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INSTITUT DE MATHÉMATIQUES DE BORDEAUX
UNIVERSITÉ DE BORDEAUX

On the Dynamics of Some Complex Fluids

par
Francesco De Anna

THÈSE

pour obtenir le grade de

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Dietro ogni problema c'è un'opportunità.

Galileo Galilei

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Résumé

Dans le cadre de cette thèse, on s'intéresse à la dynamique de quelques fluides complexes. D'une part on étudie la dynamique des cristaux liquides nématiques, en utilisant les modèles proposés par

- Ericksen et Leslie,
- Beris et Edwards,
- Qian et Sheng.

D'autre part, on analyse un fluide complexe dont la dynamique dépend de la température et qui est modélisée par

- le système de Boussinesq.

Les cristaux liquides sont des matériaux avec une phase de la matière intermédiaire entre les liquides et les solides qui sont des phases plus connues. Dans cette thèse, on s'intéresse à l'étude du problème de Cauchy associé à chaque système modélisant leurs hydrodynamiques. Tout d'abord on obtient des résultats d'existence et d'unicité de solutions faibles ou classiques, solutions qui sont globales en temps. Ensuite, on analyse la propagation de la régularité des données initiales pour ces solutions.

Le cadre fonctionnel adopté pour les données initiales est celui des espaces de Besov homogènes, généralisant des classes d'espaces mieux connues : les espaces de Sobolev homogènes et les espaces de Hölder.

Le système Ericksen-Leslie est considéré dans la version simplifiée proposée par F. Lin et C. Liu, version qui préserve les principales difficultés du système initial. On étudie ce problème en dimension supérieure ou égale à deux. On considère le système dans le cas inhomogène, c'est-à-dire avec une densité variable. De plus, on s'intéresse au cas d'une densité de faible régularité qui est autorisée à présenter des discontinuités. Donc, le résultat que l'on démontre peut être mis en relation avec la dynamique des mélanges de nématiques non miscibles. On démontre l'existence globale en temps de solutions faibles de régularité invariante par changement d'échelle, en supposant une condition de petitesse sur les données initiales dans des espaces de Besov critiques. On démontre aussi l'unicité de ces solutions si de plus on suppose une condition supplémentaire de régularité pour les données initiales.

Le système Beris-Edwards est analysé dans le cas bidimensionnel. On obtient l'existence et l'unicité de solutions faibles globales en temps, lorsque les données initiales sont dans des espaces de Sobolev spécifiques (sans condition de petitesse). Le niveau de régularité de ces espaces fonctionnels est adapté pour bien définir les solutions faibles. L'unicité est une question délicate et demande une estimation doublement logarithmique pour une norme sur la différence entre deux solutions dans un espace de Banach convenable. Le lemme d'Osgood permet alors de conclure à l'unicité de la solution. On obtient également un résultat de propagation de régularité d'indice positif.

Afin de prendre en compte l'inertie des molécules, on considère aussi le modèle proposé par Qian et Sheng, et on étudie le cas de la dimension supérieure ou égale à deux. Ce système montre une caractéristique structurale spécifique, plus précisément la présence d'un terme inertiel, ce qui génère des difficultés significatives. On démontre l'existence d'une fonctionnelle de Lyapunov et

l'existence et l'unicité de solutions classiques globales en temps, en considérant des données initiales petites.

Enfin, on analyse le système de Boussinesq et on montre l'existence et l'unicité de solutions globales en temps. On considère la viscosité en fonction de la température en supposant simplement que la température initiale soit bornée, tandis que la vitesse initiale est dans des espaces de Besov avec indice de régularité critique. Les données initiales ont une composante verticale grande et satisfont à une condition de petitesse spécifique sur les composantes horizontales: elles doivent être exponentiellement petites par rapport à la composante verticale.

Mots clés: Cristaux liquides nématiques, système Ericksen-Leslie, système Beris-Edwards, système Qian-Sheng, système Boussinesq, densité variable, viscosité variable, théorie de Littlewood-Paley, espaces de Besov, analyse harmonique, inégalités logarithmiques, régularisation du noyau de la chaleur.

Abstract

The present thesis is devoted to the dynamics of specific complex fluids. On the one hand we study the dynamics of the so-called nematic liquid crystals, through the models proposed by

- Ericksen and Leslie,
- Beris and Edwards,
- Qian and Sheng.

On the other hand we analyze the dynamics of a temperature-dependent complex fluid, whose dynamics is governed by

- the Boussinesq system.

Nematic liquid crystals are materials exhibiting a state of matter between an ordinary fluid and a solid. In this thesis we are interested in studying the Cauchy problem associated to each system modelling their hydrodynamics. At first, we establish some well-posedness results, such as existence and uniqueness of global-in-time weak or classical solutions. Moreover we also analyze some dynamical behaviours of these solutions, such as propagations of both higher and lower regularities.

The general framework for the initial data is that of Besov spaces, which extend the most widely known classes of Sobolev and Hölder spaces.

The Ericksen-Leslie system is studied in a simplified form proposed by F. Lin and C. Liu, which retains the main difficulties of the original one. We consider both a two-dimensional and a three-dimensional space-domain. We assume the density to be no constant, i.e. the inhomogeneous case, moreover we allow it to present discontinuities along an interface so that we can describe a mixture of liquid crystal materials with different densities. We prove the existence of global-in-time weak solutions under smallness conditions on the initial data in critical homogeneous Besov spaces. These solutions are invariant under the scaling behaviour of the system. We also show that the uniqueness holds under a tiny extra-regularity for the initial data.

The Beris-Edwards system is analyzed in a two-dimensional space-domain. We achieve existence and uniqueness of global-in-time weak solutions when the initial data belongs to specific Sobolev spaces (without any smallness condition). The regularity of these functional spaces is suitable in order to well define a weak solution. We achieve the uniqueness result through a specific analysis, controlling the norm of the difference between two weak solutions and performing a delicate double-logarithmic estimate. Then, the uniqueness holds thanks to the Osgood lemma. We also achieve a result about regularity propagation.

The Qian-Sheng model is analyzed in a space-domain with dimension greater or equal than two. In this case, we emphasize some important characteristics of the system, especially the presence of an inertial term, which generates significant difficulties. We perform the existence of a Lyapunov functional and the existence and uniqueness of classical solutions under a smallness condition for the initial data.

Finally we deal with the well-posedness of the Boussinesq system. We prove the existence of global-in-time weak solutions when the space-domain has a dimension greater or equal than two. We deal with the case of a viscosity dependent on the temperature. The initial temperature is just supposed to be bounded, while the initial velocity belongs to some critical Besov Space. The initial data have a large vertical component while the horizontal components fulfil a specific smallness conditions: they are exponentially smaller than the vertical component.

Keywords: Nematic liquid crystal, Ericksen-Leslie system, Beris-Edwards system, Qian-Sheng system, Boussinesq system, variable viscosity, Littlewood-Paley theory, Besov spaces, harmonic analysis, logarithmic estimates, regularizing effects for the heat kernel.

Contents

I	Dynamics of liquid crystals	1
Chapter 1	Introduction (<i>Version Française</i>)	3
	Résultats pour le système d'Ericksen-Leslie	3
	Énoncés des principaux résultats	4
	Résultats pour le système de Beris-Edwards	6
	Énoncés des principaux résultats	7
	Résultats pour le système de Qian-Sheng	10
	Résultats pour le système de Boussinesq	11
Chapter 2	Introduction	15
	The Discovery of Liquid crystals	15
	Meshophases of liquid crystals	16
	Nematic liquid crystal	16
	Smectic liquid crystal	16
	Cholesteric	18
	Some examples	18
Chapter 3	Dynamic theory of nematics	19
	The fluid behaviour	19
	The Ericksen-Leslie theory	21
	Development of the theory	21
	The Oseen-Frank energy density	22
	The balance laws	24
	The general Ericksen-Leslie equations	26
	A useful simplification	27

The Beris-Edwards dynamic theory	29
The de Gennes order tensor	30
The bulk free energy	32
The elastic free energy	33
The Beris-Edwards system	33
The Qian-Sheng dynamic theory	36
Chapter 4 Contributions of the thesis	39
Contribution to the Ericksen-Leslie theory	39
Statement of the main results	40
Contribution to the Beris-Edwards theory	42
Statement of the main results	43
Contribution to the Qian-Sheng theory	46
Contribution to the Boussinesq theory	48
Chapter 5 Ericksen-Leslie system	51
Introduction and main results	51
A short review of some results on the nematic liquid crystal theory	52
Main results	55
Preliminaries	60
Smooth initial data	63
Existence of a Global Solution	74
Lagrangian Coordinates	76
Uniqueness	77
Uniqueness: the smooth case	78
Uniqueness: the general case	83
Estimates	87
Technical Results for the Heat and Stokes equations	90
Chapter 6 Corotational Beris-Edwards model	93
Introduction and main results	93
The Order Tensor Theory	94
The Q-Tensor System	95

Some Developments in the order tensor Theory	96
Main Results	96
Preliminaries and Notations	98
Sobolev and Besov Spaces	98
Homogeneous Paradiifferential Calculus	99
The Frobenius Norm	100
Weak Solutions	100
The Difference Between Two Solutions	107
Uniqueness	108
Regularity Propagation	116
Technical tools	125
A bound for the Friedrichs scheme	127
Estimate	128
Chapter 7 General Beris-Edwards system	131
Introduction	131
The energy decay, apriori estimates and scaling	135
The existence of weak solutions	139
The uniqueness of weak solutions	144
Littlewood-Paley theory	144
The proof of the uniqueness	150
Technical results	178
Chapter 8 Qian-Sheng system	183
Introduction	183
Physical aspects	184
Main results	186
The singular potential	188
Explicit equation and conservation laws for the inviscid model	189
Explicit equations and conservation laws for the viscous models	190
Commutator and a-priori estimates	192
Classical solutions	197

II	The Boussinesq system	209
Chapter 9	A well-posedness result	211
	Introduction	211
	Preliminaries	216
	Existence of solutions for smooth initial data	219
	Weak solutions: the smooth case	232
	The general case: smooth initial data	234
	Weak solutions: the general case	243
	Inequalities	243
III	Appendix	249
Chapter 10	Appendix	251
	Relation between the Oseen-Frank energy density and the Gibbs free energy density . . .	251
	The Oseen-Frank identity	253
	Some specifics about the Ericksen-Leslie system	253
	Uniaxial behaviour of the Qian-Sheng system	257

Part I

Dynamics of liquid crystals

Chapter 1

Introduction (*Version Française*)

Le principal objet de cette thèse est l'étude de plusieurs équations aux dérivées partielles provenant de la dynamique des cristaux liquides de type nématique. Plus précisément, on montre que les systèmes d'Ericksen-Leslie, Beris-Edwards et Qian-Sheng sont bien posés et on étudie la propagation des régularités des solutions associées.

On renvoie le lecteur au prochain chapitre pour la dérivation de ces systèmes et une introduction avec plus d'explications. Dans ce chapitre, on donne une présentation générale des résultats obtenus. En premier lieu, on s'intéresse au problème de Cauchy associé à chacun des systèmes considérés, pour des conditions initiales convenables, c'est-à-dire prises dans des espaces fonctionnels bien choisis, de sorte que le modèle présente aussi un intérêt physique. Par exemple, dans le cas du système Ericksen-Leslie, le résultat que l'on démontre peut être mis en relation avec la dynamique des mélanges de nématiques non miscibles.

On considère à la fois le cas bidimensionnel et le cas N -dimensionnel, pour $N \geq 3$. D'une part, on montre l'existence et l'unicité de solutions dans le cas N -dimensionnel, en utilisant principalement des outils d'analyse harmonique. D'autre part, on démontre l'existence et l'unicité de solutions faibles ou classiques dans le cas bidimensionnel, au moyen cette fois-ci d'estimations d'énergies, d'analyse de Fourier, et du calcul para-différentiel.

On étudie aussi la propagation des régularités, principalement dans le système proposé par Beris-Edwards. Ici, les méthodes employées sont principalement des techniques issues de l'analyse de Fourier et des estimations de type logarithmique.

On commence la présentation par celle du modèle d'Ericksen-Leslie.

1.1 Résultats pour le système d'Ericksen-Leslie

Durant les années 50-60, J.L. Ericksen [41] et F. Leslie [69] ont développé la théorie la plus répandue jusqu'à aujourd'hui pour la modélisation de la dynamique des cristaux liquides du type nématique. Ils ont considéré que les molécules ont une orientation locale, ce qui est représenté mathématiquement par des fonctions prenant leurs valeurs dans l'espace des *directors*, c'est-à-dire dans la sphère unité. Leur évolution dans le temps est représentée par une équation dont les inconnues sont des champs de vecteurs unitaires d qui sont transportés et tournés par un champ de vitesse u . La vitesse du centre de gravité des molécules satisfait un système de Navier-Stokes incompressible couplé à un tenseur de contraintes généré par la présence des *directors*.

On considère la version simplifiée du système Ericksen-Leslie proposée par F. Lin [74, 75, 80]:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla \Pi = -\operatorname{div}(\nabla d \odot \nabla d) & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d & \mathbb{R}_+ \times \mathbb{R}^N, \\ |d| = 1 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u, \rho, d)|_{t=0} = (u_0, \rho_0, d_0) & \mathbb{R}^N. \end{array} \right. \quad (1.1)$$

C'est un système couplé entre les équations de Navier-Stokes (inhomogènes et incompressibles) et le flot de la chaleur harmonique sur la sphère transporté par la vitesse u . On rappelle que les fonctions considérées sont définies par :

$$\begin{aligned} \rho &= \rho(t, x) \in \mathbb{R}_+ && \text{désigne la densité,} \\ u &= u(t, x) \in \mathbb{R}^N && \text{représente le champ de vitesse,} \\ \Pi &= \Pi(t, x) \in \mathbb{R} && \text{est la pression,} \\ d &= d(t, x) \in \mathbb{S}^{N-1} && \text{est le } \textit{director}, \end{aligned}$$

chacune dépendant de la variable de temps $t \in \mathbb{R}_+$ et de la variable d'espace $x \in \mathbb{R}^N$. On considère l'espace entier \mathbb{R}^N comme domaine pour la variable x , pour une dimension N quelconque supérieure ou égale à deux. Le symbole $\nabla d \odot \nabla d$ désigne la matrice de taille $N \times N$ dont la coordonnée en position (i, j) est donnée par :

$$(\nabla d \odot \nabla d)_{ij} := \partial_i d \cdot \partial_j d = d_{k,i} d_{k,j},$$

pour $i, j = 1, \dots, N$.

Il est important d'observer que F. Lin a proposé ce système simplifié pour un fluide homogène, c'est-à-dire pour une densité constante. La version inhomogène que l'on considère ici est intéressante en particulier pour modéliser un mélange de cristaux liquides de densités différentes.

1.1.1 Énoncés des principaux résultats

Avant d'énoncer notre principal résultat dont on donne la preuve dans le chapitre 5, il nous faut décrire les espaces fonctionnels dans lesquels nos données initiales sont définies. Notre intérêt est d'imposer une faible régularité sur la densité initiale ρ_0 , de sorte que l'on puisse autoriser des discontinuités. On la suppose donc simplement bornée :

$$\rho_0 \in L^\infty(\mathbb{R}^N).$$

Maintenant, on peut remarquer que le système (1.1) a une propriété en commun avec les équations de Navier-Stokes classiques. Précisément, si (ρ, u, d) satisfait (1.1) pour les conditions initiales (ρ_0, u_0, d_0) , alors pour tout λ positif, les fonctions

$$(\rho, u, d)_\lambda(t, x) := (\rho(t, x), \lambda u(\lambda^2 t, \lambda x), d(\lambda^2 t, \lambda x))$$

sont aussi solutions, mais pour les conditions initiales $(\rho_0(\lambda x), \lambda u_0(\lambda x), d_0(\lambda x))$. Ainsi, une caractéristique importante du système (1.1) est que le gradient du *director* ∇d et la vitesse u ont un changement d'échelle équivalent. Il est donc naturel de prendre u_0 et ∇d_0 dans le même espace

fonctionnel.

De plus, d'autres heuristiques (que l'on détaille dans le chapitre 5) suggèrent que les espaces fonctionnels les plus adaptés pour $(u_0, \nabla d_0)$ sont les *espaces de Besov homogènes* avec un indice de régularité critique. Plus précisément, on suppose

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^s.$$

pour des indices $p, r \in [1, \infty]$ convenables et un indice de régularité $s \in \mathbb{R}$ critique. Ces espaces fonctionnels sont définis grâce à la théorie de Littlewood-Paley (on renvoie le lecteur à la section 5.3 pour une description complète). Il est peut-être intéressant de remarquer que les espaces de Besov homogènes généralisent des classes d'espaces mieux connues: les espaces de Sobolev homogènes et les espaces de Hölder. Cette généralisation se voit à travers les relations

$$\dot{B}_{2,2}^s = \dot{H}^s, \quad \dot{B}_{\infty,\infty}^\sigma = \dot{C}^\sigma,$$

pour $s \in \mathbb{R}_+$ et $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$.

Notre résultat principal à propos du modèle Ericksen-Leslie peut être résumé par l'énoncé suivant :

Théorème 1.1.1. *Supposons que les données initiales $(\rho_0, u_0, \nabla d_0)$ remplissent la condition de petitesse :*

$$\left\| \frac{1}{\rho_0} - 1 \right\|_{L^\infty} + \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \leq c_0,$$

pour une constante c_0 suffisamment petite, avec $1 < p < N$ et $1 < r < \infty$, alors le système (1.1) admet des solutions faibles globales en temps. Si de plus on suppose une condition supplémentaire de régularité sur $(u_0, \nabla d_0)$, précisément

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}, \quad (1.2)$$

pour un petit ε positif, alors une telle solution est unique.

Dans l'énoncé ci-dessus, il n'y a pas d'information sur l'espace fonctionnel dans lequel vivent nos solutions faibles. Par souci de clarté, on renvoie ce type d'informations au chapitre 5, ainsi qu'un énoncé plus détaillé du Théorème 1.1.1. Toutefois, on peut anticiper le fait que les espaces fonctionnels dans lesquels on va chercher nos solutions faibles sont de deux types : si les données initiales sont assez lisses, on considère des espaces du type Lebesgue $L^{q_1}(0, T; L^{q_2}(\mathbb{R}^N))$, pour de bons choix de q_1 et q_2 dépendant de r , tandis que si les données initiales manquent de régularité, l'espace fonctionnel est celui de Serrin, qui sont des espaces anisotropes de Lebesgue avec des poids en temps.

De plus, il est important de mentionner que la condition supplémentaire de régularité que l'on impose dans l'énoncé du théorème 1.1.1 joue un rôle crucial dans la preuve d'unicité. En effet, on travaille sur la preuve d'unicité en reformulant le système (1.1) dans les coordonnées de Lagrange, suivant l'approche de Danchin et Mucha dans [27] pour les équations de Navier-Stokes inhomogènes. Plus précisément, la régularité supplémentaire dont on dispose maintenant sur le champ de la vitesse permet de construire l'application de flot $\psi(t, x)$, i.e.

$$\psi(t, x) = x + \int_0^t u(s, \psi(s, x)) ds.$$

L'existence d'une telle application est due à la régularité du type Lipschitz que le champ de vitesse u possède, grâce à l'hypothèse supplémentaire sur les données initiales. Les coordonnées de Lagrange

simplifient en un sens notre problème, assurant que la densité ρ est constante dans les variables lagrangiennes, puisque qu'elle est régie par l'équation de transport.

1.2 Résultats pour le système de Beris-Edwards

Cette thèse est aussi dévouée au problème de Cauchy relié au système de Beris-Edwards. On prend pour domaine l'espace \mathbb{R}^2 tout entier, de sorte que l'hydrodynamique des matériaux nématiques est décrite par

$$\begin{cases} \partial_t Q + u \cdot \nabla Q + S(\nabla u, Q) = H(Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = \operatorname{div}\{\tau + \sigma\} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = 0 = (u_0, Q_0) & \mathbb{R}^2. \end{cases} \quad (1.3)$$

Ici $Q = Q(t, x)$ est le de Gennes tenseur (section 3.3.1), $u = u(t, x) \in \mathbb{R}^3$ représente le champ de vitesse, $\Pi = \Pi(t, x) \in \mathbb{R}$ désigne la pression, le tout dépendant des variables de temps $t \in \mathbb{R}_+$ et d'espace $x \in \mathbb{R}^2$.

$H(Q)$ est défini par

$$H(Q) = L\Delta Q - aQ + b \left(Q^2 - \operatorname{tr}\{Q^2\} \frac{\operatorname{Id}}{3} \right) - cQ \operatorname{tr}\{Q^2\},$$

avec a , b et c des constants qui dépendent du matériau. C'est un tenseur dépendant de Q et de son gradient, représentant la contribution de l'énergie élastique et de l'énergie de "bulk". Enfin, τ et σ sont respectivement les parties symétriques et antisymétriques du tenseur des contraintes additionnel, dont la formule est décrite par

$$\begin{aligned} \tau &:= -\xi \left(Q + \frac{1}{3} \operatorname{Id} \right) H(Q) - \xi H(Q) \left(Q + \frac{1}{3} \operatorname{Id} \right) + 2\xi \left(Q + \frac{1}{3} \operatorname{Id} \right) Q H(Q) - L \nabla Q \odot \nabla Q, \\ \sigma &:= Q H(Q) - H(Q) Q = [Q, H(Q)]. \end{aligned}$$

Le terme $S(\nabla u, Q)$ tient compte du fait que les molécules sont transportées, tournées et alignées par le flot et est défini par:

$$S(\nabla u, Q) := (\xi A + \Omega) \left(Q + \frac{1}{3} \operatorname{Id} \right) + \left(Q + \frac{1}{3} \operatorname{Id} \right) (\xi A - \Omega) - 2\xi \left(Q + \frac{1}{3} \operatorname{Id} \right) \operatorname{tr}(Q \nabla u),$$

avec $A = (\nabla u + {}^t \nabla u)/2$ et $\Omega = (\nabla u - {}^t \nabla u)/2$. Le paramètre ξ est une constante spécifique aux cristaux liquides. D'ordinaire, il prend des valeurs petites, on peut donc négliger sa contribution. Dans cette situation, le système (1.3) se présente sous la forme suivante :

$$\begin{cases} \dot{Q} - \Omega Q + Q \Omega - \Gamma L^2 \Delta Q = -\Gamma \left(aQ - b \left(Q^2 - \operatorname{tr}\{Q^2\} \frac{\operatorname{Id}}{3} \right) + \operatorname{ctr}\{Q^2\} Q \right) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = L \operatorname{div} \{ Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q \} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (1.4)$$

où le point désigne la dérivée matérielle $\partial_t + u \cdot \nabla$ et $\nabla Q \odot \nabla Q$ est un tenseur 3×3 , dont la composante (i, j) est donnée par

$$(\nabla Q \odot \nabla Q)_{ij} = \operatorname{tr}\{\partial_i Q \partial_j Q\} = \partial_i Q_{\alpha\beta} \partial_j Q_{\alpha\beta}.$$

Ensuite on divise notre étude en deux sous-cas :

- le cas corotationnel, quand ξ est nul,
- le cas général, lorsque ξ est positif (ou nul).

On démontre l'existence et l'unicité de solutions faibles aussi bien pour ξ nul que pour ξ non nul, ainsi que des résultats de propagation de régularité pour le cas corotationnel. Afin de comprendre le cadre fonctionnel dans lequel chercher nos solutions faibles, il peut être intéressant de remarquer que le champ de vitesse u et le gradient du tenseur d'ordre ∇Q ont le même changement d'échelle, ce qui est similaire à l'un des systèmes Ericksen-Leslie. Si (u, Q) est une solution de (1.3) avec les conditions initiales (u_0, Q_0) , alors

$$(u, Q)_\lambda(t, x) := (\lambda u(\lambda^2 t, \lambda x), Q(\lambda^2 t, \lambda x)), \quad \lambda > 0,$$

est toujours une solution pour un système avec une densité de "bulk" rééchelonnée, avec cette fois-ci pour données initiales $(\lambda u_0(\lambda x), Q_0(\lambda x))$. Dans le cas nématique uni-axial, ce comportement peut être vu comme une conséquence de celui du système Ericksen-Leslie. En effet, le tenseur d'ordre Q prend ici la forme

$$Q(t, x) = s \left(d(t, x) \otimes d(t, x) - \frac{Id}{3} \right),$$

de sorte que le "scaling behaviour" de Q se déduit de celui du champ des *directors*

$$d_\lambda(t, x) := d(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

1.2.1 Énoncés des principaux résultats

Le cas corotationnel

D'après les remarques précédentes, il est naturel de prendre nos données initiales u_0 et ∇Q_0 dans le même espace fonctionnel. De plus, puisqu'on travaille en dimension deux d'espace, les solutions faibles les plus adaptées sont celles de Leray. Alors, on prend les données initiales dans des espaces de Sobolev, c'est-à-dire

$$(u_0, Q_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2).$$

Notre résultat d'existence et d'unicité peut être énoncé comme suit :

Théorème 1.2.1. *Supposons (u_0, Q_0) dans $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ et supposons que le paramètre ξ soit nul. Alors le système (1.4) admet une unique solution faible (u, Q) , globale en temps, et qui satisfait*

$$\begin{aligned} u &\in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^1(\mathbb{R}^2)) \cap L^2(0, T, \dot{H}^2(\mathbb{R}^2)), \end{aligned} \tag{1.5}$$

pour tout réel positif T .

Le principal intérêt du Théorème 1.2.1 concerne l'unicité, puisque l'existence a été traitée par Paicu et Zarnescu dans [99]. Cependant, on étudie aussi cette question d'existence, mais avec une approche différente de la leur : on utilise une méthode entre le schéma de Friedrichs et le théorème de point fixe de Schaefer.

Les principales difficultés qui apparaissent lors de la preuve d'unicité de la solution pour le système (1.4) sont liées à l'équation de la quantité de mouvement. On peut essentiellement imaginer le système comme un système de Navier-Stokes fortement perturbé. Pour les équations de Navier-Stokes classiques, il est bien connu que l'unicité des solutions faibles en dimension deux découle d'arguments plutôt standards, tandis qu'il s'agit d'un problème ouvert majeur en dimension trois.

Les systèmes étendus que l'on traite se situent dans une position intermédiaire, puisque la perturbation produite par la présence du tenseur des contraintes additionnel entraîne de réelles difficultés techniques liées surtout au manque du contrôle d'un nombre suffisant de dérivées sur la vitesse u . Une façon plutôt classique de gérer ce problème est d'utiliser une norme faible pour estimer la différence entre deux solutions, c'est-à-dire une norme définie sur des espaces moins réguliers que ceux contenant les solutions. Cette approche a déjà été utilisée dans le contexte Navier-Stokes classique, dans [48] et [88].

Dans notre cas, pour des raisons techniques, on considère un espace de Sobolev homogène d'indice négatif, précisément $\dot{H}^{-1/2}$. Le fait que la donnée initiale de la différence des deux solutions soit zéro (i.e. $(\delta u, \delta Q)_{t=0} = 0$) nous aide à contrôler la différence dans un espace homogène de faible régularité. Pour contrôler les termes non linéaires on utilise une loi qui montre que le produit est un opérateur borné opérant sur les espaces suivants :

$$\dot{H}^s(\mathbb{R}^2) \times \dot{H}^t(\mathbb{R}^2) \rightarrow \dot{H}^{s+t-\frac{1}{2}}(\mathbb{R}^2),$$

pour tout réel s et t , avec $|s|, |t| < 1$ et tels que $s + t$ est strictement positif. Il faut alors remarquer que le fait d'évaluer la différence à un niveau de régularité $s = 0$, c'est-à-dire dans $L^2(\mathbb{R}^2)$, permettrait seulement d'établir un résultat d'unicité "weak-strong", comme dans [99]. En se plaçant dans un espace de Sobolev avec un indice négatif, \dot{H}^s avec $s \in (-1, 0)$, on peut montrer l'unicité de solutions faibles. On s'attend à avoir une preuve similaire dans n'importe quel \dot{H}^s avec $s \in (-1, 0)$, et notre choix $s = -1/2$ est juste pour la clarté de la présentation.

Dans la preuve d'unicité, notre but principal est d'obtenir une inégalité de type Gronwall. En effet, l'unicité se réduit alors à une estimation du type suivant :

$$\Phi'(t) \leq \chi(t)\Phi(t),$$

où $\Phi(t)$ est la norme de la différence entre deux solutions, et χ est à priori dans $L^1_{loc}(\mathbb{R}_+)$.

D'autres difficultés propres à ce système viennent s'ajouter. Celles-ci sont de deux types :

- contrôler les termes avec les dérivées maximales, c'est-à-dire les dérivées les plus grandes en u qui apparaissent dans l'équation de Q , et les dérivées les plus grandes en Q qui apparaissent dans l'équation de u ,
- contrôler les grandes puissances de Q , comme par exemple $Q \text{tr}\{Q^2\}$ dans l'énergie de "bulk".

On traite la première difficulté en prenant en compte les caractéristiques spécifiques du système, qui permettent *d'éliminer les termes les plus compliqués*. En ce qui concerne la seconde, on la contourne par des arguments d'analyse harmonique, qui mènent à l'inégalité de Gronwall mentionnée plus haut.

Un autre résultat important de cette thèse dans le cas corotationnel, est un résultat de propagation de régularité. On prend notre donnée initiale dans un espace de Sobolev inhomogène, avec un indice de régularité positif, i.e

$$(u_0, \nabla Q_0) \in H^s(\mathbb{R}^2), \quad \text{avec } s > 0.$$

Ensuite, on étudie la propagation de la norme homogène de Sobolev \dot{H}^s . La première étape est assez classique, on considère des estimations d'énergie dans les espaces \dot{H}^s . Toutefois, le fait de considérer $s > 0$ enlève une caractéristique importante du système : on ne peut plus éliminer les termes les plus difficiles. Ainsi, on doit contrôler tous les termes, mais en le faisant, un premier

problème apparaît : il faut contrôler la norme $L^\infty(\mathbb{R}^d)$ de la solution $(u, \nabla Q)$. Une façon habituelle de gérer ce problème est d'utiliser les inclusions de Sobolev classiques

$$H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2),$$

mais celui-ci impose un indice de régularité $s > 1$, de sorte qu'en procédant de cette façon, on s'attend à propager seulement les hautes régularités, comme dans [99]. Ce problème est résolu en contrôlant différemment les hautes fréquences des basses fréquences de notre solution. On localise la transformée de Fourier de notre solution sur une boule d'un rayon convenable N , en utilisant une fonction de "cut-off". Ensuite, d'une part, on contrôle la norme $L^\infty(\mathbb{R}^2)$ de notre solution localisée sur les basses fréquences pour l'inégalité de Bernstein, et d'autre part, on estime les hautes fréquences en utilisant le fait que les solutions ont une régularité convenable. Enfin, en faisant un choix approprié de N , la propagation des régularités se réduit à une estimation logarithmique

$$\Phi'(t) \leq \chi(t)\Phi(t)\left(1 + \ln \Phi(t)\right),$$

où $\Phi(t)$ désigne la norme \dot{H}^s de notre solution et χ est à priori dans $L^1_{loc}(\mathbb{R}_+)$. Cette estimation conduit à un contrôle à croissance doublement exponentielle en temps de $\Phi(t)$. Notre résultat de propagation s'énonce alors:

Théorème 1.2.2. *Si ξ est nul et (u_0, Q_0) est dans $H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, pour un réel positif s , alors la solution donnée par le Théorème 4.2.1 satisfait*

$$\begin{aligned} u &\in L^\infty(0, T; H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^{s+1}(\mathbb{R}^2)) \cap L^2(0, T, \dot{H}^{s+2}(\mathbb{R}^2)). \end{aligned}$$

Le cas général

Dans le cas général, i.e. lorsque le paramètre ξ est supposé positif ou nul, on montre l'existence et l'unicité de solutions faibles. Ceci est résumé dans l'énoncé suivant :

Théorème 1.2.3. *Supposons que (u_0, Q_0) soit dans $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ et que ξ soit positif. Alors le système (1.3) admet une unique solution faible (u, Q) , globale en temps et qui satisfait*

$$\begin{aligned} u &\in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^1(\mathbb{R}^2)) \cap L^2(0, T, \dot{H}^2(\mathbb{R}^2)), \end{aligned}$$

pour tout T positif.

Malgré la simplicité de l'énoncé, la preuve met en jeu une analyse profonde, notamment des termes qui apparaissent lorsque ξ est non nul.

Remarquons que dans ce cas aussi, le principal résultat du Théorème 4.2.3 réside dans l'unicité, la preuve de l'existence étant une révision des arguments de [98].

La présence des termes relatifs à ξ accroît les principales difficultés associés à l'unicité des solutions pour les systèmes (1.3). Par exemple, on doit contrôler les puissances les plus hautes de Q qui interagissent avec u , comme $Q \operatorname{tr}\{\nabla u Q\}$. Même si on procède de la même manière que dans le cas corotationnel, en utilisant la norme faible $\dot{H}^{-1/2}$ pour estimer la différence entre deux solutions faibles, nous allons aboutir à une estimation du type double-logarithmique, qui conduit

à l'unicité à l'aide du lemme d'Osgood. En effet, l'unicité se réduit à une estimation du type :

$$\Phi'(t) \leq \chi(t)\Phi(t) \left(1 + \ln \left(1 + e + \frac{1}{\Phi(t)} \right) + \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \ln \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \right),$$

où $\Phi(t)$ désigne les normes $\dot{H}^{-1/2}$ de nos solutions et χ est a priori dans $L^1_{loc}(\mathbb{R}_+)$. La preuve se base sur une technique spécifique d'analyse harmonique, localisant la transformée de Fourier de notre solution dans une boule de rayon N et en estimant séparément les basses fréquences des hautes fréquences.

Dans ces estimations, on utilise des inclusions de Sobolev bien choisies, et on écrit explicitement les constantes associées à chacune d'entre eux. Ensuite, une relation spécifique mettant en jeu ces constantes, le rayon N , et $\Phi(t)$ permet enfin d'obtenir l'estimation double-logarithmique attendue.

1.3 Résultats pour le système de Qian-Sheng

Les derniers résultats de cette thèse relatifs à la dynamique des cristaux liquides concernent l'existence et l'unicité de solutions pour le modèle proposé par Qian et Sheng, dans un espace à d dimensions, pour $d = 2, 3$.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \operatorname{div} \{ \sigma + \sigma' \} & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ J\ddot{Q} + \mu_1(\dot{Q} - [\Omega, Q]) - \Delta Q = -\mathcal{L} \frac{\partial \psi_B}{\partial Q} + \mu_2 A & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^d. \end{cases} \quad (1.6)$$

Ici, les fonctions sont les mêmes que dans le système de Beris-Edwards : $u \in \mathbb{R}^d$ est le champ de vitesse, $Q \in \mathbb{R}^{d \times d}$ est le tenseur de de Gennes, le scalaire p désigne la pression, A est la partie symétrique de ∇u , et Ω est la partie antisymétrique de ∇u . On va définir la densité de l'énergie de "bulk" $\psi_B(Q)$ dans (3.25), alors que les tenseurs des contraintes σ et σ' dans (3.34) et (3.36). Toutes ces fonctions dépendent des variables de temps $t \in \mathbb{R}_+$ et d'espace $x \in \mathbb{R}^d$.

La nouveauté la plus importante dans ce modèle est la contribution d'inertie $J\ddot{Q}$ dans l'équation du tenseur de de Gennes Q . Ce terme apparaît en considérant la contribution de l'énergie cinétique rotationnelle, précisément

$$\frac{J}{2} |\dot{Q}|^2.$$

D'ordinaire, cette densité est négligeable, puisque la densité d'inertie J prend généralement de très petites valeurs. Prendre en considération ce terme rend le système (1.6) plus difficile à traiter en comparaison au modèle proposé par Beris et Edwards. En effet, l'équation de Q devient une équation du type hyperbolique, alors que celle qui apparaît quand $J = 0$ est de type parabolique. Ainsi, on ne peut espérer aucun *effet régularisant* pour le tenseur Q .

Notre premier résultat concerne le comportement dissipatif relatif au système (1.6). On démontre l'existence d'une fonctionnelle de Lyapunov $E(t) = E(u(t), Q(t))$, qui correspond à l'énergie totale du matériel. L'énoncé est le suivant :

Théorème 1.3.1. *Supposant quelques restrictions sur les coefficients de viscosité (voir Théorème 8.1.1), le système (1.6) admet la fonctionnelle de Lyapunov suivante :*

$$E(t) := \int_{\mathbb{R}^d} \frac{1}{2} \left(|u|^2 + J|\dot{Q}|^2 + \frac{1}{2} |\nabla Q|^2 + \psi_B(Q) \right)$$

avec $\psi_B(Q)$, la densité d'énergie de "bulk" donnée par (3.25).

Notre second résultat concerne l'existence et l'unicité de solutions *classiques* pour le système (1.6). On travaille dans des espaces de Sobolev $H^s(\mathbb{R}^d)$, pour s suffisamment grand. Notre résultat s'énonce ainsi :

Théorème 1.3.2. *Supposons $(u_0, Q_0) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}$ dans $H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$ avec s supérieur à $d/2$. Supposant quelques restrictions sur les coefficients de viscosité (voir Théorème 8.1.1), et supposant que les normes des données initiales $\|v_0\|_{H^s}$ et $\|Q_0\|_{H^{s+1}}$ soient assez petites, alors il existe une unique solution forte (v, Q) du système (1.6). Cette solution est globale en temps et satisfait*

$$\begin{aligned} u &\in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)), \\ Q &\in L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)) \quad \text{avec} \quad \dot{Q} \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)). \end{aligned}$$

1.4 Résultats pour le système de Boussinesq

Bien que les résultats que nous avons exposés jusqu'à présent concernent la dynamique des cristaux liquides, dans cette thèse on considère aussi le système de Boussinesq. Dans le chapitre 9, on étudie l'évolution d'un fluide dont la viscosité dépend de la température et l'hydrodynamique est modélisée par le système suivant

$$\begin{cases} \partial_t \theta + \operatorname{div}(\theta u) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta)D(u)) + \nabla \Pi = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \theta)|_{t=0} = (\bar{u}, \bar{\theta}) & \mathbb{R}^d, \end{cases} \quad (1.7)$$

où d, θ, u, Π et \mathcal{M} désignent respectivement la dimension, la température, la vitesse, la pression et le tenseur de déformation donné par $D(u) := (\nabla u + {}^t \nabla u)/2$.

Ce système est un couplage entre une équation de transport qui gouverne l'évolution de la température et les équations de Navier-Stokes homogènes qui modélisent la vitesse du fluide. Notons qu'il s'agit ici d'un cas spécifique du système de Boussinesq général où la viscosité $\nu(\theta)$ dépend de la température (on renvoie le lecteur au chapitre (9) pour plus de détails). Ce cas particulier nous permet de décrire quelques phénomènes géophysiques, notamment lorsque la viscosité du fluide a tendance à décroître quand sa température augmente.

En suivant l'approche utilisée pour le système de Ericksen-Leslie, on considère une température initiale qui peut présenter des discontinuités, de sorte qu'on puisse modéliser un mélange entre des fluides non miscibles et de températures différentes.

Notre résultat principal concerne l'existence globale en temps des solutions faibles du système (1.7) sous certaines conditions naturelles sur les données initiales. Avant d'énoncer le résultat, on rappelle brièvement quelques propriétés de ce système.

Comme le système de Navier-Stokes classique, le système (1.7) admet un changement d'échelle particulier : si $(u(t, x), \theta(t, x))$ est solution de (1.7) de donnée initiale $(\bar{u}(x), \bar{\theta}(x))$, alors les fonctions suivantes

$$(u, \theta)_\lambda(t, x) := (\lambda u(\lambda^2 t, \lambda x), \theta(\lambda^2 t, \lambda x)), \quad \lambda > 0$$

sont encore solutions de (1.7) de données initiales $(\lambda \bar{u}(\lambda x), \lambda \bar{\theta}(x))$.

Par conséquent, il est assez naturel de considérer des données initiales dans des espaces de Banach de normes invariantes par ce changement d'échelle. En premier lieu, on exige seulement que $\bar{\theta}$ soit bornée, i.e.

$$\bar{\theta} \in L^\infty(\mathbb{R}^d),$$

ce qui permet à la température de présenter des discontinuités. Ensuite, on considère une vitesse initiale appartenant à un espace de Besov homogène avec un indice de régularité critique, i.e.

$$u_0 \in \dot{B}_{p,r}^{\frac{d}{p}-1}(\mathbb{R}^d),$$

pour des valeurs appropriées $p, r \in [1, \infty]$.

Sous une hypothèse spécifique de petitesse sur les données initiales, plus précisément, en supposant que

$$\eta := (\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\} \quad (1.8)$$

soit suffisamment petit, où $\bar{u}^h := (\bar{u}^1, \dots, \bar{u}^{d-1})$ sont les composantes horizontales de la vitesse et \bar{u}^d sa composante verticale, on démontre l'existence globale entemps des solutions faibles pour le système (1.7). De (1.8), on voit que la viscosité est supposée proche d'une constante positive, qu'on impose égal à 1 par souci de clarté. De plus, la composante verticale \bar{u}^d peut être large si la fluctuation de la viscosité $\nu(\theta_0) - 1$ et les composantes horizontales \bar{u}^h sont suffisamment petites (exponentiellement petites par rapport à \bar{u}^d).

Ce type de condition initiale a déjà été considéré dans la littérature, par exemple dans [60] et [30] dans le cadre du système de Navier-Stokes inhomogène.

Les méthodes utilisées dans la preuve de l'existence des solutions faibles pour le système (1.7) sont similaires à celles qu'on utilise pour le système de Ericksen-Leslie, en considérant une caractérisation particulière des espaces de Besov homogènes et le théorème maximale de régularité pour le noyau de la chaleur. Cependant, il y a quelques difficultés spécifiques du système (1.7) :

- l'anisotropie sur la condition de petitesse des données initiales exige de nouvelles méthodes afin de contrôler les normes des solutions,
- la basse régularité sur la viscosité diminue le nombre de dérivés qu'on peut contrôler sur la vitesse u .

Dû à l'anisotropie sur la condition de petitesse (1.8), les composantes horizontales u^h et la composante verticale u^d de la vitesse doivent être estimées séparément. La structure algébrique du système de Navier-Stokes joue un rôle important dans nos estimations.

En utilisant la condition de divergence nulle sur la vitesse et la structure algébrique des termes non-linéaires, on observe que l'équation sur la composante verticale est du type parabolique-linéaire où les coefficients dépendent des composantes horizontales. Pour cette raison, on n'impose pas de condition de petitesse sur \bar{u}^d .

Ensuite, on étudie l'équation sur les composantes horizontales en analysant deux types de termes non-linéaires : d'une part les termes bilinéaires sur les composantes horizontales et, d'autre part, les interactions entre les composantes horizontales et verticales. À cause de la non-linéarité, une condition de petitesse sur les composantes horizontales de la vitesse initiale est nécessaire pour résoudre l'équation globalement en temps. De plus, la contribution donnée par la composante verticale conduit à une amplification exponentielle de cette condition de petitesse, donc l'estimation qu'on obtient est de la même structure que η donnée par (1.8).

Enfin, la condition de basse régularité sur la température initiale implique une basse régularité aussi sur $\nu(\theta)$, c'est pour cela que le terme $\operatorname{div}(\nu(\theta)\mathcal{M})$ doit être vu au sens des distributions. Ainsi, en écrivant l'équation sur u sous forme intégrale, le terme précédent devient

$$\int_0^t \operatorname{div} S(t-s)((\nu(\theta) - 1)\mathcal{M})(s) ds,$$

où $S(\cdot)$ est le semi-groupe de la chaleur. Pour cela, on a besoin de l'effet régularisant du noyau de la chaleur pour contrôler la vitesse u . C'est pour cette heuristique que le théorème de régularité maximale de la chaleur permet d'estimer seulement u et ∇u dans des espaces de Lebesgue du type $L^q(0, T; L^{q_2}(\mathbb{R}^d))$, tandis que Δu prend un sens simplement distributionnel.

Notre résultat principal s'énonce alors :

Théorème 1.4.1. *Soit $(\bar{\theta}, \bar{u})$ dans $L^\infty(\mathbb{R}^d) \times \dot{B}_{p,r}^{d/p-1}(\mathbb{R}^d)$ avec p dans $(1, d)$ et r dans $(1, \infty)$. Si on suppose*

$$\eta := (\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\} \leq c_0$$

pour une petite constante positive c_0 , alors le système (1.7) admet une solution faible globale en temps. De plus, on obtient les estimations suivantes :

$$\|(u^h, \nabla u^h)\|_{\mathfrak{X}} \leq C_1 \eta, \quad \|(u^d, \nabla u^d)\|_{\mathfrak{X}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3,$$

pour certaines constantes positives C_1, C_2 et C_3 , et où \mathfrak{X} est un espace de Banach approprié (voir 9.1.4).

L'espace fonctionnel \mathfrak{X} est un espace du type Lebesgue à poids en temps (on renvoie le lecteur au chapitre 9, Théorème 9.1.3 et Théorème 9.1.4, pour une description détaillée).

Chapter 2

Introduction

2.1 The Discovery of Liquid crystals

Generally, the physical state of a material can be determined by the motion degree of freedom about its molecules. Certainly, anyone knows the three most widespread physical states of matter, namely solid, liquid and gas. If the movement degree of freedom is almost zero, namely the forces which act on the molecules do not allow any kind of movement, forcing the material structure to be confined in a specific order, then we are classifying a solid material. If such degree still preserves a strong intermolecular force but it is not able to restrict the molecules to lie on a regular organization, then we are considering a fluid state of matter. Finally in the gas phase the forces and the distance between the molecules are weak and large respectively, so that the material is not confined and it is able to extend its volume.

However this classification is not quite accurate. Indeed some organic materials do not exhibit a single transition from solid to liquid, but rather several intermediate transitions involving new phases. At lower temperatures these materials become solid, while at higher temperatures they become isotropic liquids. However, when cooling down, we encounter phase transitions to the liquid crystalline forms, often referred to as mesophases, or also *mesomorphic phases* (mesomorphic: of intermediate form). Materials exhibiting this type of intermediate state of matter are called Liquid Crystals.

Liquid crystals were discovered in 1888, when the Austrian botanist Reinitzer [102], working in the Institute of Plant Physiology at the University of Prague, observed a particular phenomena. He was performing experiments on cholesteryl benzoate, a cholesterol based compound, in order to establish the correct formula and molecular weight of cholesterol. Trying to precisely determine the melting point (an important indicator of the purity of a substance), he heated up the temperature and observed that the material became a cloudy liquid at 145.5°C while at 178.5°C the cloudiness suddenly disappeared and the substance turned into a clear liquid. As a first deduction, Reinitzer believed that this phenomena was due to impurities in the material, however further purifications did not change this particular behavior property. He was reporting what now we denote a cholesteric liquid crystal (or chiral nematic liquid crystal).

Astonished by his discovery, Reinitzer sent two samples and a letter to the German physicist Otto Lehmann, an expert in crystal optics. Reinitzer believed that his observation had some relations with the research of the physicist. Eventually, Lehmann [65] determined that the cloudy liquid had a specific type of molecular order while the transparent liquid at higher temperature had the characteristic isotropic state of all common liquids. He finally realized that the cloudy liquid was a new state of matter and he called it *liquid crystal*, since it retained properties of both liquids

and solids. Nowadays such term is commonly used.

An other relevant contribution in the liquid crystal study was given by Vorländer in 1907 [111]. He was able to derive the rule of the most extended molecular shape of material presenting at least two melting points: the constitutive molecules must present a rod-like structure. Such discovery was to be of great importance in order to develop the mathematical theory modelling liquid crystal also from a geometrical point of view.

2.2 Meshophases of liquid crystals

Some liquid crystal materials can present a considerable range of mesophases. This phenomena is known as *polymorphism*. In the 1922, Friedel [46] classified different liquid crystal mesophases into three main categories: *nematic*, *smectic* and *cholesteric*. In this sections we present the main features for each of them, referring the reader to [107] and [33] for more details.

2.2.1 Nematic liquid crystal

The nematic liquid crystal mesophase is mainly characterized by molecules which present an alignment along a privileged direction, as shown in Figure 2.1. Commonly, this direction is called the *anisotropic axis*. In this mesophase, the molecules do not have a specific long-range positional distribution. Indeed the center of mass of the molecules can freely translate, as in a common isotropic fluid. Most of nematic liquid crystals are *uniaxial*, namely one can observe a rotational symmetry along the anisotropic axis. However some liquid crystals are *biaxial*, meaning that in addition the molecules orient along a secondary axes.



Figure 2.1: A schematic representation of nematic molecules.

The word nematic derives from the Greek $\nu\eta\mu\alpha$ (Greek: *nema*), which means thread. Indeed, in a nematic sample, one can often observe thread-like defects, namely lines of singularity in the alignment of the constituent molecules.

We present now a short history about the development of nematic liquid crystals. For more details we refer the reader to [107]. The first nematic-type liquid crystal was discovered by Gatterman and Ritschke [49] in the 1890. They synthesized the so called *p-azoxyanisole* (PAA), a material which does not naturally occur. In the 1969 Kelker and Scheurle [64] successfully managed to synthesize the 4-methoxybenzylidene-4'-butylane (MBBA). It was the first nematic exhibiting stability under a room temperature, however, despite this interesting property, such material was not considered suitable for applications. In the 1973 Gray and collaborators [51] [52] obtained a nematic-type material, useful to display technological purposes, that is the 4-pentyl-4'-cyanobiphenyl(5CB).

The reader should keep in mind that the main results of this thesis concern the dynamics of nematic-type liquid crystals, which are the most interesting from an applicative point of view. Indeed, nowadays, the most of displays and monitors are often composed by a mixture of nematic materials.

2.2.2 Smectic liquid crystal

In this phase, molecules show a degree of translational order which is not present in the nematic. In the smectic state, the molecules maintain the general orientational order of nematics, but also tend to align themselves in layers or planes as depicted in Figure 2.2. Since smectics are more



Figure 2.2: The smectic state of liquid crystal: starting from the left-hand side, smectic A, smectic C and smectic C* mesophases, respectively.

ordered than nematics, usually this mesophase occurs at a lower temperature with respect to the ones of materials exhibiting a nematic state.

The word smectic derives from the Greek word $\sigma\mu\eta\gamma\gamma\mu\alpha$, namely soap. Indeed each layer can slide over another one, thus smectic liquid crystals present some mechanical properties which are common in soaps.

The smectic mesophases can also be classified into three main categories, as illustrated in Figure 2.2: *smectic A*, *smectic C*, *smectic C**.

When smectic A mesophase occurs, the constituent molecules have a stratified positional order and, on average, an orientation along the layer normal (left-hand image of Figure 2.2). In each layers, the centres of gravity present no long-range order, which means that every stratification dynamics is that of a two-dimensional isotropic liquid. This lamellar-type alignment was detected by several macroscopic effects, some of them known since the beginning of the 20th century, for instance by Friedel and Grandjean in [46, 47, 50].

The smectic C mesophase preserves the layer-type alignment of smectic A materials, however the preferred axis is tilted away from the layer-normal (see the image in the middle of Figure 2.2). As a consequence the material is optically biaxial. The angle between the orientation and the layer normal is commonly called the *smectic C tilt angle* or *smectic cone angle* and it usually depends on temperature.

The Smectic C* phase, also called *chiral smectic C*, exhibits the same layer-type alignment and orientation of smectic C, except that the two-dimensional stratifications evolve in an helical configuration, proceeding along the layer normal, as shown on the right-hand side image of Figure 2.2. This phase can occur when the constitutive molecules are *enantiomorphic*, namely the mirror image cannot be reoriented so as to appear identical to the starting structure. The first compound fulfilling these specifications was synthesized by Liébert, Strzelecki and Keller [90] and is known by the acronym DOBAMBC, which stands for *(S)-(-)-p'-decyloxybenzylidene p'-amino 2-methylbutyl c*.

Up to now, smectic liquid crystal have received few attentions for application, nevertheless smectic C* materials exhibit a spontaneous polarization, a property which is not present in the other mesophases. Such feature is due to the material's ferroelectricity, as explained by P. Oswald and P. Pieranski in [95] and it has recently attracted interest for displays technology.

2.2.3 Cholesteric

The cholesteric mesophase, also called *chiral nematic*, exhibits properties similar to the nematic one, except for a particular helical structure: the constitutive molecules distribute themselves into two-dimensional nematic like layers, whose orientations twist along a continuous helical pattern, as shown in Figure 2.3. This helical structure is due to the *chiral* characteristic of the constituent molecules, namely their mirror image has an identical composition which can not be superposed to the starting configuration.

An important parameter of cholesteric liquid crystal is the so called *pitch of cholesteric*. It is determined measuring the distance between two nematic-type layers, once the axis of orientation rotates through a full circle. The periodicity length of the cholesteric is half of this distance, because, on average, the molecules have a lack of polarity, so there is no physical difference between a molecule in a certain configuration and the same rotated of 180° .



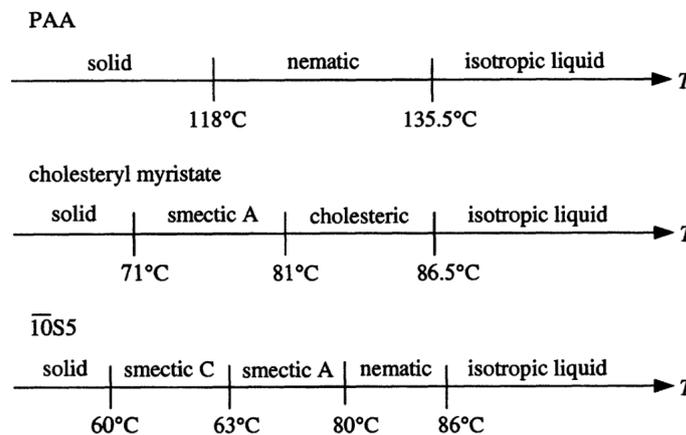
Figure 2.3: the helical structure of a chiral liquid crystal.

The constituent molecules of cholesterics can twist both with a left-hand or right-hand side, however at a given fixed temperature, a sample of such material produces helical structure always in the same sense. Anyway there exists cholesterics that change the orientation of the helix by modifying the temperature, as explained in [33].

Many cholesteric liquid crystals derive from cholesterol (which is not a liquid crystal), so that the name. However, as exposed by Collings in [21], some cholesterics have no correlation with cholesterol, thus the second denomination chiral nematic.

2.2.4 Some examples

Some liquid crystal materials can show several type of mesophases, a phenomena called *polymorphism*, and the phase transitions mainly depend on temperature. We present here a phase diagram, taken from [107], pag. 7. We can observe the phase transitions of three different types of liquid crystals: PAA, cholesteryl myristate and $\overline{10S5}$.



Chapter 3

Dynamic theory of nematics

One of the main purposes of this thesis is to give a contribution to the dynamic theory of nematic liquid crystals. At first, it seems convenient to recall the most widespread theories modelling the time-evolution of nematics, namely the Ericksen-Leslie theory, the Beris-Edwards theory and the Qian-Sheng theory. These theories provided specific partial differential equations, which are the basis of our research.

In this chapter we are going to present an overview concerning the development of these models. Once this background will be completed, we will state the results concerning the contribution of this thesis, whose proofs will be postponed to the next sections.

3.1 The fluid behaviour

Before starting with the presentation of the cited theories modelling the nematic hydrodynamics, it is perhaps interesting to mention that all of them can be seen as generalizations of the more widespread Navier-Stokes dynamical theory for an usual isotropic fluid.

In nematic liquid crystals, the center of mass of each constituent molecule has a freely degree of translation, as a common particle in an isotropic liquid. This allows us to introduce a natural continuum variable describing the dynamic of liquid crystal, namely the *velocity field* $u(t, x)$, at a position x and for a fixed time t . We should interpret $u(t, x)$ as in the case of an usual homogeneous fluid, so that we can already present two balance laws: the *conservation of mass* and the *balance of linear momentum*. If our material occupies a three dimensional volume $U = U(t)$ with boundary $\partial U = \partial U(t)$ at a time t , then the two conservation laws are given by

$$\frac{d}{dt} \int_U \rho(t, x) dx = 0, \quad (3.1)$$

$$\frac{d}{dt} \int_U \rho(t, x) u(t, x) dx = \int_U \rho(t, x) F(t, x) dx + \int_{\partial U} \sigma d\nu, \quad (3.2)$$

respectively, where the scalar ρ stands for mass-density of the nematic material, F is the external body force per unit mass and σ is the surface force per unit area. The first equation tells us that the mass of the volume U is conserved for any time t , while the second one states that the time rate of change of linear momentum of our nematic is equal to the total acting force.

We are going to assume our nematic liquid crystal to be incompressible, that is every subsection of our material does not change volume under the pressure of the flow. This feature can be expressed as a free divergence condition on the velocity field u . Indeed, assuming the velocity field to be

smooth enough, we can construct the so-called *velocity flow* $\psi(t, x)$, determined by

$$\begin{cases} \partial_t \psi(t, x) = u(t, \psi(t, x)) & (t, x) \in [0, T) \times U_0, \\ \psi(0, x) = x & x \in U_0, \end{cases}$$

a well-posed system, thanks to the Cauchy-Lipschitz theorem. Here U_0 stands for the initial volume-configuration of the material, so that, the evolutionary trajectory of a particle $x \in U_0$ in the material is given by the time-dependent function $t \rightarrow \psi(t, x)$. Then, for any positive time t , the volume $U(t)$, corresponds to the set of every $\psi(t, x)$, such that the related particle x belongs to U_0 . Now, the incompressibility condition can be formulated in integral form as follows

$$m(V_0) = \int_{V_0} dx = \int_{V(t)} dx = m(V(t)),$$

for any sub-volume V_0 of U_0 , where m is the usual Lebesgue measure on \mathbb{R}^3 and $V(t)$ is the configuration of the volume V_0 at a time t , that is $V(t) = \psi(t, V_0)$. Thanks to a standard change of variables $x = \psi(t, y)$, we formulate both the left and right-hand sides as integrals on the domain V_0 , i.e.

$$\int_{V_0} dx = \int_{V_0} |\det J\psi(t, y)| dy,$$

where $J\psi$ stands for the Jacobian matrix of ψ . Thus, from the arbitrariness of V_0 , the flow ψ must satisfy

$$|\det J\psi(t, y)| = \det J\psi(t, y) = 1$$

at every point y in V_0 and for every time t , where the modulus is negligible because of the continuity of the Jacobian and the initial condition $\det J\psi(0, \cdot) = 1$. Deriving in time and thanks to some easy calculations, the incompressibility condition assumes the form

$$0 = \partial_t \det J\psi(t, y) = [\det J\psi(t, y)] \operatorname{tr}\{\nabla u\} = \operatorname{div} u,$$

namely a free divergence condition on the velocity field u .

Now, we want to reformulate the two balance laws (3.1) and (3.2) in point form. The conservation law for mass (3.1) in point form can be easily achieved through the Reynolds' transport theorem. Indeed, by the change of variables $x = \psi(t, y)$, (3.1) becomes

$$0 = \frac{d}{dt} \int_{U(t)} \rho(t, x) dx = \frac{d}{dt} \int_{U_0} \rho(t, \psi(t, y)) |\det J\psi(t, y)| dy.$$

Now, recalling that $\det J\psi(t, y) = 1$ and passing the time-derivative under the integral sign,

$$0 = \frac{d}{dt} \int_{U_0} \rho(t, \psi(t, y)) dy = \int_{U(t)} [\partial_t \rho + u \cdot \nabla \rho](t, x) dx.$$

Thanks to the arbitrariness of $U = U(t)$ (that is the arbitrariness of U_0), we achieve

$$\partial_t \rho(t, x) + u(t, x) \cdot \nabla \rho(t, x) = 0, \quad (3.3)$$

so that, the conservation of mass in point form corresponds to the well-known transport equation.

Now, we take care of the balance of angular momentum. Denoting by ν the normal to the boundary $\partial U(t)$, one can show by the usual tetrahedron argument (see for instance [38], section 1.6.1) that the i -th component σ_i of the surface force can be expressed in terms of the so-called *total stress tensor* σ_{ij} , i.e.

$$\sigma_i := \sum_{j=1}^3 \sigma_{ij} \nu_j.$$

Thus the balance law of linear momentum (3.2), the Reynolds' theorem and the Green's theorem lead to

$$\frac{d}{dt} \int_U \rho u \, dx = \int_U \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) \, dx = \int_U \rho F + \operatorname{div} \sigma \, dx,$$

where $u \otimes u$ stands for the quadratic tensor defined by $(u \otimes u)_{ij} = u_i u_j$. Writing this result in point form and recalling the transport equation (3.3), we obtain the more familiar momentum equation

$$\rho(\partial_t u + u \cdot \nabla u) = \rho F + \operatorname{div} \sigma. \quad (3.4)$$

In the forthcoming sections we are going to establish an explicit formula for the total stress tensor for a nematic liquid crystal, however for the sake of completeness, let us present the isotropic form of this tensor, known as the *Cauchy stress tensor*:

$$\sigma_{ij} = \delta_{ij} p + \mu A, \quad (3.5)$$

where p stands for the pressure of the fluid, δ_{ij} is the Kronecker delta, μ is the viscosity of the fluid and finally A is the *rate of strain tensor* whose formula is given by

$$A_{ij} = \left(\frac{\nabla u + {}^t \nabla u}{2} \right)_{ij} = \frac{u_{i,j} + u_{j,i}}{2}.$$

Replacing the Cauchy stress tensor (3.5) into the linear momentum equation (3.4), the dynamic of a liquid crystal in an isotropic state can be modelled by

$$\rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla p = \rho F,$$

which is the widely-known Navier-Stokes equation. It is perhaps interesting to remark that in this case the total stress tensor (3.5) is affected by friction forces. One can neglect this contribution when the viscosity is small enough. In this case the Cauchy stress tensor corresponds to $\sigma_{ij} = -\delta_{ij} p$, so that the momentum equation (3.4) reads as

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p = \rho F,$$

namely the well-known Euler equations. In the next section we will see the difference between the total stress tensor for nematics and the Cauchy stress tensor for the Navier-Stokes and Euler equations. The anisotropic orientations of the constituent molecules contribute to the linear momentum equation by an additional forcing term. This term corresponds to an additional stress tensor which depends on the variations of the molecular alignment.

3.2 The Ericksen-Leslie theory

3.2.1 Development of the theory

Usually a dynamical theory for a complex fluid should take inspiration by the related static one. Thus the early static theory for nematics proposed by Oseen [93, 94] in the 1925 and Zocher [119] in the 1927 should be considered as the starting point for the Ericksen-Leslie development. Their approach was of great importance for Franck [45], indeed, making use of their ideas, he performed a complete static theory for nematics in the 1958, theory based on the so called Oseen-Frank energy

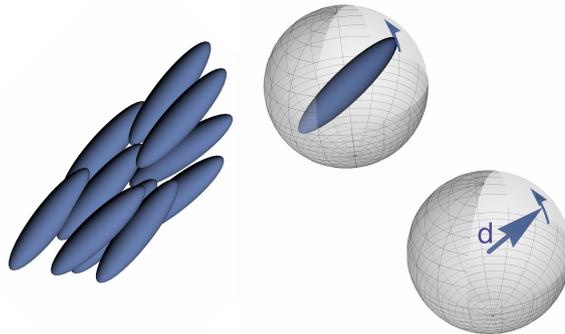


Figure 3.1: A schematic representation of a nematic liquid crystal. The unit vector d describes the average alignment of the molecules

density. An explicit formula of this energy will be exposed in section 3.2.2, since it plays a main role for the formulation of the Ericksen-Leslie system.

The first formulation of a constitutive theory for the liquid crystals dynamics dates back to the works of Anzelius [6] in 1931, however the first widely accepted model was made by Ericksen [40] in the 1961. Ericksen performed some balance laws in order to extend the static theory of nematics to their dynamical behaviour. Thus, in the 1966 [68] and in the 1968 [69], Leslie proceeded to generalise his approach, successfully obtaining some constitutive equations in order to model the dynamic evolution of nematic liquid crystal. This led to one of the most widespread and useful theory describing the dynamics of these materials, namely the well-celebrated Ericksen-Leslie theory.

Starting from these works, the interest for the dynamical behaviour of nematics increased, for instance in the 1979 Leslie [71] reviewed his theory and gave some impression of the type of problem arising, while de Gennes and Prost [33] investigated examples and applications for such materials.

In the 1992, Leslie [67] proceeded to reformulate constitutive equations for the Ericksen-Leslie theory, passing through a more comprehensible approach, well exposed also in the Stewart's work [107]. Taking inspiration by this two works, we now present a derivation of the Ericksen-Leslie theory, arising to constitutive equations for nematic materials.

3.2.2 The Oseen-Frank energy density

In this section we present the so-called Oseen-Frank energy density and we proceed similarly as to the approach proposed by Frank in [45].

As already explained in the previous chapter, a nematic liquid crystal is characterized by constituent rod-like molecules which tend to align each other along a privileged direction. Then it is natural to represent such orientation through a unit vector d , called the *director*, as depicted in figure Figure 3.1. We recall that, on average, the molecules have a lack of polarity, thus the vector d and $-d$ are equivalent in the sense that the sign has no physical meaning.

We consider the three dimensional Euclidian space \mathbb{R}^3 and we denote by $x = (x_1, x_2, x_3)$ the coordinates with respect to the canonical basis. If our nematic occupies a smooth domain U into \mathbb{R}^3 , the mean alignment at a point ζ in U is

$$d = d(\zeta) \in \mathbb{R}^3, \quad \text{with} \quad |d(\zeta)|^2 = 1.$$

Once the sign of the director d is chosen, we introduce in ζ a local right-handed Cartesian coor-

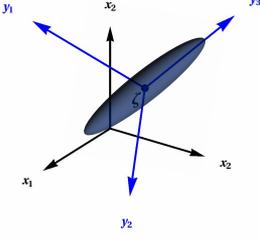


Figure 3.2: A local Cartesian coordinate system with y_3 parallel to the director.

dinate system $y := (y_1, y_2, y_3)$ as depicted in figure Figure 3.2, imposing y_3 parallel to $d(\zeta)$. Moreover we assume that our nematic liquid crystal is uniaxial, thus the y_1 and y_2 axis can be freely chosen in the plain perpendicular to $d(\zeta)$. Then the transformed coordinates y are determined via rotation and translation as follows:

$$y_i = \sum_{j=1}^3 R_{ij}(x_j - \zeta_j), \quad (3.6)$$

for any $i = 1, 2$ and 3 , where R is a suitable 3×3 -rotation matrix. We denote by \tilde{d} the director d with respect to the new coordinates y , so that its formula is given by

$$\tilde{d}_i(y) = \sum_{j=1}^3 R_{ij}d_j(x) = \sum_{j=1}^3 R_{ij}d_j(\zeta + {}^t R y),$$

where ${}^t R$ is the transposed (and thus the inverse) matrix of R .

Since $\tilde{d}(0)$ (namely $d(\zeta)$) is parallel to the y_3 -axis, then the gradient of \tilde{d} in the origin (namely in ζ) reads as follows

$$\nabla_y \tilde{d}(0) = \begin{pmatrix} \frac{\partial \tilde{d}_1}{\partial y_1}(0) & \frac{\partial \tilde{d}_1}{\partial y_2}(0) & \frac{\partial \tilde{d}_1}{\partial y_3}(0) \\ \frac{\partial \tilde{d}_2}{\partial y_1}(0) & \frac{\partial \tilde{d}_2}{\partial y_2}(0) & \frac{\partial \tilde{d}_2}{\partial y_3}(0) \\ 0 & 0 & 0 \end{pmatrix} \quad (3.7)$$

and the six components of curvature at the point ζ are given by

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{d}_1}{\partial y_1}(0) \\ \frac{\partial \tilde{d}_2}{\partial y_2}(0) \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \tilde{d}_2}{\partial y_1}(0) \\ \frac{\partial \tilde{d}_1}{\partial y_2}(0) \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \tilde{d}_1}{\partial y_3}(0) \\ \frac{\partial \tilde{d}_2}{\partial y_3}(0) \end{pmatrix}, \quad (3.8)$$

which stand for the *splay*, the *twist* and the *bend*, respectively, as illustrated in Figure 3.3.

Now, as explained by Andrienko in [5], the Gibbs free energy density w_F of a liquid crystal can be written taking in consideration six curvature strains:

$$w_F = \sum_i^6 k_i a_i + \frac{1}{2} \sum_{i,j}^6 k_{ij} a_i a_j$$

where k_i and $k_{ij} = k_{ji}$ are curvature elastic constants, and where the terms a_i are defined by

$$a_1 := s_1, \quad a_2 := t_2, \quad a_3 := b_1, \quad a_4 := -t_1, \quad a_5 := s_2 \quad \text{and} \quad a_6 := b_2.$$

Since we are considering uniaxial nematics, then the energy density must be invariant under a rotation around the axis y_3 . Thanks to this feature, considering for instance a rotation about $\pi/2$ and one about $\pi/4$, we achieve some constrictions on the curvature elastic constants, so that our energy density reads as follows:

$$w_F = k_1(s_1 + s_2) + k_2(t_1 + t_2) + \frac{k_{11}}{2}(s_1 + s_2)^2 + \frac{k_{22}}{2}(t_1 + t_2)^2 + \\ + \frac{k_{33}}{2}(b_1^2 + b_2^2) + k_{12}(s_1 + s_2)(t_1 + t_2) - (k_{22} + k_{24})(s_1 s_2 + t_1 t_2).$$

Further material considerations, as the absence of polarity, the absence of enantiomorphism, yield k_1 , k_2 and k_{12} to be null, while some specifics of the curvatures (postponed in the appendix 10.1)

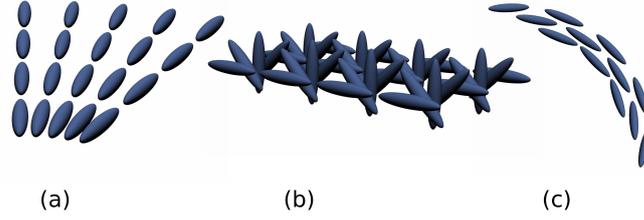


Figure 3.3: The three distinct curvature strains of a nematic liquid crystal: (a) splay, (b) twist and (c) bend.

allow the Oseen-Frank energy density to be formulated depending on the director field d and its gradient, i.e.

$$w_F(d, \nabla d) = \frac{k_{11}}{2}(\operatorname{div}d)^2 + \frac{k_{22}}{2}(d \cdot \operatorname{rot}d)^2 + \frac{k_{33}}{2}|d \wedge \operatorname{rot}d|^2 + \frac{k_{22} + k_{24}}{2}\{\operatorname{tr}\{\nabla d^2\} - (\operatorname{div}d)^2\}, \quad (3.9)$$

which is the Oseen-Frank energy density in its widespread form. As already pointed out, the first three terms represent splay, twist and bend respectively, while the last one is known as the *saddle-splay* term and it can be rewritten in a divergence form as follows:

$$\begin{aligned} \operatorname{tr}\{(\nabla d)^2\} - (\operatorname{div}d)^2 &= \sum_{i,j=1}^3 \frac{\partial d_i}{\partial x_j} \frac{\partial d_j}{\partial x_i} - \frac{\partial d_i}{\partial x_i} \frac{\partial d_j}{\partial x_j} \\ &= \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left[\frac{\partial d_i}{\partial x_j} d_j \right] - \frac{\partial^2 d_i}{\partial x_i \partial x_j} d_j - \frac{\partial}{\partial x_j} \left[\frac{\partial d_i}{\partial x_i} d_j \right] + d_j \frac{\partial^2 d_i}{\partial x_i \partial x_j} \\ &= \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left[\sum_{j=1}^3 \frac{\partial d_i}{\partial x_j} d_j \right] - \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left[\sum_{i=1}^3 \frac{\partial d_i}{\partial x_i} d_j \right] = \operatorname{div} \left\{ d \cdot \nabla d - (\operatorname{div}d)d \right\}, \end{aligned}$$

thus it acts as a surface-energy contribution. We conclude this section with the total elastic free energy of a nematic occupying a three dimensional domain U :

$$\begin{aligned} \int_U w_F(d(\zeta), \nabla d(\zeta)) d\zeta &= \int_U \frac{k_{11}}{2}(\operatorname{div}d(\zeta))^2 + \frac{k_{22}}{2}(d(\zeta) \cdot \operatorname{rot}d(\zeta))^2 + \\ &+ \frac{k_{33}}{2}|d(\zeta) \wedge \operatorname{rot}d(\zeta)|^2 + \frac{k_{22} + k_{24}}{2}\{\operatorname{tr}\{\nabla d(\zeta)^2\} - (\operatorname{div}d(\zeta))^2\} d\zeta. \end{aligned}$$

3.2.3 The balance laws

We now come back to the dynamic description of a nematic liquid crystal and this section is devoted to the derivation of the so-called Ericksen-Leslie equations. We mainly follow the structure proposed by Stewart in [107], and we refer the reader to the exhaustive review of Leslie [71] for more details concerning the physical assumptions.

We start the description of a nematic liquid crystal dynamics introducing a new continuum variable: the *local angular velocity* $w(t, x)$, which corresponds, on average, to the angular velocity of the constituent molecules at a position x and time t . Then, since director d is a unit vector field,

it fulfils:

$$\dot{d} = w \wedge d, \quad (3.10)$$

where the superposed dot is an abbreviation for the material time derivative D_t

$$\dot{d} = D_t d := \partial_t d + u \cdot \nabla d.$$

It is perhaps worth remarking that the local angular velocity does not correspond to the classical angular velocity for an isotropic fluid, also known as vorticity, whose formula is $(\text{rot } u)/2$. In a nematic liquid crystal this kinematic parameter is referred as *regional angular velocity*, and it describes the local spinning motion of the material near some point. Furthermore the difference ω between these continuum variables

$$\omega := w - \frac{1}{2} \text{rot } u \quad (3.11)$$

is called the *relative angular momentum*.

After the mass conservation law and the balance of linear momentum, the third law in the continuum theory of nematics liquid crystal is given by the *conservation of the angular momentum*, which reads as follows:

$$\frac{d}{dt} \int_U \rho(x \wedge u) dx = \int_U \rho(x \wedge F + K) dx + \int_{\partial U} (x \wedge \sigma + l) d\nu. \quad (3.12)$$

Here K is the external body moment per unit mass and l is the surface moment per unit area, while F is the external body force per unit mass introduced in (3.2). Several terms related to the relative angular momentum ω and the director d are intrinsically hidden inside the above equation.

It is perhaps worth remarking that in this formulation we have not taken in consideration the so called *director inertial density*, a material parameter which is usually negligible in the most of nematics. However, in this thesis we will also investigate the no-null inertial case, whose presence changes the internal structure of the constitutive equations (see section 3.4).

Now we handle with the balance law of the angular momentum. Using the notation of the Levi-Civita symbols, (3.12) becomes

$$\frac{d}{dt} \int_U \rho \epsilon_{ijk} x_j u_k dx = \int_U \rho \epsilon_{ijk} x_j F_k + \rho K_i dx + \int_{\partial U} \epsilon_{ijk} x_j \sigma_k + l_i d\nu. \quad (3.13)$$

As for the surface force for unit area, the external body moment per unit mass l can be rewritten in tensor form through $l_i = l_{ij} \nu_j$, where l_{ij} is the so called *couple stress tensor* and we recall that ν is the normal to the boundary ∂U . Similarly $\sigma_k = \sigma_{kp,p} \nu_l$. Moreover, observing that

$$D_t(\epsilon_{ijk} x_j u_k) = \epsilon_{ijk} \dot{x}_j u_k + \epsilon_{ijk} x_j \dot{u}_k = \epsilon_{ijk} u_j u_k + \epsilon_{ijk} x_j \dot{u}_k = \epsilon_{ijk} x_j \dot{u}_k,$$

where we have used $\dot{x}_j = \partial_t x_j + u_s \partial_{x_s} x_j = u_j$, we can rewrite (3.13) as follows:

$$\begin{aligned} \int_U \rho \epsilon_{ijk} x_j \dot{u}_k dx &= \int_U \rho \epsilon_{ijk} x_j F_k + \rho K_i dx + \int_U (\epsilon_{ijk} x_j \sigma_{kp})_{,p} + l_{ij,j} dx \\ &= \int_U \rho \epsilon_{ijk} x_j F_k + \rho K_i dx + \int_U \epsilon_{ijk} \sigma_{kj} + \epsilon_{ijk} x_j \sigma_{kp,p} + l_{ij,j} dx. \end{aligned}$$

Reorganizing the above equality we get

$$\int_U \epsilon_{ijk} x_j \underbrace{(\rho \dot{u}_k - \rho F_k - \sigma_{kp,p})}_{=0} dx = \int_U (\rho K_i + \epsilon_{ijk} \sigma_{jk} + l_{ij,j}) dx,$$

where the left-hand side is null thanks to the linear momentum equation (3.4). Finally, from the arbitrariness of U the angular momentum equation in point form reads as follows:

$$\rho K_i + \epsilon_{ijk} \sigma_{jk} + l_{ij,j} = 0. \quad (3.14)$$

If the external body moment K and the couple stress tensor l are null, then the momentum equation determines the total stress tensor σ to exactly corresponds to the Cauchy stress tensor (3.5), reducing our dynamical system to the classical Navier-Stokes equations.

Now, we want to express both the total stress tensor σ and the couple stress tensor l in terms of the nematic continuum variables, the director d and the velocity field u . In order to do that, we need to introduce the so-called Ericksen work postulate.

3.2.4 The general Ericksen-Leslie equations

It is useful to introduce some terminology and some kinematic quantities. We have already defined rate of strain tensor A as the symmetric tensor of the velocity gradient. Then we denote by Ω the so called *vorticity tensor*, which corresponds to the skew-adjoint tensor of the velocity gradient, that is:

$$\Omega := \frac{\nabla u - {}^t \nabla u}{2}. \quad (3.15)$$

Always following the structure of [107] and [67], we consider the vector field \mathcal{N} given by

$$\mathcal{N} := \omega \wedge d,$$

where we recall that ω stands for the relative angular velocity (3.11). Now, recalling that $2\Omega v = (\text{rot } u) \wedge v$ for any vector v , then we can reformulate the vector field \mathcal{N} as

$$\mathcal{N} = w \wedge d - \frac{1}{2}(\text{rot } u) \wedge d = \dot{d} - \Omega d,$$

where we have also used equality (3.10). Following the terminology of Truesdell and Noll [110], \mathcal{N} stands for the *co-rotational time flux* of the director d . The co-rotational time flux \mathcal{N} gives an important contribution into the total stress tensor σ . More precisely it plays an important role in the so called *Leslie stress tensor*, and for specific information on this point the reader is referred to the appendix.

In order to achieve an explicit formulation of the total and couple stress tensors, Ericksen [40,41] and Leslie [67] introduced a work postulate. It states as follows:

the rate at which forces and moments do work on a volume of nematic goes into changes in either the above stored energy or the kinetic energy, or is lost in viscous dissipation.

Then, their work postulate is expressed as follows:

$$\int_U \rho(F \cdot u + K \cdot w) dx + \int_{\partial U} (\sigma \cdot u + l \cdot w) d\nu = \frac{D}{Dt} \int_U \left(\frac{1}{2} \rho |u|^2 + w_F \right) dx + \int_U \mathcal{D} dx, \quad (3.16)$$

where w_F is the Oseen-Frank energy density (3.2.2) and \mathcal{D} is the *rate of viscous dissipation per unit volume*, also known as dissipation function. Thanks to this postulate and several structural remarks (postponed in the appendix), one can finally achieve the sequent formulation for the total stress tensor σ and the couple stress tensor l

$$\sigma_{ij} = -p \delta_{ij} - \frac{\partial w_F}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij},$$

$$l_{ij} = \epsilon_{islt} d_s \frac{\partial w_F}{\partial d_{l,j}},$$

where $\tilde{\sigma}$ stands for the Leslie stress tensor also called *viscous stress*, whose formula is

$$\tilde{\sigma}_{ij} = \alpha_1 A_{lk} d_k d_l d_i d_j + \alpha_2 d_i \mathcal{N}_j + \alpha_3 \mathcal{N}_i d_j + \alpha_4 A_{ij} + \alpha_5 d_j A_{ik} d_k + \alpha_6 d_i A_{jk} d_k. \quad (3.17)$$

Here, the coefficients $\alpha_1, \dots, \alpha_6$ are known as the *Leslie viscosity coefficients*. The reader should observe that if the Oseen-Frank energy $w_F(d, \nabla d)$ is null (as in the case of a constant director d) and the viscous stress reduces to $\alpha_4 A$, then the relative total stress tensor σ corresponds to the Cauchy stress tensor (3.5), while the couple stress l vanishes. This is the case of an isotropic liquid crystal which dynamics is governed by the Navier-Stokes equations.

Taking into account the above expressions into (3.2) and (3.4), we can finally perform the widely celebrated Ericksen-Leslie equations, which reads as follows

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) = \operatorname{div} \sigma, \\ \operatorname{div} u = 0, \\ d \wedge (\tilde{g} + h) = 0, \\ |d|^2 = 1 \end{cases} \quad (3.18)$$

where $\tilde{g} = -\gamma_1 \mathcal{N} - \gamma_2 A n$, with γ_1 and γ_2 parameters dependent on the viscosity coefficients, and with h as the *molecular field*

$$h = \operatorname{div} \left\{ \frac{\partial w_F}{\partial \nabla d} \right\} - \frac{\partial w_F}{\partial d}.$$

3.2.5 A useful simplification

The Ericksen-Leslie equations (3.18) are particularly complicated as they contain a large number of terms, so there are few mathematical studies of the full systems (for more details see chapter 5). Nevertheless the main mathematical difficulties are still present in the following simplified model, a version of which was originally proposed by Fanghua Lin around the 1990 in [74, 75, 80]:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u + \nabla p = -\lambda \operatorname{div}(\nabla d \odot \nabla d) \\ \operatorname{div} u = 0 \\ \partial_t d + u \cdot \nabla d - \gamma \Delta d = \gamma |\nabla d|^2 d \\ |d| = 1 \end{cases} \quad (3.19)$$

This is a strongly coupled system between the inhomogeneous and incompressible Navier-Stokes equation and the transported heat flow of harmonic maps into sphere. We recall that here the continuum variables are denoted as follows:

$$\begin{aligned} \rho &= \rho(t, x) \in \mathbb{R}_+ \text{ denotes the density, } u = u(t, x) \in \mathbb{R}^3 \text{ represents the velocity field,} \\ p &= p(t, x) \in \mathbb{R} \text{ is the pressure and } d = d(t, x) \in \mathbb{S}^2 \text{ is the director,} \end{aligned}$$

all depending on the time variable $t \in \mathbb{R}_+$ and on the space variables $x \in \mathbb{R}^3$. The symbol $\nabla d \odot \nabla d$ is a tensor whose components are given by

$$(\nabla d \odot \nabla d)_{ij} = \partial_i d \cdot \partial_j d, \quad (3.20)$$

while \mathbb{S}^2 is the two-dimensional unit-sphere in \mathbb{R}^3 .

The general Ericksen-Leslie equations (3.18) and the above system have a strong interconnection. For instance we can consider a simplified version of the Oseen-Frank energy (3.9), imposing the three coefficients k_{11} , k_{22} , k_{33} to coincide to a positive constant λ whose value depends on the material, and assuming the remaining coefficient k_{24} to be null. Under this assumptions the Oseen-Frank energy density assumes the usual form of an elastic-type energy density

$$w_F(\nabla d) = \frac{\lambda}{2} d_{i,j} d_{i,j} = \frac{\lambda}{2} |\nabla d|^2. \quad (3.21)$$

In this case, the *steady-state* formulation of the general system (3.18) and its simplified version (3.19) coincide (where with steady-state we mean the absence of fluid motion, imposing the velocity field to be null $u \equiv 0$). Indeed, the angular momentum equation of (3.18) becomes

$$d \wedge \left(-\gamma_1 d_t + \lambda \Delta d \right) = 0,$$

that is

$$-\gamma_1 d_t + \lambda \Delta d + \beta d = 0,$$

where the scalar β is the Lagrangian multiplier due to the constraint $|d| = 1$. Its value can be calculated multiplying the above equation by d , i.e.

$$\beta = \frac{\gamma_1}{2} \frac{d}{dt} |d|^2 - \lambda \Delta d \cdot d = -\lambda d_{k,j} d_k = -\lambda (d_{k,j} d_k)_{,j} + \lambda d_{k,j} d_{k,j} = \lambda |\nabla d|^2, \quad (3.22)$$

where in the last equality we have used $2d_{k,j} d_k = \partial_j |d|^2 = 0$. Imposing $\gamma = \lambda/\gamma_1$, we deduce that the director equations of the general and the simplified Ericksen-Leslie systems are equivalent, namely they read as follows:

$$\partial_t d - \gamma \Delta d = \gamma |\nabla d|^2 d.$$

It is perhaps interesting remarking that if we also assume the director d to not depend on time, which means d to be a stationary solution, the above angular momentum coincides with

$$-\Delta d = |\nabla d|^2 d,$$

namely *the harmonic map into the sphere* whose solutions are minimizers of the elastic energy

$$\int_U |\nabla d|^2,$$

for a domain U and suitable boundary conditions, under the restriction d to returns value into the sphere, i.e. $|d| = 1$.

However, in general it is not natural to have a flow of liquid crystals for which there is no motion in the fluid itself, i.e. u is null. Thus, it is of major interest taking in consideration the action of the velocity field u . In this case, the simplified Ericksen-Leslie system (3.19) arises from additional conditions on the Leslie viscosity: we always assume the Oseen-Frank energy to read as the elastic energy (3.21) and we also impose α_1 , α_2 , α_3 , α_5 and α_6 to be null in the viscous stress tensor $\tilde{\sigma}$ (3.17). This yields the viscous stress $\tilde{\sigma}$ to be equal to $\alpha_4 A$, so that the balance of linear momentum reads as follows

$$\rho(\partial_t u + u \cdot \nabla u) = \operatorname{div} \sigma = -\nabla p + \lambda \operatorname{div} \{ \nabla d \odot \nabla d \} + \frac{\alpha_4}{2} \Delta u.$$

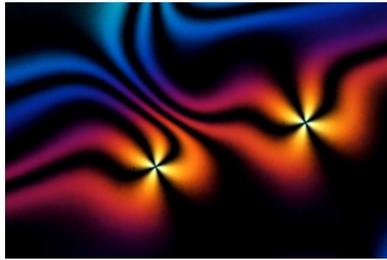


Figure 3.4: Pair of point defects, called *Boojums* in a nematic film¹.

Finally, Fanghua Lin [77] neglected the action of the vortex stretching term $-\Omega d$ to the corotational time flux \mathcal{N} , simplifying \mathcal{N} as the material derivative of the director

$$\mathcal{N} = \dot{d}.$$

Hence, the balance of angular momentum in the Ericksen-Leslie system (3.18) is equivalent to

$$-\gamma_1 \mathcal{N} + \lambda \Delta d + \beta d = -\gamma_1 \dot{d} + \lambda \Delta d + \beta d = 0, \quad (3.23)$$

with β always the Lagrangian multiplier related to the restriction $|d| = 1$. Observing that the dot-product between the director d and its material derivative is null

$$d \cdot \mathcal{N} = d \cdot \dot{d} = \frac{1}{2} \frac{D}{Dt} [|d|^2] = 0,$$

we can compute the value of β proceeding as in (3.22), i.e. $\beta = \lambda |\nabla d|^2$. Then equation (3.23) becomes

$$\gamma_1 (\partial_t d + u \cdot \nabla d) - \lambda \Delta d = \lambda |\nabla d|^2 d,$$

namely the balance of angular momentum of the simplified system (3.19).

The simplified system (3.19) is more handy than the original equations, yet it exhibits the main analytic challenges, as we will see in one of the main contributions of this thesis concerning a well-posedness result for this simplified version of Ericksen-Leslie.

3.3 The Beris-Edwards dynamic theory

In the previous chapter we have seen that the nematic phase is most naturally described by a director field d which returns value into the sphere. Moreover we have already remarked that the constituent molecules present a lack of polarity, on average, so that there is no physical difference between d and its opposite $-d$. This sign arbitrariness makes the Ericksen-Leslie theory presenting some deficiency on modelling some physical phenomena, especially the so-called *defects*.

Defects are among one of the most important visual patterns associated with nematic liquid crystals. Polarized light microscopy techniques are the simplest way to observe these physical phenomena, thanks to sudden and localised changes in the intensity of the light, as illustrated in Figure 3.4.

If defects occur in a nematic material, it means that the director d is not a smooth function in

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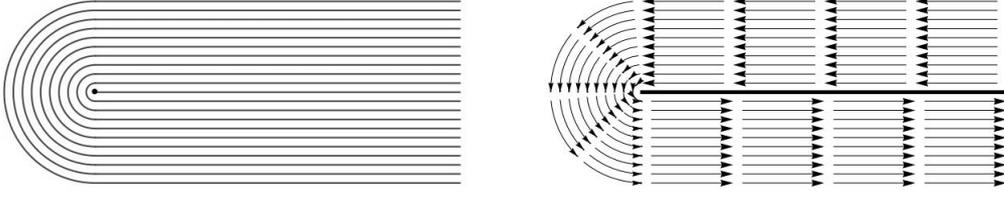


Figure 3.5: On the left-hand side a point disclination occurs around the molecular-orientation lines. On the right-hand side the choice of a direction generates a discontinuity on the director field as in the case of a line disclination.

space, and it presents discontinuities along the associated patterns. These discontinuities can be located at a point, a line or on a surface.

The most common defects are the point and the lines defects, known as *disclinations* (from the Frank terminology *disinclinations* [45]). Nevertheless, some liquid crystal samples may contain surface defects, known as sheet defects, which are physically unstable since they tend to smear out into continuously distorted regions of finite thicknesses, called *walls*. For more details concerning defects we refer the reader to the book by the Gennes and Prost [33].

For instance, a first deficiency that occurs in the director model is that of considering a point disinclination (see for instance [9]), as illustrated in figure Figure 3.5. The choice of a sign for the director d can generate discontinuities which have a no-physical relation with defects. Indeed figure Figure 3.5 is an example of a non-orientable field which is continuous in every point except on the defect. Thus, the natural idea of defining defects as discontinuities of the director field is not completely well-posed.

We present now an alternative approach describing the nematic dynamics, which also removes the orientation problem described above. This approach is based on the concept of order tensor, introduced by de Gennes [31, 33], in order to phrase a Landau-Ginzburg-type theory for nematic materials.

3.3.1 The de Gennes order tensor

Let us assume that our nematic material lies on a three-dimensional domain U at a time t . For any position x in U , instead of considering an exact direction on the sphere \mathbb{S}^2 , we establish the probability that the director field $d(t, x)$ belongs to some measurable subset \mathcal{A} of the sphere, as depicted in Figure 3.6. Then, we introduce a continuously distributed probability $\mathcal{P} = \mathcal{P}_{t,x}$ on \mathbb{S}^2 , driven by a density f through

$$\mathcal{P}(\mathcal{A}) = \int_{P \in \mathcal{A}} f(P) d\nu(P) =: \int_{\mathcal{A}} df(P),$$

where A is a Lebesgue measurable set on the unit sphere.

As always, we assume the constituent molecules to be unpolar on average. Then, the no-physical difference between the extremities can be expressed as a symmetric constriction on the probability:

$$\mathcal{P}(\mathcal{A}) = \mathcal{P}(-\mathcal{A})$$

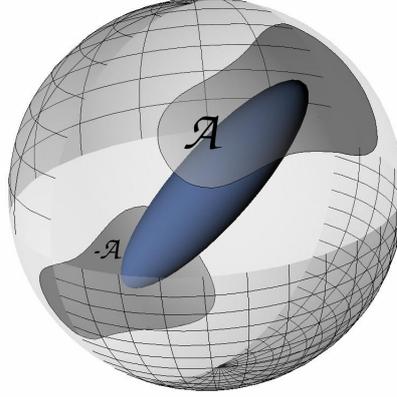


Figure 3.6: Instead of fixing a precise position into the sphere, the de Gennes theory considers the probability of the director to belong to a set \mathcal{A}

for any measurable subset \mathcal{A} . Note that because of this symmetry, the first momentum of the probability \mathcal{P} vanishes:

$$\int_{P \in \mathbb{S}^2} P df(P) = 0.$$

Thus the first nontrivial information comes from the second order momentum M , which is believed to be the most important contribution to the energy [80],

$$M := \int_{P \in \mathbb{S}^2} P \otimes P df(P) = \left(\int_{P \in \mathbb{S}^2} P_i P_j df(P) \right)_{i,j=1,2,3} \in \mathcal{M}_3(\mathbb{R}),$$

where $\mathcal{M}_3(\mathbb{R})$ denotes the 3×3 matrices with real coefficients. It yields a symmetric matrix M with trace $\text{tr}M = 1$.

In the presence of an isotropic liquid, the orientation of the molecules is uniform in every direction, hence in this case the probability \mathcal{P}_0 is given by

$$\mathcal{P}_0(\mathcal{A}) = \frac{1}{4\pi} \int_{P \in \mathcal{A}} d\nu(P),$$

so that the corresponding second momentum tensor M_0 is exactly $\text{Id}/3$. We denote by Q the difference between a general M and M_0

$$Q = M - \frac{\text{Id}}{3} = \int_{P \in \mathbb{S}^2} \left[P \otimes P - \frac{\text{Id}}{3} \right] df(P),$$

obtaining a tensor which is known as the *de Gennes order parameter tensor* or simply the *order tensor*. Thus we can interpret Q as a deviation of the second moment tensor from its isotropic value.

From the definition, it is clear that the order parameter tensor takes value in the space of three-dimensional matrices which are traceless and symmetric

$$S_0^{(3)} := \left\{ Q \in \mathcal{M}_3(\mathbb{R}), \quad Q = {}^t Q, \quad \text{tr}\{Q\} = 0 \right\}.$$

This is a five-dimensional Hilbert space, equipped with the inner-product induced by $\mathcal{M}_3(\mathbb{R})$:

$$\langle Q, \tilde{Q} \rangle_{S_0^{(3)}} := Q_{ij}\tilde{Q}_{ji} = Q_{ij}\tilde{Q}_{ij} = \text{tr}\{Q\tilde{Q}\},$$

for any tensors Q and \tilde{Q} in \mathfrak{S}_0 .

The order tensor Q is the new continuum variable modelling the nematic dynamics, replacing the director d . Then, it is perhaps important to recall its dependence on time and space $Q = Q(t, x)$.

An order tensor Q can describe three different types of structure, thanks to its eigenvalues:

- (i) If Q admits three equal eigenvalues, namely Q is null, then the resulting material is an isotropic fluid.
- (ii) If Q admits two equal eigenvalues, then the material is optically uniaxial with the distinguished eigen-direction as optic axis.
- (iii) If Q admits three distinct eigenvalues, then the material is biaxial.

The uniaxial order tensor Q is the most common in nematic materials. Moreover a polar decomposition leads these tensors to be expressed as follows:

$$Q = s \left(d \otimes d - \frac{\text{Id}}{3} \right), \quad \text{with } |d| = 1, \quad (3.24)$$

where d is exactly the director of the Ericksen-Leslie theory. Here the scalar quantity s is a measure of the degree of alignment of molecules.

We remark that the choice of (3.24) as new continuum variable allows us to overcome the defect-representation difficulty presented in Figure 3.5. Indeed, now the point disclination can easily be described as discontinuity of the order tensor parameter associated do the director d .

As already pointed out in the introduction, most of nematic liquid crystals are uniaxial. Nevertheless some of them are biaxial, presenting an additional orientation along a secondary axes. In this case the order tensor Q is of type (iii) and can be expressed as

$$Q = s_1 \left(d \otimes d - \frac{\text{Id}}{3} \right) + s_2 \left(m \otimes m - \frac{\text{Id}}{3} \right), \quad \text{with } |d| = 1, \quad |m| = 1 \quad \text{and} \quad s_1 \neq s_2.$$

The parameters s_1 and s_2 are not null, while d and m are two distinct eigenvectors with eigenvalues $(2s_1 - s_2)/3$ and $(2s_2 - s_1)/3$, respectively. Finally, the third eigenvalue is given by $-(s_1 + s_2)/3$.

De Gennes proposed two free-energy types governing nematic materials. The first one is a Landau-type potential, the *bulk energy*, a polynomial in the invariants of the order tensor Q , which dictates the preferred phase of the material. The second contribution is given by an elastic free-energy density in the form of a quadratic expression in the gradient of Q , describing any distortion of the structure of the material. The following sections describe the components of these energies.

3.3.2 The bulk free energy

The bulk free energy, also known as the *Landau-de Gennes thermotropic energy* [92], is a potential function describing which state the liquid crystal tends to be in, i.e. uniaxial, biaxial, or isotropic. The isotropic state $Q = 0$ should be minimizer at high temperature, while at low temperature there should appear uniaxial minimizers.

Up to fourth order, the most general form of the energy density is

$$\psi_B(Q) := \frac{a}{2}|Q|^2 - \frac{b}{3}\text{tr}\{Q^3\} + \frac{c}{4}|Q|^4, \quad (3.25)$$

where a, b and c are temperature and material dependent constants.

It is perhaps important to remark that this is a truncated Taylor expansion on $Q = 0$ [104]. Indeed the fourth order is the lowest that is suitable to describe a phase transition. Indeed it allows the free-energy density to have two distinct minima (for more details we refer the reader to [106]).

In order to have the bulk energy (3.25) bounded from below, c needs to be a positive constant. Moreover, for an uniaxial nematic, the order tensor assumes the form (3.24), thus by a direct calculation $\text{tr}\{Q^3\} = 2s^3/9$ which is positive assuming $s > 0$. By contrast, if the molecules are aligned on average perpendicular to the axis of symmetry then $\text{tr}\{Q^3\}$ is negative. In order to distinguish those two states b needs to be different from zero. Finally the constant a does not present a structural restriction.

3.3.3 The elastic free energy

The elastic energy density of a liquid crystal corresponds to the induced energy which occurs when distorting the order tensor in space. It is commonly taken to depend on Q and its gradient ∇Q . Moreover, as explained in [92], this energy must remain unchanged if we apply a rotation or a translation to the material. Thus, the second order elastic energy can involve only four invariant terms, that is

$$|\nabla Q|^2 = Q_{\alpha\beta,i}Q_{\alpha\beta,i}, \quad |\text{div } Q|^2 = Q_{\alpha i,i}Q_{\alpha j,j}, \quad Q_{\alpha i,j}Q_{\alpha j,i}, \quad Q_{ij}Q_{\alpha\beta,i}Q_{\alpha\beta,j},$$

and commonly it assumes the following form

$$\psi_e(Q, \nabla Q) := \frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2}|\text{div } Q|^2 + \frac{L_3}{2}Q_{\alpha i,j}Q_{\alpha j,i} + \frac{L_4}{2}Q_{ij}Q_{\alpha\beta,i}Q_{\alpha\beta,j}. \quad (3.26)$$

with L_1, L_2, L_3 and L_4 material-dependent constants, typically very small compared with the bulk constants a, b and c .

Usually, the constant L_4 is taken equal to zero, since it may cause the elastic energy to be not bounded from below (see for instance [8]), so that ψ_e is often written in the following form:

$$\psi_e(Q, \nabla Q) := \frac{L_1}{2}|\nabla Q|^2 + \frac{L_2}{2}|\text{div } Q|^2 + \frac{L_3}{2}Q_{\alpha i,j}Q_{\alpha j,i}.$$

However, the L_4 -term is necessary to distinguish splay and bend deformations [106], performing the full anisotropy characteristics of the Oseen-Frank energy density (3.9).

3.3.4 The Beris-Edwards system

There exist several models which describe the dynamics of liquid crystals making use of the Q order tensor. A comparison of these models can be found in [105] and in this section we present the so-called Beris-Edwards system.

This model was presented by Beris and Edwards in 1994 and it has been largely adopted in literature both for numerical and analytical studies ([1, 35, 54]). Their theory is derived formulating on a macroscopic scale the Poisson-bracket method, as extensively exposed in their book [12].

There is a strictly correlation between this model and the Ericksen-Leslie one: if we assume the order tensor to be uniaxial as in (3.24), the Beris-Edwards system coincides with the Ericksen-Leslie system.

The system models the evolution of liquid crystal molecules together with the underlying flow, through a parabolic-type system coupling an incompressible Navier-Stokes system with a nonlinear convection-diffusion system. The local orientation of the molecules is described through the order tensor Q . The evolution of the Q 's is driven by the free energy of the molecules as well as the transport, distortion and alignment effects caused by the flow. The flow field u satisfies a forced incompressible Navier-Stokes system, with the forcing provided by the additional, non-Newtonian stress caused by the molecules orientations, thus expressed in terms of Q .

The evolution of Q is given by:

$$\partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) = \Gamma H(Q), \quad (3.27)$$

with Γ a material-dependent positive constant. For a nonnegative velocity field u , the molecules are transported by the flow (as indicated by the convective derivative $\partial_t + u \cdot \nabla$) as well as being tumbled and aligned by the flow, fact described by the term

$$S(\nabla u, Q) := (\xi A + \Omega)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)(\xi A - \Omega) - 2\xi(Q + \frac{1}{3}Id)\text{tr}(Q\nabla u), \quad (3.28)$$

where $A = (\nabla u + {}^t\nabla u)/2$ is the rate of strain tensor and $\Omega = (\nabla u - {}^t\nabla u)/2$ is the vorticity tensor. The constant ξ is specific to the liquid crystal material and measures the ratio between the tumbling and the aligning effect that shear flow would exert over the liquid crystal director.

We also denote by H , the Q -dependent tensor

$$H(Q) := -\mathcal{L} \left[\frac{\partial \psi_B(Q)}{\partial Q} + \text{div} \frac{\partial \psi_e(Q)}{\partial \nabla Q} \right] \quad \text{with} \quad \left[\text{div} \frac{\partial \psi_e(Q)}{\partial \nabla Q} \right]_{\alpha\beta} = \sum_{\gamma=1}^3 \partial_\gamma \frac{\partial \psi_e(Q)}{\partial (\partial_\gamma Q_{\alpha\beta})},$$

where ψ_B and ψ_e are the bulk and elastic energy densities, formulated in (3.25) and (3.26) respectively, and denoting with \mathcal{L} the projection onto the space of trace-free matrices.

In this thesis we consider the most widespread elastic energy density, that is

$$\psi_e(\nabla Q) = \frac{L}{2} |\nabla Q|^2, \quad (3.29)$$

imposing $L := L_1$, and neglecting the other elastic coefficients L_2 , L_3 and L_4 . Hence H can be explicitly formulated as follows:

$$H(Q) = L\Delta Q - aQ + b \left(Q^2 - \text{tr}\{Q^2\} \frac{\text{Id}}{3} \right) - cQ\text{tr}\{Q^2\}. \quad (3.30)$$

The vector field u satisfies the forced incompressible Navier-Stokes system

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p &= \lambda \text{div} \{ \tau + \sigma \} \\ \text{div} u &= 0 \end{aligned}$$

where $\nu, \lambda > 0$ with λ measuring the ratio of the elastic effects (produced by the liquid crystal molecules) to that of the diffusive effects. The forcing is provided by the additional stress caused by the presence of the liquid crystal molecules, more specifically we have the symmetric part τ of

the additional stress tensor:

$$\tau := -\xi\left(Q + \frac{1}{3}Id\right)H(Q) - \xi H(Q)\left(Q + \frac{1}{3}Id\right) + 2\xi\left(Q + \frac{1}{3}Id\right)QH(Q) - L\nabla Q \odot \nabla Q, \quad (3.31)$$

where the tensor $\nabla Q \odot \nabla Q$ is defined as

$$(\nabla Q \odot \nabla Q)_{ij} := Q_{\alpha\beta,i}Q_{\alpha\beta,j} = \text{tr}\{\partial_i Q \partial_j Q\}.$$

Moreover the antisymmetric part σ of the stress tensor is given by

$$\sigma := QH(Q) - H(Q)Q = [Q, H(Q)]. \quad (3.32)$$

Summarizing, the Beris-Edwards equations read as follows

$$\left\{ \begin{array}{l} \partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) - \Gamma L \Delta Q = -aQ + b(Q^2 - \text{tr}\{Q^2\}\frac{Id}{3}) - cQ\text{tr}\{Q^2\}, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \lambda \text{div}\{QH(Q) - H(Q)Q\} + \\ \quad + \lambda \text{div}\left\{-\xi\left(Q + \frac{Id}{3}\right)H(Q) - \xi H(Q)\left(Q + \frac{Id}{3}\right) + 2\xi\left(Q + \frac{Id}{3}\right)\text{tr}\{QH(Q)\} - L\nabla Q \odot \nabla Q\right\}, \\ \text{div } u = 0. \end{array} \right. \quad (3.33)$$

3.4 The Qian-Sheng dynamic theory

We have seen that both Ericksen-Leslie (3.18) and Beris-Edwards (3.33) describe the continuum theory of nematic through a strongly coupled system between two parabolic-type equations. In the Ericksen-Leslie case, this structure is more clear when the simplified version (3.19) occurs. However this intrinsic characteristic is achieved neglecting a physical contribution, namely the inertia of the constituent molecules.

Let us come back to the work postulate (3.16) proposed by Ericksen and Leslie:

$$\int_U \rho(F \cdot u + K \cdot w) dx + \int_{\partial U} (\sigma \cdot u + l \cdot w) d\nu = \frac{D}{Dt} \int_U \left(\frac{1}{2} \rho |u|^2 + w_F \right) dx + \int_U \mathcal{D} dx,$$

The law explains how the rate at which force and moments do work is stored in the total-material energy or is lost in viscous dissipation. However, the above total energy considers only the kinetic and the Oseen-Frank energies contribution, meaning that we have neglected the action of the rotational kinetic energy

$$\frac{J}{2} \int_U |\dot{d}|^2.$$

Here J stands for the *moment of inertial density*. In most circumstances the rotational kinetic energy is negligible, imposing J to vanish, nevertheless it can play a non-trivial role when the anisotropic axis of the constituent molecules is subject to large accelerations. In this situation, the work-postulated must take in consideration the inertial contribution (see [70]) as follows:

$$\int_U \rho(F \cdot u + K \cdot w) dx + \int_{\partial U} (\sigma \cdot u + l \cdot w) d\nu = \frac{D}{Dt} \int_U \left(\frac{1}{2} \rho |u|^2 + J |\dot{d}|^2 + w_F \right) dx + \int_U \mathcal{D} dx.$$

In this case Ericksen-Leslie system becomes

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) = \operatorname{div} \sigma, \\ \operatorname{div} u = 0, \\ d \wedge (J \ddot{d} + \tilde{g} + h) = 0, \\ |d|^2 = 1, \end{cases}$$

hence the angular momentum equation change from a parabolic to a hyperbolic-type equation.

One of the main results of this thesis concerns the contribution of the inertial term, in the setting of the order parameter tensor. We proceed considering an alternative formalism to the one of Beris and Edwards. Indeed, in the 1998 Qian and Sheng [101] arrived at a system of evolution equations for both the velocity field and the order tensor field, taking inspiration with the analogy to the balance laws of Ericksen and Leslie.

The governing momentum and order evolution equations are

$$\dot{u}_i = \partial_j (-p \delta_{ji} + \sigma_{ji} + \sigma'_{ji}),$$

with the free-divergence condition $\operatorname{div} u = u_{k,k} = 0$, and

$$J \ddot{Q}_{ij} = h_{ij} + h'_{ij} - \lambda \delta_{ij} - \epsilon_{ijk} \lambda_k.$$

Here, the distortion stress tensor σ is given by

$$\sigma_{ji} := -\frac{\partial\psi_e}{\partial(Q_{\alpha\beta,j})}Q_{\alpha\beta,i}, \quad (3.34)$$

with ψ_e stands for the elastic energy

$$\psi_e(\nabla Q) := \frac{L}{2}|\nabla Q|^2. \quad (3.35)$$

Moreover, the elastic molecular tensor h is defined as

$$h_{ij} := -\frac{\partial\psi_B}{\partial Q_{ij}} + \partial_k \frac{\partial\psi_e}{\partial(Q_{ij,k})}$$

Furthermore the viscous stress σ' and the viscous molecular field h' are given by

$$\begin{aligned} \sigma'_{ij} &:= \beta_1 Q_{ij} Q_{lk} A_{lk} + \beta_4 A_{ij} + \beta_5 Q_{il} A_{lj} + \beta_6 Q_{jl} A_{li} \\ &\quad + \frac{1}{2} \mu_2 \mathcal{N}_{ij} - \mu_1 Q_{il} \mathcal{N}_{lj} + \mu_1 Q_{jl} \mathcal{N}_{li}, \\ h'_{ij} &:= -\frac{1}{2} \mu_2 A_{ij} - \mu_1 N_{ij} \end{aligned} \quad (3.36)$$

where $\beta_1, \beta_4, \beta_5, \beta_6, \mu_1$ and μ_2 are viscosity coefficients. A is the rate of strain tensor $(\nabla u + {}^t\nabla u)/2$ and \mathcal{N} is the time rate of change of Q_{ij} with respect to the background fluid angular velocity, whose formula is

$$\mathcal{N} = \dot{Q} - \Omega Q + Q \Omega,$$

where Ω is the vorticity tensor. The Qian and Sheng equations in a no-index form reads as follows:

$$\left\{ \begin{aligned} \partial_t u + u \cdot \nabla u - \frac{\beta_4}{2} \Delta u + \nabla p &= \operatorname{div} \{ -\nabla Q \odot \nabla Q + \beta_1 Q \operatorname{tr}\{QA\} + \\ &\quad + \beta_5 A Q + \beta_6 Q A \} + \operatorname{div} \{ \frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \}, \\ \operatorname{div} u &= 0, \\ J \ddot{Q} + \mu_1 (\dot{Q} - [\Omega, Q]) - \Delta Q &= -\mathcal{L} \frac{\partial\psi_B}{\partial Q} + \frac{\mu_2}{2} A. \end{aligned} \right. \quad (3.37)$$

Chapter 4

Contributions of the thesis

The present thesis is mainly devoted to the analysis of several partial differential equations arising from the dynamics of nematic liquid crystals. More precisely, we perform some well-posedness results and dynamic behaviours for the simplified Ericksen-Leslie system (3.19), the Beris-Edwards model (3.33) and the Qian-Sheng system (3.37), which we have introduced in the previous chapter.

In this chapter, we present an exhaustive overview about the achieved results. First, it is worth mentioning that one of the main contributions of this thesis concerns the study of the Cauchy problem associated to each system, up to suitable initial data. These initial data belong to suitable functional spaces, the properties of which make each model interesting also from a physical point of view. In the Ericksen-Leslie case, for instance, we consider an initial density that can present discontinuities along an interface, thus our result can be related to the dynamics of immiscible-nematics mixture.

This thesis studies the dynamics of nematic materials which occupy a two-dimensional or an N -dimensional space-domain, where N is an integer greater than three. More precisely we consider the Beris-Edwards system in the two-dimensional whole space, while the Ericksen-Leslie and the Qian-Sheng system are defined in the N -dimensional whole space.

On the one hand, we prove existence and uniqueness of weak or classical solutions in the two-dimensional case, making use of energy estimates, Fourier analysis and the paradifferential calculus. On the other, we prove well-posedness results in the N -dimensional framework, mainly adopting tools of harmonic analysis.

The thesis is interested also to the propagation of regularity for the initial data, especially under the dynamics proposed by Beris and Edwards. Here, we mainly use techniques which are mostly based on Fourier analysis and logarithmic-type estimates.

