct two paths W_i^1 and W_i^2 in each pair (X_i^*, Y_i^*) . Then we combine all these paths with P_i 's and Q_i 's to obtain a Hamiltonian cycle in G . At last we fix the length of W_i^1 and W_i^2 in each pair to make sure that x and y have distance $\frac{n}{2}$ on this Hamiltonian cycle.

For each $1 \leq i \leq k$, we choose any even integers l_i^1, l_i^2 such that $4 \leq l_i^1, l_i^2 \leq 2|X_i^*| - 4$ and $l_i^1 + l_i^2 = 2|X_i^*|$. We will fix these integers later.

For $2 \leq i \leq k$, using Lemma 5.12, we construct two paths W_i^1 and W_i^2 in the pair (X_i^*, Y_i^*) such that

- (a) W_i^1 has end vertices x_i^1 and y_i^1 with $|V(W_i^1)| = l_i^1$;

(b) W_i^2 has end vertices x_i^2 and y_i^2 with $|V(W_i^2)| = l_i^2$.

For $i = 1$, we construct two paths W_1^1 and W_1^2 in the pair (X_1^*, Y_1^*) such that

(c) W_1^1 has end vertices x_1^1 and y_1^1 with $|V(W_1^1)| = l_1^1$;

(d) W_1^2 has end vertices x_1^2 and y_1^2 with $|V(W_1^2)| = l_1^1$.

Then

$$C = P_1 \cup \left(\bigcup_{i=2}^k (W_i^1 \cup P_i) \right) \cup W_1^1 \cup Q_1 \cup \left(\bigcup_{i=2}^k (W_i^2 \cup Q_i) \right) \cup W_1^2$$

is a Hamiltonian cycle in G .

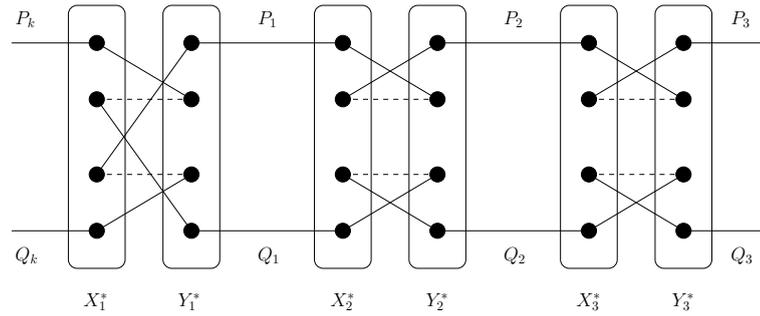


FIGURE 5.3: Construction of the Hamiltonian cycle C

To finish our proof, we need to make sure that x and y have distance $\frac{n}{2}$ on C . Our Hamiltonian cycle is constructed in a bipartite graph $(\bigcup X_i) \cup (\bigcup Y_i)$ (take $\bigcup X_i$ as a part and $\bigcup Y_i$ the other one). Since x behaves like a vertex in X_1 and y behaves like a vertex in X_t , the distance of x and y on C should be an even number. Recall that we assume that $\frac{n}{2}$ is even. So there is no parity problem in this non-extremal case.

Claim 5.26. *We can properly choose the value of l_i^1 ($2 \leq i \leq t$) such that $\text{dist}_C(x, y) = \frac{n}{2}$.*

Proof. We consider this path $P := P_1 \cup (\bigcup_{i=2}^t (W_i^1 \cup P_i))$ of the Hamiltonian cycle. We need the distance of x and y on C to be $\frac{n}{2}$, so the vertex number between x and y on P should be $\frac{n}{2} - 1$. For the vertices between x and y on P , the only vertex not belong to W_i^1 ($2 \leq i \leq t$) is w_x . Thus we need to make sure

$$\frac{n}{2} - 1 = \sum_{i=2}^t l_i^1 + 1. \quad (5.5)$$

Since by Lemma 5.12, l_i^1 can be any even integer such that $4 \leq l_i^1 \leq 2|X_i^*| - 4$. By $|X_i^*| \geq (1 - \frac{d}{2})L$, we can choose l_i^1 such that $\sum_{i=2}^t l_i^1$ can be any even integer with the

following bound,

$$4(t-1) \leq \sum_{i=2}^t l_i^1 \leq 2(t-1)\left(1 - \frac{d}{2}\right)L - 4(t-1) = 2(t-1)\left(\left(1 - \frac{d}{2}\right)L - 2\right).$$

We know that $t \leq k = \frac{l}{2}$, then $4(t-1) < 2l$. Since $l \leq M_0$ in the Regularity Lemma and n is sufficiently large (let $n \geq 4M_0 + 4$), we can say that $4(t-1) < 2l \leq 2M_0 \leq \frac{n}{2} - 2$.

By equation 5.4, we also know $t-1 \geq \frac{1-3d}{2}l$, so

$$\begin{aligned} 2(t-1)\left(\left(1 - \frac{d}{2}\right)L - 2\right) &\geq (1-3d)\left(1 - \frac{d}{2}\right)lL - 2(1-3d)l \\ &\geq \left(1 - \frac{7}{2}d\right)(1-2\epsilon)n - 2l \\ &\geq (1-4d)n - 2M_0 \geq \frac{3}{4}n - 2M_0, \end{aligned}$$

provided $4\epsilon \leq d \leq \frac{1}{16}$ and $l \leq M_0$. Since n is sufficiently large (let $n \geq 8M_0$), we can say that $2(t-1)\left(\left(1 - \frac{d}{2}\right)L - 2\right) \geq \frac{3}{4}n - 2M_0 \geq \frac{n}{2}$.

So we can choose the values of l_i^1 ($2 \leq i \leq t$) such that $\sum_{i=2}^t l_i^1 = \frac{n}{2} - 2$ satisfying equation 5.5.

We choose l_i^1 ($2 \leq i \leq t$) such that equation 5.5 holds and arbitrarily choose the even integers l_1^1, l_i^1 ($t < i \leq k$) with the conditions in Lemma 5.12. Thus $\text{dist}_C(x, y) = \frac{n}{2}$. \square

We now complete the proof of the non-extremal case 1: $\frac{n}{2}$ is even and X (the cluster friendly to y) is denoted by Y_t . We prove other non-extremal cases in the next subsection.

5.3.5 Other non-extremal cases

We discuss the other non-extremal cases. Suppose $\frac{n}{2}$ is even and X (the cluster friendly to y) is denoted by X_t ($1 \leq t \leq k$) in the second step of the proof above. We call this the non-extremal case 2. It seems that the method above doesn't work. Since our Hamiltonian cycle is constructed in a bipartite graph $(\bigcup X_i) \cup (\bigcup Y_i)$ and x (respectively y) behaves like a vertex in X_1 (respectively Y_t), we cannot locate x and y with the even distance $\frac{n}{2}$ on the Hamiltonian cycle.

We need some tricks to change the parity. Recall that, in the second step of the proof above, subsection 5.3.2, we construct $P_2 = y_2^1 x_3^1$ and $Q_2 = y_2^2 x_3^2$. Since $\delta(G) \geq \frac{n}{2} + 1$, any two vertices have at least two common neighbors in G . Suppose that y_2^1, x_3^1 have

a common neighbor u_1 and y_2^2, x_3^2 have a common neighbor $u_2 \neq u_1$, we claim that we can choose these vertices such that $u_1 \notin \{y_2^2, x_3^2\}, u_2 \notin \{y_2^1, x_3^1\}$. To see this, by Lemma 5.3, there are at least $(d - \epsilon)L$ vertices of Y_2 can be chosen as y_2^1 and y_2^2 , and at least $(d - \epsilon)L - \epsilon L$ vertices of X_3 can be chosen as x_3^1 and x_3^2 . Assume that we have chosen y_2^1, x_3^1 and one of its common neighbors u_1 . There are at least $(d - \epsilon)L - 2$ vertices can be chosen as y_2^2 , chose two of them, say y_1 and y_2 . For each y_i , there are at least $(d - \epsilon)L - \epsilon L - 2$ vertices can be chosen as x_3^2 , chose two of them, say x_1 and x_2 , such that $y_i x_i$ is an edge, for $i = 1, 2$. If y_i and x_i have a common neighbor not in $\{u_1, y_2^1, x_3^1\}$, then chose this common neighbor as u_2 and y_i as y_2^2, x_i as x_3^2 . Now assume that all the common neighbors of y_i and x_i are in $\{u_1, y_2^1, x_3^1\}$. Since y_i and x_i have at least two common neighbors, there is a vertex in $\{u_1, y_2^1, x_3^1\}$, say u_1 , is a common neighbor both of y_1, x_1 and y_2, x_2 . Without loss of generality, assume that x_3^1 is an another common neighbor of y_1 and x_1 . Now we chose y_2 as new y_2^1, x_2 as new x_3^1, u_1 also be u_1 , and chose y_1 as y_2^2, x_1 as y_2^1 , and the vertex chosen as x_3^1 at first now is chosen as u_2 .

Now we chose new paths of P_2 and Q_2 in step 2 of the proof, which will change the parity as we need.

First, assume that u_1 and u_2 are both different with x and y . Note that x is friendly to Y_1 and y is friendly to X_t , if we use the same method to construct the Hamiltonian cycle, then x and y is connected, as shown in Claim 5.26, by a path $P := P_1 \cup (\bigcup_{i=2}^t (W_i^1 \cup P_i))$ of the Hamiltonian cycle, which is of odd length now. If we replace $P_2 = y_2^1 x_3^1$ and $Q_2 = y_2^2 x_3^2$ by $P_2 = y_2^1 u_1 x_3^1$ and $Q_2 = y_2^2 u_2 x_3^2$, respectively, then by the same method to construct the Hamiltonian cycle, x and y will be connected by a path of even length on the Hamiltonian cycle. By a similar calculation as in proof of Claim 5.26, we can make sure that the distance between x and y on this cycle is the even number $\frac{n}{2}$.

Second, assume that we cannot find these required $y_2^1, x_3^1, y_2^2, x_3^2$ such that u_1 and u_2 are both different with x and y . We choose y_2^1 to be friendly to X_2 and X_3 . There are at least $(1 - 2\epsilon)L$ possible choices for y_2^1 and it is similar for y_2^2 . We choose x_3^1 to be a neighbor of y_2^1 and friendly to Y_2 and Y_3 . There are at least $(d - \epsilon)L - 2\epsilon L = (d - 3\epsilon)L$ possible choices for x_3^1 and it is similar for x_3^2 . By the assumption, y_2^1 and x_3^1 have one of x and y as a common neighbor, and it is same for y_2^2 and x_3^2 . That means all the vertices that can be chosen as y_2^1 and x_3^1 or can be chosen as y_2^2 and x_3^2 must be the neighbors of x or y . So x or y should have at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and also in X_3 . If y has at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and in X_3 , we change the choice of X and choose Y_2 to be the X . By the degree of y in Y_2 we can join y to $X = Y_2$ and we also join x in $Y = Y_1$ as before. Now x and y both behave like vertices in $(\bigcup X_i)$. Otherwise, x has at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and in X_3 . We change the choice of Y and choose X_3 to be the Y . Since we can join x to $Y = X_3$ and join y to $X = X_t$, x and y both

behave like vertices in $(\bigcup Y_i)$. By the choice of X and Y , we give a new notation for all clusters and continue the proof as before. Although the calculation in Claim 5.25 and Claim 5.26 will have some minor differences, it won't affect the conclusion.

We consider the cases when $\frac{n}{2}$ is odd. Actually in the second step of the proof, if the selected clusters X and Y belong to the different parts of the bipartite graph $(\bigcup X_i) \cup (\bigcup Y_i)$, there is no parity problem, the proof is same to non-extremal case 1, except the differences of the notations. If the selected clusters X and Y belong to the same part of the bipartite graph $(\bigcup X_i) \cup (\bigcup Y_i)$, there is a parity problem, the proof is same to non-extremal case 2, except the differences of the notations.

5.4 Extremal cases

5.4.1 Extremal case 1

Suppose G is a graph on n vertices with $\delta(G) \geq \frac{n}{2} + 1$ and there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$. We suppose $\alpha \leq (\frac{1}{9})^3$. Let $\alpha_1 = \alpha^{\frac{1}{3}}$ and $\alpha_2 = \alpha^{\frac{2}{3}}$. So $\alpha_1 \geq 9\alpha_2$.

We need the following lemma to continue our proof.

Lemma 5.27. *If G is in extremal case 1, then G contains a balanced spanning bipartite subgraph G^* with parts U_1, U_2 and G^* has the following properties:*

(a) *there is a vertex set W such that there exist vertex-disjoint 2-paths (paths of length two) in G^* with the vertices of W as the internal vertices (not the end vertices) in each 2-path and $|W| \leq \alpha_2 n$;*

(b) *$\deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \notin W$.*

Proof. For $i = 1, 2$, let $V_i^* = \{v \in V_i : \deg(v, V_{3-i}) \geq (1 - \alpha_1)\frac{n}{2}\}$.

We claim that $|V_i - V_i^*| \leq \alpha_2 \frac{n}{2}$. Otherwise

$$d(V_1, V_2) < \frac{\frac{\alpha_2 n}{2}(1 - \alpha_1)\frac{n}{2} + \frac{n}{2}(\frac{1}{2} - \frac{\alpha_2}{2})n}{(\frac{n}{2})^2} = \alpha_2(1 - \alpha_1) + (1 - \alpha_2) = 1 - \alpha,$$

which is a contradiction. So $|V_i^*| \geq (1 - \alpha_2)\frac{n}{2}$.

