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Periods of the motivic fundamental groupoid of P1{0, μN,∞

Claire Glanois

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UNIVERSITÉ PIERRE ET MARIE CURIE

École Doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline: Mathématiques

Periods of the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$.

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Aux inconnues,
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'*Prairie de tous mes instants, ils ne peuvent me fouler.*
Leur voyage est mon voyage et je reste obscurité.'

R. Char

Abstract

In this thesis, following F. Brown's point of view, we look at the Hopf algebra structure of motivic cyclotomic multiple zeta values, which are motivic periods of the fundamental groupoid of $\mathbb{P}^1 \diagdown \{0, \mu_N, \infty\}$. By application of a surjective *period* map (which, under Grothendieck's period conjecture, is an isomorphism), we deduce results (such as generating families, identities, etc.) on cyclotomic multiple zeta values, which are complex numbers. The coaction of this Hopf algebra (explicitly given by a combinatorial formula from A. Goncharov and F. Brown's works) is the dual of the action of a so-called *motivic* Galois group on these specific motivic periods. This entire study was actually motivated by the hope of a Galois theory for periods, which should extend the usual Galois theory for algebraic numbers.

In the first part, we focus on the case of motivic multiple zeta values ($N = 1$) and Euler sums ($N = 2$). In particular, we present new bases for motivic multiple zeta values: one via motivic Euler sums, and another (depending on an analytic conjecture) which is known in the literature as the Hoffman $\star$ basis; under a general motivic identity that we conjecture, these bases are identical.

In the second part, we apply some Galois descents ideas to the study of these periods, and examine how periods of the fundamental groupoid of $\mathbb{P}^1 \diagdown \{0, \mu_{N'}, \infty\}$ are embedded into periods of $\pi_1(\mathbb{P}^1 \diagdown \{0, \mu_N, \infty\})$, when $N' \mid N$. After giving some general criteria for any N, we focus on the cases $N = 2, 3, 4, \text{`}6\text{'}, 8$, for which the motivic fundamental group generates the category of mixed Tate motives on $\mathcal{O}_N[\frac{1}{N}]$ (unramified if $N = 6$). For those N, we are able to construct Galois descents explicitly, and extend P. Deligne's results.

`Key words`: *Periods, Polylogarithms, multiple zeta values, Mixed Tate Motives, cyclotomic field, Hopf algebra, Motivic fundamental group, Galois Descent.*

Résumé.

A travers ce manuscrit, en s'inspirant du point de vue adopté par F. Brown, nous examinons la structure d'algèbre de Hopf des multizêtas motiviques cyclotomiques, qui sont des périodes motiviques du groupoïde fondamental de $\mathbb{P}^1 \diagdown \{0, \mu_N, \infty\}$. Par application d'un morphisme *période* surjectif (isomorphisme sous la conjecture de Grothendieck), nous pouvons déduire des résultats (tels des familles génératrices, des identités, etc.) sur ces nombres complexes que sont les multizêtas cyclotomiques. La coaction de cette algèbre de Hopf (explicite par une formule combinatoire due aux travaux de A.B. Goncharov et F. Brown) est duale à l'action d'un dénommé *groupe de Galois motivique* sur ces périodes motiviques. Ces recherches sont ainsi motivées par l'espoir d'une théorie de Galois pour les périodes, étendant la théorie de Galois usuelle pour les nombres algébriques.

Dans un premier temps, nous nous concentrons sur les multizêtas ($N = 1$) et les sommes d'Euler ($N = 2$) motiviques. En particulier, de nouvelles bases pour les multizetas motiviques sont présentées: une via les sommes d'Euler motiviques, et une seconde (sous une conjecture analytique) qui est connue sous le nom de *Hoffman* $\star$; soulignons que sous une identité motivique générale que nous conjecturons également, ces bases sont identiques.

Dans un second temps, nous appliquons des idées de descentes galoisiennes à l'étude de ces périodes, en regardant notamment comment les périodes du groupoïde fondamental de $\mathbb{P}^1 \diagdown \{0, \mu_{N'}, \infty\}$ se plongent dans les périodes de $\pi_1(\mathbb{P}^1 \diagdown \{0, \mu_N, \infty\})$, lorsque $N' \mid N$. Après avoir fourni des critères généraux (quel que soit N), nous nous tournons vers les cas $N = 2, 3, 4, \text{`}6\text{'}, 8$, pour lesquels le groupoïde fondamental motivique engendre la catégorie des motifs de Tate mixtes sur $\mathcal{O}_N[\frac{1}{N}]$ (non ramifié si $N = 6$). Pour ces valeurs, nous sommes en mesure d'expliciter les descentes galoisiennes, et d'étendre les résultats de P. Deligne.

`Mots clés`: *Périodes, Polylogarithmes, multizêtas, corps cyclotomiques, Motifs de Tate Mixtes, algèbre de Hopf, groupe motivique fondamental, descente galoisienne.*

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Chapter 1

Introduction

'Qui est-ce ? Ah, très bien, faites entrer l'infini.'

Aragon

1.1 Motivation

1.1.1 Periods

A ***period*** [I] denotes a complex number that can be expressed as an integral of an algebraic function over an algebraic domain.[II] They form the algebra of periods $\mathcal{P}$, fundamental class of numbers between algebraic numbers $\overline{\mathbb{Q}}$ and complex numbers.
The study of these integrals is behind a large part of algebraic geometry, and its connection with number theory, notably via L-functions [III]; and many of the constants which arise in mathematics, transcendental number theory or in physics turn out to be periods, which motivates the study of these particular numbers.

`Examples:`

- The following numbers are periods:

$$\sqrt{2} = \int_{2x^2 \leq 1} dx \ , \quad \pi = \int_{x^2+y^2 \leq 1} dxdy \quad \text{and} \quad \log(z) = \int_1^z \frac{dx}{x}, z > 1, z \in \overline{\mathbb{Q}}.$$

- Famous -alleged transcendental- numbers which conjecturally are not periods:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \ , \quad \gamma = \lim_{n\to\infty} \left(-ln(n) + \sum_{k=1}^{n} \frac{1}{k}\right) \text{ or } \quad \frac{1}{\pi}.$$

It can be more useful to consider the ring of extended periods, by inverting π:

$$\widehat{\mathcal{P}} := \mathcal{P}\left[\frac{1}{\pi}\right].$$

[I] For an enlightening survey, see the reference article [59].

[II] We can equivalently restrict to integral of rational functions over a domain in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients, by introducing more variables.

[III] One can associate a L− function to many arithmetic objects such as a number field, a modular form, an elliptic curve, or a Galois representation. It encodes its properties, and has wonderful (often conjectural) meromorphic continuation, functional equations, special values, and non-trivial zeros (Riemann hypothesis).

- Multiple polylogarithms at algebraic arguments (in particular cyclotomic multiple zeta values), by their representation as iterated integral given below, are periods. Similarly, special values of Dedekind zeta function $\zeta_F(s)$ of a number field, of L-functions, of hypergeometric series, modular forms, etc. are (conjecturally at least) periods or extended periods.

- Periods also appear as Feynman integrals: Feynman amplitudes $I(D)$ can be written as a product of Gamma functions and meromorphic functions whose coefficients of its Laurent series expansion at any integer D are periods (cf. [7]), where D is the dimension of spacetime.

Although most periods are transcendental, they are constructible; hence, the algebra $\mathcal{P}$ is countable, and any period contains only a finite amount of information. Conjecturally (by Grothendieck's conjecture), the only relations between periods comes from the following rules of elementary integral calculus[I]:

(i) Additivity (of the integrand and of the integration domain)

(ii) Invertible changes of variables

(iii) Stokes's formula.

Another way of viewing a period $\int_\gamma \boldsymbol{\omega}$ is via a comparison between two cohomology theories: the algebraic *De Rham* cohomology, and the singular (*Betti*) cohomology. More precisely, let X a smooth algebraic variety defined over $\mathbb{Q}$ and Y a closed subvariety over $\mathbb{Q}$.

- On the one hand, the *algebraic* De Rham cohomology $H^\bullet_{dR}(X)$ is the hypercohomology of the sheaf of algebraic (Kähler) differentials on X. If X is affine, it is defined from the de Rham complex $\Omega^\bullet(X)$ which is the cochain complex of global algebraic (Kähler) differential forms on X, with the exterior derivative as differential. Recall that the *classical* k^{th} de Rham cohomology group is the quotient of smooth closed k-forms on the manifold $\mathrm{X}_{/\mathbb{C}}$ modulo the exact k-forms on X.
  Given ω a closed algebraic n-form on X whose restriction on Y is zero, it defines an equivalence class $[\omega]$ in the relative de Rham cohomology groups $H^n_{dR}(X,Y)$, which are finite-dimensional $\mathbb{Q}-$ vector space.

- On the other hand, the Betti homology $H^B_\bullet(X)$ is the homology of the chain complex induced by the boundary operation of singular chains on the manifold $X(\mathbb{C})$; Betti cohomology groups $H^n_B(X,Y) = H^B_n(X,Y)^\vee$ are the dual $\mathbb{Q}$ vector spaces (taking here coefficients in $\mathbb{Q}$, not $\mathbb{Z}$).
  Given γ a singular n chain on $X(\mathbb{C})$ with boundary in $Y(\mathbb{C})$, it defines an equivalence class $[\gamma]$ in the relative Betti homology groups $H^B_n(X,Y) = H^n_B(X,Y)^\vee$.[II]

Furthermore, there is a comparison isomorphism between relative de Rham and relative Betti cohomology (due to Grothendieck, coming from the integration of algebraic differential forms on singular chains):

$$\mathrm{comp}_{B,dR} : H^\bullet_{dR}(X,Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^\bullet_B(X,Y) \otimes_{\mathbb{Q}} \mathbb{C}.$$

[I] However, finding an algorithm to determine if a real number is a period, or if two periods are equal seems currently out of reach; whereas checking if a number is algebraic, or if two algebraic numbers are equal is rather "easy" (with "LLL"-type reduction algorithm, resp. by calculating the g.c.d of two vanishing polynomials associated to each).

[II] Relative homology can be calculated using the following long exact sequence:

$$\cdots \to H_n(Y) \to H_n(X) \to H_n(X,Y) \to H_{n-1}(Y) \to \cdots .$$

By pairing a basis of Betti homology to a basis of de Rham cohomology, we obtain the *matrix of periods*, which is a square matrix with entries in $\mathcal{P}$ and determinant in $\sqrt{\mathbb{Q}^*}(2i\pi)^{\mathbb{N}^*}$; i.e. its inverse matrix has its coefficients in $\widehat{\mathcal{P}}$. Then, up to the choice of these two basis:

The period $\int_\gamma \omega$ is the coefficient of this pairing $\langle [\gamma], \mathrm{comp}_{B,dR}([\omega])\rangle$.

`Example`: Let $X = \mathbb{P}^1 \diagdown \{0, \infty\}$, $Y = \emptyset$ and γ_0 the counterclockwise loop around 0:

$$H_i^B(X) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ \mathbb{Q}\,[\gamma_0] & \text{if } i = 1 \\ 0 & \text{else .} \end{cases} \quad \text{and} \quad H_{dR}^i(X) = \begin{cases} \mathbb{Q} & \text{if } i = 0 \\ \mathbb{Q}\left[\frac{d}{dx}\right] & \text{if } i = 1 \\ 0 & \text{else .} \end{cases}$$

Since $\int_{\gamma_0} \frac{dx}{x} = 2i\pi$, $2i\pi$ is a period; as we will see below, it is a period of the Lefschetz motive $\mathbb{L} := \mathbb{Q}(-1)$.

Viewing periods from this cohomological point of view naturally leads to the definition of ***motivic periods*** given below [I], which form an algebra $\mathcal{P}^{\mathfrak{m}}$, equipped with a period homomorphism:

$$\mathrm{per} : \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}.$$

A variant of Grothendieck's conjecture, which is a presently inaccessible conjecture in transcendental number theory, predicts that it is an isomorphism.
There is an action of a so-called motivic Galois group $\mathcal{G}$ on these motivic periods as we will see below in §2.1. If Grothendieck's period conjecture holds, this would hence extend the usual Galois theory for algebraic numbers to periods (cf. [2]).

In this thesis, we will focus on motivic (cyclotomic) multiple zeta values, defined in §2.3, which are motivic periods of the motivic (cyclotomic) fundamental group, defined in §2.2. Their images under this period morphism are the (cyclotomic) multiple zeta values; these are fascinating examples of periods, which are introduced in the next section (see also [3]).

1.1.2 Multiple zeta values

The Zeta function is known at least since Euler, and finds itself nowadays, in its various generalized forms (multiple zeta values, Polylogarithms, Dedekind zeta function, L-functions, etc), at the crossroad of many different fields as algebraic geometry (with periods and motives), number theory (notably with modular forms), topology, perturbative quantum field theory (with Feynman diagrams, cf. [60]), string theory, etc. Zeta values at even integers are known since Euler to be rational multiples of even powers of π:

Lemma.

$$\textit{For } n \geq 1, \quad \zeta(2n) = \frac{\mid B_{2n} \mid (2\pi)^{2n}}{2(2n)!}, \textit{ where } B_{2n} \textit{ is the } 2n^{th} \textit{ Bernoulli number.}$$

However, the zeta values at odd integers already turn out to be quite interesting periods:

Conjecture. $\pi, \zeta(3), \zeta(5), \zeta(7), \cdots$ *are algebraically independent.*

[I] The definition of a motivic period is given in §2.4 in the context of a category of Mixed Tate Motives. In general, one can do with Hodge theory to define $\mathcal{P}^{\mathfrak{m}}$, which is not strictly speaking *motivic*, once we specify that the mixed Hodge structures considered come from the cohomology of algebraic varieties.

This conjecture raises difficult transcendental questions, rather out of reach; currently we only know $\zeta(3) \notin \mathbb{Q}$ (Apéry), infinitely many odd zeta values are irrational (Rivoal), or other quite partial results (Zudilin, Rivoal, etc.); recently, F. Brown paved the way for a pursuit of these results, in [23].

Multiple zeta values relative to the Nth roots of unity μ_N, which we shall denote by $\mathbf{MZV}_{\boldsymbol{\mu_N}}$ are defined by: [I]

$$\zeta\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p \end{pmatrix} := \sum_{0<k_1<k_2\cdots<k_p} \frac{\epsilon_1^{k_1}\cdots\epsilon_p^{k_p}}{k_1^{n_1}\cdots k_p^{n_p}},\ \epsilon_i\in\mu_N,\ n_i\in\mathbb{N}^*,\ (n_p,\epsilon_p)\neq(1,1). \quad (1.1)$$

The *weight*, often denoted w below, is defined as $\sum n_i$, the *depth* is the length p, whereas the *height*, usually denoted h, is the number of n_i greater than 1. The weight is conjecturally a grading, whereas the depth is only a filtration. Denote also by $\boldsymbol{\mathcal{Z}^N}$ the $\mathbb{Q}$-vector space spanned by these multiple zeta values relative to μ_N.
These MZV_{μ_N} satisfy both *shuffle* $\sqcup\!\sqcup$ relation (coming from the integral representation below) and *stuffle* $*$ relation (coming from this sum expression), which turns $\mathcal{Z}^N$ into an algebra. These relations, for $N=1$, are conjectured to generate all the relations between MZV if we add the so-called Hoffman (regularized double shuffle) relation; cf. [25], [77] for a good introduction to this aspect. However, the literature is full of other relations among these (cyclotomic) multiple zeta values: cf. [4], [38], [72], [68], [70], [85], [86]. Among these, we shall require the so-called pentagon resp. *hexagon* relations (for $N=1$, cf. [39]), coming from the geometry of moduli space of genus 0 curves with 5 ordered marked points $X=\mathcal{M}_{0,5}$ resp. with 4 marked points $X=\mathcal{M}_{0,4}=\mathbb{P}^1\diagdown\{0,1,\infty\}$ and corresponding to a contractible path in X; hexagon relation (cf. Figure 4.1) is turned into an *octagon* relation (cf. Figure 4.2) for $N>1$ (cf. [38]) and is used below in §4.2.

One crucial point about multiple zeta values, is their *integral representation*[II], which makes them clearly *periods* in the sense of Kontsevich-Zagier. Let us define first the following iterated integrals and differential forms, with $a_i\in\{0,\mu_N\}$:

$$\boldsymbol{I(0;a_1,\dots,a_n;1)} := \int_{0<t_1<\cdots<t_n<1} \frac{dt_1\cdots dt_n}{(t_1-a_1)\cdots(t_n-a_n)} = \int_0^1 \omega_{a_1}\dots\omega_{a_n},\ \text{with } \omega_a := \frac{dt}{t-a}. \quad (1.2)$$

In this setting, with $\eta_i := (\epsilon_i\dots\epsilon_p)^{-1}\in\mu_N$, $n_i\in\mathbb{N}^{*\,\mathrm{III}}$:

$$\zeta\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p \end{pmatrix} = (-1)^p I(0;\eta_1,\mathbf{0}^{n_1-1},\eta_2,\mathbf{0}^{n_2-1},\dots,\eta_p,\mathbf{0}^{n_p-1};1). \quad (1.3)$$

Remarks:

[I] Beware, there is no consensus on the order for the arguments of these MZV: sometimes the summation order is reversed.

[II] Obtained by differentiating, considering there variables $z_i\in\mathbb{C}$, since:

$$\frac{d}{dz_p}\zeta\begin{pmatrix} n_1,\dots,n_p \\ z_1,\dots,z_{p-1},z_p \end{pmatrix} = \begin{cases} \frac{1}{z_p}\zeta\begin{pmatrix} n_1,\dots,n_p-1 \\ z_1,\dots,z_{p-1},z_p \end{pmatrix} & \text{if } n_p\neq 1 \\ \frac{1}{1-z_p}\zeta\begin{pmatrix} n_1,\dots,n_{p-1} \\ z_1,\dots,z_{p-1}z_p \end{pmatrix} & \text{if } n_p=1. \end{cases}$$

[III] The use of bold in the iterated integral writing indicates a repetition of the corresponding number, as 0 here.

- Multiple zeta values can be seen as special values of generalized multiple polylogarithms, when ϵ_i are considered in $\mathbb{C}$[I]. First, notice that in weight 2, $Li_1(z) := \zeta\binom{1}{z}$ is the logarithm $-\log(1-z)$. Already the dilogarithm, in weight 2, $Li_2(z) := \zeta\binom{2}{z} = \sum_{k>0}\frac{z^k}{k^2}$, satisfies nice functional equations[II] and arises in many places such as in the Dedekind zeta value $\zeta_F(2)$ for F an imaginary quadratic field, in the Borel regulator in algebraic K-theory, in the volume of hyperbolic manifolds, etc.; cf. [40]; some of these connections can be generalized to higher weights.

- Recall that an iterated integral of closed (real or complex) differential $1-$forms ω_i along a path γ on a 1-dimensional (real or complex) differential manifold M is homotopy invariant, cf. [27]. If $M = \mathbb{C}\diagdown\{a_1,\dots,a_N\}$ [III] and ω_i are meromorphic closed $1-$forms, with at most simple poles in a_i, and $\gamma(0) = a_1$, the iterated integral $I = \int_\gamma \omega_1\cdots\omega_n$ is divergent. The divergence being polynomial in $\log\epsilon$ ($\epsilon \ll 1$) [IV], we define the iterated integral I as the constant term, which only depends on $\gamma'(0)$. This process is called *regularization*, we need to choose the tangential base points to properly define the integral. Later, we will consider the straight path *dch* from 0 to 1, with tangential base point $\overrightarrow{1}$ at 0 and $\overrightarrow{-1}$ at 1, denoted also $\overrightarrow{1}_0, \overrightarrow{-1}_1$ or simply $\overrightarrow{01}$ for both.

`Notations`: In the case of *multiple zeta values* (i.e. $N=1$) resp. of *Euler sums* (i.e. $N=2$), since $\epsilon_i \in \{\pm 1\}$, the notation is simplified, using $z_i \in \mathbb{Z}^*$:

$$\zeta(z_1,\dots,z_p) := \zeta\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p\end{pmatrix} \text{ with } \begin{pmatrix} n_i \\ \epsilon_i\end{pmatrix} := \begin{pmatrix} \mid z_i \mid \\ sign(z_i)\end{pmatrix}. \tag{1.4}$$

Another common notation in the literature is the use of *overlines* instead of negative arguments, i.e.: $z_i := \begin{cases} n_i & \text{if } \epsilon_i = 1 \\ \overline{n_i} & \text{if } \epsilon_i = -1 \end{cases}$.

1.2 Contents

In this thesis, we mainly consider the ***motivic*** versions of these multiple zeta values, denoted $\boldsymbol{\zeta^{\mathfrak{m}}}(\cdot)$ and shortened $\mathbf{MMZV}_{\boldsymbol{\mu_N}}$ and defined in §2.3. They span a $\mathbb{Q}$-vector space $\boldsymbol{\mathcal{H}^N}$ of motivic multiple zetas relative to μ_N. There is a surjective homomorphism, called the *period map*, which is conjectured to be an isomorphism (this is a special case of the period conjecture):

$$\mathbf{per} : w : \quad \mathcal{H}^N \to \mathcal{Z}^N \ , \quad \zeta^{\mathfrak{m}}(\cdot) \mapsto \zeta(\cdot). \tag{1.5}$$

Working on the motivic side, besides being conjecturally identical to the complex numbers side, turns out to be somehow simpler, since motivic theory provides a Hopf Algebra structure as we will see throughout this thesis. Notably, each identity between motivic MZV_{μ_N} implies an identity for their periods; a motivic basis for $\mathrm{MMZV}\mu_N$ is hence a generating family (conjecturally basis) for $\mathrm{MZV}\mu_N$.

Indeed, on the side of motivic multiple zeta values, there is an action of a *motivic Galois group* $\boldsymbol{\mathcal{G}}$[V], which, passing to the dual, factorizes through a *coaction* $\boldsymbol{\Delta}$ as we will see in

[I] The series is absolutely convergent for $\mid \epsilon_i \mid < 1$, converges also for $\mid \epsilon_i \mid = 1$ if $n_p > 1$. Cf. [73] for an introduction.

[II] As the functional equations with $Li_2\left(\frac{1}{z}\right)$ or $Li_2(1-z)$ or the famous five terms relation, for its sibling, the Bloch Wigner function $D(z) := Im\left(Li_2(z) + \log(\mid z \mid)\log(1-z)\right)$:

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

[III] As for cyclotomic MZV, with $a_i \in \mu_N \cup \{0\}$; such an $I = \int_\gamma \omega_1 \dots \omega_n$ is a multivalued function on M.

[IV] More precisely, we can prove that $\int_\epsilon^1 \gamma^*(\omega_1)\cdots\gamma^*(\omega_n) = \sum_{i=0}\alpha_i(\epsilon)\log^i(\epsilon)$, with $\alpha_i(\epsilon)$ holomorphic in $\epsilon = 0$; $\alpha_0(\epsilon)$ depends only on $\gamma'(0)$.

[V] Later, we will define a category of Mixed Tate Motives, which will be a tannakian category: consequently equivalent to a category of representation of a group $\mathcal{G}$; cf. §2.1.

§2.4. This coaction, which is given by an explicit combinatorial formula (Theorem (2.4.2), [Goncharov, Brown]), is the keystone of this PhD. In particular, it enables us to prove linear independence of MMZV, as in the theorem stated below (instead of adding yet another identity to the existing zoo of relations between MZV), and to study Galois descents. From this, we deduce results about numbers by applying the period map.

This thesis is structured as follows:

Chapter 2 sketches the background necessary to understand this work, from Mixed Tate Motives to the Hopf algebra of motivic multiple zeta values at μ_N, with some specifications according the values of N, and results used throughout the rest of this work. The combinatorial expression of the coaction (or of the weight graded *derivation operators* D_r extracted from it, (2.45)) is the cornerstone of this work. We shall also bear in mind Theorem 2.4.4 stating which elements are in the kernel of these derivations), which sometimes allows to lift identities from MZV to motivic MZV, up to rational coefficients, as we will see throughout this work.
`Nota Bene`: A *motivic relation* is indeed stronger; it may hence require several relations between MZV in order to lift an identity to motivic MZV. An example of such a behaviour occurs with some Hoffman $\star$ elements, in Lemma 4.4.3.

Chapter 3 explains the main results of this PhD, ending with a wider perspective and possible future paths.

Chapter 4 focuses on the cases $N = 1$, i.e. multiple zeta values and $N = 2$, i.e. Euler sums, providing some new bases:

(*i*) First, we introduce *Euler* $\sharp$ *sums*, variants of Euler sums, defined in §2.3 as in (1.3), replacing each $\omega_{\pm 1}$ by $\omega_{\pm\sharp} := 2\omega_{\pm 1} - \omega_0$, except for the first one and prove:

Theorem. *Motivic Euler* $\sharp$ *sums with only positive odd and negative even integers as arguments are unramified: i.e. motivic multiple zeta values.*

By application of the period map above:

Corollary. *Each Euler* $\sharp$ *sums with only positive odd and negative even integers as arguments is unramified, i.e.* $\mathbb{Q}$ *linear combination of multiple zeta values.*

Moreover, we can extract a basis from this family:

Theorem. $\{\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, 2a_1+3, \ldots, 2a_{p-1}+3, -(2a_p+2)\right), a_i \geq 0\}$ *is a graded basis of the space of motivic multiple zeta values.*

By application of the period map:

Corollary. *Each multiple zeta value is a* $\mathbb{Q}$ *linear combination of elements of the same weight in* $\{\zeta^{\sharp}\left(2a_0+1, 2a_1+3, \ldots, 2a_{p-1}+3, -(2a_p+2)\right), a_i \geq 0\}$.

(*ii*) We also prove the following, where Euler $\star$ sums are defined (cf. §2.3) as in (1.3), replacing each $\omega_{\pm 1}$ by $\omega_{\pm\star} := \omega_{\pm 1} - \omega_0$, except the first:

Theorem. *If the analytic conjecture (*4.4.4*) holds, then the motivic Hoffman* $\star$ *family* $\{\zeta^{\star,\mathfrak{m}}(\{2,3\}^{\times})\}$ *is a basis of* $\mathcal{H}^1$*, the space of MMZV.*

(*iii*) Conjecturally, the two previous basis, namely the Hoffman * family and the Euler$^\sharp$ family, are the same. Indeed, we conjecture a generalized motivic Linebarger-Zhao equality (Conjecture 4.5.1) which expresses each motivic multiple zeta $\star$ value as a motivic Euler $\sharp$ sum. It extends the Two One formula [Ohno-Zudilin], the Three One Formula [Zagier], and Linebarger Zhao formula, and applies to motivic MZV. If this conjecture holds, then (*i*) implies that the Hoffman* family is a basis.

Such results on linear independence of a family of motivic MZV are proved recursively, once we have found the *appropriate level* filtration on the elements; ideally, the family considered is stable under the derivations [I]; the filtration, as we will see below, should correspond to the *motivic depth* defined in §2.4.3, and decrease under the derivations [II]; if the derivations, modulo some spaces, act as a deconcatenation on these elements, linear independence follows naturally from this recursion. Nevertheless, to start this procedure, we need an analytic identity[III], which is left here as a conjecture in the case of the Hoffman $\star$ basis. This conjecture is of an entirely different nature from the techniques developed in this thesis. We expect that it could be proved using analytic methods along the lines of [82], [63].

Chapter 5 applies ideas of Galois descents on the motivic side. Originally, the notion of Galois descent was inspired by the question: which linear combinations of Euler sums are *unramified*, i.e. multiple zeta values?[IV] More generally, looking at the motivic side, one can ask which linear combinations of MMZV_{μ_N} lie in $\mathrm{MMZV}_{\mu_{N'}}$ for N' dividing N. This is what we call *descent* (the first level of a descent) and can be answered by exploiting the motivic Galois group action. General descent criteria are given; in the particular case of $N = 2, 3, 4, \text{‘6’}, 8$[V], Galois descents are made explicit and our results lead to new bases of MMZV relative to $\mu_{N'}$ in terms of a basis of MMZV relative to μ_N, and in particular, a new proof of P. Deligne's results [31].
Going further, we define ramification spaces which constitute a tower of intermediate spaces between the elements in MMZV_{μ_N} and the whole space of $\mathrm{MMZV}_{\mu_{N'}}$. This is summed up in §3.2 and studied in detail Chapter 5 or article [42].
Moreover, as we will see below, these methods enable us to construct the motivic periods of categories of mixed Tate motives which cannot be reached by standard methods: i.e. are not simply generated by a motivic fundamental group.

Chapter 6 gathers some applications of the coaction, from maximal depth terms, to motivic identities, via unramified motivic Euler sums; other potential applications of these Galois ideas to the study of these periods are still waiting to be further investigated.

`Consistency:`
Chapter 2 is fundamental to understand the tools and the proofs of both Chapter 4, 5 and 6 (which are independent between them), but could be skimmed through before the reading of the main results in Chapter 3. The proofs of Chapter 4 are based on the results of Annexe $A.1$, but could be read independently.

[I] If the family is not *a priori stable* under the coaction, we need to incorporate in the recursion an hypothesis on the coefficients which appear when we express the right side with the elements of the family.

[II] In the case of Hoffman basis ([20]), or Hoffman $\star$ basis (Theorem 4.4.1) it is the number of 3, whereas in the case of Euler $\sharp$ sums basis (Theorems 4.3.2), it is the depth minus one; for the *Deligne* basis given in Chapter 5 for $N = 2, 3, 4, \text{‘6’}, 8$, it is the usual depth. The filtration by the level has to be stable under the coaction, and more precisely, the derivations D_r decrease the level on the elements of the conjectured basis, which allows a recursion.

[III] Where F. Brown in [20], for the Hoffman basis, used the analytic identity proved by Zagier in [82], or [63].

[IV] This was already a established question, studied by Broadhurst (which uses the terminology *honorary*) among others. Notice that this issue surfaces also for motivic Euler sums in some results in Chapter 3 and 5.

[V] `Nota Bene`: $N = \text{‘6’}$ is a special case; the quotation marks indicate here that we restrict to *unramified* MMZV cf. §2.1.1.

Chapter 2

Background

2.1 Motives and Periods

Here we sketch the motivic background where the motivic iterated integrals (and hence this work) mainly take place; although most of it can be taken as a black box. Nevertheless, some of the results coming from this rich theory are fundamental to our proofs.

2.1.1 Mixed Tate Motives

Motives in a nutshell. Motives are supposed to play the role of a universal (and algebraic) cohomology theory (see [1]). This hope is partly nourished by the fact that, between all the classical cohomology theories (de Rham, Betti, l-adique, crystalline), we have comparison isomorphisms in characteristic 0 [I]. More precisely, the hope is that there should exist a tannakian (in particular abelian, tensor) category of motives $\mathcal{M}(k)$, and a functor $\mathrm{Var}_k \xrightarrow{h} \mathcal{M}(k)$ such that:

For each Weil cohomology[II]: $\mathrm{Var}_k \xrightarrow{H} \mathrm{Vec}_k$, *there exists a realization map* w_H *such that the following commutes*:

$$\begin{tikzcd} \mathrm{Var}_k \arrow[r, "h"] \arrow[d, "\forall H"] & \mathcal{M}(k) \arrow[dl, "\exists w_H"] \\ \mathrm{Vec}_K & \end{tikzcd} ,$$

where h *satisfy properties such as* $h(X \times Y) = h(X) \oplus h(Y)$, $h(X \coprod Y) = h(X) \otimes h(Y)$. The realizations functors are conjectured to be full and faithful (conjecture of periods of Grothendieck, Hodge conjecture, Tate conjecture, etc.)[III].

To this end, Voedvosky (cf. [76]) constructed a triangulated category of Mixed Motives $DM^{\mathrm{eff}}(k)_{\mathbb{Q}}$, with rational coefficients, equipped with tensor product and a functor:

$$M_{gm} : \mathrm{Sch}_{/k} \to DM^{\mathrm{eff}} \text{ satisfying some properties such as:}$$

Kunneth $M_{gm}(X \times Y) = M_{gm}(X) \otimes M_{gm}(Y)$.

$\mathbb{A}^1$-invariance $M_{gm}(X \times \mathbb{A}^1) = M_{gm}(X)$.

[I] Even in positive characteristic, $\dim H^i(X)$ does not depend on the cohomology chosen among these.

[II] This functor should verify some properties, such as Kunneth formula, Poincare duality, etc. as the classic cohomology theories.
If we restrict to smooth projective varieties, SmProj_k, we can construct such a category, the category of pure motives $\mathcal{M}^{pure}(k)$ starting from the category of correspondence of degree 0. For more details, cf. [57].

[III] In the case of Mixed Tate Motives over number fields as seen below, Goncharov proved it for Hodge and l-adique Tate realizations, from results of Borel and Soule.

Mayer Vietoris $M_{gm}(U \cap V) \to M_{gm}(U) \otimes M_{gm}(V) \to M_{gm}(U \cup V) \to M_{gm}(U \cap V)[1]$, U, V open, is a distinguished triangle.[I]

Gysin $M_{gm}(X \setminus Z) \to M_{gm}(X) \to M_{gm}(Z)(c)[2c] \to M_{gm}(X \setminus Z)[1]$, X smooth, Z smooth, closed, of codimension c, is a distinguished triangle.

We would like to extract from the triangulated category $DM^{\text{eff}}(k)_{\mathbb{Q}}$ an abelian category of Mixed Tate Motives over k[II]. However, we still are not able to do it in the general case, but it is possible for some triangulated tensor subcategory of type Tate, generated by $\mathbb{Q}(n)$ with some properties.

Remark: $\mathbb{L} := \mathbb{Q}(-1) = H^1(\mathbb{G}_{\mathfrak{m}}) = H^1(\mathbb{P}^1 \setminus \{0, \infty\})$ which is referred to as the *Lefschetz motive*, is a pure motive, and has period $(2i\pi)$. Its dual is the so-called *Tate motive* $\mathbb{T} := \mathbb{Q}(1) = \mathbb{L}^\vee$. More generally, let us define $\mathbb{Q}(-n) := \mathbb{Q}(-1)^{\otimes n}$ resp. $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ whose periods are in $(2i\pi)^n\mathbb{Q}$ resp. $(\frac{1}{2i\pi})^n\mathbb{Q}$, hence extended periods in $\widehat{P}$; we have the decomposition of the motive of the projective line: $h(\mathbb{P}^n) = \oplus_{k=0}^n \mathbb{Q}(-k)$.

Mixed Tate Motives over a number field. Let first define, for k a number field, the category $\mathcal{DM}(k)_{\mathbb{Q}}$ from $\mathcal{DM}^{\text{eff}}(k)_{\mathbb{Q}}$ by formally "inverting" the Tate motive $\mathbb{Q}(1)$, and then $\mathcal{DMT}(k)_{\mathbb{Q}}$ as the smallest triangulated full subcategory of $\mathcal{DM}(k)_{\mathbb{Q}}$ containing $\mathbb{Q}(n), n \in \mathbb{Z}$ and stable by extension.
By the vanishing theorem of Beilinson-Soule, and results from Levine (cf. [61]), there exists:[III]

> A tannakian *category of Mixed Tate motives* over k with rational coefficients, $\mathcal{MT}(k)_{\mathbb{Q}}$ and equipped with a weight filtration W_r indexed by even integers such that $gr^W_{-2r}(M)$ is a sum of copies of $\mathbb{Q}(r)$ for $M \in \mathcal{MT}(k)$, i.e.,
> Every object $M \in \mathcal{MT}(k)_{\mathbb{Q}}$ is an iterated extension of Tate motives $\mathbb{Q}(n), n \in \mathbb{Z}$.

such that (by the works of Voedvodsky, Levine [61], Bloch, Borel (and K-theory), cf. [34]):

$$\begin{array}{ll} \text{Ext}^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(k)_{\mathbb{Q}} \otimes \mathbb{Q} \cong & \left\{ \begin{array}{ll} k^* \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } n = 1. \\ \mathbb{Q}^{r_1+r_2} & \text{if } n > 1 \text{ odd} \\ \mathbb{Q}^{r_2} & \text{if } n > 1 \text{ even} \end{array} \right. . \\ \text{Ext}^i_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong 0 & \text{if } i > 1 \text{ or } n \leq 0. \end{array}$$

Here, r_1 resp r_2 stand for the number of real resp. complex (and non real, up to conjugate) embeddings from k to $\mathbb{C}$.
In particular, the weight defines a canonical fiber functor:

$$\begin{array}{ll} \omega : & \mathcal{MT}(k) \to \text{Vec}_{\mathbb{Q}} \\ & M \mapsto \oplus \omega_r(M) \end{array} \qquad \text{with} \left\{ \begin{array}{l} \omega_r(M) := \text{Hom}_{\mathcal{MT}(k)}(\mathbb{Q}(r), gr^W_{-2r}(M)) \\ \text{i.e.} \quad gr^W_{-2r}(M) = \mathbb{Q}(r) \otimes \omega_r(M). \end{array} \right.$$

The category of Mixed Tate Motives over k, since tannakian, is equivalent to the category of representations of the so-called ***motivic Galois group*** $\boldsymbol{\mathcal{G}^{\mathcal{MT}}}$ of $\mathcal{MT}(k)$ [IV]:

[I] Distinguished triangles in $DMT^{\text{eff}}(k)$, i.e. of type Tate, become exact sequences in $\mathcal{MT}(k)$.
[II] A way would be to define a t structure on this category, and the heart of the t-structure, by Bernstein, Beilinson, Deligne theorem is a full admissible abelian sub-category.
[III] A *tannakian* category is abelian, k-linear, tensor rigid (autoduality), has an exact faithful fiber functor, compatible with $\otimes$ structures, etc. Cf. [33] about Tannakian categories.
[IV] With the equivalence of category between A Comodules and Representations of the affine group scheme Spec(A), for A a Hopf algebra. Note that Rep($\mathbb{G}_m$) is the category of k-vector space $\mathbb{Z}$-graded of finite dimension.

$$\boxed{\mathcal{MT}(k)_{\mathbb{Q}} \cong \mathrm{Rep}_k \mathcal{G}^{\mathcal{MT}} \cong \mathrm{Comod}\,(\mathcal{O}(\mathcal{G}^{\mathcal{MT}})) \quad \text{where } \mathcal{G}^{\mathcal{MT}} := \mathrm{Aut}^{\otimes}\omega. \qquad (2.1)}$$

The motivic Galois group $\mathcal{G}^{\mathcal{MT}}$ decomposes as, since ω graded:

$$\mathcal{G}^{\mathcal{MT}} = \mathbb{G}_m \ltimes \mathcal{U}^{\mathcal{MT}}, \quad \text{i.e.} \quad 1 \to \mathcal{U}^{\mathcal{MT}} \to \mathcal{G}^{\mathcal{MT}} \leftrightarrows \mathbb{G}_m \to 1 \quad \textit{is an exact sequence,}$$

$$\textit{where } \mathcal{U}^{\mathcal{MT}} \textit{ is a pro-unipotent group scheme defined over } \mathbb{Q}.$$

The action of $\mathbb{G}_m$ is a grading, and $\mathcal{U}^{\mathcal{MT}}$ acts trivially on the graded pieces $\omega(\mathbb{Q}(n))$.

Let $\mathfrak{u}$ denote the completion of the pro-nilpotent graded Lie algebra of $\mathcal{U}^{\mathcal{MT}}$ (defined by a limit); $\mathfrak{u}$ is free and graded with negative degrees from the $\mathbb{G}_m$-action. Furthermore[I]:

$$\mathfrak{u}^{ab} \cong \bigoplus \mathrm{Ext}^1_{\mathcal{MT}}(\mathbb{Q}(0), \mathbb{Q}(n))^{\vee} \text{ in degree } n. \qquad (2.2)$$

Hence the *fundamental Hopf algebra* is [II]:

$$\mathcal{A}^{\mathcal{MT}} := \mathcal{O}(\mathcal{U}^{\mathcal{MT}}) \cong (U^{\wedge}(\mathfrak{u}))^{\vee} \cong T(\oplus_{n\geq 1}\mathrm{Ext}^1_{\mathcal{MT}_N}(\mathbb{Q}(0), \mathbb{Q}(n))^{\vee}). \qquad (2.3)$$

Hence, by the Tannakian dictionary (2.1): $\mathcal{MT}(k)_{\mathbb{Q}} \cong \mathrm{Rep}^{gr}\mathcal{U}^{\mathcal{MT}} \cong \mathrm{Comod}^{gr}\mathcal{A}^{\mathcal{MT}}$.

Once an embedding $\sigma : k \hookrightarrow \mathbb{C}$ fixed, Betti cohomology leads to a functor *Betti realization*:

$$\omega_{B_\sigma} : \mathcal{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}, \quad M \mapsto M_\sigma.$$

De Rham cohomology leads similarly to the functor *de Rham realization*:

$$\omega_{dR} : \mathcal{MT}(k) \to \mathrm{Vec}_k, \quad M \mapsto M_{dR}\ , \quad M_{dR} \text{ weight graded.}$$

Beware, the de Rham functor ω_{dR} here is not defined over $\mathbb{Q}$ but over k and $\omega_{dR} = \omega \otimes_{\mathbb{Q}} k$, so the de Rham realization of an object M is $M_{dR} = \omega(M) \otimes_{\mathbb{Q}} k$.
Between all these realizations, we have comparison isomorphisms, such as:

$$M_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow[\sim]{\mathrm{comp}_{dR,\sigma}} M_{dR} \otimes_{k,\sigma} \mathbb{C} \text{ with its inverse } \mathrm{comp}_{\sigma,dR}.$$

$$M_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow[\sim]{\mathrm{comp}_{\omega,B_\sigma}} M_\omega \otimes_{\mathbb{Q}} \mathbb{C} \text{ with its inverse } \mathrm{comp}_{B_\sigma,\omega}.$$

[I] Since $\mathrm{Ext}^2_{\mathcal{MT}}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$, which implies $\forall M$, $H^2(\mathfrak{u}, M) = 0$, hence $\mathfrak{u}$ free. Moreover, $(\mathfrak{u}^{ab}) = (\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = H_1(\mathfrak{u};\mathbb{Q})$, then, for $\mathcal{U}$ unipotent:

$$\left(\mathfrak{u}^{ab}\right)^{\vee}_{m-n} \cong \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}}}(\mathbb{Q}(n), \mathbb{Q}(m)).$$

[II] Recall the anti-equivalence of Category, between Hopf Algebra and Affine Group Schemes:

$$\begin{array}{ccc} k - Alg^{op} & \xrightarrow{\sim} & k - \mathrm{AffSch} \\ \uparrow & & \uparrow \\ k - \mathrm{HopfAlg}^{op} & \xrightarrow{\sim} & k - \mathrm{AffGpSch} \\ & & \\ A & \longmapsto & \mathrm{Spec}\ A \\ & & \\ \mathcal{O}(G) & \longleftarrow\!\shortmid & G : R \mapsto \mathrm{Hom}_k(\mathcal{O}(G), R) \end{array}.$$

It comes from the fully faithful Yoneda functor $C^{op} \to \mathrm{Fonct}(C, \mathrm{Set})$, leading to an equivalence of Category if we restrict to Representable Functors: $k - \mathrm{AffGpSch} \cong \mathrm{RepFonct}\,(\mathrm{Alg}^{\,op}, Gp)$. Properties for Hopf algebra are obtained from Affine Group Scheme properties by 'reversing the arrows' in each diagram.
Remark that G is unipotent if and only if A is commutative, finite type, connected and filtered.

Define also, looking at tensor-preserving isomorphisms:

$$\begin{array}{lll} \mathcal{G}_B := \mathrm{Aut}^{\otimes}(\omega_B), & \text{resp. } \mathcal{G}_{dR} := \mathrm{Aut}^{\otimes}(\omega_{dR}) & \\ P_{\omega,B} := \mathrm{Isom}^{\otimes}(\omega_B, \omega), & \text{resp. } P_{B,\omega} := \mathrm{Isom}^{\otimes}(\omega, \omega_B), & (\mathcal{G}^{\mathcal{MT}}, \mathcal{G}_B) \text{ resp. } (\mathcal{G}_B, \mathcal{G}^{\mathcal{MT}}) \text{ bitorsors .} \end{array}$$

Comparison isomorphisms above define $\mathbb{C}$ points of these schemes: $\mathrm{comp}_{\omega,B} \in P_{B,\omega}(\mathbb{C})$.

Remarks: By (2.1):[I]

> A Mixed Tate motive over a number field is uniquely defined by its de Rham realization, a vector space M_{dR}, with an action of the motivic Galois group $\mathcal{G}^{\mathcal{MT}}$.

`Example:` For instance $\mathbb{Q}(n)$, as a Tate motive, can be seen as the vector space $\mathbb{Q}$ with the action $\lambda \cdot x := \lambda^n x$, for $\lambda \in \mathbb{Q}^* = \mathrm{Aut}(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q})$.

Mixed Tate Motives over $\mathcal{O}_S$. Before, let's recall for k a number field and $\mathcal{O}$ its ring of integers, *archimedian values* of k are associated to an embedding $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$, such that:

$$\mid x \mid := \mid \sigma(x) \mid_\infty \text{ , where } \mid \cdot \mid_\infty \text{ is the usual absolute value,}$$

and *non archimedian values* are associated to non-zero prime ideals of $\mathcal{O}$[II]:

$$v_{\mathfrak{p}} : k^\times \to \mathbb{Z}, \quad v_{\mathfrak{p}}(x) \text{ is the integer such that } x\mathcal{O}_{\mathfrak{p}} = \mathfrak{p}^{v_{\mathfrak{p}}(x)}\mathcal{O}_{\mathfrak{p}} \text{ for } x \in k^\times.$$

For S a finite set of absolute values in k containing all archimedian values, the *ring of S-integers*:

$$\mathcal{O}_S := \{x \in k \mid v(x) \geq 0 \text{ for all valuations } v \notin S\}.$$

Dirichlet unit's theorem generalizes for $\mathcal{O}_S^\times$, abelian group of type finite: [III]

$$\mathcal{O}_S^\times \cong \mu(K) \times \mathbb{Z}^{\text{ card }(S)-1}.$$

`Examples`:

- Taking S as the set of the archimedian values leads to the usual ring of integers $\mathcal{O}$, and would lead to the unramified category of motives $\mathcal{MT}(\mathcal{O})$ below.
- For $k = \mathbb{Q}$, p prime, with $S = \{v_p, \mid \cdot \mid_\infty\}$, we obtain $\mathbb{Z}\left[\frac{1}{p}\right]$. Note that the definition does not allow to choose $S = \{\cup_{\substack{q \text{prime} \\ q \neq p}} v_q, \mid \cdot \mid_\infty\}$, which would lead to the localization $\mathbb{Z}_{(p)} := \{x \in \mathbb{Q} \mid v_p(x) \geq 0\}$.

Now, let us define the categories of Mixed Tate Motives which interest us here:

Definition. $\mathcal{MT}_\Gamma$: *For Γ sub-vector space of $Ext^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong k^* \otimes \mathbb{Q}$:*

$\mathcal{MT}_\Gamma$: *the tannakian subcategory formed by objects M such that each subquotient E of M:*

$$0 \to \mathbb{Q}(n+1) \to E \to \mathbb{Q}(n) \to 0 \quad \Rightarrow \quad [E] \in \Gamma \subset Ext^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1))^{\mathrm{IV}}.$$

[I] The different cohomologies should be viewed as interchangeable realizations. Etale chomology, with the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (cf [3]) is related to the number N_p of points of reduction modulo p. For Mixed Tate Motives (and conjecturally only for those) N_p are polynomials modulo p, which is quite restrictive.

[II] $\mathcal{O}$ is a Dedekind domain, $\mathcal{O}_{\mathfrak{p}}$ a discrete valuation ring whose prime ideals are prime ideals of $\mathcal{O}$ which are included in $(\mathfrak{p})\mathcal{O}_{\mathfrak{p}}$.

[III] It will be used below, for dimensions, in 2.31. Here, card $(S) = r_1 + r_2 +$ card (non-archimedian places); as usual, r_1, r_2 standing for the number of real resp. complex (and non real, and up to conjugate) embeddings from k to $\mathbb{C}$; $\mu(K)$ is the finite cyclic group of roots of unity in K.

[IV] $\mathrm{Ext}^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathrm{Ext}^1_{\mathcal{MT}(k)}(\mathbb{Q}(n), \mathbb{Q}(n+1))$.

$\boldsymbol{\mathcal{MT}(\mathcal{O}_S)}$**:** *The category of mixed Tate motives unramified in each finite place $v \notin S$:*

$$\mathcal{MT}(\mathcal{O}_S) := \mathcal{MT}_\Gamma, \quad \textit{for } \Gamma = \mathcal{O}_S^* \otimes \mathbb{Q}.$$

Extension groups for these categories are then identical to those of $\mathcal{MT}(k)$ except:

$$\mathrm{Ext}^1_{\mathcal{MT}_\Gamma}(\mathbb{Q}(0), \mathbb{Q}(1)) = \Gamma, \quad \text{resp.} \quad \mathrm{Ext}^1_{\mathcal{MT}(\mathcal{O}_S)}(\mathbb{Q}(0), \mathbb{Q}(1)) = K_1(\mathcal{O}_S) \otimes \mathbb{Q} = \mathcal{O}_S^* \otimes \mathbb{Q}. \tag{2.4}$$

Cyclotomic Mixed Tate Motives. In this thesis, we focus on the cyclotomic case and consider the following categories, and sub-categories, for k_N the Nth cyclotomic field, $\mathcal{O}_N := \mathbb{Z}[\xi_N]$ its ring of integers, with ξ_N a primitive N^{th} root of unity:

$$\begin{array}{ll} \boldsymbol{\mathcal{MT}_{N,M}} & := \mathcal{MT}\left(\mathcal{O}_N\left[\frac{1}{M}\right]\right). \\ \boldsymbol{\mathcal{MT}_{\Gamma_N}}, & \text{with } \Gamma_N \text{ the } \mathbb{Q}\text{-sub vector space of } \left(\mathcal{O}\left[\frac{1}{N}\right]\right)^* \otimes \mathbb{Q} \\ & \text{generated by } \{1-\zeta_N^a\}_{0<a<N} \text{ (modulo torsion).} \end{array}$$

Hence:

$$\mathcal{MT}(\mathcal{O}_N) \subsetneq \mathcal{MT}_{\Gamma_N} \subset \mathcal{MT}\left(\mathcal{O}_N\left[\frac{1}{N}\right]\right)$$

The second inclusion is an equality if and only if N has all its prime factors inert[I], since:

$$\Gamma_N := \begin{cases} \left(\mathcal{O}\left[\frac{1}{p}\right]\right)^* \otimes \mathbb{Q} & \text{if } N = p^r \\ (\mathcal{O}^* \otimes \mathbb{Q}) \oplus \left(\oplus_{\substack{p \text{ prime} \\ p|N}} \langle p \rangle \otimes \mathbb{Q}\right) & \text{else .} \end{cases} . \tag{2.5}$$

The motivic cyclotomic MZV lie in the subcategory $\mathcal{MT}_{\Gamma_N}$, as we will see more precisely in §2.3.

`Notations:` We may sometimes drop the M (or even N), to lighten the notations:[II]:

$$\mathcal{MT}_N := \begin{cases} \mathcal{MT}_{N,N} & \text{if } N = 2,3,4,8 \\ \mathcal{MT}_{6,1} & \text{if } N = \text{`6'}. \end{cases}$$

2.1.2 Motivic periods

Let $\mathcal{M}$ a tannakian category of mixed Tate motives. Its *algebra of motivic periods* is defined as (cf. [32], [24], and [22], §2):

$$\boldsymbol{\mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}} := \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{M}}(\omega, \omega_B)) = \mathcal{O}(P_{B,\omega}).$$

A ***motivic period*** denoted as a triplet $[\boldsymbol{M}, \boldsymbol{v}, \boldsymbol{\sigma}]^{\mathfrak{m}}$, element of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}$, is constructed from a motive $M \in$ Ind $(\mathcal{M})$, and classes $v \in \omega(M)$, $\sigma \in \omega_B(M)^\vee$. It is a function $P_{B,\omega} \to \mathbb{A}^1$, which, on its rational points, is given by:

$$P_{B,\omega}(\mathbb{Q}) \to \mathbb{Q}\ , \quad \alpha \mapsto \langle \alpha(v), \sigma \rangle. \tag{2.6}$$

[I] I.e. each prime p dividing N, generates $(\mathbb{Z}/m\mathbb{Z})^*$, for m such as $N = p^{v_p(N)}m$. It could occur only in the following cases: $N = p^s, 2p^s, 4p^s, p^s q^k$, with extra conditions in most of these cases such as: 2 is a primitive root modulo p^s etc.

[II] For instance, $\mathcal{MT}_3$ is the category $\mathcal{MT}\left(\mathcal{O}_3\left[\frac{1}{3}\right]\right)$.

Its *period* is obtained by the evaluation on the complex point $\mathrm{comp}_{B,dR}$:

$$\begin{array}{lcl} \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}} & \to & \mathbb{C} \\ [M,v,\sigma]^{\mathfrak{m}} & \mapsto & \langle \mathrm{comp}_{B,dR}(v\otimes 1),\sigma\rangle. \end{array} \tag{2.7}$$

`Example`: The first example is the *Lefschetz motivic period*: $\mathbb{L}^{\mathfrak{m}} := [H^1(\mathbb{G}_m),[\frac{dx}{x}],[\gamma_0]]^{\mathfrak{m}}$, period of the Lefschetz motive $\mathbb{L}$; it can be seen as the *motivic* $(2i\pi)^{\mathfrak{m}}$; this notation appears below.

This construction can be generalized for any pair of fiber functors ω_1, ω_2 leading to:

Motivic periods of type (ω_1,ω_2), which are in the following algebra of motivic periods:

$$\boldsymbol{\mathcal{P}^{\omega_1,\omega_2}_{\mathcal{M}}} := \mathcal{O}\left(P_{\omega_1,\omega_2}\right) = \mathcal{O}\left(\mathrm{Isom}^{\otimes}(\omega_2,\omega_1)\right).$$

Remarks:

- The groupoid structure (composition) on the isomorphisms of fiber functors on $\mathcal{M}$, by dualizing, leads to a coalgebroid structure on the spaces of motivic periods:

$$\mathcal{P}^{\omega_1,\omega_3}_{\mathcal{M}} \to \mathcal{P}^{\omega_1,\omega_2}_{\mathcal{M}} \otimes \mathcal{P}^{\omega_2,\omega_3}_{\mathcal{M}}.$$

- Any structure carried by these fiber functors (weight grading on ω_{dR}, complex conjugation on ω_B, etc.) is transmitted to the corresponding ring of periods.

`Examples`:

- For (ω,ω_B), it comes down to (our main interest) $\mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}$ as defined in (2.6). By the last remark, $\mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}$ inherits a weight grading and we can define (cf. [22], §2.6):

$\boldsymbol{\mathcal{P}^{\mathfrak{m},+}_{\mathcal{M}}} \subset \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}$, the ring of *geometric periods*, is generated by periods of motives with non-negative weights: $\left\{[M,v,\sigma]^{\mathfrak{m}} \in \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}} \mid W_{-1}M = 0\right\}$.

- The ring of periods of type (ω,ω) is $\mathcal{P}^{\omega}_{\mathcal{M}} := \mathcal{O}\left(\mathrm{Aut}^{\otimes}(\omega)\right) = \mathcal{O}\left(\mathcal{G}^{\mathcal{MT}}\right)$.[I]
  Unipotent variants of these periods are defined when restricting to the unipotent part $\mathcal{U}^{\mathcal{MT}}$ of $\mathcal{G}^{\mathcal{MT}}$, and intervene below (in 2.25):

$$\boldsymbol{\mathcal{P}^{\mathfrak{a}}_{\mathcal{M}}} := \mathcal{O}\left(\mathcal{U}^{\mathcal{MT}}\right) = \mathcal{A}^{\mathcal{MT}}, \quad \text{the fundamental Hopf algebra.}$$

  They correspond to the notion of framed objects in mixed Tate categories, cf. [44]. By restriction, there is a map:

$$\mathcal{P}^{\omega}_{\mathcal{M}} \to \mathcal{P}^{\mathfrak{a}}_{\mathcal{M}}.$$

By the remark above, there is a coaction:

$$\boldsymbol{\Delta}^{\mathfrak{m},\boldsymbol{\omega}} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}} \to \mathcal{P}^{\omega}_{\mathcal{M}} \otimes \mathcal{P}^{\mathfrak{m}}_{\mathcal{M}}.$$

Moreover, composing this coaction by the augmentation map $\epsilon : \mathcal{P}^{\mathfrak{m},+}_{\mathcal{M}} \to (\mathcal{P}^{\mathfrak{m},+}_{\mathcal{M}})_0 \cong \mathbb{Q}$, leads to the morphism (details in [22], §2.6):

$$\boldsymbol{\pi}_{\mathfrak{a},\mathfrak{m}} : \quad \mathcal{P}^{\mathfrak{m},+}_{\mathcal{M}} \to \mathcal{P}^{\mathfrak{a}}_{\mathcal{M}}, \tag{2.8}$$

[I] In the case of a mixed Tate category over $\mathbb{Q}$, as $\mathcal{MT}(\mathbb{Z})$, this is equivalent to the *De Rham periods* in $\mathcal{P}^{\mathfrak{dR}}_{\mathcal{M}} := \mathcal{O}\left(\mathrm{Aut}^{\otimes}(\omega_{dR})\right)$, defined in [22]; however, for other cyclotomic fields k considered later ($N>2$), we have to consider the canonical fiber functor, since it is defined over k.

which is, on periods of a motive M such that $W_{-1}M = 0$: $\quad [M, v, \sigma]^{\mathfrak{m}} \to [M, v, {}^t c(\sigma)]^{\mathfrak{a}}$,

where c is defined as the composition : $\quad M_\omega \twoheadrightarrow gr_0^W M_\omega = W_0 M_\omega \xrightarrow{\mathrm{comp}_{B,\omega}} W_0 M_B \hookrightarrow M_B$.

Bear in mind also the non-canonical isomorphisms, compatible with weight and coaction ([22], Corollary 2.11) between those $\mathbb{Q}$ algebras:

$$\mathcal{P}_{\mathcal{M}}^{\mathfrak{m}} \cong \mathcal{P}_{\mathcal{M}}^{\mathfrak{a}} \otimes_{\mathbb{Q}} \mathbb{Q}\left[(\mathbb{L}^{\mathfrak{m}})^{-1}, \mathbb{L}^{\mathfrak{m}}\right], \quad \text{and} \quad \mathcal{P}_{\mathcal{M}}^{\mathfrak{m},+} \cong \mathcal{P}_{\mathcal{M}}^{\mathfrak{a}} \otimes_{\mathbb{Q}} \mathbb{Q}\left[\mathbb{L}^{\mathfrak{m}}\right]. \tag{2.9}$$

In particular, $\pi_{\mathfrak{a},\mathfrak{m}}$ is obtained by sending $\mathbb{L}^{\mathfrak{m}}$ to 0.

In the case of a category of mixed Tate motive $\mathcal{M}$ defined over $\mathbb{Q}$, [I] the complex conjugation defines the *real Frobenius* $\mathcal{F}_\infty : M_B \to M_B$, and induces an involution on motivic periods $\mathcal{F}_\infty : \mathcal{P}_{\mathcal{M}}^{\mathfrak{m}} \to \mathcal{P}_{\mathcal{M}}^{\mathfrak{m}}$. Furthermore, $\mathbb{L}^{\mathfrak{m}}$ is anti invariant by $\mathcal{F}_\infty$ (i.e. $\mathcal{F}_\infty(\mathbb{L}^{\mathfrak{m}}) = -\mathbb{L}^{\mathfrak{m}}$). Then, let us define:

$\boldsymbol{\mathcal{P}_{\mathcal{M},\mathbb{R}}^{\mathfrak{m},+}}$ the subset of $\mathcal{P}_{\mathcal{M}}^{\mathfrak{m},+}$ invariant under the real Frobenius $\mathcal{F}_\infty$, which, by (2.9) satisfies :

$$\mathcal{P}_{\mathcal{M}}^{\mathfrak{m},+} \cong \mathcal{P}_{\mathcal{M},\mathbb{R}}^{\mathfrak{m},+} \oplus \mathcal{P}_{\mathcal{M},\mathbb{R}}^{\mathfrak{m},+} . \mathbb{L}^{\mathfrak{m}} \quad \text{and} \quad \mathcal{P}_{\mathcal{M},\mathbb{R}}^{\mathfrak{m},+} \cong \mathcal{P}_{\mathcal{M}}^{\mathfrak{a}} \otimes_{\mathbb{Q}} \mathbb{Q}\left[(\mathbb{L}^{\mathfrak{m}})^2\right]. \tag{2.10}$$

Motivic Galois theory. The ring of motivic periods $\mathcal{P}_{\mathcal{M}}^{\mathfrak{m}}$ is a bitorsor under Tannaka groups $(\mathcal{G}^{\mathcal{MT}}, \mathcal{G}_B)$. If Grothendieck conjecture holds, via the period isomorphism, there is therefore a (left) action of the motivic Galois group $\mathcal{G}^{\mathcal{MT}}$ on periods.
More precisely, for each period p there would exist:

- (i) well defined conjugates: elements in the orbit of $\mathcal{G}^{\mathcal{MT}}(\mathbb{Q})$.
- (ii) an algebraic group over $\mathbb{Q}$, $\mathcal{G}_p = \mathcal{G}^{\mathcal{MT}} / Stab(p)$, where $Stab(p)$ is the stabilizer of p; $\mathcal{G}_p$, the Galois group of p, transitively permutes the conjugates.

`Examples`:

- For π for instance, the Galois group corresponds to $\mathbb{G}_m$. Conjugates of π are in fact $\mathbb{Q}^*\pi$, and the associated motive would be the Lefschetz motive $\mathbb{L}$, motive of $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$, as seen above.
- For $\log t$, $t > 0$, $t \in \mathbb{Q} \setminus \{-1, 0, 1\}$, this is a period of the Kummer motive in degree 1:[II]

  $$K_t := M_{gm}(X, \{1, t\}) \in \mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \text{ , where } X := \mathbb{P}^1 \setminus \{0, \infty\}.$$

  Since a basis of $H_1^B(X, \{1, t\})$ is $[\gamma_0]$, $[\gamma_{1,t}]$ with γ_t the straight path from 1 to t, and a basis of $H_{dR}^1(X, \{1, t\})$ is $[dx]$, $\left[\frac{dx}{x}\right]$, the period matrix is:

  $$\begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q}\log(t) & 2i\pi\mathbb{Q} \end{pmatrix}.$$

  The conjugates of $\log t$ are $\mathbb{Q}^* \log t + \mathbb{Q}$, and its Galois group is $\mathbb{Q}^* \ltimes \mathbb{Q}$.

[I] As, in our concerns, $\mathcal{MT}_N$ above with $N = 1, 2$; in these exceptional (real) cases, we want to keep track of only even Tate twists.
[II] Remark the short exact sequence: $0 \to \mathbb{Q}(1) \to H_1(X, \{1, t\}) \to \mathbb{Q}(0) \to 0$.

- Similarly for zeta values $\zeta(n)$, n odd in $\mathbb{N}^* \setminus \{1\}$ which are periods of a mixed Tate motive over $\mathbb{Z}$ (cf. below): its conjugates are $\mathbb{Q}^*\zeta(n)+\mathbb{Q}$, and its Galois group is $\mathbb{Q}^* \ltimes \mathbb{Q}$. Grothendieck's conjecture implies that $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent. More precisely, $\zeta(n)$ is a period of $E_n \in \mathcal{MT}(\mathbb{Q})$, where:

$$0 \to \mathbb{Q}(n) \to E_n \to \mathbb{Q}(0) \to 0.$$

  Notice that for even n, by Borel's result, $\mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$, which implies $E_n = \mathbb{Q}(0) \oplus \mathbb{Q}(n)$, and hence $\zeta(n) \in (2i\pi)^n\mathbb{Q}$.

- More generally, multiple zeta values at roots of unity μ_N occur as periods of mixed Tate motives over $\mathbb{Z}[\xi_N]\left[\frac{1}{N}\right]$, ξ_N primitive N^{th} root of unity. The motivic Galois group associated to the algebra $\mathcal{H}^N$ generated by MMZV_{μ_N} is conjectured to be a quotient of the motivic Galois group $\mathcal{G}^{\mathcal{MT}_N}$, equal for some values of N: $N = 1, 2, 3, 4, 8$ for instance, as seen below. We expect MZV to be simple examples in the conjectural Galois theory for transcendental numbers.

Remark: By K-theory results above, non-zero Ext groups for $\mathcal{MT}(\mathbb{Q})$ are:

$$\mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q}^* \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{p \text{ prime}}\mathbb{Q} & \text{if } n = 1 \\ \mathbb{Q} & \text{if } n \text{ odd} > 1. \end{cases}$$

Generators of these extension groups correspond exactly to periods $\log(p)$, p prime in degree 1 and $\zeta(odd)$ in degree odd > 1, which are periods of $\mathcal{MT}(\mathbb{Q})$.

2.2 Motivic fundamental group

Prounipotent completion. Let Π be a group freely generated by $\gamma_0, \ldots, \gamma_N$. The completed Hopf algebra $\widehat{\Pi}$ is defined by:

$$\widehat{\Pi} := \varprojlim \mathbb{Q}[\Pi] \big/ I^n, \qquad \text{where } I := \langle \gamma - 1, \gamma \in \Pi \rangle \text{ is the augmentation ideal.}$$

Equipped with the completed coproduct Δ such that elements of Π are primitive, it is isomorphic to the Hopf algebra of non commutative formal series:[I]

$$\widehat{\Pi} \xrightarrow[\gamma_i \mapsto \exp(e_i)]{\sim} \mathbb{Q}\langle\langle e_0, \ldots, e_N \rangle\rangle.$$

The *prounipotent completion* of Π is an affine group scheme Π^{un}:

$$\boldsymbol{\Pi^{un}}(R) = \{x \in \widehat{\Pi} \widehat{\otimes} R \mid \Delta x = x \otimes x\} \cong \{S \in R\langle\langle e_0, \ldots, e_N \rangle\rangle^\times \mid \Delta S = S \otimes S, \epsilon(S) = 1\}, \tag{2.11}$$

i.e. the set of non-commutative formal series with $N+1$ generators which are group-like for the completed coproduct for which e_i are primitive.
It is dual to the shuffle $\sqcup\!\sqcup$ relation between the coefficients of the series S^{II}. Its affine ring of regular functions, is the Hopf algebra (filtered, connected) for the shuffle product, and deconcatenation coproduct:

$$\boldsymbol{\mathcal{O}(\Pi^{un})} = \varinjlim \left(\mathbb{Q}[\Pi] \big/ I^{n+1}\right)^\vee \cong \mathbb{Q}\left\langle e^0, \ldots, e^N \right\rangle. \tag{2.12}$$

[I] Well defined inverse since the log converges in $\widehat{\Pi}$; $exp(e_i)$ are then group-like for Δ. Notice that the Lie Algebra of the group of group-like elements is formed by the primitive elements and conversely; besides, the universal enveloping algebra of primitive elements is the whole Hopf algebra.

[II] It is a straightforward verification that the relation $\Delta S = S \otimes S$ implies the shuffle $\sqcup\!\sqcup$ relation between the coefficients of S.

Motivic Fundamental pro-unipotent groupoid. [I] The previous construction can be applied to $\pi_1(X,x)$, resp. $\pi_1(X,x,y)$, if assumed free, the fundamental group resp. groupoid of X with base point x, resp. x,y, rational points of X, an algebraic variety over $\mathbb{Q}$; the groupoid $\pi_1(X,x,y)$, is a bitorsor formed by the homotopy classes of path from x to y.

From now, let's turn to the case $X_N := \mathbb{P}^1 \diagdown \{0,\infty,\mu_N\}$. There, the group $\pi_1(X_N,x)$ is freely generated by γ_0 and $(\gamma_\eta)_{\eta\in\mu_N}$, the loops around 0 resp. $\eta\in\mu_N$.[II]
Chen's theorem implies here that we have a perfect pairing:

$$\mathbb{C}[\pi_1(X_N,x,y)]\diagup I^{n+1} \otimes \mathbb{C}\langle \omega_0, (\omega_\eta)_{\eta\in\mu_N}\rangle_{\leq n} \to \mathbb{C}. \tag{2.13}$$

In order to define the motivic π_1^{un}, let us introduce (cf. [45], Theorem 4.1):

$$Y^{(n)} := \cup_i Y_i, \text{ where } \quad \begin{array}{l} Y_0 := \{x\}\times X^{n-1} \\ Y_i := X^{i-1}\times\Delta\times X^{n-i-1}, \quad \Delta\subset X\times X \text{ the diagonal } . \\ Y_n := X^{n-1}\times\{y\} \end{array} \tag{2.14}$$

Then, by Beilinson theorem ([45], Theorem 4.1), coming from $\gamma\mapsto[\gamma(\Delta_n)]$:

$$H_k(X^n,Y^{(n)}) \cong \begin{cases} \mathbb{Q}[\pi_1(X,x,y)]\diagup I^{n+1} & \text{for } k=n \\ 0 & \text{for } k<n \end{cases} .$$

The left side defines a mixed Tate motive and:

$$\mathcal{O}(\pi_1^{un}(X,x,y)) \xrightarrow{\sim} \varinjlim_n H^n(X^n,Y^{(n)}). \tag{2.15}$$

By (2.15), $\mathcal{O}\left(\pi_1^{un}(X,x,y)\right)$ defines an Ind object [III] in the category of Mixed Tate Motives over k, since $Y_I^{(n)} := \cap Y_i^{(n)}$ is the complement of hyperplanes, hence of type Tate:

$$\boxed{\mathcal{O}\left(\pi_1^{un}(\mathbb{P}^1\diagdown\{0,\infty,\mu_N\},x,y)\right) \in \text{Ind } \mathcal{MT}(k).} \tag{2.16}$$

We denote it $\boldsymbol{\mathcal{O}\left(\pi_1^{\mathfrak{m}}(X,x,y)\right)}$, and $\mathcal{O}\left(\pi_1^{\omega}(X,x,y)\right)$, $\mathcal{O}\left(\pi_1^{dR}(X,x,y)\right)$, $\mathcal{O}\left(\pi_1^{B}(X,x,y)\right)$ its realizations, resp. $\boldsymbol{\pi_1^{\mathfrak{m}}(X)}$ for the corresponding $\mathcal{MT}(k)$-groupoid scheme, called the ***motivic fundamental groupoid***, with the composition of path.

Remark: The pairing (2.13) can be thought in terms of a perfect pairing between homology and de Rham cohomology, since (Wojtkowiak [80]):

$$H^n_{dR}(X^n,Y^{(n)}) \cong k_N\langle\omega_0,\ldots,\omega_N\rangle_{\leq n}.$$

The construction of the prounipotent completion and then the motivic fundamental groupoid would still work for the case of *tangential base points* , cf. [34], §3[IV]. Let us denote λ_N the

[I] "*Esquisse d'un programme*"[50], by Grothendieck, vaguely suggests to study the action of the absolute Galois group of the rational numbers $\mathrm{Gal}(\overline{\mathbb{Q}}\diagup\mathbb{Q})$ on the étale fundamental group $\pi_1^{et}(\mathcal{M}_{g,n})$, where $\mathcal{M}_{g,n}$ is the moduli space of curves of genus g and n ordered marked points. In the case of $\mathcal{M}_{0,4}=\mathbb{P}^1\diagdown\{0,1,\infty\}$, Deligne proposed to look instead (analogous) at the pro-unipotent fundamental group $\pi_1^{un}(\mathbb{P}^1\diagdown\{0,1,\infty\})$. This motivates also the study of multiple zeta values, which arose as periods of this fundamental group.

[II] Beware, since $\pi_1(X,x,y)$ is not a group, we have to pass first to the dual in the previous construction:

$$\pi_1^{un}(X,x,y) := \mathrm{Spec}\left(\varinjlim\left(\mathbb{Q}[\pi_1]\diagup I^{n+1}\right)^\vee\right).$$

[III] Ind objects of a category $\mathcal{C}$ are inductive filtered limit of objects in $\mathcal{C}$.

[IV] I.e. here non-zero tangent vectors in a point of $\{0,\mu_N,\infty\}$ are seen as "base points at infinite". Deligne explained how to replace ordinary base points with tangential base points.

straight path between 0 and ξ_N, a primitive root of unity. In the following, we will particularly consider the tangential base points $\overrightarrow{0\xi_N} := (\overrightarrow{1}_0, \overrightarrow{-1}_{\xi_N})$, defined as $(\lambda'_N(0), -\lambda'_N(1))$; but similarly for each $x, y \in \mu_N \cup \{0, \infty\}$, such that ${}_x\lambda_y$ the straight path between x, y in in $\mathbb{P}^1(\mathbb{C}\setminus\{0, \mu_N, \infty\})$, we associate the tangential base points $\overrightarrow{xy} := ({}_x\lambda'_y(0), -{}_x\lambda'_y(1))$[I]. Since the motivic torsor of path associated to such tangential basepoints depends only on x, y (cf. [34], §5) we will denote it ${}_x\Pi^{\mathfrak{m}}_y$. This leads to a groupoid structure via ${}_x\Pi^{\mathfrak{m}}_y \times_y \Pi^{\mathfrak{m}}_z \to_x \Pi^{\mathfrak{m}}_z$: cf. Figure 2.1 and [34].
In fact, by Goncharov's theorem, in case of these tangential base points, the motivic torsor of path corresponding has good reduction outside N and (cf. [34], §4.11):

$$\mathcal{O}\left({}_x\Pi^{\mathfrak{m}}_y\right) \in \text{ Ind } \mathcal{MT}_{\Gamma_N} \subset \text{ Ind } \mathcal{MT}\left(\mathcal{O}_N\left[\frac{1}{N}\right]\right). \tag{2.17}$$

The case of ordinary base points, lying in Ind $\mathcal{MT}(k)$, has no such good reduction.
In summary, from now, we consider, for $x, y \in \mu_N \cup \{0\}$[II]:

> *The motivic bitorsors of path* ${}_x\Pi^{\mathfrak{m}}_y := \pi^{\mathfrak{m}}_1(X_N, \overrightarrow{xy})$ on $X_N := \mathbb{P}^1 - \{0, \mu_N, \infty\}$ with tangential basepoints given by $\overrightarrow{xy} := (\lambda'(0), -\lambda'(1))$ where λ is the straight path from x to y, $x \neq -y$.

Let us denote ${}_x\Pi_y :=_x \Pi^{\omega}_y$, resp. ${}_x\Pi^{dR}_y$, ${}_x\Pi^{B}_y$ its ω, resp. de Rham resp. Betti realizations. In particular, Chen's theorem implies that we have an isomorphism:

$${}_0\Pi^B_1 \otimes \mathbb{C} \xrightarrow{\sim} {}_0\Pi_1 \otimes \mathbb{C}.$$

Therefore, the motivic fundamental group above boils down to:

(*i*) The affine group schemes ${}_x\Pi^B_y$, $x, y \in \mu_N \cup \{0, \infty\}$, with a groupoid structure. The Betti fundamental groupoid is the pro-unipotent completion of the ordinary topological fundamental groupoid, i.e. corresponds to $\pi^{un}_1(X, x, y)$ above.