Let us briefly describe the structure of this chapter. In the next section we begin our presentation, starting from the Ericksen-Leslie model for nematics. In section 4.2 we take into consideration the dynamics proposed by Beris and Edwards. Here we separately consider two cases, on the one hand we analyze the general system, on the other we take into consideration a simplified version, the *co-rotational* system, which retains the main features of the original one. Finally, in section 4.3 we present the results concerning the Qian-Sheng model.

4.1 Contribution to the Ericksen-Leslie theory

As already exposed in the introduction, in the 50s and 60s, J.L. Ericksen [41] and F. Leslie [69] developed the most widely accepted model describing the dynamics of nematic liquid crystals. The

rod-like molecules have a local orientation, fact modelled mathematically through functions taking values into the space of directors that is the unit sphere. Their time evolution is described by an equation for unit-length vector fields d that are transported and rotated by a velocity field u . The velocity of the centers of masses of the molecules obeys an incompressible Navier-Stokes system with an additional stress tensor generated by the presence of the directors.

We consider the simplified version of the Ericksen-Leslie system introduced in section 3.2.5:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla \Pi = -\operatorname{div}(\nabla d \odot \nabla d) & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d & \mathbb{R}_+ \times \mathbb{R}^N, \\ |d| = 1 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u, \rho, d)|_{t=0} = (u_0, \rho_0, d_0) & \mathbb{R}^N. \end{array} \right. \quad (4.1)$$

This is a strongly coupled system between the inhomogeneous and incompressible Navier-Stokes equation and the transported heat flow of harmonic maps into sphere. We recall that the continuum variables are defined as follows:

$$\begin{aligned} \rho &= \rho(t, x) \in \mathbb{R}_+ && \text{denotes the density,} \\ u &= u(t, x) \in \mathbb{R}^N && \text{represents the velocity field,} \\ \Pi &= \Pi(t, x) \in \mathbb{R} && \text{is the pressure,} \\ d &= d(t, x) \in \mathbb{S}^{N-1} && \text{is the director,} \end{aligned}$$

all depending on the time variable $t \in \mathbb{R}_+$ and on the space variables $x \in \mathbb{R}^N$. We consider the whole space \mathbb{R}^N as space-domain, with a general dimension N greater or equal to two. The symbol $\nabla d \odot \nabla d$ denotes the $N \times N$ matrix whose (i, j) -th entry is given by

$$(\nabla d \odot \nabla d)_{ij} := \partial_i d \cdot \partial_j d = d_{k,i} d_{k,j},$$

for $i, j = 1, \dots, N$. The positive constants ν , λ and γ introduced in section 3.2.5 has been imposed equal to 1, for the sake of simplicity.

It is perhaps important to remark that the simplified system proposed by F. Lin is for homogeneous fluids, that is with constant density. The inhomogeneous version we consider here is relevant particularly for modelling a mixture of liquid crystals with different densities. Most importantly, perhaps it also allows to give an interpretation of defect patterns as discontinuities in the density. This defect interpretation is not new in literature, for instance we cite [34] of De Matteis and Virga, despite it concerns a density-dependent Korteweg fluid. Our main contribution is to prove a well-posedness result that allows for the existence of these discontinuities, since we deal with densities that are only bounded.

4.1.1 Statement of the main results

In order to announce our main result whose proof is given in chapter 5, it is perhaps important to define the functional spaces where our initial data are defined.

Our interest is to impose a regularity on the initial density ρ_0 as low as possible, so that we can allow discontinuities along an interface. Thus our initial density is only assumed to be bounded

$$\rho_0 \in L^\infty(\mathbb{R}^N).$$

Now, we can remark that the liquid crystal system (4.1) has a scaling property, like the classical Navier-Stokes equations. Namely, if (ρ, u, d) solves (4.1) with initial data (ρ_0, u_0, d_0) , then for every positive λ , the functions

$$(\rho, u, d)_\lambda(t, x) := (\rho(t, x), \lambda u(\lambda^2 t, \lambda x), d(\lambda^2 t, \lambda x))$$

also provide a solution but with initial data $(\rho_0(\lambda x), \lambda u_0(\lambda x), d_0(\lambda x))$. Thus, an important feature of system (4.1) is that of the gradient of the director field ∇d and the velocity field u have an equivalent “scaling behaviour”. Then, it is natural to consider u_0 and ∇d_0 into the same functional framework.

Moreover, other heuristics (extensively exposed in chapter 5) suggest that the most suitable functional spaces for $(u_0, \nabla d_0)$ are that of *homogeneous Besov* spaces with a critical index of regularity. More precisely, we assume

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^s.$$

for suitable integrability indexes p, r in $[1, \infty]$ and a critical regularity index s in \mathbb{R} . These functional spaces are defined through the so-called Littlewood-Paley theory (we refer the reader to section 5.3 for a complete description). It is perhaps interesting to remark that the homogeneous Besov spaces extend the most widely known classes of homogeneous Sobolev and Hölder spaces, through the relations

$$\dot{B}_{2,2}^s = \dot{H}^s, \quad \dot{B}_{\infty,\infty}^\sigma = \dot{C}^\sigma,$$

for any $s \in \mathbb{R}$ and for any $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$.

Then, our main results concerning the Ericksen-Leslie dynamic theory can be summarized in the following statement:

Theorem 4.1.1. *Let us assume that the initial data $(\rho_0, u_0, \nabla d_0)$ fulfil the smallness condition*

$$\left\| \frac{1}{\rho_0} - 1 \right\|_{L_x^\infty} + \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \leq c_0,$$

for a small enough positive constant c_0 , with $1 < p < N$ and $1 < r < \infty$, then system (4.1) admits weak solutions. If moreover we assume an extra tiny regularity on $(u_0, \nabla d_0)$, namely

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}, \tag{4.2}$$

for a small positive ε , then such solutions are unique.

In the above statement, there is no information about the functional space where our weak solutions live. Specific information and an extended version of Theorem 4.1.1 are postponed to chapter 5, for the sake of simplicity. Nevertheless, we anticipate that the functional setting where to look for weak solutions are of two types: if the initial data are smooth enough, we consider Lebesgue-type spaces $L^{q_1}(0, T; L^{q_2}(\mathbb{R}^N))$, for suitable r -dependent indexes q_1 and q_2 , while if the initial data exhibit low regularity, the functional space is that of the Serrin’s weighted in time class.

Moreover, it is perhaps worth mentioning that the extra regularity we impose in Theorem 4.1.1 plays a major part in proving the uniqueness result. Indeed, we handle the uniqueness part reformulating system (4.1) in Lagrangian coordinates, following the approach of Danchin and Mucha in [27] for the inhomogeneous Navier-Stokes equations. More precisely, the extra regularity allows to construct the flow map $\psi(t, x)$, i.e.

$$\psi(t, x) = x + \int_0^t u(s, \psi(s, x)) ds, .$$

The existence of such map is due to the Lipschitz space-regularity the velocity field u achieves, thanks to the extra condition on the initial data. The Lagrangian coordinates simplify in some way our problem, granting the density ρ to be constant, since it is governed by a transport equation.

4.2 Contribution to the Beris-Edwards theory

This thesis is also devoted to the Cauchy problem associated to the Beris-Edwards system. We consider the two dimensional whole space \mathbb{R}^2 as space-domain, so that the hydrodynamics of nematic materials is described by

$$\begin{cases} \partial_t Q + u \cdot \nabla Q + S(\nabla u, Q) = H(Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = \operatorname{div}\{\tau + \sigma\} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = 0 = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (4.3)$$

Here $Q = Q(t, x)$ is the order tensor introduced in section 3.3.1, $u = u(t, x) \in \mathbb{R}^2$ represents the velocity field, $\Pi = \Pi(t, x) \in \mathbb{R}$ stands for the pressure, everything depending on time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}^2$.

$H(Q)$ has been defined in (3.30), it is a tensor dependent on Q and ∇Q , and it is a forcing term related to the contribution given by the elastic and the bulk energies. Finally, τ and σ are the symmetric and antisymmetric part of the additional stress tensor respectively, whose formula has been proposed in (3.31) and (3.32).

It is perhaps interesting to remark that one can assume the system to have a three-dimensional target despite the two-dimensional domain. This situation occurs when physically there is no dependence on one of the three spacial directions.

As expressed in definition (3.28), the term $S(\nabla u, Q)$ depends also on a parameter ξ . This is a constant specific to the liquid crystal material. Usually ξ assumes small values, thus one can neglect its contribution. In this situation, system (4.3) reads as follows:

$$\begin{cases} \dot{Q} - \Omega Q + Q \Omega - \Gamma L^2 \Delta Q = -\Gamma(aQ - b(Q^2 - \operatorname{tr}\{Q^2\} \frac{\operatorname{Id}}{2}) + \operatorname{ctr}\{Q^2\}Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = L \operatorname{div}\{Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q\} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (4.4)$$

where the superposed dot is the material derivative $\partial_t + u \cdot \nabla$ and $\nabla Q \odot \nabla Q$ is a 2×2 -tensor, whose (i, j) -th component is given by

$$(\nabla Q \odot \nabla Q)_{ij} = \operatorname{tr}\{\partial_i Q \partial_j Q\} = \partial_i Q_{\alpha\beta} \partial_j Q_{\alpha\beta}.$$

We divide our study into two subcases:

- the corotational case, when ξ is null,
- the general case, when ξ is positive (or null).

We prove existence and uniqueness of weak solutions both for ξ different and equal to zero and we also prove a result about regularity propagation for the corotational case.

In order to understand the functional framework where to look for weak solutions, it is perhaps interesting to remark that the velocity field u and the gradient of the order tensor ∇Q have the same scaling behaviour, which is similar to the one of the Ericksen-Leslie system (3.19). If (u, Q) is solution for (4.3) with initial data (u_0, Q_0) , then

$$(u, Q)_\lambda(t, x) := (\lambda u(\lambda^2 t, \lambda x), Q(\lambda^2 t, \lambda x)), \quad \lambda > 0,$$

is still solution with new initial data $(\lambda u_0(\lambda x), Q_0(\lambda x))$. In the uniaxial nematic case, this scaling behaviour can be seen as a consequence of the Ericksen-Leslie's one. Indeed, here the order tensor Q assumes the form

$$Q(t, x) = s \left(d(t, x) \otimes d(t, x) - \frac{Id}{3} \right),$$

so that the scaling behaviour of Q turns out from the director field ones:

$$d_\lambda(t, x) := d(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

4.2.1 Statement of the main results

The corotational case

According to the previous considerations, it is natural to take our initial data u_0 and ∇Q_0 in the same functional space. Moreover, since we consider a two-dimensional space-domain, the more suitable type of weak solutions are the Leray's ones. Then the initial data are taken in Sobolev functional spaces, that is

$$(u_0, Q_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2).$$

Our well-posedness result can be summarized in the following theorem.

Theorem 4.2.1. *Assume that (u_0, Q_0) belongs to $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and let us suppose the parameter ξ to be null. Then system (4.4) admits an unique weak solution (u, Q) , which is global in time and it satisfies*

$$\begin{aligned} u &\in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^1(\mathbb{R}^2)) \cap L^2(0, T; \dot{H}^2(\mathbb{R}^2)), \end{aligned} \quad (4.5)$$

for any positive real time T .

We remark that, although the solution $(u, \nabla Q)$ is defined globally in time, the $L^2(\mathbb{R}^2)$ -norm (respectively $\dot{H}^1(\mathbb{R}^2)$ -norm) is only locally bounded (respectively $L^2_{loc}(\mathbb{R}_+)$). In the usual Navier-Stokes equations, classical energy estimates for Leray's weak solutions allow a global in time L^2 -integrability for the \dot{H}^1 -norm of u and a global in time bound for the $L^2(\mathbb{R}^2)$ -norm. In the Beris-Edwards system we lose this feature and this is mainly due to a structural difficulty.

The order tensor equation is driven by the variational derivative of the bulk energy density $\psi_B(Q)$, up to projection to the set of null trace matrices, that is

$$\mathcal{L} \frac{\partial \psi_b(Q)}{\partial Q} = -\Gamma \left(aQ - b(Q^2 - \text{tr}\{Q^2\} \frac{Id}{2}) + \text{ctr}\{Q^2\}Q \right).$$

When applying a classical energy estimate, the bulk energy occurs as additional term to the solution's norms. If the bulk energy could only interact as a positive term, then we would achieve the classical global control in time, however we can only expect the bulk density to be bounded from below (imposing the constant $c > 0$). This also requires to estimate the bulk energy contribution, however we will see that the best control one can achieve has an exponential grow in time.

The main contribution of Theorem 4.2.1 is about uniqueness, as the existence part was dealt by Paicu and Zarnescu in [99]. Nevertheless we approach also to the existence part with a different method to the author's ones: we make use of a coupled technique between the Friedrichs scheme and the Schaefer's fixed point theorem.

The main difficulties associated with treating the uniqueness of solutions for systems (4.4) are related to the presence of the momentum equation. One can essentially think of the system as a highly non-trivial perturbation of the Navier-Stokes equations. It is known that for Navier-Stokes alone, the uniqueness of weak solutions in dimension two can be achieved through rather standard arguments, while in three dimension it is a major open problem.

The extended systems that we deal with have an intermediary position, as the perturbation produced by the presence of the additional stress-tensor generates significant technical difficulties related in the first place to the weak norms available for the u term. A rather common way of dealing with this issue is by using a weak norm for estimating the difference between the two weak solutions, a norm that is below the natural spaces in which the weak solutions are defined. This approach was used before in the context of the usual Navier-Stokes system in [48] and [88]

In our case for technical convenience we use a homogeneous Sobolev space, namely $\dot{H}^{-1/2}$. The fact that the initial data for the difference is zero (i.e. $(\delta u, \delta Q)_{t=0} = 0$) helps in controlling the difference in such a low regularity space. Moreover, one of the main reasons for choosing the homogeneous setting is a specific product law which shows that the product is a bounded operator acting in the following spaces:

$$\dot{H}^s(\mathbb{R}^2) \times \dot{H}^t(\mathbb{R}^2) \rightarrow H^{s+t-\frac{1}{2}}(\mathbb{R}^2),$$

for any real s and t with $|s|, |t| < 1$ such that $s+t$ is positive. It is worth remarking that evaluating the difference at regularity level $s = 0$ i.e. in $L^2(\mathbb{R}^2)$, would only allow to prove a weak-strong uniqueness result, along the lines of [99]. Working in a negative Sobolev space, \dot{H}^s with $s \in (-1, 0)$ allows to capture the uniqueness of weak solutions. We expect that a similar proof would work in any \dot{H}^s with $s \in (-1, 0)$ and our choice $s = -1/2$ is just for convenience.

In the uniqueness proof, our main work is to obtain a Gronwall-type inequality. Indeed the uniqueness reduces to an estimate of the following type:

$$\Phi'(t) \leq \chi(t)\Phi(t),$$

where $\Phi(t)$ is the norm of the difference between two solutions and χ is a-priori in $L^1_{loc}(\mathbb{R}_+)$.

In addition there are some difficulties that are specific to this system. These are of two different types, being related to:

- Controlling the “extraneous” maximal derivatives: that is the highest derivatives in u that appear in the Q equation and the highest derivatives in Q that appear in the u equation,
- Controlling the high powers of Q , such as $Q \text{tr}\{Q^2\}$ in the bulk energy contribution.

The first difficulty is dealt with by taking into account the specific feature of the coupling that allows for the *cancellation of the worst terms*, when considering certain physically meaningful combination of terms. For what concerns the second difficulty, this is overcome by delicate harmonic analysis arguments leading to the usual Gronwall inequality mentioned before.

Another important contribution of this thesis in the corotational setting is a result about regularity propagation. We consider our initial data to belong to a nonhomogenous Sobolev space with

positive index of regularity, i.e.

$$(u_0, \nabla Q_0) \in \dot{H}^s(\mathbb{R}^2), \quad \text{with } s > 0.$$

Then we study the propagation of the homogeneous Sobolev norm \dot{H}^s . The first step is rather standard, namely we consider an \dot{H}^s -energy estimate. However, considering $s > 0$ leads to losing an important structural feature of the system, namely the cancellation of the worst terms. Thus, it is necessary to make a control for each of them, but doing that, a first problem appears: the control of the $L^\infty(\mathbb{R}^d)$ -norm for the solution $(u, \nabla Q)$. A rather common way of dealing with this problem is by using a classical Sobolev embedding

$$H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2),$$

but this requires an index of regularity $s > 1$, so that proceeding along this line, we can expect to propagate only higher regularities, in the same line of [99]. This difficulty is dealt with separately controlling the high and the low frequencies of our solution. We localise the Fourier transform of our solution into a ball with a suitable radius N , making use of a cut-off function. Then on the one-hand we control the $L^\infty(\mathbb{R}^2)$ -norm of our solution localised in the low frequencies, and on the other-hand we estimate the higher ones. Then, choosing a suitable radius N , the regularity propagation reduces to a delicate one-logarithmic estimate

$$\Phi'(t) \leq \chi(t)\Phi(t)\left(1 + \ln \Phi(t)\right),$$

where $\Phi(t)$ stands for the \dot{H}^s -norms of our solution and χ is a priori in $L^1_{loc}(\mathbb{R}_+)$. This estimate leads to a double-exponential control of the growth-in-time of $\Phi(t)$. Then our regularity-propagations result reads as follows:

Theorem 4.2.2. *If ξ is null and (u_0, Q_0) belongs to $H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, for a positive real s , then the solution given by Theorem 4.2.1 fulfils*

$$\begin{aligned} u &\in L^\infty(0, T; H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^{s+1}(\mathbb{R}^2)) \cap L^2(0, T; \dot{H}^{s+2}(\mathbb{R}^2)). \end{aligned}$$

The general case

In the general framework, i.e. when the parameter ξ is assumed to be greater or equal to zero, we performed existence and uniqueness of weak solutions. This result can be resumed in the following statement:

Theorem 4.2.3. *Assume that (u_0, Q_0) belongs to $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and let the parameter ξ be positive. Then system (4.3) admits an unique weak solution (u, Q) , which is global in time and it satisfies*

$$\begin{aligned} u &\in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\ Q &\in C(\mathbb{R}_+, H^1(\mathbb{R}^2)) \cap L^2(0, T; \dot{H}^2(\mathbb{R}^2)), \end{aligned}$$

for any positive real time T .

Despite the simplicity of the statement, the proof requires a deep analysis with a specific study of the terms appearing when ξ is not null.

Let us remark that also in this case the main contribution of Theorem 4.2.3 is about uniqueness, as the existence part is just a fairly straightforward revisit of the arguments in [98].

The presence of the ξ -terms increases the main difficulties associated with treating the uniqueness of solutions for systems (4.3). For instance we have to control the high powers of Q interacting with u such as $Q\text{tr}\{\nabla u Q\}$. Although we proceed along the same line of the corotational case, by using the weak norm $\dot{H}^{-1/2}$ for estimating the difference between the two weak solutions, now our main goal is to obtain a delicate-double logarithmic estimate that leads to the Osgood lemma. Indeed the uniqueness reduces to an estimate of the following type:

$$\Phi'(t) \leq \chi(t)\Phi(t) \left(1 + \ln \left(1 + e + \frac{1}{\Phi(t)} \right) + \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \ln \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \right),$$

where $\Phi(t)$ stands for the $\dot{H}^{-1/2}$ -norms of our solution and χ is a priori in $L^1_{loc}(\mathbb{R}_+)$. The proof is based on a specific harmonic technique, localising the Fourier transform of our solution with a ball of radius N and separately estimating both the low frequencies and the high frequencies.

In these estimates, we make use of suitable Sobolev embeddings and we explicitly write the constant of embedding for each of them. Then, a specific relation between these constants, the radius N and $\Phi(t)$ finally allows to achieve the cited double-logarithmic estimate.

4.3 Contribution to the Qian-Sheng theory

As a last contribution to the dynamics of liquid crystals, this thesis is also devoted to the well-posedness of the Qian-Sheng system in a d -dimensional setting, for $d = 2, 3$.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \text{div}\{\sigma + \sigma'\} & \mathbb{R}_+ \times \mathbb{R}^d, \\ \text{div } u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ J\ddot{Q} + \mu_1(\dot{Q} - [\Omega, Q]) - \Delta Q = -\mathcal{L}\frac{\partial\psi_B}{\partial Q} + \mu_2 A & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^d. \end{cases} \quad (4.6)$$

Here, the continuum variables are the same of the Beris-Edwards system: $u \in \mathbb{R}^d$ is the velocity field, $Q \in \mathbb{R}^{d \times d}$ is the order tensor, the scalar p stands for the pressure, A is the rate of strain tensor and Ω is the vorticity tensor. The bulk energy density $\psi_B(Q)$ has been defined in (3.25), while the stress tensor σ and σ' are in (3.34) and (3.36). Everything depends on the time variable $t \in \mathbb{R}_+$ and the space variables $x \in \mathbb{R}^d$.

It is worth recalling that the most important novelty in this model is the inertial contribution $J\ddot{Q}$ in the order tensor equation. This term comes from when considering the contribution of the rotational kinetic energy density, namely

$$\frac{J}{2}|\dot{Q}|^2.$$

Usually this energy density is negligible, since the inertial density J commonly assumes very tiny values. Taking into consideration this term makes system (4.6) more challenging than the model proposed by Beris and Edwards. Indeed the order tensor equation becomes a hyperbolic-type equation, in contrast to the parabolic one occurring when $J = 0$. Thus, we can not expect any kind of *regularizing effects* for the order tensor Q .

Our first result concerns a dissipative behaviour related to system (4.6). More precisely we prove the existence of a Lyapunov functional $E(t) = E(u(t), Q(t))$, which correspond to the total energy of the material, that is the kinetic energy, the bulk energy and the rotational kinetic energy. The statement reads as follows

Theorem 4.3.1. *Under some restriction on the viscosity coefficients (see Theorem 8.1.1), system (4.6) admits the following Lyapunov functional:*

$$E(t) := \int_{\mathbb{R}^d} \frac{1}{2} \left(|u|^2 + J|\dot{Q}|^2 + \frac{1}{2}|\nabla Q|^2 + \psi_B(Q) \right)$$

with $\psi_B(Q)$, the bulk energy density (3.25).

Considering smooth solutions, we prove that the time derivative of E does not assume positive value:

$$\frac{d}{dt} E(t) \leq 0, \quad \text{for any } t \in \mathbb{R}_+.$$

It is worth remarking that we can not construct weak solutions making use of this Lyapunov functional, as in the case of Beris-Edwards. This is due to the hyperbolic structure of the order tensor equation, which does not allow regularizing effects on Q . More precisely the highest number of derivative in Q we can control in $L^2(\mathbb{R}^d)$ is one, namely $\|\nabla Q\|_{L^2(\mathbb{R}^d)}$. This leads to some difficulties when constructing weak solutions, mainly due to the nonlinear forcing term

$$(\nabla Q \odot \nabla Q)_{ij} = \partial_i Q_{\alpha\beta} \partial_j Q_{\alpha\beta},$$

in the stress tensor σ . Indeed, the most common way to construct weak solutions is by a compactness method, so that this nonlinear term should present some troubles when considering its weak-limit convergence.

Then, our second result concerns the existence and uniqueness of *classical* solutions for system (4.6). We work in a $H^s(\mathbb{R}^d)$ -Sobolev spaces, for a sufficient large positive s . Our statement reads as follows:

Theorem 4.3.2. *Let us assume $(u_0, Q_0) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}$ to belong to $H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$ with s greater than $d/2$. Up to suitable conditions on the viscosity coefficients (see Theorem 8.1.2) and supposing the initial norms $\|u_0\|_{H^s}$ and $\|Q_0\|_{H^{s+1}}$ to be small enough, then there exists a unique solution (v, Q) of system (4.6). This solution is global in time and it fulfils*

$$\begin{aligned} u &\in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)), \\ Q &\in L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)) \quad \text{with} \quad \dot{Q} \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)). \end{aligned}$$

In the existence part, thanks to a Friedrichs-type scheme, we construct approximate solutions which satisfy uniform estimates. These estimate turn out from a rather standard H^s -energy control of the equations.

The reader should observe that in the above theorem, we impose a smallness condition on the initial data (u_0, Q_0) also in the two-dimensional setting $d = 2$. It is known that in the simplest case of the Navier-Stokes equations, existence and uniqueness of classical and weak solutions are achieved for any initial data, while in the case of a three dimensional setting it is still a major problem. Our system has an intermediate difficulty, because of the presence of the forcing terms related to the order tensor Q . In a $L^2(\mathbb{R}^2)$ -energy level these terms are handled by specific features of the coupling which allow their cancellation. However, in a H^s -setting this characteristic is lost and we need to estimate each forcing term related to the stress tensors.

Our main work on proving the existence of classical solutions is to obtain an uniform estimate for our approximate solutions, that is

$$\Phi'(t) + \Psi(t) \leq C\Phi(t)\Psi(t), \tag{4.7}$$

where C is a suitable positive constant, Φ is the bounded in time H^s -norms of our solution and Ψ is the integrable in time H^s -norms. Then, a rather standard argument (see Lemma 8.5.1) allows to propagate the smallness condition on the initial data (i.e. on $\Phi(0)$). This leads the right-hand side of the above equation to be absorbed by the left-hand side, which allow to have an uniform control of the norms. Then we can construct our classical solution, through a compactness method.

The uniqueness of our solutions is proven evaluating the difference between two solutions at a regularity level $s = 0$, i.e. in $L^2(\mathbb{R}^2)$. Our work is mainly to obtain an estimate that leads to the Gronwall lemma. Here the main difficulties are handled taking into account a specific feature of the coupling system related to the difference of the two solutions. This feature allows the cancellation of the worst term when considering certain physically meaningful combinations.

4.4 Contribution to the Boussinesq theory

Although this introduction and the main results we have exposed until now concern the dynamics of liquid crystals, this thesis is also devoted to the so-called Boussinesq system. In chapter 9 we study the evolution of a fluid presenting a temperature-dependent viscosity, the hydrodynamics of which is modelled by the following system:

$$\begin{cases} \partial_t \theta + \operatorname{div}(\theta u) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta)D(u)) + \nabla \Pi = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \theta)|_{t=0} = (\bar{u}, \bar{\theta}) & \mathbb{R}^d, \end{cases} \quad (4.8)$$

where d is the dimension, θ is the temperature, u stands for the velocity, Π is the pressure and finally $D(u)$ is the rate of strain tensor $D(u) := (\nabla u + {}^t\nabla u)/2$, all depending on the time variable $t \in \mathbb{R}_+$ and on the space variables $x \in \mathbb{R}^d$.

This is a strongly coupled system between a transport equation, governing the evolution of the temperature, and the homogeneous Navier-Stokes equations modelling the velocity of the fluid. The temperature-dependent viscosity $\nu(\theta)$ leads to a specific sub-case of the general Boussinesq system (we refer the reader to chapter 9 for more details). This condition allows us to describe several geophysical phenomena, for instance it is well known that the viscosity of a fluids tends to decrease (or, alternatively, its fluidity tends to increase) as its temperature increases.

Moreover, following the approach used for the Ericksen-Lelie system, our interest is to consider an initial temperature with discontinuities along an interface, so that our model assumes also a physical meaning when describing a mixture of immiscible fluids with different temperatures.

Our main result concerns the existence of global weak solutions for system (4.8) under specific and natural conditions on the initial data. Before stating our main result, let us briefly describe some features of system (4.8).

As the classical Navier-Stokes equations, system (4.8) admits a particular scaling behaviour: if $(u(t, x), \theta(t, x))$ solves (4.8) with initial data $(\bar{u}(x), \bar{\theta}(x))$, then the following rescaled functions

$$(u, \theta)_\lambda(t, x) := (\lambda u(\lambda^2 t, \lambda x), \theta(\lambda^2 t, \lambda x)), \quad \lambda > 0$$

are still solutions of (4.8) with $(\lambda \bar{u}(\lambda x), \lambda \bar{\theta}(x))$ as new initial data. Then, it is natural to consider initial data in Banach spaces whose norms are invariant under the above scaling behaviour. First

we assume $\bar{\theta}$ only to be bounded, that is

$$\bar{\theta} \in L^\infty(\mathbb{R}^d),$$

whose low regularity allows the initial temperature to present discontinuities. Then, we consider an initial velocity in an homogeneous Besov space with critical index of regularity, i.e.

$$u_0 \in \dot{B}_{p,r}^{\frac{d}{p}-1}(\mathbb{R}^d),$$

for suitable p and r in $[1, \infty]$.

We prove the existence of weak solutions for system (4.8) under a specific smallness condition on the initial data, that is when

$$\eta := \left(\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} \right) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\} \quad (4.9)$$

is assumed to be sufficiently small, where $\bar{u}^h := (\bar{u}^1, \dots, \bar{u}^{d-1})$ are the horizontal coordinates of the velocity field, while \bar{u}^d stands for the vertical one. Let us analyse this type of smallness condition. First the viscosity is supposed close to a positive constant, and we impose it equal to 1 for the sake of simplicity. Moreover the initial velocity field can present a large vertical component \bar{u}^d , if the initial fluctuation of the viscosity $\nu(\theta_0) - 1$ and the horizontal components of the initial velocity \bar{u}^h are small enough (exponentially small when compared to the vertical component \bar{u}^d). This fact is described by the exponential term in (4.9), which can assume large values, multiplied by a small enough quantity in order to have η small enough. This type of initial condition is not new in literature, for instance we cite [60] and [30], devoted to the well-posedness of the inhomogeneous Navier-Stokes equations.

When proving the existence of weak solutions, we follow an approach similar to the one used for the Ericksen-Leslie system, considering a useful characterization of the homogeneous Besov spaces and the maximal regularity theorem for the heat kernel. Nevertheless, there are some difficulties that are specific to system (4.8):

- the anisotropic smallness condition, which requires some particular new ideas and technical tools when controlling the norms of our solutions,
- the low regularity of the viscosity, which decreases the maximal number of derivative on u we can control.

The anisotropic smallness condition (4.9) requires to separately estimate the horizontal coordinates u^h and the vertical coordinate u^d of the velocity field. At first, it is worth remarking that the algebraical structure of the Navier-Stokes system plays a main role when propagating the bound for the norms of our solutions.

Using the divergence free condition and the special algebraical structure of the non-linear term, we notice that the equation on the vertical component is a linear parabolic equation whose coefficients depends on the horizontal components. This yields the vertical component to not require any type of smallness condition on \bar{u}^d .

As second step, we analyse the equation on the horizontal coordinates together with the underlying non-linear terms. These are of two types: on the one hand there are bilinear terms in the horizontal coordinates, on the other hand there are coupling terms as interactions between the horizontal components and the vertical one. Because of this non-linearity, it is necessary a smallness condition on the horizontal coordinates of the velocity field in order to solve the equation. Moreover, the

contribution given by the vertical coordinate leads to an exponential amplification of this smallness condition, so that the bound we achieve has the same structure of η in (4.9).

Now, the assumption on the initial temperature does not allow $\nu(\theta)$ to exhibit enough regularity in order to read $\operatorname{div}(\nu(\theta)\mathcal{M})$ in a strong sense. Then, writing the equation on u in a mild formulation, the previous term leads to

$$\int_0^t \operatorname{div} S(t-s)((\nu(\theta) - 1)\mathcal{M})(s) ds,$$

where S is the heat kernel. This means that we already need a regularizing effect of the heat kernel when controlling the velocity field u . Because of this heuristics we deduce that the maximal regularity theorem permits an estimate only of u and ∇u in a $L^{q_1}(0, T; L^{q_2}(\mathbb{R}^d))$ -Lebesgue space, while Δu assumes only a distributional sense.

Our main result concerning the Boussinesq theorem can be summarized in the following statement:

Theorem 4.4.1. *Let us assume our initial data $(\bar{\theta}, \bar{u})$ to belong to $L^\infty(\mathbb{R}^d) \times \dot{B}_{p,r}^{d/p-1}(\mathbb{R}^d)$ with p in $(1, d)$ and r in $(1, \infty)$. Supposing*

$$\eta := (\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\} \leq c_0$$

for a small positive constant c_0 , then system (4.8) admits a weak solutions which is global in time. Moreover, denoting by \mathfrak{X} the Banach space where the components of the velocity field live, we have the following control:

$$\|(u^h, \nabla u^h)\|_{\mathfrak{X}} \leq C_1 \eta, \quad \|(u^d, \nabla u^d)\|_{\mathfrak{X}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3,$$

for some suitable positive constants C_1, C_2 and C_3 .

The functional framework \mathfrak{X} where we look for our solutions is a Lebesgue-type functional space with specific weight-in-time (we refer the reader to chapter 9, Theorem 9.1.3 and Theorem 9.1.4, for a complete description).

Chapter 5

Ericksen-Leslie system

In this chapter we present the results of the following manuscript:

F. De Anna, *Global solvability of the inhomogeneous Ericksen-Leslie system with only bounded density*, submitted (2015)

5.1 Introduction and main results

In the 50s and 60s, J.L. Ericksen [41] and F. Leslie [69] developed the most widely accepted model describing the dynamics of nematic liquid crystals. This is a material exhibiting a state of matter between the ordinary isotropic liquids and the solids. Its main characteristic feature is that the rod-like molecules have a local orientation, fact modelled mathematically through functions taking values into the space of “directors” that is the unit sphere. Their time evolution is described by an equation for unit-length vector fields that are transported and rotated by a velocity field. The velocity of the centers of masses of the molecules obeys an incompressible Navier-Stokes system with an additional stress tensor generated by the presence of the directors.

The equations are particularly complicated as they contain a large number of terms, so there are few mathematical studies of the full systems (for more details see next section). Nevertheless the main mathematical difficulties are still present in the following simplified model in the whole space, a version of which was originally proposed by F. Lin in [74, 75, 80]:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \nu \Delta u + \nabla \Pi = -\lambda \operatorname{div}(\nabla d \odot \nabla d) & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t d + u \cdot \nabla d - \gamma \Delta d = \gamma |\nabla d|^2 d & \mathbb{R}_+ \times \mathbb{R}^N, \\ |d| = 1 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u, \rho, d)|_{t=0} = (u_0, \rho_0, d_0) & \mathbb{R}^N. \end{array} \right. \quad (5.1)$$

This is a strongly coupled system between the inhomogeneous and incompressible Navier-Stokes equation and the transported heat flow of harmonic maps into sphere. Here

$\rho = \rho(t, x) \in \mathbb{R}_+$ denotes the density, $u = u(t, x) \in \mathbb{R}^N$ represents the velocity field,
 $\Pi = \Pi(t, x) \in \mathbb{R}$ is the pressure and $d = d(t, x) \in \mathbb{S}^{N-1}$ is the director

all depending on the time variable $t \in \mathbb{R}_+$ and on the space variables $x \in \mathbb{R}^N$, with $N \geq 2$. The symbol $\nabla d \odot \nabla d$ denotes the $N \times N$ matrix whose (i, j) -th entry is given by $\partial_i d \cdot \partial_j d$, for

$i, j = 1, \dots, N$. The positive constants ν , λ and γ stand for the viscosity, the competition between kinetic energy and potential energy and respectively the microscopic elastic relaxation time for the molecular orientation field.

Let us note that the original Ericksen-Leslie as well as the simplification proposed by F. Lin are for homogenous fluids. The inhomogeneous version we consider here is relevant particularly for modelling a mixture of liquid crystals with different densities. Most importantly perhaps it also allows to give an interpretation of defect patterns as discontinuities in the density and our main contribution is to prove a well-posedness result that allows for the existence of these discontinuities, since we deal with densities that are only bounded.

A presentation of the previous literature is provided in the next subsection, while the last subsection of the introduction contains a non-technical exposition of the main results.

5.1.1 A short review of some results on the nematic liquid crystal theory

We present here a short (and of course incomplete) overview of the literature, referring the reader to the recent review of F. Lin and C. Wang [82] for more details on current developments in the hydrodynamics of nematic liquid crystals.

The well-posedness of the general Ericksen-Leslie equations was studied by Fanghua Lin and Chun Liu in [79] where they proved the existence of weak and strong solutions under certain specific restrictions. The results were further improved by Wu, Xu and Liu in [117].

The homogeneous version of system (5.1) was introduced by Fanghua Lin (see [74, 75, 80]) as a simplification (see the appendix in [77]) of the Ericksen-Leslie equations. The simplified system (5.1) is simpler than the original Ericksen-Leslie equations, yet it exhibits the main analytic challenges of the original system.

The homogeneous case

In [77] F. Lin and C. Liu proposed the homogeneous version of the system (5.1), in which they replaced the most challenging term $|\nabla d|^2 d$ with one given by $f(d) = \nabla F(d)$. Their motivation was to simplify the nonlinear term $|\nabla d|^2 d$ in the director equation, the term which encodes the constraint $d \in \mathbb{S}^{N-1}$. This corresponds to a Ginzburg-Landau type of penalisation for the unit-length constraint, by setting for instance $F(d) = \frac{(1-|d|^2)^2}{4\varepsilon^2}$ where ε is a positive small parameter.

They studied the wellposedness of the system, establishing the following basic energy law:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\|u\|^2 + \lambda \|\nabla d\|^2 + 2\lambda F(d)) dx = - \int_{\Omega} (\nu \|\nabla u\|^2 + \lambda \gamma \|\Delta d - f(d)\|^2) dx.$$

Then, with a modified Galerkin method, they were able to prove the existence of a weak solution. They also obtained uniqueness in the two dimensional case and, for large enough viscosity, also in the three dimensional case. Furthermore they proved a stability result for the equilibria.

Later, in [78] F. Lin and C. Liu obtained a partial regularity result for the system, showing that a suitable weak solution has the potentially singular set of one-dimensional Hausdorff measure zero in space-time.

Afterwards, the more challenging case when one keeps the $d|\nabla d|^2$ term, and works with unit-length vector fields d , was first considered in F. Lin, J. Lin and C. Wang in [76]. They assumed two-dimensional domains, but allowed for the director to take values in $3D$ that is in \mathbb{S}^2 . They proved both interior and boundary regularity theorems under a smallness condition, which allowed to

obtain the existence of global weak solutions on a bounded smooth domain. Such solutions were shown to be smooth except possibly for a finite set of times.

More recently, in [83], F. Lin and C. Wang established the existence of a global in time weak solution for the homogeneous version of (5.1), in the three dimensional setting. In here the initial condition u_0 was taken to satisfy the natural assumptions, namely free divergence and in $L^2(\Omega, \mathbb{R}^3)$, while the initial director field d_0 was assumed to be in $H^1(\Omega, \mathbb{R}^3)$. However, they made a special restriction on d_0 by assuming that d_0 has a positive vertical component $d_0^3(x) > 0$, for a.e $x \in \Omega$.

In [81] F. Lin and C. Wang developed some uniqueness results. In the two dimensional case they proved that uniqueness holds provided that u belongs to the classical energy space $L_t^\infty L_x^2 \cap L_t^2 H_x^1$, the gradient of the pressure $\nabla \Pi$ is in $L_t^{4/3} L_x^{4/3}$ and the director field d is in $L_t^\infty \dot{H}_x^1 \cap L_t^2 \dot{H}_x^2$. In the three dimensional case they proved a similar result under stronger assumptions, namely $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1 \cap C([0, T], L_x^N)$, $\Pi \in L_t^{N/2} L_x^{N/2}$ and $d \in L_t^2 \dot{H}_x^1 \cap C([0, T], \dot{W}^{1, N})$, where \dot{H}_x^1 and $\dot{W}^{1, N}$ denote the classical homogeneous Sobolev spaces on \mathbb{R}^N .

Recently, Hieber, Nesensohn, Prüss and Schade [55] proposed an alternative approach, namely to consider the system as a quasilinear parabolic equation, proving the existence and uniqueness of strong solutions on a maximal time interval. They also showed that the equilibria are normally stable, i.e. for an initial data close to equilibria, there exists a global solution which converges exponentially in time to an equilibrium. Moreover they proved the analytic regularity of their solutions.

The inhomogeneous case

In [63], Jiang and Tan considered the system (5.1) on a three dimensional bounded domain with $f(d) = \nabla F(d)$ instead of $|\nabla d|^2 d$. They derived the global existence of weak solutions assuming that the boundary is smooth enough, and a specific relation between the initial density and the initial velocity holds, namely $u_0 = 0$ whenever $\rho_0 = 0$ and $|u_0|^2/\rho_0 \in L^1(\Omega)$.

In [118], Zhou, Fan and Nakamura established an existence and uniqueness result for the two dimensional inhomogeneous system (5.1) on a smooth bounded domain, for arbitrary initial velocity u_0 and small ∇d_0 in L^2 and initial density ρ_0 . We emphasise that in here the initial density ρ_0 was assumed to be sufficiently smooth, namely in $W^{1, r}(\Omega)$, with $r \in (2, \infty)$.

In [72], J. Li considered the system (5.1) on a bounded domain Ω in two or three dimension, with Dirichlet boundary conditions. Assuming regularity on the initial density, namely $\rho_0 \in H^1(\Omega)$, Li proved the existence of a unique global strong solution provided the data are small in the L_x^2 -setting.

In the compressible case results were obtained in [114], by D. Wang and Yu, and also [61] by F. Jiang, S. Jiang and D. Wang.

Statement of the main results

At first, let us observe that system (5.1) contains (formally imposing the molecular orientation field to be constant) the incompressible inhomogeneous Navier-Stokes equations, thus we cannot expect to obtain better results than those of this sub-system. We mention the paper of Huang, Paicu, and Zhang [60] where the authors established the existence and uniqueness of solutions in the whole space as well as the paper of Danchin and Zhang [30] where similar results are obtained in the half-space setting. In this work we aim to extend their results to the liquid crystal framework.

We immediately observe that the presence of the additional equations, for the director field, requires a significant update of the strategy used in the inhomogeneous Navier-Stokes setting. Indeed now we consider a coupled system between the inhomogeneous Navier-Stokes equation

and the director field equation. The director field equation is not only a classical parabolic-type equation (which could be dealt with as in [60] and [30]), but it is a transported heat flow of harmonic maps into the sphere, namely d solves a parabolic-type equation and is constrained to belong to \mathbb{S}^{N-1} . This restriction generates a challenging nonlinear term, namely $|\nabla d|^2 d$, which is known to be capable of generating finite-time singularities (for the equation where $u = 0$).

For instance, although it is natural to assume that u and ∇d belong to the same functional space (because of their scaling, see Remark 5.1.1 below), the presence of such a nonlinear term will require additional control on ∇d . Moreover this term will also generate difficulties when we will construct approximate solutions (see for instance Remark 5.2.3). Another significant challenge, specific to the system, and which is not present in the inhomogeneous Navier-Stokes equation, will appear in the proof of the uniqueness result. Here we will use a non-standard Lagrangian formulation, in which we will formally take the d and ∇d as independent unknowns, see the beginning of Section 5.6.

There is no loss of generality in taking the constant viscosity $\nu = 1$ in (5.1). Similarly we impose the constants λ and γ to be 1, for the convenience of the reader. Thanks to the properties of the transport equation, for smooth enough u we have that if $\rho_0 > 0$ then this will hold for all times and thus we can define $a := 1/\rho - 1$ and reformulate the system (5.1) as

$$\left\{ \begin{array}{ll} \partial_t a + \operatorname{div}(au) = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t u + u \cdot \nabla u + (1+a)\{\nabla \Pi - \Delta u\} = -(1+a)\operatorname{div}(\nabla d \odot \nabla d) & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d & \mathbb{R}_+ \times \mathbb{R}^N, \\ |d| = 1 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u, a, d)|_{t=0} = (u_0, a_0, d_0) & \mathbb{R}^N, \end{array} \right. \quad (5.2)$$

Remark 5.1.1. *The liquid crystal system (5.2) has a scaling property, like the classical Navier-Stokes. Namely, if $(a, u, d, \nabla \Pi)$ solves (5.2) with initial data (a_0, u_0, d_0) , then for every positive λ , the functions*

$$(a, u, d, \nabla \Pi)_\lambda := (a(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), d(\lambda^2 t, \lambda x), \lambda^2 \nabla \Pi(\lambda^2 t, \lambda x))$$

also provide a solution but with initial data $(a_0(\lambda x), \lambda u_0(\lambda x), d_0(\lambda x))$.

It is thus natural to consider the initial data in a Banach space which has a norm invariant under the previous scaling. Moreover, we note that ∇d_0 has the same scaling as u_0 , thus it is natural to take them in the same functional space. An example of scaling-invariant space is $(a_0, u_0, \nabla d_0) \in L_x^\infty \times \dot{B}_{p,r}^{N/p-1} \times \dot{B}_{p,r}^{N/p-1}$, where $\dot{B}_{p,r}^{N/p-1}$ stands for the homogeneous Besov space (see the next section for more details and for the definition of Besov spaces).

We are going to consider an initial data of this type and we note that the case of bounded density allows discontinuities along an interface. This is important from a physical point of view as it can describe a mixture of liquid crystal materials with different densities and it is also relevant to defect patterns, when interpreting defects as discontinuities in density.

In this work we will consider initial data of the following type:

$$a_0 \in L_x^\infty, \quad (u_0, \nabla d_0) \in \dot{B}_{p,r}^{\frac{N}{p}-1} \quad \text{with} \quad d_0 : \mathbb{R}^N \rightarrow \mathbb{S}^{N-1} \quad \text{and} \quad \operatorname{div} u_0 = 0, \quad (5.3)$$

where $\dot{B}_{p,r}^{N/p-1}$ is the critical homogeneous Besov space, with indexes $1 < r < \infty$ and $1 < p < N$.

From now on we suppose that our initial data verifies the following smallness condition:

$$\eta := \|a_0\|_{L_x^\infty} + \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \leq c_0, \quad (5.4)$$

where c_0 is a positive constant, small enough.

Remark 5.1.2. *We will be working with weak solutions, as these are the only ones compatible with the initial density being only L^∞ .*

However, let us note that our weak solutions are constructed in critical scaling invariant spaces which is the classical framework for strong solutions for classical Navier-Stokes system.

In our situation the smallness conditions on the initial data are natural as one can check in the papers ([26], [60] and [100]), concerning the inhomogeneous Navier-Stokes equations. Indeed this case is unlike the Leray solutions for the homogeneous Navier-Stokes system in the 2D or 3D situation where no smallness conditions is required on the weak solution but the solution is constructed in the L_x^2 setting which is the space of the conserved energy.

Our main results (Theorem 5.2.2, Theorem 5.2.4, Theorem 5.2.6 and Theorem 5.2.7) can be summarized in the following statement:

Theorem 5.1.3. *Let us assume that the initial data $(a_0, u_0, \nabla d_0)$ fulfil the smallness condition (5.4), with $1 < p < N$ and $1 < r < \infty$, then system (5.2) admits weak solutions. If moreover we assume an extra tiny regularity on $(u_0, \nabla d_0)$, namely*

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}, \quad (5.5)$$

then such solutions are unique.

Let us now briefly describe the structure of this work. In the next section we state the technical versions of the previously mentioned result. In section 5.3 we briefly recall some properties and characterizations of Besov spaces, and prove in detail some technical Lemmas and Theorems concerning the regularizing effects of the heat kernel.

In the section 5.4 we prove the existence of solutions for system (5.2) with initial data more regular than in (5.3). Such results will play an important role in the proofs of our main results, both for the existence part (with regularized initial data), and the uniqueness part (allowing to reformulate (5.2) in Lagrangian coordinates). Section 5.5 is devoted to the proof of the existence part, split into two cases, namely Theorem 5.2.2 and Theorem 5.2.4. Regularizing the initial data we construct a sequence of approximate solutions and we pass to the limit thanks to some uniform estimates.

In section 5.6 and section 5.7 we present the uniqueness results. We impose a little bit more regularity on the initial data, which allows us to obtain the Lagrangian coordinates. Thus, we are able to prove the uniqueness of the solution for system (5.2) on a small initial time-interval. Then we conclude by a bootstrap method, obtaining the uniqueness in two different cases, namely Theorem 5.2.6 and 5.2.7. Finally in section 5.8 we prove, for the convenience of the reader, some technical results which are useful in the main proofs.

5.2 Main results

As in the case of the inhomogeneous Navier-Stokes equation, we can not assume u with a better regularity than $\tilde{L}_t^1 \dot{B}_{p,r}^{N/p+1}$ (see [7] for a complete explanation of such space). Hence, the product

$a\Delta u_L$ between L_x^∞ and $\dot{B}_{p,r}^{N/p-1}$ assumes a distributional sense only if $p < N$, where u_L is the free solution of the heat equation with u_0 as initial datum. This explains the restriction for p . If the index r is supposed to be equal to 1 then we expect to obtain a velocity field to be in $L_t^1 \mathcal{L}ip_x$ (where $L_t^1 \mathcal{L}ip_x$ stands for $L^1(\mathbb{R}_+, \mathcal{L}ip(\mathbb{R}^N))$) which is very useful to solve the transport equation on the density by Lagrangian coordinates. Our condition $r > 1$ is general enough to include the case of non-Lipschitz velocity field.

Before introducing our main Theorems, let us explain the meaning of weak solution for system (5.2).

Definition 5.2.1. *(a, u, d) is a weak solution of (5.2) if $|d| = 1$ almost everywhere and*

▷ *for any test function $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ the following equalities are well-defined and fulfilled:*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^N} a(t, x) (\partial_t \phi(t, x) + u(t, x) \cdot \nabla \phi(t, x)) dt dx + \int_{\mathbb{R}^N} a_0(x) \phi(0, x) dx = 0$$

$$\int_{\mathbb{R}^N} u \cdot \nabla \phi = 0.$$

▷ *for any vector valued function $\Phi = (\Phi_1, \dots, \Phi_N) \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$ the following identities are well-defined and satisfied:*

$$\int_{\mathbb{R}_+ \times \mathbb{R}^N} u \cdot \partial_t \Phi - \{u \cdot \nabla u + (1+a)[\nabla \Pi - \Delta u + \operatorname{div}\{\nabla d \odot \nabla d\}]\} \cdot \Phi + \int_{\mathbb{R}^N} u_0 \cdot \Phi(0, \cdot) = 0,$$

$$\int_{\mathbb{R}_+ \times \mathbb{R}^N} d \cdot \partial_t \Phi - \{u \cdot \nabla d - \Delta d - |\nabla d|^2 d\} \cdot \Phi + \int_{\mathbb{R}^N} d_0 \cdot \Phi(0, \cdot) = 0.$$

The functional framework: the smooth case

The maximal regularity Theorem (see Theorem 5.3.2) and the characterization of the homogeneous Besov spaces (see Theorem 5.3.11) play an important role for the study of (5.2), since we can reformulate the second and the third equations of (5.2) in the following integral form:

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \{ -u \cdot \nabla u - (1+a)\nabla \Pi + a\Delta u - (1+a)\operatorname{div}(\nabla d \odot \nabla d) \}(s) ds,$$

$$d(t) = e^{t\Delta} d_0 + \int_0^t e^{(t-s)\Delta} \{ -u \cdot \nabla d + |\nabla d|^2 d \}(s) ds.$$

It is reasonable to suppose the solution having the same regularity as for the linear heat equation given by the heat kernel convoluted with the initial data. Moreover, due to the low regularity of the initial density, which is supposed to be a general bounded function, the transport equation on the density forces us to suppose a only bounded. Finally, the classical maximal regularizing effect for heat kernel (see Theorem 5.3.2) suggests us to look for a solution in a $L_t^r L_x^q$ setting. Now in the simpler case where u just solves the heat equation with initial data u_0 , having Δu in $L_t^r L_x^q$ is equivalent to $u_0 \in \dot{B}_{q,r}^{N/q-1}$ on the condition $N/q - 1 = 2 - 2/r$ (see Corollary 5.3.11.1). From the immersion $\dot{B}_{p,r}^{N/p-1} \hookrightarrow \dot{B}_{q,r}^{N/q-1}$ for every $q \geq p$, we understand that this strategy requires $p \leq Nr/(3r - 2)$. Furthermore, since the velocity field u may be seen as solution of the Stokes system

$$\partial_t u - \Delta u + \nabla \Pi = -u \cdot \nabla u + a(\Delta u - \nabla \Pi) - \operatorname{div}\{\nabla d \odot \nabla d\}, \quad \operatorname{div} u = 0,$$

it turns out that

$$\|(\partial_t u, \nabla^2 u, \nabla \Pi)\|_{L_t^r L_x^q} \lesssim \|u_0\|_{\dot{B}_{q,r}^{\frac{N}{p}-1}} + \|(u \cdot \nabla u, a \Delta u, a \nabla \Pi, (1+a) \operatorname{div}\{\nabla d \odot \nabla d\})\|_{L_t^r L_x^q},$$

where we have applied the characterization of the homogeneous Besov Space (Corollary (5.3.11.1) and the Maximal-Regularity Theorem (Theorem 5.3.2). Here the first relation we expect between the regularities of u and d , namely $u \cdot \nabla u = \operatorname{div}(u \otimes u)$, Δu and $\operatorname{div}\{\nabla d \odot \nabla d\}$ to be in the same $L_t^r L_x^q$ space. Thus, according to the previous remark, it is natural to look for a solution such that u and ∇d fulfill the same functional properties and this explains why we suppose ∇d_0 and u_0 belonging to the same critical Besov space $\dot{B}_{p,r}^{N/p-1}$.

According to the above heuristics, imposing $q = Nr/(3r - 2)$, we aim to find a solution in the following space: $a, d \in L_{t,x}^\infty$ and $(u, \nabla d, \nabla \Pi) \in \mathfrak{X}_{r,T}$, where

$$\mathfrak{X}_{r,T} := \left\{ (u, \nabla d, \nabla \Pi) \quad \text{with} \quad \nabla d \in L_t^{3r} L_x^{\frac{3Nr}{3r-2}}, \quad (u, \nabla d) \in L_T^{2r} L_x^{\frac{Nr}{r-1}}, \right. \\ \left. \nabla(u, \nabla d) \in L_T^{2r} L_x^{\frac{Nr}{2r-1}} \cap L_t^r L_x^{\frac{2Nr}{r-1}}, \quad (\nabla^2 u, \nabla^3 d, \nabla \Pi) \in L_T^r L_x^{\frac{Nr}{3r-2}} \right\}.$$

We also define the following norm

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{X}_{r,T}} := \|\nabla d\|_{L_T^{3r} L_x^{\frac{3Nr}{3r-2}}} + \|\nabla(u, \nabla d)\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla(u, \nabla d)\|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}} + \\ + \|(u, \nabla d)\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|(\nabla^2 u, \nabla^3 d, \nabla \Pi)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}},$$

and impose $\mathfrak{X}_r = \mathfrak{X}_{r,\infty}$. Thus, our first result reads as follows:

Theorem 5.2.2. *Let $1 < r < 2$ and $p \in (1, Nr/(3r - 2)]$. Suppose that the initial data (a_0, u_0, d_0) are determined by (5.3). There exists a positive constant c_0 such that, if*

$$\eta := \|a_0\|_{L_x^\infty} + \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \leq c_0,$$

is fulfilled, then there exists a global weak solution $(a, u, d, \nabla \Pi)$ of (5.2), such that $(u, \nabla d, \nabla \Pi) \in \mathfrak{X}_r$, $(u, \nabla d) \in L_t^2 L_x^\infty$ and $a, d \in L_{t,x}^\infty$. Furthermore $\|a\|_{L_{t,x}^\infty} \leq \|a_0\|_{L_x^\infty}$ and the following inequality is satisfied:

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{X}_r} + \|(u, \nabla d)\|_{L_t^2 L_x^\infty} \lesssim \eta. \quad (5.6)$$

Remark 5.2.3. *In this first theorem we have supposed the constriction $1 < r < 2$. To explain this condition, we anticipate that the proof will be based on an iterate scheme which includes the following one:*

$$\partial_t d^n - \Delta d^n = |\nabla d^{n-1}|^2 d^{n-1} - u^{n-1} \cdot \nabla d^{n-1}.$$

The condition $(u^n, \nabla d^n)_\mathbb{N} \subset \mathfrak{X}_r$ is not enough to easily control the $L_{t,x}^\infty$ -norm of $(d^n)_\mathbb{N}$ (moreover the structure of the iterate scheme doesn't yields $d^n \equiv 1$ almost everywhere). Thus, we have added the condition $(u^n, \nabla d^n)_\mathbb{N} \subset L_t^2 L_x^\infty$ which requires $(u_0, \nabla d_0) \in \dot{B}_{\infty,2}^{-1}$. We conjecture that such restriction is not necessary, however we have imposed it to simplify the proof for the reader. Indeed the case $r \geq 2$ is treated in our second result, Theorem 5.2.4.

The functional framework: the general case

As we have already pointed out, the choice of a $L_t^r L_x^q$ functional setting requires the condition $p \leq Nr/(3r - 2)$. The more general case $1 < p < N$ can be handled by the addition of a

weight in time. Indeed the simpler case where u just solves the heat equation with initial data u_0 , having $u_0 \in \dot{B}_{p_1, \bar{r}}^{N/p_1-1}$ for some $p_1 \in [p, N)$ and $\bar{r} \in [r, \infty]$ is equivalent to impose $t^{1/2(3-N/p_1)} \Delta u$ in $L^{\bar{r}}(\mathbb{R}_+, dt/t)$. Hence, with similar heuristics proposed in the first case, adding such weights in time, we aim to find a solution in the following functional space: $a, d \in L_{t,x}^\infty$ and $(u, \nabla d, \nabla \Pi) \in \mathfrak{Y}_{r,T}$, where $\mathfrak{Y}_{r,T} = \mathfrak{Y}_{r,T}(p_1, p_2, p_3)$ is the set of $(u, \nabla d, \nabla \Pi)$ such that

$$\left\{ \begin{aligned} t^{\gamma_1}(u, \nabla d) &\in L_t^{2r} L_x^{p_3}, & t^{\gamma_2}(u, \nabla d) &\in L_t^\infty L_x^{p_3}, & t^{\gamma_3} \nabla d &\in L_t^{2r} L_x^{3p_1}, & t^{\gamma_4} \nabla d &\in L_t^\infty L_x^{3p_1}, \\ t^{\beta_1} \nabla(u, \nabla d) &\in L_t^{2r} L_x^{p_2}, & t^{\beta_2} \nabla(u, \nabla d) &\in L_t^\infty L_x^{p_2}, & t^{\beta_3} \nabla(u, \nabla d) &\in L_t^{2r} L_x^{\frac{p_3}{2}}, \\ t^{\beta_4} \nabla(u, \nabla d) &\in L_t^\infty L_x^{\frac{p_3}{2}}, & t^{\alpha_1}(\nabla^2 u, \nabla^3 d, \nabla \Pi) &\in L_t^{2r} L_x^{p_1}, & t^{\alpha_2}(\nabla^2 u, \nabla^3 d, \nabla \Pi) &\in L_t^r L_x^{p_1} \end{aligned} \right\}.$$

where we have fixed $\max\{p, Nr/(2r-1)\} < p_1 < N$, $Nr/(r-1) < p_3 \leq \infty$ and p_2 such that $1/p_1 = 1/p_2 + 1/p_3$. Furthermore, the weight in time exponents are defined by

$$\begin{aligned} \alpha_1 &:= \frac{1}{2} \left(3 - \frac{N}{p_1} \right) - \frac{1}{2r}, & \beta_1 &:= \frac{1}{2} \left(2 - \frac{N}{p_2} \right) - \frac{1}{2r}, & \gamma_1 &:= \frac{1}{2} \left(1 - \frac{N}{p_3} \right) - \frac{1}{2r}, & \gamma_2 &:= \frac{1}{2} \left(1 - \frac{N}{p_3} \right), \\ \alpha_2 &:= \frac{1}{2} \left(3 - \frac{N}{p_1} \right) - \frac{1}{r}, & \beta_2 &:= \frac{1}{2} \left(2 - \frac{N}{p_2} \right), & \gamma_3 &:= \frac{1}{2} \left(1 - \frac{N}{3p_1} \right) - \frac{1}{2r}, & \gamma_4 &:= \frac{1}{2} \left(1 - \frac{N}{3p_1} \right), \\ & & \beta_3 &:= \frac{1}{2} \left(2 - \frac{2N}{p_3} \right) - \frac{1}{2r}, & \beta_4 &:= \frac{1}{2} \left(2 - \frac{2N}{p_3} \right), \end{aligned}$$

We also denote by $\|(u, d, \nabla \Pi)\|_{\mathfrak{Y}_{r,T}}$ the following norm:

$$\begin{aligned} &\|t^{\beta_1} \nabla(u, \nabla d)\|_{L_T^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla(u, \nabla d)\|_{L_T^\infty L_x^{p_2}} + \|t^{\beta_3} \nabla(u, \nabla d)\|_{L_T^{2r} L_x^{\frac{p_3}{2}}} + \\ &+ \|t^{\beta_4} \nabla(u, \nabla d)\|_{L_T^\infty L_x^{\frac{p_3}{2}}} + \|t^{\gamma_1}(u, \nabla d)\|_{L_T^{2r} L_x^{p_3}} + \|t^{\gamma_2}(u, \nabla d)\|_{L_T^\infty L_x^{p_3}} + \|t^{\gamma_3} \nabla d\|_{L_T^{2r} L_x^{3p_1}} + \\ &+ \|t^{\gamma_4} \nabla d\|_{L_T^\infty L_x^{3p_1}} + \|t^{\alpha_1}(\nabla^2 u, \nabla \Pi)\|_{L_T^{2r} L_x^{p_1}} + \|t^{\alpha_2}(\nabla^2 u, \nabla^3 d, \nabla \Pi)\|_{L_T^r L_x^{p_1}} \end{aligned}$$

and impose $\mathfrak{Y}_r := \mathfrak{Y}_{r,\infty}$. Hence our second and more general result concerning the existence of a solution reads as follows:

Theorem 5.2.4. *Let $1 < r < \infty$ and $p \in (1, N)$. Suppose that the initial data (a_0, u_0, d_0) are determined by (5.3). There exists a positive constant c_0 such that, if*

$$\eta := \|a_0\|_{L_x^\infty} + \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \leq c_0,$$

is fulfilled, then there exists a global weak-solution (a, u, d) of (5.2), such that $(a, d) \in L_{t,x}^\infty$ and $(u, \nabla d, \nabla \Pi)$ belongs to \mathfrak{Y}_r . Furthermore $\|a\|_{L_{t,x}^\infty} \leq \|a_0\|_{L_x^\infty}$, $|d(t, x)| = 1$ for almost every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$ and the following inequality is satisfied:

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{Y}_r} \lesssim \eta.$$

Uniqueness

In order to recover the uniqueness of the constructed global weak-solutions, we need to add an extra regularity on the initial data for the velocity field and the director field. Namely we add to (5.3) the following hypotheses

$$(u_0, \nabla d_0) \in \dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon},$$

for a sufficient small positive constant ε . With this extra-regularity, we are able to obtain the velocity field u to be in $L_{t,loc}^1 \mathcal{L}ip_x$. This allows us to reformulate system (5.1) in Lagrangian

coordinates. Such coordinates simplify in some way our problem, granting the density a to be constant, since it is governed by a transport equation. Therefore, we proceed in the same line of [60], proving the uniqueness of the constructed solutions for a initial time interval. Thus we conclude by a bootstrap method in order to recover the global uniqueness.

First, let us introduce the functional frameworks. Fixing the value of ε in $(0, 1)$ and assuming the constriction $r < 2/(2 - \varepsilon)$, we define the following space:

$$\mathfrak{X}_r^\varepsilon := \left\{ (u, \nabla d, \nabla \Pi) \quad \text{with} \quad \begin{aligned} \nabla d &\in L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}, & \nabla(u, \nabla d) &\in L_t^{\frac{2}{2-\varepsilon}} L_x^\infty, \\ \nabla(u, \nabla d) &\in L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}, & (\nabla^2 u, \nabla^3 d, \nabla \Pi) &\in L_t^r L_x^{\frac{Nr}{(3-\varepsilon)r-2}} \end{aligned} \right\}.$$

Remark 5.2.5. *Let us immediately remark that for $r < 2/(2 - \varepsilon)$ we get $Nr/((3 - \varepsilon)r - 2) > N$. Thus, it will be possible to apply the Sobolev inequality in order to get the velocity field to be Lipschitz in space, which plays an important role, as we have already mentioned.*

The first uniqueness result of this article reads as follows:

Theorem 5.2.6. *Let ε be a positive constant in $(0, 1)$. Suppose that the hypotheses of Theorem 5.2.2 are satisfied with $r < 2/(2 - \varepsilon)$ and let (a, u, d) be the solution generated. Let us assume that $(u_0, \nabla d_0)$ also belongs to $\dot{B}_{N,p}^{N/p-1+\varepsilon}$, then we have $(u, \nabla d, \nabla \Pi) \in \mathfrak{X}_r^\varepsilon$ with*

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{X}_r^\varepsilon} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{2}{3}\varepsilon}}.$$

and the uniqueness holds in this functional framework.

As we have already exposed, the results of Theorem 5.2.2 and 5.2.6 require the constriction $1 < p \leq Nr/(3r - 2)$. Here, the existence and the uniqueness hold in a setting of type $L_t^r L_x^q$. However, to recover the uniqueness for the general case $1 < p < N$, we need again to add a weight in time. More precisely, fixing $q_1 \in (N, N/(1 - \varepsilon))$, $q_3 > Nr/((1 - \varepsilon)r - 1)$ and imposing q_2 such that $1/q_1 = 1/q_2 + 1/q_3$, we define the space:

$$\mathfrak{Y}_{r,T}^\varepsilon = \mathfrak{Y}_{r,T}^\varepsilon(q_1, q_2, q_3) := \left\{ (u, \nabla d, \nabla \Pi) \quad \text{with} \quad \begin{aligned} t^{\gamma_1^\varepsilon}(u, \nabla d) &\in L_t^{2r} L_x^{q_3}, \\ t^{\gamma_2^\varepsilon}(u, \nabla d) &\in L_t^\infty L_x^{q_3}, \quad t^{\gamma_3^\varepsilon} \nabla d \in L_t^{2r} L_x^{3q_1}, \quad t^{\gamma_4^\varepsilon} \nabla d \in L_t^\infty L_x^{3q_1}, \quad t^{\beta_1^\varepsilon} \nabla(u, \nabla d) \in L_t^{2r} L_x^{q_2}, \\ t^{\beta_2^\varepsilon} \nabla(u, \nabla d) &\in L_t^\infty L_x^{q_2}, \quad t^{\alpha_1^\varepsilon}(\nabla^2 u, \nabla^3 d, \nabla \Pi) \in L_t^{2r} L_x^{q_1}, \quad t^{\alpha_2^\varepsilon}(\nabla^2 u, \nabla^3 d, \nabla \Pi) \in L_t^r L_x^{q_1} \end{aligned} \right\},$$

where the exponents of the weights in time are defined by

$$\begin{aligned} \alpha_1^\varepsilon &:= \frac{1}{2} \left(3 - \frac{N}{q_1} - \varepsilon \right) - \frac{1}{2r}, & \beta_1^\varepsilon &:= \frac{1}{2} \left(2 - \frac{N}{q_2} - \varepsilon \right) - \frac{1}{2r}, & \gamma_1^\varepsilon &:= \frac{1}{2} \left(1 - \frac{N}{q_3} - \varepsilon \right) - \frac{1}{2r}, \\ \alpha_2^\varepsilon &:= \frac{1}{2} \left(3 - \frac{N}{q_1} - \varepsilon \right) - \frac{1}{r}, & \beta_2^\varepsilon &:= \frac{1}{2} \left(2 - \frac{N}{q_2} - \varepsilon \right), & \gamma_2^\varepsilon &:= \frac{1}{2} \left(1 - \frac{N}{q_3} - \varepsilon \right), \\ \gamma_3^\varepsilon &:= \frac{1}{2} \left(1 - \frac{N}{3q_1} - \varepsilon \right) - \frac{1}{2r} & \gamma_4^\varepsilon &:= \frac{1}{2} \left(1 - \frac{N}{3q_1} - \varepsilon \right). \end{aligned}$$

Therefore, our main uniqueness result reads as follows:

Theorem 5.2.7. *Let us assume that the hypotheses of Theorem 5.2.4 are satisfied and suppose also that $(u_0, \nabla d_0) \in \dot{B}_{p,r}^{N/p-1+\varepsilon}$ for a positive ε bounded by $\min\{1/r, 1 - 1/r, N/p - 1\}$. Then the solution (a, u, d) determined by Theorem 5.2.4 fulfills also $(u, \nabla d, \nabla \Pi) \in \mathfrak{Y}_r^\varepsilon$ and we have*

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{Y}_r^\varepsilon} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}}.$$

Moreover, such solution is unique in this functional framework.

Remark 5.2.8. *Let us observe that if $(a, u, d, \nabla \Pi)$ is a weak solution of (5.2), then, $\omega := (|d|^2 - 1)/2$ is a weak solution of the following heat equation with a linear perturbation and with null initial datum:*

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + |\nabla d|^2 \omega - \Delta \omega = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \omega = 0 & \mathbb{R}^N. \end{cases}$$

Since the unique solution of such system is $\omega \equiv 0$, then the condition $|d(t, x)| = 1$ almost everywhere, is already determined by the fourth equation of (5.2). Hence, from here on, (5.2) is going to be analyzed without $|d| = 1$ which is already given by the remaining equations of the system. Such feature is not preserved by the approximate systems (5.13) and (5.14) of (5.2), hence in the third section we cannot utilize $|d^n(t, x)| = 1$ almost everywhere.

5.3 Preliminaries

This section is devoted to the study of several regularizing effects for the heat kernel, which will be useful for the proof of the main theorems. At first step let us recall the well-known Hardy-Littlewood-Sobolev inequality (see [7]. Theorem 1.7).

Theorem 5.3.1 (Hardy-Littlewood-Sobolev inequality). *Let f belongs to L_x^p , with $1 < p < \infty$, $\alpha \in]0, N[$ and suppose $r \in]0, \infty[$ satisfies*

$$\frac{1}{p} + \frac{\alpha}{N} = 1 + \frac{1}{r}.$$

Then

$$|\cdot|^{-\alpha} * f \in L_x^r,$$

and there exists a positive constant C such that

$$\| |\cdot|^{-\alpha} * f \|_{L_x^r} \leq C \|f\|_{L_x^p}.$$

A direct consequence is the following corollary (see [66], Theorem 2.4)

Corollary 5.3.1.1. *Let f belongs to L_x^p , with $1 < p < N$ and let $(\sqrt{-\Delta})^{-1}$ be the Riesz potential, defined by*

$$(\sqrt{-\Delta})^{-1} f := \mathcal{F}^{-1} \left(\frac{1}{|\xi|} \hat{f} \right).$$

Then $(\sqrt{-\Delta})^{-1}$ is a bounded operator from L_x^p to $L_x^{p^*}$ with $p^* = pN/(N - p)$

Let us now enunciate the well-known $L^p L^q$ -Maximal Regularity Theorem, whose proof is available in [66].

Theorem 5.3.2 (Maximal $L_t^p L_x^q$ regularity for the heat kernel). *Let $T \in (0, \infty]$, $1 < p, q < \infty$ and $f \in L^p(0, T; L_x^q)$. Let the operator \mathcal{A} be defined by*

$$\mathcal{A}f(t, \cdot) := \int_0^t \Delta e^{(t-s)\Delta} f(s, \cdot) ds.$$

Then \mathcal{A} is a bounded operator from $L^p(0, T; L_x^q)$ to $L^p(0, T; L_x^q)$.

Lemma 5.3.3. *Let $T \in (0, \infty]$, $1 < r_1, r_2 < \infty$, $q_1 \in [1, \infty]$ and $q_2 \in [q_1, \infty]$, such that*

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) < \frac{1}{2}. \quad (5.7)$$

Let the operator \mathcal{B} be defined by

$$\mathcal{B}f(t, \cdot) := \int_0^t \nabla e^{(t-s)\Delta} f(s, \cdot) ds.$$

Then, we have that \mathcal{B} is a bounded operator from $L^{r_1}(0, T; L_x^{q_1})$ with values to $L^{r_2}(0, T; L_x^{q_2})$, if the following equality is fulfilled:

$$\frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} + \frac{1}{r_2}. \quad (5.8)$$

Proof. At first let us observe that, if K denotes the heat kernel, than for all $1 \leq \lambda \leq \infty$ we have

$$\|\nabla K(t, \cdot)\|_{L_x^\lambda} = \frac{1}{t^{\frac{d}{2} \frac{1}{\lambda} + \frac{1}{2}}} \|K(1, \cdot)\|_{L_x^\lambda}. \quad (5.9)$$

Observe that, for every $t \in \mathbb{R}_+$,

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s) ds \right\|_{L_x^{q_2}} \leq \int_0^t \|\nabla K(t-s, \cdot) * f(s, \cdot)\|_{L_x^{q_2}} ds \leq \int_0^t \|\nabla K(t-s)\|_{L_x^{\tilde{q}}} \|f(s)\|_{L_x^{q_1}} ds,$$

with $1/\tilde{q} + 1/q_1 = 1/q_2 + 1$. Thus, by (5.9), we obtain

$$\|\mathcal{B}f(t)\|_{L_x^{q_2}} = [1_{[0, T]}(s)|s|^{-\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) - \frac{1}{2}} *_s 1_{[0, T]}(s) \|f(s)\|_{L_x^{q_1}}](t)$$

and by virtue of Theorem 5.3.1 we conclude the proof of Lemma 5.3.3. \square

Moreover we have also

Lemma 5.3.4. *\mathcal{B} is a bounded operator from $L^r(0, T; L_x^p)$ to $L^r(0, T; L_x^{p^*})$, for any $T > 0$, with $1 < p < N$ and $p^* = pN/(N-p)$.*

Proof. It is a direct consequence of Theorem 5.3.2 and Corollary 5.3.1.1, observing that

$$\mathcal{B}f(t) := - \int_0^t \Delta e^{(t-s)\Delta} (\sqrt{-\Delta})^{-1} Rf(s) ds,$$

where R is the Riesz transform $Rf = \frac{\nabla}{\sqrt{-\Delta}} f := \mathcal{F}^{-1}(i \frac{\xi}{|\xi|} \hat{f})$, which is a bounded operator from L_x^q to L_x^q for any $q \in (1, \infty)$. \square

Lemma 5.3.5. *Let $T \in (0, \infty]$, $1 < r_1, r_2 < \infty$, $q_1 \in [1, \infty]$ and $q_2 \in [q_1, \infty]$, such that*

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) < 1. \quad (5.10)$$

Let $f \in L_T^{r_1} L_x^{q_1}$ and let the operator \mathcal{C} be defined by

$$\mathcal{C}f(t, \cdot) := \int_0^t e^{(t-s)\Delta} f(s, \cdot) ds,$$

Then, \mathcal{C} is a bounded operator from $L^{r_1}(0, T; L_x^{q_1})$ with values to $L^{r_2}(0, T; L_x^{q_2})$, where

$$\frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = 1 + \frac{1}{r_2}. \quad (5.11)$$

Proof. The proof is basically equivalent to the previous one, observing that

$$\|K(t, \cdot)\|_{L_x^\lambda} = \frac{1}{t^{\frac{N}{2} \frac{1}{\lambda'}}} \|K(1, \cdot)\|_{L_x^\lambda}, \quad (5.12)$$

for every $1 \leq \lambda \leq \infty$. \square

The next Theorem is a variation of Theorem 5.3.2 for functions which belong to some $L_t^p L_x^q$ -space, up to a weight in time. Its proof has already been presented in [60] by Lemma 3.1.

Theorem 5.3.6. *Let $T \in]0, \infty]$, $1 < \bar{r}, q < \infty$ and $\alpha \in (0, 1 - 1/\bar{r})$. Let the operator \mathcal{A} be defined as in Theorem 5.3.2. Suppose that $t^\alpha f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^q)$. Then $t^\alpha \mathcal{A}f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^q)$ and there exists $C > 0$ such that*

$$\|t^\alpha \mathcal{A}f(t)\|_{L^{\bar{r}}(0, T; L_x^q)} \leq C \|t^\alpha f(t)\|_{L^{\bar{r}}(0, T; L_x^q)}.$$

The next two Lemmas are a particular case of Lemma 5.8.1 and Lemma 5.8.2, therefore the proof is postponed to section 5.8.

Lemma 5.3.7. *Let the operator \mathcal{C} be defined as in Lemma 5.3.5. Consider $T \in (0, \infty]$, $1 < \bar{r} < \infty$, and moreover suppose that q, \tilde{q} satisfy $N/2 < q < N$, $N < \tilde{q} \leq \infty$. Let α, γ and $\bar{\gamma}$ be defined by*

$$\alpha := \frac{1}{2} \left(3 - \frac{N}{q} \right) - \frac{1}{\bar{r}}, \quad \gamma := \frac{1}{2} \left(1 - \frac{N}{\tilde{q}} \right) - \frac{1}{\bar{r}} \quad \text{and} \quad \bar{\gamma} := \frac{1}{2} \left(1 - \frac{N}{\tilde{q}} \right).$$

If $t^\alpha f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^q)$ then $t^\gamma \mathcal{C}f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^{\tilde{q}})$. Furthermore there exists a constant $C = C(q, \tilde{q}, \bar{r}) > 0$ such that

$$\|t^\gamma \mathcal{C}f(t)\|_{L^{\bar{r}}(0, T; L_x^{\tilde{q}})} \leq C \|t^\alpha f(t)\|_{L^{\bar{r}}(0, T; L_x^q)}.$$

Moreover, if $\bar{r} > 2$ and $N\bar{r}/(2\bar{r} - 2) < q$, then $t^{\bar{\gamma}} \mathcal{C}f(t)$ belongs to $L_T^\infty L_x^{\tilde{q}}$ and there exists a positive constant $\bar{C} = \bar{C}(q, \tilde{q}, \bar{r})$ such that

$$\|t^{\bar{\gamma}} \mathcal{C}f(t)\|_{L_T^\infty L_x^{\tilde{q}}} \leq \bar{C} \|t^\alpha f(t)\|_{L^{\bar{r}} L_x^q}.$$

Lemma 5.3.8. *Let the operators \mathcal{B} be defined as in Lemma 5.3.3. Consider $T \in (0, \infty]$, $1 < \bar{r} < \infty$, and moreover suppose that q, \bar{q} satisfy $N/2 < q < N$ and $q \leq \bar{q}$ such that $1/q - 1/\bar{q} < 1/N$. Let α be defined as in Lemma 5.3.7 and β and $\bar{\beta}$ be defined by*

$$\bar{\beta} := \frac{1}{2} \left(2 - \frac{N}{\bar{q}} \right) \quad \text{and} \quad \beta := \frac{1}{2} \left(1 - \frac{N}{\bar{q}} \right) - \frac{1}{\bar{r}}.$$

If $t^\alpha f(t)$ belongs to $L_T^{\bar{r}} L_x^q$ then $t^\beta \mathcal{B}f(t)$ belongs to $L_T^{\bar{r}} L_x^{\bar{q}}$ and there exists a positive constant $C = C(q, \bar{q}, \bar{r})$ such that

$$\|t^\beta \mathcal{B}f(t)\|_{L_T^{\bar{r}} L_x^{\bar{q}}} \leq C \|t^\alpha f(t)\|_{L_T^{\bar{r}} L_x^q}.$$

Moreover, if $\bar{r} > 2$, $N\bar{r}/(2\bar{r} - 2) < q$ and $\bar{q} < Nr$ then $t^{\bar{\beta}} \mathcal{B}f(t)$ belongs to $L_T^\infty L_x^{\bar{q}}$ and there exists a

positive constant $\bar{C} = \bar{C}(q, \bar{q}, \bar{r})$ such that

$$\|t^{\bar{\beta}} \mathcal{B}f(t)\|_{L_T^\infty L_x^{\bar{q}}} \leq \bar{C} \|t^\alpha f(t)\|_{L_T^{\bar{r}} L_x^q}.$$

For the main properties of homogeneous Besov Spaces we refer to [7]. However, let us briefly recall the definition and two important results which characterize such spaces in relation to the heat kernel.

Definition 5.3.9. Let χ be a smooth nonincreasing radial function which has support in $B(0, 4/3)$ and such that $\chi \equiv 1$ on $B(0, 1)$. Imposing $\varphi_q(\xi) := \chi(\xi 2^{-q-1}) - \chi(\xi 2^{-q})$ for every $q \in \mathbb{Z}$, we define the homogeneous Littlewood-Paley dyadic block $\dot{\Delta}_q$ by

$$\dot{\Delta}_q u := \mathcal{F}^{-1}(\varphi_q \mathcal{F}u),$$

where u is a temperate distribution and \mathcal{F} is the Fourier transform on \mathbb{R}^N .

The homogeneous Besov Space is defined as follows:

Definition 5.3.10. For any $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$, let us define $\dot{B}_{p,r}^s$ as the set of tempered distribution f such that

$$\|f\|_{\dot{B}_{p,r}^s} := \|2^{sq} \|\dot{\Delta}_q f\|_{L_x^p}\|_{L^r(\mathbb{Z})}$$

and for all smooth compactly supported function θ on \mathbb{R}^N we have

$$\lim_{\lambda \rightarrow +\infty} \theta(\lambda D)f = 0 \quad \text{in } L^\infty(\mathbb{R}^N).$$

Theorem 5.3.11 (Characterization of Homogeneous Besov Spaces). Let s be a negative real number and $(p, r) \in [1, \infty]^2$. u belongs to $\dot{B}_{p,r}^s$ if and only if $e^{t\Delta}u$ belongs to L_x^p for almost every $t \in \mathbb{R}_+$ and

$$t^{-\frac{s}{2}} \|e^{t\Delta}u\|_{L_x^p} \in L^r\left(\mathbb{R}_+; \frac{dt}{t}\right).$$

Moreover, there exists a positive constant C such that

$$\frac{1}{C} \|u\|_{\dot{B}_{p,r}^s} \leq \left\| \left\| t^{-\frac{s}{2}} e^{t\Delta}u \right\|_{L_x^p} \right\|_{L^r(\mathbb{R}_+; \frac{dt}{t})} \leq C \|u\|_{\dot{B}_{p,r}^s}.$$

An immediate consequence is the following Corollary:

Corollary 5.3.11.1. Let $p \in [1, \infty]$ and $r \in [1, \infty)$. u belongs to $\dot{B}_{p,r}^{-\frac{2}{r}}$ if and only if $e^{t\Delta}u \in L_t^r L_x^p$. Moreover, there exists a positive constant C such that

$$\frac{1}{C} \|u\|_{\dot{B}_{p,r}^{-\frac{2}{r}}} \leq \|e^{t\Delta}u\|_{L_t^r L_x^p} \leq C \|u\|_{\dot{B}_{p,r}^{-\frac{2}{r}}}.$$

Theorem 5.3.12. Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s , the space \dot{B}_{p_1, r_1}^s is continuously embedded in $\dot{B}_{p_2, r_2}^{s-N(1/p_1-1/p_2)}$.

5.4 Smooth initial data

In order to prove Theorem 5.2.2 and Theorem 5.2.4, in this section we are going to establish the global existence of a solution for system (5.2), considering more regular initial data. More precisely

we are going to consider an initial Lipschitz density and moreover we suppose the initial velocity and the initial director field with a little bit more regularity with respect to the one of Theorem 5.2.2 and Theorem 5.2.4.

Proposition 5.4.1. *Let $\varepsilon \in (0, 1)$, $r \in (1, 2/(2-\varepsilon)]$, $p \in (1, Nr/(3r-2)]$. Suppose that the initial condition (5.3) is fulfilled and moreover $\nabla a_0 \in L_x^\infty$ and $(u_0, \nabla d_0) \in \dot{B}_{p,r}^{N/p-1+\varepsilon}$. If the smallness condition (5.4) holds, then (5.2) admits a global weak-solution (a, u, d) such that $(a, d) \in L_{t,x}^\infty$, ∇a belongs to $L_{t,loc}^\infty L_x^\infty$, $(u, \nabla d, \nabla \Pi)$ belongs to $\mathfrak{X}_r \cap \mathfrak{X}_r^\varepsilon$ and $(u, \nabla d)$ belongs to $L_t^2 L_x^\infty$, with*

$$\begin{aligned} \|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{X}_r} + \|(u, \nabla d)\|_{L_t^2 L_x^\infty} &\lesssim \eta \\ \|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{X}_r^\varepsilon} &\lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{N/p-1+\varepsilon}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{N/p-1+\frac{2}{3}\varepsilon}}^3. \end{aligned}$$

Proof. The mainly idea is to proceed by an iterate scheme with a similar approach as the one proposed by Danchin and Mucha in [30]. We solve a sequence of linear systems which comes from (5.2) and we prove that their solutions converge to the one we are looking for. We set $(a^0, u^0, \nabla d^0, \nabla \Pi^0) = (0, 0, 0, 0)$ and we solve inductively the following two systems:

$$\begin{cases} \partial_t a^n + u^{n-1} \cdot \nabla a^n = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ a^n|_{t=0} = a_0 & \mathbb{R}^N, \end{cases} \quad (5.13)$$

$$\begin{cases} \partial_t d^n - \Delta d^n = |\nabla d^{n-1}|^2 d^{n-1} - u^{n-1} \cdot \nabla d^{n-1} & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t u^n + u^{n-1} \cdot \nabla u^n - \Delta u^n + \nabla \Pi^n = F^n & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u^n = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u^n, d^n)|_{t=0} = (u_0, d_0) & \mathbb{R}^N, \end{cases} \quad (5.14)$$

where F^n is defined by:

$$F^n := (1 + a^n) \operatorname{div}(\nabla d^n \odot \nabla d^n) + a^n (\Delta u^{n-1} - \nabla \Pi^{n-1}). \quad (5.15)$$

The global existence and uniqueness of a solution to (5.13) is standard, since u^{n-1} belongs to $L_{loc}^1 \mathcal{L}ip$ and so it is possible to construct the Lagrangian coordinates. Furthermore,

$$\|a^n\|_{L_{t,x}^\infty} \leq \|a_0\|_{L_x^\infty} \quad (5.16)$$

and

$$\|\nabla a^n(t)\|_{L_x^\infty} \leq \|\nabla a_0\|_{L_x^\infty} \exp \left\{ \int_0^t \|\nabla u^{n-1}(s)\|_{L_x^\infty} ds \right\} \quad (5.17)$$

are fulfilled for every natural number n . For system (5.14) we apply Proposition 5.9.1. We have already pointed out in Remark 5.2.8 that (5.14) does not yield $|d^n| = 1$ almost everywhere, while such constriction is given by the fourth equation of (5.2).

Let us prove by induction that the following inequalities are satisfied for every $n \in \mathbb{N}$:

$$\|d^n\|_{L_{t,x}^\infty} \leq (1 + C\eta)e^{C\eta}, \quad (5.18)$$

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{X}_r} + \|(u^n, \nabla d^n)\|_{L_t^2 L_x^\infty} \lesssim \eta, \quad (5.19)$$

where C is a positive constant. We consider initially system (5.14) and we want to estimate d^n .

By the mild formulation for the heat equation, we obtain that

$$d^n(t) = e^{t\Delta}d_0 + \int_0^t e^{(t-s)\Delta} \{ -u^{n-1} \cdot \nabla d^{n-1} + |\nabla d^{n-1}|^2 d^n \}(s) ds. \quad (5.20)$$

Hence, we deduce

$$\begin{aligned} \|d^n(t)\|_{L_x^\infty} &\leq \|d_0\|_{L_x^\infty} + \int_0^t \|u^{n-1}(s)\|_{L_x^\infty} \|\nabla d^{n-1}(s)\|_{L_x^\infty} + \|\nabla d^{n-1}(s)\|_{L_x^\infty}^2 \|d^n(s)\|_{L_x^\infty} ds \\ &\leq 1 + \|u^{n-1}\|_{L_t^2 L_x^\infty} \|\nabla d^{n-1}\|_{L_t^2 L_x^\infty} + \|\nabla d^{n-1}(s)\|_{L_x^\infty}^2 \|d^n(s)\|_{L_x^\infty} ds \\ &\leq 1 + C\eta + \int_0^t \|\nabla d^{n-1}(s)\|_{L_x^\infty}^2 \|d^n(s)\|_{L_x^\infty} ds. \end{aligned} \quad (5.21)$$

Applying the Gronwall inequality and by the induction hypotheses, we obtain (5.18). We want now to estimate ∇d^n and u^n . From (5.20) we get

$$\nabla d^n(t) = e^{t\Delta} \nabla d_0 + \int_0^t e^{(t-s)\Delta} \nabla \{ u^{n-1} \cdot \nabla d^{n-1} + |\nabla d^{n-1}|^2 d^n \}(s) ds \quad (5.22)$$

First, let us estimate ∇d^n in $L_t^{3r} L_x^{3Nr/(3r-2)}$. By Corollary 5.3.11.1 and Lemma 5.3.3, with $r_1 = 6r/5$, $r_2 = 3r$, $q_1 = 3Nr/(6r-5)$ and $q_2 = 3Nr/(3r-2)$ which verify (5.7) and (5.8), namely

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} - \frac{1}{2r} < \frac{1}{2} \quad \text{and} \quad \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{5}{6r} + \frac{1}{2} - \frac{1}{2r} = \frac{1}{2} + \frac{1}{r_2}, \quad (5.23)$$

we get

$$\|\nabla d^n\|_{L_t^{3r} L_x^{\frac{3Nr}{3r-2}}} \lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|(u^{n-1}, \nabla d^{n-1})\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla d^{n-1}\|_{L_t^{3r} L_x^{\frac{3Nr}{3r-2}}} \lesssim \eta \quad (5.24)$$

Moreover, applying Corollary 5.3.11.1, Lemma 5.3.3, Lemma 5.3.4, Lemma 5.3.5 and Theorem 5.3.2, we obtain

$$\begin{aligned} &\|\nabla d^n\|_{L_t^2 L_x^\infty} + \|\nabla d^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \|\nabla^2 d^n\|_{L_t^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla^2 d^n\|_{L_t^{2r} L_x^{\frac{Nr}{2(r-1)}}} + \|\nabla^3 d^n\|_{L_t^{3r} L_x^{\frac{Nr}{3r-2}}} \lesssim \\ &\lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{2r-1}}} \|\nabla d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \\ &\quad + \|\nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{2r-1}}} \|\nabla d^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|d^n\|_{L_{t,x}^\infty} + \|\nabla d^{n-1}\|_{L_t^{3r} L_x^{\frac{3Nr}{3r-2}}}^2 \|\nabla d^n\|_{L_t^{3r} L_x^{\frac{3Nr}{3r-2}}}. \end{aligned} \quad (5.25)$$

We have used Lemma 5.3.5 with $r_1 = r$, $r_2 = 2r$ (respectively $r_2 = 2$), $q_1 = Nr/(3r-2)$ and $q_2 = Nr/(r-1)$ (respectively $q_2 = \infty$), which fulfill the conditions (5.10) and (5.11), namely

$$\begin{aligned} \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) &= 1 - \frac{1}{2r} < 1 & \text{(resp. } \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) &= \frac{3}{2} - \frac{1}{r} < 1 \text{ since } r < 2), \\ \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) &= 1 + \frac{1}{2r} = 1 + \frac{1}{r_2} & \text{(resp. } \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) &= \frac{3}{2} = 1 + \frac{1}{r_2}). \end{aligned}$$

Moreover, $(Nr/(3r-2))^* = Nr/(2(r-1))$ for Lemma 5.3.4 and the constants of Lemma 5.3.3 are determined by $r_1 = r$, $q_1 = Nr/(3r-2)$, $r_2 = 2r$ and $q_2 = Nr/(2r-1)$ which satisfy the conditions

(5.7) and (5.8), that is

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} - \frac{1}{2r} < \frac{1}{2} \quad \text{and} \quad \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{r} + \frac{1}{2} - \frac{1}{2r} = \frac{1}{2} + \frac{1}{r_2}.$$

The hypotheses (5.19) for $n - 1$ allows us to absorb all the terms on the right-hand side of (5.25) with index n by the left-hand side (for η small enough). Hence, (5.19) is true at least for the terms related to d . Now, let us estimate the remaining terms. By the Mild formulation for the Stokes equation, we get

$$u^n(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \{ -u^{n-1} \cdot \nabla u^n(s) + F^n(s) \} ds \quad (5.26)$$

where \mathbb{P} is the well known Leray projector. Moreover, applying div to the second equation of (5.14), we get $-\Delta \Pi^n = \{-u^{n-1} \cdot \nabla u^n + F^n\}$, which yields

$$\nabla \Pi^n = RR \cdot \{ -u^{n-1} \cdot \nabla u^n + F^n \}, \quad (5.27)$$

where R is the Riesz transform (see Lemma 5.3.4). Since \mathbb{P} and R are bounded operators from L^p to L^p , for every $1 < p < \infty$, applying Corollary 5.3.11.1, Lemma 5.3.3, Lemma 5.3.4, Lemma 5.3.5 (with the constants r_1, r_2, q_1 and q_2 as in (5.25)) and Theorem 5.3.2, we deduce that

$$\begin{aligned} & \|\nabla u^n\|_{L_t^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla u^n\|_{L_t L_x^{\frac{Nr}{2(r-1)}}} + \|u^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \|u^n\|_{L_t^2 L_x^\infty} + \|(\nabla^2 u^n, \nabla \Pi^n)\|_{L_t L_x^{\frac{Nr}{3r-2}}} \lesssim \\ & \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla u^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \|F^n\|_{L_t L_x^{\frac{Nr}{3r-2}}} \lesssim \eta \|\nabla u^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \eta. \end{aligned} \quad (5.28)$$

As in the previous estimates, the term of index n in the right-hand side can be absorbed by the left-hand side, obtaining finally (5.19).

Now, let us observe that $\|\nabla d^n\|_{L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}} \lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{2\varepsilon}{3}}}$, for every $n \in \mathbb{N}$. Indeed, by induction and recalling the Mild formulation of d^n (5.20), we get

$$\begin{aligned} \|\nabla d^n\|_{L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}} & \lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{2\varepsilon}{3}}} + \|(u^{n-1}, \nabla d^{n-1})\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla d^{n-1}\|_{L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}} \\ & \lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{2\varepsilon}{3}}}, \end{aligned}$$

where we have used Corollary 5.3.11.1 and Lemma 5.3.3, with $r_1 = 6r/5$, $r_2 = 3r$, $q_1 = 3Nr/((6-\varepsilon)r-5)$ and $q_2 = 3Nr/((3-\varepsilon)r-2)$ which verify (5.7) and (5.8), as in (5.23). Thus, in the same line of (5.25) we get

$$\begin{aligned} & \|\nabla^2 d^n\|_{L_t^{\frac{2}{2-\varepsilon}} L_x^\infty} + \|\nabla^2 d^n\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} + \|\nabla^3 d^n\|_{L_t L_x^{\frac{Nr}{(3-\varepsilon)r-2}}} \lesssim \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \\ & + \|u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} + \|\nabla u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} \|\nabla d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} + \\ & + \|\nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} \|\nabla d^n\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|d^n\|_{L_{t,x}^\infty} + \|\nabla d^{n-1}\|_{L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}}^2 \|\nabla d^n\|_{L_t^{3r} L_x^{\frac{3Nr}{(3-\varepsilon)r-2}}}. \end{aligned}$$

Here, we have used Theorem 5.3.2 and Lemma 5.3.3 with $r_1 = r$, $q_1 = Nr/((3-\varepsilon)r-2)$, $r_2 = 2r$ (respectively $r_2 = 2/(2-\varepsilon)$) and $q_2 = Nr/((2-\varepsilon)r-1)$ (respectively $q_2 = \infty$), which satisfy the conditions (5.7) and (5.8), namely

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} - \frac{1}{2r} < \frac{1}{2} \quad (\text{resp. } \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{3-\varepsilon}{2} - \frac{1}{r} < \frac{1}{2} \text{ since } r < \frac{2}{2-\varepsilon}),$$

$$\frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} + \frac{1}{2r} = 1 + \frac{1}{r_2} \quad (\text{resp. } \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} + \frac{2-\varepsilon}{2} = \frac{1}{2} + \frac{1}{r_2}).$$

Furthermore, in the same line of (5.28), we get

$$\begin{aligned} & \|\nabla u^n\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} + \|\nabla u^n\|_{L_t^{\frac{2}{2-\varepsilon}} L_x^\infty} + \|(\nabla^2 u^n, \nabla \Pi^n)\|_{L_t^r L_x^{\frac{Nr}{(3-\varepsilon)r-2}}} \lesssim \\ & \lesssim \|u_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \|u^{n-1}\|_{L_t^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla u^n\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} + \\ & \quad + \eta \|\nabla(u^n, \nabla d^{n-1})\|_{L_t^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} + \eta \|(\nabla^2 u^{n-1}, \nabla \Pi^{n-1})\|_{L_t^r L_x^{\frac{Nr}{(3-\varepsilon)r-2}}}. \end{aligned}$$

Summarizing the previous consideration, we get by induction

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{X}_r^\varepsilon} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \|\nabla d_0\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{2}{3}\varepsilon}}, \quad n \in \mathbb{N}. \quad (5.29)$$

At last, arguing as in the proof of (5.29), we get also the following inequality

$$\begin{aligned} & \|\nabla d^n\|_{L_t^{\frac{12r}{4-3\varepsilon}} L_x^{\frac{3Nr}{3r-2}}} + \|\nabla(u^n, \nabla d^n)\|_{L_t^{\frac{4r}{2-\varepsilon}} L_x^{\frac{Nr}{2r-1}}} + \|(\nabla^2 u^n, \nabla \Pi^n)\|_{L_t^{\frac{4r}{4-\varepsilon}} L_x^{\frac{Nr}{3r-2}}} \\ & \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\frac{\varepsilon}{2r}}} \lesssim \eta + \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}}. \end{aligned} \quad (5.30)$$

Here, we need Lemma 5.3.5 with $r_1 = 4r/(4-\varepsilon)$, $r_2 = 4r/(2-\varepsilon)$, $q_1 = Nr/(3r-2)$ and $q_2 = Nr/(r-1)$, which fulfill the conditions (5.10) and (5.11), namely

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = 1 - \frac{1}{2r} < 1 \quad \text{and} \quad \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{4-\varepsilon}{4r} + 1 - \frac{1}{2r} = 1 + \frac{2-\varepsilon}{4r} = 1 + \frac{1}{r_2},$$

We need also Lemma 5.3.3 with $r_1 = 4r/(4-\varepsilon)$, $q_1 = Nr/(3r-2)$, $r_2 = 4r/(2-\varepsilon)$ and $q_2 = Nr/(2r-1)$, which satisfy the conditions (5.7) and (5.8),

$$\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{1}{2} - \frac{1}{2r} < \frac{1}{2} \quad \text{and} \quad \frac{1}{r_1} + \frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) = \frac{4-\varepsilon}{4r} + \frac{1}{2} - \frac{1}{2r} = \frac{1}{2} + \frac{2-\varepsilon}{4r} = \frac{1}{2} + \frac{1}{r_2}.$$

Now we claim that, for every $T > 0$, d^n is a Cauchy sequence in $L_T^\infty L_x^\infty$, $(u^n, \nabla d^n, \nabla \Pi_n)_\mathbb{N}$ is a Cauchy sequence in $\mathfrak{X}_{r,T}$ and $(u^n, \nabla d^n)_\mathbb{N}$ is a Cauchy sequence in $L_T^2 L_x^\infty$. Denoting $\delta u^n := u^{n+1} - u^n$, $\delta a^n := a^{n+1} - a^n$, $\delta d^n := d^{n+1} - d^n$, $\delta \Pi^n := \Pi^{n+1} - \Pi^n$, we define

$$\delta U^n(T) := \|\delta d^n\|_{L_T^\infty L_x^\infty} + \|(\delta u^n, \nabla \delta d^n, \nabla \delta \Pi^n)\|_{\mathfrak{X}_{r,T}} + \|(\delta u^n, \nabla \delta d^n)\|_{L_T^2 L_x^\infty}$$

We want to prove that $\sum_{n \in \mathbb{N}} \delta U^n(T)$ is finite. First, let us consider δa^n , which is solution of the following system

$$\begin{cases} \partial_t \delta a^n + u^n \cdot \nabla \delta a^n = -\delta u^{n-1} \nabla a^n & \mathbb{R}_+ \times \mathbb{R}^N, \\ \delta a^n|_{t=0} = 0 & \mathbb{R}^N. \end{cases} \quad (5.31)$$

Using standard estimates for the transport equation, we obtain that

$$\begin{aligned} \|\delta a^n(t)\|_{L_x^\infty} & \leq \int_0^t \|\delta u^{n-1}(s)\|_{L_x^\infty} \|\nabla a^n(s)\|_{L_x^\infty} ds \leq \|\delta u^{n-1}\|_{L^1(0,t;L_x^\infty)} \|\nabla a_0\|_{L_x^\infty} \times \\ & \quad \times \exp \left\{ \int_0^t \|\nabla u^n(s)\|_{L_x^\infty} ds \right\} \leq \bar{C}(T) \delta U^{n-1}(t) \|\nabla a_0\|_{L_x^\infty}, \end{aligned} \quad (5.32)$$

for every $t \in (0, T)$, where $\bar{C}(T) := T^{\frac{1}{2}} \exp \{ T^{\frac{\varepsilon}{2}} (\| (u_0, \nabla d_0) \|_{\dot{B}_{p,r}^{N/p-1+\varepsilon}} + \| \nabla d_0 \|_{\dot{B}_{p,r}^{N/p-1+\varepsilon}}^3) \}$. Considering δd^n we observe that it is solution of

$$\begin{cases} \partial_t \delta d^n - \Delta \delta d^n = \delta H^n & \mathbb{R}_+ \times \mathbb{R}^N, \\ \delta d^n|_{t=0} = 0 & \mathbb{R}^N, \end{cases} \quad (5.33)$$

where

$$\begin{aligned} \delta H^n &:= -\delta u^{n-1} \cdot \nabla d^n - u^{n-1} \cdot \nabla \delta d^{n-1} + \nabla \delta d^{n-1} \odot \nabla d^n d^n + \\ &+ \nabla d^{n-1} \odot \nabla \delta d^{n-1} d^n + \nabla d^{n-1} \odot \nabla d^{n-1} \delta d^n. \end{aligned}$$

Thus $\| \delta d^n(t) \|_{L_x^\infty} \leq \| \delta H^n \|_{L_T^1 L_x^\infty} \lesssim \eta \| (\delta u^{n-1}, \nabla \delta d^{n-1}) \|_{L_T^2 L_x^\infty} + \eta^2 \| \delta d^n(t) \|_{L_T^\infty L_x^\infty}$, which yields

$$\| \delta d^n \|_{L_T^\infty L_x^\infty} \lesssim \eta \delta U^{n-1}(T). \quad (5.34)$$

Arguing exactly as in the proof of inequality (5.24) and (5.25), we obtain

$$\begin{aligned} & \| \nabla \delta d^n \|_{L_T^{3r} L_x^{\frac{3Nr}{3r-2}}} + \| \nabla \delta d^n \|_{L_T^2 L_x^\infty} + \| \nabla \delta d^n \|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \| \nabla^2 \delta d^n \|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \| \nabla^2 \delta d^n \|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}} + \\ & + \| \nabla^3 \delta d^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \| \delta H^n \|_{L_T^{\frac{6}{5}r} L_x^{\frac{3Nr}{6r-2}}} + \| \nabla \delta H^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \eta (\delta U^n(T) + \delta U^{n-1}(T)). \end{aligned}$$

In order to conclude our estimate we have to bound the terms related to δu^n , which is solution of

$$\begin{cases} \partial_t \delta u^n + u^n \cdot \nabla \delta u^n + \delta u^{n-1} \cdot \nabla u^n - \Delta \delta u^n + \nabla \delta \Pi^n = \delta F^n & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} \delta u^n = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \delta u^n|_{t=0} = 0 & \mathbb{R}^N, \end{cases} \quad (5.35)$$

where $\delta F^n := F^{n+1} - F^n$. First, let us observe that

$$\begin{aligned} & \| \nabla \delta u^n \|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \| \nabla \delta u^n \|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}} + \| \delta u^n \|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \| \delta u^n \|_{L_T^2 L_x^\infty} + \\ & + \| (\nabla^2 \delta u^n, \nabla \delta \Pi^n) \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \eta (\| \delta u^{n-1} \|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \| \nabla \delta u^n \|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}}) + \| \delta F^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}}. \end{aligned}$$

Hence, denoting by $G^n := \operatorname{div}(\nabla d^n \odot \nabla d^n)$, we have that

$$\begin{aligned} \| \delta F^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} & \lesssim \| \delta a^n(G^n, \Delta u^n, \nabla \Pi^n) \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \| \delta G^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \\ & + \eta \| (\Delta \delta u^{n-1}, \nabla \delta \Pi^{n-1}) \|_{L_T^r L_x^{\frac{Nr}{3r-2}}}. \end{aligned}$$

Since $\| \delta G^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \eta (\| \nabla \delta d^n \|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \| \nabla^2 \delta d^n \|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}})$, and recalling (5.30) and (5.32),

$$\begin{aligned} \| \delta F^n \|_{L_T^r L_x^{\frac{Nr}{3r-2}}} & \lesssim \| (G^n, \Delta u^n, \nabla \Pi^n) \|_{L_T^{\frac{4r}{4-\varepsilon}} L_x^{\frac{Nr}{3r-2}}} \| \delta a^n \|_{L_T^{\frac{4r}{\varepsilon}} L_x^\infty} + \delta U^n(T) \eta \\ & \lesssim \bar{C}(T) \| \nabla a_0 \|_{L_x^\infty} \| \delta U^{n-1} \|_{L_T^{\frac{4r}{\varepsilon}}} + \delta U^n(T) \eta. \end{aligned}$$

Summarizing the previous considerations and supposing η small enough, we obtain

$$\delta U^n(T) \lesssim \eta \delta U^{n-1}(T) + \bar{C}(T) \|\nabla a_0\|_{L_x^\infty} \|\delta U^{n-1}\|_{L^{\frac{4r}{\varepsilon}}(0,T)}. \quad (5.36)$$

We claim that there exists $C(T) > 0$ and $K(T) > 0$ such that, for all $t \in [0, T]$, and for all $n \in \mathbb{N}$

$$\delta U^n(t) \leq C(T) \eta^{\frac{n}{2}} \exp \left\{ K(T) \frac{t}{\sqrt{\eta}} \right\}. \quad (5.37)$$

We are going to prove it by induction and the base case is trivial, since it is sufficient to find $C(T) > 0$ such that, for all $t \in [0, T]$, $\delta U^0(t) \leq C(T)$. Then it is fulfilled $\delta U^0(t) \leq C(T) \exp \left\{ K(T) t / \sqrt{\eta} \right\}$, for all $K(T) > 0$ and for all $t \in [0, T]$. Passing through the induction hypotheses and by (5.36), we have that there exists $\tilde{C} > 0$ such that, for all $t \in [0, T]$

$$\begin{aligned} \delta U^n(t) &\leq \tilde{C} \left[\eta \delta U^{n-1}(t) + \bar{C}(T) \|\nabla a_0\|_{L_x^\infty} \|\delta U^{n-1}\|_{L^{\frac{4r}{\varepsilon}}(0,t)} \right] \\ &\leq \tilde{C} \left[\eta C(T) \eta^{\frac{n-1}{2}} \exp \left\{ K(T) \frac{t}{\sqrt{\eta}} \right\} + \right. \\ &\quad \left. + \bar{C}(T) \|\nabla a_0\|_{L_x^\infty} C(T) \eta^{\frac{n-1}{2}} \left(\int_0^t \exp \left\{ \frac{4r}{\varepsilon} K(T) \frac{s}{\sqrt{\eta}} \right\} ds \right)^{\frac{\varepsilon}{4r}} \right] \\ &\leq \tilde{C} \left(\sqrt{\eta} C(T) + \frac{\varepsilon^{\frac{\varepsilon}{4r}}}{(4r K(T))^{\frac{\varepsilon}{4r}} \eta^{\frac{1}{2} - \frac{\varepsilon}{4r}}} C(T) \bar{C}(T) \|\nabla a_0\|_{L_x^\infty} \right) \eta^{\frac{n}{2}} \exp \left\{ K(T) \frac{t}{\sqrt{\eta}} \right\}. \end{aligned}$$

Choosing $K(T) > 0$ big enough and supposing η small enough, we can assume

$$\tilde{C} \left(\sqrt{\eta} C(T) + \frac{\varepsilon^{\frac{\varepsilon}{4r}}}{(4r K(T))^{\frac{\varepsilon}{4r}} \eta^{\frac{1}{2} - \frac{\varepsilon}{4r}}} C(T) \bar{C}(T) \|\nabla a_0\|_{L_x^\infty} \right) \leq C(T),$$

which finally yields (5.36). It is now immediate to conclude that $(d^n)_{\mathbb{N}}$, $(\nabla(u^n, \nabla d^n))_{\mathbb{N}}$ and moreover $(u^n, \nabla d^n, \nabla \Pi^n)_{\mathbb{N}}$ are Cauchy sequences in $L_T^\infty L_x^\infty$, $L_T^2 L_x^\infty$ and $\mathfrak{X}_{r,T}$, respectively. Furthermore, resuming (5.32), we deduce that $(a^n)_{\mathbb{N}}$ is a Cauchy sequence in $L_T^\infty L_x^\infty$. Granted with these convergence results and recalling the inequalities (5.16), (5.19) and 5.29 we conclude that the limit $(a, u, \nabla d, \nabla \Pi)$ fulfills the property of the Proposition.

Finally, recalling that, for every positive integer n , (a^n, u^n, d^n) is solution of (5.13) and (5.14), passing through the limit, we deduce that (a, u, d) is solution of (5.2) with (a_0, u_0, d_0) as initial data, and this completes the proof of Proposition 5.4.1. \square

Proposition 5.4.2. *Let $r \in (1, \infty)$, $p \in (1, N)$. Suppose that the initial data fulfill (5.3) and moreover $\nabla a_0 \in L_x^\infty$ and $(u_0, \nabla d_0) \in \dot{B}_{p,r}^{N/p-1+\varepsilon}$ with $\varepsilon < \min\{1/r, 1 - 1/r, N/p - 1\}$. If the smallness condition (5.4) holds, then there exists a global weak-solution (a, u, d) with the same property of Theorem 5.2.4. Moreover $\nabla a \in L_{t,loc}^\infty L_x^\infty$, $(u, \nabla d, \nabla \Pi)$ belongs to $\mathfrak{Y}_r^\varepsilon$ and*

$$\|(u, \nabla d, \nabla \Pi)\|_{\mathfrak{Y}_r^\varepsilon} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,1}^{\frac{N}{p}-1+\varepsilon}}. \quad (5.38)$$

Remark 5.4.3. *The condition (5.38) ensures the velocity field to be in $L_{t,loc}^1 \text{Lip}_x$. Indeed, a classical Gagliardo-Nirenberg interpolation inequality*

$$\|\nabla f\|_{L_x^\infty} \lesssim \|f\|_{L_x^{q_3}}^{1-\theta} \|\nabla^2 f\|_{L_x^{q_1}}^\theta \quad \theta = \frac{Nq_1 + q_1q_3}{Nq_1 + 2q_1q_3 - Nq_3} = \frac{1}{1 + \frac{q_1q_3 - Nq_3}{q_1q_3 + Nq_1}} \in \left[\frac{1}{2}, 1 \right],$$

allows us to obtain the following estimate for every positive T :

$$\|\nabla(u, \nabla d)\|_{L_T^1 L_x^\infty} \lesssim \|(t^{-\gamma_1^\varepsilon}, t^{-\alpha_1^\varepsilon})\|_{L_T^{(2r)'}} \left(\|t^{\gamma_1^\varepsilon}(u, \nabla d)\|_{L_T^{2r} L_x^{q_3}} + \|t^{\alpha_1^\varepsilon} \nabla^2(u, \nabla d)\|_{L_T^{2r} L_x^{q_1}} \right) < \infty.$$

As already mentioned, such condition permits the existence of the flow for the velocity field, hence we can reformulate system (5.1) through Lagrangian coordinates (see Section 6). Adding a weight in time, we can increase the time integrability by

$$\|t^{\alpha_1^\varepsilon}(\nabla u, \nabla^2 d)\|_{L_T^{2r} L_x^\infty} \lesssim T^{\alpha_1^\varepsilon - \gamma_1^\varepsilon} \|t^{\gamma_1^\varepsilon}(u, \nabla d)\|_{L_T^{2r} L_x^{q_3}} + \|t^{\alpha_1^\varepsilon} \nabla^2(u, \nabla d)\|_{L_T^{2r} L_x^{q_1}} < \infty,$$

observing that $\alpha_1^\varepsilon - \gamma_1^\varepsilon$ is positive. These estimates are going to be useful to prove the uniqueness for the solution of (5.1) in the $\mathfrak{Y}_r \cap \mathfrak{Y}_r^\varepsilon$ functional framework.