For any vertex $v \in V_i - V_i^*$, if $\deg(v, V_i) \geq (1 - \alpha_1)\frac{n}{2}$, we also add v to V_{3-i}^* . We denote the two resulting sets by V_i' ($i = 1, 2$) and let $V_0 = V - V_1' - V_2'$. We have

$|V_0| \leq \alpha_2 n$. For every vertex v in V'_i ,

$$\deg(v, V'_{3-i}) \geq (1 - \alpha_1) \frac{n}{2} - \alpha_2 \frac{n}{2}. \quad (5.6)$$

For every vertex u in V_0 ,

$$\deg(u, V'_i) > \left(\frac{n}{2} - (1 - \alpha_1) \frac{n}{2}\right) - \alpha_2 \frac{n}{2} \geq (\alpha_1 - \alpha_2) \frac{n}{2}.$$

First, we assume $|V'_1|, |V'_2| \leq \frac{n}{2}$. Let $W = V_0$ and we add all the vertices in V_0 to V'_1 and V'_2 such that the final two sets are of the same size. We denote the final two sets by U_1 and U_2 corresponding to V'_1 and V'_2 respectively. Let $W_1 = U_1 - V'_1$ and $W_2 = U_2 - V'_2$. Thus $W = W_1 \cup W_2$. Since for each vertex $u \in W_i$, $\deg(u, V'_{3-i}) > (\alpha_1 - \alpha_2) \frac{n}{2} \geq 2\alpha_2 n \geq 2|W_i|$, we can greedily choose two neighbors of u in V'_{3-i} such that the neighbors of all the vertices of W_i are distinct ($i = 1, 2$). So W, U_1, U_2 are what we need. The degree condition is $\deg_{G^*}(v) \geq (1 - \alpha_1 - \alpha_2) \frac{n}{2}$ by equation 5.6 for all $v \notin W$.

Second, without loss of generality we assume $|V'_1| > \frac{n}{2}$. Let W_1 be the set of vertices $v \in V'_1$ such that $\deg(v, V'_1) \geq \alpha_1 \frac{n}{2}$.

If $|W_1| \geq |V'_1| - \frac{n}{2}$, we take W to be the set of all vertices of V_0 and arbitrary $|V'_1| - \frac{n}{2}$ vertices of W_1 . Note that

$$|W| = |V_0| + |V'_1| - \frac{n}{2} \leq \frac{n}{2} - |V_2^*| \leq \alpha_2 \frac{n}{2}.$$

Let $U_1 = V'_1 - W$ and $U_2 = V'_2 \cup W$. Then for every vertex $u \in W$, we have

$$\deg(u, U_1) > (\alpha_1 - \alpha_2) \frac{n}{2} - \alpha_2 \frac{n}{2} \geq \alpha_2 n \geq 2|W|.$$

Similarly, we can greedily choose two neighbors of u in U_1 such that the neighbors of all the vertices of W are distinct. The degree condition is $\deg_{G^*}(v) \geq (1 - \alpha_1 - \alpha_2) \frac{n}{2} - \alpha_2 \frac{n}{2} = (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ by equation 5.6 for all $v \notin W$.

Now we assume $|W_1| < |V'_1| - \frac{n}{2}$. Let $U_1 = V'_1 - W_1$ and $U_2 = V'_2 \cup V_0 \cup W_1$. Let $t = |U_1| - \frac{n}{2}$, so $t \leq \alpha_2 \frac{n}{2}$. Considering the induced graph $G[U_1]$, we know that

$$\delta(G[U_1]) \geq \delta(G) - |U_2| \geq \frac{n}{2} + 1 - \left(\frac{n}{2} - t\right) \geq t + 1;$$

$$\Delta(G[U_1]) \leq \alpha_1 \frac{n}{2}.$$

Suppose $G[U_1]$ has a biggest family of vertex-disjoint 2-paths on a vertex set S and the number of those vertex-disjoint 2-paths is s . We consider the number of edges

between S and $G[U_1] - S$. So

$$t\left(\frac{n}{2} - 3s\right) \leq \delta(G[U_1])(|U_1| - 3s) \leq 3s\Delta(G[U_1]) \leq 3s\alpha_1\frac{n}{2}.$$

We can get

$$s \geq \frac{\frac{n}{2}t}{3(t + \alpha_1\frac{n}{2})} \geq \frac{\frac{n}{2}t}{3(\alpha_2\frac{n}{2} + \alpha_1\frac{n}{2})} \geq \frac{t}{3(\alpha_2 + \alpha_1)} > t. \quad (5.7)$$

So $G[U_1]$ has at least t vertex-disjoint 2-paths. We choose t vertex-disjoint 2-paths in $G[U_1]$ and move the internal vertices of all these vertex-disjoint 2-paths to U_2 . Now $|U_1| = |U_2| = \frac{n}{2}$. Let W be the union of $V_0 \cup W_1$ and all these internal vertices. For any vertex $u \in V_0 \cup W_1$,

$$\deg(u, U_1) - 3\alpha_2\frac{n}{2} > (\alpha_1 - \alpha_2)\frac{n}{2} - \alpha_2\frac{n}{2} - 3\alpha_2\frac{n}{2} \geq 2|V_0 \cup W_1|.$$

We can find vertex-disjoint 2-paths in $G[U_1, V_0 \cup W_1]$ with all the vertices of $u \in V_0 \cup W_1$ as internal vertices such that these 2-paths are all vertex-disjoint with those existing 2-paths. And $\deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \notin W$ as before. \square

Now we construct the desired Hamiltonian cycle in G . In the proof of Lemma 5.27, we know that the 2-paths are greedily chosen, so we assume that x, y won't be any end vertices of those 2-paths (actually in the last part of the proof of Lemma 5.27, we can also assume those moved 2-paths won't have x, y as the end vertices by equation 5.7. But x, y can be the internal vertex of a 2-path.

First, assume $\frac{n}{2}$ is odd. By Lemma 5.27, we obtain a spanning bipartite graph G^* .

Sub-case 1: suppose x, y are in different parts of G^* , say $x \in U_1, y \in U_2$.

Assume $W \neq \emptyset$ and $x, y \notin W$. We need the following claim to string all the vertices of W in a path.

Claim 5.28. *We can construct a path P with end vertices $x_1 \in U_1$ and $y_1 \in U_2$ such that P contains all the vertices of W and $|V(P)| = 4|W|$.*

Proof. Partition $W = W_1 \cup W_2$ with $W_1 = W \cap U_1$ and $W_2 = W \cap U_2$. Suppose that $W_1 = \{w_1, w_2, \dots, w_t\}$ and the two end vertices of the 2-path containing w_i are a_i, b_i ($1 \leq i \leq t$). Since

$$\deg_{G^*}(a_{i+1}) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}, \quad \deg_{G^*}(b_i) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$$

by Lemma 5.27, a_{i+1} and b_i have at least $(1 - 2\alpha_1 - 4\alpha_2)\frac{n}{2}$ common neighbors in G^* ($1 \leq i \leq t - 1$). We greedily choose $c_i \in U_1$ which is a common neighbor of a_{i+1}, b_i

($1 \leq i \leq t-1$). Since $|W| \leq \alpha_2 n$, we can choose all these c_i 's such that they are distinct. Let $P_1 = a_1 w_1 b_1 c_1 a_2 w_2 b_2 \dots c_{t-1} a_t w_t b_t$. P_1 contains all the vertices of W_1 and $|V(P_1)| = 4|W_1| - 1$. Similarly, we can construct another path P_2 which contains all the vertices of W_2 and $|V(P_2)| = 4|W_2| - 1$. Suppose the end vertices of P_2 are $u, v \in U_1$. We choose an unused neighbor of v in U_2 , denoted v' , and choose a common unused neighbor of v', b_t in U_1 , denoted u' . This is possible because all the vertices of $V(G^*) - V(P_1) - V(P_2)$ have degree at least $(1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - 4\alpha_2 n \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2}$.

Let $P = P_1 \cup P_2 \cup \{b_t u', u' v', v' v\}$, which is the path we need. $|V(P)| = 4|W| \leq 4\alpha_2 n$. We denote the end vertices of P by $x_1 \in U_1$ and $y_1 \in U_2$. \square

Let $U_1^* = U_1 - V(P)$ and $U_2^* = U_2 - V(P)$. By the proof of Claim 5.28, we know that $|U_1^*| = |U_2^*|$. For any vertex $u \in U_1^*$,

$$\deg_{U_2^*}(u) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - 4\alpha_2 n = (1 - \alpha_1 - 10\alpha_2)\frac{n}{2},$$

and for any vertex $v \in U_2^*$, similarly we have

$$\deg_{U_1^*}(v) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2}.$$

We choose two unused neighbors of x (respectively y), denoted y_2, y_3 (respectively x_2, x_3), choose an unused common neighbor of y_1, y_2 in U_1^* , denoted x_4 , and choose an unused neighbor of x_2 in U_2^* , denoted y_4 . Let

$$U_1' = (U_1^* - \{x, x_2, x_4\}) \cup \{x_1\}, U_2' = (U_2^* - \{y, y_1, y_2\}) \cup \{y_1\}.$$

We have $n' = |U_1'| = |U_2'| < \frac{n}{2}$. For any vertex u in U_1' and any vertex v in U_2' , we have

$$\deg_{U_2'}(u) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3, \deg_{U_1'}(v) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3.$$

Since n can be sufficiently large, we can say $(1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3 \geq (1 - \alpha_1 - 11\alpha_2)\frac{n}{2}$. For any vertex u in U_1' , we have

$$\deg_{U_2'}(u) \geq (1 - \alpha_1 - 11\alpha_2)\frac{n}{2} \geq (1 - \alpha_1 - 11\alpha_2)n',$$

and for any vertex v in U_2' , similarly we have

$$\deg_{U_1'}(v) \geq (1 - \alpha_1 - 11\alpha_2)n'.$$

By Lemma 5.6, we can get that (U_1', U_2') is $(\sqrt{\alpha_1 + 11\alpha_2}, 1 - \alpha_1 - 11\alpha_2)$ -super-regular. Since $\alpha \leq (\frac{1}{9})^3$, $1 - \alpha_1 - 11\alpha_2 \geq \frac{2}{3}$, we can say (U_1', U_2') is $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular.

Applying Lemma 5.12 to the pair (U'_1, U'_2) , we construct two vertex-disjoint paths P_1 and P_2 such that the end vertices of P_1 are x_1, y_4 , the end vertices of P_2 are x_3, y_3 and $|V(P_i)|$ is l^i ($i = 1, 2$). We denote P_3 to be the path $P_3 := y_1x_4y_2xy_3$ and P_4 to be the path $P_4 := x_3yx_2y_4$. Then

$$C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$$

is a Hamiltonian cycle in G (see Figure 5.4, Sub-case 1). We fix $l^1 = \frac{n}{2} - 1 - (|V(P)| - 2 + 4) = \frac{n}{2} - |V(P)| - 3$ and $l^2 = 2n' - l^1$. Thus it is not hard to see that x and y have distance $\frac{n}{2}$ on C .

Now assume that at least one of x, y is in W , without loss of generality, we say $x \in W$. We can similarly construct a path P with end vertices $x_1 \in U_1$ and $y_1 \in U_2$ such that P contains all the vertices of $W - \{x, y\}$ as in Claim 5.28. We need to make sure P won't use the vertices of the 2-path which contains x . It is possible because we always greedily choose the vertices to construct P . So we can still find the unused neighbors of x and finish the proof as before.

We also need to consider $W = \emptyset$. In G^* we choose two neighbors of x , denoted y_1, y_2 , and two neighbors of y , denoted x_1, x_2 . Let $U'_1 = U_1 - \{x\}$ and $U'_2 = U_2 - \{y\}$. By Lemma 5.6 and Lemma 5.27, we can say (U'_1, U'_2) is $(\sqrt{\alpha_1 + 3\alpha_2}, \frac{2}{3})$ -super-regular. Let $l^1 = l^2 = \frac{n}{2} - 1$. By Lemma 5.12, we construct two vertex-disjoint paths such that the end vertices of P_1 are x_1, y_1 , the end vertices of P_2 are x_2, y_2 and $|V(P_i)|$ is equal to l^i ($i = 1, 2$). Let $P_3 := y_1xy_2$ and $P_4 := x_1yx_2$. So $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is our desired Hamiltonian cycle.

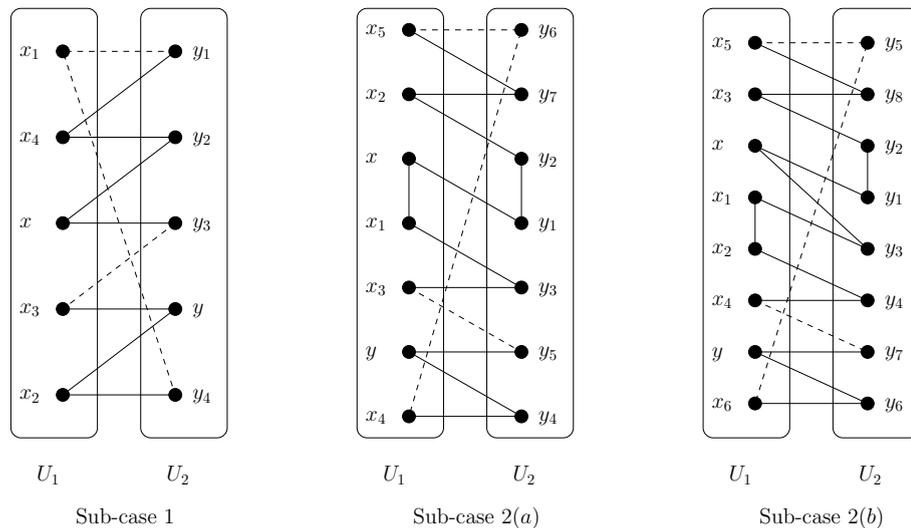


FIGURE 5.4: Extremal case 1.