(*ii*) $\Pi(X) = \pi^{\omega}_1(X)$, the affine group scheme over $\mathbb{Q}$. It does not depend on x, y since the existence of a canonical de Rham path between x and y implies a canonical isomorphism $\Pi(X) \cong_x \Pi(X)_y$; however, the action of the motivic Galois group $\mathcal{G}$ is sensitive to the tangential base points x, y.

(*iii*) a canonical comparison isomorphism of schemes over $\mathbb{C}$, $\text{comp}_{B,\omega}$.

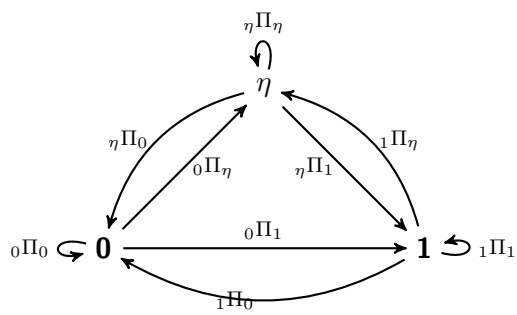


Figure 2.1: Part of the Fundamental groupoid Π.
This picture however does not represent accurately the tangential base points.

[I] In order that the path does not pass by 0, we have to exclude the case where $x = -y$ if N even.
[II] ${}_x\Pi^{\mathfrak{m}}_y$ is a bitorsor under $({}_x\Pi^{\mathfrak{m}}_x, {}_y\Pi^{\mathfrak{m}}_y)$.

Moreover, the dihedral group[I] $Di_N = \mathbb{Z}/2\mathbb{Z} \ltimes \mu_N$ acts on $X_N = \mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$: the group with two elements corresponding to the action $x \mapsto x^{-1}$ and the cyclic group μ_N acting by $x \mapsto \eta x$. Notice that for $N = 1, 2, 4$, the group of projective transformations $X_N \to X_N$ is larger than Di_N, because of special symmetries, and detailed in $A.3$. [II]
The dihedral group Di_N acts then on the motivic fundamental groupoid $\pi_1^{\mathfrak{m}}(X, x, y)$, $x, y \in \{0\} \cup \mu_N$ by permuting the tangential base points (and its action is respected by the motivic Galois group):

$$\text{For} \quad \sigma \in Di_N, \quad {}_x\Pi_y \to_{\sigma.x} \Pi_{\sigma.y}$$

The group scheme $\mathcal{V}$ of automorphisms on these groupoids ${}_x\Pi_y$, respecting their structure, i.e.:

- groupoid structure, i.e. the compositions ${}_x\Pi_y \times_y \Pi_z \to_x \Pi_z$,
- μ_N-equivariance as above,
- inertia: the action fixes $\exp(e_x) \in_x \Pi_x(\mathbb{Q})$,

is isomorphic to (cf. [34], §5 for the detailed version):

$$\begin{array}{l} \mathcal{V} \cong_0 \Pi_x \\ a \mapsto a \cdot_0 1_x \end{array} . \tag{2.18}$$

In particular, the *Ihara action* defined in (2.42) corresponds via this identification to the composition law for these automorphisms, and then can be computed explicitly. Its dual would be the combinatorial coaction Δ used through all this work.

In consequence of these equivariances, we can restrict our attention to:

$${}_0\Pi^{\mathfrak{m}}_{\xi_N} := \pi_1^{\mathfrak{m}}(X_N, \overrightarrow{0\xi_N}) \text{ or equivalently at } {}_0\Pi^{\mathfrak{m}}_1.$$

Keep in mind, for the following, that ${}_0\Pi_1$ is the functor:

$${}_0\Pi_1 : R \text{ a } \mathbb{Q}-\text{algebra } \mapsto \left\{ S \in R\langle\langle e_0, (e_\eta)_{\eta \in \mu_N} \rangle\rangle^\times | \Delta S = S \otimes S \quad \epsilon(S) = 1 \right\}, \tag{2.19}$$

whose affine ring of regular functions is the graded (Hopf) algebra for the shuffle product:

$$\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\left\langle e^0, (e^\eta)_{\eta \in \mu_N} \right\rangle. \tag{2.20}$$

The Lie algebra of ${}_0\Pi_1(R)$ would naturally be the primitive series ($\Delta S = 1 \otimes S + S \otimes 1$).

Let us denote $dch^B_{0,1} =_0 1^B_1$, the image of the straight path (*droit chemin*) in ${}_0\Pi^B_1(\mathbb{Q})$, and $dch^{dR}_{0,1}$ or Φ_{KZ_N} the corresponding element in ${}_0\Pi_1(\mathbb{C})$ via the Betti-De Rham comparison isomorphism:

$$\boldsymbol{\Phi_{KZ_N}} := dch^{dR}_{0,1} := \mathrm{comp}_{dR,B}({}_01^B_1) = \sum_{W \in \{e_0, (e_\eta)_{\eta \in \mu_N}\}^\times} \zeta_{\sqcup\!\sqcup}(w) w \quad \in \mathbb{C}\langle\langle e_0, (e_\eta)_{\eta \in \mu_N} \rangle\rangle, \tag{2.21}$$

[I] Symmetry group of a regular polygon with N sides.
[II] Each homography ϕ defines isomorphisms:

$$\begin{array}{ccc} {}_a\Pi_b & \xrightarrow[\sim]{\phi} & {}_{\phi(a)}\Pi_{\phi(b)} \\ f(e_0, e_1, \ldots, e_n) & \mapsto & f(e_{\phi(0)}, e_{\phi(1)}, \ldots, e_{\phi(n)}) \end{array} \quad \text{and, passing to the dual } \mathcal{O}({}_{\phi(a)}\Pi_{\phi(b)}) \xrightarrow[\sim]{\phi^\vee} \mathcal{O}({}_a\Pi_b).$$

where the correspondence between MZV and words in e_0, e_η is similar to the iterated integral representation (1.3), with η_i. It is known as the *Drinfeld's associator* and arises also from the monodromy of the famous Knizhnik−Zamolodchikov differential equation.[I]

Category generated by $\pi_1^{\mathfrak{m}}$. Denote by:

> $\boldsymbol{\mathcal{MT}'_N}$ the full Tannakian subcategory of $\mathcal{MT}_N$ generated by the fundamental groupoid,

(i.e. generated by $\mathcal{O}(\pi_1^{\mathfrak{m}}(X_N, \overrightarrow{01}))$ by sub-objects, quotients, $\otimes$, $\oplus$, duals) and let:

- $\mathcal{G}^N = \mathbb{G}_m \ltimes \mathcal{U}^N$ its motivic *Galois group* defined over $\mathbb{Q}$,
- $\mathcal{A}^N = \mathcal{O}(\mathcal{U}^N)$ its *fundamental Hopf algebra*,
- $\mathcal{L}^N := \mathcal{A}^N_{>0}/\mathcal{A}^N_{>0} \cdot \mathcal{A}^N_{>0}$ the Lie *coalgebra of indecomposable elements*.

`Nota Bene`: $\mathcal{U}^N$ is the quotient of $\mathcal{U}^{\mathcal{MT}}$ by the kernel of the action on ${}_0\Pi_1$: i.e. $\mathcal{U}^N$ acts faithfully on ${}_0\Pi_1$.

Remark: In the case of $N = 1$ (by F. Brown in [20]), or $N = 2, 3, 4, \text{`6'}, 8$ (by P. Deligne, in [31], proven in a dual point of view in Chapter 5), these categories $\mathcal{MT}'_N$ and $\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{N}\right])$ are equal. More precisely, for $\xi_N \in \mu_N$ a fixed primitive root, the following motivic torsors of path are sufficient to generate the category:

$\boldsymbol{N = 2, 3, 4}$**:** $\Pi^{\mathfrak{m}}(\mathbb{P}^1 \diagdown \{0, 1, \infty\}, \overrightarrow{0\xi_N})$ generates $\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{N}\right])$.

$\boldsymbol{N =}$ **'6':** [II] $\Pi^{\mathfrak{m}}(\mathbb{P}^1 \diagdown \{0, 1, \infty\}, \overrightarrow{0\xi_6})$ generates $\mathcal{MT}(\mathcal{O}_6)$.

$\boldsymbol{N = 8}$**:** $\Pi^{\mathfrak{m}}(\mathbb{P}^1 \diagdown \{0, \pm 1, \infty\}, \overrightarrow{0\xi_6})$ generates $\mathcal{MT}(\mathcal{O}_8\left[\frac{1}{2}\right])$.

However, if N has a prime factor which is non inert, the motivic fundamental group, is in the proper subcategory $\mathcal{MT}_{\Gamma_N}$ and hence can not generate $\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{N}\right])$.

2.3 Motivic Iterated Integrals

Taking from now $\mathcal{M} = \mathcal{MT}'_N$, $M = \mathcal{O}(\pi_1^{\mathfrak{m}}(\mathbb{P}^1 - \{0, \mu_N, \infty\}, \overrightarrow{xy}))$, the definition of motivic periods (2.6) leads to motivic iterated integrals relative to μ_N. Indeed:

> A ***motivic iterated integral*** is the triplet $I^{\mathfrak{m}}(x; w; y) := \left[\mathcal{O}\left(\Pi^{\mathfrak{m}}\left(X_N, \overrightarrow{xy}\right)\right), w, {}_x dch_y^B\right]^{\mathfrak{m}}$ where $w \in \omega(M)$, ${}_x dch_y^B$ is the image of the straight path from x to y in $\omega_B(M)^\vee$ and whose period is:
>
> $$\mathrm{per}(I^{\mathfrak{m}}(x; w; y)) = I(x; w; y) = \int_x^y w = \langle \mathrm{comp}_{B,dR}(w \otimes 1), {}_x dch_y^B \rangle \in \mathbb{C}. \tag{2.22}$$

[I] Indeed, for $N = 1$, Drinfeld associator is equal to $G_1^{-1}G_0$, where G_0, G_1 are solutions, with certain asymptotic behavior at 0 and 1 of the Knizhnik−Zamolodchikov differential equation:

$$\frac{d}{dz}G(z) = \left(\frac{e_0}{z} + \frac{e_1}{1-z}\right) G(z).$$

[II] The quotation marks around 6 underlines that we consider the unramified category in this case.

Remarks:

- There, $w \in \omega(\mathcal{O}({}_x\Pi_y^{\mathfrak{m}})) \cong \mathbb{Q}\langle \omega_0, (\omega_\eta)_{\eta\in\mu_N}\rangle$ where $\omega_\eta := \frac{dt}{t-\eta}$. Similarly to 1.2, let:

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) := I^{\mathfrak{m}}(a_0; \omega_{\boldsymbol{a}}; a_{n+1}), \qquad \text{where } \omega_{\boldsymbol{a}} := \omega_{a_1}\cdots\omega_{a_n}, \text{ for } a_i \in \{0\}\cup\mu_N \tag{2.23}$$

- The Betti realization functor ω_B depends on the embedding $\sigma : k \hookrightarrow \mathbb{C}$. Here, by choosing a root of unity, we fixed the embedding σ.

For $\mathcal{M}$ a category of Mixed Tate Motives among $\mathcal{MT}_N, \mathcal{MT}_{\Gamma_N}$ resp. $\mathcal{MT}'_N$, let introduce the graded $\mathcal{A}^{\mathcal{M}}$-comodule, with trivial coaction on $\mathbb{L}^{\mathfrak{m}}$ (degree 1):

$$\boldsymbol{\mathcal{H}^{\mathcal{M}}} := \mathcal{A}^{\mathcal{M}} \otimes \begin{cases} \mathbb{Q}\left[(\mathbb{L}^{\mathfrak{m}})^2\right] & \text{for } N = 1,2 \\ \mathbb{Q}\left[\mathbb{L}^{\mathfrak{m}}\right] & \text{for } N > 2. \end{cases} \subset \mathcal{O}(\mathcal{G}^{\mathcal{M}}) = \mathcal{A}^{\mathcal{M}} \otimes \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}, (\mathbb{L}^{\mathfrak{m}})^{-1}]. \tag{2.24}$$

`Nota Bene`: For $N > 2$, it corresponds to the geometric motivic periods, $\mathcal{P}_{\mathcal{M}}^{\mathfrak{m},+}$ whereas for $N = 1, 2$, it is the subset $\mathcal{P}_{\mathcal{M},\mathbb{R}}^{\mathfrak{m},+}$ invariant by the real Frobenius; cf. (2.9), (2.10).
For $\mathcal{M} = \mathcal{MT}'_N$, we will simply denote it $\mathcal{H}^N := \mathcal{H}^{\mathcal{MT}'_N}$. Moreover:

$$\mathcal{H}^N \subset \mathcal{H}^{\mathcal{MT}_{\Gamma_N}} \subset \mathcal{H}^{\mathcal{MT}_N}.$$

Cyclotomic iterated integrals of weight n are periods of π_1^{un} (of X^n relative to $Y^{(n)}$): [I]

> Any motivic iterated integral $I^{\mathfrak{m}}$ relative to μ_N is an element of $\mathcal{H}^N$, which is the graded $\mathcal{A}^N-$ comodule generated by these motivic iterated integrals relative to μ_N.

In a similar vein, define:

- $\boldsymbol{I^{\omega}}$: A motivic period of type (ω, ω), in $\mathcal{O}(\mathcal{G})$:

$$I^{\omega}(x; w; y) = \left[\mathcal{O}\left({}_x\Pi_y^{\mathfrak{m}}\right), w, {}_x 1_y^{\omega}\right]^{\omega}, \qquad \text{where } \begin{cases} w \in \omega(\mathcal{O}\left({}_x\Pi_y^{\mathfrak{m}}\right)) \\ {}_x 1_y^{\omega} \in \omega(M)^{\vee} = \mathcal{O}\left({}_x\Pi_y\right)^{\vee} \end{cases}. \tag{2.25}$$

  where ${}_x 1_y^{\omega} \in \mathcal{O}\left({}_x\Pi_y\right)^{\vee}$ is defined by the augmentation map $\epsilon : \mathbb{Q}\langle e^0, (e^{\eta})_{\eta\in\mu_N}\rangle \to \mathbb{Q}$, corresponding to the unit element in ${}_x\Pi_y$. This defines a function on $\mathcal{G} = \mathrm{Aut}^{\otimes}(\omega)$, given on the rational points by $g \in \mathcal{G}(\mathbb{Q}) \mapsto \langle g\omega, \epsilon\rangle \in \mathbb{Q}$.

- $\boldsymbol{I^{\mathfrak{a}}}$: the image of I^{ω} in $\mathcal{A} = \mathcal{O}(\mathcal{U})$, by the projection $\mathcal{O}(\mathcal{G}) \twoheadrightarrow \mathcal{O}(\mathcal{U})$. These *unipotent* motivic periods are the objects studied by Goncharov, which he called motivic iterated integrals; for instance, $\zeta^{\mathfrak{a}}(2) = 0$.

- $\boldsymbol{I^{\mathfrak{l}}}$: the image of $I^{\mathfrak{a}}$ in the coalgebra of indecomposables $\mathcal{L} := \mathcal{A}_{>0}/\mathcal{A}_{>0}.\mathcal{A}_{>0}$.[II]

Remark: It is similar (cf. [20]) to define $\mathcal{H}^N$, as $\mathcal{O}({}_0\Pi_1)/J$, with:

- $J \subset \mathcal{O}({}_0\Pi_1)$ is the biggest graded ideal $\subset \ker per$ closed by the coaction Δ^c, corresponding to the ideal of motivic relations, i.e.:

$$\Delta^c(J) \subset \mathcal{A} \otimes J + J\mathcal{A} \otimes \mathcal{O}({}_0\Pi_1).$$

[I] Notations of (2.14). Cf. also (2.17). The case of tangential base points requires blowing up to get rid of singularities. Most interesting periods are often those whose integration domain meets the singularities of the differential form.

[II] Well defined since $\mathcal{A} = \mathcal{O}(\mathcal{U})$ is graded with positive degrees.

· the ⧢-homomorphism: per : $\mathcal{O}({}_0\Pi_1) \to \mathbb{C}$, $e^{a_1}\cdots e^{a_n} \mapsto I(0;a_1,\dots,a_n;1) := \int_{dch}\omega$.

Once the motivic iterated integrals are defined, motivic cyclotomic multiple zeta values follows, as usual (cf. 1.2):

Motivic multiple zeta values relative to μ_N are defined by, for $\epsilon_i \in \mu_N, k \geq 0, n_i > 0$

$$\boldsymbol{\zeta_k^{\mathfrak{m}}}\begin{pmatrix}\boldsymbol{n_1,\dots,n_p}\\ \boldsymbol{\epsilon_1,\dots,\epsilon_p}\end{pmatrix} := (-1)^p I^{\mathfrak{m}}\left(0;\mathbf{0}^k,(\epsilon_1\cdots\epsilon_p)^{-1},\mathbf{0}^{n_1-1},\cdots,(\epsilon_i\cdots\epsilon_p)^{-1},\mathbf{0}^{n_i-1},\cdots,\epsilon_p^{-1},\mathbf{0}^{n_p-1};1\right) \tag{2.26}$$

An *admissible* (motivic) MZV is such that $(n_p,\epsilon_p) \neq (1,1)$; otherwise, they are defined by shuffle regularization, cf. (2.28) below; the versions $\boldsymbol{\zeta_k^{\mathfrak{a}}}(\cdots)$, or $\boldsymbol{\zeta_k^{\mathfrak{l}}}(\cdots)$ are defined similarly, from $I^{\mathfrak{a}}$ resp. $I^{\mathfrak{l}}$ above. The roots of unity in the iterated integral will often be denoted by $\eta_i := (\epsilon_i\cdots\epsilon_p)^{-1}$

From (2.8), there is a surjective homomorphism called the ***period map***, conjectured to be isomorphism:

$$\text{per} : \mathcal{H} \to \mathcal{Z}\ ,\ \zeta^{\mathfrak{m}}\begin{pmatrix}n_1,\dots,n_p\\ \epsilon_1,\dots,\epsilon_p\end{pmatrix} \mapsto \zeta\begin{pmatrix}n_1,\dots,n_p\\ \epsilon_1,\dots,\epsilon_p\end{pmatrix}. \tag{2.27}$$

`Nota Bene:` Each identity between motivic cyclotomic multiple zeta values is then true for cyclotomic multiple zeta values and in particular each result about a basis with motivic MZV implies the corresponding result about a generating family of MZV by application of the period map.
Conversely, we can sometimes *lift* an identity between MZV to an identity between motivic MZV, via the coaction (as in [20], Theorem 3.3); this is discussed below, and illustrated throughout this work in different examples or counterexamples, as in Lemma 4.4.3. It is similar in the case of motivic Euler sums ($N = 2$). We will see (Theorem 2.4.4) that for other roots of unity there are several rational coefficients which appear at each step (of the coaction calculus) and prevent us from concluding by identification.

Properties. Motivic iterated integrals satisfy the following properties:

(i) $I^{\mathfrak{m}}(a_0;a_1) = 1$.

(ii) $I^{\mathfrak{m}}(a_0;a_1,\cdots a_n;a_{n+1}) = 0$ if $a_0 = a_{n+1}$.

(iii) Shuffle product:[I]

$$\zeta_k^{\mathfrak{m}}\begin{pmatrix}n_1,\dots,n_p\\ \epsilon_1,\dots,\epsilon_p\end{pmatrix} = (-1)^k\sum_{i_1+\cdots+i_p=k}\binom{n_1+i_1-1}{i_1}\cdots\binom{n_p+i_p-1}{i_p}\zeta^{\mathfrak{m}}\begin{pmatrix}n_1+i_1,\dots,n_p+i_p\\ \epsilon_1,\dots,\epsilon_p\end{pmatrix}. \tag{2.28}$$

(iv) Path composition:

$$\forall x \in \mu_N\cup\{0\}\,, I^{\mathfrak{m}}(a_0;a_1,\dots,a_n;a_{n+1}) = \sum_{i=1}^{n} I^{\mathfrak{m}}(a_0;a_1,\dots,a_i;x)I^{\mathfrak{m}}(x;a_{i+1},\dots,a_n;a_{n+1}).$$

[I]Product rule for iterated integral in general is:

$$\int_\gamma \phi_1\cdots\phi_r\cdot\int_\gamma \phi_{r+1}\cdots\phi_{r+s} = \sum_{\sigma\in Sh_{r,s}}\int_\gamma \phi_{\sigma^{-1}(1)}\cdots\phi_{\sigma^{-1}(r+s)},$$

where $Sh_{r,s} \subset \mathfrak{S}_{r+s}$ is the subset of permutations which respect the order of $\{1,\dots,r\}$ and $\{r+1,\dots,r+s\}$. Here, to define the non convergent case, (iii) is sufficient, paired with the other rules.

(v) Path reversal: $I^{\mathfrak{m}}(a_0;a_1,\dots,a_n;a_{n+1}) = (-1)^n I^{\mathfrak{m}}(a_{n+1};a_n,\dots,a_1;a_0)$.

(vi) Homothety: $\forall \alpha \in \mu_N, I^{\mathfrak{m}}(0;\alpha a_1,\dots,\alpha a_n;\alpha a_{n+1}) = I^{\mathfrak{m}}(0;a_1,\dots,a_n;a_{n+1})$.

Remark: These relations, for the multiple zeta values relative to μ_N, and for the iterated integrals $I(a_0;a_1,\cdots a_n;a_{n+1})$ (1.3), are obviously all easily checked.

It has been proven that motivic iterated integrals verify stuffle $*$ relations, but also pentagon, and hexagon (resp. octagon for $N>1$) ones, as iterated integral at μ_N. In depth 1, by Deligne and Goncharov, the only relations satisfied by the motivic iterated integrals are distributions and conjugation relations, stated in §2.4.3.

Motivic Euler $\star$, $\sharp$ sums. Here, assume that $N=1$ or 2.[I] In the motivic iterated integrals above, $I^{\mathfrak{m}}(\cdots,a_i,\cdots)$, a_i were in $\{0,\pm 1\}$. We can extend by linearity to $a_i \in \{\pm\star,\pm\sharp\}$, which corresponds to a $\omega_{\pm\star}$, resp. $\omega_{\pm\sharp}$ in the iterated integral, with the differential forms:

$$\boldsymbol{\omega}_{\pm\star} := \omega_{\pm 1} - \omega_0 = \frac{dt}{t(\pm t-1)} \quad \text{and} \quad \boldsymbol{\omega}_{\pm\sharp} := 2\omega_{\pm 1} - \omega_0 = \frac{(t\pm 1)dt}{t(t\mp 1)}.$$

It means that, by linearity, for A,B sequences in $\{0,\pm 1,\pm\star,\pm\sharp\}$:

$$I^{\mathfrak{m}}(A,\pm\star,B) = I^{\mathfrak{m}}(A,\pm 1,B) - I^{\mathfrak{m}}(A,0,B), \text{ and } I^{\mathfrak{m}}(A,\pm\sharp,B) = 2I^{\mathfrak{m}}(A,\pm 1,B) - I^{\mathfrak{m}}(A,0,B). \tag{2.29}$$

$\boldsymbol{\zeta}^{\star,\mathfrak{m}}$: *Motivic Euler $\star$ Sums* are defined by a similar integral representation as MES (1.3), with $\omega_{\pm\star}$ replacing the $\omega_{\pm 1}$, except the first one, which stays a $\omega_{\pm 1}$.
Their periods, Euler $\star$ sums, which are already common in the literature, can be written as a summation similar than for Euler sums replacing strict inequalities by large ones:

$$\zeta^{\star}(n_1,\dots,n_p) = \sum_{0<k_1\le k_2\le\cdots\le k_p} \frac{\epsilon_1^{k_1}\cdots\epsilon_p^{k_p}}{k_1^{|n_1|}\cdots k_p^{|n_p|}}, \quad \epsilon_i := sign(n_i), \quad n_i \in \mathbb{Z}^*, n_p \neq 1.$$

$\boldsymbol{\zeta}^{\sharp,\mathfrak{m}}$: *Motivic Euler $\sharp$ Sums* are defined by a similar integral representation as MES (1.3), with $\omega_{\pm\sharp}$ replacing the $\omega_{\pm 1}$, except the first one, which stays a $\omega_{\pm 1}$.

They are both $\mathbb{Q}$-linear combinations of multiple Euler sums, and appear in Chapter 4, via new bases for motivic MZV (Hoffman $\star$, or with Euler $\sharp$ sums) and in the Conjecture 4.5.1.

Dimensions. Algebraic K-theory provides an *upper bound* for the dimensions of motivic cyclotomic iterated integrals, since:

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{MT}_{N,M}}(\mathbb{Q}(0),\mathbb{Q}(1)) = (\mathcal{O}_{k_N}[\tfrac{1}{M}])^* \otimes \mathbb{Q} \\
&\mathrm{Ext}^1_{\mathcal{MT}_{\Gamma_N}}(\mathbb{Q}(0),\mathbb{Q}(1)) = \Gamma_N \\
&\mathrm{Ext}^1_{\mathcal{MT}_{N,M}}(\mathbb{Q}(0),\mathbb{Q}(n)) = \mathrm{Ext}^1_{\mathcal{MT}_{\Gamma}}(\mathbb{Q}(0),\mathbb{Q}(n)) = K_{2n-1}(k_N)\otimes\mathbb{Q} && \text{for } n>1. \\
&\mathrm{Ext}^i_{\mathcal{MT}_{N,M}}(\mathbb{Q}(0),\mathbb{Q}(n)) = \mathrm{Ext}^i_{\mathcal{MT}_{\Gamma}}(\mathbb{Q}(0),\mathbb{Q}(n)) = 0 && \text{for } i>1 \text{ or } n\le 0.
\end{aligned} \tag{2.30}$$

Let $n_{\mathfrak{p}_M}$ denote the number of different prime ideals above the primes dividing M, ν_N the number of primes dividing N and φ Euler's indicator function. For $M|N$ (cf. [8]), using

[I] Detailed definitions of these $\star$ and $\sharp$ versions are given in §4.1.

Dirichlet S-unit theorem when $n=1$:

$$\dim K_{2n-1}(\mathcal{O}_{k_N}[1/M])\otimes\mathbb{Q} = \begin{cases} 1 & \text{if } N=1 \text{ or } 2, \text{ and } n \text{ odd }, (n,N)\neq(1,1). \\ 0 & \text{if } N=1 \text{ or } 2, \text{ and } n \text{ even }. \\ \frac{\varphi(N)}{2}+n_{\mathfrak{p}_M}-1 & \text{if } N>2, n=1. \\ \frac{\varphi(N)}{2} & \text{if } N>2, n>1. \end{cases} \tag{2.31}$$

The numbers of generators in each degree, corresponding to the categories $\mathcal{MT}_{N,M}$ resp. $\mathcal{MT}_{\Gamma_N}$, differ only in degree 1:

$$\begin{array}{lll} \text{In degree } >1: & b_N := b_{N,M} = b_{\Gamma_N} = \frac{\varphi(N)}{2} & \\ \text{In degree } 1: & a_{N,M} := \frac{\varphi(N)}{2}+n_{\mathfrak{p}_M}-1 \quad \text{whereas} \quad a_{\Gamma_N} := \frac{\varphi(N)}{2}+\nu(N)-1. \end{array} \tag{2.32}$$

`Nota Bene`: The following formulas in this paragraph can be applied for the categories $\mathcal{MT}_{N,M}$ resp. $\mathcal{MT}_{\Gamma_N}$, replacing a_N by $a_{N,M}$ resp. a_{Γ_N}.

In degree 1, for $\mathcal{MT}_{M,N}$, only the units modulo torsion matters whereas for the category $\mathcal{MT}_{\Gamma_N}$, only the cyclotomic units modulo torsion matters in degree 1, cf. §2.4.3. Recall that cyclotomic units form a subgroup of finite index in the group of units, and generating families for cyclotomic units modulo torsion are (cf. [5])[I]:

$$\begin{array}{ll} \text{For } N=p^s: & \left\{\frac{1-\xi_N^a}{1-\xi_N}, a\wedge p=1\right\}, \quad \text{where } a\wedge b := gcd(a,b). \\ \text{For } N=\prod_i p_i^{s_i}=\prod q_i: & \left\{\frac{1-\xi_{q_i}^a}{1-\xi_{q_i}}, a\wedge p_i=1\right\}\cup\{1-\xi_d^a, \quad a\wedge d=1, d\mid N, d\neq q_i\} \end{array}.$$

Results on cyclotomic units determine depth 1 weight 1 results for MMZV_{μ_N} (cf. §.2.4.3).

Knowing dimensions, we lift (2.2) to a non-canonical isomorphism with the free Lie algebra:

$$\mathfrak{u}^{\mathcal{MT}} \underset{n.c}{\cong} L := \mathbb{L}_{\mathbb{Q}}\left\langle \left(\sigma_1^j\right)_{1\leq j\leq a_N}, \left(\sigma_i^j\right)_{1\leq j\leq b_N}, i>1\right\rangle \quad \sigma_i \text{ in degree } -i. \tag{2.33}$$

The generators σ_i^j of the graded Lie algebra $\mathfrak{u}$ are indeed non-canonical, only their classes in the abelianization are.[II] For the fundamental Hopf algebra, with $f_i^j=(\sigma_i^j)^\vee$ in degree j:

$$\mathcal{A}^{\mathcal{MT}} \underset{n.c}{\cong} A := \mathbb{Q}\left\langle \left(f_1^j\right)_{1\leq j\leq a_N}, \left(f_i^j\right)_{1\leq j\leq b_N}, i>1\right\rangle. \tag{2.35}$$

$\mathcal{A}^{\mathcal{MT}}$ is a cofree commutative graded Hopf algebra cogenerated by a_N elements $f_1^\bullet$ in degree 1, and b_N elements $f_r^\bullet$ in degree $r>1$.

The comodule $\mathcal{H}^N\subseteq\mathcal{O}({}_0\Pi_1)$ embeds, non-canonically[III], into $\mathcal{H}^{\mathcal{MT}_N}$ and hence:

$$\mathcal{H}^N \xhookrightarrow[n.c.]{\phi^N} H^N := \mathbb{Q}\left\langle \left(f_1^j\right)_{1\leq j\leq a_N}, \left(f_r^j\right)_{\substack{r>1\\1\leq j\leq b_N}}\right\rangle \otimes \mathbb{Q}[g_1]. \tag{2.36}$$

[I] If we consider cyclotomic units in $\mathbb{Z}[\xi_N]\left[\frac{1}{M}\right]$, with $M=\prod r_i$, r_i prime power, we have to add $\{1-\xi_{r_i}\}$.

[II] In other terms, this means:

$$H_1(\mathfrak{u}^{\mathcal{MT}};\mathbb{Q})\cong\bigoplus_{i,j \text{ as above}}[\sigma_i^j]\mathbb{Q}, \quad H_i^B(\mathfrak{u}^{\mathcal{MT}};\mathbb{Q})=0 \text{ for } i>1. \tag{2.34}$$

[III] We can fix the image of algebraically independent elements with trivial coaction.
For instance, for $N=3$, we can choose to send: $\zeta^{\mathrm{m}}\binom{r}{j}\overset{\phi}{\longmapsto} f_r$, and $(2i\pi)^{\mathrm{m}}\overset{\phi}{\longmapsto} g_1$.

`Nota Bene:` This comodule embedding is an isomorphism for $N = 1, 2, 3, 4, \text{`6'}, 8$ (by F. Brown [20] for $N = 1$, by Deligne [31] for the other cases; new proof in Chapter 5), since the categories $\mathcal{MT}'_N$, $\mathcal{MT}(\mathcal{O}\left[\frac{1}{N}\right])$ and $\mathcal{MT}_{\Gamma_N}$ are equivalent. However, for some other N, such as N prime greater than 5, it is not an isomorphism.
Looking at the dimensions $d_n^N := \dim \mathcal{H}_n^{\mathcal{MT}_N}$:

Lemma 2.3.1. *For $N > 2$, d_n^N satisfies two (equivalent) recursive formulas*[I]*:*

$$\begin{aligned} d_n^N &= 1 + a_N d_{n-1} + b_N \textstyle\sum_{i=2}^n d_{n-i} \\ d_n^N &= (a_N+1)d_{n-1} + (b_N - a_N)d_{n-2} \quad \textit{with} \begin{cases} d_0 = 1 \\ d_1 = a_N + 1 \end{cases} . \end{aligned}$$

Hence the Hilbert series for the dimensions of $\mathcal{H}^{\mathcal{MT}}$ is:

$$h_N(t) := \sum_k d_k^N t^k = \frac{1}{1 - (a_N+1)t + (a_N - b_N)t^2}.$$

In particular, these dimensions (for $\mathcal{H}^{\mathcal{MT}_{\Gamma_N}}$) are an upper bound for the dimensions of motivic MZV_{μ_N} (i.e. of $\mathcal{H}^N$), and hence of MZV_{μ_N} by the period map. In the case $N = p^r$, $p \geq 5$ prime, this upper bound is conjectured to be not reached; for other N however, this bound is still conjectured to be sharp (cf. §3.4).

`Examples:`

- For the unramified category $\mathcal{MT}(\mathcal{O}_N)$:

$$d_n = \frac{\varphi(N)}{2} d_{n-1} + d_{n-2}.$$

- For $M \mid N$ such that all primes dividing M are inert, $n_{\mathfrak{p}_M} = \nu(N)$. In particular, it is the case if $N = p^r$:

$$\text{For } \mathcal{MT}\left(\mathcal{O}_{p^r}\left[\frac{1}{p}\right]\right), \quad d_n = \left(\frac{\varphi(N)}{2} + 1\right)^n.$$

Let us detail the cases $N = 2, 3, 4, \text{`6'}, 8$ considered in Chapter 5:

$N \backslash\ d_n^N$	A	Dimension relation d_n^N	Hilbert series
$N = 1^{\mathrm{II}}$	1 generator in each odd degree > 1 $\mathbb{Q}\langle f_3, f_5, f_7, \cdots\rangle$	$d_n = d_{n-3} + d_{n-2}$, $d_2 = 1$, $d_1 = 0$	$\frac{1}{1-t^2-t^3}$
$N = 2^{\mathrm{III}}$	1 generator in each odd degree ≥ 1 $\mathbb{Q}\langle f_1, f_3, f_5, \cdots\rangle$	$d_n = d_{n-1} + d_{n-2}$ $d_0 = d_1 = 1$	$\frac{1}{1-t-t^2}$
$N = 3, 4$	1 generator in each degree ≥ 1 $\mathbb{Q}\langle f_1, f_2, f_3, \cdots\rangle$	$d_k = 2d_{k-1} = 2^k$	$\frac{1}{1-2t}$
$N = 8$	2 generators in each degree ≥ 1 $\mathbb{Q}\langle f_1^1, f_1^2, f_2^1, f_2^2, \cdots\rangle$	$d_k = 3d_{k-1} = 3^k$	$\frac{1}{1-3t}$
$N = 6$ $\mathcal{MT}(\mathcal{O}_6\left[\frac{1}{6}\right])$	1 in each degree > 1, 2 in degree 1 $\mathbb{Q}\langle f_1^1, f_1^2, f_2, f_3, \cdots\rangle$	$d_k = 3d_{k-1} - d_{k-2}$, $d_1 = 3$	$\frac{1}{1-3t+t^2}$
$N = 6$ $\mathcal{MT}(\mathcal{O}_6)$	1 generator in each degree > 1 $\mathbb{Q}\langle f_2, f_3, f_4, \cdots\rangle$	$d_k = 1 + \sum_{i\geq 2} d_{k-i}$ $= d_{k-1} + d_{k-2}$	$\frac{1}{1-t-t^2}$

[I] Those two recursive formulas, although equivalent, leads to two different perspective for counting dimensions.

2.4 Motivic Hopf algebra

2.4.1 Motivic Lie algebra.

Let $\mathfrak{g}$ the free graded Lie algebra generated by $e_0, (e_\eta)_{\eta\in\mu_N}$ in degree -1. Then, the completed Lie algebra $\mathfrak{g}^\wedge$ is the Lie algebra of ${}_0\Pi_1(\mathbb{Q})$ and the universal enveloping algebra $U\mathfrak{g}$ is the cocomutative Hopf algebra which is the graded dual of $O({}_0\Pi_1)$:

$$(U\mathfrak{g})_n = (\mathbb{Q}e_0 \oplus_{\eta\in\mu_N} \mathbb{Q}e_\eta)^{\otimes n} = (O({}_0\Pi_1)^\vee)_n. \tag{2.38}$$

The product is the concatenation, and the coproduct is such that e_0, e_η are primitive.

Considering the motivic version of the Drinfeld associator:

$$\Phi^{\mathfrak{m}} := \sum_w \zeta^{\mathfrak{m}}(w)w \in \mathcal{H}\langle\langle e_0, e_\eta\rangle\rangle \text{ , where :} \tag{2.39}$$

$$\zeta^{\mathfrak{m}}(e_0^n e_{\eta_1} e_0^{n_1-1}\cdots e_{\eta_p} e_0^{n_p-1}) = \zeta_n^{\mathfrak{m}}\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p\end{pmatrix} \text{ with } \begin{array}{l}\epsilon_p := \eta_p^{-1} \\ \epsilon_i := \eta_i^{-1}\eta_{i+1}\end{array}.$$

`Nota Bene:` This motivic Drinfeld associator satisfies the double shuffle relations, and, for $N=1$, the associator equations defined by Drinfeld (pentagon and hexagon), replacing $2\pi i$ by the Lefschetz motivic period $\mathbb{L}^{\mathfrak{m}}$; for $N>1$, an octagon relation generalizes this hexagon relation, as we will see in §4.2.2.
Moreover, it defines a map:

$$\oplus\mathcal{H}_n^\vee \to U\mathfrak{g} \quad \text{which induces a map: } \oplus\,\mathcal{L}_n^\vee \to U\mathfrak{g}.$$

Define $\mathfrak{g}^{\mathfrak{m}}$, the *Lie algebra of motivic elements* as the image of $\oplus\mathcal{L}_n^\vee$ in $U\mathfrak{g}$:[I]

$$\oplus\,\mathcal{L}_n^\vee \xrightarrow{\sim} \mathfrak{g}^{\mathfrak{m}} \hookrightarrow U\mathfrak{g}. \tag{2.40}$$

The Lie algebra $\mathfrak{g}^{\mathfrak{m}}$ is equipped with the Ihara bracket given precisely below. Notice that for the cases $N = 1, 2, 3, 4, '6', 8$, $\mathfrak{g}^{\mathfrak{m}}$ is non-canonically isomorphic to the free Lie algebra L defined in (2.33), generated by $(\sigma_i)'s$.

Ihara action. As said above, the group scheme $\mathcal{V}$ of automorphisms of ${}_x\Pi_y, x, y \in \{0, \mu_N\}$ is isomorphic to ${}_0\Pi_1$ (2.18), and the group law of automorphisms leads to the Ihara action. More precisely, for $a \in_0 \Pi_1$ (cf. [34]):

$$\begin{array}{llllll}
\text{The action on } {}_0\Pi_0: & \langle a\rangle_0: & {}_0\Pi_0 & \to & {}_0\Pi_0 & \\
 & & \exp(e_0) & \mapsto & \exp(e_0) & \\
 & & \exp(e_\eta) & \mapsto & ([\eta]\cdot a)\exp(e_\eta)([\eta]\cdot a)^{-1} & \\
\text{Then, the action on } {}_0\Pi_1: & \langle a\rangle: & {}_0\Pi_1 & \to & {}_0\Pi_1 & \\
 & & b & \mapsto & \langle a\rangle_0(b)\cdot a &
\end{array} \tag{2.41}$$

[II]For $N=1$, Broadhurst and Kreimer made a more precise conjecture for dimensions of multiple zeta values graded by the depth, which transposes to motivic ones:

$$\sum \dim(gr_d^{\mathfrak{D}}\mathcal{H}_n^1)s^n t^d = \frac{1+\mathbb{E}(s)t}{1-\mathbb{O}(s)t+\mathbb{S}(s)t^2-\mathbb{S}(s)t^4}, \quad \text{where } \begin{array}{l}\mathbb{E}(s) := \frac{s^2}{1-s^2} \\ \mathbb{O}(s) := \frac{s^3}{1-s^2} \\ \mathbb{S}(s) := \frac{s^{12}}{(1-s^4)(1-s^6)}\end{array} \tag{2.37}$$

where $\mathbb{E}(s)$, resp. $\mathbb{O}(s)$, resp. $\mathbb{S}(s)$ are the generating series of even resp. odd simple zeta values resp. of the space of cusp forms for the full modular group $PSL_2(\mathbb{Z})$. The coefficient $\mathbb{S}(s)$ of t^2 can be understood via the relation between double zetas and cusp forms in [41]; The coefficient $\mathbb{S}(s)$ of t^4, underlying exceptional generators in depth 4, is now also understood by the recent work of F. Brown [21], who gave an interpretation of this conjecture via the homology of an explicit Lie algebra.

[III]For $N=2$, the dimensions are Fibonacci numbers.

[I]The action of the Galois group $\mathcal{U}^{\mathcal{MT}}$ turns $\mathcal{L}$ into a coalgebra, and hence $\mathfrak{g}^{\mathfrak{m}}$ into a Lie algebra.

This action is called the ***Ihara action***:

$$\begin{array}{rccl} \circ: & {}_0\Pi_1 \times_0 \Pi_1 & \to & {}_0\Pi_1 \\ & (a,b) & \mapsto & a \circ b := \langle a \rangle_0(b) \cdot a. \end{array} \tag{2.42}$$

At the Lie algebra level, it defines the *Ihara bracket* on $Lie({}_0\Pi_1)$:

$$\{a,b\} := a \circ b - b \circ a. \tag{2.43}$$

`Nota Bene:` The dual point of view leads to a combinatorial coaction Δ^c, which is the keystone of this work.

2.4.2 Coaction

The motivic Galois group $\mathcal{G}^{\mathcal{MT}_N}$ and hence $\mathcal{U}^{\mathcal{MT}}$ acts on the de Rham realization ${}_0\Pi_1$ of the motivic fundamental groupoid (cf. [34], §4.12). It is fundamental, since the action of $\mathcal{U}^{\mathcal{MT}}$ is compatible with the structure of ${}_x\Pi_y$ (groupoid, μ_N equivariance and inertia), that this action factorizes through the Ihara action, using the isomorphism $\mathcal{V} \cong_0 \Pi_1$ (2.18):

$$\begin{tikzcd} \mathcal{U}^{\mathcal{MT}} \times_0 \Pi_1 \arrow[r] \arrow[d] & {}_0\Pi_1 \arrow[d, "\sim"] \\ {}_0\Pi_1 \times_0 \Pi_1 \arrow[r, "\circ"] & {}_0\Pi_1 \end{tikzcd}$$

Since $\mathcal{A}^{\mathcal{MT}} = \mathcal{O}(\mathcal{U}^{\mathcal{MT}})$, this action gives rise by duality to a coaction: $\Delta^{\mathcal{MT}}$, compatible with the grading, represented below. By the previous diagram, the combinatorial coaction Δ^c (on words on $0, \eta \in \mu_N$), which is explicit (the formula being given below), factorizes through $\Delta^{\mathcal{MT}}$. Remark that $\Delta^{\mathcal{MT}}$ factorizes through $\mathcal{A}$, since $\mathcal{U}$ is the quotient of $\mathcal{U}^{\mathcal{MT}}$ by the kernel of its action on ${}_0\Pi_1$. By passing to the quotient, it induces a coaction Δ on $\mathcal{H}$:

$$\begin{tikzcd} \mathcal{O}({}_0\Pi_1) \arrow[r, "\Delta^c"] \arrow[d, "\sim"] & \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{O}({}_0\Pi_1) \arrow[d] \\ \mathcal{O}({}_0\Pi_1) \arrow[r, "\Delta^{\mathcal{MT}}"] \arrow[d] & \mathcal{A}^{\mathcal{MT}} \otimes_{\mathbb{Q}} \mathcal{O}({}_0\Pi_1) \arrow[d] \\ \mathcal{H} \arrow[r, "\Delta"] & \mathcal{A} \otimes \mathcal{H}. \end{tikzcd}$$

The coaction for motivic iterated integrals is given by the following formula, due to A. B. Goncharov (cf. [44]) for $\mathcal{A}$ and extended by F. Brown to $\mathcal{H}$ (cf. [20]):

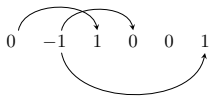
Theorem. *The coaction $\Delta : \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$ is given by the combinatorial coaction Δ^c:*

$$\Delta^c I^{\mathfrak{m}}(a_0; a_1, \cdots a_n; a_{n+1}) =$$

$$\sum_{k; i_0 = 0 < i_1 < \cdots < i_k < i_{k+1} = n+1} \left(\prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \cdots a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \cdots a_{i_k}; a_{n+1}).$$

Remark: It has a nice geometric formulation, considering the a_i as vertices on a half-circle:

$$\Delta^c I^{\mathfrak{m}}(a_0; a_1, \cdots a_n; a_{n+1}) = \sum_{\substack{\text{polygons on circle} \\ \text{with vertices } (a_{i_p})}} \prod_p I^{\mathfrak{a}} \left(\begin{array}{c} \text{arc between consecutive vertices} \\ \text{from } a_{i_p} \text{ to } a_{i_{p+1}} \end{array} \right) \otimes I^{\mathfrak{m}}(\text{ vertices }).$$

`Example`: In the reduced coaction[I] of $\zeta^{\mathfrak{m}}(-1,3) = I^{\mathfrak{m}}(0;-1,1,0,0;1)$, there are 3 non zero cuts: 

.. Hence:

$$\begin{aligned}&\Delta'(I^{\mathfrak{m}}(0;-1,1,0,0;1))\\ &= I^{\mathfrak{a}}(0;-1;1)\otimes I^{\mathfrak{m}}(0;1,0,0;1)+I^{\mathfrak{a}}(-1;1;0)\otimes I^{\mathfrak{m}}(0;-1,0,0;1)+I^{\mathfrak{a}}(-1;1,0,0;1)\otimes I^{\mathfrak{m}}(0;-1;1)\end{aligned}$$

I.e, in terms of motivic Euler sums, using the properties of motivic iterated integrals (§2.3):

$$\Delta'(\zeta^{\mathfrak{m}}(-1,3)) = \zeta^{\mathfrak{a}}(-1)\otimes\zeta^{\mathfrak{m}}(3) - \zeta^{\mathfrak{a}}(-1)\otimes\zeta^{\mathfrak{m}}(-3) + (\zeta^{\mathfrak{a}}(3)-\zeta^{\mathfrak{a}}(-3))\otimes\zeta^{\mathfrak{m}}(-1).$$

Define for $r \geq 1$, the *derivation operators*:

$$\boldsymbol{D_r} : \mathcal{H} \to \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}, \tag{2.44}$$

composite of $\Delta' = \Delta^c - 1\otimes id$ with $\pi_r \otimes id$, where π_r is the projection $\mathcal{A} \to \mathcal{L} \to \mathcal{L}_r$.

`Nota Bene:` It is sufficient to consider these weight-graded derivation operators to keep track of all the information of the coaction.

According to the previous theorem, the action of D_r on $I^{\mathfrak{m}}(a_0;a_1,\cdots a_n;a_{n+1})$ is:

$$\boxed{\begin{aligned}&D_r I^{\mathfrak{m}}(a_0;a_1,\cdots a_n;a_{n+1}) = \\ &\sum_{p=0}^{n-1} I^{\mathfrak{l}}(a_p;a_{p+1},\cdots a_{p+r};a_{p+r+1})\otimes I^{\mathfrak{m}}(a_0;a_1,\cdots a_p,a_{p+r+1}\cdots a_n;a_{n+1}).\end{aligned}} \tag{2.45}$$

Remarks

- Geometrically, it is equivalent to keep in the previous coaction only the polygons corresponding to an unique cut of (interior) length r between two elements of the iterated integral.
- These maps D_r are derivations:

$$D_r(XY) = (1\otimes X)D_r(Y) + (1\otimes Y)D_r(X).$$

- This formula is linked with the equation differential satisfied by the iterated integral $I(a_0;\cdots;a_{n+1})$ when the $a_i's$ vary (cf. [44])[II]:

$$dI(a_0;\cdots;a_{n+1}) = \sum dI(a_{i-1};a_i;a_{i+1})I(a_0;\cdots\widehat{a_i}\cdots;a_{n+1}).$$

`Example`: By the previous example:

$$\begin{aligned}D_3(\zeta^{\mathfrak{m}}(-1,3)) &= (\zeta^{\mathfrak{a}}(3)-\zeta^{\mathfrak{a}}(-3))\otimes\zeta^{\mathfrak{m}}(-1)\\ D_1(\zeta^{\mathfrak{m}}(-1,3)) &= \zeta^{\mathfrak{a}}(-1)\otimes(\zeta^{\mathfrak{m}}(3)-\zeta^{\mathfrak{m}}(-3))\end{aligned}$$

[I] $\Delta'(x) := \Delta(x) - 1\otimes x - x\otimes 1$
[II] Since $I(a_{i-1};a_i;a_{i+1}) = \log(a_{i+1}-a_i) - \log(a_{i-1}-a_i)$.

2.4.3 Depth filtration

The inclusion of $\mathbb{P}^1\diagdown\{0,\mu_N,\infty\} \subset \mathbb{P}^1\diagdown\{0,\infty\}$ implies the surjection for the de Rham realizations of fundamental groupoid:

$$ {}_0\Pi_1 \to \pi_1^{dR}(\mathbb{G}_m, \overrightarrow{01}). \tag{2.46}$$

Looking at the dual, it corresponds to the inclusion of:

$$\mathcal{O}\left(\pi_1^{dR}(\mathbb{G}_m, \overrightarrow{01})\right) \cong \mathbb{Q}\left\langle e^0\right\rangle \hookrightarrow \mathcal{O}\left({}_0\Pi_1\right) \cong \mathbb{Q}\left\langle e^0, (e^\eta)_\eta\right\rangle. \tag{2.47}$$

This leads to the definition of an increasing *depth filtration* $\mathcal{F}^{\mathfrak{D}}$ on $\mathcal{O}({}_0\Pi_1)$[I] such that:

$$\boldsymbol{\mathcal{F}_p^{\mathfrak{D}}\mathcal{O}({}_0\Pi_1)} := \left\langle \text{ words } w \text{ in } e^0, e^\eta, \eta \in \mu_N \text{ such that } \sum_{\eta\in\mu_N} deg_{e^\eta} w \leq p \right\rangle_{\mathbb{Q}}. \tag{2.48}$$

This filtration is preserved by the coaction and thus descends to $\mathcal{H}$ (cf. [21]), on which:

$$\mathcal{F}_p^{\mathfrak{D}}\mathcal{H} := \left\langle \zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\dots,n_r \\ \epsilon_1,\dots,\epsilon_r\end{pmatrix}, r \leq p\right\rangle_{\mathbb{Q}}. \tag{2.49}$$

In the same way, we define $\mathcal{F}_p^{\mathfrak{D}}\mathcal{A}$ and $\mathcal{F}_p^{\mathfrak{D}}\mathcal{L}$. Beware, the corresponding grading on $\mathcal{O}({}_0\Pi_1)$ is not motivic and the depth is not a grading on $\mathcal{H}^{\mathrm{II}}$. The graded spaces $gr_p^{\mathfrak{D}}$ are defined as the quotient $\mathcal{F}_p^{\mathfrak{D}}/\mathcal{F}_{p-1}^{\mathfrak{D}}$.
Similarly, there is an increasing depth filtration on $U\mathfrak{g}$, considering the degree in $\{e_\eta\}_{\eta\in\mu_N}$, which passes to the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$(2.40) such that the graded pieces $gr_{\mathfrak{D}}^r\mathfrak{g}^{\mathfrak{m}}$ are dual to $gr_r^{\mathfrak{D}}\mathcal{L}$.
In depth 1, there are canonical elements:[III]

$$\overline{\sigma}_i^{(\eta)} := (\text{ad } e_0)^{i-1}(e_\eta) \in gr_{\mathfrak{D}}^1\mathfrak{g}^{\mathfrak{m}}. \tag{2.50}$$

They satisfy the distribution and conjugation relations stated below.

Depth 1. In depth 1, it is known for $\mathcal{A}$ (cf. [34] Theorem 6.8):

Lemma 2.4.1 (Deligne, Goncharov)**.** *The elements $\zeta^{\mathfrak{a}}(r;\eta)$ are subject only to the following relations in $\mathcal{A}$:*

Distribution

$$\forall d|N\ ,\ \forall\eta\in\mu_{\frac{N}{d}}\ ,\ (\eta,r)\neq(1,1)\ ,\ \zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \eta\end{pmatrix} = d^{r-1}\sum_{\epsilon^d=\eta}\zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \epsilon\end{pmatrix}.$$

Conjugation

$$\zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \eta\end{pmatrix} = (-1)^{r-1}\zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \eta^{-1}\end{pmatrix}.$$

[I] It is the filtration dual to the filtration given by the descending central series of the kernel of the map 2.46; it can be defined also from the cokernel of 2.47, via the decontatenation coproduct.

[II] For instance: $\zeta^{\mathfrak{m}}(3) = \zeta^{\mathfrak{m}}(1,2)$.

[III] For $N=1$, there are only the $\overline{\sigma}_{2i+1} := (\text{ad}e_0)^{2i}(e_1) \in gr_{\mathfrak{D}}^1\mathfrak{g}^{\mathfrak{m}}$, $i>0$ and the subLie algebra generated by them is not free, which means also there are other "exceptional" generators in higher depth, cf. [20].
For $N = 2,3,4,$'6', 8, when keeping η_i as in Lemma 5.2.1, $(\overline{\sigma}_i^{(\eta_i)})$ then generate a free Lie algebra in $gr_{\mathfrak{D}}\mathfrak{g}$.

Remark: More generally, distribution relations for MZV relative to μ_N are:

$$\forall d|N, \quad \forall \epsilon_i \in \mu_{\frac{N}{d}}\ , \quad \zeta\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p \end{pmatrix} = d^{\sum n_i - p} \sum_{\eta_1^d=\epsilon_1} \cdots \sum_{\eta_p^d=\epsilon_p} \zeta\begin{pmatrix} n_1,\dots,n_p \\ \eta_1,\dots,\eta_p \end{pmatrix}.$$

They are deduced from the following identity:

$$\text{For } d|N, \epsilon \in \mu_{\frac{N}{d}}\ , \ \sum_{\eta^d=\epsilon} \eta^n = \begin{cases} d\epsilon^{\frac{n}{d}} & \text{if } d|n \\ 0 & \text{else .} \end{cases}$$

These relations are obviously the analogues of those satisfied by the cyclotomic units modulo torsion.

In weight $r > 1$, a basis for $gr_1^{\mathfrak{D}}\mathcal{A}$ is formed by depth 1 MMZV at primitive roots up to conjugation. However, MMZV_{μ_N} of weight 1, $\zeta^{\mathfrak{m}}\left(\begin{smallmatrix} 1 \\ \xi_N^a \end{smallmatrix}\right) = -\log(1-\xi_N^a)$, are more subtle. For instance (already in [29]):

Lemma. *A $\mathbb{Z}$-basis for $\mathcal{A}_1$ is hence:*

$\boldsymbol{N = p^r}$**:** $\quad \left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^k \end{smallmatrix}\right) \quad a \wedge p = 1 \quad 1 \le a \le \frac{p-1}{2}\right\}.$

$\boldsymbol{N = pq}$**:** *With $p < q$ primes:*

$$\left\{\left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^k \end{smallmatrix}\right) \quad a \wedge p = 1 \quad 1 \le a \le \frac{p-1}{2}\right\} \bigcup_{a \in (\mathbb{Z}/q\mathbb{Z})^* / \langle -1, p\rangle} \left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^{ap} \end{smallmatrix}\right)\right\} \setminus \left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^{a} \end{smallmatrix}\right)\right\}\right.$$

$$\left. \bigcup_{a \in (\mathbb{Z}/p\mathbb{Z})^* / \langle -1, q\rangle} \left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^{aq} \end{smallmatrix}\right)\right\} \setminus \left\{\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^{a} \end{smallmatrix}\right)\right\}\right\}$$

Remarks:

- Indeed, for $N = pq$, a phenomenon of loops occurs: orbits via the action of p and -1 on $(\mathbb{Z}/q\mathbb{Z})^*$, resp. of q and -1 on $(\mathbb{Z}/p\mathbb{Z})^*$. Consequently, for each loop we have to remove a primitive root $\zeta\left(\begin{smallmatrix} 1 \\ \xi^a \end{smallmatrix}\right)$ and add the non primitive $\zeta\left(\begin{smallmatrix} 1 \\ \xi^{ap} \end{smallmatrix}\right)$ to the basis.[I] The situation for N a product of primes would be analogue, considering different orbits associated to each prime; we just have to pay more attention when orbits intersect, for the choice of the representatives a: avoid to withdraw or add an element already chosen for previous orbits.
- Depth 1 results also highlight a nice behavior in the cases $N = 2, 3, 4, \text{`}6\text{'}, 8$: primitive roots of unity modulo conjugation form a basis (as in the case of prime powers) and if we restrict (for dimension reasons) for non primitive roots to 1 (or ± 1 for $N = 8$), it is annihilated in weight 1 and in weight > 1 modulo p.
- In weight 1, there always exists a $\mathbb{Z}$- basis.[II]

`Example`: For $N = 34$, relations in depth 1, weight 1 lead to two orbits, with $(a) := \zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi_N^a \end{smallmatrix}\right)$:

$$\begin{array}{ll} (2) = (16) + (1) & (6) = (3) + (14) \\ (16) = (8) + (9) & (14) = (7) + (10) \\ (8) = (4) + (13) & (10) = (5) + (12) \\ (4) = (2) + (15) & (12) = (11) + (6) \end{array},$$

[I] Cardinal of an orbit $\{\pm ap^i \mod N\}$ is either the order of p modulo q, if odd, or half of the order of p modulo q, if even.

[II] Conrad and Zhao conjectured ([29]) there exists a basis of MZV_{μ_N} for the $\mathbb{Z}$-module spanned by MZV_{μ_N} for each N and fixed weight w, except $N = 1$, $w = 6, 7$.

Hence a basis could be:

$$\left\{\zeta^{\mathfrak{a}}\begin{pmatrix}1\\ \xi_{34}^k\end{pmatrix}, k\in\{5,7,9,11,13,15,\mathbf{2},\mathbf{6}\}\right\}.$$

Motivic depth. The *motivic depth* of an element in $\mathcal{H}^{\mathcal{MT}_N}$ is defined, via the correspondence (2.36), as the degree of the polynomial in the (f_i). [I] It can also be defined recursively as, for $\mathfrak{Z}\in\mathcal{H}^N$:

$$\begin{array}{lll}\mathfrak{Z} \text{ of motivic depth } 1 & \text{if and only if} & \mathfrak{Z}\in\mathcal{F}_1^{\mathfrak{D}}\mathcal{H}^N.\\ \mathfrak{Z} \text{ of motivic depth } \leq p & \text{if and only if} & (\forall r<n, D_r(\mathfrak{Z}) \text{ of motivic depth } \leq p-1).\end{array}$$

For $\mathfrak{Z}=\zeta^{\mathfrak{m}}\binom{n_1,\dots,\epsilon_p}{\epsilon_1,\dots,\epsilon_p}\in\mathcal{H}^N$ of motivic depth $p_{\mathfrak{m}}$, we clearly have the inequalities:

$$\text{depth } p\geq p_c\geq p_{\mathfrak{m}} \text{ motivic depth,}\qquad \text{where } p_c \text{ is the smallest } i \text{ such that } \mathfrak{Z}\in\mathcal{F}_i^{\mathfrak{D}}\mathcal{H}^N.$$

`Nota Bene:` For $N=2,3,4,$'6', 8, $p_{\mathfrak{m}}$ always coincide with p_c, whereas for $N=1$, they may differ.

2.4.4 Derivation space

Translating (2.44) for cyclotomic MZV:

Lemma 2.4.2.

$$D_r:\mathcal{H}_n\to\mathcal{L}_r\otimes\mathcal{H}_{n-r}$$

$$D_r\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n_1,\dots,n_p\\ \epsilon_1,\dots,\epsilon_p\end{pmatrix}\right)=\delta_{r=n_1+\cdots+n_i}\zeta^{\mathfrak{l}}\begin{pmatrix}n_1,\cdots n_i\\ \epsilon_1,\dots,\epsilon_i\end{pmatrix}\otimes\zeta^{\mathfrak{m}}\begin{pmatrix}n_{i+1},\cdots,n_p\\ \epsilon_{i+1},\dots,\epsilon_p\end{pmatrix}$$

$$+\sum_{\substack{1\leq i<j\leq p\\ \{r\leq\sum_{k=i}^j n_k-1\}}}\left[\delta_{\sum_{k=i+1}^j n_k\leq r}\zeta^{\mathfrak{l}}_{r-\sum_{k=i+1}^j n_k}\begin{pmatrix}n_{i+1},\dots,n_j\\ \epsilon_{i+1},\dots,\epsilon_j\end{pmatrix}+(-1)^r\delta_{\sum_{k=i}^{j-1} n_k\leq r}\zeta^{\mathfrak{l}}_{r-\sum_{k=i}^{j-1} n_k}\begin{pmatrix}n_{j-1},\cdots n_i,\\ \epsilon_{j-1}^{-1},\dots,\epsilon_i^{-1}\end{pmatrix}\right]$$

$$\otimes\zeta^{\mathfrak{m}}\begin{pmatrix}\cdots,\sum_{k=i}^j n_k-r,\cdots\\ \cdots,\prod_{k=i}^j\epsilon_k,\cdots\end{pmatrix}\qquad(2.51)$$

Proof. Straightforward from (2.44), passing to MZV_{μ_N} notation. □

A key point is that the Galois action and hence the coaction respects the weight grading and the depth filtration[II]:

$$D_r(\mathcal{H}_n)\subset\mathcal{L}_r\otimes_{\mathbb{Q}}\mathcal{H}_{n-r}.$$
$$D_r(\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n)\subset\mathcal{L}_r\otimes_{\mathbb{Q}}\mathcal{F}_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-r}.$$

Indeed, the depth filtration is motivic, i.e.:

$$\Delta(\mathcal{F}_n^{\mathfrak{D}}\mathcal{H})\subset\sum_{p+q=n}\mathcal{F}_p^{\mathfrak{D}}\mathcal{A}\otimes\mathcal{F}_q^{\mathfrak{D}}\mathcal{H}.$$

Furthermore, $\mathcal{F}_0^{\mathfrak{D}}\mathcal{A}=\mathcal{F}_0^{\mathfrak{D}}\mathcal{L}=0$. Therefore, the right side of $\Delta(\bullet)$ is in $\mathcal{F}_q^{\mathfrak{D}}\mathcal{H}$, with $q<n$. This feature of the derivations D_r (decreasing the depth) will enable us to do some recursion on depth through this work.

Passing to the depth-graded, define:

$$gr_p^{\mathfrak{D}}D_r:gr_p^{\mathfrak{D}}\mathcal{H}\to\mathcal{L}_r\otimes gr_{p-1}^{\mathfrak{D}}\mathcal{H},\text{ as the composition }(id\otimes gr_{p-1}^{\mathfrak{D}})\circ D_{r|gr_p^{\mathfrak{D}}\mathcal{H}}.$$

By Lemma 2.4.2, all the terms appearing in the left side of $gr_p^{\mathfrak{D}}D_{2r+1}$ have depth 1. Hence, let consider from now the derivations $D_{r,p}$:

[I] Beware, ϕ is non-canonical, but the degree is well defined.
[II] Notice that $(\mathcal{F}_0^{\mathfrak{D}}\mathcal{L}=0$.

Lemma 2.4.3.

$$\boldsymbol{D_{r,p}} : gr_p^{\mathfrak{D}}\mathcal{H} \to gr_1^{\mathfrak{D}}\mathcal{L}_r \otimes gr_{p-1}^{\mathfrak{D}}\mathcal{H}$$

$$D_{r,p}\left(\zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\dots,n_p \\ \epsilon_1,\dots,\epsilon_p\end{pmatrix}\right) = (\text{A0})\ \delta_{r=n_1}\ \zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_1\end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} n_2,\cdots \\ \epsilon_2,\cdots\end{pmatrix}$$

$$(\text{A})\ \ +\sum_{i=2}^{p-1}\delta_{n_i\le r<n_i+n_{i-1}-1}(-1)^{r-n_i}\binom{r-1}{r-n_i}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_i\end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} \cdots, n_i+n_{i-1}-r,\cdots \\ \cdots,\epsilon_{i-1}\epsilon_i,\cdots\end{pmatrix}$$

$$(\text{B})\ \ -\sum_{i=1}^{p-1}\delta_{n_i\le r<n_i+n_{i+1}-1}(-1)^{n_i}\binom{r-1}{r-n_i}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_i^{-1}\end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} \cdots, n_i+n_{i+1}-r,\cdots \\ \cdots,\epsilon_{i+1}\epsilon_i,\cdots\end{pmatrix}$$

$$(\text{C})\ \ +\sum_{i=2}^{p-1}\delta_{\substack{r=n_i+n_{i-1}-1\\ \epsilon_{i-1}\epsilon_i\neq 1}}\left((-1)^{n_i}\binom{r-1}{n_i-1}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_{i-1}^{-1}\end{pmatrix} + (-1)^{n_{i-1}-1}\binom{r-1}{n_{i-1}-1}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_i\end{pmatrix}\right)$$
$$\otimes\zeta^{\mathfrak{m}}\begin{pmatrix} \cdots,1,\cdots \\ \cdots,\epsilon_{i-1}\epsilon_i,\cdots\end{pmatrix}$$

$$(\text{D})\ \ +\delta_{n_p\le r<n_p+n_{p-1}-1}(-1)^{r-n_p}\binom{r-1}{r-n_p}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_p\end{pmatrix} \otimes \zeta^{\mathfrak{m}}\begin{pmatrix} \cdots, n_{p-1}+n_p-r \\ \cdots,\epsilon_{p-1}\epsilon_p\end{pmatrix}$$

$$(\text{D'})\ +\delta_{\substack{r=n_p+n_{p-1}-1\\ \epsilon_{p-1}\epsilon_p\neq 1}}(-1)^{n_{p-1}}\left(\binom{r-1}{n_p-1}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_{p-1}^{-1}\end{pmatrix} - \binom{r-1}{n_{p-1}-1}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_p\end{pmatrix}\right)\otimes\zeta^{\mathfrak{m}}\begin{pmatrix} \cdots,1 \\ \cdots\epsilon_{p-1}\epsilon_p\end{pmatrix}.$$

Remarks:

- The terms of type (D, D'), corresponding to a *deconcatenation*, play a particular role since modulo some congruences (using depth 1 result for the left side of the coaction), we will get rid of the other terms in the cases $N = 2, 3, 4, \text{'6'}, 8$ for the elements in the basis. In the dual point of view of Lie algebra, like in Deligne article [31] or Wojtkowiak [79], this corresponds to showing that the Ihara bracket $\{,\}$ on these elements modulo some vector space reduces to the usual bracket $[,]$. More generally, in other case of basis, as Hoffman one's for $N = 1$, the idea is still to find an appropriate filtration on the conjectural basis, such that the coaction in the graded space acts on this family, modulo some space, as the deconcatenation, as for the f_i alphabet. Indeed, on H (2.36), the weight graded part of the coaction, D_r is defined by:

$$\begin{array}{llll} D_r: & H_n & \longrightarrow & L_r\otimes H_{n-r} \quad \text{such that :} \\ & f_{i_1}^{j_1}\cdots f_{i_k}^{j_k} & \longmapsto & \begin{cases} f_{i_1}^{j_1}\otimes f_{i_2}^{j_2},\dots,f_{i_k}^{j_k} & \text{if } i_1=r. \\ 0 & \text{else .}\end{cases}\end{array} \tag{2.52}$$

- One fundamental feature for a family of motivic multiple zeta values (which makes it "natural" and simple) is the *stability* under the coaction. For instance, if we look at the following families which appears in Chapter 5:

$$\zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\cdots n_{p-1},n_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\end{pmatrix} \quad \text{with } \epsilon_p\in\mu_N \quad \text{primitive} \quad \text{and } (\epsilon_i)_{i<p} \quad \text{non primitive.}$$

  If N is a power of a prime, this family is stable via the coaction. [I] It is also stable via the Galois action if we only need to take 1 as a non primitive (1-dimensional case), as for $\mathcal{MT}(\mathcal{O}_6)$.

[I] Since in this case, (non primitive) $\cdot$ (non primitive) = non primitive and non primitive $\cdot$ primitive = primitive root. Note also, for dimensions reasons, if we are looking for a basis in this form, we should have $N-\varphi(N)\ge\frac{\varphi(N)}{2}$, which comes down here to the case where N is a power of 2 or 3.

Proof. Straightforward from 2.4.2, using the properties of motivic iterated integrals previously listed (§2.3). Terms of type (A) correspond to cuts from a 0 (possibly the very first one) to a root of unity, (B) terms from a root of unity to a 0, (C) terms between two roots of unity and (D,D') terms are the cuts ending by the last 1, called *deconcatenation terms.* □

Derivation space. By Lemma 2.4.1 (depth 1 results), once we have chosen a basis for $gr_1^{\mathfrak{D}}\mathcal{L}_r$, composed by some $\zeta^{\mathfrak{a}}(r_i;\eta_i)$, we can well define: [I]

(*i*) For each (r_i,η_i):

$$\boldsymbol{D^{\eta_i}_{r_i,p}} : gr_p^{\mathfrak{D}}\mathcal{H} \to gr_{p-1}^{\mathfrak{D}}\mathcal{H}, \tag{2.53}$$

as the composition of $D_{r_i,p}$ followed by the projection:

$$\pi^{\eta} : gr_1^{\mathfrak{D}}\mathcal{L}_r \otimes gr_{p-1}^{\mathfrak{D}}\mathcal{H} \to gr_{p-1}^{\mathfrak{D}}\mathcal{H}, \qquad \zeta^{\mathfrak{m}}(r;\epsilon)\otimes X \mapsto c_{\eta,\epsilon,r}X,$$

with $c_{\eta,\epsilon,r}\in\mathbb{Q}$ the coefficient of $\zeta^{\mathfrak{m}}(r;\eta)$ in the decomposition of $\zeta^{\mathfrak{m}}(r;\epsilon)$ in the basis.

(*ii*)

$$\mathscr{D}_{\boldsymbol{r,p}} \text{ as the set of } D^{\eta_i}_{r_i,p} \text{ for } \zeta^{\mathfrak{m}}(r_i,\eta_i) \text{ in the chosen basis of } gr_1^{\mathfrak{D}}\mathcal{A}_r. \tag{2.54}$$

(*iii*) The *derivation set* $\mathscr{D}$ as the (disjoint) union: $\mathscr{D} := \sqcup_{r>0}\{\mathscr{D}_r\}$.

Remarks:

- In the case $N=2,3,4,$'6', the cardinal of $\mathscr{D}_{r,p}$ is one (or 0 if r even and $N=2$, or if $(r,N)=(1,6)$), whereas for $N=8$ the space generated by these derivations is 2-dimensional, generated by $D_r^{\xi_8}$ and $D_r^{-\xi_8}$ for instance.
- Doing the same procedure for the n.c. Hopf comodule H defined in (2.36), isomorphic to $\mathcal{H}^{\mathcal{MT}_N}$, since the coproduct on H is the deconcatenation (2.52), leads to the following derivations operators:

$$\begin{array}{llll} D_r^j: & H_n & \to & H_{n-r} \\ & f_{i_1}^{j_1}\cdots f_{i_k}^{j_k} & \mapsto & \begin{cases} f_{i_2}^{j_2},\dots,f_{i_k}^{j_k} & \text{if } j_1=j \text{ and } i_1=r. \\ 0 & \text{else .} \end{cases} \end{array} .$$

Now, consider the following application, depth graded version of the derivations above, fundamental for several linear independence results in §4.3 and Chapter 5:

$$\boldsymbol{\partial_{n,p}} := \oplus_{\substack{r<n \\ D\in\mathscr{D}_{r,p}}} D : gr_p^{\mathfrak{D}}\mathcal{H}_n \to \oplus_{r<n}\left(gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-r}\right)^{\oplus \text{ card } \mathscr{D}_{r,p}} \tag{2.55}$$

Kernel of $\boldsymbol{D_{<n}}$. A key point for the use of these derivations is the ability to prove some relations (and possibly lift some from MZV to motivic MZV) up to rational coefficients. This comes from the following theorem, looking at primitive elements:

Theorem 2.4.4. *Let $D_{<n} := \oplus_{r<n}D_r$, and fix a basis $\{\zeta^{\mathfrak{a}}\binom{n}{\eta_j}\}$ of $gr_1^{\mathfrak{D}}\mathcal{A}_n$. Then:*

$$\ker D_{<n}\cap\mathcal{H}_n^N = \begin{cases} \mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{1} & \textit{for } N=1,2 \textit{ and } n\neq 1. \\ \oplus\mathbb{Q}\pi^{\mathfrak{m}}\bigoplus_{1\le j\le a_N}\mathbb{Q}\zeta^{\mathfrak{m}}\binom{1}{\eta_j}. & \textit{for } N>2, n=1. \\ \oplus\mathbb{Q}(\pi^{\mathfrak{m}})^n\bigoplus_{1\le j\le b_N}\mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{\eta_j}. & \textit{for } N>2, n>1. \end{cases} .$$

[I] Without passing to the depth-graded, we could also define D_r^{η} as $D_r:\mathcal{H}\to gr_1^{\mathfrak{D}}\mathcal{L}_r\otimes\mathcal{H}$ followed by $\pi_r^{\eta}\otimes id$ where $\pi^{\eta}: gr_1^{\mathfrak{D}}\mathcal{L}_r\to\mathbb{Q}$ is the projection on $\zeta^{\mathfrak{m}}\binom{r}{\eta}$, once we have fixed a basis for $gr_1^{\mathfrak{D}}\mathcal{L}_r$; and define as above $\mathscr{D}_r$ as the set of the $D_{r,p}^{\eta}$, for $\zeta^{\mathfrak{m}}(r,\eta)$ in the basis of $gr_1^{\mathfrak{D}}\mathcal{A}_r$.