Proof of Proposition 5.4.2. Proceeding with the same strategy of Proposition 5.4.1, we consider the sequence of solutions for the systems (5.13) and (5.14). We claim by induction that such solutions belong to the same space of Theorem 5.2.4 and moreover that

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{Y}_r} \lesssim \eta. \quad (5.39)$$

At first, let us observe that $\|e^{t\Delta} d_0\|_{L_x^\infty} \leq \|d_0\|_{L_x^\infty} \leq 1$. Furthermore let us recall the Mild formulations

$$d^n(t) = e^{t\Delta} d_0 + \int_0^t e^{(t-s)\Delta} \{|\nabla d^{n-1}|^2 d^n - u^{n-1} \cdot \nabla d^{n-1}\}(s) ds, \quad (5.40)$$

by Lemma 5.8.3, with $\bar{r} = r$, $q = p_3/2 > Nr/(2r-2)$ and $\sigma = 2\gamma_1 = 1 - N/p_3 - 1/r$,

$$\|d^n(t)\|_{L_{t,x}^\infty} \leq 1 + C_r \|s^{2\gamma_1} u^{n-1} \cdot \nabla d^{n-1}(s)\|_{L_s^r L_x^{\frac{p_3}{2}}} + C_r \left(\int_0^t s^{2r\gamma_1} \|\nabla d^{n-1}(s)\|_{L_x^{p_3}}^{2r} \|d^n(s)\|_{L_x^\infty}^r ds \right)^{\frac{1}{r}},$$

for every $t \in \mathbb{R}_+$, where C_r is a suitable positive constant. Thus, by the induction hypotheses and the Gronwall inequality, we deduce

$$\|d^n(t)\|_{L_x^\infty}^r \leq 2^{r-1} (1 + C_r \eta^2)^r \exp \left\{ 2^{r-1} C_r \eta^{2r} \right\} \leq \bar{C}_r, \quad (5.41)$$

where \bar{C}_r is a suitable positive constant which depends only by r . Furthermore, using standard estimates for the transport equation, we have $\|a^n\|_{L_{t,x}^\infty} \leq \|a_0\|_{L_x^\infty}$ and for all $t > 0$,

$$\|\nabla a^n(t)\|_{L_{t,x}^\infty} \leq \|\nabla a_0\|_{L_x^\infty} \exp \left\{ \int_0^t \|\nabla u^{n-1}(s)\|_{L_x^\infty} ds \right\}. \quad (5.42)$$

First, let us remark that, by Theorem 5.3.11 and Theorem 5.3.12,

$$\begin{aligned} & \|t^{\gamma_1} e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^\infty L_x^{p_3}} + \|t^{\gamma_3} e^{t\Delta} \nabla d_0\|_{L_t^{2r} L_x^{3p_1}} + \\ & + \|t^{\gamma_4} e^{t\Delta} \nabla d_0\|_{L_t^\infty L_x^{3p_1}} + \|t^{\beta_1} \nabla e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^\infty L_x^{p_2}} + \\ & + \|t^{\beta_3} \nabla e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^{2r} L_x^{\frac{p_3}{2}}} + \|t^{\beta_4} \nabla e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^\infty L_x^{\frac{p_3}{2}}} + \|t^{\alpha_1} \nabla^2 e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^{2r} L_x^{p_1}} + \\ & + \|t^{\alpha_1} \nabla^2 e^{t\Delta}(u_0, \nabla d_0)\|_{L_t^r L_x^{p_1}} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p_3,r}^{\frac{N}{p_3}-1} \cap \dot{B}_{3p_1,r}^{\frac{N}{3p_1}-1} \cap \dot{B}_{p_2,r}^{\frac{N}{p_2}-1} \cap \dot{B}_{\frac{p_3}{2},r}^{\frac{2N}{2}-1} \cap \dot{B}_{p_1,r}^{\frac{N}{p_1}-1}} \\ & \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} \lesssim \eta. \end{aligned}$$

We claim now that $t^{\gamma_1} \nabla d^n$ belongs to $L_t^{2r} L_x^{p_3}$, $t^{\gamma_2} \nabla d^n$ to $L_t^\infty L_x^{p_3}$, $t^{\gamma_3} \nabla d^n$ to $L_t^{2r} L_x^{3p_1}$ and $t^{\gamma_4} \nabla d^n$ to $L_t^\infty L_x^{3p_1}$. At first, denoting by $T^n := \nabla(|\nabla d^{n-1}|^2 d^n - u^{n-1} \cdot \nabla d^{n-1})$, we have that $t^{\alpha_1} T^n$ belongs to $L_t^{2r} L_x^{p_1}$, thanks to the induction hypotheses and (5.41). Moreover, applying Lemma 5.3.7 with $\bar{r} = 2r$, $q := p_1 > Nr/(2r-1) = N\bar{r}/(2\bar{r}-2)$, $\tilde{q} := p_3 > Nr/(r-1) > N$ and the same Lemma with $q := p_1$ and $\tilde{q} := 3p_1 > 3Nr/(2r-1) > N$, we finally obtain

$$\|t^{\gamma_1} \nabla d^n\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} \nabla d^n\|_{L_t^\infty L_x^{p_3}} + \|t^{\gamma_3} \nabla d^n\|_{L_t^{2r} L_x^{3p_1}} + \|t^{\gamma_4} \nabla d^n\|_{L_t^\infty L_x^{3p_1}} \lesssim \eta + \|t^{\alpha_1} T^n\|_{L_t^{2r} L_x^{p_1}}. \quad (5.43)$$

Similarly, by Lemma 5.3.8 with $q := p_1 \in (N/2, N)$ and $\bar{q} := p_2$ (respectively $\bar{q} := p_3/2$), so that $1/q - 1/\bar{q} = 1/p_3 < (r-1)/Nr < 1/N$ (respectively $1/q - 1/\bar{q} < (2r-1)/Nr - 2(r-1)/Nr = 1/Nr < 1/N$), we deduce that

$$\|t^{\beta_1} \nabla^2 d^n\|_{L_t^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla^2 d^n\|_{L_t^\infty L_x^{p_2}} + \|t^{\beta_3} \nabla^2 d^n\|_{L_t^{2r} L_x^{\frac{p_3}{2}}} + \|t^{\beta_4} \nabla^2 d^n\|_{L_t^\infty L_x^{\frac{p_3}{2}}} \lesssim \eta + \|t^{\alpha_1} T^n\|_{L_t^{2r} L_x^{p_1}}. \quad (5.44)$$

Finally, by Theorem 5.3.6, with $\bar{r} := 2r$ (respectively $\bar{r} := r$), $\alpha := \alpha_1$ (respectively $\alpha := \alpha_2$) and $q := p_1$, it turns out

$$\|t^{\alpha_1} \nabla^3 d^n\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} \nabla^3 d^n\|_{L_t^\infty L_x^{p_1}} \lesssim \eta + \|t^{\alpha_1} T^n\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} T^n\|_{L_t^\infty L_x^{p_1}}. \quad (5.45)$$

Summarizing (5.43), (5.44), (5.45), developing the right-hand side related to T^n by

$$\begin{aligned} \|t^{\alpha_1} T^n\|_{L_t^{2r} L_x^{p_1}} &\lesssim \|t^{\gamma_1} \nabla d^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_2} \nabla^2 d^{n-1}\|_{L_t^\infty L_x^{p_2}} + \|t^{\gamma_3} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3p_1}} \|t^{\gamma_4} \nabla d^{n-1}\|_{L_t^\infty L_x^{3p_1}} \times \\ &\times \|t^{\gamma_4} \nabla d^n\|_{L_t^\infty L_x^{3p_1}} + \|t^{\gamma_1} u^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_2} \nabla^2 d^{n-1}\|_{L_t^\infty L_x^{p_2}} + \|t^{\beta_1} \nabla u^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\gamma_2} \nabla d^{n-1}\|_{L_t^\infty L_x^{p_2}} \end{aligned}$$

and by

$$\begin{aligned} \|t^{\alpha_2} T^n\|_{L_t^\infty L_x^{p_1}} &\lesssim \|t^{\gamma_1} \nabla d^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_1} \nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_3} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3p_1}} \|t^{\gamma_3} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3p_1}} \times \\ &\times \|t^{\gamma_4} \nabla d^n\|_{L_t^\infty L_x^{3p_1}} + \|t^{\gamma_1} u^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_1} \nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{p_2}} + \|t^{\beta_1} \nabla u^{n-1}\|_{L_t^{2r} L_x^{p_2}} \|t^{\gamma_1} \nabla d^{n-1}\|_{L_t^{2r} L_x^{p_3}} \end{aligned}$$

we can absorb the terms with index n by the left-hand side of (5.43), (5.44) and (5.45), thus it results that (5.39) is true at least for the terms related to d .

Now, let us take the velocity field into account. At first, we recall that u^n fulfills

$$u^n(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}\{-u^{n-1} \cdot \nabla u^n(s) + F^n(s)\} ds, \quad (5.46)$$

with $F^n = (1 + a^n) \operatorname{div}(\nabla d^n \odot \nabla d^n) + a^n(\Delta u^{n-1} - \nabla \Pi^{n-1})$. Arguing as for (5.43), (5.44) we obtain

$$\begin{aligned} \|t^{\gamma_1} u^n\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u^n\|_{L_t^\infty L_x^{p_3}} + \|t^{\beta_1} \nabla u^n\|_{L_t^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla u^n\|_{L_t^\infty L_x^{p_2}} + \\ + \|t^{\beta_3} \nabla u^n\|_{L_t^{2r} L_x^{\frac{p_3}{2}}} + \|t^{\beta_4} \nabla u^n\|_{L_t^\infty L_x^{\frac{p_3}{2}}} \lesssim \eta + \|t^{\alpha_1} F^n\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_1} u^{n-1} \cdot \nabla u^n\|_{L_t^{2r} L_x^{p_1}}. \end{aligned} \quad (5.47)$$

Proceeding as for (5.45) and recalling that $\nabla \Pi^n = RR \cdot \{-u^{n-1} \cdot \nabla u^n + F^n\}$, we have also

$$\begin{aligned} \|t^{\alpha_1} (\nabla^2 u^n, \nabla \Pi^n)\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} (\nabla^2 u^n, \nabla \Pi^n)\|_{L_t^\infty L_x^{p_1}} \lesssim \eta + \|t^{\alpha_1} F^n\|_{L_t^{2r} L_x^{p_1}} + \\ + \|t^{\alpha_1} u^{n-1} \cdot \nabla u^n\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} F^n\|_{L_t^\infty L_x^{p_1}} + \|t^{\alpha_2} u^{n-1} \cdot \nabla u^n\|_{L_t^\infty L_x^{p_1}} \end{aligned} \quad (5.48)$$

Thus, developing the terms related to F^n by

$$\begin{aligned} \|t^{\alpha_1} F^n\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} F^n\|_{L_t^r L_x^{p_1}} &\lesssim (1 + \|a_0\|_{L_x^\infty}) \|t^{\beta_1} \nabla^2 d^n\|_{L^{2r} L_x^{p_2}} \|t^{\beta_3} \nabla^2 d^n\|_{L^\infty L_x^{p_3}} + \\ &+ \|a_0\|_{L_x^\infty} (\|t^{\alpha_1} (\Delta u^{n-1}, \nabla \Pi^{n-1})\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2} (\Delta u^{n-1}, \nabla \Pi^{n-1})\|_{L_t^r L_x^{p_1}}) \end{aligned}$$

and summarizing (5.47) and (5.48), we can absorb the terms with index n by the left-hand side, so that (5.39) is fulfilled for all the indexes $n \in \mathbb{N}$.

Moreover, applying Lemma 5.3.7 to (5.40) and (5.46), with $\bar{r} := 2r$ of $q := p_1$ and $\tilde{q} := \infty$, we get also

$$\|t^{\frac{1}{2} - \frac{1}{2r}} (u^n, \nabla d^n)\|_{L_t^{2r} L_x^\infty} + \|t^{\frac{1}{2}} (u^n, \nabla d^n)\|_{L_t^\infty L_x^\infty} \lesssim \eta + \|t^{\alpha_1} (T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{p_1}} \lesssim \eta.$$

Now, we claim by induction that $(u^n, \nabla d^n, \nabla \Pi^n)$ belongs to $\mathfrak{Y}_r^\varepsilon$ and moreover

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{Y}_r^\varepsilon} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}}, \quad (5.49)$$

for every $n \in \mathbb{N}$ (uniformly). Recalling (5.40) and (5.46) and thanks to Lemma 5.3.7 with $\bar{r} := 2r$, $q = p_1 \in (N\bar{r}/(2\bar{r} - 2), N)$, $\tilde{q} = q_3 > Nr/((1 - \varepsilon)r - 1) > N$ (since $\varepsilon < 1 - 1/r$) or $\tilde{q} = 3q_1 > 3N$,

$$\begin{aligned} \|t^{\gamma_1^\varepsilon - \varepsilon} (u^n, \nabla d^n)\|_{L_t^{2r} L_x^{q_3}} + \|t^{\gamma_3^\varepsilon - \varepsilon} \nabla d^n\|_{L_t^{2r} L_x^{3q_1}} + \|t^{\gamma_4^\varepsilon - \varepsilon} \nabla d^n\|_{L_t^\infty L_x^{3q_1}} &\lesssim \\ &\lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1}} + \|t^{\alpha_1} (T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{p_1}}, \end{aligned}$$

for every $n \in \mathbb{N}$. Hence, since $\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{Y}_r^\varepsilon} \lesssim \eta$, we deduce the following uniformly estimate:

$$\|t^{\gamma_1^\varepsilon - \varepsilon} (u^n, \nabla d^n)\|_{L_t^{2r} L_x^{q_3}} + \|t^{\gamma_3^\varepsilon - \varepsilon} \nabla d^n\|_{L_t^{2r} L_x^{3q_1}} + \|t^{\gamma_4^\varepsilon - \varepsilon} \nabla d^n\|_{L_t^\infty L_x^{3q_1}} \lesssim \eta. \quad (5.50)$$

We still proceed by induction and the base case $(0, 0, 0) \in \mathfrak{Y}_r^\varepsilon$ is trivial. Now, let us assume that $(u^{n-1}, \nabla d^{n-1}, \nabla \Pi^{n-1})$ belongs to $\mathfrak{Y}_r^\varepsilon$. At first, since $(u_0, \nabla d_0) \in \dot{B}_{q_3, 2r}^{N/q_3 - 1 + \varepsilon}$, $\nabla(u_0, \nabla d_0) \in \dot{B}_{q_2, 2r}^{N/q_2 - 2 + \varepsilon}$ and $\nabla^2(u_0, \nabla d_0) \in \dot{B}_{q_1, 2r}^{N/q_1 - 3 + \varepsilon}$, by Theorem 5.3.11 we get that the conditions for u and ∇d determined by $\mathfrak{Y}_r^\varepsilon$ are satisfied by $e^{t\Delta} u_0$ and $e^{t\Delta} \nabla d_0$. Now, arguing as for proving (5.43) and (5.47), by Lemma 5.8.1 with $\bar{r} := 2r$, $q := q_1 \in (N, N/(1 - \varepsilon))$ and $\tilde{q} := q_3 > q$ or $\tilde{q} := 3q_1 > q$, we obtain that

$$\begin{aligned} \|t^{\gamma_1^\varepsilon} (u^n, \nabla d^n)\|_{L_t^{2r} L_x^{q_3}} + \|t^{\gamma_2^\varepsilon} (u^n, \nabla d^n)\|_{L_t^\infty L_x^{q_3}} + \|t^{\gamma_3^\varepsilon} \nabla d^n\|_{L_t^{2r} L_x^{3q_1}} + \\ + \|t^{\gamma_4^\varepsilon} \nabla d^n\|_{L_t^\infty L_x^{3q_1}} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \|t^{\alpha_1} (T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{q_1}} \end{aligned} \quad (5.51)$$

Moreover by Lemma 5.8.2 with $\bar{r} := 2r$, $q := q_1$ and $\bar{q} := q_2$, so that $1/q - 1/\bar{q} = 1/q_3 < 1/N$, we have

$$\begin{aligned} \|t^{\beta_1^\varepsilon} \nabla (u^n, \nabla d^n)\|_{L_t^{2r} L_x^{q_2}} + \|t^{\beta_2^\varepsilon} \nabla (u^n, \nabla d^n)\|_{L_t^\infty L_x^{q_2}} &\lesssim \\ &\lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \|t^{\alpha_1} (T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{q_1}} \end{aligned} \quad (5.52)$$

Finally, by Theorem 5.3.6 with $\bar{r} := 2r$ (respectively $\bar{r} := r$) and $\alpha := \alpha_1^\varepsilon$ (respectively $\alpha := \alpha_2^\varepsilon$),

we get

$$\begin{aligned} & \|t^{\alpha_1^\varepsilon}(\nabla^2 u^n, \nabla^3 d^n \nabla \Pi)\|_{L_t^{2r} L_x^{q_1}} + \|t^{\alpha_2^\varepsilon}(\nabla^2 u^n, \nabla^3 d^n \nabla \Pi)\|_{L_t^r L_x^{q_1}} \lesssim \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}} + \\ & \quad + \|t^{\alpha_1^\varepsilon}(T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{q_1}} + \|t^{\alpha_2^\varepsilon}(T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^r L_x^{q_1}} \end{aligned} \quad (5.53)$$

Thus, summarizing (5.51), (5.52), (5.53) we develop the terms on the right-hand sides as follows

$$\begin{aligned} & \|t^{\alpha_1^\varepsilon}(T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^{2r} L_x^{q_1}} + \|t^{\alpha_2^\varepsilon}(T^n, F^n, u^{n-1} \cdot \nabla u^n)\|_{L_t^r L_x^{q_1}} \lesssim \|t^{\gamma_1^\varepsilon - \varepsilon} \nabla d^{n-1}\|_{L_t^{2r} L_x^{q_3}} \times \\ & \quad \times \|t^{\beta_2^\varepsilon} \nabla^2 d^{n-1}\|_{L_t^\infty L_x^{q_2}} + \|t^{\gamma_3^\varepsilon - \varepsilon} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3q_1}} \|t^{\gamma_4^\varepsilon - \varepsilon} \nabla d^{n-1}\|_{L_t^\infty L_x^{3q_1}} \|t^{\gamma_4^\varepsilon} \nabla d^n\|_{L_t^\infty L_x^{3q_1}} + \\ & \quad + \|t^{\gamma_1^\varepsilon - \varepsilon} u^{n-1}\|_{L_t^{2r} L_x^{q_3}} \|t^{\beta_2^\varepsilon} \nabla^2 d^{n-1}\|_{L_t^\infty L_x^{q_2}} + \|t^{\beta_1^\varepsilon} \nabla u^{n-1}\|_{L_t^{2r} L_x^{q_3}} \|t^{\gamma_2^\varepsilon - \varepsilon} \nabla d^{n-1}\|_{L_t^\infty L_x^{q_2}} + \\ & \quad + \|t^{\gamma_1} \nabla d^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_1} \nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_3} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3p_1}} \|t^{\gamma_3} \nabla d^{n-1}\|_{L_t^{2r} L_x^{3p_1}} \times \\ & \quad \times \|t^{\gamma_4} \nabla d^n\|_{L_t^\infty L_x^{3p_1}} + \|t^{\gamma_1} u^{n-1}\|_{L_t^{2r} L_x^{p_3}} \|t^{\beta_1} \nabla^2 d^{n-1}\|_{L_t^{2r} L_x^{p_2}} + \|t^{\beta_1} \nabla u^{n-1}\|_{L_t^{2r} L_x^{p_2}} \times \\ & \quad \times \|t^{\gamma_1} \nabla d^{n-1}\|_{L_t^{2r} L_x^{p_3}} + (1 + \|a_0\|_{L_x^\infty}) \|t^{\beta_1} \nabla^2 d^n\|_{L_t^{2r} L_x^{q_2}} \|t^{\gamma_3 - \varepsilon} \nabla d^n\|_{L_t^\infty L_x^{q_3}} + \\ & \quad + \|a_0\|_{L_x^\infty} (\|t^{\alpha_1^\varepsilon}(\Delta u^{n-1}, \nabla \Pi^{n-1})\|_{L_t^{2r} L_x^{p_1}} + \|t^{\alpha_2^\varepsilon}(\Delta u^{n-1}, \nabla \Pi^{n-1})\|_{L_t^r L_x^{p_1}}). \end{aligned}$$

Thus, recalling (5.50), we can absorb all the terms with index n by the left-hand side of (5.51), (5.52) and (5.53), finally obtaining (5.49), thanks to the induction hypotheses.

Now let us observe that, by Remark 5.4.3, for every $T > 0$ there exists $\hat{C}(T) > 0$ such that

$$\|\nabla u^n\|_{L_T^1 L_x^\infty} \lesssim \hat{C}(T) \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}}.$$

To conclude the proof we want to show that $(a^n, d^n, u^n)_\mathbb{N}$ is a Cauchy sequence in the considered spaces. The strategy is similar to the last part of Theorem 5.4.1. Denoting $\delta u^n := u^{n+1} - u^n$ and so on for δd^n , δa^n and $\delta \Pi^n$, for all $T > 0$ we define

$$\delta U^n(T) := \|(\delta u^n, \nabla \delta d^n, \nabla \Pi^n)\|_{\mathfrak{Y}_{r,T}} + \|\delta d^n\|_{L_T^\infty L_x^\infty} + \|t^{\frac{1}{2}} \delta u^n\|_{L_T^\infty L_x^\infty}.$$

We want to prove that $\sum_{n \in \mathbb{N}} \delta U^n(T)$ is finite. Let us consider δa^n which is solution of

$$\partial_t \delta a^n + u^n \cdot \nabla \delta a^n = -\delta u^{n-1} \nabla a^n \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N \quad \text{with } \delta a^n|_{t=0} = 0 \quad \text{in } \mathbb{R}^N.$$

By standard estimates for the transport equation and by (5.42), we obtain

$$\|\delta a^n(t)\|_{L_x^\infty} \leq \int_0^t \|\delta u^{n-1}(s)\|_{L_x^\infty} \|\nabla a^n(s)\|_{L_x^\infty} ds \leq \bar{C}(T) \left(\int_0^T \frac{1}{s^{\frac{3}{4}}} ds \right)^{\frac{2}{3}} \|\delta U^{n-1}\|_{L_T^3 L_x^3} \|\nabla a_0\|_{L_x^\infty}, \quad (5.54)$$

where $\bar{C}(T) = \exp\{\hat{C}(T) \|(u_0, \nabla d_0)\|_{\dot{B}_{p,r}^{\frac{N}{p}-1+\varepsilon}}\}$. Considering δd^n , we recall that is solution of

$$\partial_t \delta d^n - \Delta \delta d^n = \delta H^n \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N, \quad \delta d^n|_{t=0} = 0 \quad \text{in } \mathbb{R}^N,$$

with $\delta H^n = -\delta u^{n-1} \cdot \nabla d^n - u^{n-1} \cdot \nabla \delta d^{n-1} + \nabla \delta d^{n-1} \odot \nabla d^n + \nabla d^{n-1} \odot \nabla \delta d^{n-1} + \nabla d^{n-1} \odot \nabla d^{n-1} \delta d^n$. Hence, by Lemma 5.8.3 with $\sigma = 2\gamma_1$ and $q = p_3/2$, we get

$$\|\delta d^n(t)\|_{L_x^\infty} \leq \|s^{2\gamma_1} \delta H^n\|_{L^r(0,t; L_x^{\frac{p_3}{2}})} \lesssim \eta \delta U^{n-1}(T) + \eta^2 \|\delta d^n\|_{L_T^\infty L_x^\infty},$$

for every $t \in (0, T)$. Taking the sup on $t \in (0, T)$, we obtain

$$\|\delta d^n\|_{L_T^\infty L_x^\infty} \lesssim \eta \delta U^{n-1}(T). \quad (5.55)$$

Moreover, recalling that δu^n is solution of

$$\partial_t \delta u^n - \Delta \delta u^n + \nabla \delta \Pi^n = \delta F^n - u^n \cdot \nabla \delta u^n - \delta u^{n-1} \cdot \nabla u^n, \quad \operatorname{div} \delta u^n = 0, \quad \delta u^n|_{t=0} = 0,$$

where the explicit formula of δF^n is

$$\begin{aligned} \delta F^n = & \delta a^n (\operatorname{div}\{\nabla d^n \odot \nabla d^n\} + \Delta u^n - \nabla \Pi^n) + (1 + a^{n+1}) \operatorname{div}\{\nabla \delta d^n \odot \nabla d^n + \\ & + \nabla d^{n+1} \odot \nabla \delta d^n\} + a^n (\Delta \delta u^{n-1} - \nabla \delta \Pi^{n-1}), \end{aligned}$$

which yields that

$$\begin{aligned} \|(\delta u^n, \nabla \delta d^n, \nabla \delta \Pi^n)\|_{\mathfrak{Y}_r} + \|t^{\frac{1}{2}} \delta u^n\|_{L_{T,x}^\infty} & \lesssim \|t^{\alpha_1} (\nabla \delta H^n, \delta F^n, u^n \cdot \nabla \delta u^n, \delta u^{n-1} \cdot \nabla u^n)\|_{L_T^{2r} L_x^{p_1}} \\ + \|t^{\alpha_2} (\nabla \delta H^n, \delta F^n, u^n \cdot \nabla \delta u^n, \delta u^{n-1} \cdot \nabla u^n)\|_{L_T^r L_x^{p_1}} & \lesssim \eta (\delta U^{n-1}(T) + \delta U^n(T) + \|\delta a^n\|_{L_{T,x}^\infty}). \end{aligned} \quad (5.56)$$

Summarizing (5.55) and (5.56), it turns out that

$$\delta U^n(T) \lesssim \eta \delta U^{n-1}(T) + \eta \|\delta a^n\|_{L_T^\infty L_x^\infty}$$

and thanks to (5.54), we finally obtain

$$\delta U^n(T) \lesssim \eta \delta U^{n-1}(T) + \bar{C}(T) T^{\frac{1}{6}} \|\delta U^{n-1}\|_{L_T^3} \|\nabla a_0\|_{L_x^\infty}.$$

Such inequality is strictly similar to (5.36), hence we can conclude the proof of the proposition arguing exactly as in the last part of the proof of proposition 5.4.1. \square

5.5 Existence of a Global Solution

Let us now tackle the proof to the existence part of our main results, namely Theorem 5.2.2 and Theorem 5.2.4. Thanks to the dyadic partition we regularize the initial velocity u_0 and the initial molecular orientation d_0 , while we regularize the initial density a_0 by a family of mollifiers. The key is to use the existence results and the estimates of the previous section, constructing a family of solutions for (5.2) with the regularized initial data. Due to the low regularity of a_0 , it is not possible to prove the strong convergence of such approximate solutions. Hence, we shall focus on a compactness method, along the same line of [30] and [60].

Let $(\chi_n)_\mathbb{N}$ be a family of mollifiers, we define $a_{0,n} := \chi_n * a_0$, for every $n \in \mathbb{N}$. $a_{0,n}$ belongs to $W_x^{1,\infty}$ and its L_x^∞ -norm is bounded by $\|a_0\|_{L_x^\infty}$. Moreover, $(a_{0,n})_\mathbb{N}$ weak* converges to a_0 up to a subsequence (which we still denote by $(a_{0,n})_\mathbb{N}$). Since d_0 belongs to L_x^∞ , which is a subset of $\dot{B}_{\infty,\infty}^0$, and u_0 belongs to $\dot{B}_{p,r}^{N/p-1}$, we cut the low and the high frequencies in the following way:

$$u_{0,n} := \sum_{|k| \leq n} \dot{\Delta}_k u_0, \quad d_{0,n} := \sum_{|k| \leq n} \dot{\Delta}_k d_0.$$

Each term $d_{0,n}$ belongs to L_x^∞ with norm bounded by 1. Moreover $u_{0,n}$ and $\nabla d_{0,n}$ belong to $\dot{B}_{p,1}^s$ for every real number s . In addition, the smallness condition (5.4) is still valid for $(a_{0,n}, u_{0,n}, \nabla d_{0,n})$.

Proof of Theorem 5.2.2. As already pointed out, $u_{0,n}$ and $\nabla d_{0,n}$ belong to $\dot{B}_{p,1}^s$, for every real number s , in particular for $s = N/p - 1$ and $s = N/p + 1$. The hypotheses of Proposition 5.4.1 are fulfilled, hence it determines (u^n, d^n, a^n) solution of (5.2) with $u_{0,n}$, $d_{0,n}$ and $a_{0,n}$ as initial data. Furthermore we get the following uniform estimates for the norms of such solutions:

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{X}_r} \lesssim \eta$$

for every $n \in \mathbb{N}$. By these inequalities and the momentum equation of (5.2), $(\partial_t u^n)_{\mathbb{N}}$ is a bounded sequence in $L_t^r L_x^{Nr/(3r-2)}$ and $(\partial_t d^n)_{\mathbb{N}}$ is a bounded sequence in $L_t^r L_x^{Nr/(2r-2)}$. Thus, applying Ascoli-Arzelà Theorem, we conclude that there exists a subsequence of $(u^n, d^n, a^n)_{\mathbb{N}}$ (which we still denote by $(u^n, d^n, a^n)_{\mathbb{N}}$) and some $(u, d, a, \nabla \Pi)$ with $a, d \in L_{t,x}^\infty$ and $(u, \nabla d, \nabla \Pi) \in \mathfrak{X}_r$ such that

$$\begin{aligned} (a^n, d^n) &\rightharpoonup (a, d) && \text{weak } * \text{ in } L_{t,x}^\infty, \\ \nabla d^n &\rightharpoonup \nabla d && \text{weakly in } L_t^{3r} L_x^{\frac{3Nr}{3r-2}}, L_t^{2r} L_x^{\frac{Nr}{r-1}}, L_t^2 L_x^\infty, \\ \nabla(u^n, \nabla d^n) &\rightharpoonup \nabla(u, \nabla d) && \text{weakly in } L_t^{2r} L_x^{\frac{Nr}{2r-1}}, L_t^r L_x^{\frac{Nr}{r-1}}, \end{aligned}$$

with in addition

$$\nabla(\nabla u^n, \Pi^n) \rightharpoonup \nabla(\nabla u, \Pi) \quad \text{weakly in } L_t^r L_x^{\frac{Nr}{3r-2}}$$

and

$$u^n \rightarrow u \quad \text{strongly in } L_{t,loc}^r L_{x,loc}^{\frac{Nr}{r-1} - \bar{\varepsilon}},$$

for all positive $\bar{\varepsilon}$ small enough. The last strongly convergence is due to an interpolation result, observing that, for every $T > 0$, the sequence $(u_n - e^{t\Delta} u_{0,n})_{\mathbb{N}}$ is uniformly bounded and equicontinuous in $C([0, T], L_x^{Nr/(3r-2)})$ and moreover $(e^{t\Delta} u_{0,n})_{\mathbb{N}}$ converges to $e^{t\Delta} u_0$ strongly in $L_t^r L_x^{Nr/(r-1)}$ (since $(u_{0,n})_{\mathbb{N}}$ converges to u_0 strongly in $\dot{B}_{p,r}^{N/r-1}$). We deduce that $u^n \cdot \nabla d^n$ and $u^n \cdot \nabla u^n$ converge to $u \cdot \nabla d$ and $u \cdot \nabla u$ respectively. Then, it is sufficient to prove that $a^n(\Delta u^n + \nabla \Pi^n)$ converge to $a(\Delta u + \nabla \Pi)$ in the distributional sense, in order to conclude that (u, d, a) is a solution for (5.2) with initial data (u_0, d_0, a_0) . Toward this, we shall follow [30] and [60], proving that $(a^n)_{\mathbb{N}}$ strongly converges to a in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^N)$ for any $m < \infty$. thanks to the transport equation of (5.2) we have

$$\partial_t (a^n)^2 + u^n \cdot \nabla (a^n)^2 = 0,$$

which yields

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

where ω is the weak $*$ limit of $((a^n)^2)_{\mathbb{N}}$ (up to a subsequence). Moreover, by a mollifying method as that in [37], we infer that

$$\partial_t a^2 + \operatorname{div}(u a^2) = 0.$$

Thus

$$\begin{cases} \partial_t (a^2 - \omega) + \operatorname{div}\{u(a^2 - \omega)\} = 0 & \mathbb{R}_+ \times \mathbb{R}^N \\ (a^2 - \omega)|_{t=0} = 0, \end{cases}$$

and from the uniqueness of the transport equation (see [37]) we conclude that $a^2 - \omega = 0$ almost everywhere. We deduce that $(\|a^n\|_{L^2(\Omega)})_{\mathbb{N}}$ converges to $\|a\|_{L^2(\Omega)}$, for every Ω bounded subset of $\mathbb{R}_+ \times \mathbb{R}^N$, hence $(a^n)_{\mathbb{N}}$ strongly converges to a in $L^2(\Omega)$. By interpolation, we deduce that $(a^n)_{\mathbb{N}}$ strongly converges to a in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^N)$ for any $m < \infty$ and this completes the proof to the existence part of Theorem 5.2.2. \square

Proof of Theorem 5.2.4. We proceed along the same line of the previous proof, using Proposition 5.4.2 instead of Proposition 5.4.1. We get the following uniform estimates for the sequence of the approximate solutions:

$$\|(u^n, \nabla d^n, \nabla \Pi^n)\|_{\mathfrak{Y}_r} \lesssim \eta.$$

Since $\alpha_2 r' = (1/2(3 - N/p_1) - 1/r)r' < (1 - 1/r)r' = 1$, $(\nabla^2 u^n, \nabla \Pi^n)_{\mathbb{N}} = (t^{-\alpha_2} t^{\alpha_2} (\nabla^2 u^n, \nabla \Pi^n)_{\mathbb{N}})$ is uniformly bounded in $L_T^{\tau_1} L_x^{p_1}$, where τ_1 belongs to $(1, r/(1 + \alpha_2 r))$ and $T > 0$. Similarly $(\nabla u^n, \nabla^2 d^n)_{\mathbb{N}}$ and $(u^n, \nabla d^n)_{\mathbb{N}}$ are uniformly bounded in $L_T^{\tau_2} L_x^{p_2}$ and $L_T^{\tau_3} L_x^{p_3}$ respectively, where τ_2 belongs to $(1, 2r/(1 + \beta_1 2r))$ and $\tau_3 \in (1, 2r/(1 + \gamma_1 2r))$. It is not restrictive to choose τ_2 and τ_3 such that $1/\tau_4 := 1/\tau_2 + 1/\tau_3$ is less than 1. Hence $(\partial_t u^n)_{\mathbb{N}}$ is uniformly bounded in $L^{\tau_1}(0, T; L_x^{p_1})$ which yields that $(u^n - e^{t\Delta} u_{0,n})_{\mathbb{N}}$ is uniformly bounded and equicontinuous in $C([0, T], L_x^{p_1})$. Moreover $(e^{t\Delta} u_{0,n})_{\mathbb{N}}$ converges to $e^{t\Delta} u_0$ in $L^{\tau_3}(0, T; L^{p_3})$. Hence, by Ascoli-Arzelà Theorem, we conclude that, up to extraction, the sequence $(u^n, d^n, a^n, \nabla \Pi^n)_{\mathbb{N}}$ converges to some $(u, d, a, \nabla \Pi)_{\mathbb{N}}$ such that a, d belong to $L_{t,x}^{\infty}$ and $(u, \nabla d, \nabla \Pi) \in \mathfrak{Y}_r$. The convergence is in the following sense:

$$\begin{aligned} (a^n, d^n) &\rightharpoonup (a, d) && \text{weak } * \text{ in } L_{t,loc}^{\infty}, \\ \nabla d^n &\rightharpoonup \nabla d && \text{weakly in } L_{t,loc}^{\tau_3} L_x^{p_3}, \\ \nabla(u^n, \nabla d^n) &\rightharpoonup \nabla(u, \nabla d) && \text{weakly in } L_{t,loc}^{\tau_2} L_x^{p_2}, \end{aligned}$$

with in addition

$$\nabla(\nabla u^n, \Pi^n) \rightharpoonup \nabla(\nabla u, \Pi) \quad \text{weakly in } L_{t,loc}^{\tau_1} L_x^{p_1}$$

and

$$u^n \rightarrow u \quad \text{strongly in } L_{t,loc}^{\tau_3} L_{x,loc}^{p_3 - \varepsilon},$$

for all positive ε small enough. Finally we can repeat the argument at the end of the proof of Theorem 5.2.2, concluding the existence part of Theorem 5.2.4. \square

5.6 Lagrangian Coordinates

The uniqueness result is basically based on the Lagrangian coordinates concept. The key is to rewrite system (5.2) under such coordinates, obtaining a new formulation which allows the uniqueness in the functional framework of the main Theorems. This strategy has already been treated by Danchin and Mucha in [27] on a subfamily of (5.2), namely the incompressible Navier-Stokes equations with variable density in the whole space. We claim to extend it to the general simplified Ericksen-Leslie system. Before going on, in this section we recall some mainly results concerning the Lagrangian coordinates.

Let $T \in (0, \infty]$, we consider a vector field u in $L_T^1 \mathcal{L}ip_x$. The flow X of u is defined as the solution of the following ordinary differential equation:

$$\begin{cases} \partial_t X(t, y) = u(t, X(t, y)) & (t, y) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ X(0, y) = y & y \in \mathbb{R}^n. \end{cases}$$

The unique solution is granted by Cauchy-Lipschitz Theorem. Defining $v(t, y) = u(t, X(t, y))$ we get the following relation between the Eulerian coordinates x and the Lagrangian coordinates y :

$$x = X(t, y) = y + \int_0^t v(s, y) ds.$$

Furthermore, fixing $t' \in \mathbb{R}$, let $\tilde{X} = \tilde{X}(t', t, x)$ be the unique solution of

$$\begin{cases} \partial_t \tilde{X}(t', t, x) = u(t, X(t', t, x)) & (t, x) \in (t', \infty) \times \mathbb{R}^n, \\ \tilde{X}(t', t', x) = x & y \in \mathbb{R}^n. \end{cases}$$

Then $Y = Y(t, x) = \tilde{X}(-t, 0, x)$ is the inverse map of X . Setting $D := {}^t\nabla$, we get $A(t, y) := (D_y X)^{-1}(t, y) = D_x Y(t, X(t, y))$ and moreover

$$\|A(t) - \text{Id}\|_{L_x^\infty} \leq \int_0^t \|D_y v(s)\|_{L_x^\infty} ds.$$

Assuming that u has $L_T^1 \mathcal{L}ip_x$ -norm small enough, we obtain that the right-hand side of the previous inequality is less than 1. Thus $A(t, y)$ is determined by

$$A(t, y) = D_x Y(t, X(t, y)) = (\text{Id} + (D_y X(t, y) - \text{Id}))^{-1} = \sum_{k=-\infty}^{\infty} (-1)^k \left(\int_0^t D_y v(s, y) ds \right)^k.$$

Furthermore

$$(\nabla_x u)(t, X(t, y)) = {}^t A(t, y) \nabla_y v(t, y), \quad (\text{div}_x u)(t, X(t, y)) = \text{div}_y \{A(t, y) v(t, y)\},$$

Setting $b(t, y) := a(t, X(t, y))$, $P(t, y) := \Pi(t, X(t, y))$ and moreover

$$\omega(t, y) := d(t, X(t, y)), \quad h(t, y) := (\nabla_x d)(t, X(t, y)), \quad (5.57)$$

system (5.2) becomes

$$\begin{cases} \partial_t b = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t v - (1+b) \text{div}_y \{A {}^t A \nabla_y v\} + {}^t A \nabla_y P = -(1+b) \text{div}_y \{A h \odot h\} & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t \omega - {}^t A : \nabla h = |h|^2 \omega & \mathbb{R}_+ \times \mathbb{R}^N, \\ \text{div}({}^t A v) = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ h = {}^t A \nabla_y \omega & \mathbb{R}_+ \times \mathbb{R}^N, \\ (v, b, \omega)|_{t=0} = (u_0, a_0, d_0) & \mathbb{R}^N, \end{cases}$$

which is the Lagrangian formulation. Moreover, taking the derivative in x to the third equation of (5.2)

$$\partial_t \nabla d + u \cdot \nabla^2 d + \nabla u \cdot \nabla d - \Delta \nabla d = 2 \nabla d \cdot \nabla^2 d + |\nabla d|^2 \nabla d,$$

thus, h is solution of

$$\partial_t h + ({}^t A \nabla_y v) \cdot \nabla_y h - \text{div}_y (A {}^t A \nabla_y h) = 2h \cdot \nabla_y h \omega + |h|^2 h.$$

5.7 Uniqueness

This section is devoted to the proof of Theorem 5.2.6 and Theorem 5.2.7. For $i = 1, 2$, let (u_i, d_i, a_i) be two solutions of (5.2) satisfying the condition of Theorem 5.2.4. Let X_i be the flow generated by u_i , for $i = 1, 2$, and (v_i, ω_i, b_i) the Lagrangian formulations of the solutions. At first, let us observe that $b_1 \equiv b_2 \equiv a_0$, thus setting $\delta v := v_1 - v_2$, $\delta \omega := \omega_1 - \omega_2$ and $\delta P := P_1 - P_2$, we observe

that $(\delta v, \delta \omega, \delta h, \delta P)$ is solution for

$$\begin{cases} \partial_t \delta v - \Delta \delta v + \nabla \delta P = a_0(\Delta \delta v - \nabla \delta P) + \delta f_1 + \delta f_2 + \delta f_3 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t \delta \omega = \delta f_4 + \delta f_5 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t \delta h - \Delta \delta h = \delta f_6 + \delta f_7 + \delta f_8 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div}\{\delta v\} = \delta g & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t \delta g = \delta R & \mathbb{R}_+ \times \mathbb{R}^N, \\ (\delta v, \delta \omega, \delta h)|_{t=0} = (0, 0, 0) & \mathbb{R}^N, \end{cases} \quad (5.58)$$

where

$$\begin{aligned} \delta f_1 &:= (1 + a_0)[(\operatorname{Id} - {}^t A_2)\nabla \delta P - \delta A \nabla P_1], \\ \delta f_2 &:= (1 + a_0)\operatorname{div}\{(A_2 {}^t A_2 - \operatorname{Id})\nabla \delta v + (A_2 {}^t A_2 - A_1 {}^t A_1)\nabla v_1\}, \\ \delta f_3 &:= (1 + a_0)\operatorname{div}\{\delta A(h_2 \odot h_2) + A_1(\delta h \odot h_2) + A_1(h_1 \odot \delta h)\}, \\ \delta f_4 &:= \delta h \cdot h_2 \omega_2 + h_1 \cdot \delta h \omega_2 + |h_1|^2 \delta \omega, \\ \delta f_5 &:= {}^t \delta A : \nabla h_2 + {}^t A_1 : \nabla \delta h, \end{aligned}$$

and

$$\begin{aligned} \delta f_6 &:= -{}^t \delta A \nabla v_2 \cdot h_2 - {}^t A_1 \nabla \delta v \cdot h_2 - {}^t A_1 \nabla v_1 \cdot \delta h, \\ \delta f_7 &:= \operatorname{div}\{(A_2 {}^t A_2 - \operatorname{Id})\nabla \delta h + (A_2 {}^t A_2 - A_1 {}^t A_1)\nabla h_1\}, \\ \delta f_8 &:= 2\delta h \cdot \nabla h_2 \omega_2 + 2h_1 \cdot \nabla \delta h \omega_2 + h_1 \cdot \nabla h_1 \delta \omega, \\ \delta g &:= (\operatorname{Id} - {}^t A_2) : \nabla \delta v + {}^t \delta A : \nabla v_1, \\ \delta R &:= \partial_t[(\operatorname{Id} - A_2)\delta v] - \partial_t[\delta A v_1]. \end{aligned}$$

In what follows, we will use repeatedly the following identity:

$$\delta A(t) = \left(\int_0^t D\delta v(\tau) d\tau \right) \left(\sum_{k \geq 1} \sum_{0 \leq j < k} C_1^j(t) C_2^{k-1-j}(t) \right), \quad (5.59)$$

with

$$C_i(t) := \int_0^t Dv_i(\tau), \quad \text{for } i = 1, 2.$$

5.7.1 Uniqueness: the smooth case

Let us assume that $1 < p < Nr/(3r-2)$, $\varepsilon \in (0, 1]$ and $r \in (1, 2/(2-\varepsilon))$. We suppose our initial data $(u_0, \nabla d_0)$ to be in $\dot{B}_{p,r}^{N/p-1+\varepsilon} \cap \dot{B}_{p,r}^{N/p-1}$ and we want to prove that the solution for (5.1), given by Theorem 5.2.2, is unique. First, let us observe that our solution belongs to the functional framework of Theorem 5.2.6, thanks to proposition 5.4.1. Now, let us tackle the proof of the uniqueness. We need the following Lemma

Lemma 5.7.1. *Let $T > 0$ and let us assume that $f, \nabla g$, and R belong to $L_T^r L_x^{Nr/(3r-2)}$. Then*

$$\begin{cases} \partial_t v - \Delta v + \nabla P = f & (0, T) \times \mathbb{R}^N, \\ \operatorname{div} v = g & (0, T) \times \mathbb{R}^N, \\ \partial_t g = \operatorname{div} R & (0, T) \times \mathbb{R}^N, \\ v|_{t=0} = 0 & \mathbb{R}^N, \end{cases}$$

admits a unique solution such that

$$\begin{aligned} & \|v\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|v\|_{L_T^2 L_x^\infty} + \|\nabla v\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla v\|_{L_T^{2r} L_x^{\frac{Nr}{2(r-1)}}} + \\ & + \|(\partial_t v, \nabla^2 v, \nabla \Pi)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \|(f, \nabla g, R)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}}. \end{aligned}$$

Proof. Applying $-\operatorname{div}$ to the first equation, we get

$$-\Delta P = \operatorname{div} \{R - f - \nabla g\},$$

which yields $\nabla P = RR \cdot \{R - f - \nabla g\}$. Hence, $\|\nabla P\|_{L_x^q} \leq \|R, f, \nabla g\|_{L_x^q}$, for every $q \in (1, \infty)$. Moreover, v is determined by

$$v(t) = \int_0^t e^{(t-s)\Delta} (f - \nabla P)(s) ds,$$

thus by Lemmas 5.3.3, 5.3.5 and Theorem 5.3.2, we obtain the required estimate. \square

Thus, recalling system (5.58), we get

$$\begin{aligned} & \|\delta v\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\delta v\|_{L_T^2 L_x^\infty} + \|\nabla \delta v\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla \delta v\|_{L_T^{2r} L_x^{\frac{Nr}{2(r-1)}}} + \|\delta v\|_{L_T^2 L_x^\infty} + \\ & + \|(\partial_t \delta v, \nabla^2 \delta v, \nabla \delta \Pi)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \|(\delta f_1, \delta f_2, \delta f_3, \nabla \delta g, \delta R)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}}, \end{aligned}$$

where we have also used that $\|b_0\|_{L_x^\infty} = \|a_0\|_{L_x^\infty} \leq \eta$. Furthermore by the second equation of (5.58), we get $\delta \omega \in L_T^\infty L_x^{Nr/2(r-1)}$ and

$$\|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \lesssim \|(\delta f_4, \delta f_5)\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}},$$

and by Theorem 5.3.2, Lemmas 5.3.3 and 5.3.5, we get

$$\begin{aligned} & \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\delta h\|_{L_T^{3r} L_x^{\frac{3Nr}{3r-2}}} + \|\delta h\|_{L_T^2 L_x^\infty} + \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \\ & + \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2(r-1)}}} + \|\nabla^2 \delta h\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \|(\delta f_6, \delta f_7, \delta f_8)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}}. \end{aligned}$$

Summarizing the previous inequality, we need to control the right-hand side of

$$\begin{aligned} & \|(\delta v, \delta h, \delta P)\|_{\mathfrak{X}_{r,T}} + \|\partial_t v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \lesssim \\ & \lesssim \|(\delta f_1, \delta f_2, \delta f_3, \delta f_6, \delta f_7, \delta f_8, \nabla \delta g, \delta R)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|(\delta f_4, \nabla \delta f_5, \delta f_6)\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}}. \end{aligned}$$

We are going to estimate each of these terms step by step. Moreover, in what follows we will use

that

$$\begin{aligned} \|(\nabla^2 v_i, \nabla^2 h_i, \nabla P_i)\|_{L_T^r L_x^N} &\lesssim \|(\nabla^2 v_i, \nabla^2 h_i, \nabla P_i)\|_{L_T^r L_x^{\frac{Nr}{(3-\varepsilon)r-2}}} + \|(\nabla^2 v_i, \nabla^2 h_i, \nabla P_i)\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \\ &< \infty, \\ \|\nabla(v_i, h_i)\|_{L_T^{2r} L_x^{Nr}} &\lesssim \|\nabla(v_i, h_i)\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} + \|\nabla(v_i, h_i)\|_{L_T^{2r} L_x^{\frac{Nr}{(2-\varepsilon)r-1}}} < \infty, \\ \|\nabla A_i\|_{L_T^\infty L_x^{\frac{Nr}{(3-\varepsilon)r-2}}} &\lesssim \|\nabla A_i\|_{L_T^\infty L_x^N} + \|\nabla A_i\|_{L_T^\infty L_x^{\frac{Nr}{3r-2}}} < \infty, \end{aligned}$$

for $i = 1, 2$.

Bounds for δf_1 . From the definition of δf_1 , we readily get

$$\|\delta f_1\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \|\text{Id} - {}^t A_2\|_{L_T^\infty L_x^\infty} \|\nabla \delta P\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \|\nabla P\|_{L_T^{2r} L_x^N},$$

where $Nr/(2r-2)$ is the Lebesgue exponent in the critical Sobolev embedding

$$W_x^{1, \frac{Nr}{3r-2}} \hookrightarrow L_x^{\frac{Nr}{2(r-1)}}.$$

Consequently, because $T < 1$, recalling (5.59), we obtain

$$\|\delta f_1\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \|\nabla v_2\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} \|\nabla \delta P\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|\nabla^2 \delta v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \|\nabla P\|_{L_T^{2r} L_x^{\frac{Nr}{3r-2}}}.$$

Thus there exists a continuous function $t \rightarrow \chi_1(t)$, which goes to 0 for $t \rightarrow 0$ and

$$\|\delta f_1\|_{L_{rT} L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_1(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.60)$$

Bounds for δf_2 . From the definition of δf_2 and observing that $A_2 {}^t A_2 - A_1 {}^t A_1 = \delta A {}^t A_2 + A_1 {}^t \delta A$, we deduce that

$$\begin{aligned} \|\delta f_2\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\nabla(A_2 {}^t A_2)\|_{L_T^\infty L_x^N} \|\nabla \delta v\|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}} + (\|A_2 {}^t A_2\|_{L_T^\infty L_x^\infty} + 1) \|\nabla^2 \delta v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} (\|\nabla A_2\|_{L_T^\infty L_x^N} + \|\nabla A_1\|_{L_T^\infty L_x^N}) \|\nabla v_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} + \|\nabla \delta A\|_{L_T^\infty L_x^{\frac{Nr}{3r-2}}} (\|A_2\|_{L_T^\infty L_x^\infty} + \\ &+ \|A_1\|_{L_T^\infty L_x^\infty}) \|\nabla v_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} + \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} (\|A_2\|_{L_T^\infty L_x^\infty} + \|A_1\|_{L_T^\infty L_x^\infty}) \|\nabla^2 v_1\|_{L_T L_x^N} \end{aligned}$$

Hence, there exists a continuous function $t \rightarrow \chi_2(t)$ which goes to 0 for $t \rightarrow 0$, such that

$$\|\delta f_2\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_2(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.61)$$

Bounds for δf_3 From the definition of δf_3 we get

$$\begin{aligned}
\|\delta f_3\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \\
&\|\nabla \delta A\|_{L_T^\infty L_x^{\frac{Nr}{3r-2}}} \|h_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty}^2 + \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2r-1}}} \|\nabla h_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \\
&+ \|\nabla A_1\|_{L_T^\infty L_x^N} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\nabla A_1\|_{L_T^\infty L_x^N} \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \\
&+ \|A_1\|_{L_T^\infty L_x^\infty} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|A_1\|_{L_T^\infty L_x^\infty} \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \\
&+ \|\nabla A_1\|_{L_T^\infty L_x^N} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\nabla A_1\|_{L_T^\infty L_x^N} \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \\
&+ \|A_1\|_{L_T^\infty L_x^\infty} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|A_1\|_{L_T^\infty L_x^\infty} \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-2}}}
\end{aligned}$$

Hence, there exists $\chi_3(t)$ such that

$$\|\delta f_3\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_3(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.62)$$

Bounds for δf_6 From the definition of δf_6 we get

$$\begin{aligned}
\|\delta f_6\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2r-1}}} \|\nabla v_2\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} \|h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \\
&+ \|A_1\|_{L_T^\infty L_x^\infty} \|\nabla \delta v\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|h_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|A_1\|_{L_T^\infty L_x^\infty} \|\nabla v_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-2}}},
\end{aligned}$$

Thus

$$\|\delta f_6\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_6(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}, \quad (5.63)$$

for an oportune continuous function $\chi_7(t)$ which goes to 0 when $t \rightarrow 0$.

Bounds for δf_7 . From the definition of δf_7 we have

$$\begin{aligned}
\|\delta f_7\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\nabla A_2\|_{L_T^\infty L_x^N} \|A_2\|_{L_T^\infty L_x^\infty} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2(r-1)}}} + (\|A_2^t A_2\|_{L_T^\infty L_x^\infty} + 1) \|\nabla^2 \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{3r-2}}} + \\
&+ \|\nabla \delta A\|_{L_T^\infty L_x^{\frac{Nr}{3r-2}}} (\|A_1\|_{L_T^\infty L_x^\infty} + \|A_2\|_{L_T^\infty L_x^\infty}) \|\nabla h_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} + \\
&+ \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} (\|\nabla A_1\|_{L_T^\infty L_x^N} + \|\nabla A_2\|_{L_T^\infty L_x^N}) \|\nabla h_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} + \\
&+ \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} (\|A_1\|_{L_T^\infty L_x^\infty} + \|A_2\|_{L_T^\infty L_x^\infty}) \|\nabla h_1\|_{L_T L_x^N},
\end{aligned}$$

which yields that there exists a continuous function $\chi_7(t) \geq 0$, with $\chi_7(0) = 0$, such that

$$\|\delta f_7\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_7(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.64)$$

Bounds for δf_8 . From the definition of δf_8 we get

$$\begin{aligned} \|\delta f_8\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla h_2\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|\omega_2\|_{L_T^\infty L_x^\infty} + \\ &+ \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla \delta h\|_{L_T^{2r} L_x^{\frac{Nr}{2r-1}}} \|\omega_2\|_{L_T^\infty L_x^\infty} + \|h_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\nabla h_1\|_{L_T^{2r} L_x^{Nr}} \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \end{aligned}$$

which yields that there exists a continuous function $\chi_8(t) \geq 0$, with $\chi_8(0) = 0$, such that

$$\|\delta f_8\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_8(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}} + \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \right\}. \quad (5.65)$$

Bounds for $\nabla \delta g$. By the definition of δg we get

$$\begin{aligned} \|\nabla \delta g\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\nabla A_2\|_{L_T^\infty L_x^N} \|\nabla \delta v\|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}} + \|\text{Id} - A_2\|_{L_T^\infty L_x^\infty} \|\nabla^2 \delta v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \\ &+ \|\nabla \delta A\|_{L_T^\infty L_x^{\frac{Nr}{3r-2}}} \|\nabla v_1\|_{L_T^{\frac{2}{2-\varepsilon}} L_x^\infty} + \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \|\nabla^2 v_1\|_{L_T^r L_x^N}. \end{aligned}$$

We deduce that there exists a continuous function $\chi_g(t)$ with $\chi_g(0) = 0$ such that

$$\|\nabla \delta g\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_g(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.66)$$

Bounds for δR . From the definition of δR we have

$$\begin{aligned} \|\delta R\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} &\lesssim \|\nabla v_2\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\delta v\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\text{Id} - A_2\|_{L_T^\infty L_x^\infty} \|\partial_t \delta v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \\ &+ \|\nabla \delta v\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|v_1\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} + \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \|\partial_t v_1\|_{L_T^r L_x^N}, \end{aligned}$$

Thus, there exists a continuous function $\chi_R(t)$ with $\chi_R(0) = 0$ such that

$$\|\delta R\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \lesssim \chi_R(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}} + \|\partial_t \delta v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} \right\}. \quad (5.67)$$

Bounds for δf_4 . From the definition of δf_4 it follows

$$\begin{aligned} \|\delta f_4\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}} &\lesssim \|\delta h\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|(h_1, h_2)\|_{L_T^{2r} L_x^{\frac{Nr}{r-1}}} \|\omega_2\|_{L_T^\infty L_x^\infty} + \\ &+ \|h_1\|_{L_T^2 L_x^\infty}^2 \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}}. \end{aligned}$$

Therefore, we obtain

$$\|\delta f_4\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}} \lesssim \chi_4(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}} + \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \right\}. \quad (5.68)$$

Bounds for δf_5 . From the definition of δf_5 it follows

$$\|\delta f_5\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}} \lesssim \|\delta A\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \|\nabla h_2\|_{L_t^{\frac{2}{2-\varepsilon}} L_x^\infty} + t^{1-\frac{1}{r}} \|A_1\|_{L_T^\infty L_x^\infty} \|\nabla \delta h\|_{L_T^r L_x^{\frac{Nr}{2(r-1)}}}$$

Therefore, we obtain

$$\|\delta f_5\|_{L_T^1 L_x^{\frac{Nr}{2(r-1)}}} \lesssim \chi_5(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{X}_{r,T}}. \quad (5.69)$$

Summarizing (5.60), (5.61), (5.62), (5.63), (5.64), (5.65), (5.66), (5.67), (5.68) and (5.69), we deduce that there exists a continuous function $\chi(t) = \sum_i \chi_i(t)$ which assume 0 for $t = 0$, such that

$$\begin{aligned} & \|(\delta v, \delta h, \delta P)\|_{\mathfrak{X}_{r,T}} + \|\partial_t v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \lesssim \\ & \lesssim \chi(T) \left\{ \|(\delta v, \delta h, \delta P)\|_{\mathfrak{X}_{r,T}} + \|\partial_t v\|_{L_T^r L_x^{\frac{Nr}{3r-2}}} + \|\delta \omega\|_{L_T^\infty L_x^{\frac{Nr}{2(r-1)}}} \right\}, \end{aligned}$$

which yields the uniqueness of the solution to (5.2) on a sufficiently small interval. Then uniqueness part can be completed by a bootstrap method.

5.7.2 Uniqueness: the general case

Now let us consider the general case $1 < p < N$, $\varepsilon \in (0, \min\{1/r, 1 - 1/r, N/p - 1\}]$ and our initial data $(u_0, \nabla d_0)$ in $\dot{B}_{p,r}^{N/p-1+\varepsilon} \cap \dot{B}_{p,r}^{N/p-1}$. We want to prove that the solution for (5.1), given by Theorem 5.2.4, is unique. Let us observe that our solution belongs to the functional framework of Theorem 5.2.6, thanks to proposition 5.4.2. We also recall Remark 5.4.3 for the Lipschitz-estimates and suppose $T < 1$. In order to prove the uniqueness we need the following Lemma

Lemma 5.7.2. *Let $\alpha_1, \beta_i, \gamma_j$ and p_1, p_2, p_3 be defined by Theorem 5.2.4, for $i = 1, 2$ and $j = 1, 2, 3$. If $t^{\alpha_1} f, t^{\alpha_1} \nabla g$ and $t^{\alpha_1} R$ belong to $L_T^{2r} L_x^{p_1}$, then*

$$\begin{cases} \partial_t v - \Delta v + \nabla P = f & (0, T) \times \mathbb{R}^N, \\ \operatorname{div} v = g & (0, T) \times \mathbb{R}^N, \\ \partial_t g = \operatorname{div} R & (0, T) \times \mathbb{R}^N, \\ v|_{t=0} = 0 & \mathbb{R}^N, \end{cases}$$

has a unique solution such that

$$\begin{aligned} & \|t^{\gamma_1} v\|_{L_T^{2r} L_x^{p_3}} + \|t^{\gamma_2} v\|_{L_T^\infty L_x^{p_3}} + \|t^{\gamma_3} v\|_{L_T^{2r} L_x^{3p_1}} + \|t^{\gamma_4} v\|_{L_T^\infty L_x^{3p_1}} + \|t^{\beta_1} v\|_{L_T^{2r} L_x^{p_2}} + \\ & + \|t^{\beta_2} v\|_{L_T^\infty L_x^{p_2}} + \|t^{\alpha_1} (\partial_t v, \nabla^2 v, \nabla P)\|_{L_T^{2r} L_x^{p_1}} \lesssim \|t^{\alpha_1} (f, \nabla g, R)\|_{L_T^{2r} L_x^{p_1}} \end{aligned} \quad (5.70)$$

Proof. The proof is basically equivalent to the one of Lemma 5.7.1 □

By (5.58) and the previous Lemma, it follows that

$$\begin{aligned} & \|t^{\beta_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla \delta v\|_{L_T^\infty L_x^{p_2}} + \|t^{\gamma_1} \delta v\|_{L_T^{2r} L_x^{p_3}} + \|t^{\gamma_2} \delta v\|_{L_T^\infty L_x^{p_3}} + \|t^{\gamma_3} \delta v\|_{L_T^{2r} L_x^{3p_1}} + \\ & + \|t^{\gamma_4} \delta v\|_{L_T^\infty L_x^{3p_1}} + \|t^{\alpha_1} (\partial_t \delta v, \nabla^2 \delta v, \nabla \delta P)\|_{L_T^{2r} L_x^{p_1}} \lesssim \|t^{\alpha_1} (\delta f_1, \delta f_2, \delta f_3, \nabla \delta g, \delta R)\|_{L_T^{2r} L_x^{p_1}}, \end{aligned}$$

where we have also used that $\|b_0\|_{L_x^\infty} = \|a_0\|_{L_x^\infty} \leq \eta$. Furthermore, by the second equation of (5.58) we get $t^{\alpha_1} \delta \omega \in L_T^\infty L_x^{p_1^*}$, where $p_1^* = p_1 N / (N - p_1)$ is the Lebesgue exponent in the critical Sobolev embedding

$$W_x^{1,p_1} \hookrightarrow L_x^{p_1^*}.$$

Moreover

$$\|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1^*}} \lesssim \|t^{\alpha_1} (\delta f_4, \delta f_5)\|_{L_T^1 L_x^{p_1^*}}.$$

By Theorem 5.3.6, Lemma 5.3.7 and Lemma 5.3.8, we get

$$\begin{aligned} & \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_3}} + \|t^{\gamma_2} \delta h\|_{L_T^\infty L_x^{p_3}} + \|t^{\gamma_3} \delta h\|_{L_T^{2r} L_x^{3p_1}} + \|t^{\gamma_4} \delta h\|_{L_T^\infty L_x^{3p_1}} + \\ & + \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} + \|t^{\beta_2} \nabla \delta h\|_{L_T^\infty L_x^{p_2}} + \|t^{\alpha_1} \nabla^2 \delta h\|_{L_T^{2r} L_x^{p_1}} \lesssim \|t^{\alpha_1} (\delta f_6, \delta f_7, \delta f_8)\|_{L_T^{2r} L_x^{p_1}}. \end{aligned}$$

Summarizing the last inequalities, we deduce that we have to control the right-hand side of

$$\begin{aligned} & \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\alpha_1} \partial_t \delta v\|_{L_T^{2r} L_x^{p_1}} + \|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1^*}} \lesssim \|t^{\alpha_1} (\delta f_1, \delta f_2, \nabla \delta f_3)\|_{L_T^{2r} L_x^{p_1}} + \\ & + \|t^{\alpha_1} (\delta f_6, \delta f_7, \delta f_8, \nabla \delta g, \delta R)\|_{L_T^{2r} L_x^{p_1}} + \|t^{\alpha_1} (\delta f_4, \delta f_5)\|_{L_T^1 L_x^{p_1^*}}. \end{aligned} \quad (5.71)$$

Let us now estimate the right-hand side of (5.71) term by term.

Remark 5.7.3. *In what follows, we will use repeatedly the following estimates:*

$$\begin{aligned} & t^{\alpha_1} < t^{\alpha_1^\varepsilon} \quad \text{for } t \leq T < 1, \\ & \|\nabla \delta A\|_{L_T^\infty L_x^{p_1}} \lesssim \|\nabla^2 \delta v\|_{L_T^1 L_x^{p_1}} \lesssim \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}}, \\ & \|\delta A\|_{L_T^\infty L_x^{p_2}} \lesssim \|\nabla \delta v\|_{L_T^1 L_x^{p_2}} \lesssim \|t^{\beta_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_2}}. \end{aligned}$$

Moreover, if we consider $p_3 = \infty$ we get also the following estimate

$$\|t^{\frac{1}{2} - \frac{1}{2r}} h_1\|_{L_T^{2r} L_x^\infty} + \|t^{\frac{1}{2}} h_1\|_{L_T^\infty L_x^\infty} < \infty.$$

Bounds for $t^{\alpha_1} \delta f_1$. From the definition of δf_1 , we readily get

$$\|t^{\alpha_1} \delta f_1\|_{L_T^{2r} L_x^{p_1}} \lesssim \|\text{Id} - {}^t A_2\|_{L_T^\infty L_x^\infty} \|t^{\alpha_1} \nabla \delta P\|_{L_T^{2r} L_x^{p_1}} + \|\delta A\|_{L_T^\infty L_x^{p_1^*}} \|t^{\alpha_1} \nabla P\|_{L_T^{2r} L_x^N},$$

where $p_1^* = p_1 N / (N - p_1)$ is the Lebesgue exponent in the critical Sobolev embedding

$$W_x^{1,p_1} \hookrightarrow L_x^{p_1^*}.$$

Consequently, recalling (5.59) and observing that $L_x^N \hookrightarrow L_x^{p_1} \cap L_x^{q_1}$, we obtain

$$\|t^{\alpha_1} \delta f_1\|_{L_T^{2r} L_x^{p_1}} \lesssim \|t^{\alpha_1^\varepsilon} \nabla v_2\|_{L_T^{2r} L_x^\infty} \|t^{\alpha_1} \nabla \delta P\|_{L_T^{2r} L_x^{p_1}} + \|\nabla^2 \delta v\|_{L_T^1 L_x^{p_1}} \|t^{\alpha_1} \nabla P\|_{L_T^{2r} L_x^{p_1}}^\theta \|t^{\alpha_1} \nabla P\|_{L_T^{2r} L_x^{q_1}}^{1-\theta},$$

for θ determined by $1/N = \theta/p_1 + (1-\theta)/q_1$. We get

$$\begin{aligned} & \|t^{\alpha_1} \delta f_1\|_{L_T^{2r} L_x^{p_1}} \lesssim \\ & \lesssim \|t^{\alpha_1^\varepsilon} \nabla v_2\|_{L_T^{2r} L_x^\infty} \|t^{\alpha_1} \nabla \delta P\|_{L_T^{2r} L_x^{p_1}} + \|\nabla^2 \delta v\|_{L_T^1 L_x^{p_1}} \|t^{\alpha_1} \nabla P\|_{L_T^{2r} L_x^{p_1}}^\theta \|t^{\alpha_1} \nabla P\|_{L_T^{2r} L_x^{q_1}}^{1-\theta}, \\ & \lesssim \|t^{\alpha_1^\varepsilon} \nabla v_2\|_{L_T^{2r} L_x^\infty} \|t^{\alpha_1} \nabla \delta P\|_{L_T^{2r} L_x^{p_1}} + \|t^{-\alpha_1}\|_{L_T^{(2r)'}} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}} \\ & \lesssim \|t^{\alpha_1^\varepsilon} \nabla v_2\|_{L_T^{2r} L_x^\infty} \|t^{\alpha_1} \nabla \delta P\|_{L_T^{2r} L_x^{p_1}} + T^{1-\alpha_1(2r)'} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}} \end{aligned}$$

Thus there exists a continuous function $t \rightarrow \chi_1(t)$, which goes to 0 for $t \rightarrow 0$ and

$$\|t^{\alpha_1} \delta f_1\|_{L_{2rT} L_x^{p_1}} \lesssim \chi_1(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} \quad (5.72)$$

Bounds for $t^{\alpha_1} \delta f_2$. From the definition of δf_2 and observing that $A_2 {}^t A_2 - A_1 {}^t A_1 = \delta A {}^t A_2 + A_1 {}^t \delta A$, we deduce

$$\begin{aligned} \|t^{\alpha_1} \delta f_2\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|\nabla(A_2 {}^t A_2)\|_{L_T^\infty L_x^N} \|t^{\alpha_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_1}^*} + (\|A_2 {}^t A_2\|_{L_T^\infty L_x^\infty} + 1) \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{p_1}^*} (\|\nabla A_2\|_{L_T^\infty L_x^N} + \|\nabla A_1\|_{L_T^\infty L_x^N}) \|t^{\alpha_1} \nabla v_1\|_{L_T^{2r} L_x^\infty} + \\ &+ \|\nabla \delta A\|_{L_T^\infty L_x^{p_1}} (\|A_2\|_{L_T^\infty L_x^\infty} + \|A_1\|_{L_T^\infty L_x^\infty}) \|t^{\alpha_1} \nabla v_1\|_{L_T^{2r} L_x^\infty} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{p_1}^*} (\|A_2\|_{L_T^\infty L_x^\infty} + \|A_1\|_{L_T^\infty L_x^\infty}) \|t^{\alpha_1} \nabla^2 v_1\|_{L_T^{2r} L_x^N} \end{aligned}$$

Again by $L_x^N \hookrightarrow L_x^{p_1} \cap L_x^{q_1}$ and the critical Sobolev embedding, there exists a continuous function $t \rightarrow \chi_2(t)$ which goes to 0 for $t \rightarrow 0$, such that

$$\|t^{\alpha_1} \delta f_2\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_2(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}. \quad (5.73)$$

Bounds for $t^{\alpha_1} \delta f_3$ From the definition of δf_3 we get

$$\begin{aligned} \|t^{\alpha_1} \delta f_3\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|\nabla \delta A\|_{L_T^\infty L_x^{p_1}} \|t^{\frac{1}{2} - \frac{1}{2r}} h_1\|_{L_T^{2r} L_x^\infty} \|t^{\frac{1}{2}} h_1\|_{L_T^\infty L_x^\infty} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{p_1}^*} \left(\|t^{\beta_1} \nabla h_1\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_1\|_{L_T^\infty L_x^{p_3}} + \|t^{\beta_1} \nabla h_1\|_{L_T^{2r} L_x^{q_2}} \|t^{\gamma_2} h_1\|_{L_T^\infty L_x^{q_3}} \right) + \\ &+ \|\nabla A_1\|_{L_T^\infty L_x^N} \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_2\|_{L_T^\infty L_x^{p_3}} + \|\nabla A_1\|_{L_T^\infty L_x^N} \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\beta_2} h_2\|_{L_T^\infty L_x^{p_3}} + \\ &+ \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_2\|_{L_T^\infty L_x^{p_3}} + \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\beta_2} \nabla h_2\|_{L_T^\infty L_x^{p_3}} + \\ &+ \|\nabla A_1\|_{L_T^\infty L_x^N} \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_1\|_{L_T^\infty L_x^{p_3}} + \|\nabla A_1\|_{L_T^\infty L_x^N} \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\beta_2} h_1\|_{L_T^\infty L_x^{p_3}} + \\ &+ \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_1\|_{L_T^\infty L_x^{p_3}} + \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\beta_2} \nabla h_1\|_{L_T^\infty L_x^{p_3}} \end{aligned}$$

Hence, arguing exactly as for (5.72) and (5.73), there exists $\chi_3(t)$ such that

$$\|t^{\alpha_1} \nabla \delta f_3\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_3(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}. \quad (5.74)$$

Bounds for $t^{\alpha_1} \delta f_6$ From the definition of δf_6 we get

$$\begin{aligned} \|t^{\alpha_1} \delta f_6\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|t^{\beta_1} \delta A\|_{L_T^\infty L_x^{p_2}} \|t^{\alpha_1} \nabla v_2\|_{L_T^{2r} L_x^\infty} \|t^{\gamma_2} h_2\|_{L_T^\infty L_x^{p_3}} + \\ &+ \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\beta_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} h_2\|_{L_T^\infty L_x^{p_3}} + \|A_1\|_{L_T^\infty L_x^\infty} \|t^{\beta_2} \nabla v_1\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_3} \delta h\|_{L_T^\infty L_x^{p_3}}, \end{aligned}$$

Thus

$$\|t^{\alpha_1} \delta f_6\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_6(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}, \quad (5.75)$$

for an oportune continuous function $\chi_6(t)$ which goes to 0 when $t \rightarrow 0$.

Bounds for $t^{\alpha_1} \delta f_7$. From the definition of δf_7 we have

$$\begin{aligned} \|t^{\alpha_1} \delta f_7\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|\nabla A_2\|_{L_T^\infty L_x^N} \|A_2\|_{L_T^\infty L_x^\infty} \|t^{\alpha_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_1}^*} + (\|A_2 {}^t A_2\|_{L_T^\infty L_x^\infty} + 1) \|t^{\alpha_1} \nabla^2 \delta h\|_{L_T^{2r} L_x^{p_1}} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{p_1}^*} \|\nabla(A_1, A_2)\|_{L_T^\infty L_x^N} \|t^{\alpha_1} \nabla h_1\|_{L_T^{2r} L_x^\infty} + \\ &+ \|\delta A\|_{L_T^\infty L_x^{p_1}^*} (\|A_1\|_{L_T^\infty L_x^\infty} + \|A_2\|_{L_T^\infty L_x^\infty}) \|t^{\alpha_1} \nabla^2 h_1\|_{L_T^{2r} L_x^N}, \end{aligned}$$

which yields that there exists a continuous function $\chi_7(t) \geq 0$, with $\chi_7(0) = 0$, such that

$$\|t^{\alpha_1} \delta f_7\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_7(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}. \quad (5.76)$$

Bounds for $t^{\alpha_1} \delta f_8$. From the definition of δf_8 we get

$$\begin{aligned} \|t^{\alpha_1} \delta f_8\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|t^{\beta_1} \delta h\|_{L_T^{2r} L_x^{p_3}} \|t^{\beta_2} \nabla h_2\|_{L_T^\infty L_x^{p_2}} \|\omega_2\|_{L_T^\infty L_x^\infty} + \|t^{\beta_2} h_1\|_{L_T^\infty L_x^{p_3}} \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \times \\ &\quad \times \|\omega_2\|_{L_T^\infty L_x^\infty} + \|t^{\beta_2} h_1\|_{L_T^\infty L_x^{p_3}} \|t^{\beta_2} \nabla h_1\|_{L_T^\infty L_x^{p_2}} \|t^{-\frac{1}{2} - \frac{1}{2r}}\|_{L_T^{\frac{2r}{r-1} - \varepsilon}} \|t^{\frac{1}{2}} \delta \omega\|_{L_T^\infty L_x^\infty} \end{aligned}$$

For $\bar{\varepsilon}$ small enough, that is,

$$\|t^{\alpha_1} \delta f_8\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_9(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\frac{1}{2}} \delta \omega\|_{L_T^\infty L_x^\infty} \right\}. \quad (5.77)$$

Bounds for $t^{\alpha_1} \nabla \delta g$. By the definition of δg we get

$$\begin{aligned} \|t^{\alpha_1} \nabla \delta g\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|\nabla A_2\|_{L_T^\infty L_x^N} \|t^{\alpha_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_1}^*} + \|\text{Id} - A_2\|_{L_T^\infty L_x^\infty} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}} + \\ &\quad + \|\nabla \delta A\|_{L_T^\infty L_x^{p_1}} \|t^{\alpha_1} \nabla v_1\|_{L_T^{2r} L_x^\infty} + \|\delta A\|_{L_T^\infty L_x^{p_1}^*} \|t^{\alpha_1} \nabla^2 v_1\|_{L_T^{2r} L_x^N}. \end{aligned}$$

We deduce that there exists a continuous function $\chi_g(t)$ with $\chi_g(0) = 0$ such that

$$\|t^{\alpha_1} \nabla \delta g\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_g(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}. \quad (5.78)$$

Bounds for $t^{\alpha_1} \delta R$. From the definition of δR we have

$$\begin{aligned} \|t^{\alpha_1} \delta R\|_{L_T^{2r} L_x^{p_1}} &\lesssim \|t^{\beta_2} \nabla v_2\|_{L_T^\infty L_x^{p_2}} \|t^{\gamma_1} \delta v\|_{L_T^{2r} L_x^{p_3}} + \|\text{Id} - A_2\|_{L_T^\infty L_x^\infty} \|t^{\alpha_1} \partial_t \delta v\|_{L_T^{2r} L_x^{p_1}} + \\ &\quad + \|t^{\beta_1} \nabla \delta v\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} v_1\|_{L_T^\infty L_x^{p_3}} + \|\delta A\|_{L_T^{2r} L_x^{p_1}^*} \|t^{\alpha_1} \partial_t v_1\|_{L_T^{2r} L_x^N}, \end{aligned}$$

Thus, there exists a continuous function $\chi_R(t)$ with $\chi_R(0) = 0$ such that

$$\|t^{\alpha_1} \delta R\|_{L_T^{2r} L_x^{p_1}} \lesssim \chi_R(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\alpha_1} \partial_t \delta v\|_{L_T^{2r} L_x^{p_1}} \right\}. \quad (5.79)$$

Bounds for $t^{\alpha_1} \delta f_4$. From the definition of δf_4 it follows

$$\begin{aligned} \|t^{\alpha_1} \delta f_4\|_{L_T^1 L_x^{p_1}^*} &\lesssim \|t^{\beta_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} (h_1, h_2)\|_{L_T^\infty L_x^{p_3}} \|(\omega_1, \omega_2)\|_{L_T^\infty L_x^\infty} + \|t^{\gamma_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \times \\ &\quad \times \|t^{\gamma_2} \nabla (h_1, h_2)\|_{L_T^\infty L_x^{p_3}} \|(\omega_1, \omega_2)\|_{L_T^\infty L_x^\infty} + \|t^{\beta_1} \delta h\|_{L_T^{2r} L_x^{p_2}} \|t^{\gamma_2} (h_1, h_2)\|_{L_T^\infty L_x^{p_3}} \|\nabla (\omega_1, \omega_2)\|_{L_T^{2r} L_x^\infty} + \\ &\quad + \|t^{\frac{1}{2}} h_1\|_{L_T^\infty L_x^\infty}^2 \|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1}^*} \end{aligned}$$

Therefore, we obtain

$$\|t^{\alpha_1} \delta f_4\|_{L_T^1 L_x^{p_1}^*} \lesssim \chi_4(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1}^*} \right\}. \quad (5.80)$$

Bounds for $t^{\gamma_1} \delta f_5$. By definition

$$\begin{aligned} \|t^{\gamma_1} \delta f_5\|_{L_T^1 L_x^{p_1}^*} &\lesssim \|t^{\alpha_1} \delta A\|_{L_T^\infty L_x^{p_1}^*} \|\nabla h_2\|_{L_T^1 L_x^\infty} + \|A_1\|_{L^2 r_T L_x^\infty} \|t^{\alpha_1} \nabla \delta h\|_{L_T^{2r} L_x^{p_1}^*} \\ &\lesssim \|t^{\alpha_1} \nabla^2 \delta v\|_{L_T^{2r} L_x^{p_1}} \|\nabla h_2\|_{L_T^1 L_x^\infty} + \|A_1\|_{L^2 r_T L_x^\infty} \|t^{\alpha_1} \nabla^2 \delta h\|_{L_T^{2r} L_x^{p_1}} \end{aligned}$$

hence

$$\|t^{\alpha_1} \delta f_5\|_{L_T^1 L_x^{p_1^*}} \lesssim \chi_5(T) \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}}. \quad (5.81)$$

with χ_5 as the previous functions.

Summarizing points (5.72), (5.73), (5.74), (5.75), (5.76), (5.77), (5.78), (5.79), (5.80) and (5.81), we finally obtain

$$\begin{aligned} & \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\alpha_1} \partial_t \delta v\|_{L_T^{2r} L_x^{p_1}} + \|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1^*}} \lesssim \\ & \chi(T) \left\{ \|(\delta v, \delta h, \nabla \delta P)\|_{\mathfrak{Y}_{r,T}} + \|t^{\alpha_1} \delta \omega\|_{L_T^\infty L_x^{p_1^*}} + \|t^{\alpha_1} \partial_t \delta v\|_{L_T^{2r} L_x^{p_1}} \right\} \end{aligned}$$

where χ stands for $\sum_i \chi_i$. Thus, for T sufficiently small, the left-hand side has to be 0. This proves the uniqueness at least in small time interval. Then uniqueness part can be completed by a bootstrap method. This concludes the proof of Theorem 5.2.6.

5.8 Estimates

Lemma 5.8.1. *Let the operator \mathcal{C} be defined as in Lemma 5.3.5. Consider $T \in (0, \infty]$, $\varepsilon \geq 0$ small enough, $1 < \bar{r} < \infty$, and moreover suppose that q, \tilde{q} satisfy $N/2 < q < N/(1 - \varepsilon)$, $\max\{N, q\} < \tilde{q} \leq \infty$. Let α^ε , γ^ε and $\bar{\gamma}^\varepsilon$ be defined by*

$$\alpha^\varepsilon := \frac{1}{2} \left(3 - \frac{N}{q} - \varepsilon \right) - \frac{1}{\bar{r}}, \quad \gamma^\varepsilon := \frac{1}{2} \left(1 - \frac{N}{\tilde{q}} - \varepsilon \right) - \frac{1}{\bar{r}} \quad \text{and} \quad \bar{\gamma}^\varepsilon := \frac{1}{2} \left(1 - \frac{N}{\tilde{q}} - \varepsilon \right).$$

If $t^{\alpha^\varepsilon} f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^q)$ then $t^{\gamma^\varepsilon} \mathcal{C}f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^{\tilde{q}})$. Furthermore there exist $C_\varepsilon = C_\varepsilon(q, \tilde{q}, \bar{r}) > 0$ such that

$$\|t^{\gamma^\varepsilon} \mathcal{C}f(t)\|_{L^{\bar{r}}(0, T; L_x^{\tilde{q}})} \leq C_\varepsilon \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0, T; L_x^q)}. \quad (5.82)$$

Moreover, if $\bar{r} > 2$ and $N\bar{r}/(2\bar{r} - 2) < q$, then $t^{\bar{\gamma}^\varepsilon} \mathcal{C}f(t)$ belongs to $L^\infty(0, T; L_x^{\tilde{q}})$ and there exists a positive constant $\bar{C}_\varepsilon = \bar{C}_\varepsilon(q, \tilde{q}, \bar{r})$ such that

$$\|t^{\bar{\gamma}^\varepsilon} \mathcal{C}f(t)\|_{L^\infty(0, T; L_x^{\tilde{q}})} \leq \bar{C}_\varepsilon \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0, T; L_x^q)}. \quad (5.83)$$

Proof. Recalling (5.12) we have

$$\|t^{\gamma^\varepsilon} \mathcal{C}f(t)\|_{L_x^{\tilde{q}}} \lesssim \int_0^t \frac{1}{|t-s|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}} \|f(s)\|_{L_x^q} ds.$$

Defining $F(s) := \|s^{\alpha^\varepsilon} f(s)\|_{L_x^q}$, by a change of variable $s = t\tau$ and because $\gamma^\varepsilon - \alpha^\varepsilon + 1 = (1/q - 1/\tilde{q})N/2$, we get that

$$t^{\gamma^\varepsilon} \|\mathcal{C}f(t)\|_{L_x^{\tilde{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}} \tau^{-\alpha^\varepsilon} F(t\tau) d\tau.$$

Applying Minkowski inequality, we deduce that

$$\|t^{\gamma^\varepsilon} \mathcal{C}f(t)\|_{L_t^{\bar{r}} L_x^{\tilde{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}} \tau^{-\alpha^\varepsilon} \left(\int_0^{T\tau} F(t\tau)^{\bar{r}} dt \right)^{\frac{1}{\bar{r}}} d\tau,$$

which yields

$$\|t^{\gamma^\varepsilon} \mathcal{C}f(t)\|_{L_t^{\bar{r}} L_x^{\bar{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}})}} \tau^{-\alpha^\varepsilon-\frac{1}{\bar{r}}} d\tau \|t^{\alpha^\varepsilon} f(t)\|_{L^{r_1}(0,T;L_x^{q_1})}.$$

Thus, because $(1/q - 1/\bar{q})N/2 < 1$ and $0 < 1/2(3 - N/q - \varepsilon) < 1$, we obtain inequality (5.82).

On the other hand, observing that

$$t^{\bar{\gamma}^\varepsilon} \|\mathcal{C}f(t)\|_{L_x^{\bar{q}}} \lesssim \int_0^t \frac{t^{\bar{\gamma}^\varepsilon} s^{-\alpha^\varepsilon}}{|t-s|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}})}} F(s) ds \lesssim \left(\int_0^t \frac{t^{\bar{\gamma}^\varepsilon \bar{r}'} s^{-\alpha^\varepsilon \bar{r}'}}{|t-s|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}}) \bar{r}'}} ds \right)^{\frac{1}{\bar{r}'}} \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}.$$

By a change of variable $s = t\tau$ and because $\bar{\gamma}^\varepsilon \bar{r}' - \alpha^\varepsilon \bar{r}' - (1/q - 1/\bar{q})N\bar{r}'/2 + 1 = 0$, we obtain

$$t^{\bar{\gamma}^\varepsilon} \|\mathcal{C}f(t)\|_{L_x^{\bar{q}}} \lesssim \left(\int_0^1 \frac{\tau^{-\alpha^\varepsilon \bar{r}'}}{|1-\tau|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}}) \bar{r}'}} d\tau \right)^{\frac{1}{\bar{r}'}} \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}.$$

Since $q > N\bar{r}/(2\bar{r} - 2)$ yields $(1/q - 1/\bar{q})N/2 < 1/\bar{r}'$ and $q < N/(1 - \varepsilon)$ implies $\alpha^\varepsilon \bar{r}' < 1$, we obtain (5.83), which completes the proof of the Lemma. \square

Lemma 5.8.2. *Let the operators \mathcal{B} be defined as in Lemma 5.3.3. Consider $T \in (0, \infty]$, $\varepsilon \geq 0$ small enough, $1 < \bar{r} < \infty$, and moreover suppose that q, \bar{q} satisfy $N/2 < q < N/(1 - \varepsilon)$ and $q \leq \bar{q}$ such that $1/q - 1/\bar{q} < 1/N$. Let α^ε be defined as in Lemma 5.8.1 and β^ε and $\bar{\beta}^\varepsilon$ be defined by*

$$\bar{\beta}^\varepsilon := \frac{1}{2} \left(2 - \frac{N}{\bar{q}} - \varepsilon \right) \quad \text{and} \quad \beta^\varepsilon := \frac{1}{2} \left(1 - \frac{N}{\bar{q}} - \varepsilon \right) - \frac{1}{\bar{r}}.$$

If $t^{\alpha^\varepsilon} f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^{\bar{q}})$ then $t^{\beta^\varepsilon} \mathcal{B}f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^{\bar{q}})$ and there exists a positive constant $C_\varepsilon = C_\varepsilon(q, \bar{q}, \bar{r})$ such that

$$\|t^{\beta^\varepsilon} \mathcal{B}f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})} \leq \bar{C}_\varepsilon \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}. \quad (5.84)$$

Moreover, if $\bar{r} > 2$, $N\bar{r}/(2\bar{r} - 2) < q$ and $\bar{q} < N\bar{r}$ then $t^{\bar{\beta}^\varepsilon} \mathcal{B}f(t)$ belongs to $L^\infty(0, T; L_x^{\bar{q}})$ and there exists a positive constant $\bar{C}_\varepsilon = \bar{C}_\varepsilon(q, \bar{q}, \bar{r})$ such that

$$\|t^{\bar{\beta}^\varepsilon} \mathcal{B}f(t)\|_{L^\infty(0,T;L_x^{\bar{q}})} \leq C_\varepsilon \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}. \quad (5.85)$$

Proof. At first, recalling (5.9), we get that

$$t^{\beta^\varepsilon} \|\mathcal{B}f(t)\|_{L_x^{\bar{q}}} \lesssim t^{\beta^\varepsilon} \int_0^t \frac{1}{|t-s|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}})+\frac{1}{2}}} \|f(s)\|_{L_x^q} ds.$$

Defining $F(s) := \|s^{\alpha^\varepsilon} f(s)\|_{L_x^q}$, by a change of variable $s = t\tau$ and because $\beta^\varepsilon - \alpha^\varepsilon + 1 = 1/2 + (1/q - 1/\bar{q})N/2$, we get that

$$t^{\beta^\varepsilon} \|\mathcal{B}f(t)\|_{L_x^{\bar{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{\bar{q}}-\frac{1}{\bar{q}})+\frac{1}{2}}} \tau^{-\alpha^\varepsilon} F(t\tau) d\tau.$$

Applying Minkowski inequality, we deduce that

$$\|t^{\beta^\varepsilon} \mathcal{B}f(t)\|_{L_t^{\bar{r}} L_x^{\bar{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\bar{q}})+\frac{1}{2}}} \tau^{-\alpha^\varepsilon} \left(\int_0^{T\tau} F(t\tau)^{\bar{r}} dt \right)^{\frac{1}{\bar{r}}} d\tau, \quad (5.86)$$

which yields

$$\|t^{\beta^\varepsilon} \mathcal{B}f(t)\|_{L_t^{\bar{r}} L_x^{\bar{q}}} \lesssim \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\bar{q}})+\frac{1}{2}}} \tau^{-\alpha^\varepsilon - \frac{1}{\bar{r}}} d\tau \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}.$$

Because $0 < (1/q - 1/\bar{q})N/2 + 1/2 < 1$ and $0 < 1/2(3 - N/q - \varepsilon) < 1$, we deduce inequality (5.84).

For the second inequality, proceeding in a similar way of the previous Lemma, we obtain that

$$t^{\beta^\varepsilon} \|\mathcal{B}f(t)\|_{L_x^{\bar{q}}} \lesssim \left(\int_0^t \frac{t^{\beta^\varepsilon \bar{r}'} s^{-\alpha^\varepsilon \bar{r}'}}{|t-s|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\bar{q}})\bar{r}'+\frac{1}{2}\bar{r}'}} ds \right)^{\frac{1}{\bar{r}'}} \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}$$

By a change of variable $s = t\tau$ and because $\beta^\varepsilon \bar{r}' - \alpha^\varepsilon \bar{r}' - (1/q - 1/\bar{q})N\bar{r}'/2 - \bar{r}'/2 + 1 = 0$, we obtain

$$t^{\beta^\varepsilon} \|\mathcal{B}f(t)\|_{L_x^{\bar{q}}} \lesssim \left(\int_0^1 \frac{\tau^{-\alpha^\varepsilon \bar{r}'}}{|1-\tau|^{\frac{N}{2}(\frac{1}{q}-\frac{1}{\bar{q}})\bar{r}'+\frac{1}{2}\bar{r}'}} d\tau \right)^{\frac{1}{\bar{r}'}} \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}.$$

Since by the hypotheses we can deduce $\alpha_1^{\varepsilon \bar{r}'} < 1$ and $(1/q - 1/\bar{q})N\bar{r}'/2 + \bar{r}'/2 < 1$ then there exists $C_\varepsilon > 0$ such that

$$\|t^{\beta^\varepsilon} \mathcal{B}f(t)\|_{L^\infty(0,T;L_x^{\bar{q}})} \leq C_\varepsilon \|t^{\alpha^\varepsilon} f(t)\|_{L^{\bar{r}}(0,T;L_x^{\bar{q}})}$$

□

Lemma 5.8.3. *Let $1 < \bar{r} < \infty$, $q > N\bar{r}/(2\bar{r} - 2)$ and $\sigma := 1 - N/(2q) - 1/\bar{r}$. Let us suppose that $t^\sigma f$ belongs to $L^{\bar{r}}(0,T;L_x^{\bar{q}})$ with $T \in (0, \infty]$. Then $\mathcal{C}f$ belongs to $L^\infty(0,T;L_x^\infty)$ and for every $t \in (0, T)$*

$$\|\mathcal{C}f(t)\|_{L_x^\infty} \leq C_{\bar{r}} \|s^\sigma f\|_{L^{\bar{r}}(0,t;L_x^{\bar{q}})},$$

where $C_{\bar{r}}$ is a positive constant dependent only by \bar{r} .

Proof. Recalling (5.12) we get

$$\|\mathcal{C}f(t)\|_{L_x^\infty} \leq \int_0^t \frac{1}{|t-s|^{\frac{N}{2}\frac{1}{q}\sigma}} F(s) ds,$$

for every $t \in (0, T)$, where $F(s) = s^\sigma \|f(s)\|_{L_x^\infty}$. By the change of variable $s = t\tau$ we obtain

$$\|\mathcal{C}f(t)\|_{L_x^\infty} \leq \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}\frac{1}{q}\sigma}} F(t\tau) t^{1-\frac{N}{2}\frac{1}{q}\sigma} ds = \int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}\frac{1}{q}\sigma}} F(t\tau) t^{\frac{1}{\bar{r}}} ds.$$

Hence, by Hölder inequality, it follows

$$\|\mathcal{C}f(t)\|_{L_x^\infty} \leq \left(\int_0^1 \frac{1}{|1-\tau|^{\frac{N}{2}\frac{1}{q}\bar{r}'}} d\tau \right)^{\frac{1}{\bar{r}'}} \left(\int_0^1 |F(t\tau)|^{\bar{r}} t d\tau \right)^{\frac{1}{\bar{r}}}.$$

Since $\bar{r}'N/(2q) < 1$, we finally get

$$\|\mathcal{C}f(t)\|_{L_x^\infty} \leq C_{\bar{r}} \left(\int_0^1 |F(t\tau)|^r t d\tau \right)^{\frac{1}{r}} = C_{\bar{r}} \|s^\sigma f\|_{L^{\bar{r}}(0,t;L_x^q)}.$$

□

Finally we enunciate the following Lemma, which proof is basically equivalent to the previous one.

Lemma 5.8.4. *Let $2 < \bar{r} < \infty$, $q > N\bar{r}/(\bar{r} - 2)$ and $\sigma := (1 - N/q)1/2 - 1/\bar{r}$. Let us suppose that $t^\sigma f$ belongs to $L^{\bar{r}}(0, T; L_x^q)$ with $T \in (0, \infty]$. Then $\mathcal{B}f$ and $t^{-1/2}\mathcal{C}f$ belong to $L^\infty(0, T; L_x^\infty)$ and for every $t \in (0, T)$*

$$\|\mathcal{B}f(t), t^{-\frac{1}{2}}\mathcal{C}f(t)\|_{L_x^\infty} \leq C_{\bar{r}} \|s^\sigma f\|_{L^{\bar{r}}(0,t;L_x^q)},$$

where $C_{\bar{r}}$ is a positive constant dependent only by \bar{r} .

5.9 Technical Results for the Heat and Stokes equations

We consider the following system, composed by an Heat equation and a free Stokes equation with a linear perturbation:

$$\begin{cases} \partial_t u + v \cdot \nabla u - \Delta u + \nabla \Pi = f_1 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \partial_t d - \Delta d = f_2 & \mathbb{R}_+ \times \mathbb{R}^N, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^N, \\ (u, d)|_{t=0} = (u_0, d_0) & \mathbb{R}^N, \end{cases} \quad (5.87)$$

where $d_0 \in L_x^\infty$ and $(u_0, \nabla d_0)$ belongs to $\dot{B}_{p,r}^{N/p-1}$ with $1 < p < N$ and $1 < r < \infty$. Propositions 5.9.1 and Proposition 5.9.2 concern the existence of a solution $(u, d, \nabla \Pi)$, which belong to $\mathfrak{X}_{r,T}$ and $\mathfrak{Y}_{r,T}$ respectively. For p less than (or equal to) the critical exponent $Nr/(3r - 2)$ we can solve our system in a functional framework based only on some regularizing effects for the heat kernel in $L^p L^q$ spaces. However, if p exceeds this critical value, in order to handle this less of regularity we have the add a weight in time.

Proposition 5.4.1 requires the following result:

Proposition 5.9.1. *Let $1 < r \leq 2$ and $1 < p \leq Nr/(3r - 2)$. Suppose that $f_1, \nabla f_2$ belong to $L_T^r L_x^{N3/(3r-2)}$, $f_2 \in L_T^1 L_x^q \cap L_T^r L_x^{Nr/2(r-1)}$, ∇f_2 belongs to $L_T^{6r/5} L_x^{3Nr/(6r-2)}$. Assume that v belongs to $L_T^{2r} L_x^{Nr/(r-1)}$ and its norm is small enough. Let us assume that d_0 takes value in \mathbb{S}^{N-1} , $u_0, \nabla d_0$ belong to $\dot{B}_{p,r}^{N/p-1}$ and condition (5.4) is satisfied. Then there exists $(u, d, \nabla \Pi)$ solution of (5.87) such that d belongs to $L_T^\infty L_x^\infty$, $(u, \nabla d, \nabla \Pi)$ belongs to $\mathfrak{X}_{r,T}$ and $(u, \nabla d)$ belongs to $L_T^2 L_x^\infty$.*

Proof. The case of the simple heat equation in d is provided by the Mild formulation, namely

$$d(t) = e^{t\Delta} d_0 + \int_0^t e^{(t-s)\Delta} f_2(s) ds = e^{t\Delta} d_0 + \mathcal{C}f_2(t). \quad (5.88)$$

We immediately get $d \in L_T^\infty L_x^\infty$ and its norm is bounded by $\|d_0\|_{L_x^\infty} + \int_0^t \|f_2(s)\|_{L_x^\infty} ds$. Moreover, by Corollary 5.3.11.1, because $\nabla d_0 \in \dot{B}_{p,r}^{N/p-1} \hookrightarrow \dot{B}_{Nr/(r-1), 2r}^{-1/r}$, we deduce that $e^{t\Delta} \nabla d_0$ belongs $L_T^{2r} L_x^{Nr/(r-1)}$. the integral part $\nabla \mathcal{C}f_2(t) = \mathcal{C} \nabla f_2(t)$ is handled by Lemma 5.3.5 with $r_1 = r$, $r_2 = 2r$ $q_1 = Nr/(3r - 2)$ and $q_2 = Nr/(r - 1)$. Similarly, because $\nabla^2 d_0$ belongs to $\dot{B}_{Nr/(2r-1), 2r}^{-1/r} \cap$

$\dot{B}_{Nr/(2r-2),r}^{-2/r}$, we get $\nabla e^{t\Delta}\nabla d_0 \in L_T^{2r}L_x^{Nr/2r-1} \cap L_T^rL_x^{Nr/(2r-2)}$ and by Lemma 5.3.3 we obtain $\mathcal{B}\nabla f_2 \in L_T^{2r}L_x^{Nr/2r-1} \cap L_T^rL_x^{Nr/(2r-2)}$. Observing also that $\dot{B}_{p,r}^{N/r-1}$ is embedded in $\dot{B}_{3Nr/(3r-2),3r}^{-2/3r}$, again by Corollary 5.3.11.1 we get that $e^{t\Delta}\nabla d_0$ belongs to $L_T^{3r}L_x^{3Nr/(3r-2)}$. The same property is fulfilled by $\mathcal{B}f_2 = \mathcal{C}\nabla f_2$, using Lemma 5.3.3 with $r_1 = 6r/5$ and $q_1 = 3Nr/(6r-5)$. At last, since $\nabla^3 d_0 \in \dot{B}_{Nr/(3r-2),r}^{-2/r}$ we deduce again by Corollary 5.3.11.1 that $\nabla^2 e^{t\Delta}\nabla d_0$ belongs to $L_T^rL_x^{Nr/(3r-2)}$, while the same result is allowed for $\mathcal{A}\nabla f_2$ by Theorem 5.3.2. Hence, $(u, \nabla d, \nabla \Pi)$ belongs to $\mathfrak{X}_{r,T}$ at least for the terms related to d . Furthermore, since ∇d_0 belongs to $\dot{B}_{\infty,2}^{-1}$ (here the necessary condition $r \leq 2$) we get $e^{t\Delta}\nabla d_0 \in L_T^2L_x^\infty$ and by Lemma 5.3.3 with $r_1 = r$, $r_2 = 2$, $q_1 = Nr/(r-1)$ and $q_2 = \infty$, we deduce that $\mathcal{B}f_2$ belongs to $L_T^2L_x^\infty$, that is $\nabla d \in L_T^2L_x^\infty$. Concerning the Stokes equation with the v -linear perturbation the mainly idea is to use the Fixed-Point Theorem on the space $\tilde{\mathfrak{X}}_{r,T}$ determined by

$$\tilde{\mathfrak{X}}_{r,T} := \{ (u, \nabla \Pi) \text{ such that } (u, d, \nabla \Pi) \in \mathfrak{X}_{r,T} \}.$$

Indeed, let $(\omega_i, \nabla P_i)$ belong to $\tilde{\mathfrak{X}}_{r,T}$, for $i = 1, 2$, and let us define

$$\begin{aligned} u_i(t) &:= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} \{ -v \cdot \nabla \omega_i - \nabla P_i + f_1(s) \} ds, \\ \nabla \Pi_i &:= -RR \cdot \{ v \cdot \nabla \omega_i + f_1 \}, \end{aligned} \quad (5.89)$$

then we have $(u_i, \nabla P_i) \in \tilde{\mathfrak{X}}_{r,T}$, by the same techniques used for d . Moreover, subtracting in i , $\delta u := u_1 - u_2$, $\delta \nabla \Pi := \nabla \Pi_1 - \nabla \Pi_2$, $\delta \omega := \omega_1 - \omega_2$ and $\delta \nabla P := \nabla P_1 - \nabla P_2$, we get

$$\|(\delta u, \delta \nabla \Pi)\|_{\tilde{\mathfrak{X}}_{r,T}} \lesssim \|v\|_{L_T^{2r}L_x^{Nr/(r-1)}} \|(\delta \omega, \delta P)\|_{\tilde{\mathfrak{X}}_{r,T}}.$$

Thus, by the Fixed-Point Theorem, on the condition $\|v\|_{L_T^{2r}L_x^{Nr/(r-1)}}$ small enough, there exists $(u, d, \nabla \Pi)$ solution for (5.87), with the properties described by the statement. This concludes the proof of Proposition 5.9.1. \square

Now we extend the range of r to $(1, \infty)$ and we consider an index of integrability p greater than the critical $Nr/(3r-2)$. As already mentioned, here the addition of a weight in time is necessary. The following result is used in proposition 5.4.2.

Proposition 5.9.2. *Let $1 < r < \infty$ and $Nr/(3r-2) < p < N$. Recalling the notation of Theorem 5.2.4, let us suppose that $t^{\alpha_1}(f_1, \nabla f_2)$ belongs to $L_T^{2r}L_x^{p_1}$ and $t^{2\gamma_1}f_2$ belongs to $L_T^rL_x^{p_3/2}$. Assume that $t^{\gamma_1} \in L_T^{2r}L_x^{p_3}$ and its norm is small enough. Let d_0 and u_0 be defined as in Proposition 5.9.1. Then there exists $(u, d, \nabla \Pi) \in \mathfrak{Y}_{r,T}$ solution of (5.87), with $d \in L_T^\infty L_x^\infty$.*

Proof. The proof is basically equivalent to the one of Proposition 5.9.1. At first, by (5.88) and Lemma 5.8.3, we get

$$\|d\|_{L_T^\infty L_x^\infty} \lesssim \|d_0\|_{L_x^\infty} + \|t^{2\gamma_1}f_2\|_{L_T^r L_x^{p_3/2}}.$$

Recalling Theorem 5.3.11, by $\nabla d_0 \in \dot{B}_{p_3,2r}^{N/p_3-1}$, $\nabla^2 d_0 \in \dot{B}_{p_2,2r}^{N/p_2-1}$ and $\nabla^3 d_0 \in \dot{B}_{p_1,2r}^{N/p_1-1}$, we get that $t^{\gamma_1}e^{t\Delta}\nabla d_0 \in L_T^{2r}L_x^{p_3}$, $t^{\beta_1}\nabla e^{t\Delta}\nabla d_0 \in L_T^{2r}L_x^{p_2}$ and $t^{\alpha_1}e^{t\Delta}\nabla d_0 \in L_T^{2r}L_x^{p_1}$. Similarly $t^{\gamma_2}e^{t\Delta}\nabla d_0 \in L_T^\infty L_x^{p_3}$ and we get also $t^{\beta_2}\nabla e^{t\Delta}\nabla d_0 \in L_T^\infty L_x^{p_2}$. Because $\nabla d_0 \in \dot{B}_{3p_1,2r}^{N/(3p_1)-1}$ we get $t^{\gamma_3}e^{t\Delta}\nabla d_0 \in L_T^{2r}L_x^{3p_1}$ and $t^{\gamma_4}e^{t\Delta}\nabla d_0$ belongs to $L_T^\infty L_x^{3p_1}$.

Using Lemma 5.8.1 and Lemma 5.8.2 with $\varepsilon = 0$, $q = p_1$, $\tilde{q} = p_2$, $\bar{q} = p_3$ or $\bar{q} = 3p_1$, we deduce the previous results for $\mathcal{C}\nabla f_2$ instead of $e^{t\Delta}\nabla d_0$ (observing also that $\nabla\mathcal{C} = \mathcal{B}$ and $\nabla^2\mathcal{C} = \mathcal{A}$). Thus ∇d fulfils all the condition imposed by $\mathfrak{Y}_{r,T}$.

To conclude the proof, we use the Fixed-Point Theorem. Denoting $\tilde{Y}_{r,T}$ the set composed by the couples $(u, \nabla\Pi)$ such that $(u, d, \nabla\Pi)$ belongs to $\mathfrak{Y}_{r,T}$, we consider $(\omega_i, \nabla P_i) \in \tilde{Y}_{r,T}$, for $i = 1, 2$. Thus, defining $(u_i, \nabla\Pi_i)$ by (5.89), we have

$$\|(\delta u, \delta\nabla\Pi)\|_{\tilde{X}_{r,T}} \lesssim \|t^{\gamma_1}v\|_{L_T^{2r}L_x^{p_3}} \|(\delta\omega, \delta P)\|_{\tilde{X}_{r,T}},$$

hence there exists $(u, d, \nabla\Pi) \in \mathfrak{Y}_{r,T}$ solution of (5.87), and this concludes the proof. \square

Chapter 6

Corotational Beris-Edwards model

In this chapter we present the results of the following manuscript:

F. De Anna, *A Global 2D well-posedness result on the order tensor liquid crystal theory*, submitted (2015)

The reader should consider the results achieved in this chapter as an introduction to the ones of chapter 7, where we study the general Beris-Edwards system. The complexity of the general system requires a very deep analysis, making use of particular structural features. In this chapter, the difficulties coming from the corotational Beris-Edwards system are more handleable, then they should make the reader familiar with some specifics of the system, which will be useful when considering the general model in Chapter 7.

6.1 Introduction and main results

The theory of liquid crystal materials has attracted much attention over the recent decades. Generally, the physical state of a material can be determined by the motion degree of freedom about its molecules. Certainly, the widespread physical states of matter are the solid, the liquid and the gas ones. If the movement degree of freedom is almost zero, namely the forces which act on the molecules don't allow any kind of movement, forcing the material structure to be confined in a specific order, then we are classifying a solid material. If such degree still preserves a strong intermolecular force but it is not able to restrict the molecules to lie on a regular organization, then we are considering a fluid state of matter. Finally in the gas phase the forces and the distance between the molecules are weak and large respectively, so that the material is not confined and it is able to extend its volume.

However, some materials possess some common liquid features as well as some solid properties, namely the liquid crystals. As the name suggests, a liquid crystal is a compound of fluid molecules, which has a state of matter between the ordinary liquid one and the crystal solid one. The molecules have not a positional order but they assume an orientation which can be modified by the velocity flow. At the same time a variation of the alignment can induce a velocity field as well. In a common liquid (more correctly an isotropic liquid) if we consider the orientation of a single molecule then we should see the random variation of its position. Nevertheless, in a crystal liquid, we see an amount of orientational order.

It is well-documented that liquid crystals have been well-known for more than a century, however they have received a growth in popularity and much study only in recent decades, since they have attracted more attention thanks to their potential applications (see for instance [90]).

Commonly, in literature the liquid crystals are categorized by three sub-families, namely the nematics, the cholesterics and the smectics. On a nematic liquid crystal, the molecules have the same alignment with a preferred direction, however their positions are not correlated. On a cholesteric liquid crystal we have a foliation of the material where, on each plaque, the molecules orient themselves along the same direction (which can depend on the foliation). As in the nematic case, a cholesteric liquid crystal doesn't require any kind of relation between the positions of the molecules. At last, on a smectic liquid crystal we have still a privileged direction for all the molecules, as in the nematic case, however the position of them is bonded by a stratification. In addition to the orientational ordering, the molecules lie in layers.

6.1.1 The Order Tensor Theory

A first mathematical approach to model the generic liquid crystals was proposed by Ericksen [41] and Leslie [69] over the period of 1958 through 1968. Even if they presented a system which has been extensively studied in literature, for instance in [79] and [117], several mathematical challenges and difficulties reside in such model. Hence, in 1994, Baris and Edwards [12] proposed an alternative approach based on the concept of order Q-tensor, that one can find also in physical literature, for example [35] and [109]. The reader can find an exhaustive introduction to the Q-tensor Theory in a recent paper of Mottram and Newton [92], however we present here some hints in order to introduce the Q-tensor system.