Sub-case 2: suppose x, y are in the same part of G^* , without loss of generality, say $x, y \in U_1$.

Since the construction in Sub-case 1 is always in the bipartite graph (U_1, U_2) and $\frac{n}{2}$ is odd, it seems that the same method doesn't work in this case. Actually we need some edges in $G[U_1]$ and $G[U_2]$ to change the parity.

Assume $W \neq \emptyset$ and $x, y \notin W$ (if one of x, y is in W , the discussion is almost the same as discussed before). Since $\delta(G) \geq \frac{n}{2} + 1$, x should have a neighbor in U_1 .

Assume this neighbor, denoted x_1 , is not y . We choose a neighbor of x in $U_2 - W$, denoted y_1 , and choose a neighbor of y_1 in U_2 , denoted y_2 . Whether x_1 is in W or not, we can find an unused neighbor of x_1 in $U_2 - W$, denoted y_3 , and choose an unused neighbor of y_3 in $U_1 - W$, denoted x_3 . Whether y_2 is in W or not, we can find an unused neighbor of y_2 in $U_1 - W$, denoted x_2 . We choose two unused neighbors of y in $U_2 - W$, denoted y_4, y_5 , and an unused neighbor of y_4 in $U_1 - W$, denoted x_4 . Since $\deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \notin W$, it is possible to choose all these vertices as discussed in Sub-case 1. By the same method of Claim 5.28, we can construct a path P with end vertices $x_5 \in U_1$ and $y_6 \in U_2$ such that P contains all the unused vertices of W and $|V(P)| \leq 4|W|$. Since the vertices used in P are greedily chosen, we can assume that P won't use any existing chosen vertices. We choose a common unused neighbor of x_2, x_5 in $U_2 - W$, denoted y_7 . Let

$$U'_1 = U_1 - V(P) - \{x, y, x_1, x_2\}, U'_2 = (U_2 - V(P) - \{y_1, y_2, y_3, y_4, y_7\}) \cup \{y_6\}.$$

Note that $|U'_1| = |U'_2|$, let $|U'_1| = |U'_2| = n'$. By Lemma 5.6 and n is sufficiently large, (U'_1, U'_2) is a $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular pair. Applying Lemma 5.12 to the pair (U'_1, U'_2) , we can construct two paths P_1 and P_2 such that the end vertices of P_1 are x_4, y_6 , the end vertices of P_2 are x_3, y_5 and $|V(P_i)| = l^i$ ($i = 1, 2$). Let $P_3 := x_5y_7x_2y_2y_1xx_1y_3x_3$ and $P_4 := x_4y_4yy_5$. Then

$$C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$$

is a Hamiltonian cycle in G (see Figure 5.4, Sub-case 2(a)). We fix $l^1 = \frac{n}{2} - 1 - (|V(P)| + 5 - 1) = \frac{n}{2} - |V(P)| - 5$ and $l^2 = 2n' - l^1$. It is not hard to see that x and y have distance $\frac{n}{2}$ on C . Here we omit all the calculations about α , because it is almost same as Sub-case 1.

Now assume that y is the only neighbor of x in U_1 but y has a neighbor which is not x in U_1 , then the proof is similar to the proof in the last paragraph if we deal with y first. We assume that y is the only neighbor of x in U_1 and x is the only neighbor of y in U_1 . We choose a neighbor of x in $U_2 - W$, denoted y_1 . Since $\deg_G(y_1) \geq \frac{n}{2} + 1$, y_1 has a neighbor in U_2 , denoted y_2 . We choose another unused neighbor of x in $U_2 - W$,

denoted y_3 , and a neighbor of y_3 in $U_1 - W$, denoted x_1 . Since $\deg_G(x_1) \geq \frac{n}{2} + 1$, x_1 has a neighbor in U_1 , denoted x_2 . By our assumption, x_2 should not be either of x and y . We choose an unused neighbor of y_2 in $U_1 - W$, denoted x_3 , an unused neighbor of x_2 in $U_2 - W$, denoted y_4 , and an unused neighbor of y_4 in $U_1 - W$, denoted x_4 . By the same method of Claim 5.28, we construct a path P with end vertices $x_5 \in U_1$ and $y_5 \in U_2$ such that P contains all the unused vertices of W and $|V(P)| \leq 4|W|$. We choose two unused neighbors of y in $U_2 - V(P)$, denoted y_6, y_7 , and choose a neighbor of y_6 in $U_1 - V(P)$, denoted x_6 , and choose a common unused neighbor of x_3, x_5 in $U_2 - V(P)$, denoted y_8 . Let

$$U'_1 = U_1 - V(P) - \{x, y, x_1, x_2, x_3\}, U'_2 = (U_2 - V(P) - \{y_1, y_2, y_3, y_4, y_6, y_8\}) \cup \{y_5\}.$$

Note that $|U'_1| = |U'_2|$, let $|U'_1| = |U'_2| = n'$. By Lemma 5.6 and n is sufficiently large, (U'_1, U'_2) is a $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular pair. Applying Lemma 5.12 to the pair (U'_1, U'_2) , we can construct two paths P_1 and P_2 such that the end vertices of P_1 are x_6, y_5 , the end vertices of P_2 are x_4, y_7 and $|V(P_i)| = l^i$ ($i = 1, 2$). Let $P_3 := x_5 y_8 x_3 y_2 y_1 x y_3 x_1 x_2 y_4 x_4$ and $P_4 := x_6 y_6 y y_7$. Then

$$C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$$

is a Hamiltonian cycle in G (see Figure 5.4, Sub-case 2(b)). Let $l^1 = \frac{n}{2} - |V(P)| - 5$ and $l^2 = 2n' - l^1$. Thus x and y have distance $\frac{n}{2}$ on C . We can say all the choices of the vertices are possible because of the minimum degree of G^* . We omit all these similar calculations here.

If W is empty, we take the path P be an edge. The rest of the proof is the same as above.

At the end we need to consider the case when $\frac{n}{2}$ is even. Actually if x, y are in the same part of G^* , the proof is similar to Sub-case 1, and if x, y are in different parts of G^* , the proof is similar to Sub-case 2.

5.4.2 Extremal case 2

Suppose G is a graph on n vertices with $\delta(G) \geq \frac{n}{2} + 1$ and there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$. We suppose $\alpha \leq (\frac{1}{9})^3$. Let $\alpha_1 = \alpha^{\frac{1}{3}}$ and $\alpha_2 = \alpha^{\frac{2}{3}}$.

We also need a similar lemma as Lemma 5.27.

Lemma 5.29. *If G is in extremal case 2, then $V(G)$ can be partitioned into two balanced parts U_1 and U_2 such that*

(a) there is a set $W_1 \subseteq U_1$ (respectively $W_2 \subseteq U_2$) such that there exist vertex-disjoint 2-paths in $G[U_1]$ (respectively $G[U_2]$) with the vertices of W_1 (respectively W_2) as the internal vertices in each 2-path and $|W_1| \leq \alpha_2 \frac{n}{2}$ (respectively $|W_2| \leq \alpha_2 \frac{n}{2}$);

(b) $\deg_{G[U_1]}(u) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $u \in U_1 - W_1$ and $\deg_{G[U_2]}(v) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $v \in U_2 - W_2$.

Proof. The argument is similar to the proof of Lemma 5.27. For some similar claims, we just give them without proofs.

For $i = 1, 2$, let $V_i^* = \{v \in V_i : \deg(v, V_i) \geq (1 - \alpha_1) \frac{n}{2}\}$. We can claim that $|V_i - V_i^*| \leq \alpha_2 \frac{n}{2}$ since otherwise $d(V_1, V_2) > \frac{\alpha_1 \frac{n}{2} \cdot \alpha_2 \frac{n}{2}}{\frac{n}{2} \cdot \frac{n}{2}} = \alpha$, a contradiction.

For any vertex $v \in V_i - V_i^*$, if $\deg(v, V_{3-i}) \geq (1 - \alpha_1) \frac{n}{2}$, we also add it to V_{3-i}^* . We denote the final two sets by V_i' ($i = 1, 2$) and let $V_0 = V - V_1' - V_2'$. Thus $|V_0| \leq \alpha_2 n$. For every vertex v in V_i' , $\deg(v, V_i') \geq (1 - \alpha_1) \frac{n}{2} - \alpha_2 \frac{n}{2}$ ($i = 1, 2$). For every vertex u in V_0 , $\deg(u, V_i') \geq (\frac{n}{2} - (1 - \alpha_1) \frac{n}{2}) - \alpha_2 \frac{n}{2} \geq (\alpha_1 - \alpha_2) \frac{n}{2}$ ($i = 1, 2$).

First, we assume $|V_1'|, |V_2'| \leq \frac{n}{2}$. Then we add all the vertices in V_0 to V_1' and V_2' such that the final two sets are of the same size. Denote the final two sets by U_1 and U_2 . Let $W_1 = U_1 - V_1'$ and $W_2 = U_2 - V_2'$, then $V_0 = W_1 \cup W_2$. Since for each vertex $u \in W_1$, $\deg(u, V_1') \geq (\alpha_1 - \alpha_2) \frac{n}{2} \geq 2\alpha_2 n \geq 2|V_0|$, we can greedily choose two neighbors of u in V_1' such that the neighbors of all the vertices in W_1 are distinct. So W_1 and U_1 are what we need. It is same to find 2-paths in $G[U_2]$. The degree conclusion also holds.

Second, without loss of generality we assume $|V_1'| > \frac{n}{2}$. Let V_1^0 be the set of vertices $v \in V_1'$ such that $\deg(v, V_2') \geq \alpha_1 \frac{n}{2}$.

If $|V_1^0| \geq |V_1'| - \frac{n}{2}$, we take W_2 to be the set of all vertices of V_0 and $|V_1'| - \frac{n}{2}$ vertices of V_1^0 and W_1 to be an empty set. Let $U_1 = V_1' - W_2$ and $U_2 = V_2' \cup W_2$. So $|W_2| \leq \alpha_2 \frac{n}{2}$. For every vertex $u \in W_2$, we have $\deg(u, V_2') \geq (\alpha_1 - \alpha_2) \frac{n}{2} - \alpha_2 \frac{n}{2} \geq \alpha_2 n \geq 2|W_2|$. Thus we can greedily choose two neighbors of u in V_2' such that the neighbors of all the vertices in W_2 are distinct. U_1, U_2, W_1, W_2 are what we need.

Now we assume $|V_1^0| < |V_1'| - \frac{n}{2}$. Let $U_1 = V_1' - V_1^0$ and $U_2 = V_2' \cup V_0 \cup V_1^0$. Let $t = |U_1| - \frac{n}{2}$, so $t \leq \alpha_2 \frac{n}{2}$. We consider the bipartite graph (U_1, V_2') . Suppose that (U_1, V_2') has a biggest family of vertex-disjoint 2-paths on a vertex set S , such that the internal vertices of these 2-paths are in U_1 and the end vertices of these 2-paths are in V_2' . Let $S = S_1 \cup S_2$ with the internal vertex set $S_1 \subseteq U_1$ and the end vertex set $S_2 \subseteq V_2'$. Suppose $|S_1| = s$, $|S_2| = 2s$. We use δ^* to denote the minimum degree of vertices of V_2' in (U_1, V_2') . So $\delta^* \geq \frac{n}{2} + 1 - (\frac{n}{2} - t - 1) = t + 2$. We use Δ^* to denote the maximum

degree of vertices of U_1 in (U_1, V_2') . So $\Delta^* < \alpha_1 \frac{n}{2}$. Then

$$\delta^*(|V_2'| - 2s) \leq e(V_2' - S_2, U_1) \leq s(\Delta^* - 2) + \left(\frac{n}{2} + t - s\right).$$

By some calculations, we can get

$$\begin{aligned} s &\geq \frac{(t+2)|V_2'| - \left(\frac{n}{2} + t\right)}{\Delta^* + 2t + 1} \\ &\geq \frac{(t+2)(1 - \alpha_2)\frac{n}{2} - \frac{n}{2} - \alpha_2\frac{n}{2}}{\alpha_1\frac{n}{2} + 1 + 2\alpha_2\frac{n}{2}} \\ &= \frac{(t+1)(1 - \alpha_2)\frac{n}{2} - 2\alpha_2\frac{n}{2}}{(\alpha_1 + 2\alpha_2)\frac{n}{2} + 1} \end{aligned}$$

Since n can be sufficiently large, we can conclude $s > t$.

We pick t vertex-disjoint 2-paths with the internal vertex set $S_1 \subseteq U_1$ and move the vertices of S_1 into U_2 . Now we get $|U_1| = |U_2| = \frac{n}{2}$. Let $W_2 = V_0 \cup V_1^0 \cup S_1$ and W_1 be an empty set. For every vertex $u \in V_0 \cup V_1^0$, $\deg_{G[U_2]}(u) \geq (\alpha_1 - \alpha_2)\frac{n}{2} - \alpha_2\frac{n}{2} \geq 2|W_2|$. We can greedily find disjoint 2-paths in $G[V_0 \cup V_1^0, V_2']$ with all the vertices of $V_0 \cup V_1^0$ as internal vertices such that these 2-paths are all disjoint with the existing 2-paths. U_1, U_2, W_1, W_2 are what we need. \square

For a graph G in extremal case 2, we apply Lemma 5.29 to G and get a partition of $V(G) = U_1 \cup U_2$ with the properties in Lemma 5.29.