Proof. It comes from the injective morphism of graded Hopf comodule (2.36), isomorphism for $N = 1, 2, 3, 4, \text{`6'}, 8$:

$$\phi : \mathcal{H}^N \xrightarrow[\sim]{n.c} H^N := \mathbb{Q}\left\langle \left(f_i^j\right)\right\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\left[g_1\right].$$

Indeed, for H^N, the analogue statement is obviously true, for $\Delta' = 1 \otimes \Delta + \Delta \otimes 1$:

$$\ker \Delta' \cap H_n = \oplus_j f_n^j \oplus g_1^n.$$

□

Corollary 2.4.5. *Let $D_{<n} := \oplus_{r<n} D_r$.*[I] *Then:*

$$\ker D_{<n} \cap \mathcal{H}_n^N = \begin{cases} \mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{1} & \text{for } N = 1, 2. \\ \mathbb{Q}(\pi^{\mathfrak{m}})^n \oplus \mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{\xi_N} & \text{for } N = 3, 4, \text{`6'}. \\ \mathbb{Q}(\pi^{\mathfrak{m}})^n \oplus \mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{\xi_8} \oplus \mathbb{Q}\zeta^{\mathfrak{m}}\binom{n}{-\xi_8} & \text{for } N = 8. \end{cases}.$$

In particular, by this result (for $N = 1, 2$), proving an identity between motivic MZV (resp. motivic Euler sums), amounts to:

1. Prove that the coaction is identical on both sides, computing D_r for $r > 0$ smaller than the weight. If the families are not stable under the coaction, this step would require other identities.

2. Use the analytic corresponding result for MZV (resp. Euler sums) to deduce the remaining rational coefficient; if the analytic equivalent is unknown, we can at least evaluate numerically this rational coefficient.

Some examples are given in §6.3 and §4.4.3.
Another important use of this corollary, is the decomposition of (motivic) multiple zeta values into a conjectured basis, which has been explained by F. Brown in [19].[II]
However, for greater N, several rational coefficients appear at each step, and we would need linear independence results before concluding.

[I] For $N = 1$, we restrict to r odd > 1; for $N = 2$ we restrict to r odd; for $N =$ '6' we restrict to $r > 1$.

[II] He gave an exact numerical algorithm for this decomposition, where, at each step, a rational coefficient has to be evaluated; hence, for other roots of unity, the generalization, albeit easily stated, is harder for numerical experiments.

Chapter 3

Results

3.1 Euler $\star, \sharp$ sums *[Chapter 4]*

In Chapter 4, we focus on motivic Euler sums ($N = 2$), shortened ES, and motivic multiple zeta values ($N = 1$), with in particular some new bases for the vector space of MMZV: one with Euler $\sharp$ sums and, under an analytic conjecture, the so-called *Hoffman $\star$ family.* These two variants of Euler sums are (cf. Definition 4.1.1):

Euler $\star$ sums corresponds to the analogue multiple sums of ES with $\leq$ instead of strict inequalities. It verifies:

$$\zeta^\star(n_1, \ldots, n_p) = \sum_{\circ = \text{`+' or ,}} \zeta(n_1 \circ \cdots \circ n_p). \tag{3.1}$$

`Notation:` This ' + ' operation on $n_i \in \mathbb{Z}$, is a summation of absolute values, while signs are multiplied.
These have already been studied in many papers: [9], [56], [58], [65], [69], [86].

Euler $\sharp$ sums are, similarly, linear combinations of MZV but with 2-power coefficients:

$$\zeta^\sharp(n_1, \ldots, n_p) = \sum_{\circ = \text{`+' or ,}} 2^{p-n_+} \zeta(n_1 \circ \cdots \circ n_p), \quad \text{with } n_+ \text{ the number of } +. \tag{3.2}$$

We also pave the way for a motivic version of a generalization of a Linebarger and Zhao's equality (Conjecture 4.5.1) which expresses each motivic multiple zeta $\star$ as a motivic Euler $\sharp$ sums; under this conjecture, Hoffman $\star$ family is a basis, identical to the one presented with Euler sums $\sharp$.

The first (naive) idea, when looking for a basis for the space of multiple zeta values, is to choose:

$$\{\zeta(2n_1 + 1, 2n_2 + 1, \ldots, 2n_p + 1)(2i\pi)^{2s}, n_i \in \mathbb{N}^*, s \in \mathbb{N}\}.$$

However, considering Broadhurst-Kreimer conjecture (2.37), the depth filtration clearly does *not* behave so nicely in the case of MZV [I] and already in weight 12, they are not linearly independent:

$$28\zeta(9,3) + 150\zeta(7,5) + 168\zeta(5,7) = \frac{5197}{691}\zeta(12).$$

[I] Remark, as we will see in Chapter 5, or as we can see in [31] that for $N = 2, 3, 4, \text{`6'}, 8$, the depth filtration is dual of the descending central series of $\mathcal{U}$, and, in that sense, does *behave well.* For instance, the following family is indeed a basis of motivic Euler sums:

$$\{\zeta^{\mathrm{m}}(2n_1 + 1, 2n_2 + 1, \ldots, 2n_{p-1} + 1, -(2n_p + 1))(\mathbb{L}^{\mathrm{m}})^{2s}, n_i \in \mathbb{N}, s \in \mathbb{N}\}.$$

Consequently, in order to find a basis of motivic MZV, we have to:

`Either`: Allow *higher* depths, as the Hoffman basis (proved by F Brown in [20]), or the $\star$ analogue version:

$$\texttt{Hoffman}\ \star \quad : \{\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, 3, \mathbf{2}^{a_1}, \ldots, 3, \mathbf{2}^{a_p}\right), a_i \geq 0\}.$$

The analogous real family Hoffman $\star$ was also conjectured (in [56], Conjecture 1) to be a basis of the space of MZV. Up to an analytic conjecture (4.4.4), we prove (in §4.4) that the motivic Hoffman $\star$ family is a basis of $\mathcal{H}^1$, the space of motivic MZV[I]. In this case, the notion of *motivic depth* (explained in §2.4.3) is the number of 3, and is here in general much smaller than the depth.

`Or`: Pass by motivic Euler sums, as the Euler $\sharp$ basis given below; it is also another illustration of the descent idea of Chapter 5: roughly, it enables to reach motivic periods in $\mathcal{H}^{N'}$ coming from above, i.e. via motivic periods in $\mathcal{H}^N$, for $N' \mid N$.

More precisely, let look at the following motivic Euler $\sharp$ sums:

Theorem. *The motivic Euler sums $\zeta^{\sharp,\mathfrak{m}}\left(\{\overline{even}, odd\}^{\times}\right)$ are motivic geometric periods of $\mathcal{MT}(\mathbb{Z})$. Hence, they are $\mathbb{Q}$ linear combinations of motivic multiple zeta values.*[II]

`Notations`: Recall that an overline $\overline{x}$ corresponds to a negative sign, i.e. $-x$ in the argument. Here, the family considered is a family of Euler $\sharp$ sums with only positive odd and negative even integers for arguments.
This motivic family is even a generating family of motivic MZV from which we are able to extract a basis:

Theorem. *A basis of $\mathcal{P}^{\mathfrak{m},+}_{\mathcal{MT}(\mathbb{Z}),\mathbb{R}} = \mathcal{H}^1$, the space of motivic multiple zeta values is:*

$$\{\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2}\right)\ ,\ a_i \geq 0\}.$$

The proof is based on the good behaviour of this family with respect to the coaction and the depth filtration; the suitable filtration corresponding to the *motivic depth* for this family is the usual depth minus 1.
By application of the period map, combining these results:

Corollary. *Each Euler sum $\zeta^{\sharp}\left(\{\overline{even}, odd\}^{\times}\right)$ (i.e. with positive odd and negative even integers for arguments) is a $\mathbb{Q}$ linear combination of multiple zeta values of the same weight. Conversely, each multiple zeta value of depth $< d$ is a $\mathbb{Q}$ linear combination of elements $\zeta^{\sharp}\left(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2}\right)$, of the same weight with $a_i \geq 0$, $p \leq d$.*

Remarks:

- Finding a *good* basis for the space of motivic multiple zeta values is a fundamental question. Hoffman basis may be unsatisfactory for various reasons, while this basis with Euler sums (linear combinations with 2 power coefficients) may appear slightly more natural, in particular since the motivic depth is here the depth minus 1. However, both of those two basis are not basis of the $\mathbb{Z}$ module and the primes appearing in the determinant of the *passage matrix*[III] are growing rather fast.[IV]

[I] Up to this analytic statement, 4.4.4, the Hoffman $\star$ family is then a generating family for MZV.

[II] Since, by [20], we know that Frobenius invariant geometric motivic periods of $\mathcal{MT}(\mathbb{Z})$ are $\mathbb{Q}$ linear combinations of motivic multiple zeta values.

[III] The inverse of the matrix expressing the considered basis in term of a $\mathbb{Z}$ basis.

[IV] Don Zagier has checked this for small weights with high precision; he suggested that the primes involved in the case of this basis could have some predictable features, such as being divisor of $2^n - 1$.

- Looking at how periods of $\mathcal{MT}(\mathbb{Z})$ embeds into periods of $\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])$, is a fragment of the Galois descent ideas of Chapter 5.
  Euler sums which belong to the $\mathbb{Q}$-vector space of multiple zeta values, sometimes called *honorary*, have been studied notably by D. Broadhurst (cf. [11]) among others. We define then *unramified* motivic Euler sums as motivic ES which are $\mathbb{Q}$-linear combination of motivic MZV, i.e. in $\mathcal{H}^1$. Being unramified for a motivic period implies that its period is unramified, i.e. honorary; some examples of unramified motivic ES are given in §6.2, or with the family above. In Chapter 5, we give a criterion for motivic Euler sums to be unramified 5.1.3, which generalizes for some other roots of unity; by the period map, this criterion also applies to Euler sums.

- For these two theorems, in order to simplify the coaction, we crucially need a motivic identity in the coalgebra $\mathcal{L}$, proved in §4.2, coming from the octagon relation pictured in Figure 4.3. More precisely, we need to consider the linearized version of the anti-invariant part by the Frobenius at infinity of this relation, in order to prove this hybrid relation (Theorem 4.2.3), for $n_i \in \mathbb{N}^*$, $\epsilon_i \in \pm 1$:
  $$\zeta^{\mathfrak{l}}_k \begin{pmatrix} n_0, \cdots, n_p \\ \epsilon_0, \dots, \epsilon_p \end{pmatrix} + \zeta^{\mathfrak{l}}_{n_0+k} \begin{pmatrix} n_1, \dots, n_p \\ \epsilon_1, \dots, \epsilon_p \end{pmatrix} \equiv (-1)^{w+1} \left(\zeta^{\mathfrak{l}}_k \begin{pmatrix} n_p, \dots, n_0 \\ \epsilon_p, \dots, \epsilon_0 \end{pmatrix} + \zeta^{\mathfrak{l}}_{k+n_p} \begin{pmatrix} n_{p-1}, \dots, n_0 \\ \epsilon_{p-1}, \dots, \epsilon_0 \end{pmatrix} \right).$$
  Thanks to this hybrid relation, and the antipodal relations presented in §4.2.1, the coaction expression is considerably simplified in Appendix A.1.

Theorem. *If the analytic conjecture (*4.4.4*) holds, then the motivic Hoffman $\star$ family $\{\zeta^{\star,\mathfrak{m}}(\{2,3\}^\times)\}$ is a basis of $\mathcal{H}^1$, the space of MMZV.*

`Nota Bene:` A MMZV $\star$, in the depth graded, is obviously equal to the corresponding MMZV. However, the motivic Hoffman (i.e. with only 2 and 3) multiple zeta ($\star$) values are almost all zero in the depth graded (the *motivic depth* there being the number of 3). Hence, the analogous result for the non $\star$ case[I], proved by F. Brown, does not make the result in the $\star$ case anyhow simpler.

Denote by $\mathcal{H}^{2,3}$ the $\mathbb{Q}$-vector space spanned by the motivic Hoffman $\star$ family. The idea of the proof is similar as in the non-star case done by Francis Brown. We define an increasing filtration $\mathcal{F}^L_\bullet$ on $\mathcal{H}^{2,3}$, called the *level*, such that:[II]

$$\mathcal{F}^L_l \mathcal{H}^{2,3} \text{ is spanned by } \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_l}), \text{ with less than “l” 3.}$$

One key feature is that the vector space $\mathcal{F}^L_l \mathcal{H}^{2,3}$ is stable under the action of $\mathcal{G}$.
The linear independence is then proved thanks to a recursion on the level and on the weight, using the injectivity of a map ∂ where ∂ came out of the level and weight-graded part of the coaction Δ (cf. §4.4.1). The injectivity is proved via 2-adic properties of some coefficients with Conjecture 4.4.4.
One noteworthy difference is that, when computing the coaction on the motivic MZV*, some motivic MZV** arise, which are a non convergent analogue of MZV* and have to be renormalized. Therefore, where F. Brown in the non-star case needed an analytic formula proven by Don Zagier ([82]), we need some slightly more complicated identities (in Lemma 4.4.3) because the elements involved, such as $\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a, 3, \mathbf{2}^b)$ for instance, are not of depth 1 but are linear combinations of products of depth 1 motivic MZV times a power of π.

These two bases for motivic multiple zeta values turn to be identical, when considering this conjectural motivic identity, more generally:

[I] I.e. that the motivic Hoffman family is a basis of the space of MMZV, cf [19].

[II] Beware, this notion of level is different than the level associated to a descent in Chapter 5. It is similar as the level notion for the Hoffman basis, in F. Brown paper's [20]. It corresponds to the motivic depth, as we will see through the proof.

Conjecture. *For $a_i, c_i \in \mathbb{N}^*$, $c_i \neq 2$,*

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, c_1, \cdots, c_p, \mathbf{2}^{a_p}\right) = \tag{3.3}$$

$$(-1)^{1+\delta_{c_1}} \zeta^{\sharp,\mathfrak{m}}\left(\pm(2a_0+1-\delta_{c_1}), \mathbf{1}^{c_1-3}, \cdots, \mathbf{1}^{c_i-3}, \pm(2a_i+3-\delta_{c_i}-\delta_{c_{i+1}}), \ldots, \pm(2a_p+2-\delta_{c_p})\right).$$

where the sign $\pm$ is always $-$ for an even argument, $+$ for an odd one, $\delta_c = \delta_{c=1}$, Kronecker symbol, and $\mathbf{1}^n := \mathbf{1}^{min(0,n)}$ is a sequence of n 1 if $n \in \mathbb{N}$, an empty sequence else.

This conjecture expresses each motivic MZV* as a linear combination of motivic Euler sums, which gives another illustration of the Galois descent between the Hopf algebra of motivic MZV and the Hopf algebra of motivic Euler sums.

`Nota Bene`: Such a *motivic relation* between MMZV$_{\mu_N}$ is stronger than its analogue between MZV$_{\mu_N}$ since it contains more information; it implies many other relations because of its Galois conjugates. This explain why its is not always simple to lift an identity from MZV to MMZV from the Theorem 2.4.5. If the family concerned is not stable via the coaction, such as (iv) in Lemma 4.4.3, we may need other analytic equalities before concluding.

This conjecture implies in particular the following motivic identities, whose analogue for real Euler sums are proved as indicated in the brackets[I]:

Two-One [For $c_i = 1$, Ohno Zudilin: [69]]:

$$\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 1, \cdots, 1, \mathbf{2}^{a_p}) = -\zeta^{\sharp,\mathfrak{m}}\left(\overline{2a_0}, 2a_1+1, \ldots, 2a_{p-1}+1, 2a_p+1\right). \tag{3.4}$$

Three-One [For c_i alternatively 1 and 3, Zagier conjecture, proved in [9]]

$$\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 1, \mathbf{2}^{a_1}, 3 \cdots, 1, \mathbf{2}^{a_{p-1}}, 3, \mathbf{2}^{a_p}) = -\zeta^{\sharp,\mathfrak{m}}\left(\overline{2a_0}, \overline{2a_1+2}, \ldots, \overline{2a_{p-1}+2}, \overline{2a_p+2}\right). \tag{3.5}$$

Linebarger-Zhao $\star$ [With $c_i \geq 3$, Linebarger Zhao in [65]]:

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, c_1, \cdots, c_p, \mathbf{2}^{a_p}\right) = -\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, \mathbf{1}^{c_1-3}, \cdots, \mathbf{1}^{c_i-3}, 2a_i+3, \ldots, \overline{2a_p+2}\right) \tag{3.6}$$

In particular, restricting to all $c_i = 3$:

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_p}\right) = -\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, 2a_1+3, \ldots, 2a_{p-1}+3, \overline{2a_p+2}\right). \tag{3.7}$$

`Nota Bene`: Hence the previous conjecture (4.37) implies that the motivic Hoffman $\star$ is a basis, since we proved the right side of (4.37) is a basis:

$$\text{Conjecture 4.5.1} \quad \Longrightarrow \quad \text{Hoffman } \star \text{ is a basis of MMZV.}$$

`Examples:` The previous conjecture would give such relations:

$$\zeta^{\star,\mathfrak{m}}(2,2,3,3,2) = -\zeta^{\sharp,\mathfrak{m}}(5,3,-4) \quad \zeta^{\star,\mathfrak{m}}(5,6,2) = -\zeta^{\sharp,\mathfrak{m}}(1,1,1,3,1,1,1,-4)$$
$$\zeta^{\star,\mathfrak{m}}(1,6) = \zeta^{\sharp,\mathfrak{m}}(-2,1,1,1,-2) \quad \zeta^{\star,\mathfrak{m}}(2,4,1,2,2,3) = -\zeta^{\sharp,\mathfrak{m}}(3,1,-2,-6,-2).$$

[I]Beware, only the identity for real Euler sums is proved; the motivic analogue stays a conjecture.

3.2 Galois Descents *[Chapter 5]*

There, we study Galois descents for categories of mixed Tate motives $\mathcal{MT}_{\Gamma_N}$, and how periods of $\pi_1^{un}(X_{N'})$ are embedded into periods of $\pi_1^{un}(X_N)$ for $N' \mid N$. Indeed, for each N, N' with $N'|N$ there are the motivic Galois group $\mathcal{G}^{\mathcal{MT}_N}$ acting on $\mathcal{H}^{\mathcal{MT}_N}$ and a Galois descent between $\mathcal{H}^{\mathcal{MT}_{N'}}$ and $\mathcal{H}^{\mathcal{MT}_N}$, such that:

$$(\mathcal{H}^{\mathcal{MT}_N})^{\mathcal{G}^{N/N'}} = \mathcal{H}^{\mathcal{MT}_{N'}}.$$

Since for $N = 2, 3, 4, \text{`}6\text{'}, 8$, the categories $\mathcal{MT}_N$ and $\mathcal{MT}'_N$ are equal, this Galois descent has a parallel for the motivic fundamental group side; we will mostly neglect the difference in this chapter:

$$\begin{tikzcd}
\mathcal{H}^N \arrow[r, hook, "\sim", "n.c"'] & \mathcal{H}^{\mathcal{MT}_N} \\
\mathcal{H}^{N'} \arrow[u, "\mathcal{G}^{N/N'}"] \arrow[r, hook, "\sim", "n.c"'] & \mathcal{H}^{\mathcal{MT}_{N'}} \arrow[u, "\mathcal{G}^{\mathcal{MT}}_{N/N'}"'] \\
\mathbb{Q}[i\pi^{\mathfrak{m}}] \arrow[u, "\mathcal{U}^{N'}"] \arrow[r, hook, "\sim"] & \mathbb{Q}[i\pi^{\mathfrak{m}}] \arrow[u, "\mathcal{U}^{\mathcal{MT}_{N'}}"'] \\
\mathbb{Q} \arrow[u, "\mathbb{G}_m"] \arrow[uu, bend left=60, "\mathcal{G}^N"] & \mathbb{Q} \arrow[u, "\mathbb{G}_m"'] \arrow[uu, bend right=60, "\mathcal{G}^{\mathcal{MT}_N}"']
\end{tikzcd}$$

Figure 3.1: Galois descents, $N = 2, 3, 4, \text{`}6\text{'}, 8$ (level 0).[I]

`Nota Bene:` For $N' = 1$ or 2, $i\pi^{\mathfrak{m}}$ has to be replaced by $\zeta^{\mathfrak{m}}(2)$ or $(\pi^{\mathfrak{m}})^2$, since we consider, in $\mathcal{H}^{N'}$ only periods invariant by the Frobenius $\mathcal{F}_\infty$. In the descent between $\mathcal{H}^N$ and $\mathcal{H}^{N'}$, we require hence invariance by the Frobenius in order to keep only those periods; this condition get rid of odd powers of $i\pi^{\mathfrak{m}}$.

The first section of Chapter 5 gives an overview for the Galois descents valid for any N: a criterion for the descent between $\mathrm{MMZV}_{\mu_{N'}}$ and MMZV_{μ_N} (Theorem 5.1.1), a criterion for being unramified (Theorem 5.1.2), and their corollaries. The conditions are expressed in terms of the derivations D_r, since they reflect the Galois action. Indeed, looking at the descent between $\mathcal{MT}_{N,M}$ and $\mathcal{MT}_{N',M'}$, sometimes denoted $(\lceil) = (k_N/k_{N'}, M/M')$, it has possibly two components:

- The change of cyclotomic fields $k_N/k_{N'}$; there, the criterion has to be formulated in the depth graded.
- The change of ramification M/M', which is measured by the 1 graded part of the coaction i.e. D_1 with the notations of §2.4.

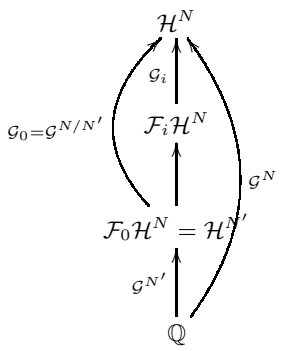
The second section specifies the descents for $N \in \{2, 3, 4, \text{`}6\text{'}, 8\}$ [II] represented in Figure 3.3, and 3.4. In particular, this gives a basis of motivic multiple zeta values relative to $\mu_{N'}$ via motivic multiple zeta values relative to μ_N, for these descents considered, $N' \mid N$. It also gives a new proof of Deligne's results ([31]): the category of mixed Tate motives over $\mathcal{O}_{k_N}[1/N]$, for $N \in \{2, 3, 4, \text{`}6\text{'}, 8\}$ is spanned by the motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ with an explicit basis; as claimed in §2.2, we can even restrict to a smaller

[I] The (non-canonical) horizontal isomorphisms have to be chosen in a compatible way.
[II] As above, the quotation marks underline that we consider the unramified category for $N = \text{`}6\text{'}$.

fundamental groupoid.
Let us present our results further and fix a descent $(\lceil) = (k_N/k_{N'}, M/M')$ among these considered (in Figures 3.3, 3.4), between the category of mixed Tate motives of $\mathcal{O}_N[1/M]$ and $\mathcal{O}_{N'}[1/M']$.[I] Each descent $(\lceil)$ is associated to a subset $\mathscr{D}^{\lceil} \subset \mathscr{D}$ of derivations, which represents the action of the Galois group $\mathcal{G}^{N/N'}$. It defines, recursively on i, an increasing motivic filtration $\mathcal{F}_i^{\lceil}$ on $\mathcal{H}^N$ called *motivic level*, stable under the action of $\mathcal{G}^{\mathcal{MT}_N}$:

$$\texttt{Motivic level:} \begin{cases} \mathcal{F}_{-1}^{\lceil}\mathcal{H}^N := 0 \\ \boldsymbol{\mathcal{F}_i^{\lceil}} \text{ the largest submodule of } \mathcal{H}^N \text{ such that } \mathcal{F}_i^{\lceil}\mathcal{H}^N/\mathcal{F}_{i-1}^{\lceil}\mathcal{H}^N \text{ is killed by } \mathscr{D}^{\lceil}. \end{cases}.$$

The 0th level $\mathcal{F}_0^{\lceil}\mathcal{H}^N$, corresponds to invariants under the group $\mathcal{G}^{N/N'}$ while the ith level $\mathcal{F}_i^{\lceil}$, can be seen as the ith *ramification space* in generalized Galois descents. Indeed, they correspond to a decreasing filtration of ith ramification Galois groups $\mathcal{G}_i$, which are the subgroups of $\mathcal{G}^{N/N'}$ which acts trivially on $\mathcal{F}^i\mathcal{H}^N$.[II]



$$\begin{aligned} &(\mathcal{H}^N)^{\mathcal{G}_i} = \mathcal{F}_i\mathcal{H}^N \\ &\mathcal{G}^{N/N'} = \mathcal{G}_0 \quad \supset \mathcal{G}_1 \quad \supset \cdots \quad \supset \mathcal{G}_i \cdots \\ &\mathcal{H}^{N'} = \mathcal{F}_0\mathcal{H}^N \quad \subset \mathcal{F}_1\mathcal{H}^N \quad \subset \cdots \quad \subset \mathcal{F}_i\mathcal{H}^N \cdots . \end{aligned} \tag{3.9}$$

Figure 3.2: Representation of a Galois descent.

Those ramification spaces constitute a tower of intermediate spaces between the elements in MMZV_{μ_N} and the whole space of $\mathrm{MMZV}_{\mu_{N'}}$.

Let define the quotients associated to the motivic level:

$$\boldsymbol{\mathcal{H}^{\geq i}} := \mathcal{H}/\mathcal{F}_{i-1}\mathcal{H}\ , \quad \mathcal{H}^{\geq 0} = \mathcal{H}.$$

[I] Usually, the indication of the descent (in the exponent) is omitted when we look at a specific descent.

[II] *On* ***ramification groups*** *in usual Galois theory*: let L/K a Galois extension of local fields. By Hensel's lemma, $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ and the ith ramification group is defined as:

$$G_i := \{g \in \mathrm{Gal}(L/K) \mid v(g(\alpha) - \alpha) > i\}\,, \quad \text{where} \begin{cases} v \text{ is the valuation on } L \\ \mathfrak{p} = \{x \in L \mid v(x) > 0\} \text{ maximal ideal for } L \end{cases}. \tag{3.8}$$

Equivalently, this condition means g acts trivially on $\mathcal{O}_L/\mathfrak{p}^{i+1}$, i.e. $g(x) \equiv x \pmod{\mathfrak{p}^{i+1}}$. This decreasing filtration of normal subgroups corresponds, by the Galois fundamental theorem, to an increasing filtration of Galois extensions:

$$\begin{aligned} G_0 = \ &\mathrm{Gal}(L/K) \supset G_1 \supset G_2 \supset \cdots \supset G_i \cdots \\ K = K_0 &\subset K_1 \subset K_2 \subset \cdots \subset K_i \cdots \end{aligned}$$

G_1, the inertia subgroup, corresponds to the subextension of minimal ramification.

The descents considered are illustrated by the following diagrams:

$$\begin{tikzcd}
 & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_8[\frac{1}{2}])} & \\
 & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_4[\frac{1}{2}])} \arrow[u, "\mathcal{F}_0^{k_8/k_4,2/2}"] & \boxed{\mathcal{H}^{\mathcal{MT}(\mathcal{O}_4)}} \arrow[l, "\mathcal{F}_0^{k_4/k_4,2/1}"'] \arrow[uul, bend right, "\mathcal{F}_0^{k_8/\mathbb{Q},2/1}"'] \\
\mathcal{H}^{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])} \arrow[u, "\mathcal{F}_0^{k_4/\mathbb{Q},2/2}"] & & \mathcal{H}^{\mathcal{MT}(\mathbb{Z})}, \arrow[ul, "\mathcal{F}_0^{k_4/\mathbb{Q},2/1}"] \arrow[l, "\mathcal{F}_0^{\mathbb{Q}/\mathbb{Q},2/1}"]
\end{tikzcd}$$

Figure 3.3: The cases $N = 1, 2, 4, 8$.

$$\begin{tikzcd}
 & & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_6)} \\
\mathcal{H}^{\mathcal{MT}(\mathcal{O}_3[\frac{1}{3}])} & & \boxed{\mathcal{H}^{\mathcal{MT}(\mathcal{O}_3)}} \arrow[ll, "\mathcal{F}_0^{k_3/k_3,3/1}"'] \arrow[u, dotted] \\
\boxed{\mathcal{H}^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}} \arrow[u, "\mathcal{F}_0^{k_3/\mathbb{Q},3/3}"] & & \mathcal{H}^{\mathcal{MT}(\mathbb{Z})} \arrow[ul, "\mathcal{F}_0^{k_3/\mathbb{Q},3/1}"] \arrow[uu, bend right, "\mathcal{F}_0^{k_6/\mathbb{Q},1/1}"']
\end{tikzcd}$$

Figure 3.4: The cases $N = 1, 3, \text{‘}6\text{’}$.

Remarks:

- The vertical arrows represent the change of field and the horizontal arrows the change of ramification. The full arrows are the descents made explicit in Chapter 5.
  More precisely, for each arrow $A \overset{\mathcal{F}_0}{\leftarrow} B$ in the above diagrams, we give a basis $\mathcal{B}_n^A$ of $\mathcal{H}_n^A$, and a basis of $\mathcal{H}_n^B = \mathcal{F}_0\mathcal{H}_n^A$ in terms of the elements of $\mathcal{B}_n^A$; similarly for the higher level of these filtrations.
- The framed spaces $\mathcal{H}^{\cdots}$ appearing in these diagrams are not known to be associated to a fundamental group and there is presently no other known way to reach these (motivic) periods. For instance, we obtain by descent, a basis for $\mathcal{H}_n^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}$ in terms of the basis of $\mathcal{H}_n^{\mathcal{MT}(\mathcal{O}_3[\frac{1}{3}])}$.

`Example: Descent between Euler sums and MZV.` The comodule $\mathcal{H}^1$ embeds, non-canonically, into $\mathcal{H}^2$. Let first point out that:[I] $D_1(\mathcal{H}^1) = 0$; the Galois descent between $\mathcal{H}^2$ and $\mathcal{H}^1$ is precisely measured by D_1:

Theorem. *Let $\mathfrak{z} \in \mathcal{H}^2$, a motivic Euler sum. Then:*

$$\mathfrak{z} \in \mathcal{H}^1, \text{ i.e. is a motivic MZV} \iff D_1(\mathfrak{z}) = 0 \text{ and } D_{2r+1}(\mathfrak{z}) \in \mathcal{H}^1.$$

[I] Since all the motivic iterated integrals with only $0, 1$ of length 1 are zero by properties stated in §2.3, hence the left side of D_1, defined in (2.45), would always cancel.

This is a useful recursive criterion to determine if a (motivic) Euler sum is in fact a (motivic) multiple zeta value. It can be generalized for other roots of unity, as we state more precisely in §5.1. These unramified motivic Euler sums are the 0th-level of the filtration by the motivic level here defined as:

$$\mathcal{F}_i\mathcal{H}^2 \text{ is the largest sub-module such that } \mathcal{F}_i/\mathcal{F}_{i-1} \text{ is killed by } D_1.$$

Results. More precisely, for $N \in \{2, 3, 4, \text{`6'}, 8\}$, we define a particular family $\mathcal{B}^N$ of motivic multiple zeta values relative to μ_N with different notions of ***level*** on the basis elements, one for each Galois descent considered above:

$$\mathcal{B}^N := \left\{ \zeta^{\mathfrak{m}} \begin{pmatrix} x_1, \cdots x_{p-1}, x_p \\ \epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_p \xi_N \end{pmatrix} (2\pi i)^{s,\mathfrak{m}} \ , \ x_i \in \mathbb{N}^*, s \geq 0, \begin{cases} x_i \geq 1 \text{ odd } , \ \epsilon_i = 1 \text{ and } s \text{ even if } N = 2 \\ x_i \geq 1 \ , \ \epsilon_i = 1 \text{ if } N = 3, 4 \\ x_i > 1 \ , \ \epsilon_i = 1 \text{ if } N = 6 \\ x_i \geq 1 \ , \ \epsilon_i = \pm 1 \text{ if } N = 8 \end{cases} . \right\} \tag{3.10}$$

Denote by $\mathcal{B}_{n,p,i}$ the subset of elements with weight n, depth p and level i.

`Examples`:

$\cdot N = 2$**:** The basis for motivic Euler sums: $\mathcal{B}^2 := \left\{ \zeta^{\mathfrak{m}} \left(\begin{smallmatrix} 2y_1+1, \ldots, 2y_p+1 \\ 1, 1, \ldots, 1, -1 \end{smallmatrix} \right) \zeta^{\mathfrak{m}}(2)^s, y_i \geq 0, s \geq 0 \right\}$. The level for the descent from $\mathcal{H}^2$ to $\mathcal{H}^1$ is defined as the number of $y_i's$ equal to 0.

$\cdot N = 4$**:** The basis is: $\mathcal{B}^4 := \left\{ \zeta^{\mathfrak{m}} \left(\begin{smallmatrix} x_1, \ldots, x_p \\ 1, 1, \ldots, 1, \sqrt{-1} \end{smallmatrix} \right) (2\pi i)^{s,\mathfrak{m}}, s \geq 0, x_i > 0 \right\}$.

The level is: $\begin{array}{l} \cdot \text{ the number of even } x_i's \text{ for the descent from } \mathcal{H}^4 \text{ to } \mathcal{H}^2 \\ \cdot \text{ the number of even } x_i's + \text{ the number of } x_i's \text{ equal to 1 for the descent from } \mathcal{H}^4 \text{ to } \mathcal{H}^1 \end{array}$.

$\cdot N = 8$**:** the level includes the number of $\epsilon_i's$ equal to -1, etc.

The quotients $\mathcal{H}^{\geq i}$, respectively filtrations $\mathcal{F}_i$ associated to the descent $\lceil$, will match with the sub-families (level restricted) $\mathcal{B}_{n,p,\geq i}$, respectively $\mathcal{B}_{n,p,\leq i}$. Indeed, we prove: [I]

Theorem. *With* $\mathbb{Z}_{1[P]} := \left\{ \frac{a}{1+bP}, a, b \in \mathbb{Z} \right\}$ *where* $P := \begin{cases} 2 & \text{for } N = 2, 4, 8 \\ 3 & \text{for } N = 3, \text{`6'} \end{cases}$.

- $\mathcal{B}_{n,\leq p,\geq i}$ *is a basis of* $\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}$ *and* $\mathcal{B}_{n,\cdot,\geq i}$ *a basis of* $\mathcal{H}_n^{\geq i}$.
- $\mathcal{B}_{n,p,\geq i}$ *is a basis of* $gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}$ *on which it defines a* $\mathbb{Z}_{1[P]}$*-structure:*

  Each $\zeta^{\mathfrak{m}} \left(\begin{smallmatrix} z_1, \ldots, z_p \\ \epsilon_1, \ldots, \epsilon_p \end{smallmatrix} \right)$ *decomposes in* $gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}$ *as a* $\mathbb{Z}_{1[P]}$*-linear combination of* $\mathcal{B}_{n,p,\geq i}$ *elements.*
- *We have the two split exact sequences in bijection:*

$$0 \longrightarrow \mathcal{F}_i\mathcal{H}_n \longrightarrow \mathcal{H}_n \overset{\pi_{0,i+1}}{\to} \mathcal{H}_n^{\geq i+1} \longrightarrow 0$$

$$0 \to \langle \mathcal{B}_{n,\cdot,\leq i} \rangle_{\mathbb{Q}} \to \langle \mathcal{B}_n \rangle_{\mathbb{Q}} \to \langle \mathcal{B}_{n,\cdot,\geq i+1} \rangle_{\mathbb{Q}} \to 0.$$

- *A basis for the filtration spaces* $\mathcal{F}_i\mathcal{H}_n$*:*

$$\cup_p \left\{ \mathfrak{Z} + cl_{n,\leq p,\geq i+1}(\mathfrak{Z}), \mathfrak{Z} \in \mathcal{B}_{n,p,\leq i} \right\},$$

where $cl_{n,\leq p,\geq i} : \langle \mathcal{B}_{n,p,\leq i-1} \rangle_{\mathbb{Q}} \to \langle \mathcal{B}_{n,\leq p,\geq i} \rangle_{\mathbb{Q}}$ *such that* $\mathfrak{Z} + cl_{n,\leq p,\geq i}(\mathfrak{Z}) \in \mathcal{F}_{i-1}\mathcal{H}_n$.

[I] Cf. Theorem 5.2.4 slightly more precise.

· *A basis for the graded space $gr_i\mathcal{H}_n$:*

$$\cup_p \left\{\mathfrak{Z} + cl_{n,\leq p,\geq i+1}(\mathfrak{Z}), \mathfrak{Z} \in \mathcal{B}_{n,p,i}\right\}.$$

`Nota Bene`: The morphism $cl_{n,\leq p,\geq i+1}$ satisfying those conditions is unique.

The linear independence is obtained first in the depth graded, and the proof relies on the bijectivity of the following map $\partial^{i,\lceil}_{n,p}$ by an argument involving 2 or 3 adic properties:[I]

$$\partial^{i,\lceil}_{n,p} : gr^{\mathfrak{D}}_p \mathcal{H}^{\geq i}_n \to \oplus_{r<n} \left(gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i-1}_{n-r}\right)^{\oplus c^{\lceil}_r} \oplus_{r<n} \left(gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i}_{n-r}\right)^{\oplus c^{\backslash\lceil}_r}, \; c^{\lceil}_r, c^{\backslash\lceil}_r \in \mathbb{N}, \qquad (3.11)$$

which is obtained from the depth and weight graded part of the coaction, followed by a projection for the left side (by depth 1 results), and by passing to the level quotients ((5.7)). Once the freeness obtained, the generating property is obtained from counting dimensions, since K-theory gives an upper bound for the dimensions.

This main theorem generalizes in particular a result of P. Deligne ([31]), which we could formulate by different ways: [II]

Corollary. · *The map $\mathcal{G}^{\mathcal{MT}} \to \mathcal{G}^{\mathcal{MT}'}$ is an isomorphism.*

· *The motivic fundamental group $\pi_1^{\mathfrak{m}}\left(\mathbb{P}^1\diagdown\{0,\mu_N,\infty\},\overline{0\xi_N}\right)$ generates the category of mixed Tate motives $\mathcal{MT}_N$.*

· *$\mathcal{B}_n$ is a basis of $\mathcal{H}^N_n$, the space of motivic MZV relative to μ_N.*

· *The geometric (and Frobenius invariant if $N=2$) motivic periods of $\mathcal{MT}_N$ are $\mathbb{Q}$-linear combinations of motivic MZV relative to μ_N (unramified for $N=$ '6').*

Remarks:

· For $N=$ '6' the result remains true if we restrict to iterated integrals relative not to all 6th roots of unity but only to these relative to primitive roots.

· We could even restrict to: $\begin{array}{ll} \pi_1^{\mathfrak{m}}\left(\mathbb{P}^1\diagdown\{0,1,\infty\},\overline{0\xi_N}\right) & \text{for } N=2,3,4,\text{'6'} \\ \pi_1^{\mathfrak{m}}\left(\mathbb{P}^1\diagdown\{0,\pm1,\infty\},\overline{0\xi_N}\right) & \text{for } N=8 \end{array}$.

The previous theorem also provides the Galois descent from $\mathcal{H}^{\mathcal{MT}_N}$ to $\mathcal{H}^{\mathcal{MT}_{N'}}$:

Corollary. *A basis for $MMZV_{\boldsymbol{\mu_{N'}}}$ is formed by $MMZV_{\boldsymbol{\mu_N}} \in \mathcal{B}^N$ of level 0 each corrected by a $\mathbb{Q}$-linear combination of $MMZV_{\boldsymbol{\mu_N}}$ of level greater than or equal to 1:*

$$\textit{Basis of } \mathcal{H}^{N'}_n : \quad \left\{\mathfrak{Z} + cl_{n,\cdot,\geq 1}(\mathfrak{Z}), \mathfrak{Z} \in \mathcal{B}^N_{n,\cdot,0}\right\}.$$

Remark: Descent can be calculated explicitly in small depth, less than or equal to 3, as we explain in the Appendix $A.2$. In the general case, we could make the part of maximal depth of $cl(\mathfrak{Z})$ explicit (by inverting a matrix with binomial coefficients) but motivic methods do not enable us to describe the other coefficients for terms of lower depth.

[I] The first $c^{\lceil}_r$ components of $\partial^{i,\lceil}_{n,p}$ correspond to the derivations in $\mathscr{D}^{\lceil}$ associated to the descent, which hence decrease the motivic level.

[II] The basis $\mathcal{B}$, in the cases where $N \in \{3,4,8\}$ is identical to P. Deligne's in [31]. For $N=2$ (resp. $N=$ '6' unramified) it is a linear basis analogous to his algebraic basis which is formed by Lyndon words in the odd (resp. ≥ 2) positive integers (with $\ldots 5 \leq 3 \leq 1$); a Lyndon word being strictly smaller in lexicographic order than all of the words formed by permutation of its letters. Deligne's method is roughly dual to this point of view, working in Lie algebras, showing the action is faithful and that the descending central series of $\mathcal{U}$ is dual to the depth filtration.

`Example,` $N=2$: A basis for motivic multiple zeta values is formed by:

$$\left\{ \zeta^{\mathfrak{m}}\left(2x_1+1,\ldots,\overline{2x_p+1}\right)\zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{\exists i, y_i=0\\ q\leq p}} \alpha^{\mathbf{x}}_{\mathbf{y}}\zeta^{\mathfrak{m}}(2y_1+1,\ldots,\overline{2y_q+1})\zeta^{\mathfrak{m}}(2)^s \ , \ x_i>0, \alpha^{\mathbf{x}}_{\mathbf{y}}\in\mathbb{Q} \right\}.$$

Starting from a motivic Euler sum with odds greater than 1, we add some *corrections terms*, in order to get an element in $\mathcal{H}^1$, the space of MMZV. At this level, correction terms are motivic Euler sums with odds, and at least one 1 in the arguments; i.e. they are of level ≥ 1 with the previous terminology. For instance, the following linear combination is a motivic MZV:

$$\zeta^{\mathfrak{m}}(3,3,\overline{3}) + \frac{774}{191}\zeta^{\mathfrak{m}}(1,5,\overline{3}) - \frac{804}{191}\zeta^{\mathfrak{m}}(1,3,\overline{5}) - 6\zeta^{\mathfrak{m}}(3,1,\overline{5}) + \frac{450}{191}\zeta^{\mathfrak{m}}(1,1,\overline{7}).$$

3.3 Miscellaneous Results *[Chapter 6]*

Chapter 6 is devoted on the Hopf algebra structure of motivic multiple zeta values relative to μ_N, particularly for $N=1,2$, presenting various uses of the coaction, and divided into sections as follows:

1. An important use of the coaction, is the decomposition of (motivic) multiple zeta values into a conjectured basis, as explained in [19]. It is noteworthy to point out that the coaction always enables us to determine the coefficients of the maximal depth terms. We consider in §6.1 two simple cases, in which the space $gr^{\mathfrak{D}}_{max}\mathcal{H}_n$ is 1 dimensional:

   (i) For $N=1$, when the weight is a multiple of 3 ($w=3d$), such that the depth $p>d$:[I]
   $$gr^{\mathfrak{D}}_p\mathcal{H}_{3d} = \mathbb{Q}\zeta^{\mathfrak{m}}(3)^d.$$

   (ii) For $N=2,3,4$, when weight equals depth:
   $$gr^{\mathfrak{D}}_p\mathcal{H}_p = \mathbb{Q}\zeta^{\mathfrak{m}}\begin{pmatrix}1\\ \xi_N\end{pmatrix}^p.$$

   The corresponding Lie algebra, called the *diagonal Lie algebra*, has been studied by Goncharov in [45], [46].

   In these cases, we are able to determine the projection:
   $$\vartheta : gr^{\mathfrak{D}}_{max}\mathcal{H}^N_n \to \mathbb{Q},$$
   either via the linearized Ihara action $\underline{\circ}$, or via the dual point of view of infinitesimal derivations D_r. For instance, for (i) ($N=1$, $w=3d$), it boils down to look at:
   $$\frac{D_3^{\circ d}}{d!} \quad \text{or} \quad \exp_\circ(\overline{\sigma}_3), \text{ where } \overline{\sigma}_{2i+1} = (-1)^i(\mathrm{ad}e_0)^{2i}(e_1)^{\mathrm{II}}.$$

   In general, the space $gr^{\mathfrak{D}}_{max}\mathcal{H}^N_n$ is more than 1-dimensional; nevertheless, these methods could be generalized.

[I] This was a question asked for by D. Broadhurst: an algorithm, or a formula for the coefficient of $\zeta(3)^d$ of such a MZV, when decomposed in Deligne basis.

[II] These $\overline{\sigma}_{2i+1}$ are the generators of $gr^{\mathfrak{D}}_1\mathfrak{g}^{\mathfrak{m}}$, the depth 1 graded part of the motivic Lie algebra; cf. (2.50).

2. Using criterion 5.1.3, we provide in the second section infinite families of honorary motivic multiple zeta values up to depth 5, with specified alternating odd or even integers. It was inspired by some isolated examples of honorary multiple zeta values found by D. Broadhurst[I], such as $\zeta(\overline{8},5,\overline{8}),\zeta(\overline{8},3,\overline{10}),\zeta(3,\overline{6},3,\overline{6},3)$, where we already could observe some patterns of even and odd. Investigating this trail in a systematic way, looking for any general families of unramified (motivic) Euler sums (without linear combinations first), we arrive at the families presented in §6.2, which unfortunately, stop in depth 5. However, this investigation does not cover the unramified $\mathbb{Q}$-linear combinations of motivic Euler sums, such as those presented in Chapter 4, Theorem 4.3.1 (motivic Euler $\sharp$ sums with positive odds and negative even integers).

3. By Corollary 2.4.5, we can lift some identities between MZV to motivic MZV (as in [20], Theorem 4.4), and similarly in the case of Euler sums. Remark that, as we will see for depth 1 Hoffman $\star$ elements (Lemma 4.4.3), the lifting may not be straightforward, if the family is not stable under the coaction. In this section §6.3, we list some identities that we are able to lift to motivic versions, in particular some *Galois trivial* elements[II] or product of simple zetas, and sum identities.

Remark: The stability of a family on the coaction is a precious feature that allows to prove easily (by recursion) properties such as linear independence[III], Galois descent features (unramified for instance), identities (§6.3), etc.

3.4 And Beyond?

For most of N values, the situation concerning the periods of $\mathcal{MT}_{\Gamma_N}\subset\mathcal{MT}(\mathcal{O}_N[\frac{1}{N}])$ is still hazy, although it has been studied in several articles, notably by Goncharov ([45],[46], [47][IV]) and Zhao: some bounds on dimensions, tables in small weight, and other results and thoughts on cyclotomic MZV can be seen in [85], [84], [29].

`Nota Bene`: As already pointed out, as soon as N has a non inert prime factor p[V], $\mathcal{MT}_{\Gamma_N}\subsetneq\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{N}\right])$. Hence, some motivic periods of $\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{N}\right])$ are not motivic iterated integrals on $\mathcal{P}^1\diagdown\{0,\mu_N,\infty\}$ as considered above; already in weight 1, there are more generators than the logarithms of cyclotomic units $\log^{\mathfrak{m}}(1-\xi_N^a)$.

Nevertheless, we can *a priori* split the situation (of $\mathcal{MT}_{\Gamma_N}$) into two main schemes:

(i) As soon as N has two distinct prime factors, or N power of 2 or 3, it is commonly believed that the motivic fundamental group $\pi_1^{\mathfrak{m}}(\mathbb{P}^1\diagdown\{0,\infty,\mu_N\},\overrightarrow{01})$ generates $\mathcal{MT}_{\Gamma_N}$,

---

[I] Those emerged when looking at the *depth drop phenomena*, cf. [14].

[II] Galois trivial here means that the unipotent part of the Galois group acts trivially, not $\mathbb{G}_m$; hence not strictly speaking Galois trivial.

[III] If we find an appropriate filtration respected by the coaction, and such as the 0 level elements are Galois-trivial, it corresponds then to the motivic depth filtration; for the Hoffman ($\star$) basis it is the number of 3; for the Euler $\sharp$ sums basis, it is the number of odds, also equal to the depth minus 1; for Deligne basis relative to μ_N, $N=2,3,4,$ '6', 8, it is the usual depth.

[IV] Goncharov studied the structure of the fundamental group of $\mathbb{G}_m\diagdown\mu_N$ and made some parallels with the topology of some modular variety for $GL_{m,/\mathbb{Q}}$, $m>1$ notably. He also proved, for $N=p\geq 5$, that the following morphism, given by the Ihara bracket, is not injective:

$$\beta:\bigwedge^2 gr_1^{\mathfrak{D}}\mathfrak{g}_1^{\mathfrak{m}}\to gr_2^{\mathfrak{D}}\mathfrak{g}_2^{\mathfrak{m}}\quad\text{and}\quad\begin{array}{ll}\dim\bigwedge^2 gr_1^{\mathfrak{D}}\mathfrak{g}_1^{\mathfrak{m}} & =\frac{(p-1)(p-3)}{8}\\ \dim\ker\beta & =\frac{p^2-1}{24}\\ \dim Im\beta & =\dim gr^{\mathfrak{D}}\mathfrak{g}_2^{\mathfrak{m}}=\frac{(p-1)(p-5)}{12}\\ \dim gr^{\mathfrak{D}}\mathfrak{g}_3^{\mathfrak{m}} & \geq\frac{(p-5)(p^2-2p-11)}{48}.\end{array}$$

Note that $gr_2^{\mathfrak{D}}\mathfrak{g}^{\mathfrak{m}}$ corresponds to the space generated by $\zeta^{\mathfrak{m}}\left(\begin{smallmatrix}1,1\\ \epsilon_1,\epsilon_2\end{smallmatrix}\right)$ quotiented by dilogarithms $\zeta^{\mathfrak{m}}\left(\begin{smallmatrix}2\\ \epsilon\end{smallmatrix}\right)$, modulo torsion.

[V] In particular, as soon as $N\neq p^s,2p^s,4p^s,p^sq^k$ for p,q odd prime since $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic $\Leftrightarrow m=2,4,p^k,2p^k$.

even though no suitable basis has been found. Also, in these cases, Zhao conjectured there were non standard relations[I]. Nevertheless, in the case of N power of 2 or power of 3, there seems to be a candidate for a basis (3.12) and some linearly independent families were exhibited:

$$\zeta^{\mathfrak{m}}\begin{pmatrix} n_1, \cdots n_{p-1}, n_p \\ \epsilon_1, \ldots, \epsilon_{p-1}, \epsilon_p \end{pmatrix} \quad \text{with } \epsilon_p \in \mu_N \quad \text{primitive} \quad \text{and } (\epsilon_i)_{i<p} \quad \text{non primitive.} \tag{3.12}$$

Indeed, when N is a power of 2 or 3, linearly independent subfamilies of 3.12, keeping $\frac{3}{4}$ resp. $\frac{2}{3}$ generators in degree 1, and all generators in degree $r > 1$ are presented in [79] (in a dual point of view of the one developed here).

`Nota Bene`: Some subfamilies of 3.12, restricting to $\{\epsilon_i = 1, x_i \geq 2\}$ (here ϵ_p still as above) can be easily proven (via the coaction, by recursion on depth) to be linearly independent for any N; if N is a prime power, we can widen to $x_i \geq 1$, and for N even to $\epsilon_i \in \{\pm 1\}$; nevertheless, these families are considerably *small*.

(ii) For $N = p^s$, p prime greater than 5, there are *missing periods*: i.e. it is conjectured that the motivic fundamental group $\pi_1^{\mathfrak{m}}(\mathbb{P}^1 \diagdown \{0, \infty, \mu_N\}, \overrightarrow{01})$ does not generate $\mathcal{MT}_{\Gamma_N}$. For $N = p \geq 5$, it can already be seen in weight 2, depth 2. More precisely, (taking the dual point of view of Goncharov in [46]), the following map is not surjective:

$$\begin{array}{llll} D_1 : & gr_2^{\mathfrak{D}}\mathcal{A}_2 & \to & \mathcal{A}_1 \otimes \mathcal{A}_1 \\ & \zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1,1 \\ \xi^a, \xi^b \end{smallmatrix}\right) & \mapsto & (a) \otimes (b) + \delta_{a+b\neq 0}((b) - (a)) \otimes (a+b) \end{array}, \text{ where } \boldsymbol{(a)} := \zeta^{\mathfrak{a}}\begin{pmatrix} 1 \\ \xi^a \end{pmatrix}. \tag{3.13}$$

These missing periods were a motivation for instance to introduce Aomoto polylogarithms (in [36])[II].

Another idea, in order to reach these missing periods would be to use Galois descents: coming from a category above, in order to arrive at the category underneath, in the manner of Chapter 5. For instance, missing periods for $N = p$ prime > 5, could be reached via a Galois descent from the category $\mathcal{MT}_{\Gamma_{2p}}$ [III]. First, let point out that this category has the same dimensions than $\mathcal{MT}_p$ in degree > 1, and has one more generator in degree 1, corresponding to $\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^p \end{smallmatrix}\right)$. Furthermore, for p prime, the descent between $\mathcal{H}^{2p}$ and $\mathcal{H}^p$ is measured by D_1^p, the component of D_1 associated to $\zeta^{\mathfrak{a}}\left(\begin{smallmatrix} 1 \\ \xi^p \end{smallmatrix}\right)$:

$$\text{Let } \mathfrak{Z} \in \mathcal{H}^{2p}, \text{ then } \mathfrak{Z} \in \mathcal{H}^p \Leftrightarrow \begin{cases} D_1^p(\mathfrak{Z}) = 0 \\ D_r(\mathfrak{Z}) \in \mathcal{H}^p \end{cases}$$

The situation is pictured by:

$$\begin{tikzcd} \mathcal{H}^{2p} := \mathcal{H}^{\Gamma_{2p}} \arrow[r, hook] & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_{2p}[\frac{1}{2p}])} \\ \mathcal{H}^p := \mathcal{H}^{\Gamma_p} = \mathcal{H}^{\mathcal{MT}(\mathcal{O}_p[\frac{1}{p}])} \arrow[u, "D_1^p"] \arrow[r, equal] & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_{2p}[\frac{1}{p}])} \\ \mathcal{H}^{\mathcal{MT}(\mathcal{O}_p)} \arrow[u, "\{D_1^{2a} - D_1^a\}_{2 \leq a \leq \frac{p-1}{2}}"] \arrow[r, equal] & \mathcal{H}^{\mathcal{MT}(\mathcal{O}_{2p})} \end{tikzcd} . \tag{3.14}$$

[I] Non standard relations are these which do not come from distribution, conjugation, and regularised double shuffle relation, cf. [84]

[II] Aomoto polylogarithms generalize the previous iterated integrals, with notably differential forms such as $\frac{dt_i}{t_i - t_{i+1} - a_i}$; there is also a coaction acting on them.

[III] This category is equal to $\mathcal{MT}(\mathcal{O}_{2p}([\frac{1}{2p}]))$ iff 2 is a primitive root modulo p. Some conditions on p necessary or sufficient are known: this implies that $p \equiv 3, 5 \pm \mod 8$; besides, if $p \equiv 3, 11 \mod 16$, it is true, etc.

`Example, for N=5`: A basis of $gr_p^{\mathfrak{D}}\mathcal{A}_1$ corresponds to the logarithms of the roots of unity ξ^1, ξ^2; here, $\xi = \xi_5$ is a primitive fifth root of unity. Moreover, the image of $D_1 : \mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathcal{A}_1$ on $\zeta^{\mathfrak{a}}\left(\begin{smallmatrix}1,1\\ \xi_5^a,\xi_5^b\end{smallmatrix}\right)$ is (cf. 3.13):

$$\mathrm{Vec}_{\mathbb{Q}}\left\langle (1)\otimes(1), (2)\otimes(2), (1)\otimes(2)+(2)\otimes(1)\right\rangle .$$

We notice that one dimension is missing (3 instead of 4). Allowing the use of tenth roots of unity, adding for instance here in depth 2, $\zeta^{\mathfrak{a}}\left(\begin{smallmatrix}1,1\\ \xi_{10}^1,\xi_{10}^2\end{smallmatrix}\right)$ recovers the surjection for D_1. Since we have at our disposal criterion to determine if a $\mathrm{MMZV}_{\mu_{10}}$ is in $\mathcal{H}^5$, we could imagine constructing a base of $\mathcal{H}^5$ from tenth roots of unity.
`Nota Bene`: More precisely, we have the following spaces, descents and dimensions:

$$\begin{array}{ccc}
\mathcal{H}^{\mathcal{MT}(\mathcal{O}_{10}[\frac{1}{10}])} = \mathcal{H}^{\Gamma_{10}} & & \\
\uparrow{\scriptstyle D_1^5} & & \\
\mathcal{H}^{\mathcal{MT}(\mathcal{O}_{10}[\frac{1}{5}])} = \mathcal{H}^{\mathcal{MT}(\mathcal{O}_5[\frac{1}{5}])} = \mathcal{H}^{\Gamma_5} & & d_n = 2d_{n-1} + 3d_{n-2} = 3d_{n-1} \\
\uparrow{\scriptstyle D_1^4+D_1^2} & & \\
\mathcal{H}^{\mathcal{MT}(\mathcal{O}_5)} = \mathcal{H}^{\mathcal{MT}(\mathcal{O}_{10})} & & d'_n = 2d'_{n-1} + d'_{n-2}
\end{array} \tag{3.15}$$

Remarks:

- Recently (in [15]), Broadhurst made some conjectures about *multiple Landen values*, i.e. periods associated to the ring of integers of the real subfield of $\mathbb{Q}(\xi_5)$, i.e. $\mathbb{Z}[\rho]$, with $\rho := \frac{1+\sqrt{5}}{2}$, the golden ratio[I]. Methods presented through this thesis could be transposed in such context.

- It also worth noticing that, for $N = p > 5$, modular forms obstruct the freeness of the Lie algebra $gr_{\mathfrak{D}}\mathfrak{g}^{\mathfrak{m}}$[II], as in the case of $N = 1$ (cf. [21]). Indeed, for $N = 1$ one can associate, to each cuspidal form of weight n, a relation between weight n double and simple multiple zeta values, cf. [41]. Notice that, on the contrary, for $N = 2, 3, 4, 8$, $gr_{\mathfrak{D}}\mathfrak{g}^{\mathfrak{m}}$ is free. This fascinating connection with modular forms still waits to be explored for cyclotomic MZV. [III]

- In these cases where $gr_{\mathfrak{D}}\mathfrak{g}^{\mathfrak{m}}$ is not free, since we have to turn towards other basis (than 3.12), we may remember the Hoffman basis (of $\mathcal{H}_n^1$, cf [20]): $\{\zeta^{\mathfrak{m}}(\{2,3\}^{\times})\}_{\text{weight } n}$, whose dimensions verify $d_n = d_{n-2} + d_{n-3}$. Looking at dimensions in Lemma 2.3.1, two cases bring to mind a basis in the ***Hoffman's way***:

  (*i*) For $\mathcal{MT}(\mathcal{O}_N)$, since $d_n = \frac{\varphi(N)}{2}d_{n-1} + d_{n-2}$, this suggests to look for a basis with **1** (with $\frac{\varphi(N)}{2}$ choices of Nth roots of unity) and **2** (1 choice of Nth roots of unity).

  (*ii*) For $\mathcal{MT}\left(\mathcal{O}_{p^r}\left[\frac{1}{p}\right]\right)$, where $p \mid N$ and p inert, since $d_n = \left(\frac{\varphi(N)}{2}+1\right)^n$, this suggests a basis with only 1 above, and $(\frac{\varphi(N)}{2}+1)$ choices of Nth roots of unity; in particular if $N = p^k$.

[I] He also looked at the case of the real subfield of $\mathbb{Q}(\xi_7)$ in his latest article: [16]

[II] Goncharov proved that the subspace of cuspidal forms of weight 2 on the modular curve $X_1(p)$ (associated to $\Gamma_1(p)$), of dimension $\frac{(p-5)(p-7)}{24}$ embeds into $\ker\beta$, for $N = p \geq 11$ which leaves another part of dimension $\frac{p-3}{2}$.

[III] We could hope also for an interpretation, in these cyclotomic cases, of exceptional generators and relations in the Lie algebra, in the way of [21] for $N = 1$.

`Example`: For $N=2$, the recursion relation for dimensions $d_n = d_{n-1} + d_{n-2}$ of $\mathcal{H}_n^2$ suggests, *in the Hoffman's way*, a basis composed of motivic Euler sums with only 1 and 2. For instance, the following are candidates conjectured to be a basis, supported by numerical computations:

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\dots,n_{p-1},n_p \\ 1,\dots,1,-1\end{pmatrix}, n_i\in\{2,1\}\right\},\ \text{OR}\ \left\{\zeta^{\mathfrak{m}}\begin{pmatrix} 1,\cdots 1, \\ \boldsymbol{s},-1\end{pmatrix}\zeta^{\mathfrak{m}}(2)^{\bullet}, \boldsymbol{s}\in\{\{1\},\{-1,-1\}\}^*\right\}.$$

However, there is not a nice *suitable* filtration[I] corresponding to the motivic depth which would allow a recursive proof [II].

[I] In the second case, it appears that we could proceed as follows to show the linear independence of these elements, where p equals 1+ the number of 1 in the E_n element: Prove that, for $x\in E_{n,p}$ there exists a linear combination $cl(x)\in E_{n,>p}$ such that $x+cl(x)\in\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n$, and then that $\{x+cl(x), x\in E_{n,p}\}$ is precisely a basis for $gr_p^{\mathfrak{D}}\mathcal{H}_n$, considering, for $2r\leq n-p$:

$$D_{2r+1}: gr_p^{\mathfrak{D}}\mathcal{H}_n\to gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-2r-1}.$$

[II] A suitable filtration, whose level 0 would be the power of π, level 1 would be linear combinations of $\zeta(odd)\cdot\zeta(2)^{\bullet}$, etc.; as in proofs in §4.5.1.

Chapter 4

MZV $\star$ and Euler $\sharp$ sums

`Contents:` After introducing motivic Euler $\star$, and $\sharp$ sums, with some useful motivic relations (antipodal and hybrid), the third section focuses on some specific Euler $\sharp$ sums, starting by a broad subfamily of *unramified* elements (i.e. which are motivic MZV) and extracting from it a new basis for $\mathcal{H}^1$. The fourth section deals with the Hoffman star family, proving it is a basis of $\mathcal{H}^1$, up to an analytic conjecture (4.4.4). In Appendix §4.7, some missing coefficients in Lemma 4.4.3, although not needed for the proof of the Hoffman $\star$ Theorem 4.4.1, are discussed. The last section presents a conjectured motivic equality (4.5.1) which turns each motivic MZV $\star$ into a motivic Euler $\sharp$ sums of the previous honorary family; in particular, under this conjecture, the two previous bases are identical. The proofs here are partly based on results of Annexe §A.1, which themselves use relations presented in §4.2.

4.1 Star, Sharp versions

Here are the different variants of motivic Euler sums (MES) used in this chapter, where a $\pm\star$ resp. $\pm\sharp$ in the notation below $I(\cdots)$ stands for a $\omega_{\pm\star}$ resp. $\omega_{\pm\sharp}$ in the iterated integral:[I]

Definition 4.1.1. *Using the expression in terms of motivic iterated integrals (*1.3*), motivic Euler sums are, with $n_i \in \mathbb{Z}^*$, $\epsilon_i := sign(n_i)$:*

$$\zeta_k^{\mathfrak{m}}(n_1,\dots,n_p) := (-1)^p I^{\mathfrak{m}}\left(0;0^k,\epsilon_1\cdots\epsilon_p,0^{|n_1|-1},\dots,\epsilon_i\cdots\epsilon_p,0^{|n_i|-1},\dots,\epsilon_p,0^{|n_p|-1};1\right). \tag{4.1}$$

With the differentials: $\omega_{\pm\star} := \omega_{\pm 1} - \omega_0 = \dfrac{dt}{t(\pm t-1)}$, $\qquad \omega_{\pm\sharp} := 2\omega_{\pm 1} - \omega_0 = \dfrac{(t\pm 1)dt}{t(t\mp 1)}$,

MES $\star$ *are defined similarly than* (4.1) *with $\omega_{\pm\star}$ (instead of $\omega_{\pm 1}$), ω_0 and a $\omega_{\pm 1}$ at the beginning:*

$$\zeta_k^{\star,\mathfrak{m}}(n_1,\dots,n_p) := (-1)^p I^{\mathfrak{m}}\left(0;0^k,\epsilon_1\cdots\epsilon_p,0^{|n_1|-1},\epsilon_2\cdots\epsilon_p\star,0^{|n_2|-1},\dots,\epsilon_p\star,0^{|n_p|-1};1\right).$$

MES $\star\star$ *similarly with only $\omega_{\pm\star},\omega_0$ (including the first):*

$$\zeta_k^{\star\star,\mathfrak{m}}(n_1,\dots,n_p) := (-1)^p I^{\mathfrak{m}}\left(0;0^k,\epsilon_1\cdots\epsilon_p\star,0^{|n_1|-1},\epsilon_2\cdots\epsilon_p\star,0^{|n_2|-1},\dots,\epsilon_p\star,0^{|n_p|-1};1\right).$$

MES $\sharp$ *with $\omega_{\pm\sharp},\omega_0$ and a $\omega_{\pm 1}$ at the beginning:*

$$\zeta_k^{\sharp,\mathfrak{m}}(n_1,\dots,n_p) := 2(-1)^p I^{\mathfrak{m}}\left(0;0^k,\epsilon_1\cdots\epsilon_p,0^{|n_1|-1},\epsilon_2\cdots\epsilon_p\sharp,0^{|n_2|-1},\dots,\epsilon_p\sharp,0^{|n_p|-1};1\right).$$

[I] Possibly regularized with (2.28).

MES $\sharp\sharp$ *similarly with only* $\omega_{\pm\sharp}, \omega_0$ *(including the first):*

$$\zeta_k^{\sharp\sharp,\mathfrak{m}}(n_1,\dots,n_p) := (-1)^p I^{\mathfrak{m}}\left(0;0^k,\epsilon_1\cdots\epsilon_p\sharp,0^{|n_1|-1},\epsilon_2\cdots\epsilon_p\sharp,0^{|n_2|-1},\dots,\epsilon_p\sharp,0^{|n_p|-1};1\right).$$

Remarks:

- The Lie algebra of the fundamental group $\pi_1^{dR}(\mathbb{P}^1\smallsetminus\{0,1,\infty\}) = \pi_1^{dR}(\mathcal{M}_{0,4})$ is generated by e_0, e_1, e_∞ with the only condition than $e_0+e_1+e_\infty = 0$[I]. If we keep e_0 and e_∞ as generators, instead of the usual e_0, e_1, it leads towards MMZV ** up to a sign, instead of MMZV since $-\omega_0+\omega_1-\omega_\star = 0$. We could also choose e_1 and e_∞ as generators, which leads to another version of MMZV that has not been much studied yet. These versions are equivalent since each one can be expressed as $\mathbb{Q}$ linear combination of another one.

- By linearity and ⧢-regularisation (2.28), all these versions ($\star$, $\star\star$, $\sharp$ or $\sharp\sharp$) are $\mathbb{Q}$-linear combination of motivic Euler sums. Indeed, with n_+ the number of $+$ among $\circ$:

$$\begin{array}{lcll}
\zeta^{\star,\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{\circ=\text{`}+\text{' or ,}} & \zeta^{\mathfrak{m}}(n_1\circ\cdots\circ n_p)\\
\zeta^{\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{\circ=\text{`}+\text{' or ,}}(-1)^{n_+} & \zeta^{\star,\mathfrak{m}}(n_1\circ\cdots\circ n_p)\\
\zeta^{\sharp,\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{\circ=\text{`}+\text{' or ,}} 2^{p-n_+} & \zeta^{\mathfrak{m}}(n_1\circ\cdots\circ n_p)\\
\zeta^{\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{\circ=\text{`}+\text{' or ,}}(-1)^{n_+}2^{-p} & \zeta^{\sharp,\mathfrak{m}}(n_1\circ\cdots\circ n_p)\\
\zeta^{\star\star,\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{i=0}^{p-1} & \zeta^{\star,\mathfrak{m}}_{|n_1|+\cdots+|n_i|}(n_{i+1},\cdots,n_p)\\
 & = & \sum_{\substack{\circ=\text{`}+\text{' or ,}\\ i=0}}^{p-1} & \zeta^{\mathfrak{m}}_{|n_1|+\cdots+|n_i|}(n_{i+1}\circ\cdots\circ n_p)\\
\zeta^{\sharp\sharp,\mathfrak{m}}(n_1,\dots,n_p) & = & \sum_{i=0}^{p-1} & \zeta^{\sharp,\mathfrak{m}}_{|n_1|+\cdots+|n_i|}(n_{i+1},\cdots,n_p)\\
 & = & \sum_{\substack{\circ=\text{`}+\text{' or ,}\\ i=0}}^{p-1} 2^{p-i-n_+} & \zeta^{\mathfrak{m}}_{|n_1|+\cdots+|n_i|}(n_{i+1}\circ\cdots\circ n_p)\\
\zeta^{\star,\mathfrak{m}}(n_1,\dots,n_p) & = & \zeta^{\star\star,\mathfrak{m}}(n_1,\dots,n_p) & -\zeta^{\star\star,\mathfrak{m}}_{|n_1|}(n_2,\dots,n_p)\\
\zeta^{\sharp,\mathfrak{m}}(n_1,\dots,n_p) & = & \zeta^{\sharp\sharp}(n_1,\dots,n_p) & -\zeta^{\sharp\sharp,\mathfrak{m}}_{|n_1|}(n_2,\dots,n_p)
\end{array}$$

`Notation:` Beware, the '$+$' here is on $n_i\in\mathbb{Z}^*$ is a summation of absolute values while signs are multiplied:

$$n_1\text{`}+\text{'}\cdots\text{`}+\text{'}n_i \to sign(n_1\cdots n_i)(|n_1|+\cdots+|n_i|).$$

`Examples:` Expressing them as $\mathbb{Q}$ linear combinations of motivic Euler sums[II]:

$$\begin{array}{lcl}
\zeta^{\star,\mathfrak{m}}(2,\overline{1},3) & = & -I^{\mathfrak{m}}(0;-1,0,-\star,\star,0,0;1)\\
 & = & \zeta^{\mathfrak{m}}(2,\overline{1},3)+\zeta^{\mathfrak{m}}(\overline{3},3)+\zeta^{\mathfrak{m}}(2,\overline{4})+\zeta^{\mathfrak{m}}(\overline{6})\\
\zeta^{\sharp,\mathfrak{m}}(2,\overline{1},3) & = & -2I^{\mathfrak{m}}(0;-1,0,-\sharp,\sharp,0,0;1)\\
 & = & 8\zeta^{\mathfrak{m}}(2,\overline{1},3)+4\zeta^{\mathfrak{m}}(\overline{3},3)+4\zeta^{\mathfrak{m}}(2,\overline{4})+2\zeta^{\mathfrak{m}}(\overline{6})\\
\zeta^{\star\star,\mathfrak{m}}(2,\overline{1},3) & = & -I^{\mathfrak{m}}(0;-\star,0,-\star,\star,0,0;1)\\
 & = & \zeta^{\star,\mathfrak{m}}(2,\overline{1},3)+\zeta^{\star,\mathfrak{m}}_2(\overline{1},3)+\zeta^{\star,\mathfrak{m}}_3(3)\\
 & = & \zeta^{\star,\mathfrak{m}}(2,\overline{1},3)+\zeta^{\star,\mathfrak{m}}(\overline{3},3)+3\zeta^{\star,\mathfrak{m}}(\overline{2},4)+6\zeta^{\star,\mathfrak{m}}(1,5)-10\zeta^{\star,\mathfrak{m}}(6)\\
 & = & 11\zeta^{\mathfrak{m}}(\overline{6})+2\zeta^{\mathfrak{m}}(\overline{3},3)+\zeta^{\mathfrak{m}}(2,\overline{4})+\zeta^{\mathfrak{m}}(2,\overline{1},3)+3\zeta^{\mathfrak{m}}(\overline{2},4)+6\zeta^{\mathfrak{m}}(\overline{1},5)-10\zeta^{\mathfrak{m}}(6)
\end{array}$$

[I] For the case of motivic Euler sums, it is the Lie algebra generated by $e_0, e_1, e_{-1}, e_\infty$ with the only condition than $e_0+e_1+e_{-1}+e_\infty = 0$; similarly for other roots of unity with e_η. Note that e_i corresponds to the class of the residue around i in $H^1_{dR}(\mathbb{P}^1\smallsetminus\{0,\mu_N,\infty\})^\vee$.

[II] To get rid of the 0 in front of the MZV, as in the last example, we use the shuffle regularisation 2.28.

Stuffle. One of the most famous relations between cyclotomic MZV, the *stuffle* relation, coming from the multiplication of series, has been proven to be *motivic* i.e. true for cyclotomic MMZV, which was a priori non obvious. [I] In particular:

Lemma 4.1.2.

$$\zeta^{\mathfrak{m}}\begin{pmatrix} a_1,\dots,a_r \\ \alpha_1,\dots,\alpha_r \end{pmatrix}\zeta^{\mathfrak{m}}\begin{pmatrix} b_1,\dots,b_s \\ \beta_1,\dots,\beta_s \end{pmatrix} = \sum_{\substack{\binom{c_j}{\gamma_j}=\binom{a_i}{\alpha_i},\binom{b_{i'}}{\beta_{i'}} \text{ or } \binom{a_i+b_{i'}}{\alpha_i\beta_{i'}} \\ \text{order } (a_i),(b_i) \text{ preserved}}} \zeta^{\mathfrak{m}}\begin{pmatrix} c_1,\dots,c_m \\ \gamma_1,\dots,\gamma_m \end{pmatrix}.$$

$$\zeta^{\star,\mathfrak{m}}\begin{pmatrix} a_1,\dots,a_r \\ \alpha_1,\dots,\alpha_r \end{pmatrix}\zeta^{\star,\mathfrak{m}}\begin{pmatrix} b_1,\dots,b_s \\ \beta_1,\dots,\beta_s \end{pmatrix} = \sum_{\substack{\binom{c_j}{\gamma_j}=\binom{a_i}{\alpha_i},\binom{b_i}{\beta_i} \text{ or } \binom{a_i+b_{i'}}{\alpha_i\beta_{i'}} \\ \text{order } (a_i),(b_i) \text{ preserved}}} (-1)^{r+s+m}\zeta^{\star,\mathfrak{m}}\begin{pmatrix} c_1,\dots,c_m \\ \gamma_1,\dots,\gamma_m \end{pmatrix}.$$

$$\zeta^{\sharp,\mathfrak{m}}\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{\alpha} \end{pmatrix}\zeta^{\sharp,\mathfrak{m}}\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{\beta} \end{pmatrix} = \sum_{\substack{\binom{c_j}{\gamma_j}=\begin{pmatrix} a_i+\sum_{l=1}^k a_{i+l}+b_{i'}+l \\ \alpha_i\prod_{l=1}^k \alpha_{i+l}\beta_{i'}+l \end{pmatrix} \text{ or } \begin{pmatrix} b_{i'}+\sum_{l=1}^k a_{i+l}+b_{i'}+l \\ \beta_{i'}\prod_{l=1}^k \alpha_{i+l}\beta_{i'}+l \end{pmatrix} \\ k\geq 0,\ \text{order } (a_i),(b_i) \text{ preserved}}} (-1)^{\frac{r+s-m}{2}}\zeta^{\sharp,\mathfrak{m}}\begin{pmatrix} c_1,\dots,c_m \\ \gamma_1,\dots,\gamma_m \end{pmatrix}.$$

Remarks:

- In the depth graded, stuffle corresponds to shuffle the sequences $\binom{\boldsymbol{a}}{\boldsymbol{\alpha}}$ and $\binom{\boldsymbol{b}}{\boldsymbol{\beta}}$.
- Other identities mixing the two versions could also be stated, such as

$$\zeta^{\star,\mathfrak{m}}\begin{pmatrix} a_1,\dots,a_r \\ \alpha_1,\dots,\alpha_r \end{pmatrix}\zeta^{\mathfrak{m}}\begin{pmatrix} b_1,\dots,b_s \\ \beta_1,\dots,\beta_s \end{pmatrix} = \sum_{\substack{\binom{c_j}{\gamma_j}=\binom{a_i}{\alpha_i},\binom{b_{i'}}{\beta_{i'}} \text{ or } \begin{pmatrix} (\sum_{l=1}^k a_{i+l})+b_{i'} \\ (\prod_{l=1}^k \alpha_{i+l})\beta_{i'} \end{pmatrix} \\ k\geq 1,\text{order } (a_i),(b_i) \text{ preserved}}} \zeta^{\mathfrak{m}}\begin{pmatrix} c_1,\dots,c_m \\ \gamma_1,\dots,\gamma_m \end{pmatrix}.$$

4.2 Relations in $\mathcal{L}$

4.2.1 Antipode relation

In this part, we are interested in some Antipodal relations for motivic Euler sums in the coalgebra $\mathcal{L}$, i.e. modulo products. To explain quickly where they come from, let's go back to two combinatorial Hopf algebra structures.