Let us assume that our material lies on a domain Ω of \mathbb{R}^3 . A first natural strategy to model the molecules orientation is to introduce a vector field d , the so called director field (see for instance [80], which returns value on \mathbb{S}^2 , the boundary of the unit sphere on \mathbb{R}^3 . Here $d(t, x)$ is a specific vector for any fixed time and for any $x \in \Omega$. Instead of considering a precise vector $d(t, x)$ on \mathbb{S}^2 , an alternative approach is to establish the probability that this vector belongs to some measurable subset $\mathcal{A} \subseteq \mathbb{S}^2$. Therefore we introduce a continuously distributed measure \mathcal{P} on \mathbb{S}^2 , driven by a density ρ

$$\mathcal{P}(\mathcal{A}) = \int_{P \in \mathcal{A}} \rho(P) d\sigma(P) = \int_{P \in \mathcal{A}} d\rho(P).$$

We supposed the molecules to be unpolar, so that there is no difference between the extremities of them, so mathematically the probability $\mathcal{P}(\mathcal{A})$ is always equal to $\mathcal{P}(-\mathcal{A})$, which yields that the first order momentum vanishes:

$$\int_{P \in \mathbb{S}^2} \rho(P) d\sigma = 0.$$

Now considering the second order momentum tensor, given by

$$M := \int_{P \in \mathbb{S}^2} P \otimes P d\rho(P) = \left(\int_{P \in \mathbb{S}^2} P_i P_j d\rho(P) \right)_{i,j=1,2,3} \in \mathcal{M}_3(\mathbb{R}),$$

where $\mathcal{M}_3(\mathbb{R})$ denotes the 3×3 matrices with real coefficients, we observe that M is a symmetric matrix and it has trace $\text{tr}M = 1$.

In the presence of an isotropic liquid, the orientation of the molecules is uniform in every direction, hence in this case the probability \mathcal{P}_0 is given by

$$\mathcal{P}_0(\mathcal{A}) = \int_{P \in \mathcal{A}} d\sigma(P),$$

so that the corresponding second order momentum M_0 is exactly $\text{Id}/3$. We denote by Q the difference between a general M and M_0 obtaining a tensor which is known as the de Gennes order

parameter tensor. Roughly speaking, Q interprets the deviation between a general liquid crystal and an isotropic one. From the definition, it is straightforward that Q is a symmetric tensor and moreover it has null trace. If Q assumes the form $s_+(d \otimes d - \text{Id}/3)$, where s_+ is a suitable constant, then the system which models the liquid crystal (and we are going to present) reduces to the widespread Ericksen-Leslie system (see for instance [12]).

6.1.2 The Q-Tensor System

The present work is devoted to the global solvability issue for the following system as an evolutionary model for the liquid crystal hydrodynamics:

$$\begin{cases} \partial_t Q + u \cdot \nabla Q - \Omega Q + Q \Omega = \Gamma H(Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = L \text{div} \{ Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q \} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \text{div } u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (P)$$

Here $Q = Q(t, x) \in \mathcal{M}_3(\mathbb{R})$ denotes the order tensor, $u = u(t, x) \in \mathbb{R}^3$ represents the velocity field, $\Pi = \Pi(t, x) \in \mathbb{R}$ stands for the pressure, everything depending on the time variable $t \in \mathbb{R}$ and on the space variable $x \in \mathbb{R}^2$. The symbol $\nabla Q \odot \nabla Q$ denotes the 3×3 matrix whose (i, j) -th entry is given by $\text{tr}(\partial_i Q \partial_j Q)$, for $i, j = 1, 2, 3$. Moreover Γ , ν and L are three positive constants.

The left hand side of the order tensor equation is composed by a classical transport time derivative while, defining Ω as the antisymmetric matrix $\Omega := (\nabla u - {}^t \nabla u)1/2$, $Q \Omega - \Omega Q$ is an Oldroyd time derivative and describes how the flow gradient rotates and stretches the order parameter. On the right-hand-side, $H(Q)$ denotes

$$H(Q) := \underbrace{-aQ + b\left(Q^2 - \text{tr}(Q^2)\frac{\text{Id}}{3}\right) - c \text{tr}(Q^2)Q + L \Delta Q}_{P(Q)},$$

and P is the so called Landau-de Gennes thermotropic forces (more precisely it is a truncated Taylor expansion about the original one, see for instance [104]). Here a , b and c are real constant, and from here on we are going to assume c to be positive.

In reality, (P) is a simplification of a more general system. More precisely, fixing a real $\xi \in [0, 1]$, we consider

$$\begin{cases} \partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) = \Gamma H(Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = \text{div} \{ \tau + \sigma \} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \text{div } u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (P_\xi)$$

where $S(\nabla u, Q)$ stands for

$$S(Q, \nabla u) := (\xi D + \Omega)\left(Q + \frac{\text{Id}}{3}\right) + \left(Q + \frac{\text{Id}}{3}\right)(\xi D - \Omega) - 2\xi\left(Q + \frac{\text{Id}}{3}\right)\text{tr}(Q \nabla u),$$

with $D := (\nabla u + {}^t \nabla u)1/2$. Moreover τ and σ are the symmetric and antisymmetric part of the the additional stress tensor respectively, namely

$$\tau := -\xi\left(Q + \frac{\text{Id}}{3}\right)H(Q) - \xi H(Q)\left(Q + \frac{\text{Id}}{3}\right) +$$

$$+ 2\xi(Q + \frac{\text{Id}}{3})\text{tr}\{QH(Q)\} - L\{\nabla Q \odot \nabla Q + \frac{\text{Id}}{3}|Q|^2\},$$

$$\sigma := QH(Q) - H(Q)Q.$$

Here ξ is a molecular parameter which describes the rapport between the tumbling and aligning effect that a shear flow exert over the liquid crystal directors. From here on, we are going to consider the simplest case $\xi = 0$, which leads to system (P).

Before going on, let us recall what we mean by a weak solution of system (P).

Definition 6.1.1. *Let Q_0 and u_0 be a 3×3 matrix a 3-vector respectively, whose components belong to $L^2(\mathbb{R}^2)$. We say that (u, Q) is a weak solution for (P) if u belongs to $L_{loc}^\infty(\mathbb{R}_+, L_x^2) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}^1)$, Q belongs to $C(\mathbb{R}_+, H^1) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}^2)$ and (P) is fulfilled in the distributional sense.*

6.1.3 Some Developments in the order tensor Theory

Although the Q-tensor theory has received more attention in several disciplines as Physics [90], numerical analysis [86], mathematical analysis [92], the solvability study of the related system has not received numerous investigations, yet. We recall here some recent results.

in [113], D. Wang, X. Xu and C. Yu developed the existence and long time dynamics of globally defined weak solution. In their paper, system (P) is considered in the compressible and inhomogeneous setting, the fluid density ρ not necessarily constant, described by a transport equation, and moreover a pressure dependent on ρ .

In [42] J. Fahn and T. Ozawa proved some regularity criteria for a local strong solution of system (P).

In [99], M. Paicu and A. Zarnescu first showed the existence of a Lyapunov functional for system (P). Then they proved the existence of a weak solution thanks to a Friedrichs scheme. They also showed the propagation of higher regularity, namely $H^s(\mathbb{R}^2) \times H^{1+s}(\mathbb{R}^2)$ for (u, Q) , with $s > 1$. At last they established an uniqueness result on the condition that one of the two considered solutions is a strong-solution, that is they proved the so-called weak-strong uniqueness.

In [98] M. Paicu and A. Zarnescu proved existence of weak solutions for system (P_ξ) when ξ is a general value of $[0, \xi_0]$ for some $0 < \xi_0 < 1$.

In [53] F. G. Guillén-González and L. A. Rodríguez-Bellido established the existence and uniqueness of a local in time weak solution on a bounded domain. They also gave a regularity criterion which yields such solutions to be global in time. Moreover they proved the global existence and uniqueness of strong solutions provided a viscosity large enough.

In [54] F. G. Guillén-González and L. A. Rodríguez-Bellido proved the existence of global in time weak-solutions, an uniqueness criteria and a maximum principle for Q . They also established the traceless and symmetry for Q , for any weak solution.

6.1.4 Main Results

Article [99] is probably one of the best-known research interesting the solvability of (P), globally in time and in the whole space. Nevertheless their results can be improved and this work is mainly devoted to this purpose.

First Paicu and Zarnescu proved an uniqueness result on the condition that at least one of the considered solutions is a strong solution. This is due to the necessity to control $(u(t), \nabla Q(t))$ in $L^\infty(\mathbb{R}^2)$, which leads to control $(u(t), \nabla Q(t))$ in $H^s(\mathbb{R}^2)$ with $s > 1$, thanks to the Sobolev Embedding $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Assuming $s > 1$, they imposed one of the two solutions to be a

classical solution. Their approach is to estimate the difference between two solutions in the same functional space the solutions belong to, i.e. in an $L^2(\mathbb{R}^2)$ -setting.

In this work, we improve their result making use of a strategy which is inspired by [48] and [88]. Indeed, since the difference between two solutions has null initial datum, then it is possible to estimate such difference in a functional space with a lower regularity than the one related to the existence part, namely in an $H^{-1/2}(\mathbb{R}^2)$ -functional framework. We will see that this allows us to avoid the problem of controlling the $L^\infty(\mathbb{R}^2)$ norm, so that we are able to prove the following result:

Theorem 6.1.2. *Let us assume that system (P) admits two weak solutions (u_i, Q_i) , $i = 1, 2$, in the sense of definition 6.1.1. Then such solutions are equal, $(u_1, Q_1) \equiv (u_2, Q_2)$.*

The second (and last) gap concerns the propagation of regularity. Paicu and Zarnescu consider initial data (u_0, Q_0) in $H^s(\mathbb{R}^2) \times H^{1+s}(\mathbb{R}^2)$, with s greater than 1. Then, they are able to prove that such high-regularity is preserved by the related solution of (P). Denoting by

$$f(t) := \|u(t)\|_{\dot{H}^s}^2 + \|\nabla Q(t)\|_{\dot{H}^s}^2, \quad g(t) := \|\nabla u(t)\|_{\dot{H}^s}^2 + \|\Delta Q(t)\|_{\dot{H}^s}^2,$$

the major part of their proof releases on the Osgood lemma, applied on an inequality of the following type:

$$\frac{d}{dt}f(t) + g(t) \leq C f(t) \ln\{e + f(t)\}, \quad t \in \mathbb{R}_+,$$

for a suitable positive constant C . However such estimate requires again to control the norm $\|(u(t), \nabla Q(t))\|_{L^\infty}$ by $\|(u(t), \nabla Q(t))\|_{H^s}$, and this is true only if s is greater than 1. We fix such lack, namely we extend the propagation for $0 < s$, passing through an alternative approach. Indeed we control the L^∞ -norm by a different method (see Lemma 6.7.2 and (6.38)). Thus, our second result reads as follows:

Theorem 6.1.3. *Assume that (u_0, Q_0) belongs to $H^s(\mathbb{R}^2) \times H^{1+s}(\mathbb{R}^2)$, with $0 < s$. Then, the solution (u, Q) given by Theorem 6.1.4 fulfills*

$$(u, \nabla Q) \in L_{t,loc}^\infty \dot{H}^s(\mathbb{R}^2) \cap L_{t,loc}^2 \dot{H}^{s+1}(\mathbb{R}^2).$$

Now, we have also chosen to perform an existence result, for the sake of completeness, although it was proven by Paicu and Zarnescu. Nevertheless, here we use an alternative approach. Indeed in [99], the authors utilize a Friedrichs scheme, regularizing every equation of (P), while our method is based on a coupled technique between the Friedrichs scheme and the Schaefer's fixed point theorem, regularizing only the momentum equation of (P). This method is inspired by [77], where F. Lin use a modified Galerkin method coupled with the Schauder fixed point theorem, in the proof of an existence result. Then our last result reads as follows:

Theorem 6.1.4. *Assume that (u_0, Q_0) belongs to $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, then system (P) admits a global in time weak solution (u, Q) , in the sense of definition 6.1.1.*

Let us briefly describe the structure of this chapter: in the next section we recall some classical tools which are useful for our proofs, in section 6.3 we deal with Theorem 6.1.4, the existence of weak solutions, in section 6.4 and 6.5 we establish Theorem 6.1.2, i.e. such solutions are unique, and finally in section 6.6 we deal with Theorem (6.1.3), proving the propagation of regularities. We put forward in section 6.7 some technical details, for the sake of simplicity.

6.2 Preliminaries and Notations

In this section we illustrate some widely recognized mathematical tools and moreover we report some notations which are going to be extensively utilized in this research.

6.2.1 Sobolev and Besov Spaces

First, let us introduce the spaces we are going to work with (we refer the reader to [7] for an exhaustive study and more details) . We recall the well-known definition of Homogeneous Sobolev space \dot{H}^s and Non-Homogeneous Sobolev Space H^s :

Definition 6.2.1. *Let $s \in \mathbb{R}$, the Homogeneous Sobolev Space \dot{H}^s (also denoted $\dot{H}^s(\mathbb{R}^2)$) is the space of tempered distributions $u \in \mathcal{S}'$ over \mathbb{R}^2 , the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^2)$ and it fulfills*

$$\|u\|_{\dot{H}^s} := \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

Moreover u belongs to the Non-Homogeneous Sobolev Space H^s (or $H^s(\mathbb{R}^2)$) if $\hat{u} \in L^2_{loc}(\mathbb{R}^2)$ and

$$\|u\|_{H^s} := \int_{\mathbb{R}^2} (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

H^s is an Hilbert space for any real s , while \dot{H}^s requires $s < d/2$, otherwise it is Pre-Hilbert. Their inner products are

$$\langle u, v \rangle_{H^s} = \int_{\mathbb{R}^2} (1 + |\xi|)^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi \quad \text{and} \quad \langle u, v \rangle_{\dot{H}^s} = \int_{\mathbb{R}^2} |\xi|^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi,$$

respectively. Even if such dot-products are the most common ones, from here on we are going to use the ones related to the Besov Spaces (at least for the homogeneous case). Hence, first we need to define them. In order to do that, it is fundamental to introduce the Dyadic Partition. Let $\chi = \chi(\xi)$ be a smooth function whose support is inside the the ball $|\xi| \leq 1$. Let us assume that χ is identically equal to 1 in $|\xi| \leq 3/4$, then, imposing $\varphi_q(\xi) := \chi(\xi 2^{-q-1}) - \chi(\xi 2^{-q})$ for any $q \in \mathbb{Z}$, we define the Homogeneous Littlewood-Paley Block $\dot{\Delta}_q$ by

$$\dot{\Delta}_q f := \mathcal{F}^{-1}(\varphi_q \hat{f}) \in \mathcal{S}', \quad \text{for any } f \in \mathcal{S}'.$$

Moreover we denote by \dot{S}_j the operator $\sum_{q \leq j-1} \dot{\Delta}_q$, for any $j \in \mathbb{Z}$. We can now present the definition of Homogeneous Besov Space

Definition 6.2.2. *For any $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$, we define $\dot{B}^s_{p,r}$ as the set of tempered distribution f such that*

$$\|f\|_{\dot{B}^s_{p,r}} := \|2^{sq} \|\dot{\Delta}_q f\|_{L^p_x} \|l^r(\mathbb{Z})\|$$

and for all smooth compactly supported function θ on \mathbb{R}^2 we have

$$\lim_{\lambda \rightarrow +\infty} \theta(\lambda D) f = 0 \quad \text{in } L^\infty(\mathbb{R}^2).$$

It is straightforward that the space $\dot{B}^s_{2,2}$ and \dot{H}^s coincides for any real s , and their norms are

equivalent, so we will use the following abuse of notation from here on:

$$\langle u, v \rangle_{\dot{H}^s} := \langle u, v \rangle_{\dot{B}_{2,2}^s} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q u, \dot{\Delta}_q v \rangle_{L_x^2},$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the common inner product of $L_x^2 := L^2(\mathbb{R}^2)$.

A profitable feature of the Homogeneous Besov space with negative index s is the following one (see Proposition 2.33 of [7])

Proposition 6.2.3. *Let $s < 0$ and $1 \leq p, r \leq \infty$. Then u belongs to $\dot{B}_{p,r}^s$ if and only if*

$$(2^{qs} \|\dot{S}_q u\|_{L_x^p})_{q \in \mathbb{Z}} \in L^r(\mathbb{Z}).$$

Moreover there exists two positive constant c_s and C_S such that

$$c_s \|u\|_{\dot{B}_{p,r}^s} \leq \| (2^{qs} \|\dot{S}_q u\|_{L_x^p})_{q \in \mathbb{Z}} \|_{L^r(\mathbb{Z})} \leq C_s \|u\|_{\dot{B}_{p,r}^s}.$$

6.2.2 Homogeneous Paradifferential Calculus

In this subsection we give some hints about how the product acts between \dot{H}^s and \dot{H}^t , for some appropriate real s and t . We present several tools which will play a major part in all our proofs. First, let us begin with the following Theorem, whose proof is put forward in the appendix:

Theorem 6.2.4. *Let s and t be two real numbers such that $|s|$ and $|t|$ belong to $[0, 1)$. Let us assume that $s + t$ is positive, then for every $a \in \dot{H}^s$ and for every $b \in \dot{H}^t$, the product ab belongs to \dot{H}^{s+t-1} and there exists a positive constant (not dependent on a and b) such that*

$$\|ab\|_{\dot{H}^{s+t-1}} \leq C \|a\|_{\dot{H}^s} \|b\|_{\dot{H}^t}$$

One of the main reasons we should consider the Besov formulation $\dot{B}_{2,2}^s$ of the homogeneous Sobolev space \dot{H}^s is the so-called Bony decomposition:

$$fg = \dot{T}_f g + \dot{T}_g h + \dot{R}(f, g), \quad \text{with} \quad \dot{T}_f g := \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g \quad \text{and} \quad \dot{R}(f, g) := \sum_{q \in \mathbb{Z}, |l| \leq 1} \dot{\Delta}_q f \dot{\Delta}_{q+l} g.$$

This decomposition is quite helpful when analyzing $\dot{\Delta}_q(fg)$ for some integer q . In order to deal the most challenging terms, in this work we will make use of a reformulation of the Bony decomposition. We present it in its matrix form. Let q be an integer, and A, B be $N \times N$ matrices, whose components are homogeneous temperate distributions, we denote by

$$\begin{aligned} \mathcal{J}_q^1(A, B) &:= \sum_{|q-q'| \leq 5} [\dot{\Delta}_q, \dot{S}_{q'-1} A] \dot{\Delta}_{q'} B, & \mathcal{J}_q^3(A, B) &:= \dot{S}_{q-1} A \dot{\Delta}_q B, \\ \mathcal{J}_q^2(A, B) &:= \sum_{|q-q'| \leq 5} (\dot{S}_{q'-1} A - \dot{S}_{q-1} A) \dot{\Delta}_q \dot{\Delta}_{q'} B, & \mathcal{J}_q^4(A, B) &:= \sum_{q' \geq q-5} \dot{\Delta}_q (\dot{\Delta}_{q'} A \dot{S}_{q'+2} B), \end{aligned} \tag{6.1}$$

then the following product law for AB , is satisfied:

$$\dot{\Delta}_q(AB) = \mathcal{J}_q^1(A, B) + \mathcal{J}_q^2(A, B) + \mathcal{J}_q^3(A, B) + \mathcal{J}_q^4(A, B), \tag{6.2}$$

for any integer q .

6.2.3 The Frobenius Norm

Before beginning with the proofs of our main results, let us give the following remark:

Remark 6.2.5. *The most common inner product defined on $\mathcal{M}_3(\mathbb{R})$ (the 3×3 real matrices) is determined by:*

$$A \cdot B = \sum_{i,j=1}^3 A_{ij}B_{ij} = \text{tr}\{{}^tAB\}, \quad \text{for any } A, B \in \mathcal{M}_3(\mathbb{R}).$$

Hence, if at least one of the two matrices is symmetric, for instance A , then we obtain

$$A \cdot B = \text{tr}\{AB\}, \quad (6.3)$$

which determines the well-known Frobenius norm of a matrix $|A| := \sqrt{\text{tr}\{A^2\}}$. Since any solution (u, Q) for (P) fulfills

$$Q(t, x) \in S_0 := \{A \in \mathcal{M}_3(\mathbb{R}), \text{tr}\{A\} = 0 \quad \text{and} \quad {}^tA = A\},$$

for almost every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$ (see [99] and [54]), then from here on we will repeatedly use (6.3).

Moreover, we will use the symbol \lesssim (instead of \leq) which is defined as follows: for any non-negative real numbers a and b , $a \lesssim b$ if and only if there exists a positive constant C (not dependent on a and b) such that $a \leq Cb$.

6.3 Weak Solutions

This section deals with the existence of weak solutions for (P) in the sense of definition 6.1.1. As we have already explained, we are going to proceed with a coupled method between the Friedrichs scheme and the Schaefer's Theorem. Hence, before going on, let us recall the widely recognized Schaefer's fixed point Theorem

Theorem 6.3.1. *Let Ψ be a continuous and compact mapping of a Banach Space X into itself, such that the set $\{x \in X : x = \lambda \Psi x \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded. Then T has a fixed point.*

First, we introduce one of the key ingredients of our proofs, namely the mollifying operator J_n defined by

$$\mathcal{F}(J_n f)(\xi) = 1_{[\frac{1}{n}, n]}(\xi) \quad \text{for } \xi \in \mathbb{R}_\xi^2,$$

which erases the high and the low frequencies.

We claim the existence and uniqueness of a solution for the following system

$$\begin{cases} \partial_t Q + (J_n u \cdot \nabla Q) - J_n \Omega Q + Q J_n \Omega = \Gamma H(Q) & [0, T) \times \mathbb{R}^2, \\ \partial_t u + J_n \mathcal{P}(J_n u \cdot \nabla J_n u) - \nu \Delta u = L J_n \mathcal{P} \text{div} \{ Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q \} & [0, T) \times \mathbb{R}^2, \\ \text{div } u = 0 & [0, T) \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{cases} \quad (P_n)$$

where \mathcal{P} stands for the Leray projector operator, which is determined by

$$\mathcal{F}\{\mathcal{P}f\}(\xi) := \hat{f}(\xi) - \frac{\xi}{|\xi|} \frac{\xi}{|\xi|} \cdot \hat{f}(\xi), \quad \text{for } f \in (L_x^p)^2, \quad \text{with } 1 < p < \infty,$$

and T is a positive real number. It is well known that \mathcal{P} is a bounded operator of $(L_x^p)^2$ into itself when $p \in (1, \infty)$.

Remark 6.3.2. We say (u, Q) is a weak solution of the problem (P_n) , provided that

$$u \in C([0, T], L_x^2) \cap L^2(0, T; \dot{H}^1), \quad Q \in C([0, T]; H^1) \cap L^2(0, T; \dot{H}^2)$$

and (P_n) is valid in the distributional sense.

The following proposition plays a major part in our main proof, since it allows us to control the L_x^p -norm of Q only by Q_0 .

Proposition 6.3.3. Suppose that $u \in C([0, T], L_x^2) \cap L^2(0, T; \dot{H}^1)$ and moreover that $Q \in C([0, T], H^1) \cap L^2(0, T; \dot{H}^2)$ is a weak solution of

$$\partial_t Q + u \cdot \nabla Q - \Omega Q + Q \Omega - \Gamma L \Delta Q = \Gamma P(Q) \quad \text{in } [0, T) \times \mathbb{R}^2, \quad \text{and} \quad Q_{t=0} = Q_0 \in H^1.$$

Then, for every $2 \leq q < \infty$, the following estimate is fulfilled

$$\|Q(t)\|_{L_x^q} + q \int_{\mathbb{R}^2} \text{tr}\{Q(t)^2\}^{\frac{q}{2}-1} |\nabla Q(t)|^2 dx \leq \|Q_0\|_{H^1} \exp\{Ct\},$$

for a suitable positive constant C dependent only on q, Γ, a, b and c .

Proof. Fixing $p \in (1, \infty)$, We multiply both left and right-hand side by $2pQ \text{tr}\{Q^2\}^{p-1}$, we take the trace and we integrate in \mathbb{R}^2 , obtaining that

$$\begin{aligned} \frac{d}{dt} \|Q(t)\|_{L_x^{2p}}^{2p} - \Gamma 2Lp \langle Q(t) \text{tr}\{Q(t)^2\}^{p-1}, \Delta Q(t) \rangle_{L_x^2} &= \\ = 2\Gamma p \int_{\mathbb{R}^2} \text{tr}\{Q(t)^2\}^{p-1} \text{tr}\{P(Q(t, x))Q(t, x)\} dx, \end{aligned}$$

for almost every $t \in (0, T)$, where we have used $\text{div } u = 0$ and $\text{tr}\{Q\Omega Q - \Omega Q^2\} = 0$. First, analyzing the second term on the left-hand side, integrating by parts, we determine the following identity:

$$\begin{aligned} -\langle 2pQ \text{tr}\{Q^2\}^{p-1}, \Delta Q \rangle_{L^2} &= \\ = \sum_{i=1}^2 \left[2p \int_{\mathbb{R}^2} \text{tr}\{Q^2\}^{p-1} \text{tr}\{(\partial_i Q)^2\} + 2p \int_{\mathbb{R}^2} \partial_i [\text{tr}\{Q^2\}^{p-1}] \text{tr}\{Q \partial_i Q\} \right] \\ = 2p \int_{\mathbb{R}^2} \text{tr}\{Q^2\}^{p-1} |\nabla Q|^2 + 4p(p-1) \int_{\mathbb{R}^2} \text{tr}\{Q^2\}^{p-2} |\nabla [\text{tr}\{Q^2\}]|^2 \geq 0, \end{aligned}$$

which allows us to obtain

$$\frac{d}{dt} \|Q(t)\|_{L^{2p}}^{2p} + 2p \int_{\mathbb{R}^2} \text{tr}\{Q^2\}^{p-1} |\nabla Q|^2 \leq \Gamma \int_{\mathbb{R}^2} 2p \text{tr}\{Q^2\} \text{tr}\{P(Q(t, x))Q(t, x)\} dx.$$

Now, we deal with the right-hand side by a direct computation, observing that

$$\begin{aligned} \int_{\mathbb{R}^2} \text{tr}\{Q^2\}^{p-1} \text{tr}\{P(Q)Q\} dx &= \Gamma \int_{\mathbb{R}^2} \left[-a \text{tr}\{Q^2\}^p + b \text{tr}\{Q^2\}^{p-1} \text{tr}\{Q^3\} - c \text{tr}\{Q^2\}^{p+1} \right] \\ &\lesssim \|Q\|_{L_x^{2p}}^{2p} - \frac{c}{2} \|Q\|_{L_x^{2(p+1)}}^{2(p+1)} \lesssim \|Q\|_{L_x^{2p}}^{2p}, \end{aligned}$$

where we have used the following feature about a symmetric matrix with null trace:

$$\left| \int_{\mathbb{R}^2} \operatorname{tr}\{Q^2\}^{p-1} \operatorname{tr}\{Q^3\} \right| \leq \varepsilon \|Q\|_{L^{2(p+1)}}^{2(p+1)} + \frac{1}{\varepsilon} \|Q\|_{L^{2p}}^{2p},$$

for a positive real ε , small enough. Indeed, if Q has λ_1 , λ_1 , and $-\lambda_1 - \lambda_2$ as eigenvalues, we achieve that $\operatorname{tr}\{Q^3\} = -3\lambda_1\lambda_2(\lambda_1 + \lambda_2)$ and $\operatorname{tr}\{Q^2\} = 2(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2)$, hence

$$\begin{aligned} |\operatorname{tr}\{Q^3\}| &\lesssim \varepsilon \lambda_1^2 \lambda_2^2 + \frac{1}{\varepsilon} (\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2) \\ &\lesssim \varepsilon (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2)^2 + \frac{1}{\varepsilon} (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) \lesssim \varepsilon \operatorname{tr}\{Q^2\}^2 + \frac{1}{\varepsilon} \operatorname{tr}\{Q^2\}. \end{aligned}$$

Therefore, we deduce that

$$\left| \int_{\mathbb{R}^2} \operatorname{tr}\{Q^2\}^{p-1} \operatorname{tr}\{Q^3\} \right| \lesssim \varepsilon \int_{\mathbb{R}^N} \operatorname{tr}\{Q^2\}^{(p+1)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \operatorname{tr}\{Q^2\}^p. \quad (6.4)$$

Summarizing the previous consideration, we get $\frac{1}{2} \frac{d}{dt} \|Q(t)\|_{L_x^{2p}}^{2p} \lesssim \|Q(t)\|_{L_x^{2p}}^{2p}$, so that the statement is proved, thanks to the Gronwall's inequality. \square

Now, let us focus on one of the main theorems of this section, which reads as follows:

Theorem 6.3.4. *Let n be a positive integer and assume that (u_0, Q_0) belongs to $L_x^2 \times H_x^1$. Then, system (P_n) admits a unique local weak solution.*

Proof. The key method of the proof relies on the Schauder's Theorem. We define the compact operator Ψ from $C([0, T], L_x^2) \cap L^2(0, T; \dot{H}^1)^2$ to itself as follows: $(\Psi(u), Q) =: (\tilde{u}, Q)$ is the unique weak solution (in the sense of remark 6.3.2) of the following Cauchy problem:

$$\begin{cases} \partial_t Q + (J_n u \cdot \nabla Q) - J_n \Omega Q + Q J_n \Omega = \Gamma H(Q) & [0, T] \times \mathbb{R}^2, \\ \partial_t \tilde{u} + J_n \mathcal{P} (J_n \tilde{u} \cdot \nabla J_n \tilde{u}) - \nu \Delta \tilde{u} = L J_n \mathcal{P} \operatorname{div} \{ Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q \} & [0, T] \times \mathbb{R}^2, \\ \operatorname{div} \tilde{u} = 0 & [0, T] \times \mathbb{R}^2, \\ (\tilde{u}, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2. \end{cases}$$

We claim that the hypotheses of the Schauder's Theorem are fulfilled, namely Ψ is a compact mapping of $X := C([0, T], L_x^2) \cap L^2(0, T; \dot{H}^1)$ into itself, and the set

$$\{u = \lambda \Psi(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. First we deal with the compactness of Ψ . Considering a bounded family \mathcal{F} of X , we claim that the closure of $\Psi(\mathcal{F})$ is compact in X . If we prove that $\Psi(\mathcal{F})$ is an uniformly bounded and equicontinuous family of $C([0, T]; L_x^2)$ and moreover that $\{\Psi(u)(t) \text{ with } t \in [0, T] \text{ and } u \in \mathcal{F}\}$ is a compact set of L_x^2 , then the result is at least valid as Ψ mapping of X into $C([0, T], L^2)$, thanks to the Arzelà-Ascoli Theorem. Multiplying the first equation by $Q - \Delta Q$ and integrating in \mathbb{R}^2 , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\|\nabla Q\|_{L_x^2}^2 + \|Q\|_{L_x^2}^2 \right] + \Gamma L (\|\nabla Q\|_{L_x^2}^2 + \|\Delta Q\|_{L_x^2}^2) &= \int_{\mathbb{R}^2} \left[\operatorname{tr}\{(J_n \Omega Q - Q J_n \Omega) \Delta Q\} - \right. \\ &\quad \left. - \operatorname{tr}\{(J_n u \cdot \nabla Q) \Delta Q\} \right] - \Gamma L \int_{\mathbb{R}^2} \left[a \operatorname{tr}\{Q \Delta Q\} - b \operatorname{tr}\{Q^2 \Delta Q\} + c \operatorname{tr}\{Q \Delta Q\} \operatorname{tr}\{Q^2\} \right] + \\ &\quad + \Gamma \int_{\mathbb{R}^2} \left[a \operatorname{tr}\{Q^2\} - b \operatorname{tr}\{Q^3\} + c \operatorname{tr}\{Q^2\}^2 \right], \end{aligned}$$

almost everywhere in $(0, T)$, which allows us to achieve

$$\begin{aligned} \frac{d}{dt} \left[\|\nabla Q(t)\|_{L_x^2}^2 + \|Q(t)\|_{L_x^2}^2 \right] + \Gamma L (\|\nabla Q\|_{L_x^2}^2 + \|\Delta Q\|_{L_x^2}^2) &\leq \\ &\leq C_n (1 + \|u(t)\|_{L_x^2}^2) (\|Q(t)\|_{L^2}^2 + \|Q(t)\|_{L_x^6}^6 + \|\nabla Q(t)\|_{L_x^2}^2) + \frac{\Gamma L}{100} \|\Delta Q(t)\|_{L_x^2}^2, \end{aligned}$$

where C_n is a positive constant dependent on n . Therefore, thanks to Proposition 6.3.3, we realize that the family composed by $Q = Q(u)$ as u ranges on \mathcal{F} is a bounded family in $C([0, T]; H^1) \cap L^2(0, T; \dot{H}^2)$. Now, multiplying the second equation by \tilde{u} we get the following equality:

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_{L_x^2}^2 + \nu \|\nabla \tilde{u}(t)\|_{L_x^2}^2 = L \int_{\mathbb{R}^2} \text{tr}\{(\nabla Q \odot \nabla Q + Q \Delta Q - \Delta Q Q) \nabla \tilde{u}\}(t, x) dx =: F(t),$$

for almost every $t \in (0, T)$. Thus it turns out that

$$\begin{aligned} \frac{d}{dt} \|\tilde{u}(t)\|_{L_x^2}^2 + \nu \|\nabla \tilde{u}(t)\|_{L_x^2}^2 &\leq |F(t)| \leq \|\nabla Q\|_{L_x^4}^2 \|\nabla \tilde{u}\|_{L_x^2} + \|Q\|_{L_x^2} \|\Delta Q\|_{L_x^2} \|\nabla \tilde{u}\|_{L_x^\infty} \\ &\leq C_n (\|Q(t)\|_{L_x^2}^2 + \|\nabla Q(t)\|_{L_x^2}^2) \|\Delta Q(t)\|_{L_x^2}^2 + \frac{\nu}{100} \|\nabla \tilde{u}(t)\|_{L_x^2}^2, \end{aligned} \quad (6.5)$$

where $C_n > 0$ depends on n . Here, we have used the feature $J_n \tilde{u} = \tilde{u}$, which comes from the uniqueness of the solution for the second equation, so that $\|\nabla \tilde{u}\|_{L_x^\infty} \leq C_n \|\nabla \tilde{u}\|_{L_x^2}$. Summarizing the previous considerations and thanks to the Gonwall's inequality we discover that $\Psi(\mathcal{F})$ is a bounded family in X , so in $C([0, T], L^2)$. Moreover, from (6.5) and the previous result, it turns out that $|F(t)|$ is bounded on $[0, T]$, uniformly in $u \in \mathcal{F}$. Hence $\Psi(\mathcal{F})$ is an equicontinuous family of $C([0, T]; L_x^2)$. Finally, because $J_n \tilde{u} = \tilde{u}$, we get that $\{\Psi(u)(t) \text{ with } t \in [0, T] \text{ and } u \in \mathcal{F}\}$ is a subset of a bounded L_x^2 -family composed by functions with Fourier-transform supported in the annulus $\mathcal{C}(1/n, n)$, which is a compact family of L_x^2 . Summarizing all the previous consideration, we get that $\Psi(\mathcal{F})$ is compact in $C([0, T], L_x^2)$ thanks to the Arzelà-Ascoli Theorem.

It remains to prove that $\Psi(\mathcal{F})$ is compact in $L^2(0, T; \dot{H}^1)$, so that Ψ is a compact mapping of X into itself. Since $J_n \Psi(u(t)) = \Psi(u(t))$ for every $u \in \mathcal{F}$ and $t \in (0, T)$, the precompactness of $\Psi(\mathcal{F})$ in $L^2(0, T; \dot{H}^1)$ is equivalent to the precompactness of $\Psi(\mathcal{F})$ in $L^2((0, T) \times \mathbb{R}^2)$. Recalling that $\Psi(\mathcal{F})$ is precompact in $C([0, T], L_x^2)$ which is embedded in $L^2((0, T) \times \mathbb{R}^2)$ (for T finite), then we determine the result, so that, in conclusion Ψ is a compact operator from X to itself.

Now, we deal with the Schaefer's Theorem hypotheses, namely the set

$$\{u = \lambda \Psi(u) \text{ for some } \lambda \in (0, 1)\}$$

is a bounded family of X . First, we point out that if $u = \lambda \Psi(u)$, then the couple (u, Q) is a solution for

$$\begin{cases} \partial_t Q + \lambda J_n u \cdot \nabla Q - \lambda J_n \Omega Q + \lambda Q J_n \Omega = \Gamma H(Q) & [0, T] \times \mathbb{R}^2, \\ \partial_t u + J_n \mathcal{P}(J_n u \cdot \nabla J_n u) - \nu \Delta u = L J_n \mathcal{P} \text{div} \{Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q\} & [0, T] \times \mathbb{R}^2, \\ \text{div } u = 0 & [0, T] \times \mathbb{R}^2, \\ (u, Q)_{t=0} = (u_0, Q_0) & \mathbb{R}^2. \end{cases}$$

We multiply the first equation by $Q - \Delta Q$, the second equation by u , we integrate everything in \mathbb{R}^2 and we sum the results, obtaining:

$$\frac{d}{dt} \left[\|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|u\|_{L^2}^2 \right] + \Gamma L \|\nabla Q\|_{L^2}^2 + \Gamma L \|\Delta Q\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 =$$

$$\begin{aligned} & \lambda \langle J_n u \cdot \nabla Q, Q - \Delta Q \rangle_{L^2} + \lambda \langle J_n \Omega Q - Q J_n \Omega, Q - \Delta Q \rangle_{L^2} + \Gamma \langle P(Q), Q - \Delta Q \rangle_{L^2} - \\ & - \langle J_n u \cdot \nabla J_n u, \nabla J_n u \rangle_{L^2} + L \langle Q \Delta Q - \Delta Q Q, \nabla J_n u \rangle_{L^2} + L \langle \nabla Q \odot \nabla Q, \nabla J_n u \rangle_{L^2}. \end{aligned}$$

According to $\|J_n u\|_{L^\infty} + \|\nabla J_n u\|_{L^\infty} \leq C_n \|u\|_{L^2}$, up to a positive constant C_n dependent on n , it is not computationally demanding to achieve the following estimate:

$$\begin{aligned} \frac{d}{dt} \left[\|Q\|_{L_x^2}^2 + \|\nabla Q\|_{L_x^2}^2 + \|u\|_{L_x^2}^2 \right] + \Gamma L \|\Delta Q\|_{L_x^2}^2 + \nu \|\nabla u\|_{L_x^2}^2 & \leq \\ & \leq \tilde{C}_n \|\nabla Q\|_{L_x^2}^2 \left[\|Q\|_{L_x^2}^2 + \|\nabla Q\|_{L_x^2}^2 + \|u\|_{L_x^2}^2 \right] + \frac{\nu}{100} \|\nabla u\|_{L_x^2}^2 + \frac{\Gamma L}{100} \|\Delta Q\|_{L_x^2}^2. \end{aligned}$$

Now, we recall that ∇Q is in bounded in $L^2(0, T, L_x^2)$ thanks to Proposition 6.3.3 (imposing $q = 2$). Therefore, thanks to the Gronwall's inequality, we detect the following estimate:

$$\begin{aligned} & \|Q\|_{L^\infty(0, T; L_x^2)}^2 + \|\nabla Q\|_{L^\infty(0, T; L_x^2)}^2 + \|u\|_{L^\infty(0, T; L_x^2)}^2 + \\ & + \|\Delta Q\|_{L^2(0, T; L_x^2)}^2 + \|\nabla u\|_{L^2(0, T; L_x^2)}^2 \lesssim \|(u_0, Q_0, \nabla Q_0)\|_{L_x^2} e^{C_n \|Q_0\|_{H^1} e^{CT}}, \end{aligned}$$

so that, the family $\{u = \lambda \Psi(u) \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded in X . Hence, applying the Schaefer's fixed point Theorem, we conclude that there exists a fixed point for Ψ , namely there exists a weak solution (u, Q) (in the sense of remark 6.3.2) for the system (P_n) . \square

Remark 6.3.5. *In the previous proof T has only to be bounded, and it has no correlation with the initial data, so that the solution (u^n, Q^n) of system (P_n) , given by Proposition 6.3.3, it should be supposed to belong to*

$$C(\mathbb{R}_+, L_x^2) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}^1) \times C(\mathbb{R}_+, H^1) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}^2).$$

We are now able to prove our main existence result, namely Theorem 6.1.4.

Proof of Theorem 6.1.4. Let us fix a positive real T and let (u^n, Q^n) be the solution of (P_n) given by Proposition 6.3.3, for any positive integer n . We analyse such solutions in order to develop some n -uniform bound for their norms, which will allow us to apply some classical methods about compactness and weakly convergence.

We multiply the first equation of (P_n) by $Q^n - L\Delta Q^n$, the second one by u^n , we integrate in \mathbb{R}^2 and finally we sum the results, obtaining the following identity

$$\begin{aligned} \frac{d}{dt} \left[\|u^n\|_{L_x^2} + \|Q^n\|_{L_x^2} + \Gamma L \|\nabla Q^n\|_{L_x^2}^2 \right] + \nu \|\nabla u^n\|_{L_x^2} + \Gamma L \|\nabla Q^n\|_{L_x^2} + \Gamma L^2 \|\Delta Q^n\|_{L_x^2} & = \\ = \underbrace{-\langle u^n \cdot \nabla Q^n, Q^n \rangle_{L_x^2}}_{=0} + \underbrace{L \langle u^n \cdot \nabla Q^n, \Delta Q^n \rangle_{L_x^2}}_{\mathcal{B}} + \underbrace{\langle \Omega^n Q^n - Q^n \Omega^n, Q^n \rangle_{L_x^2}}_{=0} - & \\ - \underbrace{L \langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{L_x^2}}_{\mathcal{A}} + \Gamma \langle P(Q^n), Q^n \rangle_{L_x^2} - \Gamma L \langle P(Q^n), \Delta Q^n \rangle_{L_x^2} - & \\ - \underbrace{\langle u^n \cdot \nabla u^n, u^n \rangle_{L_x^2}}_{=0} - \underbrace{L \langle Q^n \Delta Q^n - \Delta Q^n Q^n, \nabla u^n \rangle_{L_x^2}}_{\mathcal{AA}} - \underbrace{L \langle \operatorname{div}\{\nabla Q^n \odot \nabla Q^n\}, u^n \rangle_{L_x^2}}_{\mathcal{BB}}. & \quad (6.6) \end{aligned}$$

First, let us observe that $\mathcal{A} + \mathcal{AA} = 0$ thanks to Lemma 6.7.1. Moreover $\langle u^n \cdot \nabla Q^n, Q^n \rangle_{L_x^2}$ and $\langle u^n \cdot \nabla u^n, u^n \rangle_{L_x^2}$ are null, because of the divergence-free condition of u^n , while $\langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{L_x^2}$ is zero since Q^n is symmetric. Furthermore $\mathcal{B} + \mathcal{BB} = 0$ since the following identity is satisfied:

$$\operatorname{tr}\{u^n \cdot \nabla Q^n \Delta Q^n\} = \operatorname{div}\{\nabla Q^n \odot \nabla Q^n\} \cdot u^n - \operatorname{div}\{u^n (|\partial_1 Q^n|^2 + |\partial_2 Q^n|^2)\}.$$

Recalling (6.4) with $p = 1$, it turns out that

$$\Gamma \langle P(Q^n), Q^n \rangle_{L^2} \lesssim \|Q^n\|_{L_x^2}^2 - \frac{c}{2} \|Q^n\|_{L_x^4}^4 \leq \|Q^n\|_{L_x^2}^2,$$

while, by a direct computation and thanks to Proposition 6.3.3, we deduce

$$\begin{aligned} \Gamma L \langle P(Q^n), \Delta Q^n \rangle_{L_x^2} &\lesssim \|\nabla Q^n\|_{L_x^2}^2 + \|Q^n\|_{L^6}^3 \|\Delta Q^n\|_{L_x^2} \\ &\lesssim \|\nabla Q^n\|_{L_x^2}^2 + \|Q_0\|_{H^1}^6 e^{6Ct} + C_{\Gamma,L} \|\Delta Q^n\|_{L_x^2}^2, \end{aligned}$$

where C is positive real constant, not dependent on n and $C_{\Gamma,L} > 0$ is a suitable small enough constant which will allow to absorb $\|\Delta Q^n\|_{L_x^2}^2$ by the left-hand side of (6.6). Thus, summarizing the previous considerations, we get

$$\begin{aligned} \frac{d}{dt} \left[\|u^n\|_{L_x^2}^2 + \|Q^n\|_{L_x^2}^2 + \Gamma L \|\nabla Q^n\|_{L_x^2}^2 \right] + \nu \|\nabla u^n\|_{L^2}^2 + \Gamma L^2 \|\Delta Q^n\|_{L_x^2}^2 &\lesssim \\ &\lesssim \|Q^n\|_{L_x^2}^2 + \|\nabla Q^n\|_{L_x^2}^2 + \|Q_0\|_{H^1}^6 e^{6Ct}, \end{aligned}$$

which yields

$$\begin{aligned} \|(u^n, Q^n, \nabla Q^n)\|_{L^\infty(0,T;L_x^2)} + \|(\nabla u^n, \Delta Q^n)\|_{L^2(0,T;L_x^2)} &\lesssim \\ &\lesssim (\|u_0\|_{L_x^2} + \|Q_0\|_{L_x^2} + \|Q_0\|_{H^1}^6) \exp\{\tilde{C}t\}, \end{aligned} \quad (6.7)$$

for a suitable positive constant \tilde{C} , independent on n .

Thanks to the previous control, we carry out to pass to the limit as n goes to $+\infty$, and we claim to found a weak solution for system (P) . We fix at first a bounded domain Ω of \mathbb{R}^2 , with a smooth enough boundary. At first we claim that $(Q^n)_{\mathbb{N}}$ is a Cauchy sequence in $C([0, T], L^2(\Omega))$, and the major part of the proof releases in the Arzelà-Ascoli Theorem. We have already proven that $(Q^n)_{\mathbb{N}}$ is bounded in such space, moreover, since $Q^n(t)$ belongs to $H^1(\Omega)$ which is compactly embedded in $L^2(\Omega)$, we get that $\{Q^n(t) : n \in \mathbb{N} \text{ and } t \in [0, T]\}$ is a compact set of $L^2(\Omega)$. Moreover, observing that

$$\begin{aligned} \|\partial_t Q^n\|_{L^2(\Omega)} &\leq \|u^n\|_{L_x^4} \|\nabla Q^n\|_{L_x^4} + \|\nabla u^n\|_{L_x^2} \|Q^n\|_{L_x^\infty} + \|P(Q^n)\|_{L_x^2} \\ &\leq \|u^n\|_{L_x^2}^{\frac{1}{2}} \|\nabla u^n\|_{L_x^2}^{\frac{1}{2}} \|\nabla Q^n\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q^n\|_{L_x^2}^{\frac{1}{2}} + \|\nabla u^n\|_{L_x^2} \|Q^n\|_{H^2} + \|Q^n\|_{L_x^2} + \|Q^n\|_{L_x^4}^2 + \|Q^n\|_{L_x^6}^3. \end{aligned}$$

Therefore, it turns out that $(\partial_t Q^n)_{\mathbb{N}}$ is an uniformly bounded sequence in $L^1(0, T; L_x^2)$ which yields that $(Q^n)_{\mathbb{N}}$ is uniformly equicontinuous in $C([0, T], L_x^2)$, so that, applying the Arzelà-Ascoli Theorem, there exists $Q \in C([0, T], L_x^2)$ such that Q^n strongly converges to Q , up to a subsequence. Moreover, thanks to (6.7), we also obtain that ∇Q and ΔQ belong to $L^\infty(0, T; L_x^2)$ and $L^2(0, T; L_x^2)$ respectively, and we have:

$$\nabla Q^n \rightharpoonup \nabla Q \quad w - L^2(0, T; L_x^2) \quad \text{and} \quad \Delta Q^n \rightharpoonup \Delta Q \quad w - L^2(0, T; L_x^2),$$

up to a subsequence. Now, let us fix a bounded smooth domain Ω of \mathbb{R}^2 . Then $\nabla Q^n(t)$ weakly converges to $\nabla Q(t)$ in $H^1(\Omega)$, for almost every $t \in (0, T)$, up to a subsequence, so that, from the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that $\nabla Q^n(t)$ strongly converges to $\nabla Q(t)$ in $L^2(\Omega)$, for almost every $t \in (0, T)$. Moreover $\|\nabla Q^n - \nabla Q\|_{L^2(\Omega)}$ belongs to $L^\infty(0, T)$ and its norm

is uniformly bounded in n . Hence applying the dominated convergence Theorem, we get

$$\lim_{n \rightarrow \infty} \int_0^T \|\nabla Q^n(t) - \nabla Q(t)\|_{L^2}^2 dt = \int_0^T \lim_{n \rightarrow \infty} \|\nabla Q^n(t) - \nabla Q(t)\|_{L^2}^2 dt = 0,$$

namely ∇Q^n strongly converges to ∇Q in $L^2(0, T; L^2(\Omega))$. Since ∇Q^n is bounded in the Lebesgue-space $L^2(0, T; L_x^6)$ (from the embedding $H^1 \hookrightarrow L_x^6$) we get also that ∇Q^n weakly converges to ∇Q in $w - L^2(0, T; L_x^6)$, so that ∇Q^n strongly converges to ∇Q in $L^2(0, T; L^4(\Omega))$ by interpolation. This range of convergences shows that $\nabla Q \odot \nabla Q$ and $Q\Delta Q - \Delta Q$ are the limits of $\nabla Q^n \odot \nabla Q^n$ and $Q^n\Delta Q^n - \Delta Q^n Q^n$, as n goes to infinity, respectively in $L^1(0, T; L^4(\Omega))$ and $L^1(0, T; L^{4/3}(\Omega))$. The strongly convergence of $P(Q^n)$ to $P(Q)$ in $L^2(0, T; L^2(\Omega))$ is straightforward, while, with a similar strategy, we are able to prove the existence of $u \in L^\infty(0, T; L_x^2)$ with $\nabla u \in L^2(0, T; L_x^2)$ such that u^n strongly converges to u in $L^2(0, T; L^4(\Omega))$ and ∇u^n weakly converges to ∇u in $L^2(0, T; L^2(\Omega))$ (everything up to a subsequence). Hence $u^n \cdot \nabla u^n$ and $\Omega^n Q^n - Q^n \Omega^n$ weakly converges in $L^1(0, T; L^{4/3}(\Omega))$ to $u \cdot \nabla u$ and $\Omega Q - Q \Omega$ respectively. Finally $u^n \cdot \nabla Q^n$ strongly converges to $u \cdot \nabla Q$ in $L^1(0, T; L^2(\Omega))$.

Now, $J_n \phi$ strongly converges to ϕ in $L^\infty(0, T; L_x^p)$, for any $\phi \in \mathfrak{D}((0, T) \times \Omega)$ and for any $1 \leq p < \infty$. Considering all the previous convergences and since (u^n, Q^n) is a weak solution of (P_n) , namely

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^N} \text{tr}\{Q^n \partial_t \Psi\} - \int_{\mathbb{R}^N} \text{tr}\{Q_0 \Psi(0, \cdot)\} + \int_0^T \int_{\mathbb{R}^N} \text{tr}\{(u^n \cdot \nabla Q^n) \Psi\} + \\ + \int_0^T \int_{\mathbb{R}^N} \text{tr}\{(\Omega^n Q^n - Q^n \Omega^n) \Psi\} = \Gamma \int_0^T \int_{\mathbb{R}^N} \text{tr}\{H(Q^n) \Psi\}, \end{aligned}$$

for every $N \times N$ -matrix Ψ with coefficients in $\mathfrak{D}([0, T] \times \Omega)$ and

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^N} u^n \cdot \partial_t \psi - \int_{\mathbb{R}^N} u_0 \cdot \psi(0, \cdot) + \int_0^T \int_{\mathbb{R}^N} (u^n \cdot \nabla u^n) \cdot \mathcal{P} J_n \psi - \nu \int_0^T \int_{\mathbb{R}^N} u^n \cdot \Delta \psi = \\ = -L \int_0^T \int_{\mathbb{R}^N} [Q^n \Delta Q^n - \Delta Q^n Q^n - \nabla Q^n \odot \nabla Q^n] \cdot \mathcal{P} J_n \nabla \psi, \end{aligned}$$

for any N -vector ψ with coefficients in $\mathfrak{D}([0, T] \times \Omega)$, we pass through the limit as n goes to ∞ , obtaining

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^N} \text{tr}\{Q \partial_t \Psi\} - \int_{\mathbb{R}^N} \text{tr}\{Q_0 \Psi(0, \cdot)\} + \int_0^T \int_{\mathbb{R}^N} \text{tr}\{(u \cdot \nabla Q) \Psi\} + \\ + \int_0^T \int_{\mathbb{R}^N} \text{tr}\{(\Omega Q - Q \Omega) \Psi\} = \Gamma \int_0^T \int_{\mathbb{R}^N} \text{tr}\{H(Q) \Psi\} \end{aligned}$$

and

$$\begin{aligned} - \int_0^T \int_{\mathbb{R}^N} u \cdot \partial_t \psi - \int_{\mathbb{R}^N} u_0 \cdot \psi(0, \cdot) + \int_0^T \int_{\mathbb{R}^N} (u \cdot \nabla u) \cdot \mathcal{P} \psi - \nu \int_0^T \int_{\mathbb{R}^N} u \cdot \Delta \psi = \\ = -L \int_0^T \int_{\mathbb{R}^N} [Q \Delta Q - \Delta Q Q - \nabla Q \odot \nabla Q] \cdot \mathcal{P} \nabla \psi. \end{aligned}$$

From the arbitrariness of T and Ω , we finally achieve that (u, Q) is a weak solution for (P) in the sense of definition 6.1.1. \square

6.4 The Difference Between Two Solutions

This section is devoted to an important remark which plays a major part in our uniqueness result. We deal with the difference between two weak solutions (u_i, Q_i) , $i = 1, 2$, of (P) in the sense of definition 6.1.1. Denoting by $(\delta u, \delta Q)$ the difference between the first and the second one, we claim that such element belongs to a lower regular space than the one the solutions belong to.

Proposition 6.4.1. *For any finite positive T , δu and $\nabla \delta Q$ belong to $L^\infty(0, T; \dot{H}^{-1/2})$.*

Remark 6.4.2. *In virtue of Proposition 6.4.1 and since $(\nabla \delta u, \Delta \delta Q)$ belongs to $L_t^2 L_x^2$ then*

$$(\nabla \delta u, \Delta \delta Q) \in L^2(0, T; \dot{H}^{-1/2}),$$

for any finite positive T , thanks to a classical real interpolation method:

$$\begin{aligned} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\nabla \delta u\|_{\dot{H}^{-\frac{3}{2}}}^{\frac{1}{3}} \|\nabla \delta u\|_{L_x^2}^{\frac{2}{3}} \lesssim \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} + \|\nabla \delta u\|_{L_x^2}, \\ \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\Delta \delta Q\|_{\dot{H}^{-\frac{3}{2}}}^{\frac{1}{3}} \|\Delta \delta Q\|_{L_x^2}^{\frac{2}{3}} \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} + \|\Delta \delta Q\|_{L_x^2}. \end{aligned}$$

Proof of Proposition 6.4.1. Fixing $T > 0$ we are going to prove that δu belongs to the space $L^\infty(0, T; \dot{H}^{-1/2})$ and δQ belongs to $L^\infty(0, T; \dot{H}^{1/2})$. We denote by f_1 and f_2

$$\begin{aligned} f_1 := & -u_1 \cdot \nabla Q_1 + u_2 \cdot \nabla Q_2 + \Omega_1 Q_1 - \Omega_2 Q_2 - Q_1 \Omega_1 + Q_2 \Omega_2 + \\ & + \Gamma \left\{ \frac{b}{3} (Q_1^2 - Q_2^2 - \text{tr}\{Q_1^2 - Q_2^2\} \frac{\text{Id}}{3}) - c \text{tr}\{Q_1^2\} Q_1 + c \text{tr}\{Q_2^2\} Q_2 \right\}, \end{aligned}$$

$$\begin{aligned} f_2 := & \mathcal{P} \left[-\text{div}\{u_1 \otimes u_1 - u_2 \otimes u_2\} + L \text{div} \{ Q_1 \Delta Q_1 - Q_2 \Delta Q_2 - \right. \\ & \left. - \Delta Q_1 Q_1 + \Delta Q_2 Q_2 - \nabla Q_1 \odot \nabla Q_1 + \nabla Q_2 \odot \nabla Q_2 \right], \end{aligned}$$

respectively. Then δQ and δu are weak solutions of the following Cauchy Problems:

$$\partial_t \delta Q - \Gamma L \Delta \delta Q + \Gamma a \delta Q = f_1 \quad \text{and} \quad \partial_t \delta u - \nu \Delta \delta u = f_2 \quad \text{in} \quad [0, T) \times \mathbb{R}^2,$$

with null initial data. By the classical Theory of Evolutionary Parabolic Equation, it is sufficient to prove that f_1 and f_2 belong to $L^2(0, T; \dot{H}^{-1/2})$ and $L^2(0, T; \dot{H}^{-3/2})$ respectively in order to obtain

$$\|(\delta u, \nabla \delta Q)\|_{L^\infty(0, T; \dot{H}^{-\frac{1}{2}})} \lesssim \|f_1\|_{L^2(0, T; \dot{H}^{-\frac{1}{2}})} + \|f_2\|_{L^2(0, T; \dot{H}^{-\frac{3}{2}})},$$

and conclude the proof. We start by f_1 and Theorem 6.2.4 plays a major part. For any $i = 1, 2$, we get

$$\begin{aligned} \|u_i \cdot \nabla Q_i\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|u_i\|_{\dot{H}^{\frac{1}{2}}} \|\nabla Q_i\|_{L_x^2} \lesssim \|u_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla u_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla Q_i\|_{L_x^2} \in L^4(0, T), \\ \|\Omega_i Q_i\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\nabla u_i\|_{L_x^2} \|Q_i\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\nabla u_i\|_{L_x^2} \|Q_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla Q_i\|_{L_x^2}^{\frac{1}{2}} \in L^2(0, T), \\ \|Q_i^2\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|Q_i\|_{\dot{H}^{\frac{1}{2}}} \|Q_i\|_{L_x^2} \lesssim \|Q_i\|_{L_x^2} \|\nabla Q_i\|_{L_x^2}^{\frac{1}{2}} \|Q_i\|_{L_x^2}^{\frac{1}{2}} \in L^\infty(0, T), \\ \|\text{tr}\{Q_i^2\} Q_i\|_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|Q_i^2\|_{L_x^2} \|Q_i\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\nabla Q_i\|_{L_x^2}^{\frac{3}{2}} \|Q_i\|_{L_x^2}^{\frac{3}{2}} \in L^\infty(0, T). \end{aligned}$$

Then, summarizing the previous estimates, we deduce that f_1 belongs to $L^2(0, T; \dot{H}^{-1/2})$. Now, let

us handle the terms of f_2 :

$$\begin{aligned} \|\operatorname{div}\{u_i \otimes u_i\}\|_{\dot{H}^{-\frac{3}{2}}} &\lesssim \|u_i \otimes u_i\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|u_i\|_{\dot{H}^{\frac{1}{2}}} \|u_i\|_{L_x^2} \\ &\lesssim \|u_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla u_i\|_{L_x^2}^{\frac{1}{2}} \|u_i\|_{L_x^2} \in L^4(0, T), \\ \|\operatorname{div}\{Q_i \Delta Q_i\}\|_{\dot{H}^{-\frac{3}{2}}} &\lesssim \|Q_i \Delta Q_i\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|Q_i\|_{\dot{H}^{\frac{1}{2}}} \|\Delta Q_i\|_{L_x^2} \\ &\lesssim \|Q_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla Q_i\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_i\|_{L_x^2} \in L^\infty(0, T), \end{aligned}$$

$$\begin{aligned} \|\operatorname{div}\{\nabla Q_i \odot \nabla Q_i\}\|_{\dot{H}^{-\frac{3}{2}}} &\lesssim \|\nabla Q_i \odot \nabla Q_i\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla Q_i\|_{\dot{H}^{\frac{1}{2}}} \|\nabla Q_i\|_{L_x^2} \\ &\lesssim \|\nabla Q_i\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_i\|_{L_x^2}^{\frac{1}{2}} \|\nabla Q_i\|_{L_x^2} \in L^4(0, T), \end{aligned}$$

which finally implies that f_2 belongs to $L^2(0, T; \dot{H}^{-\frac{3}{2}})$. This concludes the proof of Proposition 6.4.1. \square

6.5 Uniqueness

In this section we present our first original result. We are going to prove Theorem 6.1.2, namely the uniqueness of the weak solutions, given by Theorem 6.1.4. We implement the uniqueness result of Paicu and Zarnescu in [99], concerning the weak-strong uniqueness. Indeed the authors suppose that at least one of the solutions is a classical solution. The leading cause of such restriction relies on the choice to control the difference between two solutions in an L_x^2 -setting. However, this requires to estimate the L_x^∞ -norm of one of the solutions, $\|(u, \nabla Q)\|_{L_x^\infty}$, for instance by a Sobolev embedding, therefore the necessity to put $(u(t), \nabla Q(t))$ in some \dot{H}^s with $s > 1$, for any real t .

In this article we overcome this drawback, performing the uniqueness of weak solutions, thanks to an alternative approach which is inspired by [48] and [88]. The main idea is to evaluate the difference between two weak solutions in a functional space which is less regular than L_x^2 . Considering two weak solutions $(u_1, \nabla Q_1)$ and $(u_2, \nabla Q_2)$, we define $(\delta u, \delta Q)$ as the difference between the first one and the second one. It is straightforward that such difference is a weak solution for the following system:

$$\begin{cases} \partial_t \delta Q + \delta u \cdot \nabla Q_1 + u_2 \cdot \nabla \delta Q - \delta S(\nabla u, Q) - \Gamma L \Delta \delta Q = \Gamma \delta P(Q) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \delta u + \delta u \cdot \nabla u_1 + u_2 \cdot \nabla \delta u - \nu \Delta \delta u + \nabla \delta \Pi = L \operatorname{div} \left\{ \delta Q \Delta Q_1 + \right. \\ \quad \left. + Q_2 \Delta \delta Q - \Delta \delta Q Q_1 - \Delta Q_2 \delta Q - \nabla \delta Q \odot \nabla Q_1 - \nabla Q_2 \odot \nabla \delta Q \right\} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} \delta u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (\delta u, \delta Q)_{t=0} = (0, 0) & \mathbb{R}^2, \end{cases} \quad (\delta P)$$

where we have also defined

$$\delta \Omega := \Omega_1 - \Omega_2, \quad \delta \Pi := \Pi_1 - \Pi_2, \quad \delta P(Q) := P(Q_1) - P(Q_2).$$

and moreover

$$\delta S(Q, \nabla u) := \Omega_1 Q_1 - Q_1 \Omega_1 + \Omega_2 Q_2 - Q_2 \Omega_2 = \delta Q \delta \Omega - \delta \Omega \delta Q + \delta \Omega Q_2 - Q_2 \delta \Omega + \Omega_2 \delta Q - \delta Q \Omega_2.$$

Recalling the previous subsection, we take the $\dot{H}^{-1/2}$ -inner product between the first equation of

(δP) and $-L\Delta\delta Q$ and moreover we consider the scalar product in $\dot{H}^{-1/2}$ between the second one and δu :

$$\begin{aligned}
& \frac{d}{dt} \left[\frac{1}{2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + L \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \right] + \nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \Gamma L^2 \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 = \\
& = L\Gamma \langle \delta P(Q), \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - L \langle \delta u \cdot \nabla Q_1, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle u_2 \cdot \nabla\delta Q, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} + \\
& + L \langle \delta S(Q, \nabla u), \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - L \langle \nabla\delta Q \odot \nabla Q_1, \nabla\delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle \nabla Q_2 \odot \nabla\delta Q, \nabla\delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \\
& - \langle \delta u \cdot \nabla u_1, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \langle u_2 \cdot \nabla\delta u, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle \delta Q \Delta\delta Q - \Delta\delta Q \delta Q, \nabla\delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + \\
& + L \langle Q_2 \Delta\delta Q - \Delta\delta Q Q_2, \nabla\delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle \delta Q \Delta Q_2 - \Delta Q_2 \delta Q, \nabla\delta u \rangle_{\dot{H}^{-\frac{1}{2}}}. \tag{6.8}
\end{aligned}$$

Denoting by $\Phi(t) = \frac{1}{2} \|\delta u(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + L \|\nabla\delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2$ we claim that

$$\frac{d}{dt} \Phi(t) \leq \chi(t) \Phi(t), \quad \text{for almost every } t \in \mathbb{R}_+,$$

where $\chi \geq 0$ belongs to $L^1_{loc}(\mathbb{R}_+)$. Hence, uniqueness holds thanks to the Gronwall Lemma and since $\Phi(0)$ is null. Thus, we need to analyze every term of the right-hand side of (6.8). From here on $C_{\Gamma,L}$ and C_ν are suitable positive constants which will be determined in the end of the proof.

Simpler Terms

First, we begin evaluating every term which is handleable by Theorem 6.2.4.

Estimate of $\Gamma L \langle \delta P(Q), \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$

From the definition of $\delta P(Q)$, and since $\text{tr}\{\Delta Q\}$ is null, we need to control

$$\begin{aligned}
\Gamma L \langle \delta P(Q), \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &= -\Gamma L a \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + \Gamma L b \langle \delta Q Q_1 + Q_2 \delta Q, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \\
&- \Gamma L c \langle \delta Q \text{tr}\{Q_1^2\}, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - \Gamma L c \langle \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\} Q_1, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

We overcome the second term in the right hand-side of the equality as follows:

$$\begin{aligned}
\Gamma L b \langle \delta Q Q_1 + Q_2 \delta Q, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{L_x^2}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Furthermore, we observe that

$$\begin{aligned}
\Gamma L c \langle \delta Q \text{tr}\{Q_1^2\}, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|Q_1^2\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|Q_1\|_{L_x^4}^2 \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|Q_1\|_{L_x^2} \|\nabla Q_1\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|Q_1\|_{L_x^2}^2 \|\nabla Q_1\|_{L_x^2}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2
\end{aligned}$$

and moreover

$$\begin{aligned}
& \Gamma L c \langle \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\} Q_1, \Delta\delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \left(\|Q_1\|_{L_x^2}^2 + \|Q_2\|_{L_x^2} \|Q_1\|_{L_x^2} \right) \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{L_x^4}^2 \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{L_x^2} \|\nabla(Q_1, Q_2)\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla(Q_1, Q_2)\|_{L_x^2}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Finally, summarizing the previous inequality, we get

$$\Gamma L \langle \delta P(Q), \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla(Q_1, Q_2)\|_{L_x^2}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2$$

Estimate of $L \langle \delta u \cdot \nabla Q_1, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \delta u \cdot \nabla Q_1, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\delta u\|_{\dot{H}^{-\frac{1}{4}}} \|\nabla Q_1\|_{\dot{H}^{\frac{3}{4}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\nabla Q_1\|_{L_x^2}^{\frac{1}{4}} \times \\ &\times \|\Delta Q_1\|_{L_x^2}^{\frac{3}{4}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla Q_1\|_{L_x^2}^{\frac{2}{3}} \|\Delta Q_1\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Estimate of $L \langle u_2 \cdot \nabla \delta Q, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle u_2 \cdot \nabla \delta Q, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|u_2\|_{\dot{H}^{\frac{3}{4}}} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{4}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|u_2\|_{L_x^2}^{\frac{1}{4}} \|\nabla u_2\|_{L_x^2}^{\frac{3}{4}} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{5}{4}} \\ &\lesssim C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|u_2\|_{L_x^2}^{\frac{2}{3}} \|\nabla u_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Estimate of $L \langle \delta Q \delta \Omega - \delta \Omega \delta Q, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \delta Q \delta \Omega - \delta \Omega \delta Q, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\delta \Omega\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla(u_1, u_2)\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Estimate of $L \langle \Omega_2 \delta Q - \delta Q \Omega_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \Omega_2 \delta Q - \delta Q \Omega_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\Omega_2\|_{L_x^2} \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla u_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Estimate of $L \langle \nabla \delta Q \odot \nabla Q_1, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \nabla \delta Q \odot \nabla Q_1, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{4}}} \|\nabla Q_1\|_{\dot{H}^{\frac{3}{4}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\nabla Q_1\|_{L_x^2}^{\frac{1}{4}} \|\Delta Q_1\|_{L_x^2}^{\frac{3}{4}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla Q_1\|_{L_x^2}^{\frac{2}{3}} \|\Delta Q_1\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

Estimate of $L \langle \nabla Q_2 \odot \nabla \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \nabla Q_2 \odot \nabla \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{4}}} \|\nabla Q_2\|_{\dot{H}^{\frac{3}{4}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\nabla Q_2\|_{L_x^2}^{\frac{1}{4}} \|\Delta Q_2\|_{L_x^2}^{\frac{3}{4}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla Q_2\|_{L_x^2}^{\frac{2}{3}} \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

Estimate of $\langle \delta u \cdot \nabla u_1, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\langle \delta u \cdot \nabla u_1, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta u\|_{\dot{H}^{\frac{1}{2}}} \|\nabla u_1\|_{L_x^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla u_1\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2$$

Estimate of $\langle u_2 \cdot \nabla \delta u, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\langle u_2 \cdot \nabla \delta u, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta u\|_{\dot{H}^{\frac{1}{2}}} \|\nabla u_2\|_{L^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla u_2\|_{L^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2$$

Estimate of $L \langle \delta Q \Delta \delta Q - \Delta \delta Q \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \delta Q \Delta \delta Q - \Delta \delta Q \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\Delta(Q_1, Q_2)\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\Delta(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Estimate of $L \langle \delta Q \Delta Q_2 - \Delta Q_2 \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$

$$\begin{aligned} L \langle \delta Q \Delta Q_2 - \Delta Q_2 \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\Delta Q_2\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

The Residual Terms

Now we deal with the terms in the right-hand side of (6.8) which we have not evaluated yet, namely

$$L \langle \delta \Omega Q_2 - Q_2 \delta \Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle Q_2 \Delta \delta Q - \Delta \delta Q Q_2, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}. \quad (6.9)$$

Here, the difference between the two solutions appears with the higher derivative-order, more precisely the inner product is driven by $\nabla \delta u$ (i.e. $\delta \Omega$) and $\Delta \delta Q$. This clearly generates a drawback if we want to analyze every remaining term, proceeding as the previous estimates. Let us remark that if we consider the L_x^2 -inner product instead of the $\dot{H}^{-1/2}$ -one, then this last sum is null, thanks to Lemma 6.7.1. However the $\dot{H}^{-1/2}$ -setting force us to analyze such sum, and we overcome the described obstacle, first considering the equivalence between $\dot{H}^{-1/2}$ and $\dot{B}_{2,2}^{-1/2}$, and moreover thanks to decomposition (6.1), namely

$$\mathcal{J}_q^1(A, B) := \sum_{|q-q'|\leq 5} [\dot{\Delta}_q, \dot{S}_{q'-1} A] \dot{\Delta}_{q'} B, \quad \mathcal{J}_q^3(A, B) := \dot{S}_{q-1} A \dot{\Delta}_q B,$$

$$\mathcal{J}_q^2(A, B) := \sum_{|q-q'|\leq 5} (\dot{S}_{q'-1} A - \dot{S}_{q-1} A) \dot{\Delta}_q \dot{\Delta}_{q'} B, \quad \mathcal{J}_q^4(A, B) := \sum_{q' \geq q-5} \dot{\Delta}_q (\dot{\Delta}_{q'} A \dot{S}_{q'+2} B),$$

with $\dot{\Delta}_q(AB) = \mathcal{J}_q^1(A, B) + \mathcal{J}_q^2(A, B) + \mathcal{J}_q^3(A, B) + \mathcal{J}_q^4(A, B)$, for any integer q .

First, let us begin with

$$\begin{aligned} L \langle \delta \Omega Q_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &= \sum_{q \in \mathbb{Z}} 2^{-q} L \langle \dot{\Delta}_q(\delta \Omega Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\ &= \sum_{q \in \mathbb{Z}} \sum_{i=1}^4 2^{-q} L \langle \mathcal{J}_q^i(\delta \Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}. \end{aligned}$$

First we separately study the case $i = 1, 2, 4$. The term related to $i = 3$ is the challenging one and we are not able to evaluate it. However, we will see how such term is going to be erased. Let us begin with $i = 1$ then

$$\begin{aligned} \mathcal{I}_q^1 &:= 2^{-q} L \langle \mathcal{J}_q^1(\delta \Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = L 2^{-q} \sum_{|q-q'|\leq 5} \langle [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \delta \Omega, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\ &\lesssim \sum_{|q-q'|\leq 5} 2^{-q} \| [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \delta \Omega \|_{L_x^2} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2}. \end{aligned}$$

Hence, applying the commutator estimate (see Lemma 2.97 in [7]) we get

$$\begin{aligned}
\mathcal{I}_q^1 &\lesssim \sum_{|q-q'|\leq 5} 2^{-2q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^4} \|\dot{\Delta}_{q'} \delta \Omega\|_{L_x^4} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \\
&\lesssim \sum_{|q-q'|\leq 5} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^2}^{\frac{1}{2}} \|\dot{S}_{q'-1} \Delta Q_2\|_{L_x^2}^{\frac{1}{2}} 2^{-\frac{q'}{2}} \|\dot{\Delta}_{q'} \delta u\|_{L_x^4} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \\
&\lesssim \sum_{|q-q'|\leq 5} \|\nabla Q_2\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_2\|_{L_x^2}^{\frac{1}{2}} \|\dot{\Delta}_{q'} \delta u\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2},
\end{aligned}$$

which finally yields

$$\begin{aligned}
\mathcal{I}_q^1 &\lesssim \|\nabla Q_2\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_2\|_{L_x^2}^{\frac{1}{2}} \|\delta u\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\nabla Q_2\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_2\|_{L_x^2}^{\frac{1}{2}} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}},
\end{aligned}$$

that is

$$\begin{aligned}
L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^1(\delta \Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \\
&\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned} \tag{6.10}$$

Now, let us handle the case $i = 2$. We argue as before:

$$\begin{aligned}
\mathcal{I}_q^2 &:= 2^{-q} L \langle \mathcal{J}_q^2(\delta \Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&= L 2^{-q} \sum_{|q-q'|\leq 5} \langle (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \dot{\Delta}_q \dot{\Delta}_{q'} \delta \Omega, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&\lesssim 2^{-q} \|(\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2)\|_{L_x^\infty} \|\dot{\Delta}_q \dot{\Delta}_{q'} \delta \Omega\|_{L_x^2} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2},
\end{aligned}$$

so that, observing that $\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2$ fulfills

$$\|\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2\|_{L_x^\infty} \lesssim 2^{-2q} \|\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2\|_{L_x^\infty} \lesssim 2^{-q} \|\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2\|_{L_x^2},$$

then we obtain

$$\begin{aligned}
\mathcal{I}_q^2 &\lesssim 2^{-2q} \sum_{|q-q'|\leq 5} \|(\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2)\|_{L_x^2} \|\dot{\Delta}_q \dot{\Delta}_{q'} \delta \Omega\|_{L_x^2} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \\
&\lesssim 2^{-2q} \sum_{|q-q'|\leq 5} \|\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q'} \delta \Omega\|_{L_x^2} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \\
&\lesssim \sum_{|q-q'|\leq 5} 2^{-\frac{q'}{2}} \|\dot{\Delta}_{q'} \delta u\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \|\Delta Q_2\|_{L_x^2}.
\end{aligned}$$

Thus, it turns out that

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^2(\delta \Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \tag{6.11}$$

Now, we take into consideration the case $i = 4$. Here we will use a convolution method and the Young inequality, since the sum in q' is not finite. Then, let us observe that

$$\begin{aligned} \mathcal{I}_q^4 &:= 2^{-q} L \langle \mathcal{J}_q^4(\delta\Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = L 2^{-q} \sum_{q-q' \leq 5} \langle \dot{\Delta}_{q'} Q_2 \dot{S}_{q'+2} \delta\Omega, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\ &\lesssim 2^{-q} \sum_{q-q' \leq 5} \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|\dot{S}_{q'+2} \delta\Omega\|_{L_x^2} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2}. \end{aligned}$$

Observing that

$$\begin{aligned} \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} &\lesssim 2^{q'} \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} \lesssim 2^{-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2}, \\ \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} &\lesssim 2^q \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2}, \end{aligned}$$

it turns out

$$\begin{aligned} \mathcal{I}_q^4 &\lesssim 2^{-q} \sum_{q-q' \leq 5} 2^{-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'+2} \delta\Omega\|_{L_x^2} 2^q \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \\ &\lesssim \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta\Omega\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \\ &\lesssim \|\Delta Q_2\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta\Omega\|_{L_x^2}. \end{aligned}$$

Then, by convolution, the Young inequality and Proposition 6.2.3, we finally obtain

$$\begin{aligned} L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(\delta\Omega, Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \|\Delta Q_2\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned} \quad (6.12)$$

Summarizing (6.10), (6.11) and (6.12) and recalling the definition of $J_q^3(\delta\Omega, Q_2)$, we finally get

$$\begin{aligned} L \langle \delta\Omega Q_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} \delta\Omega \dot{\Delta}_q Q_2, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \\ &\lesssim \tilde{\chi}_1 \Phi + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned}$$

where $\tilde{\chi}_1$ belongs to $L_{loc}^1(\mathbb{R}_+)$. Hence, we need to analyze

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} \delta\Omega \dot{\Delta}_q Q_2, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}$$

and this term is going to disappear by a simplification.

Now we handle the term $\langle Q_2 \delta\Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$ of (6.9). Observing that it can be rewritten as $\langle {}^t(Q_2 \delta\Omega), {}^t \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$, that is $-\langle \delta\Omega Q_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}$ then we proceed as in the previous estimates, obtaining

$$\begin{aligned} L \langle \delta\Omega Q_2 - Q_2 \delta\Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} \delta\Omega \dot{\Delta}_q Q_2 - \dot{\Delta}_q Q_2 \dot{S}_{q-1} \delta\Omega, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} & \\ &\lesssim \tilde{\chi} \Phi + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned} \quad (6.13)$$

so that it remains to control

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} \delta \Omega \dot{\Delta}_q Q_2 - \dot{\Delta}_q Q_2 \dot{S}_{q-1} \delta \Omega, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}. \quad (6.14)$$

Now, we focus on $L \langle Q_2 \Delta \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$ of (6.9) and we use again decomposition (6.1) as follows

$$\begin{aligned} L \langle Q_2 \Delta \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &= L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q (Q_2 \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &= L \sum_{q \in \mathbb{Z}} \sum_{i=1}^4 2^{-q} \langle \mathcal{J}_q^i(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2}. \end{aligned}$$

As before, we estimate the terms related to $i = 1, 2, 4$ while when $i = 3$ the associated term is going to be erased. When $i = 1$ we get

$$\begin{aligned} L 2^{-q} \langle \mathcal{J}_q^1(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &= L \sum_{|q-q'| \leq 5} 2^{-q} \langle [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \Delta \delta Q, \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \Delta \delta Q \|_{L_x^2} \| \dot{\Delta}_q \nabla \delta u \|_{L_x^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-2q} \| \dot{S}_{q'-1} \nabla Q_2 \|_{L_x^4} \| \dot{\Delta}_{q'} \Delta \delta Q \|_{L_x^4} \| \dot{\Delta}_q \nabla \delta u \|_{L_x^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| \dot{S}_{q'-1} \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{S}_{q'-1} \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{\Delta}_{q'} \nabla \delta Q \|_{L_x^4} \| \dot{\Delta}_q \nabla \delta u \|_{L_x^2} \\ &\lesssim \sum_{|q-q'| \leq 5} \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{\Delta}_{q'} \nabla \delta Q \|_{L_x^2} 2^{-\frac{q}{2}} \| \dot{\Delta}_q \nabla \delta u \|_{L_x^2} \end{aligned}$$

Hence, taking the sum as $q \in \mathbb{Z}$,

$$\begin{aligned} L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^1(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &\lesssim \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \nabla \delta Q \|_{L_x^2} \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \nabla \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \| \nabla Q_2 \|_{L_x^2}^2 \| \Delta Q_2 \|_{L_x^2}^2 \| \nabla \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

We evaluate the term related to $i = 2$ as follows:

$$\begin{aligned} L 2^{-q} \langle \mathcal{J}_q^2(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &= L 2^{-q} \langle (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \dot{\Delta}_q \dot{\Delta}_{q'} \Delta \delta Q, \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \|_{L_x^\infty} \| \dot{\Delta}_q \dot{\Delta}_{q'} \Delta \delta Q \|_{L_x^2} \| \dot{\Delta}_q \nabla \delta u \|_{L_x^2}, \end{aligned}$$

so that

$$\begin{aligned}
L2^{-q}\langle \mathcal{J}_q^2(Q_2, \Delta\delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} & \\
&\lesssim \sum_{|q-q'|\leq 5} 2^{-2q} \|(\dot{S}_{q'-1}\Delta Q_2 - \dot{S}_{q-1}\Delta Q_2)\|_{L_x^2} \|\dot{\Delta}_q \dot{\Delta}_{q'} \Delta\delta Q\|_{L_x^2} \|\dot{\Delta}_q \nabla \delta u\|_{L_x^2} \\
&\lesssim \sum_{|q-q'|\leq 5} 2^{-2q} \|\dot{S}_{q'-1}\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_q \Delta\delta Q\|_{L_x^2} \|\dot{\Delta}_q \nabla \delta u\|_{L_x^2} \\
&\lesssim \sum_{|q-q'|\leq 5} 2^{-\frac{q}{2}} \|\dot{\Delta}_{q'} \delta u\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \Delta\delta Q\|_{L_x^2} \|\Delta Q_2\|_{L_x^2}
\end{aligned}$$

Thus, taking the sum in q , it turns out that

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^2(Q_2, \Delta\delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2$$

At last, when $i = 4$,

$$\begin{aligned}
L2^{-q}\langle \mathcal{J}_q^4(Q_2, \Delta\delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &= L2^{-q} \sum_{q-q' \leq 5} \langle \dot{\Delta}_{q'} Q_2 \dot{S}_{q'+2} \Delta\delta Q, \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\
&\lesssim 2^{-q} \sum_{q-q' \leq 5} \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|\dot{S}_{q'+2} \Delta\delta Q\|_{L_x^2} \|\dot{\Delta}_q \nabla \delta u\|_{L_x^2} \\
&\lesssim 2^{-q} \sum_{q-q' \leq 5} 2^{-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'+2} \Delta\delta Q\|_{L_x^2} 2^q \|\dot{\Delta}_q \delta u\|_{L_x^2} \\
&\lesssim \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \Delta\delta Q\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \delta u\|_{L_x^2} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \Delta\delta Q\|_{L_x^2}
\end{aligned}$$

Hence, by convolution, the Young inequalities and Proposition 6.2.3, we obtain

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(Q_2, \Delta\delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &\lesssim \|\Delta Q_2\|_{L_x^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Since $\langle \Delta\delta Q Q_2, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} = \langle {}^t(\Delta\delta Q Q_2), {}^t \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} = \langle Q_2 \Delta\delta Q, {}^t \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$, then we proceed as for estimating $\langle Q_2 \Delta\delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}$, so that we obtain the following control

$$\begin{aligned}
L \langle Q_2 \Delta\delta Q - \Delta\delta Q Q_2, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta\delta Q - \dot{\Delta}_q \Delta\delta Q \dot{S}_{q-1} Q_2 \delta \Omega, \dot{\Delta}_q \nabla u \rangle_{L_x^2} & \\
&\lesssim \tilde{\chi}_2 \Phi + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2
\end{aligned} \tag{6.15}$$

where χ_2 belongs to $L_{loc}^1(\mathbb{R}_+)$. Now, the term we need to erase is

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta\delta Q - \dot{\Delta}_q \Delta\delta Q \dot{S}_{q-1} Q_2 \delta \Omega, \dot{\Delta}_q \nabla u \rangle_{L_x^2}. \tag{6.16}$$

Thus, summing (6.14) and (6.16), we obtain

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \left\{ \langle \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta \Omega - \dot{\Delta}_q \delta \Omega \dot{S}_{q-1} Q_2, \Delta \dot{\Delta}_q \delta Q \rangle_{L_x^2} + \langle \dot{S}_{q-1} Q_2 \Delta \dot{\Delta}_q \delta Q - \Delta \dot{\Delta}_q \delta Q \dot{S}_{q-1} Q_2, \nabla \delta u \rangle_{L_x^2} \right\},$$

which is a series with every coefficients null, thanks to Lemma 6.7.1. In virtue of this last result, recalling (6.13) and (6.15), we finally obtain

$$\begin{aligned} L \langle \delta \Omega Q_2 - Q_2 \delta \Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle Q_2 \Delta \delta Q - \Delta \delta Q Q_2, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\ \lesssim \tilde{\chi} \Phi + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Conclusion

Recalling (6.8) and summarizing all the estimate of the previous two sub-sections, we conclude that there exists a function χ which belongs to $L^1_{loc}(\mathbb{R}_+)$ such that

$$\frac{d}{dt} \Phi(t) + \nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \Gamma L^2 \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \lesssim \chi(t) \Phi(t) + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2$$

for almost every $t \in \mathbb{R}_+$. Thus, choosing $C_{\Gamma, L}$ and C_ν small enough, we absorb the last two terms in the right-hand side by the left-hand side, finally obtaining

$$\frac{d}{dt} \left[\frac{1}{2} \|\delta u(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + L \|\nabla \delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 \right] \lesssim \chi \left[\frac{1}{2} \|\delta u(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + L \|\nabla \delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 \right].$$

Since the initial datum is null and thanks to the Gronwall inequality, we deduce that $(\delta u, \nabla \delta Q) = 0$ which yields $(\delta u, \delta Q) = 0$, since $\delta Q(t)$ decays to 0 at infinity for almost every t . Hence, we have finally achieved the uniqueness of the weak solution for system (P).

6.6 Regularity Propagation

We now handle the propagation of low regularity, namely we prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Let us consider the following sequence of system:

$$\left\{ \begin{array}{ll} \partial_t Q^n + J_n \mathcal{P}(J_n u^n \nabla J_n Q^n) - J_n \mathcal{P}(J_n \Omega^n J_n Q^n) + \\ \quad + J_n \mathcal{P}(J_n Q^n J_n \Omega^n) - \Gamma L \Delta J_n Q^n = P^n(Q^n) & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u^n + J_n \mathcal{P}(J_n u^n \nabla J_n u^n) - \nu \Delta J_n u^n = \\ \quad = \Gamma L \operatorname{div} J_n \mathcal{P}\{J_n Q^n \Delta J_n Q^n - \Delta J_n Q^n J_n Q^n - \nabla J_n Q^n \odot \nabla J_n Q^n\} & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u^n = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (u^n, Q^n)|_{t=0} = (u_0, Q_0) & \mathbb{R}^2, \end{array} \right. \quad (\tilde{P}_n)$$

where

$$P^n(Q^n) := -a J_n Q^n + b \left[J_n (J_n Q^n J_n Q^n) - \operatorname{tr}\{J_n (J_n Q^n J_n Q^n)\} \frac{\operatorname{Id}}{3} \right] - c J_n Q^n \operatorname{tr}\{J_n (J_n Q^n J_n Q^n)\}.$$

Moreover we recall that J_n is the regularizing operator defined by

$$\hat{J}_n f(\xi) = 1_{[\frac{1}{n}, n]}(\xi) \hat{f}(\xi)$$

and \mathcal{P} stands for the Leray projector. The Friedrichs scheme related to (\tilde{P}_n) is not much different to the (P_n) -one, however here the Q -tensor equation has been regularized, as well. System (P_n) has been utilized in [99] and the authors have proven the existence of a strong solution (u^n, Q^n) which converges to a weak solution for (P) , as n goes to ∞ (up to a subsequence). Thanks to our uniqueness result, Theorem 6.1.2, we deduce that such solution is exactly the one determined by Theorem 6.1.4 and it is unique. Hence, instead of proceeding by a priori estimate (as in [99]), we formalize our proof, evaluating directly the (\tilde{P}_n) -scheme. We will establish some estimates, which are uniformly in n , which yields that the weak-solution of (P) fulfills them as well. This is only a strategy in order to formalize the a priori-estimate, while the major part of our proof releases on the inequalities we are going to proof.

Since $(J_n u^n, J_n Q^n) = (u^n, Q^n)$ (by uniqueness), then $(u^n(t), Q^n(t))$ belongs to $H^{1+s} \times H^{2+s}$ for almost every $t \in \mathbb{R}_+$ and for every $n \in \mathbb{N}$. We apply $\dot{\Delta}_q$ to the first and the second equations of (\tilde{P}_n) , then we apply $\langle \cdot, \dot{\Delta}_q u^n \rangle_{L^2}$ to the first one and $-L \langle \cdot, \dot{\Delta}_q \Delta Q^n \rangle_{L^2}$ to the second one, obtaining the following identity:

$$\begin{aligned} & \frac{d}{dt} \left[\|\dot{\Delta}_q u^n\|_{L^2}^2 + L \|\dot{\Delta}_q \nabla Q^n\|_{L^2}^2 \right] + \nu \|\dot{\Delta}_q \nabla u^n\|_{L^2}^2 + \Gamma L^2 \|\dot{\Delta}_q \Delta Q^n\|_{L^2}^2 \\ &= \langle \dot{\Delta}_q (\Delta Q^n Q^n - Q^n \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} - \langle \dot{\Delta}_q (u^n \cdot \nabla u^n), \dot{\Delta}_q u^n \rangle_{L^2} + \\ & \quad + \langle \dot{\Delta}_q (\nabla Q^n \odot \nabla Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} + L \langle \dot{\Delta}_q (u^n \cdot \nabla Q^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} + \\ & \quad + L \langle \dot{\Delta}_q (\Omega^n Q^n - Q^n \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} - L \langle \dot{\Delta}_q P^n(Q^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2}. \end{aligned}$$

Multiplying both left-hand and the right-hand sides by 2^{2qs} and taking the sum as $q \in \mathbb{Z}$ we obtain

$$\begin{aligned} & \frac{d}{dt} \left[\|u^n\|_{\dot{H}^s}^2 + L \|\nabla Q^n\|_{\dot{H}^s}^2 \right] + \nu \|\nabla u^n\|_{\dot{H}^s}^2 + \Gamma L^2 \|\Delta Q^n\|_{\dot{H}^s}^2 \\ &= L \langle \Delta Q^n Q^n - Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s} - \langle u^n \cdot \nabla u^n, u^n \rangle_{\dot{H}^s} + L \langle \nabla Q^n \odot \nabla Q^n, \nabla u^n \rangle_{\dot{H}^s} + \\ & \quad + L \langle u^n \cdot \nabla Q^n, \Delta Q^n \rangle_{\dot{H}^s} + L \langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s} - L \langle P^n(Q^n), \Delta Q^n \rangle_{\dot{H}^s}. \end{aligned} \quad (6.17)$$

The key part of our proof relies on the Osgood inequality, therefore we need to estimate all the terms of the right-hand side of (6.17). First, let us proceed estimating the easier terms.

Estimate of $\langle u^n \cdot \nabla u^n, u^n \rangle_{\dot{H}^s}$

We begin with $\langle \dot{\Delta}_q (u^n \cdot \nabla u^n), \dot{\Delta}_q u^n \rangle_{L^2}$, with $q \in \mathbb{Z}$. Passing through the Bony decomposition

$$\begin{aligned} & \langle \dot{\Delta}_q (u^n \cdot \nabla u^n), \dot{\Delta}_q u^n \rangle_{L^2} = \\ &= \underbrace{\sum_{|q-q'| \leq 5} \left\langle \sum_{i=1}^2 \dot{\Delta}_q T_{u_i^n} \partial_i u^n + \dot{\Delta}_q T_{\partial_i u^n} u_i^n, \dot{\Delta}_q u^n \right\rangle_{L^2}}_{\mathcal{A}_q} + \underbrace{\sum_{q' \geq q-5} \left\langle \sum_{i=1}^2 \dot{\Delta}_q R(u_i^n, \partial_i u^n), \dot{\Delta}_q u^n \right\rangle_{L^2}}_{\mathcal{B}_q} \end{aligned}$$

We handle the term \mathcal{A}_q as follows:

$$\mathcal{A}_q \lesssim \sum_{|q-q'| \leq 5} \left[\|\dot{S}_{q'-1} u^n\|_{L^\infty} \|\dot{\Delta}_{q'} \nabla u^n\|_{L^2} + \|\dot{S}_{q'-1} \nabla u^n\|_{L^\infty} \|\dot{\Delta}_{q'} u^n\|_{L^2} \right] \|\dot{\Delta}_q u^n\|_{L^2}$$

$$\begin{aligned}
&\lesssim \sum_{|q-q'|\leq 5} \left[\|\dot{S}_{q'-1}u^n\|_{L^\infty} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} + 2^{q'} \|\dot{S}_{q'-1}u^n\|_{L^\infty} \|\dot{\Delta}_{q'}u^n\|_{L^2} \right] \|\dot{\Delta}_q u^n\|_{L^2} \\
&\lesssim \sum_{|q-q'|\leq 5} \left[\|\dot{S}_{q'-1}u^n\|_{L^\infty} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} + \|\dot{S}_{q'-1}u^n\|_{L^\infty} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} \right] \|\dot{\Delta}_q u^n\|_{L^2} \\
&\lesssim \|u^n\|_{L^\infty} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{|q-q'|\leq 5} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2},
\end{aligned}$$

so that, multiplying by 2^{2sq} and taking the sum as $q \in \mathbb{Z}$,

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{A}_q \lesssim \|u^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left\{ 2^{2qs} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{|q-q'|\leq 5} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} \right\} \lesssim \|u^n\|_{L^\infty} \|u^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}. \quad (6.18)$$

The control of \mathcal{B}_q relies on convolution and the Young inequality, indeed

$$\mathcal{B}_q \lesssim \sum_{\substack{q' \geq q-5 \\ |l| \leq 1}} \|\dot{\Delta}_{q'+l}u^n\|_{L^\infty} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} \|\dot{\Delta}_q u^n\|_{L^2} \lesssim \|u^n\|_{L^\infty} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{q' \geq q-5} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2},$$

hence

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{B}_q &\lesssim \|u^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left\{ 2^{2qs} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{q' \geq q-5} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} \right\} \\
&\lesssim \|u^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left\{ 2^{qs} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{q' \geq q-5} 2^{(q-q')s} 2^{q's} \|\dot{\Delta}_{q'}\nabla u^n\|_{L^2} \right\} \\
&\lesssim \|u^n\|_{L^\infty} \|u^n\|_{\dot{H}^s} \sum_{q \in \mathbb{Z}} \left\{ 2^{qs} \|\dot{\Delta}_q u^n\|_{L^2} \sum_{q' \in \mathbb{Z}} 2^{(q-q')s} 1_{(-\infty, 5)}(q-q') b_{q'} \right\},
\end{aligned}$$

where $(b_{q'})_{\mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$. Thus, we obtain

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{B}_q \lesssim \|u^n\|_{L^\infty} \|u^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}, \quad (6.19)$$

thanks to the Young inequality. Finally, summarizing (6.18) and (6.19), we obtain

$$\langle u^n \cdot \nabla u^n, u^n \rangle_{\dot{H}^s} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q (u^n \cdot \nabla u^n), \dot{\Delta}_q u^n \rangle_{L^2} \lesssim \|u^n\|_{L^\infty} \|u^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}. \quad (6.20)$$

Estimate of $\langle u^n \cdot \nabla Q^n, \Delta Q^n \rangle_{\dot{H}^s}$

Arguing exactly as for proving (6.20), we obtain

$$\langle u^n \cdot \nabla Q^n, \Delta Q^n \rangle_{\dot{H}^s} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q (u^n \cdot \nabla Q^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \lesssim \|u^n\|_{L^\infty} \|\nabla Q^n\|_{\dot{H}^s} \|\Delta Q^n\|_{\dot{H}^s}. \quad (6.21)$$

Estimate of $\langle \nabla Q^n \odot \nabla Q^n, \nabla u^n \rangle_{\dot{H}^s}$

We keep on our control, evaluating the term $\langle \dot{\Delta}_q (\nabla Q^n \odot \nabla Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2}$, with $q \in \mathbb{Z}$. The explicit

integral formula of such term is the following one:

$$\begin{aligned} & \int_{\mathbb{R}^2} \sum_{i,k=1}^2 \dot{\Delta}_q(\operatorname{tr}\{\partial_i Q \partial_k Q\}) \dot{\Delta}_q \partial_k u_i^n = \int_{\mathbb{R}^2} \sum_{i,k=1}^2 \sum_{j,l=1}^3 \dot{\Delta}_q[\partial_i Q_{jl}^n \partial_k Q_{lj}^n] \dot{\Delta}_q \partial_k u_i^n \\ & = \underbrace{\int_{\mathbb{R}^2} \sum_{i,k=1}^2 \sum_{j,l=1}^3 \dot{\Delta}_q[\dot{T}_{\partial_i Q_{jl}^n} \partial_k Q_{lj}^n + \dot{T}_{\partial_k Q_{lj}^n} \partial_i Q_{jl}^n] \dot{\Delta}_q \partial_k u_i^n}_{\mathcal{C}_q} + \underbrace{\int_{\mathbb{R}^2} \sum_{i,k,j,l} \dot{\Delta}_q \dot{R}(\partial_i Q_{jl}^n, \partial_k Q_{lj}^n) \dot{\Delta}_q \partial_k u_i^n}_{\mathcal{D}_q}, \end{aligned}$$

where we have used the Bony decomposition again. First, let us observe that

$$\begin{aligned} \mathcal{C}_q & \lesssim \sum_{|q-q'|\leq 5} \|S_{q-1} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ & \lesssim \|\nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{|q-q'|\leq 5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2}, \end{aligned}$$

which yields

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{C}_q & \lesssim \|\nabla Q^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} 2^{2qs} \left\{ \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{|q-q'|\leq 5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right\} \\ & \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}. \end{aligned} \quad (6.22)$$

Moreover, considering \mathcal{D}_q , we get

$$\begin{aligned} \mathcal{D}_q & \lesssim \sum_{\substack{q' \geq q-5 \\ |l| \leq 5}} \|\dot{\Delta}_{q'+l} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ & \lesssim \|\nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2}, \end{aligned}$$

so that, proceeding as in the proof of (6.19),

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{D}_q & \lesssim \|\nabla Q^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left\{ 2^{qs} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{q' \in \mathbb{Z}} 2^{(q'-q)s} 2^{q's} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right\} \\ & \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}, \end{aligned} \quad (6.23)$$

thanks to the Young inequality. Thus, summarizing (6.22) and (6.23), we achieve

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q(\nabla Q^n \odot \nabla Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}, \quad (6.24)$$

Estimate of $\langle \Delta Q^n Q^n - Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s}$

Now, we carry out of $\langle \Delta Q^n Q^n - Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s}$. This is the first non trivial term to evaluate. We choose to use the decomposition (6.1), presented in the preliminaries, instead of the classical Bony decomposition (which we have used until now). We will remark the presence of a term inside such decomposition, which is hard to control. However we will see that such drawback is going to

be erased. Let us begin controlling $\langle Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s}$:

$$\begin{aligned} \langle Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s} &= \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q(Q^n \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} \\ &= \sum_{q \in \mathbb{Z}} \sum_{i=1}^4 2^{2qs} \langle \mathcal{J}_q^i(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2}. \end{aligned}$$

where \mathcal{J}_q^i has been defined by (6.1), for $i = 1, \dots, 4$. When $i = 1$, we point out that

$$\begin{aligned} \langle \mathcal{J}_q^1(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} &= \sum_{|q-q'| \leq 5} \langle [\dot{\Delta}_q, \dot{S}_{q'-1} Q^n] \dot{\Delta}_{q'} \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-q'} \|\dot{S}_{q'-1} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_{q'} \Delta Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2}. \end{aligned}$$

which yields

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^1(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla Q^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}. \quad (6.25)$$

On the other hand, for $i = 2$, we proceed as follows:

$$\begin{aligned} \langle \mathcal{J}_q^2(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} &= \sum_{|q-q'| \leq 5} \langle (\dot{S}_{q'-1} Q^n - \dot{S}_{q-1} Q^n) \dot{\Delta}_q \dot{\Delta}_{q'} \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} \\ &\lesssim \sum_{|q-q'| \leq 5} \|\dot{S}_{q'-1} Q^n - \dot{S}_{q-1} Q^n\|_{L^\infty} \|\dot{\Delta}_q \dot{\Delta}_{q'} \Delta Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ &\lesssim \sum_{|q-q'| \leq 5} \|\dot{S}_{q'-1} \nabla Q^n - \dot{S}_{q-1} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2}, \end{aligned}$$

which yields

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^2(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla Q^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}. \quad (6.26)$$

The case $i = 4$ is handled as follows:

$$\begin{aligned} \langle \mathcal{J}_q^4(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} &= \sum_{q' \geq q-5} \langle \dot{\Delta}_q [\dot{\Delta}_{q'} Q^n \dot{S}_{q'+2} \Delta Q^n], \dot{\Delta}_q \nabla u^n \rangle_{L^2} \\ &\lesssim \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} Q^n\|_{L^2} \|\dot{S}_{q'+2} \Delta Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ &\lesssim \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \|\dot{S}_{q'+2} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2}. \end{aligned}$$

Therefore, multiplying by 2^{2qs} and taking the sum as $q \in \mathbb{Z}$,

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^4(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} &\lesssim \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left(2^{qs} \|\dot{\Delta}_q \nabla u^n\|_{L^2} \sum_{q' \in \mathbb{Z}} 2^{(q-q')s} \mathbf{1}_{(-\infty, 5)}(q-q') 2^{q's} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right) \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \left\{ \sum_{q \in \mathbb{Z}} \left(\sum_{q' \in \mathbb{Z}} 2^{(q-q')s} \mathbf{1}_{(-\infty, 5)}(q-q') 2^{q's} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right)^2 \right\}^{\frac{1}{2}}, \end{aligned}$$

so that, by convolution and the Young inequality

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^4(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}. \quad (6.27)$$

It remains to control the term related to \mathcal{J}_q^3 , namely

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^3(Q^n, \Delta Q^n), \dot{\Delta}_q \nabla u^n \rangle_{L^2} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} \quad (6.28)$$

As already remarked in the beginning, such term presents some difficulties. For instance, fixing $q \in \mathbb{Z}$ in the sum, the more natural estimate is the following one:

$$\langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} \leq \|\dot{S}_{q-1} Q^n\|_{L^\infty} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \|\dot{\Delta}_q \nabla u^n\|_{L^2}.$$

The presence of the low frequencies \dot{S}_{q-1} in the first norm doesn't permit to transport a gradient to Q^n , so the best expectation is the following one:

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} \lesssim \|Q^n\|_{L^\infty} \|\Delta Q^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}.$$

Of course such inequality is not useful for our purpose, i.e. an Osgood type inequality. For example there isn't a term that appears in the time derivative of the left-hand side of (6.17). Even if there exists a way to overcome such challenging evaluation, we will see that (6.28) is going to be erased.

Now, let us keep on our control. We have to examine $\langle \Delta Q^n Q^n, \nabla u^n \rangle_{\dot{H}^s}$. Observing that an equivalent formulation is $\langle Q^n \Delta Q^n, {}^t \nabla u^n \rangle_{\dot{H}^s}$ (Q^n and ΔQ^n are symmetric matrices) we recompute the previous inequality (with ${}^t \nabla u$ instead of ∇u), so that

$$\sum_{q \in \mathbb{Z}} \sum_{i=1,2,4} 2^{2qs} \langle \mathcal{J}_q^i(Q^n, \Delta Q^n), \dot{\Delta}_q {}^t \nabla u^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla u^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}. \quad (6.29)$$

As before, \mathcal{J}_q^3 is an inflexible term, so that, recalling (6.28), we need to erase what follows:

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \left\{ \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} - \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n, \dot{\Delta}_q {}^t \nabla u^n \rangle_{L^2} \right\} = \\ = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n - \dot{\Delta}_q \Delta Q^n \dot{S}_{q-1} Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2}. \end{aligned} \quad (6.30)$$

Estimate of $\langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s}$

Now, let us continue estimating $\langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s}$. The strategy as the same organization of the previous evaluation. We begin analyzing $\langle Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s}$

$$\langle Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q(Q^n \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} = \sum_{q \in \mathbb{Z}} \sum_{i=1}^4 2^{2qs} \langle \mathcal{J}_q^i(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2}.$$

First, considering $i = 1$ and $q \in \mathbb{Z}$, we get

$$\begin{aligned} \langle \mathcal{J}_q^1(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} &= \sum_{|q-q'| \leq 5} \langle [\dot{\Delta}_q, \dot{S}_{q'-1} Q^n] \dot{\Delta}_{q'} \Omega^n, \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \|\dot{S}_{q'-1} \nabla Q^n\|_{L^\infty} \|\dot{\Delta}_{q'} \nabla u^n\|_{L^2} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \\ &\lesssim \|\nabla Q^n\|_{L^\infty} \|\dot{\Delta}_q \nabla Q^n\|_{L^2} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'} \nabla u^n\|_{L^2}. \end{aligned}$$

therefore, taking the sum as $q \in \mathbb{Z}$,

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^1(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty} \|\nabla Q^n\|_{\dot{H}^s} \|\nabla u^n\|_{\dot{H}^s}. \quad (6.31)$$

By a similar method as for proving (6.31) or (6.26), the case $i = 2$ produces

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^2(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \lesssim \|\nabla Q^n\|_{L^\infty}^2 \|\nabla Q^n\|_{\dot{H}^s}^2 + \frac{\nu}{100} \|\nabla u^n\|_{\dot{H}^s}^2,$$

while, for $i = 4$, we get

$$\begin{aligned} \langle \mathcal{J}_q^4(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} &= \sum_{q' \geq q-5} \langle \dot{\Delta}_q [\dot{\Delta}_{q'} Q^n \dot{S}_{q'+2} \Omega^n], \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \\ &\lesssim \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} Q^n\|_{L^2} \|\dot{S}_{q'+2} \Omega^n\|_{L^\infty} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \\ &\lesssim \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \|\dot{S}_{q'+2} u^n\|_{L^\infty} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \\ &\lesssim \|u^n\|_{L^\infty} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \sum_{q' \geq q-5} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2}. \end{aligned}$$

Thus, multiplying by 2^{2qs} and taking the sum as $q \in \mathbb{Z}$, we realize that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^4(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} &\lesssim \\ &\lesssim \|u^n\|_{L^\infty} \sum_{q \in \mathbb{Z}} \left(2^{qs} \|\dot{\Delta}_q \Delta Q^n\|_{L^2} \sum_{q' \in \mathbb{Z}} 2^{(q-q')s} 1_{(-\infty, 5)}(q-q') 2^{q's} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right) \\ &\lesssim \|u^n\|_{L^\infty} \|\Delta Q^n\|_{\dot{H}^s} \left[\sum_{q \in \mathbb{Z}} \left(\sum_{q' \in \mathbb{Z}} 2^{(q-q')s} 1_{(-\infty, 5)}(q-q') 2^{q's} \|\dot{\Delta}_{q'} \nabla Q^n\|_{L^2} \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

so that, passing through the Young inequality,

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^4(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \lesssim \|u^n\|_{L^\infty} \|\Delta Q^n\|_{\dot{H}^s} \|\nabla Q^n\|_{\dot{H}^s}. \quad (6.32)$$

As the reader has already understood, the challenging term is the one related to \mathcal{J}_q^3 , that is

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \langle \mathcal{J}_q^3(Q^n, \Omega^n), \dot{\Delta}_q \Delta Q^n \rangle_{L^2} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{S}_{q-1} Q^n, \dot{\Delta}_q \Omega^n, \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \quad (6.33)$$

As (6.30), we are not capable to control it, so we claim that such obstacle is going to be simplified. Going on, we observe that $\langle \Omega^n Q^n, \Delta Q^n \rangle_{\dot{H}^s}$ can be reformulated as $\langle Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s}$, which we have just analyzed. Hence we need to control (6.33) twice, that is

$$\sum_{q \in \mathbb{Z}} 2^{2qs} 2 \langle S_{q-1} Q^n \dot{\Delta}_q \Omega^n, \dot{\Delta}_q \Delta Q^n \rangle_{L^2} = \sum_{q \in \mathbb{Z}} 2^{2qs} \langle S_{q-1} Q^n \dot{\Delta}_q \Omega^n - \dot{\Delta}_q \Omega^n S_{q-1} Q^n, \dot{\Delta}_q \Delta Q^n \rangle_{L^2}, \quad (6.34)$$

The Simplification

Recalling (6.30) and (6.34), we have not evaluated

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{2qs} \{ \langle \dot{S}_{q-1} Q^n \dot{\Delta}_q \Delta Q^n - \dot{\Delta}_q \Delta Q^n \dot{S}_{q-1} Q^n, \dot{\Delta}_q \nabla u^n \rangle_{L^2} + \\ + \langle S_{q-1} Q^n \dot{\Delta}_q \Omega^n - \dot{\Delta}_q \Omega^n S_{q-1} Q^n, \dot{\Delta}_q \Delta Q^n \rangle_{L^2} \}, \end{aligned}$$

yet. However, this is a series whose coefficients are null, thanks to Theorem 6.7.1. Hence, we have overcome all the previous lacks, so that the following inequality is fulfilled:

$$\begin{aligned} \langle \Delta Q^n Q^n - Q^n \Delta Q^n, \nabla u^n \rangle_{\dot{H}^s} - \langle \Omega^n Q^n - Q^n \Omega^n, \Delta Q^n \rangle_{\dot{H}^s} \lesssim \\ \lesssim \|(u^n, \nabla Q^n)\|_{L^\infty} \|(\nabla u^n, \Delta Q^n)\|_{\dot{H}^s} \|(u^n, \nabla Q^n)\|_{\dot{H}^s}. \end{aligned} \quad (6.35)$$

Estimate of $\langle \mathcal{P}(Q^n), \Delta Q^n \rangle_{\dot{H}^s}$

Finally, the last term to estimate is $\langle \mathcal{P}(Q^n), \Delta Q^n \rangle_{\dot{H}^s}$. Such evaluation is not a problematic, however it is computationally demanding, therefore we put forward in section 6.9 the proof of the following inequality:

$$\langle \mathcal{P}(Q^n), \Delta Q^n \rangle_{\dot{H}^s} \lesssim (1 + \|Q^n\|_{H^2} + \|Q^n\|_{H^2}^2) \|\nabla Q^n\|_{\dot{H}^s}^2, \quad (6.36)$$

where we remind that H^2 is a non-homogeneous Sobolev Space.