First, assume x and y are in different parts in the partition of $V(G)$, without loss of generality, we say that $x \in U_1$ and $y \in U_2$. Since $\delta(G) \geq \frac{n}{2} + 1$, x (respectively y) should have at least two neighbors in U_2 (respectively U_1). Denote a neighbor of x in U_2 by x_1 and a neighbor of y in U_1 by y_1 such that $x_1 \neq y$ and $y_1 \neq x$. Since the 2-paths are all greedily chosen in Lemma 5.29, we can assume that x, y, x_1, y_1 are not the end vertices of those 2-paths.

Claim 5.30. *There is a Hamiltonian path in $G[U_1]$ with end vertices x and y_1 .*

Proof. Whether x and y_1 are in W_1 or not, we can find a neighbor of x in $U_1 - W_1$, denoted u , and a neighbor of y_1 in $U_1 - W_1$, denoted v . Suppose $W_1 - \{x, y_1\} = \{w_1, w_2, \dots, w_t\}$ and the two end vertices of the 2-path containing w_i are a_i, b_i . Since $\deg_{G[U_1]}(a_i) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ and $\deg_{G[U_1]}(b_i) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$, we can greedily choose $c_i \in U_1$ which is a common neighbor of a_{i+1} and b_i ($1 \leq i \leq t-1$). Moreover we can choose all these c_i to be distinct. We also greedily choose c_t which is a common neighbor of b_t and v . Then $P_1 = a_1 w_1 b_1 c_1 a_2 w_2 b_2 c_2 \dots b_{t-1} c_{t-1} a_t w_t b_t c_t v$ is a path containing all

the vertices of those 2-paths (except the 2-paths containing x, y_1 , if x, y_1 are in W_1). $|V(P_1)| = 4t + 1 \leq 4\alpha_2 \frac{n}{2} + 1$.

Let $U^* = (U_1 - V(P) - \{x, y_1\}) \cup \{a_1\}$. We consider the induced subgraph $G[U^*]$. For any vertex $w \in U^*$, $deg_{G[U^*]}(w) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2} - 4\alpha_2 \frac{n}{2} - 2$. Since n is sufficiently large and $\alpha \leq (\frac{1}{9})^3$, $deg_{G[U^*]}(w) \geq (1 - \alpha_1 - 7\alpha_2) \frac{n}{2} > \frac{n}{4} + 1 \geq \frac{|U^*|}{2} + 1$ for any vertex $w \in U^*$. So $G[U^*]$ is Hamiltonian-connected. We can find a path P_2 in $G[U^*]$ with end vertices u, a_1 containing all the vertices of U^* . Then $H_1 = \{xu\} \cup P_1 \cup P_2 \cup \{vy_1\}$ is a Hamiltonian path in $G[U_1]$ with end vertices x and y_1 . \square

By the same method, we can construct a Hamiltonian path H_2 in $G[U_2]$ with end vertices y and x_1 . So

$$C = \{xx_1, yy_1\} \cup H_1 \cup H_2$$

is a Hamiltonian cycle in G such that $dist_C(x, y) = \frac{n}{2}$ (see Figure 5.5 (a)).

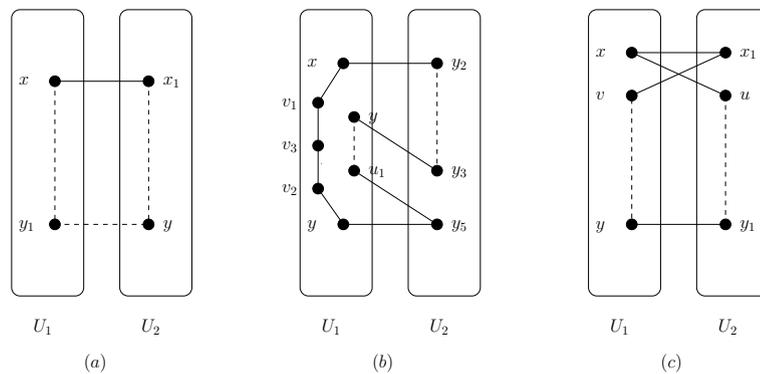


FIGURE 5.5: Extremal case 2.

Now assume x and y are in the same part in the partition of $V(G)$, without loss of generality, say $x, y \in U_1$. Since $\delta(G) \geq \frac{n}{2} + 1$, x and y should have at least two neighbors in U_2 . We choose a neighbor of x in U_2 , denoted x_1 , and a neighbor of y in U_2 , denoted y_1 , such that $x_1 \neq y_1$. Since the 2-paths are all greedily chosen in Lemma 5.29, we can assume that x, y, x_1, y_1 are not the end vertices of those 2-paths.

Assume there is a vertex $u \in U_2 - \{x_1, y_1\}$ such that it has two neighbors $u_1, u_2 \in U_1 - \{x, y\}$. We also assume that u_1, u_2 are not the end vertices of the 2-paths. Whether x, u_2 are in W_1 or not, we claim that we can find a path of length at most four with end vertices x and u_2 in $G[U_1]$. Indeed, the worst case is when x, u_2 are both in W_1 . We can find a neighbor of x in $U_1 - W_1$, denoted v_1 , and a neighbor of u_2 in $U_1 - W_1$, denoted v_2 . We choose a common neighbor of v_1, v_2 in $G[U_1]$, denoted v_3 . $xv_1v_3v_2u_2$ is a path of length four with end vertices x and u_2 in $G[U_1]$. Then we can construct a path with end vertices x_1 and u_1 , denote it by $P_1 = x_1xv_1v_3v_2u_2uu_1$. By the same method

in the proof of Claim 5.30, we construct a path P_2 in $G[U_1]$ with end vertices u_1 and y , containing all the vertices of $U_1 - V(P_1)$. In $G[U_2]$, by the same method in the proof of Claim 5.30, we can construct a path P_3 with end vertices x_1 and y_1 , containing all the vertices of $U_2 - \{u\}$. So

$$C = P_1 \cup P_2 \cup P_3 \cup \{yy_1\}$$

is a Hamiltonian cycle in G such that $\text{dist}_C(x, y) = \frac{n}{2}$ (see Figure 5.5 (b)).

Assume there is a vertex $u \in U_2 - \{x_1, y_1\}$ such that only one of the neighbors of u in U_1 is equal to x or y , without loss of generality, we assume that u_1, u_2 , the two neighbors of u , satisfy that $u_2 = x$ but $u_1 \neq y$. Let $P_1 = x_1xuu_1$. The rest construction is the same as in the last paragraph.

At last we assume that in U_1 , the neighbors of all the vertices of $U_2 - \{x_1, y_1\}$ are x and y . That means any vertex in $U_1 - \{x, y\}$ is adjacent to x_1 and y_1 . We choose a neighbor of x in U_2 , denoted u , and a neighbor of x_1 in U_1 , denoted v . We construct a path $P_1 = vx_1xu$. By the same method in the proof of Claim 5.30, we can construct a path P_2 in $G[U_1]$ with end vertices v and y , containing all the vertices of $U_1 - \{x\}$, and a path P_3 in $G[U_2]$ with end vertices u and y_1 , containing all the vertices of $U_2 - \{x_1\}$. So

$$C = P_1 \cup P_2 \cup P_3 \cup \{yy_1\}$$

is a Hamiltonian cycle in G such that $\text{dist}_C(x, y) = \frac{n}{2}$ (see Figure 5.5 (c)).

5.5 Concluding remarks and further work

In this chapter, we gave a proof of Enomoto's conjecture for graphs of sufficiently large order. Actually, our approach also works for proving Conjecture 1.51, we will show this in the next chapter.

Note that, our proof is for graphs of sufficiently large order, but how large is it? That is the common question when we use Regularity Lemma. We try to prove the Theorem 5.2 without using Regularity Lemma, but unfortunately, we did not succeed yet. Recently, there are some works on how the use of the Regularity Lemma and the Blow-up Lemma can be avoided in certain extremal problems of dense graphs (see [65]). This gives us positive sign to find a way to prove Theorem 5.2 avoiding using the Regularity Lemma in future.

Chapter 6

Distributing pairs of vertices on Hamiltonian cycles

In this chapter we give a proof of Conjecture 1.51 for graphs of sufficiently large order. It is kind of continue work of Chapter 5. To make it easier to read, we state again Conjecture 1.51 here.

Conjecture 6.1. [33] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

Our main result is the Theorem 1.53. We state again this theorem here.

Theorem 6.2. [47] *There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = k$.*

6.1 Introduction

In Chapter 5, we gave a proof of Enomoto's conjecture for graphs of sufficiently large order.

Theorem 6.3 (Theorem 5.2). *There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G , there is a Hamiltonian cycle C of G such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.*

Here we will prove that Faudree-Li conjecture is true for graphs of sufficiently large order by showing Theorem 6.2.

The main idea and the main tools of the proof of Theorem 6.2 and Theorem 5.2 are similar, but there are also some differences since in Theorem 5.2, any two given vertices are required to be located on a Hamiltonian cycle of distance half of the order of the graph, while in Theorem 6.2 the distance is extended from half of the order of the graph to two. We don't want to just point out the differences between the proofs of Theorem 6.2 and Theorem 5.2, to make this chapter complete, we will give the whole proof of Theorem 6.2. We will follow all the notations, such as *balanced partition*, *reduced graph* et al. as in Chapter 5.

6.2 Outline of the proof

In our proof for Theorem 6.2 we will use the Regularity Lemma-Blow-up Lemma method as many other studies. Some claims and lemmas in our proof are similar as in Chapter 5. We will give these claims and lemmas without proofs here.

We first consider the graphs of even order n . We assume that n is sufficiently large and fix the following sequence of parameters,

$$0 < \epsilon \ll d \ll \alpha \ll 1. \quad (6.1)$$

By Theorem 1.48 and Theorem 6.3, we assume that the required distance k satisfying

$$\frac{n}{6} < k < \frac{n}{2}. \quad (6.2)$$

We define two extremal cases as follows.

Extremal Case 1: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$.

Extremal Case 2: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$.

We will prove Theorem 6.2 for the non-extremal case in Section 6.3 and for the extremal cases in Section 6.4. The proof will include some sub-cases discussions because of the parity of k .

When n is odd, we pick a vertex $z \neq x$ or y of the graph G and consider the induced graph $G[V(G) - z]$. Since the order of the graph $G[V(G) - z]$ is even, we use the similar proof as above in $G[V(G) - z]$ and obtain a Hamiltonian cycle in $G[V(G) - z]$.

Then we “insert” z into this cycle and make sure x, y have distance k on the resulting Hamiltonian cycle of G . In Section 6.3 and 6.4, the proof will include all the discussions when n is odd.

6.3 Non-extremal case

6.3.1 The graph order n is even

Step 1. Applying the Regularity Lemma.

Let G be a graph of even order which is not either of the extremal cases. We apply the Regularity Lemma to G with parameters ϵ, d, m_0 to obtain a partition of G into clusters V_1, \dots, V_l and an exception set V_0 , and a spanning subgraph G' . Assume that l is even, if not, we move the vertices of one of the clusters into V_0 to make l be an even number. Now $|V_0| \leq 2\epsilon n$ and $lL \geq (1 - 2\epsilon)n$. Let $s := \frac{l}{2}$.

Let R be the reduced graph of G .

The following claim, which is proved in Chapter 5, shows that R inherits the minimum degree condition.

Claim 6.4. $\delta(R) \geq (\frac{1}{2} - 2d)l$.

As shown in Chapter 5, R is Hamiltonian.

Let C_R be a Hamiltonian cycle in R . We choose two distinct clusters X, Y which are as close as possible on C_R such that x is friendly to Y and y is friendly to X . The following claim is proved in Chapter 5.

Claim 6.5. *We can choose distinct clusters X and Y such that x is friendly to Y , y is friendly to X and $\text{dist}_{C_R}(X, Y) \leq 3dl$.*

By Claim 6.5 we choose these two clusters X and Y and give a new notation for all clusters except V_0 as follows. We choose a direction of C_R , which is along the longer path from Y to X on C_R (there are two paths from Y to X on C_R , and we choose the longer one). Starting from Y , we denote the clusters by $Y_1, X_2, Y_2, X_3, Y_3, \dots, X_s, Y_s, X_1$ along this direction. Y is denoted by Y_1 and X is denoted by a X_i or a Y_i .

Step 2. Constructing paths to connect Y_i and X_{i+1} (with $X_{s+1} = X_1$) for $1 \leq i \leq s$.

In this step, the parity of k and the new notation of X will affect our constructions. We first assume that k is even and X is denoted by Y_t for some $1 < t \leq s$. We call X_i, Y_i *partners* of each other ($1 \leq i \leq s$).