First recall that if A is a graded connected bialgebra, there exists an unique antipode S (leading to a Hopf algebra structure)[II], which is the graded map defined by:

$$S(x) = -x - \sum S(x_{(1)}) \cdot x_{(2)}, \tag{4.2}$$

where $\cdot$ is the product and using Sweedler notations for the coaction:

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum x_{(1)} \otimes x_{(2)} = \Delta'(x) + 1 \otimes x + x \otimes 1.$$

Hence, in the quotient $A/A_{>0} \cdot A_{>0}$:

$$S(x) \equiv -x.$$

[I] The stuffle for these motivic iterated integrals can be deduced from works by Goncharov on mixed Hodge structures, but was also proved in a direct way by G. Racinet, in his thesis, or I. Souderes in [75] via blow-ups. Remark that shuffle relation, coming from the iterated integral representation is clearly *motivic*.

[II] It comes from the usual required relation for the antipode in a Hopf algebra, but because it is graded and connected, we can apply the formula recursively to construct it, in an unique way.

4.2.1.1 The ⧢ Hopf algebra

Let $X = \{a_1, \cdots, a_n\}$ an alphabet and $A_{⧢} := \mathbb{Q}\langle X^{\times}\rangle$ the $\mathbb{Q}$-vector space generated by words on X, i.e. non commutative polynomials in a_i. It is easy to see that $A_{⧢}$ is a Hopf algebra with the ⧢ shuffle product, the deconcatenation coproduct Δ_D and antipode $S_{⧢}$:

$$\Delta_D(a_{i_1} \cdots a_{i_n}) = \sum_{k=0}^{n} a_{i_1} \cdots a_{i_k} \otimes a_{i_{k+1}} \cdots a_{i_n}. \tag{4.3}$$

$$S_{⧢}(a_{i_1} \cdots a_{i_n}) = (-1)^n a_{i_n} \cdots a_{i_1}. \tag{4.4}$$

$A_{⧢}$ is even a connected graded Hopf algebra, called the *shuffle Hopf algebra*; the grading coming from the degree of polynomial. By the equivalence of category between $\mathbb{Q}$-Hopf algebra and $\mathbb{Q}$-Affine Group Scheme, it corresponds to:

$$G = \mathrm{Spec} A_{⧢} : R \to \mathrm{Hom}(\mathbb{Q}\langle X\rangle, R) = \{S \in R\langle\langle a_i\rangle\rangle \mid \Delta_{⧢} S = S \widehat{\otimes} S, \epsilon(S) = 1\}, \tag{4.5}$$

where $\Delta_{⧢}$ is the coproduct dual to the product ⧢:

$$\Delta_{⧢}(a_{i_1} \cdots a_{i_n}) = (1 \otimes a_{i_1} + a_{i_1} \otimes 1) \cdots (1 \otimes a_{i_n} + a_{i_n} \otimes 1).$$

Let restrict now to $X = \{0, \mu_N\}$; our main interest in this Chapter is $N = 2$, but it can be extended to other roots of unity. The shuffle relation for motivic iterated integral relative to μ_N:

$$I^{\mathfrak{m}}(0; \cdot; 1) \text{ is a morphism of Hopf algebra from } A_{⧢} \text{ to } (\mathbb{R}, \times) : \tag{4.6}$$
$$I^{\mathfrak{m}}(0; w; 1) I^{\mathfrak{m}}(0; w'; 1) = I^{\mathfrak{m}}(0; w ⧢ w'; 1) \text{ with } w, w' \text{ words in } X.$$

Lemma 4.2.1 (**Antipode ⧢**). *In the coalgebra $\mathcal{L}$, with w the weight, $\bullet$ standing for $MMZV_{\mu_N}$, or $\star\star$ ($N = 2$) resp. $\sharp\sharp$-version ($N = 2$):*

$$\zeta_{n-1}^{\bullet,\mathfrak{l}} \begin{pmatrix} n_1, \ldots, n_p \\ \epsilon_1, \ldots, \epsilon_p \end{pmatrix} \equiv (-1)^{w+1} \zeta_{n_p-1}^{\bullet,\mathfrak{l}} \begin{pmatrix} n_{p-1}, \ldots, n_1, n \\ \epsilon_{p-1}^{-1}, \ldots, \epsilon_1^{-1}, \epsilon \end{pmatrix} \text{ where } \epsilon := \epsilon_1 \cdot \ldots \cdot \epsilon_p.$$

This formula stated for any N is slightly simpler in the case $N = 1, 2$ since $n_i \in \mathbb{Z}^*$:

$$\text{Antipode ⧢} : \begin{array}{l} \zeta_{n-1}^{\bullet,\mathfrak{l}}(n_1, \ldots, n_p) \equiv (-1)^{w+1} \zeta_{|n_p|-1}^{\bullet,\mathfrak{l}}(n_{p-1}, \ldots, n_1, sign(n_1 \cdots n_p) n) \\ I^{\mathfrak{l}}(0; X; \epsilon) \equiv (-1)^w I^{\mathfrak{l}}(\epsilon; \widetilde{X}; 0) \equiv (-1)^{w+1} I^{\mathfrak{l}}(0; \widetilde{X}; \epsilon) \end{array}. \tag{4.7}$$

Here X is any word in $0, \pm 1$ or $0, \pm\star$ or $0, \pm\sharp$, and $\widetilde{X}$ denotes the *reversed* word.

Proof. For motivic iterated integrals, as said above:

$$S_{⧢}(I^{\mathfrak{m}}(0; a_1, \ldots, a_n; 1)) = (-1)^n I^{\mathfrak{m}}(0; a_n, \ldots, a_1; 1),$$

which, in terms of the MMZV_{μ_N} notation is:

$$S_{⧢}\left(\zeta_{n-1}^{\bullet,\mathfrak{l}} \begin{pmatrix} n_1, \ldots, n_p \\ \epsilon_1, \ldots, \epsilon_p \end{pmatrix}\right) \equiv (-1)^w \zeta_{n_p-1}^{\bullet,\mathfrak{l}} \begin{pmatrix} n_{p-1}, \ldots, n_1, n \\ \epsilon_{p-1}^{-1}, \ldots, \epsilon_1^{-1}, \epsilon \end{pmatrix} \text{ where } \epsilon := \epsilon_1 \cdot \ldots \cdot \epsilon_p.$$

Then, if we look at the antipode recursive formula (4.2) in the coalgebra $\mathcal{L}$, for $a_i \in \{0, \mu_N\}$:

$$S_{⧢}(I^{\mathfrak{l}}(0; a_1, \ldots, a_n; 1)) \equiv -I^{\mathfrak{l}}(0; a_1, \ldots, a_n; 1).$$

This leads to the lemma above. The ⧢-antipode relation can also be seen at the level of iterated integrals as the path composition modulo products followed by a reverse of path. ☐

4.2.1.2 The $*$ Hopf algebra

Let $Y = \{\cdots, y_{-n}, \ldots, y_{-1}, y_1, \cdots, y_n, \cdots\}$ an infinite alphabet and $A_* := \mathbb{Q}\langle Y^\times\rangle$ the non commutative polynomials in y_i with rational coefficients, with $y_0 = 1$ the empty word. Similarly, it is a graded connected Hopf algebra called the *stuffle Hopf algebra*, with the stuffle $*$ product and the following coproduct:[I]

$$\Delta_{D*}(y_{n_1}\cdots y_{n_p}) = \sum y_{n_1}\cdots y_{n_i} \otimes y_{n_{i+1}}, \ldots, y_{n_p}, \quad n_i \in \mathbb{Z}^*. \tag{4.8}$$

`Nota Bene`: Remark that here we restricted to Euler sums, $N = 2$, but it could be extended for other roots of unity, for which stuffle relation has been stated in §4.1.
The completed dual is the Hopf algebra of series $\mathbb{Q}\langle\langle Y\rangle\rangle$ with the coproduct:

$$\Delta_*(y_n) = \sum_{\substack{k=0 \\ sgn(n)=\epsilon_1\epsilon_2}}^{|n|} y_{\epsilon_1 k} \otimes y_{\epsilon_2(n-k)}.$$

Now, let introduce the notations:[II]

$$(y_{n_1}\cdots y_{n_p})^\star := \sum_{\substack{1=i_0<i_1<\cdots<i_{k-1}\leq i_{k+1}=p \\ k\geq 0}} y_{n_{i_0}\text{`+'}\cdots\text{`+'}n_{i_1-1}}\cdots y_{n_{i_j}\text{`+'}\cdots\text{`+'}n_{i_{j+1}-1}}\cdots y_{n_{i_k}\text{`+'}\cdots\text{`+'}n_{i_{k+1}}}.$$

$$(y_{n_1}\cdots y_{n_p})^\sharp := \sum_{\substack{1=i_0<i_1<\cdots<i_{k-1}\leq i_{k+1}=p \\ k\geq 0}} 2^{k+1} y_{n_{i_0}\text{`+'}\cdots\text{`+'}n_{i_1-1}}\cdots y_{n_{i_j}\text{`+'}\cdots\text{`+'}n_{i_{j+1}-1}}\cdots y_{n_{i_k}\text{`+'}\cdots\text{`+'}n_{i_{k+1}}},$$

where $n_i \in \mathbb{Z}^*$ and the operation ' + ' indicates that signs are multiplied whereas absolute values are summed. It is straightforward to check that:

$$\Delta_{D*}(w^\star) = (\Delta_{D*}(w))^\star, \quad \text{and} \quad \Delta_{D*}(w^\sharp) = (\Delta_{D*}(w))^\sharp. \tag{4.9}$$

As said above, the relation stuffle is motivic:

$$\zeta^{\mathfrak{m}}(\cdot) \text{ is a morphism of Hopf algebra from } A_* \text{ to } (\mathbb{R}, \times).$$

Lemma 4.2.2 (**Antipode** $*$). *In the coalgebra $\mathcal{L}$, with $n_i \in \mathbb{Z}^*$*

$$\zeta^{\mathfrak{l}}_{n-1}(n_1, \ldots, n_p) \equiv (-1)^{p+1}\zeta^{\star,\mathfrak{l}}_{n-1}(n_p, \ldots, n_1).$$

$$\zeta^{\sharp,\mathfrak{l}}_{n-1}(n_1, \ldots, n_p) \equiv (-1)^{p+1}\zeta^{\sharp,\mathfrak{l}}_{n-1}(n_p, \ldots, n_1).$$

Proof. By recursion, using the formula (4.2), and the following identity (left to the reader):

$$\sum_{i=0}^{p-1}(-1)^i(y_{n_i}\cdots y_{n_1})^\star * (y_{n_{i+1}}\cdots y_{n_p}) = -(-1)^p(y_{n_p}\cdots y_{n_1})^\star,$$

we deduce the antipode S_*:

$$S_*(y_{n_1}\cdots y_{n_p}) = (-1)^p(y_{n_p}\cdots y_{n_1})^\star.$$

Similarly:

$$S_*((y_{n_1}\cdots y_{n_p})^\sharp) = -\sum_{i=0}^{n-1} S_*((y_{n_1}\cdots y_{n_i})^\sharp) * (y_{n_{i+1}}\cdots y_{n_p})^\sharp$$

[I] For the ⧢ algebra, we had to use the notation in terms of iterated integrals, with $0, \pm 1$, but for the $*$ stuffle relation, it is more natural with the Euler sums notation, which corresponds to $y_{n_i}, n_i \in \mathbb{Z}$.

[II] Here $\star$ resp. $\sharp$ refers naturally to the Euler $\star$ resp. $\sharp$, sums, as we see in the next lemma. Beware, it is not a $*$ homomorphism.

$$= -\sum_{i=0}^{n-1}(-1)^i(y_{n_i}\cdots y_{n_1})^\sharp * (y_{n_{i+1}}\cdots y_{n_p})^\sharp = (-1)^p(y_{n_p}\cdots y_{n_1})^\sharp.$$

Then, we deduce the lemma, since $\zeta^{\mathfrak{m}}(\cdot)$ is a morphism of Hopf algebra. Moreover, the formula (4.2) in the coalgebra $\mathcal{L}$ gives that:

$$S(\zeta^{\mathfrak{l}}(\mathbf{s})) \equiv -\zeta^{\mathfrak{l}}(\mathbf{s}).$$

□

4.2.2 Hybrid relation in $\mathcal{L}$

In this part, we look at a new relation called *hybrid relation* between motivic Euler sums in the coalgebra $\mathcal{L}$, i.e. modulo products, which comes from the motivic version of the octagon relation (for $N > 1$, cf. [38]) [I]

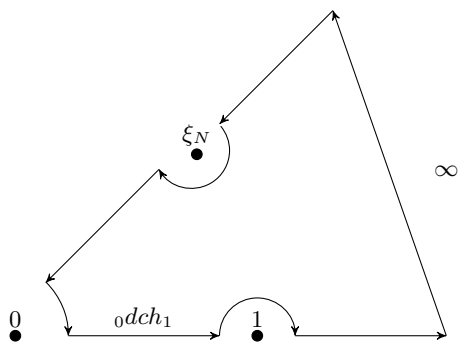


Figure 4.2: Octagon relation, $N > 1$:
$\Phi(e_0, e_1, \ldots, e_n)e^{\frac{2i\pi e_1}{N}}\Phi(e_\infty, e_1, e_n, \ldots, e_2)^{-1}e^{\frac{2i\pi e_\infty}{N}}\Phi(e_\infty, e_n, \ldots, e_1)e^{\frac{2i\pi e_n}{N}}\Phi(e_0, e_n, e_1, \ldots, e_{n-1})^{-1}e^{\frac{2i\pi e_0}{N}}$
$= 1$

This relation is motivic, and hence valid for the motivic Drinfeld associator $\Phi^{\mathfrak{m}}$ (2.39), replacing $2i\pi$ by the Lefschetz motivic period $\mathbb{L}^{\mathfrak{m}}$.

Let focus on the case $N = 2$ and recall that the space of motivic periods of $\mathcal{MT}\left(\mathbb{Z}[\frac{1}{2}]\right)$ decomposes as (cf. 2.10):

$$\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])} = \mathcal{H}^2 \oplus \mathcal{H}^2.\mathbb{L}^{\mathfrak{m}}, \quad \text{where} \quad \begin{array}{l} \mathcal{H}^2 \text{ is } \mathcal{F}_\infty \text{ invariant} \\ \mathcal{H}^2.\mathbb{L}^{\mathfrak{m}} \text{ is } \mathcal{F}_\infty \text{ anti-invariant} \end{array}. \tag{4.10}$$

For the motivic Drinfeld associator, seeing the path in the Riemann sphere, it becomes:

[I]

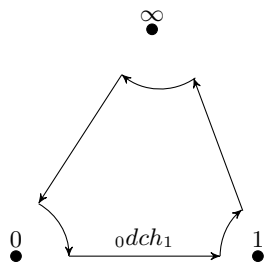


Figure 4.1: For $N = 1$, Hexagon relation: $e^{i\pi e_0}\Phi(e_\infty, e_0)e^{i\pi e_\infty}\Phi(e_1, e_\infty)e^{i\pi e_1}\Phi(e_0, e_1) = 1$.

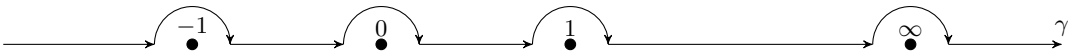


Figure 4.3: Octagon relation, $N = 2$ with $e_0 + e_1 + e_{-1} + e_\infty = 0$:
$e^{\frac{\mathbb{L}^{\mathfrak{m}} e_{-1}}{2}} \Phi^{\mathfrak{m}}(e_0, e_{-1}, e_1)^{-1} e^{\frac{\mathbb{L}^{\mathfrak{m}} e_0}{2}} \Phi^{\mathfrak{m}}(e_0, e_1, e_{-1}) e^{\frac{\mathbb{L}^{\mathfrak{m}} e_1}{2}} \Phi^{\mathfrak{m}}(e_\infty, e_1, e_{-1})^{-1} e^{\frac{\mathbb{L}^{\mathfrak{m}} e_\infty}{2}} \Phi^{\mathfrak{m}}(e_\infty, e_{-1}, e_1) = 1.$

Let $X = \mathbb{P}^1 \setminus \{0, \pm 1, \infty\}$. The action of the *real Frobenius* $\boldsymbol{\mathcal{F}_\infty}$ on $X(\mathbb{C})$ is induced by complex conjugation. The real Frobenius acts on the Betti realization $\pi^B(X(\mathbb{C}))$[I], and induces an involution on motivic periods, compatible with the Galois action:

$$\mathcal{F}_\infty : \mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])} \to \mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}.$$

The Lefschetz motivic period $\mathbb{L}^{\mathfrak{m}}$ is anti-invariant by $\mathcal{F}_\infty$:

$$\mathcal{F}_\infty \mathbb{L}^{\mathfrak{m}} = -\mathbb{L}^{\mathfrak{m}},$$

whereas terms corresponding to real paths in Figure 4.3, such as Drinfeld associator terms, are obviously invariant by $\mathcal{F}_\infty$.

The linearized $\mathcal{F}_\infty$-anti-invariant part of this octagon relation leads to the following hybrid relation.

Theorem 4.2.3. *In the coalgebra $\mathcal{L}^2$, with $n_i \in \mathbb{Z}^*$, w the weight:*

$$\zeta^{\mathfrak{l}}_k(n_0, n_1, \ldots, n_p) + \zeta^{\mathfrak{l}}_{|n_0|+k}(n_1, \ldots, n_p) \equiv (-1)^{w+1} \left(\zeta^{\mathfrak{l}}_k(n_p, \ldots, n_1, n_0) + \zeta^{\mathfrak{l}}_{k+|n_p|}(n_{p-1}, \ldots, n_1, n_0) \right),$$

Equivalently, in terms of motivic iterated integrals, for X any word in $\{0, \pm 1\}$, with $\widetilde{X}$ the reversed word, we obtain both:

$$I^{\mathfrak{l}}(0; 0^k, \star, X; 1) \equiv I^{\mathfrak{l}}(0; X, \star, 0^k; 1) \equiv (-1)^{w+1} I^{\mathfrak{l}}(0; 0^k, \star, \widetilde{X}; 1),$$

$$I^{\mathfrak{l}}(0; 0^k, -\star, X; 1) \equiv I^{\mathfrak{l}}(0; -X, -\star, 0^k; 1) \equiv (-1)^{w+1} I^{\mathfrak{l}}(0; 0^k, -\star, -\widetilde{X}; 1)$$

The proof is given below, firstly for $k = 0$, using octagon relation (Figure 4.3). The generalization for any $k > 0$ is deduced directly from the shuffle regularization (2.28).

Remarks:

- This theorem implies notably the famous *depth-drop phenomena* when weight and depth have not the same parity (cf. Corollary 4.2.4).
- Equivalently, this statement is true for X any word in $\{0, \pm\star\}$. Recall that (2.29), by linearity:
$$I^{\mathfrak{m}}(\ldots, \pm\star, \ldots) := I^{\mathfrak{m}}(\ldots, \pm 1, \ldots) - I^{\mathfrak{m}}(\ldots, 0, \ldots).$$
- The point of view adopted by Francis Brown in [21], and its use of commutative polynomials (also seen in Ecalle work) can be applied in the coalgebra $\mathcal{L}$ and leads to a new proof of Theorem 4.2.3 in the case of MMZV, i.e. $N = 1$, sketched in Appendix $A.4$; it uses the stuffle relation and the antipode shuffle. Unfortunately, generalization for motivic Euler sums of this proof is not clear, because of this commutative polynomial setting.

Since Antipode $*$ relation expresses $\zeta^{\mathfrak{l}}_{n-1}(n_1, \ldots, n_p) + (-1)^p \zeta^{\mathfrak{l}}_{n-1}(n_p, \ldots, n_1)$ in terms of smaller depth (cf. Lemma 4.2.2), when weight and depth have not the same parity, it turns out that a (motivic) Euler sum can be expressed by smaller depth:[II]

[I] It is compatible with the groupoid structure of π^B, and the local monodromy.

[II] Erik Panzer recently found a new proof of this depth drop result for MZV at roots of unity, which appear as a special case of some functional equations of polylogarithms in several variables.

Corollary 4.2.4. *If $w+p$ odd, a motivic Euler sum in $\mathcal{L}$ is reducible in smaller depth:*

$$2\zeta^{\mathfrak{l}}_{n-1}(n_1,\dots,n_p) \equiv$$

$$-\zeta^{\mathfrak{l}}_{n+|n_1|-1}(n_2,\dots,n_p)+(-1)^p\zeta^{\mathfrak{l}}_{n+|n_p|-1}(n_{p-1},\dots,n_1)+\sum_{\substack{\circ=+ \text{ or }, \\ \text{at least one } +}}(-1)^{p+1}\zeta^{\mathfrak{l}}_{n-1}(n_p\circ\cdots\circ n_1).$$

Proof of Theorem 4.2.3 First, the octagon relation (Figure 4.3) is equivalent to:

Lemma 4.2.5. *In $\mathcal{P}^{\mathfrak{m}}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}\langle\langle e_0,e_1,e_{-1}\rangle\rangle$, with $e_0+e_1+e_{-1}+e_\infty=0$:*

$$\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1})e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}\Phi^{\mathfrak{m}}(e_{-1},e_0,e_\infty)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}\Phi^{\mathfrak{m}}(e_1,e_\infty,e_0)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}=1, \tag{4.11}$$

Hence, the linearized octagon relation is:

$$-e_0\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)+\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)e_0+(e_0+e_{-1})\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)-\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)(e_0+e_{-1})$$
$$-e_1\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)+\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)e_1\equiv 0. \tag{4.12}$$

Proof. · Let's first remark that:

$$\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1})=\Phi^{\mathfrak{m}}(e_1,e_0,e_\infty)^{-1}.$$

Indeed, the coefficient in the series $\Phi^{\mathfrak{m}}(e_1,e_0,e_\infty)$ of a word $e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_r}e_0^{a_r}$, where $\eta_i\in\{\pm1\}$ is (cf. §4.6):

$$I^{\mathfrak{m}}\left(0;(\omega_1-\omega_{-1})^{a_0}(-\omega_{\mu_1})(\omega_1-\omega_{-1})^{a_1}\cdots(-\omega_{\mu_r})(\omega_1-\omega_{-1})^{a_r};1\right) \ \texttt{with}\ \mu_i:=\begin{cases}-\star & \texttt{if}\ \eta_i=1\\ -1 & \texttt{if}\ \eta_i=-1\end{cases}.$$

Let introduce the following homography $\phi_{\tau\sigma}=\phi_{\tau\sigma}-1$ (cf. Annexe $(A.3)$):

$$\phi_{\tau\sigma}=\phi_{\tau\sigma}^{-1}:t\mapsto\frac{1-t}{1+t}:\begin{cases}-\omega_\star\mapsto\omega_\star\\ -\omega_1\mapsto\omega_{-\star}\\ \omega_{-1}-\omega_1\mapsto-\omega_0\\ \omega_{-1}\mapsto-\omega_{-1}\\ \omega_{-\star}\mapsto-\omega_1\end{cases}.$$

If we apply $\phi_{\tau\sigma}$ to the motivic iterated integral above, it gives: $I^{\mathfrak{m}}\left(1;\omega_0^{a_0}\omega_{\eta_1}\omega_0^{a_1}\cdots\omega_{\eta_r}\omega_0^{a_r};0\right)$. Hence, summing over words w in e_0,e_1,e_{-1}:

$$\Phi^{\mathfrak{m}}(e_1,e_0,e_\infty)=\sum I^{\mathfrak{m}}(1;w;0)w$$

Therefore:

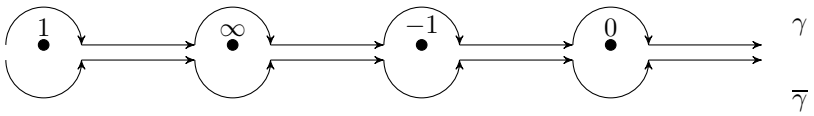
$$\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1})\Phi^{\mathfrak{m}}(e_1,e_0,e_\infty)=\sum_{w,w=uv}I^{\mathfrak{m}}(0;u;1)I^{\mathfrak{m}}(1;u;0)w=1.$$

We used the composition formula for iterated integral to conclude, since for w non empty, $\sum_{w=uv}I^{\mathfrak{m}}(0;u;1)I^{\mathfrak{m}}(1;u;0)=I^{\mathfrak{m}}(0;w;0)=0$.
Similarly:

$$\Phi^{\mathfrak{m}}(e_0,e_{-1},e_1)=\Phi^{\mathfrak{m}}(e_{-1},e_0,e_\infty)^{-1},\quad\text{and}\quad\Phi^{\mathfrak{m}}(e_\infty,e_1,e_{-1})=\Phi^{\mathfrak{m}}(e_1,e_\infty,e_0)^{-1}.$$

The identity 4.11 follows from 4.3.

· Let consider both paths on the Riemann sphere γ and $\overline{\gamma}$, its conjugate: [I]



Applying $(id - \mathcal{F}_\infty)$ to the octagon identity 4.11 [II] leads to:

$$\begin{aligned}&\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1})e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}\Phi^{\mathfrak{m}}(e_{-1},e_0,e_\infty)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}\Phi^{\mathfrak{m}}(e_1,e_\infty,e_0)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}\\ &-\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1})e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}\Phi^{\mathfrak{m}}(e_{-1},e_0,e_\infty)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}\Phi^{\mathfrak{m}}(e_1,e_\infty,e_0)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}=0.\end{aligned} \tag{4.13}$$

By (4.10), the left side of (4.13), being anti-invariant by $\mathcal{F}_\infty$, lies in $\mathcal{H}^2\cdot\mathbb{L}^{\mathfrak{m}}\langle\langle e_0,e_1,e_{-1}\rangle\rangle$. Consequently, we can divide it by $\mathbb{L}^{\mathfrak{m}}$ and consider its projection $\pi^{\mathcal{L}}$ in the coalgebra $\mathcal{L}\langle\langle e_0,e_1,e_{-1}\rangle\rangle$, which gives firstly:

$$\begin{aligned}&\Phi^{\mathfrak{l}}(e_0,e_1,e_{-1})\pi^{\mathcal{L}}\left((\mathbb{L}^{\mathfrak{m}})^{-1}\left[e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}-e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}\right]\right)\\ &+\pi^{\mathcal{L}}\left((\mathbb{L}^{\mathfrak{m}})^{-1}\left[e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}-e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}\right]\right)\\ &+\pi^{\mathcal{L}}\left((\mathbb{L}^{\mathfrak{m}})^{-1}\left[e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}-e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}\right]\right)\\ &+\pi^{\mathcal{L}}\left((\mathbb{L}^{\mathfrak{m}})^{-1}\left[e^{\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)e^{\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}-e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_0}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_{-1}}{2}}e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_\infty}{2}}\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)e^{-\frac{\mathbb{L}^{\mathfrak{m}}e_1}{2}}\right]\right)=0\end{aligned} \tag{4.14}$$

The first line is zero (since $e_0+e_1+e_{-1}+e_\infty=0$) whereas each other line will contribute by two terms, in order to give (4.12). Indeed, the projection $\pi^{\mathcal{L}}(x)$, when seeing x as a polynomial (with only even powers) in $\mathbb{L}^{\mathfrak{m}}$, only keep the constant term; hence, for each term, only one of the exponentials above e^x contributes by its linear term i.e. x, while the others contribute simply by 1. For instance, if we examine carefully the second line of (4.14), we get:

$$\begin{aligned}&= e_0\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)+\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)(e_{-1}+e_\infty+e_1)\\ &\quad -(-e_0)\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)-\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)(-e_{-1}-e_\infty-e_1)\ .\\ &= 2\left[e_0\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)-\Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty)e_0\right]\end{aligned}$$

Similarly, the third line of (4.14) is equal to $(e_0+e_{-1})\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)-\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)(e_0+e_{-1})$ and the last line is equal to $-e_1\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)+\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)e_1$. Therefore, (4.14) is equivalent to (4.12), as claimed.

□

This linearized octagon relation 4.12, while looking at the coefficient of a specific word in $\{e_0,e_1,e_{-1}\}$, provides an identity between some $\zeta^{\star\star,\mathfrak{l}}(\bullet)$ and $\zeta^{\mathfrak{l}}(\bullet)$ in the coalgebra $\mathcal{L}$. The different identities obtained in this way are detailed in the §4.6. In the following proof of Theorem 4.2.3, two of those identities are used.

[I] Path γ corresponds to the cycle σ, $1\mapsto\infty\mapsto-1\mapsto0\mapsto1$ (cf. in Annexe A.3). Beware, in the figure, the position of both path is not completely accurate in order to distinguish them.

[II] The identity 4.11 corresponds to the path γ whereas applying $\mathcal{F}_\infty$ to the path γ corresponds to the path $\overline{\gamma}$ represented.

Proof of Theorem 4.2.3. The identity with MMZV_{μ_2} is equivalent to, in terms of motivic iterated integrals:[I]

$$I^{\mathfrak{l}}(0;0^k,\star,X;1)\equiv I^{\mathfrak{l}}(0;X,\star,0^k;1) \text{ and } I^{\mathfrak{l}}(0;0^k,-\star,X;1)\equiv I^{\mathfrak{l}}(0;-X,-\star,0^k;1).$$

Furthermore, by shuffle regularization formula (2.28), spreading the first 0 further inside the iterated integrals, the identity $I^{\mathfrak{l}}(0;\mathbf{0}^k,\star,X;1)\equiv(-1)^{w+1}I^{\mathfrak{l}}(0;\mathbf{0}^k,\star,\widetilde{X};1)$ boils down to the case $k=0$.
The notations are as usual: $\epsilon_i=\mathrm{sign}(n_i)$, $\epsilon_i=\eta_i\eta_{i+1}$,$\epsilon_p=\eta_p$, $n_i=\epsilon_i(a_i+1)$.

(*i*) In (4.12), if we look at the coefficient of a specific word in $\{e_0,e_1,e_{-1}\}$ ending and beginning with e_{-1} (as in §4.6), only two terms contribute, i.e.:

$$e_{-1}\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)-\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)e_{-1} \tag{4.15}$$

The coefficient of $e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}$ in $\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1)$ is $(-1)^{n+p}\zeta^{\star\star,\mathfrak{m}}_{n_0-1}(n_1,\cdots,n_{p-1},-n_p)$.[II] Hence, the coefficient in (4.15) (as in (4.12)) of the word $e_{-1}e_0^{a_0}e_{\eta_1}\cdots e_{\eta_p}e_0^{a_p}e_{-1}$ is:

$$\zeta^{\star\star,\mathfrak{l}}_{|n_0|-1}(n_1,\cdots,-n_p,1)-\zeta^{\star\star,\mathfrak{l}}(n_0,n_1,\cdots,n_{p-1},-n_p)=0,\qquad \text{with}\prod_{i=0}^{p}\epsilon_i=1.$$

In terms of iterated integrals, reversing the first one with Antipode ⧢, it is:

$$I^{\mathfrak{l}}\left(0;-X,\star;1\right)\equiv I^{\mathfrak{l}}\left(0;\star,-X;1\right),\ \text{with } X:=0^{n_0-1}\eta_1 0^{n_1-1}\cdots\eta_p 0^{n_p-1}.$$

Therefore, since X can be any word in $\{0,\pm\star\}$, by linearity this is also true for any word X in $\{0,\pm1\}$: $I^{\mathfrak{l}}\left(0;X,\star;1\right)\equiv I^{\mathfrak{l}}\left(0;\star,X;1\right)$.

(*ii*) Now, let look at the coefficient of a specific word in $\{e_0,e_1,e_{-1}\}$ beginning by e_1, and ending by e_{-1}. Only two terms in the left side of (4.12) contribute, i.e.:

$$-e_1\Phi^{\mathfrak{l}}(e_1,e_\infty,e_0)-\Phi^{\mathfrak{l}}(e_\infty,e_{-1},e_1)e_{-1} \tag{4.16}$$

The coefficient in this expression of the word $e_1e_0^{a_0}e_{\eta_1}\cdots e_{\eta_p}e_0^{a_p}e_{-1}$ is:

$$\zeta^{\star\star,\mathfrak{l}}_{|n_0|-1}(n_1,\cdots,n_p,-1)-\zeta^{\star\star,\mathfrak{l}}(n_0,n_1,\cdots,n_p)=0,\qquad \text{with}\prod_{i=0}^{p}\epsilon_i=-1.$$

In terms of iterated integrals, reversing the first one with Antipode ⧢, it is:

$$I^{\mathfrak{l}}\left(0;-X,-\star;1\right)\equiv I^{\mathfrak{l}}\left(0;-\star,X;1\right).$$

Therefore, since X can be any word in $\{0,\pm\star\}$, by linearity this is also true for any word X in $\{0,\pm1\}$.

□

For Euler ⋆⋆ sums.

Corollary 4.2.6. *In the coalgebra $\mathcal{L}^2$, with $n_i\in\mathbb{Z}^*$, $n\geq1$:*

$$\zeta^{\star\star,\mathfrak{l}}_{n-1}(n_1,\ldots,n_p)\equiv(-1)^{w+1}\zeta^{\star\star,\mathfrak{l}}_{n-1}(n_p,\ldots,n_1). \tag{4.17}$$

Motivic Euler ⋆⋆ sums of depth p in $\mathcal{L}$ form a dihedral group of order $p+1$:

$$\text{(Shift)}\qquad \zeta^{\star\star,\mathfrak{l}}_{|n|-1}(n_1,\ldots,n_p)\equiv\zeta^{\star\star,\mathfrak{l}}_{|n_1|-1}(n_2,\ldots,n_p,n)\qquad \textit{where } sgn(n):=\prod_i sgn(n_i).$$

[I] Indeed, if $\prod_{i=0}^{p}\epsilon_i=1$, it corresponds to the first case, whereas if $\prod_{i=0}^{p}\epsilon_i$, we need the second case.
[II] The expressions of those associators are more detailed in the proof of Lemma 4.6.1.

Indeed, these two identities lead to a dihedral group structure of order $p+1$: (4.17), respectively SHIFT, correspond to the action of a reflection resp. of a cycle of order p on motivic Euler $\star\star$ sums of depth p in $\mathcal{L}$.

Proof. Writing $\zeta^{\star\star,\mathfrak{m}}$ as a sum of Euler sums:

$$\zeta^{\star\star,\mathfrak{m}}_{n-1}(n_1,\dots,n_p)=\sum_{i=1}^{p}\zeta^{\mathfrak{m}}_{n-1+|n_1|+\cdots+|n_{i-1}|}(n_i\circ\cdots\circ n_p)=\sum_{\substack{r\\A_i}}\Big(\zeta^{\mathfrak{m}}_{n-1}(A_1,\dots,A_r)+\zeta^{\mathfrak{m}}_{n-1+|A_1|}(A_2,\dots,A_r)\Big),$$

where the last sum is over $(A_i)_i$ such that each A_i is a non empty "sum" of consecutive $(n_j)'s$, preserving the order; the absolute value being summed whereas the sign of the n_i involved are multiplied; moreover, $\mid A_1 \mid\geq\mid n_1 \mid$ resp. $\mid A_r \mid\geq\mid n_p \mid$.
Using Theorem (4.2.3) in the coalgebra $\mathcal{L}$, the previous equality turns into:

$$(-1)^{w+1}\sum_{\substack{r\\A_i}}\Big(\zeta^{\mathfrak{l}}_{n-1}(A_r,\dots,A_1)+\zeta^{\mathfrak{l}}_{n-1+|A_r|}(A_{r-1},\dots,A_1)\Big)\equiv(-1)^{w+1}\zeta^{\star\star,\mathfrak{m}}_{n-1}(n_p,\dots,n_1).$$

The identity SHIFT is obtained as the composition of Antipode ⧢ (4.7) and the first identity of the corollary. □

For Euler ♯♯ sums.

Corollary 4.2.7. *In the coalgebra $\mathcal{L}$, for $n\in\mathbb{N}$, $n_i\in\mathbb{Z}^*$, $\epsilon_i:=sgn(n_i)$:*

REVERSE

$$\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1,\dots,n_p)+(-1)^w\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_p,\dots,n_1)\equiv\begin{cases}0 & \text{if } w+p \text{ even}\\ \zeta^{\sharp,\mathfrak{l}}_n(n_1,\dots,n_p) & \text{if } w+p \text{ odd}\end{cases}.$$

SHIFT $\zeta^{\sharp\sharp,\mathfrak{l}}_{n-1}(n_1,\dots,n_p)\equiv\zeta^{\sharp\sharp,\mathfrak{l}}_{|n_1|-1}(n_2,\dots,n_p,\epsilon_1\cdots\epsilon_p\cdot n)$, *for $w+p$ even.*

CUT $\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1,\cdots,n_p)\equiv\zeta^{\sharp\sharp,\mathfrak{l}}_{n+|n_p|}(n_1,\cdots,n_{p-1})$, *for $w+p$ odd.*

MINUS $\zeta^{\sharp\sharp,\mathfrak{l}}_{n-i}(n_1,\cdots,n_p)\equiv\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1,\cdots,n_{p-1},\epsilon_p(\mid n_p\mid -i))$, *for* $\begin{matrix}w+p \text{ odd}\\ i\leq\min(n,\mid n_p\mid)\end{matrix}$.

SIGN $\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1,\cdots,n_{p-1},n_p)\equiv\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1,\cdots,n_{p-1},-n_p)$, *for $w+p$ odd.*
This identity, in terms of motivic iterated integral implies that for X any sequence of $0,\pm\sharp$, such that $w+p$ odd:

$$I^{\mathfrak{l}}(-1;X;1)\equiv 0.$$

REMARK: In the coaction of Euler sums, terms with $\overline{1}$ can appear[I], which are clearly not motivic multiple zeta values. The left side corresponding to such a term in the coaction part $D_{2r+1}(\cdot)$ is $I^{\mathfrak{l}}(1;X;-1)$, X odd weight with $0,\pm\sharp$. It is worth underlying that, for the $\sharp$ family with $\{\overline{even},odd\}$, these terms disappear by SIGN, since by constraint on parity, X will always be of even depth for such a cut. This $\sharp$ family is then more suitable for an unramified criterion, cf. §4.3.

Proof. These are consequences of the hybrid relation in Theorem 4.2.3.

- REVERSE: Writing $\zeta^{\sharp\sharp,\mathfrak{l}}$ as a sum of Euler sums:

$$\zeta^{\sharp\sharp,\mathfrak{m}}_k(n_1,\dots,n_p)+(-1)^w\zeta^{\sharp\sharp,\mathfrak{m}}_k(n_p,\dots,n_1)$$

[I]More precisely, using the notations of Lemma $A.1.1$, a $\overline{1}$ can appear in terms of the type $T_{\epsilon,-\epsilon}$ for a cut between ϵ and $-\epsilon$.

$= \sum_{i=1}^{p} 2^{p-i+1-n_+} \zeta^{\mathfrak{m}}_{k+n_1+\cdots+n_{i-1}}(n_i \circ \cdots \circ n_p) + (-1)^w 2^{i-n_+} \zeta^{\mathfrak{m}}_{k+n_p+\cdots+n_{i+1}}(n_i \circ \cdots \circ n_1)$

$= \sum_{\substack{r \\ A_i}} 2^{r-1} \left(2\zeta^{\mathfrak{m}}_k(A_1, \ldots, A_r) + 2(-1)^w \zeta^{\mathfrak{m}}_k(A_r, \ldots, A_1) + \zeta^{\mathfrak{m}}_{k+A_1}(A_2, \ldots, A_r) + (-1)^w \zeta^{\mathfrak{m}}_{k+A_r}(A_{r-1}, \ldots, A_1) \right)$

where the sum is over (A_i) such that each A_i is a non empty "sum" of consecutive $(n_j)'s$, preserving the order; i.e. absolute values of n_i are summed whereas signs are multiplied; moreover, A_1 resp. A_r are no less than n_1 resp. n_p.
By Theorem 4.2.3, the previous equality turns into, in $\mathcal{L}$:

$$\sum_{\substack{r \\ A_i}} 2^{r-1} \left(\zeta^{\mathfrak{l}}_k(A_1, \ldots, A_r) + (-1)^w \zeta^{\mathfrak{l}}_k(A_r, \ldots, A_1) \right)$$

$$\equiv 2^{-1} \left(\zeta^{\sharp,\mathfrak{l}}_k(n_1, \ldots, n_p) + (-1)^w \zeta^{\sharp,\mathfrak{l}}_k(n_p, \ldots, n_1) \right) \equiv 2^{-1} \zeta^{\sharp,\mathfrak{l}}_k(n_1, \ldots, n_p) \left(1 + (-1)^{w+p+1} \right).$$

By the Antipode $\star$ relation applied to $\zeta^{\sharp,\mathfrak{l}}$, it implies the result stated, splitting the cases $w+p$ even and $w+p$ odd.

- Shift: Obtained when combining Reverse and Antipode ⧢, when $w+p$ even.
- Cut: Reverse in the case $w+p$ odd implies:

$$\zeta^{\sharp\sharp,\mathfrak{l}}_{n+|n_1|}(n_2, \ldots, n_p) + (-1)^w \zeta^{\sharp\sharp,\mathfrak{l}}_n(n_p, \ldots, n_1) \equiv 0,$$

  Which, reversing the variables, gives the Cut rule.
- Minus follows from Cut since, by Cut both sides are equal to $\zeta^{\sharp\sharp,\mathfrak{l}}_{n-i+|n_p|}(n_1, \cdots, n_{p-1})$.
- In Cut, the sign of n_p does not matter, hence, using Cut in both directions, with different signs leads to Sign:

$$\zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1, \ldots, n_p) \equiv \zeta^{\sharp\sharp,\mathfrak{l}}_{n+|n_p|}(n_1, \ldots, n_{p-1}) \equiv \zeta^{\sharp\sharp,\mathfrak{l}}_n(n_1, \ldots, -n_p).$$

  Note that, translating in terms of iterated integrals, it leads to, for X any sequence of $0, \pm\sharp$, with $w+p$ odd:

$$I^{\mathfrak{l}}(0; X; 1) \equiv I^{\mathfrak{l}}(0; -X; 1),$$

  where $-X$ is obtained from X after exchanging $\sharp$ and $-\sharp$. Moreover, $I^{\mathfrak{l}}(0; -X; 1) \equiv I^{\mathfrak{l}}(0; X; -1) \equiv -I^{\mathfrak{l}}(-1; X; 0)$. Hence, we obtain, using the composition rule of iterated integrals modulo product:

$$I^{\mathfrak{l}}(0; X; 1) + I^{\mathfrak{l}}(-1; X; 0) \equiv I^{\mathfrak{l}}(-1; X; 1) \equiv 0.$$

□

4.3 Euler $\sharp$ sums

Let's consider more precisely the following family, appearing in Conjecture 4.5.1, ith only positive odd and negative even integers for arguments:

$$\zeta^{\sharp,\mathfrak{m}} \left(\{\overline{\text{even}}, \text{odd}\}^{\times} \right).$$

In the iterated integral, this condition means that we see only the following sequences:

$$\epsilon 0^{2a} \epsilon, \quad \text{or} \quad \epsilon 0^{2a+1} - \epsilon, \quad \text{with} \quad \epsilon \in \{\pm\sharp\}.$$

Theorem 4.3.1. *The motivic Euler sums $\zeta^{\sharp,\mathfrak{m}}(\{\overline{even}, odd\}^{\times})$ are motivic geometric$^+$ periods of $\mathcal{MT}(\mathbb{Z})$.*
Hence, they are $\mathbb{Q}$ linear combinations of motivic multiple zeta values.

The proof, in §4.3.2, relies mainly upon the stability under the coaction of this family.
This motivic family is even a generating family of motivic MZV:

Theorem 4.3.2. *The following family is a basis of $\mathcal{H}^1$:*

$$\mathcal{B}^\sharp := \left\{\zeta^{\sharp,\mathfrak{m}}(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2})\ ,\ a_i \geq 0\right\}.$$

First, it is worth noticing that this subfamily is also stable under the coaction.

REMARK: It is conjecturally the same family as the Hoffman star family $\zeta^\star(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_p})$, by Conjecture (4.5.1).

For that purpose, we use the increasing *depth filtration* $\mathcal{F}^{\mathfrak{D}}$ on $\mathcal{H}^2$ such that (cf. §2.4.3):

$$\mathcal{F}^{\mathfrak{D}}_p\mathcal{H}^2 \text{ is generated by Euler sums of depth smaller than } p.$$

Note that it is not a grading, but we define the associated graded as the quotient $gr^{\mathfrak{D}}_p := \mathcal{F}^{\mathfrak{D}}_p / \mathcal{F}^{\mathfrak{D}}_{p-1}$. The vector space $\mathcal{F}^{\mathfrak{D}}_p\mathcal{H}$ is stable under the action of $\mathcal{G}$. The linear independence of this $\sharp$ family is proved below thanks to a recursion on the depth and on the weight, using the injectivity of a map ∂ where ∂ came out of the depth and weight-graded part of the coaction Δ.

4.3.1 Depth graded Coaction

In Chapter 2, we defined the depth graded derivations $D_{r,p}$ (cf. 2.4.3), and $D^{-1}_{r,p}$ (2.53) after the projection on the right side, using depth 1 results:

$$gr^{\mathcal{D}}_1\mathcal{L}_{2r+1} = \mathbb{Q}\zeta^{\mathfrak{l}}(2r+1).$$

Let look at the following maps, whose injectivity is fundamental to the Theorem 4.3.2:

$$D^{-1}_{2r+1,p} : gr^{\mathfrak{D}}_p\mathcal{H}_n \to gr^{\mathfrak{D}}_{p-1}\mathcal{H}_{n-2r-1}.$$

$$\partial_{<n,p} := \oplus_{2r+1<n} D^{-1}_{2r+1,p}.$$

Their explicit expression is:

Lemma 4.3.3. [II]

$$\begin{aligned}
&D^{-1}_{2r+1,p}\left(\zeta^{\sharp,\mathfrak{m}}(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2})\right) = \\
&\qquad \delta_{r=a_0}\frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2r+2}\zeta^{\sharp,\mathfrak{m}}(2a_1+3, \cdots, \overline{2a_p+2}) \\
&+ \sum_{\substack{0\leq i\leq p-2,\\ r=a_{i+1}+a_i+1-\alpha}}^{\alpha\leq a_i} \frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2a_{i+1}+2}\zeta^{\sharp,\mathfrak{m}}(\cdots, 2a_{i-1}+3, 2\alpha+3, 2a_{i+2}+3, \cdots, \overline{2a_p+2}) \\
&+ \sum_{\substack{1\leq i\leq p-1,\\ r=a_{i-1}+a_i+1-\alpha}}^{\alpha\leq a_i} \frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2a_{i-1}+2}\zeta^{\sharp,\mathfrak{m}}(\cdots, 2a_{i-2}+3, 2\alpha+3, 2a_{i+1}+3, \cdots, \overline{2a_p+2}) \\
&\quad + (\text{DECONCATENATION}) \sum_{\substack{\alpha\leq a_p\\ r=a_{p-1}+a_p+1-\alpha}} 2\binom{2r}{2a_p+1}\zeta^{\sharp,\mathfrak{m}}(\cdots, 2a_{p-1}+3, \overline{2\alpha+2}). \qquad (4.18)
\end{aligned}$$

[II] To be accurate, the term $i=0$ in the first sum has to be understood as:

$$\frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2a_1+2}\zeta^{\sharp,\mathfrak{m}}(2\alpha+3, 2a_2+3, \cdots, \overline{2a_p+2}).$$

Meanwhile the terms $i=1$, resp. $i=p$ in the second sum have to be understood as:

$$\frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2a_0+2}\zeta^{\sharp,\mathfrak{m}}(2\alpha+3, 2a_2+3, \cdots, \overline{2a_p+2}) \quad \text{resp.} \quad \frac{2^{2r+1}}{1-2^{2r}}\binom{2r}{2a_{p-1}+2}\zeta^{\sharp,\mathfrak{m}}(\cdots, 2a_{p-2}+3, \overline{2\alpha+2}).$$

Proof. Looking at the Annexe $A.1$ expression for D_{2r+1}, we obtain for $D_{2r+1,p}$ keeping only the cuts of depth one (removing exactly one non zero element):

$$\begin{aligned}
&D_{2r+1,p}\zeta^{\sharp,\mathfrak{m}}(2a_0+1,2a_1+3,\cdots,2a_{p-1}+3,\overline{2a_p+2}) = \\
&\sum_{\substack{i,\alpha\le a_i\\ r=a_{i+1}+a_i+1-\alpha}} 2\zeta^{\mathfrak{l}}_{2a_i-2\alpha}(2a_{i+1}+3)\otimes\zeta^{\sharp,\mathfrak{m}}(\cdots,2a_{i-1}+3,2\alpha+3,2a_{i+2}+3,\cdots,\overline{2a_p+2}) \\
+&\sum_{\substack{i,\alpha\le a_i\\ r=a_{i-1}+a_i+1-\alpha}} 2\zeta^{\mathfrak{l}}_{2a_i-2\alpha}(2a_{i-1}+3)\otimes\zeta^{\sharp,\mathfrak{m}}(\cdots,2a_{i-2}+3,2\alpha+3,2a_{i+1}+3,\cdots,\overline{2a_p+2}) \\
&\qquad\qquad + \sum_{\substack{\alpha\le a_p\\ r=a_{p-1}+a_p+1-\alpha}} 2\zeta^{\mathfrak{l}}_{2a_{p-1}-2\alpha+1}(\overline{2a_p+2})\otimes\zeta^{\sharp,\mathfrak{m}}(\cdots,2a_{p-1}+3,\overline{2\alpha+2}).
\end{aligned}$$

To lighten the result, some cases at the borders ($i=0$, or $i=p$) have been included in the sum, being fundamentally similar (despite some index problems). These are clarified in the previous footnote[II].
In particular, with notations of the Lemma $A.1.1$, $T_{0,0}$ terms can be neglected as they decrease the depth by at least 2; same for the $T_{0,\epsilon}$ and $T_{\epsilon,0}$ for cuts between ϵ and $\pm\epsilon$. To obtain the lemma, it remains to check the coefficient of $\zeta^{\mathfrak{l}}(\overline{2r+1})$ for each term in the left side thanks to the known identities:

$$\zeta^{\mathfrak{l}}(2r+1) = \frac{-2^{2r}}{2^{2r}-1}\zeta^{\mathfrak{l}}(\overline{2r+1}) \quad \text{and} \quad \zeta^{\mathfrak{l}}_{2r+1-a}(a) = (-1)^{a+1}\binom{2r}{a-1}\zeta^{\mathfrak{l}}(2r+1).$$

□

4.3.2 Proofs of Theorem 4.3.1 and 4.3.2

Proof of Theorem 4.3.1. By Corollary 5.1.2, we can prove it in two steps:

- First, checking that $D_1(\cdot)=0$ for this family, which is rather obvious by Lemma 5.2.8 since there is no sequence of the type $\{0,\epsilon,-\epsilon\}$ or $\{\epsilon,-\epsilon,0\}$ in the iterated integral.
- Secondly, we can use a recursion on weight to prove that $D_{2r+1}(\cdot)$, for $r>0$, are unramified. Consequently, using recursion, this follows from the following statement:

  The family $\zeta^{\sharp,\mathfrak{m}}(\{\overline{\text{even}},+\text{odd}\}^{\times})$ is stable under D_{2r+1}.

  This is proved in Lemma $A.1.3$, using the relations of §4.2 in order to simplify the *unstable cuts*, i.e. the cuts where a sequence of type $\epsilon,0^{2a+1},\epsilon$ or $\epsilon,0^{2a},-\epsilon$ appears; indeed, these cuts give rise to a even or to a $\overline{\text{odd}}$ in the $\sharp$ Euler sum.

One fundamental observation about this family, used in Lemma $A.1.3$ is: for a subsequence of odd length from the iterated integral, because of these patterns of $\epsilon,\mathbf{0}^{2a},\epsilon$, or $\epsilon,\mathbf{0}^{2a+1},-\epsilon$, we can put in relation the depth p, the weight w and s the number of sign changes among the $\pm\sharp$:

$$w\equiv p-s \pmod 2.$$

It means that if we have a cut $\epsilon_0,\cdots\epsilon_{p+1}$ of odd weight, then:

Either: Depth p is odd, s even, $\epsilon_0=\epsilon_{p+1}$, Or: Depth p is even, s odd, $\epsilon_0=-\epsilon_{p+1}$.

□

Proof of Theorem 4.3.2. By a cardinality argument, it is sufficient to prove the linear independence of the family, which is based on the injectivity of $\partial_{<n,p}$. Let us define: [I]

[I] Sub-$\mathbb{Q}$ vector space of $\mathcal{H}^1$ by previous Theorem.

$\mathcal{H}^{odd\sharp}$: $\mathbb{Q}$-vector space generated by $\zeta^{\sharp,\mathfrak{m}}(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2})$.

The first thing to remark is that $\mathcal{H}^{odd\sharp}$ is stable under these derivations, by the expression obtained in Lemma $A.1.4.$:

$$D_{2r+1}(\mathcal{H}_n^{odd\sharp}) \subset \mathcal{L}_{2r+1} \otimes \mathcal{H}_{n-2r-1}^{odd\sharp},$$

Now, let consider the restriction on $\mathcal{H}^{odd\sharp}$ of $\partial_{<n,p}$ and prove:

$$\partial_{<n,p} : gr_p^{\mathfrak{D}} \mathcal{H}_n^{odd\sharp} \to \oplus_{2r+1<n} gr_{p-1}^{\mathfrak{D}} \mathcal{H}_{n-2r-1}^{odd\sharp} \text{ is bijective.}$$

The formula (4.18) gives the explicit expression of this map. Let us prove more precisely:

$$M_{n,p}^{\mathfrak{D}} \text{ the matrix of } \partial_{<n,p} \text{ on } \left\{\zeta^{\sharp,\mathfrak{m}}(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2})\right\} \text{ in terms of } \left\{\zeta^{\sharp,\mathfrak{m}}(2b_0+1, 2b_1+3, \cdots, 2b_{p-2}+3, \overline{2b_{p-1}+2})\right\} \text{ is invertible.}$$

`Nota Bene`: The matrix $M_{n,p}^{\mathfrak{D}}$ is well (uniquely) defined provided that the $\zeta^{\sharp,\mathfrak{m}}$ of the second line are linearly independent. So first, we have to consider the formal matrix associated $\mathbb{M}_{n,p}^{\mathfrak{D}}$ defined explicitly (combinatorially) by the formula for the derivations given, and prove $\mathbb{M}_{n,p}^{\mathfrak{D}}$ is invertible. Afterwards, we could state that $M_{n,p}^{\mathfrak{D}}$ is well defined and invertible too since equal to $\mathbb{M}_{n,p}^{\mathfrak{D}}$.

Proof. The invertibility comes from the fact that the (strictly) smallest terms 2-adically in (4.18) are the deconcatenation ones, which is an injective operation. More precisely, let $\widetilde{M}_{n,p}^{\mathfrak{D}}$ be the matrix $\mathbb{M}_{n,p}$ where we have multiplied each line corresponding to D_{2r+1} by (2^{-2r}). Then, order elements on both sides by lexicographical order on $(a_p, \ldots, a_0)$, resp. $(r, b_{p-1}, \ldots, b_0)$, such that the diagonal corresponds to $r = a_p+1$ and $b_i = a_i$ for $i < p$. The 2 -adic valuation of all the terms in (4.18) (once divided by 2^{2r}) is at least 1, except for the deconcatenation terms since:

$$v_2\left(2^{-2r+1}\binom{2r}{2a_p+1}\right) \leq 0 \Longleftrightarrow v_2\left(\binom{2r}{2a_p+1}\right) \leq 2r-1.$$

Then, modulo 2, only the deconcatenation terms remains, so the matrix $\widetilde{M}_{n,p}^{\mathfrak{D}}$ is triangular with 1 on the diagonal. This implies that $\det(\widetilde{M}_{n,p}^{\mathfrak{D}}) \equiv 1 \pmod 2$, and in particular is non zero: the matrix $\widetilde{M}_{n,p}^{\mathfrak{D}}$ is invertible, and so does $\mathbb{M}_{n,p}^{\mathfrak{D}}$. □

This allows us to complete the proof since it implies:

The elements of $\mathcal{B}^{\sharp}$ are linearly independent.

Proof. First, let prove the linear independence of this family of the same depth and weight, by recursion on p. For depth 0, this is obvious since $\zeta^{\mathfrak{m}}(\overline{2n})$ is a rational multiple of π^{2n}. Assuming by recursion on the depth that the elements of weight n and depth $p-1$ are linearly independent, since $M_{n,p}^{\mathfrak{D}}$ is invertible, this means both that the $\zeta^{\sharp,\mathfrak{m}}(2a_0+1, 2a_1+3, \cdots, 2a_{p-1}+3, \overline{2a_p+2})$ of weight n are linearly independent and that $\partial_{<n,p}$ is bijective, as announced before.
The last step is just to realize that the bijectivity of $\partial_{<n,l}$ also implies that elements of different depths are also linearly independent. The proof could be done by contradiction: by applying $\partial_{<n,p}$ on a linear combination where p is the maximal depth appearing, we arrive at an equality between same level elements. □

□

4.4 Hoffman $\star$

Theorem 4.4.1. *If the analytic conjecture (4.4.4) holds, then the motivic Hoffman $\star$ family $\{\zeta^{\star,\mathfrak{m}}(\{2,3\}^{\times})\}$ is a basis of $\mathcal{H}^1$, the space of MMZV.*

For that purpose, we define an increasing filtration $\mathcal{F}^L_\bullet$ on $\mathcal{H}^{2,3}$, called **level**, such that:

$$\mathcal{F}^L_l\mathcal{H}^{2,3} \text{ is spanned by } \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0},3,\cdots,3,\mathbf{2}^{a_p}), \text{ with less than 'l' 3.} \tag{4.19}$$

It corresponds to the motivic depth for this family, as we see through the proof below and the coaction calculus.

Sketch. The vector space $\mathcal{F}^L_l\mathcal{H}^{2,3}$ is stable under the action of $\mathcal{G}$ (4.21). The linear independence of the Hoffman $\star$ family is proved below (§4.4.2) thanks to a recursion on the level and on the weight, using the injectivity of a map ∂^L where ∂^L came out of the level and weight-graded part of the coaction Δ (cf. 4.4.2). The injectivity is proved via 2-adic properties of some coefficients conjectured in 4.4.4.
Indeed, when computing the level graded coaction (cf. Lemma 4.4.2) on the Hoffman $\star$ elements, looking at the left side, some elements appear, such as $\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)$ but also $\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)$. These are not always of depth 1 as we could expect,[I] but at least are abelians: product of motivic simple zeta values, as proved in Lemma 4.4.3.
To prove the linear independence of Hoffman $\star$ elements, we will then need to know some coefficients appearing in Lemma 4.4.3 (or at least the 2-adic valuation) of $\zeta(weight)$ for each of these terms, conjectured in 4.4.4, which is the only missing part of the proof, and can be solved at the analytic level.

4.4.1 Level graded coaction

Let use the following form for a MMZV*, gathering the 2:

$$\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0},c_1,\cdots,c_p,\mathbf{2}^{a_p}),\quad c_i\in\mathbb{N}^*, c_i\neq 2.$$

This writing is suitable for the Galois action (and coaction) calculus, since by the antipode relations (§4.2), much of the cuts from a 2 to a 2 get simplified (cf. Annexe §A.1).
For the Hoffman family, with only 2 and 3, the expression obtained is:[II]

$$\begin{aligned}
&D_{2r+1}\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0},3,\cdots,3,\mathbf{2}^{a_p})\\
&= \delta_{2r+1}\sum_{i<j}\left[\begin{array}{lll}
+ & \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_{i+1}},3,\cdots,3,\mathbf{2}^{\leq a_j}) & \otimes\ \zeta^{\star,\mathfrak{m}}(\cdots,3,\mathbf{2}^{1+a_i+\leq a_j},3,\cdots)\\
- & \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{\leq a_i},3,\cdots,3,\mathbf{2}^{a_{j-1}}) & \otimes\ \zeta^{\star,\mathfrak{m}}(\cdots,3,\mathbf{2}^{1+a_j+\leq a_i},3,\cdots)\\
+ & \left(\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_{i+1}},3,\cdots,\mathbf{2}^{a_j},3)+\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{<a_i},3,\cdots,\mathbf{2}^{a_j},3)\right) & \otimes\ \zeta^{\star,\mathfrak{m}}(\cdots,3,\mathbf{2}^{<a_i},3,\mathbf{2}^{a_{j+1}},3,\cdots)\\
- & \left(\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_{j+1}},3,\cdots,3)+\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{<a_j},3,\cdots,3)\right) & \otimes\ \zeta^{\star,\mathfrak{m}}(\cdots,3,\mathbf{2}^{a_{i-1}},3,\mathbf{2}^{<a_j},3,\cdots)
\end{array}\right]\\
&\quad + \delta_{2r+1}\quad \left(\zeta^{\star,\mathfrak{l}}(\mathbf{2}^{a_0},3,\cdots,3,\mathbf{2}^{\leq a_i})-\zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{\leq a_i},3,\cdots,3,\mathbf{2}^{a_0})\right)\ \otimes\ \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{\leq a_i},3,\cdots)\\
&\quad + \delta_{2r+1}\quad \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{\leq a_j},3,\cdots,3,\mathbf{2}^{a_p})\ \otimes\ \zeta^{\star,\mathfrak{m}}(\cdots,3,\mathbf{2}^{\leq a_j}).
\end{aligned} \tag{4.20}$$

In particular, the coaction on the Hoffman $\star$ elements is stable.
By the previous expression (4.20), we see that each cut (of odd length) removes at least one 3. It means that the level filtration is stable under the action of $\mathcal{G}$ and:

$$D_{2r+1}(\mathcal{F}^L_l\mathcal{H}^{2,3})\subset\mathcal{L}_{2r+1}\otimes\mathcal{F}^L_{l-1}\mathcal{H}^{2,3}_{n-2r-1}. \tag{4.21}$$

[I] As for the Hoffman non $\star$ case done by Francis Brown, using a result of Don Zagier for level 1.
[II] Cf. Lemma A.1.2; where δ_{2r+1} means here that the left side has to be of weigh $2r+1$.

Then, let consider the level graded derivation:

$$gr_l^L D_{2r+1} : gr_l^L \mathcal{H}_n^{2,3} \to \mathcal{L}_{2r+1} \otimes gr_{l-1}^L \mathcal{H}_{n-2r-1}^{2,3}. \tag{4.22}$$

If we restrict ourselves to the cuts in the coaction that remove exactly one 3 in the right side, the formula (4.20) leads to:

$gr_l^L D_{2r+1} \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_p}) =$

$$\begin{array}{ll} & -\delta_{a_0<r\leq a_0+a_1+2} \quad \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, 3, \mathbf{2}^{r-a_0-2}) \quad \otimes \quad \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0+a_1+1-r}, 3, \cdots) \\ \sum_{i<j} & \left[\begin{array}{l} \delta_{r\leq a_i} \quad \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r) \qquad \otimes \left(\zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i-1}+a_i-r+1}, 3, \cdots) - \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i+1}+a_i-r+1}, 3, \cdots)\right) \\ + \left(\delta_{r=a_i+2}\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i}, 3) + \delta_{r<a_i+a_{i-1}+3}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{r-a_i-3}, 3, \mathbf{2}^{a_i}, 3)\right) \otimes \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_i+a_{i-1}-r+1}, 3, \mathbf{2}^{a_{i+1}}, 3, \cdots) \\ - \left(\delta_{r=a_i+2}\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i}, 3) + \delta_{r<a_i+a_{i+1}+3}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{r-a_i-3}, 3, \mathbf{2}^{a_i}, 3)\right) \otimes \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i-1}}, 3, \mathbf{2}^{a_i+a_{i+1}-r+1}, 3, \cdots) \end{array} \right] \\ \text{(D)} & +\delta_{a_p+1\leq r\leq a_p+a_{p-1}+1} \quad \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{r-a_p-1}, 3, \mathbf{2}^{a_p}) \quad \otimes \quad \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_p+a_{p-1}-r+1}). \end{array} \tag{4.23}$$

By the antipode ⧢ relation (cf. 4.7):

$$\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^a, 3, \mathbf{2}^b, 3) = \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^b, 3, \mathbf{2}^{a+1}) = \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{b+1}, 3, \mathbf{2}^{a+1}) - \zeta^{\star,\mathfrak{l}}(\mathbf{2}^{b+1}, 3, \mathbf{2}^{a+1}).$$

Then, by Lemma 4.4.3, all the terms appearing in the left side of $gr_l^L D_{2r+1}$ are product of simple MZV, which turns into, in the coalgebra $\mathcal{L}$ a rational multiple of $\zeta^{\mathfrak{l}}(2r+1)$:

$$gr_l^L D_{2r+1}(gr_l^L \mathcal{H}_n^{2,3}) \subset \mathbb{Q}\zeta^{\mathfrak{l}}(2r+1) \otimes gr_{l-1}^L \mathcal{H}_{n-2r-1}^{2,3}.$$

Sending $\zeta^{\mathfrak{l}}(2r+1)$ to 1 with the projection $\pi : \mathbb{Q}\zeta^{\mathfrak{l}}(2r+1) \to \mathbb{Q}$, we can then consider:

- $\boldsymbol{\partial_{r,l}^L} : gr_l^L \mathcal{H}_n^{2,3} \to gr_{l-1}^L \mathcal{H}_{n-2r-1}^{2,3}$, defined as the composition

  $$\partial_{r,l}^L := gr_l^L \partial_{2r+1} := m \circ (\pi \otimes id)(gr_l^L D_r) : \quad gr_l^L \mathcal{H}_n^{2,3} \to \mathbb{Q} \otimes_{\mathbb{Q}} gr_{l-1}^L \mathcal{H}_{n-2r-1}^{2,3} \to gr_{l-1}^L \mathcal{H}_{n-2r-1}^{2,3}.$$

- $\boldsymbol{\partial_{<n,l}^L} := \oplus_{2r+1<n} \partial_{r,l}^L$.

The injectivity of this map is the keystone of the Hoffman* proof. Its explicit expression is:

Lemma 4.4.2. $\partial_{r,l}^L(\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_p})) =$

$$\begin{array}{ll} -\delta_{a_0<r\leq a_0+a_1+2}\widetilde{B}^{a_0+1,r-a_0-2} & \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0+a_1+1-r}, 3, \cdots) \\ +\sum_{i<j}\left(\delta_{r\leq a_i} C_r \,(\right. & \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i-1}+a_i-r+1}, 3, \cdots) - \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i+1}+a_i-r+1}, 3, \cdots)\big) \\ +\delta_{a_i+2\leq r\leq a_i+a_{i-1}+2}\widetilde{B}^{a_i+1,r-a_i-2} & \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_i+a_{i-1}-r+1}, 3, \mathbf{2}^{a_{i+1}}, 3, \cdots) \\ -\delta_{a_i+2\leq r\leq a_i+a_{i+1}+2}\widetilde{B}^{a_i+1,r-a_i-2} & \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_{i-1}}, 3, \mathbf{2}^{a_i+a_{i+1}-r+1}, 3, \cdots)\big) \\ \text{(D)} + \delta_{a_p+1\leq r\leq a_p+a_{p-1}+1} B^{r-a_p-1,a_p} & \zeta^{\star,\mathfrak{m}}(\cdots, 3, \mathbf{2}^{a_p+a_{p-1}-r+1}), \end{array}$$

$$\textit{with } \widetilde{B}^{a,b} := B^{a,b} C_{a+b+1} - A^{a,b}.$$

Proof. Using Lemma 4.4.3 for the left side of $gr_p^L D_{2r+1}$, and keeping just the coefficients of ζ^{2r+1}, we obtain easily this formula. In particular:

$\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^a, 3, \mathbf{2}^b) = \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a+1}, 3, \mathbf{2}^b) - \zeta^{\star,\mathfrak{l}}(\mathbf{2}^{a+1}, 3, \mathbf{2}^b) = \widetilde{B}^{a+1,b}\zeta^{\mathfrak{l}}(\overline{2a+2b+5})$.
$\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^a, 3, \mathbf{2}^b, 3) = \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^b, 3, \mathbf{2}^{a+1}) = \widetilde{B}^{b+1,a+1}\zeta^{\mathfrak{l}}(\overline{2a+2b+7})$.

□

4.4.2 Proof of Theorem 4.4.1

Since the cardinal of the Hoffman $\star$ family in weight n is equal to the dimension of $\mathcal{H}_n^1$, [I] it remains to prove that the Hoffman $\star$ elements are linearly independent. We will use a recursion on the level and the map defined above, $\partial^L_{<n,l}$.

Let prove the following statement:

$$\partial^L_{<n,l} : gr^L_l \mathcal{H}^{2,3}_n \to \oplus_{2r+1<n} gr^L_{l-1}\mathcal{H}^{2,3}_{n-2r-1} \text{ is bijective.} \tag{4.24}$$

Using the Conjecture (assumed for this theorem) 4.4.4, regarding the 2-adic valuation of these coefficients, with $r = a + b + 1$:[II]

$$\begin{cases} C_r = \frac{2^{2r+1}}{2r+1} & \Rightarrow v_2(C_r) = 2r+1. \\ \widetilde{B}^{a,b} := B^{a,b}C_r - A^{a,b} = 2^{2r+1}\left(\frac{1}{2r+1} - \frac{\binom{2r}{2a}}{2^{2r}-1}\right) & \Rightarrow v_2(\widetilde{B}^{a,b}) \geq 2r+1. \\ B^{a,b}C_r = C_r - 2\binom{2r}{2b+1} & \Rightarrow v_2(B^{0,r-1}C_r) = 2 + v_2(r) \leq v_2(B^{a,b}C_r) < 2r+1. \end{cases} \tag{4.25}$$

The deconcatenation terms in $\partial^L_{<n,l}$, which correspond to the terms with $B^{a,b}C_r$ are then the smallest 2-adically, which is crucial for the injectivity.

Now, define a matrix $M_{n,l}$ as the matrix of $\partial^L_{<n,l}$ on $\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_l})$ in terms of $\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{b_0}, 3, \cdots, 3, \mathbf{2}^{b_{l-1}})$; even if up to now, we do not know that these families are linearly independent. We order the elements on both sides by lexicographical order on $(a_l, \ldots, a_0)$, resp. $(r, b_{l-1}, \ldots, b_0)$, such that the diagonal corresponds to $r = a_l$ and $b_i = a_i$ for $i < l$ and claim:

The matrix $M_{n,l}$ of $\partial^L_{<n,l}$ on the Hoffman $\star$ elements is invertible.

Proof. Indeed, let $\widetilde{M}_{n,l}$ be the matrix $M_{n,l}$ where we have multiplied each line corresponding to D_{2r+1} by $(2^{-v_2(r)-2})$. Then modulo 2, because of the previous computations on the 2-adic valuations of the coefficients, only the deconcatenations terms remains. Hence, with the previous order, the matrix is, modulo 2, triangular with 1 on the diagonal; the diagonal being the case where $B^{0,r-1}C_r$ appears. This implies that $\det(\widetilde{M}_{n,l}) \equiv 1 \pmod 2$, and in particular is non zero. Consequently, the matrix $\widetilde{M}_{n,l}$ is invertible and so does $M_{n,l}$. □

The Hoffman $\star$ elements are linearly independent.

Proof. Let first prove it for the Hoffman $\star$ elements of a same level and weight.
Level 0 is obvious: $\zeta^{\star,\mathfrak{m}}(2)^n$ is a rational multiple of $(\pi^{\mathfrak{m}})^{2n}$.
Assuming by recursion on the level that the Hoffman $\star$ elements of weight n and level $l-1$ are linearly independent, since $M_{n,l}$ is invertible, this means both that the Hoffman $\star$ elements of weight n and level l are linearly independent and that $\partial^L_{<n,l}$ is bijective, as announced in (4.24).
The last step is to realize that the bijectivity of $\partial^L_{<n,l}$ also implies that Hoffman $\star$ elements of different levels are linearly independent. Indeed, proof can be done by contradiction: applying $\partial^L_{<n,l}$ to a linear combination of Hoffman $\star$ elements, l being the maximal number of 3, we arrive at an equality between same level elements, and at a contradiction. □

[I] Obviously same recursive relation: $d_n = d_{n-2} + d_{n-3}$
[II] The last inequality comes from the fact that $v_2(\binom{2r}{2b+1}) < 2r$.

4.4.3 Analytic conjecture

Here are the equalities needed for Theorem 4.4.1, known up to some rational coefficients:

Lemma 4.4.3. *With w, resp. d and ht denoting the weight, resp. the depth and the height. Let's recall first:*

$$\begin{array}{rcll} \zeta^{\mathfrak{m}}(\overline{r}) & = & (2^{1-r}-1) & \zeta^{\mathfrak{m}}(r). \\ \zeta^{\mathfrak{m}}(2n) & = & \frac{|B_n|2^{3n-1}3^n}{(2n)!} & \zeta^{\mathfrak{m}}(2)^n. \end{array} \tag{4.26}$$

(i) $\zeta^{\star,\mathfrak{m}}(\mathbf{2}^n) = -2\zeta^{\mathfrak{m}}(\overline{2n}) = \frac{(2^{2n}-2)6^n}{(2n)!}|B_{2n}|\zeta^{\mathfrak{m}}(2)^n$.