The Final Step

Summarizing the equality (6.17) and the inequalities (6.20), (6.21), (6.24), (6.35) and (6.36), we deduce

$$\begin{aligned} \frac{d}{dt} \left[\|u^n\|_{\dot{H}^s}^2 + L \|\nabla Q^n\|_{\dot{H}^s}^2 \right] + \nu \|\nabla u^n\|_{\dot{H}^s}^2 + \Gamma L^2 \|\Delta Q^n\|_{\dot{H}^s}^2 \lesssim \\ \lesssim \|(u^n, \nabla Q^n)\|_{L^\infty} \|(u^n, \nabla Q^n)\|_{\dot{H}^s} \|(u^n, \nabla Q^n)\|_{\dot{H}^{1+s}} + (1 + \|Q^n\|_{H^2} + \|Q^n\|_{H^2}^2) \|\nabla Q^n\|_{\dot{H}^s}^2. \end{aligned} \quad (6.37)$$

We define $\Phi(t) := \|u^n\|_{\dot{H}^s}^2 + \|\nabla Q^n\|_{\dot{H}^s}^2$ and $\Psi(t) := \|\nabla u^n\|_{\dot{H}^s}^2 + \|\Delta Q^n\|_{\dot{H}^s}^2$, so that (6.37) yields

$$\begin{aligned} \Phi'(t) + \Psi(t) \lesssim \|(u^n(t), \nabla Q^n(t))\|_{L^\infty} \|(u^n(t), \nabla Q^n(t))\|_{\dot{H}^s} \|(u^n(t), \nabla Q^n(t))\|_{\dot{H}^{1+s}} + \\ + (1 + \|Q^n(t)\|_{H^2} + \|Q^n(t)\|_{H^2}^2) \Phi(t). \end{aligned}$$

Then, fixing a positive integer $N = N(t)$, we apply Lemma 6.7.2, obtaining

$$\begin{aligned} \Phi' + \Psi \lesssim & \left\{ \|(u^n, \nabla Q^n)\|_{L^2} + \sqrt{N} \|(u^n, \nabla Q^n)\|_{\dot{H}^1} + 2^{-Ns} \|(u^n, \nabla Q^n)\|_{\dot{H}^{s+1}} \right\} \times \\ & \times \|(u^n, \nabla Q^n)\|_{\dot{H}^s} \|(u^n, \nabla Q^n)\|_{\dot{H}^{1+s}} + (1 + \|Q^n\|_{H^2} + \|Q^n\|_{H^2}^2) \Phi. \end{aligned} \quad (6.38)$$

For simplicity, let us define

$$f_1 := \|(u^n, \nabla Q^n)\|_{L^2}^2 + 1 + \|Q^n\|_{H^2} + \|Q^n\|_{H^2}^2, \quad f_2 := \|(u^n, \nabla Q^n)\|_{\dot{H}^1}^2,$$

hence (6.38) implies

$$\Phi'(t) + \Psi(t) \leq C \{ f_1(t) \Phi(t) + N f_2(t) \Phi(t) + 2^{-Ns} \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s} \Psi(t) \}, \quad (6.39)$$

for a positive constant C . Now, choosing $N(t)$ to be a positive integer which fulfills

$$\frac{1}{s} \log_2 \{ 2 + 4C + \Phi(t) \} \leq N(t) \leq \frac{1}{s} \log_2 \{ 2 + 4C + \Phi(t) \} + 1$$

it turns out from (6.39)

$$\Phi'(t) + \Psi(t) \leq C \{ f_1(t) \Phi(t) + f_2(t) \Phi(t) (\frac{1}{s} \log_2 \{ 2 + 4C + \Phi(t) \} + 1) \} + \frac{1}{2} \Psi(t),$$

so that, finally, increasing the value of C , we obtain

$$\Phi'(t) + \Psi(t) \leq C (f_1(t) + f_2(t)) \Phi(t) \log_2 \{ 2 + 4C + \Phi(t) \}, \quad (6.40)$$

which yields

$$\Phi'(t) \leq \frac{C}{\ln 2} (f_1(t) + f_2(t)) (2 + 4C + \Phi(t)) \ln \{ 2 + 4C + \Phi(t) \}.$$

By integrating this differential inequality, we obtain

$$2 + 4C + \Phi(t) \leq (2 + 4C + \Phi(0)) \exp \left\{ \frac{C}{\ln 2} \int_0^t (f_1(s) + f_2(s)) ds \right\}.$$

Recalling the definition of Φ , f_1 and f_2 , we obtain

$$\begin{aligned} \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 & \leq \\ & \leq (2 + 4C + \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2) \exp \left\{ \frac{C}{\ln 2} \int_0^t (\|(u^n(s), \nabla Q^n(s))\|_{L^2}^2 + 1 + \|Q^n(s)\|_{H^2} + \|Q^n(s)\|_{H^2}^2) ds \right\}, \end{aligned}$$

so that, thanks to Proposition 6.8.1

$$\|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 \leq (2 + 4C + \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2)^{\tilde{C}} (\|Q_0\|_{H^1} + \|u_0\|_{L_x^2}) e^{CT},$$

for some suitable positive constants \tilde{C} and C . Moreover, integrating (6.40) in time, we get

$$\int_0^t \Psi(s) ds \leq \Phi(0) + C \int_0^t (f_1(t) + f_2(t)) \Phi(t) \log_2 \{ 2 + 4C + \Phi(t) \},$$

that is

$$\begin{aligned} \int_0^t \|(u^n, \nabla Q^n)(\tau)\|_{\dot{H}^{s+1}}^2 d\tau &\leq \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2 + C \int_0^t \{ \|(u^n, \nabla Q^n)\|_{L^2}^2 + 1 + \|Q^n\|_{H^2} + \\ &+ \|Q^n\|_{H^2}^2 \}(\tau) d\tau \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 \log_2 \{ 2 + 4C + \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 \} \\ &\leq \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2 + C(t, \|u_0\|_{L_x^2}, \|Q_0\|_{H^1}) \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 \log_2 \{ 2 + 4C + \|(u^n, \nabla Q^n)(t)\|_{\dot{H}^s}^2 \}, \end{aligned}$$

where $C(t, \|u_0\|_{L_x^2}, \|Q_0\|_{H^1})$ is a suitable positive constant, given by Proposition 6.8.1. Since such estimates are uniform in n , we pass to the limit as n goes to ∞ , obtaining

$$\begin{aligned} \|(u, \nabla Q)\|_{L_T^\infty \dot{H}^s}^2 &\leq \\ &\leq (2 + 4C + \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2) \tilde{C}(\|Q_0\|_{H^1} + \|u_0\|_{L_x^2}) e^{CT}, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \|(u, \nabla Q)(\tau)\|_{\dot{H}^{s+1}}^2 d\tau &\leq \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2 + C(t, \|u_0\|_{L_x^2}, \|Q_0\|_{H^1}) \times \\ &\times (2 + 4C + \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2) \tilde{C}(\|Q_0\|_{H^1} + \|u_0\|_{L_x^2}) e^{CT} \times \\ &\times \log_2 \{ 2 + 4C + (2 + 4C + \|(u_0, \nabla Q_0)\|_{\dot{H}^s}^2) \tilde{C}(\|Q_0\|_{H^1} + \|u_0\|_{L_x^2}) e^{CT} \}, \end{aligned}$$

where (u, Q) is solution of (P) with (u_0, Q_0) as initial data. This concludes the proof of Theorem (6.1.3). \square

6.7 Thecnincal tools

Lemma 6.7.1. *Let Q_1 and Q_2 be two 3×3 symmetric matrices with entries in $H^2(\mathbb{R}^2)$. Assume that u is a 3-vector with components in $H^1(\mathbb{R}^2)$ and let Ω be the 3×3 matrix defined by $1/2(\nabla u - {}^t \nabla u)$. Then the following identity is satisfied:*

$$\int_{\mathbb{R}^2} \text{tr}\{(\Omega Q_2 - Q_2 \Omega) \Delta Q_1\} + \int_{\mathbb{R}^2} \text{tr}\{(\Delta Q_1 Q_2 - Q_1 \Delta Q_2) \nabla u\} = 0$$

Proof. By a direct computation

$$\begin{aligned} \int_{\mathbb{R}^2} \text{tr}\{(\Omega Q_2 - Q_2 \Omega) \Delta Q_1\} &= \int_{\mathbb{R}^2} [\text{tr}\{\Omega Q_2 \Delta Q_1\} - \text{tr}\{Q_2 \Omega \Delta Q_1\}] = \int_{\mathbb{R}^2} [\text{tr}\{\Omega Q_2 \Delta Q_1\} - \\ &- \text{tr}\{\Delta Q_1 {}^t \Omega Q_2\}] = 2 \int_{\mathbb{R}^2} \text{tr}\{\Omega Q_2 \Delta Q_1\} = \int_{\mathbb{R}^2} \text{tr}\{\nabla u Q_2 \Delta Q_1 - {}^t \nabla u Q_2 \Delta Q_1\} = \\ &= \int_{\mathbb{R}^2} \text{tr}\{(Q_1 \Delta Q_2 - \Delta Q_1 Q_2) \nabla u\}, \end{aligned}$$

which concludes the proof of the Lemma. \square

Lemma 6.7.2. *Let f be a function in $H^1 \cap \dot{H}^{1+s}$ with $s > 0$. Then, there exists $C > 0$ such that*

$$\|f\|_{L^\infty} \leq C(\|f\|_{L^2} + \sqrt{N} \|f\|_{H^1} + 2^{-Ns} \|f\|_{\dot{H}^{1+s}}),$$

for any positive integer N .

Proof. Let us fix $N > 0$. Then $f = \dot{S}_{N+1}f + (\text{Id} - \dot{S}_{N+1})f$ fulfills

$$\|f\|_{L^\infty} \leq \|\dot{S}_{N+1}f\|_{L^\infty} + \underbrace{\left\| \sum_{q \geq N} \dot{\Delta}_q f \right\|_{L^\infty}}_{\mathcal{A}} \leq \underbrace{\sum_{q < N} \|\dot{\Delta}_q f\|_{L^\infty}}_{\mathcal{A}} + \underbrace{\sum_{q \geq N} \|\dot{\Delta}_q f\|_{L^\infty}}_{\mathcal{B}}.$$

First, let us analyze \mathcal{A} :

$$\begin{aligned} \sum_{q < N} \|\dot{\Delta}_q f\|_{L^\infty} &= \sum_{q \leq 0} \|\dot{\Delta}_q f\|_{L^\infty} + \sum_{q=1}^N \|\dot{\Delta}_q f\|_{L^\infty} \lesssim \sum_{q \leq 0} 2^q \|\dot{\Delta}_q f\|_{L^2} + \sum_{q=1}^N 2^q \|\dot{\Delta}_q f\|_{L^2} \\ &\lesssim \sum_{q \leq 0} \|\dot{\Delta}_q f\|_{L^2} + \sqrt{N} \|f\|_{\dot{H}^1} \lesssim \|f\|_{L^2} + \sqrt{N} \|f\|_{\dot{H}^1}. \end{aligned}$$

Finally, from the definition of \mathcal{B}

$$\sum_{q \geq N} \|\dot{\Delta}_q f\|_{L^\infty} = \sum_{q \geq N} 2^q \|\dot{\Delta}_q f\|_{L^2} = \sum_{q \geq N} 2^{-sq} 2^{q(1+s)} \|\dot{\Delta}_q f\|_{L^2} \lesssim 2^{-Ns} \|f\|_{\dot{H}^{1+s}},$$

which concludes the proof of the lemma. \square

Proof. proof of Theorem 6.2.4 At first we identify the Sobolev Spaces \dot{H}^s and \dot{H}^t with the Besov Spaces $\dot{B}_{2,2}^s$ and $\dot{B}_{2,2}^t$ respectively. We claim that ab belongs to $\dot{B}_{2,2}^{s+t-N/2}$ and

$$\|ab\|_{\dot{B}_{2,2}^{s+t-N/2}} \leq C \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t},$$

for a suitable positive constant.

We decompose the product ab through the Bony decomposition, namely $ab = \dot{T}_a b + \dot{T}_b a + R(a, b)$, where

$$\dot{T}_a b := \sum_{q \in \mathbb{Z}} \dot{\Delta}_q a \dot{S}_{q-1} b, \quad \dot{T}_b a := \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} a \dot{\Delta}_q b, \quad \dot{R}(a, b) := \sum_{\substack{q \in \mathbb{Z} \\ |\nu| \leq 1}} \dot{\Delta}_q a \dot{\Delta}_{q+\nu} b.$$

For any $q \in \mathbb{Z}$, we have

$$\begin{aligned} 2^{q(s+t-\frac{N}{2})} \|(\dot{\Delta}_q \dot{T}_a b, \dot{\Delta}_q \dot{T}_b a)\|_{L^2} &\lesssim \\ &\lesssim \sum_{|q-q'| \leq 5} 2^{q's} \|\dot{\Delta}_{q'} a\|_{L^2} 2^{q'(t-\frac{N}{2})} \|\dot{S}_{q-1} b\|_{L^\infty} + \sum_{|q-q'| \leq 5} 2^{q'(s-\frac{N}{2})} \|\dot{S}_{q-1} a\|_{L^\infty} 2^{q't} \|\dot{\Delta}_q b\|_{L^2}, \end{aligned}$$

so that we determine the following feature

$$\begin{aligned} \|(\dot{T}_a b, \dot{T}_b a)\|_{\dot{B}_{2,2}^{s+t-\frac{N}{2}}} &\leq \|(\dot{T}_a b, \dot{T}_b a)\|_{\dot{B}_{2,1}^{s+t-\frac{N}{2}}} \\ &\lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{\infty,2}^{t-\frac{N}{2}}} + \|a\|_{\dot{B}_{\infty,2}^{s-\frac{N}{2}}} \|b\|_{\dot{B}_{2,2}^t} \lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t}, \end{aligned}$$

where we have used the embedding $\dot{B}_{2,2}^\sigma \hookrightarrow \dot{B}_{\infty,2}^{\sigma-N/2}$, for any $\sigma \in \mathbb{R}$ and Proposition 6.2.3.

In order to conclude the proof, we have to handle the rest $\dot{R}(a, b)$. By a direct computation, for

any $q \in \mathbb{Z}$,

$$2^{(t+s)q} \|\dot{\Delta}_q \dot{R}(a, b)\|_{L^1} \leq \sum_{\substack{q' \geq q-5 \\ |\nu| \leq 1}} 2^{(q-q')(s+t)} 2^{q's} \|\dot{\Delta}_{q'} a\|_{L^2} 2^{(q'+\nu)t} \|\dot{\Delta}_{q'+\nu} a\|_{L^2},$$

so that, thanks to the Young inequality, we deduce

$$\|\dot{R}(a, b)\|_{\dot{B}_{2,2}^{s+t-\frac{N}{2}}} \lesssim \|\dot{R}(a, b)\|_{\dot{B}_{1,1}^{s+t}} \lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t},$$

where we have used the embedding $\dot{B}_{1,1}^{s+t} \hookrightarrow \dot{B}_{2,2}^{s+t-N/2}$ and moreover that $\sum_{q \leq 5} 2^{q(s+t)}$ is finite, since $s+t$ is positive. \square

6.8 A bound for the Friedrichs scheme

This section is devoted to a specific bound for the norms of the solutions of system \tilde{P}_n . This result has already been presented in [99]. We propose here the proof for the sake of completeness.

Proposition 6.8.1. *Let (u^n, Q^n) be a solution for system \tilde{P}_n with initial data $(u_0, Q_0) \in L_x^2 \times H^1$. Then*

$$\|(Q^n, \nabla Q^n, u^n)\|_{L^\infty(0,T;L_x^2)}^2 + \|(\Delta Q^n, \nabla u^n)\|_{L^2(0,T;L_x^2)}^2 \lesssim (\|Q_0\|_{H^1} + \|u_0\|_{L_x^2}) e^{CT} \quad (6.41)$$

Proof. We multiply the first equation by $Q^n - L\Delta Q^n$, while the momentum equation by u^n and we integrate everything in \mathbb{R}^2 , obtaining

$$\begin{aligned} & \frac{d}{dt} [\|Q^n\|_{L_x^2}^2 + L\|\nabla Q^n\|_{L_x^2}^2 + \|u^n\|_{L_x^2}^2] + \nu\|\nabla u^n\|_{L_x^2}^2 + \Gamma L\|\nabla Q^n\|_{L_x^2}^2 + \|\Delta Q^n\|_{L_x^2}^2 + c\|Q^n\|_{L_x^4}^4 = \\ & = -a\|Q^n\|_{L_x^2}^2 + b\langle (Q^n)^2, Q^n \rangle_{L_x^2} + aL\|\nabla Q^n\|_{L_x^2}^2 - bL\langle (Q^n)^2, \Delta Q^n \rangle_{L_x^2} + cL\langle Q^n \text{tr}\{(Q^n)^2\}, \Delta Q^n \rangle_{L_x^2}. \end{aligned}$$

At first, recalling that $c > 0$ and integrating by part, we get

$$cL\langle Q^n \text{tr}\{(Q^n)^2\}, \Delta Q^n \rangle_{L_x^2} = -cL \int_{\mathbb{R}^2} |\nabla Q^n|^2 \text{tr}\{(Q^n)^2\} - cL \int_{\mathbb{R}^2} \left| \nabla \frac{|Q^n|^2}{2} \right|^2 \leq 0.$$

This yields that

$$\begin{aligned} & \frac{d}{dt} [\|Q^n\|_{L_x^2}^2 + L\|\nabla Q^n\|_{L_x^2}^2 + \|u^n\|_{L_x^2}^2] + \nu\|\nabla u^n\|_{L_x^2}^2 + \Gamma L\|\nabla Q^n\|_{L_x^2}^2 + \Gamma L^2\|\Delta Q^n\|_{L_x^2}^2 + \\ & \quad + cL \int_{\mathbb{R}^2} |\nabla Q^n|^2 \text{tr}\{(Q^n)^2\} + cL \int_{\mathbb{R}^2} \left| \nabla \frac{|Q^n|^2}{2} \right|^2 + c\|Q^n\|_{L_x^4}^4 \\ & \leq -a\|Q^n\|_{L_x^2}^2 + aL\|\nabla Q^n\|_{L_x^2}^2 + c_{\Gamma,L}\|\Delta Q^n\|_{L_x^2}^2 + b\langle (Q^n)^2, Q^n \rangle_{L_x^2} - bL\langle (Q^n)^2, \Delta Q^n \rangle_{L_x^2} \end{aligned}$$

Assuming $c_{\Gamma,L}$ small enough, we can absorb $c_{\Gamma,L}\|\Delta Q^n\|_{L_x^2}^2$ by the left-hand side. Moreover, still

integrating by part

$$\begin{aligned} -bL\langle(Q^n)^2, \Delta Q^n\rangle_{L_x^2} &= -bL\langle(Q^n)^2, \Delta Q^n\rangle_{L_x^2} \int_{\mathbb{R}^2} \sum_{i=1}^2 \operatorname{tr}\{(\partial_i Q^n)^2 Q^n\} \\ &\leq \frac{c}{100} \int_{\mathbb{R}^2} |\nabla Q^n|^2 \operatorname{tr}\{Q^2\} + C\|Q^n\|_{L_x^2}^2, \end{aligned}$$

for a suitable positive constant C . Finally, proceeding as for proving (6.4) when $p = 1$, we get

$$b\langle(Q^n)^2, Q^n\rangle_{L_x^2} \leq \frac{c}{100}\|Q^n\|_{L^4}^4 + \frac{100}{c}\|Q^n\|_{L_x^2}^2.$$

Summarizing, we finally deduce

$$\begin{aligned} \frac{d}{dt} [\|Q^n\|_{L_x^2}^2 + L\|\nabla Q^n\|_{L_x^2}^2 + \|u^n\|_{L_x^2}^2] + \nu\|\nabla u^n\|_{L_x^2}^2 + \Gamma L\|\nabla Q^n\|_{L_x^2}^2 + \Gamma L^2\|\Delta Q^n\|_{L_x^2}^2 + \\ + cL \int_{\mathbb{R}^2} |\nabla Q^n|^2 \operatorname{tr}\{(Q^n)^2\} + cL \int_{\mathbb{R}^2} \left| \nabla \frac{|Q^n|^2}{2} \right|^2 + c\|Q^n\|_{L_x^4}^4 \\ \lesssim \|Q^n\|_{L_x^2}^2 + \|\nabla Q^n\|_{L_x^2}^2, \end{aligned}$$

which yields (6.41). \square

6.9 Estimate

The purpose of this section is to estimate $\langle \mathcal{P}(Q^n), \Delta Q^n \rangle_{\dot{H}^s}$. In order to facilitate the reader, we are not going to indicate the index n , from here on. We have to examine

$$\begin{aligned} \langle \mathcal{P}(Q), \Delta Q \rangle_{\dot{H}^s} &= \langle -aQ + b[Q^2 - \operatorname{tr}\{Q^2\} \frac{\operatorname{Id}}{3}] - \operatorname{ctr}\{Q^2\}Q, \Delta Q \rangle_{\dot{H}^s} \\ &= \langle -aQ + bQ^2 - \operatorname{ctr}\{Q^2\}Q, \Delta Q \rangle_{\dot{H}^s}, \end{aligned}$$

where $\langle \operatorname{tr}\{Q^2\} \operatorname{Id}, \Delta Q \rangle_{\dot{H}^s} = 0$ since ΔQ has null trace. It is trivial that

$$-\langle aQ, \Delta Q \rangle_{\dot{H}^s} \lesssim \|\nabla Q\|_{\dot{H}^s}. \quad (6.42)$$

Now, let us consider $b\langle Q^2, \Delta Q \rangle_{\dot{H}^s}$. By definition we have

$$\begin{aligned} b\langle Q^2, \Delta Q \rangle_{\dot{H}^s} &= b \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q [Q^2], \dot{\Delta}_q \Delta Q \rangle_{L^2} \\ &= b \sum_{q \in \mathbb{Z}} 2^{2qs} [\underbrace{2 \langle \dot{\Delta}_q T_Q Q, \dot{\Delta}_q \Delta Q \rangle_{L^2}}_{\mathcal{A}_q} + \underbrace{\langle \dot{\Delta}_q \dot{R}(Q, Q), \dot{\Delta}_q \Delta Q \rangle_{L^2}}_{\mathcal{B}_q}] \end{aligned}$$

We concentrate on \mathcal{A}_q , getting

$$\mathcal{A}_q \leq \sum_{|q-q'| \leq 5} \|\dot{S}_{q'-1} Q \dot{\Delta}_{q'} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \lesssim \|Q\|_{L^\infty} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'} \nabla Q\|_{L^2},$$

so that

$$b \sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{A}_q \lesssim \|Q\|_{L^\infty} \|\nabla Q\|_{\dot{H}^s}^2. \quad (6.43)$$

Now, analyzing \mathcal{B}_q , we observe that

$$\mathcal{B}_q \leq \sum_{\substack{q' \geq q-5 \\ |l| \leq 1}} \|\dot{\Delta}_{q'} Q \dot{\Delta}_{q'+l} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \lesssim \|Q\|_{L^\infty} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{q' \geq q-5} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla Q\|_{L^2},$$

so that

$$b \sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{B}_q \lesssim \|Q\|_{L^\infty} b \sum_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{q' \in \mathbb{Z}} 2^{(q-q')(s+1)} 1_{(-\infty, 5)}(q-q') \|\dot{\Delta}_{q'} \nabla Q\|_{L^2}.$$

Thus, by convolution and young inequality $b \sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{B}_q \lesssim \|Q\|_{L^\infty} \|\nabla Q\|_{\dot{H}^s}^2$ and, recalling (6.43),

$$b \langle Q^2, \Delta Q \rangle_{\dot{H}^s} \lesssim \|Q\|_{L^\infty} \|\nabla Q\|_{\dot{H}^s}^2. \quad (6.44)$$

Now, it remains to examine $c \langle Q \text{tr}\{Q^2\}, \Delta Q \rangle_{\dot{H}^s}$. The procedure is quietly similar to the previous one. At first we use the Bony decomposition as follows:

$$\begin{aligned} \langle Q \text{tr}\{Q^2\}, \Delta Q \rangle_{\dot{H}^s} &= \sum_{q \in \mathbb{Z}} 2^{2qs} \langle \dot{\Delta}_q(Q \text{tr}\{Q^2\}), \dot{\Delta}_q \Delta Q \rangle_{L^2} \\ &= \sum_{q \in \mathbb{Z}} 2^{2qs} \left[\underbrace{\langle \dot{\Delta}_q \dot{T}_Q(\text{tr}\{Q^2\} \text{Id}), \dot{\Delta}_q \Delta Q \rangle_{L^2}}_{\mathcal{A}_q} + \underbrace{\langle \dot{\Delta}_q \dot{T}_{\text{tr}\{Q^2\}} \text{Id} Q, \dot{\Delta}_q \Delta Q \rangle_{L^2}}_{\mathcal{B}_q} \right. \\ &\quad \left. + \underbrace{\langle \dot{\Delta}_q \dot{R}(Q, \text{tr}\{Q^2\} \text{Id}), \dot{\Delta}_q \Delta Q \rangle_{L^2}}_{\mathcal{C}_q} \right] \end{aligned}$$

First, we concentrate on \mathcal{A}_q , the more computationally demanding term, obtaining

$$\begin{aligned} \mathcal{A}_q &\leq \sum_{|q-q'| \leq 5} \|\dot{S}_{q'-1} Q \dot{\Delta}_{q'}(\text{tr}\{Q^2\} \text{Id})\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \lesssim \|Q\|_{L^\infty} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'}(Q^2)\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \\ &\lesssim \|Q\|_{L^\infty} \sum_{|q-q'| \leq 5} \left[\underbrace{2 \|\dot{\Delta}_{q'} \dot{T}_Q Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2}}_{I_{q,q'}} + \underbrace{\|\dot{\Delta}_{q'} \dot{R}(Q, Q)\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2}}_{II_{q,q'}} \right] \end{aligned}$$

The term I_q is the simpler one, indeed

$$I_{q,q'} \lesssim \sum_{|q'-q''| \leq 5} \|\dot{S}_{q''-1} Q \dot{\Delta}_{q''} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \lesssim \|Q\|_{L^\infty} \sum_{|q'-q''| \leq 5} \|\dot{\Delta}_{q''} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2},$$

so that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} \|Q\|_{L^\infty} \sum_{|q-q'| \leq 5} I_{q,q'} &\lesssim \|Q\|_{L^\infty}^2 \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} \sum_{|q'-q''| \leq 5} \|\dot{\Delta}_{q''} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \\ &\lesssim \|Q\|_{L^\infty}^2 \sum_{q \in \mathbb{Z}} \sum_{|q-q''| \leq 10} \|\dot{\Delta}_{q''} \nabla Q\|_{L^2} \|\dot{\Delta}_q \nabla Q\|_{L^2} \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}. \end{aligned}$$

We overcome the term $II_{q,q'}$ as follows:

$$\begin{aligned}
II_{q,q'} &\lesssim \|\dot{\Delta}_q \Delta Q\|_{L^2} \sum_{\substack{q'' \geq q'-5 \\ |l| \leq 1}} \|\dot{\Delta}_{q''} Q\|_{L^2} \|\dot{\Delta}_{q''+l} Q\|_{L^\infty} \\
&\lesssim \|Q\|_{L^\infty} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{q'' \geq q'-5} 2^{q-q''} \|\dot{\Delta}_{q''} \nabla Q\|_{L^2},
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} \|Q\|_{L^\infty} \sum_{|q-q'| \leq 5} II_{q,q'} &\lesssim \|Q\|_{L^\infty}^2 \sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{|q-q'| \leq 5} \sum_{q'' \geq q'-5} 2^{q-q''} \|\dot{\Delta}_{q''} \nabla Q\|_{L^2} \\
&\lesssim \|Q\|_{L^\infty}^2 \sum_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{q'' \geq q-10} 2^{(q-q'')(s+1)} 2^{q''s} \|\dot{\Delta}_{q''} \nabla Q\|_{L^2},
\end{aligned}$$

so that, by convolution and Young inequality $\sum_{q \in \mathbb{Z}} \|Q\|_{L^\infty} \sum_{|q-q'| \leq 5} II_{q,q'} \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}$. Summarizing the previous inequalities, we get $\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{A}_q \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}^2$. In order to examine \mathcal{B}_q it is sufficient to observe that

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{B}_q &\lesssim \sum_{q \in \mathbb{Z}} 2^{2qs} \sum_{|q-q'| \leq 5} \|\dot{S}_{q'-1}(\text{tr}\{Q^2\} \text{Id}) \dot{\Delta}_{q'} Q\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \\
&\lesssim \|Q^2\|_{L^\infty} \sum_{q \in \mathbb{Z}} 2^{2qs} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'} \nabla Q\|_{L^2} \|\dot{\Delta}_q \nabla Q\|_{L^2} \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}^2.
\end{aligned}$$

It remains indeed \mathcal{C}_q , which is straightforward, indeed

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{C}_q &\lesssim \sum_{q \in \mathbb{Z}} 2^{2qs} \sum_{\substack{q' \geq q-5 \\ |l| \leq 1}} \|\dot{\Delta}_{q'} Q \dot{\Delta}_{q'+l}(Q^2)\|_{L^2} \|\dot{\Delta}_q \Delta Q\|_{L^2} \\
&\lesssim \|Q\|_{L^\infty}^2 \sum_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q \nabla Q\|_{L^2} \sum_{q' \geq q-5} 2^{(q-q')(s+1)} \|\dot{\Delta}_{q'} \nabla Q\|_{L^2},
\end{aligned}$$

thus, by convolution and the Young inequality, $\sum_{q \in \mathbb{Z}} 2^{2qs} \mathcal{C}_q \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}^2$. Summarizing, we finally get $c\langle Q \text{tr}\{Q^2\}, \Delta Q \rangle_{\dot{H}^s} \lesssim \|Q\|_{L^\infty}^2 \|\nabla Q\|_{\dot{H}^s}^2$ and recalling (6.42)-(6.44), we finally obtain

$$\langle \mathcal{P}(Q), \Delta Q \rangle_{\dot{H}^s} \lesssim (1 + \|Q\|_{L^\infty} + \|Q\|_{L^\infty}^2) \|\nabla Q\|_{\dot{H}^s}^2 \lesssim (1 + \|Q\|_{H^2} + \|Q\|_{H^2}^2) \|\nabla Q\|_{\dot{H}^s}^2,$$

where the last inequality is due to the embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Hence, inequality (6.36) is proven.

Chapter 7

General Beris-Edwards system

In this chapter we present the results of the following paper:

F. De Anna, A. Zarnescu (accepted), *Uniqueness of weak solutions of the full coupled Navier-Stokes and Q-tensor system in 2D*, Comm. Math. Sci., (2016)

7.1 Introduction

The main aim of this work is to prove the uniqueness of weak solutions for a type of coupled Navier-Stokes and Q-tensor systems proposed in [12] and studied numerically and analytically in [1, 35, 54, 58, 98]. This type of system models nematic liquid crystals and provides in a certain sense an extension of the classical Ericksen-Leslie model [35], whose uniqueness of weak-solutions was proved in [115]. In the remainder of this introduction we will briefly present the equations and we state our main result.

The system models the evolution of liquid crystal molecules together with the underlying flow, through a parabolic-type system coupling an incompressible Navier-Stokes system with a nonlinear convection-diffusion system. The local orientation of the molecules is described through a function Q taking values from $\mathbb{R}_+ \times \Omega \subset \mathbb{R}_+ \times \mathbb{R}^d, d = 2, 3$ into the set of so-called d -dimensional Q -tensors that is

$$S_0^{(d)} \stackrel{\text{def}}{=} \left\{ Q \in \mathbb{M}^{d \times d}; Q_{ij} = Q_{ji}, \text{tr}(Q) = 0, i, j = 1, \dots, d \right\}$$

(the most relevant physical situations being $d = 2, 3$). The evolution of the Q 's is driven by a gradient flow of the free energy of the molecules as well as the transport, distortion and alignment effects caused by the flow. The flow field $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ satisfies a forced incompressible Navier-Stokes system, with the forcing provided by the additional, non-Newtonian stress caused by the molecules orientations, thus expressed in terms of Q . We restrict ourselves to the case $\Omega = \mathbb{R}^d$ and work with non-dimensional quantities. The evolution of Q is given by:

$$\partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) = -\Gamma \frac{\partial \mathcal{F}_e}{\partial Q}$$

with $\Gamma > 0$. Here

$$\mathcal{F}_e(Q) = \int_{\mathbb{R}^d} \frac{L}{2} |\nabla Q|^2 + \left(\frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \right) dx \quad (7.1)$$

is the free energy of the liquid crystal molecules and $\frac{\partial \mathcal{F}_e}{\partial Q}$ denotes the variational derivative. The L, a, b, c constants are specific to the material with:

$$L > 0 \text{ and } a, b, c \in \mathbb{R}, c > 0 \quad (7.2)$$

If $u = 0$ the Q -tensor equation would simply be a gradient flow of the free energy. For $u \neq 0$ the molecules are transported by the flow (as indicated by the convective derivative $\partial_t + u \cdot \nabla$) as well as being tumbled and aligned by the flow, fact described by the term

$$S(\nabla u, Q) := (\xi D + \Omega)(Q + \frac{1}{d}Id) + (Q + \frac{1}{d}Id)(\xi D - \Omega) - 2\xi(Q + \frac{1}{d}Id)\text{tr}(Q\nabla u) \quad (7.3)$$

where $D := \frac{1}{2}(\nabla u + {}^t\nabla u)$ and $\Omega := \frac{1}{2}(\nabla u - {}^t\nabla u)$ are, respectively, the symmetric part and the antisymmetric part, of the velocity gradient matrix ∇u . The constant ξ is specific to the liquid crystal material.

The flow satisfies the forced Navier-Stokes system:

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= \nu \Delta u + \nabla p + \lambda \text{div}\{\tau + \sigma\} \\ \text{div } u &= 0 \end{aligned}$$

where $\nu, \lambda > 0$ with λ measuring the ratio of the elastic effects (produced by the liquid crystal molecules) to that of the diffusive effects. The forcing is provided by the additional stress caused by the presence of the liquid crystal molecules, more specifically we have the symmetric part of the additional stress tensor:

$$\tau := \left[-\xi \left(Q + \frac{1}{d}Id \right) H - \xi H \left(Q + \frac{1}{d}Id \right) + 2\xi \left(Q + \frac{1}{d}Id \right) QH - L\nabla Q \odot \nabla Q \right]$$

and the antisymmetric part:

$$\sigma := QH - HQ$$

where we denoted

$$H := -\frac{\partial \mathcal{F}_e}{\partial Q} = L\Delta Q - aQ + b[Q^2 - \frac{\text{tr}(Q^2)}{d}Id] - cQ\text{tr}(Q^2) \quad (7.4)$$

Summarising we have the coupled system:

$$\begin{aligned} \partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) &= \Gamma(L\Delta Q - aQ + b[Q^2 - \frac{\text{tr}(Q^2)}{d}Id] - cQ\text{tr}(Q^2)) \\ \partial_t u + (u \cdot \nabla)u &= \nu \Delta u + \nabla p + \lambda \nabla \cdot (QH - HQ) + \lambda \nabla \cdot \left[-\xi \left(Q + \frac{1}{d}Id \right) H - \right. \\ &\quad \left. - \xi H \left(Q + \frac{1}{d}Id \right) + 2\xi \left(Q + \frac{1}{d}Id \right) QH - L\nabla Q \odot \nabla Q \right] \\ \text{div } u &= 0 \end{aligned} \quad (7.5)$$

where $\Gamma, L, \nu, c > 0$, $a, b \in \mathbb{R}$. Let us observe that this is a slight extension of the system considered in [98], where $\lambda = 1$. However, this does not create any major difficulties compared to equations in [98] but it is more relevant from a physical point of view.

The main result of this work is the uniqueness of weak solutions, which are defined in a rather standard manner:

Definition 7.1.1. *A pair (Q, u) is called a weak solution of the system (7.5), subject to initial data*

$$Q(0, x) = \bar{Q}(x) \in H^1(\mathbb{R}^d; S_0^{(d)}), u(0, x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \text{div } \bar{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \quad (7.6)$$

if $Q \in L_{loc}^\infty(\mathbb{R}_+; H^1) \cap L_{loc}^2(\mathbb{R}_+; H^2)$, $u \in L_{loc}^\infty(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^1)$ and for every compactly supported $\varphi \in C^\infty([0, \infty) \times \mathbb{R}^d; S_0^{(d)})$, $\psi \in C^\infty([0, \infty) \times \mathbb{R}^d; \mathbb{R}^d)$ with $\text{div } \psi = 0$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} (-Q \cdot \partial_t \varphi - \Gamma L \Delta Q \cdot \varphi) - Q \cdot u \nabla_x \varphi \, dx \, dt -$$

$$\begin{aligned}
& - \int_0^\infty \int_{\mathbb{R}^d} (\xi D + \Omega)(Q + \frac{1}{d} Id) \cdot \varphi + (Q + \frac{1}{d} Id)(\xi D - \Omega) \cdot \varphi \, dx dt - \\
& - \int_0^\infty \int_{\mathbb{R}^d} 2\xi(Q + \frac{1}{d} Id) \operatorname{tr}(Q \nabla u) \cdot \varphi \, dx dt = \int_{\mathbb{R}^d} \bar{Q}(x) \cdot \varphi(0, x) \, dx + \\
& + \Gamma \int_0^\infty \int_{\mathbb{R}^d} \left\{ -aQ + b[Q^2 - \frac{\operatorname{tr}(Q^2)}{d} Id] - cQ \operatorname{tr}(Q^2) \right\} \cdot \varphi \, dx dt
\end{aligned} \tag{7.7}$$

and

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^d} -u \partial_t \psi - u_\alpha u_\beta \partial_\alpha \psi_\beta + \nu \nabla u \nabla \psi \, dt \, dx - \int_{\mathbb{R}^d} \bar{u}(x) \psi(0, x) \, dx \\
& = L\lambda \int_0^\infty \int_{\mathbb{R}^d} Q_{\gamma\delta, \alpha} Q_{\gamma\delta, \beta} \psi_{\alpha, \beta} - Q_{\alpha\gamma} \Delta Q_{\gamma\beta} \psi_{\alpha, \beta} + \Delta Q_{\alpha\gamma} Q_{\gamma\beta} \psi_{\alpha, \beta} \, dx \, dt + \\
& + \xi\lambda \int_0^\infty \int_{\mathbb{R}^d} \left(Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{d} \right) H_{\gamma\beta} \psi_{\alpha, \beta} + H_{\alpha\gamma} \left(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{d} \right) \psi_{\alpha, \beta} \, dx dt - \\
& - 2\xi\lambda \int_0^\infty \int_{\mathbb{R}^d} \left(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{d} \right) Q_{\gamma\delta} H_{\gamma\delta} \psi_{\alpha, \beta} \, dx \, dt.
\end{aligned} \tag{7.8}$$

We can now state our main result, which is the existence and uniqueness of weak solutions:

Theorem 7.1.2. *Let $d = 2, 3$ and take*

$$Q(0, x) = \bar{Q}(x) \in H^1(\mathbb{R}^d; S_0^{(d)}), \quad u(0, x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \quad \operatorname{div} \bar{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

Then system (7.5) admits a global weak solution. Moreover if $d = 2$, then uniqueness holds.

Remark 7.1.3. *With minor modifications to the proof, that are left to the interested reader, the result also holds when the system is 2D in the domain but 3D in the target, which physically corresponds to a situation where there is no dependence in one of the three spatial directions.*

The main part of the theorem is about uniqueness, as the existence part is just a fairly straightforward revisit of the arguments in [98]. The main difficulties associated with treating the system (7.5) are related to the presence of the Navier-Stokes part. One can essentially think of the system as a highly non-trivial perturbation of a Navier-Stokes system. It is known that for Navier-Stokes alone the uniqueness of weak solutions in 2D can be achieved through rather standard arguments, while in 3D it is a major open problem.

The extended system that we deal with has an intermediary position, as the perturbation produced by the presence of the additional stress-tensor generates significant technical difficulties related in the first place to the weak norms available for the u term. A rather common way of dealing with this issue is by using a weak norm for estimating the difference between the two weak solutions, a norm that is below the natural spaces in which the weak solutions are defined. This approach was used before in the context of the related Leslie-Ericksen model [73] as well as for the usual Navier-Stokes system in [48] and [88].

In our case, for technical convenience we use a homogeneous Sobolev space, namely $\dot{H}^{-\frac{1}{2}}$. The fact that the initial data for the difference is zero (i.e. $(\delta u, \delta Q)_{t=0} = 0$) helps in controlling the difference in such a low regularity space. However, one of the main reasons for choosing the homogeneous setting is a specific product law, see Proposition 7.4.4 in section 7.5. The mentioned theorem shows that the product is a bounded operator from $\dot{H}^s(\mathbb{R}^2) \times \dot{H}^t(\mathbb{R}^2)$ into $\dot{H}^{s+t-1}(\mathbb{R}^2)$, for any $|s|, |t| \leq 1$ such that $s+t$ is positive. We note that evaluating the difference at regularity level $s = 0$ i.e. in L^2 , would only allow to prove a weak-strong uniqueness result, along the lines

of [99]. Working in a negative Sobolev space, \dot{H}^s with $s \in (-1, 0)$ allows to capture the uniqueness of weak solutions. We expect that a similar proof would work in any \dot{H}^s with $s \in (-1, 0)$ and our choice $s = -\frac{1}{2}$ is just for convenience.

Our main work is to obtain the delicate double-logarithmic type estimates that lead to an Osgood lemma, a generalization of the Gronwall inequality (see [7], Lemma 3.4). Indeed the uniqueness reduces to an estimate of the following type:

$$\Phi'(t) \leq \chi(t) \left\{ \Phi(t) + \Phi(t) \ln \left(1 + e + \frac{1}{\Phi(t)} \right) + \Phi(t) \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \ln \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \right\},$$

where $\Phi(t)$ is the norm of the difference between two solutions and χ is a priori in L^1_{loc} .

In addition to these there are some difficulties that are specific to this system. These are of two different types, being related to:

- controlling the “extraneous” maximal derivatives: that is the highest derivatives in u that appear in the Q equation and the highest derivatives in Q that appear in the u equation,
- controlling the high powers of Q , such as $Q \operatorname{tr}(Q^2)$ in particular those that interact with u terms (such as $Q \operatorname{tr}(Q \nabla u)$).

The first difficulty is dealt with by taking into account the specific feature of the coupling that allows for the *cancellation of the worst terms*, when considering certain physically meaningful combinations of terms. This feature is explored in the next section where we revisit and revise the existence proof from [98]. In what concerns the second difficulty, this is overcome by delicate harmonic analysis arguments leading to the double logarithmic estimates mentioned before.

The work is organised as follows: in the next section we revisit the existence arguments done in cite [98], providing a slight adaptation to our case and a minor correction to one of the estimates used there. The main benefit of this section is that it exhibits in a simple setting a number of cancellations that are later-on crucial for the uniqueness argument. In the third section we start by introducing a number of technical harmonic analysis tools related to the Littlewood-Paley theory and then use them in the proof of our main result. Some standard but perhaps less-known tools, together with some more technical estimate are postponed in section 7.5.

Notations and conventions

Let $S_0^{(d)} \subset \mathbb{M}^{d \times d}$ denote the space of Q -tensors in dimension d , i.e.

$$S_0^{(d)} \stackrel{\text{def}}{=} \left\{ Q \in \mathbb{M}^{d \times d}; Q_{ij} = Q_{ji}, \operatorname{tr}(Q) = 0, i, j = 1, \dots, d \right\}$$

We use the Einstein summation convention, that is we assume summation over repeated indices.

We define the Frobenius norm of a matrix $|Q| \stackrel{\text{def}}{=} \sqrt{\operatorname{tr} Q^2} = \sqrt{Q_{\alpha\beta} Q_{\alpha\beta}}$ and define Sobolev spaces of Q -tensors in terms of this norm. For instance $H^1(\mathbb{R}^d, S_0^{(d)}) \stackrel{\text{def}}{=} \{Q : \mathbb{R}^d \rightarrow S_0^{(d)}, \int_{\mathbb{R}^d} |\nabla Q(x)|^2 + |Q(x)|^2 dx < \infty\}$ where $|\nabla Q|^2(x) \stackrel{\text{def}}{=} Q_{\alpha\beta,\gamma}(x) Q_{\alpha\beta,\gamma}(x)$ with $Q_{\alpha\beta,\gamma} \stackrel{\text{def}}{=} \partial_\gamma Q_{\alpha\beta}$. For $A, B \in S_0^{(d)}$ we denote $A : B = \operatorname{tr}(AB)$, $|A| = \sqrt{\operatorname{tr}(A^2)}$ and $\|(A, B)\|_X = \|A\|_X + \|B\|_X$, for any suitable Banach space X . We also denote $\Omega_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2} (\partial_\beta u_\alpha - \partial_\alpha u_\beta)$, $u_{\alpha,\beta} \stackrel{\text{def}}{=} \partial_\beta u_\alpha$ and $(\nabla Q \odot \nabla Q)_{ij} = Q_{\alpha\beta,i} Q_{\alpha\beta,j}$.

7.2 The energy decay, apriori estimates and scaling

In the absence of the flow, when $u = 0$ in the equations (7.5), the free energy is a Lyapunov functional of the system. If $u \neq 0$ we still have a Lyapunov functional for (7.5) but this time one that includes the kinetic energy of the system. These estimates provide as usually the basis for obtaining apriori estimates for the system. The propositions in this section show this and their proofs follow closely the ones of the similar propositions in [98] where they were done for the case $\lambda = 1$. The reason for including them is to display in relatively simple setting the cancellations that will appear again in the proof of the uniqueness theorem but in a much more complicated framework. We have:

Proposition 7.2.1. *The system (7.5) has a Lyapunov functional:*

$$E(t) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} |u|^2(t, x) dx + \int_{\mathbb{R}^d} \frac{L\lambda}{2} |\nabla Q|^2(t, x) dx + \lambda \int_{\mathbb{R}^d} \left(\frac{a}{2} \text{tr}(Q^2(t, x)) - \frac{b}{3} \text{tr}(Q^3(t, x)) + \frac{c}{4} \text{tr}^2(Q^2(t, x)) \right) dx \quad (7.9)$$

If $d = 2, 3$ and (Q, u) is a smooth solution of (7.5) such that $Q \in L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d))$ and $u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d))$ then, for all $t < T$, we have:

$$\begin{aligned} \frac{d}{dt} E(t) &= -\nu \int_{\mathbb{R}^d} |\nabla u|^2 dx \\ &\quad - \Gamma \lambda \int_{\mathbb{R}^d} \text{tr} \left(L\Delta Q - aQ + b[Q^2 - \frac{\text{tr}(Q^2)}{d} Id] - cQ \text{tr}(Q^2) \right)^2 dx \leq 0 \end{aligned} \quad (7.10)$$

Proof. We multiply the first equation in (7.5) to the right by $-\lambda H$, take the trace, integrate over \mathbb{R}^d and by parts and sum with the second equation multiplied by u and integrated over \mathbb{R}^d and by parts (let us observe that because of our assumptions on Q and u we do not have boundary terms, when integrating by parts). We obtain:

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u|^2 + \frac{L\lambda}{2} |\nabla Q|^2 + \lambda \left(\frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \right) dx \\ &+ \nu \int_{\mathbb{R}^d} |\nabla u|^2 dx + \Gamma \lambda \int_{\mathbb{R}^d} \text{tr} \left(L\Delta Q L - aQ + b[Q^2 - \frac{\text{tr}(Q^2)}{d} Id] - cQ \text{tr}(Q^2) \right)^2 dx \\ &= \lambda \underbrace{\int_{\mathbb{R}^d} u \cdot \nabla Q_{\alpha\beta} \left(-aQ_{\alpha\beta} + b[Q_{\alpha\gamma} Q_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{d} \text{tr}(Q^2)] - cQ_{\alpha\beta} \text{tr}(Q^2) \right) dx}_{\stackrel{\text{def}}{=} \mathcal{I}} \\ &+ \lambda \underbrace{\int_{\mathbb{R}^d} (-\Omega_{\alpha\gamma} Q_{\gamma\beta} + Q_{\alpha\gamma} \Omega_{\gamma\beta}) \left(-aQ_{\alpha\beta} + b[Q_{\alpha\delta} Q_{\delta\beta} - \frac{\delta_{\alpha\beta}}{d} \text{tr}(Q^2)] - cQ_{\alpha\beta} \text{tr}(Q^2) \right) dx}_{\stackrel{\text{def}}{=} \mathcal{II}} \\ &\quad - \lambda \xi \underbrace{\int_{\mathbb{R}^d} \left(Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{d} \right) D_{\gamma\beta} H_{\alpha\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{J}_1} - \lambda \xi \underbrace{\int_{\mathbb{R}^d} D_{\alpha\gamma} \left(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{d} \right) H_{\alpha\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{J}_2} \end{aligned}$$

$$\begin{aligned}
& +2\lambda\xi \underbrace{\int_{\mathbb{R}^d} (Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{d}) H_{\alpha\beta} \text{tr}(Q\nabla u) dx}_{\stackrel{\text{def}}{=} \mathcal{J}_3} + L\lambda \underbrace{\int_{\mathbb{R}^d} u_\gamma Q_{\alpha\beta,\gamma} \Delta Q_{\alpha\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{A}} \\
& \quad - \frac{L\lambda}{2} \underbrace{\int_{\mathbb{R}^d} u_{\alpha,\gamma} Q_{\gamma\beta} \Delta Q_{\alpha\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{B}} + \frac{L\lambda}{2} \underbrace{\int_{\mathbb{R}^d} u_{\gamma,\alpha} Q_{\gamma\beta} \Delta Q_{\alpha\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{C}} \\
& \quad + \frac{L\lambda}{2} \underbrace{\int_{\mathbb{R}^d} Q_{\alpha\gamma} u_{\gamma,\beta} \Delta Q_{\alpha\beta} dx}_{\mathcal{C}} - \frac{L\lambda}{2} \underbrace{\int_{\mathbb{R}^d} Q_{\alpha\gamma} u_{\beta,\gamma} \Delta Q_{\alpha\beta} dx}_{\mathcal{B}} \\
& \quad + L\lambda \underbrace{\int_{\mathbb{R}^d} Q_{\gamma\delta,\alpha} Q_{\gamma\delta,\beta} u_{\alpha,\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{AA}} - L\lambda \underbrace{\int_{\mathbb{R}^d} Q_{\alpha\gamma} \Delta Q_{\gamma\beta} u_{\alpha,\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{CC}} \\
& \quad + L\lambda \underbrace{\int_{\mathbb{R}^d} \Delta Q_{\alpha\gamma} Q_{\gamma\beta} u_{\alpha,\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{BB}} + \lambda\xi \underbrace{\int_{\mathbb{R}^d} (Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{d}) H_{\gamma\beta} u_{\alpha,\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{JJ}_1} \\
& + \lambda\xi \underbrace{\int_{\mathbb{R}^d} H_{\alpha\gamma} (Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{d}) u_{\alpha,\beta} dx}_{\stackrel{\text{def}}{=} \mathcal{JJ}_2} - 2\lambda\xi \underbrace{\int_{\mathbb{R}^d} (Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{d}) u_{\alpha,\beta} \text{tr}(QH) dx}_{\stackrel{\text{def}}{=} \mathcal{JJ}_3} \\
& = -L\lambda \underbrace{\int_{\mathbb{R}^d} u_{\alpha,\gamma} Q_{\gamma\beta} \Delta Q_{\alpha\beta} dx}_{2\mathcal{B}} + L\lambda \underbrace{\int_{\mathbb{R}^d} u_{\gamma,\alpha} Q_{\gamma\beta} \Delta Q_{\alpha\beta} dx}_{2\mathcal{C}} \\
& \quad - L\lambda \underbrace{\int_{\mathbb{R}^d} Q_{\alpha\gamma} \Delta Q_{\gamma\beta} u_{\alpha,\beta} dx}_{\mathcal{CC}} + L\lambda \underbrace{\int_{\mathbb{R}^d} \Delta Q_{\alpha\gamma} Q_{\gamma\beta} u_{\alpha,\beta} dx}_{\mathcal{BB}} = 0 \tag{7.11}
\end{aligned}$$

where $\mathcal{I} = 0$ (since $\nabla \cdot u = 0$), $\mathcal{II} = 0$ (since $Q_{\alpha\beta} = Q_{\beta\alpha}$) and for the second equality we used

$$\begin{aligned}
& \underbrace{\int_{\mathbb{R}^d} u_\gamma Q_{\alpha\beta,\gamma} \Delta Q_{\alpha\beta} dx}_{\mathcal{A}} + \underbrace{\int_{\mathbb{R}^d} Q_{\gamma\delta,\alpha} Q_{\gamma\delta,\beta} u_{\alpha,\beta} dx}_{\mathcal{AA}} = \int_{\mathbb{R}^d} u_\gamma Q_{\alpha\beta,\gamma} \Delta Q_{\alpha\beta} dx \\
& - \int_{\mathbb{R}^d} Q_{\gamma\delta,\alpha} Q_{\gamma\delta,\beta\beta} u_\alpha dx - \int_{\mathbb{R}^d} Q_{\gamma\delta,\alpha\beta} Q_{\gamma\delta,\beta} u_\alpha dx = \int_{\mathbb{R}^d} \frac{1}{2} Q_{\gamma\delta,\beta} Q_{\gamma\delta,\beta} u_{\alpha,\alpha} dx = 0
\end{aligned}$$

together with $Q_{\alpha\alpha} = H_{\alpha\alpha} = u_{\alpha,\alpha} = 0$, $\mathcal{J}_3 = \mathcal{JJ}_3$ and

$$\begin{aligned}
\mathcal{J}_1 + \mathcal{J}_2 & = \int_{\mathbb{R}^d} \frac{1}{2} Q_{\alpha\gamma} u_{\gamma,\beta} H_{\alpha\beta} + \frac{1}{2} Q_{\alpha\gamma} u_{\beta,\gamma} H_{\alpha\beta} dx \\
& + \int_{\mathbb{R}^d} \frac{1}{2} u_{\alpha,\gamma} Q_{\gamma\beta} H_{\alpha\beta} + \frac{1}{2} u_{\gamma,\alpha} Q_{\gamma\beta} H_{\alpha\beta} dx + \frac{2}{d} \int_{\mathbb{R}^d} D_{\alpha\beta} H_{\alpha\beta} dx \\
& = \int_{\mathbb{R}^d} \frac{1}{2} (Q_{\alpha\gamma} u_{\gamma,\beta} H_{\alpha\beta} + u_{\gamma,\alpha} Q_{\gamma\beta} H_{\alpha\beta}) + \frac{1}{2} (Q_{\alpha\gamma} u_{\beta,\gamma} H_{\alpha\beta} + u_{\alpha,\gamma} Q_{\gamma\beta} H_{\alpha\beta}) dx \\
& \quad + \frac{1}{d} \int_{\mathbb{R}^d} (u_{\alpha,\beta} + u_{\beta,\alpha}) H_{\alpha\beta} dx = \int_{\mathbb{R}^d} H_{\beta\alpha} Q_{\alpha\gamma} u_{\gamma,\beta} + Q_{\gamma\alpha} H_{\alpha\beta} u_{\beta,\gamma} dx \\
& \quad + \frac{2}{d} \int_{\mathbb{R}^d} u_{\alpha,\beta} H_{\alpha\beta} dx = \mathcal{JJ}_1 + \mathcal{JJ}_2.
\end{aligned}$$

Finally, the last equality in (7.11) is a consequence of the straightforward identities $2\mathcal{B} + \mathcal{B}\mathcal{B} = 2\mathcal{C} + \mathcal{C}\mathcal{C} = 0$. \square

It can be easily checked that the system has a scaling, namely we have:

Lemma 7.2.2. *Let (Q, u, p) be a solution of (7.5). Then letting*

$$u_\delta(t, x) \stackrel{\text{def}}{=} \delta u(\delta x, \delta^2 t), \quad Q_\delta(t, x) \stackrel{\text{def}}{=} Q(\delta x, \delta^2 t), \quad p_\delta(t, x) \stackrel{\text{def}}{=} \delta^2 p(\delta x, \delta^2 t) \quad (7.12)$$

we have that $(Q_\delta, u_\delta, p_\delta)$ satisfy (7.5) with $F(Q) = -aQ + b[Q^2 - \frac{\text{tr}(Q^2)}{d}Id] - cQ\text{tr}(Q^2)$ replaced by $F_\delta(Q_\delta) = \delta^2 \left[-aQ_\delta + b[(Q_\delta)^2 - \frac{\text{tr}(Q_\delta^2)}{d}] - cQ_\delta\text{tr}(Q_\delta^2) \right]$. We note that, in dimension two, the space $\dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ is invariant by the scaling.

In the following we assume that there exists a smooth solution of (7.5) and obtain estimates on the behaviour of various norms.

Proposition 7.2.3. *Let (Q, u) be a smooth solution of (7.5) in dimension $d = 2$ or $d = 3$, with restriction (7.2), and smooth initial data $(\bar{Q}(x), \bar{u}(x))$, that decays fast enough at infinity so that we can integrate by parts in space (for any $t \geq 0$) without boundary terms. We assume that $|\xi| < \xi_0$ where ξ_0 is an explicitly computable constant, scale invariant, depending on $a, b, c, d, \Gamma, \nu, \lambda$.*

For $(\bar{Q}, \bar{u}) \in H^1 \times L_x^2$, we have

$$\|Q(t, \cdot)\|_{H^1} \leq C_1 + \bar{C}_1 e^{\bar{C}_1 t} \|\bar{Q}\|_{H^1}, \forall t \geq 0 \quad (7.13)$$

with C_1, \bar{C}_1 depending on $(a, b, c, d, \Gamma, L, \nu, \bar{Q}, \bar{u})$. Moreover

$$\|u(t, \cdot)\|_{L_x^2}^2 + \nu \int_0^t \|\nabla u\|_{L_x^2}^2 \leq C_1. \quad (7.14)$$

Proof. We denote:

$$X_{\alpha\beta} \stackrel{\text{def}}{=} L\Delta Q_{\alpha\beta} - cQ_{\alpha\beta}\text{tr}(Q^2), \quad \alpha, \beta = 1, 2, 3.$$

Then equation (7.10) becomes:

$$\begin{aligned} & \frac{d}{dt} E(t) + \nu \|\nabla u\|_{L_x^2}^2 + \Gamma\lambda L^2 \|\Delta Q\|_{L_x^2}^2 + \Gamma\lambda c^2 \|Q\|_{L^6}^6 \\ & - 2cL\Gamma\lambda \int_{\mathbb{R}^d} \Delta Q_{\alpha\beta} Q_{\alpha\beta} \text{tr}(Q^2) dx + a^2\Gamma\lambda \|Q\|_{L_x^2}^2 + b^2\Gamma\lambda \int_{\mathbb{R}^d} \text{tr} \left(Q^2 - \frac{\text{tr}(Q^2)}{d} \right)^2 dx \\ & \leq 2a\Gamma\lambda \underbrace{\int_{\mathbb{R}^d} \text{tr}(XQ) dx}_{\stackrel{\text{def}}{=} \mathcal{I}} - 2b\Gamma\lambda \underbrace{\int_{\mathbb{R}^d} \text{tr}(XQ^2) dx}_{\stackrel{\text{def}}{=} \mathcal{J}} + 2ab\Gamma\lambda \int_{\mathbb{R}^d} \text{tr}(Q^3) dx. \end{aligned} \quad (7.15)$$

Integrating by parts we have:

$$\begin{aligned} -2cL\Gamma\lambda \int_{\mathbb{R}^d} \Delta Q_{\alpha\beta} Q_{\alpha\beta} \text{tr}(Q^2) dx &= 2cL\Gamma\lambda \int_{\mathbb{R}^d} Q_{\alpha\beta, k} Q_{\alpha\beta, k} \text{tr}(Q^2) dx \\ & \quad + 2cL\Gamma\lambda \int_{\mathbb{R}^d} Q_{\alpha\beta, k} Q_{\alpha\beta} \partial_k (\text{tr}(Q^2)) dx \\ &= 2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q|^2 \text{tr}(Q^2) dx + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla (\text{tr}(Q^2))|^2 dx \geq 0 \end{aligned} \quad (7.16)$$

(where for the last inequality we used the assumption (7.2) and $L, \Gamma, \lambda > 0$). One can easily see that

$$\mathcal{I} = -\frac{L}{2}\|\nabla Q\|_{L_x^2}^2 - c\|Q\|_{L^4}^4. \quad (7.17)$$

On the other hand, for any $\varepsilon > 0$ and $\tilde{C} = \tilde{C}(\varepsilon, c)$ an explicitly computable constant, we have:

$$\begin{aligned} \mathcal{J} &= L \int_{\mathbb{R}^d} Q_{\alpha\beta, kk} Q_{\alpha\gamma} Q_{\gamma\beta} dx - c \int_{\mathbb{R}^d} \text{tr}(Q^2) \text{tr}(Q^3) dx \leq -L \int_{\mathbb{R}^d} Q_{\alpha\beta, k} Q_{\alpha\gamma, k} Q_{\gamma\beta} dx \\ &\quad - L \int_{\mathbb{R}^d} Q_{\alpha\beta, k} Q_{\alpha\gamma} Q_{\gamma\beta, k} dx + \int_{\mathbb{R}^d} \text{tr}(Q^2) \left(\frac{\tilde{C}}{\varepsilon} \text{tr}(Q^2) + \varepsilon \text{tr}^2(Q^2) \right) dx \\ &\leq L\varepsilon \int_{\mathbb{R}^d} |\nabla Q|^2 \text{tr}(Q^2) dx + \frac{\tilde{C}}{\varepsilon} \|\nabla Q\|_{L_x^2}^2 + \int_{\mathbb{R}^d} \text{tr}(Q^2) \left(\frac{\tilde{C}}{\varepsilon} \text{tr}(Q^2) + \varepsilon \text{tr}^2(Q^2) \right) dx \end{aligned}$$

Using the last three relations in (7.15) we obtain:

$$\begin{aligned} \frac{d}{dt} E(t) + \nu \|\nabla u\|_{L_x^2}^2 + \Gamma \lambda L^2 \|\Delta Q\|_{L_x^2}^2 + c^2 \Gamma \lambda \|Q\|_{L^6}^6 + a^2 \Gamma \lambda \|Q\|_{L_x^2}^2 \\ + 2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q|^2 \text{tr}(Q^2) dx + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla(\text{tr}(Q^2))|^2 dx \\ \leq 2|a|\Gamma\lambda \left(\frac{L}{2} \|\nabla Q\|_{L_x^2}^2 + c\|Q\|_{L^4}^4 \right) + 2|b|\Gamma\lambda L\varepsilon \int_{\mathbb{R}^d} |\nabla Q|^2 \text{tr}(Q^2) dx \\ + 2|b|\Gamma\lambda \frac{\tilde{C}}{\varepsilon} \|\nabla Q\|_{L_x^2}^2 + 2|b|\Gamma\lambda \int_{\mathbb{R}^d} \text{tr}(Q^2) \left(\frac{\tilde{C}}{\varepsilon} \text{tr}(Q^2) + \varepsilon \text{tr}^2(Q^2) \right) dx \\ + 2|ab|\Gamma\lambda (\varepsilon \|Q\|_{L_x^2}^2 + \frac{\tilde{C}}{\varepsilon} \|Q\|_{L^4}^4) \end{aligned}$$

Taking ε small enough we can absorb all the terms with an epsilon coefficient on the right into the left hand side, and we are left with

$$\begin{aligned} \frac{d}{dt} E(t) + \nu \|\nabla u\|_{L_x^2}^2 + \Gamma \lambda L^2 \|\Delta Q\|_{L_x^2}^2 + \Gamma \lambda c^2 \|Q\|_{L^6}^6 \\ + \Gamma \lambda a^2 \|Q\|_{L_x^2}^2 + 2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q|^2 \text{tr}(Q^2) dx \\ + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla(\text{tr}(Q^2))|^2 dx \leq \bar{C} \left(\|\nabla Q\|_{L_x^2}^2 + \|Q\|_{L^4}^4 \right), \end{aligned} \quad (7.18)$$

with $\bar{C} = \bar{C}(a, b, c)$.

The last relation is not yet enough because the Q terms without derivatives in $E(t)$ are not summing to a positive number. However, let us note that, if $a > 0$ we obtain the a-priori estimates by using the inequality $\text{tr}(Q^3) \leq \frac{3}{8} \text{tr}(Q^2) + \text{tr}(Q^2)^2$. If $a \leq 0$ we have to estimate separately $\|Q\|_{L_x^2}$ and this ask for a smallness condition for ξ .

We need to control in some sense low frequencies of Q . To this end, we multiply the first equation in (7.5) by Q , take the trace, integrate over \mathbb{R}^d and by parts and we obtain:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |Q|^2(t, x) dx = \Gamma \left(-L \int_{\mathbb{R}^d} |\nabla Q|^2 dx - a \int_{\mathbb{R}^d} |Q(x)|^2 dx + \right.$$

$$\begin{aligned}
& +b \int_{\mathbb{R}^d} \operatorname{tr}(Q^3)dx - c \int_{\mathbb{R}^d} |Q|^4 dx) + \underbrace{\int_{\mathbb{R}^d} \operatorname{tr}(\Omega Q^2 - Q\Omega Q)dx}_{\stackrel{\text{def}}{=} \mathcal{I}} \\
& + \xi \underbrace{\int_{\mathbb{R}^d} D_{\alpha\gamma}(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{d})Q_{\alpha\beta} + (Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{d})D_{\gamma\beta}Q_{\alpha\beta} - 2(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{d})Q_{\alpha\beta}\operatorname{tr}(Q\nabla u)dx}_{\stackrel{\text{def}}{=} \mathcal{II}}.
\end{aligned}$$

Recalling that Q is symmetric we have $\mathcal{I} = 0$. Also:

$$\begin{aligned}
|\mathcal{II}| &= |2\xi| \left| \int_{\mathbb{R}^d} \frac{1}{d} D_{\alpha\beta}Q_{\alpha\beta} + D_{\alpha\gamma}Q_{\gamma\beta}Q_{\beta\alpha} - Q_{\alpha\beta}Q_{\alpha\beta}\operatorname{tr}(Q\nabla u)dx \right| \\
&\leq C(d) \int_{\mathbb{R}^d} \varepsilon |\nabla u|^2 dx + \int_{\mathbb{R}^d} \frac{|\xi|^2}{\varepsilon} (|Q|^2 + |Q|^6) dx
\end{aligned}$$

Thus we get:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |Q|^2 dx \leq C(d)\varepsilon \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{|\xi|^2}{\varepsilon} \int_{\mathbb{R}^d} |Q|^2 + |Q|^6 dx + \hat{C} \int_{\mathbb{R}^d} |Q|^2 + |Q|^4 dx \quad (7.19)$$

with $\hat{C} = \hat{C}(a, b) > 0$. Let us observe now that there exists $M = M(a, b, c)$ large enough, so that

$$\frac{M}{2} \operatorname{tr}(Q^2) + \frac{c}{8} \operatorname{tr}^2(Q^2) \leq (M + \frac{a}{2}) \operatorname{tr}(Q^2) - \frac{b}{3} \operatorname{tr}(Q^3) + \frac{c}{4} \operatorname{tr}^2(Q^2) \quad (7.20)$$

for any $Q \in S_0$. Multiplying the equation (7.19) by M and adding to (7.18) we obtain:

$$\begin{aligned}
& \frac{d}{dt} (E(t) + M\|Q\|_{L_x^2}^2) + \nu \|\nabla u\|_{L_x^2}^2 + \Gamma\lambda L^2 \|\Delta Q\|_{L_x^2}^2 + \Gamma\lambda c^2 \|Q\|_{L^6}^6 + a^2 \|Q\|_{L_x^2}^2 \\
& + 2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q|^2 \operatorname{tr}(Q^2) dx + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla (\operatorname{tr}(Q^2))|^2 dx \\
& \leq \bar{C} \left(\|\nabla Q\|_{L_x^2}^2 + \|Q\|_{L^4}^4 \right) + MC(d)\varepsilon \int_{\mathbb{R}^d} |\nabla u|^2 dx \\
& + \frac{M|\xi|^2}{\varepsilon} \int_{\mathbb{R}^d} |Q|^2 + |Q|^6 dx + M\hat{C} \int_{\mathbb{R}^d} |Q|^2 + |Q|^4 dx
\end{aligned} \quad (7.21)$$

We chose ε small enough so that $MC(d)\varepsilon < \nu$. Finally we make the assumption that $|\xi|$ is small enough, depending on a, b, c, d, ν so that $\frac{M|\xi|^2}{\varepsilon} \leq \Gamma\lambda c^2$. Then taking into account equation (7.20) we obtain the claimed relation (7.13). \square

We note that the ξ small hypothesis is necessary because we are in infinite domain, for example, in the periodic domain, we can add a constant to the functional and get the a priori L^p estimates without any smallness condition on ξ .

7.3 The existence of weak solutions

The next proposition follows closely the similar result in [98] where it was done for $\lambda = 1$. The purpose for including it here is to provide an alternative approximation system thus correcting the proof in [98] and also to show how the cancellations that appeared previously in the derivation of the energy law still survive at the approximate level but with some differences, phenomenon which

will appear in a much more complex setting in the proof of uniqueness in the next section.

Proposition 7.3.1. *For $d = 2, 3$ there exists a weak solution (Q, u) of the system (7.5) subject to initial conditions (7.6). The solution (Q, u) is such that $Q \in L_{loc}^\infty(\mathbb{R}_+; H^1) \cap L_{loc}^2(\mathbb{R}_+; H^2)$ and $u \in L_{loc}^\infty(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^1)$.*

Proof. As first step of the construction of weak solutions for the system (7.5) we construct for any fixed $\varepsilon > 0$ a global weak solution

$$Q_\varepsilon \in L_{loc}^\infty(\mathbb{R}_+; H^1) \cap L_{loc}^2(\mathbb{R}_+; H^2), \quad u_\varepsilon \in L_{loc}^\infty(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^1)$$

for the modified system obtained by mollifying the coefficients of the equation for the Q tensor and by adding to the equation of the velocity a regularizing term. This term is needed in order to estimate some "bad" terms which does not disappear in an energy estimate. For the simplicity of the notations, we drop the indices ε and we denote the solution $(Q_\varepsilon, u_\varepsilon)$ by (Q, u) .

$$\left\{ \begin{array}{l} \partial_t Q + (R_\varepsilon u) \nabla Q - \left((R_\varepsilon (\xi D + \Omega)) (Q + \frac{1}{d} Id) \right) \\ - \left((Q + \frac{1}{d} Id) R_\varepsilon (\xi D - \Omega) \right) \\ + 2\xi \left((Q + \frac{1}{d} Id) \text{tr}(Q \nabla R_\varepsilon u) \right) = \Gamma H \\ \partial_t u + (R_\varepsilon u) \nabla u - \nu \Delta u + \nabla p = -\varepsilon \mathcal{P} R_\varepsilon \left(\sum_{l,m=1}^d \nabla Q_{lm} (R_\varepsilon u \cdot \nabla Q_{lm}) |R_\varepsilon u \nabla Q| \right) \\ + \varepsilon \mathcal{P} \nabla \cdot R_\varepsilon \left(\nabla R_\varepsilon u |\nabla R_\varepsilon u|^2 \right) - \lambda \xi \nabla \cdot R_\varepsilon \left((Q + \frac{1}{d} Id) H \right) - \xi \mathcal{P} \nabla \cdot R_\varepsilon \left(H (Q + \frac{1}{d} Id) \right) \\ + 2\lambda \xi \nabla \cdot R_\varepsilon \left((Q + \frac{1}{d} Id) (QH) \right) - L \lambda R_\varepsilon (\nabla \cdot \text{tr}(\nabla Q \nabla Q)) \\ + L \lambda \mathcal{P} \nabla \cdot R_\varepsilon (Q \Delta Q - \Delta Q Q) \\ (Q, u)|_{t=0} = (R_\varepsilon \bar{Q}, R_\varepsilon \bar{u}). \end{array} \right.$$

where R_ε is the convolution operator with the kernel $\varepsilon^{-d} \chi(\varepsilon^{-1} \cdot)$.

In order to construct the global weak solution for this system, we use the classical Friedrich's scheme. We define the mollifying operator

$$\widehat{J_n f}(\xi) \stackrel{\text{def}}{=} 1_{\{2^{-n} \leq |\xi| \leq 2^n\}} \hat{f}(\xi).$$

We consider the approximating system:

$$\left\{ \begin{array}{l} \partial_t Q^{(n)} + J_n \left(R_\varepsilon J_n u^n \nabla J_n Q^{(n)} \right) - J_n \left((\xi J_n R_\varepsilon D^{(n)} + J_n R_\varepsilon \Omega^{(n)}) (J_n Q^{(n)} + \frac{1}{d} Id) \right) \\ - J_n \left((J_n Q^{(n)} + \frac{1}{d} Id) (\xi J_n R_\varepsilon D^{(n)} - J_n R_\varepsilon \Omega^{(n)}) \right) \\ + 2\xi J_n \left((J_n Q^{(n)} + \frac{1}{d} Id) \text{tr}(J_n Q^{(n)} \nabla J_n R_\varepsilon u^{(n)}) \right) = \Gamma \tilde{H}^{(n)} \\ \partial_t u^n + \mathcal{P} J_n (\mathcal{P} J_n R_\varepsilon u^n \nabla \mathcal{P} J_n u^n) - \nu \Delta \mathcal{P} J_n u^{(n)} = \\ - \varepsilon \mathcal{P} J_n R_\varepsilon \left(\sum_{l,m=1}^d \nabla J_n Q_{lm}^{(n)} \left(R_\varepsilon J_n u^n \cdot \nabla J_n Q_{lm}^{(n)} \right) |R_\varepsilon J_n u^n \nabla J_n Q^{(n)}| \right) \\ + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon \left(\nabla R_\varepsilon J_n u^{(n)} |\nabla R_\varepsilon J_n u^{(n)}|^2 \right) \\ - \lambda \xi \mathcal{P} \nabla \cdot J_n \left((J_n Q^{(n)} + \frac{1}{d} Id) \tilde{H}^{(n)} \right) - \lambda \xi \mathcal{P} \nabla \cdot J_n \left(\tilde{H}^{(n)} (J_n Q^{(n)} + \frac{1}{d} Id) \right) \\ + 2\lambda \xi \mathcal{P} \nabla \cdot J_n \left((J_n Q^{(n)} + \frac{1}{d} Id) (J_n Q^{(n)} \tilde{H}^{(n)}) \right) - L \lambda \mathcal{P} J_n (\nabla \cdot \text{tr}(J_n Q^{(n)} \nabla J_n Q^{(n)})) \\ + L \lambda \mathcal{P} \nabla \cdot J_n (J_n Q^{(n)} \Delta J_n Q^{(n)} - \Delta J_n Q^{(n)} J_n Q^{(n)}) \end{array} \right. \quad (7.22)$$

where \mathcal{P} denotes the Leray projector onto divergence-free vector fields, M is a positive constant, and $\tilde{H}^{(n)} \stackrel{\text{def}}{=} L\Delta J_n Q^{(n)} - aJ_n Q^{(n)} + bJ_n[(J_n Q^{(n)} J_n Q^{(n)}) - \frac{\text{tr}(J_n Q^{(n)} J_n Q^{(n)})}{d} Id] - cJ_n \left(J_n Q^{(n)} |J_n Q^{(n)}|^2 \right)$.

We take as initial data $(J_n R_\varepsilon \bar{Q}, J_n R_\varepsilon \bar{u})$.

The system above can be regarded as an ordinary differential equation in L^2 verifying the conditions of the Cauchy-Lipschitz theorem. Thus it admits a unique maximal solution $(Q^{(n)}, u^{(n)}) \in C^1([0, T_n]; L^2(\mathbb{R}^d; \mathbb{R}^{d \times d}) \times L^2(\mathbb{R}^d, \mathbb{R}^d))$. As we have $(\mathcal{P} J_n)^2 = \mathcal{P} J_n$ and $J_n^2 = J_n$ the pair $(J_n Q^{(n)}, \mathcal{P} J_n u^{(n)})$ is also a solution of (7.22). By uniqueness we have $(J_n Q^{(n)}, \mathcal{P} J_n u^{(n)}) = (Q^{(n)}, u^{(n)})$ hence $(Q^{(n)}, u^{(n)}) \in C^1([0, T_n], H^\infty)$ and $(Q^{(n)}, u^{(n)})$ satisfy the system:

$$\left\{ \begin{array}{l} \partial_t Q^{(n)} + J_n \left(R_\varepsilon u^n \nabla Q^{(n)} \right) - J_n \left((\xi R_\varepsilon D^{(n)} + R_\varepsilon \Omega^{(n)}) (Q^{(n)} + \frac{1}{d} Id) \right) \\ - J_n \left((Q^{(n)} + \frac{1}{d} Id) (\xi R_\varepsilon D^{(n)} - R_\varepsilon \Omega^{(n)}) \right) + 2\xi J_n \left((Q^{(n)} + \frac{1}{d} Id) \text{tr}(Q^{(n)} \nabla R_\varepsilon u^n) \right) = \Gamma \bar{H}^{(n)} \\ \partial_t u^n + \mathcal{P} J_n (R_\varepsilon u^n \nabla u^n) - \nu \Delta u^{(n)} = \\ - \varepsilon \mathcal{P} J_n \left(\sum_{l,m=1}^d \nabla Q_{lm}^{(n)} \left(R_\varepsilon u^n \cdot \nabla Q_{lm}^{(n)} \right) |R_\varepsilon u^n \nabla Q^{(n)}| \right) \\ + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon \left(\nabla R_\varepsilon u^{(n)} | \nabla R_\varepsilon u^{(n)}|^2 \right) \\ - \lambda \xi \mathcal{P} \nabla \cdot J_n \left((Q^{(n)} + \frac{1}{d} Id) \bar{H}^{(n)} \right) - \lambda \xi \mathcal{P} \nabla \cdot J_n \left(\bar{H}^{(n)} (Q^{(n)} + \frac{1}{d} Id) \right) \\ + 2\lambda \xi \mathcal{P} \nabla \cdot J_n \left((Q^{(n)} + \frac{1}{d} Id) (Q^{(n)} \bar{H}^{(n)}) \right) \\ - L\lambda \mathcal{P} J_n (\nabla \cdot \text{tr}(\nabla Q^{(n)} \nabla Q^{(n)})) + L\lambda \mathcal{P} \nabla \cdot J_n (Q^{(n)} \Delta Q^{(n)} - \Delta Q^{(n)} Q^{(n)}) \end{array} \right. \quad (7.23)$$

where

$$\bar{H}^{(n)} \stackrel{\text{def}}{=} L\Delta Q^{(n)} - aQ^{(n)} + bJ_n[(Q^{(n)} Q^{(n)}) - \frac{\text{tr}(J_n(Q^{(n)} Q^{(n)}))}{d} Id] - cJ_n(Q^{(n)} |Q^{(n)}|^2).$$

The initial data is $(J_n \bar{Q}, J_n \bar{u})$. We recall now a few properties of J_n :

Lemma 7.3.2. *The operators \mathcal{P} and J_n are selfadjoint in L^2 . Moreover J_n and $\mathcal{P} J_n$ are also idempotent and J_n commutes with distributional derivatives.*

We proceed in a manner analogous to the proof of Proposition 7.2.1 and multiply the first equation in (7.23) by $-\lambda \bar{H}^{(n)}$, take the trace, integrate over \mathbb{R}^d and by parts, and add to the second equation multiplied by $u^{(n)}$. Let us observe that almost all the cancellations in the proof of (7.2.1) hold, except for a few terms that need to be estimated separately. We also have some more new terms that we added in the regularization, terms that control the ones which do not cancel. Thus we have:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \text{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx \\ & + \nu \int_{\mathbb{R}^d} |\nabla u^n|^2 dx + \Gamma \lambda \int_{\mathbb{R}^d} \text{tr} \left[J_n \left(L\Delta Q^{(n)} - aQ^{(n)} + b[(Q^{(n)})^2 \right. \right. \\ & \quad \left. \left. - \frac{\text{tr}((Q^{(n)})^2)}{3} Id] - cQ^{(n)} |Q^{(n)}|^2 \right) \right]^2 dx \\ & + \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon u \nabla Q^{(n)}|^3 dx + \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon \nabla u^n|^4 dx \leq \end{aligned}$$

$$\begin{aligned}
&\leq \lambda \int_{\mathbb{R}^d} J_n \left(R_\varepsilon u^n \cdot \nabla Q_{\alpha\beta}^{(n)} \right) J_n \left(bQ_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx \\
&+ \lambda \int_{\mathbb{R}^d} J_n \left(-R_\varepsilon \Omega_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} + Q_{\alpha\gamma}^{(n)} R_\varepsilon \Omega_{\gamma\beta}^{(n)} \right) J_n \left(bQ_{\alpha\delta}^{(n)} Q_{\delta\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx \quad (7.24)
\end{aligned}$$

hence :

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \text{tr}(Q^{(n)})^3 \right. \\
&+ \frac{c}{4} |Q^{(n)}|^4 \Big) dx + \nu \int_{\mathbb{R}^d} |\nabla u^n|^2 dx + \Gamma \lambda \int_{\mathbb{R}^d} L^2 |\Delta Q^{(n)}|^2 dx + \Gamma \lambda a^2 \int_{\mathbb{R}^d} |Q^{(n)}|^2 dx \\
&+ C(b^2, d, \Gamma, \lambda) \int_{\mathbb{R}^d} |Q^{(n)}|^4 dx + \Gamma \lambda c^2 \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx \\
&+ \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon u \nabla Q^{(n)}|^3 dx + \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon \nabla u^n|^4 dx \leq \underbrace{2\Gamma \lambda c \int_{\mathbb{R}^d} L \Delta Q^{(n)} \cdot Q^{(n)} |Q^{(n)}|^2 dx}_{\stackrel{\text{def}}{=} \mathcal{I}} \\
&- 2\Gamma \lambda \int_{\mathbb{R}^d} L \Delta Q^{(n)} \cdot \left(-aQ^{(n)} + bJ_n \left[(Q^{(n)})^2 - \frac{\text{tr}(Q^{(n)})^2}{d} Id \right] \right) dx \\
&- 2\Gamma \lambda \int_{\mathbb{R}^d} cQ^{(n)} |Q^{(n)}|^2 \cdot \left(aQ^{(n)} - bJ_n \left[(Q^{(n)})^2 - \frac{\text{tr}(Q^{(n)})^2}{d} Id \right] \right) dx \\
&+ \lambda \underbrace{\int_{\mathbb{R}^d} J_n \left(R_\varepsilon u^n \cdot \nabla Q_{\alpha\beta}^{(n)} \right) J_n \left(bQ_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx}_{\stackrel{\text{def}}{=} \mathcal{II}} \\
&+ C \int_{\mathbb{R}^d} |R_\varepsilon \nabla u^n|^2 |Q^{(n)}|^2 dx + \frac{\Gamma c^2}{8} \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx + C \int_{\mathbb{R}^d} |Q^{(n)}|^4 dx.
\end{aligned}$$

We have that

$$\begin{aligned}
\mathcal{II} &= \int_{\mathbb{R}^d} \left(R_\varepsilon u^n \cdot \nabla Q_{\alpha\beta}^{(n)} \right) J_n \left(bQ_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx \\
&\leq \left(\frac{4}{\Gamma c^2} + \frac{1}{4C(b^2, d, \Gamma)} \right) \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^2 dx \\
&+ \frac{C(b^2, d, \Gamma)}{2} \|Q^{(n)}\|_{L^4}^4 + \frac{\Gamma c^2}{8} \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx \\
&\leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^3 dx + C(\varepsilon, b^2, c^2, d, \Gamma) \int_{\mathbb{R}^d} \sum_{l,m=1}^d |R_\varepsilon u^n \cdot \nabla Q_{lm}^{(n)}| dx \\
&+ \frac{C(b^2, c^2, d, \Gamma)}{2} \|Q^{(n)}\|_{L^4}^4 + \frac{\Gamma c^2}{8} \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx \\
&\leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^3 dx + C_1(\varepsilon, b^2, c^2, d, \Gamma) \int_{\mathbb{R}^d} |u^n|^2 dx \\
&+ C_2(\varepsilon, b, c, d^2, \Gamma) \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 dx + \frac{C(b^2, d, \Gamma)}{2} \|Q^{(n)}\|_{L^4}^4 \\
&+ \frac{\Gamma c^2}{8} \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx \quad (7.25)
\end{aligned}$$

Using the fact that $\mathcal{I} \leq 0$ and the estimate for \mathcal{II} shown before, we replace in (7.24) and obtain:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \text{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx \\ & + \nu \int_{\mathbb{R}^d} |\nabla u^n|^2 dx + \Gamma \lambda \int_{\mathbb{R}^d} L^2 |\Delta Q^{(n)}|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^3 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla R_\varepsilon u^n|^4 dx \\ & \leq \int_{\mathbb{R}^d} |Q^{(n)}|^2 + |Q^{(n)}|^4 dx + C \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 dx + C(\varepsilon) \int_{\mathbb{R}^d} |u^{(n)}|^2 dx. \end{aligned}$$

This estimate does not readily provide bounds on $Q^{(n)}$ because the term $\frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\text{tr}(Q^{(n)})^3 + \frac{c}{4}|Q^{(n)}|^4$ could be negative. In order to obtain H^1 estimates we proceed as in the proof of Proposition 7.2.3. We put the proof in the chapter 7.5 by Proposition 7.5.1. We can continue to proceed as in the proof of Proposition 7.2.3 and in fact in this case because of the first two regularizing terms on the right hand side of the u^n equation in (7.23) we do not need the ξ small assumption. These estimates allow us to conclude that $T_n = \infty$ and we also get the following apriori bounds:

$$\begin{aligned} \sup_n \|\nabla R_\varepsilon u^n\|_{L^4(0,T;L^4)}, \sup_n \|R_\varepsilon u^n \cdot \nabla Q^{(n)}\|_{L^3(0,T;L^3)} &\leq C(\varepsilon) \\ \sup_n \|Q^{(n)}\|_{L^2(0,T;H^2) \cap L^\infty(0,T;H^1)} &< \infty, \\ \sup_n \|u^n\|_{L^\infty(0,T;L^2) \cap L^2(0,T;H^1)} &< \infty, \end{aligned} \quad (7.26)$$

for any $T < \infty$. By the bounds which can be obtained by using the equation on $\partial_t(Q^{(n)}, u^n)$ in some $L^\infty_{loc}(H^{-N})$ for large enough N , we get, by classical local compactness Aubin-Lions lemma, on a subsequence, that:

$$\begin{aligned} Q^{(n)} &\rightharpoonup Q \text{ in } L^2(0, T; H^2) \text{ and } Q^{(n)} \rightarrow Q \text{ in } L^2(0, T; H_{loc}^{2-\delta}), \forall \delta > 0 \\ Q^{(n)}(t) &\rightharpoonup Q(t) \text{ in } H^1 \text{ for all } t \in \mathbb{R}_+ \\ u^n &\rightharpoonup u \text{ in } L^2(0, T; H^1) \text{ and } u^n \rightarrow u \text{ in } L^2(0, T; H_{loc}^{1-\delta}), \forall \delta > 0 \\ u^n(t) &\rightharpoonup u(t) \text{ in } L^2 \text{ for all } t \in \mathbb{R}_+ \end{aligned}$$

Thus we can pass to the limit and obtain a weak solution of the approximating system:

$$\left\{ \begin{aligned} & \partial_t Q^{(\varepsilon)} + R_\varepsilon u^\varepsilon \nabla Q^{(\varepsilon)} - (\xi R_\varepsilon D^\varepsilon + R_\varepsilon \Omega^\varepsilon)(Q^{(\varepsilon)} + \frac{1}{d} Id) + \left((Q^{(\varepsilon)} + \frac{1}{d} Id)(\xi R_\varepsilon D^\varepsilon - R_\varepsilon \Omega^\varepsilon) \right) \\ & - 2\xi \left((Q^{(\varepsilon)} + \frac{1}{d} Id) \text{tr}(Q^{(\varepsilon)} \nabla u^\varepsilon) \right) = \Gamma H^\varepsilon \\ & \partial_t u^\varepsilon + \mathcal{P} R_\varepsilon u^\varepsilon \nabla u^\varepsilon = -\varepsilon \mathcal{P} R_\varepsilon \left(\sum_{l,m=1}^d \nabla Q_{lm} (R_\varepsilon u \cdot \nabla Q_{lm}) |R_\varepsilon u \nabla Q| \right) \\ & + \varepsilon \mathcal{P} \nabla \cdot R_\varepsilon \left(R_\varepsilon \nabla u |R_\varepsilon \nabla u|^2 \right) - \lambda \xi \mathcal{P} \nabla \cdot R_\varepsilon \left((Q^{(\varepsilon)} + \frac{1}{d} Id) H^\varepsilon \right) \\ & - \lambda \xi \mathcal{P} \nabla \cdot R_\varepsilon \left(H^\varepsilon (Q^{(\varepsilon)} + \frac{1}{d} Id) \right) + 2\lambda \xi \mathcal{P} \nabla \cdot R_\varepsilon \left((Q^{(\varepsilon)} + \frac{1}{d} Id) (Q^{(\varepsilon)} H^\varepsilon) \right) \\ & - L\lambda \mathcal{P} (\nabla \cdot R_\varepsilon \text{tr}(\nabla Q^{(\varepsilon)} \odot \nabla Q^{(\varepsilon)})) + L\lambda \mathcal{P} \nabla \cdot R_\varepsilon (Q^{(\varepsilon)} \Delta Q^{(\varepsilon)} - \Delta Q^{(\varepsilon)} Q^{(\varepsilon)}) + \nu \Delta u^\varepsilon \end{aligned} \right. \quad (7.27)$$

where we recall that $H = L\Delta Q^{(\varepsilon)} - aQ^{(\varepsilon)} + b[(Q^{(\varepsilon)})^2 - \frac{\text{tr}((Q^{(\varepsilon)})^2)}{d} Id] - cQ^{(\varepsilon)} \text{tr}((Q^{(\varepsilon)})^2)$. The initial data for the limit system is $(R_\varepsilon \bar{Q}, R_\varepsilon \bar{u})$.

One can easily see that the solutions of (7.27) are smooth, first by obtaining C^∞ regularity for the first Q equations, by bootstrapping the regularity improvement provided by the linear heat

equation, and then the regularity for the u equation, by bootstrapping the regularity improvement provided by a linear advection equation. For this system we can proceed as in the case of apriori estimates and obtain the same estimates, independent of ε because the solutions are smooth and all the cancellations that were used in the apriori estimates also hold here. In particular we obtain:

$$\begin{aligned} \sup_{\varepsilon} \|Q^{(\varepsilon)}\|_{L^\infty(0,T;H^1) \cap L^2(0,T;H^2)} &< \infty, \\ \sup_{\varepsilon} \|u^\varepsilon\|_{L^\infty(0,T;L^2) \cap L^2(0,T;H^1)} &< \infty \end{aligned} \quad (7.28)$$

for any $T < \infty$. Taking into account those bounds and also the bounds which can be obtained by using the equation on $\partial_t(Q^\varepsilon, u^\varepsilon)$ in some $L^p_{loc}(H^{-N})$ for large enough N , we get, by classical local compactness Aubin-Lions lemma and by weak convergence arguments, that there exists a $Q \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2)$ and a $u \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1)$ so that, on a subsequence, we have:

$$\begin{aligned} Q^{(\varepsilon)} &\rightharpoonup Q \text{ in } L^2(0, T; H^2) \text{ and } Q^{(\varepsilon)} \rightarrow Q \text{ in } L^2(0, T; H^{2-\delta}_{loc}), \forall \delta > 0 \\ Q^{(\varepsilon)}(t) &\rightharpoonup Q(t) \text{ in } H^1 \text{ for all } t \in \mathbb{R}_+ \\ u^\varepsilon &\rightharpoonup u \text{ in } L^2(0, T; H^1) \text{ and } u^\varepsilon \rightarrow u \text{ in } L^2(0, T; H^{1-\delta}_{loc}), \forall \delta > 0 \\ u^\varepsilon(t) &\rightharpoonup u(t) \text{ in } L^2 \text{ for all } t \in \mathbb{R}_+ \end{aligned} \quad (7.29)$$

These convergences allow us to pass to the limit in the weak solutions of the system (7.27) to obtain a weak solution of (7.5), namely (7.7), (7.8). Of all the terms there is only one type that is slightly difficult to treat in passing to the limit, namely:

$$L \int_0^\infty \int_{\mathbb{R}^d} \partial_\beta \left(Q_{\alpha\gamma}^{(\varepsilon)} \Delta Q_{\gamma\beta}^{(\varepsilon)} - \Delta Q_{\alpha\gamma}^{(\varepsilon)} Q_{\gamma\beta}^{(\varepsilon)} \right) \psi_\alpha dx dt = -L \int_0^\infty \int_{\mathbb{R}^d} \left(Q_{\alpha\gamma}^{(\varepsilon)} \Delta Q_{\gamma\beta}^{(\varepsilon)} - \Delta Q_{\alpha\gamma}^{(\varepsilon)} Q_{\gamma\beta}^{(\varepsilon)} \right) \cdot \psi_{\alpha,\beta} dx dt.$$

Taking into account that ψ is compactly supported and the convergences (7.29) one can easily pass to the limit the terms $\psi_{\alpha,\beta} Q_{\alpha\gamma}^{(\varepsilon)}$ and $\psi_{\alpha,\beta} \Delta Q_{\gamma\beta}^{(\varepsilon)}$ strongly in $L^2(0, T; L^2)$. Relations (7.29) give that $\Delta Q_{\gamma\beta}^{(\varepsilon)}$, $\Delta Q_{\alpha\gamma}^{(\varepsilon)}$ converges weakly in $L^2(0, T; L^2)$. Thus we get convergence to the limit term

$$\begin{aligned} &L \int_0^\infty \int_{\mathbb{R}^d} \partial_\beta (Q_{\alpha\gamma} \Delta Q_{\gamma\beta}) \psi_\alpha dx dt - L \int_0^\infty \int_{\mathbb{R}^d} \partial_\beta (\Delta Q_{\alpha\gamma}) Q_{\gamma\beta} \psi_\alpha dx dt \\ &= -L \int_0^T \int_{\mathbb{R}^d} (\Delta Q_{\gamma\beta}) (\partial_\beta \psi_\alpha Q_{\alpha\gamma}) dx dt + L \int_0^T \int_{\mathbb{R}^d} (\Delta Q_{\alpha\gamma}) (\partial_\beta \psi_\alpha Q_{\gamma\beta}) dx dt. \end{aligned}$$

Using also the uniform bound of $\varepsilon \|R_\varepsilon u^\varepsilon \nabla Q^\varepsilon\|_{L^3}^3$ it is easy to check that $\varepsilon \int |R_\varepsilon u^\varepsilon \nabla Q^\varepsilon|^2 \nabla Q^\varepsilon \cdot R_\varepsilon \mathcal{P} \varphi dx dt$ converges to zero. A similar result holds for the ε -term $\varepsilon \mathcal{P} \nabla \cdot (R_\varepsilon \nabla u |R_\varepsilon \nabla u|^2)$. \square

7.4 The uniqueness of weak solutions

We start with a number of technical tools that are crucial for our proof.

7.4.1 Littlewood-Paley theory

We define \mathcal{C} to be the ring of center 0, of small radius 1/2 and great radius 2. There exist two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0, 1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \forall \xi \in \mathbb{R}^d \quad (7.30)$$

$$|p - q| \geq 5 \Rightarrow \text{Supp } \varphi(2^{-q}\cdot) \cap \text{Supp } \varphi(2^{-p}\cdot) = \emptyset. \quad (7.31)$$

For instance, one can take $\chi \in \mathcal{D}(B(0, 1))$ such that $\chi \equiv 1$ on $B(0, 1/2)$ and take

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi).$$

Then, we are able to define the Littlewood-Paley decomposition. Let us denote by \mathcal{F} the Fourier transform on \mathbb{R}^d . Let $h, \tilde{h}, \dot{\Delta}_q, \dot{S}_q$ ($q \in \mathbb{Z}$) be defined as follows:

$$\begin{aligned} h &= \mathcal{F}^{-1}\varphi \quad \text{and} \quad \tilde{h} = \mathcal{F}^{-1}\chi, \\ \dot{\Delta}_q u &= \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int h(2^q y)u(x - y)dy, \\ \dot{S}_q u &= \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) = 2^{qd} \int \tilde{h}(2^q y)u(x - y)dy. \end{aligned}$$

We recall that for two appropriately smooth functions a and b we have the Bony's paraproduct decomposition [13]:

$$ab = \dot{T}_a b + \dot{T}_b a + \dot{R}(a, b)$$

where

$$\dot{T}_a b = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} a \dot{\Delta}_q b, \quad \dot{T}_b a = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} b \dot{\Delta}_q a, \quad \text{and} \quad \dot{R}(a, b) = \sum_{\substack{q \in \mathbb{Z}, \\ i \in \{0, \pm 1\}}} \dot{\Delta}_q a \dot{\Delta}_{q+i} b.$$

Then we have

$$\dot{\Delta}_q(ab) = \dot{\Delta}_q \dot{T}_a b + \dot{\Delta}_q \dot{T}_b a + \dot{\Delta}_q \dot{R}(a, b) = \dot{\Delta}_q \dot{T}_a b + \dot{\Delta}_q \tilde{R}(a, b),$$

where $\tilde{R}(a, b) = \dot{T}_b a + \dot{R}(a, b) = \sum_{q \in \mathbb{Z}} \dot{S}_{q+2} b \dot{\Delta}_q a$. Moreover:

$$\begin{aligned} \dot{\Delta}_q(ab) &= \sum_{|q'-q| \leq 5} \dot{\Delta}_q(\dot{S}_{q'-1} a \dot{\Delta}_{q'} b) + \sum_{q' > q-5} \dot{\Delta}_q(\dot{S}_{q'+2} b \dot{\Delta}_{q'} a) \\ &= \sum_{|q'-q| \leq 5} [\dot{\Delta}_q, \dot{S}_{q'-1} a] \dot{\Delta}_{q'} b + \sum_{|q'-q| \leq 5} \dot{S}_{q'-1} a \dot{\Delta}_q \dot{\Delta}_{q'} b + \sum_{q' > q-5} \dot{\Delta}_q(\dot{S}_{q'+2} b \dot{\Delta}_{q'} a) \\ &= \sum_{|q'-q| \leq 5} [\dot{\Delta}_q, \dot{S}_{q'-1} a] \dot{\Delta}_{q'} b + \sum_{|q'-q| \leq 5} (\dot{S}_{q'-1} a - \dot{S}_{q-1} a) \dot{\Delta}_q \dot{\Delta}_{q'} b \\ &\quad + \sum_{q' > q-5} \dot{\Delta}_q(\dot{S}_{q'+2} b \dot{\Delta}_{q'} a) + \underbrace{\sum_{|q'-q| \leq 5} \dot{S}_{q-1} a \dot{\Delta}_q \dot{\Delta}_{q'} b}_{=\dot{S}_{q-1} a \dot{\Delta}_q b} \end{aligned} \quad (7.32)$$

In terms of this decomposition we can express the Sobolev norm of an element u in the (nonhomogeneous!) space H^s as:

$$\|u\|_{H^s} = (\|\dot{S}_0 u\|_{L^2}^2 + \sum_{q \in \mathbb{N}} 2^{2qs} \|\dot{\Delta}_q u\|_{L^2}^2)^{1/2}$$

These are a particular case of the general nonhomogeneous Besov spaces $B_{p,r}^s$, for $s \in \mathbb{R}, p, r \in [1, \infty]^2$ consisting of all tempered distributions u such that:

$$\|u\|_{B_{p,r}^s} := \begin{cases} (\|\dot{S}_0 u\|_{L^p}^r + \sum_{q \in \mathbb{N}} 2^{rqs} \|\dot{\Delta}_q u\|_{L^p}^r)^{1/r} & \text{if } r < \infty \\ \max(\|\dot{S}_0 u\|_{L^p}, \sup_{q \in \mathbb{N}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p}) & \text{if } r = \infty \end{cases}$$

which reduces to the nonhomogeneous Sobolev space H^s for $p = r = 2$.

Similarly we also have the norm of the *homogenous* Sobolev spaces \dot{H}^s :

$$\|u\|_{\dot{H}^s} = \left(\sum_{q \in \mathbb{Z}} 2^{2qs} \|\dot{\Delta}_q u\|_{L^2}^2 \right)^{1/2}$$

and the homogenous Besov spaces $\dot{B}_{p,r}^s$ for $s \in \mathbb{R}, p, r \in [1, \infty]^2$ consisting of all the homogeneous tempered distributions u such that:

$$\|u\|_{\dot{B}_{p,r}^s} \stackrel{def}{=} \begin{cases} \left\| \left(\sum_{q \in \mathbb{Z}} 2^{rqs} \|\dot{\Delta}_q u\|_{L^p}^r \right)^{\frac{1}{r}} \right\| & \text{if } r < \infty \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p} & \text{if } r = \infty \end{cases}$$

which reduces to the homogeneous Sobolev space \dot{H}^s for $p = r = 2$.

Let us note that the homogeneous Besov spaces have somewhat better product rules, and this specificity encoded in Proposition 7.4.4 will be very useful in our subsequent estimates.

Furthermore we will need the following characterisation of the homogeneous norms, in terms of operators $\dot{S}_q u$:

Lemma 7.4.1. [Prop. 2.33] , [7] *Let $s < 0$ and $p, r \in [1, \infty]^2$. A tempered distribution u belongs to $\dot{B}_{p,r}^s$ if and only if:*

$$(2^{qs} \|\dot{S}_q u\|_{L^p})_{q \in \mathbb{Z}} \in l^r$$

and for some constant C depending only on the dimension d we have:

$$C^{-|s|+1} \|u\|_{\dot{B}_{p,r}^s} \leq \| (2^{qs} \|\dot{S}_q u\|_{L^p})_q \|_{l^r} \leq C \left(1 + \frac{1}{|s|}\right) \|u\|_{\dot{B}_{p,r}^s}$$

We will use the following well-known estimates:

Lemma 7.4.2. ([18], [19])

(i) *(Bernstein inequalities)*

$$2^{-q} \|\nabla \dot{S}_q u\|_{L_x^p} \leq C \|u\|_{L_x^p}, \forall 1 \leq p \leq \infty$$

$$c \|\dot{\Delta}_q u\|_{L_x^p} \leq 2^{-q} \|\dot{\Delta}_q \nabla u\|_{L_x^p} \leq C \|\dot{\Delta}_q u\|_{L_x^p}, \forall 1 \leq p \leq \infty$$

(ii) *(Bernstein inequalities)*

$$\|\dot{\Delta}_q u\|_{L_x^b} \leq 2^{d(\frac{1}{a}-\frac{1}{b})q} \|\dot{\Delta}_q u\|_{L_x^a}, \text{ for } b \geq a \geq 1$$

$$\|\dot{S}_q u\|_{L_x^b} \leq 2^{d(\frac{1}{a}-\frac{1}{b})q} \|\dot{S}_q u\|_{L_x^a}, \text{ for } b \geq a \geq 1$$

(ii) *(commutator estimate)*

$$\|[\dot{\Delta}_q, u]v\|_{L_x^p} \leq C 2^{-q} \|\nabla u\|_{L_x^r} \|v\|_{L_x^s} \quad (7.33)$$

with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. The constant C depends only on the function φ used in defining $\dot{\Delta}_q$ but not on p, r, s .

Proof. For the commutator estimate we begin by writing

$$[\dot{\Delta}_q, u]v(x) = \dot{\Delta}_q(uv)(x) - u(x)\dot{\Delta}_q v(x) = 2^{qd} \int h(2^q y)(u(x-y) - u(x))v(x-y)dy$$

$$\begin{aligned}
&= 2^{qd} \int_{\mathbb{R}^d} \int_0^1 \frac{\partial}{\partial \tau} \left\{ h(2^q y) u(x - \tau y) v(x - y) \right\} dy d\tau \\
&= -2^{qd} \int_0^1 \int_{\mathbb{R}^d} h(2^q y) y \cdot \nabla u(x - \tau y) v(x - y) dy d\tau \\
&= -2^{-q} \int_0^1 \int_{\mathbb{R}^d} \tilde{h}_{2^q}(y) \cdot \nabla u(x - \tau y) v(x - y) dy d\tau,
\end{aligned}$$

where $\tilde{h}(y) := yh(y) \in \mathcal{S}(R^d)^d$ and $\tilde{h}_\lambda(y) := \lambda^d \tilde{h}(\lambda y)$. Using the Cauchy-Schwartz inequality and a change of variables, we get

$$\begin{aligned}
|[\dot{\Delta}_q, u]v(x)| &\leq 2^{-q} \int_0^1 \left(\int_{\mathbb{R}^d} |\tilde{h}_{2^q}(y)| |\nabla u(x - \tau y)|^{\frac{r}{p}} dy \right)^{\frac{p}{r}} d\tau \left(\int_{\mathbb{R}^d} |\tilde{h}_{2^q}(y)| |v(x - y)|^{\frac{s}{p}} dy \right)^{\frac{p}{s}} \\
&= 2^{-q} \int_0^1 \left(\int_{\mathbb{R}^d} \frac{|\tilde{h}_{2^q \tau^{-1}}(y)|}{\tau^d} |\nabla u(x - y)|^{\frac{r}{p}} dy \right)^{\frac{p}{r}} d\tau \left(\int_{\mathbb{R}^d} |\tilde{h}_{2^q}(y)| |v(x - y)|^{\frac{s}{p}} dy \right)^{\frac{p}{s}} \\
&= 2^{-q} \int_0^1 \left(\frac{|\tilde{h}_{2^q \tau^{-1}}|}{\tau^d} * |\nabla u|^{\frac{r}{p}}(x) \right)^{\frac{p}{r}} d\tau \left(|\tilde{h}_{2^q}| * |v|^{\frac{s}{p}}(x) \right)^{\frac{p}{s}}
\end{aligned}$$

Taking the L^p norm in the x variable, using the Cauchy-Schwartz inequality in the x variable and convolution estimates we obtain

$$\begin{aligned}
\|[\dot{\Delta}_q, u]v\|_{L_x^p} &\leq 2^{-q} \int_0^1 \left\| \left(\frac{|\tilde{h}_{2^q \tau^{-1}}|}{\tau^d} * |\nabla u|^{\frac{r}{p}}(x) \right)^{\frac{p}{r}} \right\|_{L_x^p} d\tau \left\| \left(|\tilde{h}_{2^q}| * |v|^{\frac{s}{p}}(x) \right)^{\frac{p}{s}} \right\|_{L_x^s} \\
&\leq 2^{-q} \left(\int_0^1 \left\| \frac{|\tilde{h}_{2^q \tau^{-1}}|}{\tau^d} * |\nabla u|^{\frac{r}{p}} \right\|_{L_x^p}^{\frac{p}{r}} d\tau \right) \left\| |\tilde{h}_{2^q}| * |v|^{\frac{s}{p}} \right\|_{L_x^p}^{\frac{p}{s}} \\
&\leq 2^{-q} \int_0^1 \frac{\|\tilde{h}_{2^q \tau^{-1}}\|_{L_x^1}^{\frac{p}{r}}}{\tau^d} d\tau \|\nabla u\|_{L_x^r} \|\tilde{h}_{2^q}\|_{L_x^1}^{\frac{p}{s}} \|v\|_{L_x^s} \\
&\leq 2^{-q} \|\tilde{h}_{2^{-q}}\|_{L^1}^{\frac{p}{r}} \|\tilde{h}_{2^{-q}}\|_{L^1}^{\frac{p}{s}} \|\nabla u\|_{L_x^r} \|v\|_{L_x^s}.
\end{aligned}$$

Now, since

$$\|\tilde{h}_{2^{-q}}\|_{L_x^1} = \int_{\mathbb{R}^d} 2^{-qd} |\tilde{h}(2^{-q}x)| dx = \int_{\mathbb{R}^d} |\tilde{h}(y)| dy = \|\tilde{h}\|_{L_x^1},$$

we finally obtain

$$\|[\dot{\Delta}_q, u]v\|_{L_x^p} \leq 2^{-q} \|\tilde{h}\|_{L_x^1}^{\frac{p}{r}} \|\tilde{h}\|_{L_x^1}^{\frac{p}{s}} \|\nabla u\|_{L_x^r} \|v\|_{L_x^s} = \|\tilde{h}\|_{L_x^1} 2^{-q} \|\nabla u\|_{L_x^r} \|v\|_{L_x^s}$$

so the constant in the inequality is $C = \|\tilde{h}\|_{L^1}$ and it does not depend on p, r, s . \square

We will also make use of a Bernstein-type inequality evolving the operator \dot{S}_q .

Lemma 7.4.3. *there exist two positive constants \tilde{c} and \tilde{C} such that*

$$\tilde{c} \|(\dot{S}_q - \dot{S}_{q'})u\|_{L_x^p} \leq 2^{-q} \|(\dot{S}_q - \dot{S}_{q'})\nabla u\|_{L_x^p} \leq \tilde{C} \|(\dot{S}_q - \dot{S}_{q'})u\|_{L_x^p}, \forall 1 \leq p \leq \infty,$$

for any integers q and q' with $|q - q'| \leq 5$.

Proof. First, we consider new localizer functions as follows:

$$\tilde{\varphi}_q(\xi) := \frac{1}{10} \sum_{|q-j| \leq 10} \varphi_j(\xi) \quad \text{and} \quad \tilde{\chi}(\xi) := \begin{cases} \sum_{q \leq -1} \tilde{\varphi}_q(\xi) & \text{if } \xi \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

so that (7.30) and (7.31) are satisfied with $\tilde{\varphi}$ and $\tilde{\chi}$ instead of φ and χ . Then defining the new homogeneous dyadic block $\dot{\Delta}_q$ in the same line of Δ_q , we have

$$\dot{\Delta}_q(\dot{S}_q - \dot{S}_{q'})u = \frac{1}{10} \sum_{|q-j| \leq 10} \dot{\Delta}_j(\dot{S}_q - \dot{S}_{q'})(u) = \frac{1}{10}(\dot{S}_q - \dot{S}_{q'})(u).$$

Then the inequality turns out from (i) of Lemma 7.4.2, making use of $\dot{\Delta}_q$ instead of Δ_q . \square

Now, we prove an useful product law between homogeneous Sobolev which will play a main role in our estimates.

Proposition 7.4.4. *Let s and t be two real numbers such that $|s|$ and $|t|$ belong to $[0, d/2)$. Let us assume that $s+t$ is positive, then for every $a \in \dot{H}^s(\mathbb{R}^d)$ and for every $b \in \dot{H}^t(\mathbb{R}^d)$, the product ab belongs to $\dot{H}^{s+t-d/2}$ and there exists a positive constant (not dependent by a and b) such that*

$$\|ab\|_{\dot{H}^{s+t-d/2}} \leq C \|a\|_{\dot{H}^s} \|b\|_{\dot{H}^t}$$

Proof. At first we identify the Sobolev Spaces \dot{H}^s and \dot{H}^t with the Besov Spaces $\dot{B}_{2,2}^s$ and $\dot{B}_{2,2}^t$ respectively. We claim that ab belongs to $\dot{B}_{2,2}^{s+t-d/2}$ and

$$\|ab\|_{\dot{B}_{2,2}^{s+t-d/2}} \leq C \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t},$$

for a suitable positive constant.

We decompose the product ab through the Bony decomposition, namely $ab = \dot{T}_a b + \dot{T}_b a + R(a, b)$, where

$$\dot{T}_a b := \sum_{q \in \mathbb{Z}} \dot{\Delta}_q a \dot{S}_{q-1} b, \quad \dot{T}_b a := \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} a \dot{\Delta}_q b, \quad \dot{R}(a, b) := \sum_{\substack{q \in \mathbb{Z} \\ |\nu| \leq 1}} \dot{\Delta}_q a \dot{\Delta}_{q+\nu} b.$$

For any $q \in \mathbb{Z}$, we have

$$\begin{aligned} & 2^{q(s+t-\frac{d}{2})} \|(\dot{\Delta}_q \dot{T}_a b, \dot{\Delta}_q \dot{T}_b a)\|_{L^2} \lesssim \\ & \lesssim \sum_{|q-q'| \leq 5} 2^{q's} \|\dot{\Delta}_q a\|_{L_x^2} 2^{q'(t-\frac{d}{2})} \|\dot{S}_{q-1} b\|_{L^\infty} + \sum_{|q-q'| \leq 5} 2^{q'(s-\frac{d}{2})} \|\dot{S}_{q-1} a\|_{L_x^\infty} 2^{q't} \|\dot{\Delta}_q b\|_{L^2}, \end{aligned}$$

hence

$$\begin{aligned} \|(\dot{T}_a b, \dot{T}_b a)\|_{\dot{B}_{2,2}^{s+t-d/2}} & \leq \|(\dot{T}_a b, \dot{T}_b a)\|_{\dot{B}_{2,1}^{s+t-d/2}} \\ & \lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{\infty,2}^{t-d/2}} + \|a\|_{\dot{B}_{\infty,2}^{s-d/2}} \|b\|_{\dot{B}_{2,2}^t} \lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t}, \end{aligned}$$

where we have used the embedding $\dot{B}_{2,2}^\sigma \hookrightarrow \dot{B}_{\infty,2}^{\sigma-d/2}$, for any $\sigma \in \mathbb{R}$ and moreover the following

norm-equivalence

$$\|u\|_{\dot{B}_{p,r}^{\tilde{\sigma}}} \approx \|(2^{\tilde{\sigma}} \|S_q u\|_{L_x^p})_{q \in \mathbb{Z}}\|_{l^r(\mathbb{Z})}, \quad u \in \dot{B}_{p,r}^{\tilde{\sigma}},$$

for any $1 \leq p, r \leq \infty$ and $\tilde{\sigma} < 0$.