By Claim 6.5, $\text{dist}_{C_R}(Y_1, Y_t) = l - 2t + 2 \leq 3dl$. So

$$t - 1 \geq \frac{1 - 3d}{2}l. \quad (6.3)$$

Let $w_x, y_1^1 \in Y_1$ be two neighbors of x such that w_x is friendly to X_2 and y_1^1 is friendly to X_1 . Since x is friendly to Y_1 , x has at least $(d - \epsilon)L$ neighbors in Y_1 . By Lemma 5.3, at least $(d - \epsilon)L - \epsilon L$ vertices can be chosen as w_x (also true for y_1^1). Let $x_2^1 \in X_2$ be a neighbor of w_x such that x_2^1 is friendly to Y_2 . By Lemma 5.3, at least $(d - \epsilon)L - \epsilon L$ vertices can be chosen as x_2^1 . We obtain a path $P_1 = y_1^1 x w_x x_2^1$ connecting Y_1 and X_2 . We call this procedure *joining* x to Y_1 and x behaves like a vertex in X_1 . Let $y_1^2 \in Y_1$ be a vertex friendly to X_1 and let $x_2^2 \in X_2$ be a neighbor of y_1^2 such that x_2^2 is friendly to Y_2 . By Lemma 5.3, this choice of y_1^2 and x_2^2 is possible and make them be distinct with the vertices in P_1 . We obtain another path $Q_1 = y_1^2 x_2^2$. Similarly we can construct paths $P_t = y_t^1 y_w x_{t+1}^1$ and $Q_t = y_t^2 x_{t+1}^2$, where $y_t^1, y_t^2 \in Y_t$ is friendly to X_t , $x_{t+1}^1, x_{t+1}^2 \in X_{t+1}$ is friendly to Y_{t+1} and $w_y \in Y_t$. We call this procedure *joining* y to Y_t and x behaves like a vertex in X_t . For $1 \leq i \leq s$ and $i \neq 1, t$, let $y_i^1 \in Y_i$ (resp. $y_i^2 \in Y_i$) be a vertex friendly to X_i and $x_{i+1}^1 \in X_{i+1}$ (resp. $x_{i+1}^2 \in X_{i+1}$) be a neighbor of y_i^1 (resp. y_i^2) such that x_{i+1}^1 (resp. x_{i+1}^2) is friendly to Y_{i+1} . Here $x_1^1 = x_{s+1}^1$ and $x_1^2 = x_{s+1}^2$. Let $P_i = y_i^1 x_{i+1}^1$ and $Q_i = y_i^2 x_{i+1}^2$ ($1 \leq i \leq s$ and $i \neq 1, t$). By Lemma 5.3, we can make sure all these paths are vertex-disjoint.

In summary, we have constructed paths P_i and Q_i , which are vertex-disjoint and connect Y_i and X_{i+1} ($1 \leq i \leq s$), x is on P_1 and y is on P_t . Each end vertex of these paths is friendly to its cluster's partner. We use INT to denote the vertex set of all internal vertices on all P_i 's and Q_i 's. Now $INT = \{x, y, w_x, w_y\}$.

For every $1 \leq i \leq s$, let

$$X'_i := \{u \in X_i : \text{deg}(u, Y_i) \geq (d - \epsilon)L\}, Y'_i := \{v \in Y_i : \text{deg}(v, X_i) \geq (d - \epsilon)L\}.$$

Since (X_i, Y_i) is ϵ -regular, we have $|X'_i|, |Y'_i| \geq (1 - \epsilon)L$. We move all the vertices in $X_i - X'_i$ and $Y_i - Y'_i$ to V_0 . If $|X'_i| \neq |Y'_i|$, say $|X'_i| > |Y'_i|$, we choose an arbitrary subset of X'_i of size $|Y'_i|$ and still name it X'_i . We still denote the set $V_0 \cup \bigcup_{i=1}^k (X_i - X'_i) \cup \bigcup_{i=1}^k (Y_i - Y'_i)$ by V_0 . We also remove all the vertices in INT out of V_0 , X'_i and Y'_i . This may cause that some (X'_i, Y'_i) is not balanced. For example, if $w_x \in Y'_1$ and we remove it from Y'_1 , then (X'_1, Y'_1) be not balanced. In this example, to make sure (X'_1, Y'_1) is balanced, we move one vertex in X'_1 to V_0 . We do the same operations for all

the vertices in INT . Since $|INT| = 4$ and n is sufficiently large, at most $\epsilon L + 4 \leq 2\epsilon n$ vertices are moved to V_0 . We derive that $|V_0| \leq 4\epsilon n$ and $|X'_i| = |Y'_i| \geq (1 - \epsilon)L - 1$ ($1 \leq i \leq s$) in this step. Since $\epsilon L \geq \epsilon \frac{(1-2\epsilon)n}{l} \geq \frac{\epsilon(1-2\epsilon)}{M_0}n$ and n is sufficiently large, we say $\epsilon L \geq 1$. Thus $|X'_i| = |Y'_i| \geq (1 - 2\epsilon)L$ ($1 \leq i \leq s$). The minimum degree in each pair is at least $(d - \epsilon)L - \epsilon L - 1 \geq (d - 3\epsilon)L$. We also can say (X'_i, Y'_i) is 2ϵ -regular by Slicing Lemma.

Now assume that k is even and X is denoted by X_t for some $1 \leq t \leq s$. We construct $P_2 = y_2^1 x_3^1$ and $Q_2 = y_2^2 x_3^2$ as before. Since $\delta(G) \geq \frac{n}{2} + 1$, any two vertices have at least two common neighbors. Suppose that y_2^1, x_3^1 have a common neighbor u_1 and y_2^2, x_3^2 have a common neighbor $u_2 \neq u_1$. First, if u_1 and u_2 are both different with x and y , then we choose $P_2 = y_2^1 u_1 x_3^1$ and $Q_2 = y_2^2 u_2 x_3^2$. We join x to Y_1 , join y to X_t and continue all the other constructions of P_i 's and Q_i 's as before. In this case x is on P_1 and y is on P_{t-1} . Second, assume that we cannot find these required vertices $y_2^1, x_3^1, y_2^2, x_3^2$ such that u_1 and u_2 are both different with x and y . We choose y_2^1, y_2^2 to be friendly to X_2 and X_3 . There are at least $(1 - 2\epsilon)L$ possible choices for y_2^1 and it is similar for y_2^2 . We choose x_3^1 to be a neighbor of y_2^1 and friendly to Y_2 and Y_3 . There are at least $(d - \epsilon)L - 2\epsilon L = (d - 3\epsilon)L$ possible choices for x_3^1 and it is similar for x_3^2 . By the assumption every these possible $y_2^1, x_3^1, y_2^2, x_3^2$ should be neighbors of x or y . So x or y should have at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and also in X_3 . If y has at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and in X_3 , we change the choice of X and choose Y_2 to be the X . Here $X = Y_2$ and $Y = Y_1$, so $\text{dist}_{C_R}(X, Y) = 2$. Otherwise, x has at least $\frac{1}{2}(d - 3\epsilon)L$ neighbors in Y_2 and in X_3 . We change the choice of Y and choose X_3 to be the Y . In this case $X = X_t$ and $Y = X_3$, so $\text{dist}_{C_R}(X, Y) \leq 3dl + 3$ by Claim 6.5. We give a new notation for all clusters except V_0 according to the new choice of X, Y and construct P_i 's, Q_i 's as before. In this new notation, we still say we join x to Y_1 and join y to Y_t for a new t . We need to mention that (6.3) may have slight difference as follows, but it won't affect the calculation in the following Claim 6.7.

We also need to consider the case when k is odd. Actually if k is odd and X is denoted by $X_{t'}$ for some $1 \leq t' \leq s$, the discussions in this case is similar as in the case when k is even and X is denoted by Y_t for some $1 \leq t \leq s$. And if k is odd and X is denoted by $Y_{t'}$ for some $1 \leq t' \leq s$, the discussions in this case is similar as in the case when k is even and X is denoted by X_t for some $1 \leq t \leq s$. We omit all these similar arguments here.

Step 3. Extending Q_i 's ($1 \leq i \leq s$) by using all the vertices of V_0 .

If a vertex v is friendly to a cluster X , we denote this relation by $v \sim X$. If two clusters X and Y are a regular pair, we denote this relation by $X \sim Y$. Given two

vertices $u, v \in V(G)$, a u, v -chain of length $2s$ with distinct clusters $A_1, B_1, \dots, A_s, B_s$ is $u \sim A_1 \sim B_1 \sim \dots \sim A_s \sim B_s \sim v$ and $\{A_j, B_j\} = \{X_i, Y_i\}$ for some $1 \leq i \leq k$. We can say that for each pair of vertices in V_0 we have the following claim, which is proved in Chapter 5.

Claim 6.6. *For each pair of vertices $\{u, v\}$ in V_0 , we can find u, v -chains of length at most four such that every cluster is used in at most $\frac{d}{10}L$ chains.*

Now we extend Q_i 's ($1 \leq i \leq s$). By Claim 6.6, we construct chains of length at most four for each pair such that every cluster is used in at most $\frac{d}{10}L$ chains. We deal with the vertices of V_0 pair by pair. Assume that we deal with the pair (u, v) now. Denote $X_i^* = X_i' - INT$ and $Y_i^* = Y_i' - INT$.

We first consider the case when the u, v -chain has length two. Without loss of generality, assume that this chain is $u \sim X_i \sim Y_i \sim v$, for some $1 \leq i \leq s$. Let $w_1 \in Y_i^*$ be a neighbor of x_i^2 and let $w_2 \in X_i^*$ be a neighbor of u such that w_1 and w_2 are adjacent. By Slicing Lemma, we can say this choice is possible. Let $w_3 \in X_i^*$ be another neighbor of u . To include u , we extend Q_{i-1} to $Q_{i-1} \cup \{x_i^2 w_1, w_1 w_2, w_2 u, u w_3\}$. We still denote this new path by Q_{i-1} and denote the end vertex w_3 of this path by the new x_i^2 . Since u behaves like a vertex in Y_i on Q_{i-1} , we call this procedure *inserting u into Y_i* by extending Q_{i-1} . Similarly, let $w_4 \in X_i^*$ be a neighbor of y_i^2 and let $w_5, w_6 \in Y_i^*$ be two neighbors of v such that w_4 and w_5 are adjacent. To include v , we extend Q_i to $Q_i \cup \{y_i^2 w_4, w_4 w_5, w_5 v, v w_6\}$. We still denote this new path by Q_i and denote the end vertex w_6 of this path by the new y_i^2 . Since v behaves like a vertex in X_i on Q_i , we call this procedure *inserting v into X_i* by extending Q_i . And we update the set INT . Indeed, three vertices of X_i^* are added to INT (also for Y_i^*) and totally eight vertices are added to INT including u, v .

Now we consider that the u, v -chain has length four. Without loss of generality, we assume that the chain is $u \sim X_i \sim Y_i \sim X_j \sim Y_j \sim v$, for some i, j . We extend the path Q_{i-1} by inserting u into Y_i . We choose a vertex of Y_i^* which is friendly to X_j and insert it into Y_j to extend Q_{j-1} . At last we extend the path Q_j by inserting v into X_j . Meanwhile, we update the set INT . Indeed, two vertices of X_i^* are added to INT (also for Y_i^*) and three vertices of X_j^* are added to INT (also for Y_j^*). So totally twelve vertices are added into INT including u, v .

In each time inserting the vertex pair (u, v) , the cluster pair (X_i^*, Y_i^*) is still balanced and at most three vertices of a cluster in the chain are used. So

$$|X_i^*| = |Y_i^*| \geq (1 - 2\epsilon)L - 3\frac{d}{10}L \geq (1 - \frac{d}{2})L$$

provided $\epsilon < \frac{d}{10}$.

For each vertex $u \in X_i^*$, we have

$$\deg(u, Y_i^*) \geq (d - 3\epsilon)L - 3\frac{d}{10}L \geq \frac{d}{2}L$$

provided $\epsilon < \frac{d}{15}$. And it is the same for the degree of any vertex in Y_i^* .

By Slicing Lemma, we can say the pair (X_i^*, Y_i^*) is $(2\epsilon, \frac{d}{2})$ -super-regular ($1 \leq i \leq s$). We continue this process till there is no vertices left in V_0 .

Step 4. Constructing the desired Hamiltonian cycle.

In this step, we use Lemma 5.12 to construct two paths W_i^1 and W_i^2 in each pair (X_i^*, Y_i^*) ($1 \leq i \leq s$). Then we combine all these paths with P_i 's and Q_i 's to obtain a Hamiltonian cycle in G . At last we adjust the length of W_i^1 and W_i^2 in each pair to make sure that x and y have distance k on this Hamiltonian cycle.

For each $1 \leq i \leq s$, we choose any even integers l_i^1, l_i^2 such that $4 \leq l_i^1, l_i^2 \leq 2|X_i^*| - 4$ and $l_i^1 + l_i^2 = 2|X_i^*|$. We will adjust these integers later.

For $2 \leq i \leq s$, by Lemma 5.12, we construct two paths W_i^1 and W_i^2 in the pair (X_i^*, Y_i^*) such that

- (a) W_i^1 has end vertices x_i^1 and y_i^1 with $|V(W_i^1)| = l_i^1$;
- (b) W_i^2 has end vertices x_i^2 and y_i^2 with $|V(W_i^2)| = l_i^2$.

And for $i = 1$, we construct two paths W_1^1 and W_1^2 in the pair (X_1^*, Y_1^*) such that

- (c) W_1^1 has end vertices x_1^1 and y_1^1 with $|V(W_1^1)| = l_1^1$;
- (d) W_1^2 has end vertices x_1^2 and y_1^2 with $|V(W_1^2)| = l_1^2$.

It is not hard to see

$$C = P_1 \cup \left(\bigcup_{i=2}^k (W_i^1 \cup P_i) \right) \cup W_1^1 \cup Q_1 \cup \left(\bigcup_{i=2}^k (W_i^2 \cup Q_i) \right) \cup W_1^2$$

is a Hamiltonian cycle in G (See Figure 6.1).