(ii) $\zeta_1^{\star,\mathfrak{m}}(\mathbf{2}^n) = -2\sum_{r=1}^n \zeta^{\mathfrak{m}}(2r+1)\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r})$.

(iii)

$$\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^n) = \sum_{d\leq n}\sum_{\substack{w(\boldsymbol{m})=2n\\ ht(\boldsymbol{m})=d(\boldsymbol{m})=d}} 2^{2n-2d}\zeta^{\mathfrak{m}}(\boldsymbol{m}) \tag{4.27}$$

$$= \sum_{2n=\sum s_k(2i_k+1)+2S\atop i_k\neq i_j}\left(\prod_{k=1}^p \frac{C_{i_k}^{s_k}}{s_k!}\zeta^{\mathfrak{m}}(\overline{2i_k+1})^{s_k}\right) D_S\zeta^{\mathfrak{m}}(2)^S.$$

$$\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^n) = -\sum_{d\leq n}\sum_{\substack{w(\boldsymbol{m})=2n+1\\ ht(\boldsymbol{m})=d(\boldsymbol{m})=d}} 2^{2n+1-2d}\zeta^{\mathfrak{m}}(\boldsymbol{m}) \tag{4.28}$$

$$= \sum_{2n+1=\sum s_k(2i_k+1)+2S\atop i_k\neq i_j}\left(\prod_{k=1}^p \frac{C_{i_k}^{s_k}}{s_k!}\zeta^{\mathfrak{m}}(\overline{2i_k+1})^{s_k}\right) D_S\zeta^{\mathfrak{m}}(2)^S$$

(iv) $\zeta^{\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b) = \sum A_r^{a,b}\zeta^{\mathfrak{m}}(\overline{2r+1})\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r})$.

(v)

$$\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b) = \sum_{w=\sum s_k(2i_k+1)+2S\atop i_k\neq i_j} B^{a,b}_{\substack{i_1,\cdots,i_p\\ s_1\cdots s_p}}\left(\prod_{k=1}^p \frac{C_{i_k}^{s_k}}{s_k!}\zeta^{\mathfrak{m}}(\overline{2i_k+1})^{s_k}\right) D_S\zeta^{\mathfrak{m}}(2)^S. \tag{4.29}$$

$$\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b) = D^{a,b}\zeta^{\mathfrak{m}}(2)^{\frac{w}{2}} + \sum_{w=\sum s_k(2i_k+1)+2S\atop i_k\neq i_j} B^{a,b}_{\substack{i_1,\cdots,i_p\\ s_1\cdots s_p}}\left(\prod_{k=1}^p \frac{C_{i_k}^{s_k}}{s_k!}\zeta^{\mathfrak{m}}(\overline{2i_k+1})^{s_k}\right) D_S\zeta^{\mathfrak{m}}(2)^S. \tag{4.30}$$

Where:

- $C_r = \frac{2^{2r+1}}{2r+1}$, D_S *explicit*[I] *and with the following constraint:*

$$A_r^{a,b} = A_r^{a,r-a-1} + C_r\left(B^{r-b-1,b} - B^{r-a-1,a} + \delta_{r\leq b} - \delta_{r\leq a}\right). \tag{4.31}$$

- *The recursive formula for B-coefficients, where $B^{x,y} := B^{x,y}_{\substack{x+y+1\\1}}$ and $r < a+b+1$:*

$$\begin{array}{rcl} B^{a,b}_{\substack{r\\1}} & = & \delta_{r\leq b} - \delta_{r<a} + B^{r-b-1,b} + \frac{D^{a-r-1,b}}{a+b-r+1} + \delta_{r=a}\frac{2(2^{2b+1}-1)6^{b+1}|B_{2b+2}|}{(2b+2)!D_{b+1}}. \\ B^{a,b}_{\substack{i_1,\cdots,i_p\\ s_1\cdots s_p}} & = & \begin{cases} \delta_{i_1\leq b} - \delta_{i_1<a} + B^{i_1-b-1,b} + B^{a-i_1-1,b}_{\substack{i_1,\ldots,i_p\\ s_1-1,\ldots,s_p}} & \text{for } \sum s_k \text{ odd} \\ \delta_{i_1\leq b} - \delta_{i_1\leq a} + B^{i_1-b-1,b} + B^{a-i_1,b}_{\substack{i_1,\ldots,i_p\\ s_1-1,\ldots,s_p}} & \text{else .} \end{cases} \end{array} \tag{4.32}$$

[I] Cf. Proof.

Before giving the proof, here is the (analytic) conjecture remaining on some of these coefficients, sufficient to complete the Hoffman $\star$ basis proof (cf. Theorem 4.4.1):

Conjecture 4.4.4. *The equalities* (v) *are satisfied for real MZV, with:*

$$B^{a,b} = 1 - \frac{2}{C_{a+b+1}} \binom{2a+2b+2}{2b+1}.$$

Remarks:

- This conjecture is of an entirely different nature from the techniques developed in this thesis. We can expect that it can proved using analytic methods as the usual techniques of identifying hypergeometric series, as in [82], or [63].
- The equality (iv) is already proven in the analytic case by Ohno-Zagier (cf.[56], [82]), with the values of the coefficient $A_r^{a,b}$ given below. Nevertheless, as we will see through the proofs below, to explicit the coefficients for the (stronger) motivic identity (iv), we need to prove the other identities in (v).
- We will use below a result of Ohno and Zagier on sums of MZV of fixed weight, depth and height to conclude for the coefficients for (iii).

Theorem 4.4.5. *If the analytic conjecture (*4.4.4*) holds, the equalities* (iv), (v) *are true in the motivic case, with the same values of the coefficients. In particular:*

$$A_r^{a,b} = 2\left(-\delta_{r=a} + \binom{2r}{2a}\right)\frac{2^{2r}}{2^{2r}-1} - 2\binom{2r}{2b+1}.$$

Proof. Remind that if we know a motivic equality up to one unknown coefficient (of $\zeta(weight)$), the analytic result analogue enables us to conclude on its value by Corollary 2.4.5.
Let assume now, in a recursion on n, that we know $\{B^{a,b}, D^{a,b}, B^{a,b}_{\substack{i_1\cdots i_p\\ s_1\cdots s_p}}\}_{a+b+1<n}$ and consider (a,b) such that $a+b+1=n$. Then, by (4.32), we are able to compute the $B^{a,b}_{\substack{\mathbf{i}\\ \mathbf{s}}}$ with $(s,i) \neq (1,n)$. Using the analytic (v) equality, and Corollary 2.4.5, we deduce the only remaining unknown coefficient $B^{a,b}$ resp. $D^{a,b}$ in (v).
Lastly, by recursion on n we deduce the $A_r^{a,b}$ coefficients: let assume they are known for $a+b+1<n$, and take (a,b) with $a+b+1=n$. By the constraint (4.31), since we already know B and C coefficients, we deduce $A_r^{a,b}$ for $r<n$. The remaining coefficient, $A_n^{a,b}$, is obtained using the analytic (iv) equality and Corollary 2.4.5. □

`Proof of` **Lemma** 4.4.3. :

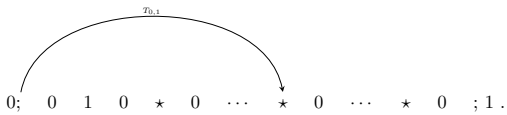
Proof. Computing the coaction on these elements, by a recursive procedure, we are able to prove these identities up to some rational coefficients, with the Corollary 2.4.5. When the analytic analogue of the equality is known (such as $(i), (ii), (iii)$) for MZV, we may conclude on the value of the remaining rational coefficient of $\zeta^{\mathfrak{m}}(\omega)$ by identification.
However, if the family is not stable under the coaction (as for (iv)), knowing the analytic case (by Ohno-Zagier) will not be enough.
The following proof refers to the formulation of D_{2r+1} of Lemma $A.1.1$: cuts of length $2r+1$ among the sequence of $0, 1$, or $\star$ corresponding to the iterated integral writing; there are different kind of cuts (according their extremities), and a cut may bring out two terms ($T_{0,0}$ and $T_{0,\star}$ etc.), as in the Lemma $A.1.1$.
The simplifications are illustrated by the diagrams, where some arrows (term of a cut) of the same color get simplified together, by rules specified in Annexe §$A.1$.

(i) The corresponding iterated integral:

$$I^{\mathfrak{m}}(0;1,0,\star,0\cdots,\star,0;1).$$

The only possible cuts of odd length are between two $\star$ ($T_{0,\star}$ and $T_{\star,0}$) or $T_{1,0}$ from the first 1 to a $\star$, or $T_{0,1}$ from a $\star$ to the last 1. By SHIFT (A.2), these cuts get simplified two by two. Since $D_{2r+1}(\cdot)$, for $2r+1<2n$ are all zero, it belongs to $\mathbb{Q}\zeta^{\mathfrak{m}}(2n)$ (Corollary 2.4.5).
Using the (known) analytic equality, we conclude about the value of the coefficient.

(ii) It is very similar to the previous case: using SHIFT $(A.2)$, it remains only the cut:



$$D_{2r+1}(\zeta_1^{\star,\mathfrak{m}}(\mathbf{2}^n))=\zeta_1^{\mathfrak{l},\star}(\mathbf{2}^r)\otimes\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r})=-2\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r}).$$

The last equality being deduced from the recursive hypothesis (smaller weight). The analytic equality (coming from the Zagier-Ohno formula, and the ⧢ regulation) enables us to conclude on the value of the remaining coefficient of $\zeta^{\mathfrak{m}}(2n+1)$.

(iii) By ⧢ regularisation, we can express each Euler sum $\star\star$ as a linear combination of Euler sums. For both Euler sums considered in (iii), it is:

$$\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^n)=\sum_{k_i\text{ even}}\zeta^{\mathfrak{m}}_{2n-\sum k_i}(k_1,\cdots,k_p)=\sum_{n_i\geq 2}\left(\sum_{\substack{k_i\text{ even}\\ k_i\leq n_i}}\binom{n_1-1}{k_1-1}\cdots\binom{n_d-1}{k_d-1}\right)\zeta^{\mathfrak{m}}(n_1,\cdots,n_d).$$

Using the multi-binomial formula:

$$2^{\sum m_i}=\sum_{l_i\leq m_i}\binom{m_1}{l_1}(1-(-1))^{l_1}\cdots\binom{m_d}{l_d}(1-(-1))^{l_d}=2^d\sum_{\substack{l_i\leq m_i\\ l_i\text{ odd}}}\binom{m_1}{l_1}\cdots\binom{m_d}{l_d}.$$

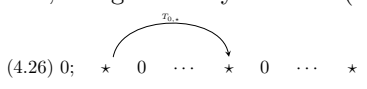
Thus:

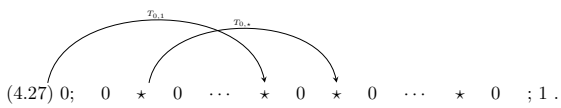
$$\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^n)=\sum_{d\leq n}\sum_{\substack{w(\mathbf{m})=2n\\ ht(\mathbf{m})=d(\mathbf{m})=d}}2^{2n-2d}\zeta^{\mathfrak{m}}(\mathbf{m}).$$

Similarly for (4.27), since:

$$\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^n)=\sum_{k_i\text{ even}}\zeta^{\mathfrak{m}}_{2n+1-\sum k_i}(k_1,\cdots,k_p)=\sum_{d\leq n}\sum_{\substack{w(\mathbf{m})=2n\\ ht(\mathbf{m})=d(\mathbf{m})=d}}2^{2n-2d}\zeta^{\mathfrak{m}}(\mathbf{m}).$$

Now, using still only SHIFT $(A.2)$, it remains the following cuts:





With a recursion on n for both (4.26), (4.27), we deduce:

$$D_{2r+1}(\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^n))=\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^r)\otimes\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{n-r})=C_r\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r-1}).$$

$$D_{2r+1}(\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^n))=\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r)\otimes\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r})=C_r\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r}).$$

To find the remaining coefficients, we need the analytic result corresponding, which is a consequence of the sum relation for MZV of fixed weight, depth and height, by Ohno and Zagier ([68], Theorem 1), via the hypergeometric functions.

Using [68], the generating series of these sums is, with $\alpha, \beta = \frac{x+y\pm\sqrt{(x+y)^2-4z}}{2}$:

$$\begin{aligned}\phi_0(x,y,z): &= \textstyle\sum_{\substack{s\leq d\\ w\geq d+s}} \left(\sum \zeta(\mathbf{k})\right) x^{w-d-s} y^{d-s} z^{s-1} \\ &= \tfrac{1}{xy-z}\left(1-\exp\left(\textstyle\sum_{m=2}^{\infty} \tfrac{\zeta(m)}{m}(x^m+y^m-\alpha^m-\beta^m)\right)\right).\end{aligned}$$

From this, let express the generating series of both $\zeta^{\star\star}(\mathbf{2}^n)$ and $\zeta_1^{\star\star}(\mathbf{2}^n)$:

$$\phi(x) := \sum_w \left(\sum_{\substack{ht(\mathbf{k})=d(\mathbf{k})=d\\ w\geq 2d}} 2^{w-2d}\zeta(\mathbf{k})\right) x^{w-2} = \phi_0(2x,0,x^2).$$

Using the result of Ohno and Don Zagier:

$$\phi(x) = \frac{1}{x^2}\left(\exp\left(\sum_{m=2}^{\infty}\frac{2^m-2}{m}\zeta(m)x^m\right)-1\right).$$

Consequently, both $\zeta^{\star\star}(\mathbf{2}^n)$ and $\zeta_1^{\star\star}(\mathbf{2}^n)$ can be written explicitly as polynomials in simple zetas. For $\zeta^{\star\star}(\mathbf{2}^n)$, by taking the coefficient of x^{2n-2} in $\phi(x)$:

$$\zeta^{\star\star}(\mathbf{2}^n) = \sum_{\substack{\sum m_i s_i = 2n\\ m_i\neq m_j}} \prod_{i=1}^{k}\left(\frac{1}{s_i!}\left(\zeta(m_i)\frac{2^{m_i}-2}{m_i}\right)^{s_i}\right).$$

Gathering the zetas at even arguments, it turns into:

$$\zeta^{\star\star}(\mathbf{2}^n) = \sum_{\substack{\sum(2i_k+1)s_k+2S=2n\\ i_k\neq i_j}} \prod_{i=1}^{p}\left(\frac{1}{s_k!}\left(\zeta(2i_k+1)\frac{2^{2i_k+1}-2}{2i_k+1}\right)^{s_k}\right) d_S\zeta(2)^S,$$

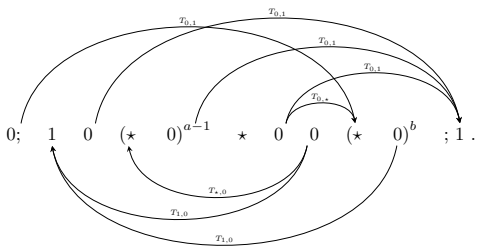
$$\text{where } d_S := 3^S\cdot 2^{3S}\sum_{\substack{\sum m_i s_i = S\\ m_i\neq m_j}} \prod_{i=1}^{k}\left(\frac{1}{s_i!}\left(\frac{\mid B_{2m_i}\mid (2^{2m_i-1}-1)}{2m_i(2m_i)!}\right)^{s_i}\right). \tag{4.33}$$

It remains to turn $\zeta(odd)$ into $\zeta(\overline{odd})$ by 4.26 to fit the expression of the Lemma:

$$\zeta^{\star\star}(\mathbf{2}^n) = \sum_{\substack{\sum(2i_k+1)s_k+2S=2n\\ i_k\neq i_j}} \prod_{i=1}^{p}\left(\frac{1}{s_k!}\left(c_{i_k}\zeta(\overline{2i_k+1})\right)^{s_k}\right) d_S\zeta(2)^S, \text{ where } c_r = \frac{2^{2r+1}}{2r+1}.$$

It is completely similar for $\zeta_1^{\star\star}(\mathbf{2}^n)$: by taking the coefficient of x^{2n-3} in $\phi(x)$, we obtained the analytic analogue of (4.25), with the same coefficients d_S and c_r.
Now, using these analytic results for (4.26), (4.27), by recursion on the weight, we can identify the coefficient D_S and C_r with resp. d_S and c_r, since there is one unknown coefficient at each step of the recursion.

(iv) After some simplifications by Antipodes rules (§A.1), only the following cuts remain:



This leads to the formula:

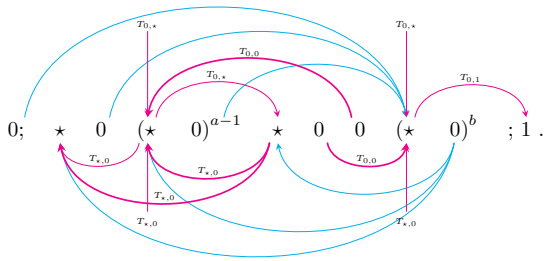
$$D_{2r+1}(\zeta^{\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) = \big(\zeta^{\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^{r-a-1})+$$
$$(\delta_{r\leq b}-\delta_{r\leq a})\,\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^r)+\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{r-b-1},3,\mathbf{2}^b)-\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{r-a-1},3,\mathbf{2}^a)\big)\otimes\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r}).$$

In particular, the Hoffman $\star$ family is not stable under the coaction, so we need first to prove (v), and then:

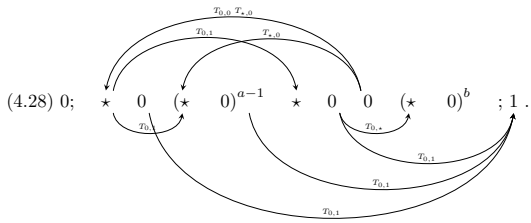
$$D_{2r+1}(\zeta^{\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) = \big(A_r^{a,r-a-1}+C_r\left(B^{r-b-1,b}-B^{r-a-1,a}+\delta_{r\leq b}-\delta_{r\leq a}\right)\big)\,\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n-r}).$$

It leads to the constraint (4.31) given above for A coefficients. To specify the values of A coefficients, we would need not only the analytic Ohno Zagier formula (known) but also the analogue of (v) formulas for the analytic case; as stated in Conjecture 4.4.4.

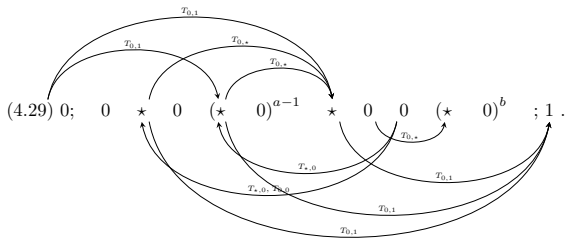
(v) By Annexe rules, the following cuts get simplified (colors by colors), above with below-[I]



Indeed, cyan arrows get simplified by Antipode ⊔⊔, $T_{0,0}$ resp. $T_{0,\star}$ above with $T_{0,0}$ resp. $T_{\star,0}$ below; magenta ones by Shift $(A.2)$, term above with the term below shifted by two on the left. It remains the following cuts for (4.28):



In a very similar way, the simplifications lead to the following remaining terms:



Then, the derivations reduce to:

$$D_{2r+1}(\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) = \Big((\delta_{r\leq b}-\delta_{r\leq a})\,\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r)+\delta_{r>b}\zeta^{\star\star,\mathfrak{m}-\mathfrak{l}}(\mathbf{2}^{r-b-1},3,\mathbf{2}^b)\Big)\otimes\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r})+$$
$$+\delta_{r\leq a-1}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r)\otimes\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^{a-r-1},3,\mathbf{2}^b)+\delta_{r=a}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^a)\otimes\zeta_2^{\star\star,\mathfrak{m}}(\mathbf{2}^b).$$

[I] The vertical arrows indicates a cut from the $\star$ to a $\star$ of the same group.

$$D_{2r+1}(\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) = \Big((\delta_{r\leq b}-\delta_{r\leq a})\,\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r)+\zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{r-b-1},3,\mathbf{2}^b)\Big)\otimes\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r})+$$
$$+\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^r)\otimes\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{a-r},3,\mathbf{2}^b).$$

With a recursion on w for both:

$$\begin{aligned} D_{2r+1}(\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) &= C_r\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes \\ &\big(\big(\delta_{r\leq b}-\delta_{r<a}+B_r^{r-b-1,b}\big)\,\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r})+\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^{\boldsymbol{a-r-1}},3,\mathbf{2}^b)+\delta_{r=a}\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{b+1})\big)\,. \end{aligned}$$

$$D_{2r+1}(\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^a,3,\mathbf{2}^b)) = C_r\zeta^{\mathfrak{l}}(\overline{2r+1})\otimes\big(\big(\delta_{r\leq b}-\delta_{r\leq a}+B_r^{r-b-1,b}\big)\,\zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^{n-r})+\zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{a-r},3,\mathbf{2}^b)\big)\,.$$

This leads to the recursive formula (4.32) for B.

□

4.5 Motivic generalized Linebarger Zhao Conjecture

We conjecture the following motivic identities, which express each motivic MZV $\star$ as a motivic Euler $\sharp$ sum:

Conjecture 4.5.1. *For $a_i, c_i \in \mathbb{N}^*$, $c_i \neq 2$,*

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0},c_1,\cdots,c_p,\mathbf{2}^{a_p}\right) = (-1)^{1+\delta_{c_1}}\zeta^{\sharp,\mathfrak{m}}\left(B_0,\mathbf{1}^{c_1-3},\cdots,\mathbf{1}^{c_i-3},B_i,\ldots,B_p\right),$$

where $\begin{cases} B_0 := \pm(2a_0+1-\delta_{c_1}) \\ B_i := \pm(2a_i+3-\delta_{c_i}-\delta_{c_{i+1}}) \\ B_p := \pm(2a_p+2-\delta_{c_p}) \end{cases}$ *, with* $\pm := \begin{cases} -\ \textit{if}\ \mid B_i \mid\ \textit{even} \\ +\ \textit{if}\ \mid B_i \mid\ \textit{odd} \end{cases}$ *, and* $\delta_c := \delta_{c=1}$, *the Kronecker symbol.*
and $\mathbf{1}^n := \mathbf{1}^{min(0,n)}$ is a sequence of n 1 if $n \in \mathbb{N}$, an empty sequence else.

Remarks:

- Motivic Euler $\sharp$ sums appearing on the right side have already been proven to be unramified in §4.3, i.e. MMZV.
- This conjecture implies that the motivic Hoffman $\star$ family is a basis, since it corresponds here to the motivic Euler $\sharp$ sum family proved to be a basis in Theorem 4.3.2: cf. (4.37).
- The number of sequences of consecutive 1 in $\zeta^\star$, n_1 is linked with the number of even in $\zeta^\sharp$, n_e, here by the following formula:

  $$n_e = 1+2n_1-2\delta_{c_p}-\delta_{c_1}.$$

  In particular, when there is no 1 in the MMZV $\star$, there is only one even (at the end) in the Euler sum $\sharp$. There are always at least one even in the Euler sums.

Special cases of this conjecture, which are already proven for real Euler sums (references indicated in the braket), but remain conjectures in the motivic case:

Two-One [Ohno Zudilin, [69].]

$$\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0},1,\cdots,1,\mathbf{2}^{a_p}) = -\zeta^{\sharp,\mathfrak{m}}\left(\overline{2a_0},2a_1+1,\ldots,2a_{p-1}+1,2a_p+1\right). \tag{4.34}$$

Three-One [Broadhurst et alii, [9].] [I]

$$\zeta^{\star,\mathfrak{m}}(\mathbf{2}^{a_0},1,\mathbf{2}^{a_1},3\cdots,1,\mathbf{2}^{a_{p-1}},3,\mathbf{2}^{a_p}) = -\zeta^{\sharp,\mathfrak{m}}\left(\overline{2a_0},\overline{2a_1+2},\ldots,\overline{2a_{p-1}+2},\overline{2a_p+2}\right). \tag{4.35}$$

[I] The Three-One formula was conjectured for real Euler sums by Zagier, proved by Broadhurst et alii in [9].

Linebarger-Zhao⋆ [Linebarger Zhao, [65]] With $c_i \geq 3$:

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, c_1, \cdots, c_p, \mathbf{2}^{a_p}\right) = -\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, \mathbf{1}^{c_1-3}, \cdots, \mathbf{1}^{c_i-3}, 2a_i+3, \ldots, \overline{2a_p+2}\right) \tag{4.36}$$

In particular, when all $c_i = 3$:

$$\zeta^{\star,\mathfrak{m}}\left(\mathbf{2}^{a_0}, 3, \cdots, 3, \mathbf{2}^{a_p}\right) = -\zeta^{\sharp,\mathfrak{m}}\left(2a_0+1, 2a_1+3, \ldots, 2a_{p-1}+3, \overline{2a_p+2}\right). \tag{4.37}$$

`Examples`: Particular identities implied by the previous conjecture, sometimes known for MZV and which could then be proven for motivic Euler sums directly with the coaction:

- $\zeta^{\star,\mathfrak{m}}(1, \{2\}^n) = 2\zeta^{\mathfrak{m}}(2n+1)$.
- $\zeta^{\star,\mathfrak{m}}(1, \{2\}^a, 1, \{2\}^b) = \zeta^{\sharp\mathfrak{m}}(2a+1, 2b+1) = 4\zeta^{\mathfrak{m}}(2a+1, 2b+1) + 2\zeta^{\mathfrak{m}}(2a+2b+2)$.
- $\zeta^{\mathfrak{m}}(n) = -\zeta^{\sharp,\mathfrak{m}}(\{1\}^{n-2}, -2) = -\sum_{\substack{w(\boldsymbol{k})=n \\ \boldsymbol{k}\text{admissible}}} \mathbf{2}^p \zeta^{\mathfrak{m}}(k_1, \ldots, k_{p-1}, -k_p)$.
- $\zeta^{\star,\mathfrak{m}}(\{2\}^n) = \sum_{\substack{\boldsymbol{k}\in\{\text{ even }\}^\times \\ w(\boldsymbol{k})=2n}} \zeta^{\mathfrak{m}}(\boldsymbol{k}) = -2\zeta^{\mathfrak{m}}(-2n)$.

We paved the way for the proof of Conjecture 4.5.1, bringing it back to an identity in $\mathcal{L}$:

Theorem 4.5.2. *Let assume:*

(*i*) *The analytic version of* 4.5.1 *is true.*

(*ii*) *In the coalgebra* $\mathcal{L}$*, i.e. modulo products, for odd weights:*

$$\zeta^{\sharp,\mathfrak{l}}_{B_0-1}(\mathbf{1}^{\gamma_1}, \cdots, \mathbf{1}^{\gamma_p}, B_p) \equiv \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0-1}, c_1, \cdots, \mathbf{2}^{a_p}) - \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, c_1-1, \ldots, \mathbf{2}^{a_p}), \tag{4.38}$$

$$\textit{with } c_1 \geq 3,\ a_0 > 0,\ \gamma_i = c_i - 3 + 2\delta_{c_i} \textit{ and } \begin{cases} B_0 = 2a_0 + 1 - \delta_{c_1} \\ B_i = 2a_i + 3 - \delta_{c_i} - \delta_{c_{i+1}} \\ B_p = 2a_p + 3 - \delta_{c_p} \end{cases}.$$

Then:

I. Conjecture 4.5.1 *is true, for motivic Euler sums.*

II. In the coalgebra $\mathcal{L}$*, for odd weights, with* $c_1 \geq 3$ *and the previous notations:*

$$\zeta^{\sharp,\mathfrak{l}}(\mathbf{1}^{\gamma_1}, \cdots, \mathbf{1}^{\gamma_p}, B_p) \equiv -\zeta_1^{\star,\mathfrak{l}}(c_1-1, \mathbf{2}^{a_1}, c_2, \cdots, c_p, \mathbf{2}^{a_p}). \tag{4.39}$$

`ADDENDUM`: The hypothesis (*i*) is proved: J. Zhao deduced it from its Theorem 1.4 in [86].

Remark: The (*ii*) hypothesis should be proven either directly via the various relations in $\mathcal{L}$ proven in §4.2 (as for 4.39), or using the coaction, which would require the analytic identity corresponding. Beware, (*ii*) would only be true in $\mathcal{L}^2$, not in $\mathcal{H}^2$.

Proof. To prove this equality 1. at a motivic level by recursion, we would need to proof that the coaction is equal on both side, and use the conjecture analytic version of the same equality. We prove I and II successively, in a same recursion on the weight:

I. Using the formulas of the coactions D_r for these families (Lemma A.1.2 and A.1.4), we can gather terms in both sides according to the right side, which leads to three types:

$$\begin{array}{lllll} (a) & \zeta^{\star,\mathfrak{m}}(\cdots, \mathbf{2}^{a_i}, \alpha, \mathbf{2}^{\beta}, c_{j+1}, \cdots) & \longleftrightarrow & \zeta^{\sharp,\mathfrak{m}}(B_0 \cdots, B_i, 1^{\gamma}, B, 1^{\gamma_{j+1}}, \ldots, B_p) \\ (b) & \zeta^{\star,\mathfrak{m}}(\cdots, \mathbf{2}^{a_{i-1}}, c_i, \mathbf{2}^{\beta}, c_{j+1}, \cdots) & \longleftrightarrow & \zeta^{\sharp,\mathfrak{m}}(B_0 \cdots, B_{i-1}, 1^{\gamma_i}, B, 1^{\gamma_{j+1}}, \ldots, B_p) \\ (c) & \zeta^{\star,\mathfrak{m}}(\cdots, c_i, \mathbf{2}^{\beta}, \alpha, \mathbf{2}^{a_j}, \cdots) & \longleftrightarrow & \zeta^{\sharp,\mathfrak{m}}(B_0 \cdots, 1^{\gamma_{i+1}}, B, 1^{\gamma}, B_{j+1}, \ldots, B_p) \end{array},$$

with $\gamma = \alpha - 3$ and $B = 2\beta + 3 - \delta_{c_{j+1}}$, or $B = 2\beta + 3 - \delta_{c_i} - \delta_{c_{j+1}}$ for (*b*).
The third case, antisymmetric of the first case, may be omitted below. By recursive hypothesis, these right sides are equal and it remains to compare the left sides associated:

(a) On the one hand, by lemma $A.1.2$, the left side corresponding:

$$\delta_{\substack{3\le\alpha\le c_{i+1}-1\\0\le\beta a_j}}\zeta^{\star,\mathfrak{l}}_{c_{i+1}-\alpha}-(\mathbf{2}^{a_j-\beta},\dots,\mathbf{2}^{a_{i+1}}).$$

On the other hand (Lemma $A.1.4$), the left side is:

$$-\delta_{\substack{2\le B\le B_j\\0\le\gamma\le\gamma_{i+1}-1}}\zeta^{\sharp,\mathfrak{l}}(B_j-B+1,1^{\gamma_j},\dots,1^{\gamma_{i+1}-\gamma-1}).$$

They are both equal, by 4.39, where $c_{i+1}-\alpha+2$ corresponds to c_1 and is greater than 3.

(b) By lemma $A.1.2$, the left side corresponding for $\zeta^\star$:

$$\begin{array}{llll}
- & \delta_{c_i>3}\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i},\dots,\mathbf{2}^{a_j-\beta-1}) & + & \delta_{c_j>3}\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_j},\dots,\mathbf{2}^{a_i-\beta-1})\\
- & \delta_{c_i=1}\zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_j-\beta},\dots,\mathbf{2}^{a_i}) & + & \delta_{c_{j+1}=1}\zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i-\beta},\dots,\mathbf{2}^{a_j})\\
+ & \delta_{\substack{c_{i+1}=1\\\beta>a_i}}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i+a_j-\beta},\dots,\mathbf{2}^{a_{i+1}}) & - & \delta_{\substack{c_j=1\\\beta>a_j}}\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i+a_j-\beta},\dots,\mathbf{2}^{a_j-1})\\
- & \delta_{a_j<\beta\le a_i+a_j+1}\zeta^{\star\star,\mathfrak{l}}_{c_j-2}(\mathbf{2}^{a_j-1},\dots,\mathbf{2}^{a_i+a_j-\beta+1}) & + & \delta_{a_i<\beta\le a_j+a_i+1}\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-2}(\mathbf{2}^{a_{i+1}},\dots,\mathbf{2}^{a_i+a_j-\beta+1}).
\end{array}$$

It should correspond to (using still lemma $A.1.4$), with $B_k=2a_k+3-\delta_{c_k}-\delta_{c_{k+1}}$, $\gamma_k=c_k-3+2\delta_{c_k}$ and $B=2\beta+3-\delta_{c_i}-\delta_{c_{j+1}}$:

$$\Big(\delta_{B_i<B}\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i+B_j-B}(1^{\gamma_j},\dots,1^{\gamma_{i+1}})-\delta_{B_j<B}\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i+B_j-B}(1^{\gamma_{i+1}},\dots,1^{\gamma_j})$$

$$+\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i-B}(1^{\gamma_{i+1}},\dots,B_j)-\zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\dots,B_i)\Big).$$

The first line has even depth, while the second line has odd depth, as noticed in Lemma $A.1.4$. Let distinguish three cases, and assume $a_i<a_j$:[I]

(i) When $\beta<a_i<a_j$, we should have:

$$\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i-B}(1^{\gamma_{i+1}},\dots,B_j)-\zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\dots,B_i)\text{ equal to:}\tag{4.40}$$

$$\begin{array}{llll}
-\delta_{c_i>3} & \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_j-\beta-1},\dots,\mathbf{2}^{a_i}) & -\delta_{c_i=1} & \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_j-\beta},\dots,\mathbf{2}^{a_i})\\
+\delta_{c_{j+1}>3} & \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i-\beta-1},\dots,\mathbf{2}^{a_j}) & +\delta_{c_{j+1}=1} & \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i-\beta},\dots,\mathbf{2}^{a_j})
\end{array}$$

· Let first look at the case where $c_i>3,c_{j+1}>3$. Renumbering the indices, using Shift for odd depth for the second line, it is equivalent to, with $\alpha=\beta+1$, $B_p=2a_p+3,B_0=2a_0+3$:

$$\begin{array}{lll}
 & \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0-\alpha},c_1,\cdots,c_p,\mathbf{2}^{a_p}) & -\ \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0},c_1,\cdots,c_p,\mathbf{2}^{a_p-\alpha})\\
\equiv & \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B}(1^{\gamma_1},\cdots,1^{\gamma_p},B_p) & -\ \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-B}(B_0,1^{\gamma_1},\cdots,1^{\gamma_p})\\
\equiv & \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-1}(B_0-B+1,1^{\gamma_1},\cdots,1^{\gamma_p}) & -\ \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-B}(B_0,1^{\gamma_1},\cdots,1^{\gamma_p})\\
\equiv & \zeta^{\sharp,\mathfrak{l}}_{B_p-1}(B_0-B+1,1^{\gamma_1},\cdots,1^{\gamma_p}) & -\ \zeta^{\sharp,\mathfrak{l}}_{B_p-B}(B_0,1^{\gamma_1},\cdots,1^{\gamma_p}).
\end{array}$$

This boils down to (4.38) applied to each $\zeta_2^{\star\star}$, since by Shift ($A.2$) the two terms of the type $\zeta_1^{\star\star}$ get simplified.

· Let now look at the case where $c_i=1,c_{j+1}>3$ [II]; hence $B_i=2a_i+2-\delta_{c_{i+1}}$, $B=2\beta+2$. In a first hand, we have to consider:

$$\zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_i-\beta-1},c_{i+1},\cdots,c_j,\mathbf{2}^{a_j})-\zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_j-\beta},c_j,\cdots,c_{i+1},\mathbf{2}^{a_i}).$$

[I] The case $a_j<a_i$ is anti-symmetric, hence analogue.
[II] The case $c_{j+1}=1,c_i>3$ being analogue, by symmetry.

By renumbering indices in 4.40, the correspondence boils down here to the following $\diamond = \bowtie$, where $B_0 = 2a_0 + 3 - \delta_{c_1}$, $B_i = 2a_i + 3 - \delta_{c_i} - \delta_{c_{i+1}}$, $B = 2\beta + 2$:

$$(\diamond) \quad \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0-\beta}, c_1, \cdots, c_p, \mathbf{2}^{a_p}) - \zeta^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0+1}, c_1, \cdots, c_p, \mathbf{2}^{a_p-\beta})$$

$$(\bowtie) \quad \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1}(1^{\gamma_1}, \ldots, 1^{\gamma_p}, B_p) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-B}(1^{\gamma_p}, \ldots, 1^{\gamma_1}, B_0+1).$$

Turning in $(\diamond)$ the second term into a $\zeta^{\star,\mathfrak{l}}(2, \cdots) + \zeta_2^{\star\star,\mathfrak{l}}(\cdots)$, and applying the identity (4.38) for both terms $\zeta_2^{\star\star,\mathfrak{l}}(\cdots)$ leads to

$$(\diamond) \begin{cases} +\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0-\beta+1}, c_1-1, \cdots, c_p, \mathbf{2}^{a_p}) & -\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0+1}, c_1-1, \cdots, c_p, \mathbf{2}^{a_p-\beta}) & (\diamond_{\mathbf{1}}) \\ +\zeta^{\sharp,\mathfrak{l}}_{B_0-B+1}(\mathbf{1}^{\gamma_1}, \cdots, \mathbf{1}^{\gamma_p}, B_p) & -\zeta^{\sharp,\mathfrak{l}}_{B_0-1}(\mathbf{1}^{\gamma_1}, \cdots, \mathbf{1}^{\gamma_p}, B_p - B + 2) & (\diamond_{\mathbf{2}}) \\ -\zeta^{\star,\mathfrak{l}}(\mathbf{2}^{a_0+1}, c_1, \cdots, c_p, \mathbf{2}^{a_p-\beta}) & & (\diamond_{\mathbf{3}}) \end{cases}$$

The first line, $(\diamond_1)$ by Shift is zero. We apply Antipode $*$ on the terms of the second line, then turn each into a difference $\zeta_n^{\sharp\sharp}(m, \cdots) - \zeta_{n+m}^{\sharp\sharp}(\cdots)$; the terms of the type $\zeta_{n+m}^{\sharp\sharp}(\cdots)$, are identical and get simplified:

$$(\diamond_{\mathbf{2}}) \begin{array}{lll} \equiv & \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1}(B_p, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) & -\zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1+B_p}(\mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) \\ & -\zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-1}(B_p - B + 2, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) & +\zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1+B_p}(\mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) \\ \equiv & \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1}(B_p, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) & -\zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-1}(B_p - B + 2, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}). \end{array}$$

Furthermore, applying the recursion hypothesis (I.), i.e. conjecture 4.5.1 on $(\diamond_3)$, and turn it into a difference of $\zeta^{\sharp\sharp}$:

$$(\diamond_{\mathbf{3}}) \begin{array}{ll} & -\zeta^{\star,\mathfrak{l}}(\mathbf{2}^{a_0+1}, c_1, \cdots, c_p, \mathbf{2}^{a_p-\beta}) \\ \equiv & -\zeta^{\sharp,\mathfrak{l}}(B_p - B + 1, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}, B_0) \\ \equiv & -\zeta^{\sharp\sharp,\mathfrak{l}}(B_p - B + 1, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}, B_0) + \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-B+1}(\mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}, B_0) \end{array}$$

When adding $(\diamond_{\mathbf{2}})$ and $(\diamond_{\mathbf{3}})$ to get $(\diamond)$, the two last terms (odd depth) being simplified by Shift, it remains:

$$(\diamond) \quad \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0-B+1}(B_p, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}) - \zeta^{\sharp\sharp,\mathfrak{l}}(B_p - B + 1, \mathbf{1}^{\gamma_p}, \cdots, \mathbf{1}^{\gamma_1}, B_0).$$

This, applying Antipode $*$ to the first term, Cut and Shift to the second, corresponds to $(\bowtie)$.

(ii) When $\beta > a_j > a_i$, we should have:

$$\begin{array}{ll} & -\zeta^{\star\star,\mathfrak{l}}_{c_j-2}(\mathbf{2}^{a_{j-1}}, \ldots, \mathbf{2}^{a_i+a_j-\beta+1}) \quad +\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-2}(\mathbf{2}^{a_{i+1}}, \ldots, \mathbf{2}^{a_i+a_j-\beta+1}) \\ \equiv & +\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i+B_j-B}(1^{\gamma_j}, \ldots, 1^{\gamma_{i+1}}) \quad -\zeta^{\sharp\sharp,\mathfrak{l}}_{B_i+B_j-B}(1^{\gamma_{i+1}}, \ldots, 1^{\gamma_j}). \end{array}$$

Using Shift $(A.2)$ for the first line, and renumbering the indices, it is equivalent to, with $c_1, c_p \geq 3$ and $a_0 > 0$:

$$\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, c_1 - 1, \cdots, c_p) - \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, c_1, \cdots, c_p - 1) \tag{4.41}$$

$$\equiv \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0+2}(1^{\gamma_1}, \cdots, 1^{\gamma_p}) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_0+2}(1^{\gamma_p}, \cdots, 1^{\gamma_1}) \equiv \zeta^{\sharp,\mathfrak{l}}_{B_0+2}(1^{\gamma_1}, \ldots, 1^{\gamma_p}).$$

The last equality comes from Corollary 4.2.7, since depth is even. By (4.42) applied on each term of the first line

$$\zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, c_1 - 1, \cdots, c_p) - \zeta_1^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0}, c_1, \cdots, c_p - 1)$$

$$\equiv \zeta_2^{\star\star,\mathfrak{l}}(\mathbf{2}^{a_0-1}, c_1, \cdots, c_p) + \zeta^{\sharp\mathfrak{l}}_{2a_0}(3, 1^{\gamma_p}, \cdots, 1^{\gamma_1}) - \zeta_2^{\star\star,\mathfrak{l}}(c_p, \cdots, c_1, \mathbf{2}^{a_0-1}) - \zeta_2^{\sharp\sharp,\mathfrak{l}}(2a_0+1, 1^{\gamma_p}, \cdots, 1^{\gamma_1}).$$

By Antipode ⧢, the $\zeta^{\star\star}$ get simplified, and by the definition of $\zeta^{\sharp\sharp}$, the previous equality is equal to:

$$\equiv -\zeta^{\sharp\mathfrak{l}}_{2a_0+3}(1^{\gamma_p},\cdots,1^{\gamma_1})+\zeta^{\sharp\sharp\mathfrak{l}}_{2a_0}(3,1^{\gamma_p},\cdots,1^{\gamma_1})+\zeta^{\sharp\sharp\mathfrak{l}}_{2a_0+4}(1^{\gamma_p-1},\cdots,1^{\gamma_1})$$
$$-\zeta^{\sharp\sharp\mathfrak{l}}_{2}(2a_0+1,1^{\gamma_p},\cdots,1^{\gamma_1})+\zeta^{\sharp\sharp\mathfrak{l}}_{2a_0+3}(1^{\gamma_r},\cdots,1^{\gamma_1}).$$

Then, by Shift ($A.2$), the second and fourth term get simplified while the third and fifth term get simplified by Cut ($A.3$). It remains:

$$-\zeta^{\sharp,\mathfrak{l}}_{2a_0+3}(1^{\gamma_p},\cdots,1^{\gamma_1}),\qquad \text{which leads straight to 4.41.}$$

(iii) When $a_i<\beta<a_j$, we should have:

$$-\zeta^{\star\star,\mathfrak{l}}_{2}(\mathbf{2}^{a_i},\ldots,\mathbf{2}^{a_j-\beta-1})+\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-2}(\mathbf{2}^{a_{i+1}},\ldots,\mathbf{2}^{a_i+a_j-\beta+1})$$
$$\equiv \zeta^{\sharp\sharp,\mathfrak{l}}_{B_i+B_j-B}(1^{\gamma_j},\ldots,1^{\gamma_{i+1}})-\zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\ldots,B_i).$$

Using resp. Antipode Shift ($A.2$) for the first line, and re-ordering the indices, it is equivalent to, with $c_1\geq 3$, $B_0=2a_0+1-\delta c_1$ here:

$$\zeta^{\star\star,\mathfrak{l}}_{2}(\mathbf{2}^{a_0-1},c_1,\cdots,c_p,\mathbf{2}^{a_p})-\zeta^{\star\star,\mathfrak{l}}_{1}(\mathbf{2}^{a_0},c_1-1,\cdots,c_p,\mathbf{2}^{a_p}) \tag{4.42}$$
$$\equiv \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-1}(B_0,1^{\gamma_1},\cdots,1^{\gamma_p})-\zeta^{\sharp\sharp,\mathfrak{l}}_{B_0+B_p-1}(1^{\gamma_p},\cdots,1^{\gamma_1})\equiv\zeta^{\sharp,\mathfrak{l}}_{B_0-1}(1^{\gamma_1},\cdots,1^{\gamma_p},B_p).$$

This matches with the identity 4.38; the last equality coming from Shift since depth is odd.

(c) Antisymmetric of the first case.

II. Let us denote the sequences $\mathbf{X}=\mathbf{2}^{a_1},\ldots,\mathbf{2}^{a_p}$ and $\mathbf{Y}=\mathbf{1}^{\gamma_1-1},B_1,\cdots,\mathbf{1}^{\gamma_p}$.
We want to prove that:

$$\zeta^{\sharp,\mathfrak{l}}(1,\mathbf{Y},B_p)\equiv-\zeta^{\star,\mathfrak{l}}_{1}(c_1-1,\mathbf{X}) \tag{4.43}$$

Relations used are mostly these stated in §4.2. Using the definition of $\zeta^{\star\star}$:

$$\begin{aligned}-\zeta^{\star,\mathfrak{l}}_{1}(c_1-1,\mathbf{X}) &\equiv -\zeta^{\star\star,\mathfrak{l}}_{1}(c_1-1,\mathbf{X})+\zeta^{\star\star,\mathfrak{l}}_{c_1}(\mathbf{X})\\ &\equiv -\zeta^{\star\star,\mathfrak{l}}(1,c_1-1,\mathbf{X})+\zeta^{\star,\mathfrak{l}}(1,c_1-1,\mathbf{X})+\zeta^{\star\star,\mathfrak{l}}(c_1,\mathbf{X})-\zeta^{\star,\mathfrak{l}}(c_1,\mathbf{X})\\ &\equiv \zeta^{\star,\mathfrak{l}}(1,c_1-1,\mathbf{X})-\zeta^{\star,\mathfrak{l}}(c_1,\mathbf{X})-\zeta^{\star,\mathfrak{l}}(c_1-1,\mathbf{X},1).\end{aligned} \tag{4.44}$$

There, the first and third term in the second line, after applying Shift, have given the last $\zeta^{\star}$ in the last line.
Using then Conjecture 4.5.1, in terms of MMZV$^{\sharp}$, then MMZV$^{\sharp\sharp}$, it gives:

$$\begin{aligned}&\zeta^{\sharp,\mathfrak{l}}(2,\mathbf{Y},B_p-1)+\zeta^{\sharp,\mathfrak{l}}(1,1,\mathbf{Y},B_p-1)+\zeta^{\sharp,\mathfrak{l}}(1,\mathbf{Y},B_p-1,1)\\ &\qquad\equiv\zeta^{\sharp\sharp,\mathfrak{l}}(2,\mathbf{Y},B_p-1)-\zeta^{\sharp\sharp,\mathfrak{l}}_{2}(\mathbf{Y},B_p-1)+\zeta^{\sharp\sharp,\mathfrak{l}}(1,1,\mathbf{Y},B_p-1)\\ &\qquad\quad-\zeta^{\sharp\sharp,\mathfrak{l}}_{1}(1,\mathbf{Y},B_p-1)+\zeta^{\sharp\sharp,\mathfrak{l}}(1,\mathbf{Y},B_p-1,1)-\zeta^{\sharp\sharp,\mathfrak{l}}_{1}(\mathbf{Y},B_p-1,1)\end{aligned} \tag{4.45}$$

First term (odd depth)[I] is simplified with the last, by Schift. Fifth term (even depth) get simplified by Cut with the fourth term. Hence it remains two terms of even depth:

$$\equiv-\zeta^{\sharp\sharp,\mathfrak{l}}_{2}(\mathbf{Y},B_p-1)+\zeta^{\sharp\sharp,\mathfrak{l}}(1,1,\mathbf{Y},B_p-1)\equiv-\zeta^{\sharp\sharp,\mathfrak{l}}_{1}(\mathbf{Y},B_p)+\zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-1}(1,1,\mathbf{Y}),$$

where Minus resp. Cut have been applied. This matches with (4.43) since, by Shift :

$$\equiv-\zeta^{\sharp\sharp,\mathfrak{l}}_{1}(\mathbf{Y},B_p)+\zeta^{\sharp\sharp,\mathfrak{l}}(1,\mathbf{Y},B_p)\equiv\zeta^{\sharp,\mathfrak{l}}(1,\mathbf{Y},B_p).$$

[I] Since weight is odd, we know also depth parity of these terms.

The case $c_1 = 3$ slightly differs since (4.44) gives, by recursion hypothesis I.(4.5.1):

$$-\zeta_1^{\star,\mathfrak{l}}(2,\mathbf{X}) \equiv \zeta^{\sharp,\mathfrak{l}}(B_1+1,\mathbf{Y}',B_p-1) + \zeta^{\sharp,\mathfrak{l}}(1,B_1,\mathbf{Y}',B_p-1) + \zeta^{\sharp,\mathfrak{l}}(B_1,\mathbf{Y}',B_p-1,1),$$

where $\mathbf{Y}' = \mathbf{1}^{\gamma_2},\cdots,\mathbf{1}^{\gamma_p}$, odd depth. Turning into MES$^{\sharp\sharp}$, and using identities of §4.2 in the same way than above, leads to the result. Indeed, from:

$$\begin{aligned}\equiv\ & \zeta^{\sharp\sharp,\mathfrak{l}}(B_1+1,\mathbf{Y}',B_p-1) + \zeta^{\sharp\sharp,\mathfrak{l}}(1,B_1,\mathbf{Y}',B_p-1) + \zeta^{\sharp\sharp,\mathfrak{l}}(B_1,\mathbf{Y}',B_p-1,1)\\ & -\zeta^{\sharp\sharp,\mathfrak{l}}_{B_1+1}(\mathbf{Y}',B_p-1) - \zeta_1^{\sharp\sharp,\mathfrak{l}}(B_1,\mathbf{Y}',B_p-1) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_1}(\mathbf{Y}',B_p-1,1)\end{aligned}$$

First and last terms get simplified via Shift, while third and fifth term get simplified by Cut; besides, we apply minus for second term, and minus for the fourth term, which are both of even depth. This leads to 4.39, using again Shift for the first term:

$$\begin{aligned}&\equiv \zeta^{\sharp\sharp,\mathfrak{l}}_{B_p-1}(1,B_1,\mathbf{Y}') - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_1}(\mathbf{Y}',B_p)\\ &\equiv \zeta^{\sharp\sharp,\mathfrak{l}}(B_1,\mathbf{Y}',B_p) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_1}(\mathbf{Y}',B_p)\\ &\equiv \zeta^{\sharp,\mathfrak{l}}(B_1,\mathbf{Y}',B_p).\end{aligned}$$

□

4.6 Appendix 1: From the linearized octagon relation

The identities in the coalgebra $\mathcal{L}$ obtained from the linearized octagon relation 4.12:

Lemma 4.6.1. *In the coalgebra $\mathcal{L}$, $n_i \in \mathbb{Z}^*$:*

(i) $\zeta^{\star\star,\mathfrak{l}}(n_0,\cdots,n_p) = (-1)^{w+1}\zeta^{\star\star,\mathfrak{l}}(n_p,\cdots,n_0)$.

(ii) $\zeta^{\mathfrak{l}}(n_0,\cdots,n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}(n_0,\cdots,n_p) + (-1)^p\zeta^{\star\star\mathfrak{l}}_{n_p}(n_{p-1},\cdots,n_1,n_0) = 0$.

(iii)

$$\begin{aligned}&\zeta^{\mathfrak{l}}_{n_0-1}(n_1,\cdots,n_p) - \zeta^{\mathfrak{l}}_{n_0}(n_1,\cdots,n_{p-1},\epsilon_p(\mid n_p\mid +1)) =\\ &\qquad (-1)^w\left[\zeta^{\star\star,\mathfrak{l}}_{n_0-1}(n_1,\cdots,n_p) - \zeta^{\star\star,\mathfrak{l}}_{n_0}(n_1,\cdots,n_{p-1},\epsilon_p(\mid n_p\mid +1))\right].\end{aligned}$$

Proof. The sign of n_i is denoted ϵ_i as usual. First, we remark that, with $\eta_i = \pm 1$, $n_i = \epsilon_i(a_i+1)$, and $\epsilon_i = \eta_i\eta_{i+1}$:

$$\begin{aligned}\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1) &= \sum I^{\mathfrak{m}}\left(0;(-\omega_0)^{a_0}(-\omega_{-\eta_1\star})(-\omega_0)^{a_1}\cdots(-\omega_{-\eta_p\star})(-\omega_0)^{a_p};1\right)e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\\ &= \sum(-1)^{n+p}\zeta^{\star\star,\mathfrak{m}}_{n_0-1}(n_1,\cdots,n_{p-1},-n_p)\,e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\end{aligned}$$

Similarly, with $\mu_i := \begin{cases}\star & \texttt{if}\ \eta_i = 1\\ 1 & \texttt{if}\ \eta_i = -1\end{cases}$, applying the homography $\phi_{\tau\sigma}$ to get the second line:

$$\begin{aligned}\Phi^{\mathfrak{m}}(e_{-1},e_0,e_\infty) &= \sum I^{\mathfrak{m}}\left(0;(\omega_1-\omega_{-1})^{a_0}\omega_{\mu_1}(\omega_1-\omega_{-1})^{a_1}\cdots\omega_{\mu_p}(\omega_1-\omega_{-1})^{a_p};1\right)e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\\ \Phi^{\mathfrak{l}}(e_{-1},e_0,e_\infty) &= \sum(-1)^p I^{\mathfrak{m}}\left(0;0^{a_0}\omega_{-\eta_1}0^{a_1}\cdots\omega_{-\eta_p}0^{a_p};1\right)e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\\ &= \sum\zeta^{\mathfrak{m}}_{n_0-1}(n_1,\cdots,n_{p-1},-n_p)\,e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\end{aligned}.$$

Lastly, still using $\phi_{\tau\sigma}$, with here $\mu_i := \begin{cases}\star & \texttt{if}\ \eta_i = 1\\ 1 & \texttt{if}\ \eta_i = -1\end{cases}$:

$$\begin{aligned}\Phi^{\mathfrak{m}}(e_1,e_\infty,e_0) &= \sum I^{\mathfrak{m}}\left(0;(\omega_{-1}-\omega_1)^{a_0}\omega_{\mu_1}(\omega_{-1}-\omega_1)^{a_1}\cdots\omega_{\mu_p}(\omega_{-1}-\omega_1)^{a_p};1\right)e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\\ \Phi^{\mathfrak{l}}(e_1,e_\infty,e_0) &= \sum(-1)^{w+1}I^{\mathfrak{m}}\left(0;0^{a_0}\omega_{\eta_1\star}0^{a_1}\cdots\omega_{\eta_p\star}0^{a_p};1\right)e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\\ &= \sum(-1)^{n+p+1}\zeta^{\star\star,\mathfrak{m}}_{n_0-1}(n_1,\cdots,n_{p-1},n_p)\,e_0^{a_0}e_{\eta_1}e_0^{a_1}\cdots e_{\eta_p}e_0^{a_p}\end{aligned}.$$

(i) This case is the one used in Theorem 4.2.3. This identity is equivalent to, in terms of iterated integrals, for X any sequence of $\{0, \pm 1\}$ or of $\{0, \pm\star\}$:

$$\begin{cases} I^{\mathfrak{l}}(0; 0^k, \star, X; 1) & = & I^{\mathfrak{l}}(0; X, \star, 0^k; 1) & \text{if } \prod_{i=0}^{p} \epsilon_i = 1 \Leftrightarrow \eta_0 = 1 \\ I^{\mathfrak{l}}(0; 0^k, -\star, X; 1) & = & I^{\mathfrak{l}}(0; -X, -\star, 0^k; 1) & \text{if } \prod_{i=0}^{p} \epsilon_i = -1 \Leftrightarrow \eta_0 = -1 \end{cases}$$

The first case is deduced from 4.12 when looking at the coefficient of a word beginning and ending by e_{-1} (or beginning and ending by e_1), whereas the second case is obtained from the coefficient of a word beginning by e_{-1} and ending by e_1, or beginning by e_1 and ending by e_{-1}.

(ii) Let split into two cases, according to the sign of $\prod \epsilon_i$:

- In 4.12, when looking at the coefficient of a word beginning by e_1 and ending by e_0, only these three terms contribute:

$$\Phi^{\mathfrak{l}}(e_{-1}, e_0, e_\infty)e_0 - \Phi^{\mathfrak{l}}(e_\infty, e_{-1}, e_1)e_0 - e_1\Phi^{\mathfrak{l}}(e_1, e_\infty, e_0).$$

Moreover, the coefficient of $e_{-1}e_0^{a_0}e_{\eta_1}\cdots e_{\eta_p}e_0^{a_p+1}$ is, using the expressions above for $\Phi^{\mathfrak{l}}(\cdot)$:

$$(-1)^p I^{\mathfrak{l}}(0; -1, -X; 1) + (-1)^{w+1} I^{\mathfrak{l}}(0; -\star, -X_\star; 1) + (-1)^w I^{\mathfrak{l}}(0; X_\star, 0; 1) = 0,$$
$$\text{where} \quad \begin{array}{l} X := \omega_0^{a_0}\omega_{\eta_1}\cdots\omega_{\eta_p}\omega_0^{a_p} \\ X_\star := \omega_0^{a_0}\omega_{\eta_1\star}\cdots\omega_{\eta_p\star}\omega_0^{a_p} \end{array}.$$

In terms of motivic Euler sums, it is, with $\prod \epsilon_i = 1$:

$$\zeta^{\mathfrak{l}}(n_0, \cdots, -n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}(n_0, \cdots, -n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}_{n_0-1}(n_1, \cdots, n_{p-1}, \epsilon_p(\mid n_p \mid +1)) = 0.$$

Changing n_p into $-n_p$, and applying Antipode ⧢ to the last term, it gives, with now $\prod \epsilon_i = -1$:

$$\zeta^{\mathfrak{l}}(n_0, \cdots, n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}(n_0, \cdots, n_p) + (-1)^p\zeta^{\star\star\mathfrak{l}}_{n_p}(n_{p-1}, \cdots, n_1, n_0) = 0.$$

- Similarly,for the coefficient of a word beginning by e_{-1} and ending by e_0, only these three terms contribute:

$$\Phi^{\mathfrak{l}}(e_{-1}, e_0, e_\infty)e_0 - \Phi^{\mathfrak{l}}(e_\infty, e_{-1}, e_1)e_0 + e_{-1}\Phi^{\mathfrak{l}}(e_\infty, e_{-1}, e_1).$$

Similarly than above, it leads to the identity, with $\prod \epsilon_i = -1$:

$$\zeta^{\mathfrak{l}}(n_0, \cdots, -n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}(n_0, \cdots, -n_p) + (-1)^{w+p}\zeta^{\star\star\mathfrak{l}}_{n_0-1}(n_1, \cdots, n_{p-1}, -\epsilon_p(\mid n_p \mid +1)) = 0.$$

Changing n_p into $-n_p$, and applying Antipode ⧢ to the last term, it gives, with now $\prod \epsilon_i = 1$:

$$\zeta^{\mathfrak{l}}(n_0, \cdots, n_p) + (-1)^{w+p+1}\zeta^{\star\star\mathfrak{l}}(n_0, \cdots, n_p) + (-1)^p\zeta^{\star\star\mathfrak{l}}_{n_p}(n_{p-1}, \cdots, n_1, n_0) = 0.$$

(iii) When looking at the coefficient of a word beginning by e_0 and ending by e_0 in 4.12, only these three terms contribute:

$$-e_0\Phi^{\mathfrak{l}}(e_{-1}, e_0, e_\infty) + \Phi^{\mathfrak{l}}(e_{-1}, e_0, e_\infty)e_0 + e_0\Phi^{\mathfrak{l}}(e_\infty, e_{-1}, e_1) - \Phi^{\mathfrak{l}}(e_\infty, e_{-1}, e_1)e_0.$$

If we identify the coefficient of the word $e_0^{a_0+1}e_{-\eta_1}\cdots e_{-\eta_p}e_0^{a_p+1}$, it leads straight to the identity (iii).

Remark: Looking at the coefficient of words beginning by e_0 and ending by e_1 or e_{-1} in 4.12 would lead to the same identity than the second case. □

4.7 Appendix 2: Missing coefficients

In Lemma 4.4.3, the coefficients $D_{a,b}$ appearing (in (v)) are (the only one which are) not conjectured. These values are not required for the proof of Theorem 4.4.1. However, let present here a table of values in small weights for them. Let consider, the coefficient corresponding to $\zeta^\star(\mathbf{2}^n)$ instead of $\zeta^\star(2)^n$, which is (by (i) in Lemma 4.4.3), with $n = a+b+1$:

$$\widetilde{D}^{a,b} := \frac{(2n)!}{6^n \mid B_{2n} \mid (2^{2n}-2)} D^{a,b} \quad \text{and} \quad \widetilde{D}_n := \frac{(2n)!}{6^n \mid B_{2n} \mid (2^{2n}-2)} D_n. \tag{4.46}$$

We have an expression (4.33) for D_n, albeit not very elegant, which would give:

$$\widetilde{D}_n = \frac{2^{2n}(2n)!}{(2^{2n}-2) \mid B_{2n} \mid} \sum_{\substack{\sum m_i s_i = n \\ m_i \neq m_j}} \prod_{i=1}^{k} \left(\frac{1}{s_i!} \left(\frac{\mid B_{2m_i} \mid (2^{2m_i-1}-1)}{2m_i(2m_i)!} \right)^{s_i} \right). \tag{4.47}$$

Here is a table of values for $\widetilde{D}_n$ and $\widetilde{D}^{k,n-k-1}$ in small weights:

$\cdot \diagdown n$	2	3	4	5
$\widetilde{D_n}$	$\frac{19}{2^3-1}$	$\frac{275}{2^5-1}$	$\frac{11813}{3(2^7-1)}$	$\frac{783}{7}$
$\widetilde{D}_{k,n-1-k}$	$\frac{-12}{7}$	$\frac{-84}{31}, \frac{160}{31}$	$\frac{1064}{127}, \frac{-1680}{127}, \frac{-9584}{381}$	$\frac{189624}{2555}, \frac{-137104}{2555}, \frac{-49488}{511}, \frac{-17664}{511}$

$\cdot \diagdown n$	6	7	8	9
$\widetilde{D_n}$	$\frac{581444793}{691(2^{11}-1)}$	$\frac{263101079}{21(2^{13}-1)}$	$\frac{6807311830555}{3617(2^{15}-1)}$	$\frac{124889801445461}{43867(2^{17}-1)}$

The denominators of $\widetilde{D_n}, \widetilde{D}_{k,n-1-k}$ can be written as $(2^{2n-1}-1)$ times the numerator of the Bernoulli number B_{2n}. No formula has been found yet for their numerators, that should involve binomial coefficients. These coefficients are related since, by shuffle:

$$\begin{aligned} & \zeta_2^{\star\star,\mathfrak{m}}(\mathbf{2}^n) + \textstyle\sum_{k=0}^{n-1} \zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^k, 3, \mathbf{2}^{n-k-1}) && = 0 \\ \text{Hence :} \quad & \zeta^{\star\star,\mathfrak{m}}(\mathbf{2}^{n+1}) - \zeta^{\star,\mathfrak{m}}(\mathbf{2}^{n+1}) \textstyle\sum_{k=0}^{n-1} \zeta_1^{\star\star,\mathfrak{m}}(\mathbf{2}^k, 3, \mathbf{2}^{n-k-1}) && = 0. \end{aligned}$$

Identifying the coefficients of $\zeta^\star(\mathbf{2}^n)$ in formulas $(iii), (v)$ in Lemma 4.4.3 leads to:

$$1 - \widetilde{D_n} = \sum_{k=0}^{n-1} \widetilde{D}_{k,n-1-k}. \tag{4.48}$$

Chapter 5

Galois Descents

`Contents:` The first section gives the general picture (for any N), sketching the Galois descent ideas. The second section focuses on the cases $N = 2, 3, 4, \text{`6'}, 8$, defining the filtrations by the motivic level associated to each descent, and displays both results and proofs. Some examples in small depth for are given in the Annexe §A.2.

`Notations`: For a fixed N, let $k_N := \mathbb{Q}(\xi_N)$, where $\xi_N \in \mu_N$ is a primitive N^{th} root of unity, and $\mathcal{O}_N$ is the ring of integers of k_N. The subscript or exponent N will be omitted when it is not ambiguous. For the general case, the decomposition of N is denoted $N = \prod q_i = \prod p_i^{\alpha}$.

5.1 Overview

Change of field. As said in Chapter 3, for each N, N' with $N'|N$, the Galois action on $\mathcal{H}_N$ and $\mathcal{H}_{N'}$ is determined by the coaction Δ. More precisely, let consider the following descent[I], `assuming` $\phi_{N'}$ `is an isomorphism of graded Hopf comodule`: [II]

$$\begin{tikzcd}
\mathcal{H}^N \arrow[r, hook, "\phi_N", "n.c"'] & \mathcal{H}^{\mathcal{MT}_{\Gamma_N}} \\
\mathcal{H}^{N'} \arrow[u, "\mathcal{G}^{N/N'}"] \arrow[r, hook, "\phi_{N'}", "\sim", "n.c"'] & \mathcal{H}^{\mathcal{MT}_{\Gamma_{N'}}} \arrow[u, "\mathcal{G}^{\mathcal{MT}}_{N/N'}"']
\end{tikzcd}$$

Let choose a basis for $gr_1\mathcal{L}_r^{\mathcal{MT}_{N'}}$, and extend it into a basis of $gr_1\mathcal{L}_r^{\mathcal{MT}_N}$:

$$\left\{\zeta^{\mathfrak{m}}(r;\eta'_{i,r})\right\}_i \subset \left\{\zeta^{\mathfrak{m}}(r;\eta'_{i,r})\right\} \cup \left\{\zeta^{\mathfrak{m}}(r;\eta_i)\right\}_{1\leq i\leq c_r},$$

$$\text{where} \quad c_r = \begin{cases} a_N - a_{N'} = \frac{\varphi(N)-\varphi(N')}{2} + p(N) - p(N') & \text{if } r = 1 \\ b_N - b_{N'} = \frac{\varphi(N)-\varphi(N')}{2} & \text{if } r > 1 \end{cases}.$$

Then, once this basis fixed, let split the set of derivations $\mathscr{D}^N$ into two parts (cf. §2.4.4), one corresponding to $\mathcal{H}^{N'}$:

$$\mathscr{D}^N = \mathscr{D}^{\backslash\lceil} \uplus \mathscr{D}^{\lceil} \quad \text{where} \quad \begin{cases} \mathscr{D}^{\backslash\lceil} = \mathscr{D}^{N'} := \cup_r \left\{D_r^{\eta'_{i,r}}\right\}_{1\leq i\leq c_r} \\ \mathscr{D}^{\lceil} := \cup_r \left\{D_r^{\eta_{i,r}}\right\}_{1\leq i\leq c_r} \end{cases}. \tag{5.1}$$

[I] More generally, there are Galois descents $(\lceil) = (k_N/k_{N'}, M/M')$ from $\mathcal{H}^{\mathcal{MT}\left(\mathcal{O}_{k_N}[\frac{1}{M}]\right)}$, to $\mathcal{H}^{\mathcal{MT}\left(\mathcal{O}_{k_{N'}}[\frac{1}{M'}]\right)}$, with $N' \mid N$, $M' \mid M$, with a set of derivations $\mathscr{D}^{\lceil} \subset \mathscr{D}^N$ associated.

[II] Conjecturally as soon as $N' \neq p^r$, $p \geq 5$. Proven for $N' = 1, 2, 3, 4, \text{`6'}, 8$.

`Examples:`

· For the descent from $\mathcal{MT}_3$ to $\mathcal{MT}_1$: $\mathscr{D}^{(k_3/\mathbb{Q},3/1)} = \left\{D_1^{-1}, D_{2r}^{\xi_3}, r>0\right\}$.

· For the descent from $\mathcal{MT}_8$ to $\mathcal{MT}_4$: $\mathscr{D}^{(k_8/k_4,2/2)} = \left\{D_r^{\xi_8} - D_r^{-\xi_8}, r>0\right\}$.

· For the descent from $\mathcal{MT}_9$ to $\mathcal{MT}_3$: $\mathscr{D}^{(k_9/k_3,3/3)} = \left\{D_r^{\xi_9} - D_r^{-\xi_9^4}, D_r^{\xi_9} - D_r^{-\xi_9^7} r>0\right\}$.[I]

Theorem 5.1.1. *Let $N' \mid N$ such that $\mathcal{H}^{N'} \cong \mathcal{H}^{\mathcal{MT}_{\Gamma_N}}$.*
Let $\mathfrak{Z} \in gr_p^{\mathfrak{D}}\mathcal{H}_n^N$, depth graded MMZV relative to μ_N.
Then $\mathfrak{Z} \in gr_p^{\mathfrak{D}}\mathcal{H}^{N'}$, i.e. $\mathfrak{Z}$ is a depth graded MMZV relative to $\mu_{N'}$ modulo smaller depth if and only if:

$$\left(\forall r<n, \forall D_{r,p} \in \mathscr{D}_r^{\lceil}, \quad D_{r,p}(\mathfrak{Z}) = 0\right) \quad \textit{and} \quad \left(\forall r<n, \forall D_{r,p} \in \mathscr{D}^{\setminus\lceil}, \quad D_{r,p}(\mathfrak{Z}) \in gr_{p-1}^{\mathfrak{D}}\mathcal{H}^{N'}\right).$$

Proof. In the (f_i) side, the analogue of this theorem is pretty obvious, and the result can be transported via ϕ, and back since $\phi_{N'}$ isomorphism by assumption. □

This is a very useful recursive criterion (derivation strictly decreasing weight and depth) to determine if a (motivic) multiple zeta value at μ_N is in fact a (motivic) multiple zeta value at $\mu_{N'}$, modulo smaller depth terms; applying it recursively, it could also take care of smaller depth terms. This criterion applies for motivic MZV_{μ_N}, and by period morphism is deduced for MZV_{μ_N}.

Change of Ramification. If the descent has just a ramified part, the criterion can be stated in a non depth graded version. Indeed, there, since only weight 1 matters, to define the derivation space $\mathcal{D}^{\lceil}$ as above (5.1), we need to choose a basis for $\mathcal{O}_N^* \otimes \mathbb{Q}$, which we complete with $\left\{\xi_N^{\frac{N}{q_i}}\right\}_{i\in I}$ into a basis for Γ_N. Then, with $N = \prod p_i^{\alpha_i} = \prod q_i$:

Theorem 5.1.2. *Let $\mathfrak{Z} \in \mathcal{H}_n^N \subset \mathcal{H}^{\mathcal{MT}_{\Gamma_N}}$, MMZV relative to μ_N.*
Then $\mathfrak{Z} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}$ unramified if and only if:

$$\left(\forall i \in I, D_1^{\xi^{\frac{N}{q_i}}}(\mathfrak{Z}) = 0\right) \quad \textit{and} \quad \left(\forall r<n, \forall D_r \in \mathscr{D}^{\setminus\lceil}, \quad D_r(\mathfrak{Z}) \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}\right).$$

`Nota Bene`: Intermediate descents and change of ramification, keeping part of some of the weight 1 elements $\left\{\xi_N^{\frac{N}{q_i}}\right\}$ could also be stated.

`Examples`:

$N = 2$: As claimed in the introduction, the descent between $\mathcal{H}^2$ and $\mathcal{H}^1$ is precisely measured by D_1:[II]

Corollary 5.1.3. *Let $\mathfrak{Z} \in \mathcal{H}^2 = \mathcal{H}^{\mathcal{MT}_2}$, a motivic Euler sum.*
Then $\mathfrak{Z} \in \mathcal{H}^1 = \mathcal{H}^{\mathcal{MT}_1}$, i.e. $\mathfrak{Z}$ is a motivic multiple zeta value if and only if:

$$D_1(\mathfrak{Z}) = 0 \quad \textit{and} \quad D_{2r+1}(\mathfrak{Z}) \in \mathcal{H}^1.$$

[I] By the relations in depth 1, since:

$$\zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \xi_9^3 \end{pmatrix} = 3^{r-1}\left(\zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \xi_9^1 \end{pmatrix} + \zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \xi_9^4 \end{pmatrix} + \zeta^{\mathfrak{a}}\begin{pmatrix} r \\ \xi_9^7 \end{pmatrix}\right) \quad \text{etc.}$$

[II] $\mathscr{D}^{(\mathbb{Q}/\mathbb{Q},2/1)} = \left\{D_1^{-1}\right\}$ with the above notations; and D_1^{-1} is here simply denoted D_1 .

$\boldsymbol{N = 3, 4, 6}$**:**

Corollary 5.1.4. *Let $N \in \{3, 4, 6\}$ and $\mathfrak{Z} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N[\frac{1}{N}])}$, a motivic MZV_{μ_N}. Then $\mathfrak{Z}$ is unramified, $\mathfrak{Z} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}$ if and only if:*

$$D_1(\mathfrak{Z}) = 0 \text{ and } \quad D_r(\mathfrak{Z}) \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}.$$

$\boldsymbol{N = p^r}$**:** A basis for $\mathcal{O}^N \otimes \mathbb{Q}$ is formed by: $\left\{\frac{1-\xi^k}{1-\xi}\right\}_{\substack{k\wedge p=1\\ 0<k\leq\frac{N}{2}}}$, which corresponds to

$$\text{a basis of } \mathcal{A}_1^{\mathcal{MT}(\mathcal{O}_N)} \quad : \left\{\zeta^{\mathfrak{m}}\begin{pmatrix}1\\ \xi^k\end{pmatrix} - \zeta^{\mathfrak{m}}\begin{pmatrix}1\\ \xi\end{pmatrix}\right\}_{\substack{k\wedge p=1\\ 0<k\leq\frac{N}{2}}}.$$

It can be completed in a basis of $\mathcal{A}_1^N$ with $\zeta^{\mathfrak{m}}\begin{pmatrix}1\\ \xi^1\end{pmatrix}$. [I] However, if we consider the basis of $\mathcal{A}_1^N$ formed by primitive roots of unity up to conjugates, the criterion for the descent could also be stated as follows:

Corollary 5.1.5. *Let $N = p^r$ and $\mathfrak{Z} \in \mathcal{H}^{\mathcal{MT}_{\Gamma_N}} = \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N[\frac{1}{p}])}$, relative to μ_N*[II]*. Then $\mathfrak{Z}$ is unramified, $\mathfrak{Z} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}$ if and only if:*

$$\sum_{\substack{k\wedge p=1\\ 0<k\leq\frac{N}{2}}} D_1^{\xi_N^k}(\mathfrak{Z}) = 0 \quad \text{and} \quad \forall\begin{cases} r>1\\ 1<k\leq\frac{N}{2}\\ k\wedge p=1\end{cases}, \quad D_r^{\xi_N^k}(\mathfrak{Z}) \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_N)}.$$

5.2 Descents for $N = 2, 3, 4, '6', 8$.

5.2.1 Depth 1

Let start with depth 1 results, deduced from Lemma 2.4.1 (from [31]), fundamental to initiate the recursion later.