In order to conclude the proof, we have to handle the rest $\dot{R}(a, b)$. By a direct computation, for any $q \in \mathbb{Z}$,

$$2^{(t+s)q} \|\dot{\Delta}_q \dot{R}(a, b)\|_{L_x^1} \leq \sum_{\substack{q' \geq q-5 \\ |\nu| \leq 1}} 2^{(q-q')(s+t)} 2^{q's} \|\dot{\Delta}_{q'} a\|_{L_x^2} 2^{(q'+\nu)t} \|\dot{\Delta}_{q'+\nu} a\|_{L_x^2},$$

so that, thanks to the Young inequality, we deduce

$$\|\dot{R}(a, b)\|_{\dot{B}_{2,2}^{s+t-\frac{d}{2}}} \lesssim \|\dot{R}(a, b)\|_{\dot{B}_{1,1}^{s+t}} \lesssim \|a\|_{\dot{B}_{2,2}^s} \|b\|_{\dot{B}_{2,2}^t},$$

where we have used the embedding $\dot{B}_{1,1}^{s+t} \hookrightarrow \dot{B}_{2,2}^{s+t-d/2}$ and moreover $\sum_{q \leq 5} 2^{q(s+t)} < \infty$ since $s+t$ is positive. \square

Let us finally present two specific Sobolev inequalities. The first one allows a bound for the L^∞ -norm of the low frequencies of a suitable function, while the second one is interpolation-type inequality. The explicit formula of the constants of embedding will play a key role when establishing the delicate double-logarithmic inequality.

Proposition 7.4.5. *Let N be a positive real number and f a function in H^1 . Then $\dot{S}_N f$ belongs to L_x^∞ and*

$$\|\dot{S}_N f\|_{L_x^\infty} \lesssim \|f\|_{L_x^2} + \sqrt{N} \|\nabla f\|_{L_x^2} \lesssim (1 + \sqrt{N}) \|(f, \nabla f)\|_{L_x^2}.$$

Proof. We split $\dot{S}_N f$ into two parts, namely $\dot{S}_N f = \sum_{q < 0} \dot{\Delta}_q f + \sum_{0 \leq q < N} \dot{\Delta}_q f$. First we observe that

$$\left\| \sum_{q < 0} \dot{\Delta}_q f \right\|_{L_x^\infty} \leq \sum_{q < 0} \|\dot{\Delta}_q f\|_{L_x^\infty} \lesssim \sum_{q < 0} 2^q \|\dot{\Delta}_q f\|_{L_x^2} \lesssim \left(\sum_{q < 0} 2^q \right) \|f\|_{L_x^2}.$$

Similarly, considering the second term, we get

$$\begin{aligned} \left\| \sum_{0 < q \leq N} \dot{\Delta}_q f \right\|_{L_x^\infty} &\leq \sum_{0 < q \leq N} \|\dot{\Delta}_q f\|_{L_x^\infty} \lesssim \sum_{0 < q \leq N} 2^q \|\dot{\Delta}_q f\|_{L_x^2} \\ &\lesssim \sum_{0 < q \leq N} \|\dot{\Delta}_q \nabla f\|_{L_x^2} \lesssim \left(\sum_{0 < q \leq N} 1 \right)^{\frac{1}{2}} \left(\sum_{0 < q \leq N} \|\dot{\Delta}_q \nabla f\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \sqrt{N} \|f\|_{\dot{H}^1}, \end{aligned}$$

which concludes the proof of the Theorem. \square

The following Lemma plays a main role in the uniqueness result of Theorem 7.1.2, more precisely inequality (7.34) is the key for the double-logarithmic estimate.

Lemma 7.4.6. *There exists a positive constant C such that for any $p \in [1, \infty)$ the following inequality is satisfied:*

$$\|f\|_{L^{2p}(\mathbb{R}^2)} \leq C \sqrt{p} \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{p}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{p}} \quad (7.34)$$

Proof. The proof of this lemma was presented in [96] (lemma 4.3) and we report it here, for the sake of simplicity. thanks to Sobolev embeddings, we have

$$\|f\|_{L^{2p}(\mathbb{R}^2)} \leq C \sqrt{p} \|f\|_{\dot{H}^{1-\frac{1}{p}}(\mathbb{R}^2)}. \quad (7.35)$$

Moreover, since $\dot{H}^{1-1/p}(\mathbb{R}^2)$ is an interpolation space between $L^2(\mathbb{R}^2)$ and $\dot{H}^1(\mathbb{R}^2)$, the following inequality is satisfied:

$$\|f\|_{\dot{H}^{1-1/p}(\mathbb{R}^2)} \leq \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{p}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1-\frac{1}{p}},$$

which leads to (7.34), together with (7.35). \square

7.4.2 The proof of the uniqueness

In this section we provide the proof of the uniqueness result for the weak solutions of system (7.5). The main idea is to evaluate the difference between two weak solutions in a functional space which is less regular than L_x^2 such as $\dot{H}^{-\frac{1}{2}}$. Such strategy is not new in literature, for instance we recall [48] and [88]. We now provide the uniqueness part of the proof of Theorem 7.1.2.

Proof. Let us consider two weak solutions (u_1, Q_1) and (u_2, Q_2) of system (7.5). We denote $\delta u := u_1 - u_2$ and $\delta Q := Q_1 - Q_2$ while $\delta S(Q, \nabla u)$ stands for $S(Q_1, \nabla u_1) - S(Q_2, \nabla u_2)$. Similarly, we define $\delta H(Q)$, $\delta F(Q)$, $\delta \tau$ and $\delta \sigma$. Thus $(\delta u, \delta Q)$ is a weak solution of

$$\begin{cases} \partial_t \delta Q - L \Delta \delta Q = \delta S(Q, \nabla u) + \Gamma \delta H(Q) - \delta u \cdot \nabla Q_1 - u_2 \cdot \nabla \delta Q & \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \delta u - \Delta \delta u + \nabla \delta \Pi = L \operatorname{div}\{\delta \tau + \delta \sigma\} - \delta u \cdot \nabla u_1 - u_2 \cdot \nabla \delta u & \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} \delta u = 0 & \mathbb{R}_+ \times \mathbb{R}^2, \\ (\delta u, \delta Q)_{t=0} = (0, 0) & \mathbb{R}^2. \end{cases} \quad (7.36)$$

First, let us explicitly state $\delta S(Q, \nabla u)$, $\delta F(Q)$, $\delta \tau$ and $\delta \sigma$ in terms of δQ and δu , namely:

$$\begin{aligned} \delta S(Q, \nabla u) &= +(\xi \delta D + \delta \Omega) \delta Q + (\xi \delta D + \delta \Omega) \left(Q_2 + \frac{\operatorname{Id}}{2} \right) + (\xi D_2 + \Omega_2) \delta Q + \delta Q (\xi \delta D - \delta \Omega) + \\ &+ \left(Q_2 + \frac{\operatorname{Id}}{2} \right) (\xi \delta D - \delta \Omega) + \delta Q (\xi D_2 - \Omega_2) - 2\xi \delta Q \operatorname{tr}(\delta Q \nabla \delta u) - 2\xi \delta Q \operatorname{tr}(\delta Q \nabla u_2) + \\ &- 2\xi \delta Q \operatorname{tr}(Q_2 \nabla \delta u) - 2\xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \operatorname{tr}(\delta Q \nabla \delta u) - 2\xi \delta Q \operatorname{tr}(Q_2 \nabla u_2) + \\ &- 2\xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \operatorname{tr}(\delta Q \nabla u_2) - 2\xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \operatorname{tr}(Q_2 \nabla \delta u), \end{aligned}$$

$$\begin{aligned} \delta F(Q) &= -a \delta Q + b \left(Q_1 \delta Q + \delta Q Q_2 \right) - b \operatorname{tr}\{\delta Q Q_1 + Q_2 \delta Q\} \frac{\operatorname{Id}}{2} - \\ &- c \left[\delta Q \operatorname{tr}\{Q_1^2\} + Q_2 \operatorname{tr}\{\delta Q Q_1 + Q_2 \delta Q\} \right] \end{aligned}$$

$$\delta H(Q) = \delta F(Q) + L \Delta \delta Q.$$

$$\begin{aligned} \delta \tau &= -\xi \delta Q F(Q_1) - \xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \delta F(Q) - L \xi \delta Q \Delta \delta Q - L \xi \delta Q \Delta Q_2 - L \xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \Delta \delta Q + \\ &- \xi F(Q_1) \delta Q - \xi \delta F(Q) \left(Q_2 + \frac{\operatorname{Id}}{2} \right) - L \xi \Delta \delta Q \delta Q - L \xi \Delta Q_2 \delta Q - L \xi \Delta \delta Q \left(Q_2 + \frac{\operatorname{Id}}{2} \right) + \\ &+ 2\xi \delta Q \operatorname{tr}\{Q_1 F(Q_1)\} + 2\xi Q_2 \operatorname{tr}\{\delta Q F(Q_1)\} + \\ &+ 2\xi Q_2 \operatorname{tr}\{Q_2 \delta F(Q)\} + 2L \xi \delta Q \operatorname{tr}\{\delta Q \Delta \delta Q\} + 2L \xi \delta Q \operatorname{tr}\{\delta Q \Delta Q_2\} + 2L \xi \delta Q \operatorname{tr}\{Q_2 \Delta \delta Q\} + \\ &+ 2L \xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \operatorname{tr}\{\delta Q \Delta \delta Q\} + 2L \xi \delta Q \operatorname{tr}\{Q_2 \Delta Q_2\} + 2L \xi \left(Q_2 + \frac{\operatorname{Id}}{2} \right) \operatorname{tr}\{\delta Q \Delta Q_2\} + \end{aligned}$$

$$\begin{aligned}
& + 2L\xi \left(Q_2 + \frac{\text{Id}}{2} \right) \text{tr}\{Q_2 \Delta \delta Q\} - L\nabla \delta Q \odot \nabla Q_1 - L\nabla Q_2 \odot \nabla \delta Q - \\
& - L \frac{\text{Id}}{2} \text{tr}\{\delta Q Q_1\} - L \frac{\text{Id}}{2} \text{tr}\{Q_2 \delta Q\}
\end{aligned}$$

$$\begin{aligned}
\delta \sigma = & \delta Q F(Q_1) + Q_2 \delta F(Q) - F(Q_1) \delta Q - \delta F(Q) Q_2 + L \delta Q \Delta \delta Q + L Q_2 \Delta \delta Q + L \delta Q \Delta Q_2 \\
& - L \Delta \delta Q \delta Q - L \Delta Q_2 \delta Q - L \Delta \delta Q Q_2
\end{aligned}$$

Taking the inner product in $\dot{H}^{-1/2}$ of the first equation with $-L\lambda\Delta\delta Q$ and adding to it the scalar product in $\dot{H}^{-1/2}$ of the second one by $\frac{1}{\lambda}\delta u$ we get:

$$\begin{aligned}
& \frac{d}{dt} \left[\frac{1}{2\lambda} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + L \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \right] + \frac{\nu}{\lambda} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \Gamma L^2 \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 = \\
& - L \langle (\xi \delta D + \delta \Omega) \delta Q, \Delta \delta Q \rangle - \underbrace{L \xi \langle \delta D Q_2, \Delta \delta Q \rangle}_{\mathcal{A}_1} - \underbrace{L \langle \delta \Omega Q_2, \Delta \delta Q \rangle}_{\mathcal{B}_1} - \underbrace{L \xi \langle \frac{\delta D}{2}, \Delta \delta Q \rangle}_{\mathcal{C}_1} \\
& - \underbrace{L \langle \frac{\delta \Omega}{2}, \Delta \delta Q \rangle}_{\mathcal{D}_1} - L \langle (\xi D_2 + \Omega_2) \delta Q, \Delta \delta Q \rangle - L \langle \delta Q (\xi \delta D - \delta \Omega), \Delta \delta Q \rangle - \underbrace{L \xi \langle Q_2 \delta D, \Delta \delta Q \rangle}_{\mathcal{A}_2} \\
& + \underbrace{L \langle Q_2 \delta \Omega, \Delta \delta Q \rangle}_{\mathcal{B}_2} - \underbrace{L \xi \langle \frac{\delta D}{2}, \Delta \delta Q \rangle}_{\mathcal{C}_2} + \underbrace{L \langle \frac{\delta \Omega}{2}, \Delta \delta Q \rangle}_{\mathcal{D}_2} - L \langle \delta Q (\xi D_2 - \Omega_2), \Delta \delta Q \rangle \\
& + 2L\xi \langle \delta Q \text{tr}(\delta Q \nabla \delta u), \Delta \delta Q \rangle + 2L\xi \langle \delta Q \text{tr}(\delta Q \nabla u_2), \Delta \delta Q \rangle + 2L\xi \langle \delta Q \text{tr}(Q_2 \nabla \delta u), \Delta \delta Q \rangle \\
& + 2L\xi \langle Q_2 \text{tr}(\delta Q \nabla \delta u), \Delta \delta Q \rangle + \underbrace{2L\xi \langle \frac{\text{Id}}{2} \text{tr}(\delta Q \nabla \delta u), \Delta \delta Q \rangle}_{=0} + 2L\xi \langle \delta Q \text{tr}(Q_2 \nabla u_2), \Delta \delta Q \rangle \\
& + 2L\xi \langle Q_2 \text{tr}(\delta Q \nabla u_2), \Delta \delta Q \rangle + \underbrace{2L\xi \langle \frac{\text{Id}}{2} \text{tr}(\delta Q \nabla u_2), \Delta \delta Q \rangle}_{=0} + \underbrace{2L\xi \langle Q_2 \text{tr}(Q_2 \nabla \delta u), \Delta \delta Q \rangle}_{\mathcal{E}_1} \\
& + \underbrace{2L\xi \langle \frac{\text{Id}}{2} \text{tr}(Q_2 \nabla \delta u), \Delta \delta Q \rangle}_{=0} + La\Gamma \langle \delta Q, \Delta \delta Q \rangle - Lb\Gamma \langle Q_1 \delta Q + \delta Q Q_2, \Delta \delta Q \rangle \\
& + \underbrace{Lb\Gamma \langle \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\} \frac{\text{Id}}{2}, \Delta \delta Q \rangle}_{=0} + Lc\Gamma \langle \delta Q \text{tr}\{Q_1^2\}, \Delta \delta Q \rangle \\
& + Lc\Gamma \langle Q_2 \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\}, \Delta \delta Q \rangle + L \langle \delta u \cdot \nabla Q_1, \Delta \delta Q \rangle \\
& + L \langle u_2 \cdot \nabla \delta Q, \Delta \delta Q \rangle - a\xi \langle \delta Q Q_1, \nabla \delta u \rangle + b\xi \langle \delta Q Q_1^2, \nabla \delta u \rangle - b\xi \langle \delta Q \text{tr}(Q_1^2) \frac{\text{Id}}{2}, \nabla \delta u \rangle \tag{7.37} \\
& - c\xi \langle \delta Q \text{tr}(Q_1^2) Q_1, \nabla \delta u \rangle - a\xi \langle (Q_2 + \frac{\text{Id}}{2}) \delta Q, \nabla \delta u \rangle \\
& + b\xi \langle (Q_2 + \frac{\text{Id}}{2}) (Q_1 \delta Q + \delta Q Q_2), \nabla \delta u \rangle - b\xi \langle \frac{Q_2}{2} \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\}, \nabla \delta u \rangle \\
& - \underbrace{b\xi \langle \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\} \frac{\text{Id}}{9}, \nabla \delta u \rangle}_{=0} - c\xi \langle (Q_2 + \frac{\text{Id}}{2}) \delta Q \text{tr}\{Q_1^2\}, \nabla \delta u \rangle \\
& - c\xi \langle (Q_2 + \frac{\text{Id}}{2}) Q_2 \text{tr}\{\delta Q Q_1 + Q_2 \delta Q\}, \nabla \delta u \rangle + L\xi \langle \delta Q \Delta \delta Q, \nabla \delta u \rangle + L\xi \langle \delta Q \Delta Q_2, \nabla \delta u \rangle \\
& + \underbrace{L\xi \langle Q_2 \Delta \delta Q, \nabla \delta u \rangle}_{\mathcal{A}_3} + \underbrace{L\xi \langle \frac{\Delta \delta Q}{2}, \nabla \delta u \rangle}_{\mathcal{C}_3} - a\xi \langle Q_1 \delta Q, \nabla \delta u \rangle
\end{aligned}$$

$$\begin{aligned}
& -L\langle \delta Q \Delta Q_2, \nabla \delta u \rangle + L\langle \Delta \delta Q \delta Q, \nabla \delta u \rangle + \underbrace{L\langle \Delta \delta Q Q_2, \nabla \delta u \rangle}_{\mathcal{B}_4} \\
& + L\langle \Delta Q_2 \delta Q, \nabla \delta u \rangle - \langle u_2 \cdot \nabla \delta u, \delta u \rangle - \langle \delta u \cdot \nabla u_1, \delta u \rangle.
\end{aligned}$$

Denoting

$$\Phi(t) := 1/(2\lambda) \|\delta u(t)\|_{\dot{H}^{-1/2}}^2 + L \|\nabla \delta Q(t)\|_{\dot{H}^{-1/2}}^2$$

we will aim to show that Φ satisfies the inequality

$$\Phi'(t) \leq \chi(t) \mu(\Phi(t)), \quad (7.38)$$

where μ is an Osgood modulus of continuity (see [7], Definition 3.1), given by

$$\mu(r) := r + r \ln \left(1 + e + \frac{1}{r}\right) + r \ln \left(1 + e + \frac{1}{r}\right) \ln \ln \left(1 + e + \frac{1}{r}\right). \quad (7.39)$$

with $\chi \in L^1_{loc}$ a priori. We are going to find a double-logarithmic estimate, hence thanks to the Osgood Lemma (see [7], Lemma 3.4) and since $\Phi(0)$ is null, we get that $\Phi \equiv 0$, which yields the uniqueness of the solution for system (7.5).

First, let us observe following simplifications of (7.37):

$$0 = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \mathcal{D}_1 + \mathcal{D}_2 = \mathcal{F}_1 + \mathcal{F}_2.$$

The key method we use to obtain the desired estimates is the para-differential calculus decomposition summarized in the following:

Remark 7.4.7. *Let q be an integer, and A, B be $d \times d$ matrices whose components are homogeneous temperate distributions. We are going to use the following notation:*

$$\begin{aligned}
\mathcal{J}_q^1(A, B) & := \sum_{|q-q'| \leq 5} [\dot{\Delta}_q, \dot{S}_{q'-1} A] \dot{\Delta}_{q'} B, & \mathcal{J}_q^3(A, B) & := \dot{S}_{q-1} A \dot{\Delta}_q B, \\
\mathcal{J}_q^2(A, B) & := \sum_{|q-q'| \leq 5} (\dot{S}_{q'-1} A - \dot{S}_{q-1} A) \dot{\Delta}_q \dot{\Delta}_{q'} B, & \mathcal{J}_q^4(A, B) & := \sum_{q' \geq q-5} \dot{\Delta}_q (\dot{\Delta}_{q'} A \dot{S}_{q'+2} B).
\end{aligned}$$

Then we can decompose the product AB as follows

$$\dot{\Delta}_q(AB) = \mathcal{J}_q^1(A, B) + \mathcal{J}_q^2(A, B) + \mathcal{J}_q^3(A, B) + \mathcal{J}_q^4(A, B) \quad (7.40)$$

for any integer q .

Moreover from now on we will use the notation \lesssim as follows: for any non-negative real numbers a and b , we denote $a \lesssim b$ if and only if there exists a positive constant C (independent of a and b) such that $a \leq Cb$.

Estimate of $\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4$

Let us begin analyzing the terms $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 of (7.37). First, we observe that

$$\mathcal{A}_2 = -L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q(Q_2 \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = -L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \sum_{i=1}^4 \langle \mathcal{J}_q^i(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}$$

Now, when $i = 1$, we have

$$\begin{aligned}
2^{-q} \langle \mathcal{J}_q^1(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &= \sum_{|q-q'| \leq 5} 2^{-q} \langle [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \delta D, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \delta D \|_{L_x^2} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-2q} \| \dot{S}_{q'-1} \nabla Q_2 \|_{L_x^4} \| \dot{\Delta}_{q'} \delta D \|_{L_x^4} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| \dot{S}_{q'-1} \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{S}_{q'-1} \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{\Delta}_{q'} \delta u \|_{L_x^4} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \dot{\Delta}_{q'} \delta u \|_{L_x^2} 2^{-\frac{q}{2}} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2},
\end{aligned} \tag{7.41}$$

for every $q \in \mathbb{Z}$. Hence, we get

$$\begin{aligned}
-L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^1(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \delta u \|_{L_x^2} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \| \nabla Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \Delta Q_2 \|_{L_x^2}^{\frac{1}{2}} \| \delta u \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \| \nabla Q_2 \|_{L_x^2}^2 \| \Delta Q_2 \|_{L_x^2}^2 \| \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2,
\end{aligned} \tag{7.42}$$

where we have used the following interpolation inequality:

$$\| \delta u \|_{L_x^2} \leq \| \delta u \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \delta u \|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} = \| \delta u \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}}.$$

When $i = 2$, the following inequalities are fulfilled:

$$\begin{aligned}
2^{-q} \langle \mathcal{J}_q^2(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &= \sum_{|q-q'| \leq 5} 2^{-q} \langle (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \dot{\Delta}_q \dot{\Delta}_{q'} \delta D, \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-q} \| (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \|_{L_x^\infty} \| \dot{\Delta}_q \dot{\Delta}_{q'} \delta D \|_{L_x^2} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-2q} \| (\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2) \|_{L_x^2} \| \dot{\Delta}_q \dot{\Delta}_{q'} \delta D \|_{L_x^2} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-2q} \| \Delta Q_2 \|_{L_x^2} \| \dot{\Delta}_{q'} \delta D \|_{L_x^2} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \\
&\lesssim \sum_{|q-q'| \leq 5} 2^{-\frac{q'}{2}} \| \dot{\Delta}_{q'} \delta u \|_{L_x^2} 2^{-\frac{q}{2}} \| \dot{\Delta}_q \Delta \delta Q \|_{L_x^2} \| \Delta Q_2 \|_{L_x^2},
\end{aligned} \tag{7.43}$$

for any $q \in \mathbb{Z}$. Thus, it turns out that

$$-L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^2(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \lesssim \| \Delta Q_2 \|_{L_x^2}^2 \| \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2 \tag{7.44}$$

The term corresponding to $i = 3$ cannot be estimated as before. We will see that this challenging term will be simplified. Finally, when $i = 4$, we have

$$\begin{aligned}
2^{-q} \langle \mathcal{J}_q^4(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &= L 2^{-q} \sum_{q-q' \leq 5} \langle \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \dot{S}_{q'+2} \delta D], \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&\lesssim 2^{-q} \sum_{q-q' \leq 5} \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \\
&\lesssim 2^{-q} \sum_{q-q' \leq 5} 2^{-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} 2^q \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \\
&\lesssim \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta D\|_{L_x^2},
\end{aligned} \tag{7.45}$$

for any $q \in \mathbb{Z}$. Hence,

$$\begin{aligned}
&-L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \sum_{q \in \mathbb{Z}} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \nabla \delta Q\|_{L_x^2} \sum_{q' \in \mathbb{Z}} 2^{\frac{q-q'}{2}} 1_{(-\infty, 5]}(q-q') 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \left(\sum_{q \in \mathbb{Z}} \left| \sum_{q' \in \mathbb{Z}} 2^{q-q'} 1_{(-\infty, 5]}(q-q') 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and by convolution

$$\begin{aligned}
&\left(\sum_{q \in \mathbb{Z}} \left| \sum_{q' \in \mathbb{Z}} 2^{q-q'} 1_{(-\infty, 5]}(q-q') 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \delta D\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{q \leq 5} 2^q \right) \left(\sum_{q \in \mathbb{Z}} 2^{-q} \|\dot{S}_q \delta D\|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}},
\end{aligned}$$

so that

$$\begin{aligned}
-L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \|\Delta Q_2\|_{L_x^2} \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned} \tag{7.46}$$

Summarizing, it remains to control

$$\mathcal{A}_1 + \mathcal{A}_3 + \mathcal{A}_4 - L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^3(Q_2, \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}.$$

Now, observing that

$$\mathcal{A}_1 = -L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q (\delta D Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = -L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q^t (\delta D Q_2), \dot{\Delta}_q^t \Delta \delta Q \rangle_{L_x^2}$$

$$= -L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q(Q_2 \delta D), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = \mathcal{A}_2,$$

we estimate \mathcal{A}_1 with the previous inequalities, so that it remains to control

$$\begin{aligned} \mathcal{A}_3 + \mathcal{A}_4 - 2L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \mathcal{J}_q^3(Q_2, \delta D) &= \mathcal{A}_3 + \mathcal{A}_4 - \\ &- 2L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta D \dot{\Delta}_q \Delta \delta Q \}. \end{aligned}$$

Now, let us consider $\mathcal{A}_3 = L\xi \langle Q_2 \Delta \delta Q, \nabla \delta u \rangle$. We proceed along the lines used before, namely we use the decomposition given by (7.40):

$$\mathcal{A}_3 = L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q(Q_2 \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} = L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \sum_{i=1}^4 \langle \mathcal{J}_q^i(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2}.$$

When $i = 1$, proceeding as for (7.41), we have

$$\begin{aligned} &2^{-q} \langle \mathcal{J}_q^1(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &\lesssim \|\nabla Q_2\|_{L_x^2}^{\frac{1}{2}} \|\Delta Q_2\|_{L_x^2}^{\frac{1}{2}} \sum_{|q-q'| \leq 5} \|\dot{\Delta}_{q'} \nabla \delta Q\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \nabla \delta u\|_{L_x^2}, \end{aligned}$$

thus, considering the sum over $q \in \mathbb{Z}$ as in (7.42), we deduce that

$$\begin{aligned} L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^1(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &+ C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.47)$$

Proceeding as for proving (7.43), when $i = 2$, we get

$$2^{-q} \langle \mathcal{J}_q^2(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \lesssim \sum_{|q-q'| \leq 5} 2^{-\frac{q}{2}} \|\dot{\Delta}_{q'} \delta u\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q \Delta \delta Q\|_{L_x^2} \|\Delta Q_2\|_{L_x^2}$$

for every $q \in \mathbb{Z}$. Thus, as in (7.44), it turns out that

$$L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^2(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \quad (7.48)$$

Finally, with the same strategy as for (7.45), we observe that

$$\begin{aligned} &2^{-q} \langle \mathcal{J}_q^4(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &\lesssim \|\Delta Q_2\|_{L_x^2} 2^{-q} \|\dot{\Delta}_q \delta u\|_{L_x^2}^2 \sum_{q-q' \leq 5} 2^{\frac{q-q'}{2}} 2^{-\frac{q'+2}{2}} \|\dot{S}_{q'+2} \Delta \delta Q\|_{L_x^2}, \end{aligned}$$

hence, as for (7.46), we obtain

$$\begin{aligned} L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} &\lesssim \|\Delta Q_2\|_{L_x^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.49)$$

Summarizing all the previous considerations, we note that it remains to control

$$\begin{aligned} & \mathcal{A}_4 + L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \left[\langle \mathcal{J}_q^3(Q_2, \Delta\delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} - 2 \int_{\mathbb{R}^2} \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta D \dot{\Delta}_q \Delta \delta Q \} \right] = \\ & = \mathcal{A}_4 + L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} \left[\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \nabla \delta u \} - 2 \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta D \dot{\Delta}_q \Delta \delta Q \} \right]. \end{aligned}$$

We handle the last term \mathcal{A}_4 arguing as for \mathcal{A}_3 , since \mathcal{A}_4 is given by

$$\begin{aligned} L\xi \langle \dot{\Delta}_q (\Delta \delta Q Q_2), \dot{\Delta}_q \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &= L\xi \langle \dot{\Delta}_q (Q_2 \Delta \delta Q), {}^t \dot{\Delta}_q \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\ &= L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \sum_{i=1}^4 \langle \mathcal{J}_q^i(Q_2, \Delta \delta Q), \dot{\Delta}_q {}^t \nabla \delta u \rangle_{L_x^2}. \end{aligned}$$

The terms related to $i = 1, 2, 4$ are estimated similarly as \mathcal{A}_3 . Hence it remains to evaluate

$$\begin{aligned} & L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \left\{ \langle \mathcal{J}_q^3(Q_2, \Delta \delta Q), \dot{\Delta}_q {}^t \nabla \delta u \rangle_{L_x^2} + \right. \\ & \left. + \int_{\mathbb{R}^2} \left[\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \nabla \delta u \} - 2 \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta D \dot{\Delta}_q \Delta \delta Q \} \right] \right\} = \\ & = 2L\xi \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} \left[\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \delta D \} - \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta D \dot{\Delta}_q \Delta \delta Q \} \right] = 0, \end{aligned}$$

which is a null series since the trace acts on symmetric matrices, so that we can permute their order.

Estimate of $\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$

Now we want to estimate $\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$, namely

$$-L \langle \delta \Omega Q_2 - Q_2 \delta \Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} - L \langle Q_2 \Delta \delta Q - \Delta \delta Q Q_2, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}}.$$

First let us consider

$$\begin{aligned} \mathcal{B}_2 &= L \langle Q_2 \delta \Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \\ &= L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q (\delta \Omega Q_2), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} = L \sum_{q \in \mathbb{Z}} 2^{-q} \sum_{i=1}^4 \langle \mathcal{J}_q^i(Q_2, \delta \Omega), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2}. \end{aligned}$$

Proceeding exactly as for proving (7.42), (7.44) and (7.46), with $\delta \Omega$ instead of δD , the following estimates are obtained:

$$\begin{aligned} L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^1(Q_2, \delta \Omega), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &\quad + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2, \\ L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^2(Q_2, \delta \Omega), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} &\lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned}$$

and

$$L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^4(Q_2, \delta\Omega), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \lesssim \|\Delta Q_2\|_{L_x^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2.$$

Now observing that

$$\mathcal{B}_1 = -L \langle \delta\Omega Q_2, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} = -L \langle {}^t(\delta\Omega Q_2), {}^t \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} = L \langle Q_2 \delta\Omega, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} = \mathcal{B}_2,$$

it remains to control

$$\begin{aligned} & \mathcal{B}_3 + \mathcal{B}_4 + 2L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \mathcal{J}_q^3(Q_2, \delta\Omega), \dot{\Delta}_q \Delta \delta Q \rangle_{L_x^2} \\ &= \mathcal{B}_3 + \mathcal{B}_4 + 2L \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta\Omega \dot{\Delta}_q \Delta \delta Q \}. \end{aligned}$$

Now, we turn to \mathcal{B}_3 :

$$\begin{aligned} -\mathcal{B}_3 &= L \langle Q_2 \Delta \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} = L \sum_{q \in \mathbb{Z}} 2^{-q} \langle \dot{\Delta}_q (Q_2 \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} \\ &= L \sum_{q \in \mathbb{Z}} 2^{-q} \sum_{i=1}^4 \langle \mathcal{J}_q^i(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2}. \end{aligned}$$

We remark that $\mathcal{B}_3 = -\mathcal{A}_3/\xi$, hence the terms related to $i = 1, 2, 4$ are estimated as in (7.47), (7.48) and (7.49). Thus it remains to control

$$\begin{aligned} & \mathcal{B}_4 + L \sum_{q \in \mathbb{Z}} 2^{-q} [\langle \mathcal{J}_q^3(Q_2, \Delta \delta Q), \dot{\Delta}_q \nabla \delta u \rangle_{L_x^2} + 2 \int_{\mathbb{R}^2} \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta\Omega \dot{\Delta}_q \Delta \delta Q \}] = \\ &= \mathcal{B}_4 + L \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} [\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \nabla \delta u \} + 2 \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta\Omega \dot{\Delta}_q \Delta \delta Q \}]. \end{aligned}$$

Observing that $\mathcal{B}_4 = -\mathcal{A}_4/\xi$ we argue as for \mathcal{B}_3 , hence it remains to evaluate

$$\begin{aligned} & L \sum_{q \in \mathbb{Z}} 2^{-q} \left\{ \langle \mathcal{J}_q^3(Q_2, \Delta \delta Q) \dot{\Delta}_q {}^t \nabla \delta u \rangle_{L_x^2} + \right. \\ & \quad \left. + \int_{\mathbb{R}^2} [\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \nabla \delta u \} + 2 \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta\Omega \dot{\Delta}_q \Delta \delta Q \}] \right\} = \\ &= 2L \sum_{q \in \mathbb{Z}} 2^{-q} \int_{\mathbb{R}^2} [\text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \Delta \delta Q \dot{\Delta}_q \delta\Omega \} + \text{tr} \{ \dot{S}_{q-1} Q_2 \dot{\Delta}_q \delta\Omega \dot{\Delta}_q \Delta \delta Q \}] = 0, \end{aligned}$$

where for the cancellation we used that $\dot{S}_{q-1} Q_2$ and $\dot{\Delta}_q \Delta \delta Q$ are symmetric while $\dot{\Delta}_q \delta\Omega$ is skew-symmetric.

One-logarithmic Estimates

In this subsection, we evaluate the terms of (7.37) which are related to the single-logarithmic term of the equality (7.38).

Estimate of $\langle \delta Q \text{tr} \{ Q_2 \nabla u_2 \}, \Delta \delta Q \rangle$. Let us fix a positive real number $N > 0$ and split the considered

term into two parts, the high and the low frequencies:

$$\begin{aligned} & \langle \delta Q \operatorname{tr}\{Q_2 \nabla u_2\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \\ &= \langle \delta Q \operatorname{tr}\{\dot{S}_N Q_2 \nabla u_2\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} + \langle \delta Q \operatorname{tr}\left\{\sum_{q \geq N} \dot{\Delta}_q Q_2 \nabla u_2\right\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}}. \end{aligned}$$

At first we deal with the low frequencies, observing that

$$\begin{aligned} \langle \delta Q \operatorname{tr}\{\dot{S}_N Q_2 \nabla u_2\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} &\lesssim \|\delta Q \operatorname{tr}\{\dot{S}_N Q_2 \nabla u_2\}\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\dot{S}_N Q_2 \nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\dot{S}_N Q_2\|_{L_x^\infty} \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}, \end{aligned}$$

hence, by Theorem 7.4.5, we get

$$\begin{aligned} & \langle \delta Q \operatorname{tr}\{\dot{S}_N Q_2 \nabla u_2\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} (\|Q_2\|_{L_x^2} + \sqrt{N} \|\nabla Q_2\|_{L_x^2}) \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim (1 + N) \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|\nabla u_2\|_{L_x^2}^2 + C_\Gamma \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

For the high frequencies, we proceed as follows:

$$\begin{aligned} & \langle \delta Q \operatorname{tr}\left\{\sum_{q \geq N} \dot{\Delta}_q Q_2 \nabla u_2\right\}, \Delta \delta Q \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta Q \operatorname{tr}\left\{\sum_{q \geq N} \dot{\Delta}_q Q_2 \nabla u_2\right\}\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\delta Q\|_{\dot{H}^{\frac{3}{4}}} \left\|\sum_{q \geq N} \dot{\Delta}_q Q_2 \nabla u_2\right\|_{\dot{H}^{-\frac{1}{4}}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|(Q_1, Q_2)\|_{L_x^2}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L_x^2}^{\frac{3}{4}} \left\|\sum_{q \geq N} \dot{\Delta}_q Q_2\right\|_{\dot{H}^{\frac{3}{4}}} \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|(Q_1, Q_2)\|_{L_x^2}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L_x^2}^{\frac{3}{4}} \left(\sum_{q \geq N} 2^{\frac{3}{4}q} \|\dot{\Delta}_q Q_2\|_{L_x^2}\right) \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|(Q_1, Q_2)\|_{L_x^2}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L_x^2}^{\frac{3}{4}} \left(\sum_{q \geq N} 2^{-\frac{q}{4}} \|\dot{\Delta}_q \nabla Q_2\|_{L_x^2}\right) \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|(Q_1, Q_2)\|_{L_x^2}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L_x^2}^{\frac{3}{4}} \left(\sum_{q \geq N} 2^{-\frac{q}{4}}\right) \|\nabla Q_2\|_{L_x^2} \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim 2^{-\frac{N}{4}} \|(Q_1, Q_2)\|_{L_x^2}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L_x^2}^{\frac{3}{4}} \|\nabla Q_2\|_{L_x^2} \|\nabla u_2\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}. \end{aligned}$$

Now, fixing $t > 0$ arbitrary, and taking $N = N(t) := \lceil \ln(1 + e + 1/\Phi(t)) \rceil > 0$, where $\lceil \cdot \rceil$ is the ceiling function, we get

$$\begin{aligned} & \langle \delta Q(t) \operatorname{tr}\{Q_2(t) \nabla u_2(t)\}, \Delta \delta Q(t) \rangle_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|(Q_2, \nabla Q_2)(t)\|_{L_x^2}^2 \|\nabla u_2(t)\|_{L_x^2}^2 \Phi(t) \ln\left(1 + e + \frac{1}{\Phi(t)}\right) + \\ &+ \|(Q_1, Q_2)(t)\|_{L_x^2}^{\frac{1}{2}} \|\nabla(Q_1, Q_2)(t)\|_{L_x^2}^{\frac{3}{2}} \|\nabla Q_2(t)\|_{L_x^2}^2 \|\nabla u_2(t)\|_{L_x^2}^2 \Phi(t) + C_\Gamma \|\Delta \delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Thus we have obtained a one-logarithmic term of (7.38). Similarly, we can handle the estimate of the following elements:

$$\begin{aligned}
& +2L\xi\langle\delta Q\operatorname{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}+2L\xi\langle\delta Q\operatorname{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}+ \\
& +2L\xi\langle\delta Q\operatorname{tr}(Q_2\nabla\delta u),\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}+2L\xi\langle Q_2\operatorname{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}+ \\
& +2L\xi\langle Q_2\operatorname{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}-2L\xi\langle\delta Q\operatorname{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}- \\
& -2L\xi\langle\delta Q\operatorname{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}-2L\xi\langle\delta Q\operatorname{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}- \\
& -2L\xi\langle Q_2\operatorname{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}-2L\xi\langle\delta Q\operatorname{tr}\{Q_2\Delta Q_2\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}- \\
& -2L\xi\langle Q_2\operatorname{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

Double-Logarithmic Estimates

In this subsection, we perform the most challenging estimate. Now, we want to control $\mathcal{E}_1 + \mathcal{E}_2$, namely

$$\begin{aligned}
\mathcal{E}_1 + \mathcal{E}_2 &= 2L\xi\left(\langle Q_2\operatorname{tr}\{Q_2\nabla\delta u\},\Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}}-\langle Q_2\operatorname{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}}\right) \\
&= 2L\xi\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\operatorname{tr}\{\dot{\Delta}_q(Q_2\operatorname{tr}\{Q_2\nabla\delta u\})\dot{\Delta}_q\Delta\delta Q-\dot{\Delta}_q(Q_2\operatorname{tr}\{Q_2\Delta\delta Q\})\dot{\Delta}_q\nabla\delta u\} \\
&= 2L\xi\sum_{i=1}^4\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\operatorname{tr}\{\mathcal{J}_q^i(Q_2,\operatorname{tr}\{Q_2\nabla\delta u\})\operatorname{Id}\dot{\Delta}_q\Delta\delta Q-\mathcal{J}_q^i(Q_2,\operatorname{tr}\{Q_2\Delta\delta Q\})\operatorname{Id}\dot{\Delta}_q\nabla\delta u\}.
\end{aligned} \tag{7.50}$$

We will see that there are elements inside this decomposition that generate the double-logarithmic term in (7.38). We proceed by considering the indexes $i = 1, 2, 3, 4$, step by step.

Estimate of \mathcal{J}_q^1 . We start with the term of (7.50) related to $i = 1$, passing through the following decomposition:

$$\begin{aligned}
& \sum_{j=1}^4\sum_{|q-q'|\leq 5}\int_{\mathbb{R}_2}\operatorname{tr}\{([\dot{\Delta}_q,\dot{S}_{q'-1}Q_2]\operatorname{tr}\{\mathcal{J}_{q'}^j(Q_2,\nabla\delta u)\})\operatorname{Id}\dot{\Delta}_q\Delta\delta Q- \\
& -([\dot{\Delta}_q,\dot{S}_{q'-1}Q_2]\operatorname{tr}\{\mathcal{J}_{q'}^j(Q_2,\Delta\delta Q)\})\operatorname{Id}\dot{\Delta}_q\nabla\delta u\}.
\end{aligned} \tag{7.51}$$

When $j = 1$, we have

$$\begin{aligned}
\mathcal{I}_1^1(q,q',q'') &:= \int_{\mathbb{R}_2}\left\{\left([\dot{\Delta}_q,\dot{S}_{q'-1}Q_2]\operatorname{tr}\{[\dot{\Delta}_{q'},\dot{S}_{q''-1}Q_2]\dot{\Delta}_{q''}\nabla\delta u\})\operatorname{Id}\right)\dot{\Delta}_q\Delta\delta Q+ \\
& -\left([\dot{\Delta}_q,\dot{S}_{q'-1}Q_2]\operatorname{tr}\{[\dot{\Delta}_{q'},\dot{S}_{q''-1}Q_2]\dot{\Delta}_{q''}\Delta\delta Q\})\operatorname{Id}\right)\dot{\Delta}_q\nabla\delta u\} \\
&\lesssim 2^{-q}\|\dot{S}_{q'-1}\nabla Q_2\|_{L_x^\infty}2^{-q'}\|\dot{S}_{q''-1}\nabla Q_2\|_{L_x^\infty}\|\dot{\Delta}_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L_x^2}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L_x^2} \\
&\lesssim 2^{-q-q'}2^{\frac{q'}{2}}\|\dot{S}_{q'-1}\nabla Q_2\|_{L_x^4}2^{\frac{q''}{2}}\|\dot{S}_{q''-1}\nabla Q_2\|_{L_x^4}2^{q''}\times \\
& \quad \times\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L_x^2}2^q\|\dot{\Delta}_q(\delta u,\nabla\delta Q)\|_{L_x^2},
\end{aligned}$$

which yields

$$\mathcal{I}_1^1(q, q', q'') \lesssim 2^{\frac{3q''}{2} - \frac{q'}{2}} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2}. \quad (7.52)$$

Hence, taking the sum, we deduce that

$$\begin{aligned} 2L\xi \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} \sum_{|q'-q''| \leq 5} 2^{-q} \mathcal{I}_1^1(q, q', q'') &\lesssim \\ &\lesssim \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} \sum_{|q'-q''| \leq 5} 2^{-q} 2^{\frac{3q''}{2} - \frac{q'}{2}} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \times \\ &\quad \times \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \sum_{q \in \mathbb{Z}} \sum_{|q-q''| \leq 10} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{L_x^2}^2 \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.53)$$

Now, when $j = 2$ in (7.51), we remark that

$$\begin{aligned} \mathcal{I}_2^1(q, q', q'') &:= \\ &\int_{\mathbb{R}_2} \text{tr} \left\{ \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2) \dot{\Delta}_{q'} \dot{\Delta}_{q''} \nabla \delta u \} \text{Id} \right) \dot{\Delta}_q \Delta \delta Q + \right. \\ &\quad \left. + \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2) \dot{\Delta}_{q'} \dot{\Delta}_{q''} \Delta \delta Q \} \text{Id} \right) \dot{\Delta}_q \nabla \delta u \right\} \\ &\lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} \|\dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2\|_{L_x^\infty} \times \\ &\quad \times \|\dot{\Delta}_q \dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} 2^{-q'} \|\dot{S}_{q''-1} \nabla Q_2 - \dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} \times \\ &\quad \times \|\dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{-q-q'} 2^{\frac{q'}{2}} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^4} 2^{\frac{q''}{2}} \|\dot{S}_{q''-1} \nabla Q_2\|_{L_x^4} \times \\ &\quad \times 2^{q''} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} 2^q \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim 2^{\frac{3q''}{2} - \frac{q'}{2}} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2}, \end{aligned}$$

which is equivalent to (7.52). Hence, proceeding as in (7.53), we get

$$\begin{aligned} 2L\xi \sum_{q \in \mathbb{Z}} \sum_{\substack{|q-q'| \leq 5 \\ |q'-q''| \leq 5}} 2^{-q} \mathcal{I}_2^1(q, q', q'') &\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 \\ &\quad + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Concerning the term of (7.51) related to $j = 4$, we have

$$\begin{aligned}
\mathcal{I}_4^1(q, q', q'') &:= \int_{\mathbb{R}_2} \operatorname{tr} \left\{ \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \operatorname{tr} \{ \dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \nabla \delta u) \} \operatorname{Id} \right) \dot{\Delta}_q \Delta \delta Q - \right. \\
&\quad \left. - \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \operatorname{tr} \{ \dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \Delta \delta Q) \} \operatorname{Id} \right) \dot{\Delta}_q \nabla \delta u \right\} \\
&\lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} \|\dot{\Delta}_q (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \nabla \delta u, \dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} 2^{q \times} \\
&\quad \times \|\dot{\Delta}_q (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \nabla \delta u, \dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \Delta \delta Q)\|_{L_x^1} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q''} Q_2\|_{L_x^2} \|\dot{S}_{q''+2} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} 2^{-2q''} \|\dot{\Delta}_{q''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q''+2} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim 2^{-2q''} 2^{q'} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} \Delta Q_2\|_{L_x^2} 2^{q''} \|\dot{S}_{q''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim 2^{q'-q''} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|\dot{S}_{q''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2}
\end{aligned} \tag{7.54}$$

Hence

$$\begin{aligned}
2L\xi \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} \sum_{q'' \geq q'-5} 2^{-q} \mathcal{I}_4^1(q, q', q'') &\lesssim \\
&\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \sum_{q \in \mathbb{Z}} 2^{-\frac{q}{2}} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \times \\
&\quad \times \sum_{|q-q'| \leq 5} 2^{\frac{q'-q}{2}} \sum_{q'' \geq q'-5} 2^{\frac{q'-q''}{2}} 2^{-\frac{q''}{2}} \|\dot{S}_{q''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \\
&\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \sum_{q \in \mathbb{Z}} 2^{-\frac{q}{2}} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \times \\
&\quad \times \sum_{q'' \geq q-10} 2^{\frac{q-q''}{2}} 2^{-\frac{q''}{2}} \|\dot{S}_{q''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \\
&\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
&\quad \times \left[\sum_{q \in \mathbb{Z}} \left| \sum_{q-q'' \leq 10} 2^{\frac{q-q''}{2}} 2^{-\frac{q''}{2}} \|\dot{S}_{q''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \right|^2 \right]^{\frac{1}{2}} \\
&\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \left(\sum_{q \leq 10} 2^{\frac{q}{2}} \right) \left(\sum_{q \in \mathbb{Z}} 2^{-q} \|\dot{S}_q (\delta u, \nabla \delta Q)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\
&\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned} \tag{7.55}$$

Concerning (7.51), it remains to control the term related to $j = 3$. We fix $0 < \varepsilon \leq 5/6$ and we consider the low frequencies $q \leq N$, for some suitable positive $N \geq 1$ (so that $1 + \sqrt{N} < 2\sqrt{N}$):

$$\begin{aligned}
\mathcal{I}_3^1(q, q') &:= \int_{\mathbb{R}_2} \operatorname{tr} \left\{ \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \operatorname{tr} \{ \dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \nabla \delta u \} \operatorname{Id} \right) \dot{\Delta}_q \Delta \delta Q - \right. \\
&\quad \left. - \left([\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \operatorname{tr} \{ \dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \Delta \delta Q \} \operatorname{Id} \right) \dot{\Delta}_q \nabla \delta u \right\} \\
&\lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^{\frac{2}{\varepsilon}}} \|\dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \nabla \delta u, \dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \Delta \delta Q\|_{L_x^{\frac{2}{1-\varepsilon}}} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2}
\end{aligned}$$

$$\lesssim \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^{\frac{2}{\varepsilon}}} \|\dot{S}_{q'-1} Q_2\|_{L_x^\infty} 2^{-q} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}.$$

Thanks to Theorem 7.4.5, we get

$$\begin{aligned} \|\dot{S}_{q'-1} Q_2\|_{L_x^\infty} &\lesssim (1 + \sqrt{q'-1}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \\ &\lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \lesssim \sqrt{N} \|(Q_2, \nabla Q_2)\|_{L_x^2}, \end{aligned}$$

hence $\mathcal{I}_3^1(q, q')$ is bounded by

$$\mathcal{I}_3^1(q, q') \lesssim \sqrt{N} \|(Q_2, \nabla Q_2)\|_{L_x^{\frac{2}{\varepsilon}}} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}. \quad (7.56)$$

Now, we will need the following inequality, which will finally lead to the delicate double-logarithmic estimate:

$$\|(Q_2, \nabla Q_2)\|_{L_x^{\frac{2}{\varepsilon}}} \leq \frac{C}{\sqrt{\varepsilon}} \|(Q_2, \nabla Q_2)\|_{L_x^2}^\varepsilon \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^{1-\varepsilon},$$

This is a consequence of Lemma 7.4.6, imposing $p = 1/\varepsilon$, where C is a positive constant independent of ε and Q_2 . We will see that the double-logarithmic term comes out of a suitable choice of ε in terms of N . Again, using Lemma 7.4.6 we have

$$\begin{aligned} \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}} &\leq \frac{C}{1-\varepsilon} \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^{1-\varepsilon} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^\varepsilon \\ &\leq 6C \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^{1-\varepsilon} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^\varepsilon, \end{aligned}$$

since $\varepsilon \leq 5/6$. Hence (7.56) becomes

$$\begin{aligned} \mathcal{I}_3^1(q, q') &\lesssim \sqrt{\frac{N}{\varepsilon}} \|(Q_2, \nabla Q_2)\|_{L_x^2}^{1+\varepsilon} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^{1-\varepsilon} \times \\ &\quad \times \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^{1-\varepsilon} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^\varepsilon \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}, \end{aligned} \quad (7.57)$$

thus, since $ab \leq a^{2/(1-\varepsilon)} + b^{2/(1+\varepsilon)}$, we deduce

$$\begin{aligned} \mathcal{I}_3^1(q, q') &\lesssim \left(\frac{N}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \|(Q_2, \nabla Q_2)\|_{L_x^2}^{\frac{2(1+\varepsilon)}{1-\varepsilon}} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^2 + \\ &\quad + \min\{C_\nu, C_\Gamma\} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^{\frac{2\varepsilon}{1+\varepsilon}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^{\frac{2}{1+\varepsilon}} \\ &\lesssim \left(\frac{N}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}} \|(Q_2, \nabla Q_2)\|_{L_x^2}^{\frac{2(1+\varepsilon)}{1-\varepsilon}} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^2 + \\ &\quad + \min\{C_\nu, C_\Gamma\} \left(\|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^2 + \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^2 \right). \end{aligned}$$

Imposing $\varepsilon = (1 + \ln N)^{-1}$ and observing that $\frac{1}{1-\varepsilon} = 1 + 1/\ln N$

$$N^{\frac{1}{1-\varepsilon}} = N N^{\frac{1}{\ln N}} = eN, \quad \varepsilon^{-\frac{1}{1-\varepsilon}} = \varepsilon^{-1} \varepsilon^{-\frac{\varepsilon}{1-\varepsilon}} = (1 + \ln N) e^{\frac{\varepsilon}{1-\varepsilon} \ln \frac{1}{\varepsilon}} \lesssim (1 + \ln N),$$

we obtain:

$$\mathcal{I}_3^1(q, q') \lesssim N(1 + \ln N) \max\{\|(Q_2, \nabla Q_2)\|_{L_x^2}^6, 1\} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^2 +$$

$$+ \min \left\{ C_\nu, C_{\Gamma,L} \right\} \left(\|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2}^2 + \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2}^2 \right),$$

which yields

$$\begin{aligned} & \sum_{q \leq N} \sum_{|q-q'| \leq 5} 2^{-q} \mathcal{I}_3^1(q, q') \\ & \lesssim N(1 + \ln N) \max \left\{ \|(Q_2, \nabla Q_2)\|_{L_x^2}^6, 1 \right\} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ & \quad + C_\nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

For the high frequencies, namely for $q > N \geq 1$, we proceed as follows:

$$\begin{aligned} \mathcal{I}_3^1(q, q') & \lesssim 2^{-q} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^\infty} \times \\ & \quad \times \|\dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \nabla\delta u, \dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \Delta\delta Q\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim 2^{-q} (1 + \sqrt{q'}) \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|\dot{S}_{q'-1} Q_2\|_{L_x^\infty} \times \\ & \quad \times \|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim 2^{-q} (1 + \sqrt{q'})^2 \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \times \\ & \quad \times \|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim q' \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla\delta Q)\|_{L_x^2}, \end{aligned}$$

which implies

$$\begin{aligned} & \sum_{q > N} \sum_{|q-q'| \leq 5} 2^{-q} \mathcal{I}_3^1(q, q') \lesssim \\ & \lesssim \sum_{q > N} \sum_{|q-q'| \leq 5} 2^{-2q} q' \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \times \\ & \quad \times \|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla\delta Q)\|_{L_x^2} \\ & \lesssim \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \times \\ & \quad \times \|(\delta u, \nabla\delta Q)\|_{L_x^2} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \sum_{q > N} \sum_{|q-q'| \leq 5} 2^{-\frac{3}{2}q + \frac{1}{2}q'} \\ & \lesssim 2^{-\frac{N}{2}} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla\delta Q)\|_{L_x^2} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}. \end{aligned}$$

Summarizing, we get

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} 2^{-q} \mathcal{I}_3^1(q, q') \\ & \lesssim N(1 + \ln N) \max \left\{ \|(Q_2, \nabla Q_2)\|_{L_x^2}^6, 1 \right\} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ & \quad + 2^{-N} \|(\nabla Q_2, \Delta Q_2)\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|(u_1, u_2, \nabla Q_1, \nabla Q_2)\|_{L_x^2}^2 + \\ & \quad + C_\nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \tag{7.58}$$

Choosing $N = N(t) := \lceil \ln(1 + e + 1/\Phi(t)) \rceil$ (thus $\varepsilon < 1/(1 + \ln \ln\{1 + e\}) < 5/6$) where with $\lceil \cdot \rceil$

we denote the ceiling function, relation (7.58) implies

$$\begin{aligned}
\sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} 2^{-q} \mathcal{I}_1^3(q, q') &\lesssim \max \{ \| (Q_2, \nabla Q_2) \|_{L_x^2}^6, 1 \} \times \\
&\times \| (\nabla Q_2, \Delta Q_2) \|_{L_x^2}^2 \| (\delta u, \nabla \delta Q) \|_{\dot{H}^{-\frac{1}{2}}}^2 \ln \left(e + \frac{1}{\Phi(t)} \right) \left(1 + \ln \ln \left(e + \frac{1}{\Phi(t)} \right) \right) + \\
&+ \| (\nabla Q_2, \Delta Q_2) \|_{L_x^2}^2 \| (Q_2, \nabla Q_2) \|_{L_x^2}^2 \| (u_1, u_2, \nabla Q_1, \nabla Q_2) \|_{L_x^2}^2 + \\
&+ \Phi(t) + C_\nu \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned} \tag{7.59}$$

Estimate of \mathcal{J}_q^2 Now, we handle the term of (7.50) related to $i = 2$, namely

$$\begin{aligned}
\sum_{j=1}^4 \sum_{|q-q'| \leq 5} \int_{\mathbb{R}_2} \text{tr} \{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q \mathcal{J}_{q'}^j(Q_2, \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q + \\
(\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q \mathcal{J}_{q'}^j(Q_2, \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \}.
\end{aligned} \tag{7.60}$$

When $j = 1$, we have

$$\begin{aligned}
\mathcal{I}_1^2(q, q', q'') &:= \int_{\mathbb{R}_2} \text{tr} \{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q ([\dot{\Delta}_{q'}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q + \\
&- (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q ([\dot{\Delta}_{q'}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \} \\
&\lesssim \| \dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2 \|_{L_x^\infty} \times \\
&\quad \times \| \dot{\Delta}_q ([\dot{\Delta}_{q'}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)) \|_{L_x^2} \| \dot{\Delta}_q (\nabla \delta u, \Delta \delta Q) \|_{L_x^2} \\
&\lesssim 2^{-\frac{q}{2}} \| \dot{S}_{q'-1} \nabla Q_2 - \dot{S}_{q-1} \nabla Q_2 \|_{L_x^4} 2^{\frac{q}{2}} \times \\
&\quad \times \| \dot{\Delta}_q ([\dot{\Delta}_{q'}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)) \|_{L_x^{\frac{4}{3}}} \| \dot{\Delta}_q (\nabla \delta u, \Delta \delta Q) \|_{L_x^2} \\
&\lesssim \| \nabla Q_2 \|_{L_x^4} 2^{-q'} \| \dot{S}_{q''-1} \nabla Q_2 \|_{L_x^4} \| \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q) \|_{L_x^2} \| \dot{\Delta}_q (\nabla \delta u, \Delta \delta Q) \|_{L_x^2} \\
&\lesssim 2^{-q'+q''+q} \| \nabla Q_2 \|_{L_x^2} \| \Delta Q_2 \|_{L_x^2} \| \dot{\Delta}_{q''} (\delta u, \nabla \delta Q) \|_{L_x^2} \| \dot{\Delta}_q (\delta u, \nabla \delta Q) \|_{L_x^2}.
\end{aligned}$$

Since $|q - q'| \leq 5$ and $|q' - q''| \leq 5$ then $-q' + q'' + q \simeq 3q''/2 - q'/2$, so that the last inequality is equivalent to (7.52). Hence, proceeding as in (7.53), we get

$$\begin{aligned}
2L\xi \sum_{q \in \mathbb{Z}} \sum_{\substack{|q-q'| \leq 5 \\ |q'-q''| \leq 5}} 2^{-q} \mathcal{I}_1^2(q, q', q'') &\lesssim \| \nabla Q_2 \|_{L_x^2}^2 \| \Delta Q_2 \|_{L_x^2}^2 \| (\delta u, \nabla \delta Q) \|_{\dot{H}^{-\frac{1}{2}}}^2 \\
&+ C_\nu \| \nabla \delta u \|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \| \Delta \delta Q \|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

When $j = 2$, we observe that

$$\begin{aligned}
\mathcal{I}_2^2(q, q', q'') &:= \\
&\int_{\mathbb{R}_2} \text{tr} \{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2) \dot{\Delta}_{q'} \dot{\Delta}_{q''} \nabla \delta u \} \dot{\Delta}_q \Delta \delta Q + \\
&- (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2) \dot{\Delta}_{q'} \dot{\Delta}_{q''} \Delta \delta Q \} \dot{\Delta}_q \nabla \delta u \} \\
&\lesssim \| \dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2 \|_{L_x^\infty} \| \dot{S}_{q''-1} Q_2 - \dot{S}_{q'-1} Q_2 \|_{L_x^\infty} \times
\end{aligned}$$

$$\begin{aligned}
& \times \|\dot{\Delta}_{q'} \dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|L_x^2\| \dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{-\frac{q+q'}{2}} \|\dot{S}_{q'-1} \nabla Q_2 - \dot{S}_{q-1} \nabla Q_2\|_{L_x^4} \|\dot{S}_{q''-1} \nabla Q_2 - \dot{S}_{q'-1} \nabla Q_2\|_{L_x^4} \times \\
& \quad \times \|\dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{\frac{q'}{2} + \frac{q}{2}} \|\nabla Q_2\|_{L_x^4}^2 \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{\frac{q'}{2} + \frac{q}{2}} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2}.
\end{aligned}$$

Since $|q - q'| \leq 5$ and $|q' - q''| \leq 5$ then $q'/2 + q/2 \simeq 3q''/2 - q'/2$, so that the last inequality is equivalent to (7.52). Hence, proceeding as in (7.53), we get

$$\begin{aligned}
2L\xi \sum_{q \in \mathbb{Z}} \sum_{\substack{|q-q'| \leq 5 \\ |q'-q''| \leq 5}} 2^{-q} \mathcal{I}_2^2(q, q', q'') & \lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
& + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

When $j = 4$:

$$\begin{aligned}
\mathcal{I}_4^2(q, q', q'') & := \int_{\mathbb{R}_2} \left\{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q + \right. \\
& \quad \left. - (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \right\} \\
& \lesssim \|\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} (\nabla \delta u, \Delta \delta Q))\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{q'} \|\dot{S}_{q'-1} \nabla Q_2 - \dot{S}_{q-1} \nabla Q_2\|_{L_x^2} \times \\
& \quad \times \|\dot{\Delta}_{q'} (\dot{\Delta}_{q''} Q_2 \dot{S}_{q''+2} (\nabla \delta u, \Delta \delta Q))\|_{L_x^1} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{q'-q''} \|\nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q''+2}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{q'-q''} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|\dot{S}_{q''+2}(\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2},
\end{aligned}$$

which is equivalent to the last inequality of (7.54). Thus, arguing as in (7.55), we deduce

$$\begin{aligned}
2L\xi \sum_{q \in \mathbb{Z}} \sum_{\substack{|q-q'| \leq 5 \\ q'' \geq q'-5}} 2^{-q} \mathcal{I}_4^2(q, q', q'') & \lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
& + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

When $j = 3$ we fix a real number $N > 1$ and we consider the low frequencies $q' \leq N$ as follows

$$\begin{aligned}
\mathcal{I}_3^2(q, q') & := \int_{\mathbb{R}_2} \text{tr} \left\{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q (\dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q + \right. \\
& \quad \left. - (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \text{tr} \{ \dot{\Delta}_q (\dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \right\} \\
& \lesssim \|\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_q (\dot{S}_{q'-1} Q_2 \dot{\Delta}_{q'} (\nabla \delta u, \Delta \delta Q))\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{-q} \|\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim \|\dot{S}_{q'-1} \Delta Q_2 - \dot{S}_{q-1} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2}.
\end{aligned} \tag{7.61}$$

If $q' \leq 1$ then $\|\dot{S}_{q'-1}Q_2\|_{L_x^\infty} \lesssim 2^{\frac{q'}{2}}\|\dot{S}_{q'-1}Q_2\|_{L_x^2} \leq \|Q_2\|_{L_x^2}$, while if $1 < q' \leq N$ we have

$$\|\dot{S}_{q'-1}Q_2\|_{L_x^\infty} \lesssim (\|Q_2\|_{L_x^2} + \sqrt{q'-1}\|\nabla Q_2\|_{L_x^2}) \lesssim (\|Q_2\|_{L_x^2} + \sqrt{N}\|\nabla Q_2\|_{L_x^2}),$$

thanks to Theorem 7.4.5. Therefore, we deduce that

$$\begin{aligned} \mathcal{I}_3^2(q, q') &\lesssim \|\Delta Q_2\|_{L_x^2} (\|Q_2\|_{L_x^2} + \sqrt{N}\|\nabla Q_2\|_{L_x^2}) \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim (1+N)\|\Delta Q_2\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2}^2 \\ &\quad + C_\nu \|\dot{\Delta}_{q'} \nabla \delta u\|_{L_x^2}^2 + C_{\Gamma, L} \|\dot{\Delta}_{q'} \Delta \delta Q\|_{L_x^2}^2. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{q' \leq N} \sum_{|q'-q| \leq 5} 2^{-q} \mathcal{I}_3^2(q, q') &\lesssim (1+N)\|\Delta Q_2\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &\quad + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.62)$$

For the high frequencies $q' > N$ we get,

$$\begin{aligned} \mathcal{I}_3^2(q, q') &\lesssim \|\dot{S}_{q'-1}Q_2 - \dot{S}_{q-1}Q_2\|_{L_x^\infty} \|\dot{\Delta}_q(\dot{S}_{q'-1}Q_2 \dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q))\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{-q} \|\dot{S}_{q'-1}\Delta Q_2 - \dot{S}_{q-1}\Delta Q_2\|_{L_x^2} \|\dot{S}_{q'-1}Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{\frac{q'-q}{2}} \|\Delta Q_2\|_{L_x^2} (1 + \sqrt{q'-1}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim (1 + \sqrt{q'-1}) \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}, \end{aligned} \quad (7.63)$$

therefore

$$\begin{aligned} \sum_{q' > N} \sum_{|q'-q| \leq 5} 2^{-q} \mathcal{I}_3^2(q, q') &\lesssim 2^{-N} \|\Delta Q_2\|_{L_x^2} \|\nabla Q_2\|_{L_x^2} \|(\delta u, \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim 2^{-2N} \|\Delta Q_2\|_{L_x^2}^2 \|\nabla Q_2\|_{L_x^2}^2 \|(\delta u, \delta Q)\|_{L_x^2}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.64)$$

Summarizing (7.62) and (7.64), we get

$$\begin{aligned} \sum_{q' \in \mathbb{Z}} \sum_{|q'-q| \leq 5} 2^{-q} \mathcal{I}_3^2(q, q') &\lesssim (1+N)\|\Delta Q_2\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &\quad + 2^{-2N} \|\Delta Q_2\|_{L_x^2}^2 \|\nabla Q_2\|_{L_x^2}^2 \|(\delta u, \delta Q)\|_{L_x^2}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned} \quad (7.65)$$

Now we define $N := \lceil \ln\{e + 1/\Phi(t)\}/2 \rceil$, obtaining

$$\begin{aligned} \sum_{q \in \mathbb{Z}} \sum_{|q'-q| \leq 5} 2^{-q} \mathcal{I}_3^2(q, q') &\lesssim \\ &\|\Delta Q_2(t)\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)(t)\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q(t))\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &\quad + C_{\Gamma, L} \|\Delta \delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\Delta Q_2(t)\|_{L_x^2}^2 \|\nabla Q_2(t)\|_{L_x^2}^2 \|(\delta u, \delta Q)(t)\|_{L_x^2}^2 \left(1 + \ln\left(e + \frac{1}{\Phi(t)}\right)\right). \end{aligned} \quad (7.66)$$

Estimate of \mathcal{J}_q^3 Now, let us deal with the term of (7.50) related to $i = 3$, namely

$$\begin{aligned} & \int_{\mathbb{R}_2} \operatorname{tr} \{ \dot{S}_{q-1} Q_2 \operatorname{tr} \{ \dot{\Delta}_q (Q_2 \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q - \dot{S}_{q-1} Q_2 \operatorname{tr} \{ \dot{\Delta}_q (Q_2 \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \} = \\ & = \sum_{j=1}^4 \int_{\mathbb{R}_2} \operatorname{tr} \{ \dot{S}_{q-1} Q_2 \operatorname{tr} \{ \mathcal{J}_{q'}^j(Q_2, \nabla \delta u) \} \dot{\Delta}_q \Delta \delta Q - \dot{S}_{q-1} Q_2 \operatorname{tr} \{ \mathcal{J}_{q'}^j(Q_2, \Delta \delta Q) \} \dot{\Delta}_q \nabla \delta u \}. \end{aligned} \quad (7.67)$$

Let us consider $j = 1$ and define

$$\begin{aligned} \mathcal{I}_1^3(q, q') := & \int_{\mathbb{R}_2} \operatorname{tr} \left\{ \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \nabla \delta u \right\} \dot{\Delta}_q \Delta \delta Q - \right. \\ & \left. - \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \Delta \delta Q \right\} \dot{\Delta}_q \nabla \delta u \right\}. \end{aligned}$$

We proceed as for proving (7.57): we fix a positive real $\varepsilon \in (0, 5/6]$ and we consider the low frequencies $q \leq N$, for a suitable positive $N \geq 1$.

$$\begin{aligned} \mathcal{I}_1^3(q, q') &= \int_{\mathbb{R}_2} \operatorname{tr} \left\{ \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \nabla \delta u \right\} \dot{\Delta}_q \Delta \delta Q \right. \\ & \quad \left. - \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ [\dot{\Delta}_q, \dot{S}_{q'-1} Q_2] \dot{\Delta}_{q'} \Delta \delta Q \right\} \dot{\Delta}_q \nabla \delta u \right\} \\ &\lesssim 2^{-q'} \|\dot{S}_{q-1} Q_2\|_{L_x^\infty} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^{\frac{2}{\varepsilon}}} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim (1 + \sqrt{N}) 2^{q-q'} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^{\frac{2}{\varepsilon}}} \times \\ & \quad \times \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim \sqrt{\frac{N}{\varepsilon}} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{S}_{q'-1} \nabla Q_2\|_{L_x^2}^\varepsilon \|\dot{S}_{q'-1} \Delta Q_2\|_{L_x^2}^{1-\varepsilon} \times \\ & \quad \times \|\dot{\Delta}_{q'}(\delta u, \nabla \delta Q)\|_{L_x^2}^{1-\varepsilon} \|\dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^\varepsilon \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}, \end{aligned}$$

which is equivalent to the last inequality of (7.57). Hence, arguing as for proving (7.59), we get

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{|q-q'| \leq 5} 2^{-q} \mathcal{I}_1^3(q, q') \lesssim \\ & \lesssim \max \left\{ \|(Q_2, \nabla Q_2)\|_{L_x^2}^6, 1 \right\} \times \\ & \quad \times \left(\|\nabla Q_2, \Delta Q_2\|_{L_x^2}^2 \|\delta u, \nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \left(1 + \ln \ln \left(1 + e + \frac{1}{\Phi(t)} \right) \right) \right) + \\ & + \|\nabla Q_2, \Delta Q_2\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|(u_1, u_2, \nabla Q_1, \nabla Q_2)\|_{L_x^2}^2 \Phi(t) \\ & \quad + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned}$$

Further on, when $j = 2$ in (7.67), let us consider the low frequencies $q \leq N$:

$$\begin{aligned} \mathcal{I}_2^3(q, q') &:= \int_{\mathbb{R}_2} \operatorname{tr} \left\{ \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \dot{\Delta}_q \dot{\Delta}_{q'} \nabla \delta u \right\} \dot{\Delta}_q \Delta \delta Q + \right. \\ & \quad \left. - \dot{S}_{q-1} Q_2 \operatorname{tr} \left\{ (\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2) \dot{\Delta}_{q'} \dot{\Delta}_{q'} \Delta \delta Q \right\} \dot{\Delta}_q \nabla \delta u \right\} \\ &\lesssim \|\dot{S}_{q-1} Q_2\|_{L_x^\infty} \|\dot{S}_{q'-1} Q_2 - \dot{S}_{q-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_q \dot{\Delta}_{q'}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \end{aligned}$$

$$\lesssim \|\dot{S}_{q-1}Q_2\|_{L_x^\infty} \|\dot{S}_{q'-1}\Delta Q_2 - \dot{S}_{q-1}\Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\delta u, \nabla\delta Q)\|_{L_x^2},$$

which is as the last inequalities of (7.61) (recalling that $q \sim q'$). Moreover for the high frequencies $q > N$

$$\begin{aligned} & \mathcal{I}_2^3(q, q') \\ & \lesssim \|\dot{S}_{q-1}Q_2\|_{L_x^\infty} \|\dot{S}_{q'-1}Q_2 - \dot{S}_{q-1}Q_2\|_{L_x^\infty} \|\dot{\Delta}_q\dot{\Delta}_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim (1 + \sqrt{q-1}) \|(Q_2, \nabla Q_2)\|_{L_x^2} 2^{-q} \|\dot{S}_{q'-1}\Delta Q_2 - \dot{S}_{q-1}\Delta Q_2\|_{L_x^2} \times \\ & \quad \times \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim (1 + \sqrt{q-1}) \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla\delta Q)\|_{L_x^2} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}, \end{aligned}$$

which is the equivalent to the last inequality (7.63). Hence, arguing as for proving (7.66), we get

$$\begin{aligned} & \sum_{q \in \mathbb{Z}} \sum_{|q'-q| \leq 5} 2^{-q} \mathcal{I}_2^3(q, q') \lesssim \|\Delta Q_2(t)\|_{L_x^2}^2 \|(Q_2, \nabla Q_2)(t)\|_{L_x^2}^2 \times \\ & \quad \times \|(\delta u, \nabla\delta Q(t))\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla\delta u(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta\delta Q(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ & \quad + \|\Delta Q_2(t)\|_{L_x^2}^2 \|\nabla Q_2(t)\|_{L_x^2}^2 \|(\delta u, \delta Q)(t)\|_{L_x^2}^2 \Phi(t) \left(1 + \ln\left(1 + e + \frac{1}{\Phi(t)}\right)\right). \end{aligned}$$

Now, when $j = 3$ in (7.67), we observe that

$$\begin{aligned} \mathcal{I}_3^3(q) & := \int_{\mathbb{R}_2} \left\{ \text{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\nabla\delta u\} \text{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\Delta\delta Q\} \right. \\ & \quad \left. - \text{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\Delta\delta Q\} \text{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\nabla\delta u\} \right\} = 0, \end{aligned}$$

for any $q \in \mathbb{Z}$. Thus it remains to control the $j = 4$ term, namely

$$\begin{aligned} \mathcal{I}_4^3(q, q') & := \int_{\mathbb{R}_2} \left\{ \dot{S}_{q-1}Q_2 \text{tr}\{(\dot{\Delta}_q(\dot{\Delta}_{q'}Q_2\dot{S}_{q'+2}\nabla\delta u))\} \dot{\Delta}_q\Delta\delta Q \right. \\ & \quad \left. - \dot{S}_{q-1}Q_2 \text{tr}\{(\dot{\Delta}_q(\dot{\Delta}_{q'}Q_2\dot{S}_{q'+2}\Delta\delta Q))\} \dot{\Delta}_q\nabla\delta u \right\} \\ & \lesssim \|\dot{S}_{q-1}Q_2\|_{L_x^\infty} \|\dot{\Delta}_q(\dot{\Delta}_{q'}Q_2\dot{S}_{q'+2}(\nabla\delta u, \Delta\delta Q))\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2}. \end{aligned}$$

At first let us consider the low frequencies $q \leq N$, with $N > 1$:

$$\begin{aligned} \mathcal{I}_4^3(q, q') & \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{\Delta}_{q'}Q_2\|_{L_x^\infty} \times \\ & \quad \times \|\dot{S}_{q'+2}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} 2^{-q'} \|\dot{\Delta}_{q'}\Delta Q_2\|_{L_x^2} \|\dot{S}_{q'+2}(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim (1 + \sqrt{N}) \|\nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q'}\Delta Q_2\|_{L_x^2} \|\dot{S}_{q'+2}(\delta u, \nabla\delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2} \\ & \lesssim (1 + \sqrt{N}) 2^{\frac{q'}{2}} \|\nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q'}\Delta Q_2\|_{L_x^2} \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\dot{\Delta}_q(\nabla\delta u, \Delta\delta Q)\|_{L_x^2}, \end{aligned}$$

which yields

$$\begin{aligned}
& \sum_{q \leq N} \sum_{q' \geq q-5} 2^{-q} \mathcal{I}_4^3(q, q') \\
& \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
& \quad \times \sum_{q \leq N} \sum_{q' \geq q-5} 2^{\frac{q'}{2}-q} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
& \quad \times \sum_{q \in \mathbb{Z}} 2^{-\frac{q}{2}} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \sum_{q' \geq q-5} 2^{\frac{q'-q}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \\
& \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
& \quad \times \left(\sum_{q' \in \mathbb{Z}} \left| \sum_{q \in \mathbb{Z}} 2^{\frac{q-q'}{2}} 1_{(-\infty, 5]}(q-q') \|\dot{\Delta}_q \Delta Q_2\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}},
\end{aligned}$$

thus by convolution

$$\begin{aligned}
& \sum_{q \leq N} \sum_{q' \geq q-5} 2^{-q} \mathcal{I}_4^3(q, q') \\
& \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim (1 + N) \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 \\
& \quad + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

For the high frequencies, $q > N$,

$$\begin{aligned}
& \sum_{q \geq N} \sum_{q' \geq q-5} 2^{-q} \mathcal{I}_2^3(q, q') \lesssim \sum_{q \geq N} \sum_{q' \geq q-5} 2^{-q} (1 + \sqrt{q-1}) \|\dot{S}_{q-1}(Q_2, \nabla Q_2)\|_{L_x^2} \times \\
& \quad \times \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|\dot{S}_{q'+2}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \times \\
& \quad \times \sum_{q \geq N} 2^{-\frac{q}{2}} (1 + \sqrt{q}) \sum_{q' \geq q-5} 2^{\frac{q'-q}{2}} \|\dot{\Delta}_{q'} \nabla Q_2\|_{L_x^2} 2^{-\frac{q'}{2}} \|\dot{S}_{q'+2}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
& \quad \times 2^{-\frac{N}{2}} \left(\sum_{q \in \mathbb{Z}} \left| \sum_{q' \geq q-5} 2^{\frac{q'-q}{2}} \|\dot{\Delta}_{q'} \nabla Q_2\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

so that, by convolution

$$\begin{aligned}
& \sum_{q \geq N} \sum_{q' \geq q-5} 2^{-q} \mathcal{I}_2^3(q, q') \\
& \lesssim 2^{-N} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

Summarizing, we get

$$\begin{aligned} \sum_{q \in \mathbb{Z}} \sum_{q' \geq q-5} 2^{-q} \mathcal{I}_2^3(q, q') &\lesssim (1+N) \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &+ C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + \\ &+ 2^{-2N} \|(Q_2, \nabla Q_2)\|_{L_x^2}^2 \|\nabla Q_2\|_{L_x^2}^2 \|(\nabla \delta u, \Delta \delta Q)\|_{L_x^2}^2, \end{aligned}$$

which is similar to (7.65), hence we can conclude as in (7.66).

Estimate of \mathcal{J}_q^4 Now, we handle the last term of (7.50), which is related to $i = 4$, namely

$$\begin{aligned} &\sum_{q' \geq q-5} \int_{\mathbb{R}_2} \text{tr} \{ \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q'+2}(Q_2 \nabla \delta u) \}] \dot{\Delta}_q \Delta \delta Q - \\ &\quad - \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q'+2}(Q_2 \Delta \delta Q) \}] \dot{\Delta}_q \nabla \delta u \} \\ &= \sum_{q' \leq q-5} \sum_{q'' \leq q'+1} \int_{\mathbb{R}_2} \text{tr} \{ \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q'}(Q_2 \nabla \delta u) \}] \dot{\Delta}_q \Delta \delta Q \\ &\quad - \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q'}(Q_2 \Delta \delta Q) \}] \dot{\Delta}_q \nabla \delta u \} \\ &= \sum_{j=1}^4 \sum_{q' \leq q-5} \sum_{q'' \leq q'+1} \int_{\mathbb{R}_2} \text{tr} \{ \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \mathcal{J}_{q''}^j(Q_2, \nabla \delta u) \}] \dot{\Delta}_q \Delta \delta Q - \\ &\quad - \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \mathcal{J}_{q''}^j(Q_2, \Delta \delta Q) \}] \dot{\Delta}_q \nabla \delta u \}. \end{aligned} \quad (7.68)$$

First, we consider the term related to $j = 1$, that is

$$\begin{aligned} \mathcal{I}_1^4(q, q', q'', q''') &:= \int_{\mathbb{R}_2} \text{tr} \{ \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ [\dot{\Delta}_{q''}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} \nabla \delta u \}] \dot{\Delta}_q \Delta \delta Q + \\ &\quad - \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ [\dot{\Delta}_{q''}, \dot{S}_{q''-1} Q_2] \dot{\Delta}_{q''} \Delta \delta Q \}] \dot{\Delta}_q \nabla \delta u \} \\ &\lesssim \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ [\dot{\Delta}_{q''}, S_{q''-1} Q_2] \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q) \}]\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^q \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ [\dot{\Delta}_{q''}, S_{q''-1} Q_2] \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q) \}]\|_{L_x^1} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q-q''} \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|\dot{S}_{q''-1} \nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q-q'-q''} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q''-1} \nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q-q'-q''+q'''} \|\Delta Q_2\|_{L_x^2} \|\nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q'''} (\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2}. \end{aligned} \quad (7.69)$$

Hence, taking the sum in q, q', q'' and q''' (and observing that $|q'' - q'''| \leq 5$), we get

$$\begin{aligned} &\sum_{q \in \mathbb{Z}} \sum_{q' \geq q-5} \sum_{q'' \leq q'+1} \sum_{|q'''-q''| \leq 5} 2^{-q} \mathcal{I}_1^4(q, q', q'', q''') \lesssim \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \sum_{q, q', q''} 2^{\frac{q}{2}-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} (\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\ &\quad \times \sum_{q \in \mathbb{Z}} 2^{\frac{q}{2}} \sum_{q' \geq q-5} 2^{-q'} \sum_{q'' \leq q'+1} 2^{\frac{q''}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''} (\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \end{aligned}$$

$$\begin{aligned}
& \times \sum_{q'' \in \mathbb{Z}} 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q''+1} 2^{\frac{q''}{2}-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \sum_{q \leq q'+5} 2^{\frac{q}{2}} \\
& \lesssim \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
& \quad \times \sum_{q'' \in \mathbb{Z}} 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q''+1} 2^{\frac{q''}{2}-\frac{q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \\
& \lesssim \|\nabla Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \left(\sum_{q'' \in \mathbb{Z}} \left| \sum_{q' \geq q''+1} 2^{\frac{q''}{2}-\frac{q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}} \\
& \lesssim \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \tag{7.70}
\end{aligned}$$

When $j = 2$ in (7.68), we observe that

$$\begin{aligned}
\mathcal{I}_2^4(q, q', q'', q''') & := \int_{\mathbb{R}_2} \text{tr} \left\{ \dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q''-1} Q_2) \dot{\Delta}_{q''} \dot{\Delta}_{q'''} \nabla \delta u \} \right] \dot{\Delta}_q \Delta \delta Q - \right. \\
& \quad \left. - \dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q''-1} Q_2) \dot{\Delta}_{q''} \dot{\Delta}_{q'''} \Delta \delta Q \} \right] \dot{\Delta}_q \nabla \delta u \right\} \\
& \lesssim \|\dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q''-1} Q_2) \dot{\Delta}_{q''} \dot{\Delta}_{q'''} (\nabla \delta u, \Delta \delta Q) \} \right]\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^q \|\dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ (\dot{S}_{q''-1} Q_2 - \dot{S}_{q''-1} Q_2) \dot{\Delta}_{q''} \dot{\Delta}_{q'''} (\nabla \delta u, \Delta \delta Q) \} \right]\|_{L_x^1} \times \\
& \quad \times \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^q \|\dot{\Delta}_{q'} Q_2\|_{L_x^\infty} \|(\dot{S}_{q''-1} Q_2 - \dot{S}_{q''-1} Q_2)\|_{L_x^2} \times \\
& \quad \times \|\dot{\Delta}_{q''} \dot{\Delta}_{q'''} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{q-q'-q''} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|(\dot{S}_{q''-1} \nabla Q_2 - \dot{S}_{q''-1} \nabla Q_2)\|_{L_x^2} \times \\
& \quad \times \|\dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^{q-q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} (\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2},
\end{aligned}$$

which is equivalent to the last inequality of (7.69) (since $|q'' - q'''| \leq 5$). Hence, arguing as for proving (7.70), the following estimate holds:

$$\begin{aligned}
& \sum_{q \in \mathbb{Z}} \sum_{q' \geq q-5} \sum_{q'' \leq q'+1} \sum_{|q'''-q''| \leq 5} 2^{-q} \mathcal{I}_2^4(q, q', q'', q''') \lesssim \\
& \lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Now, let us analyze the term in (7.68) related to $j = 3$. Assuming $q'' \leq N$ for a suitable positive N , we get

$$\begin{aligned}
\mathcal{I}_3^4(q, q', q'') & := \int_{\mathbb{R}_2} \text{tr} \left\{ \dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q''-1} Q_2 \dot{\Delta}_{q''} \nabla \delta u \} \right] \dot{\Delta}_q \Delta \delta Q - \right. \\
& \quad \left. \dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q''-1} Q_2 \dot{\Delta}_{q''} \Delta \delta Q \} \right] \dot{\Delta}_q \nabla \delta u \right\} \\
& \lesssim \|\dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q''-1} Q_2 \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q) \} \right]\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^q \|\dot{\Delta}_q \left[\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{S}_{q''-1} Q_2 \dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q) \} \right]\|_{L_x^1} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
& \lesssim 2^q \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} \|\dot{S}_{q''-1} Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q''} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2}
\end{aligned}$$

$$\begin{aligned}
&\lesssim 2^q \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim (1 + \sqrt{N}) 2^{\frac{3q}{2} + \frac{3q''}{2} - 2q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \times \\
&\quad \times 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{q'' \leq N} \sum_{q' \geq q'' - 1} \sum_{q \leq q' + 5} 2^{-q} \mathcal{I}_3^4(q, q', q'') \lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
&\quad \times \sum_{q'' \leq N} 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q'' - 1} 2^{\frac{3q''}{2} - 2q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \sum_{q \leq q' + 5} 2^{\frac{q}{2}} \\
&\lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
&\quad \times \sum_{q'' \leq N} 2^{-\frac{q''}{2}} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q'' - 1} 2^{\frac{3q''}{2} - \frac{3q'}{2}} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \\
&\lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \times \\
&\quad \times \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \left(\sum_{q'' \in \mathbb{Z}} \left| \sum_{q' \geq q'' - 1} 2^{\frac{3}{2}q'' - \frac{3}{2}q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \right|^2 \right)^{\frac{1}{2}} \\
&\lesssim (1 + \sqrt{N}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}$$

Considering the high frequencies $q'' > N$

$$\begin{aligned}
\mathcal{I}_3^4(q, q', q'') &\lesssim \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr}\{\dot{S}_{q''} Q_2 \dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\}]\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim 2^q \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr}\{\dot{S}_{q''} Q_2 \dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\}]\|_{L_x^1} \|\dot{\Delta}_q(\delta u, \nabla \delta Q)\|_{L_x^2} \\
&\lesssim 2^q \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} \|\dot{S}_{q''} Q_2\|_{L_x^\infty} \|\dot{\Delta}_{q''}(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim 2^{\frac{3q}{2} - 2q'} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} (1 + \sqrt{q''}) \|(Q_2, \nabla Q_2)\|_{L_x^2} \times \\
&\quad \times 2^{q''} \|\dot{\Delta}_{q''}(\delta u, \nabla \delta Q)\|_{L_x^2} 2^{-\frac{q}{2}} \|\dot{\Delta}_q(\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\
&\lesssim (1 + \sqrt{q''}) 2^{\frac{3q}{2} + q'' - 2q'} \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}},
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{q'' > N} \sum_{q' \geq q'' - 1} \sum_{q \leq q' + 5} 2^{-q} \mathcal{I}_3^4(q, q', q'') \lesssim \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{L_x^2} \times \\
&\quad \times \sum_{q'' > N} (1 + \sqrt{q''}) 2^{q''} \sum_{q' \geq q'' - 1} 2^{-2q'} \sum_{q \leq q' + 5} 2^{\frac{q}{2}} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{L_x^2} \times \\
&\quad \times \sum_{q'' > N} (1 + \sqrt{q''}) 2^{q''} \sum_{q' \geq q'' - 1} 2^{-2q' + \frac{q'}{2}} \\
&\lesssim \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{L_x^2} \times \\
&\quad \times \sum_{q'' > N} (1 + \sqrt{q''}) 2^{-\frac{q''}{2}}
\end{aligned}$$

$$\lesssim \|\Delta Q_2\|_{L_x^2} \|(Q_2, \nabla Q_2)\|_{L_x^2} \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|(\delta u, \nabla \delta Q)\|_{L_x^2} 2^{-\frac{N}{2}},$$

Summarizing the last inequalities we obtain an estimate similar to (7.65), so that we can conclude arguing as in (7.66). Finally, it remains to examine when $j = 4$, as last term. Let us define

$$\begin{aligned} \mathcal{I}_4^4(q, q', q'', q''') &:= \int_{\mathbb{R}^2} \text{tr} \left\{ \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q''} (\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} \nabla \delta u) \}] \dot{\Delta}_q \Delta \delta Q + \right. \\ &\quad \left. - \dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q''} (\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} \Delta \delta Q) \}] \dot{\Delta}_q \nabla \delta u \right\} \\ &\lesssim \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q''} [\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} (\nabla \delta u, \Delta \delta Q)] \}]\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^q \|\dot{\Delta}_q [\dot{\Delta}_{q'} Q_2 \text{tr} \{ \dot{\Delta}_{q''} [\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} (\nabla \delta u, \Delta \delta Q)] \}]\|_{L_x^1} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^q \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} [\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} (\nabla \delta u, \Delta \delta Q)]\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q+q''} \|\dot{\Delta}_{q'} Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} [\dot{\Delta}_{q'''} Q_2 \dot{S}_{q'''+2} (\nabla \delta u, \Delta \delta Q)]\|_{L_x^1} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q+q''} \|\dot{\Delta}_{q'} \Delta Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} Q_2\|_{L_x^2} \|\dot{S}_{q'''+2} (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2} \\ &\lesssim 2^{q-q'+q''-q'''} \|\dot{\Delta}_{q'} \nabla Q_2\|_{L_x^2} \|\dot{\Delta}_{q''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \|\dot{\Delta}_q (\nabla \delta u, \Delta \delta Q)\|_{L_x^2}. \end{aligned}$$

Hence, taking the sum in q, q', q'' and q''' , we get

$$\begin{aligned} &\sum_{q \in \mathbb{Z}} \sum_{q' \geq q-5} \sum_{q'' \leq q'-1} \sum_{q''' \geq q''+5} 2^{-q} \mathcal{I}_4^4(q, q', q'', q''') \lesssim \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \times \\ &\quad \times \sum_{q \in \mathbb{Z}} \sum_{q' \geq q-5} \sum_{q'' \leq q'-1} \sum_{q''' \geq q''+5} 2^{\frac{q}{2}-q'+q''-q'''} \|\dot{\Delta}_{q'''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \times \\ &\quad \times \sum_{q''' \in \mathbb{Z}} \sum_{q'' \leq q'''-5} 2^{q''-q'''} \|\dot{\Delta}_{q'''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q''+1} 2^{-q'} \sum_{q \leq q'+5} 2^{\frac{q}{2}} \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \times \\ &\quad \times \sum_{q''' \in \mathbb{Z}} \sum_{q'' \leq q'''-5} 2^{q''-q'''} \|\dot{\Delta}_{q'''} \Delta Q_2\|_{L_x^2} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \sum_{q' \geq q''+1} 2^{-\frac{q'}{2}} \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \times \\ &\quad \times \sum_{q''' \in \mathbb{Z}} \sum_{q'' \leq q'''-5} 2^{\frac{q''-q'''}{2}} \|\dot{\Delta}_{q'''} \Delta Q_2\|_{L_x^2} 2^{-\frac{q'''}{2}} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \sum_{q''' \in \mathbb{Z}} \|\dot{\Delta}_{q'''} \Delta Q_2\|_{L_x^2} 2^{-\frac{q'''}{2}} \|\dot{S}_{q'''+2} (\delta u, \nabla \delta Q)\|_{L_x^2} \\ &\lesssim \|(\nabla \delta u, \Delta \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla Q_2\|_{L_x^2} \|\Delta Q_2\|_{L_x^2} \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|\nabla Q_2\|_{L_x^2}^2 \|\Delta Q_2\|_{L_x^2}^2 \|(\delta u, \nabla \delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and this concludes the estimates of the term $\mathcal{E}_1 + \mathcal{E}_2$.

Remaining Terms

For the sake of completeness, now we analyze all the remaining terms. However we point out that they are going to be estimates using simply just Proposition 7.4.4, hence they are not a challenging

drawback. For instance, let us observe that

$$\begin{aligned} & L\langle(\xi\delta D + \delta\Omega)\delta Q, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} + L\langle(\xi D_2 + \Omega_2)\delta Q, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} + L\langle\delta Q(\xi\delta D + \delta\Omega), \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} + \\ & \quad + L\langle\delta Q(\xi D_2 + \Omega_2), \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\nabla(u_1, u_2)\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \times \\ & \quad \times \|\nabla(u_1, u_2)\|_{L_x^2} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla(u_1, u_2)\|_{L_x^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Moreover $La\Gamma\langle\delta Q, \Delta\delta Q\rangle_{\dot{H}^{-1/2}} \lesssim \|\delta Q\|_{\dot{H}^{-1/2}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-1/2}}^2$ and

$$\begin{aligned} & Lb\Gamma\langle Q_1\delta Q + \delta QQ_2, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|Q_1\delta Q + \delta QQ_2\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|(Q_1, Q_2)\|_{L_x^2} \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2} \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Furthermore, by a direct computation, we get

$$\begin{aligned} & Lc\Gamma\langle\delta Q\text{tr}\{Q_1^2\}, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} + Lc\Gamma\langle Q_2\text{tr}\{Q_1\delta Q + \delta QQ_2\}, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1^2, Q_2^2)\|_{L_x^2} \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|(Q_1, Q_2)\|_{L_x^4}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla(Q_1, Q_2)\|_{L_x^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and

$$\begin{aligned} & L\langle\delta u \cdot \nabla Q_1, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} + L\langle u_2 \cdot \nabla\delta Q, \Delta\delta Q\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(u_2, \nabla Q_1)\|_{\dot{H}^{\frac{3}{4}}} \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{4}}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(u_2, \nabla Q_1)\|_{L_x^2}^{\frac{1}{4}} \|(\nabla u_2, \Delta Q_1)\|_{L_x^2}^{\frac{3}{4}} \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|(\Delta\delta u, \Delta\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(u_2, \nabla Q_1)\|_{L_x^2}^{\frac{2}{3}} \|(\nabla u_2, \Delta Q_1)\|_{L_x^2}^2 \|(\delta u, \nabla\delta Q)\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma,L} \|\Delta\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Moreover $a\xi\langle\delta QQ_1, \nabla\delta u\rangle_{\dot{H}^{-1/2}} \lesssim \|\delta Q\|_{\dot{H}^{1/2}} \|Q_1\|_{L^2} \|\nabla\delta u\|_{\dot{H}^{-1/2}} \lesssim \|Q_1\|_{L^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-1/2}}^2 + C_\nu \|\nabla\delta u\|_{\dot{H}^{-1/2}}^2$,

$$\begin{aligned} & b\xi\langle\delta Q(Q_1^2 - \text{tr}\{Q_1^2\}\frac{\text{Id}}{2}), \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|Q_1^2\|_{L_x^2} \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|Q_1\|_{L_x^4}^2 \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|Q_1\|_{L_x^2} \|\nabla Q_1\|_{L_x^2} \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|Q_1\|_{L_x^2}^2 \|\nabla Q_1\|_{L_x^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and

$$\begin{aligned} & c\xi\langle\delta Q\text{tr}(Q_1^2)Q_1, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|Q_1^2\|_{L_x^2} \|Q_1\|_{L^\infty} \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|Q_1\|_{L_x^2}^2 \|\nabla Q_1\|_{L_x^2}^2 \|Q_1\|_{\dot{H}^2}^2 \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Now, $a\xi\langle(Q_2 + \text{Id}/2)\delta Q, \nabla\delta u\rangle_{\dot{H}^{-1/2}} \lesssim (\|Q_2\|_{L_x^2}^2 + 1)\|\nabla\delta Q\|_{\dot{H}^{-1/2}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-1/2}}^2$ and

$$\begin{aligned} & b\xi\langle(Q_2 + \frac{\text{Id}}{2})(Q_1\delta Q + \delta QQ_2), \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} - b\xi\langle Q_2\text{tr}\{Q_1\delta Q + \delta QQ_2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \\ & \lesssim (\|Q_2\|_{L_x^\infty} + 1)\|(Q_1, Q_2)\|_{L_x^2}\|\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim (\|Q_2\|_{H^2} + 1)\|(Q_1, Q_2)\|_{L_x^2} \times \\ & \times \|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim (\|Q_2\|_{H^2} + 1)^2\|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Equivalently, we get

$$\begin{aligned} & c\xi\langle(Q_2 + \frac{\text{Id}}{2})\delta Q\text{tr}\{Q_1^2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + c\xi\langle(Q_2 + \frac{\text{Id}}{2})Q_2\text{tr}\{\delta QQ_1 + Q_2\delta Q\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \\ & \lesssim \|Q_2\|_{L^\infty}\|(Q_1^2, Q_2^2)\|_{L_x^2}\|\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|Q_2\|_{H^2}\|(Q_1, Q_2)\|_{L_x^4}^2\|\nabla\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|Q_2\|_{H^2}^2\|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla(Q_1, Q_2)\|_{L_x^2}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and moreover

$$\begin{aligned} & L\xi\langle\delta Q\Delta\delta Q, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + L\xi\langle\delta Q\Delta Q_2, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|\Delta(Q_1, Q_2)\|_{L_x^2}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|\Delta(Q_1, Q_2)\|_{L_x^2}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

We can similarly control the terms from $-a\xi\langle Q_1\delta Q, \nabla\delta u\rangle_{\dot{H}^{-1/2}}$ to $L\xi\langle\Delta Q_2, \delta Q, \nabla\delta\rangle_{\dot{H}^{-1/2}}$ in (7.37), proceeding as in the previous estimates. Furthermore

$$\begin{aligned} & 2a\xi\langle\delta Q\text{tr}\{Q_1^2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + 2a\xi\langle Q_2\text{tr}\{\delta QQ_1\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + 2a\xi\langle Q_2\text{tr}\{Q_2\delta Q\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|(Q_1^2, Q_2^2)\|_{L_x^2}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla(Q_1, Q_2)\|_{L_x^2}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2, \end{aligned}$$

$$\begin{aligned} & 2b\xi\langle\delta Q\text{tr}\{Q_1^3\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + 2b\xi\langle Q_2\text{tr}\{\delta QQ_2^2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + \\ & + 2b\xi\langle Q_2\text{tr}\{Q_2(\delta QQ_1 + Q_2\delta Q)\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla(Q_1, Q_2)\|_{L_x^2}^2\|(Q_1, Q_2)\|_{H^2}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and also

$$\begin{aligned} & 2c\xi\langle\delta Q\text{tr}\{Q_1^2\}^2, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + 2c\xi\langle Q_2\text{tr}\{\delta QQ_1 + Q_2\delta Q\}\text{tr}\{Q_1^2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + \\ & + 2c\xi\langle Q_2\text{tr}\{Q_2^2\}\text{tr}\{\delta QQ_1 + Q_2\delta Q\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|(Q_1^4, Q_2^4)\|_{L_x^2}\|\delta Q\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^4\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla(Q_1, Q_2)\|_{L_x^2}^3\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ & \lesssim \|(Q_1, Q_2)\|_{L_x^2}^2\|\nabla(Q_1, Q_2)\|_{L_x^2}^6\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2. \end{aligned}$$

Furthermore, we observe that

$$2L\xi\langle\delta Q\text{tr}\{\delta Q\Delta\delta Q\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} + 2L\xi\langle\delta Q\text{tr}\{\delta Q\Delta Q_2\}, \nabla\delta u\rangle_{\dot{H}^{-\frac{1}{2}}} +$$

$$\begin{aligned}
& + 2L\xi \langle \delta Q \operatorname{tr}\{Q_2 \Delta \delta Q\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + 2L\xi \langle Q_2 \operatorname{tr}\{\delta Q \Delta \delta Q\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + \\
& + 2L\xi \langle \delta Q \operatorname{tr}\{Q_2 \Delta Q_2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + 2L\xi \langle Q_2 \operatorname{tr}\{\delta Q \Delta Q_2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|\Delta(Q_1, Q_2)\|_{L_x^2} \|(Q_1, Q_2)\|_{L_x^\infty} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\Delta(Q_1, Q_2)\|_{L_x^2}^2 \|(Q_1, Q_2)\|_{H^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2
\end{aligned}$$

and

$$\begin{aligned}
& L \langle \nabla \delta Q \odot \nabla Q_1, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + L \langle \nabla Q_2 \odot \nabla \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{4}}} \|\nabla(Q_1, Q_2)\|_{\dot{H}^{\frac{3}{4}}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\nabla \delta Q\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L^2}^{\frac{1}{4}} \|\Delta(Q_1, Q_2)\|_{L^2}^{\frac{3}{4}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{4}} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{4}} \|\nabla(Q_1, Q_2)\|_{L^2}^{\frac{1}{4}} \|\Delta(Q_1, Q_2)\|_{L^2}^{\frac{3}{4}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla(Q_1, Q_2)\|_{L^2}^{\frac{3}{2}} \|\Delta(Q_1, Q_2)\|_{L^2}^2 \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Moreover

$$\begin{aligned}
& La \langle \delta Q Q_1, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + a \langle Q_2 \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \\
& - a \langle Q_1 \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|(Q_1, Q_2)\|_{L^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{L^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{L^2}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2, \\
& -Lb \langle \delta Q(Q_1^2 - \operatorname{tr}\{Q_1^2\} \frac{\operatorname{Id}}{3}), \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + b \langle (Q_1^2 - \operatorname{tr}\{Q_1^2\} \frac{\operatorname{Id}}{3}) \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \\
& - b \langle Q_2(Q_1 \delta Q + \delta Q Q_2 - \operatorname{tr}\{Q_1 \delta Q + \delta Q Q_2\} \frac{\operatorname{Id}}{3}), \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \\
& - b \langle (Q_1 \delta Q + \delta Q Q_2 - \operatorname{tr}\{Q_1 \delta Q + \delta Q Q_2\} \frac{\operatorname{Id}}{3}) \delta Q, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1^2, Q_2^2)\|_{L^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|(Q_1, Q_2)\|_{L^4}^2 \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{L^2} \|\nabla(Q_1, Q_2)\|_{L^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{L^2}^2 \|\nabla(Q_1, Q_2)\|_{L^2}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2,
\end{aligned}$$

and

$$\begin{aligned}
& Lc \langle \delta Q Q_1 \operatorname{tr}\{Q_1^2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} + c \langle Q_2 \delta Q \operatorname{tr}\{Q_1^2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - \\
& - c \langle Q_1 \delta Q \operatorname{tr}\{Q_1^2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} - c \langle \delta Q Q_2 \operatorname{tr}\{Q_1^2\}, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\delta Q\|_{\dot{H}^{\frac{1}{2}}} \|(Q_1, Q_2)\|_{L_x^\infty} \|(Q_1^2, Q_2^2)\|_{L_x^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}} \|(Q_1, Q_2)\|_{H^2} \|(Q_1, Q_2)\|_{L_x^4}^2 \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\
& \lesssim \|\nabla \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \|(Q_1, Q_2)\|_{H^2}^2 \|(Q_1, Q_2)\|_{L^2}^2 \|\nabla(Q_1, Q_2)\|_{L^2}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2.
\end{aligned}$$

Finally

$$\begin{aligned} \langle u_2 \cdot \nabla \delta u, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} &= -\langle u_2 \otimes \delta u, \nabla \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|u_2\|_{\dot{H}^{\frac{1}{2}}} \|\delta u\|_{L^2} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|u_2\|_{L^2}^{\frac{1}{2}} \|\nabla u_2\|_{L^2}^{\frac{1}{2}} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^{\frac{3}{2}} \lesssim \|u_2\|_{L^2}^2 \|\nabla u_2\|_{L^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 \end{aligned}$$

and

$$\langle \delta u \cdot \nabla u_1, \delta u \rangle_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\delta u\|_{\dot{H}^{\frac{1}{2}}} \|\nabla u_1\|_{L_x^2} \|\delta u\|_{\dot{H}^{-\frac{1}{2}}} \lesssim C_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \|\nabla u_1\|_{L_x^2}^2 \|\delta u\|_{\dot{H}^{-\frac{1}{2}}}^2.$$

Conclusion

Recalling (7.37) and summarizing all the the previous estimates, we conclude that there exists a function χ which belongs to $L_{loc}^1(\mathbb{R}_+)$ such that

$$\frac{d}{dt} \Phi(t) + \nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + \Gamma L^2 \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2 \lesssim \chi(t) \mu(\Phi(t)) + c_\nu \|\nabla \delta u\|_{\dot{H}^{-\frac{1}{2}}}^2 + C_{\Gamma, L} \|\Delta \delta Q\|_{\dot{H}^{-\frac{1}{2}}}^2$$

where μ is the Osgood modulus of continuity defined in (7.39). Hence, choosing $C_{\Gamma, L}$ and C_ν small enough from the beginning, we can absorb the last two terms on the right-hand side by the left-hand side, obtaining (7.38). We deduce that $\Phi \equiv 0$, thanks to the Osgood Lemma and the null initial data $\Phi(0) = 0$. Thus, $(\delta u, \nabla \delta Q)$ is identically zero and δQ as well, since $\delta Q(t)$ decades to 0 at infinity for almost every t .

□

7.5 Thecnical results

Proposition 7.5.1. *Let $(Q^{(n)}, u^n)$ be a smooth solution of (7.23) in dimension $d = 2$ or $d = 3$, with restriction (7.2), and smooth initial data $(\bar{Q}(x), \bar{u}(x))$, that decays fast enough at infinity so that we can integrate by parts in space (for any $t \geq 0$) without boundary terms. We assume that $|\xi| < \xi_0$ where ξ_0 is an explicitly computable constant, scale invariant, depending on $a, b, c, d, \Gamma, \nu, \lambda$.*

For $(\bar{Q}, \bar{u}) \in H^1 \times L^2$, we have

$$\|Q^{(n)}(t, \cdot)\|_{H^1} \leq C_1 + \bar{C}_1 e^{\bar{C}_1 t} \|\bar{Q}\|_{H^1}, \forall t \geq 0 \quad (7.71)$$

with C_1, \bar{C}_1 depending on $(a, b, c, d, \Gamma, L, \nu, \bar{Q}, \bar{u})$. Moreover

$$\|u^n(t, \cdot)\|_{L^2}^2 + \nu \int_0^t \|\nabla u^n\|_{L^2}^2 \leq C_1 \quad (7.72)$$

Proof. We denote:

$$X_{\alpha\beta}^n \stackrel{\text{def}}{=} L \Delta Q_{\alpha\beta}^{(n)} - c Q_{\alpha\beta}^{(n)} \text{tr}((Q^{(n)})^2), \quad \alpha, \beta = 1, 2, 3. \quad (7.73)$$

Multiplying the first equation in (7.23) by $-\lambda \bar{H}^n$ and the second one by u^n , taking the trace

and integrating over \mathbb{R}^d , we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \text{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx \\
& + \nu \|\nabla u^n\|_{L^2}^2 + \Gamma \lambda L^2 \|\Delta Q^{(n)}\|_{L^2}^2 \\
& + \Gamma \lambda c^2 \|J_n(Q^{(n)} \text{tr}\{Q^{(n)}\})\|_{L^2}^2 - 2cL\Gamma\lambda \int_{\mathbb{R}^d} \Delta Q_{\alpha\beta}^{(n)} Q_{\alpha\beta}^{(n)} \text{tr}\{(Q^{(n)})^2\} dx + a^2\Gamma\lambda \|Q^{(n)}\|_{L^2}^2 \\
& + b^2\Gamma\lambda \int_{\mathbb{R}^d} \text{tr}\left\{J_n\left((Q^{(n)})^2 - \frac{\text{tr}\{(Q^{(n)})^2\}}{d}\right)^2\right\} dx \\
& + \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon u \nabla Q^{(n)}|^3 dx + \varepsilon \int_{\mathbb{R}^d} |R_\varepsilon \nabla u^n|^4 dx \\
& \leq 2a\Gamma\lambda \underbrace{\int_{\mathbb{R}^d} \text{tr}\{X^n Q^{(n)}\} dx}_{\stackrel{\text{def}}{=} \mathcal{I}_n} - 2b\Gamma\lambda \underbrace{\int_{\mathbb{R}^d} \text{tr}\{X^n (Q^{(n)})^2\} dx}_{\stackrel{\text{def}}{=} \mathcal{J}_n} \\
& + 2ab\Gamma\lambda \underbrace{\int_{\mathbb{R}^d} \text{tr}\{(Q^{(n)})^3\} dx + \lambda \int_{\mathbb{R}^d} J_n \left(R_\varepsilon u^n \cdot \nabla Q_{\alpha\beta}^{(n)} \right) J_n \left(bQ_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx}_{\stackrel{\text{def}}{=} \mathcal{II}} \\
& + \lambda \int_{\mathbb{R}^d} J_n \left(-R_\varepsilon \Omega_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} + Q_{\alpha\gamma}^{(n)} R_\varepsilon \Omega_{\gamma\beta}^{(n)} \right) J_n \left(bQ_{\alpha\delta}^{(n)} Q_{\delta\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx
\end{aligned} \tag{7.74}$$

Integrating by parts we have:

$$\begin{aligned}
& -2cL\Gamma\lambda \int_{\mathbb{R}^d} \Delta Q_{\alpha\beta}^{(n)} Q_{\alpha\beta}^{(n)} \text{tr}\{(Q^{(n)})^2\} dx = 2cL\Gamma\lambda \int_{\mathbb{R}^d} Q_{\alpha\beta,k}^{(n)} Q_{\alpha\beta,k}^{(n)} \text{tr}\{(Q^{(n)})^2\} dx \\
& + 2cL\Gamma\lambda \int_{\mathbb{R}^d} Q_{\alpha\beta,k}^{(n)} Q_{\alpha\beta}^{(n)} \partial_k \left(\text{tr}\{(Q^{(n)})^2\} \right) dx = 2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \text{tr}\{(Q^{(n)})^2\} dx \\
& \quad + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla \left(\text{tr}\{(Q^{(n)})^2\} \right)|^2 dx \geq 0
\end{aligned} \tag{7.75}$$

(where for the last inequality we used the assumption (7.2) and $L, \Gamma, \lambda > 0$). One can easily see that

$$\mathcal{I}_n = -\frac{L}{2} \|\nabla Q^{(n)}\|_{L^2}^2 - c \|Q^{(n)}\|_{L^4}^4 \tag{7.76}$$

and moreover

$$\begin{aligned}
& \lambda \int_{\mathbb{R}^d} J_n \left(-R_\varepsilon \Omega_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} + Q_{\alpha\gamma}^{(n)} R_\varepsilon \Omega_{\gamma\beta}^{(n)} \right) J_n \left(bQ_{\alpha\delta}^{(n)} Q_{\delta\beta}^{(n)} - cQ_{\alpha\beta}^{(n)} |Q^{(n)}|^2 \right) dx \leq \\
& \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon \nabla u^n|^4 dx + C(\varepsilon) \int_{\mathbb{R}^d} |Q^{(n)}|^4 dx + \frac{\Gamma c^2}{2} \int_{\mathbb{R}^d} |J_n(Q^{(n)} |Q^{(n)}|^2)|^2 dx
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$ and $\tilde{C} = \tilde{C}(\varepsilon, c)$ an explicitly computable constant, we have:

$$\begin{aligned}
\mathcal{J}_n & = L \int_{\mathbb{R}^d} Q_{\alpha\beta,k}^{(n)} Q_{\alpha\gamma}^{(n)} Q_{\gamma\beta}^{(n)} dx - c \int_{\mathbb{R}^d} \text{tr}\{(Q^{(n)})^2\} \text{tr}\{(Q^{(n)})^3\} dx \\
& \leq -L \int_{\mathbb{R}^d} Q_{\alpha\beta,k}^{(n)} Q_{\alpha\gamma,k}^{(n)} Q_{\gamma\beta}^{(n)} dx - L \int_{\mathbb{R}^d} Q_{\alpha\beta,k}^{(n)} Q_{\alpha\gamma}^{(n)} Q_{\gamma\beta,k}^{(n)} dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^d} \operatorname{tr}\{(Q^{(n)})^2\} \left(\frac{\tilde{C}}{\varepsilon} \operatorname{tr}\{(Q^{(n)})^2\} + \varepsilon \operatorname{tr}^2\{(Q^{(n)})^2\} \right) dx \\
& \leq L\varepsilon \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \operatorname{tr}\{(Q^{(n)})^2\} dx + \frac{\tilde{C}}{\varepsilon} \|\nabla Q^{(n)}\|_{L^2}^2 \\
& + \int_{\mathbb{R}^d} \operatorname{tr}\{(Q^{(n)})^2\} \left(\frac{\tilde{C}}{\varepsilon} \operatorname{tr}\{(Q^{(n)})^2\} + \varepsilon \operatorname{tr}^2\{(Q^{(n)})^2\} \right) dx
\end{aligned}$$

Using the last four relations in (7.74) and considering (7.25) we obtain:

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \operatorname{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx \\
& \quad + \nu \|\nabla u^n\|_{L^2}^2 + \Gamma \lambda L^2 \|\Delta Q^{(n)}\|_{L^2}^2 \\
& + \frac{\Gamma \lambda c^2}{2} \|J_n(Q^{(n)} \operatorname{tr}\{Q^{(n)}\})\|_{L^2}^2 + a^2 \Gamma \lambda \|Q^{(n)}\|_{L^2}^2 + 2cL\Gamma \lambda \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \operatorname{tr}\{(Q^{(n)})^2\} dx \\
& + cL\Gamma \lambda \int_{\mathbb{R}^d} |\nabla (\operatorname{tr}\{(Q^{(n)})^2\})|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^3 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla R_\varepsilon u^n|^4 dx \\
& \leq 2|a|\Gamma \lambda \left(\frac{L}{2} \|\nabla Q^{(n)}\|_{L^2}^2 + c\|Q^{(n)}\|_{L^4}^4 \right) \\
& \quad + 2|b|\Gamma \lambda L\varepsilon \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \operatorname{tr}\{(Q^{(n)})^2\} dx + 2|b|\Gamma \lambda \frac{\tilde{C}}{\varepsilon} \|\nabla Q^{(n)}\|_{L^2}^2 \\
& \quad + 2|b|\Gamma \lambda \int_{\mathbb{R}^d} \operatorname{tr}\{(Q^{(n)})^2\} \left(\frac{\tilde{C}}{\varepsilon} \operatorname{tr}\{(Q^{(n)})^2\} + \varepsilon \operatorname{tr}^2\{(Q^{(n)})^2\} \right) dx \\
& \quad + 2|ab|\Gamma \lambda (\varepsilon \|Q^{(n)}\|_{L^2}^2 + (C(\varepsilon) + \frac{\tilde{C}}{\varepsilon}) \|Q^{(n)}\|_{L^4}^4)
\end{aligned}$$

Taking ε small enough we can absorb all the terms with an ε coefficient on the right into the left hand side, and we are left with

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \operatorname{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx \\
& + \nu \|\nabla u^n\|_{L^2}^2 + \Gamma \lambda L^2 \|\Delta Q^{(n)}\|_{L^2}^2 + \frac{\Gamma \lambda c^2}{2} \|J_n(Q^{(n)} \operatorname{tr}\{Q^{(n)}\})\|_{L^2}^2 + \Gamma \lambda a^2 \|Q^{(n)}\|_{L^2}^2 \\
& + 2cL\Gamma \lambda \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \operatorname{tr}\{(Q^{(n)})^2\} dx + cL\Gamma \lambda \int_{\mathbb{R}^d} |\nabla (\operatorname{tr}\{(Q^{(n)})^2\})|^2 dx \\
& \leq \bar{C} \left(\|\nabla Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4 \right)
\end{aligned}$$

with $\bar{C} = \bar{C}(a, b, c)$.