Now we adjust the values of l_i^1 and l_i^2 ($1 \leq i \leq s$) to make sure that x and y have distance k on the Hamiltonian cycle.

Claim 6.7. *We can properly choose the value of l_i^1 ($2 \leq i \leq s$) such that $\text{dist}_C(x, y) = k$.*

Proof. Without loss of generality, we consider the case when k is even and X is denoted by Y_t in the beginning of the second step. Since x is on P_1 and y is on P_t , we consider

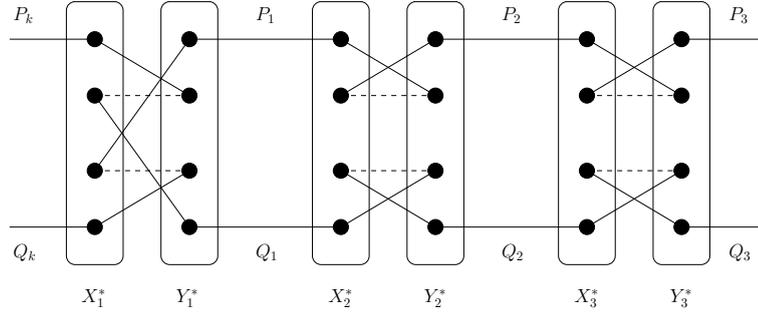


FIGURE 6.1: Construction of the Hamiltonian cycle

the path $P := P_1 \cup (\bigcup_{i=2}^t (W_i^1 \cup P_i))$ to make sure x and y have distance k on this path. That means the number of vertices between x and y on P should be $k - 1$. Among the vertices between x and y on P , the only vertex not belong to W_i^1 ($2 \leq i \leq t$) is w_x . Thus we need to make sure

$$k - 1 = \sum_{i=2}^t l_i^1 + 1. \quad (6.4)$$

Since k is even and all these l_i^1 's are also even, there is no parity problem. We say that $\sum_{i=2}^t l_i^1$ can be any even value satisfying

$$\frac{n}{6} - 2 < \sum_{i=2}^t l_i^1 < \frac{n}{2} - 2. \quad (6.5)$$

Since by Lemma 5.12, l_i^1 can be any even integer such that $4 \leq l_i^1 \leq 2|X_i^*| - 4$. By $|X_i^*| \geq (1 - \frac{d}{2})L$, we can choose l_i^1 such that $\sum_{i=2}^t l_i^1$ can be any even integer with the following bound,

$$4(t-1) \leq \sum_{i=2}^t l_i^1 \leq 2(t-1)(1 - \frac{d}{2})L - 4(t-1) = 2(t-1)((1 - \frac{d}{2})L - 2).$$

We know that $t \leq s = \frac{l}{2}$, then $4(t-1) < 2l$. Since $l \leq M_0$ in the Regularity Lemma and n is sufficiently large (let $n \geq 12(M_0 + 1)$), we can say that $4(t-1) < 2l \leq 2M_0 \leq \frac{n}{6} - 2$.

By (6.3), we also know $t - 1 \geq \frac{1-3d}{2}l$, so

$$\begin{aligned} 2(t-1)((1 - \frac{d}{2})L - 2) &\geq (1 - 3d)(1 - \frac{d}{2})lL - 2(1 - 3d)l \\ &\geq (1 - \frac{7}{2}d)(1 - 2\epsilon)n - 2l \\ &\geq (1 - 4d)n - 2M_0 \geq \frac{3}{4}n - 2M_0, \end{aligned}$$

provided $4\epsilon \leq d \leq \frac{1}{16}$ and $l \leq M_0$. Since n is sufficiently large (let $n \geq 8M_0$), we can say that $2(t-1)((1-\frac{d}{2})L-2) \geq \frac{3}{4}n - 2M_0 \geq \frac{n}{2}$.

Thus we have proved that $\sum_{i=2}^t l_i^1$ can be any even value satisfying (6.5). By our assumption (6.2), $\frac{n}{6} < k < \frac{n}{2}$. So it is possible to choose l_i^1 ($2 \leq i \leq t$) such that $\sum_{i=2}^t l_i^1$ satisfying (6.4).

We choose l_i^1 ($2 \leq i \leq t$) such that (6.4) holds and arbitrarily choose the even integers l_1^1, l_i^1 ($t < i \leq k$) with the conditions in Lemma 5.12. Thus $dist_C(x, y) = k$. \square

6.3.2 The graph order n is odd

We pick a vertex $z \neq x$ or y and assume that $G[V-z]$ is a graph which is not either of the extremal cases. The order of the induced graph $G[V-z]$ is $n-1$ and $\delta(G[V-z]) \geq \frac{n-1}{2} + 1$. We apply the proof above to $G[V-z]$ and obtain a Hamiltonian cycle C in $G[V-z]$ such that $dist_C(x, y) = k$. We denote the longer path between x and y on C by $P(x, y)$, i.e. the distance of x and y on $P(x, y)$ is $n-1-k$. Since $\delta(G) \geq \frac{n}{2} + 1$, z has at least two neighbors such that they are adjacent on C .

If there exist two vertices $u, v \in P(x, y)$ such that they are neighbors of z and adjacent in $P(x, y)$, then replacing the edge uv in C by the path uzv we can obtain a Hamiltonian cycle C^* in G such that $dist_{C^*}(x, y) = k$.

Otherwise, this kind of two vertices u, v are on the shorter path between x and y on C . We make the shorter path between x, y on C be the longer one by adjusting the values of l_i^1 's. Precisely, we consider the path $P = P_1 \cup (\bigcup_{i=2}^t (W_i^1 \cup P_i))$ as in Claim 6.7 and make sure $dist_P(x, y) = n-1-k$.

If $uv \in E(P_i)$ for some $1 \leq i \leq t$, we need to make sure that

$$n - k - 2 = \sum_{i=2}^t l_i^1 + 1. \tag{6.6}$$

Since n is sufficiently large, we can say $4(t-1) \leq \frac{n}{2} - 2$ and $2(t-1)((1-\frac{d}{2})L-2) \geq \frac{5}{6}n - 2$ similarly as in Claim 6.7. So $\sum_{i=2}^t l_i^1$ can be any even value satisfying $\frac{n}{2} - 2 < \sum_{i=2}^t l_i^1 < \frac{5}{6}n - 2$. Since $\frac{n}{6} < k < \frac{n}{2}$, we can choose the value of l_i^1 ($2 \leq i \leq t$) satisfying (6.6). By Lemma 5.12, we construct new paths in the pairs of clusters according to the new choice of l_i^1 's and obtain a new Hamiltonian cycle C' in $G[V-z]$. Now the edge uv is in the longer path between x and y on C' . Replacing uv by uzv in C' , we get our desired Hamiltonian cycle in G .

Now assume that $uv \in E(W_i^1)$ for some $2 \leq i \leq t$, without loss of generality, we say $uv \in E(W_2^1)$. We fix the values of l_2^1, l_2^2 and the paths W_2^1, W_2^2 , and adjust the values of l_i^1 's ($3 \leq i \leq t$) to make sure $\text{dist}_P(x, y) = n - 1 - k$. By some similar calculations as above, we can choose the values of l_i^1 's ($3 \leq i \leq t$) satisfying (6.6). Then by the new constructions of paths in the pairs (X_i^*, Y_i^*) according to the new choice of l_i^1 ($3 \leq i \leq t$), we can get a new Hamiltonian cycle C'' in $G[V - z]$ such that the edge uv is in the longer path between x and y on C'' . Replacing uv by uzv in C'' , we get a new Hamiltonian cycle of G such that the distance between x and y on this cycle is k .

6.4 Extremal cases

6.4.1 Extremal case 1

6.4.1.1 The graph order n is even

Suppose G is a graph on even n vertices with $\delta(G) \geq \frac{n}{2} + 1$ and there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$. Assume that $\alpha \leq (\frac{1}{9})^3$. Let $\alpha_1 = \alpha^{\frac{1}{3}}$ and $\alpha_2 = \alpha^{\frac{2}{3}}$. So $\alpha_1 \geq 9\alpha_2$. We need some preparations in Chapter 5 to continue our proof.

We have the following lemma and claim in Chapter 5.

Lemma 6.8. *If G is in extremal case 1, then G contains a balanced spanning bipartite subgraph G^* with parts U_1, U_2 and G^* has the following properties:*

(a) *there is a vertex set W such that there exist vertex-disjoint 2-paths (paths of length two) in G^* with the vertices of W as the internal vertices (not the end vertices) in each 2-path and $|W| \leq \alpha_2 n$;*

(b) *$\deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \notin W$.*

Claim 6.9. *We can construct a path P with end vertices $u \in U_1$ and $v \in U_2$ such that P contains all the vertices of W and $|V(P)| \leq 4|W| \leq 4\alpha_2 n$.*

Applying Lemma 6.8 to G , we obtain the graph G^* and all those properties in Lemma 6.8. In the proof of Lemma 6.8, we know that the 2-paths are greedily chosen, so we assume that x, y won't be any end vertices of those 2-paths. But x, y can be the internal vertex of a 2-path.

First we assume that k is odd and x, y are in different parts of G^* , say $x \in U_1$ and $y \in U_2$.

If $W = \emptyset$, let $y_1, y_2 \in U_2$ be two neighbors of x and let $x_1, x_2 \in U_1$ be two neighbors of y . By (b) in Lemma 6.8, it is possible to choose those vertices. Let $U'_1 = U_1 - \{x\}$, $U'_2 = U_2 - \{y\}$ and $n' = |U'_1| = |U'_2|$. By Lemma 6.8, $\deg_{U'_2}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - 1$ for any vertex $v \in U'_1$. Since n is sufficiently large, we have $\deg_{U'_2}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - 1 \geq (1 - \alpha_1 - 3\alpha_2)\frac{n}{2} \geq (1 - \alpha_1 - 3\alpha_2)n'$ for any vertex $v \in U'_1$. Since $\alpha \leq (\frac{1}{9})^3$, we have $1 - \alpha_1 - 3\alpha_2 \geq \frac{2}{3}$. Similarly, we have $\deg_{U'_1}(u) \geq (1 - \alpha_1 - 3\alpha_2)n'$ for any vertex $u \in U'_2$. By Lemma 5.6, we can say (U'_1, U'_2) is $(\sqrt{\alpha_1 + 3\alpha_2}, \frac{2}{3})$ -super-regular. Let $l^1 = k - 1$ and $l^2 = 2n' - k + 1$. By Lemma 5.12, we construct two vertex-disjoint paths P_1 and P_2 such that the end vertices of P_1 are x_1, y_1 , the end vertices of P_2 are x_2, y_2 and $|V(P_i)|$ is equal to l^i ($i = 1, 2$). Let $P_3 := y_1x_1y_2$ and $P_4 := x_1y_2x_2$. So $C = P_1 \cup P_2 \cup P_3 \cup P_4$ is a Hamiltonian cycle of G . Moreover the distance of x and y on C is k .

If $W \neq \emptyset$, by Claim 6.9 we construct a path P with end vertices $x_1 \in U_1$ and $y_1 \in U_2$ such that P contains all the vertices of W and $|V(P)| \leq 4|W|$. Since P is greedily constructed, we make sure that x and y are not included in P whether x and y are in W or not. If x or y is in W , we also make sure the vertices on the 2-paths containing x or y are not included in P . Let $U_1^* = U_1 - V(P)$ and $U_2^* = U_2 - V(P)$. By the proof of Claim 6.9, we can say $|U_1^*| = |U_2^*|$. For any vertex $u \in U_1^*$, $\deg_{U_2^*}(u) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2} - 4\alpha_2n = (1 - \alpha_1 - 10\alpha_2)\frac{n}{2}$ and for any vertex $v \in U_2^*$, $\deg_{U_1^*}(v) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2}$. Let $y_2, y_3 \in U_2^*$ be two neighbors of x and let $x_2, x_3 \in U_1^*$ be two neighbors of y . We choose a common unused neighbor of y_1, y_2 in U_1^* , denoted x_4 , and choose an unused neighbor of x_2 in U_2^* , denoted y_4 . Let $U'_1 = (U_1^* - \{x, x_2, x_4\}) \cup \{x_1\}$, $U'_2 = U_2^* - \{y, y_1, y_2\}$ and $n' = |U'_1| = |U'_2| \leq \frac{n}{2}$. For any vertex u in U'_1 , $\deg_{U'_2}(u) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3$ and for any vertex v in U'_2 , $\deg_{U'_1}(v) \geq (1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3$. Since n can be sufficiently large, we can say $(1 - \alpha_1 - 10\alpha_2)\frac{n}{2} - 3 \geq (1 - \alpha_1 - 11\alpha_2)\frac{n}{2}$. For any vertex u in U'_1 , $\deg_{U'_2}(u) \geq (1 - \alpha_1 - 11\alpha_2)\frac{n}{2} \geq (1 - \alpha_1 - 11\alpha_2)n'$ and for any vertex v in U'_2 , $\deg_{U'_1}(v) \geq (1 - \alpha_1 - 11\alpha_2)n'$. Since $\alpha \leq (\frac{1}{9})^3$, $1 - \alpha_1 - 11\alpha_2 \geq \frac{2}{3}$. By Lemma 5.6, we can say (U'_1, U'_2) is $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular. Let $l^1 = k - 1$ and $l^2 = 2n' - k + 1$. Applying Lemma 5.12 to the pair (U'_1, U'_2) , we construct two vertex-disjoint paths P_1 and P_2 such that the end vertices of P_1 are x_3, y_3 , the end vertices of P_2 are x_1, y_4 and $|V(P_i)|$ is l^i ($i = 1, 2$). We denote P_3 to be the path $P_3 := y_1x_4y_2x_3$ and P_4 to be the path $P_4 := x_3y_3x_2y_4$. Then $C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$ is a Hamiltonian cycle in G (See Figure 6.2. (1)). And $\text{dist}_C(x, y) = k$.