Lemma 5.2.1. *The basis for $gr_1^{\mathfrak{D}}\mathcal{A}$ is:*

$$\left\{\zeta^{\mathfrak{a}}(r;\xi) \text{ such that } \begin{cases} r>1 \text{ odd} & \text{if } N=1\\ r \text{ odd} & \text{if } N=2\\ r>0 & \text{if } N=3,4\\ r>1 & \text{if } N=6\end{cases}\right\}$$

For $N = 8$, the basis for $gr_1^{\mathfrak{D}}\mathcal{A}_r$ is two dimensional, for all $r > 0$:

$$\{\zeta^{\mathfrak{a}}(r;\xi), \zeta^{\mathfrak{a}}(r;-\xi)\}.$$

Let explicit these relations in depth 1 for $N = 2, 3, 4, '6', 8$, since we would use some p-adic properties of the basis elements in our proof:

For $N = 2$**:** The distribution relation in depth 1 is:

$$\zeta^{\mathfrak{a}}\begin{pmatrix}2r+1\\ 1\end{pmatrix} = (2^{-2r}-1)\zeta^{\mathfrak{a}}\begin{pmatrix}2r+1\\ -1\end{pmatrix}.$$

[I] With the previous theorem notations, $\mathcal{D}^{\lceil} = \{D_1^{\xi}\}$ whereas $\mathcal{D}^{\setminus\lceil} = \{D_1^{\xi^k} - D_1^{\xi}\}_{\substack{k\wedge p=1\\ 1<k\leq\frac{N}{2}}} \cup_{r>1} \{D_r^{\xi^k}\}_{\substack{k\wedge p=1\\ 0<k\leq\frac{N}{2}}}$; where D_1^{ξ} has to be understood as the projection of the left side over $\zeta^{\mathfrak{a}}\left(\frac{1}{\xi}\right)$ in respect to the basis above of $\mathcal{H}_1^{\mathcal{MT}(\mathcal{O}_N)}$ more $\zeta^{\mathfrak{a}}\left(\frac{1}{\xi}\right)$. This leads to a criterion equivalent to (5.1.5).

[II] For instance a MMZV relative to μ_N. Beware, for $p > 5$, there could be other periods.

For $N=3$**:**

$$\zeta^{\mathfrak{l}}\begin{pmatrix}2r+1\\1\end{pmatrix}\left(1-3^{2r}\right)=2\cdot 3^{2r}\zeta^{\mathfrak{l}}\begin{pmatrix}2r+1\\\xi\end{pmatrix}\qquad \zeta^{\mathfrak{l}}\begin{pmatrix}2r\\1\end{pmatrix}=0\qquad \zeta^{\mathfrak{l}}\begin{pmatrix}r\\\xi\end{pmatrix}=(-1)^{r-1}\zeta^{\mathfrak{l}}\begin{pmatrix}r\\\xi^{-1}\end{pmatrix}.$$

For $N=4$**:**

$$\begin{array}{rclrcl}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\1\end{smallmatrix}\right)\left(1-2^{r-1}\right) & = & 2^{r-1}\cdot\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-1\end{smallmatrix}\right)\text{ for } r\neq 1 & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & = & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r\\-1\end{smallmatrix}\right)=0\\ \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\-1\end{smallmatrix}\right) & = & 2^{2r+1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\\xi\end{smallmatrix}\right) & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi\end{smallmatrix}\right) & = & (-1)^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi^{-1}\end{smallmatrix}\right).\end{array}$$

For $N=6$**:**

$$\begin{array}{rclrcl}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\1\end{smallmatrix}\right)\left(1-2^{r-1}\right) & = & 2^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-1\end{smallmatrix}\right)\text{ for } r\neq 1 & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & = & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r\\-1\end{smallmatrix}\right)=0\\ \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\-1\end{smallmatrix}\right) & = & \frac{2\cdot 3^{2r}}{1-3^{2r}}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\\xi\end{smallmatrix}\right) & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi^2\end{smallmatrix}\right) & = & \frac{2^{r-1}}{1-(-2)^{r-1}}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi\end{smallmatrix}\right).\\ \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi\end{smallmatrix}\right) & = & (-1)^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi^{-1}\end{smallmatrix}\right) & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-\xi\end{smallmatrix}\right) & = & (-1)^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-\xi^{-1}\end{smallmatrix}\right).\end{array}$$

For $N=8$**:**

$$\begin{array}{rclrcl}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\1\end{smallmatrix}\right) & = & \frac{2^{r-1}}{(1-2^{r-1})}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-1\end{smallmatrix}\right)\text{ for } r\neq 1 & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & = & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r\\-1\end{smallmatrix}\right)=0\\ \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-i\end{smallmatrix}\right) & = & 2^{r-1}\left(\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\xi\end{smallmatrix}\right)+\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-\xi\end{smallmatrix}\right)\right) & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\-1\end{smallmatrix}\right) & = & 2^{2r+1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r+1\\i\end{smallmatrix}\right)\\ \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\pm\xi\end{smallmatrix}\right) & = & (-1)^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\\pm\xi^{-1}\end{smallmatrix}\right) & \zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\i\end{smallmatrix}\right) & = & (-1)^{r-1}\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}r\\-i\end{smallmatrix}\right)\end{array}$$

5.2.2 Motivic Level filtration

Let fix a descent $(\lceil)=(k_N/k_{N'},M/M')$ from $\mathcal{H}^{\mathcal{MT}(\mathcal{O}_{k_N}[\frac{1}{M}])}$, to $\mathcal{H}^{\mathcal{MT}(\mathcal{O}_{k_{N'}}[\frac{1}{M'}])}$, with $N'\mid N$, $M'\mid M$, among these considered in this section, represented in Figures 3.3, 3.4. Let us define a motivic level increasing filtration $\mathcal{F}^{\lceil}$ associated, from the set of derivations associated to this descent, $\mathscr{D}^{\lceil}\subset\mathscr{D}^N$, defined in (5.1).

Definition 5.2.2. *The filtration by the* ***motivic level*** *associated to a descent* $(\lceil)$ *is defined recursively on* $\mathcal{H}^N$ *by:*

- $\mathcal{F}^{\lceil}_{-1}\mathcal{H}^N=0$.
- $\mathcal{F}^{\lceil}_{i}\mathcal{H}^N$ *is the largest submodule of* $\mathcal{H}^N$ *such that* $\mathcal{F}^{\lceil}_{i}\mathcal{H}^N/\mathcal{F}^{\lceil}_{i-1}\mathcal{H}^N$ *is killed by* $\mathscr{D}^{\lceil}$*, i.e. is in the kernel of* $\oplus_{D\in\mathscr{D}^{\lceil}}D$.

It's a graded Hopf algebra's filtration:

$$\mathcal{F}_i\mathcal{H}.\mathcal{F}_j\mathcal{H}\subset\mathcal{F}_{i+j}\mathcal{H}\ ,\quad \Delta(\mathcal{F}_n\mathcal{H})\subset\sum_{i+j=n}\mathcal{F}_i\mathcal{A}\otimes\mathcal{F}_j\mathcal{H}.$$

The associated graded is denoted: $gr^{\lceil}_i$ and the quotients, coalgebras compatible with Δ:

$$\mathcal{H}^{\geq 0}:=\mathcal{H}\ ,\ \boldsymbol{\mathcal{H}^{\geq i}}:=\mathcal{H}/\mathcal{F}_{i-1}\mathcal{H}\text{ with the projections :}\qquad \forall j\geq i\ ,\ \pi_{i,j}:\mathcal{H}^{\geq i}\to\mathcal{H}^{\geq j}.\quad(5.2)$$

Note that, via the isomorphism ϕ, the motivic filtration on $\mathcal{H}^{\mathcal{MT}_N}$ corresponds to[I]:

$$\mathcal{F}^{\lceil}_i\mathcal{H}^{\mathcal{MT}_N}\longleftrightarrow\left\langle x\in H^N\mid Deg^{\lceil}(x)\leq i\right\rangle_{\mathbb{Q}},\qquad(5.3)$$

where $Deg^{\lceil}$ is the degree in $\left\{\{f^j_r\}_{b_{N'}<j\leq b_N}\underset{r>1}{,}\{f^j_1\}_{a_{N'}<j\leq a_N}\right\}$, which are the images of the complementary part of $gr_1\mathcal{L}^{\mathcal{MT}_{N'}}$ in the basis of $gr_1\mathcal{L}^{\mathcal{MT}_N}$.

[I] In particular, remark that $\dim\mathcal{F}^{\lceil}_i\mathcal{H}^{\mathcal{MT}_N}_n$ are known.

`Example`: For the descent between $\mathcal{H}^{\mathcal{MT}_2}$ and $\mathcal{H}^{\mathcal{MT}_1}$, since $gr_1\mathcal{L}^{\mathcal{MT}_2} = \left\langle \zeta^{\mathfrak{m}}(-1), \{\zeta^{\mathfrak{m}}(2r+1)\}_{r>0} \right\rangle$:

$\mathcal{F}_i\mathcal{H}^{\mathcal{MT}_2} \quad \underset{\sim}{\xrightarrow{\phi}} \quad \langle x \in \mathbb{Q}\langle f_1, f_3, \cdots\rangle \otimes \mathbb{Q}[f_2] \mid Deg_{f_1}(x) \leq i\rangle_{\mathbb{Q}}$, where $Deg_{f_1} =$ degree in f_1.

By definition of these filtrations:

$$D^{\eta}_{r,p}\left(\mathcal{F}_i\mathcal{H}_n\right) \subset \begin{cases} \mathcal{F}_{i-1}\mathcal{H}_{n-r} & \text{if } D^{\eta}_{r,p} \in \mathscr{D}^{\lceil}_r \\ \mathcal{F}_i\mathcal{H}_{n-r} & \text{if } D^{\eta}_{r,p} \in \mathscr{D}^{\setminus\lceil}_r \end{cases} . \tag{5.4}$$

Similarly, looking at $\partial_{n,p}$ (cf. 2.55):

$$\partial_{n,p}(\mathcal{F}_{i-1}\mathcal{H}_n) \subset \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\mathcal{F}_{i-2}\mathcal{H}_{n-r}\right)^{\text{card } \mathscr{D}^{\lceil}_r} \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\mathcal{F}_{i-1}\mathcal{H}_{n-r}\right)^{\text{card } \mathscr{D}^{\setminus\lceil}_r} . \tag{5.5}$$

This allows us to pass to quotients, and define $D^{\eta,i,\lceil}_{n,p}$ and $\partial^{i,\lceil}_{n,p}$:

$$\boldsymbol{D^{\eta,i,\lceil}_{n,p}} : gr^{\mathfrak{D}}_p\mathcal{H}^{\geq i}_n \to \begin{cases} gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i-1}_{n-r} & \text{if } D^{\eta}_{r,p} \in \mathscr{D}^{\lceil}_r \\ gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i}_{n-r} & \text{if } D^{\eta}_{r,p} \in \mathscr{D}^{\setminus\lceil}_r \end{cases} \tag{5.6}$$

$$\boxed{\boldsymbol{\partial^{i,\lceil}_{n,p}} : gr^{\mathfrak{D}}_p\mathcal{H}^{\geq i}_n \to \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i-1}_{n-r}\right)^{\text{card } \mathscr{D}^{\lceil}_r} \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i}_{n-r}\right)^{\text{card } \mathscr{D}^{\setminus\lceil}_r} . \quad (5.7)}$$

The bijectivity of this map is essential to the results stated below.

5.2.3 General Results

In the following results, the filtration considered $\mathcal{F}_i$ is the filtration by the motivic level associated to the (fixed) descent $\lceil$ while the index i, in $\mathcal{B}_{n,p,i}$ refers to the level notion for elements in $\mathcal{B}$ associated to the descent $\lceil$.[I]
We first obtain the following result on the depth graded quotients, for all $i \geq 0$, with:

$$\mathbb{Z}_{1[P]} := \frac{\mathbb{Z}}{1+P\mathbb{Z}} = \left\{\frac{a}{1+bP}, a, b \in \mathbb{Z}\right\} \text{ with } \begin{array}{ll} P=2 & \text{for } N=2,4,8 \\ P=3 & \text{for } N=3,6 \end{array} .$$

Lemma 5.2.3. ·

$\mathcal{B}_{n,p,\geq i}$ is a linearly free family of $gr^{\mathfrak{D}}_p\mathcal{H}^{\geq i}_n$ and defines a $\mathbb{Z}_{1[P]}$ structure :

Each element $\mathfrak{Z} = \zeta^{\mathfrak{m}}\left(\begin{smallmatrix} z_1,\ldots,z_p \\ \epsilon_1,\ldots,\epsilon_p \end{smallmatrix}\right) \in \mathcal{B}_{n,p}$ decomposes in a $\mathbb{Z}_{1[P]}$-linear combination of $\mathcal{B}_{n,p,\geq i}$ elements, denoted $cl_{n,p,\geq i}(\mathfrak{Z})$ in $gr^{\mathfrak{D}}_p\mathcal{H}^{\geq i}_n$, which defines, in an unique way:

$$cl_{n,p,\geq i} : \langle\mathcal{B}_{n,p,\leq i-1}\rangle_{\mathbb{Q}} \to \langle\mathcal{B}_{n,p,\geq i}\rangle_{\mathbb{Q}}.$$

· *The following map $\partial^{i,\lceil}_{n,p}$ is bijective:*

$$\partial^{i,\lceil}_{n,p} : gr^{\mathfrak{D}}_p\langle\mathcal{B}_{n,\geq i}\rangle_{\mathbb{Q}} \to \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\langle\mathcal{B}_{n-1,\geq i-1}\rangle_{\mathbb{Q}}\right)^{\oplus \text{ card } \mathcal{D}^{\lceil}_r} \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\langle\mathcal{B}_{n-2r-1,\geq i}\rangle_{\mathbb{Q}}\right)^{\oplus \text{ card } \mathcal{D}^{\setminus\lceil}_r} .$$

Before giving the proof, in the next section, let present its consequences such as bases for the quotient, the filtration and the graded spaces for each descent considered:

[I] Precisely defined, for each descent in §5.2.5.

Theorem 5.2.4. *(i) $\mathcal{B}_{n,\leq p,\geq i}$ is a basis of $\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i} = \mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}}$.*

(ii)
- *$\mathcal{B}_{n,p,\geq i}$ is a basis of $gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i} = gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}}$ on which it defines a $\mathbb{Z}_{1[P]}$-structure: Each element $\mathfrak{Z} = \zeta^{\mathfrak{m}}\left(\begin{smallmatrix} z_1,\ldots,z_p \\ \epsilon_1,\ldots,\epsilon_p \end{smallmatrix}\right)$ decomposes in a $\mathbb{Z}_{1[P]}$-linear combination of $\mathcal{B}_{n,p,\geq i}$ elements, denoted $cl_{n,p,\geq i}(\mathfrak{Z})$ in $gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}$, which defines in an unique way:*

  $cl_{n,p,\geq i} : \langle \mathcal{B}_{n,p,\leq i-1}\rangle_{\mathbb{Q}} \to \langle \mathcal{B}_{n,p,\geq i}\rangle_{\mathbb{Q}}$ such that $\mathfrak{Z} + cl_{n,p,\geq i}(\mathfrak{Z}) \in \mathcal{F}_{i-1}\mathcal{H}_n + \mathcal{F}_{p-1}^{\mathfrak{D}}\mathcal{H}_n$.
- *The following map is bijective:*

$$\partial_{n,p}^{i,\lceil} : gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i} \to \oplus_{r<n}\left(gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-1}^{\geq i-1}\right)^{\oplus\ card\ \mathcal{D}_r^{\lceil}} \oplus_{r<n}\left(gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-r}^{\geq i}\right)^{\oplus\ card\ \mathcal{D}_r^{\setminus\lceil}}.$$

- *$\mathcal{B}_{n,\cdot,\geq i}$ is a basis of $\mathcal{H}_n^{\geq i} = \mathcal{H}_n^{\geq i,\mathcal{MT}}$.*

(iii) We have the two split exact sequences in bijection:

$$0 \longrightarrow \mathcal{F}_i\mathcal{H}_n \longrightarrow \mathcal{H}_n \overset{\pi_{0,i+1}}{\to} \mathcal{H}_n^{\geq i+1} \longrightarrow 0$$

$$0 \to \langle \mathcal{B}_{n,\cdot,\leq i}\rangle_{\mathbb{Q}} \to \langle \mathcal{B}_n\rangle_{\mathbb{Q}} \to \langle \mathcal{B}_{n,\cdot,\geq i+1}\rangle_{\mathbb{Q}} \to 0.$$

The following map, defined in an unique way:

$$cl_{n,\leq p,\geq i} : \langle \mathcal{B}_{n,p,\leq i-1}\rangle_{\mathbb{Q}} \to \langle \mathcal{B}_{n,\leq p,\geq i}\rangle_{\mathbb{Q}} \text{ such that } \mathfrak{Z} + cl_{n,\leq p,\geq i}(\mathfrak{Z}) \in \mathcal{F}_{i-1}\mathcal{H}_n.$$

(iv) A basis for the filtration spaces $\mathcal{F}_i\mathcal{H}_n^{\mathcal{MT}} = \mathcal{F}_i\mathcal{H}_n$:

$$\cup_p \left\{\mathfrak{Z} + cl_{n,\leq p,\geq i+1}(\mathfrak{Z}), \mathfrak{Z} \in \mathcal{B}_{n,p,\leq i}\right\}.$$

(v) A basis for the graded space $gr_i\mathcal{H}_n^{\mathcal{MT}} = gr_i\mathcal{H}_n$:

$$\cup_p \left\{\mathfrak{Z} + cl_{n,\leq p,\geq i+1}(\mathfrak{Z}), \mathfrak{Z} \in \mathcal{B}_{n,p,i}\right\}.$$

The proof is given in §5.2.4, and the notion of level resp. motivic level, some consequences and specifications for $N = 2, 3, 4, '6', 8$ individually are provided in §5.2.5. Some examples in small depth are displayed in Appendice $A.2$.

`Consequences, level` $i = 0$:

- The level 0 of the basis elements $\mathcal{B}^N$ forms a basis of $\mathcal{H}^N = \mathcal{H}^{\mathcal{MT}_N}$, for $N = 2, 3, 4, '6', 8$. This gives a new proof (dual) of Deligne's result (in [31]).
  The level 0 of this filtration is hence isomorphic to the following algebras:[I]

$$\mathcal{F}_0^{k_N/k_{N'},M/M'}\mathcal{H}^{\mathcal{MT}_N} = \mathcal{F}_0^{k_N/k_{N'},M/M'}\mathcal{H}^N = \mathcal{H}^{\mathcal{MT}_{N',M'}} = \text{"}\mathcal{H}^{N',M'}\text{"}.$$

  Hence the inclusions in the following diagram are here isomorphisms:

$$\begin{tikzcd}
\mathcal{F}_0^{k_N/k_{N'},M/M'}\mathcal{H}^{\mathcal{MT}_N} & \mathcal{H}^{\mathcal{MT}_{N'}} \arrow[l, hook'] \\
\mathcal{F}_0^{k_N/k_{N'},M/M'}\mathcal{H}^N \arrow[u, hook] & \mathcal{H}^{N'} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}.$$

[I] The equalities of the kind $\mathcal{H}^{\mathcal{MT}_N} = \mathcal{H}^N$ are consequences of the previous theorem for $N = 2, 3, 4, '6', 8$, and by F. Brown for $N = 1$ (cf. [20]). Moreover, we have inclusions of the kind $\mathcal{H}^{\mathcal{MT}_{N'}} \subseteq \mathcal{F}_0^{k_N/k_{N'},M/M'}\mathcal{H}^{\mathcal{MT}_N}$ and we deduce the equality from dimensions at fixed weight.

· It gives, considering such a descent $(k_N/k_{N'}, M/M')$, a basis for $\mathcal{F}^0\mathcal{H}^N = \mathcal{H}^{\mathcal{MT}_{N',M'}}$ in terms of the basis of $\mathcal{H}^N$. For instance, it leads to a new basis for motivic multiple zeta values in terms of motivic Euler sums, or motivic MZV_{μ_3}.
Some others 0-level such as $\mathcal{F}_0^{k_N/k_N,P/1}$, $N = 3, 4$ which should reflect the descent from $\mathcal{MT}(\mathcal{O}_N\left[\frac{1}{P}\right])$ to $\mathcal{MT}(\mathcal{O}_N)$ are not known to be associated to a fundamental group, but the previous result enables to reach them. We obtain a basis for:

- $\boldsymbol{\mathcal{H}^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}}$ in terms of the basis of $\mathcal{H}^{\mathcal{MT}(\mathcal{O}_3[\frac{1}{3}])}$.
- $\boldsymbol{\mathcal{H}^{\mathcal{MT}(\mathcal{O}_3)}}$ in terms of the basis of $\mathcal{H}^{\mathcal{MT}(\mathcal{O}_3[\frac{1}{3}])}$.
- $\boldsymbol{\mathcal{H}^{\mathcal{MT}(\mathcal{O}_4)}}$ in terms of the basis of $\mathcal{H}^{\mathcal{MT}(\mathcal{O}_4[\frac{1}{4}])}$.

5.2.4 Proofs

As proved below, Theorem 5.2.4 boils down to the Lemma 5.2.3. Remind the map $\partial_{n,p}^{i,\lceil}$:

$$\partial_{n,p}^{i,\lceil} : gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i} \to \oplus_{r<n}\left(gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-r}^{\geq i-1}\right)^{\text{card } \mathscr{D}_r^{\lceil}} \oplus_{r<n}\left(gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-r}^{\geq i}\right)^{\text{card } \mathscr{D}_r^{\setminus\lceil}}.$$

We will look at its image on $\mathcal{B}_{n,p,\geq i}$ and prove both the injectivity of $\partial_{n,p}^{i,\lceil}$ as considered in Lemma 5.2.3, and the linear independence of these elements $\mathcal{B}_{n,p,\geq i}$.

Proof of Lemma 5.2.3 for $N = 2$: The formula (2.4.3) for $D_{2r+1,p}^{-1}$ on $\mathcal{B}$ elements:[I]

$$D_{2r+1,p}^{-1}\left(\zeta^{\mathfrak{m}}(2x_1+1,\dots,\overline{2x_p+1})\right) = \frac{2^{2r}}{1-2^{2r}}\delta_{r=x_1}\cdot\zeta^{\mathfrak{m}}(2x_2+1,\dots,\overline{2x_p+1})$$
$$\frac{2^{2r}}{1-2^{2r}}\left\{\begin{array}{l}\sum_{i=1}^{p-2}\delta_{x_{i+1}\leq r<x_i+x_{i+1}}\binom{2r}{2x_{i+1}}\\ -\sum_{i=1}^{p-1}\delta_{x_i\leq r<x_i+x_{i+1}}\binom{2r}{2x_i}\end{array}\right. \cdot\zeta^{\mathfrak{m}}\left(\cdots,2x_{i-1}+1,2(x_i+x_{i+1}-r)+1,2x_{i+2}+1,\cdots\right)$$
$$(\text{D})\quad +\delta_{x_p\leq r\leq x_p+x_{p-1}}\binom{2r}{2x_p}\cdot\zeta^{\mathfrak{m}}\left(\cdots,2x_{p-2}+1,\overline{2(x_{p-1}+x_p-r)+1}\right) \tag{5.8}$$

Terms of type (D) play a particular role since they correspond to deconcatenation for the coaction, and will be the terms of minimal 2-adic valuation.
$D_{1,p}^{-1}$ acts as a deconcatenation on this family:

$$D_{1,p}^{-1}\left(\zeta^{\mathfrak{m}}(2x_1+1,\dots,\overline{2x_p+1})\right) = \left\{\begin{array}{ll}0 & \text{if } x_p\neq 0\\ \zeta^{\mathfrak{m}}(2x_1+1,\dots,\overline{2x_{p-1}+1}) & \text{if } x_p=0.\end{array}\right. \tag{5.9}$$

For $N = 2$, $\partial_{n,p}^{i}$ (5.7) is simply:

$$\partial_{n,p}^{i} : gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i} \to gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-1}^{\geq i-1} \oplus_{1<2r+1\leq n-p+1} gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-2r-1}^{\geq i}. \tag{5.10}$$

Let prove all statements of Lemma 5.2.3, recursively on the weight, and then recursively on depth and on the level, from $i = 0$.

Proof. By recursion hypothesis, weight being strictly smaller, we assume that:

$$\mathcal{B}_{n-1,p-1,\geq i-1} \oplus_{1<2r+1\leq n-p+1} \mathcal{B}_{n-2r-1,p-1,\geq i} \text{ is a basis of}$$
$$gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-1}^{\geq i-1,\mathcal{B}} \oplus_{1<2r+1\leq n-p+1} gr_{p-1}^{\mathfrak{D}}\mathcal{H}_{n-2r-1}^{\geq i,\mathcal{B}}.$$

[I] Using identity: $\zeta^{\mathfrak{a}}(\overline{2r+1}) = (2^{-2r}-1)\zeta^{\mathfrak{a}}(2r+1)$. Projection on $\zeta^{\mathfrak{l}}(\overline{2r+1})$ for the left side.

CLAIM: The matrix $M^i_{n,p}$ of $\left(\partial^{i,\lceil}_{n,p}(z)\right)_{z\in\mathcal{B}_{n,p,\geq i}}$ on these spaces is invertible.

`Nota Bene:` Here $D_1^{-1}(z)$, resp. $D^{-1}_{2r+1,p}(z)$ are expressed in terms of $\mathcal{B}_{n-1,p-1,\geq i-1}$ resp. $\mathcal{B}_{n-2r-1,p-1,\geq i}$.
It will prove both the bijectivity of $\partial^{i,\lceil}_{n,p}$ as considered in the lemma and the linear independence of $\mathcal{B}_{n,p,\geq i}$. Let divide $M^i_{n,p}$ into four blocks, with the first column corresponding to elements of $\mathcal{B}_{n,p,\geq i}$ ending by 1:

	$x_p = 0$	$x_p > 0$
$D_{1,p}$	M1	M2
$D_{>1,p}$	M3	M4

According to (5.9), $D^{-1}_{1,p}$ is zero on the elements not ending by 1, and acts as a deconcatenation on the others. Therefore, M3 $= 0$, so $M^i_{n,p}$ is lower triangular by blocks, and the left-upper-block M1 is diagonal invertible. It remains to prove the invertibility of the right-lower-block $\widetilde{M} := M4$, corresponding to $D^{-1}_{>1,p}$ and to the elements of $\mathcal{B}_{n,p,\geq i}$ not ending by 1.

Notice that in the formula (5.8) of $D_{2r+1,p}$, applied to an element of $\mathcal{B}_{n,p,\geq i}$, most of terms appearing have a number of 1 greater than i but there are also terms in $\mathcal{B}_{n-2r-1,p-1,i-1}$, with exactly $(i-1)$ "1" for type A,B,C only. We will make disappear the latter modulo 2, since they are 2-adically greater.
More precisely, using recursion hypothesis (in weight strictly smaller), we can replace them in $gr_{p-1}\mathcal{H}^{\geq i}_{n-2r-1}$ by a $\mathbb{Z}_{\text{odd}}$-linear combination of elements in $\mathcal{B}_{n-2r-1,p-1,\geq i}$, which does not lower the 2-adic valuation. It is worth noticing that the type D elements considered are now always in $\mathcal{B}_{n-2r-1,p-1,\geq i}$, since we removed the case $x_p = 0$.

Once done, we can construct the matrix $\widetilde{M}$ and examine its entries.
Order elements of $\mathcal{B}$ on both sides by lexicographic order of its "reversed" elements:

$$(x_p, x_{p-1}, \cdots, x_1) \text{ for the colums, } (r, y_{p-1}, \cdots, y_1) \text{ for the rows.}$$

Remark that, with such an order, the diagonal corresponds to the deconcatenation terms: $r = x_p$ and $x_i = y_i$.
Referring to (5.8), and by the previous remark, we see that $\widetilde{M}$ has all its entries of 2-adic valuation positive or equal to zero, since the coefficients in (5.8) are in $2^{2r}\mathbb{Z}_{\text{odd}}$ (for types A,B,C) or of the form $\mathbb{Z}_{\text{odd}}$ for types D,D'. If we look only at the terms with 2-adic valuation zero, (which comes to consider $\widetilde{M}$ modulo 2), it only remains in (5.8) the terms of type (D,D'), that is:

$$D_{2r+1,p}(\zeta^{\mathfrak{m}}(2x_1+1,\ldots,\overline{2x_p+1})) \equiv \delta_{r=x_p+x_{p-1}}\binom{2r}{2x_p}\zeta^{\mathfrak{m}}(2x_1+1,\ldots,2x_{p-2}+1,\overline{1})$$
$$+\,\delta_{x_p\leq r<x_p+x_{p-1}}\binom{2r}{2x_p}\zeta^{\mathfrak{m}}(2x_1+1,\ldots,2x_{p-2}+1,\overline{2(x_{p-1}+x_p-r)+1}) \pmod 2.$$

Therefore, modulo 2, with the order previously defined, it remains only an upper triangular matrix $(\delta_{x_p\leq r})$, with 1 on the diagonal ($\delta_{x_p=r}$, deconcatenation terms). Thus, $\det\widetilde{M}$ has a 2-adic valuation equal to zero, and in particular can not be zero, that's why $\widetilde{M}$ is invertible.

The $\mathbb{Z}_{odd}$ structure is easily deduced from the fact that the determinant of $\widetilde{M}$ is odd, and the observation that if we consider $D_{2r+1,p}(\zeta^{\mathfrak{m}}(z_1,\ldots,z_p))$, all the coefficients are integers. $\square$

Proof of Lemma 5.2.3 for other N. These cases can be handled in a rather similar way than the case $N=2$, except that the number of generators is different and that several descents are possible, hence there will be several notions of level and filtrations by the motivic level, one for each descent. Let fix a descent $\lceil$ and underline the differences in the proof:

Proof. In the same way, we prove by recursion on weight, depth and level, that the following map is bijective:

$$\partial^{i,\lceil}_{n,p} : gr^{\mathfrak{D}}_p \langle \mathcal{B}_{n,\geq i}\rangle_{\mathbb{Q}} \to \oplus_{r<n} \left(gr^{\mathfrak{D}}_{p-1}\langle \mathcal{B}_{n-1,\geq i-1}\rangle_{\mathbb{Q}}\right)^{\oplus \text{ card } \mathscr{D}^{\lceil}_r} \oplus_{r<n}\left(gr^{\mathfrak{D}}_{p-1}\langle \mathcal{B}_{n-2r-1,\geq i}\rangle_{\mathbb{Q}}\right)^{\oplus \text{ card } \mathscr{D}^{\setminus\lceil}_r}.$$

I.e the matrix $M^i_{n,p}$ of $\left(\partial^i_{n,p}(z)\right)_{z\in\mathcal{B}_{n,p,\geq i}}$ on $\oplus_{r<n}\mathcal{B}^{\text{card } \mathscr{D}^{\lceil}_r}_{n-r,p-1,\geq i-1} \oplus_{r<n} \mathcal{B}^{\text{card } \mathscr{D}^{\setminus\lceil}_r}_{n-r,p-1,\geq i}$ [I] is invertible.

As before, by recursive hypothesis, we replace elements of level $\leq i$ appearing in $D^i_{r,p}$, $r\geq 1$ by $\mathbb{Z}_{1[P]}$-linear combinations of elements of level $\geq i$ in the quotient $gr^{\mathfrak{D}}_{p-1}\mathcal{H}^{\geq i}_{n-r}$, which does not decrease the P-adic valuation.
Now looking at the expression for $D_{r,p}$ in Lemma 2.4.3, we see that on the elements considered, [II] the left side is:

$$\text{Either } \zeta^{(l)}\binom{r}{1} \text{ for type A,B,C} \qquad \text{Or } \zeta^{(l)}\binom{r}{\xi} \text{ for Deconcatenation terms.}$$

Using results in depth 1 of Deligne and Goncharov (cf. §2.4.3), the deconcatenation terms are P-adically smaller.
`For instance`, for $N=$ '6', r odd:

$$\zeta^{\mathfrak{l}}(r;1) = \frac{2\cdot 6^{r-1}}{(1-2^{r-1})(1-3^{r-1})}\zeta^{\mathfrak{l}}(r;\xi), \quad \text{and} \quad v_3\left(\frac{2\cdot 6^{r-1}}{(1-2^{r-1})(1-3^{r-1})}\right) > 0.$$

`Nota Bene:` For $N=8$, D_r has two independent components, D^{ξ}_r and $D^{-\xi}_r$. We have to distinguish them, but the statement remains similar since the terms appearing in the left side are either $\zeta^{(l)}\binom{r}{\pm 1}$, or deconcatenation terms, $\zeta^{(l)}\binom{r}{\pm\xi}$, 2-adically smaller by §4.1.
Thanks to congruences modulo P, only the deconcatenation terms remain:

$$D_{r,p}\left(\zeta^{\mathfrak{m}}\begin{pmatrix} x_1,\dots,x_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi\end{pmatrix}\right) =$$

$$\delta_{x_p\leq r\leq x_p+x_{p-1}-1}(-1)^{r-x_p}\binom{r-1}{x_p-1}\zeta^{\mathfrak{l}}\begin{pmatrix} r \\ \epsilon_p\xi\end{pmatrix}\otimes\zeta^{\mathfrak{m}}\begin{pmatrix} x_1,\dots,x_{p-2},x_{p-1}+x_p-r \\ \epsilon_1,\cdots,\epsilon_{p-2},\epsilon_{p-1}\epsilon_p\xi\end{pmatrix} \pmod P.$$

As in the previous case, the matrix being modulo P triangular with 1 on the diagonal, has a determinant congruent at 1 modulo P, and then, in particular, is invertible.

□

`EXAMPLE for` $\boldsymbol{N=2}$**:** Let us illustrate the previous proof by an example, for weight $n=9$, depth $p=3$, level $i=0$, with the previous notations.
Instead of $\mathcal{B}_{9,3,\geq 0}$, we will restrict to the subfamily (corresponding to $\mathcal{A}$):

$$\mathcal{B}^0_{9,3,\geq 0} := \left\{\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}) \text{ of weight } 9\right\} \subset$$

$$\mathcal{B}_{9,3,\geq 0} := \left\{\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})\zeta^{\mathfrak{m}}(2)^s \text{ of weight } 9\right\}$$

[I] Elements in arrival space are linearly independent by recursion hypothesis.
[II] i.e. of the form $\zeta^{\mathfrak{m}}\begin{pmatrix} x_1,\dots,x_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi_N\end{pmatrix}$, with $\epsilon_i\in\pm 1$ for $N=8$, $\epsilon_i=1$ else.

Note that $\zeta^{\mathfrak{m}}(2)$ being trivial under the coaction, the matrix $M_{9,3}$ is diagonal by blocks following the different values of s and we can prove the invertibility of each block separately; here we restrict to the block $s=0$. The matrix $\widetilde{M}$ considered represents the coefficients of:

$$\zeta^{\mathfrak{m}}(\overline{2r+1})\otimes\zeta^{\mathfrak{m}}(2x+1,\overline{2y+1}) \quad \text{in} \quad D_{2r+1,3}(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})).$$

The chosen order for the columns, resp. for the rows [I] is the lexicographic order applied to (c,b,a) resp. to (r,y,x). Modulo 2, it only remains the terms of type D,D', that is:

$$D_{2r+1,3}(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}))\equiv\delta_{c\leq r\leq b+c}\binom{2r}{2c}\zeta^{\mathfrak{m}}(2a+1,\overline{2(b+c-r)+1}) \pmod 2.$$

With the previous order, $\widetilde{M}_{9,3}$ is then, modulo 2:[II]

$D_r,\zeta\backslash\ \zeta$	$7,1,\overline{1}$	$5,3,\overline{1}$	$3,5,\overline{1}$	$1,7,\overline{1}$	$5,1,\overline{3}$	$3,3,\overline{3}$	$1,5,\overline{3}$	$3,1,\overline{5}$	$1,3,\overline{5}$	$1,1,\overline{7}$
$D_1,\zeta^{\mathfrak{m}}(7,\overline{1})$	1	0	0	0	0	0	0	0	0	0
$D_1,\zeta^{\mathfrak{m}}(5,\overline{3})$	0	1	0	0	0	0	0	0	0	0
$D_1,\zeta^{\mathfrak{m}}(3,\overline{5})$	0	0	1	0	0	0	0	0	0	0
$D_1,\zeta^{\mathfrak{m}}(1,\overline{7})$	0	0	0	1	0	0	0	0	0	0
$D_3,\zeta^{\mathfrak{m}}(5,\overline{7})$	0	0	0	0	1	0	0	0	0	0
$D_3,\zeta^{\mathfrak{m}}(3,\overline{3})$	0	0	0	0	0	1	0	0	0	0
$D_3,\zeta^{\mathfrak{m}}(1,\overline{5})$	0	0	0	0	0	0	1	0	0	0
$D_5,\zeta^{\mathfrak{m}}(3,\overline{1})$	0	0	0	0	0	$\binom{4}{2}$	0	1	0	0
$D_5,\zeta^{\mathfrak{m}}(1,\overline{3})$	0	0	0	0	0	0	$\binom{4}{2}$	0	1	0
$D_7,\zeta^{\mathfrak{m}}(1,\overline{1})$	0	0	0	0	0	0	$\binom{6}{2}$	0	$\binom{6}{4}$	1

As announced, $\widetilde{M}$ modulo 2 is triangular with 1 on the diagonal, thus obviously invertible.

Proof of the Theorem 5.2.4.

Proof. This Theorem comes down to the Lemma 5.2.3 proving the freeness of $\mathcal{B}_{n,p,\geq i}$ in $gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}$ defining a $\mathbb{Z}_{odd}$-structure:

(i) By this Lemma, $\mathcal{B}_{n,p,\geq i}$ is linearly free in the depth graded, and $\partial_{n,p}^{i,\lceil}$, which decreases strictly the depth, is bijective on $\mathcal{B}_{n,p,\geq i}$. The family $\mathcal{B}_{n,\leq p,\geq i}$, all depth mixed is then linearly independent on $\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}\subset\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}}$: easily proved by application of $\partial_{n,p}^{i,\lceil}$. By a dimension argument, since $\dim\mathcal{F}_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}} = \text{card}\,\mathcal{B}_{n,\leq p,\geq i}$, we deduce the generating property.

(ii) By the lemma, this family is linearly independent, and by (i) applied to depth $p-1$,

$$gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i}\subset gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}}.$$

Then, by a dimension argument, since $\dim gr_p^{\mathfrak{D}}\mathcal{H}_n^{\geq i,\mathcal{MT}} = \text{card}\,\mathcal{B}_{n,p,\geq i}$ we conclude on the generating property. The $\mathbb{Z}_{odd}$ structure has been proven in the previous lemma. By the bijectivity of $\partial_{n,p}^{i,\lceil}$ (still previous lemma), which decreases the depth, and using the freeness of the elements of a same depth in the depth graded, there is no linear relation between elements of $\mathcal{B}_{n,\cdot,\geq i}$ of different depths in $\mathcal{H}_n^{\geq i}\subset\mathcal{H}_n^{\geq i\mathcal{MT}}$. The family considered is then linearly independent in $\mathcal{H}_n^{\geq i}$. Since card $\mathcal{B}_{n,\cdot,\geq i}=\dim\mathcal{H}_n^{\geq i,\mathcal{MT}}$, we conclude on the equality of the previous inclusions.

[I] I.e. for $\zeta^{\mathfrak{m}}(2a+1,2b+1,2c+1)$ resp. for $(D_{2r+1,3},\zeta^{\mathfrak{m}}(2x+1,\overline{2y+1}))$.

[II] Notice that the first four rows are exact: no need of congruences modulo 2 for D_1 because it acts as a deconcatenation on the base.

(*iii*) The second exact sequence is obviously split since $\mathcal{B}_{n,\cdot,\geq i+1}$ is a subset of $\mathcal{B}_n$. We already know that $\mathcal{B}_n$ is a basis of $\mathcal{H}_n$ and $\mathcal{B}_{n,\cdot,\geq i+1}$ is a basis of $\mathcal{H}_n^{\geq i+1}$. Therefore, it gives a map $\mathcal{H}_n \leftarrow \mathcal{H}_n^{\geq i+1}$ and split the first exact sequence.
The construction of $cl_{n,\leq p,\geq i}(x)$, obtained from $cl_{n,p,\geq i}(x)$ applied repeatedly, is the following:

$$x \in \mathcal{B}_{n,\cdot,\leq i-1} \text{ is sent on } \bar{x} \in \mathcal{H}_n^{\geq i} \cong \langle \mathcal{B}_{n,\leq p,\geq i}\rangle_{\mathbb{Q}} \text{ by the projection } \pi_{0,i} \text{ and so } x - \bar{x} \in \mathcal{F}_{i-1}\mathcal{H}.$$

Notice that the problem of making $cl(x)$ explicit boils down to the problem of describing the map $\pi_{0,i}$ in the bases $\mathcal{B}$.

(*iv*) By the previous statements, these elements are linearly independent in $\mathcal{F}_i\mathcal{H}_n^{MT}$. Moreover, their cardinal is equal to the dimension of $\mathcal{F}_i\mathcal{H}_n^{MT}$. It gives the basis announced, composed of elements $x \in \mathcal{B}_{n,\cdot,\leq i}$, each corrected by an element denoted $cl(x)$ of $\langle \mathcal{B}_{n,\cdot,\geq i+1}\rangle_{\mathbb{Q}}$.

(*v*) By the previous statements, these elements are linearly independent in $gr_i\mathcal{H}_n$, and by a dimension argument, we can conclude.

□

5.2.5 Specified Results

5.2.5.1 The case $N = 2$.

Here, since there is only one Galois descent from $\mathcal{H}^2$ to $\mathcal{H}^1$, the previous exponents for level filtrations can be omitted, as the exponent 2 for $\mathcal{H}$ the space of motivic Euler sums. Set $\mathbb{Z}_{\text{odd}} = \left\{\frac{a}{b}\ ,\ a \in \mathbb{Z}, b \in 2\mathbb{Z}+1\right\}$, rationals having a 2-adic valuation positive or infinite. Let us define particular families of motivic Euler sums, a notion of level and of motivic level.

Definition 5.2.5. · $\mathcal{B}^2 := \left\{\zeta^{\mathfrak{m}}(2x_1+1,\ldots,2x_{p-1}+1,\overline{2x_p+1})\zeta(2)^{\mathfrak{m},k}, x_i \geq 0, k \in \mathbb{N}\right\}$.
Here, the level is defined as the number of x_i equal to zero.

· *The filtration by the motivic $(\mathbb{Q}/\mathbb{Q}, 2/1)$-level,*

$$\mathcal{F}_i\mathcal{H} := \left\{\mathfrak{Z} \in \mathcal{H},\ \textit{such that } D_1^{-1}\mathfrak{Z} \in \mathcal{F}_{i-1}\mathcal{H}\ ,\ \forall r > 0, D_{2r+1}^1\mathfrak{Z} \in \mathcal{F}_i\mathcal{H}\right\}.$$

I.e. $\mathcal{F}_i$ is the largest submodule such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is killed by D_1.

This level filtration commutes with the increasing depth filtration.

Remarks: The increasing or decreasing filtration defined from the number of 1 appearing in the motivic multiple zeta values is not preserved by the coproduct, since the number of 1 can either decrease or increase (by at the most 1) and is therefore not *motivic*.

Let list some consequences of the results in §5.2.3, which generalize in particular a result similar to P. Deligne's one (cf. [31]):

Corollary 5.2.6. *The map $\mathcal{G}^{\mathcal{MT}} \to \mathcal{G}^{\mathcal{MT}'}$ is an isomorphism.*
The elements of $\mathcal{B}_n$, $\zeta^{\mathfrak{m}}(2x_1+1,\ldots,\overline{2x_p+1})\zeta(2)^k$ of weight n, form a basis of motivic Euler sums of weight n, $\mathcal{H}_n^2 = \mathcal{H}_n^{\mathcal{MT}_2}$, and define a $\mathbb{Z}_{odd}$-structure on the motivic Euler sums.

The period map, per : $\mathcal{H} \to \mathbb{C}$, induces the following result for the Euler sums:

Each Euler sum is a $\mathbb{Z}_{odd}$-linear combination of Euler sums
$\zeta(2x_1+1,\ldots,\overline{2x_p+1})\zeta(2)^k, k \geq 0, x_i \geq 0$ of the same weight.

Here is the result on the 0th level of the Galois descent from $\mathcal{H}^1$ to $\mathcal{H}^2$:

Corollary 5.2.7.

$$\mathcal{F}_0\mathcal{H}^{\mathcal{MT}_2} = \mathcal{F}_0\mathcal{H}^2 = \mathcal{H}^{\mathcal{MT}_1} = \mathcal{H}^1.$$

A basis of motivic multiple zeta values in weight n, is formed by terms of $\mathcal{B}_n$ with 0-level each corrected by linear combinations of elements of $\mathcal{B}_n$ of level 1:

$$\mathcal{B}_n^1 := \Bigg\{ \zeta^{\mathfrak{m}}(2x_1+1,\dots,\overline{2x_p+1})\zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i\geq 0\\ \text{at least one } y_i=0}} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}(2y_1+1,\dots,\overline{2y_p+1})\zeta^{\mathfrak{m}}(2)^s +$$

$$\sum_{\substack{\text{lower depth } q<p, z_i\geq 0\\ \text{at least one } z_i=0}} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}(2z_1+1,\dots,\overline{2z_q+1})\zeta^{\mathfrak{m}}(2)^s, x_i>0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}}\in\mathbb{Q}, \Bigg\}_{\sum x_i=\sum y_i=\sum z_i=\frac{n-p}{2}-s}.$$

Honorary. About the first condition in 5.1.3 to be honorary:

Lemma 5.2.8. *Let $\zeta^{\mathfrak{m}}(n_1,\cdots,n_p)\in\mathcal{H}^2$, a motivic Euler sum, with $n_i\in\mathbb{Z}^*$, $n_p\neq 1$. Then:*

$$\forall i\ ,\ n_i\neq -1 \Rightarrow D_1(\zeta^{\mathfrak{m}}(n_1,\cdots,n_p))=0$$

Proof. Looking at all iterated integrals of length 1 in $\mathcal{L}$, $I^{\mathfrak{l}}(a;b;c)$, $a,b,c\in\{0,\pm1\}$: the only non zero ones are these with a consecutive $\{1,-1\}$ or $\{-1,1\}$ sequence in the iterated integral, with the condition that extremities are different, that is:

$$I(0;1;-1), I(0;-1;1), I(1;-1;0), I(-1;+1;0), I(-1;\pm1;1), I(1;\pm1;-1).$$

Moreover, they are all equal to $\pm\log^{\mathfrak{a}}(2)$ in the Hopf algebra $\mathcal{A}$. Consequently, if there is no -1 in the Euler sums notation, it implies that D_1 would be zero. □

Comparison with Hoffman's basis. Let compare:

(*i*) The Hoffman basis of $\mathcal{H}^1$ formed by motivic MZV with only 2 and 3 ([20])

$$\mathcal{B}^H := \{\zeta^{\mathfrak{m}}(x_1,\dots,x_k),\ \text{where } x_i\in\{2,3\}\}.$$

(*ii*) $\mathcal{B}^1$, the base of $\mathcal{H}^1$ previously obtained (Corollary 5.2.7).

Beware, the index p for $\mathcal{B}^H$ indicates the number of 3 among the x_i, whereas for $\mathcal{B}^1$, it still indicates the depth; in both case, it can be seen as the *motivic depth* (cf. §2.4.3):

Corollary 5.2.9. *$\mathcal{B}^1_{n,p}$ is a basis of $gr_p^{\mathfrak{D}}\langle\mathcal{B}^H_{n,p}\rangle_{\mathbb{Q}}$ and defines a $\mathbb{Z}_{odd}$-structure.*
I.e. each element of the Hoffman basis of weight n and with p three, $p>0$, decomposes into a $\mathbb{Z}_{odd}$-linear combination of $\mathcal{B}^1_{n,p}$ elements plus terms of depth strictly less than p.

Proof. Deduced from the previous results, with the $\mathcal{Z}_{odd}$ structure of the basis for Euler sums. □

5.2.5.2 The cases $N=3,4$.

For $N=3,4$ there are a generator in each degree ≥ 1 and two Galois descents.

Definition 5.2.10. · ***Family:*** $\mathcal{B} := \left\{\zeta^{\mathfrak{m}}\left(\begin{smallmatrix}x_1,\dots,x_p\\ 1,\dots,1,\xi\end{smallmatrix}\right)(2i\pi)^{s,\mathfrak{m}}, x_i\geq 1, s\geq 0\right\}$.

· ***Level:***

The $(k_N/k_N, P/1)$-level	*is defined as*	*the number of x_i equal to 1*
The $(k_N/\mathbb{Q}, P/P)$-level		*the number of x_i even*
The $(k_N/\mathbb{Q}, P/1)$-level		*the number of even x_i or equal to 1*

· ***Filtrations by the motivic level:*** *$\mathcal{F}^{\lceil}_{-1}\mathcal{H}^N = 0$ and $\mathcal{F}^{\lceil}_i\mathcal{H}^N$ is the largest submodule of $\mathcal{H}^N$ such that $\mathcal{F}^{\lceil}_i\mathcal{H}^N/\mathcal{F}^{\lceil}_{i-1}\mathcal{H}^N$ is killed by $\mathscr{D}^{\lceil}$, where*

$$\mathscr{D}^{\lceil} = \begin{array}{ll} \{D_1^{\xi}\} & \text{for } \lceil = (k_N/k_N, P/1) \\ \{(D_{2r}^{\xi})_{r>0}\} & \text{for } \lceil = (k_N/\mathbb{Q}, P/P) \\ \{D_1^{\xi}, (D_{2r}^{\xi})_{r>0}\} & \text{for } \lceil = (k_N/\mathbb{Q}, P/1) \end{array} .$$

Remarks:

· As before, the increasing, or decreasing, filtration that we could define by the number of 1 (resp. number of even) appearing in the motivic multiple zeta values is not preserved by the coproduct, since the number of 1 can either diminish or increase (at most 1), so is not motivic.

· An effective way of seeing those motivic level filtrations, giving a recursive criterion:

$$\mathcal{F}_i^{k_N/\mathbb{Q},P/P}\mathcal{H} = \left\{ \mathfrak{Z} \in \mathcal{H}, \text{ s. t. } \forall r > 0 \ , \ D_{2r}^{\xi}(\mathfrak{Z}) \in \mathcal{F}_{i-1}^{k_N/\mathbb{Q},P/P}\mathcal{H} \ , \forall r \geq 0 \ , \ D_{2r+1}^{\xi}(\mathfrak{Z}) \in \mathcal{F}_i^{k_N/\mathbb{Q},P/P}\mathcal{H} \right\}.$$

We deduce from the result in §5.2.3 a result of P. Deligne ($i = 0$, cf. [31]):

Corollary 5.2.11. *The elements of $\mathcal{B}^N_{n,p,\geq i}$ form a basis of $gr_p^{\mathfrak{D}}\mathcal{H}_n/\mathcal{F}_{i-1}\mathcal{H}_n$.*
In particular the map $\mathcal{G}^{\mathcal{MT}_N} \to \mathcal{G}^{\mathcal{MT}'_N}$ is an isomorphism. The elements of $\mathcal{B}^N_n$, form a basis of motivic multiple zeta value relative to μ_N, $\mathcal{H}^N_n$.

The level 0 of the filtrations considered for $N'|N \in \{3, 4\}$ gives the Galois descents:

Corollary 5.2.12. *A basis of $\mathcal{H}^{N'}_n$ is formed by elements of $\mathcal{B}^N_n$ of level 0 each corrected by linear combination of elements $\mathcal{B}^N_n$ of level ≥ 1. In particular, with ξ primitive:*

· ***Galois descent*** *from $N' = 1$ to $N = 3, 4$: A basis of motivic multiple zeta values:*

$$\mathcal{B}^{1;N} := \left\{ \zeta^{\mathfrak{m}}\begin{pmatrix} 2x_1+1, \dots, 2x_p+1 \\ 1, \dots, 1, \xi \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i \geq 0 \\ \text{at least one } y_i \text{ even or } =1}} \alpha_{\boldsymbol{x},\boldsymbol{y}} \zeta^{\mathfrak{m}}\begin{pmatrix} y_1, \dots, y_p \\ 1, \dots, 1, \xi \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s \right.$$

$$\left. + \sum_{\substack{\text{lower depth } q<p, \\ \text{at least one even or } =1}} \beta_{\boldsymbol{x},\boldsymbol{z}} \zeta^{\mathfrak{m}}\begin{pmatrix} z_1, \dots, z_q \\ 1, \dots, 1, \xi \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s \ , \ x_i > 0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q} \right\}.$$

· ***Galois descent*** *from $N' = 2$ to $N = 4$: A basis of motivic Euler sums:*

$$\mathcal{B}^{2;4} := \left\{ \zeta^{\mathfrak{m}}\begin{pmatrix} 2x_1+1, \dots, 2x_p+1 \\ 1, \dots, 1, \xi_4 \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i > 0 \\ \text{at least one even}}} \alpha_{\boldsymbol{x},\boldsymbol{y}} \zeta^{\mathfrak{m}}\begin{pmatrix} y_1, \dots, y_p \\ 1, \dots, 1, \xi_4 \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s \right.$$

$$\left. + \sum_{\substack{\text{lower depth } q<p \\ z_i>0, \text{at least one even}}} \beta_{\boldsymbol{x},\boldsymbol{z}} \zeta^{\mathfrak{m}}\begin{pmatrix} z_1, \dots, z_q \\ 1, \dots, 1, \xi_4 \end{pmatrix} \zeta^{\mathfrak{m}}(2)^s \ , \ x_i \geq 0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q} \right\}.$$

· *Similarly, replacing ξ_4 by ξ_3 in $\mathcal{B}^{2;4}$, this gives a basis of:*

$$\mathcal{F}_0^{k_3/\mathbb{Q},3/3}\mathcal{H}^3_n = \boldsymbol{\mathcal{H}_n^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}}.$$

· *A basis of* $\mathcal{F}_0^{k_N/k_N,P/1}\mathcal{H}_n^N = \boldsymbol{\mathcal{H}_n^{\mathcal{MT}(\mathcal{O}_N)}}$, *with* $N=3,4$*:*

$$\mathcal{B}^{N\ unram} := \Bigg\{ \zeta^{\mathfrak{m}}\begin{pmatrix} x_1,\dots,x_p \\ 1,\dots,1,\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i>0 \\ at\ least\ one\ 1}} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}\begin{pmatrix} y_1,\dots,y_p \\ 1,\dots,1,\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s$$

$$+ \sum_{\substack{lower\ depth\ q<p \\ z_i>0, at\ least\ one\ 1}} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}\begin{pmatrix} z_1,\dots,z_q \\ 1,\dots,1,\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s\ ,\ x_i>0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q}\Bigg\}.$$

`Nota Bene:` Notice that for the last two level 0 spaces, $\mathcal{H}_n^{\mathcal{MT}(\mathcal{O}_N)}$, $N=3,4$ and $\mathcal{H}_n^{\mathcal{MT}(\mathbb{Z}[\frac{1}{3}])}$, we still do not have another way to reach them, since those categories of mixed Tate motives are not simply generated by a motivic fundamental group.

5.2.5.3 The case $N=8$.

For $N=8$ there are two generators in each degree ≥ 1 and three possible Galois descents: with $\mathcal{H}^4$, $\mathcal{H}^2$ or $\mathcal{H}^1$.

Definition 5.2.13. · ***Family:*** $\mathcal{B} := \left\{\zeta^{\mathfrak{m}}\left(\begin{smallmatrix} x_1,\dots,x_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi\end{smallmatrix}\right)(2i\pi)^{s,\mathfrak{m}}, x_i\geq 1, \epsilon_i\in\{\pm 1\}\ s\geq 0\right\}$.

· ***Level****, denoted* i*:*

The $(k_8/k_4, 2/2)$*-level*	*is the number of*	ϵ_j *equal to* -1
The $(k_8/\mathbb{Q}, 2/2)$*-level*		ϵ_j *equal to* -1 + *even* x_j
The $(k_8/\mathbb{Q}, 2/1)$*-level*		ϵ_j *equal to* -1, + *even* x_j + x_j *equal to* 1.

· ***Filtrations by the motivic level:*** $\mathcal{F}^{\lceil}_{-1}\mathcal{H}^8 = 0$ *and* $\mathcal{F}^{\lceil}_i\mathcal{H}^8$ *is the largest submodule of* $\mathcal{H}^8$ *such that* $\mathcal{F}^{\lceil}_i\mathcal{H}^8/\mathcal{F}^{\lceil}_{i-1}\mathcal{H}^8$ *is killed by* $\mathscr{D}^{\lceil}$*, where*

$$\mathscr{D}^{\lceil} = \begin{cases} \{(D_r^{\xi} - D_r^{-\xi})_{r>0}\} & for\ \lceil = (k_8/k_4, 2/2) \\ \{(D_{2r+1}^{\xi} - D_{2r+1}^{-\xi})_{r\geq 0}, (D_{2r}^{\xi})_{r>0}, (D_{2r}^{-\xi})_{r>0}\} & for\ \lceil = (k_8/\mathbb{Q}, 2/2) \\ \{(D_{2r+1}^{\xi} - D_{2r+1}^{-\xi})_{r>0}, D_1^{\xi}, D_1^{-\xi}, (D_{2r}^{\xi})_{r>0}, (D_{2r}^{-\xi})_{r>0}\} & for\ \lceil = (k_8/\mathbb{Q}, 2/1) \end{cases}.$$

Corollary 5.2.14. *A basis of* $\mathcal{H}_n^{N'}$ *is formed by elements of* $\mathcal{B}_n^N$ *of level* 0 *each corrected by linear combination of elements* $\mathcal{B}_n^N$ *of level* ≥ 1*. In particular, with* ξ *primitive:*

8 → 1: *A basis of MMZV:*

$$\mathcal{B}^{1;8} := \Bigg\{ \zeta^{\mathfrak{m}}\begin{pmatrix} 2x_1+1,\dots,2x_p+1 \\ 1,\dots,1,\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i\ at\ least\ one\ even\ or\ =1 \\ or one \epsilon_i=-1}} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}\begin{pmatrix} y_1,\dots,y_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s$$

$$+ \sum_{q<p\ lower\ depth,\ level\ \geq 1} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}\begin{pmatrix} z_1,\dots,z_q \\ \widetilde{\epsilon}_1,\dots,\widetilde{\epsilon}_q\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s\ ,\ x_i>0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q}\Bigg\}.$$

8 → 2: *A basis of motivic Euler sums:*

$$\mathcal{B}^{2;8} := \Bigg\{ \zeta^{\mathfrak{m}}\begin{pmatrix} 2x_1+1,\dots,2x_p+1 \\ 1,\dots,1,\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s + \sum_{\substack{y_i\ at\ least\ one\ even \\ or\ one\ \epsilon_i=-1}} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}\begin{pmatrix} y_1,\dots,y_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s$$

$$+ \sum_{\substack{lower\ depth q<p \\ with\ level\geq 1}} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}\begin{pmatrix} z_1,\dots,z_q \\ \widetilde{\epsilon}_1,\dots,\widetilde{\epsilon}_q\xi\end{pmatrix}\zeta^{\mathfrak{m}}(2)^s\ ,\ x_i\geq 0, \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{Q}\Bigg\}.$$

8 → 4: *A basis of MMZV relative to μ_4:*

$$\mathcal{B}^{4;8} := \Bigg\{ \zeta^{\mathfrak{m}}\begin{pmatrix} x_1,\dots,x_p \\ 1,\dots,1,\xi \end{pmatrix}(2i\pi)^s + \sum_{\text{at least one } \epsilon_i=-1} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}\begin{pmatrix} y_1,\dots,y_p \\ \epsilon_1,\dots,\epsilon_{p-1},\epsilon_p\xi \end{pmatrix}(2i\pi)^s$$

$$+ \sum_{\text{lower depth, level } \geq 1} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}\begin{pmatrix} z_1,\dots,z_q \\ \widetilde{\epsilon}_1,\dots,\widetilde{\epsilon}_q\xi \end{pmatrix}(2i\pi)^s \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q} \Bigg\}.$$

5.2.5.4 The case $N = $ '6'.

For the unramified category $\mathcal{MT}(\mathcal{O}_6)$, there is one generator in each degree > 1 and one Galois descent with $\mathcal{H}^1$.

First, let us point out this sufficient condition for a MMZV_{μ_6} to be unramified:

Lemma 5.2.15.

Let $\zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\cdots,n_p \\ \epsilon_1,\dots,\epsilon_p \end{pmatrix} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_6[\frac{1}{6}])}$ *a motivic* MZV_{μ_6}, *such that :* [I]

$$\begin{array}{ll} & \text{Each } \eta_i \in \{1,\xi_6\} \\ \text{OR} & \text{Each } \eta_i \in \{1,\xi_6^{-1}\} \end{array} \qquad \text{Then,} \quad \zeta^{\mathfrak{m}}\begin{pmatrix} n_1,\cdots,n_p \\ \epsilon_1,\dots,\epsilon_p \end{pmatrix} \in \mathcal{H}^{\mathcal{MT}(\mathcal{O}_6)}$$

Proof. Immediate, by Corollary, 5.1.4, and with the expression of the derivations (2.4.2) since these families are stable under the coaction. □

Definition 5.2.16. · ***Family:*** $\mathcal{B} := \Big\{\zeta^{\mathfrak{m}}\left(\begin{smallmatrix} x_1,\dots,x_p \\ 1,\dots,1,\xi \end{smallmatrix}\right)(2i\pi)^{s,\mathfrak{m}}, x_i > 1, s \geq 0\Big\}$.

· ***Level:*** *The* $(k_6/\mathbb{Q},1/1)$*-level, denoted* i*, is defined as the number of even* x_j*.*

· ***Filtration by the motivic*** $(k_6/\mathbb{Q},1/1)$***-level:***

$$\mathcal{F}_{-1}^{(k_6/\mathbb{Q},1/1)}\mathcal{H}^6 = 0 \text{ and } \mathcal{F}_i^{(k_6/\mathbb{Q},1/1)}\mathcal{H}^6 \text{ is the largest submodule of } \mathcal{H}^6 \text{ such that}$$
$$\mathcal{F}_i^{(k_6/\mathbb{Q},1/1)}\mathcal{H}^6/\mathcal{F}_{i-1}^{(k_6/\mathbb{Q},1/1)}\mathcal{H}^6 \text{ is killed by } \mathscr{D}^{(k_6/\mathbb{Q},1/1)} = \Big\{D_{2r}^{\xi}, r > 0\Big\}.$$

Corollary 5.2.17. *Galois descent from* $N' = 1$ *to* $N = $ '6' *unramified. A basis of MMZV:*

$$\mathcal{B}^{1;6} := \Bigg\{ \zeta^{\mathfrak{m}}\begin{pmatrix} 2x_1+1,\dots,2x_p+1 \\ 1,\dots,1,\xi \end{pmatrix}\zeta^{\mathfrak{m}}(2)^s + \sum_{y_i \text{ at least one even}} \alpha_{\boldsymbol{x},\boldsymbol{y}}\zeta^{\mathfrak{m}}\begin{pmatrix} y_1,\dots,y_p \\ 1,\dots,1,\xi \end{pmatrix}\zeta^{\mathfrak{m}}(2)^s$$

$$+ \sum_{\text{lower depth, level } \geq 1} \beta_{\boldsymbol{x},\boldsymbol{z}}\zeta^{\mathfrak{m}}\begin{pmatrix} z_1,\dots,z_q \\ 1,\dots,1,\xi \end{pmatrix}\zeta^{\mathfrak{m}}(2)^s \ , \ \alpha_{\boldsymbol{x},\boldsymbol{y}}, \beta_{\boldsymbol{x},\boldsymbol{z}} \in \mathbb{Q}, x_i > 0 \Bigg\}.$$

Chapter 6

Miscellaneous uses of the coaction

6.1 Maximal depth terms, $gr^{\mathfrak{D}}_{\max}\mathcal{H}_n$

The coaction enables us to compute, by a recursive procedure, the coefficients of the terms of *maximal depth*, i.e. the projection on the graded $gr^{\mathfrak{D}}_{\max}\mathcal{H}_n$. In particular, let look at:

- For $N = 1$, when weight is a multiple of 3 ($w = 3d$), such as depth $p > d$:
$$gr^{\mathfrak{D}}_p \mathcal{H}_{3d} = \mathbb{Q}\zeta^{\mathfrak{m}}(3)^d.$$
- Another simple case is for $N = 2, 3, 4$, when weight equals depth, which is referred to as the *diagonal comodule*:
$$gr^{\mathfrak{D}}_p \mathcal{H}_p = \mathbb{Q}\zeta^{\mathfrak{m}}\begin{pmatrix} 1 \\ \xi_N \end{pmatrix}^p.$$

The space $gr^{\mathfrak{D}}_{\max}\mathcal{H}^N_n$ is usually more than 1 dimensional, but the methods presented below could generalize.

6.1.1 MMZV, weight $3d$.

Preliminaries: Linearized Ihara action. The linearisation of the map $\circ : U\mathfrak{g} \otimes U\mathfrak{g} \to U\mathfrak{g}$ induced by Ihara action (cf. §2.4) can be defined recursively on words by, with $\eta \in \mu_N$:

$$\underline{\circ} : \quad U\mathfrak{g} \otimes U\mathfrak{g} \to U\mathfrak{g} : \quad \begin{array}{ll} a\underline{\circ}e^n_0 & = e^n_0 a \\ a\underline{\circ}e^n_0 e_\eta w & = e^n_0([\eta].a)e_\eta w + e^n_0 e_\eta([\eta].a)^* w + e^n_0 e_\eta(a\underline{\circ}w), \end{array} \tag{6.1}$$

where $*$ stands for the following involution:

$$(a_1 \cdots a_n)^* := (-1)^n a_n \cdots a_1.$$

For this paragraph, from now, let $N = 1$ and let use the *commutative polynomial setting*, introducing the isomorphism:

$$\begin{aligned} \rho : U\mathfrak{g} &\longrightarrow \mathbb{Q}\langle Y\rangle := \mathbb{Q}\langle y_0, y_1, \ldots, y_n, \cdots\rangle \\ e^{n_0}_0 e_1 e^{n_1}_0 \cdots e_1 e^{n_p}_0 &\longmapsto y^{n_0}_0 y^{n_1}_1 \cdots y^{n_p}_p \end{aligned} \tag{6.2}$$

Remind that if $\Phi \in U\mathfrak{g}$ satisfies the linearized ⧢ relation, it means that Φ is primitive for $\Delta_{⧢}$, and equivalently that $\phi_{u⧢v} = 0$, with ϕ_w the coefficient of w in Φ. In particular, this

is verified for Φ in the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$.
This property implies for $f = \rho(\Phi)$ a translation invariance (cf. 6.2 in [21])

$$f(y_0, y_1, \cdots, y_p) = f(0, y_1 - y_0, \ldots, y_p - y_0). \tag{6.3}$$

Let consider the map:

$$\begin{aligned} &\mathbb{Q}\langle Y\rangle \longrightarrow \mathbb{Q}\langle X\rangle = \mathbb{Q}\langle x_1, \ldots, x_n, \cdots\rangle, \\ &\quad f \longmapsto \overline{f} \qquad\qquad \text{where } \overline{f}(x_1, \cdots, x_p) := f(0, x_1, \cdots, x_p). \end{aligned} \tag{6.4}$$

If f is translation invariant, $f(y_0, y_1, \ldots, y_p) = \overline{f}(y_1 - y_0, \cdots, y_p - y_0)$.
The image of $\mathfrak{g}^{\mathfrak{m}}$ under ρ is contained in the subspace of polynomial in y_i invariant by translation. Hence we can consider alternatively in the following $\phi \in \mathfrak{g}^{\mathfrak{m}}$, $f = \rho(\phi)$ or $\overline{f}$.

Since the linearized action $\underline{\circ}$ respects the $\mathcal{D}$-grading, it defines, via the isomorphism $\rho : gr^r_{\mathfrak{D}} U\mathfrak{g} \to \mathbb{Q}[y_0, \ldots, y_r]$, graded version of (6.2), a map:

$\underline{\circ} : \mathbb{Q}[y_0, \ldots, y_r] \otimes \mathbb{Q}[y_0, \ldots, y_s] \to \mathbb{Q}[y_0, \ldots, y_{r+s}]$, which in the polynomial representation is:

$$\begin{aligned} f \underline{\circ} g(y_0, \ldots, y_{r+s}) = &\sum_{i=0}^{s} f(y_i, \ldots, y_{i+r}) g(y_0, \ldots, y_i, y_{i+r+1}, \ldots, y_{r+s}) \\ &+ (-1)^{\deg f + r} \sum_{i=1}^{s} f(y_{i+r}, \ldots, y_i) g(y_0, \ldots, y_{i-1}, y_{i+r}, \ldots, y_{r+s}). \end{aligned} \tag{6.5}$$

Or via the isomorphism $\overline{\rho} : gr^r_{\mathfrak{D}} U\mathfrak{g} \to \mathbb{Q}[x_1, \ldots, x_r]$, graded version of (6.4) $\circ$ (6.2):

$$\begin{aligned} f \underline{\circ} g(x_1, \ldots, x_{r+s}) = &\sum_{i=0}^{s} f(x_{i+1} - x_i, \ldots, x_{i+r} - x_i) g(y_1, \ldots, x_i, x_{i+r+1}, \ldots, x_{r+s}) \\ &+ (-1)^{\deg f + r} \sum_{i=1}^{s} f(x_{i+r-1} - x_{i+r}, \ldots, x_i - x_{i+r}) g(x_1, \ldots, x_{i-1}, x_{i+r}, \ldots, x_{r+s}). \end{aligned} \tag{6.6}$$

Coefficient of $\zeta(3)^d$. If the weight w is divisible by 3, for motivic multiple zeta values, it boils down to compute the coefficient of $\zeta^{\mathfrak{m}}(3)^{\frac{w}{3}}$ and a recursive procedure is given in Lemma 6.1.1.

Since $gr^{\mathfrak{D}}_d \mathcal{H}^1_{3d}$ is one dimensional, generated by $\zeta^{\mathfrak{m}}(3)^d$, we can consider the projection:

$$\vartheta : gr^{\mathfrak{D}}_d \mathcal{H}^1_{3d} \to \mathbb{Q}. \tag{6.7}$$

Giving a motivic multiple zeta value $\zeta^{\mathfrak{m}}(n_1, \ldots, n_d)$, of depth d and weight $w = 3d$, there exists a rational $\alpha_{\underline{\mathbf{n}}} = \vartheta(\zeta(n_1, \ldots, n_d))$ such that:

$$\boxed{\zeta^{\mathfrak{m}}(n_1, \ldots, n_d) = \frac{\alpha_{\underline{\mathbf{n}}}}{d!} \zeta^{\mathfrak{m}}(3)^d + \text{ terms of depth strictly smaller than } d.^{\mathrm{I}}} \tag{6.8}$$

In the depth graded in depth 1, $\partial\mathfrak{g}^{\mathfrak{m}}_1$, the generators are:

$$\overline{\sigma}_{2i+1} = (-1)^i (\mathrm{ad} e_0)^{2i}(e_1).$$

We are looking at, in the depth graded:

$$\exp_{\circ}(\overline{\sigma_3}) := \sum_{n=0}^{n} \frac{1}{n!} \overline{\sigma_3} \circ \cdots \circ \overline{\sigma_3} = \sum_{n=0}^{n} \frac{1}{n!} (\mathrm{ad}(e_0)^2(e_1))^{\underline{\circ} n}. \tag{6.9}$$

In the commutative polynomial representation, via $\overline{\rho}$, since $\overline{\rho}(\overline{\sigma}_{2n+1}) = x_1^{2n}$, it becomes:

$$\sum_{n=0}^{n} \frac{1}{n!} x_1^2 \underline{\circ} (x_1^2 \underline{\circ} (\cdots (x_1^2 \underline{\circ} x_1^2) \cdots)).$$

Lemma 6.1.1. *The coefficient of $\zeta^{\mathfrak{m}}(3)^p$ in $\zeta^{\mathfrak{m}}(n_1, \ldots, n_p)$ of weight $3p$ is given recursively:*

$$\begin{aligned}
\alpha_{n_1,\ldots,n_p} &= \delta_{n_p=3}\alpha_{n_1,\ldots,n_{p-1}} \\
&\quad + \sum_{\substack{k=1 \\ n_k=1}}^{p} \left(\delta_{n_{k-1}\geq 3}\alpha_{n_1,\ldots,n_{k-1}-2,n_{k+1},\ldots,n_p} - \delta_{n_{k+1}\geq 3}\alpha_{n_1,\ldots,n_{k-1},n_{k+1}-2,\ldots,n_p}\right) \\
&\quad + 2\sum_{\substack{k=1 \\ n_k=2}}^{p} \left(-\delta_{n_{k-1}\geq 3}\alpha_{n_1,\ldots,n_{k-1}-2,n_{k+1},\ldots,n_p} + \delta_{n_{k+1}\geq 3}\alpha_{n_1,\ldots,n_{k-1},n_{k+1}-2,\ldots,n_p}\right). \qquad (6.10)
\end{aligned}$$

Remarks:

- This is proved for motivic multiple zeta values, and by the period map, it also applies to multiple zeta values.
- This lemma (as the next one, more precise) could be generalized for unramified motivic Euler sums.
- All the coefficients α are all integers.