The last relation is not yet enough because there are no positive terms. However, let us note that, if $a > 0$ we obtain the a-priori estimates by using the inequality $\operatorname{tr}\{(Q^{(n)})^3\} \leq \frac{3}{8} \operatorname{tr}\{(Q^{(n)})^2\} + \operatorname{tr}\{(Q^{(n)})^2\}^2$. If $a \leq 0$ we have to estimate separately $\|Q^{(n)}\|_{L^2}$ and this asks for a smallness condition for ξ . Indeed, proceeding as for proving (7.21), we get

$$\begin{aligned}
& \frac{d}{dt} \left[\int_{\mathbb{R}^d} \frac{1}{2} |u^n|^2 + \frac{L\lambda}{2} |\nabla Q^{(n)}|^2 + \lambda \left(\frac{a}{2} |Q^{(n)}|^2 - \frac{b}{3} \operatorname{tr}(Q^{(n)})^3 + \frac{c}{4} |Q^{(n)}|^4 \right) dx + M \|Q\|_{L^2}^2 \right] \\
& \quad + \nu \|\nabla u^n\|_{L^2}^2 + \Gamma \lambda L^2 \|\Delta Q^{(n)}\|_{L^2}^2 + \frac{\Gamma \lambda c^2}{2} \|J_n(Q^{(n)} \operatorname{tr}\{Q^{(n)}\})\|_{L^2}^2 + a^2 \|Q^{(n)}\|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}
& +2cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla Q^{(n)}|^2 \operatorname{tr}\{(Q^{(n)})^2\} dx + cL\Gamma\lambda \int_{\mathbb{R}^d} |\nabla (\operatorname{tr}\{(Q^{(n)})^2\})|^2 dx \\
& + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |R_\varepsilon u^n \cdot \nabla Q^{(n)}|^3 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla R_\varepsilon u^n|^4 dx \leq \bar{C} \left(\|\nabla Q^{(n)}\|_{L^2}^2 + \|Q^{(n)}\|_{L^4}^4 \right) + \\
& MC(d)\varepsilon \int_{\mathbb{R}^d} |\nabla u^n|^2 dx + \frac{M|\xi|^2}{\varepsilon} \int_{\mathbb{R}^d} |Q^{(n)}|^2 + |J_n(Q^{(n)}) \operatorname{tr}\{(Q^{(n)})^2\}|^2 dx \\
& \qquad \qquad \qquad + M\hat{C} \int_{\mathbb{R}^d} |Q^{(n)}|^2 + |Q^{(n)}|^4 dx
\end{aligned}$$

We chose ε small enough so that $MC(d)\varepsilon < \nu$. Finally we make the assumption that $|\xi|$ is small enough, depending on a, b, c, d, ν so that

$$\frac{M|\xi|^2}{\varepsilon} \leq \Gamma\lambda c^2.$$

Then taking into account that

$$\frac{M}{2} \operatorname{tr}\{(Q^{(n)})^2\} + \frac{c}{8} \operatorname{tr}^2\{(Q^{(n)})^2\} \leq (M + \frac{a}{2}) \operatorname{tr}\{(Q^{(n)})^2\} - \frac{b}{3} \operatorname{tr}\{(Q^{(n)})^3\} + \frac{c}{4} \operatorname{tr}^2\{(Q^{(n)})^2\}$$

we obtain the claimed relation (7.71). □

Chapter 8

Qian-Sheng system

The results of this chapter originate from a joint work with A. Zarnescu, and they will appear in a forthcoming manuscript, the title of which is *strong global solutions for the inertial Qian-Sheng model of liquid crystal*.

8.1 Introduction

The main aim of this article is to study a system describing the hydrodynamics of nematic liquid crystals in the Q-tensor framework [87, 92]. There exists several such models and we will consider the one proposed by T. Qian and P. Sheng in [101]. As most tensorial models, this one provides an extension of the classical Ericksen-Leslie model [69], in particular capturing the biaxial alignment of the molecules, a feature not available in the classical Ericksen-Leslie theory.

Our main interest in this model is due to the fact that it incorporates systematically a certain term that models inertial effects. Details about this will be provided in the *Physical aspects* subsection, below.

The inertial term is usually neglected on physical grounds, a fact that is also convenient mathematically since keeping it generates considerable analytical and numerical challenges. From a mathematical point of view the system couples a forced incompressible Navier-Stokes system modelling the flow with a hyperbolic convection-diffusion system for matrix-valued functions that model the evolution of the orientations of the nematic molecules. The inertial term is responsible for the hyperbolic character of the equation describing the orientation of the molecules.

In order to clearly show the system it is convenient to introduce some terminology. The local orientation of the molecules is described through a function Q taking values from $\Omega \subset \mathbb{R}^d$, into the set of the so-called d -dimensional Q -tensors, that is

$$S_0^{(d)} := \left\{ Q \in \mathbb{M}^{d \times d}; Q_{ij} = Q_{ji}, \operatorname{tr}(Q) = 0, i, j = 1, \dots, d \right\}$$

The evolution of the Q 's is driven by the free energy of the molecules, as well as the transport, distortion and alignment effects caused by the flow.

The velocity of the centres of masses of molecules obeys a forced incompressible Navier-Stokes system, with an additional stress tensor, a forcing term modelling the effect that the interaction of the molecules has on the dynamics of their centres of masses. Explicitly the equations, in

non-dimensional form, are:

$$\begin{aligned} \dot{v} + \nabla p - \frac{\beta_4}{2} \Delta v = \nabla \cdot \left(-\nabla Q \odot \nabla Q + \beta_1 Q \text{tr}\{QA\} + \beta_5 AQ + \beta_6 QA \right) \\ + \nabla \cdot \left(\frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \right) \end{aligned} \quad (8.1)$$

$$J\ddot{Q} + \mu_1 \dot{Q} = \Delta Q - aQ + b(Q^2 - \frac{1}{d}|Q|^2) - cQ|Q|^2 + \frac{\mu_2}{2}A + \mu_1[\Omega, Q] \quad (8.2)$$

where $\dot{f} = (\partial_t + v \cdot \nabla)f$ denotes a *material derivative* and $[A, B] := AB - BA$. Furthermore $(\nabla Q \odot \nabla Q)_{ij} := \text{tr}(\partial_i Q \partial_j Q)$ and $|Q| = \sqrt{\text{tr}(Q^2)}$.

The physical relevance of the equation and their meaning is provided in the next subsection, which can be skipped without impeding on the understanding of the remaining mathematical aspects of the paper.

8.1.1 Physical aspects

The velocity v is a free-divergence vector field, satisfying a convection-diffusion equation, with forcing provided by the pressure p , the distortion stress σ and the viscous stress σ' (here and in the following we use the Einstein summation convention, of summation over repeated indices):

$$\dot{v}_i = \partial_j (-p \delta_{ji} + \sigma_{ji} + \sigma'_{ji}), \quad (8.3)$$

$$\partial_k v_k = 0. \quad (8.4)$$

where p is the pressure. Here, the distortion stress σ is given by

$$\sigma_{ji} := -\frac{\partial \mathcal{F}}{\partial (\partial_j Q_{\alpha\beta})} \partial_i Q_{\alpha\beta}$$

where we use the simplest form of the Landau-de Gennes free energy density

$$\mathcal{F}[Q] := \frac{L}{2} |\nabla Q|^2 + \psi_B(Q)$$

modelling the spatial variations through the $\frac{L}{2} |\nabla Q|^2$ term with the nematic ordering enforced through the “bulk term” taken to be of the standard form [87, 92]

$$\psi_B(Q) = \frac{a}{2} \text{tr}\{Q^2\} - \frac{b}{3} \text{tr}\{Q^3\} + \frac{c}{4} \text{tr}^2\{Q^2\}.$$

The viscous stress σ' is given by:

$$\begin{aligned} \sigma'_{ij} := \beta_1 Q_{ij} Q_{lk} A_{lk} + \beta_4 A_{ij} + \beta_5 Q_{il} A_{lj} + \beta_6 Q_{jl} A_{li} \\ + \frac{1}{2} \mu_2 \mathcal{N}_{ij} - \mu_1 Q_{il} \mathcal{N}_{ij} + \mu_1 Q_{jl} \mathcal{N}_{li}, \end{aligned}$$

where $\beta_1, \beta_4, \beta_5, \beta_6, \mu_1$ and μ_2 are viscosity coefficients satisfying the Parodi-type relation

$$\beta_6 - \beta_5 = \mu_2$$

the A stands for the rate of strain tensor

$$A_{ij} := \frac{v_{i,j} + v_{j,i}}{2},$$

i.e. the symmetric part of the velocity gradient, while \mathcal{N} is the co-rotational time flux of Q , whose (i, j) -th component is formulated as follows

$$\mathcal{N}_{ij} := (\dot{Q} - \omega \wedge Q + Q \wedge \omega)_{ij} = \partial_t Q_{ij} + v_k \partial_k Q_{ij} - \epsilon_{ikl} \omega_k Q_{lj} - \epsilon_{jkl} \omega_k Q_{il}.$$

\mathcal{N} represents the time rate of change of Q_{ij} with respect to the background fluid angular velocity $\omega = \frac{1}{2} \nabla \wedge v$. Moreover, one can reformulate \mathcal{N} making use of the vorticity tensor Ω

$$\Omega_{ij} := \frac{v_{i,j} - v_{j,i}}{2}.$$

Indeed, one can check that

$$\mathcal{N}_{ij} = (\dot{Q} - [\Omega, Q])_{ij} = \dot{Q}_{ij} - \Omega_{il} Q_{lj} + Q_{il} \Omega_{lj},$$

since we have $\omega \wedge u = \Omega u$, for any d -dimensional vector u . For a common physical example, the MBBA material, we have the following relations between the coefficients [108]:

$$\frac{\mu_2}{\mu_1} \sim -1.92, \frac{\beta_1}{\mu_1} \sim 0.17, \frac{\beta_4}{\mu_1} \sim 0.7, \frac{\beta_5}{\mu_1} \sim 0.7, \frac{\beta_6}{\mu_1} \sim -0.79 \quad (8.5)$$

Furthermore, because the coefficient β_4 corresponds to the standard Newtonian stress tensor we can assume

$$\beta_4 > 0 \quad (8.6)$$

which fixes the signs for all the viscosities.

The evolution of the order tensor Q is driven by

$$J \ddot{Q}_{ij} = h_{ij} + h'_{ij} - \lambda \delta_{ij} - \epsilon_{ijk} \lambda_k. \quad (8.7)$$

where ϵ_{ijk} , the Levi-Civita symbol. The elastic molecular field h is

$$h_{ij} := -\frac{\partial \mathcal{F}}{\partial Q_{ij}} + \partial_k \frac{\partial \mathcal{F}}{\partial (\partial_k Q_{ij})}$$

and the viscous molecular field h' is given by:

$$h'_{ij} := -\frac{1}{2} \mu_2 A_{ij} - \mu_1 \mathcal{N}_{ij},$$

The λ, λ_k are Lagrange multiplier enforcing the tracelessness and symmetry of the tensor.

The J in (8.7) stands for the inertial density and it is taken to be greater than 0. This is consistent with the fact that J has the same sign as the inertia in the Leslie-Ericksen type of model (see Appendix B in [101]) where it is assumed to be positive (see for instance the assumption that J.L. Ericksen makes in [39]).

8.1.2 Main results

As first result we show that system (8.3)-(8.7) admits a Lyapunov functional, up to some relations on the viscosity coefficients. This is a functional exhibiting the free energy due to the director field, the kinetic energy of the fluid and the rotational kinetic energy of the director field.

Theorem 8.1.1. *Let us assume that the viscosity coefficients fulfils $\beta_5 + \beta_6 = 0$, and $\beta_1, \mu_1 \geq 0$. Let us also assume that $\beta_4 > |\mu_2|/2$ and $\mu_1 > |\mu_2|/2$. Then system (8.3)-(8.7) has a Lyapunov functional:*

$$E(t) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|v|^2 + \frac{J}{2}|\dot{Q}|^2 + \mathcal{F}[Q] \right)$$

with $\mathcal{F}[Q] = \frac{|\nabla Q|^2}{2} + \psi_B(Q)$, the Landau-de Gennes energy density. More precisely, if (v, Q) is a smooth solution such that

$$\begin{aligned} v &\in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d)), \\ Q &\in L^\infty(0, T; H^1(\mathbb{R}^d)) \quad \text{with} \quad \dot{Q} \in L^\infty(0, T; L^2(\mathbb{R}^d)), \end{aligned}$$

then, for all $t < T$, the following dissipative relation is satisfied:

$$\frac{d}{dt}E(t) + \int_{\mathbb{R}^d} |\nabla v|^2 + \int_{\mathbb{R}^d} |\mathcal{N}|^2 + \int_{\mathbb{R}^d} \text{tr}\{QA\}^2 \leq 0. \quad (8.8)$$

Let us remark that our restriction on the viscosity coefficients β_1, μ_1, β_5 and β_6 are not unnatural, as one can check in the MBBA example (8.5). The proof of Theorem 8.1.1 presents some difficulties that are specific to this system, such as controlling the “extraneous” maximal derivatives, i.e. the highest derivatives in v that appear in the Q equation and the highest derivatives in Q that appear in the v equation. We mainly handle this difficulties by taking into account the specific feature of the coupling that allows for the *cancellation of the worst terms*, when considering certain physically meaningful combination of terms.

It is worth to observe that despite this apriori estimate, one can not expect to construct weak solutions making use of this energy relation. Indeed, the most common approach in order to construct weak-solutions is making use of a compactness method for a sequence of approximated solutions. As in the classical Navier-Stokes equation, the main difficulties are related to the nonlinear terms. For instance, in system (8.3)-(8.7) the stress tensor σ_{ij} presents the nonlinear term

$$(\nabla Q \odot \nabla Q)_{ij} := Q_{\alpha\beta,i}Q_{\alpha\beta,j} = \frac{1}{2} \frac{\partial |\nabla Q|^2}{\partial Q_{\alpha\beta,i}} Q_{\alpha\beta,j} \quad (8.9)$$

Now, the highest number of derivatives of Q that relation (8.8) allows us to control is one, more precisely we can control ∇Q in $L^2(0, T; L^2(\mathbb{R}^d))$, for any positive time T . Then, the convergence we can expect for a sequence of approximation for ∇Q is only a weak convergence. Thus we can not control the convergence of the product (8.9).

One should keep in mind that a positive inertial density J leads the order tensor equation to be hyperbolic, in contrast to the parabolic structure that occurs when J is neglected. In the parabolic setting one can make use of regularizing effects, achieving a control of two derivatives of Q (i.e. ΔQ in $L^2(0, T; L^2(\mathbb{R}^d))$), which certainly allows to control the limit of a product as in (8.9). This feature is lost when J is positive, so that constructing weak solutions can not be achieved through a compactness method based on estimate (8.8).

Our second result concerns the existence and uniqueness of strong solutions for system (8.3)-(8.7), for a general dimension $d \geq 2$ (which assumes a physical interest when $d = 2, 3$). We assume

the initial data (v_0, Q_0) in a Sobolev space H^s with $s > d/2$, supposing also a smallness condition on their norms. Moreover, we consider the bulk energy $\psi_B(Q)$ in its quartic form:

$$\psi_B(Q) = \frac{a}{2}Q - \frac{b}{3}\text{tr}\{Q^3\} + \frac{c}{4}Q|Q|^2, \quad (8.10)$$

with a, b and c real constants. Then, our second result state as follows

Theorem 8.1.2. *Let us assume $(v_0, Q_0) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d}$ to belong to $H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$ with $d \geq 2$ and $s > d/2$. If the constant a fulfills $a > \frac{|\mu_2|^2}{4\beta_4}$ and moreover*

$$\eta_0 := \|v_0\|_{H^s(\mathbb{R}^d)} + \|Q_0\|_{H^{s+1}(\mathbb{R}^d)}$$

is small enough, then there exists a unique classical solution (v, Q) of (8.3)-(8.7), which is global in time and fulfills

$$\begin{aligned} v &\in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)), \\ Q &\in L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d)) \quad \text{with} \quad \dot{Q} \in L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d)). \end{aligned} \quad (8.11)$$

Moreover, there exists a positive constant C (not dependent on the solution) such that

$$\begin{aligned} \|v\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d))} + \|\nabla v\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \|Q\|_{L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \|Q\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \\ + \|\dot{Q}\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d))} + \|\dot{Q}\|_{L^2(\mathbb{R}_+; H^s(\mathbb{R}^d))} \leq C\eta_0. \end{aligned}$$

The main difficulties associated with treating system (8.3)-(8.7) on the one-hand are related to the presence of the forcing term of the Navier-Stokes part, on the other-hand they derive from the inertial term in the Q -equation. One can essentially think of the system as a highly non-trivial perturbation of a Navier-Stokes system. It is known that for Navier-Stokes alone the existence and uniqueness of classical and weak solutions in $2D$ can be achieved through rather standard arguments, while in $3D$ (or in a higher dimension) it is still a major open problem. The extended system we deal with has an intermediary position, as the perturbation produced by the presence of the additional stress-tensor generates significant technical difficulties, related in the first place to the highest derivatives in Q that appear in the u -equation. Moreover, such difficulty is accentuated by the inertial term in the hyperbolic order equation, which does not allow any kind of regularity effects for Q .

The proof of the existence part in theorem 8.1.2 is based on a rather common compactness method. First we construct a sequence of approximate solutions that are global in time, making use of a Friedrichs-type scheme. Then we pass to the limit, performing some uniform estimates in a $H^s(\mathbb{R}^d)$ -setting. Despite this standard idea, some specific difficulties occur. For instance, the $H^s(\mathbb{R}^2)$ norm does not allow the cancellation of the “worst” terms, so we can not proceed as in the proof of (8.8). This difficulty is partially dealt with by reformulating the inner product of $H^s(\mathbb{R}^2)$ as follows:

$$\langle \omega_1, \omega_2 \rangle_{L^2(\mathbb{R}^d)} + \langle \omega_1, \omega_2 \rangle_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}_\xi^d} (1 + |\xi|^{2s}) \hat{\omega}_1(\xi) \hat{\omega}_2(\xi) d\xi,$$

where $\dot{H}^s(\mathbb{R}^d)$ stands for the homogeneous Sobolev space with index s . It is straightforward that this inner product generates the same topology in $H^s(\mathbb{R}^2)$ with respect to the common one, given by

$$\langle \omega_1, \omega_2 \rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}_\xi^d} (1 + |\xi|)^{2s} \hat{\omega}_1(\xi) \hat{\omega}_2(\xi) d\xi.$$

Making use of this approach, one essentially reduces the control of the worst terms only in $\dot{H}^s(\mathbb{R}^2)$, where useful commutator estimates hold.

Our main work on proving the existence of classical solutions is to obtain an uniform estimate

for our approximate solutions, that is

$$\Phi'(t) + \Psi(t) \leq C\Phi(t)\Psi(t), \quad (8.12)$$

where C is a suitable positive constant, Φ is the bounded in time H^s -norms of our solution and Ψ is the integrable in time H^s -norms. Then, a rather standard argument (see Lemma 8.5.1) allows to propagate the smallness condition on the initial data (i.e. on $\Phi(0)$). This leads the right-hand side of the above equation to be absorbed by the left-hand side, achieving the cited uniform estimates. Finally we construct our classical solution, through a compactness method.

The uniqueness of our solutions is proven evaluating the difference between two solutions at a regularity level $s = 0$, i.e. in $L^2(\mathbb{R}^2)$. Our work is mainly to obtain an estimate that leads to the Gronwall lemma. Here the main difficulties are handled taking into account a specific feature of the coupling system related to the difference of the two solutions. This feature allows the cancellation of the worst term when considering certain physically meaningful combination of terms.

It is perhaps interesting to remark that in Theorem 8.1.1 we do not consider a positive constant c in the bulk energy density $\psi_B(Q)$. Usually, this is a necessary condition in order to have $\psi_B(Q)$ bounded from below in S_0 , however we do not need this restriction on c mainly because we are dealing with a smallness condition on the initial data. This smallness property is preserved by our solutions, so that one can heuristically think that $\psi_B(\cdot)$ take in consideration Q -tensors whose components belong to a bounded domain in $H^s(\mathbb{R}^2)$. Thus, in this functional framework, ψ_B is still bounded from below. Moreover, Theorem 8.1.1 requires the constant a to be positive. A first reason for this restriction releases again in the smallness condition, since a is the constant related to the lower power of Q , which in this contest has the same behaviour of $\psi_B(Q)$. Nevertheless the main reason for the positivity of a concerns a technical part on proving (8.12), that is the time boundary and L^2 -integrability for the H^s -norm of Q .

8.1.3 The singular potential

Ball and Majumdar [8] introduced the bulk component of the internal energy functional by means of a singular functional $\psi_B = \psi_B(Q)$ that blows up when at least one of the eigenvalues of Q approaches the limiting values $1 - \frac{1}{d}$ respectively $\frac{1}{d}$. In particular, the boundedness of the free energy enforces the boundedness of Q in L^∞ . Specifically, we set

$$\psi_B(Q) = f(Q) - G(Q) \quad \text{for } Q \in R_{\text{sym},0}^{d \times d},$$

where

$$f(Q) = \begin{cases} \inf_{\rho \in \mathcal{A}_Q} \int_{\mathbb{S}^{d-1}} \rho(\mathbf{p}) \log(\rho(\mathbf{p})) \, d\mathbf{p} & \text{if } \lambda_i[Q] \in (1/d, 1 - 1/d), i = 1, \dots, d, \\ \infty & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_Q = \left\{ \rho : \mathbb{S}^2 \rightarrow [0, \infty) \mid \rho \in L^1(\mathbb{S}^{d-1}), \int_{\mathbb{S}^{d-1}} \rho(\mathbf{p}) \, d\mathbf{p} = 1; Q = \int_{\mathbb{S}^{d-1}} (\mathbf{p} \otimes \mathbf{p} - \frac{I}{d}) \rho(\mathbf{p}) \, d\mathbf{p} \right\}.$$

The function f is the singular component of the bulk potential. In here singular refers to the fact that the domain is not the whole space (while inside the domain the function is in fact smooth). The function f enjoys the following properties that can easily be deduced from [8, Section 3, Prop. 1]:

- $f : R_{\text{sym},0}^{d \times d} \rightarrow [-K, \infty]$ is convex and lower semi-continuous, with $K \geq 0$.

- The domain of f ,

$$\mathcal{D}[f] = \{Q \in R_{\text{sym},0}^{d \times d} \mid f(Q) < \infty\} = \{Q \in R_{\text{sym},0}^{d \times d} \mid \lambda_i[Q] \in (-1/d, 1 - \frac{1}{d})\}, \quad (8.13)$$

is an open convex subset of $R_{\text{sym},0}^{d \times d}$.

- f is smooth in $\mathcal{D}[f]$.

Organization

This work is organised as follows: in the next section we prove Theorem 8.1.1 in the simplest case of every viscosity coefficients null while in section 8.3 we prove Theorem 8.1.1 in its general form. In Section 8.4 we present some apriori estimates that exhibit in a simple setting a number of cancellations that are later-on crucial for proving Theorem 8.1.2 in section 8.5.

8.2 Explicit equation and conservation laws for the inviscid model

In this section we consider the system (8.3),(8.4),(8.7) with the viscous contributions $h' = 0$, $\sigma' = 0$. Then the equations (8.3), (8.7) become:

$$\dot{v} + \nabla p = -\nabla \cdot (\nabla Q \odot \nabla Q) \quad (8.14)$$

$$J\ddot{Q} = \Delta Q - \mathcal{L} \frac{\partial \psi_B}{\partial Q} \quad (8.15)$$

with \mathcal{L} denoting the projection onto the space of trace-free matrices, hence

$$\mathcal{L} \frac{\partial \psi_B(Q)}{\partial Q} = \frac{\partial \psi_B(Q)}{\partial Q} - \frac{1}{d} \text{tr} \left(\frac{\partial \psi_B(Q)}{\partial Q} \right) I_d$$

Lemma 8.2.1. *Let (v, Q) be a smooth solution of (8.14),(8.4), (8.15) in the whole space \mathbb{R}^d , and decaying sufficiently fast at infinity. Then:*

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|v|^2}{2}(t, x) + \frac{J}{2} |\dot{Q}|^2(t, x) + \frac{1}{2} |\nabla Q|^2(t, x) + \psi_B(Q(t, x)) dx = \\ \int_{\mathbb{R}^d} \frac{|v|^2}{2}(0, x) + \frac{J}{2} |\dot{Q}|^2(0, x) + \frac{1}{2} |\nabla Q|^2(0, x) + \psi_B(Q(0, x)) dx \end{aligned} \quad (8.16)$$

Proof. We multiply the equation (8.14) by v_i integrate over the space and by parts and add to it the equation (8.15) multiplied by \dot{Q}_{ij} , integrated over the space and by parts to obtain:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2) &= \int_{\mathbb{R}^d} Q_{kl,j} Q_{kl,i} v_{i,j} + \left(\Delta Q_{ij} - \mathcal{L} \frac{\partial \psi_B}{\partial Q} \right)_{ij} \left(\partial_t Q_{ij} + v \cdot \nabla Q_{ij} \right) dx \\ &= \underbrace{\int_{\mathbb{R}^d} Q_{kl,j} Q_{kl,i} v_{i,j} - \Delta Q_{ij} v_k Q_{ij,k} dx}_{:= \mathcal{I}_1} - \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij,k} Q_{ij,k} dx}_{:= \mathcal{I}_2} \\ &\quad - \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij} \frac{\partial \psi_B}{\partial Q} dx}_{:= \mathcal{I}_3} - \underbrace{\int_{\mathbb{R}^d} v \cdot \nabla Q_{ij} \frac{\partial \psi_B}{\partial Q} dx}_{:= \mathcal{I}_4} \end{aligned} \quad (8.17)$$

We see that $\mathcal{I}_1 = \mathcal{I}_4 = 0$ hence we are left with the claimed conservation law (8.16). \square

8.3 Explicit equations and conservation laws for the viscous models

In this case we have the explicit equations:

$$\begin{aligned} \dot{v} + \nabla p - \frac{\beta_4}{2} \Delta v = & \nabla \cdot \left(-\nabla Q \odot \nabla Q + \beta_1 Q \operatorname{tr}\{QA\} + \beta_5 AQ + \beta_6 QA \right) \\ & + \nabla \cdot \left(\frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, (\dot{Q} - [\Omega, Q])] \right) \end{aligned} \quad (8.18)$$

$$J\ddot{Q} + \mu_1 \dot{Q} = \Delta Q - \mathcal{L} \frac{\partial \psi_B}{\partial Q} - \frac{\mu_2}{2} A + \mu_1 [\Omega, Q] \quad (8.19)$$

where we denoted $[A, B] := AB - BA$ with \mathcal{L} again denoting the projection onto the space of trace-free matrices, hence

$$\mathcal{L} \frac{\partial \psi_B(Q)}{\partial Q} = \frac{\partial \psi_B(Q)}{\partial Q} - \frac{\operatorname{Id}}{d} \operatorname{tr} \left(\frac{\partial \psi_B(Q)}{\partial Q} \right)$$

Remark 8.3.1. *Let us clarify that if $M(x)$ is a $d \times d$ -matrix, then $\nabla \cdot M$ stands for the vector field $(M_{ij,j})_{i=1,\dots,d}$.*

Proof of Theorem 8.1.1. We multiply the equation (8.14) by v_i integrate over the space and by parts and add to it the equation (8.15) multiplied by \dot{Q}_{ij} , integrated over the space and by parts to obtain:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2) dx &= - \int_{\mathbb{R}^d} Q_{kl,j} Q_{kl,i} v_{i,j} - \left(\Delta Q_{ij} + \mathcal{L} \frac{\partial \psi_B}{\partial Q}_{ij} \right) (\partial_t Q_{ij} + v \cdot \nabla Q_{ij}) dx \\ &\quad - \int_{\mathbb{R}^d} \sigma'_{ji} v_{i,j} dx + \int_{\mathbb{R}^d} h'_{ij} \dot{Q}_{ij} dx \\ &= - \underbrace{\int_{\mathbb{R}^d} Q_{kl,j} Q_{kl,i} v_{i,j} - \Delta Q_{ij} v_k Q_{ij,k} dx}_{:= \mathcal{S}_1} - \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij,k} Q_{ij,k} dx}_{:= \mathcal{S}_2} \\ &\quad - \underbrace{\int_{\mathbb{R}^d} \partial_t Q_{ij} \frac{\partial \psi_B}{\partial Q}_{ij} dx}_{:= \mathcal{S}_3} - \underbrace{\int_{\mathbb{R}^d} v \cdot \nabla Q_{ij} \frac{\partial \psi_B}{\partial Q}_{ij} dx}_{:= \mathcal{S}_4} \\ &\quad - \beta_1 \int_{\mathbb{R}^d} Q_{ji} Q_{lk} A_{lk} v_{i,j} dx - \beta_4 \int_{\mathbb{R}^d} A_{ji} v_{i,j} dx \\ &\quad - \beta_5 \int_{\mathbb{R}^d} Q_{jl} A_{li} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx \\ &\quad - \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} \left(\dot{Q}_{ji} - \Omega_{jk} Q_{ki} + Q_{jk} \Omega_{ki} \right) v_{i,j} dx \\ &\quad + \mu_1 \int_{\mathbb{R}^d} \left(Q_{jl} \dot{Q}_{li} - \dot{Q}_{jl} Q_{li} \right) v_{i,j} dx \\ &\quad - \mu_1 \int_{\mathbb{R}^d} \left(Q_{jl} [\Omega, Q]_{li} - [\Omega, Q]_{jl} Q_{li} \right) v_{i,j} dx \\ &\quad - \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} - \mu_1 \int_{\mathbb{R}^d} \left(\dot{Q}_{ij} - \Omega_{ik} Q_{kj} + Q_{ik} \Omega_{kj} \right) \dot{Q}_{ij} dx \end{aligned}$$

which implies

$$\begin{aligned}
 & \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + \frac{1}{2}|\nabla Q|^2) + \psi_B(Q) dx + \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \mu_1 \int_{\mathbb{R}^d} |\dot{Q}|^2 dx \\
 &= -\beta_1 \int_{\mathbb{R}^d} Q_{ji} Q_{lk} A_{lk} v_{i,j} dx - \beta_5 \int_{\mathbb{R}^d} Q_{jl} A_{li} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx \\
 &\quad - \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} (\dot{Q}_{ji} - \Omega_{jk} Q_{ki} + Q_{jk} \Omega_{ki}) v_{i,j} + \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} \\
 &\quad + \mu_1 \int_{\mathbb{R}^d} (Q_{jl} \dot{Q}_{li} - \dot{Q}_{jl} Q_{li}) v_{i,j} dx + \mu_1 \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj}) \dot{Q}_{ij} dx \\
 &\quad - \mu_1 \int_{\mathbb{R}^d} (Q_{jl} [\Omega, Q]_{li} - [\Omega, Q]_{jl} Q_{li}) v_{i,j} dx dx
 \end{aligned}$$

Now we analyse each term on the right-hand side of the equality, and we will repeatedly use that $v_{i,j} = A_{ij} + \Omega_{ij}$ and moreover that $\text{tr}\{BC\}$ is null for any B symmetric and C skew-adjoint. We begin with

$$\begin{aligned}
 -\beta_1 \int_{\mathbb{R}^d} Q_{ji} Q_{lk} A_{lk} v_{i,j} &= -\beta_1 \int_{\mathbb{R}^d} Q_{ji} Q_{lk} A_{lk} A_{ij} - \beta_1 \int_{\mathbb{R}^d} Q_{ji} Q_{lk} A_{lk} \Omega_{ij} \\
 &= -\beta_1 \int_{\mathbb{R}^d} \text{tr}\{QA\}^2 - \beta_1 \int_{\mathbb{R}^d} \text{tr}\{Q\Omega\} \text{tr}\{QA\} \\
 &= -\beta_1 \int_{\mathbb{R}^d} \text{tr}\{QA\}^2,
 \end{aligned}$$

observing that $\text{tr}\{Q\Omega\}$ is null. Now we deal with

$$\begin{aligned}
 & -\beta_5 \int_{\mathbb{R}^d} Q_{jl} A_{li} v_{i,j} dx - \beta_6 \int_{\mathbb{R}^d} Q_{il} A_{lj} v_{i,j} dx + \frac{\mu_2}{2} \int_{\mathbb{R}^d} (\Omega_{jk} Q_{ki} - Q_{jk} \Omega_{ki}) v_{i,j} dx = \\
 &= -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{QA \nabla v\} - \beta_6 \int_{\mathbb{R}^d} \text{tr}\{AQ \nabla v\} + \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q \Omega) \nabla v\} \\
 &= -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{(QA + AQ) \nabla u\} - (\beta_6 - \beta_5) \int_{\mathbb{R}^d} \text{tr}\{AQ \nabla v\} + \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{(\Omega Q - Q \Omega) A\} \\
 &= -\beta_5 \int_{\mathbb{R}^d} \text{tr}\{(QA + AQ) A\} - \mu_2 \int_{\mathbb{R}^d} \text{tr}\{AQA\} - \mu_2 \int_{\mathbb{R}^d} \text{tr}\{AQ\Omega\} + \mu_2 \int_{\mathbb{R}^d} \text{tr}\{\Omega QA\} \\
 &= -(2\beta_5 + \mu_2) \int_{\mathbb{R}^d} \text{tr}\{AQA\} = - \int_{\mathbb{R}^d} (\beta_5 + \beta_6) \text{tr}\{AQA\} - \mu_2 \int_{\mathbb{R}^d} \text{tr}\{[\Omega, Q]A\}.
 \end{aligned}$$

Hence, such term is null if we assume $\beta_5 + \beta_6 = 0$. The contribution of the remaining terms related to μ_2 is null, indeed

$$\begin{aligned}
 -\frac{1}{2} \mu_2 \int_{\mathbb{R}^d} \dot{Q}_{ji} v_{i,j} + \frac{1}{2} \mu_2 \int_{\mathbb{R}^d} A_{ij} \dot{Q}_{ij} &= -\frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{\dot{Q} \nabla v\} + \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{A \dot{Q}\} \\
 &= -\frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{\dot{Q} A\} - \underbrace{\frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{\dot{Q} \Omega\}}_{=0} - \frac{\mu_2}{2} \int_{\mathbb{R}^d} \text{tr}\{A \dot{Q}\} = -\mu_2 \int_{\mathbb{R}^d} \text{tr}\{A \dot{Q}\}.
 \end{aligned}$$

Now, let us carry out of the μ_1 -terms: first

$$+\mu_1 \int_{\mathbb{R}^d} (Q_{il} \dot{Q}_{lj} - \dot{Q}_{il} Q_{lj}) v_{i,j} dx + \mu_1 \int_{\mathbb{R}^d} (\Omega_{ik} Q_{kj} - Q_{ik} \Omega_{kj}) \dot{Q}_{ij} dx =$$

$$\begin{aligned}
&= \underbrace{\mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{(Q\dot{Q} - \dot{Q}Q)A\}}_{=0} + \mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{(Q\dot{Q} - \dot{Q}Q)\Omega\} + \mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{(\Omega Q - Q\Omega)\dot{Q}\} = \\
&= 2\mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{[\Omega, Q]\dot{Q}\},
\end{aligned}$$

and moreover

$$\begin{aligned}
-\mu_1 \int_{\mathbb{R}^d} (Q_{jl}[\Omega, Q]_{li} - [\Omega, Q]_{il}Q_{lj}) v_{i,j} &= -\mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{(Q[\Omega, Q] - [\Omega, Q]Q)\Omega\} = \\
&= -\mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{(\Omega Q - Q\Omega)[\Omega, Q]\} = -\mu_1 \int_{\mathbb{R}^d} |[\Omega, Q]|^2.
\end{aligned}$$

Thus, summarizing all the previous equalities, we get

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + |\nabla Q|^2) + \psi_B(Q) dx + \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \mu_1 \int_{\mathbb{R}^d} |\dot{Q}|^2 dx + \\
&+ \beta_1 \int_{\mathbb{R}^d} \operatorname{tr}\{QA\}^2 dx + \mu_1 \int_{\mathbb{R}^d} |[\Omega, Q]|^2 dx - 2\mu_1 \int_{\mathbb{R}^d} \operatorname{tr}\{[\Omega, Q]\dot{Q}\} dx + \mu_2 \int_{\mathbb{R}^d} \operatorname{tr}\{\mathcal{N}A\} = 0,
\end{aligned}$$

which can finally be simplify as follows:

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + |\nabla Q|^2) + \psi_B(Q) dx + \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \\
&+ \beta_1 \int_{\mathbb{R}^d} \operatorname{tr}\{QA\}^2 dx + \mu_1 \int_{\mathbb{R}^d} |\dot{Q} - [\Omega, Q]|^2 dx + \mu_2 \int_{\mathbb{R}^d} \operatorname{tr}\{\mathcal{N}A\} = 0.
\end{aligned}$$

Then, recalling that the definition of the co-rotational time flux is $\mathcal{N} = \dot{Q} - [\Omega, Q]$, we deduce

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + |\nabla Q|^2) + \psi_B(Q) dx + \\
&+ \frac{\beta_4}{2} \int_{\mathbb{R}^d} |\nabla v|^2 dx + \beta_1 \int_{\mathbb{R}^d} \operatorname{tr}\{QA\}^2 dx + \mu_1 \int_{\mathbb{R}^d} |\mathcal{N}|^2 dx = - + \mu_2 \int_{\mathbb{R}^d} \operatorname{tr}\{\mathcal{N}A\}.
\end{aligned}$$

Recalling that $\beta_4 > |\mu_2|/2$ and $\beta_1 > \mu_2/2$, we finally obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} (|v|^2 + J|\dot{Q}|^2 + |\nabla Q|^2) + \psi_B(Q) dx + \\
&+ \int_{\mathbb{R}^d} |\nabla v|^2 dx + \int_{\mathbb{R}^d} \operatorname{tr}\{QA\}^2 dx + \int_{\mathbb{R}^d} |\mathcal{N}|^2 dx \leq 0,
\end{aligned}$$

and this concludes the proof of Theorem 8.1.1. \square

8.4 Commutator and a-priori estimates

In this paragraph we first state a useful commutator estimate, which plays a major role in the proof of Theorem 8.1.1. We start clarifying some notations: we denote by $(\sqrt{-\Delta})^s$ the operator given by

$$(\sqrt{-\Delta})^s f := \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi)),$$

and by $[(\sqrt{-\Delta})^s, f \cdot \nabla]g$ the commutator operator

$$[(\sqrt{-\Delta})^s, f \cdot \nabla]g := (\sqrt{-\Delta})^s(f \cdot \nabla g) - f \cdot \nabla(\sqrt{-\Delta})^s g,$$

for any suitable vector field f and function g . Then the commutator estimate reads as follows:

Lemma 8.4.1. *Let $d \geq 2$, f be a d -dimensional vector field whose component are in $\dot{H}^{s+1}(\mathbb{R}^d)$ and g be a Sobolev function in $\dot{H}^s(\mathbb{R}^d)$, with $s > d/2$. There exist a positive constant C that does not depend on f and g , such that*

$$\|[(\sqrt{-\Delta})^s, f \cdot \nabla]g\|_{L^2(\mathbb{R}^d)} \leq C \|\nabla f\|_{\dot{H}^s(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}^d)}.$$

For the proof of Lemma 8.4.1 we refer the reader to [43], Theorem 1.2.

Now we perform some a-priori estimates, for system (8.1)-(8.2). In the next section, when we will prove the existence of classical solutions, we will make use of similar estimates when constructing approximate solutions. The purpose of the next proposition is to propose these inequalities, for the original system, in order to make the reader familiar with this approach and to present important simplifications related to the coupled equations.

Before starting, let us clarify that from here on, we consider an inhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ with $s > d/2$, equipped with the inner product

$$\langle u, v \rangle_{H^s} = \langle u, v \rangle_{L_x^2} + \langle u, v \rangle_{\dot{H}^s}, \quad \text{where } L_x^2 = L^2(\mathbb{R}^d), \quad \dot{H}^s = \dot{H}^s(\mathbb{R}^d) \quad \text{and} \quad \dot{H}^s = \dot{H}^s(\mathbb{R}^d).$$

Moreover, for any positive real functions $a(t)$ and $b(t)$, we will write $a \lesssim b$ if there exists a positive constant C which does not depend on a and b , such that $a(t) \leq Cb(t)$, for any t .

Proposition 8.4.2. *Let (v, Q) be a smooth solution of system (8.1)-(8.2), which fulfills*

$$\begin{aligned} v &\in L^\infty(\mathbb{R}_+, H^s(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+, H^{s+1}(\mathbb{R}^d)), \\ Q &\in L^\infty(\mathbb{R}_+, H^{s+1}(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+, H^{s+1}(\mathbb{R}^d)), \\ \dot{Q} &\in L^\infty(\mathbb{R}_+, H^s(\mathbb{R}^d)) \cap L^2(\mathbb{R}_+, H^s(\mathbb{R}^d)), \end{aligned}$$

with $s > d/2$. Let us assume that the constant a in the bulk energy $\psi_B(Q)$ (8.10) is positive and greater than $\frac{|\mu_2|^2}{4\beta_4}$. Then the following inequality is satisfied

$$\begin{aligned} &\|v(t)\|_{H^s(\mathbb{R}^d)}^2 + \|Q(t)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(t)\|_{H^s(\mathbb{R}^d)}^2 + \\ &+ \int_0^t \left\{ \|v(s)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^s(\mathbb{R}^d)}^2 \right\} ds \\ &\lesssim \int_0^t \left(\|v(s)\|_{H^s(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^s(\mathbb{R}^d)}^2 \right) \times \\ &\quad \times \left(\|v(s)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^{s+1}(\mathbb{R}^d)}^2 + \|Q(s)\|_{H^s(\mathbb{R}^d)}^2 \right) ds. \end{aligned} \tag{8.20}$$

Proof. Before starting, let us clarify that we denote by $c, c_{\mu_1}, c_{\beta_4}$ and c_a are small positive real constants, whose value will be determined in the end of the proof.

We begin taking the H^s -product between the momentum equation (8.1) and v , that is

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 + \beta_4 \|\nabla v\|_{H^s}^2 &= -\langle v \cdot \nabla v, v \rangle_{H^s} + \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{H^s} + \\ &+ \beta_1 \langle \text{tr}\{AQ\}Q, \nabla v \rangle_{H^s} - \beta_5 \langle QA, \nabla v \rangle_{H^s} - \beta_6 \langle AQ, \nabla v \rangle_{H^s} + \\ &- \frac{\mu_2}{2} \langle \dot{Q} - [\Omega, Q], \nabla v \rangle_{H^s} + \mu_1 \langle [Q, \dot{Q}], \nabla v \rangle_{H^s} - \mu_1 \langle [Q, [Q, \Omega]], \nabla v \rangle_{H^s} \end{aligned} \tag{8.21}$$

Now we analyze each term on the right-hand side of the above equality. First, let us observe that

$$\langle v \cdot \nabla v, v \rangle_{H^s} = \underbrace{\langle v \cdot \nabla v, v \rangle_{L_x^2}}_{=0} + \langle v \cdot \nabla v, v \rangle_{\dot{H}^s}.$$

Since $s > d/2$, then $L^\infty(\mathbb{R}^d)$ is continuously embedded in $H^{s+1}(\mathbb{R}^d)$, and by a classical Gagliardo-Nirenberg inequality we have $\|v\|_{L^\infty(\mathbb{R}^d)} \lesssim \|v\|_{L^2(\mathbb{R}^d)}^\theta \|\nabla v\|_{\dot{H}^s(\mathbb{R}^d)}^{1-\theta}$, with $\theta := d/(2s+2)$. Hence the second term on the right-hand side of the above equality can be estimated as follows:

$$\begin{aligned} |\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| &= |\langle v \otimes v, \nabla v \rangle_{\dot{H}^s}| \\ &\leq \|v\|_{L_x^\infty} \|v\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s} \\ &\lesssim \|v\|_{L_x^2}^\theta \|\nabla v\|_{\dot{H}^s}^{1-\theta} \|v\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s}, \end{aligned}$$

which yields

$$\begin{aligned} |\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| &\lesssim \|v\|_{L_x^2}^\theta \|v\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s}^{2-\theta} \\ &\lesssim \|v\|_{L_x^2}^\theta \|v\|_{\dot{H}^s}^{1-\theta} \|v\|_{\dot{H}^s}^\theta \|\nabla v\|_{\dot{H}^s}^{2-\theta} \\ &\lesssim \|v\|_{H^s} \|v\|_{\dot{H}^s}^\theta \|\nabla v\|_{\dot{H}^s}^{2-\theta}. \end{aligned} \quad (8.22)$$

Since $s > d/2 \geq 1$ then $\|v\|_{\dot{H}^s}^\theta = \|\nabla v\|_{\dot{H}^{s-1}}^\theta \leq \|\nabla v\|_{H^s}^\theta$, thus

$$|\langle v \cdot \nabla v, v \rangle_{\dot{H}^s}| \leq \|v\|_{H^s} \|\nabla v\|_{H^s}^2 \lesssim \|\nabla u\|_{H^s}^2 \|v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \quad (8.23)$$

Now, the second term on the right-hand side of (8.21) is

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{H^s} = \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} + \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{\dot{H}^s}.$$

We will see that $\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2}$ is going to be simplified, while

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{\dot{H}^s} \leq \|\nabla Q\|_{L_x^\infty} \|\nabla Q\|_{\dot{H}^s} \|\nabla v\|_{\dot{H}^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2.$$

Finally, the remaining terms on the right-hand side of (8.21) are controlled as follows:

$$\begin{aligned} \beta_1 \langle \text{tr}\{AQ\}Q, \nabla v \rangle_{H^s} &\leq \|A\|_{H^s} \|Q\|_{H^s}^2 \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2, \\ \beta_5 \langle QA, \nabla v \rangle_{H^s} + \beta_6 \langle AQ, \nabla v \rangle_{H^s} &\leq \|A\|_{H^s} \|Q\|_{H^s} \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2, \\ \frac{\mu_2}{2} \langle [\Omega, Q], \nabla v \rangle_{H^s} &\lesssim \|Q\|_{H^s} \|\nabla v\|_{H^s}^2 \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2, \\ \mu_1 \langle [Q, \dot{Q}], \nabla v \rangle_{H^s} &\lesssim \|Q\|_{H^s} \|\dot{Q}\|_{H^s} \|\nabla v\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2, \\ \mu_1 \langle [Q, [\Omega, Q]], \nabla v \rangle_{H^s} &\lesssim \|Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2. \end{aligned}$$

Thus, summarizing the previous estimates together with (8.21), we get

$$\begin{aligned} \frac{d}{dt} \|v\|_{H^s}^2 + \beta_4 \|\nabla v\|_{H^s}^2 + \frac{\mu_2}{2} \langle \dot{Q}, \nabla v \rangle_{H^s} - \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} \\ \lesssim \|\nabla v\|_{H^s}^2 (\|v\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + \|Q\|_{H^s}^2) + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \end{aligned} \quad (8.24)$$

Now, let us take the H^s -inner product between the order equation (8.2) and \dot{Q} :

$$\begin{aligned} J \langle \ddot{Q}, \dot{Q} \rangle_{H^s} + \mu_1 \|\dot{Q}\|_{H^s}^2 - \mu_1 \langle [\Omega, Q], \dot{Q} \rangle_{H^s} + \frac{1}{2} \frac{d}{dt} \|\nabla Q\|_{H^s}^2 - \langle \Delta Q, v \cdot \nabla Q \rangle_{H^s} = \\ = -a \langle Q \dot{Q} \rangle_{H^s} + b \langle Q^2, \dot{Q} \rangle_{H^s} - c \langle Q \text{tr}\{Q^2\}, \dot{Q} \rangle_{H^s} - \frac{\mu_2}{2} \langle A, \dot{Q} \rangle_{H^s}. \end{aligned} \quad (8.25)$$

We begin, observing that the first term on the left-hand side fulfills

$$J\langle \ddot{Q}, \dot{Q} \rangle_{H^s} = \frac{J}{2} \frac{d}{dt} \|\dot{Q}\|_{H^s}^2 + \langle v \cdot \nabla Q, \dot{Q} \rangle_{H^s}$$

where we have also

$$\begin{aligned} \langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{H^s} &= \underbrace{\langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{L_x^2}}_{=0} + \langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{\dot{H}^s} \\ &= \langle [(\sqrt{-\Delta})^s, v \cdot \nabla] \dot{Q}, (\sqrt{-\Delta})^s \dot{Q} \rangle_{L^2} + \underbrace{\langle v \cdot \nabla (\sqrt{-\Delta})^s \dot{Q}, (\sqrt{-\Delta})^s \dot{Q} \rangle_{L^2}}_{=0}. \end{aligned}$$

Hence, applying the commutator estimate given by Lemma 8.4.1, we deduce that

$$\langle v \cdot \nabla \dot{Q}, \dot{Q} \rangle_{H^s} \lesssim \|\nabla v\|_{\dot{H}^s} \|\dot{Q}\|_{H^s}^2 \lesssim \|\nabla v\|_{H^s}^2 \|\dot{Q}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \quad (8.26)$$

Now, we keep estimating the terms in (8.25). At first

$$\mu_1 \langle [\Omega, Q], \dot{Q} \rangle_{H^s} \lesssim \|\nabla v\|_{H^s} \|Q\|_{H^s} \|\dot{Q}\|_{H^s} \lesssim \|\nabla v\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \quad (8.27)$$

Moreover, we split $\langle \Delta Q, v \cdot \nabla Q \rangle_{H^s}$ into $\langle \Delta Q, v \cdot \nabla Q \rangle_{L_x^2} + \langle \Delta Q, v \cdot \nabla Q \rangle_{\dot{H}^s}$ and we rewrite the \dot{H}^s -term as follows

$$\begin{aligned} \langle \Delta Q, v \cdot \nabla Q \rangle_{\dot{H}^s} &= \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,ii}, (\sqrt{-\Delta})^s v_j Q_{\alpha\beta,j} \rangle_{L_x^2} = \\ &= \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s v_{j,i} Q_{\alpha\beta,j} \rangle_{L_x^2} - \langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, (\sqrt{-\Delta})^s v_j Q_{\alpha\beta,ij} \rangle_{L_x^2} =: \mathcal{J}_1 + \mathcal{J}_2, \end{aligned}$$

where we have used the index summation convention, namely if there are repeated indexes then they are under summation. Now, recalling that H^s is an algebra for $s > d/2$, we handle \mathcal{J}_1 by

$$\begin{aligned} |\mathcal{J}_1| &\leq \|Q_{\alpha\beta,i}\|_{\dot{H}^s} \|v_{j,i} Q_{\alpha\beta,j}\|_{\dot{H}^s} \\ &\leq \|Q_{\alpha\beta,i}\|_{H^s} \|v_{j,i} Q_{\alpha\beta,j}\|_{H^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{\dot{H}^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2, \end{aligned}$$

and, recalling the commutator estimate given by Lemma 8.4.1, \mathcal{J}_2 is estimated by

$$\begin{aligned} \mathcal{J}_2 &= -\langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, [(\sqrt{-\Delta})^s, v \cdot \nabla] Q_{\alpha\beta,i} \rangle_{L_x^2} - \underbrace{\langle (\sqrt{-\Delta})^s Q_{\alpha\beta,i}, v \cdot \nabla (\sqrt{-\Delta})^s Q_{\alpha\beta,i} \rangle_{L_x^2}}_{=0} \\ &\lesssim \|\nabla Q\|_{\dot{H}^s} \|\langle (\sqrt{-\Delta})^s, v \cdot \nabla \rangle Q_{\alpha\beta,i}\|_{L_x^2} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{\dot{H}^s} \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \end{aligned}$$

Summarizing, we deduce that

$$\langle \Delta Q, v \cdot \nabla Q \rangle_{\dot{H}^s} \leq |\mathcal{J}_1| + |\mathcal{J}_2| \lesssim \|\nabla Q\|_{H^s}^2 \|\nabla v\|_{H^s}^2 + c_{\beta_4} \|\nabla v\|_{H^s}^2. \quad (8.28)$$

We keep going on, estimating the term on the right-hand side of equality (8.25). At first, we have

$$-a \langle \dot{Q}, Q \rangle_{H^s} = -a \frac{d}{dt} \|Q\|_{H^s}^2 - a \langle v \cdot \nabla Q, Q \rangle_{H^s},$$

with

$$-a \langle v \cdot \nabla Q, Q \rangle_{H^s} \lesssim \|v\|_{H^s} \|\nabla Q\|_{H^s} \|Q\|_{H^s} \lesssim \|Q\|_{H^s}^2 \|v\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2,$$

furthermore

$$b \langle \dot{Q}, Q^2 \rangle_{H^s} \lesssim \|\dot{Q}\|_{H^s} \|Q\|_{H^s}^2 \lesssim \|Q\|_{H^s}^2 \|Q\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}$$

and finally

$$-c \langle \dot{Q}, Q \operatorname{tr}\{Q^2\} \rangle_{H^s} \lesssim \|\dot{Q}\|_{H^s} \|Q\|_{H^s}^3 \leq \|Q\|_{H^s}^2 (\|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2).$$

Thus, summarizing all the previous considerations, equality (8.25) yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} [J \|\dot{Q}\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + a \|Q\|_{H^s}^2] + \mu_1 \|\dot{Q}\|_{H^s}^2 - \langle v \cdot \nabla Q, \Delta Q \rangle_{L_x^2} - \frac{\mu_2}{2} \langle A, \dot{Q} \rangle_{H^s} \\ &\lesssim (\|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 + \|\nabla v\|_{H^s}^2) (\|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2) + c_{\beta_4} \|\nabla v\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2. \end{aligned} \quad (8.29)$$

Now, let us consider the H^s -inner product between the order tensor equation (8.2) and $Q/2$, namely

$$\begin{aligned}
\frac{J}{2}\langle\ddot{Q}, Q\rangle_{H^s} + \frac{\mu_1}{2}\langle\dot{Q}, Q\rangle_{H^s} - \frac{\mu_1}{2}\langle[\Omega, Q], Q\rangle_{H^s} + \|\nabla Q\|_{H^s}^2 + a\|Q\|_{H^s}^2 &= \\
= b\langle Q^2, Q\rangle_{H^s} - c\langle Q\text{tr}Q^2, Q\rangle_{H^s} + \frac{\mu_2}{2}\langle A, Q\rangle_{H^s}. & \quad (8.30)
\end{aligned}$$

We analyze $\langle\ddot{Q}, Q\rangle_{H^s}$ in the left-hand side of the above equation, first splitting it into $\langle\ddot{Q}, Q\rangle_{H^s} = \langle\ddot{Q}, Q\rangle_{L_x^2} + \langle\ddot{Q}, Q\rangle_{\dot{H}^s}$, then controlling the L_x^2 -inner product by

$$\begin{aligned}
\langle\ddot{Q}, Q\rangle_{L_x^2} &= \langle\partial_t\dot{Q}, Q\rangle_{L_x^2} + \langle v \cdot \nabla\dot{Q}, Q\rangle_{L_x^2} \\
&= \frac{d}{dt}\langle\dot{Q}, Q\rangle_{L_x^2} - \langle\dot{Q}, \partial_t Q\rangle_{L_x^2} - \langle\dot{Q}, v \cdot \nabla Q\rangle_{L_x^2} = \frac{d}{dt}\langle\dot{Q}, Q\rangle_{L_x^2} - \|\dot{Q}\|_{L_x^2}^2.
\end{aligned}$$

Moreover, we handle the \dot{H}^s -product $\langle\ddot{Q}, Q\rangle_{\dot{H}^s} = \langle\partial_t\dot{Q}, Q\rangle_{\dot{H}^s} + \langle v \cdot \nabla\dot{Q}, Q\rangle_{\dot{H}^s}$, first observing that

$$\frac{J}{2}\langle\partial_t\dot{Q}, Q\rangle_{\dot{H}^s} = \frac{J}{2}\frac{d}{dt}\langle\dot{Q}, Q\rangle_{\dot{H}^s} - \frac{J}{2}\langle\dot{Q}, \partial_t Q\rangle_{\dot{H}^s}$$

and then observing that

$$\begin{aligned}
\langle v \cdot \nabla\dot{Q}, Q\rangle_{\dot{H}^s} &= \langle(\sqrt{-\Delta})^s v \cdot \nabla\dot{Q}, (\sqrt{-\Delta})^s Q\rangle_{L_x^2} \\
&= \langle[(\sqrt{-\Delta})^s, v \cdot \nabla]\dot{Q}, (\sqrt{-\Delta})^s Q\rangle_{L_x^2} + \langle(\sqrt{-\Delta})^s\dot{Q}, v \cdot \nabla(\sqrt{-\Delta})^s Q\rangle_{L_x^2} \\
&= \langle[(\sqrt{-\Delta})^s, v \cdot \nabla]\dot{Q}, (\sqrt{-\Delta})^s Q\rangle_{L_x^2} - \\
&\quad - \langle(\sqrt{-\Delta})^s\dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla]Q\rangle_{L_x^2} - \langle\dot{Q}, v \cdot \nabla Q\rangle_{\dot{H}^s}.
\end{aligned} \quad (8.31)$$

Summarizing, we get that the \dot{H}^s -product $\langle\ddot{Q}, Q\rangle_{\dot{H}^s}$ fulfills

$$\begin{aligned}
\frac{J}{2}\langle\ddot{Q}, Q\rangle_{\dot{H}^s} &= \frac{J}{2}\frac{d}{dt}\langle\dot{Q}, Q\rangle_{\dot{H}^s} - \frac{J}{2}\|\dot{Q}\|_{\dot{H}^s}^2 + \\
&\quad + \frac{J}{2}\langle[(\sqrt{-\Delta})^s, v \cdot \nabla]\dot{Q}, (\sqrt{-\Delta})^s Q\rangle_{L_x^2} - \frac{J}{2}\langle(\sqrt{-\Delta})^s\dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla]Q\rangle_{L_x^2},
\end{aligned} \quad (8.32)$$

and moreover, the commutator estimate of Lemma 8.4.1 allows us to control the last two terms by

$$\begin{aligned}
\frac{J}{2}\langle[(\sqrt{-\Delta})^s, v \cdot \nabla]\dot{Q}, (\sqrt{-\Delta})^s Q\rangle_{L_x^2} - \frac{J}{2}\langle(\sqrt{-\Delta})^s\dot{Q}, [(\sqrt{-\Delta})^s, v \cdot \nabla]Q\rangle_{L_x^2} \\
\lesssim \|\nabla v\|_{\dot{H}^s}\|\dot{Q}\|_{\dot{H}^s}\|Q\|_{\dot{H}^s} \lesssim \|\nabla v\|_{\dot{H}^s}^2\|Q\|_{\dot{H}^s}^2 + c_{\mu_1}\|\dot{Q}\|_{\dot{H}^s}^2.
\end{aligned} \quad (8.33)$$

A further development in (8.32) releases in $\frac{d}{dt}\langle\dot{Q}, Q\rangle_{H^s}$, since this term can be rewritten as follows:

$$\frac{J}{2}\frac{d}{dt}\langle\dot{Q}, Q\rangle_{H^s} = \frac{J}{4}\frac{d}{dt}[\|\dot{Q} + Q\|_{H^s}^2 - \|\dot{Q}\|_{H^s}^2 - \|Q\|_{H^s}^2]. \quad (8.34)$$

Now, we come back to equality (8.30), and we proceed estimating the remaining terms. First,

$$\frac{\mu_1}{2}\langle\dot{Q}, Q\rangle_{H^s} = \frac{\mu_1}{2}\langle\partial_t Q, Q\rangle_{H^s} + \frac{\mu_1}{2}\langle v \cdot \nabla Q, Q\rangle_{H^s} = \frac{\mu_1}{4}\frac{d}{dt}\|Q\|_{H^s}^2 + \frac{\mu_1}{2}\langle v \cdot \nabla Q, Q\rangle_{H^s},$$

where we can easily handle the term $\langle v \cdot \nabla Q, Q\rangle_{H^s}$, since H^s is an algebra, that is

$$\frac{\mu_1}{2}\langle v \cdot \nabla Q, Q\rangle_{H^s} \lesssim \|v\|_{H^s}\|\nabla Q\|_{H^s}\|Q\|_{H^s} \lesssim \|Q\|_{H^s}^2\|v\|_{H^s}^2 + c\|\nabla Q\|_{H^s}^2.$$

Similarly, we get also

$$\frac{\mu_1}{2}\langle[\Omega, Q], Q\rangle_{H^s} \lesssim \|\nabla v\|_{H^s}\|Q\|_{H^s}^2 \lesssim \|Q\|_{H^s}^2\|Q\|_{H^s}^2 + c_{\beta_4}\|\nabla v\|_{H^s}^2,$$

$$b\langle Q^2, Q\rangle_{H^s} \lesssim \|Q\|_{H^s}^3 \lesssim \|Q\|_{H^s}^2\|Q\|_{H^s}^2 + c_a\|Q\|_{H^s}^2,$$

$$c\langle Q\text{tr}Q^2, Q\rangle_{H^s} \lesssim \|Q\|_{H^s}^2\|Q\|_{H^s}^2.$$

It remains to control the μ_2 -term in (8.30), that is $\mu_2\langle A, Q\rangle_{H^s}/2$. We fix a positive constant ε in

(0, 1) small enough, and we estimate this term as follows:

$$\frac{\mu_2}{2} \langle A, Q \rangle_{H^s} \leq \frac{|\mu_2|}{2} \|\nabla v\|_{H^s} \|Q\|_{H^s} \leq \frac{|\mu_2|^2}{8a(1-\varepsilon)} \|\nabla v\|_{H^s}^2 + a(1-\varepsilon) \|Q\|_{H^s}^2.$$

Then, summarizing all the previous considerations, equality (8.30) yields

$$\begin{aligned} & \frac{d}{dt} \left[\frac{J}{4} \|\dot{Q} + Q\|_{H^s}^2 - \frac{J}{4} \|\dot{Q}\|_{H^s}^2 + \frac{\mu_1 - J}{4} \|Q\|_{H^s}^2 \right] - \frac{J}{2} \|\dot{Q}\|_{H^s}^2 + a\varepsilon \|Q\|_{H^s}^2 - \\ & - \frac{\mu_2}{8a(1-\varepsilon)} \|\nabla v\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \lesssim \left(\|\nabla v\|_{H^s}^2 + \|Q\|_{H^s}^2 \right) \left(\|Q\|_{H^s}^2 + \|v\|_{H^s}^2 \right) + \\ & + c_{\beta_4} \|\nabla v\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c_a \|Q\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2. \end{aligned} \quad (8.35)$$

We finally take the sum between (8.24), (8.29) and (8.35). Together, the three inequalities lead to

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \|v\|_{H^s}^2 + \frac{J}{4} \|Q + \dot{Q}\|_{H^s}^2 + \frac{J}{4} \|\dot{Q}\|_{H^s}^2 + \left(a - \frac{J}{4}\right) \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \right] + \\ & + \left(\frac{\beta_4}{2} - \frac{|\mu_2|^2}{8(1-\varepsilon)a} \right) \|\nabla v\|_{H^s}^2 + \left(\mu_1 - \frac{J}{2} \right) \|\dot{Q}\|_{H^s}^2 + a\varepsilon \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 - \\ & - \langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} - \langle v \cdot \nabla Q, \Delta Q \rangle_{L_x^2} \leq C \left(c_{\beta_4} \|\nabla v\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}\|_{H^s}^2 + c_a \|Q\|_{H^s}^2 + c \|\nabla Q\|_{H^s}^2 \right) + \\ & + C \left(\|\nabla v\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \right) \left(\|v\|_{H^s}^2 + \|Q\|_{H^s}^2 + \|\dot{Q}\|_{H^s}^2 + \|\nabla Q\|_{H^s}^2 \right), \end{aligned} \quad (8.36)$$

For a suitable positive constant C . Now, observing that

$$\langle \nabla Q \odot \nabla Q, \nabla v \rangle_{L_x^2} + \langle v \cdot \nabla Q, \Delta Q \rangle_{L_x^2} = 0,$$

imposing ε small enough in order to have $\frac{\beta_4}{2} > \frac{|\mu_2|^2}{8(1-\varepsilon)a}$ and assuming c , c_{β_4} , c_{μ_1} and c_a small enough, inequality (8.36) yields (8.20). This concludes the proof of the proposition. \square

8.5 Classical solutions

This section is devoted to the proof of Theorem 8.1.2, namely we prove the existence and uniqueness of strong solutions for system (8.1)-(8.2), for small initial data. We split the proof into two parts, one concerning the existence of solutions and the other one about their uniqueness.

The existence is based on a Friedrichs's type scheme which allows to construct approximate solutions. we then proceed performing uniform estimates, with the same approach of Proposition 8.4.2.

The uniqueness is achieved controlling the difference of two solutions in an L_x^2 -framework.

Proof of Theorem 8.1.2. Existence part: In order to construct global strong solutions, we use the classical Friedrichs's scheme and we perform similar estimates to the ones of the previous section. Let us first introduce some notation. We define the mollifying operator J_n trough Fourier transform, that is

$$J_n \hat{f}(\xi) := \widehat{J_n f}(\xi) := 1_{\{2^{-n} \leq |\xi| \leq 2^n\}} \hat{f}(\xi).$$

Then, we consider the following approximate momentum equation

$$\begin{aligned}
& J_n \dot{v}^{(n)} + \frac{\beta_4}{2} \Delta J_n v^{(n)} = -\nabla \cdot \mathcal{P} \left\{ J_n (\nabla J_n Q^{(n)} \odot \nabla J_n Q^{(n)}) \right\} + \\
& + \nabla \cdot \mathcal{P} \left\{ \beta_1 J_n (J_n Q^{(n)} \operatorname{tr} \{ J_n (J_n Q^{(n)} J_n A^{(n)}) \}) + \beta_5 J_n (J_n A^{(n)} J_n Q^{(n)}) + \beta_6 J_n (J_n Q^{(n)} J_n A^{(n)}) \right\} \\
& + \nabla \cdot \mathcal{P} \left\{ \frac{\mu_2}{2} (\dot{Q}^{(n)} - J_n [J_n \Omega^{(n)}, J_n Q^{(n)}]) + \mu_1 J_n [J_n Q^{(n)}, (J_n \dot{Q}^{(n)} - J_n [J_n \Omega^{(n)}, J_n Q^{(n)}])] \right\},
\end{aligned}$$

where \mathcal{P} denotes the Leray projector onto divergence-free vector fields and where we have used the abuse of notation

$$\dot{f}^{(n)} := \partial_t f^{(n)} + J_n (J_n v^{(n)} \cdot \nabla J_n f^{(n)}) \neq \partial_t f^{(n)} + J_n v^{(n)} \cdot \nabla J_n f^{(n)}.$$

We impose also the free divergence condition $\operatorname{div} v^{(n)} = 0$. Moreover, the approximate order tensor equation reads as

$$\begin{aligned}
& J J_n \ddot{Q}^{(n)} + \mu_1 J_n \dot{Q}^{(n)} = \Delta J_n Q^{(n)} - a J_n Q^{(n)} + b J_n (J_n Q^{(n)} J_n Q^{(n)}) \\
& - b \operatorname{tr} \{ J_n (J_n Q^{(n)} J_n Q^{(n)}) \} \frac{\operatorname{Id}}{3} + c J_n (J_n Q^{(n)} \operatorname{tr} \{ J_n (J_n Q^{(n)} J_n Q^{(n)}) \}) + \frac{\mu_2}{2} J_n A^{(n)}.
\end{aligned}$$

The system above, with the initial data $(v^{(n)}, Q^{(n)})_{t=0} = J_n (v_0, Q_0)$, can be regarded as an ordinary differential equation in $L^2(\mathbb{R}^d)$ verifying the conditions of the Cauchy-Lipschitz theorem. Thus it admits a unique maximal solution $(v^{(n)}, Q^{(n)})$ in $C^1([0, T^n], L^2(\mathbb{R}^d))$. As we have $(\mathcal{P} J_n)^2 = \mathcal{P} J_n$ and $J_n^2 = J_n$, the pair $(J_n v^{(n)}, J_n Q^{(n)})$ is also a solution of the previous system. Hence, by uniqueness we get that $(J_n v^{(n)}, J_n Q^{(n)}) = (v^{(n)}, Q^{(n)})$, moreover $(v^{(n)}, Q^{(n)})$ belongs to $C^1([0, T^n], H^\infty(\mathbb{R}^d))$ and it satisfies

$$\left\{ \begin{aligned}
& \dot{v}^{(n)} + \frac{\beta_4}{2} \Delta v^{(n)} = -\nabla \cdot \mathcal{P} \left\{ J_n (\nabla Q^{(n)} \odot \nabla Q^{(n)}) \right\} + \\
& + \nabla \cdot \mathcal{P} \left\{ \beta_1 J_n (Q^{(n)} \operatorname{tr} \{ J_n (Q^{(n)} A^{(n)}) \}) + \beta_5 J_n (A^{(n)} Q^{(n)}) + \beta_6 J_n (Q^{(n)} A^{(n)}) \right\} + \\
& + \nabla \cdot \mathcal{P} \left\{ \frac{\mu_2}{2} (\dot{Q}^{(n)} - J_n [\Omega^{(n)}, Q^{(n)}]) + \mu_1 J_n [Q^{(n)}, (\dot{Q}^{(n)} - J_n [\Omega^{(n)}, Q^{(n)}])] \right\}, \\
& J \ddot{Q}^{(n)} + \mu_1 \dot{Q}^{(n)} = \Delta Q^{(n)} - a Q^{(n)} + b J_n (Q^{(n)} Q^{(n)}) \\
& \quad - b \operatorname{tr} \{ J_n (Q^{(n)} Q^{(n)}) \} \frac{\operatorname{Id}}{3} + c J_n (Q^{(n)} \operatorname{tr} \{ J_n (Q^{(n)} Q^{(n)}) \}) + \frac{\mu_2}{2} A^{(n)}, \\
& (v^{(n)}, Q^{(n)})_{t=0} = J_n (v_0, Q_0).
\end{aligned} \right. \quad (8.37)$$

We claim that for any $n \in \mathbb{N}$ the maximal time $T^n = +\infty$ and moreover that there exists a positive constant C , which does not depend on n , such that

$$\begin{aligned}
& \|v^{(n)}\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d))} + \|\nabla v^{(n)}\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \|Q^{(n)}\|_{L^\infty(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \\
& + \|Q^{(n)}\|_{L^2(\mathbb{R}_+; H^{s+1}(\mathbb{R}^d))} + \|\dot{Q}^{(n)}\|_{L^\infty(\mathbb{R}_+; H^s(\mathbb{R}^d))} + \|\dot{Q}^{(n)}\|_{L^2(\mathbb{R}_+; H^s(\mathbb{R}^d))} \leq C \eta_0.
\end{aligned} \quad (8.38)$$

We begin taking the H^s -product between the momentum equation and $v^{(n)}$, that is

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|v^{(n)}\|_{H^s}^2 + \beta_4 \|\nabla v^{(n)}\|_{H^s}^2 = -\langle J_n (v^{(n)} \cdot \nabla v^{(n)}), v^{(n)} \rangle_{H^s} + \langle J_n (\nabla Q^{(n)} \odot \nabla Q^{(n)}), \nabla v^{(n)} \rangle_{H^s} \\
& + \beta_1 \langle \operatorname{tr} \{ J_n (A^{(n)} Q^{(n)}) \} Q^{(n)}, \nabla v^{(n)} \rangle_{H^s} - \beta_5 \langle Q^{(n)} A^{(n)}, \nabla v^{(n)} \rangle_{H^s} - \\
& - \beta_6 \langle A^{(n)} Q^{(n)}, \nabla v^{(n)} \rangle_{H^s} - \frac{\mu_2}{2} \langle \dot{Q}^{(n)} - J_n [\Omega^{(n)}, Q^{(n)}], \nabla v^{(n)} \rangle_{H^s} + \\
& + \mu_1 \langle J_n [Q^{(n)}, \dot{Q}^{(n)}], \nabla v^{(n)} \rangle_{H^s} - \mu_1 \langle J_n [Q^{(n)}, J_n [Q^{(n)}, \Omega^{(n)}]], \nabla v^{(n)} \rangle_{H^s}.
\end{aligned} \quad (8.39)$$

We start analysing $\langle J_n (v^{(n)} \cdot \nabla v^{(n)}), v^{(n)} \rangle_{H^s}$ on the right-hand side. First we can rewrite it as

$$\langle J_n (v^{(n)} \cdot \nabla v^{(n)}), v^{(n)} \rangle_{H^s} = \langle v^{(n)} \cdot \nabla v^{(n)}, J_n v^{(n)} \rangle_{H^s} = -\langle v^{(n)} \otimes v^{(n)}, \nabla v^{(n)} \rangle_{H^s}$$

then, proceeding as for proving (8.22) and (8.23), we deduce

$$\begin{aligned} |\langle J_n(v^{(n)} \cdot \nabla v^{(n)}), v^{(n)} \rangle_{H^s}| &= |\langle v^{(n)} \cdot \nabla v^{(n)}, J_n v^{(n)} \rangle_{H^s}| \\ &= |\langle v^{(n)} \cdot \nabla v^{(n)}, v^{(n)} \rangle_{H^s}| \\ &\lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|v^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2. \end{aligned}$$

Now, the second term on the right-hand side of (8.39) is

$$\langle J_n(\nabla Q^{(n)} \odot \nabla Q^{(n)}), \nabla v^{(n)} \rangle_{H^s} = \langle \nabla Q^{(n)} \odot \nabla Q^{(n)}, \nabla v^{(n)} \rangle_{L_x^2} + \langle \nabla Q^{(n)} \odot \nabla Q^{(n)}, \nabla v^{(n)} \rangle_{\dot{H}^s}.$$

We will see that $\langle \nabla Q^{(n)} \odot \nabla Q^{(n)}, \nabla v^{(n)} \rangle_{L_x^2}$ is going to be simplified, while

$$\begin{aligned} |\langle \nabla Q^{(n)} \odot Q^{(n)}, \nabla v^{(n)} \rangle_{\dot{H}^s}| &\lesssim \|\nabla Q^{(n)}\|_{L_x^\infty} \|\nabla Q^{(n)}\|_{\dot{H}^s} \|\nabla v^{(n)}\|_{\dot{H}^s} \\ &\lesssim \|\nabla Q^{(n)}\|_{H^s}^2 \|\nabla Q^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2. \end{aligned}$$

The remaining term on the right-hand side of (8.39) are handled as follows:

$$\begin{aligned} \beta_1 \langle \text{tr}\{J_n(A^{(n)}Q^{(n)})\}Q^{(n)}, \nabla v^{(n)} \rangle_{H^s} &\lesssim \|J_n(A^{(n)}Q^{(n)})\|_{H^s} \|Q^{(n)}\|_{H^s} \|\nabla v^{(n)}\|_{H^s} \\ &\lesssim \|A^{(n)}\|_{H^s} \|Q^{(n)}\|_{H^s}^2 \|\nabla v^{(n)}\|_{H^s} \lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2, \end{aligned}$$

$$\begin{aligned} \beta_5 \langle Q^{(n)}A^{(n)}, \nabla v^{(n)} \rangle_{H^s} + \beta_6 \langle A^{(n)}Q^{(n)}, \nabla v^{(n)} \rangle_{H^s} &\lesssim \|A^{(n)}\|_{H^s} \|Q^{(n)}\|_{H^s} \|\nabla v^{(n)}\|_{H^s} \\ &\lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2, \end{aligned}$$

$$\begin{aligned} \frac{\mu_2}{2} \langle J_n[\Omega^{(n)}, Q^{(n)}], \nabla v^{(n)} \rangle_{H^s} &= \frac{\mu_2}{2} \langle [\Omega^{(n)}, Q^{(n)}], \nabla v^{(n)} \rangle_{H^s} \\ &\lesssim \|Q^{(n)}\|_{H^s} \|\nabla v^{(n)}\|_{H^s}^2 \\ &\lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2, \end{aligned}$$

$$\begin{aligned} \mu_1 \langle J_n[Q^{(n)}, \dot{Q}^{(n)}], \nabla v^{(n)} \rangle_{H^s} &= \mu_1 \langle [Q^{(n)}, \dot{Q}^{(n)}], \nabla v^{(n)} \rangle_{H^s} \\ &\lesssim \|Q^{(n)}\|_{H^s} \|\dot{Q}^{(n)}\|_{H^s} \|\nabla v^{(n)}\|_{H^s} \\ &\lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2 \end{aligned}$$

and finally

$$\mu_1 \langle J_n[Q^{(n)}, J_n[\Omega^{(n)}, Q^{(n)}]], \nabla v^{(n)} \rangle_{H^s} \lesssim \|Q^{(n)}\|_{H^s}^2 \|\nabla v^{(n)}\|_{H^s}^2.$$

Thus, summarizing the previous estimates together with (8.39), we get

$$\begin{aligned} \frac{d}{dt} [\|v^{(n)}\|_{H^s}^2] + \beta_4 \|\nabla v^{(n)}\|_{H^s}^2 + \frac{\mu_2}{2} \langle \dot{Q}^{(n)}, \nabla v^{(n)} \rangle_{H^s} - \langle \nabla Q^{(n)} \odot \nabla Q^{(n)}, \nabla v^{(n)} \rangle_{L_x^2} &\lesssim \\ \lesssim \|\nabla v^{(n)}\|_{H^s}^2 (\|v^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2) + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2. \end{aligned} \quad (8.40)$$

Now, let us take in consideration the H^s -inner product between the order tensor equation and $\dot{Q}^{(n)}$:

$$\begin{aligned} J\langle \ddot{Q}^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} + \mu_1 \|\dot{Q}^{(n)}\|_{H^s}^2 - \mu_1 \langle [\Omega^{(n)}, Q^{(n)}], \dot{Q}^{(n)} \rangle_{H^s} + \frac{1}{2} \frac{d}{dt} \|\nabla Q^{(n)}\|_{H^s}^2 - \\ - \langle \Delta Q^{(n)}, v^{(n)} \cdot \nabla Q^{(n)} \rangle_{H^s} = a \langle Q^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} + b \langle (Q^{(n)})^2, \dot{Q}^{(n)} \rangle_{H^s} - \\ - c \langle Q^{(n)} \text{tr}\{(Q^{(n)})^2\}, \dot{Q}^{(n)} \rangle_{H^s} + \frac{\mu_2}{2} \langle A^{(n)}, \dot{Q}^{(n)} \rangle_{H^s}. \end{aligned} \quad (8.41)$$

The first term on the left-hand side fulfills

$$\begin{aligned}
J\langle \ddot{Q}^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} &= J\langle \partial_t \dot{Q}^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} + J\langle J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)}), \dot{Q}^{(n)} \rangle_{H^s} \\
&= \frac{J}{2} \frac{d}{dt} \|\dot{Q}^{(n)}\|_{H^s}^2 + J\langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, J_n \dot{Q}^{(n)} \rangle_{H^s} \\
&= \frac{J}{2} \frac{d}{dt} \|\dot{Q}^{(n)}\|_{H^s}^2 + J\langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, \dot{Q}^{(n)} \rangle_{H^s}.
\end{aligned}$$

Moreover, proceeding as for proving (8.26) and (8.27), we get

$$J\langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} + \mu_1 \langle J_n[\Omega^{(n)}, Q^{(n)}], \dot{Q}^{(n)} \rangle_{H^s} \lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2,$$

where we have used

$$\mu_1 \langle J_n[\Omega^{(n)}, Q^{(n)}], \dot{Q}^{(n)} \rangle_{H^s} = \mu_1 \langle [\Omega^{(n)}, Q^{(n)}], J_n \dot{Q}^{(n)} \rangle_{H^s} = \mu_1 \langle [\Omega^{(n)}, Q^{(n)}], \dot{Q}^{(n)} \rangle_{H^s}.$$

Now, as for proving (8.28), we have

$$\begin{aligned}
\langle \Delta Q^{(n)}, J_n(v^{(n)} \cdot \nabla Q^{(n)}) \rangle_{\dot{H}^s} &= \langle \Delta Q^{(n)}, v^{(n)} \cdot \nabla Q^{(n)} \rangle_{\dot{H}^s} \\
&\lesssim \|\nabla Q^{(n)}\|_{H^s}^2 \|\nabla v^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2,
\end{aligned}$$

while

$$\begin{aligned}
-a \langle Q^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} &= -a \left(\langle Q^{(n)}, \partial_t Q^{(n)} \rangle_{H^s} + \langle Q^{(n)}, J_n(v^{(n)} \cdot \nabla Q^{(n)}) \rangle_{H^s} \right) \\
&= -a \|Q^{(n)}\|_{H^s}^2 - a \langle Q^{(n)}, v^{(n)} \cdot \nabla Q^{(n)} \rangle_{H^s},
\end{aligned}$$

with

$$-a \langle Q^{(n)}, v^{(n)} \cdot \nabla Q^{(n)} \rangle_{H^s} \lesssim \|Q^{(n)}\|_{H^s}^2 \|v^{(n)}\|_{H^s}^2 + c \|\nabla Q^{(n)}\|_{H^s}^2.$$

Finally

$$\begin{aligned}
b \langle J_n(Q^{(n)} Q^{(n)}), \dot{Q}^{(n)} \rangle_{H^s} &= b \langle (Q^{(n)})^2, \dot{Q}^{(n)} \rangle_{H^s} \lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2 \\
&\quad - c \langle Q^{(n)} \operatorname{tr}\{J_n(Q^{(n)} Q^{(n)})\}, \dot{Q}^{(n)} \rangle_{H^s} \lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2.
\end{aligned}$$

Summarizing the above considerations, equality (8.41) leads to

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} [J \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 + a \|Q^{(n)}\|_{H^s}^2] &+ \mu_1 \|Q^{(n)}\|_{H^s}^2 - \langle v^{(n)} \cdot \nabla Q^{(n)}, \Delta Q^{(n)} \rangle_{L_x^2} - \\
&- \frac{\mu_2}{2} \langle A^{(n)}, \dot{Q}^{(n)} \rangle_{H^s} \lesssim (\|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 + \|\nabla v^{(n)}\|_{H^s}^2) \times \\
&\quad \times (\|\dot{Q}^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2) + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2.
\end{aligned} \tag{8.42}$$

Now, let us consider the H^s -inner product between the order tensor equation and $Q^{(n)}/2$:

$$\begin{aligned}
\frac{J}{2} \langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{H^s} &+ \frac{\mu_1}{2} \langle \dot{Q}^{(n)}, Q^{(n)} \rangle_{H^s} - \frac{\mu_1}{2} \langle [\Omega^{(n)}, Q^{(n)}], Q^{(n)} \rangle_{H^s} + \\
&+ \|\nabla Q^{(n)}\|_{H^s}^2 + a \|Q^{(n)}\|_{H^s}^2 = b \langle (Q^{(n)})^2, Q^{(n)} \rangle_{H^s} - \\
&- c \langle Q^{(n)} \operatorname{tr}(Q^{(n)})^2, Q^{(n)} \rangle_{H^s} + \frac{\mu_2}{4} \langle A^{(n)}, Q^{(n)} \rangle_{H^s}.
\end{aligned} \tag{8.43}$$

We proceed as in the proof of Proposition 8.4.2, first by

$$\langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{H^s} = \langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} + \langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{\dot{H}^s},$$

then

$$\begin{aligned}
\langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} &= \langle \partial_t \dot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} + \langle J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)}), Q^{(n)} \rangle_{L_x^2} \\
&= \langle \partial_t \dot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} + \langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} \\
&= \partial_t \langle \dot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} - \langle \dot{Q}^{(n)}, \partial_t Q^{(n)} \rangle_{L_x^2} - \langle \dot{Q}^{(n)}, v^{(n)} \cdot \nabla Q^{(n)} \rangle_{L_x^2} \\
&= \partial_t \langle \dot{Q}^{(n)}, Q^{(n)} \rangle_{L_x^2} - \|\dot{Q}^{(n)}\|_{L_x^2}^2.
\end{aligned}$$

Moreover

$$\langle \partial_t \dot{Q}^{(n)}, Q^{(n)} \rangle_{\dot{H}^s} = \frac{d}{dt} \langle \dot{Q}^{(n)}, Q^{(n)} \rangle_{\dot{H}^s} - \langle \dot{Q}^{(n)}, \partial_t Q^{(n)} \rangle_{\dot{H}^s}$$

and

$$\langle J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)}), Q^{(n)} \rangle_{\dot{H}^s} = \langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, J_n Q^{(n)} \rangle_{\dot{H}^s} = \langle v^{(n)} \cdot \nabla \dot{Q}^{(n)}, Q^{(n)} \rangle_{\dot{H}^s},$$

so that, proceeding as in (8.31), (8.32), (8.33) and (8.34), we get

$$\begin{aligned} \frac{J}{2} \langle \ddot{Q}^{(n)}, Q^{(n)} \rangle_{\dot{H}^s} - \frac{J}{4} \frac{d}{dt} [\|\dot{Q}^{(n)} + Q^{(n)}\|_{H^s}^2 - \|\dot{Q}^{(n)}\|_{H^s}^2 - \|Q^{(n)}\|_{H^s}^2] \\ \lesssim \|\nabla v^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2. \end{aligned}$$

Now

$$\begin{aligned} \frac{\mu_1}{2} \langle \dot{Q}^{(n)}, Q^{(n)} \rangle_{H^s} &= \frac{\mu_1}{2} \langle \partial_t Q^{(n)}, Q^{(n)} \rangle_{H^s} + \frac{\mu_1}{2} \langle J_n(v^{(n)} \cdot \nabla Q^{(n)}), Q^{(n)} \rangle_{H^s} \\ &= \frac{\mu_1}{4} \frac{d}{dt} \|Q^{(n)}\|_{H^s}^2 + \frac{\mu_1}{2} \langle v^{(n)} \cdot \nabla Q^{(n)}, Q^{(n)} \rangle_{H^s}, \end{aligned}$$

with

$$\frac{\mu_1}{2} \langle v^{(n)} \cdot \nabla Q^{(n)}, Q^{(n)} \rangle_{H^s} \lesssim \|Q^{(n)}\|_{H^s}^2 \|v^{(n)}\|_{H^s}^2 + c \|\nabla Q^{(n)}\|_{H^s}^2.$$

Then

$$\frac{\mu_1}{2} \langle J_n[\Omega^{(n)}, Q^{(n)}], Q^{(n)} \rangle_{H^s} \lesssim \|\nabla v^{(n)}\|_{H^s} \|Q^{(n)}\|_{H^s}^2 \lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2,$$

$$\begin{aligned} b \langle J_n(Q^{(n)} Q^{(n)}), Q^{(n)} \rangle_{H^s} &= b \langle Q^{(n)} Q^{(n)}, Q^{(n)} \rangle_{H^s} \\ &\lesssim \|Q^{(n)}\|_{H^s}^3 \lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 + c_a \|Q^{(n)}\|_{H^s}^2, \\ c \langle Q^{(n)} \operatorname{tr}\{J_n(Q^{(n)} Q^{(n)})\}, Q^{(n)} \rangle_{H^s} &\lesssim \|Q^{(n)}\|_{H^s}^2 \|J_n(Q^{(n)} Q^{(n)})\|_{H^s} \\ &\lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)} Q^{(n)}\|_{H^s} \\ &\lesssim \|Q^{(n)}\|_{H^s}^2 \|Q^{(n)}\|_{H^s}^2 \end{aligned}$$

and finally

$$\frac{\mu_2}{2} \langle A^{(n)}, Q^{(n)} \rangle_{H^s} \leq \frac{\mu_2}{2} \|\nabla v^{(n)}\|_{H^s} \|Q^{(n)}\|_{H^s} \leq \frac{|\mu_2|^2}{8a(1-\varepsilon)} \|\nabla v^{(n)}\|_{H^s}^2 + a(1-\varepsilon) \|Q^{(n)}\|_{H^s}^2.$$

Then, summarizing the previous considerations together with (8.43), we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{J}{4} \|\dot{Q}^{(n)} + Q^{(n)}\|_{H^s}^2 - \frac{J}{4} \|\dot{Q}^{(n)}\|_{H^s}^2 + \frac{\mu_1 - J}{4} \|Q^{(n)}\|_{H^s}^2 \right] - \frac{J}{2} \|\dot{Q}^{(n)}\|_{H^s}^2 + a\varepsilon \|Q^{(n)}\|_{H^s}^2 - \\ - \frac{\mu_2}{8a(1-\varepsilon)} \|\nabla v^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \lesssim \left(\|\nabla v^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 \right) \left(\|Q^{(n)}\|_{H^s}^2 + \|v^{(n)}\|_{H^s}^2 \right) + \\ + c_{\beta_4} \|\nabla v^{(n)}\|_{H^s}^2 + c_{\mu_1} \|\dot{Q}^{(n)}\|_{H^s}^2 + c_a \|Q^{(n)}\|_{H^s}^2 + c \|\nabla Q^{(n)}\|_{H^s}^2. \end{aligned} \quad (8.44)$$

Finally, taking the sum between (8.40), (8.42) and (8.44) and assuming c_{β_4} , c_{μ_1} , c and c_a small enough, we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \|v^{(n)}\|_{H^s}^2 + \frac{J}{4} \|Q^{(n)} + \dot{Q}^{(n)}\|_{H^s}^2 + \frac{J}{4} \|\dot{Q}^{(n)}\|_{H^s}^2 + \left(a - \frac{J}{4}\right) \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \right] \\ + \left(\beta_4 - \frac{|\mu_2|^2}{8(1-\varepsilon)a} \right) \|\nabla v^{(n)}\|_{H^s}^2 + \left(\mu_1 - \frac{J}{2} \right) \|\dot{Q}^{(n)}\|_{H^s}^2 + a\varepsilon \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \lesssim \\ \lesssim \left(\|\nabla v^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \right) \times \\ \times \left(\|v^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2 \right), \end{aligned} \quad (8.45)$$

where we have used

$$\langle J_n(\nabla Q^{(n)} \odot \nabla Q^{(n)}), \nabla v^{(n)} \rangle_{L_x^2} + \langle J_n(v^{(n)} \cdot \nabla Q^{(n)}), \Delta Q^{(n)} \rangle_{L_x^2} = 0.$$

Now, let us define the functions $x(t)$ and $y(t)$ by

$$\begin{aligned} x(t) &:= \|\nabla v^{(n)}\|_{H^s}^2 + \|\dot{Q}^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2, \\ y(t) &:= \|v^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|Q^{(n)}\|_{H^s}^2 + \|\nabla Q^{(n)}\|_{H^s}^2. \end{aligned} \quad (8.46)$$

We make use of the following lemma:

Lemma 8.5.1. *Let y be a positive function in $W_{loc}^{1,1}(\mathbb{R}_+)$ and x a almost everywhere positive function in $L_{loc}^1(\mathbb{R}_+)$. Let us assume that*

$$y'(t) + x(t) \leq y(t)x(t), \quad (8.47)$$

for almost every t in \mathbb{R}_+ . If we assume the initial datum $y(0) = y_0 \geq 0$ small enough, then y and x belong to $L^\infty(\mathbb{R}_+)$ and $L^1(\mathbb{R}_+)$ respectively, and moreover

$$\|y\|_{L^\infty(\mathbb{R}_+)} + \|x\|_{L^1(\mathbb{R}_+)} \leq y_0.$$

Proof. Assuming $y_0 \leq 1/4$, we define $T > 0$ as the sup of $t > 0$ such that $y(t) \leq 1/2$. Then, for every $t \in [0, T]$ we get

$$y'(t) + \frac{3}{4}x(t) \leq 0,$$

then, integrating from 0 to T , we deduce

$$y(T) + \frac{3}{4} \int_0^T x(t) \leq y_0 \leq \frac{1}{2}.$$

This yields that $T = +\infty$ and that

$$\|y\|_{L^\infty(\mathbb{R}_+)} + \|x\|_{L^1(\mathbb{R}_+)} \leq y_0. \quad \square$$

With this Lemma we can finally achieve a global-in-time bound for the norms of our solutions. Recalling the definition (8.46), Lemma (8.5.1) yields

$$\begin{aligned} &\sup_{t \in \mathbb{R}_+} \left\{ \|v^{(n)}(t)\|_{H^s}^2 + \|Q^{(n)}(t)\|_{H^s}^2 + \|\dot{Q}^{(n)}(t)\|_{H^s}^2 + \|\nabla Q^{(n)}(t)\|_{H^s}^2 \right\} + \\ &+ \int_{\mathbb{R}_+} \left\{ \|\nabla v^{(n)}(t)\|_{H^s}^2 + \|\dot{Q}^{(n)}(t)\|_{H^s}^2 + \|Q^{(n)}(t)\|_{H^s}^2 + \|\nabla Q^{(n)}(t)\|_{H^s}^2 \right\} dt \\ &\lesssim \|v_0\|_{H^s}^2 + \|Q_0\|_{H^s}^2 + \|\dot{Q}_0\|_{H^s}^2 + \|\nabla Q_0\|_{H^s}^2. \end{aligned}$$

Thus, by classical compactness, weak convergence arguments and thanks to the Aubin-Lions lemma, there exists

$$Q \in L_t^\infty H^{s+1} \cap L_t^2 H^{s+1}, \quad v \in L_t^\infty H^s \cap L_t^2 H^{s+1}, \quad \text{and} \quad \omega \in L_t^\infty H^s \cap L_t^2 H^s,$$

such that, up to a subsequence, we have the following convergences

$$\begin{aligned} Q^{(n)} &\rightarrow Q & \text{strong in } & L_{t,loc}^\infty H_{loc}^{s+1-\mu} \\ \dot{Q}^{(n)} &\rightarrow \omega & \text{strong in } & L_{t,loc}^\infty H_{loc}^{s-\mu} \\ v^{(n)} &\rightarrow v & \text{strong in } & L_{t,loc}^\infty H_{loc}^{s-\mu} \\ \nabla v^{(n)} &\rightharpoonup \nabla v & \text{weak in } & L_t^2 H^s \end{aligned}$$

for a suitable small positive constant μ . Moreover, given a bounded domain Ω , $H^{s-\mu}(\Omega)$ is an algebra, then $J_n(v^{(n)} \cdot \nabla Q^{(n)})$ strongly converges to $v \cdot \nabla Q$ in $L_{t,loc}^\infty H^{s-\mu}$, as n diverges, with also $v \cdot \nabla Q \in L_t^\infty H^s$. Furthermore

$$\partial_t Q = \lim_{n \rightarrow \infty} \partial_t Q^{(n)} = \lim_{n \rightarrow \infty} \left(\dot{Q}^{(n)} - v^{(n)} \cdot \nabla Q^{(n)} \right) = \omega - v \cdot \nabla Q \in L_t^\infty H^s,$$

where the limits are considered in the distributional sense. Then, we deduce $\partial_t Q \in L_t^\infty H^s$ and $\omega = \dot{Q} \in L_t^\infty H^s$. Finally, the tensor order equation yields

$$\begin{aligned} J \partial_t \dot{Q}^{(n)} &= -J J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)}) - \mu_1 \dot{Q}^{(n)} + \mu_1 J_n[\Omega^{(n)}, Q^{(n)}] + \Delta Q^{(n)} + \mu_2 A^{(n)} - \\ &- a Q^{(n)} + b \left(J_n(Q^{(n)} Q^{(n)}) + \text{tr}\{(Q^{(n)} Q^{(n)})\} \frac{\text{Id}}{3} \right) - c J_n(Q^{(n)}) \text{tr}\{J_n(Q^{(n)} Q^{(n)})\}, \end{aligned}$$

hence, observing that

$$\begin{aligned} \|J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)})\|_{H^{s-1}(\Omega)} &\lesssim \|v^{(n)} \cdot \nabla \dot{Q}^{(n)}\|_{H^{s-1}(\Omega)} \\ &\lesssim \|\nabla \cdot \{v^{(n)} \otimes \dot{Q}^{(n)}\}\|_{H^{s-1}(\Omega)} \\ &\lesssim \|v^{(n)} \otimes \dot{Q}^{(n)}\|_{H^s(\Omega)} \lesssim \|v^{(n)}\|_{H^s(\Omega)} \|\dot{Q}^{(n)}\|_{H^s(\Omega)}, \end{aligned}$$

then $\partial_t \dot{Q}^{(n)}$ belongs to $L_{t,loc}^1 H^{s-1}$, with uniformly in n bounded seminorms. Thus

$$\partial_t \dot{Q}^{(n)} \rightharpoonup \partial_t \dot{Q} \quad \text{weakly in } L_{t,loc}^1 H^{s-1},$$

up to a subsequence. Moreover, since $J_n(v^{(n)} \otimes \dot{Q}^{(n)})$ weakly converges to $v \otimes \dot{Q}$ in $L_{t,loc}^1 H^s$, then $J_n(v^{(n)} \cdot \nabla \dot{Q}^{(n)})$ weakly converges to $v \cdot \nabla \dot{Q}$ in $L_{t,loc}^1 H^s$. Then, summarizing we deduce that $\ddot{Q}^{(n)}$ weakly converges to \ddot{Q} in $L_{t,loc}^1 H^s$.

These convergences allow us to pass to the limit in the classical solutions of (8.37), deducing that (u, Q) is classical solution of system (8.18) and (8.19).