Now we assume that k is odd and x, y are in the same part of G^* , say $x, y \in U_1$.

We also assume that $W \neq \emptyset$. Since $\delta(G) \geq \frac{n}{2} + 1$, x should have a neighbor in U_1 . Assume this neighbor, denoted x_1 , is not y . Let $y_1 \in U_2 - W$ be a neighbor of x and $y_2 \in U_2$ be a neighbor of y_1 . Whether x_1 is in W or not, we can find an unused neighbor of x_1 in $U_2 - W$, denoted y_3 , and choose an unused neighbor of y_3 in $U_1 - W$,

denoted x_3 . Whether y_2 is in W or not, we can find an unused neighbor of y_2 in $U_1 - W$, denoted x_2 . Let $y_4, y_5 \in U_2 - W$ be two neighbors of y and $x_4 \in U_1 - W$ be a neighbor of y_4 . It is possible to choose all those vertices by Lemma 6.8. By the same method of Claim 6.9, we construct a path P with end vertices $x_5 \in U_1$ and $y_6 \in U_2$ such that P contains all the unused vertices of W and $|V(P)| \leq 4|W|$. We make sure that x and y are not included in P whether x and y are in W or not and P won't use any existing chosen vertices. We choose a common unused neighbor of x_2, x_5 in $U_2 - W$, denoted y_7 . Let $U'_1 = U_1 - V(P) - \{x, y, x_1, x_2\}$, $U'_2 = (U_2 - V(P) - \{y_1, y_2, y_3, y_4, y_7\}) \cup \{y_6\}$ and $n' = |U'_1| = |U'_2|$. By Lemma 5.6 and n is sufficiently large, we can say (U'_1, U'_2) is a $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular pair. Let $l^1 = k - 3$ and $l^2 = 2n' - k + 3$. Applying Lemma 5.12 to the pair (U'_1, U'_2) , we construct two paths P_1 and P_2 such that the end vertices of P_1 are x_3, y_5 , the end vertices of P_2 are x_4, y_6 and $|V(P_i)| = l^i$ ($i = 1, 2$). Let $P_3 := x_5y_7x_2y_2y_1xx_1y_3x_3$ and $P_4 := x_4y_4yy_5$. Then $C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$ is a Hamiltonian cycle in G (See Figure 6.2. (2)). And $dist_C(x, y) = k$.

If y is the only neighbor of x in U_1 but y has a neighbor which is not x in U_1 , then we deal with y first as above. We assume that y is the only neighbor of x in U_1 and x is the only neighbor of y in U_1 . Let $y_1, y_3 \in U_2 - W$ be a neighbor of x , let $y_2 \in U_2$ be a neighbor of y_1 and let $x_1 \in U_1 - W$ be a neighbor of y_3 . Since $deg_G(x_1) \geq \frac{n}{2} + 1$, x_1 has a neighbor in U_1 , denoted x_2 . By our assumption, x_2 should not be either of x and y . Let $x_3 \in U_1 - W$ be a neighbor of y_2 , $y_4 \in U_2 - W$ be a neighbor of x_2 and $x_4 \in U_1 - W$ be a neighbor of y_4 . By the same method of Claim 6.9, we construct a path P with end vertices $x_5 \in U_1 - W$ and $y_5 \in U_2 - W$ such that P contains all the unused vertices of W and $|V(P)| \leq 4|W|$. We make sure that x and y are not included in P whether x and y are in W or not and P won't use any existing chosen vertices. We choose two unused neighbors of y in $U_2 - V(P)$, denoted y_6, y_7 , and choose a neighbor of y_6 in $U_1 - V(P)$, denoted x_6 , and choose a common unused neighbor of x_3, x_5 in $U_2 - V(P)$, denoted y_8 . Let $U'_1 = U_1 - V(P) - \{x, y, x_1, x_2, x_3\}$ and $U'_2 = (U_2 - V(P) - \{y_1, y_2, y_3, y_4, y_6, y_8\}) \cup \{y_5\}$ and $n' = |U'_1| = |U'_2|$. By Lemma 5.6 and n is sufficiently large, (U'_1, U'_2) is a $(\sqrt{\alpha_1 + 11\alpha_2}, \frac{2}{3})$ -super-regular pair as before. Let $l^1 = k - 5$ and $l^2 = 2n' - k + 5$. Applying Lemma 5.12 to the pair (U'_1, U'_2) , we construct two paths P_1 and P_2 such that the end vertices of P_1 are x_4, y_7 , the end vertices of P_2 are x_6, y_5 and $|V(P_i)| = l^i$ ($i = 1, 2$). Let $P_3 := x_5y_8x_3y_2y_1xy_3x_1x_2y_4x_4$ and $P_4 := x_6y_6yy_7$. Then $C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P$ is a Hamiltonian cycle in G (See Figure 6.2. (3)). And $dist_C(x, y) = k$.

If $W = \emptyset$, the proof is similar as above. The only difference is to take the path P be an edge with one end vertex in $U_1 - W$ and the other one in $U_2 - W$.

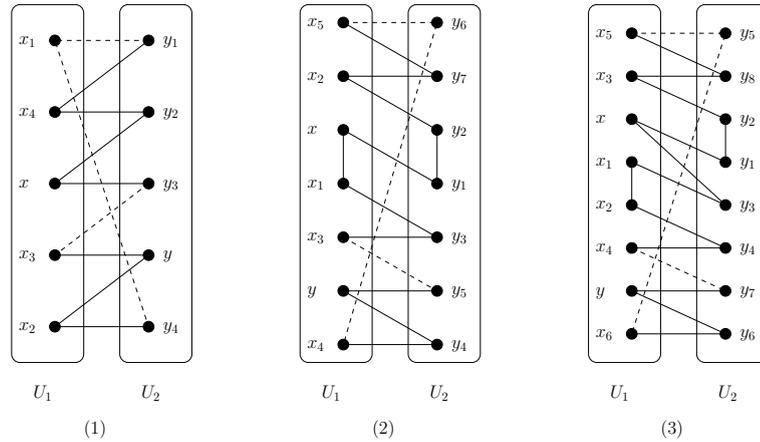


FIGURE 6.2: Extremal case 1

We also need to consider the case when k is even. The arguments are similar to the case when k is odd. There are also some cases discussions according to the position of x, y in G^* . We omit these similar proofs here.

6.4.1.2 The graph order n is odd

As in non-extremal case, we pick a vertex z . If $G[V - z]$ is in the extremal case 1, we apply the proof of the extremal case 1 when n is even to the induced graph $G[V - z]$. Then we obtain a Hamiltonian cycle C in $G[V - z]$.

If there exist two vertices u, v on the longer path joining x and y in C such that they are neighbors of z and adjacent in C , then replacing the edge uv in C by the path uzv we obtain a Hamiltonian cycle C^* in G such that $dist_{C^*}(x, y) = k$.

Otherwise, all such pairs u, v of vertices are on the shorter path joining x and y in C . After applying Lemma 6.8 to $G[V - z]$, we assume that k is odd, x, y are in different parts of G^* , say $x \in U_1$ and $y \in U_2$, and $W = \emptyset$ (the first case discussed in 5.1.1). We will show our proof in this case and the other cases are similar. Recall that we have constructed a Hamiltonian cycle $C = P_1 \cup P_2 \cup P_3 \cup P_4$ in $G[V - z]$, where P_i has length l^i ($i = 1, 2$), $P_3 = y_1xy_2$ and $P_4 = x_1yx_2$.

If $uv \in P_3$ or P_4 , we exchange the value of l^1 and l^2 , precisely, let $l^1 = 2n' - k + 1$ and $l^2 = k - 1$. Then by Lemma 5.12 we construct two vertex-disjoint paths P'_1 and P'_2 such that the end vertices of P'_1 are x_1, y_1 , the end vertices of P'_2 are x_2, y_2 and $|V(P'_i)|$ is equal to l^i ($i = 1, 2$). We construct a new Hamiltonian cycle $C^* = P'_1 \cup P'_2 \cup P_3 \cup P_4$ in $G[V - z]$. And it is not hard to see that u, v are on the longer path joining x and y in C^* . Replacing uv by uzv in C^* , we get our desired Hamiltonian cycle in G .

If $uv \in P_1$ or P_2 , u and v should be in different parts of G^* , say $u \in U_1$ and $v \in U_2$. By the minimum degree condition of G^* in Lemma 6.8, we choose a common neighbor of y_2 and v in U_1 , denoted x_3 , and a neighbor of u in U_2 , denoted y_3 . Let $U'_1 = U_1 - \{x, x_3, u\}$, $U'_2 = U_2 - \{y, y_2, v\}$ and $n' = |U'_1| = |U'_2|$. By Lemma 5.6, we can say (U'_1, U'_2) is $(\sqrt{\alpha_1 + 3\alpha_2}, \frac{2}{3})$ -super-regular. Let $l^1 = k - 1$ and $l^2 = 2n' - k + 1$. By Lemma 5.12, we construct two vertex-disjoint paths P'_1 and P'_2 such that the end vertices of P'_1 are x_1, y_1 , the end vertices of P'_2 are x_2, y_3 and $|V(P'_i)|$ is equal to l^i ($i = 1, 2$). Let $P'_3 = y_1x_2y_3x_3vuy_3$. Then $C^* = P'_1 \cup P'_2 \cup P'_3 \cup P_4$ is a new Hamiltonian cycle in $G[V - z]$ and u, v are on the longer path joining x and y in C^* . Replacing uv by uzv in C^* , we get our desired Hamiltonian cycle in G .

6.4.2 Extremal case 2

6.4.2.1 The graph order n is even

Suppose G is a graph of even n vertices with $\delta(G) \geq \frac{n}{2} + 1$ and there exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$. We suppose $\alpha \leq (\frac{1}{12})^3$. Let $\alpha_1 = \alpha^{\frac{1}{3}}$ and $\alpha_2 = \alpha^{\frac{2}{3}}$. We have a similar lemma, which is proved in Chapter 5, as Lemma 6.8 and a similar claim as Claim 6.9.

Lemma 6.10. *If G is in extremal case 2, then $V(G)$ can be partitioned into two balanced parts U_1 and U_2 such that*

(a) *there is a set $W_1 \subseteq U_1$ (resp. $W_2 \subseteq U_2$) such that there exist vertex-disjoint 2-paths in $G[U_1]$ (resp. $G[U_2]$) with the vertices of W_1 (resp. W_2) as the internal vertices in each 2-path and $|W_1| \leq \alpha_2 \frac{n}{2}$ (resp. $|W_2| \leq \alpha_2 \frac{n}{2}$);*

(b) *$\deg_{G[U_1]}(u) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $u \in U_1 - W_1$ and $\deg_{G[U_2]}(v) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $v \in U_2 - W_2$.*

Claim 6.11. *There exists a path in $G[U_1]$ including all the vertices of W_1 such that the end vertices of it are in $U_1 - W_1$ and the number of vertices on this path is no more than $4\alpha_2 \frac{n}{2}$.*

Proof. Suppose $W_1 = \{w_1, w_2, \dots, w_t\}$ and the end vertices of the 2-path containing w_i are a_i, b_i ($1 \leq i \leq t$). Since $\deg_{G[U_1]}(a_i) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ and $\deg_{G[U_1]}(b_i) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$, we greedily choose $c_i \in U_1$ which is a common neighbor of a_{i+1} and b_i ($1 \leq i \leq t - 1$). Moreover we can choose all these c_i to be distinct. Then $P = a_1w_1b_1c_1a_2w_2b_2c_2\dots b_{t-1}c_{t-1}a_tw_t b_t$ is a path containing all the vertices of W_1 . And $|V(P)| \leq 4\alpha_2 \frac{n}{2} - 1$. \square

And we need the following theorem of Williamson [101].

Theorem 6.12. [101] *If G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any $2 \leq k \leq n - 1$ and for any vertices x and y , G has a path from x to y of length k .*

We apply Lemma 6.10 to G and get a partition of $V(G) = U_1 \cup U_2$ with the properties in Lemma 6.10. In the proof of Lemma 6.10, we know that the 2-paths are greedily chosen, so we assume that x, y won't be any end vertices of those 2-paths.

Sub-case 1. x, y are in different parts, say $x \in U_1$ and $y \in U_2$.