Proof. Recursively, let consider:

$$P_{n+1}(x_1, \cdots, x_{n+1}) := x_1^2 \underline{\circ} P_n(x_1, \cdots, x_n). \qquad (6.11)$$

By the definition of the linearized Ihara action (6.5):

$$\begin{aligned}
P_{n+1}(x_1, \cdots, x_{n+1}) &= \sum_{i=0}^{n} (x_{i+1} - x_i)^2 P_n(x_1, \cdots, x_i, x_{i+2}, \ldots, x_{n+1}) \\
&\quad - \sum_{i=1}^{n} (x_{i+1} - x_i)^2 P_n(x_1, \cdots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \\
&= (x_{n+1} - x_n)^2 P_n(x_1, \ldots, x_n) + \sum_{i=0}^{n-1} (x_i - x_{i+2})(x_i + x_{i+2} - 2x_{i+1}) P_n(x_1, \cdots, x_i, x_{i+2}, \ldots, x_{n+1}).
\end{aligned}$$

Turning now towards the coefficients $c^{\mathbf{i}}$ defined by:

$$P_p(x_1, \cdots, x_p) = \sum c^{\mathbf{i}} x_1^{i_1} \cdots x_p^{i_p}, \qquad \text{we deduce:}$$

$$\begin{aligned}
c^{i_1,\ldots,i_p} &= -\delta_{\substack{i_1=0 \\ i_2\geq 2}} c^{i_2-2,i_3,\ldots,i_p} + \delta_{i_p=2} c^{i_1,\cdots,i_{p-1}} + \delta_{\substack{i_n=0 \\ i_{p-1}\geq 2}} c^{i_1,\ldots,i_{p-2},i_{p-1}-2} - 2\delta_{\substack{i_p=1 \\ i_{p-1}\geq 2}} c^{i_1,\ldots,i_{p-2},i_{p-1}-1} \\
&\quad + \sum_{\substack{k=2 \\ i_k=0}}^{p-1} \left(\delta_{i_{k-1}\geq 2} c_{i_1,\ldots,i_{k-1}-2,i_{k+1},\ldots,i_p} - \delta_{i_{k+1}\geq 2} c_{i_1,\ldots,i_{k-1},i_{k+1}-2,\ldots,i_p}\right) \\
&\quad + 2\sum_{\substack{k=2 \\ i_k=1}}^{p-1} \left(-\delta_{i_{k-1}\geq 1} c_{i_1,\ldots,i_{k-1}-2,i_{k+1},\ldots,i_p} + \delta_{i_{k+1}\geq 1} c_{i_1,\ldots,i_{k-1},i_{k+1}-2,\ldots,i_p}\right),
\end{aligned}$$

which gives the recursive formula of the lemma. □

Generalization. Another proof of the previous lemma is possible using the dual point of view with the depth-graded derivations $D_{3,p}$, looking at cuts of length 3 and depth 1.[I]
A motivic multiple zeta value of weight $3d$ and of depth $p > d$ could also be expressed as:

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_p) = \frac{\alpha_{\underline{\mathbf{n}}}}{d!}\zeta^{\mathfrak{m}}(3)^d + \text{ terms of depth strictly smaller than } d. \tag{6.12}$$

However, to compute this coefficient $\alpha_{\underline{\mathbf{n}}}$, we could not work as before in the depth graded; i.e. this time, we have to consider all the possible cuts of length 3. Then, the coefficient emerges when computing $\boldsymbol{(D_3)^{\circ d}}$.

Lemma 6.1.2. *The coefficient of $\zeta^{\mathfrak{m}}(3)^d$ in $\zeta^{\mathfrak{m}}(n_1, \dots, n_p)$ of weight $3d$ such that $p > d$, is given recursively:*

$$\begin{aligned}
\alpha_{n_1,\dots,n_p} &= \delta_{n_p=3}\alpha_{n_1,\dots,n_{p-1}} \\
&+ \sum_{\substack{k=1\\ n_k=1}}^{p} \left(\delta_{\substack{n_{k-1}\geq 3\\ k\neq 1}}\alpha_{n_1,\dots,n_{k-1}-2,n_{k+1},\dots,n_p} - \delta_{\substack{n_{k+1}\geq 3\\ k\neq p}}\alpha_{n_1,\dots,n_{k-1},n_{k+1}-2,\dots,n_p}\right) \\
&+ 2\sum_{\substack{k=1\\ n_k=2}}^{p} \left(-\delta_{n_{k-1}\geq 3}\alpha_{n_1,\dots,n_{k-1}-2,n_{k+1},\dots,n_p} + \delta_{\substack{n_{k+1}\geq 3\\ k\neq p}}\alpha_{n_1,\dots,n_{k-1},n_{k+1}-2,\dots,n_p}\right) \\
&+ \sum_{\substack{k=1\\ n_k=1, n_{k+1}=1}}^{p-1} \left(-\delta_{\substack{n_{k-1}\geq 3\\ k\neq 1}}\alpha_{n_1,\dots,n_{k-1}-1,n_{k+2},\dots,n_p} + \delta_{n_{k+2}\geq 3}\alpha_{n_1,\dots,n_{k-1},n_{k+2}-1,\dots,n_p}\right) \\
&+ \sum_{\substack{k=1\\ n_k=1, n_{k+1}=2}}^{p-1} \left(\delta_{\substack{n_{k-1}\geq 3\\ or\ k=1}}\alpha_{n_1,\dots,n_{k-1},n_{k+2},\dots,n_p} + 2\delta_{\substack{n_{k+2}\geq 2\\ k\neq p-1}}\alpha_{n_1,\dots,n_{k-1},n_{k+2},\dots,n_p}\right) \\
&+ \sum_{\substack{k=1\\ n_k=2, n_{k+1}=1}}^{p-1} \left(-2\delta_{\substack{n_{k-1}\geq 2\\ or\ k=1}}\alpha_{n_1,\dots,n_{k-1},n_{k+2},\dots,n_p} - \delta_{\substack{n_{k+2}\geq 3\\ k\neq p-1}}\alpha_{n_1,\dots,n_{k-1},n_{k+2},\dots,n_p}\right).
\end{aligned} \tag{6.13}$$

Proof. Let list first all the possible cuts of length 3 and depth 1 in a iterated integral with $\{0, 1\}$:

Contribution of a 2: $\cdots\ 1\ 0\ 1\ 0\ 0\ \cdots$ $\overset{+2}{\curvearrowleft}$, $\cdots\ 0\ 0\ 1\ 0\ 1\ \cdots$ $\overset{-2}{\curvearrowright}$

Contribution of a 1: $\cdots\ 1\ 1\ 0\ 0\ 0\ \cdots$ $\overset{-1}{\curvearrowleft}$, $\cdots\ 0\ 0\ 0\ 1\ 1\ \cdots$ $\overset{+1}{\curvearrowright}$

Contribution of a 3: $\cdots\ 1\ 0\ 0\ 1\ 0\ \cdots$ $\overset{-1}{\curvearrowleft}$, $\cdots\ 0\ 1\ 0\ 0\ 1\ \cdots$ $\overset{+1}{\curvearrowright}$.

The coefficient above the arrow is the coefficient of $\zeta^{\mathfrak{m}}(3)$ in $I^{\mathfrak{m}}(cut)$, using that:

$$\zeta_1^{\mathfrak{m}}(2) = -2\zeta^{\mathfrak{m}}(3), \quad \zeta^{\mathfrak{m}}(1,2) = \zeta^{\mathfrak{m}}(3), \quad \zeta^{\mathfrak{m}}(2,1) = -2\zeta^{\mathfrak{m}}(3), \quad \zeta_1^{\mathfrak{m}}(1,1) = \zeta^{\mathfrak{m}}(3).$$

Therefore, when there is a 1 followed or preceded by something greater than 4, the contribution is ± 1, while when there is a 2 followed or preceded by something greater than 3, the contribution is ± 2 as claimed in the lemma above. The contributions of a 3 in the third line

[I] The coefficient α indeed emerges when computing $(D_{3,p})^{\circ p}$.

when followed of preceded by something greater than 2 get simplified (except if there is a 3 at the very end); when a 3 is followed resp. preceded by a 1 however, we assimilate it to the contribution of a 1 preceded resp. followed by a 3; which leads exactly to the penultimate lemma.
Additionally to the cuts listed above:

With a 11: $\cdots\ 0\ 0\ \overset{+1}{\frown}\ 1\ 1\ 1\ \cdots$, $\cdots\ 1\ 1\ \overset{-1}{\frown}\ 1\ 0\ 0\ \cdots$

With a 21: $\cdots\ 0\ \overset{-2}{\frown}\ 1\ 0\ 1\ 1\ \cdots$, $\cdots\ 1\ \overset{-1}{\frown}\ 1\ 1\ 0\ \cdots$

With a 12: $\cdots\ 0\ \overset{+1}{\frown}\ 1\ 1\ 0\ 1\ \cdots$, $\cdots\ 1\ \overset{+2}{\frown}\ 1\ 0\ 1\ 0\ \cdots$.

This analysis leads to the given formula.

□

In particular, a sequence of the type $\boldsymbol{Y12X}$ resp. $\boldsymbol{X21Y}$ ($X \geq 2, Y \geq 3$) will imply a (3) resp. (-3) times the coefficient of the same sequence without **12**, resp. **21**.

`Examples:` Let list a few families of multiple zeta values for which we have computed explicitly the coefficient α of maximal depth:

Family	Recursion relation	Coefficient α
$\zeta^{\mathfrak{m}}(\{3\}^p)$	$\alpha_{\{3\}^p} = \alpha_{\{3\}^{p-1}}$	1
$\zeta^{\mathfrak{m}}(\{1,2\}^p)$	$\alpha_{\{1,2\}^p} = \alpha_{\{1,2\}^{p-1}}$	1
$\zeta^{\mathfrak{m}}(2,4,3\{3\}^p)$	$\alpha_{2,4,\{3\}^p} = \alpha_{2,4,\{3\}^{p-1}} + 2\alpha_{\{3\}^{p+1}}$	$2(p+1)$
$\zeta^{\mathfrak{m}}(4,2,\{3\}^p)$	$\alpha_{4,2,\{3\}^p} = 3\alpha_{4,2,\{3\}^{p-1}} - 2\alpha_{\{3\}^{p+1}}$	$-3^{p+1}+1$
$\zeta^{\mathfrak{m}}(\{3\}^p,4,2)$	$\alpha_{\{3\}^p,4,2} = -2\alpha_{\{3\}^{p+1}}$	-2
$\zeta^{\mathfrak{m}}(\{3\}^p,2,4)$	$\alpha_{\{3\}^p,2,4} = 2\alpha_{\{3\}^{p-1},2,4} - 2\alpha_{\{3\}^{p+1}}$	$(-2)^p\frac{4}{3}+\frac{2}{3}$
$\zeta^{\mathfrak{m}}(2,\{3\}^p,4)$	$\alpha_{2,\{3\}^p,4} = 2\alpha_{2,\{3\}^{p-1},4}$	2^{p+1}
$\zeta^{\mathfrak{m}}(4,\{3\}^p,2)$	$\alpha_{4,\{3\}^p,2} = -2\alpha_{4,\{3\}^{p-1},2}$	$(-2)^{p+1}$
$\zeta^{\mathfrak{m}}(1,5,\{3\}^p)$	$\alpha_{1,5,\{3\}^p} = \alpha_{1,5,\{3\}^{p-1}} - 1$	$-(p+1)$
$\zeta^{\mathfrak{m}}(\{2\}^p,\{4\}^p)$	$\alpha_{\{2\}^p,\{4\}^p} = 4\alpha_{\{2\}^{p-1},\{4\}^{p-1}}$	2^{2p-1}
$\zeta^{\mathfrak{m}}(\{2\}^p,\{3\}^a\{4\}^p)$	$\alpha_{\{2\}^p,\{3\}^a\{4\}^p} = 2^a\alpha_{\{2\}^p,\{4\}^p}$	2^{a+2p-1}
$\zeta^{\mathfrak{m}}(\{2\}^p,p+3)$	$\alpha_{\{2\}^p,p+3} = 2\alpha_{\{2\}^{p-1},p+1}$	2^p
$\zeta^{\mathfrak{m}}(2,3,4,\{3\}^p)$	$\alpha_{2,3,4,\{3\}^p} = \alpha_{2,3,4,\{3\}^{p-1}} + 2\alpha_{2,4,\{3\}^p}$	$2(p+1)(p+2)$
$\zeta^{\mathfrak{m}}(2,1,5,4,\{3\}^p)$	$\alpha_{2,1,5,4,\{3\}^p} = \alpha_{2,1,5,4,\{3\}^{p-1}} - \alpha_{2,3,4,\{3\}^p}$	$-\frac{2(p+1)(p+2)(p+3)}{3}$
$\zeta^{\mathfrak{m}}(\{2\}^a,a+3,\{3\}^b)$	$\alpha_{a;b} = 2\alpha_{a-1;b} + \alpha_{a;b-1}$	$2^a\binom{a+b}{a}$
$\zeta^{\mathfrak{m}}(\{5,1\}^n)$ with 3[I]	$\alpha = \sum_{i=1}^{2p-1}(-1)^{i-1}\alpha_{\{5,1\}^p \text{ or } p-1 \text{with } 3}$ [II]	1

For instance, for the coefficient $\alpha_{a;b;c}$ associated to $\zeta^{\mathfrak{m}}(\{3\}^a,2,\{3\}^b,4,\{3\}^c)$, the recursive relation is:

$$\alpha_{a;b;c} = \alpha_{a;b;c-1} + 2\alpha_{a;b-1;c} - 2\alpha_{a-1;b;c}, \quad \text{which leads to the formula:} \tag{6.14}$$

[I] Any $\zeta^{\mathfrak{m}}(\{5,1\}^p)$ where we have inserted some 3 in the subsequence.

[II] Either a 3 has been removed, either a $5,1$ resp. $1,5$ has been converted into a 3 (with a sign coming from if we consider the elements before or after a 1). If it ends with 3, the contribution of a 3 cancel with the contribution of the last 1.

$$\alpha_{a;b;c} = (-2)^a \sum_{l=0}^{b-1}\sum_{m=0}^{c-1} 2^l \frac{(a+l+m-1)!}{(a-1)!l!m!}\alpha_{0;b-l;c-m} + 2^b \sum_{k=0}^{a-1}\sum_{m=0}^{c-1}(-2)^k \frac{(b+k+m-1)!}{k!(b-1)!m!}\alpha_{a-k;0;c-m}$$
$$+ \sum_{l=0}^{b-1}\sum_{k=0}^{a-1}(-2)^k 2^l \frac{(k+l+c-1)!}{k!l!(c-1)!}\alpha_{a-k;b-l;0}.$$

Besides, we can also obtain very easily:

$$\alpha_{a;0;0} = (-2)^a\frac{4}{3} + \frac{2}{3}, \quad \alpha_{0;b;0} = 2^{b+1}, \quad \alpha_{0;0;c} = 2(c+1), \quad \text{and} \quad \alpha_{0;b;c} = 2^{b+1}\binom{b+c+1}{c}$$

Indeed, using $\sum_{k=0}^{n}\binom{a+k}{a} = \binom{n+a+1}{a+1}$, and $\sum_{k=0}^{n}(n-k)\binom{k}{a} = \binom{n+1}{a+2}$, we deduce:

$$\begin{aligned}\alpha_{0;b;c} &= 2\alpha_{0;b-1;c} + \alpha_{0;b;c-1} &= 2^{b+1}\left(\sum_{k=0}^{c-1}\binom{b+k-1}{b-1}(c-k+1) + \sum_{k=0}^{b-1}\binom{c+k-1}{c-1}\right)\\ &&= 2^{b+1}\left(\binom{b+c+1}{b+1} - \binom{b+c-1}{b-1} + \binom{b+c-1}{b-1}\right) = 2^{b+1}\binom{b+c+1}{c}.\end{aligned}$$

`Conjectured examples:`

Family	Conjectured coefficient α
$\zeta^{\mathfrak{m}}(\{2,4\}^p)$	α_p such that $1-\sqrt{cos(2x)} = \sum \frac{\alpha_p(-1)^{p+1}x^{2p}}{(2p)!}$
$\zeta^{\mathfrak{m}}(\{1,5\}^p)$	Euler numbers: $\frac{1}{cosh(x)} = \sum \frac{\alpha_p x^{2p}}{(2p)!}$
$\zeta^{\mathfrak{m}}(\{1,5\}^p,3)$	$(-1)^p$Euler ZigZag numbers $E_{2p+1} = 2^{2p+2}(2^{2p+2}-1)\frac{B_{2p+2}}{2p+2}$

6.1.2 $N > 1$, The diagonal algebra.

For $N = 2,3,4$, $gr_d^{\mathfrak{D}}\mathcal{H}_d$ is 1 dimensional, generated by $\zeta^{\mathfrak{m}}\binom{1}{\xi}$, where $\xi \in \mu_N$ primitive fixed, which allows us to consider the projection:

$$\vartheta^N : gr_d^{\mathfrak{D}}\mathcal{H}_d^1 \to \mathbb{Q}. \tag{6.15}$$

Giving a motivic multiple zeta value relative to μ_N, of weight d, depth d, there exists a rational such that:

$$\boxed{\zeta^{\mathfrak{m}}\begin{pmatrix}1,\dots,1\\ \epsilon_1,\dots,\epsilon_d\end{pmatrix} = \frac{\alpha_{\boldsymbol{\epsilon}}}{n!}\zeta^{\mathfrak{m}}\begin{pmatrix}1\\ \xi\end{pmatrix}^d + \text{ terms of depth strictly smaller than } d.} \tag{6.16}$$

The coefficient α being calculated recursively, using depth 1 results:

Lemma 6.1.3.

$$\alpha_{\epsilon_1,\dots,\epsilon_d} = \begin{cases} 1 & \textit{if } \boldsymbol{N \in \{2,3\}}. \\ \sum_{\substack{k=1\\ \epsilon_k\neq 1}}^{d} \beta_{\epsilon_k}\left(\delta_{\epsilon_{k-1}\epsilon_k\neq 1}\alpha_{\epsilon_1,\dots,\epsilon_{k-1}\epsilon_k,\epsilon_{k+1},\dots,\epsilon_d} - \delta_{\substack{\epsilon_{k+1}\epsilon_k\neq 1\\ k<d}}\alpha_{\epsilon_1,\dots,\epsilon_{k-1},\epsilon_{k+1}\epsilon_k,\dots,\epsilon_d}\right) & \textit{if } \boldsymbol{N = 4}. \end{cases},$$

$$\textit{with } \beta_{\epsilon_k} = \begin{cases} 2 & \textit{if } \epsilon_k = -1 \\ 1 & \textit{else} \end{cases}.$$

Proof. In regards to redundancy, the proof being in the same spirit than the previous section ($N = 1, w = 3d$), is left to the reader.[I] □

[I] The cases $N = 2,3$ correspond to the case $N = 4$ with β always equal to 1.

Remarks:

· For the following categories, the space $gr_d^{\mathfrak{D}}\mathcal{H}_d$ is also one dimensional:

$$\mathcal{MT}\left(\mathcal{O}_6\left[\frac{1}{2}\right]\right), \quad \mathcal{MT}\left(\mathcal{O}_6\left[\frac{1}{3}\right]\right), \quad \mathcal{MT}\left(\mathcal{O}_5\right), \mathcal{MT}\left(\mathcal{O}_{10}\right), \quad \mathcal{MT}\left(\mathcal{O}_{12}\right).$$

The recursive method to compute the coefficient of $\zeta^{\mathfrak{m}}\left(\begin{smallmatrix}1\\ \eta\end{smallmatrix}\right)^d$ would be similar, except that we do not know a proper basis for these spaces.

· For $N=1$, and $w \equiv 2 \mod 3$ for instance, $gr_{\max}^{\mathfrak{D}}\mathcal{H}_n$ is generated by the elements of the Euler $\sharp$ sums basis:
$\zeta^{\sharp,\mathfrak{m}}(1,\boldsymbol{s},\overline{2})$ with $\boldsymbol{s}$ composed of 3's and one 5, $\zeta^{\sharp,\mathfrak{m}}(3,3,\ldots,3,\overline{2})$ and $\zeta^{\sharp,\mathfrak{m}}(1,3,\ldots,3,\overline{4})$.

6.2 Families of unramified Euler sums.

The proof relies upon the criterion 5.1.3, which enables us to construct families of honorary multiple zeta values by iteration on the depth, with parity patterns up to depth 5.

`Notations:` The occurrences of the symbols E or O can denote arbitrary even or odd integers, whereas every repeated occurrence of symbols E_i (respectively O_i) denotes the same positive even (resp. odd) integer. Beware, here, $\overline{O},\overline{n}$ must be different from $\overline{1}$, whereas O and n may be 1. The bracket $\{\cdot,\ldots,\cdot\}$ means that every permutation of the listed elements is allowed.

Theorem 6.2.1. *The following motivic Euler sums are motivic multiple zeta values:*

◇ Depth 1*: All the depth-one Euler sums except* $\zeta^{\mathfrak{m}}(\overline{1})$..

◇ Depth 2*: All the depth-two Euler sums with odd weight (without* $\overline{1}$*).*
In even weight:

$$\zeta^{\mathfrak{m}}(\overline{O},\overline{O}), \zeta^{\mathfrak{m}}(\overline{E},\overline{E})$$

And some linear combinations:

$$\zeta^{\mathfrak{m}}(n_1,\overline{n_2})+\zeta^{\mathfrak{m}}(\overline{n_2},n_1), \zeta^{\mathfrak{m}}(n_1,\overline{n_2})+\zeta^{\mathfrak{m}}(\overline{n_1},n_2), \zeta^{\mathfrak{m}}(n_1,\overline{n_2})-\zeta^{\mathfrak{m}}(n_2,\overline{n_1}).$$

$$(2^{n_1}-1)\zeta^{\mathfrak{m}}(n_1,\overline{1})+(2^{n_1-1}-1)\zeta^{\mathfrak{m}}(\overline{1},n_1).$$

◇ Depth 3: *In even weight:*

$$\zeta^{\mathfrak{m}}(\{E,\overline{O},\overline{O}\}), \zeta^{\mathfrak{m}}(O,\overline{E},\overline{O}), \zeta^{\mathfrak{m}}(\overline{O},\overline{E},O), \zeta^{\mathfrak{m}}(\overline{O_1},\overline{E},\overline{O_1}), \zeta^{\mathfrak{m}}(O_1,\overline{E},O_1), \zeta^{\mathfrak{m}}(\overline{E_1},\overline{E},\overline{E_1}).$$

In odd weight:

$$\zeta^{\mathfrak{m}}(\{\overline{E},\overline{E},O\}), \zeta^{\mathfrak{m}}(\overline{E},\overline{O},E), \zeta^{\mathfrak{m}}(E,\overline{O},\overline{E}).$$

And some linear combinations:

$$\zeta^{\mathfrak{m}}(n_1,n_2,\overline{n_3})+(-1)^{n_1-1}\zeta^{\mathfrak{m}}(\overline{n_3},n_2,n_1) \text{ with } n_2+n_3 \text{ odd}.$$

◇ Depth 4*: In even weight:*

$$\zeta^{\mathfrak{m}}(E,\overline{O},\overline{O},E), \zeta^{\mathfrak{m}}(O,\overline{E},\overline{O},E), \zeta^{\mathfrak{m}}(O,\overline{E},\overline{E},O), \zeta^{\mathfrak{m}}(E,\overline{O},\overline{E},O),$$

$$\zeta^{\mathfrak{m}}(\overline{E},O_1,\overline{E},O_1), \zeta^{\mathfrak{m}}(O_1,\overline{E},O_1,\overline{E}), \zeta^{\mathfrak{m}}(\overline{E_1},\overline{E_2},\overline{E_1},\overline{E_2}).$$

◇ Depth 5*: In odd weight:*

$$\zeta^{\mathfrak{m}}(O_1,\overline{E_1},O_1,\overline{E_1},O_1).$$

`Examples`: These motivic Euler sums are motivic multiple zeta values:

$$\zeta^{\mathfrak{m}}(\overline{25},14,\overline{17}),\zeta^{\mathfrak{m}}(17,\overline{14},17),\zeta^{\mathfrak{m}}(\overline{24},14,\overline{24}),\zeta^{\mathfrak{m}}(6,\overline{23},\overline{17},10),\zeta^{\mathfrak{m}}(13,\overline{24},13,\overline{24},13).$$

Remarks:

- This result is true for motivic Euler sums, and implies the analogue statement for Euler sums.
- Notice that for each honorary MZV above, the reverse sum is honorary too, which was not obvious a priori, since the condition C1 below is not symmetric.
- For instance, for a motivic Euler sum in depth 2, weight n, $\mathfrak{Z}$ is unramified if and only if $D_1(\mathfrak{Z})=0$ and $D_{n-1}(\mathfrak{Z})=0$.

Sufficient condition. Let present the sufficient criterion used here to verify that a motivic Euler sum is unramified.
`Nota Bene:` This criterion is not completely necessary since it does not cover the unramified $\mathbb{Q}$-linear combinations of motivic Euler sums, such as those presented in Chapter 4 (motivic Euler $\sharp$ sums with *odd* and $-$*even*), neither some isolated symmetric examples where the unramified terms appearing in D_{2r+1} could cancel between them. However, it embrace general families of Euler sums which are unramified.

Let $\mathfrak{Z}$ an Euler sum. These four conditions are *sufficient* for $\mathfrak{Z}$ to be unramified:

C1 : No $\overline{1}$ in $\mathfrak{Z}$.
[Implies that $D_1(\mathfrak{Z})=0$.]

C2 : For each beginning $(n_1,\dots,n_i)$ of odd weight, the remaining Euler sums $\zeta(n_{i+1},\dots,n_p)$ is a MZV.
[For terms of Type A0 *in* (6.17)*].*

C3 : If a cut removes an odd-weight part (such that there is no symmetric cut possible), the remaining Euler sums (right side in Type A,B), is a MZV.
[For terms of Type A,B *in (6.17).]*

C4 : Each sub-sequence $(n_i,\dots,n_j)$ in $\mathfrak{Z}$, of even weight such that $\prod_{k=i}^{j}\epsilon_k\neq 1$ is symmetric.
[To cancel terms of Type C *in (6.17).]*

These conditions refers to the expression of the derivations D_{2r+1} (from Lemma 2.4.2):

$$\begin{aligned}
&D_{2r+1}\left(\zeta^{\mathfrak{m}}\left(n_1,\dots,n_p\right)\right)=(\text{A0})\ \delta_{2r+1=\sum_{k=1}^{i}|n_k|}\zeta^{\mathfrak{l}}(n_1,\dots,n_i)\otimes\zeta^{\mathfrak{m}}(n_{i+1},\cdots n_p)\\
&(\text{A,B})\quad \sum_{\substack{1\le i<j\le p\\ 2r+1=\sum_{k=i}^{j}|n_k|-y}}\left\{\begin{array}{l}-\delta_{2\le y\le|n_j|}\zeta^{\mathfrak{l}}_{|n_j|-y}(n_{j-1},\dots,n_{i+1},n_i)\\ +\delta_{2\le y\le|n_i|}\zeta^{\mathfrak{l}}_{|n_i|-y}(n_{i+1},\cdots,n_{j-1},n_j)\end{array}\right.\otimes\zeta^{\mathfrak{m}}(n_1,\dots,n_{i-1},\prod_{k=i}^{j}\epsilon_k\cdot y,n_{j+1},\cdots n_p).\\
&(\text{C})\ +\sum_{\substack{1\le i<j\le p\\ 2r+2=\sum_{k=i}^{j}|n_k|}}\delta_{\prod_{k=i}^{j}\epsilon_k\neq 1}\left\{\begin{array}{l}+\zeta^{\mathfrak{l}}_{|n_i|-1}(n_{i+1},\cdots,n_{j-1},n_j)\\ -\zeta^{\mathfrak{l}}_{|n_j|-1}(n_{j-1},\cdots,n_{i+1},n_i)\end{array}\right.\otimes\zeta^{\mathfrak{m}}(n_1,\dots,n_{i-1},\overline{1},n_{j+1},\cdots n_p).
\end{aligned}\tag{6.17}$$

Proof of Theorem 6.2.1. The proof is done by recursion on the depth, and using the previous criterion. D_1 is obviously 0 for all these elements, which validates already the condition (C1). It is sufficient then to prove that all the Euler sums appearing in the right side when computing $D_{2r+1}(\cdot)$, $r>0$ are unramified (i.e. multiple zeta values); it corresponds to the conditions (C2),(C3), (C4), according which term we consider.

Terms C Since Euler sums with an $\overline{1}$ can not be unramified, we have to cancel terms of type C. The symmetry condition (C4), obviously true for the elements in the theorem, get rid of these terms. Beware, for the few linear combinations of MES given in the theorem, the cuts of type (C) get simplified together (left to the reader).

Terms A,B Checking that the right side are unramified is straightforward once we have listed these right side (cf. the table below), using recursion hypothesis (since D_{2r+1} decreases the depth).
REMARK: No need to write when the right side of A,B has depth 1, since it would be obviously a multiple zeta value. In particular, depth 2 Euler sums claimed (odd weight or $(\overline{O},\overline{O}),(\overline{E},\overline{E})$) are directly proved. Idem for depth 3 Euler sums of even weight since the corresponding right sides (in A0,A,B terms) have depth ≤ 2 and odd weights, which by recursion hypothesis, is a MZV.
Therefore, let list now the possible right sides of a cut (such that there is not a symmetric cut in (6.17) that would cancel it) for Euler sums (up to reversal symmetry) of depth > 3, or depth 3, odd weight:

	Terms A0	Terms A,B
$(O,\overline{E},\overline{E})$		$(\overline{E},\overline{E})$,(O,O)
$(\overline{E},O,\overline{E})$	/	$(\overline{E},\overline{E})$
$(E,\overline{O},\overline{E})$	/	$(\overline{E},\overline{E}),(E,E)$
$(E,\overline{O},\overline{O},E)$	$(\overline{O},E)$	$(\overline{E},\overline{O},E),(E,O,E),(E,\overline{O},\overline{E}),(E,O),(O,E)$
$(O,\overline{E},\overline{O},E)$	$(\overline{E},\overline{O},E),(\overline{O},E)$	$(\overline{E},\overline{O},E),(O,E,E),(O,\overline{E},\overline{E}),(O,E)$
$(O,\overline{E},\overline{E},O)$	$(\overline{E},\overline{E},O),(\overline{E},O)$	$(\overline{E},\overline{E},O),(O,O,O),(O,\overline{E},\overline{E}),(O,E),(E,O)$
$(\overline{E},O_1,\overline{E},O_1)$	$(\overline{E},O)$	$(\overline{E},\overline{E},O),(\overline{E},O,\overline{E}),(\overline{E},\overline{O}),(E,O)$
$(\overline{E_1},\overline{E_2},\overline{E_1},\overline{E_2})$	/	$(O,\overline{E},\overline{E}),(\overline{E},O,\overline{E}),(\overline{E},\overline{E},O),(\overline{E},\overline{O}),(\overline{O},\overline{E})$
$(O_1,\overline{E_1},O_1,\overline{E_1},O_1)$	$(\overline{E_1},O_1,\overline{E_1},O_1)$, $(O_1,\overline{E},O_1)$	$(\overline{E},O_1,\overline{E},O_1),(O_1,\overline{E},\overline{E},O_1),(O_1,\overline{E},O_1,\overline{E})$, $(\overline{O},\overline{E},O),(O,\overline{E},\overline{O}),(O,O)$

The verification that these terms are unramified is straightforward by recursion.

□

Complying with Conditions C $1-4$.

Moreover, under these conditions, the only general unramified families of motivic Euler sums that we can obtain in this way (without linear combinations), are the one listed in Theorem 6.2.1*, as proved below.*

Sketch of the proof. Notice first that the condition C4 implies in particular that there are no consecutive sequences of the type (since it would create type C terms):

$$\text{SEQUENCES NOT ALLOWED}: O\overline{O},\overline{O}O,\overline{E}E,E\overline{E}.$$

It implies, from depth 3, that we can't have the sequences (otherwise one of the non allowed depth 2 sequences above will appear in A,B terms):

$$\text{SEQUENCES NOT ALLOWED}: \overline{EEO},\overline{EEE}O,\overline{EE}O\overline{E},E\overline{OE},EE\overline{O},\overline{O}EE,\overline{E}OE,\overline{EOE},\overline{OOO}.$$

Going on in this recursive way, carefully, leads to the previous theorem.

`For instance,` let $\mathfrak{Z}$ is a motivic Euler sum $\mathfrak{Z}$ with no $\overline{1}$, and let detail two cases:

Depth 3, Even weight: The right side of D_{2r+1} has odd weight and depth smaller than 2, hence is always MZV if there is no $\overline{1}$ by depth 2 results. It boils down to the condition C4: $\mathfrak{Z}$ must be either symmetric (such as O_1EO_1 or E_1EE_1 with possibly one or three overlines) either have exactly two overlines. Using the analysis above of the allowed sequences in depth 2 and 3 for condition C3,4 leads to the following:

$$(E,\overline{O},\overline{O}),(\overline{O},\overline{O},E),(O,\overline{E},\overline{O}),(\overline{O},\overline{E},O),(\overline{O},E,\overline{O}),(\overline{O_1},\overline{E},\overline{O_1}),(O_1,\overline{E},O_1),(\overline{E_1},\overline{E},\overline{E_1}).$$

Depth 4**, Even weight:** Let $\mathfrak{Z} = \zeta^{\mathfrak{m}}\binom{n_1,\dots,n_4}{\epsilon_1,\dots,\epsilon_4}$. To avoid terms of type C with a right side of depth 1: if $\epsilon_1\epsilon_2\epsilon_3 \neq 1$, either $n_1+n_2+n_3$ is odd, or $n_1 = n_3$ and $\epsilon_2 = -1$; if $\epsilon_2\epsilon_3\epsilon_4 \neq 1$, either $n_2+n_3+n_4$ is odd, or $n_2 = n_4$ and $\epsilon_3 = -1$. The following sequences are then not allowed:

$$(\overline{E},O,O,\overline{E}),(\overline{E},\overline{O},\overline{O},\overline{E}),(\overline{E},\overline{O},E,O),(\overline{E},\overline{E},O,O),(O,O,\overline{E},\overline{E}).$$

□

6.3 Motivic Identities

As we have seen above, in particular in Lemma 4.4.3, the coaction enables us to prove some identities between MMZV or MES, by recursion on the depth, up to one rational coefficient at each step. This coefficient can be deduced then, if we know the analogue identity for MZV, resp. Euler sums. Nevertheless, a *motivic identity* between MMZV (resp. MES) is stronger than the corresponding relation between real MZV (resp. Euler sums); it may hence require several relations between MZV in order to lift an identity to motivic MZV. An example of such a behaviour occurs with some Hoffman $\star$ elements, ((iv) in Lemma 4.4.3).

In this section, we list a few examples of identities, picked from the zoo of existing identities, that we are able to lift easily from Euler sums to motivic Euler sums: *Galois trivial* elements (action of the unipotent part of the Galois group being trivial), sums identities, etc.

`Nota Bene`: For other cyclotomic MMZV, we could somehow generalize this idea, but there would be several unknown coefficients at each step, as stated in Theorem 2.4.4. For $N = 3$ or 4, we have to consider all $D_r, 0 < r < n$, and there would be one resp. two (if weight even) unknown coefficients at each step ; for $N =$ '6', if unramified, considering $D_r, r > 1$, there would be also one or two unknown coefficients at each step.

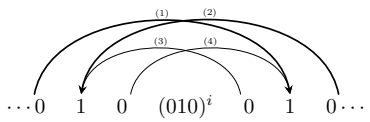
`Example:` Here is an identity known for Euler sums, proven at the motivic level by recursion on n via the coaction for motivic Euler sums (and using the analytic identity):

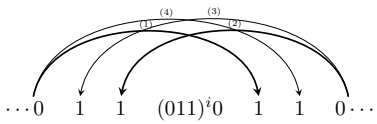
$$\zeta^{\mathfrak{m}}(\{3\}^n) = \zeta^{\mathfrak{m}}(\{1,2\}^n) = 8^n\zeta^{\mathfrak{m}}(\{1,\overline{2}\}^n). \tag{6.18}$$

Proof. These three families are stable under the coaction:

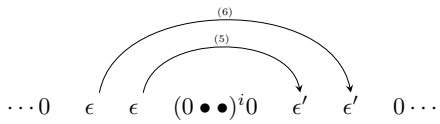
$$\begin{array}{lcll}
D_{2r+1}(\zeta^{\mathfrak{m}}(\{3\}^n)) & = & \delta_{2r+1=3s}\zeta^{\mathfrak{a}}(\{3\}^s) & \otimes\ \zeta^{\mathfrak{m}}(\{3\}^{n-s}). \\
D_{2r+1}(\zeta^{\mathfrak{m}}(\{1,2\}^n)) & = & \delta_{2r+1=3s}\zeta^{\mathfrak{a}}(\{1,2\}^s) & \otimes\ \zeta^{\mathfrak{m}}(\{1,2\}^{n-s}). \\
D_{2r+1}(\zeta^{\mathfrak{m}}(\{1,\overline{2}\}^n)) & = & \delta_{2r+1=3s}\zeta^{\mathfrak{a}}(\{1,\overline{2}\}^s) & \otimes\ \zeta^{\mathfrak{m}}(\{1,\overline{2}\}^{n-s}).
\end{array}$$

Indeed, in both case, in the diagrams below, cuts (3) and (4) are symmetric and get simplified by reversal, as cuts (1) and (2), except for last cut of type (1) which remains alone:

$\zeta^{\mathfrak{m}}(\{3\}^n)$: 

, $\zeta^{\mathfrak{m}}(\{1,2\}^n)$: 

Similarly for $\zeta^{\mathfrak{m}}(\{1,\overline{2}\}^n)$: cuts of type (3), (4) resp. (1), (2) get simplified together, except the first one, when $\epsilon = \epsilon'$ in the diagram below. The other possible cuts of odd length would be (5) and (6) below, when $\epsilon = -\epsilon'$, but each is null since antisymmetric.



, where $\bullet$ alternates between ± 1.

□

Galois trivial. The Galois action of the unipotent group $\mathcal{U}$ is trivial on $\mathbb{Q}[\mathbb{L}^{\mathfrak{m},2n}] = \mathbb{Q}[\zeta^{\mathfrak{m}}(2)^n]$. To prove an element of $\mathcal{H}_{2n}$ is a rational multiple of $\mathbb{L}^{\mathfrak{m},2n}$, it is equivalent to check it is in the kernel of the derivations D_{2r+1}, for $1 \leq 2r+1 < 2n$, by Corollary 2.4.5. We have to use the (known) analogue identities for MZV to conclude on the value of such a rational.

`Example:`

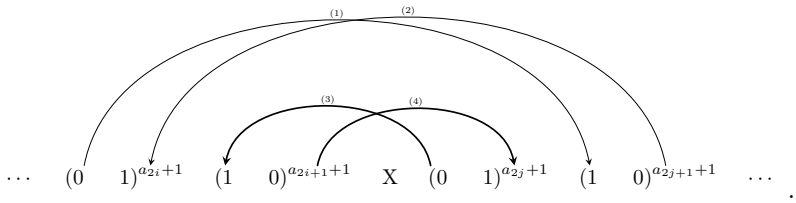
- Summing on all the possible ways to insert n **2**'s.

$$\zeta^{\mathfrak{m}}(\{1,3\}^p \text{ with n } \mathbf{2} \text{ inserted }) = \binom{2p+n}{n} \frac{\pi^{4p+2n,\mathfrak{m}}}{(2n+1)(4p+2n+1)!}. \tag{6.19}$$

- More generally, with fixed (a_i) such that $\sum a_i = n$: [I]

$$\sum_{\sigma \in \mathfrak{S}_{2p}} \zeta^{\mathfrak{m}}(2^{a_{\sigma(0)}}1, 2^{a_{\sigma(1)}}, 3, 2^{a_{\sigma(2)}}, \ldots, 1, 2^{a_{\sigma(2p-1)}}, 3, 2^{a_{\sigma(2p)}}) \in \mathbb{Q}\pi^{4p+2n,\mathfrak{m}}. \tag{6.20}$$

Proof. In order to justify why all the derivations D_{2r+1} cancel, the possible cuts of odd length are, with $X = \{01\}^{a_{2i+2}+1}\{10\}^{a_{2i+3}+1} \cdots \{01\}^{a_{2j-2}}\{10\}^{a_{2j-1}}$:



All the cuts get simplified by Antipode ⧢ 4.7, which proves the result, as follows:

- Cut (1) for $(a_0, \ldots, a_{2p})$ with Cut (2) for $(a_0, \ldots, a_{2i-1}, a_{2j+1} \cdots, a_{2i}, a_{2j+2}, \cdots, a_{2p})$.
- Similarly between (3) and (4), which get simplified considering the sequence where $(a_{2i+1}, \ldots, a_{2j})$ is reversed.

□

Polynomial in simple zetas. A way to prove that a family of (motivic) MZV are polynomial in simple (motivic) zetas, by recursion on depth:

Lemma 6.3.1. *Let $\mathfrak{Z} \in \mathcal{H}_n^1$ a motivic multiple zeta value of depth p.*
If the following conditions hold, $\forall \quad 1 < 2r+1 < n$, $m := \lfloor \frac{n}{2} \rfloor - 1$:

(i) $D_{2r+1,p}(\mathfrak{Z}) = P_r^{\mathfrak{Z}}(\zeta^{\mathfrak{m}}(3), \zeta^{\mathfrak{m}}(5), \ldots, \zeta^{\mathfrak{m}}(2m+1), \zeta^{\mathfrak{m}}(2))$,

$$\textit{with } P_r^{\mathfrak{Z}}(X_1, \cdots, X_m, Y) = \sum_{2s+\sum(2k+1)\cdot a_k = n-2r-1} \beta^r_{a_1,\ldots,a_m,s} X_1^{a_1} \cdots X_m^{a_m} Y^s.$$

(ii) *For $a_k, a_r > 0$:* $\frac{\beta^r_{a_1,\ldots,a_r-1,\cdots,a_m,s}}{a_r+1} = \frac{\beta^k_{a_1,\ldots,a_k-1,\ldots,a_m,s}}{a_k}$.

Then, $\mathfrak{Z}$ is a polynomial in depth 1 MMZV:

$$\mathfrak{Z} = \alpha\zeta^{\mathfrak{m}}(n) + \sum_{2s+\sum(2k+1)a_k = n} \alpha_{a_1,\ldots,a_m,s}\zeta^{\mathfrak{m}}(3)^{a_1} \cdots \zeta^{\mathfrak{m}}(2m+1)^{a_m}\zeta^{\mathfrak{m}}(2)^s.^{\text{II}}$$

[I] Both appears also in Charlton's article.[26].

[II] In particular, $\alpha_{a_1,\ldots,a_m,s} = \frac{\beta^r_{a_1,\ldots,a_r-1,\ldots,a_m,s}}{a_r}$ for $a_r \neq 0$.

Proof. Immediate with Corollary 2.4.5 since:

$$D_{2r+1,p}\left(\zeta^{\mathfrak{m}}(3)^{a_1}\cdots\zeta^{\mathfrak{m}}(2m+1)^{a_m}\zeta^{\mathfrak{m}}(2)^s\right)=a_r\zeta^{\mathfrak{m}}(3)^{a_1}\cdots\zeta^{\mathfrak{m}}(2r+1)^{a_r}\cdots\zeta^{\mathfrak{m}}(2m+1)^{a_m}\zeta^{\mathfrak{m}}(2)^s.$$

□

`Example`: Some examples were given in the proof of Lemma 4.4.3; the following family is polynomial in zetas [I]:

$$\zeta^{\mathfrak{m}}(\{1\}^n,m).$$

Sum formulas. Here are listed a few examples of the numerous *sum identities* known for Euler sums[II] which we can lift to motivic Euler sums, via the coaction. For these identities, as we see through the proof, the action of the Galois group is trivial; the families being stable under the derivations, we are able to lift the identity to its motivic version via a simple recursion.

Theorem 6.3.2. *Summations, if not precised are done over the admissible multi-indices, with $w(\cdot)$, resp. $d(\cdot)$, resp. $h(\cdot)$ indicating the weight, resp. the depth, resp. the height:*

(i) With fixed even (possibly negative) $\{a_i\}_{1\leq i\leq p}$ of sum $2n$:[III]

$$\sum_{\sigma\in\mathfrak{S}_p}\zeta^{\mathfrak{m}}(a_{\sigma(1)},\ldots,a_{\sigma(p)})\in\mathbb{Q}\pi^{2n,\mathfrak{m}}.$$

In particular:[IV]

$$\zeta^{\mathfrak{m}}(\{2n\}^p),\zeta^{\mathfrak{m}}(\{\overline{2n}\}^p)\in\mathbb{Q}\pi^{2np,\mathfrak{m}}.$$

More precisely, with Hoffman [54] [VI]

$$\sum_{\sum n_i=2n}\zeta^{\mathfrak{m}}(2n_1,\ldots,2n_k)=$$

$$\frac{1}{2^{2(k-1)}}\binom{2k-1}{k}\zeta^{\mathfrak{m}}(2n)-\sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor}\frac{1}{2^{2k-3}(2j+1)B_{2j}}\binom{2k-2j-1}{k}\zeta^{\mathfrak{m}}(2j)\zeta^{\mathfrak{m}}(2n-2j).$$

(ii) With Granville [49], or Zagier [83] [VI]

$$\sum_{w(\boldsymbol{k})=n,d(\boldsymbol{k})=d}\zeta^{\mathfrak{m}}(\boldsymbol{k})=\zeta^{\mathfrak{m}}(n).$$

(iii) With Aoki, Ohno [4][VI II]

$$\sum_{w(\boldsymbol{k})=n,d(\boldsymbol{k})=d}\zeta^{\star,\mathfrak{m}}(\boldsymbol{k})=\binom{n-1}{d-1}\zeta^{\mathfrak{m}}(n).$$

$$\sum_{w(\boldsymbol{k})=n,h(\boldsymbol{k})=s}\zeta^{\star,\mathfrak{m}}(\boldsymbol{k})=2\binom{n-1}{2s-1}(1-2^{1-n})\zeta^{\mathfrak{m}}(n).$$

[I] Proof method: with recursion hypothesis on coefficients, using:

$$D_{2r+1}(\zeta^{\mathfrak{m}}(\{1\}^n,m))=-\sum_{j=\max(0,2r+2-m)}^{\min(n-1,2r-1)}\zeta^{\mathfrak{l}}(\{1\}^j,2r+1-j)\otimes\zeta^{\mathfrak{m}}(\{1\}^{n-j-1},m-2r+j).$$

[II] Usually proved considering the generating function, and expressing it as a hypergeometric function.
[III] This would be clearly also true for MMZV*.
[IV] The precise coefficient is given in [11], (48) and can then be deduced also for the motivic identity.

(iv) With Le, Murakami[64][VI]

$$\sum_{w(\boldsymbol{k})=n,h(\boldsymbol{k})=s}(-1)^{d(\boldsymbol{k})}\zeta^{\mathfrak{m}}(\boldsymbol{k})=\begin{cases}0 & \text{if } n \text{ odd.}\\ \frac{(-1)^{\frac{n}{2}}\pi^{\mathfrak{m},n}}{(n+1)!}\sum_{k=0}^{\frac{n}{2}-s}\binom{n+1}{2k}(2-2^{2k})B_{2k} & \text{if } n \text{ even.}\end{cases}$$

(v) With S. Belcher (?)[VI]

$$\begin{array}{lcll}\sum_{\substack{w(\cdot)=2n\\ d(\cdot)=2p}}\zeta^{\mathfrak{m}}(odd,odd>1,odd,\dots,odd,odd>1) & = & \alpha^{n,p}\zeta^{\mathfrak{m}}(2)^n, & \alpha^{n,p}\in\mathbb{Q}\\ \sum_{\substack{w(\cdot)=2n+1\\ d(\cdot)=2p+1}}\zeta^{\mathfrak{m}}(odd,odd>1,odd,\dots,odd>1,odd) & = & \sum_{i=1}^n\beta_i^{n,p}\zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2)^{n-i}, & \beta_i^{n,p}\in\mathbb{Q}\\ \sum_{\substack{w(\cdot)=2n+1\\ d(\cdot)=2p+1}}\zeta^{\mathfrak{m}}(odd>1,odd,\dots,odd,odd>1) & = & \sum_{i=1}^n\gamma_i^{n,p}\zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2)^{n-i}, & \gamma_i^{n,p}\in\mathbb{Q}\end{array}$$

Remark: The permutation identity (i) would in particular imply that all sum of MZV at even arguments times a symmetric function of these same arguments are rational multiple of power of $\mathbb{L}^{\mathfrak{m}}$.
Many specific identities, in small depth have been already found (as Machide in [66], resp. Zhao, Guo, Lei in [51], etc.), and can be directly deduced for motivic MZV, such as:

$$\sum_{k=1}^{n-1}\zeta(2k,2n-2k)\begin{cases}1 & = & \frac{3}{4}\zeta(2n)\\ 4^k+4^{n-k} & = & (n+\frac{4}{3}+\frac{4^n}{6})\zeta(2n)\\ (2k-1)(2n-2k-1) & = & \frac{3}{4}(n-3)\zeta(2n)\end{cases}$$

$$\sum\zeta(2i,2j,2n-2i-2j)\begin{cases}1 & = & \frac{5}{8}\zeta(2n)-\frac{1}{4}\zeta(2n-2)\zeta(2)\\ ij+jk+ki & = & \frac{5n}{64}\zeta(2n)+(4n-\frac{9}{10})\zeta(2n-2)\zeta(2)\\ ijk & = & \frac{n}{128}(n-3)\zeta(2n)-\frac{1}{32}\zeta(2n-2)\zeta(2)+\frac{2n-5}{8}\zeta(2n-4)\zeta(4)\end{cases}$$

Proof. We refer to the formula of the derivations D_r in Lemma $A.1.1$. For many of these equalities, when summing over all the permutations of a certain subset, most of the cuts will get simplified two by two as followed:

$$\zeta^{\mathfrak{m}}\left(k_1,\dots,k_i,k_{i+1},\dots,k_j,k_{j+1},\cdots k_d\right)\,:\,0;\cdots 10^{k_i-1}\mathbf{1}0^{k_{i+1}-1}\cdots 0^{k_{j-1}-1}1\mathbf{0^{k_j-1}}10^{k_{j+1}-1}\cdots;1. \tag{6.21}$$

$$\zeta^{\mathfrak{m}}(k_1,\cdots,k_i,k_j,\cdots,k_{i+1},k_{j+1},\dots,k_d)\,:\,0;\cdots 10^{k_i-1}1\mathbf{0^{k_j-1}}\cdots 0^{k_{i+2}-1}10^{k_{i+1}-1}\mathbf{1}0^{k_{j+1}-1}\cdots;1. \tag{6.22}$$

It remains only the first cuts, beginning with the first 0, such as:

$$\delta_{2r+1=\sum_{j=1}^i k_j}\zeta^{\mathfrak{m}}\left(k_1,\dots,k_i\right)\otimes\zeta^{\mathfrak{m}}(k_{i+1},\dots,k_d)\,, \tag{6.23}$$

and possibly the cuts from a $k_i=1$ to k_d, if the sum is over admissible MMZV: [I]

$$-\delta_{2r+1<\sum_{j=i+1}^d k_j}\zeta^{\mathfrak{m}}\left(k_{i+1},\dots,k_{d-1},2r+1-\sum_{j=i+1}^{d-1}k_j\right)\otimes\zeta^{\mathfrak{m}}\left(k_1,\dots,k_{i-1},\sum_{j=i+1}^{d}k_j-2r\right). \tag{6.24}$$

(i) From the terms above in D_{2r+1}, (6.21), and (6.22) get simplified together, and there are no terms (6.23) since the a_i are all even. Therefore, it is in the kernel of $\oplus_{2r+1<2n}D_{2r+1}$ with even weight, hence Galois trivial.
For instance, for $\zeta^{\mathfrak{m}}(\{\overline{2n}\}^p)$, with $\epsilon,\epsilon'\in\{\pm1\}$:

[VI]The person(s) at the origin of the analytic equality for MZV, used in the proof for motivic MZV.
[I]There, beware, the MZV at the left side can end by 1.

$$\cdots \quad \epsilon \quad 0^a \quad 0 \quad \mathrm{X} \quad 0 \quad 0^a \quad \epsilon' \quad \cdots \text{ , } X = 0^{2n-a-2}\{\pm\epsilon 0^{2n-1}\}^i - \epsilon' 0^{2n-a-2}\text{, alternated sign in X.}$$

if $\epsilon' = -\epsilon$

Either, $\epsilon = \epsilon'$ and X is symmetric, and by reversal of path (cf. §A.1.1), cuts above get simplified, or $\epsilon = -\epsilon'$ and X is antisymmetric, and the cuts above still get simplified since $I^{\mathfrak{m}}(\epsilon; 0^{a+1}X; 0) = -I^{\mathfrak{m}}(0; \widetilde{X}0^{a+1}; \epsilon) = -I^{\mathfrak{m}}(0; X0^{a+1}; -\epsilon)$.

(ii) Let us denote this sum $G(n,d)$, and $G_1(n,d)$ the corresponding sum where a 1 at the end is allowed. As explained in the proof's preamble, the remaining cuts being the first ones and the one from a $k_i = 1$ to the last k_d:

$$D_{2r+1}(G(n,d)) = \sum_{i=0}^{d-1} G_1^{\mathfrak{l}}(2r+1,i) \otimes G(n-2r-1,d-i) - \sum_{i=0}^{d-1} G_1^{\mathfrak{l}}(2r+1,i) \otimes G(n-2r-1,d-i) = 0.$$

(iii) This can be proven also computing the coaction, or noticing that it can be deduced from Euler relation above, turning a MZV* into a sum of MZV of smaller depth, it turns to be:

$$\sum_{i=1}^{d} \sum_{w(\boldsymbol{k})=n, d(\boldsymbol{k})=i} \binom{n-i-1}{d-i} \zeta^{\mathfrak{m}}(\boldsymbol{k}).$$

For the Aoki-Ohno identity, using the formula for MZV $\star$, and with recursion hypothesis, we could similarly prove that the coaction is zero on these elements, and conclude with the result for MZV.

(iv) Let us denote this sum $G_-(n,s)$ and $G_{-,(1)}(n,s)$ resp. $G_{-,1}(n,s)$ the analogue sums with possibly a 1 at the end, resp. with necessarily a 1 at the end. Looking at the derivations, since we sum over all the permutations of the admissible indices, all the cuts get simplified with its symmetric cut as said above, and it remains only the beginning cut (with the first 0), and the cut from a $k_j = 1$ to the last k_d, which leads to:

$$\begin{aligned} D_{2r+1}(G_-(n,s)) &= \sum_{i=0}^{s-1} \Big(G_{-,(1)}^{\mathfrak{l}}(2r+1,i) - G_-^{\mathfrak{l}}(2r+1,i+1) - G_{-,1}^{\mathfrak{l}}(2r+1,i) \Big) \otimes G_-(n-2r-1,s-i) \\ &= \sum_{i=0}^{s-1} (G_-^{\mathfrak{l}}(2r+1,i) - G_-^{\mathfrak{l}}(2r+1,i+1)) \otimes G_-(n-2r-1,s-i). \end{aligned}$$

Using recursion hypothesis, it cancels, and thus, $G_{alt}(n,s) \in \mathbb{Q}\zeta^{\mathfrak{m}}(n)$. Using the analogue analytic equality, we conclude.

(v) For odd sequences with alternating constraints (> 1 or ≥ 1 for instance), cuts between k_i and k_j will get simplified with some symmetric terms in the sum, except possibly (when odd length), the first (i.e. from the first 1 to a first 0) and the last (i.e. from a last 0 to the very last 1) one. More precisely, with O any odd integer, possibly all different:

.

$$\begin{aligned} & D_{2r+1}\left(\sum_{\substack{w(\cdot)=2n \\ d(\cdot)=2p}} \zeta^{\mathfrak{m}}(O, O>1, \cdots, O, O>1) \right) \\ &= \sum_{i=0}^{p-1} \left(\sum_{\substack{w(\cdot)=2r+1 \\ d(\cdot)=2i+1}} \begin{matrix} +\zeta^{\mathfrak{l}}(O, O>1, \cdots, O>1, O) \\ -\zeta^{\mathfrak{l}}(O, O>1, \ldots, O>1, O) \end{matrix} \right) \otimes \sum_{\substack{w(\cdot)=2n-2r-1 \\ d(\cdot)=2p-2i-1}} \zeta^{\mathfrak{m}}(O, O>1, \ldots, O, O>1) = 0. \end{aligned}$$

.

$$D_{2r+1}\left(\sum_{\substack{w(\cdot)=2n+1\\ d(\cdot)=2p+1}} \zeta^{\mathfrak{m}}(O>1,O,\dots,O,O>1)\right)$$
$$=\sum_{i=0}^{p-1}\left(\sum_{\substack{w(\cdot)=2r+1\\ d(\cdot)=2i+1}} \zeta^{\mathfrak{l}}(O>1,O,\dots,O>1)\right)\otimes \sum_{\substack{w(\cdot)=2n-2r\\ d(\cdot)=2p-2i-1}} \zeta^{\mathfrak{m}}(O,O>1,\cdots,O,O>1).$$

By the previous identity, the right side is in $\mathbb{Q}\pi^{2n-2r}$, which proves the result claimed; it gives also the recursion for the coefficients: $\beta_r^{n,p}=\sum_{i=0}^{p-1}\beta_r^{r,i}\alpha^{n-r,p-i}$.

.

$$D_{2r+1}\left(\sum_{\substack{w(\cdot)=2n+1\\ d(\cdot)=2p+1}} \zeta^{\mathfrak{m}}(O,O>1,\dots,O>1,O)\right)=$$
$$+\sum_{i=0}^{p-1}\left(\sum_{\substack{w(\cdot)=2r+1\\ d(\cdot)=2i+1}} \zeta^{\mathfrak{m}}(O,O>1,\dots,O)\right)\otimes\sum_{\substack{w(\cdot)=2n-2r\\ d(\cdot)=2p-2i-1}} \zeta^{\mathfrak{m}}(O,O>1,\cdots,O>1)$$
$$+\sum_{i=0}^{p-1}\left(\sum_{\substack{w(\cdot)=2r+1,\\ d(\cdot)=2i+1}} \begin{matrix}+\zeta^{\mathfrak{m}}(O,O>1,\dots,O)\\ -\zeta^{\mathfrak{m}}(O,O>1,\dots,O)\end{matrix}\right)\otimes\sum_{\substack{w(\cdot)=2n-2r\\ d(\cdot)=2p-2i-1}} \zeta^{\mathfrak{m}}(O>1,O,\cdots,O>1,O)$$

As above, by recursion hypothesis, the right side of the first sum is in $\mathbb{Q}\pi^{2n-2r}$, which proves the result claimed, the second sum being 0; the rational coefficients γ are given by a recursive relation.

□

Appendix A

A.1 Coaction

The coaction formula given by Goncharov and extended by Brown for motivic iterated integrals applies to the $\star$, $\star\star$, $\sharp$ or $\sharp\sharp$ version by linearity [I]. Here is the version obtained for MMZV $\star, \star\star$, $\sharp$ or $\sharp\sharp$:[II]

Lemma A.1.1. *L being a sequence in $\{0, \pm\star\}$ resp. $\{0, \pm\sharp\}$, with possibly 1 at the beginning, $\epsilon \in \{\pm\star\}$ resp. $\in \{\pm\sharp\}$, and $s_\epsilon := sign(\epsilon)$.*

$$
\begin{aligned}
D_r I_s^{\mathfrak{m}}(0; L; 1) &= \delta_{\substack{L=A\epsilon B \\ w(A)=r}} I_k^{\mathfrak{l}}(0; A; s_\epsilon) \otimes I_{s-k}^{\mathfrak{m}}(0; s_\epsilon, B; 1) \\
&+ \sum_{\substack{L=A\epsilon B0C \\ w(B)=r}} I^{\mathfrak{l}}(s_\epsilon; B; 0) \otimes \left(\underbrace{I_s^{\mathfrak{m}}(0; A, \epsilon, 0, C; 1)}_{T_{\epsilon,0}} + \underbrace{I_s^{\mathfrak{m}}(0; A, 0, 0, C; 1)}_{T_{0,0}} \right) \\
&+ \sum_{\substack{L=A0B\epsilon C \\ w(B)=r}} I^{\mathfrak{l}}(0; B, s_\epsilon) \otimes \left(\underbrace{I_s^{\mathfrak{m}}(0; A, 0, \epsilon, C; 1)}_{T_{0,\epsilon}} + \underbrace{I_s^{\mathfrak{m}}(0; A, 0, 0, C; 1)}_{T_{0,0}} \right) \\
&+ \sum_{\substack{L=A\epsilon B\epsilon C \\ w(B)=r}} I^{\mathfrak{l}}(0; B, s_\epsilon) \otimes \left(\underbrace{I_s^{\mathfrak{m}}(0; A, \epsilon, 0, C; 1)}_{T_{\epsilon,0}} - \underbrace{I_s^{\mathfrak{m}}(0; A, 0, \epsilon, C; 1)}_{T_{0,\epsilon}} \right) \\
&+ \sum_{\substack{L=A\epsilon B-\epsilon C \\ w(B)=r}} \Bigg[I^{\mathfrak{l}}(0; B; -s_\epsilon) \otimes \underbrace{I_s^{\mathfrak{m}}(0; A, \epsilon, 0, C; 1)}_{T_{\epsilon,0}} + I^{\mathfrak{l}}(s_\epsilon; B; 0) \otimes \underbrace{I_s^{\mathfrak{m}}(0; A, 0, -\epsilon, C; 1)}_{T_{0,\epsilon}} \\
&\qquad I^{\mathfrak{l}}(s_\epsilon; B; -s_\epsilon) \otimes \underbrace{I_s^{\mathfrak{m}}(0; A, \epsilon, -\epsilon, C; 1)}_{T_{\epsilon,-\epsilon}} \Bigg].
\end{aligned}
$$

[I] Recall the identities 2.29 to turn a $\star$ (resp. $\sharp$) into a 1 (resp. two times a 1) minus a 0.

[II] *For purpose of stability*: if there is a ± 1 at the beginning, as for $\star$ or $\sharp$ versions, the cut with this first ± 1 will be let as a $T_{\pm 1,0}$ term (and not converted into a $T_{\epsilon,0}$ less a $T_{0,0}$), in order to still have a ± 1 at the beginning; whereas, if there is no ± 1 at the beginning, as for $\star\star$ or $\sharp\sharp$ version even the first cut (first line) has to be converted into a $T_{0,\epsilon}$ less a $T_{0,0}$, in order to still have a ϵ at the beginning.

Remarks:

- We will refer to these different terms T for each cut in the whole appendix when using the coaction.
- The expression of D_r for specific MMZV $\star$ and Euler $\sharp$ sums is simplified below thanks to antipodal and hybrid relations, and is fundamentally used in the proofs of Chapter 4.

Proof. The proof is straightforward from (2.45), using the linearity (with 2.29) in both directions:

(i) First, to turn ϵ into a difference of ± 1 minus 0 in order to use (2.45).

(ii) Then, in the right side, a ± 1 appeared inside the iterated integral when looking at the usual coaction formula which is turned into a sum of a term with ϵ (denoted $T_{\epsilon,0}$ or $T_{0,\epsilon}$) and a term with 0 (denoted $T_{0,0}$) by linearity of the iterated integrals and in order to end up only with $0, \epsilon$ in the right side.

Listing now the different cuts leads to the expression of the lemma, since:

- The first line corresponds to the initial cut (from the $s+1$ first 0).
- The second line corresponds to a cut either from $\pm\epsilon$ to 0; the $\pm\epsilon$ being ± 1.
- The third line corresponds to a cut from 0 to $\pm\epsilon$.
- The fourth line corresponds to cut from ϵ to ϵ, with two choices: a ϵ being fixed to 0, the other one fixed to 1. Replacing 1 by $(\epsilon)+(0)$, this leads to a $T_{0,\epsilon}$, a $T_{\epsilon,0}$ and two $T_{0,0}$ terms which get simplified together.
- The last lines correspond to cuts from ϵ to $-\epsilon$, with three possibilities: one being fixed to 0, the other one fixed to ± 1, or the first being 1, the second -1. This leads to a $T_{\epsilon,0}$, a $T_{0,-\epsilon}$ and a $T_{\epsilon,-\epsilon}$, since the $T_{0,0}$ terms get simplified.

□

A.1.1 Simplification rules

This section is devoted on the simplification of the coaction, in the case of motivic Euler sums: we gather terms in D_{2r+1} according to their right side, using relations (§4.2) between motivic iterated integrals $I^{\mathfrak{l}} \in \mathcal{L}$ to simplify the left side.

`Notations:` We use the notation of the iterated integrals inner sequences and represent a term of a cut in D_{2r+1} (referring to Lemma 4.4.2) by arrows between two elements of this sequence. The weight of the cut (which is the length of the subsequence in the iterated integral) would always be considered odd here.[I] The diagrams show which terms get simplified together: i.e. these which have same right side, but opposite left side, by the relation considered in the coalgebra $\mathcal{L}$.

Composition : The composition rule (cf. §1.6) in the coalgebra $\mathcal{L}$ boils down to:

$$I^{\mathfrak{l}}(a;X;b) \equiv -I^{\mathfrak{l}}(b;X;a), \qquad \text{with } X \text{ any sequence of } 0, \pm 1, \pm\star, \pm\sharp. \tag{A.1}$$

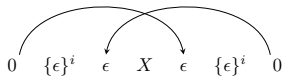
Hence, when considering a cut, we can exchange the extremities of the integral if we change of sign; this exchange is considerably used in this section, without mentioning.

[I] Since we are here only interested in motivic Euler sums, the non zero weight graded parts in the coaction are these corresponding to odd weights: D_{2r+1}, $r \geq 0$.

Antipode ⧢: 4.7 It corresponds to a reversal of path for iterated integrals:

$$I^{\mathfrak{l}}(a;X;b) \equiv (-1)^w I^{\mathfrak{l}}(b;\widetilde{X};a) \text{ for any X sequence of } 0, \pm 1, \pm\star, \pm\sharp. \text{ Hence:}$$

- If X symmetric, i.e. $\widetilde{X} = X$, these cuts get simplified, since they have same right side ($I^{\mathfrak{m}}(\cdots 0\epsilon^{i+1}0\cdots)$) while their left side are opposite:[I]

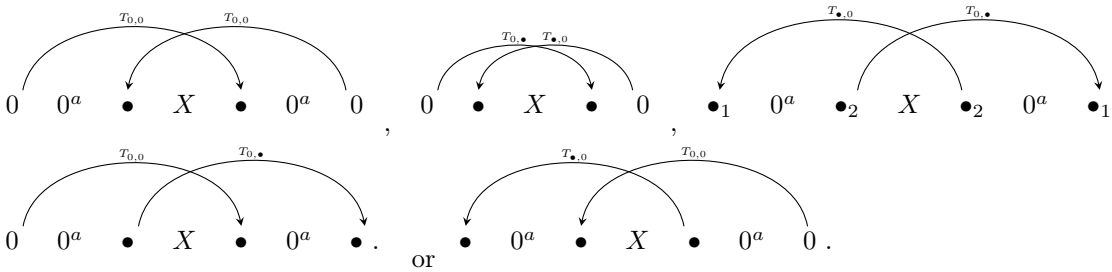


- If X antisymmetric, i.e. $\widetilde{X} = -X$, the following cut is zero[II]:

.

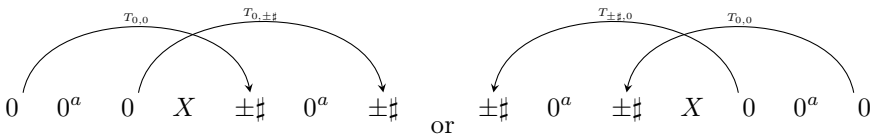
Shift For MES $\star\star$ and, when weight and depth odd for Euler $\sharp\sharp$ sums:

$$(\text{Shift})\ \zeta^{\bullet}_{n-1}(n_1,\cdots,n_p) = \zeta^{\bullet}_{n_1-1}(n_2,\cdots,n_p,n) \tag{A.2}$$



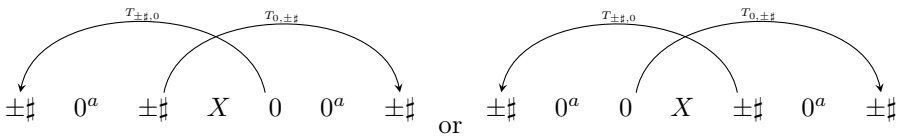
The index $\bullet_1$ and $\bullet_2$ means they can have different sign, otherwise $\bullet$ is fixed in $\pm\star, \pm\sharp$.

Cut For ES $\sharp\sharp$, with even depth[III], odd weight:



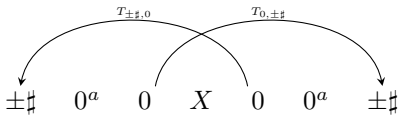
(A.3)

Cut Shifted [III] For ES $\sharp\sharp$, with even depth[III], odd weight:



(A.4)

Minus For ES $\sharp\sharp$, with even depth, odd weight:



(A.5)

[I] $I^{\mathfrak{l}}(\epsilon; X\epsilon^{i+1}; 0) \equiv -I^{\mathfrak{l}}(0;\epsilon^{i+1}\widetilde{X};\epsilon) \equiv -I^{\mathfrak{l}}(0;\epsilon^{i+1}X;\epsilon)$.

[II] Since: $I^{\mathfrak{l}}(\epsilon;X;-\epsilon) \equiv I^{\mathfrak{l}}(\epsilon;X;0) + I^{\mathfrak{l}}(0;X;-\epsilon) \equiv I^{\mathfrak{l}}(\epsilon;X;0) - I^{\mathfrak{l}}(-\epsilon;\widetilde{X};0) \equiv I^{\mathfrak{l}}(\epsilon;X;0) - I^{\mathfrak{l}}(-\epsilon;-X;0) \equiv 0$.

[III] Note that the depth considered here needed to be even is the depth of the bigger cut.

Sign For Euler $\sharp\sharp$ sums with even depth, odd weight, i.e. $X \in 0, \pm\sharp\{\}^{\times}$:

$$I^{\mathfrak{l}}(0;X;\epsilon) = I^{\mathfrak{l}}(0;X;-\epsilon). \quad \text{Hence this cut is zero :} \quad \epsilon \overset{T_{\epsilon,-\epsilon}}{\overset{\frown}{\quad X \quad}} -\epsilon \ . \tag{A.6}$$

Sign hence also means that the $\pm$ sign at one end of a cut does not matter.

A.1.2 MMZV $\star$

Let express each MMZV* as:

$$\zeta^{\star,\mathfrak{m}}(2^{a_0}, c_1, \cdots, c_p, 2^{a_p}).$$

As we will see below, this writing is suitable for the coaction expression, since most of the cuts from a 2 to another 2 get simplified by the rules above. The iterated integral corresponding:

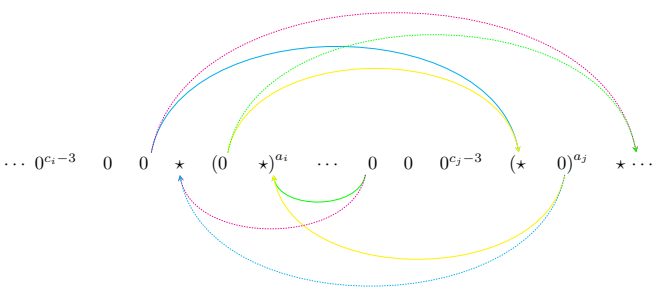
$$I\left(0;1,0,(\star,0)^{a_0-1},\star\cdots 0^{c_i-1}(\star 0)^{a_i}\star,\ldots,0^{c_j-1}(\star 0)^{a_j}\star,\ldots,0^{c_p-1}(\star 0)^{a_p};1\right). \tag{A.7}$$

Considering D_{2r+1} after some simplifications:[I]

Lemma A.1.2.

$$\begin{aligned}
&D_{2r+1}\left(\zeta^{\star,\mathfrak{m}}(2^{a_0}, c_1, \cdots, c_p, 2^{a_p})\right) = \qquad (A.8)\\
&\delta_r \sum_{i<j} \Bigg[\delta_{\substack{3\le\alpha\le c_{i+1}-1\\ 0\le\beta\le a_j}} \zeta^{\star,\mathfrak{l}}_{c_{i+1}-\alpha}(2^{a_j-\beta},\ldots,2^{a_{i+1}}) \otimes \zeta^{\star,\mathfrak{m}}(\cdots,2^{a_i},\alpha,2^{\beta},c_{j+1},\cdots)\\
&\Big(-\delta_{c_{i+1}>3}\zeta^{\star\star,\mathfrak{l}}_2(2^{a_j-\beta-1},\ldots,2^{a_{i+1}}) + \delta_{c_{j+1}>3}\zeta^{\star\star,\mathfrak{l}}_2(2^{a_j},\ldots,2^{a_{i+1}-\beta-1}) +\\
&-\delta_{c_{i+1}=1}\zeta^{\star\star,\mathfrak{l}}(2^{a_j-\beta},\ldots,2^{a_{i+1}}) + \delta_{c_{j+1}=1}\zeta^{\star\star,\mathfrak{l}}(2^{a_{i+1}-\beta},\ldots,2^{a_j})\\
&+\delta_{\substack{c_{i+2}=1\\ \beta>a_{i+1}}}\zeta^{\star\star,\mathfrak{l}}_1(2^{a_j+a_{i+1}-\beta},\ldots,2^{a_{i+2}}) - \delta_{\substack{c_j=1\\ \beta>a_j}}\zeta^{\star\star,\mathfrak{l}}_1(2^{a_{i+1}+a_j-\beta},\ldots,2^{a_{j-1}}).\\
&+\delta_{\beta>a_{i+1}}\zeta^{\star\star,\mathfrak{l}}_{c_{i+2}-2}(2^{a_{i+1}+a_j-\beta+1},\ldots,2^{a_{i+2}}) - \delta_{\beta>a_j}\zeta^{\star\star,\mathfrak{l}}_{c_j-2}(2^{a_{i+1}+a_j-\beta+1},\ldots,2^{a_{j-1}})\Big)\\
&\otimes\zeta^{\star,\mathfrak{m}}(\cdots,2^{a_i},c_{i+1},2^{\beta},c_{j+1},\cdots)\\
&-\delta_{\substack{3\le\alpha\le c_j-1\\ 0\le\beta\le a_i}}\zeta^{\star,\mathfrak{l}}_{c_j-\alpha}(2^{a_i-\beta},\ldots,2^{a_{j-1}}) \otimes \zeta^{\star,\mathfrak{m}}(\cdots,c_i,2^{\beta},\alpha,2^{a_j},\cdots)\Bigg].
\end{aligned}$$

Proof. We look at cuts of odd interior length between two elements of the sequence inside $A.7$. By Shift, the following cuts get simplified:

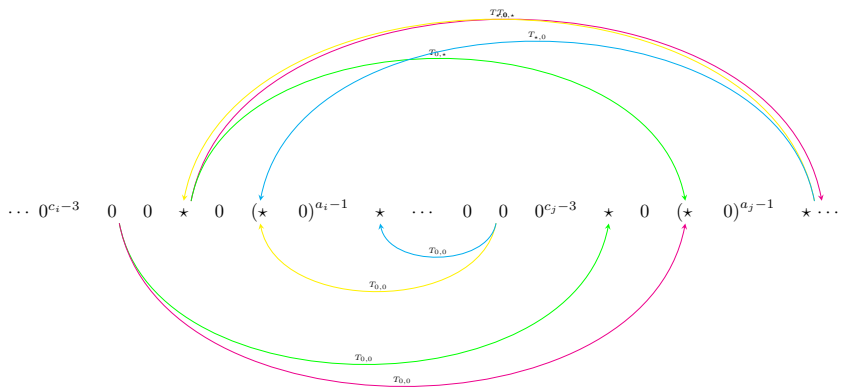


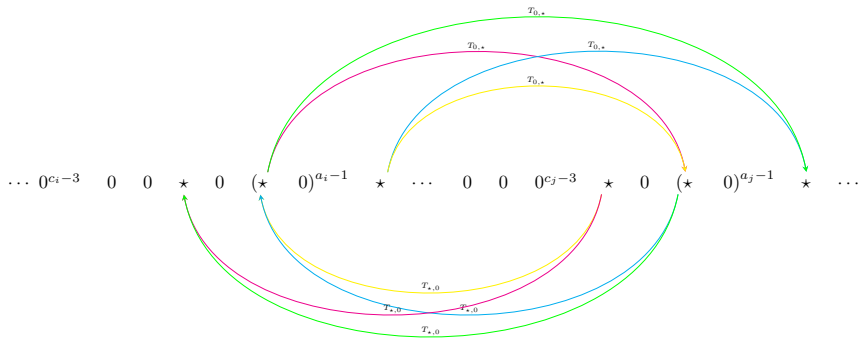
.

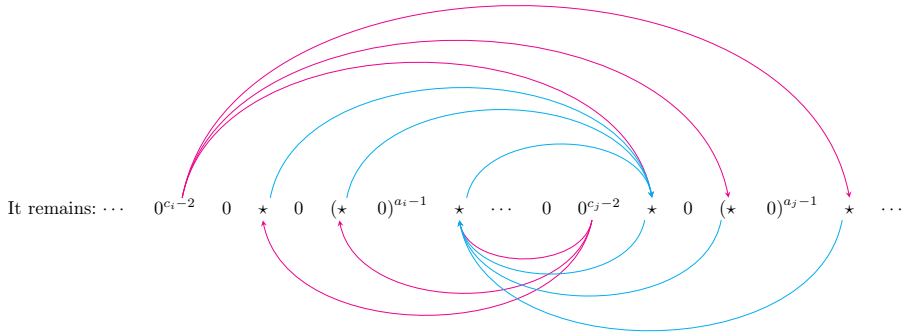
More precisely, $T_{0,0}$ resp. $T_{0,\star}$ above get simplified with $T_{0,0}$ resp. $T_{\star,0}$ below (shifted by

[I] Here δ_r underlines that the left side must have a weight equal to $2r+1$.

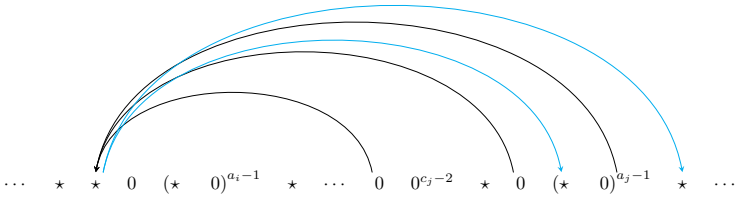
one at the right), by colors, two by two. The dotted arrows mean that in the particular case where $c_i = 1$ resp. $c_j = 1$, only $T_{0,0}$ get simplified.
The following arrows get simplified by Shift ($A.2$), still above with below and by colors:







Cyan arrows above resp. below are $T_{0,\star}$ resp. $T_{0,\star}$ terms; magenta ones above resp. below stand for $T_{0,0}$ and $T_{0,\star}$ resp. for $T_{0,0}$ and $T_{\star,0}$ terms.
If $c_i = 1$ (the case $c_j = 1$, antisymmetric, is omitted), it remains also the following cuts ($T_{\star,0}$ for black ones or $T_{0,\star}$ for cyan ones):



Gathering the remaining cuts in this diagram, according the right side:

1. For $\zeta^{\star,\mathfrak{m}}(\cdots, 2^{a_i}, \boldsymbol{\alpha}, \mathbf{2^{\beta}}, c_{j+1}, \cdots)$:

$$\left(\delta_{\substack{3\leq\alpha<c_{i+1}\\0\leq\beta<a_j}}\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-\alpha}(2^{a_{i+1}},\dots,2^{a_j-\beta}) - \left(\delta_{\substack{4\leq\alpha<c_{i+1}\\0\leq\beta<a_j}} + \delta_{\substack{\alpha=3\\0\leq\beta<a_j}}\right)\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-\alpha+2}(2^{a_{i+1}},\dots,2^{a_j-\beta-1})\right).$$

Using Shift ($A.2$) for the first term and then using the definition of $\zeta^{\star}$ it turns into:

$$\delta_{\substack{3\leq\alpha<c_{i+1}\\0\leq\beta<a_j}}\left(\zeta^{\star\star,\mathfrak{l}}_{1}(c_{i+1}-\alpha+1,2^{a_{i+1}},\dots,2^{a_j-\beta-1}) - \zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-\alpha+2}(2^{a_{i+1}},\dots,2^{a_j-\beta-1})\right)$$

$$= \delta_{\substack{3\leq\alpha<c_{i+1}\\0\leq\beta<a_j}}\zeta^{\star,\mathfrak{l}}_{1}(c_{i+1}-\alpha+1,2^{a_{i+1}},\dots,2^{a_j-\beta-1}).$$

Applying antipodes $A_{\sqcup\!\sqcup}\circ A_*\circ A_{\sqcup\!\sqcup}$:

$$= \delta_{\substack{3\leq\alpha<c_{i+1}\\0\leq\beta<a_j}}\zeta^{\star,\mathfrak{l}}_{c_{i+1}-\alpha}(2^{a_j-\beta},\dots,2^{a_{i+1}}),$$

which gives the first line in (A.8).

2. For $\zeta^{\star,\mathfrak{m}}(\cdots, 2^{a_i}, \boldsymbol{\alpha}, 2^{a_j}, c_{j+1}, \cdots)$, the corresponding left sides are:

$$-\left(\delta_{c_j+2\leq\alpha\leq c_{i+1}} + \delta_{\substack{\alpha=c_j+1\\c_{i+1}>c_j}}\right)\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}+c_j-\alpha}(2^{a_{i+1}},\dots,2^{a_j-1})$$

$$+\left(\delta_{c_{i+1}+2\leq\alpha\leq c_j} + \delta_{\substack{\alpha=c_{i+1}+1\\c_j>c_{i+1}}}\right)\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}+c_j-\alpha}(2^{a_j-1},\dots,2^{a_{i+1}})$$

$$+\delta_{3\leq\alpha<c_{i+1}}\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-\alpha}(2^{a_{i+1}},\dots,c_j) - \delta_{3\leq\alpha<c_j}\zeta^{\star\star,\mathfrak{l}}_{c_j-\alpha}(2^{a_j-1},\dots,c_{i+1})$$

Using Antipode $*$ and turning some ϵ into $'1+0'$:

$$= +\delta_{3\leq\alpha<c_{i+1}}\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}-\alpha}(2^{a_{i+1}},\dots,c_j) - \delta_{3\leq\alpha<c_j}\zeta^{\star\star,\mathfrak{l}}_{c_j-\alpha}(2^{a_j-1},\dots,c_{i+1})$$

$$+(-1)^{c_j<c_{i+1}}\delta_{\min(c_j,c_{i+1})<\alpha\leq\max(c_j,c_{i+1})}\zeta^{\star\star,\mathfrak{l}}_{c_{i+1}+c_j-\alpha}(2^{a_{i+1}},\dots,2^{a_j-1})$$

$$= \delta_{3\leq\alpha<c_{i+1}}\zeta^{\star,\mathfrak{l}}_{c_{i+1}-\alpha}(c_j,\dots,2^{a_{i+1}}) - \delta_{3\leq\alpha<c_j}\zeta^{\star,\mathfrak{l}}_{c_j-\alpha}(c_{i+1},\dots,2^{a_j-1})$$

This gives exactly the same expression than the first and fourth case for $\beta = a_i$ or a_j, and are integrated to them in (A.8).

3. For $\zeta^{\star,\mathfrak{m}}(\cdots, c_{i+1}, \mathbf{2^{\beta}}, c_{j+1}, \cdots)$: [I]

$$-\delta_{\substack{c_{i+1}>3\\0\leq\beta<a_j}}\zeta^{\star\star,\mathfrak{l}}_{2}(2^{a_j-\beta-1},\dots,2^{a_{i+1}}) + \delta_{\substack{c_{j+1}>3\\0\leq\beta<a_{i+1}}}\zeta^{\star\star,\mathfrak{l}}_{2}(2^{a_j},\dots,2^{a_{i+1}-\beta-1})$$

$$+\delta_{\substack{\beta>a_{i+1}\\c_{i+1}>3}}\zeta^{\star\star,\mathfrak{l}}_{c_{i+2}-2}(2^{a_{i+1}+a_j-\beta+1},\dots,2^{a_{i+2}}) - \delta_{\beta>a_j}\zeta^{\star\star,\mathfrak{l}}_{c_j-2}(2^{a_{i+1}+a_j-\beta+1},\dots,2^{a_j-1})$$

$$-\delta_{\substack{c_{i+1}=1\\1\leq\beta<a_j}}\zeta^{\star\star,\mathfrak{l}}(2^{a_j-\beta},\dots,2^{a_{i+1}}) + \delta_{\substack{c_{j+1}=1\\1\leq\beta<a_{i+1}}}\zeta^{\star\star,\mathfrak{l}}(2^{a_{i+1}-\beta},\dots,2^{a_j})$$

$$+\delta_{\substack{c_{i+2}=1\\\beta>a_{i+1}}}\zeta^{\star\star,\mathfrak{l}}_{1}(2^{a_j+a_{i+1}-\beta},\dots,2^{a_{i+2}}) - \delta_{\substack{c_j=1\\\beta>a_j}}\zeta^{\star\star,\mathfrak{l}}_{1}(2^{a_{i+1}+a_j-\beta},\dots,2^{a_j-1}).$$

[I] It includes the case $\alpha = 2$.

4. For $\zeta^{\star,\mathfrak{m}}(\cdots, c_i, \mathbf{2}^{\boldsymbol{\beta}}, \boldsymbol{\alpha}, 2^{a_j}, \cdots)$:

$$\left(-\delta_{\substack{2\leq\alpha\leq c_j-1\\0\leq\beta\leq a_i}}\zeta^{\star\star,\mathfrak{l}}_{c_j-\alpha}(2^{a_{j-1}},\ldots,2^{a_i-\beta}) + \left(\delta_{\substack{4\leq\alpha\leq c_j+1\\0\leq\beta\leq a_i-1}} + \delta_{\substack{\alpha=3\\0\leq\beta\leq a_i-1}}\right)\zeta^{\star\star,\mathfrak{l}}_{c_j-\alpha+2}(2^{a_{j-1}},\ldots,2^{a_i-\beta-1})\right)$$

Antisymmetric to 1.

This leads to the lemma, with the second case incorporated in the first and last line. □

A.1.3 Euler $\sharp$ sums with $\overline{\boldsymbol{even}}, \boldsymbol{odd}$

Let us consider the following family:

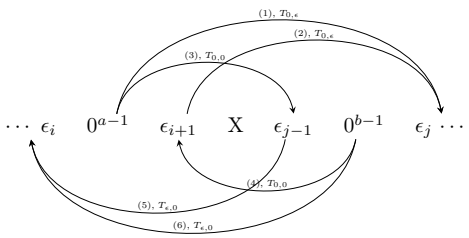
$$\zeta^{\sharp,\mathfrak{m}}\left(\{\overline{\boldsymbol{even}}, \boldsymbol{odd}\}^{\times}\right), \text{i.e. negative even and positive odd integers}$$

which, in terms of iterated integrals corresponds to, with $\epsilon \in \{\pm\sharp\}$:

$$I^{\mathfrak{m}}\left(0; \left\{\begin{array}{c} 1, \mathbf{0}^{odd}, -\sharp \\ 1, \mathbf{0}^{even}, \sharp \end{array}\right\}, \cdots, \quad \left\{\begin{array}{c} \epsilon, \mathbf{0}^{odd}, -\epsilon \\ \epsilon, \mathbf{0}^{even}, \epsilon \end{array}\right\} \quad, \cdots; 1\right). \tag{A.9}$$

Lemma A.1.3. *The family $\zeta^{\sharp\mathfrak{m}}\left(\{\overline{even}, odd\}^{\times}\right)$ is stable under the coaction.*

Proof. Looking at the possible kinds of cuts, and gathering them according the right side:



with $\left\{\begin{array}{l} X = 0^{a_1-1}\epsilon_2 0^{a_2-1}\cdots\epsilon_n 0^{a_n-1}, \\ \epsilon_{i+1}X \leftrightarrow (y_1, y_2, \ldots, y_n), \\ \text{ where } y_i = \pm a_i \text{ according to the sign of } \epsilon \end{array}\right.$.

These cuts have the same form for the right side in the coaction:

$$I^{\mathfrak{m}}(0; \cdots, \boldsymbol{\epsilon_i}\mathbf{0}^{\boldsymbol{\alpha}}\boldsymbol{\epsilon_j}, \cdots; 1).$$

Notice there would be no term $T_{\epsilon,-\epsilon}$ in a cut from ϵ to $-\epsilon$ because of Sign $(A.6)$ identity, therefore you have there all the possible cuts pictured.
A priori, cuts can create in the right side a sequence $\{\epsilon, \mathbf{0}^{even}, -\epsilon\}$ or $\left\{\epsilon, \mathbf{0}^{odd}, \epsilon\right\}$ inside the iterated integral; these cuts are the *unstable* ones, since they are out of the considered family. However, by coupling these cuts two by two, and using the rules listed at the beginning of the Annexe, the unstable cuts would all get simplified.
Indeed, let examine each of the terms $(1-6)$[I]:

[I] There is no remaining cuts between ϵ and ϵ. Notice also that the left sides of the remaining terms have an even depth.

Term	Left side	Unstable if	Simplified with
(1)	$\zeta^{\sharp\sharp,\mathfrak{l}}_{a-1-\alpha}(a_1,\dots,a_n,b)$, with $\alpha < a$.	n even	the previous cut: either (6) by Minus or (5) by Cut or (3) by Cut [I]
(2)	$\zeta^{\sharp\sharp,\mathfrak{l}}_{b-1}(a_n,\dots,a_1)$ with $\alpha = a$.	$\epsilon_{i+1} = \epsilon_j$	the previous cut: either with (5) by Shift or with (6) by Cut Shifted or with (3) by Shift.
(3)	$-\zeta^{\sharp\sharp,\mathfrak{l}}_{a+b-\alpha-1}(a_1,\dots,a_n)$ with $\alpha > b$.	n odd	the following cut: either (1) by Cut or (2) by Shift or (4) by Shift.
(4)	$\zeta^{\sharp\sharp,\mathfrak{l}}_{a+b-\alpha-1}(a_n,\dots,a_1)$ with $\alpha > a$.	n odd	the previous cut: either (6) by Cut Shifted or with (5) by Shift or with (3) by Shift.
(5)	$-\zeta^{\sharp\sharp,\mathfrak{l}}_{a-1}(a_1,\dots,a_n)$ with $\alpha = b$.	$\epsilon_i = \epsilon_{j-1}$	the following cut: either with (1) by Cut, or with (2) by Shift, or with (4) by Shift.
(6)	$-\zeta^{\sharp\sharp,\mathfrak{l}}_{b-1-\alpha}(a_n,\cdots a_1,a)$	n even	the following cut: either (1) by Minus or with (2) by Cut Shifted or with (4) by Cut Shifted

□

Derivations. Let use the writing of the Conjecture 4.4.4:

$$\zeta^{\sharp,\mathfrak{m}}(B_0,1^{\gamma_1},\dots,1^{\gamma_p},B_p) \text{ with } B_i < 0 \text{ if and only if } B_i \text{ even .} \tag{A.10}$$

`Nota Bene:` Beware, for instance B_i may be equal to 1, which implies that $\gamma_i = \gamma_{i+1} = 0$. Indeed, we look at the indices corresponding to a sequence $(2^{a_0}, c_1, \dots, c_p, 2^{a_p})$ as in the Conjecture 4.4.4:

$$\begin{array}{l} B_i = 2a_i + 3 - \delta_{c_i} - \delta_{c_{i+1}} \\ B_0 = 2a_0 + 1 - \delta_{c_1} \\ B_p = 2a_p + 2 - \delta_{c_p} \end{array}, \gamma_i := c_i - 3 + 2\delta_{c_i}, \quad \text{where} \begin{cases} a_i \geq 0 \\ c_i > 0, c_i \neq 2 \\ \delta_c := \begin{cases} 1 & \text{if } c = 1 \\ 0 & \text{else .} \end{cases} \end{cases} .$$

Lemma A.1.4.

$$D_{2r+1}\left(\zeta^{\sharp,\mathfrak{m}}(B_0,1^{\gamma_1},\dots,1^{\gamma_p},B_p)\right) = {}^{\text{II}} \tag{A.11}$$

$$\delta_r \left[-\delta_{\substack{2\leq B\leq B_j+1 \\ 0\leq\gamma\leq\gamma_{i+1}-1}} \zeta^{\sharp,\mathfrak{l}}(B_j - B + 1, 1^{\gamma_j},\dots,1^{\gamma_{i+1}-\gamma-1}) \otimes \zeta^{\sharp,\mathfrak{m}}(B_0\cdots,B_i,1^{\gamma},B,1^{\gamma_{j+1}},\dots,B_p) \right.$$

$$\left[\begin{array}{l} +\delta_{B_{i+1}<B}\zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}+B_j-B}(1^{\gamma_j},\dots,1^{\gamma_{i+2}}) \\ -\delta_{B_j<B}\zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}+B_j-B}(1^{\gamma_{i+2}},\dots,1^{\gamma_j}) \\ +\zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}-B}(1^{\gamma_{i+2}},\dots,B_j) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\dots,B_{i+1}) \end{array} \right] \otimes \zeta^{\sharp,\mathfrak{m}}(B_0\cdots,B_i,1^{\gamma_{i+1}},B,1^{\gamma_{j+1}},\dots,B_p)$$

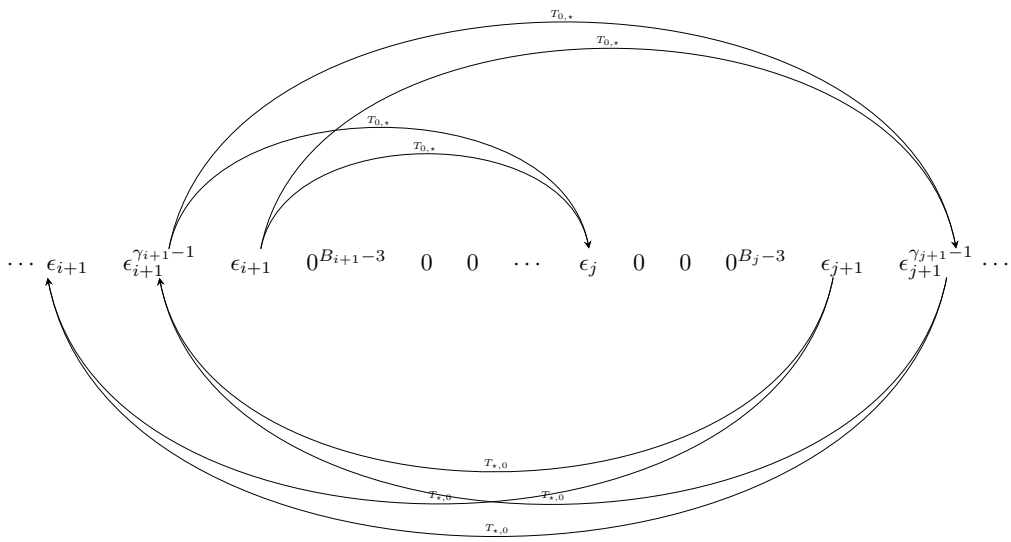
[I] It depends on the sign of $b + 1 - \alpha$ here for instance.

$$\delta_{\substack{1\leq B\leq B_{i+1}+1\\ 0\leq\gamma\leq\gamma_{j+1}-1}}\zeta^{\sharp,\mathfrak{l}}(B_{i+1}-B+1,1^{\gamma_{i+2}},\dots,1^{\gamma_{j+1}-\gamma-1})\otimes\zeta^{\sharp,\mathfrak{m}}(B_0\cdots,1^{\gamma_{i+1}},B,1^{\gamma},B_{j+1},\dots,B_p)\Big],$$

where B is positive if odd, negative if even.

Proof. `Nota Bene`: For the left side, we only look at odd weight w, and the parity of the depth d is fundamental since the relations stated above depend on the parity of $w-d$. For instance, for such a sequence $(\mathbf{1}^{\gamma_i},B_i,\dots,B_{j-1},\mathbf{1}^{\gamma_j})$ (with the previous notations), $weight-depth$ has the same parity than $\delta_{c_i}+\delta_{c_j}$.

The following cuts get simplified, with Shift, since depth is odd (B_i odd if $c_i,c_{i+1}\neq 1$):



It remains, where all the unstable cuts are simplified by the Lemma $A.1.4$, cuts that we can gather into four groups, according to the right side of the coaction:

(i) $\zeta^{\sharp,\mathfrak{m}}(B_0\cdots,B_i,1^{\gamma},B,1^{\gamma_{j+1}},\dots,B_p)$.

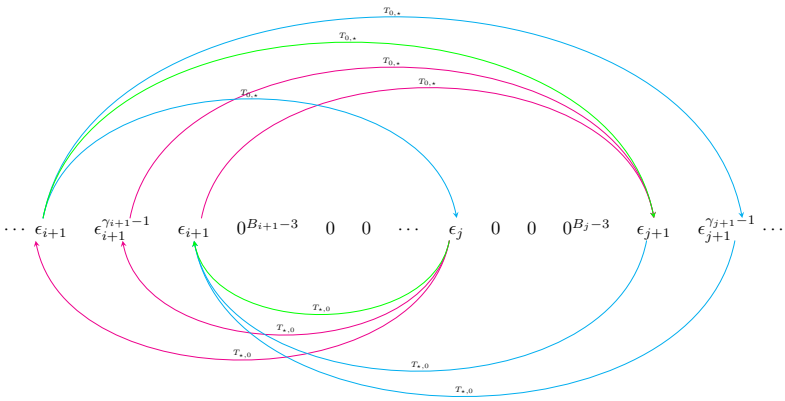
(ii) $\zeta^{\sharp,\mathfrak{m}}(B_0\cdots,B_i,1^{\gamma},B_j,\dots,B_p)$.

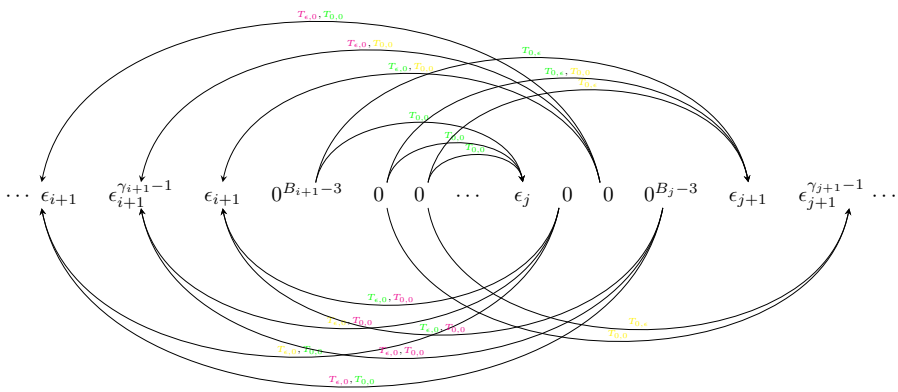
(iii) $\zeta^{\sharp,\mathfrak{m}}(B_0\cdots,B_i,1^{\gamma_{i+1}},B,1^{\gamma_{j+1}},\dots,B_p)$.

(iv) $\zeta^{\sharp,\mathfrak{m}}(B_0\cdots,1^{\gamma_{i+1}},B,1^{\gamma},B_{j+1},\dots,B_p)$.

It remains, where (iv) terms, antisymmetric of (i) ones, are omitted to lighten the diagrams:

[11] Here δ_r indicates that left side has to be of weight $2r+1$.





We just have to list the terms remaining in these two diagrams. Gathering all the terms of the coaction according to the right side as above gives:

(i) Looking at the magenta terms, with $2 \leq B \leq B_j - 1$ or $B = B_j + 1$ and $0 \leq \gamma \leq \gamma_{i+1} - 1$:

$$\zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B+1}(1^{\gamma_j},\ldots,1^{\gamma_{i+1}-\gamma-1}) - \zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\ldots,1^{\gamma_{i+1}-\gamma}) = -\zeta^{\sharp,\mathfrak{l}}(B_j-B+1,1^{\gamma_j},\ldots,1^{\gamma_{i+1}-\gamma-1})$$

With *even* depth for the first term and *odd* for the second since otherwise the cuts would be unstable and simplified by Cut; here also $c_{i+1} \neq 1$.

(ii) These match exactly with the left side of (i) for $B = B_j$ and (iv) terms for $B = B_i$.

(iii) The following cuts:

$$\delta_{B_{i+1} \geq B} \zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}-B}(1^{\gamma_{i+2}},\ldots,B_j) - \delta_{B_j \geq B} \zeta^{\sharp\sharp,\mathfrak{l}}_{B_j-B}(1^{\gamma_j},\ldots,B_{i+1}) +$$

$$\delta_{B_{i+1} < B} \zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}+B_j-B}(1^{\gamma_j},\ldots,1^{\gamma_{i+2}}) - \delta_{B_j < B} \zeta^{\sharp\sharp,\mathfrak{l}}_{B_{i+1}+B_j-B}(1^{\gamma_{i+2}},\ldots,1^{\gamma_j}).$$

The parity of $weight - depth$ for the first line is equal to the parity of $\delta_{c_{i+1}} + \delta_{c_{j+1}} + B$. Notice that if this is even, the first line has odd depth whereas the second line has even depth, and by Cut and Antipode$*$, all terms got simplified. Hence, we can restrict to B written as $2\beta + 3 - \delta_{c_{i+1}} - \delta_{c_{j+1}}$, the first line being of even depth, the second line of odd depth.

(iv) Antisymmetric of (i).

□

A.2 Galois descent in small depths, $N = 2, 3, 4, \text{‘6’}, 8$

A.2.1 $N = 2$: Depth $2, 3$

Here we have to consider only one Galois descent, from $\mathcal{H}^2$ to $\mathcal{H}^1$.
In depth 1 all the $\zeta^{\mathfrak{m}}(\overline{s})$, $s > 1$ are MMZV. Let us detail the case of depth 2 and 3 as an application of the results of Chapter 5. In depth 2, coefficients are explicit:

Lemma A.2.1. *The depth 2 part of the basis of the motivic multiple zeta values is:*

$$\left\{\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1}) - \binom{2(a+b)}{2b}\zeta^{\mathfrak{m}}(1, \overline{2(a+b)+1}), a, b > 0\right\}.$$

Proof. Indeed, we have if $a, b > 0$, $D_1(\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1})) = 0$ and for $r > 0$:

$$D_{2r+1,2}(\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1})) = \zeta^{\mathfrak{l}}(2r+1) \otimes \zeta^{\mathfrak{m}}(\overline{2(a+b-r)+1})$$
$$\left(-\delta_{a \leq r < a+b}\binom{2r}{2a} + \delta_{r=a} + \delta_{b \leq r < a+b}\binom{2r}{2b}(2^{-2r}-1) + \delta_{r=a+b}(2^{-2r}-2)\binom{2(a+b)}{2b}\right).$$

There is only the case $r = a+b$ where a term $(\zeta^{\mathfrak{m}}(\overline{1}))$ which does not belong to $\mathcal{F}_0\mathcal{H}$ appears:

$$D_{2r+1,2}(\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1})) \equiv \delta_{r=a+b}(2^{-2r}-2)\binom{2(a+b)}{2b}\zeta^{\mathfrak{l}}(2r+1) \otimes \zeta^{\mathfrak{m}}(\overline{1}) \text{ in the quotient } \mathcal{H}^{\geq 1}.$$

Referring to the previous results, we can correct $\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1})$ with terms of the same weight, same depth, and with at least one 1 (not at the end), which here corresponds only to $\zeta^{\mathfrak{m}}(1, \overline{2(a+b)+1})$.
Furthermore, the last equality being true in the quotient $\mathcal{H}^{\geq 1}$:

$$\begin{aligned} D_{2r+1,2}(\zeta^{\mathfrak{m}}(1, \overline{2(a+b)+1})) &= \zeta^{\mathfrak{l}}(2r+1) \otimes (-\delta_{r<a+b} + \delta_{r=a+b}(2^{-2r}-2))\zeta^{\mathfrak{m}}(\overline{2(a+b-r)+1}) \\ &\equiv \delta_{r=a+b}(2^{-2r}-2)\zeta^{\mathfrak{l}}(2r+1) \otimes \zeta^{\mathfrak{m}}(\overline{1}). \end{aligned}$$

According to these calculations of infinitesimal coactions:

$$\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1}) - \binom{2(a+b)}{2b}\zeta^{\mathfrak{m}}(1, \overline{2(a+b)+1}) \text{ belongs to } \mathcal{F}_0\mathcal{H} \text{ , i.e. is a MMZV.}$$

□

`Examples:` Here are some motivic multiple zeta values:

$$\zeta^{\mathfrak{m}}(3, \overline{3}) - 6\zeta^{\mathfrak{m}}(1, \overline{5}) \text{ , } \zeta^{\mathfrak{m}}(3, \overline{5}) - 15\zeta^{\mathfrak{m}}(1, \overline{7}) \text{ , } \zeta^{\mathfrak{m}}(5, \overline{3}) - 15\zeta^{\mathfrak{m}}(1, \overline{7}) \text{ , } \zeta^{\mathfrak{m}}(5, \overline{7}) - 210\zeta^{\mathfrak{m}}(1, \overline{11}).$$

Remarks:

- The corresponding Euler sums $\left\{\zeta(2a+1, \overline{2b+1}) - \binom{2(a+b)}{2b}\zeta(1, \overline{2(a+b)+1}), a, b > 0\right\}$ are a generating family of MZV in depth 2.
- Similarly, we can prove that the following elements are (resp. motivic) MZV, if no $\overline{1}$:

$\zeta(\overline{A}, \overline{B})$	$\zeta(A, \overline{B}) + \zeta(\overline{A}, B)$	if A, B odd
$\left.\begin{matrix}\zeta(A, \overline{B}) \\ \zeta(\overline{A}, B)\end{matrix}\right\}$ if $A+B$ odd	$\zeta(A, \overline{B}) + (-1)^A\binom{A+B-2}{A-1}\zeta(1, \overline{A+B-1})$	if $A+B$ even
$\zeta(\overline{1}, \overline{1}) - \frac{1}{2}\zeta(\overline{1})^2$ $\zeta(1, \overline{1}) - \frac{1}{2}\zeta(\overline{1})^2$	$\zeta(\overline{A}, B) - (-1)^A\binom{A+B-2}{A-1}\zeta(1, \overline{A+B-1})$	if $\left\{\begin{matrix} A+B \text{ even} \\ A, B \neq 1\end{matrix}\right.$

Lemma A.2.2. *The depth 2 part of the basis of $\mathcal{F}_1\mathcal{H}$ is:*

$$\left\{\zeta^{\mathfrak{m}}(2a+1, \overline{2b+1}), (a, b) \neq (0, 0)\right\}.$$

Proof. No need of correction ($\mathcal{B}_{n,2,\geq 2}$ is empty for $n \neq 2$), these elements belong to $\mathcal{F}_1\mathcal{H}$. $\square$

Lemma A.2.3. *The depth 3 part of the basis of motivic multiple zeta values is:*

$$\Big\{\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}) - \sum_{k=1}^{a+b+c} \alpha_k^{a,b,c}\zeta^{\mathfrak{m}}(1,2(a+b+c-k)+1,\overline{2k+1}) \\ -\binom{2(b+c)}{2c}\zeta^{\mathfrak{m}}(2a+1,1,\overline{2(b+c)+1}), a,b,c>0\Big\}. \quad \text{(A.12)}$$

where $\alpha_k^{a,b,c} \in \mathbb{Z}_{odd}$ are solutions of $M_3X = A^{a,b,c}$. With $A^{a,b,c}$ such that $r^{th}-$coefficient is:

$$\delta_{b\leq r<a+b}\binom{2(n-r)}{2c}\binom{2r}{2b} - \delta_{a<r<a+b}\binom{2(n-r)}{2c}\binom{2r}{2a} - \delta_{b\leq r<b+c}\binom{2(n-r)}{2a}\binom{2r}{2b}$$
$$-\delta_{r\leq a}\binom{2(n-r)}{2(b+c)}\binom{2(b+c)}{2c} + \delta_{r<b+c}\binom{2(n-r)}{2a}\binom{2(b+c)}{2c} + \delta_{c\leq r<b+c}\binom{2r}{2c}\binom{2(n-r)}{2a}(2^{-2r}-1).$$

M_3 the matrix whose $(r,k)^{th}$ coefficient is:

$$\delta_{r=a+b+c}(2^{-2r}-2)\binom{2n}{2k} + \delta_{k\leq r<n}\binom{2r}{2k}(2^{-2r}-1) - \delta_{r<n-k}\binom{2(n-r)}{2k} - \delta_{n-k\leq r<n}\binom{2r}{2(n-k)}.$$

Proof. Let $\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})$, $a,b,c>0$ fixed, and substract elements of the same weight, of depth 3 until it belongs to $gr_3\mathcal{F}_0\mathcal{H}$.
Let calculate infinitesimal coproducts referring to the formula (5.8) in the quotient $\mathcal{H}^{\geq 1}$ and use previous results for depth 2, with $n = a+b+c$:

$$D_{2r+1,3}(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})) \equiv \zeta^{\mathfrak{l}}(2r+1)\otimes\Big[\delta_{r=b+c}\binom{2(b+c)}{2c}(2^{-2r}-2)\zeta^{\mathfrak{m}}(2a+1,\overline{1})$$
$$+\zeta^{\mathfrak{m}}(1,\overline{2(n-r)+1})\Big(\delta_{a=r}\binom{2(n-r)}{2c} + \delta_{b\leq r<a+b}\binom{2r}{2b}\binom{2(n-r)}{2c} - \delta_{a\leq r<a+b}\binom{2r}{2a}\binom{2(n-r)}{2c}$$
$$-\delta_{b\leq r<b+c}\binom{2r}{2b}\binom{2(n-r)}{2a} + \delta_{c\leq r<b+c}\binom{2r}{2c}\binom{2(n-r)}{2a}(2^{-2r}-1)\Big)\Big].$$

At first, let substract $\binom{2(b+c)}{2c}\zeta(2a+1,1,\overline{2(b+c)+1})$ such that the $D_{1,2}^{-1}D_{2r+1,3}^{1}$ are equal to zero, which comes to eliminate the term $\zeta^{\mathfrak{m}}(2a+1,\overline{1})$ appearing (case $r=b+c$).
So, we are left to substract a linear combination

$$\sum_{k=1}^{a+b+c} \alpha_k^{a,b,c}\zeta^{\mathfrak{m}}(1,2(a+b+c-k)+1,\overline{2k+1})$$

such that the coefficients $\alpha_k^{a,b,c}$ are solutions of the system $M_3X = A^{a,b,c}$ where $A^{a,b,c} = (A_r^{a,b,c})_r$ satisfying in $\mathcal{H}^{\geq 1}$:

$$D_{2r+1,3}\Big(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}) - \binom{2(b+c)}{2c}\zeta(2a+1,1,\overline{2(b+c)+1})\Big) \equiv \\ A_r^{a,b,c}\zeta^{\mathfrak{l}}(2r+1)\otimes\zeta^{\mathfrak{m}}(1,\overline{2(n-r)+1}),$$

and $M_3 = (m_{r,k})_{r,k}$ matrix such that:

$$D_{2r+1,3}(\zeta^{\mathfrak{m}}(1,2(a+b+c-k)+1,\overline{2k+1})) = m_{r,k}\zeta^{\mathfrak{l}}(2r+1)\otimes\zeta^{\mathfrak{m}}(1,\overline{2(n-r)+1}).$$

This system has solutions since, according to Chapter 5 results, the matrix M_3 is invertible.[I]
Then, the following linear combination will be in $\mathcal{F}_0\mathcal{H}$:

$$\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}) - \sum_{k=1}^{a+b+c}\alpha_k^{a,b,c}\zeta^{\mathfrak{m}}(1,2(a+b+c-k)+1,\overline{2k+1}) - \binom{2(b+c)}{2c}\zeta(2a+1,1,\overline{2(b+c)+1}).$$

[I] Indeed, modulo 2, M_3 is an upper triangular matrix with 1 on diagonal.

The coefficients $\alpha_k^{a,b,c}$ belong to $\mathbb{Z}_{\text{odd}}$ since coefficients are integers, and $\det(M_3)$ is odd. Referring to the calculus of infinitesimal coactions, $A^{a,b,c}$ and M_3 are as claimed in lemma. $\square$

`Examples:`

· By applying this lemma, with $a = b = c = 1$ we obtain the following MMZV:

$$\zeta^{\mathfrak{m}}(3,3,\overline{3}) + \frac{774}{191}\zeta^{\mathfrak{m}}(1,5,\overline{3}) - \frac{804}{191}\zeta^{\mathfrak{m}}(1,3,\overline{5}) + \frac{450}{191}\zeta^{\mathfrak{m}}(1,1,\overline{7}) - 6\zeta^{\mathfrak{m}}(3,1,\overline{5}).$$

Indeed, in this case, with the previous notations:

$$M_3 = \begin{pmatrix} \frac{27}{4} & -1 & -1 \\ -\frac{53}{8} & -\frac{111}{16} & -1 \\ -\frac{1905}{64} & -\frac{1905}{64} & -\frac{127}{64} \end{pmatrix}, \quad A^{1,1,1} = \begin{pmatrix} \frac{51}{2} \\ 0 \\ 0 \end{pmatrix}.$$

· Similarly, we obtain the following motivic multiple zeta value:

$$\zeta^{\mathfrak{m}}(3,3,\overline{5}) + \frac{850920}{203117}\zeta^{\mathfrak{m}}(1,7,\overline{3}) + \frac{838338}{203117}\zeta^{\mathfrak{m}}(1,5,\overline{5}) - \frac{3673590}{203117}\zeta^{\mathfrak{m}}(1,3,\overline{7}) + \frac{20351100}{203117}\zeta^{\mathfrak{m}}(1,1,\overline{9}) - 15\zeta^{\mathfrak{m}}(3,1,\overline{7}).$$

There: $$M_3 = \begin{pmatrix} -\frac{63}{4} & 15 & -1 & -1 \\ -\frac{93}{8} & -\frac{31}{16} & -6 & -1 \\ -\frac{1009}{64} & -\frac{1905}{64} & -\frac{1023}{64} & -1 \\ -\frac{3577}{64} & -\frac{17885}{128} & -\frac{3577}{64} & -\frac{511}{256} \end{pmatrix}, \quad A^{1,1,2} = \begin{pmatrix} 210 \\ \frac{387}{8} \\ 0 \\ 0 \end{pmatrix}.$$

Lemma A.2.4. *The depth* 3 *part of the basis of* $\mathcal{F}_1\mathcal{H}$ *is:*

$$\Big\{\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1}) - \delta_{\substack{a=0 \\ or\ c=0}}(-1)^{\delta_{c=0}}\binom{2(a+b+c)}{2b}\zeta^{\mathfrak{m}}(1,1,\overline{2(a+b+c)+1})$$
$$-\delta_{c=0}\binom{2(a+b)}{2b}\zeta(1,2(a+b)+1,\overline{1}),\ \textit{at most one of } a,b,c \textit{ equals zero}\Big\}.$$

Proof. Let $\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})$ with at most one 1.

Our goal is to annihilate $D_{1,3}^{-1}$ *and* $\{D_{1,3}^{-1}\circ D_{2r+1}^{1}\}_{r>0}$*, in the quotient* $\mathcal{H}^{\geq 1}$.

Let first cancel $D_{1,3}^{-1}$: if $c \neq 0$, it is already zero; otherwise, for $c = 0$, in $\mathcal{H}^{\geq 1}$, according to the results in depth 2 for $\mathcal{F}_0$, we can substract $\binom{2(a+b)}{2a}\zeta(1,2(a+b)+1,\overline{1})$ since:

$$D_{1,3}(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{1})) \equiv \binom{2(a+b)}{2a}\zeta^{\mathfrak{m}}(1,\overline{2(a+b)+1}) \equiv \binom{2(a+b)}{2a}D_{1,3}(\zeta^{\mathfrak{m}}(1,2(a+b)+1,\overline{1})).$$

Furthermore, with $\equiv$ standing for an equality in $\mathcal{H}^{\geq 1}$:

$$\begin{aligned} D_{1,2}^{-1}D_{2r+1,3}^{1}(\zeta^{\mathfrak{m}}(2a+1,2b+1,\overline{2c+1})) &= \delta_{r=b+c}\binom{2r}{2c}(2^{-2r}-2)\zeta^{\mathfrak{m}}(\overline{2a+1}) \\ &\equiv \delta_{\substack{r=b+c \\ a=0}}\binom{2(b+c)}{2c}(2^{-2(b+c)}-2)\zeta^{\mathfrak{m}}(\overline{1}). \\ D_{1,2}^{-1}D_{2r+1,3}^{1}(\zeta^{\mathfrak{m}}(1,1,\overline{2(a+b+c)+1})) &= \delta_{r=a+b+c}(2^{-2(a+b+c)}-2)\zeta^{\mathfrak{m}}(\overline{1}). \\ D_{1,2}^{-1}D_{2r+1,3}^{1}(\zeta^{\mathfrak{m}}(1,2(a+b+c)+1,\overline{1})) &= \delta_{r=a+b+c}(2^{-2(a+b+c)}-2)\zeta^{\mathfrak{m}}(\overline{1}). \end{aligned}$$

Therefore, to cancel $D_{1,2}^{-1}\circ D_{2r+1,3}^{1}$:

· If $a = 0$ we substract $\binom{2(b+c)}{2c}\zeta^{\mathfrak{m}}(1,1,\overline{2(b+c)+1})$.

· If $c = 0$, we add $\binom{2(b+c)}{2c}\zeta^{\mathfrak{m}}(1,1,\overline{2(a+b)+1})$.

$\square$

Depth 4. The simplest example in depth 4 of MMZV obtained by this way, with $\alpha_i \in \mathbb{Q}$:

$$-\zeta^{\mathfrak{m}}(3,3,3,\overline{3})-\frac{3678667587000}{4605143289541}\zeta^{\mathfrak{m}}(1,1,1,\overline{9})+\frac{9187768536750}{4605143289541}\zeta^{\mathfrak{m}}(1,1,3,\overline{7})+\frac{41712466500}{4605143289541}\zeta^{\mathfrak{m}}(1,1,5,\overline{5})$$
$$-\frac{9160668717750}{4605143289541}\zeta^{\mathfrak{m}}(1,1,7,\overline{3})+\frac{11861255103300}{4605143289541}\zeta^{\mathfrak{m}}(1,3,1,\overline{7})+\frac{202283196216}{4605143289541}\zeta^{\mathfrak{m}}(1,3,3,\overline{5})$$
$$-\frac{993033536436}{4605143289541}\zeta^{\mathfrak{m}}(1,3,5,\overline{3})+\frac{8928106562124}{4605143289541}\zeta^{\mathfrak{m}}(1,5,1,\overline{5})-\frac{1488017760354}{4605143289541}\zeta^{\mathfrak{m}}(1,5,3,\overline{3})$$
$$-\frac{450}{191}\zeta^{\mathfrak{m}}(3,1,1,\overline{7})+\frac{804}{191}\zeta^{\mathfrak{m}}(3,1,3,\overline{5})-\frac{774}{191}\zeta^{\mathfrak{m}}(3,1,5,\overline{3})+6\zeta^{\mathfrak{m}}(3,3,1,\overline{5})$$
$$+\alpha_1\zeta^{\mathfrak{m}}(1,-11)+\alpha_2\zeta^{\mathfrak{m}}(1,-9)\zeta^{\mathfrak{m}}(2)+\alpha_3\zeta^{\mathfrak{m}}(1,-7)\zeta^{\mathfrak{m}}(2)^2+\alpha_4\zeta^{\mathfrak{m}}(1,-5)\zeta^{\mathfrak{m}}(2)^3+\alpha_5\zeta^{\mathfrak{m}}(1,-3)\zeta^{\mathfrak{m}}(2)^4.$$

A.2.2 $N = 3, 4$: Depth 2

Let us detail the case of depth 2 as an application of the results in Chapter 5 and start by defining some coefficients appearing in the next examples:

Definition A.2.5. *Set $\alpha_k^{a,b} \in \mathbb{Z}$ such that $M(\alpha_k^{a,b})_{b+1\le k\le \frac{n}{2}-1} = A^{a,b}$ with $n = 2(a+b+1)$:*

$$M := \left(\binom{2r-1}{2k-1}\right)_{b+1\le r,k\le\frac{n}{2}-1};\ A^{a,b} := \left(-\binom{2r-1}{2b}\right)_{b+1\le r\le\frac{n}{2}-1};\ \beta^{a,b} := \binom{n-2}{2b}+\sum_{k=b+1}^{a+b}\alpha_k\binom{n-2}{2k-1}.$$

`Nota Bene`: The matrix M having integers as entries and determinant equal to 1, and A having integer components, the coefficients $\alpha_k^{a,b}$ are obviously integers; the matrix M and its inverse are lower triangular with 1 on the diagonal. Furthermore:[I]:

$$\alpha_{b+i}^{a,b} = (-1)^i\binom{2b+2i-1}{2i-1}c_i,$$
$$\alpha_{b+1}^{a,b} = -(2b+1),\quad \alpha_{b+2}^{a,b} = 2\binom{2b+3}{3},\quad \alpha_{b+3}^{a,b} = -16\binom{2b+5}{5},\quad \alpha_{b+4}^{a,b} = 272\binom{2b+7}{7}.$$

Lemma A.2.6. *The depth 2 part of the basis of MMZV, for even weight $n = 2(a+b+1)$, is:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,\xi\end{pmatrix}-\beta^{a,b}\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,\xi\end{pmatrix}-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,\xi\end{pmatrix},a,b>0\right\}.$$

Proof. [II] Let $Z = \zeta^{\mathfrak{m}}(2a+1,\overline{2b+1})$ fixed, with $a,b>0$.
First we substract a linear combination of $\zeta^{\mathfrak{m}}\left(\begin{smallmatrix}n-2k,2k\\1,\xi\end{smallmatrix}\right)$ in order to cancel $\{D_{2r}\}$. It is possible since in depth 2, because $\zeta^{\mathfrak{l}}\left(\begin{smallmatrix}2r\\1\end{smallmatrix}\right)=0$:

$$D_{2r}(\zeta^{\mathfrak{m}}(x_1,\overline{x_2})) = \delta_{x_2\le 2r\le x_1+x_2-1}(-1)^{x_2}\binom{2r-1}{x_2-1}\zeta^{\mathfrak{l}}\begin{pmatrix}2r\\\xi\end{pmatrix}\otimes\zeta^{\mathfrak{m}}\begin{pmatrix}x_1+x_2-r\\\xi\end{pmatrix}.$$

Hence it is sufficient to choose α_k such that $M\alpha^{a,b} = A^{a,b}$ as in Definition A.2.5.
Now, it remains to satisfy $D_1\circ D_{2r+1}(\cdot) = 0$ (for $r = n-1$ only) in order to have an element of $\mathcal{F}_0^{k_N/\mathbb{Q},P/1}\mathcal{H}_n$. In that purpose, let substract $\beta^{a,b}\zeta^{\mathfrak{m}}(1,n-1;1,\xi)$ with $\beta^{a,b}$ as in Definition A.2.5) according to the calculation of $D_1\circ D_{2r+1}(\cdot)$, left to the reader.

□

[I] There $c_i\in\mathbb{N}$ does not depend neither on b nor on a.
[II] We omit the exponent ξ indicating the projection on the second factor of the derivations D_r, to lighten the notations.

`Examples`: The following are motivic multiple zeta values:

- $\zeta^{\mathfrak{m}}\binom{5,3}{1,\xi} - 75\zeta^{\mathfrak{m}}\binom{1,7}{1,\xi} + 3\zeta^{\mathfrak{m}}\binom{4,4}{1,\xi} - 20\zeta^{\mathfrak{m}}\binom{2,6}{1,\xi}$.
- $\zeta^{\mathfrak{m}}\binom{3,5}{1,\xi} + 15\zeta^{\mathfrak{m}}\binom{1,7}{1,\xi} + 5\zeta^{\mathfrak{m}}\binom{6,2}{1,\xi}$.
- $\zeta^{\mathfrak{m}}\binom{5,5}{1,\xi} - 350\zeta^{\mathfrak{m}}\binom{1,9}{1,\xi} + 5\zeta^{\mathfrak{m}}\binom{4,6}{1,\xi} - 70\zeta^{\mathfrak{m}}\binom{2,8}{1,\xi}$.
- $\zeta^{\mathfrak{m}}\binom{7,5}{1,\xi} + 12810\zeta^{\mathfrak{m}}\binom{1,11}{1,\xi} + 5\zeta^{\mathfrak{m}}\binom{6,6}{1,\xi} - 70\zeta^{\mathfrak{m}}\binom{4,8}{1,\xi} + 2016\zeta^{\mathfrak{m}}\binom{2,10}{1,\xi}$.
- $\zeta^{\mathfrak{m}}\binom{9,5}{1,\xi} - 685575\zeta^{\mathfrak{m}}\binom{1,13}{1,\xi} + 5\zeta^{\mathfrak{m}}\binom{8,6}{1,\xi} - 70\zeta^{\mathfrak{m}}\binom{6,8}{1,\xi} + 2016\zeta^{\mathfrak{m}}\binom{4,10}{1,\xi} - 89760\zeta^{\mathfrak{m}}\binom{2,12}{1,\xi}$.

Lemma A.2.7. *The depth 2 part of the basis of $\mathcal{F}_1^{k_N/\mathbb{Q},P/1}\mathcal{H}_n$ is for even n:*

$$\left\{\zeta^{\mathfrak{m}}\binom{2a+1,2b+1}{1,\xi} - \sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\zeta^{\mathfrak{m}}\binom{n-2k,2k}{1,\xi}, a,b\geq 0, (a,b)\neq(0,0)\right\},$$

For odd n, the part in depth 2 of the basis of $\mathcal{F}_1^{k_N/\mathbb{Q},P/1}\mathcal{H}_n$ is:

$$\left\{\zeta^{\mathfrak{m}}\binom{x_1,x_2}{1,\xi} + (-1)^{x_2+1}\binom{n-2}{x_2-1}\zeta^{\mathfrak{m}}\binom{1,n-1}{1,\xi}, x_1,x_2>1, \text{ one even, the other odd}\right\}.$$

Proof.
- For even n, we need to cancel D_{2r} (else $D_{2s}\circ D_{2r}(\cdot)\neq 0$), so we substract the same linear combination than in the previous lemma.
- For odd n, we need to cancel $D_1\circ D_{2r}$. Since $D_1\circ D_{2r}(Z) = (-1)^{x_2}\binom{n-2}{x_2-1}$, we substract $(-1)^{x_2+1}\binom{n-2}{x_2-1}\zeta^{\mathfrak{m}}(1,\overline{n-1})$.

□

Lemma A.2.8. *The depth 2 part of the basis of $\mathcal{F}_0^{k_N/\mathbb{Q},P/P}\mathcal{H}_n$ ($=\mathcal{H}_n^{\mathcal{MT}_2}$ if $N=4$) is:*

$$\left\{\zeta^{\mathfrak{m}}\binom{2a+1,2b+1}{1,\xi} - \sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\zeta^{\mathfrak{m}}\binom{n-2k,2k}{1,\xi}, a,b\geq 0\right\}.$$

Proof. To cancel D_{2r}, we substract the same linear combination than above. □

Lemma A.2.9. *The depth 2 part of the basis of $\mathcal{F}_1^{k_N/\mathbb{Q},P/P}\mathcal{H}_n$ is for even n:*

$$\left\{\zeta^{\mathfrak{m}}\binom{2a+1,2b+1}{1,\xi} - \sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\zeta^{\mathfrak{m}}\binom{n-2k,2k}{1,\xi}, a,b\geq 0,\right\},$$

And for odd n, the part in depth 2 of the basis of $\mathcal{F}_1^{k_N/\mathbb{Q},P/P}\mathcal{H}_n$ is:

$$\left\{\zeta^{\mathfrak{m}}\binom{x_1,x_2}{1,\xi}, x_1,x_2\geq 1, \text{ one even, the other odd}\right\}.$$

Proof. If n is even, to cancel $\{D_{2r}\}$, we use the same linear combination than above.
If n is odd, we already have $\zeta^{\mathfrak{m}}(x_1,x_2;1,\xi)\in\mathcal{F}_1^{k_N/\mathbb{Q},P/P}\mathcal{H}_n$.

□

A.2.3 $N = 8$: Depth 2

Let us explicit the results for the depth 2; proofs being similar (albeit longer) as in the previous sections are left to the reader; same notations than the previous case.

Lemma A.2.10. · *The depth 2 part of the basis of $MMZV_{\mu_4}$ is:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,\xi\end{pmatrix},x_i\geq 1\right\}.$$

· *The depth 2 part of the basis of motivic Euler sums is:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,\xi\end{pmatrix}\right.$$
$$\left.-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,\xi\end{pmatrix}\right)\right\}_{a,b\geq 0}$$

· *The depth 2 part of the basis of MMZV is:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,\xi\end{pmatrix}\right.$$
$$-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,\xi\end{pmatrix}\right)$$
$$\left.-\beta^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,-\xi\end{pmatrix}\right),a,b>0\right\}$$

Lemma A.2.11. · *The depth 2 part of the basis of $\mathcal{F}_1^{k_8/k_4,2/2}\mathcal{H}_n$ is, for even n:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix},\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,-\xi\end{pmatrix}-\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix},\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix},x_i\geq 1\right\}.$$

· *The depth 2 part of the basis of $\mathcal{F}_1^{k_8/\mathbb{Q},2/2}\mathcal{H}_n$ is for odd n:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,\xi\end{pmatrix},\ \textit{exactly one even } x_i\right\}.$$

The depth 2 part of the basis of $\mathcal{F}_1^{k_8/\mathbb{Q},2/2}\mathcal{H}_n$ is for even n:

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,-\xi\end{pmatrix}-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}\right)\right\}_{a,b\geq 0}$$
$$\cup\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,-\xi\end{pmatrix}-\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,-\xi\end{pmatrix}-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,-\xi\end{pmatrix}-\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}\right)\right\}_{a,b\geq 0}.$$

· *The depth 2 part of the basis of $\mathcal{F}_1^{k_8/\mathbb{Q},2/1}\mathcal{H}_n$ is for odd n:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}x_1,x_2\\-1,\xi\end{pmatrix}\right.$$
$$\left.-\gamma^{x_1,x_2}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,-\xi\end{pmatrix}\right),\ \textit{exactly one even } x_i\right\}.$$

In even weight n, depth 2 part of the basis of $\mathcal{F}_1^{k_8/\mathbb{Q},2/1}\mathcal{H}_n$ is:

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,\xi\end{pmatrix}\right\}$$
$$\cup\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}n-1,1\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-1,1\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-1,1\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-1,1\\-1,\xi\end{pmatrix}+\right.$$
$$\left.-\sum_{k=1}^{\frac{n}{2}-1}\alpha_k^{0,\frac{n}{2}-1}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,-\xi\end{pmatrix}+\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,\xi\end{pmatrix}\right)\right\}$$
$$\cup\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\\epsilon_1,\epsilon_2\xi\end{pmatrix}+\epsilon_2\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\-1,-\xi\end{pmatrix}-\beta^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\\epsilon_1,\epsilon_2\xi\end{pmatrix}+\epsilon_2\zeta^{\mathfrak{m}}\begin{pmatrix}1,n-1\\-1,-\xi\end{pmatrix}\right)\right.$$
$$\left.-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\left(\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\\epsilon_1,\epsilon_2\xi\end{pmatrix}+\epsilon_2\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\-1,-\xi\end{pmatrix}\right),a,b>0,\epsilon_i\in\{\pm1\},\epsilon_1=-\epsilon_2\right\}.$$

Where $\gamma^{x_1,x_2}=(-1)^{x_2}\binom{2r-1}{2r-x_2}$.

A.2.4 $N=$ '6': Depth 2

In depth 2, coefficients are explicit as previously:

Lemma A.2.12. *The depth 2 part of the basis of MMZV, for even weight n is:*

$$\left\{\zeta^{\mathfrak{m}}\begin{pmatrix}2a+1,2b+1\\1,\xi\end{pmatrix}-\sum_{k=b+1}^{\frac{n}{2}-1}\alpha_k^{a,b}\zeta^{\mathfrak{m}}\begin{pmatrix}n-2k,2k\\1,\xi\end{pmatrix},a,b>0\right\},$$

Proof. Proof being similar than the cases $N=3,4$ is left to the reader.

□

A.3 Homographies of $\mathbb{P}^1\diagdown\{0,\mu_N,\infty\}$

The homographies of the projective line $\mathbb{P}^1$ which permutes $\{0,\mu_N,\infty\}$, induce automorphisms $\mathbb{P}^1\diagdown\{0,\mu_N,\infty\}\to\mathbb{P}^1\diagdown\{0,\mu_N,\infty\}$. The projective space $\mathbb{P}^1\diagdown\{0,\mu_N,\infty\}$ has a dihedral symmetry, the dihedral group $Di_N=\mathbb{Z}/2\mathbb{Z}\ltimes\mu_N$ acting with $x\mapsto x^{-1}$ and $x\mapsto\eta x$. In the special case of $N=1,2,4$, and for these only, the group of homographies is bigger than the dihedral group, due to particular symmetries of the points $\mu_N\cup\{0,\infty\}$ on the Riemann sphere. Let precise these cases:

For $N=1$: The homography group is the anharmonic group generated by $z\mapsto\frac{1}{z}$ and $z\mapsto1-z$, and corresponds to the permutation group $\mathfrak{S}_3$. Precisely, projective transformations of $\mathbb{P}^1\diagdown\{0,1,\infty\}$ are:

$$\begin{array}{lll}
\phi_\tau: & t\mapsto1-t: & \begin{cases}(0,1,\infty)\mapsto(1,0,\infty)\\(\omega_0,\omega_1,\omega_\star,\omega_\sharp)\mapsto(\omega_1,\omega_0,-\omega_\star,\omega_0-\omega_\star).\end{cases}\\
\phi_c: & t\mapsto\frac{1}{1-t}: & \begin{cases}0\mapsto1\mapsto\infty\mapsto0\\(\omega_0,\omega_1,\omega_\star,\omega_\sharp)\mapsto(\omega_\star,-\omega_0,-\omega_1,-\omega_0-\omega_1)\end{cases}\\
\phi_{\tau c}: & t\mapsto\frac{t}{t-1}: & \begin{cases}(0,1,\infty)\mapsto(0,\infty,1)\\(\omega_0,\omega_1,\omega_\star)\mapsto(-\omega_\star,-\omega_1,-\omega_0)\end{cases}\\
\phi_{c\tau}: & t\mapsto\frac{1}{t}: & \begin{cases}(0,1,\infty)\mapsto(\infty,1,0)\\(\omega_0,\omega_1,\omega_\star,\omega_\sharp)\mapsto(-\omega_0,\omega_\star,\omega_1,\omega_\sharp)\end{cases}\\
\phi_{c^2}: & t\mapsto\frac{t-1}{t}: & \begin{cases}0\mapsto\infty\mapsto1\mapsto0\\(\omega_0,\omega_1,\omega_\star)\mapsto(-\omega_1,-\omega_\star,\omega_0)\end{cases}
\end{array}$$

Remark that hexagon relation (4.1) corresponds to a cycle c whereas the reflection relation corresponds to a transposition τ, and :

$$\mathfrak{S}_3=\langle c,\tau\mid c^3=id,\tau^2,c\tau c=\tau\rangle=\{1,c,c^2,\tau,\tau c,c\tau\}.$$

For $N=2$: Here, $(0,\infty,1,-1)$ has a cross ratio -1 (harmonic conjugates) and there are 8 permutations of $(0,\infty,1,-1)$ preserving its cross ratio. The homography group corresponds indeed to the group of automorphisms of a square with consecutive vertices $(0,1,\infty,-1)$, i.e. the dihedral group of degree four Di_4 defined by the presentation $\langle \sigma,\tau \mid \sigma^4=\tau^2=id, \sigma\tau\sigma=\tau\rangle$:

$$\phi_\tau : \quad t\mapsto \tfrac{1}{t} : \quad \begin{cases} \pm 1\mapsto \pm 1 \quad 0\leftrightarrow\infty \\ (\omega_0,\omega_1,\omega_\star,\omega_{-1},\omega_{-\star},\omega_{\pm\sharp})\mapsto(-\omega_0,\omega_\star,\omega_1,\omega_{-\star},\omega_{-1},\omega_{\pm\sharp}) \end{cases}$$

$$\phi_\sigma : \quad t\mapsto \tfrac{1+t}{1-t} : \quad \begin{cases} -1\mapsto 0\mapsto 1\mapsto\infty\mapsto -1 \\ (\omega_0,\omega_1,\omega_\star,\omega_{-1},\omega_{-\star})\mapsto(\omega_{-1}-\omega_1,-\omega_{-1},-\omega_1,-\omega_{-\star},-\omega_\star) \\ (\omega_\sharp,\omega_{-\sharp})\mapsto(-\omega_1-\omega_{-1},-\omega_\star-\omega_{-\star}) \end{cases}$$

$$\phi_{\sigma^2\tau} : \quad t\mapsto -t : \quad \begin{cases} -1\leftrightarrow 1 \\ (\omega_0,\omega_1,\omega_{-1},\omega_{\pm *},\omega_{\pm\sharp})\mapsto(\omega_0,\omega_{-1},\omega_1,\omega_{\mp *},\omega_{\mp\sharp}) \end{cases}$$

$$\phi_{\sigma^2} : \quad t\mapsto \tfrac{-1}{t} : \quad \begin{cases} 0\leftrightarrow\infty \quad -1\leftrightarrow 1 \\ (\omega_0,\omega_1,\omega_\star,\omega_{-1},\omega_{-\star},\omega_{\pm\sharp})\mapsto(-\omega_0,\omega_{-\star},-\omega_{-1},\omega_\star,\omega_1,\omega_{\mp\sharp}) \end{cases}$$

$$\phi_{\sigma^{-1}} : \quad t\mapsto \tfrac{t-1}{1+t} : \quad \begin{cases} 0\mapsto -1\mapsto\infty\mapsto 1\mapsto 0 \\ (\omega_0,\omega_1,\omega_{-1},\omega_\star,\omega_{-\star})\mapsto(\omega_{-1}-\omega_1,-\omega_\star,-\omega_1,-\omega_{-\star},-\omega_{-1}) \\ (\omega_\sharp,\omega_{-\sharp})\mapsto(-\omega_\star-\omega_{-\star},-\omega_1-\omega_{-1}) \end{cases}$$

$$\phi_{\tau\sigma} : \quad t\mapsto \tfrac{1-t}{1+t} : \quad \begin{cases} -1\leftrightarrow\infty \quad 0\leftrightarrow 1 \\ (\omega_0,\omega_1,\omega_\star,\omega_{-1},\omega_{-\star})\mapsto(\omega_1-\omega_{-1},-\omega_{-\star},-\omega_\star,-\omega_{-1},-\omega_1) \\ (\omega_\sharp,\omega_{-\sharp})\mapsto(-\omega_\star-\omega_{-\star},-\omega_1-\omega_{-1}) \end{cases}$$

$$\phi_{\sigma\tau} : \quad t\mapsto \tfrac{1+t}{t-1} : \quad \begin{cases} -1\leftrightarrow 0 \quad 1\leftrightarrow\infty \\ (\omega_0,\omega_1,\omega_\star,\omega_{-1},\omega_{-\star})\mapsto(\omega_1-\omega_{-1},-\omega_1,-\omega_{-1},-\omega_\star,-\omega_{-\star}) \\ (\omega_\sharp,\omega_{-\sharp})\mapsto(-\omega_1-\omega_{-1},-\omega_\star-\omega_{-\star}) \end{cases}$$

Remark that the octagon relation (4.2) comes from the cycle σ of order 4; the other permutations above could also leads to relations.

For $N=4$: $\mathbb{P}^1\diagdown\{0,1,-1,i,-i,\infty\}$ has an octahedral symmetry, and the homography group is the group of automorphisms of this octahedron placed on the Riemann sphere of vertices $(0,1,i,-1,-i,\infty)$.[I] It is composed by 48 transformations, corresponding to 24 rotational symmetries, and a reflection.

We could also look at other projective transformations: $\mathbb{P}^1\diagdown\{0,\mu_N,\infty\}\to\mathbb{P}^1\diagdown\{0,\mu_{N'},\infty\}_{N'|N}$.

`Examples:`

- $\mathbb{P}^1\diagdown\{0,-1,\infty\}\to\mathbb{P}^1\diagdown\{0,+1,\infty\}$, $t\mapsto 1+t$.
- $\mathbb{P}^1\diagdown\{0,-1,\infty\}\to\mathbb{P}^1\diagdown\{0,+1,\infty\}$, $t\mapsto \frac{1}{1+t}$.
- $\mathbb{P}^1\diagdown\{0,\pm 1,\infty\}\to\mathbb{P}^1\diagdown\{0,1,\infty\}$, $t\mapsto t^2$.

[I] Zhao showed this octahedral symmetry allows to reach the "non standard" relations which appeared in weight 3, 4 for $N=4$; non standard relations are these which do not come from distribution, conjugation, and regularised double shuffle relation, cf. [84].

A.4 Hybrid relation for MMZV

The commutative polynomial setting is briefly introduced in §6.1.1.
Let consider the following involution, which represents the Antipode ⧢ as seen in §4.2.1:

$$\boldsymbol{\sigma}: \quad \mathbb{Q}\langle Y\rangle \to \mathbb{Q}\langle Y\rangle\ , \quad f(y_0, y_1, \cdots, y_p) \mapsto (-1)^w f(y_p, y_{p-1}, \cdots, y_1), \tag{A.13}$$

with w the weight, equal to the degree of f plus p. In particular, for $f \in \rho(\mathfrak{g}^{\mathfrak{m}})$:

$$\text{Antipode } ⧢ : f + \sigma(f) = 0. \tag{A.14}$$

Note that $f^{(p)}$ denotes the part of f involving $y_0, \cdots, y_p$. We can also consider:

$$\boldsymbol{\tau}: \quad \mathbb{Q}\langle X\rangle \to \mathbb{Q}\langle X\rangle\ , \quad \overline{f}^{(p)}(x_1, \cdots, x_p) \mapsto (-1)^p \overline{f}^{(p)}(x_p \cdots, x_1). \tag{A.15}$$

The Antipode stuffle corresponds to $\tau(\overline{f}^{\star})$, where $\overline{f}^{\star}$ is defined by:

$$\overline{f}^{\star}(x_1, \ldots, x_p) := \sum_{\substack{s \leq p, i_k \\ p = \sum i_k}} f(\{x_1\}^{i_1}, \ldots, \{x_s\}^{i_s})(-1)^{d-1} \prod_{k=1}^{s} x_k^{i_k - 1}. \tag{A.16}$$

It corresponds naturally to the Euler sums $\star$ version. Then, for $\overline{f} \in \overline{\rho}(\mathfrak{g}^{\mathfrak{m}})$:

$$\text{Antipode } *: \overline{f} + \tau(\overline{f}^{\star}) = 0. \tag{A.17}$$

The *hybrid relation* (Theorem 4.2.3) for motivic multiple zeta values is equivalent to, in this setting of commutative polynomials to the following, already in some notes of F. Brown:

Theorem (F. Brown)**.** *For $\overline{f} \in \overline{\rho}(\mathfrak{g}^{\mathfrak{m}})$, the 6 terms relation holds:*

$$\overline{f}^{(p)}(x_1, \cdots, x_p) + \frac{\overline{f}^{(p-1)}(x_2 - x_1, \ldots, x_p - x_1) - \overline{f}^{(d-1)}(x_2, \cdots, x_p)}{x_1}$$
$$= (-1)^{w+1}\left(\overline{f}^{(p)}(x_p, \cdots, x_1) + \frac{\overline{f}^{(p-1)}(x_{p-1} - x_p, \ldots, x_1 - x_p) - \overline{f}^{(p-1)}(x_{p-1}, \cdots, x_1)}{x_p}\right).$$

Before giving the proof, to be convinced these statements are equivalent, let just write $\overline{f}$ as:

$$\overline{f} = \sum \alpha_{n_1, \ldots, n_k} x_1^{n_1 - 1} \cdots x_k^{n_k - 1}.$$

Then:

$$\frac{\overline{f}^{(p-1)}(x_2 - x_1, \ldots, x_p - x_1) - \overline{f}^{(p-1)}(x_2, \cdots, x_p)}{x_1}$$

$$= \sum \alpha_{n_1, \ldots, n_{p-1}} \frac{(x_2 - x_1)^{n_1 - 1} \cdots (x_p - x_1)^{n_p - 1} - x_2^{n_1 - 1} \cdots x_p^{n_p - 1}}{x_1}$$
$$= \sum \alpha_{n_1, \ldots, n_{p-1}} \sum_{\substack{1 \leq k_i \leq n_i \\ k := \sum n_i - k_i > 0}} (-1)^k x_1^{k-1} \prod_{i=1}^{d-1} \left(\binom{n_i - 1}{k_i - 1} x_{i+1}^{k_i - 1}\right)$$
$$= \sum_{\sum i_j = k} \alpha_{k_1 + i_1, \ldots, k_{p-1} + i_{p-1}} \binom{k_1 + i_1 - 1}{k_1 - 1} \cdots \binom{k_{p-1} + i_{p-1} - 1}{k_{p-1} - 1} (-1)^k x_1^{k-1} x_2^{k_1 - 1} \cdots x_p^{k_{p-1} - 1}$$

This, according to the shuffle regularization (2.28), matches exactly with the definition of $\zeta_k^{\mathfrak{m}}(k_1, \ldots, k_{p-1})$.

Proof of the previous theorem. The proof combines the shuffle relation (using that f is translation invariant notably), the linearized stuffle relation (giving a relation between depth p and depth $p-1$) and the antipode $\sqcup\!\sqcup$.
Let's take f in $\rho(\mathfrak{g}^{\mathfrak{m}})$ and let consider the difference

$$\begin{aligned} I(y_0, y_1, \cdots, y_p) &:= f^{(p)}(y_0, y_1, \cdots, y_p) + (-1)^w f^{(p)}(y_0, y_p, \cdots, y_1) \\ &= f^{(p)}(y_0, y_1, \cdots, y_p) - f^{(p)}(y_1, \cdots, y_p, y_0). \end{aligned} \tag{A.18}$$

Consider also the relation given by the linearized stuffle relation (in $\mathcal{L}$), between depth p and depth $p-1$, defining St:

$$St(y_0, y_1 \sqcup\!\sqcup y_2 \cdots y_p) := f^{(p)}(y_0, y_1 \sqcup\!\sqcup y_2 \cdots y_p), \tag{A.19}$$

Where St can then be expressed by $f^{(p-1)}$ using stuffle:

$$St(y_0, y_1 \sqcup\!\sqcup y_2 \cdots y_p) = \sum \frac{1}{y_i - y_1} \left(f^{(p-1)}(y_0, y_2, , \cdots y_{i-1}, y_1, y_{i+1}, \ldots, y_p) - f^{(p-1)}(y_0, y_2, \cdots, y_p) \right). \tag{A.20}$$

The theorem is then equivalent to the following identity

$$(\bowtie)\ I(y_0, y_1, \cdots, y_p) = (-1)^{w+1} St(y_p, y_0 \sqcup\!\sqcup y_{p-1} \cdots y_1) - St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p).$$

Indeed, looking at the previous definition ($A.20$), most of the terms of St in the right side of $(\bowtie)$ get simplified together, and it remains only:

$$(-1)^{w+1} \frac{f^{(p-1)}(y_p, \ldots, y_2, y_0) - f^{(p-1)}(y_p, \cdots, y_1)}{y_1 - y_0} - \frac{f^{(p-1)}(y_1, \ldots, y_{p-1}, y_0) - f^{(p-1)}(y_1, \cdots, y_p)}{y_p - y_0}.$$

Passing to the x_i variables, we conclude that $(\bowtie)$ is equivalent to the theorem's statement; let now prove $(\bowtie)$. By definition:

$$St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) = f^{(p)}(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) = f^{(p)}(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_{p-1}, y_p) + f^{(p)}(y_1, y_2, \ldots, y_p, y_0).$$

Doing a right shift, using the definition of I:

$$St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) \tag{A.21}$$

$$= f^{(p)}(y_p, y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_{p-1}) - I(y_p, y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_{p-1}) + f^{(p)}(y_0, y_1, y_2 \cdots, y_p) - I(y_0, y_1, y_2 \cdots, y_p).$$

Since:

$$\begin{aligned} f^{(p)}(y_p, y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_{p-1}) &= St(y_p, y_0 \sqcup\!\sqcup y_1 y_2 \cdots y_{p-1}) - f^{(p)}(y_p, y_0, y_1, y_2, \ldots, y_{p-1}) \\ f^{(p)}(y_0, y_1, y_2 \cdots, y_p) &= -I(y_p, y_0, y_1, y_2, \ldots, y_{p-1}) + f^{(p)}(y_p, y_0, y_1, y_2, \ldots, y_{p-1}). \end{aligned}$$

Then, (A.21) becomes:

$$\begin{aligned} &St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) - St(y_p, y_0 \sqcup\!\sqcup y_1 y_2 \cdots y_{p-1}) \\ &\quad = -I(y_p, y_0, y_1, y_2, \ldots, y_{p-1}) - I(y_p, y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_{p-1}) - I(y_0, y_1, y_2 \cdots, y_p). \end{aligned}$$

The sum of the first two I is $I(y_p, y_0 \sqcup\!\sqcup y_1 \cdots y_{p-1})$ which gives:

$$\begin{aligned} I(y_0, y_1, y_2 \cdots, y_p) &= -St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) + St(y_p, y_0 \sqcup\!\sqcup y_1 y_2 \cdots y_{p-1}) - I(y_p, y_0 \sqcup\!\sqcup y_1 \cdots y_{p-1}) \\ &= -St(y_1, y_0 \sqcup\!\sqcup y_2 \cdots y_p) + (-1)^{w+1} St(y_p, y_0 \sqcup\!\sqcup y_{p-1} \cdots y_1). \end{aligned} \tag{A.22}$$

The identity $(\bowtie)$ holds, and the identity of the theorem follows. $\square$

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