Since $\delta(G) \geq \frac{n}{2} + 1$, we choose $u_1 \in U_1$ such that u_1 is a neighbor of y and $u_1 \neq x$. Whether u_1 is in W_1 or not, let $u_2 \in U_1 - W_1$ be a neighbor of u_1 . Whether x is in W_1 or not, let $u_3, u_4 \in U_1 - W_1$ be two neighbors of x . Since the 2-paths are greedily chosen, we can assume that $u_1 \neq u_3, u_4$ and $u_2 \neq u_3, u_4$. By Claim 6.11, we construct a path P_1 containing all the vertices of W_1 (except x, u_1 , if they are in W_1) and we have $|V(P_1)| \leq 4\alpha_2 \frac{n}{2}$. Let $u_5, u_6 \in U_1 - W_1$ be the end vertices of P_1 and let $u_7 \in U_1 - W_1$ be a common neighbor of u_3, u_5 . Let $U_1^* = U_1 - V(P_1) - \{x, u_1, u_2, u_3, u_7\}$. Since n is sufficiently large, we have $\delta(G[U_1^*]) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2} - 4\alpha_2 \frac{n}{2} - 5 \geq (1 - \alpha_1 - 7\alpha_2) \frac{n}{2} \geq \frac{n}{4} + 1 \geq \frac{|U_1^*|}{2} + 1$ provided $\alpha \leq (\frac{1}{12})^3$. By Theorem 6.12, we construct a path P_2 with end vertices u_4 and $u_8 \in U_1^*$ such that $|V(P_2)| = \frac{n}{2} - k$. Let $U_1' = (U_1^* - V(P_2)) \cup \{u_2, u_6\}$. Then

$$\begin{aligned} \delta(G[U_1']) &\geq (1 - \alpha_1 - 7\alpha_2) \frac{n}{2} - (\frac{n}{2} - k) \\ &= k - (\alpha_1 + 7\alpha_2) \frac{n}{2} \\ &\geq \frac{k}{2} + 1 \geq \frac{|U_1'|}{2} + 1 \end{aligned}$$

provided $\alpha \leq (\frac{1}{12})^3$ and $k > \frac{n}{6}$. So we construct a Hamiltonian path P_3 in $G[U_1']$ with end vertices u_2 and u_6 . Let $u_9 \in U_2$ be a neighbor of u_7 . We claim that there exists a Hamiltonian path of $G[U_2]$ with end vertices u_9 and y .

Claim 6.13. *We can construct a Hamiltonian path of $G[U_2]$ with end vertices u_9 and y .*

Proof. By Claim 6.11, we also can construct a path Q_1 containing all the vertices of W_2 in $G[U_2]$ (except u_9 and y , if they are in W_2). Let $v_1, v_2 \in U_2 - W_2$ be the end vertices of Q_1 . Let $v_3 \in U_2 - W_2$ be a neighbor of u_9 , $v_4 \in U_2 - W_2$ be a common neighbor of v_3, v_1 and $v_5 \in U_2 - W_2$ be a neighbor of y . Let $U_2^* = (U_2 - V(Q_1) - \{u_9, v_3, v_4, y\}) \cup \{v_2\}$. Since n is sufficiently large, we have $\delta(G[U_2^*]) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2} - 4\alpha_2 \frac{n}{2} - 3 \geq (1 - \alpha_1 - 7\alpha_2) \frac{n}{2} \geq \frac{n}{4} + 1 \geq \frac{|U_2^*|}{2} + 1$ provided $\alpha \leq (\frac{1}{12})^3$. So we can construct a Hamiltonian path Q_2 of

$G[U_2^*]$ with end vertices v_2 and v_5 . Thus $Q_3 = Q_1 \cup Q_2 \cup \{u_9 v_3 v_4 v_1, y v_5\}$ is a Hamiltonian path of $G[U_2]$ with end vertices u_9 and y . \square

By Claim 6.13, we construct a Hamiltonian path P_4 of $G[U_2]$ with end vertices u_9 and y . Thus

$$C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup \{y u_1 u_2, u_8 u_9, u_4 x u_3 u_7 u_5\}$$

is a Hamiltonian cycle in G and $\text{dist}_C(x, y) = k$ (see Figure 3. (1)).

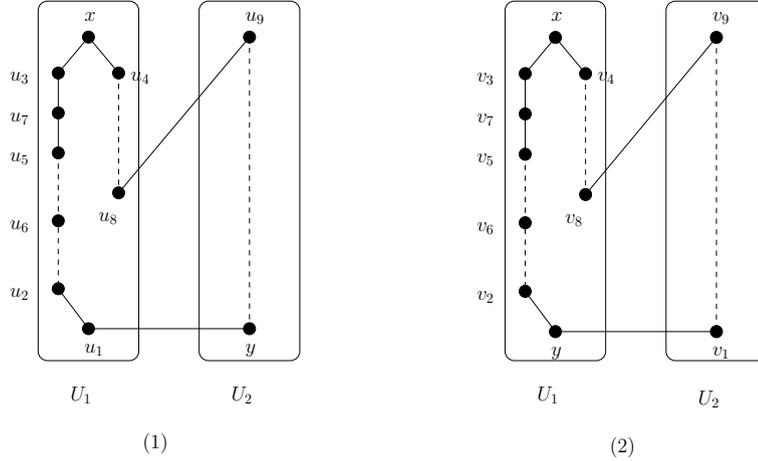


FIGURE 6.3: Extremal case 2

Sub-case 2. x and y are in the same part, say $x, y \in U_1$.

The proof is similar as above. Since $\delta(G) \geq \frac{n}{2} + 1$, we choose $v_1 \in U_2$ such that v_1 is a neighbor of y . Whether y is in W_1 or not, let $v_2 \in U_1 - W_1$ be a neighbor of y .

If $\frac{n}{2} - k - 1 \geq 1$, let $v_3, v_4 \in U_1 - W_1$ be two neighbors of x and let $v_7 \in U_1 - W_1$ be a common neighbor of v_3, v_5 . By Claim 6.11, we construct a path P'_1 with end vertices $v_5, v_6 \in U_1 - W_1$ containing all the vertices of W_1 . Let $U_1^* = U_1 - V(P'_1) - \{x, y, v_2, v_3, v_7\}$. By Theorem 6.12, we construct a path P'_2 with end vertices v_4 and $v_8 \in U_1^*$ such that $|V(P'_2)| = \frac{n}{2} - k - 1$. Let $U'_1 = (U_1^* - V(P'_2)) \cup \{v_2, v_6\}$. We construct a Hamiltonian path P'_3 in $G[U'_1]$ with end vertices v_2 and v_6 . Let $v_9 \in U_2$ be a neighbor of v_7 . We construct a Hamiltonian path P'_4 of $G[U_2]$ with end vertices v_9 and v_1 by Claim 6.13. So

$$C' = P'_1 \cup P'_2 \cup P'_3 \cup P'_4 \cup \{v_1 y v_2, v_8 v_9, v_4 x v_3 v_7 v_5\}$$

is our desired Hamiltonian cycle in G (see Figure 3. (2)).

If $\frac{n}{2} - k - 1 = 0$, let $v_{10} \in U_2$ be a neighbor of x . By Claim 6.13, we construct a Hamiltonian path P'_5 of $G[U_2]$ with end vertices v_{10} and v_1 . Let $U_1^{**} = (U_1 - V(P'_1) -$

$\{x, y, v_3, v_7\} \cup \{v_6\}$. We construct a Hamiltonian path P'_6 of $G[U_1^{**}]$ with end vertices v_2, v_6 . So

$$C^* = P'_1 \cup P'_5 \cup P'_6 \cup \{v_1yv_2, v_{10}xv_3v_7v_5\}$$

is our desired Hamiltonian cycle in G .

6.4.2.2 The graph order n is odd

Now we assume the induced graph $G[V - z]$ is in the extremal case 2. Since $G[V - z]$ is a graph of even order, we apply Lemma 6.10 to $G[V - z]$. We obtain a partition of $G[V - z]$ into two parts U_1 and U_2 with $|U_1| = |U_2| = \frac{n-1}{2}$. We can construct a Hamiltonian cycle C of $G[V - z]$ as section 5.2.1.

If there exist two vertices u, v on the longer path joining x and y in C such that they are neighbors of z and adjacent in C , then replacing the edge uv in C by the path uzv we obtain a Hamiltonian cycle C^* in G such that $dist_{C^*}(x, y) = k$.

Otherwise, this kind of two vertices u, v are on the shorter path joining x and y in C . Assume that x, y are in different parts, say $x \in U_1$ and $y \in U_2$. The proof for the other cases is similar. We will construct a new Hamiltonian cycle in $G[V - z]$. By the discussion in Sub-case 1 of section 5.2.1, we have $uv \in P_1$ or P_3 or $P_5 := xu_3u_7u_5$ or $P_6 := yu_1u_2$.

If $uv \in P_1$ or P_5 or P_6 , we fix the construction of P_1, P_4 , and construct a new path P'_2 in $G[U_1]$ with end vertices u_4 and $u_8 \in U_1$ such that $|V(P'_2)| = \frac{n-1}{2} - k + 1$. Then we also construct a new Hamiltonian path P'_3 in $G[U'_1]$ with end vertices u_2 and u_6 . So

$$C' = P_1 \cup P'_2 \cup P'_3 \cup P_4 \cup P_6 \cup \{u_8u_9, u_4xu_3u_7u_5\}$$

is a new Hamiltonian cycle in $G[V - z]$. And $dist_{C'}(x, y) = k - 1$. By replacing the edge uv in C' by the path uzv we obtain a Hamiltonian cycle in G such that x, y have distance k on this cycle.

If $uv \in P_3$, let $v_1 \in U_1$ be a common neighbor of u_6, v . After constructing P_1 , we extend P_1 to $P'_1 = P_1 \cup \{u_6v_1vu\}$. We can continue to construct the path P'_2 in $G[U_1]$ with end vertices u_4 and $u_8 \in U_1$ such that $|V(P'_2)| = \frac{n-1}{2} - k + 1$. Let $U''_1 = (U_1 - V(P'_1) - V(P'_2) - V(P_5) - \{u_1\}) \cup \{u\}$. We construct a Hamiltonian path P''_3 in $G[U''_1]$ with end vertices u_2 and u . So

$$C'' = P'_1 \cup P'_2 \cup P''_3 \cup P_4 \cup P_6 \cup \{u_8u_9, u_4xu_3u_7u_5\}$$

is a new Hamiltonian cycle in $G[V - z]$. And $\text{dist}_{C''}(x, y) = k - 1$. By replacing the edge uv in C'' by the path uzv we obtain a Hamiltonian cycle in G such that x, y have distance k on this cycle.

6.5 Concluding remarks and further work

In this chapter, we gave a proof of Faudree and Li's conjecture for graphs of sufficiently large order.

Note that our result show that for any pair of vertices, we can find a Hamiltonian cycle such that the distance between these two vertices on it is a given number (between 2 and half of the order of the graph). Can we generalize our result to 3 or more vertices with some faire distance conditions are given? This will be one of our futhuer works.

The other further work is to find a proof of Theorem 6.2 without using Regularity Lemma.

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1. A. Harutyunyan, R. Naserasr, M. Petruševski, R. Škrekovski and Q. Sun, Mapping planar graphs into the Coxeter graph, *Discrete Math.* 339(2), 839–849, 2016. (Reference [46])
2. H. Li, Y. Bai, W. He and Q. Sun, Vertex-distinguishing arc colorings of digraphs, *Discrete Applied Math.*, to appear.
3. R. Naserasr, S. Sen and Q. Sun, Walk-powers and homomorphism bounds of planar signed graphs, *Graphs Combin.*, to appear. (Reference [84])
4. W. He, H. Li and Q. Sun, Locating any two vertices on Hamiltonian cycles, submitted. (Reference [48])
5. W. He, H. Li and Q. Sun, Distributing pairs of vertices on Hamiltonian cycles, manuscript. (Reference [47])
6. W. He, R. Naserasr and Q. Sun, Cliques in walk-powers of K_4 -minor free graphs, manuscript. (Reference [49])

Titre : Une contribution à la théorie des graphes (signés) borne d'homomorphisme et hamiltonicité

Mots clés : graphe signé, cubes projectifs, homomorphisme, cycle hamiltonien

Résumé : Le problème d'homomorphisme des graphes planaires (signés) et le problème du cycle hamiltonien sont deux principaux problèmes de la théorie des graphes. Dans cette thèse, nous étudions plusieurs problèmes au sujet de ces.

En problème d'homomorphisme des graphes planaires (signés), nous prouvons que si un graphe signé cohérent de maille-déséquilibré d qui borne la classe des graphes signés cohérent de maille-déséquilibré $d+1$, il a un order au moins comme le cube $SPC(d)$. Et nous obtenons que la order du graphe optimal de maille-impair $2k+1$ qui borne tous les graphes de maille-impair $2k+1$ et ont pas graphe complet de l'ordre 4 en tant que mineur. Plus, we prouvons que le graphe Coxeter borne la

classe de tous graphes planaires de maille-impair au moins 17.

En problème du cycle hamiltonien, utilisant le Lemma de Régularité et le Blow-up Lemma, nous donnons une preuve de Enomoto conjecture pour les graphes d'ordre suffisant et nous donnons une preuve de la Faudree-Li conjecture pour les graphes d'ordre suffisant.

Title : A contribution to the theory of (signed) graph homomorphism bound and Hamiltonicity

Keywords : signed graph, projective cubes, homomorphism, Hamiltonian cycle

Abstract : The homomorphism problem of planar (signed) graphs and Hamiltonian cycle problem are two main problems in graph theory. In this thesis, we study some related topics.

For the homomorphism problem of planar (signed) graphs, we prove that if a consistent signed graph of unbalanced-girth d which bounds the class of planar consistent graphs of unbalanced-girth d , then it has the order at least as that of $SPC(d)$. And we prove give an optimal bound for the order of graph of odd-girth $2k+1$ which bounds all the graphs of odd-girth $2k+1$ and has no complete graph of order 4 as a minor. Also, we prove that the Coxeter graph bounds the class of planar graphs of

odd-girth at least 17.

For the Hamiltonian cycle problem, using Regularity Lemma and Blow-up Lemma, we give a proof of Enomoto's conjecture for graphs of sufficiently large order. We also give a proof of Faudree-Li conjecture for graphs of sufficiently large order.