Uniqueness part: We now perform the uniqueness of these solutions. Let us consider (u_1, Q_1) and (u_2, Q_2) to be strong solutions with same initial data. From here on we will use the following notation:

$$\delta Q := Q_1 - Q_2, \quad \delta \dot{Q} := \dot{Q}_1 - \dot{Q}_2, \quad \delta v := v_1 - v_2.$$

we begin the proof considering the difference between the angular momentum equations of the two solutions, namely

$$\begin{aligned} J(\delta \dot{Q})_t + v_1 \cdot \nabla \delta \dot{Q} + \delta v \cdot \nabla \dot{Q}_2 + \mu_1 \delta \dot{Q} &= \Delta \delta Q - a \delta Q + b [Q_1 \delta Q + \delta Q Q_2 + \\ &+ \text{tr}\{Q_1 \delta Q + \delta Q Q_2\} \frac{\text{Id}}{3}] - c \delta Q \text{tr}\{Q_1^2\} - c Q_2 \text{tr}\{\delta Q Q_2\} - c Q_2 \text{tr}\{Q_2 \delta Q\} + \\ &\mu_2 \delta Q + \mu_1 [\Omega_1, \delta Q] + \mu_2 [\delta \Omega, Q_2]. \end{aligned}$$

We multiply both the left and right-hand sides by $\delta \dot{Q}$, integrating over \mathbb{R}^d ,

$$\begin{aligned}
\frac{d}{dt} \left[\frac{J}{2} \|\delta\dot{Q}\|_{L_x^2}^2 + \|\nabla\delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \right] + \mu_1 \|\delta\dot{Q}\|_{L_x^2}^2 &= \langle \Delta\delta Q, v_1 \cdot \nabla\delta Q \rangle_{L_x^2} + \\
&+ \langle \Delta\delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} - \langle v_1 \cdot \nabla\delta\dot{Q}, \delta\dot{Q} \rangle_{L_x^2} - \langle \delta v \cdot \nabla\dot{Q}_2, \delta\dot{Q} \rangle_{L_x^2} - \\
&- a \langle \delta Q, v_1 \cdot \nabla\delta Q + \delta v \cdot \nabla Q_2 \rangle_{L_x^2} + b \langle Q_1 \delta Q + \delta Q Q_2, \delta\dot{Q} \rangle_{L_x^2} - \\
&- c \langle \delta Q \operatorname{tr}\{Q_1^2\} + Q_2 \operatorname{tr}\{\delta Q Q_2\} + Q_2 \operatorname{tr}\{Q_2 \delta Q\}, \delta\dot{Q} \rangle_{L_x^2} + \\
&+ \langle \delta A, \delta\dot{Q} \rangle_{L_x^2} + \mu_1 \langle [\Omega_1, \delta Q] + [\delta\Omega, Q_2], \delta\dot{Q} \rangle_{L_x^2}.
\end{aligned} \tag{8.48}$$

We perform now estimates for each term on the right-hand side. First we remark that

$$\begin{aligned}
\langle \Delta\delta Q, v_1 \cdot \nabla\delta Q \rangle_{L_x^2} &= \langle \delta Q_{\alpha\beta, jj}, (v_1)_i \delta Q_{\alpha\beta, i} \rangle_{L_x^2} \\
&= - \langle \delta Q_{\alpha\beta, j}, (v_1)_{i,j} \delta Q_{\alpha\beta, i} \rangle_{L_x^2} - \underbrace{\langle \delta Q_{\alpha\beta, j}, (v_1)_i \delta Q_{\alpha\beta, ij} \rangle_{L_x^2}}_{=0},
\end{aligned} \tag{8.49}$$

where in the second inequality we have integrated by part. Then we obtain

$$\langle \Delta\delta Q, v_1 \cdot \nabla\delta Q \rangle_{L_x^2} \lesssim \|\nabla\delta Q\|_{L_x^2} \|\delta Q\|_{L_x^2} \|\nabla v_1\|_{L_x^\infty} \lesssim \|\nabla v_1\|_{H^s} \left(\|\nabla\delta Q\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2 \right),$$

Similarly, we can proceed integrating by part also for the second term, namely

$$\begin{aligned}
\langle \Delta\delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} &= \langle \delta Q_{\alpha\beta, jj}, \delta v_i \cdot (Q_2)_{\alpha\beta, i} \rangle_{L_x^2} \\
&= - \underbrace{\langle \delta Q_{\alpha\beta, j}, \delta v_{i,j} \cdot (Q_2)_{\alpha\beta, i} \rangle_{L_x^2}}_{\mathcal{A}} - \underbrace{\langle \delta Q_{\alpha\beta, j}, \delta v_i \cdot (Q_2)_{\alpha\beta, ij} \rangle_{L_x^2}}_{\mathcal{B}}.
\end{aligned}$$

First, we control \mathcal{A} by a classical estimate:

$$\mathcal{A} \lesssim \|\nabla\delta Q\|_{L_x^2} \|\nabla\delta v\|_{L_x^2} \|\nabla Q_2\|_{L_x^\infty} \lesssim \|\nabla Q_2\|_{H^s}^2 \|\nabla\delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla\delta v\|_{L_x^2}^2.$$

The term \mathcal{B} requires a deepest analysis. First, we define the parameter θ in $(0, 1/2]$ as the minimum between $1/2$ and $s - d/2$. Thus, since ΔQ_2 belongs to $L^2(\mathbb{R}_+, H^{s-1}(\mathbb{R}^d))$, then it belongs also to $L^2(\mathbb{R}_+, H^{\theta+d/2-1}(\mathbb{R}^d))$. Then we will make use of the following Sobolev embeddings:

$$\begin{aligned}
H^{s-1}(\mathbb{R}^d) &\hookrightarrow H^{\theta+d/2-1}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{1-\theta}}(\mathbb{R}^d), \\
H^1(\mathbb{R}^d) &\hookrightarrow L^2(\mathbb{R}^d) \cap L^d(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2(1-\theta)}}(\mathbb{R}^d).
\end{aligned} \tag{8.50}$$

Then \mathcal{B} is bounded by

$$\begin{aligned}
\mathcal{B} &\lesssim \|\nabla\delta Q\|_{L_x^2} \|\delta v\|_{L_x^{\frac{2d}{d-2(1-\theta)}}} \|\Delta Q_2\|_{L_x^{\frac{d}{1-\theta}}} \lesssim \|\nabla\delta Q\|_{L_x^2} \|\delta v\|_{H^1} \|\Delta Q_2\|_{H^{\theta+\frac{d}{2}-1}} \\
&\lesssim \|\nabla\delta Q\|_{L_x^2} \|\delta v\|_{L^2} \|\Delta Q_2\|_{H^{s-1}} + \|\delta Q\|_{L_x^2} \|\nabla\delta v\|_{L^2} \|\Delta Q_2\|_{H^{s-1}} \\
&\lesssim \|\nabla Q_2\|_{H^s}^2 (\|\nabla\delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2) + c_{\beta_4} \|\nabla\delta v\|_{L_x^2} + c \|\nabla\delta Q\|_{L_x^2}.
\end{aligned}$$

Summarizing, the second term is estimated as follows:

$$\langle \Delta\delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} \lesssim \|\nabla Q_2\|_{H^s}^2 (\|\nabla\delta Q\|_{L_x^2}^2 + \|\nabla\delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2) + c_{\beta_4} \|\nabla\delta v\|_{L_x^2} + c \|\nabla\delta Q\|_{L_x^2}.$$

Now, let us observe that $\langle u_1 \cdot \nabla\delta\dot{Q}, \delta\dot{Q} \rangle_{L_x^2} = 0$ because of the free divergence condition of u_1 . Moreover, always recalling the embeddings (8.50), we have

$$\begin{aligned}
\langle \delta v \cdot \nabla\dot{Q}_2, \delta\dot{Q} \rangle_{L_x^2} &\lesssim \|\delta v\|_{L_x^{\frac{2d}{d-2(1-\theta)}}} \|\nabla\dot{Q}_2\|_{L_x^{\frac{d}{1-\theta}}} \|\delta Q\|_{L_x^2} \lesssim \|\delta v\|_{H^1} \|\nabla\dot{Q}_2\|_{H^{s-1}} \|\delta Q\|_{L_x^2} \\
&\lesssim \|\dot{Q}_2\|_{H^s} \|\delta v\|_{L^2} \|\delta Q\|_{L_x^2} + \|\dot{Q}_2\|_{H^s} \|\nabla\delta v\|_{L^2} \|\delta Q\|_{L_x^2}
\end{aligned}$$

$$\lesssim \|\dot{Q}_2\|_{H^s}^2 (\|\delta v\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2) + c_{\beta_x} \|\nabla \delta v\|_{L_x^2} + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}.$$

The remaining terms can easily be controlled by the Hölder inequality and the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. First the terms related to the parameter a fulfil

$$\begin{aligned} \langle \delta Q, v_1 \cdot \nabla \delta Q \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|v_1\|_{L^\infty} \|\nabla \delta Q\|_{L_x^2} \lesssim \|v_1\|_{H^s} (\|\delta Q\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2), \\ \langle \delta Q, \delta v \cdot \nabla Q_2 \rangle_{L_x^2} &\lesssim \|\delta Q\|_{L_x^2} \|\delta v\|_{L_x^2} \|\nabla Q_2\|_{L_x^\infty} \lesssim \|\nabla Q_2\|_{H^s} (\|\delta Q\|_{L_x^2}^2 + \|\delta v\|_{L_x^2}^2), \end{aligned}$$

the term related to b can be bounded by

$$\begin{aligned} \langle Q_1 \delta Q + \delta Q Q_2, \delta \dot{Q} \rangle_{L_x^2} &\lesssim \|(Q_1, Q_2)\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\delta \dot{Q}\|_{L_x^2} \\ &\lesssim \|(Q_1, Q_2)\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2 \end{aligned}$$

and finally the one multiplied by c is estimated by

$$\langle \delta Q \operatorname{tr}\{Q_1^2\} + Q_2 \operatorname{tr}\{\delta Q Q_2\} + Q_2 \operatorname{tr}\{Q_2 \delta Q\}, \delta \dot{Q} \rangle_{L_x^2} \lesssim \|(Q_1, Q_2)\|_{H^s}^2 (\|\delta Q\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2).$$

It remains to control the terms concerning μ_1 and μ_2 which can be handled through

$$\langle \delta A, \delta \dot{Q} \rangle_{L_x^2} \lesssim \|\delta A\|_{L_x^2} \|\delta \dot{Q}\|_{L_x^2} \lesssim \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2$$

and

$$\begin{aligned} \langle [\Omega_1, \delta Q] + [\delta \Omega, Q_2], \delta \dot{Q} \rangle_{L_x^2} &\lesssim (\|\nabla v_1\|_{H^s}^2 + \|Q_1\|_{H^s}^2) (\|\delta Q\|_{L_x^2}^2 \|\delta \dot{Q}\|_{L_x^2}^2) + \\ &\quad + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2. \end{aligned}$$

Summarizing all the previous estimates together with equality (8.48), we obtain

$$\begin{aligned} \frac{d}{dt} \left[\frac{J}{2} \|\delta \dot{Q}\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \right] + \mu_1 \|\delta \dot{Q}\|_{L_x^2}^2 &\lesssim \left(1 + \|Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 + \right. \\ &\quad \left. + \|\nabla Q_2\|_{H^s}^2 + \|Q_1\|_{H^s}^2 \right) \left(\|\delta v\|_{L_x^2}^2 + \|\delta \dot{Q}\|_{L_x^2}^2 + \|\delta Q\|_{L_x^2}^2 + \|\nabla \delta Q\|_{L_x^2}^2 \right) + \\ &\quad + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2 + c_{\mu_1} \|\delta \dot{Q}\|_{L_x^2}^2. \end{aligned} \quad (8.51)$$

Now let us take in consideration the difference between the momentum equations of the two solutions, namely

$$\begin{aligned} \partial_t \delta v + v_1 \cdot \nabla \delta v + \delta v \cdot \nabla v_2 - \frac{\beta_4}{2} \Delta \delta v &= -\nabla \cdot \left\{ \nabla \delta Q \odot \nabla Q_1 + \nabla Q_2 \odot \nabla \delta Q \right\} - \\ -\beta_1 \nabla \cdot \left\{ \operatorname{tr}\{\delta Q A_1\} Q_1 + \operatorname{tr}\{Q_2 \delta A\} Q_1 + \operatorname{tr}\{Q_2 A_2\} \delta Q \right\} &+ \beta_5 \nabla \cdot \left\{ A_1 \delta Q + \delta A Q_2 \right\} + \\ + \beta_6 \nabla \cdot \left\{ \delta Q A_1 + Q_2 \delta A \right\} + \frac{\mu_2}{2} \nabla \cdot \left\{ \delta \dot{Q} - [\delta \Omega, Q_1] - [\Omega_2, \delta Q] \right\} &+ \\ + \mu_1 \nabla \cdot \left\{ [\delta Q, (\dot{Q}_1 - [\Omega_1, Q_1])] + [Q_2, (\delta \dot{Q} - [\delta \Omega, Q_1] - [\Omega_2, \delta Q])] \right\}. & \end{aligned} \quad (8.52)$$

We proceed as before, multiplying both the left and right-hand sides by δu and integrating every-

thing over \mathbb{R}^d , obtaining

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\delta v\|_{L_x^2}^2 + \frac{\beta_4}{2} \|\nabla \delta v\|_{L_x^2}^2 = \langle \nabla \delta Q \odot \nabla Q_1 + \nabla Q_2 \odot \nabla \delta Q, \nabla \delta v \rangle_{L_x^2} + \\
& + \beta_1 \langle \text{tr}\{\delta Q A_1\} Q_1 + \text{tr}\{Q_2 A_2\} \delta Q, \nabla \delta v \rangle_{L_x^2} + \beta_1 \langle \text{tr}\{Q_2 \delta A\} Q_1, \nabla \delta u \rangle_{L_x^2} - \\
& - \beta_5 \langle A_1 \delta Q + \delta A Q_2, \nabla \delta v \rangle_{L_x^2} - \beta_6 \langle \delta Q A_1 + Q_2 \delta A, \nabla \delta v \rangle_{L_x^2} - \frac{\mu_2}{2} \langle \delta \dot{Q}, \nabla \delta v \rangle_{L_x^2} + \quad (8.53) \\
& + \frac{\mu_2}{2} \langle [\delta \Omega, Q_1] + [\Omega_2, \delta Q], \nabla \delta v \rangle_{L_x^2} - \mu_1 \langle [\delta Q, \dot{Q}_1], \nabla \delta v \rangle_{L_x^2} - \mu_1 \langle [Q_2, \delta \dot{Q}], \nabla \delta v \rangle_{L_x^2} - \\
& - \mu_1 \langle [Q_2, [\delta \Omega, Q_1] + [\Omega_2, \delta Q]], \nabla \delta v \rangle_{L_x^2} - \langle v_1 \cdot \nabla \delta v, \delta v \rangle_{L_x^2} - \langle \delta v \cdot \nabla v_2, \delta v \rangle_{L_x^2},
\end{aligned}$$

then we proceed estimating each term on the right-hand side. At first, we have

$$\begin{aligned}
\langle \nabla \delta Q \odot \nabla Q_1 + \nabla Q_2 \odot \nabla \delta Q, \nabla \delta v \rangle_{L_x^2} & \lesssim \left(\|\nabla Q_1\|_{L_x^\infty} + \|\nabla Q_2\|_{L_x^\infty} \right) \|\nabla \delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\
& \lesssim \left(\|\nabla Q_1\|_{H^s}^2 + \|\nabla Q_2\|_{H^s}^2 \right) \|\nabla \delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2,
\end{aligned}$$

while the terms concerning β_1 are handled by

$$\begin{aligned}
\langle \text{tr}\{\delta Q A_1\} Q_1 + \text{tr}\{Q_2 A_2\} \delta Q, \nabla \delta v \rangle_{L_x^2} & \lesssim \\
& \lesssim \left(\|\nabla u_1\|_{L^\infty} \|Q_1\|_{L_x^\infty} + \|\nabla u_2\|_{L^\infty} \|Q_2\|_{L_x^\infty} \right) \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\
& \lesssim \left(\|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 \right) \|\delta Q\|_{L_x^2} + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2,
\end{aligned}$$

and

$$\langle \text{tr}\{Q_2 \delta A\} Q_1, \nabla \delta v \rangle_{L_x^2} \lesssim \|Q_1\|_{L_x^\infty} \|Q_2\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 \lesssim \|Q_1\|_{H^s} \|Q_2\|_{H^s} \|\nabla \delta v\|_{L_x^2}^2.$$

Now, we bound the terms related to β_5 and β_6 as follows:

$$\begin{aligned}
\langle A_1 \delta Q + \delta A Q_2, \nabla \delta v \rangle_{L_x^2} & \lesssim \|\nabla v_1\|_{L^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} + \|Q_2\|_{L^\infty} \|\nabla \delta v\|_{L_x^2}^2 \\
& \lesssim \|\nabla v_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_2\|_{H^s}) \|\nabla \delta v\|_{L_x^2}^2,
\end{aligned}$$

$$\begin{aligned}
\langle \delta Q A_1 + Q_2 \delta A, \nabla \delta v \rangle_{L_x^2} & \lesssim \|\delta Q\|_{L_x^2} \|\nabla v_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2} + \|Q_2\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 \\
& \lesssim \|\nabla v_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_2\|_{H^s}) \|\nabla \delta v\|_{L_x^2}^2.
\end{aligned}$$

Now, keep proceeding on, we bound the terms on μ_2 by $\langle \delta \dot{Q}, \nabla \delta v \rangle_{L_x^2} \lesssim \|\delta \dot{Q}\|_{L_x^2} + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2$,

$$\begin{aligned}
\langle [\delta \Omega, Q_1] + [\Omega_2, \delta Q], \nabla \delta v \rangle_{L_x^2} & \lesssim \|Q_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2}^2 + \|\nabla v_2\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \\
& \lesssim \|\nabla v_2\|_{L_x^\infty}^2 \|\delta Q\|_{L_x^2}^2 + (c_{\beta_4} + \|Q_1\|_{L_x^\infty}) \|\nabla \delta v\|_{L_x^2}^2,
\end{aligned}$$

while the terms on μ_1 can be handled by

$$\begin{aligned}
\langle [\delta Q, \dot{Q}_1], \nabla \delta v \rangle_{L_x^2} & \lesssim \|\delta Q\|_{L_x^2} \|\dot{Q}_1\|_{L_x^\infty} \|\nabla \delta v\|_{L_x^2} \lesssim \|\dot{Q}_1\|_{H^s}^2 \|\delta Q\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2, \\
\langle [Q_2, \delta \dot{Q}], \nabla \delta v \rangle_{L_x^2} & \lesssim \|Q_2\|_{L_x^\infty} \|\delta \dot{Q}\|_{L_x^2} \|\nabla \delta v\|_{L_x^2} \lesssim \|Q_2\|_{H^s}^2 \|\delta \dot{Q}\|_{L_x^2}^2 + c_{\beta_4} \|\nabla \delta v\|_{L_x^2}^2
\end{aligned}$$

and also

$$\begin{aligned} & \langle [Q_2, [\delta\Omega, Q_1] + [\Omega_2, \delta Q]], \nabla\delta v \rangle_{L_x^2} \\ & \lesssim \|Q_2\|_{L_x^\infty} \|Q_1\|_{L_x^\infty} \|\nabla\delta v\|_{L_x^2}^2 + \|Q_2\|_{L_x^\infty} \|\nabla v_2\|_{L_x^\infty} \|\delta Q\|_{L_x^2} \|\nabla\delta v\|_{L_x^2} \\ & \lesssim \|Q_2\|_{H_x^s}^2 \|\nabla v_2\|_{H_x^s}^2 \|\delta Q\|_{L_x^2}^2 + \left(c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} \right) \|\nabla\delta v\|_{L_x^2}^2. \end{aligned}$$

Finally, let us remark that $\langle v_1 \cdot \nabla\delta v, \delta v \rangle_{L_x^2} = 0$ and

$$\langle \delta v \cdot \nabla v_2, \delta v \rangle_{L_x^2} \lesssim \|\nabla v_2\|_{L_x^\infty} \|\delta v\|_{L_x^2}^2 \lesssim \|\nabla v_2\|_{H^s} \|\delta v\|_{L_x^2}^2.$$

Thus, summarizing all the previous estimates and taking in consideration (8.52), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\delta v\|_{L_x^2}^2 + \frac{\beta_4}{2} \|\nabla\delta v\|_{L_x^2}^2 \lesssim \left\{ 1 + \|\nabla v_2\|_{H^s} + \|\nabla v_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 + \|\nabla Q_1\|_{H^s}^2 \right. \\ & \left. + \|\nabla Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 + \|\dot{Q}_1\|_{H^s}^2 \right\} \left\{ \|\delta v\|_{L_x^2}^2 + \|\nabla\delta Q\|_{L_x^2}^2 \right. \\ & \left. + \|\delta Q\|_{L_x^2}^2 + \|\dot{\delta}Q\|_{L_x^2}^2 \right\} + \left\{ c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} + \|Q_1\|_{H_x^s} + \|Q_2\|_{H_x^s} \right\} \|\nabla\delta v\|_{L_x^2}^2 \end{aligned} \quad (8.54)$$

Now, defining the functions $\Psi = \Psi(t)$ and $f = f(t)$ by

$$\begin{aligned} \Psi & := \frac{1}{2} \|\delta v\|_{L_x^2}^2 + \frac{J}{2} \|\delta\dot{Q}\|_{L_x^2}^2 + \|\nabla\delta Q\|_{L_x^2}^2 + \frac{a}{2} \|\delta Q\|_{L_x^2}^2 \\ f & := \left\{ 1 + \|Q_1\|_{H^s}^2 \|\nabla v_2\|_{H^s} + \|\nabla v_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 + \|\nabla Q_1\|_{H^s}^2 \right. \\ & \quad \left. + \|\nabla Q_2\|_{H^s}^2 + \|\nabla v_1\|_{H^s}^2 \|Q_1\|_{H^s}^2 + \|\nabla v_2\|_{H^s}^2 \|Q_2\|_{H^s}^2 + \|\dot{Q}_1\|_{H^s}^2 \right\}, \end{aligned}$$

and observing that $f \in L_{loc}^1(\mathbb{R}_+)$, we finally take the sum between (8.51) and (8.54), achieving

$$\begin{aligned} & \frac{d}{dt} \Psi + \mu_1 \|\dot{\delta}Q\|_{L_x^2} + \frac{\beta_4}{2} \|\nabla\delta v\|_{L_x^2} \lesssim f\Psi + c_{\mu_1} \|\delta\dot{Q}\|_{L_x^2} + \\ & \quad + \left\{ c_{\beta_4} + \|Q_2\|_{H_x^s} \|Q_1\|_{H_x^s} + \|Q_1\|_{H_x^s} + \|Q_2\|_{H_x^s} \right\} \|\nabla\delta v\|_{L_x^2}^2. \end{aligned}$$

Hence, assuming c_{β_4} , c_{μ_1} and the initial data small enough, we can absorb by the left-hand side the terms related to $\|\delta\dot{Q}\|_{L_x^2}$ and $\|\nabla\delta v\|_{L_x^2}$ on the right-hand side, so that the following inequality is fulfilled:

$$\frac{d}{dt} \Psi \lesssim f\Psi.$$

Then, since $\Psi(0) = 0$, the Gronwall's inequality yields Ψ to be constantly null, especially

$$\delta v = v_1 - v_2 = 0 \quad \text{and} \quad \delta Q = Q_1 - Q_2 = 0. \quad (8.55)$$

This concludes the proof of Theorem 8.1.2. \square

Part II

The Boussinesq system

Chapter 9

A well-posedness result

In this chapter we present the results of the following manuscript:

F. De Anna (accepted), *Global weak solutions for Boussinesq system with temperature dependent viscosity and bounded temperature*, Adv Differential Equ., (2016)

9.1 Introduction

The general Boussinesq system turns out from a first approximation of a coupling system related to the Navier-Stokes and the thermodynamic equations. In such approximation, if we consider the structural coefficients to be constant, as for example the viscosity, we obtain a system between two parabolic equations with linear second order operators. Nevertheless, several fluids cannot be modelled in this way, for instance if we want to study the plasma evolution. Hence it should be necessary to consider a class of quasilinear parabolic systems coming from the general Boussinesq one. This work is devoted to the global existence of solutions for the Cauchy problem related to one of these models, namely:

$$\left\{ \begin{array}{ll} \partial_t \theta + \operatorname{div}(\theta u) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta)D(u)) + \nabla \Pi = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \theta)|_{t=0} = (\bar{u}, \bar{\theta}) & \mathbb{R}^d, \end{array} \right. \quad (9.1)$$

where $D(u)$ is defined by $\nabla u + {}^t\nabla u$. Here θ , $u = (u^1, \dots, u^d)$ and Π stand for the temperature, velocity field and pressure of the fluid respectively, depending on the time variable $t \in \mathbb{R}_+ = [0, +\infty)$ and on the space variables $x \in \mathbb{R}^d$. We denote by $u^h := (u^1, \dots, u^{d-1})$ the horizontal coordinates of the velocity field, while u^d is the vertical coordinate. Furthermore $\nu(\cdot)$ stands for the viscosity coefficient, which is a smooth positive function on \mathbb{R}_+ . Such system is useful as a model to describe many geophysical phenomena, like, for example, a composed obtained by mixing several incompressible immiscible fluids. Indeed the temperature fulfills a transport equation, while the velocity flow verifies a Navier-Stokes type equation which describes the fluids evolution. We consider here the case where the viscosity depends on the temperature, which allows to characterize the immiscibility hypotheses.

Some Developments in the Boussinesq System

The general Boussinesq system, derived in [91], assumes the following form:

$$\begin{cases} \partial_t \theta + \operatorname{div}(\theta u) - \Delta \varphi(\theta) + |D|^s \theta = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta)D(u)) + \nabla \Pi = F(\theta) & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \theta)|_{t=0} = (\bar{u}, \bar{\theta}) & \mathbb{R}^d, \end{cases} \quad (9.2)$$

An exhaustive mathematical justification of such system as a model of stratified fluids (as atmosphere or oceans) is given by Danchin and He in [25]. We present here a short (and of course incomplete) overview concerning some well-posedness results.

Provided by some technical hypotheses, in [36] Díaz and Galiano established the global existence of weak solution for system 9.2 when $s = 0$. Moreover they achieved the uniqueness of such solutions in a two dimensional domain, assuming the viscosity ν to be constant.

In [56] Hmidi and Keraani study system (9.2) in a two dimensional setting, when the parameter s is null, $\varphi(\theta) = \theta$ and $F(\theta)$ stands for a Buoyancy force, more precisely they considered $F(\theta) = \theta e_2$, with e_2 the classical element of the canonical basis of \mathbb{R}^2 . They proved the global existence of weak solutions when both the initial data belong to $L^2(\mathbb{R}^2)$. Furthermore, they established the uniqueness of such solutions under an extra regularity on the initial data, namely $H^r(\mathbb{R}^2)$, for $r > 0$.

In [112] Wang and Zhang considered system (9.2) with Buoyancy force and constant viscosity, when the temperature θ satisfies

$$\partial_t \theta + \operatorname{div}(\theta u) - \operatorname{div}(k \nabla \theta) = 0,$$

where k stands for the thermal diffusivity, which also depends on the temperature. They proved existence and uniqueness of global solutions when the initial data belong to $H^r(\mathbb{R}^2)$, for $r > 0$.

In [17] Chae considered system (9.2) in two dimension, with constant viscosity and when $\varphi(\theta)$ is equal to θ or 0. In this case the author established the existence of smooth solutions.

System (9.2) has also given interest in the Euler equation framework, when the viscosity ν is supposed to be null. In this direction, Hmidi, Keerani and Rousset [57] developed the existence and uniqueness of a solutions when $s = 1$, provided that the initial velocity belongs to $\dot{B}_{\infty,1}^1 \cap \dot{W}_x^{1,p}$ while the initial temperature lives in $\dot{B}_{\infty,1}^0 \cap L_x^p$.

In [2] Abidi and Hmidi performed an existence and uniqueness result for system (9.2) in two dimension, when $\varphi \equiv 0$, $s = 0$ and the force $F(\theta) = \theta e_2$. Here, the initial velocity is supposed in $L_x^2 \cap \dot{B}_{\infty,1}^{-1}$ and the temperature belongs to $\dot{B}_{2,1}^0$.

In [29] Paicu and Danchin considered the case of constant viscosity. Given a force $F(\theta) = \theta e_2$, imposing $s = 2$ and $\phi = \theta$, the authors performed a global existence result for system (9.2), on the condition that the initial data are of Yudovich's type, namely the initial temperature is in $L_x^2 \cap \dot{B}_{p,1}^{-1}$, the initial velocity is in L_x^2 and the initial vorticity $\partial_1 \bar{u}_2 - \partial_2 \bar{u}_1$ is bounded and belongs to some Lebesgue space L_x^r with $r \geq 2$.

We mention that the case of a no-constant viscosity has also been treated in the study of the inhomogeneous incompressible Navier-Stokes equation with variable viscosity

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t(\rho u) + \operatorname{div}\{\rho u \otimes u\} - \operatorname{div}(\eta(\rho)D(u)) + \nabla \Pi = f & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \rho)|_{t=0} = (\bar{u}, \bar{\rho}) & \mathbb{R}^d. \end{cases} \quad (9.3)$$

In [3] Abidi and Paicu analysed the global well-posedness of (9.3) in certain critical Besov spaces provided that the initial velocity is small enough and the initial density is strictly close to a positive constant.

In [4] Abidi and Zhang established the existence and uniqueness of global solutions for system (9.3), on the condition that the initial velocity belongs to $H^{-2\delta} \cap H^1$, for some $\delta \in (0, 1/2)$, the initial density lives in $L_x^2 \cap \dot{W}_x^{1,r}$, with $r \in (2, 2/(1 - 2\delta))$, and $\bar{\rho} - 1$ belongs to L_x^2 .

We finally mention that in [59] Huang and Paicu investigated the time decay behavior of weak solutions for (9.3) in a two dimensional setting.

In this chapter we are going to study the global existence of solutions for the system (9.1) assuming standard and natural conditions on the initial data: the initial temperature is only assumed to be bounded and the initial velocity field is supposed to belong to certain critical homogeneous Besov space. More precisely we consider

$$\bar{\theta} \in L_x^\infty \quad \text{and} \quad \bar{u} \in \dot{B}_{p,r}^{\frac{d}{p}-1} \quad \text{with} \quad r \in (1, \infty) \quad \text{and} \quad p \in (1, d). \tag{9.4}$$

Remark 9.1.1. *As the classical Navier-Stokes equation, system (9.1) has also a scaling property, more precisely if (θ, u, Π) is a solution then, for all $\lambda > 0$,*

$$(\theta(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 \Pi(\lambda^2 t, \lambda x))$$

is also solution of (9.1), with initial data $(\bar{\theta}(\lambda x), \lambda \bar{u}(\lambda x))$. Hence it is natural to consider the initial data in a Banach space with a norm which is invariant under the previous scaling, as for instance $L_x^\infty \times \dot{B}_{p,r}^{d/p-1}$. Let us remark that this initial data type allows θ to include discontinuities along an interface, an important physical case as a model that describes a mixture of fluids with different temperatures.

From here on we suppose the viscosity ν to be a bounded smooth function, close enough to a positive constant μ , which we assume to be 1 for the sake of simplicity. Then, we assume the following small condition for the initial data to be fulfilled:

$$\eta := (\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\} \leq c_0. \tag{9.5}$$

where c_0 and c_r are two suitable positive constants. This type of initial condition is not new in literature, for instance we cite [60], where Huang, Paicu and Zhang studied an incompressible inhomogeneous fluid in the whole space with viscosity dependent on the density, and moreover [30], where Danchin and Zhang examined the same fluid typology, in the half-space setting.

Before enunciating our main results, let us recall the definition of weak solution for system (9.1):

Definition 9.1.2. *We call (θ, u, Π) a global weak solution of (9.1) if*

(i) for any test function $\varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$, the following identities are well-defined and fulfilled:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \{\theta (\partial_t \varphi + u \cdot \nabla \varphi)\}(t, x) dx dt + \int_{\mathbb{R}^d} \bar{\theta}(x) \varphi(0, x) dx = 0,$$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \{u \cdot \nabla \varphi\}(t, x) dx dt = 0,$$

(ii) for any vector valued function $\Phi = (\Phi_1, \dots, \Phi_d) \in \mathfrak{D}(\mathbb{R}_+ \times \mathbb{R}^d)^d$ the following equality is well-defined and satisfied:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \{u \cdot \partial_t \Phi - (u \cdot \nabla u) \cdot \Phi - \nu(\theta) D(u) \cdot \nabla \Phi + \Pi \operatorname{div} \Phi\}(t, x) dx dt + \int_{\mathbb{R}^d} \bar{u}(x) \cdot \Phi(0, x) dx = 0,$$

The smooth case

Some regularizing effects for the heat kernel (Theorem 9.2.2), and a useful characterization of the homogeneous Besov Spaces (Theorem 9.2.7 and Corollary 9.2.7.1) play a key role in our proof. Indeed, we can reformulate the momentum equation of (9.1) in the following integral form:

$$u(t) = e^{t\Delta} \bar{u} + \int_0^t e^{(t-s)\Delta} \{-u \cdot \nabla u + \nabla \Pi\}(s) ds + \int_0^t \operatorname{div} e^{(t-s)\Delta} \{(\nu(\theta) - 1) D(u)\}(s) ds. \quad (9.6)$$

Thus, it is reasonable to assume the velocity u having the same regularity of the convolution between the heat kernel and the initial datum \bar{u} . The Maximal Regularity Theorem suggests us to look for a solution in a $L_t^{\bar{r}} L_x^q$ functional framework. Now, in the simpler case where u just solves the heat equation with initial datum \bar{u} , having ∇u in some $L_t^{\bar{r}} L_x^q$ is equivalent to $\bar{u} \in \dot{B}_{q, \bar{r}}^{d/q-1}$ on the condition $N/q - 1 = 1 - 2/\bar{r}$. From the immersion $\dot{B}_{p, r}^{d/p-1} \hookrightarrow \dot{B}_{q, \bar{r}}^{d/q-1}$, for every $\bar{q} \geq p$ and $\bar{r} \geq r$, we deduce that this strategy requires $p \leq dr/(2r-1)$. Then, according to the above heuristics, our first result reads as follows:

Theorem 9.1.3. *Let $r \in (1, \infty)$ and $p \in (1, dr/(2r-1))$. Suppose that the initial data $(\bar{\theta}, \bar{u})$ belongs to $L_x^\infty \times \dot{B}_{p, r}^{d/p-1}$. There exist two positive constants c_0, c_r such that, if the smallness condition (9.5) is fulfilled, then there exists a global weak solution (θ, u, Π) of (9.1), in the sense of definition 9.1.2 such that*

$$u \in L_t^{2r} L_x^{\frac{dr}{r-1}}, \quad \nabla u \in L_t^{2r} L_x^{\frac{dr}{2r-1}} \cap L_t^{\bar{r}} L_x^{\frac{dr}{2(r-1)}} \quad \text{and} \quad \Pi \in L_t^r L_x^{\frac{dr}{2(r-1)}}.$$

Furthermore, the following inequalities are satisfied:

$$\begin{aligned} \|\nabla u^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla u^h\|_{L_t^{\bar{r}} L_x^{\frac{dr}{2(r-1)}}} + \|u^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq C_1 \eta, \\ \|\nabla u^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla u^d\|_{L_t^{\bar{r}} L_x^{\frac{dr}{2(r-1)}}} + \|u^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq C_2 \|\bar{u}^d\|_{\dot{B}_{p, r}^{\frac{d}{p}-1}} + C_3, \\ \|\Pi\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} &\leq C_4 \eta, \quad \|\theta\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}. \end{aligned}$$

for some positive constants C_1, C_2, C_3 and C_4 .

The general case

As we have already pointed out, the choice of a $L_t^{\bar{r}} L_x^q$ functional setting requires the condition $p < dr/(2r-1)$. The remaining case $dr/(2r-1) \leq p < d$ can be handled by the addition of a weight in time. Indeed, in the simpler case where u just solves the heat equation with initial datum \bar{u} , having u in some $\dot{B}_{p_3, \bar{r}}^{d/p_3-1}$ for some $p_3 \in (dr/(r-1), \infty)$ is equivalent to $t^{1/2(1-d/p_3)-1/\bar{r}} u \in L_t^{\bar{r}} L_x^{p_3}$. In the same line having $\nabla \bar{u}$ in a suitable Besov space $\dot{B}_{p_2, \bar{r}}^{d/p_2-1}$ is equivalent to have $t^{1/2(2-d/p_3)-1/\bar{r}} u$ in $L_t^{\bar{r}} L_x^{p_2}$. Hence, reformulating the smallness condition (9.5) by

$$\eta := \left(\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p, r}^{-1+\frac{d}{p}}} \right) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p, r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq c_0, \quad (9.7)$$

with similar heuristics proposed in the first case, our second results reads as follows:

Theorem 9.1.4. *Let p, r be two real numbers in $(2d/3, d)$ and $(1, \infty)$ respectively, such that*

$$\frac{2d}{3p} - \frac{d}{6p} < \frac{1}{2} - \frac{1}{2r}, \quad \frac{1}{r} < \frac{1}{3}\left(\frac{d}{p} - 1\right), \quad \frac{1}{r} < \frac{4}{3} - \frac{d}{p}. \tag{9.8}$$

Let us define $p_2 := 3pd/(2p + d)$ and $p_3 := 3p^*/2 = 3pd/(2d - 2p)$, so that $1/p = 1/p_2 + 1/p_3$ and

$$\alpha := \frac{1}{2}\left(3 - \frac{d}{p_1}\right) - \frac{1}{r}, \quad \beta := \frac{1}{2}\left(2 - \frac{d}{p_2}\right) - \frac{1}{2r}, \quad \gamma_1 := \frac{1}{2}\left(1 - \frac{d}{p_3}\right) - \frac{1}{2r}, \quad \gamma_2 := \frac{1}{2}\left(1 - \frac{d}{p_3}\right).$$

There exist two positive constants c_0 and c_r such that, if the smallness condition (9.7) is fulfilled, then there exists a global weak solution (θ, u, Π) of (9.1), in the sense of definition 9.1.2 such that

$$t^{\gamma_1}u \in L_t^{2r}L_x^{p_3}, \quad t^{\gamma_2}u \in L_t^\infty L_x^{p_3} \quad t^\beta \nabla u \in L_t^{2r}L_x^{p_2} \quad t^\alpha \Pi \in L_t^r L_x^{p^*}.$$

Furthermore, the following inequalities are satisfied:

$$\begin{aligned} & \|t^\alpha \nabla u^h\|_{L_t^{2r}L_x^{p^*}} + \|t^\beta \nabla u^h\|_{L_t^{2r}L_x^{p_2}} + \|t^{\gamma_1}u^h\|_{L_t^{2r}L_x^{p_3}} + \|t^{\gamma_2}u^h\|_{L_t^\infty L_x^{p_3}} \leq C_1\eta, \\ & \|t^\alpha \nabla u^d\|_{L_t^{2r}L_x^{p^*}} + \|t^\beta \nabla u^d\|_{L_t^{2r}L_x^{p_2}} + \|t^{\gamma_1}u^d\|_{L_t^{2r}L_x^{p_3}} + \|t^{\gamma_2}u^d\|_{L_t^\infty L_x^{p_3}} \leq C_2\|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3 \tag{9.9} \\ & \|t^\alpha \Pi\|_{L_t^r L_x^{p^*}} \leq C_4\eta, \quad \|\theta\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}. \end{aligned}$$

for some positive constants C_1, C_2 and C_3 .

Remark 9.1.5. *We remark that the conditions on p and r in Theorem 9.1.4 are not restrictive. Indeed, we can always embed $\dot{B}_{p,r}^{d/p-1}$ into $\dot{B}_{q,r}^{d/q-1}$ with $q \geq p$ which satisfies $q \in (2d/3, d)$ (see Theorem 9.2.8). Moreover $\dot{B}_{p,r}^{d/p-1}$ is embedded in $\dot{B}_{p,\tilde{r}}^{d/p-1}$, with $\tilde{r} \geq r$, then there is no loss of generality assuming the inequalities (9.8).*

Let us briefly describe the organization of this chapter. In the second section we recall some technical Lemmas concerning the regularizing effects for the heat kernel, as the Maximal regularity Theorem, which will play an important role in the main proofs. We also mention some results regarding the characterization of the homogeneous Besov Spaces. In the third section we prove the existence of solutions for (9.1), with stronger conditions on the initial data with respect to the ones of Theorem 9.1.3. In the fourth section we regularize our initial data and, using the results of the third section together with a compactness argument, we conclude the proof of Theorem 9.1.3. In the fifth and sixth sections we perform the proof of Theorem 9.1.4, proceeding with a similar structure of the third and fourth sections.

Remark 9.1.6. *In order to obtain the uniqueness about the solution of (9.1), the more suitable strategy is to reformulate our system by Lagrangian coordinates, following for example [60], [30] and [28]. The existence of such coordinates can be achieved supposing the velocity field with Lipschitz space condition, more precisely claiming that u belongs to $L_{loc}^1(\mathbb{R}_+; Lip_x)$, or equivalently $\nabla u \in L_{loc}^1(\mathbb{R}_+; L_x^\infty)$. If we want to obtain this condition without controlling two derivatives of u (in the same line of the existence part) and then without using Sobolev embedding, we need to bound terms like*

$$\int_0^t \Delta e^{(t-s)\Delta} \{(\nu(\theta) - 1)\nabla u\}(s) ds \tag{9.10}$$

in some $L^s(0, T; L_x^\infty)$ space, with $s > 1$. Unfortunately this is not allowed by the Maximal Regularity Theorem 9.2.2 for the heat kernel, because of the critical exponents of this spaces. Then, we need

to impose an extra regularity for the initial temperature, as $\nabla\bar{\theta} \in L_x^{l_1}$, for an opportune l_1 , in order to obtain $\nabla\theta$ in $L_{loc}^1(\mathbb{R}_+; L_x^{l_1})$ and then to split (9.10) into

$$\int_0^t \operatorname{div} e^{(t-s)\Delta} \{\nu'(\theta)\nabla\theta \cdot \nabla u\}(s) ds + \int_0^t \operatorname{div} e^{(t-s)\Delta} \{(\nu(\theta) - 1)\nabla^2 u\}(s) ds. \quad (9.11)$$

Hence we need to control the norm of $\nabla^2 u$ in some $L^{r_1}(0, T; L_x^{l_2})$, with $r_1 > 1$ and also $l_2 > d$ in order to fulfill the Morrey Theorem's hypotheses. It is necessary to do that starting from the approximate systems of the third section, however the only way to control two derivatives of the approximate solutions with some inequalities independent by the indexes $n \in \mathbb{N}$ and $\varepsilon > 0$ (present in the extra term of the perturbed transport equation) is to impose $\nabla\bar{\theta} \in L_x^{l_1}$ with $l_1 > d$. We conjecture that this is not the optimal condition for the initial data in order to obtain the uniqueness, indeed, inspired by [3], we claim that, supposing $\nabla\bar{\theta} \in L_x^d$ and $\bar{u} \in \dot{B}_{p,1}^{-1+\frac{d}{p}}$, it is possible to prove the uniqueness with the velocity field into the space

$$L_t^\infty \dot{B}_{p,1}^{-1+\frac{d}{p}} \cap L_t^1 \dot{B}_{p,1}^{1+\frac{d}{p}}.$$

However this needs to change the structure of the existence part, more precisely to change the functional space where we are looking for a solution. Since in our Theorem we suppose only the initial temperature to be bounded, then we have decided to devote this paper only to the existence part of a global weak solution for system (9.1).

9.2 Preliminaries

The purpose of this section is to present some lemmas concerning the regularizing effects for the heat kernel, which will be useful for the next sections. At first step let us recall the well-known Hardy-Littlewood-Sobolev inequality, whose proof is available in [7], Theorem 1.7.

Theorem 9.2.1 (Hardy-Littlewood-Sobolev inequality). *Let f belongs to L_x^p , with $1 < p < \infty$, $\alpha \in]0, d[$ and suppose that $r \in]0, \infty[$ satisfies $1/p + \alpha/d = 1 + 1/r$. Then $|\cdot|^{-\alpha} * f$ belongs to L_x^r and there exists a positive constant C such that $\| |\cdot|^{-\alpha} * f \|_{L_x^r} \leq C \|f\|_{L_x^p}$.*

From this Theorem we can infer the following corollary.

Corollary 9.2.1.1. *Let f belongs to L_x^p , with $1 < p < d$ and let $(\sqrt{-\Delta})^{-1}$ be the Riesz potential, defined by $(\sqrt{-\Delta})^{-1} f(\xi) := \mathcal{F}^{-1}(\hat{f}(\xi)/|\xi|)$. Then $(\sqrt{-\Delta})^{-1} f$ belongs to $L_x^{dp/(d-p)}$ and there exists a positive constant C such that $\|(\sqrt{-\Delta})^{-1} f\|_{L_x^{dp/(d-p)}} \leq C \|f\|_{L_x^p}$.*

Proof. From the equality $(\sqrt{-\Delta})^{-1} f(x) = c(|\cdot|^{-d+1} * f)(x)$, for almost every $x \in \mathbb{R}^d$ and for an appropriate constant c , the theorem is a direct consequence of Theorem 9.2.1, considering $\alpha = d - 1$. \square

One of the key ingredients used in the proof of Theorem (9.1.3) is the maximal regularity Theorem for the heat kernel. We recall here the statement (see [66], theorem 7.3).

Theorem 9.2.2 (Maximal $L^p(L^q)$ regularity for the heat kernel). *Let $T \in]0, \infty]$, $1 < p, q < \infty$ and $f \in L^p(0, T; L_x^q)$. Let the operator A be defined by*

$$Af(t, \cdot) := \int_0^t \Delta e^{(t-s)\Delta} f(s, \cdot) ds.$$

Then A is a bounded operator from $L^p(0, T; L_x^q)$ to $L^p(0, T; L_x^q)$.

If instead of Δ on the definition of the operator A we consider ∇ (the operator B of Lemmas 9.2.4 and 9.2.5) or even without derivatives (the operator C of Lemma 9.2.6), then we can obtain similar results with respect to the maximal regularity Theorem, using a direct computation. We present here the proofs. At first step let us recall two useful identities:

Remark 9.2.3. Let us denote by K the heat kernel, defined by $K(t, x) = e^{-|x|^2/(4t)}/(2\pi t)^{d/2}$, then $\|K(t, \cdot)\|_{L_x^q} = \|K(1, \cdot)\|_{L_x^q}/t^{d/(2q')}$, for all $1 \leq q < \infty$. Moreover considering the gradient of the heat kernel, $\Omega(t, x) := \nabla K(t, x) = -xK(t, x)/(2t)$, we have $\|\Omega(t, \cdot)\|_{L_x^q} = \|\Omega(1, \cdot)\|_{L_x^q}/|t|^{d/(2q')+1/2}$.

Let us denote by $R := {}^t(R_1, \dots, R_d)$, where R_j is the Riesz transform over \mathbb{R}^d , defined by

$$R_j f := \mathcal{F}^{-1} \left(-i \frac{\xi_j}{|\xi|} \hat{f} \right).$$

we recall that R_j is a bounded operator from L_x^q to itself, for every $1 < q < \infty$ (for more details we refer to [66]).

Lemma 9.2.4. Let $T \in]0, \infty]$ and $f \in L^r(0, T; L_x^p)$, with $1 < p < d$ and $1 < r < \infty$. Let the operator \mathcal{B} be defined by

$$\mathcal{B}f(t, \cdot) := \int_0^t \nabla e^{(t-s)\Delta} f(s, \cdot) ds,$$

Then \mathcal{B} is a bounded operator from $L^r(0, T; L_x^p)$ to $L^r(0, T; L_x^{\frac{dp}{d-p}})$.

Proof. From corollary 9.2.1.1 we have that, for almost every $s \in (0, T)$,

$$(\sqrt{-\Delta})^{-1} f(s) \in L_x^{\frac{dp}{d-p}}.$$

Then, reformulating \mathcal{B} by

$$\mathcal{B}f(t, \cdot) = - \int_0^t \Delta e^{(t-s)\Delta} R(\sqrt{-\Delta})^{-1} f(s, \cdot) ds,$$

we deduce, by theorem 9.2.2, that $\mathcal{B}f \in L^r(0, T; L_x^{\frac{dp}{d-p}})$ and

$$\|\mathcal{B}f\|_{L^r(0, T; L_x^{\frac{dp}{d-p}})} \leq C_1 \|R(\sqrt{-\Delta})^{-1} f\|_{L^r(0, T; L_x^{\frac{dp}{d-p}})} \leq C_2 \|f\|_{L^r(0, T; L_x^p)},$$

for some positive constants C_1 and C_2 . □

Lemma 9.2.5. Let $T \in]0, \infty]$ and $f \in L^r(0, T; L_x^p)$, with $1 < r < \infty$ and $p \in [1, \frac{dr}{r-1}]$. Let the operator \mathcal{B} be defined as in Lemma 9.2.4. Then, we have that \mathcal{B} is a bounded operator from $L^r(0, T; L_x^p)$ with values to $L^{2r}(0, T; L_x^q)$, where $1/q := 1/p - (r-1)/(dr)$.

Proof. Observe that, for every $t \in \mathbb{R}_+$,

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s) ds \right\|_{L_x^q} \leq \int_0^t \|\Omega(t-s, \cdot) * f(s, \cdot)\|_{L_x^q} ds \leq \int_0^t \|\Omega(t-s)\|_{L_x^q} \|f(s)\|_{L_x^p} ds,$$

with $1/\tilde{q} + 1/p = 1/q + 1$ or equivalently $\tilde{q}' = dr/(r-1)$. Recalling Remark 9.2.3, we obtain

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} f(s) ds \right\|_{L_x^q} \leq C \int_0^t \frac{\|f(s)\|_{L_x^p}}{|t-s|^{\frac{2r-1}{2r}}} ds \leq C \int_{\mathbb{R}} \frac{\|f(s)\|_{L_x^p}}{|t-s|^{\frac{2r-1}{2r}}} 1_{(0,T)}(s) ds.$$

Since by Theorem 9.2.1

$$|\cdot|^{-\frac{2r-1}{2r}} * \|f(\cdot)1_{(0,T)}(\cdot)\|_{L_x^p} \in L_t^{2r},$$

then there exists $\tilde{C} > 0$ such that

$$\|\mathcal{B}f\|_{L^{2r}(0,T;L_x^q)} \leq C \left\| |\cdot|^{-\frac{2r-1}{2r}} * \|f(\cdot)1_{(0,T)}(\cdot)\|_{L_x^p} \right\|_{L_t^{2r}} \leq \tilde{C} \|f\|_{L^r(0,T;L_x^p)}$$

□

Lemma 9.2.6. *Let $T \in]0, \infty]$, $r \in (1, \infty)$ and $p \in (1, \frac{dr}{2r-1})$. Let the operator \mathcal{C} be defined by*

$$\mathcal{C}f(t, \cdot) := \int_0^t e^{(t-s)\Delta} f(s, \cdot) ds,$$

Then, \mathcal{C} is a bounded operator from $L^r(0, T; L_x^p)$ to $L^{2r}(0, T; L_x^q)$, where $1/q := 1/p - (2r-1)/dr$.

Proof. For every $t \in \mathbb{R}_+$, notice that

$$\left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L_x^q} \leq \int_0^t \|K(t-s, \cdot) * f(s, \cdot)\|_{L_x^q} ds \leq \int_0^t \|K(t-s)\|_{L_x^{\tilde{q}}} \|f(s)\|_{L_x^p} ds,$$

with $1/\tilde{q} + 1/p = 1/q + 1$, that is $\tilde{q}' = dr/(2r-1)$. Recalling Remark 9.2.3, we get

$$\left\| \int_0^t e^{(t-s)\Delta} f(s) ds \right\|_{L_x^q} \leq \int_0^t \frac{\|f(s)\|_{L_x^p}}{|t-s|^{\frac{2r-1}{2r}}} ds \leq \int_{\mathbb{R}} \frac{\|f(s)\|_{L_x^p}}{|t-s|^{\frac{2r-1}{2r}}} 1_{(0,T)}(s) ds.$$

Since by Theorem 9.2.1

$$|\cdot|^{-\frac{2r-1}{2r}} * \|f(\cdot)1_{(0,T)}(\cdot)\|_{L_x^p} \in L_t^{2r},$$

then there exists $\tilde{C} > 0$ such that

$$\|\mathcal{C}f\|_{L^{2r}(0,T;L_x^q)} \leq \left\| |\cdot|^{-\frac{2r-1}{2r}} * \|f(\cdot)1_{(0,T)}(\cdot)\|_{L_x^p} \right\|_{L_t^{2r}} \leq \tilde{C} \|f\|_{L^r(0,T;L_x^p)}.$$

□

For the definition and the main properties of homogeneous Besov Spaces we refer to [7]. However let us briefly recall two results which characterize these spaces in relation to the heat kernel.

Theorem 9.2.7 (Characterization of Homogeneous Besov Spaces). *Let s be a negative real number and $(p, r) \in [1, \infty]^2$. u belongs to $\dot{B}_{p,r}^s(\mathbb{R}^d)$ if and only if $e^{t\Delta}u$ belongs to L_x^p for almost every $t \in \mathbb{R}_+$ and*

$$t^{-\frac{s}{2}} \|e^{t\Delta}u\|_{L_x^p} \in L^r\left(\mathbb{R}_+; \frac{dt}{t}\right).$$

Moreover, there exists a positive constant C such that

$$\frac{1}{C} \|u\|_{\dot{B}_{p,r}^s(\mathbb{R}^d)} \leq \left\| \left\| t^{-\frac{s}{2}} e^{t\Delta}u \right\|_{L_x^p} \right\|_{L^r(\mathbb{R}_+; \frac{dt}{t})} \leq C \|u\|_{\dot{B}_{p,r}^s(\mathbb{R}^d)}.$$

Then, imposing the index s equal to $-\frac{2}{r}$, the following Corollary is satisfied:

Corollary 9.2.7.1. *Let $p \in [1, \infty]$ and $r \in [1, \infty)$. u belongs to $\dot{B}_{p,r}^{-\frac{2}{r}}(\mathbb{R}^d)$ if and only if $e^{t\Delta}u \in L_t^r L_x^p$. Moreover, there exists a positive constant C such that*

$$\frac{1}{C} \|u\|_{\dot{B}_{p,r}^{-\frac{1}{2r}}(\mathbb{R}^d)} \leq \|e^{t\Delta}u\|_{L_t^r L_x^p} \leq C \|u\|_{\dot{B}_{p,r}^{-\frac{1}{2r}}(\mathbb{R}^d)}.$$

Finally, let us state an embedding theorem for homogeneous Besov spaces, whose proof can be found in [7] Proposition 2.20.

Theorem 9.2.8. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any real number s , the space $\dot{B}_{p_1,r_1}^s(\mathbb{R}^d)$ is continuously embedded in $\dot{B}_{p_2,r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}(\mathbb{R}^d)$.*

9.3 Existence of solutions for smooth initial data

In this section, by Proposition 9.3.2 and Theorem 9.3.3, we prove the existence of weak solutions for system (9.1), assuming a tiny extra regularity for the initial data. The proofs proceed in the same line of [30] and [26], however the novelty is to consider also an extra-term $-\mu\Delta$, with $\mu > 0$, in the transport equation. This perturbation is motivated by the necessity to control the norm of the gradient of the approximate temperature, even without a space-Lipschitz condition on the approximate velocity field. Obviously this control depends on μ . Hence we consider the following approximation of (9.1).

$$\begin{cases} \partial_t \theta + \operatorname{div}(\theta u) - \mu \Delta u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta)D(u)) + \nabla \Pi = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \theta)|_{t=0} = (\bar{u}, \bar{\theta}) & \mathbb{R}^d, \end{cases} \quad (9.12)$$

Remark 9.3.1. *Since $\operatorname{div} u = 0$, we observe that the momentum equation of system (9.12) can be reformulated as follows*

$$\begin{cases} \partial_t u^h - \Delta u^h + \nabla^h \Pi = -u^d \partial_d u^h - u^h \cdot \nabla u^h + \operatorname{div}\{(\nu(\theta) - 1)D(u)^h\} & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u^d - \Delta u^d + \partial_d \Pi = -\nabla^h u^d \cdot u^h + u^d \operatorname{div}^h u^h + \operatorname{div}\{(\nu(\theta) - 1)D(u)^d\} & \mathbb{R}_+ \times \mathbb{R}^d, \end{cases}$$

where $D(u)^h := \nabla u^h + {}^t\nabla^h u$ and $D(u)^d := \partial_d u + \nabla u^d$.

First, let us prove the existence of weak solutions for system (9.12).

Proposition 9.3.2. *Let $1 < r < \infty$ and $p \in (1, dr/(2r - 1))$. Suppose that $\bar{\theta}$ belongs to L_x^∞ and \bar{u} belongs to $\dot{B}_{p,r}^{d/p-1} \cap \dot{B}_{p,r}^{d/p-1+\varepsilon}$ with $\varepsilon < \min\{1/(2r), 1 - 1/r, 2(d/p - 2 + 1/r)\}$. If the smallness condition (9.5) holds, then there exists a global weak solution (θ, u, Π) of (9.12) such that*

$$u \in L_t^{2r} L_x^{\frac{dr}{r-1}}, \quad \nabla u \in L_t^{2r} L_x^{\frac{dr}{2r-1}} \cap L_t^r L_x^{\frac{dr}{2(r-1)}}, \quad \text{and} \quad \Pi \in L_t^r L_x^{\frac{dr}{2(r-1)}}.$$

Furthermore, the following inequalities are satisfied:

$$\begin{aligned}
& \|\nabla u^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_1 \eta, \\
& \|\nabla u^d\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3 \\
& \|\Pi\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq C_4 \eta, \quad \|\theta\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}.
\end{aligned} \tag{9.13}$$

for some suitable positive constants C_1, C_2, C_3 and C_4 which are independent by μ and ε .

Proof. First, recalling remark 9.3.1, we linearise system (9.12) as follows: we impose $(\theta_0, u_0, \Pi_0) = (0, 0, 0)$ and we consider

$$\begin{cases} \partial_t \theta_{n+1} - \mu \Delta \theta_{n+1} + \operatorname{div}(\theta_{n+1} u_n) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \theta_n|_{t=0} = \bar{\theta} & \mathbb{R}^d, \end{cases} \tag{9.14}$$

$$\begin{cases} \partial_t u_{n+1} - \Delta u_{n+1} + \nabla \Pi_{n+1} = g_{n+1} + \operatorname{div}\{(\nu(\theta_{n+1}) - 1)D(u_n)\} & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u_{n+1} = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ u_{n+1}|_{t=0} = \bar{u} & \mathbb{R}^d, \end{cases} \tag{9.15}$$

for all $n \in \mathbb{N}$, where g_{n+1} is a d -dimensional vector field, defined by

$$g_{n+1} := - \begin{pmatrix} u_n^d \partial_d u_{n+1}^h + u_n^h \cdot \nabla u_n^h \\ \nabla^h u_n^d \cdot u_{n+1}^h - u_n^d \operatorname{div}^h u_{n+1}^h \end{pmatrix} =: \begin{pmatrix} g_{n+1}^h \\ g_{n+1}^d \end{pmatrix}. \tag{9.16}$$

Moreover we denote by $D(u_n)^h := \nabla u_n^h + {}^t \nabla^h u_n$ and by $D(u_n)^d := \partial_d u_n + \nabla u_n^d$. For all $n \in \mathbb{N}$, the global existence of a weak solution $(\theta_{n+1}, u_{n+1}, \Pi_{n+1})$ of (9.14) and (9.15) is proved by induction, using Theorem 9.7.5. Thanks to this result, we have that u_{n+1} belongs to $L_t^{2r} L_x^{dr/(r-1)}$, ∇u_{n+1} belongs to $L_t^{2r} L_x^{dr/(2r-1)} \cap L_t^r L_x^{dr/(2r-2)}$, θ_{n+1} to $L_{t,x}^\infty$ and Π_{n+1} to $L_t^r L_x^{dr/(2r-2)}$.

STEP 1: ESTIMATES NOT DEPENDENT ON ε AND μ . First, the Maximal Principle for parabolic equation implies, $\|\theta_n\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}$, for any positive integer n . Now, we want to prove that

$$\begin{aligned}
& \|\nabla u_n^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_1 \eta, \\
& \|\nabla u_n^d\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3,
\end{aligned} \tag{9.17}$$

for any $n \in \mathbb{N}$ and for some suitable positive constants C_1, C_2 and C_3 . First we will show by induction that, if η is small enough then

$$\begin{aligned}
& \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq \bar{C}_1 \tilde{\eta}, \\
& \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq \bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_3,
\end{aligned} \tag{9.18}$$

for all $n \in \mathbb{N}$ and for some appropriate positive constant $\bar{C}_1, \bar{C}_2, \bar{C}_3$, where $\tilde{\eta} \leq \eta$ is defined by

$$\tilde{\eta} := \left(\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} \right) \exp \left\{ \frac{C_r}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r} \right\}. \tag{9.19}$$

Let λ be a positive real number, and let $u_{n+1,\lambda}$, $\nabla u_{n+1,\lambda}$ and $\Pi_{n+1,\lambda}$ be defined by

$$(u_{n+1,\lambda}, \nabla u_{n+1,\lambda}, \Pi_{n+1,\lambda})(t) := h_{n,\lambda}(0, t)(u_{n+1}, \nabla u_{n+1}, \Pi_{n+1})(t), \quad (9.20)$$

where, for all $0 \leq s < t < \infty$,

$$h_{n,\lambda}(s, t) := \exp \left\{ -\lambda \int_s^t \|u_n^d(\tau)\|_{L_x^{\frac{dr}{r-1}}}^{2r} d\tau - \lambda \int_s^t \|\nabla u_n^d(\tau)\|_{L_x^{\frac{dr}{2r-1}}}^{2r} d\tau \right\}. \quad (9.21)$$

Writing u_{n+1} in mild formulation, we get

$$\begin{aligned} u_{n+1}(t) = & \underbrace{e^{\frac{t\Delta}{u_L}}}_{u_L} + \underbrace{\int_0^t e^{(t-s)\Delta} \mathbb{P} g_{n+1}(s) ds}_{F_{n+1}^1(t)} + \underbrace{\int_0^t \nabla e^{(t-s)\Delta} R \cdot R \cdot \{(\nu(\theta_{n+1}) - 1)D(u_n)\}(s) ds}_{F_{n+1}^2(t)} + \\ & \underbrace{\int_0^t \operatorname{div} e^{(t-s)\Delta} \{(\nu(\theta_{n+1}) - 1)D(u_n)\}(s) ds}_{F_{n+1}^3(t)}, \end{aligned} \quad (9.22)$$

where $R := \nabla/\sqrt{-\Delta}$ is the Riesz transform ($R \cdot := \operatorname{div}/\sqrt{-\Delta}$) and $\mathbb{P} := I + R R \cdot$ is the Leray projection operator, which are bounded operators from L_x^q to L_x^q for any $q \in (1, \infty)$. Thus

$$\begin{aligned} u_{n+1,\lambda}(t) = & \underbrace{h_{n,\lambda}(0, t)u_L(t)}_{u_{L,\lambda}(t)} + \underbrace{\int_0^t h_{n,\lambda}(s, t)e^{(t-s)\Delta} \mathbb{P} g_{n+1,\lambda}(s) ds}_{F_{n+1,\lambda}^1(t)} + \underbrace{h_{n,\lambda}(0, t)F_2(t)}_{F_{n+1,\lambda}^2(t)} \\ & + \underbrace{h_{n,\lambda}(0, t)F_3(t)}_{F_{n+1,\lambda}^3(t)}, \end{aligned} \quad (9.23)$$

where $g_{n+1,\lambda}(t) = g_{n+1}(t)h_{n,\lambda}(0, t)$. First, we want to estimate $\nabla u_{n+1,\lambda}^h$ in $L_t^{2r} L_x^{dr/(2r-1)}$ and $u_{n+1,\lambda}^h$ in $L_t^{2r} L_x^{dr/(r-1)}$. We begin observing that, by Corollary 9.2.7.1 and Theorem 9.2.8,

$$\|u_{L,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\nabla u_{L,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \leq \|u_L^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\nabla u_L^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \leq C \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}, \quad (9.24)$$

for a suitable positive constant C . Furthermore, by the definition of g_{n+1} and by Lemma 9.7.1, Lemma 9.7.2, Lemma 9.2.5 and Lemma 9.2.6, we obtain

$$\begin{aligned} \|F_{n+1,\lambda}^{1,h}\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\nabla F_{n+1,\lambda}^{1,h}\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} & \leq C \left\{ \frac{1}{\lambda^{\frac{1}{4r}}} \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \right. \\ & \left. + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \frac{1}{\lambda^{\frac{1}{4r}}} \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}^{\frac{1}{2}} \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right\}. \end{aligned} \quad (9.25)$$

Furthermore, by Corollary 9.2.1.1 and Theorem 9.2.2 we also obtain

$$\begin{aligned} \|F_{n+1,\lambda}^{2,h} + F_{n+1,\lambda}^{3,h}\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} & = \left\| \int_0^t \Delta e^{(t-s)\Delta} \mathbb{P} R \cdot (\sqrt{-\Delta})^{-1} \{(\nu(\theta_{n+1}) - 1)D(u_n)\}(s) ds \right\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \\ & \leq C \|(\sqrt{-\Delta})^{-1}(\nu(\theta_{n+1}) - 1)D(u_n)\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C \|\nu - 1\|_{\infty} \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}. \end{aligned} \quad (9.26)$$

Similarly, recalling Theorem 9.2.2, we deduce that

$$\begin{aligned} \|\nabla F_{n+1,\lambda}^{2,h} + \nabla F_{n+1,\lambda}^{3,h}\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} &= \left\| \int_0^t \Delta e^{(t-s)\Delta} R \mathbb{P} R \cdot \{(\nu(\theta_{n+1}) - 1)D(u_n)\}(s) ds \right\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \\ &\leq \|\{(\nu(\theta_{n+1}) - 1)D(u_n)\}\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \leq C \|\nu - 1\|_\infty \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}. \end{aligned} \quad (9.27)$$

Summarizing (9.24), (9.25), (9.26) and (9.27), we deduce that there exists a positive constant C such that, for all $n \in \mathbb{N}$

$$\begin{aligned} &\|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \\ &\leq C \left\{ \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{1}{\lambda^{\frac{1}{4r}}} \left(\|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \right. \right. \\ &\quad \left. \left. + \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right) \right. \\ &\quad \left. + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nu - 1\|_\infty \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \right\}. \end{aligned} \quad (9.28)$$

Recalling the induction hypotheses (9.18), we fix a positive λ such that

$$C \frac{1}{\lambda^{\frac{1}{4r}}} \left(\bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \bar{C}_3 \right)^{\frac{1}{2}} = \frac{1}{4} \quad (\text{namely } \lambda := (4C)^{4r} (\bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_3)^{2r}), \quad (9.29)$$

so that we can absorb all the terms on the right-hands side of (9.28) with index $n + 1$ by the left-hand side, obtaining

$$\begin{aligned} \|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq \\ &\leq 2C \left(\|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_1 \tilde{\eta}^2 + \|\nu - 1\|_{L_x^\infty} (\bar{C}_1 \tilde{\eta} + \bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_3) \right), \end{aligned} \quad (9.30)$$

thanks to the induction hypotheses (9.18). Now we reformulate (9.30) without the index λ on the left-hand side:

$$\begin{aligned} \|\nabla u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq \sup_{t \in \mathbb{R}_+} h_{n,\lambda}(0, t)^{-1} \left(\|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right) \\ &\leq \exp \left\{ \lambda (\bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \bar{C}_3)^{2r} \right\} \left(\|\nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right), \end{aligned}$$

thanks to the second inequality of (9.18). Hence, recalling (9.29) and (9.30), we obtain the following inequality

$$\begin{aligned} \|\nabla u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq \exp \left\{ 2^{4r-1} (4C)^{4r} (\bar{C}_2^{4r} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}^{4r} + \bar{C}_3^{4r}) \right\} \times \\ &\quad \times 2C \left(\|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_1 \tilde{\eta}^2 + \|\nu - 1\|_{L_x^\infty} (\bar{C}_1 \tilde{\eta} + \bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \bar{C}_3) \right). \end{aligned}$$

Assuming that c_r of (9.5) fulfills $c_r \geq 1$ and $c_r/4 \geq 2^{4r-1} (4C)^{4r} \bar{C}_2^{4r}$, we get that the right-hand

side of the previous inequality is bounded by

$$2C \exp \left\{ 2^{4r-1} (4C)^{4r} C_3^{4r} + \frac{C_r}{4} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}}^{4r} \right\} (\|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + \bar{C}_1^2 \tilde{\eta}^2 + \|\nu - 1\|_{L_x^\infty} (\bar{C}_1 \tilde{\eta} + \bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + \bar{C}_3)) \leq 2C \exp \left\{ 2^{4r-1} (4C)^{4r} C_3^{4r} \right\} (1 + (\bar{C}_1^2 + \bar{C}_1) \tilde{\eta} + \bar{C}_2 + \bar{C}_3) \tilde{\eta},$$

where we have used $\|\nu - 1\|_\infty \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} \leq \|\nu - 1\|_\infty \exp\{\|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}}^{4r}/(4r)\}$. Imposing \bar{C}_1 big enough and η small enough in order to have

$$\exp \left\{ 2^{4r-1} (4C)^{4r} \bar{C}_3^{4r} \right\} 2C(1 + \bar{C}_2 + \bar{C}_3) < \frac{\bar{C}_1}{2} \quad \text{and} \quad \exp \left\{ 2^{4r-1} (4C)^{4r} \bar{C}_3^{4r} \right\} (\bar{C}_1 + 1) \tilde{\eta} \leq \frac{1}{2},$$

we finally obtain that the first equation of (9.18) is true for any $n \in \mathbb{N}$. Now we deal with the second equation of (9.18) and we still proceed by induction. Recalling (9.22) and proceeding in a similarly way as done in the previous estimates, the following inequality is satisfied:

$$\begin{aligned} & \|\nabla u_{n+1}^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1}^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \\ & \leq C \left\{ \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{1}{p}}} + \|g_{n+1}\|_{L_t^r L_x^{\frac{dr}{3r-2}}} + \|\nu - 1\|_\infty \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \right\}, \end{aligned}$$

for a suitable positive constant C . Hence, by the definition (9.16) of g_{n+1} , we deduce that

$$\begin{aligned} & \|\nabla u_{n+1}^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1}^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C \left\{ \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \right. \\ & + \|u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \\ & \left. + \|\nu - 1\|_\infty (\|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}) \right\}, \end{aligned}$$

so that, thanks to the induction hypotheses and the previous estimates, we bound the right hand-side by

$$(C + \bar{C}_1 \bar{C}_2 \tilde{\eta} + \|\nu - 1\|_\infty \bar{C}_2) \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + (\bar{C}_1 \bar{C}_3 + \bar{C}_1^2 \tilde{\eta} + \|\nu - 1\|_\infty (\bar{C}_1 + \bar{C}_2)) \tilde{\eta}.$$

Finally, imposing $C < \bar{C}_2$ and η small enough in order to fulfill $C + (\bar{C}_1 \bar{C}_2 + \bar{C}_2) \eta \leq \bar{C}_2$ and moreover $(\bar{C}_1 \bar{C}_3 + \bar{C}_1^2 \eta + \eta(\bar{C}_1 + \bar{C}_2)) \eta \leq \bar{C}_3$, then the second inequality of (9.18) is satisfied for any $n \in \mathbb{N}$. Now, let us prove by induction that there exist three positive constants \tilde{C}_1 , \tilde{C}_2 and \tilde{C}_3 , such that

$$\|\nabla u_{n+1}^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq \tilde{C}_1 \eta \quad \text{and} \quad \|\nabla u_{n+1}^d\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq \tilde{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + \tilde{C}_3, \quad (9.31)$$

for any positive integer n . Recalling the mild formulation 9.22 of u_{n+1} , Lemma 9.2.7.1, Corollary 9.2.7.1 and Theorem 9.2.8, it turns out that

$$\|\nabla u_L^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|\nabla F_{n+1}^{1,h}\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq C (\|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p-1}}} + \|g_{n+1}\|_{L_t^r L_x^{\frac{dr}{3r-2}}}), \quad (9.32)$$

while Theorem 9.2.2 implies

$$\|\nabla F_{n+1}^{2,h} + \nabla F_{n+1}^{3,h}\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq \|\nu - 1\|_\infty \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{dr}{2(r-1)}}}. \quad (9.33)$$

By the definition of g_{n+1} (9.16), its $L_t^r L_x^{dr/(3r-2)}$ -norm is bounded by

$$\|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n+1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}$$

Hence, thanks to the uniform estimates given by (9.18), we obtain

$$\|g_{n+1}\|_{L_t^r L_x^{\frac{dr}{3r-2}}} \leq (\bar{C}_1 \tilde{\eta} + \bar{C}_3) \bar{C}_1 \tilde{\eta} + \bar{C}_1 \bar{C}_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} \tilde{\eta} \leq (\bar{C}_1 \tilde{\eta} + \bar{C}_3 + \bar{C}_2) \bar{C}_1 \eta, \quad (9.34)$$

Furthermore, by the induction hypotheses (9.31), we remark that

$$\|\nu - 1\|_\infty \|\nabla u_n\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq \|\nu - 1\|_\infty \tilde{C}_1 \eta + \tilde{C}_2 \tilde{\eta} + \|\nu - 1\|_\infty \tilde{C}_3. \quad (9.35)$$

Thus, summarizing (9.32), (9.33), (9.34) and (9.35), we finally obtain

$$\|\nabla u_{n+1}^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq C \left\{ \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + (\bar{C}_1 \tilde{\eta} + \bar{C}_3 + \bar{C}_2) \bar{C}_1 \eta + \|\nu - 1\|_\infty \tilde{C}_1 \eta + \tilde{C}_2 \tilde{\eta} + \|\nu - 1\|_\infty \tilde{C}_3 \right\}, \quad (9.36)$$

hence, imposing $\tilde{C}_1 > C(1 + \bar{C}_1 \bar{C}_3 + \bar{C}_1 \bar{C}_2 + \tilde{C}_2 + \tilde{C}_3)$ and assuming η small enough, we get that the first inequality of (9.31) is true for any positive integer n . Now, proceeding as to prove (9.36), we get

$$\|\nabla u_{n+1}^d\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq C \left\{ \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + (\bar{C}_1 \tilde{\eta} + \bar{C}_3 + \bar{C}_2) \bar{C}_1 \eta + \|\nu - 1\|_\infty \tilde{C}_1 \eta + \tilde{C}_2 \tilde{\eta} + \|\nu - 1\|_\infty \tilde{C}_3 \right\}.$$

Hence, imposing $\tilde{C}_2 > C$, $\tilde{C}_3 > 0$ such that $C\{(\bar{C}_1 \tilde{\eta} + \bar{C}_3 + \bar{C}_2) \bar{C}_1 \eta + \|\nu - 1\|_\infty \tilde{C}_1 \eta + \tilde{C}_2 \tilde{\eta}\} < \tilde{C}_3$ and assuming η small enough, we finally establish that also the second inequality of (9.31) is true for any $n \in \mathbb{N}$.

Now, denoting $C_1 := \bar{C}_1 + \tilde{C}_1$, $C_2 := \bar{C}_2 + \tilde{C}_2$, $C_3 := \bar{C}_3 + \tilde{C}_3$ and summarizing (9.18) and (9.31), we finally obtain (9.17). To conclude this first step we observe that Π_{n+1} is determined by

$$\Pi_{n+1} := -(-\Delta)^{-\frac{1}{2}} R \cdot g_{n+1} - R \cdot R \cdot \{(\nu(\theta_{n+1}) - 1) \nabla u_n\}, \quad (9.37)$$

so that, thanks to Corollary 9.2.1.1 and (9.34), we deduce that

$$\|\Pi_{n+1}\|_{L_t^r L_x^{\frac{Nr}{2(r-1)}}} \leq C \left(\|g_{n+1}\|_{L_t^r L_x^{\frac{Nr}{3r-2}}} + \|\nu - 1\|_{L_x^\infty} \|\nabla u_n\|_{L_t^r L_x^{\frac{Nr}{2(r-1)}}} \right) \leq C_4 \eta, \quad (9.38)$$

for any $n \in \mathbb{N}$ and for a suitable positive constant C_4 .

STEP 2: ε -DEPENDENT ESTIMATES. As second step, we are going to establish some ε -dependent estimates which are useful for the third step, where we will prove that (θ^n, u^n, Π^n) is a Cauchy sequence in a suitable space. Defining $\bar{r} := 2r/(2 - \varepsilon r) > r$, then we still have $p < d\bar{r}/(2\bar{r} - 1) = 2dr/((4 + \varepsilon)r - 2)$, since ε is bounded by $2(d/p - 2 + 1/r)$. Since $\dot{B}_{p,r}^{d/p-1} \hookrightarrow \dot{B}_{p,\bar{r}}^{d/p-1}$, then there exists a positive constant C such that

$$\bar{\eta} := (\|\nu - 1\|_\infty + \|\bar{u}^h\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{p}-1}}) \exp \left\{ c_r \|\bar{u}^d\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{p}-1}}^{4r} \right\} \leq C\eta.$$

Hence, arguing exactly as for proving (9.17) with \bar{r} instead of r , we get also

$$\begin{aligned} \|\nabla u_n^h\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{2\bar{r}-1}}} + \|u_n^h\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}-1}}} &\leq C_1 \bar{\eta}, \\ \|\nabla u_n^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{2\bar{r}-1}}} + \|u_n^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}-1}}} &\leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1}} + C_3. \end{aligned} \quad (9.39)$$

First, we want to show by induction that there exists a positive constant \bar{C}_5 such that

$$\|u_n\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla u_n\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \leq \bar{C}_5 \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}}. \quad (9.40)$$

Let us remark that these norms are well defined, since $\bar{r}(1-\varepsilon)-1 > 0$ (from $\varepsilon < 1-1/r < 1-1/\bar{r}$). Recalling the mild formulation of u_{n+1} (9.22), Corollary 9.2.7.1 and Theorem 9.2.8 yield

$$\|u_L\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla u_L\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \leq C \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}},$$

for a suitable positive constant C . Moreover, thanks to Lemma 9.2.5 and Lemma 9.2.6, we get

$$\|F_{n+1}^1\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla F_{n+1}^1\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \leq C \|g_{n+1}\|_{L_t^{\bar{r}} L_x^{\frac{d\bar{r}}{(3-\varepsilon)\bar{r}-2}}}.$$

From the definition of g_{n+1} (9.16) and the estimates (9.39), we get

$$\begin{aligned} \|g_{n+1}\|_{L_t^{\bar{r}} L_x^{\frac{d\bar{r}}{(3-\varepsilon)\bar{r}-2}}} &\leq C_1 \bar{\eta} \left(\|u_n^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|u_{n+1}^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \right. \\ &\quad \left. + \|\nabla u_n^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} + \|\nabla u_{n+1}^d\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \right), \end{aligned}$$

so that, by the induction hypotheses (9.40), we have the following bound

$$\|g_{n+1}\|_{L_t^{\bar{r}} L_x^{\frac{d\bar{r}}{(3-\varepsilon)\bar{r}-2}}} \leq C_1 \bar{\eta} \left(\|u_{n+1}\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla u_{n+1}\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \right) + C_1 \bar{\eta} \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}}.$$

Moreover, thanks to Lemma 9.2.6 and Theorem 9.2.2, we get

$$\|F_{n+1}^2 + F_{n+1}^3\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla F_{n+1}^2 + \nabla F_{n+1}^3\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \leq C \|\nu - 1\|_{L_x^\infty} \|\nabla u^n\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}}.$$

Summarizing the previous estimates and absorbing the terms with indexes $n+1$ on the right side by the left-hand side, we get that there exists a positive constant C such that

$$\|u_{n+1}\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{\bar{r}(1-\varepsilon)-1}}} + \|\nabla u_{n+1}\|_{L_t^{2\bar{r}} L_x^{\frac{d\bar{r}}{(2-\varepsilon)\bar{r}-1}}} \leq (C(1 + C_1 \bar{\eta}) + \bar{C}_5 C_1 \bar{\eta}) \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}},$$

thus (9.40) is true for any positive integer n , assuming $\bar{C}_5 > 2C$ and $\bar{\eta}$ small enough. Now recalling that $\bar{r} = 2r/(2-\varepsilon r)$, (9.40) can be reformulated by

$$\|u_n\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{2dr}{(2-\varepsilon)r-2}}} + \|\nabla u_n\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{2dr}{(4-\varepsilon)r-2}}} \leq \bar{C}_5 \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}} \leq C_5 \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{\bar{r}}-1+\varepsilon}}, \quad (9.41)$$

for a suitable positive constant C_5 .

Now we want to prove the existence of a positive constant C_6 such that

$$\|u_n\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|u_n\|_{L_t^{2r} L_x^{\frac{2dr}{(2-\varepsilon)r-2}}} + \|\nabla u_n\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} + \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{2dr}{(4-\varepsilon)r-2}}} \leq C_6 \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}}. \quad (9.42)$$

Let us remark that such spaces are well defined, since $2-\varepsilon r > 0$ (from $\varepsilon < 2/r$) and $(2-\varepsilon)r-2 > 0$ (from $\varepsilon/2 < \varepsilon < 1-1/r$). Proceeding exactly as for proving (9.40), with r instead of \bar{r} and $\varepsilon/2$ instead of ε , we get

$$\|u_n\|_{L_t^{2r} L_x^{\frac{2dr}{(2-\varepsilon)r-2}}} + \|\nabla u_n\|_{L_t^{2r} L_x^{\frac{2dr}{(4-\varepsilon)r-2}}} \leq \bar{C}_6 \|\bar{u}\|_{\dot{B}_{p,\bar{r}}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}}, \quad (9.43)$$

for a suitable positive constant \bar{C}_6 . Furthermore, recalling the mild formulation of u_{n+1} (9.22), Corollary 9.2.7.1 and Theorem 9.2.8 implies

$$\|u_L\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla u_L\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} + \leq C \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}},$$

for a suitable positive constant C . Thanks to Lemma 9.2.5 and Lemma 9.2.6, we obtain

$$\|F_{n+1}^1\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla F_{n+1}^1\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} + \leq C \|g_{n+1}\|_{L_t^{\frac{2r}{2-\varepsilon r}} L_x^{\frac{dr}{3r-2}}}.$$

From the definition of g_{n+1} (9.16) and the estimates (9.17), we get that

$$\|g_{n+1}\|_{L_t^{\frac{r}{1-\varepsilon r}} L_x^{\frac{dr}{3r-2}}} \leq C_1 \eta \left(\|u_n^d\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|u_{n+1}^d\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla u_n^d\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} + \|\nabla u_{n+1}^d\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} \right),$$

so that, by the induction hypotheses of (9.42), we have the following bound

$$\|g_{n+1}\|_{L_t^{\frac{r}{1-\varepsilon r}} L_x^{\frac{dr}{3r-2}}} \leq C_1 \eta \left(\|u_{n+1}\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla u_{n+1}\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} \right) + C_1 \eta \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}}.$$

Finally, thanks to Lemma 9.2.4 and Lemma 9.2.5, we get

$$\|F_{n+1}^2 + F_{n+1}^3\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla F_{n+1}^2 + \nabla F_{n+1}^3\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} \leq C \|\nu - 1\|_{L^\infty} \|\nabla u^n\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}}.$$

Summarizing the previous estimates and absorbing the terms with indexes $n+1$ on the right side by the left-hand side, we get that there exists a positive constant C such that

$$\|u_{n+1}\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{r-1}}} + \|\nabla u_{n+1}\|_{L_t^{\frac{4r}{2-\varepsilon r}} L_x^{\frac{dr}{2r-1}}} \leq (C(1 + C_1 \bar{\eta}) + C_6 C_1 \bar{\eta}) \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}}. \quad (9.44)$$

Thus, recalling (9.43) and (9.44), we get that (9.42) is true for any $n \in \mathbb{N}$, with $C_6 > \bar{C}_6 + 2C$ and η small enough.

STEP 3. μ -DEPENDENT ESTIMATES AND CONVERGENCE OF THE SERIES. We denote by $\delta u_n := u_{n+1} - u_n$ by $\delta \nu_n := \nu(\theta_{n+1}) - \nu(\theta_n)$ and by $\delta \theta_n := \theta_{n+1} - \theta_n$, for every positive integer n . Moreover, fixing $\lambda > 0$, we define

$$\begin{aligned} \delta U_{n,\lambda}(T) := & \|\delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2})} \\ & + \|\nabla \delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla \delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}, \end{aligned}$$

where, recalling (9.21), $\delta u_{n,\lambda}(t) := \delta u_n(t)h_{n,\lambda}(0, t)$. We want to prove that the series $\sum_{n \in \mathbb{N}} \delta U_n(T)$ is finite. Denoting by $\delta g_n := g_{n+1} - g_n$, $\delta D_n := D(u_{n+1}) - D(u_n)$, then, thanks to the equality (9.22), we can formulate $\delta u_{n,\lambda} = f_{n,1} + f_{n,2} + f_{n,3}$, where

$$\begin{aligned} f_{n,1} &:= h_{n,\lambda}(0, t) \int_0^t e^{(t-s)\Delta} \mathbb{P} \delta g_n(s) ds, \\ f_{n,2} &:= h_{n,\lambda}(0, t) \int_0^t [\nabla e^{(t-s)\Delta} R \cdot R \cdot \{(\nu(\theta_n) - 1) \delta D_{n-1}\} + \\ &\quad + \operatorname{div} e^{(t-s)\Delta} \{(\nu(\theta_n) - 1) \delta D_{n-1}\}](s) ds, \\ f_{n,3} &:= h_{n,\lambda}(0, t) \left(\int_0^t \nabla e^{(t-s)\Delta} R \cdot R \cdot \{\delta \nu_n D(u_n)\}(s) ds \right. \\ &\quad \left. + h_{n,\lambda}(0, t) \int_0^t \operatorname{div} e^{(t-s)\Delta} \{\delta \nu_n D(u_n)\}(s) ds \right). \end{aligned} \quad (9.45)$$

At first step let us estimate

$$\|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})}.$$

Observing that

$$\delta g_n = - \left(\begin{array}{l} u_n^d \partial_d \delta u_n^h + \delta u_n^d \partial_d u_n^h + \delta u_{n-1}^h \cdot \nabla u_n^h + u_{n-1}^h \cdot \nabla \delta u_{n-1}^h \\ \nabla^h u_n^d \cdot \delta u_n^h + \nabla^h \delta u_n^d \cdot u_n^h - u_n^d \operatorname{div}^h \delta u_n^h - \delta u_{n-1}^d \operatorname{div}^h u_n^h \end{array} \right),$$

then, by Lemma 9.7.1 and Lemma 9.7.2, we obtain

$$\begin{aligned} &\|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \leq \\ &\leq C \left\{ \frac{1}{\lambda^{\frac{1}{4r}}} \left(\|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\partial_d \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla^h u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}^{\frac{1}{2}} \|\delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \right. \right. \\ &\quad \left. \left. + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\nabla^h \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \right) + \|\delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} \|\partial_d u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \right. \\ &\quad \left. + \|\delta u_{n-1,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_{n-1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla \delta u_{n-1,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \right. \\ &\quad \left. + \|\nabla^h \delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\delta u_{n-1,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} \|\nabla^h u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} \right\}. \end{aligned}$$

The above inequality together with (9.17) and (9.29) yield

$$\begin{aligned} &\|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \\ &\leq \frac{1}{4} \|\nabla \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + C \bar{C}_1 \tilde{\eta} \|\delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \\ &\quad + C \bar{C}_1 \tilde{C}_r \eta \left(\|\delta u_{n-1,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\nabla \delta u_{n-1}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \right) + \frac{1}{4} \|\delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \\ &\quad + C \bar{C}_1 \tilde{\eta} \|\nabla^h \delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \frac{1}{4} \|\nabla^h \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + C \bar{C}_1 \tilde{C}_r \eta \|\delta u_{n-1,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})}. \end{aligned}$$

Assuming η small enough, the previous inequality yields

$$\begin{aligned} & \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \leq \frac{1}{4} \left\{ \|\delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \right. \\ & \left. + \|\delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|\nabla \delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla \delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \right\} \end{aligned} \quad (9.46)$$

Now, let us estimate $f_{n,1}$ and $\nabla f_{n,1}$ in $L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})$ and $L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})$ respectively. Thanks to Lemma 9.7.1 and 9.7.2, the following inequality is satisfied:

$$\begin{aligned} & \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \leq \\ & \leq C \left\{ \frac{1}{\lambda^{\frac{1}{dr}}} \left(\|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\partial_d \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} + \|\nabla^h u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \right. \right. \\ & \left. + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\nabla^h \delta u_{n,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \right) + \|\delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} \|\partial_d u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \\ & \left. + \|\delta u_{n-1,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|u_{n-1}^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \|\nabla \delta u_{n-1,\lambda}^h\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} + \right. \\ & \left. + \|\nabla^h \delta u_{n,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\delta u_{n-1,\lambda}^d\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} \|\nabla^h u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right\}. \end{aligned}$$

Hence, (9.17), (9.29) and the smallness condition on η imply that

$$\begin{aligned} & \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \leq \frac{1}{4} \left\{ \|\delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \right. \\ & \left. + \|\delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla \delta u_{n,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} + \|\nabla \delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \right\}. \end{aligned} \quad (9.47)$$

Thus, summarizing (9.46) and (9.47), we obtain

$$\begin{aligned} & \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} \\ & + \|\nabla f_{n,1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \leq \frac{1}{4} \delta U_{n,\lambda}(T) + \frac{1}{4} \delta U_{n-1,\lambda}(T). \end{aligned} \quad (9.48)$$

Now, we want to estimate $f_{n,2}$ in $L^{2r}(0,T;L_x^{\frac{dr}{r-1}}) \cap L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})$ and moreover $\nabla f_{n,2}$ in $L^{2r}(0,T;L_x^{\frac{dr}{2r-1}}) \cap L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})$. From Lemma 9.2.5 and Theorem 9.2.2 we obtain

$$\begin{aligned} & \|f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2}})} + \|\nabla f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \leq \\ & \leq C \|\nu - 1\|_{\infty} \left(\|\nabla \delta u_{n-1}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla \delta u_{n-1}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \right) \\ & \leq \tilde{C}_r \eta \left(\|\nabla \delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla \delta u_{n-1,\lambda}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2}})} \right), \end{aligned}$$

hence, we deduce that

$$\begin{aligned} \|f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2})}} + \|\nabla f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \\ + \|\nabla f_{n,2}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}} \leq \bar{C}_r \eta \delta U_{n-1,\lambda}(T). \end{aligned} \quad (9.49)$$

Now we deal with $f_{n,3}$ and $\nabla f_{n,3}$. At first, since $v \in C^\infty(\mathbb{R})$ and $\|\theta_n\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}$, then there exists $\tilde{c} > 0$ (dependent on $\|\bar{\theta}\|_{L_x^\infty}$) such that $\|\delta\nu_n(t)\|_{L_x^\infty} \leq \tilde{c}\|\delta\theta_n(t)\|_{L_x^\infty}$, for almost every $t \in \mathbb{R}_+$. Moreover, by Lemma 9.2.4 and Theorem 9.2.2, we have

$$\begin{aligned} \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2})}} + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \\ + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}} \leq C \left\{ \|\delta\nu_n D(u_n)\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\delta\nu_n D(u_n)\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}} \right\}. \end{aligned}$$

Thus, recalling (9.41) and (9.42), we finally obtain

$$\begin{aligned} \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2})}} + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1}})} + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}} \\ \leq 2C\tilde{c}\|\delta\theta_n\|_{L^{\frac{4}{\varepsilon}}(0,T;L_x^\infty)} \left\{ \|\nabla u_n\|_{L_t^{\frac{4r}{2-\varepsilon}} L_x^{\frac{dr}{2r-1}}} + \|\nabla u_n\|_{L_t^{\frac{4r}{2-\varepsilon}} L_x^{\frac{2dr}{(4-\varepsilon)r-2}}} \right\} \leq \hat{C}_1(\bar{u})\|\delta\theta_n\|_{L^{\frac{4}{\varepsilon}}(0,T;L_x^\infty)}, \end{aligned} \quad (9.50)$$

where $\hat{C}_1(\bar{u}) := 2C\tilde{c}(C_5\|\bar{u}\|_{\dot{B}_{p,r}^{d/p-1+\varepsilon}} + C_6\|\bar{u}\|_{\dot{B}_{p,r}^{d/p-1+\varepsilon/2}})$. Now, let us observe that $\delta\theta_n$ is the weak solution of

$$\begin{cases} \partial_t \delta\theta_n - \mu \Delta \delta\theta_n = -\operatorname{div}(\delta\theta_n u_n) - \operatorname{div}(\delta u_{n-1} \theta_n) & \mathbb{R}_+ \times \mathbb{R}^d, \\ \delta\theta_n|_{t=0} = 0 & \mathbb{R}^d, \end{cases}$$

which implies

$$\delta\theta_n(t) = - \int_0^t \operatorname{div} e^{\mu(t-s)\Delta} \delta\theta_n(s) u_n(s) ds - \int_0^t \operatorname{div} e^{\mu(t-s)\Delta} \delta u_{n-1}(s) \theta_n(s) ds. \quad (9.51)$$

By Remark 9.2.3 we deduce then

$$\|\delta\theta_n(t)\|_{L_x^\infty} \leq \int_0^t \frac{\|\delta\theta_n(s) u_n(s)\|_{L_x^{\frac{2dr}{(2-\varepsilon)r-2}}}}{|\mu(t-s)|^{1-\frac{1}{2r}-\frac{\varepsilon}{4}}} ds + \int_0^t \frac{\|\delta u_{n-1}(s) \theta_n(s)\|_{L_x^{\frac{2dr}{(2-\varepsilon)r-2}}}}{|\mu(t-s)|^{1-\frac{1}{2r}-\frac{\varepsilon}{4}}} ds,$$

hence, defining $\alpha := (1 - 1/(2r) - \varepsilon/4)(2r)' < 1$, $\|\delta\theta_n(t)\|_{L_x^\infty}^{2r}$ is bounded by

$$2^{2r-1} \left(\int_0^t \frac{1}{|\mu(t-s)|^\alpha} ds \right)^{2r-1} \left\{ \int_0^t \|\delta\theta_n(s)\|_{L_x^\infty}^{2r} \|u_n(s)\|_{L_x^{q^*}}^{2r} ds + \int_0^t \|\bar{\theta}\|_{L_x^\infty}^{2r} \|\delta u_{n-1}(s)\|_{L_x^{q^*}}^{2r} ds \right\}.$$

Then, using the Gronwall inequality, we have

$$\begin{aligned} \|\delta\theta_n(t)\|_{L_x^\infty}^{2r} \\ \leq \left(2 \frac{(1-\alpha)t^{1-\alpha}}{\mu^\alpha} \right)^{2r-1} \|\bar{\theta}\|_{L_x^\infty}^{2r} \int_0^t \|\delta u_{n-1}(s)\|_{L_x^{\frac{2dr}{(2-\varepsilon)r-2}}}^{2r} ds \exp \left\{ \int_0^t \|u_n(s)\|_{L_x^{\frac{2dr}{(2-\varepsilon)r-2}}}^{2r} ds \right\}, \end{aligned}$$

which yields $\|\delta\theta_n(t)\|_{L^\infty} \leq \chi(t)\delta U_{n-1}(t)$, where χ is an increasing function defined by

$$\chi(t) := \left(2 \frac{(1-\alpha)t^{1-\alpha}}{\mu^\alpha}\right)^{1-\frac{1}{2r}} \exp\left\{\frac{1}{2r}C_6\|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\frac{\varepsilon}{2}}}\right\}.$$

Hence, Recalling (9.50), we deduce that

$$\begin{aligned} & \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} + \|f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(2-\varepsilon)r-2})}} + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{2dr}{(4-\varepsilon)r-2})}} \\ & + \|\nabla f_{n,3}\|_{L^{2r}(0,T;L_x^{\frac{dr}{2r-1})}} \leq \hat{C}_1(\bar{u})\chi(T)\|\delta U_{n-1}\|_{L^{\frac{4}{\varepsilon}}(0,T)}. \end{aligned} \quad (9.52)$$

Summarizing (9.48), (9.49) and (9.52) we finally deduce that

$$\delta U_{n,\lambda}(T) \leq \left(\frac{1}{3} + \frac{4}{3}\tilde{C}_r\eta\right)\delta U_{n-1,\lambda}(T) + \frac{4}{3}\hat{C}_1(\bar{u})\chi(T)\|\delta U_{n-1}\|_{L^{\frac{4}{\varepsilon}}(0,T)}, \quad (9.53)$$

Supposing η small enough, we can assume $\tilde{\mu} := (1/3 + 4\tilde{C}_r\eta/3) < 1$. Thus, fixing $T > 0$ and denoting by C_T the constant $4\hat{C}_1(\bar{u})\chi(T)\exp\{\lambda(\bar{C}_2\|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1} + \bar{C}_3})\}/3$, we get

$$\delta U_{n,\lambda}(t) \leq \tilde{\mu}\delta U_{n-1,\lambda}(t) + C_T\|\delta U_{n-1,\lambda}\|_{L^{\frac{4}{\varepsilon}}(0,t)},$$

for all $t \in [0, T]$, where we have used that χ is an increasing function. Now, let us prove by induction that there exists $C = C(T) > 0$ and $K = K(T) > 0$ such that

$$\delta U_{n,\lambda}(t) \leq C\tilde{\mu}^{\frac{n}{2}} \exp\left\{K\frac{t}{\sqrt{\tilde{\mu}}}\right\}, \quad (9.54)$$

for all $t \in [0, T]$ and for all $n \in \mathbb{N}$. The base case is trivial, since it is sufficient to find $C = C(T) > 0$ such that $\delta U_{0,\lambda}(t) \leq C$, for all $t \in [0, T]$. Then $\delta U_{0,\lambda}(t) \leq C \exp\{Kt/\tilde{\mu}\}$, for all $K > 0$. Passing to the induction, we have

$$\begin{aligned} & \delta U_{n+1,\lambda}(t) \\ & \leq \tilde{\mu}\delta U_{n-1,\lambda}(t) + C_T\|\delta U_{n-1,\lambda}\|_{L^{\frac{4}{\varepsilon}}(0,t)} \leq \sqrt{\tilde{\mu}}C\tilde{\mu}^{\frac{n+1}{2}} + C_T C\tilde{\mu}^{\frac{n}{2}} \left(\int_0^t \exp\left\{\frac{4}{\varepsilon}K\frac{s}{\sqrt{\tilde{\mu}}}\right\} ds\right)^{\frac{\varepsilon}{4}} \\ & \leq \left(\sqrt{\tilde{\mu}} + \left(\frac{\varepsilon}{4K}\right)^{\frac{4}{\varepsilon}}\tilde{\mu}^{\frac{\varepsilon}{8}-\frac{1}{2}}C_T\right)C\tilde{\mu}^{\frac{n+1}{2}} \exp\left\{K\frac{t}{\sqrt{\tilde{\mu}}}\right\}. \end{aligned}$$

Chosen $K > 0$ big enough, we finally obtain that (9.54) is true for any positive integer n . Hence, the series $\sum_{n \in \mathbb{N}} \delta u_{n,\lambda}(T)$ is convergent, for any $T \in \mathbb{R}_+$. This yields that

$$\sum_{n \in \mathbb{N}} \delta U_n(T) \leq \exp\left\{\lambda(\bar{C}_2\|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}} + \bar{C}_3})^{2r}\right\} \sum_{n \in \mathbb{N}} \delta U_{n,\lambda}(T) < \infty,$$

so that $(u_n)_{\mathbb{N}}$ and $(\nabla u_n)_{\mathbb{N}}$ are Cauchy sequences in $L^{2r}(0, T; L_x^{dr/(r-1)})$ and $L^{2r}(0, T; L_x^{\frac{dr}{2r-1}})$ respectively. Furthermore, $(\theta_n)_{\mathbb{N}}$ is a Cauchy sequence in $L^\infty((0, T) \times \mathbb{R}^d)$, since the norm $\|\delta\theta_n\|_{L^\infty((0, T) \times \mathbb{R}^d)}$ is bounded by $\chi(T)\delta U_{n-1}(T)$. Recalling also the definition of δg_n (9.16), we get

$$\sum_{n \in \mathbb{N}} \|\delta g_n\|_{L^r(0, T; L_x^{dr/(3r-2)})} < \infty,$$

for all $T > 0$. Thus $(g_n)_\mathbb{N}$ is a Cauchy sequence in $L^r(0, T; L_x^{dr/(3r-2)})$ and $((\sqrt{-\Delta})^{-1}g_n)_\mathbb{N}$ is a Cauchy sequence in $L^r(0, T; L_x^{dr/(2r-2)})$, thanks to Corollary 9.2.1.1. Recalling the mild formulation (9.45), by Lemma 9.2.4 and Theorem 9.2.2, there exist $C > 0$ such that

$$\begin{aligned} \|\nabla\delta u_n\|_{L^r(0,T;L_x^{\frac{dr}{2(r-1)}})} &\leq C\left\{\|\delta g_n\|_{L^r(0,T;L_x^{\frac{dr}{3r-2}})} + \|\nu - 1\|_\infty\|\nabla\delta u_{n-1}\|_{L^r(0,T;L_x^{\frac{dr}{2(r-1)}})} + \right. \\ &\quad \left. + \|\delta\nu_n\|_{L^\infty((0,T)\times\mathbb{R}^d)}\|\nabla u_n\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}}\right\}, \end{aligned}$$

for all $n \in \mathbb{N}$. Hence the series $\sum_{n \in \mathbb{N}} \|\nabla\delta u_n\|_{L^r(0,T;L_x^{dr/(2r-2)})}$ is finite, which implies that $(\nabla u_n)_\mathbb{N}$ is a Cauchy sequence in $L^r(0, T; L_x^{dr/(2r-2)})$. Finally $(\Pi_n)_\mathbb{N}$ is a Cauchy sequence in $L^r(0, T; L_x^{dr/(2r-2)})$, by (9.37) and this concludes the proof of the Proposition. \square

Now, let us prove that system (9.1) admits a weak solution, adding a tiny extra regularity to the initial data.

Theorem 9.3.3. *Let $1 < r < \infty$ and $p \in (1, dr/(2r - 1))$. Suppose that $\bar{\theta}$ belongs to $L_x^\infty \cap L_x^2$ and \bar{u} belongs to $\dot{B}_{p,r}^{d/p-1} \cap \dot{B}_{p,r}^{d/p-1+\varepsilon}$ with $\varepsilon < \min\{1/(2r), 1 - 1/r, 2(d/p - 2 + 1/r)\}$. If (9.5) holds, then there exists a global weak solution (θ, u, Π) of (9.12) which satisfies the properties of Theorem 9.1.3.*

Proof. By Proposition 9.3.2, there exist u_μ in $L_t^{2r} L_x^{dr/(r-1)}$ with ∇u_μ in $L_t^{2r} L_x^{dr/(2r-1)}$ and $\nabla u_\mu \in L_t^r L_x^{dr/(2r-2)}$, and also $\theta_\mu \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $\Pi_\mu \in L_t^r L_x^{2/(r-1)}$, such that $(\theta_\mu, u_\mu, \Pi_\mu)$ is weak solution of (9.12). Moreover, thanks to (9.13), we have the following weakly convergences:

$$\begin{aligned} u_{\mu_n} &\rightharpoonup u \quad w - L_t^{2r} L_x^{\frac{dr}{r-1}}, \quad \nabla u_{\mu_n} \rightharpoonup \nabla u \quad w - L_t^{2r} L_x^{\frac{dr}{2r-1}}, \quad \nabla u_{\mu_n} \rightharpoonup \nabla u \quad w - L_t^{2r} L_x^{\frac{dr}{2(r-1)}}, \\ \theta_{\mu_n} &\overset{*}{\rightharpoonup} \theta \quad w - L_{t,x}^\infty, \quad \Pi_{\mu_n} \rightharpoonup \Pi \quad w - L_t^r L_x^{\frac{dr}{2(r-1)}}, \end{aligned}$$

for a positive decreasing sequence $(\mu_n)_\mathbb{N}$ which is convergent to 0. We want to prove that (θ, u, Π) is a weak solution of (9.1). First let us observe that $\{u_\mu \mid \mu > 0\}$ is a compact set on $C([0, T]; \dot{W}_x^{-1, dr/(2r-2)})$, for all $T > 0$. Indeed, recalling the momentum equation of (9.12), $\partial_t(\sqrt{-\Delta})^{-1}u_\mu$ is uniformly bounded in $L^r(0, T; L_x^{dr/(2r-2)})$. This yields that $\{(\sqrt{-\Delta})^{-1}u_\mu \mid \mu > 0\}$ is an equicontinuous and bounded family on $C([0, T], L_x^{dr/(2r-2)})$. Hence we can assume that $(\sqrt{-\Delta})^{-1}u_{\mu_n}$ strongly converges to $(\sqrt{-\Delta})^{-1}u$ in $L^\infty(0, T; L_x^{dr/(2r-2)})$, namely u_{μ_n} strongly converges to u in $L^\infty(0, T; \dot{W}_x^{-1, dr/(2r-2)})$. We recall that $(\nabla u_{\mu_n})_\mathbb{N}$ is a bounded sequence on $L_t^r L_x^{dr/(2r-2)}$, so that $(u_{\mu_n})_\mathbb{N}$ is a bounded sequence on $L_t^r \dot{W}_x^{1, dr/(2r-2)}$. Thus, passing through the following real interpolation

$$\left[\dot{W}_x^{-1, \frac{dr}{2(r-1)}}, \dot{W}_x^{+1, \frac{dr}{2(r-1)}} \right]_{\frac{1}{2r}, 1} = \dot{B}_{\frac{dr}{2(r-1)}, 1}^{1-\frac{1}{2r}},$$

(see [11], Theorem 6.3.1), and since $\dot{B}_{dr/(2r-2), 1}^{1-\frac{1}{r}} \hookrightarrow L_x^{dr/(r-1)}$ (see [7], Theorem 2.39), we deduce that,

$$\begin{aligned} \|u_{\mu_n} - u\|_{L^{2r}(0,T;L_x^{\frac{dr}{r-1}})} &\leq C \left\| \|u_{\mu_n} - u\|_{\dot{W}_x^{-1, \frac{dr}{2(r-1)}}}^{1-\frac{1}{2r}} \|u_{\mu_n} - u\|_{\dot{W}_x^{1, \frac{dr}{2(r-1)}}}^{\frac{1}{2r}} \right\|_{L^{2r}(0,T)} \\ &\leq C \|u_{\mu_n} - u\|_{L^\infty(0,T;\dot{W}_x^{-1, \frac{dr}{2(r-1)}})}^{1-\frac{1}{2r}} \|u_{\mu_n} - u\|_{L^1(0,T;\dot{W}_x^{1, \frac{dr}{2(r-1)}})}^{\frac{1}{2r}}, \end{aligned}$$

for all $T > 0$. This implies that u_{ε_n} strongly converges to u in $L_{loc}^{2r}(\mathbb{R}_+; L_x^{\frac{dr}{r-1}})$, for all $T > 0$, and moreover that $u_{\mu_n}\theta_{\mu_n}$ and $u_{\mu_n} \cdot \nabla u_{\varepsilon_n}$ converge to $u\theta$ and $u \cdot \nabla u$, respectively, in the distributional sense. We deduce that θ is a weak solution of

$$\partial_t \theta + \operatorname{div}(\theta u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \quad \theta|_{t=0} = \bar{\theta} \quad \text{in } \mathbb{R}^d. \quad (9.55)$$

Now, we claim that $\theta_{\mu_n} \rightarrow \theta$ almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^d$, up to a subsequence. Multiplying the first equation of (9.12) by $\theta/2$ and integrating in $[0, t) \times \mathbb{R}^d$ we get

$$\|\theta_{\mu_n}(t)\|_{L_x^2}^2 + \mu_n \int_0^t \|\nabla \theta_{\mu}(s)\|_{L_x^2}^2 ds = \|\bar{\theta}\|_{L_x^2}^2,$$

which yields $\|\theta_{\mu_n}\|_{L^2((0,T) \times \mathbb{R}^d)} \leq T^{1/2} \|\bar{\theta}\|_{L_x^2}$ for any $T > 0$. Moreover, multiplying (9.55) by θ and integrating in $[0, t) \times \mathbb{R}^d$, we achieve $\|\theta(t)\|_{L_x^2} = \|\bar{\theta}\|_{L_x^2}$ for any $t \in (0, T)$, hence

$$\limsup_{n \rightarrow \infty} \|\theta_{\mu_n}\|_{L^\infty(0,T;L_x^2)} \leq T^{\frac{1}{2}} \|\bar{\theta}\|_{L_x^2} = \|\theta\|_{L^2(0,T;L_x^2)}.$$

Thus we can extract a subsequence (which we still call it θ_{μ_n}) such that θ_{μ_n} strongly converges to θ in $L_{loc}^2(\mathbb{R}_+ \times \mathbb{R}^d)$. We deduce that θ_{μ_n} converges almost everywhere to θ , up to a subsequence, and $\nu(\theta_{\mu_n})$ strongly converges to $\nu(\theta)$ in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^d)$, for every $1 \leq m < \infty$, thanks to the Dominated Convergence Theorem. Then $\nu(\theta_{\mu_n})D(u_{\mu_n})$ converges to $\nu(\theta)D(u)$ in the distributional sense.

Summarizing all the previous considerations we finally conclude that (θ, u, Π) is a weak solution of (9.1) and it satisfies the inequalities given by (9.13). \square

9.4 Weak solutions: the smooth case

In this section we present the proof of Theorem 9.1.3. Because of the low regularity of the initial temperature, by the dyadic partition we approximate our initial data and by Theorem 9.3.3 we construct a sequence of approximate solutions. First, still using the mentioned Theorem, we observe that such solutions fulfill inequalities which are dependent only on the initial data. Therefore, using a compactness argument, we establish that the approximate solutions converge, up to a subsequence, and that the limit is the solution we are looking for.

Proof of Theorem 9.1.3. Recalling the Besov embedding $L_x^\infty \hookrightarrow \dot{B}_{\infty,\infty}^0$, we define

$$\bar{\theta}_n := \chi_n \sum_{|j| \leq n} \dot{\Delta}_j \bar{\theta} \quad \text{and} \quad \bar{u}_n := \sum_{|j| \leq n} \dot{\Delta}_j \bar{u}, \quad \text{for every } n \in \mathbb{N},$$

where $\chi_n \leq 1$ is a cut-off function which has support on the ball $B(0, n) \subset \mathbb{R}^d$ and $\chi_n \equiv 1$ in $B(0, n/2)$. Thus $\bar{\theta}_n \in L_x^\infty \cap L_x^2$ and $\bar{u}_n \in \dot{B}_{p,r}^{d/p} \cap \dot{B}_{p,r}^{d/p-1+\varepsilon}$, with $\varepsilon < \min\{1/(2r), 1 - 1/r, 2(d/p - 2 + 1/r)\}$. Then, by Theorem 9.3.3, there exists (θ_n, u_n, Π_n) weak solution of

$$\begin{cases} \partial_t \theta_n + \operatorname{div}(\theta_n u_n) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u_n + u_n \cdot \nabla u_n - \operatorname{div}(\nu(\theta_n)D(u_n)) + \nabla \Pi_n = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u_n = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (\theta_n, u_n)_{t=0} = (\bar{\theta}_n, \bar{u}_n) & \mathbb{R}^d, \end{cases}$$

such that $\theta_n \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$, $u_n \in L_t^{2r} L_x^{dr/(r-1)}$, $\nabla u_n \in L_t^{2r} L_x^{dr/(2r-1)} \cap L_t^r L_x^{dr/(2r-2)}$ and moreover $\Pi_n \in L_t^r L_x^{\frac{dr}{2(r-1)}}$. Furthermore the following inequalities are satisfied:

$$\begin{aligned} & \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_1 \eta, \\ & \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2(r-1)}}} + \|\nabla u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|u_n^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + C_3, \\ & \|\Pi_n\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \leq C_4 \eta, \quad \|\theta_n\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)} \leq C \|\bar{\theta}\|_{L_x^\infty}, \end{aligned}$$

for all $n \in \mathbb{N}$ and for some positive constants C_1, C_2, C_3, C_4, C_5 and C . Then there exists a subsequence (which we still denote by $((\theta_n, u_n, \Pi_n))_{\mathbb{N}}$) and (θ, u, Π) in the same space of (θ_n, u_n, Π_n) , such that

$$\begin{aligned} u_n &\rightharpoonup u \quad w - L_t^{2r} L_x^{\frac{dr}{r-1}}, \quad \nabla u_n \rightharpoonup \nabla u \quad w - L_t^{2r} L_x^{\frac{dr}{2r-1}}, \quad \nabla u_n \rightharpoonup \nabla u \quad w - L_t^{2r} L_x^{\frac{dr}{2(r-1)}}, \\ \theta_n &\overset{*}{\rightharpoonup} \theta \quad w^* - L_{t,x}^\infty, \quad \Pi_n \rightharpoonup \Pi \quad w - L_t^r L_x^{\frac{dr}{2(r-1)}}. \end{aligned}$$

Moreover, proceeding as in Theorem 9.3.3, u_n strongly converges to u in $L_{loc,t}^{2r} L_x^{dr/(r-1)}$, so that θ is weak solution of

$$\partial_t \theta + \operatorname{div}(\theta u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad \theta|_{t=0} = \bar{\theta} \quad \text{in } \mathbb{R}^d. \tag{9.56}$$

Now, we claim that $\theta_n^2 \overset{*}{\rightharpoonup} \theta^2$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. Observing that $\|\theta^2\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)} \leq C^2 \|\bar{\theta}\|_{L_x^\infty}^2$, there exists $\omega \in L_{t,x}^\infty$ such that $\theta_n^2 \overset{*}{\rightharpoonup} \omega$ in $L_{t,x}^\infty$, up to a subsequence. Now, let us remark that θ_n^2 is weak solution of

$$\partial_t \theta_n^2 + \operatorname{div}(\theta_n^2 u_n) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad \theta_n^2|_{t=0} = \bar{\theta}^2 \quad \text{in } \mathbb{R}^d,$$

then, passing through the limit as n goes to ∞ , we deduce that ω is weak solution of

$$\partial_t \omega + \operatorname{div}(\omega u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad \omega|_{t=0} = \bar{\theta}^2 \quad \text{in } \mathbb{R}^d.$$

Moreover, multiplying (9.56) by θ , we get

$$\partial_t \theta^2 + \operatorname{div}(\theta^2 u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad \theta^2|_{t=0} = \bar{\theta}^2 \quad \text{in } \mathbb{R}^d,$$

which yields $\omega = \theta^2$, from the uniqueness of the transport equation. Summarizing the previous considerations, we deduce that $\theta_n \rightarrow \theta$ $s-L_{loc}^2(\mathbb{R}_+ \times \mathbb{R}^d)$, so that θ_n converges to θ almost everywhere in $\mathbb{R}_+ \times \mathbb{R}^d$ up to a subsequence, thus $\nu(\theta_n)$ converges to $\nu(\theta)$ almost everywhere in $\mathbb{R}_+ \times \mathbb{R}^d$. We conclude that and $\nu(\theta_n)$ strongly converges to $\nu(\theta_n)$ in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^d)$, for every $m \in [1, \infty)$, thanks to the Dominated Convergence Theorem. Therefore, passing through the limit as n goes to ∞ , we deduce that

$$\operatorname{div}(\nu(\theta_n) \nabla u_n) \rightarrow \operatorname{div}(\nu(\theta) \nabla u),$$

in the distributional sense, which allows to conclude that (θ, u, Π) is a weak solution of (9.1). \square

Remark 9.4.1. *If we replace the two first equations of system (9.1) by*

$$\partial_t \theta + \operatorname{div}(\theta u) + a\theta = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d \quad \text{and} \quad \partial_t u + u \cdot \nabla u - \operatorname{div}(\nu(\theta) D(u)) + \nabla \Pi = a\theta e_d \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d,$$

where $e_d = {}^t(0, \dots, 1) \in \mathbb{R}^d$ and a is a positive real constant, then we can adapt our strategy in order

to establish the existence of weak solutions for such new system. In the case of the original system, a term as θ_{ε_d} can be assumed only to be bounded both in time and space, hence it does not provide a time integrability, which is necessary in order to achieve the existence result. However, adding the damping term $a\theta$ to the classical transport equation, and supposing $\bar{\theta}$ to belongs to $L_x^{dr/(3r-2)}$, then

$$\|\theta(t)\|_{L_x^{\frac{dr}{3r-2}}} \leq \|\bar{\theta}\|_{L_x^{\frac{dr}{3r-2}}} \exp\{-at\},$$

for every $t \in \mathbb{R}_+$. Thus θ belongs to $L_t^r L_x^{dr/(3r-2)}$ and we can proceed as in the previous proofs, obtaining a global weak solution (θ, u, Π) which belongs to the space defined by Theorem 9.1.3. Moreover, increasing η by

$$\eta_2 := \left(\|\nu - 1\|_{\infty} + \|\bar{u}^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + a\|\bar{\theta}\|_{L_x^{\frac{dr}{3r-2}}} \right) \exp\left\{c_r \|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{4r}\right\},$$

the solution (θ, u, Π) fulfills

$$\begin{aligned} \|\nabla u^h\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla u^h\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} + \|u^h\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq C_1 \eta_2, \\ \|\nabla u^d\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla u^d\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} \|u^d\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} &\leq C_2 \left(\|\bar{u}^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + a\|\bar{\theta}\|_{L_x^{\frac{dr}{3r-2}}} \right) + C_3, \\ \|\Pi\|_{L_t^r L_x^{\frac{dr}{2(r-1)}}} &\leq C_4 \eta_2, \end{aligned}$$

for some positive constants C_1, C_2, C_3 and C_4 .

9.5 The general case: smooth initial data

As preliminary, before starting the proof of Theorem 9.1.4, we enunciate three fundamental Lemma concerning the regularizing effects of the heat kernel, which will be useful. We recall that \mathcal{B} and \mathcal{C} are defined by

$$\mathcal{B}f(t) := \int_0^t \nabla e^{(t-s)\Delta} f(s) ds, \quad \mathcal{C}f(t) := \int_0^t e^{(t-s)\Delta} f(s) ds.$$

Lemma 9.5.1. *Let us assume that $p, p_3, r, \alpha, \gamma_1, \gamma_2$ fulfill the hypotheses of Theorem 9.1.4 and let ε be a non-negative constant bounded by $\min\{1/r, 1 - 1/r, d/p - 1\}$. If $t^\alpha f(t)$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^p)$ then $t^{\gamma_1} \mathcal{C}f(t)$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^{p_3})$ and there exists a positive constant C such that*

$$\|t^{\gamma_1} \mathcal{C}f(t)\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_3})} \leq C \|t^\alpha f(t)\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^p)}.$$

Moreover, if ε is null then $t^{\gamma_2} \mathcal{C}f(t)$ belongs to $L^\infty(0, T; L_x^{p_3})$ and

$$\|t^{\gamma_2} \mathcal{C}f(t)\|_{L^\infty(0, T; L_x^{p_3})} \leq C \|t^\alpha f(t)\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^p)}.$$

Lemma 9.5.2. *Let us assume that p, p_2, r, α, β fulfill the hypotheses of Theorem 9.1.4 and let ε be a non-negative constant bounded by $\min\{1/r, 1 - 1/r, d/p - 1\}$. If $t^\alpha f(t)$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^p)$ then $t^\beta \mathcal{B}f(t)$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^{p_2})$ and there exists a positive constant C such that*

$$\|t^\beta \mathcal{B}f(t)\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_2})} \leq C \|t^\alpha f(t)\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^p)}.$$

Lemma 9.5.3. *Let us assume that $p, p_2, r, \alpha, \beta, \gamma_1, \gamma_2$ fulfill the hypotheses of Theorem 9.1.4 and let ε be a non-negative constant bounded by $\min\{1/r, 1 - 1/r, d/p - 1\}$. If $t^\beta f$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^{p_2})$ then $t^{\gamma_1} \mathcal{B}f(t)$ belongs to $L^{2r/(1-\varepsilon r)}(0, T; L_x^{p_3})$*

$$\|t^{\gamma_1} \mathcal{B}f\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_3})} \leq C \|t^\beta f\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_2})}. \tag{9.57}$$

Furthermore, if $\varepsilon = 0$ then there exists a positive C such that

$$\|t^{\gamma_2} \mathcal{B}f\|_{L^\infty(0, T; L_x^{p_3})} \leq C \|t^\beta f\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_2})}. \tag{9.58}$$

The proofs of these lemmas are a direct consequence of Remark 9.2.3. We perform the one of Lemma 9.5.3, while the others can be achieved thanks to similar procedures.

Proof of Lemma 9.5.3. We begin controlling the $L^{2r/(1-\varepsilon r)}(0, T; L_x^{p_3})$ -norm. First Remark (9.2.3) yields

$$\|t^{\gamma_1} \mathcal{B}f(t)\|_{L_x^{p_3}} \leq C \int_0^t \frac{t^{\gamma_1}}{|t-s|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}}} \|f(s)\|_{L_x^{p_2}} ds = C \int_0^1 \frac{t^{\gamma_1-\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}-\beta}}{|1-\tau|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}\tau\beta}} F(t\tau) d\tau,$$

where $F(s) := s^\beta \|f(s)\|_{L_x^{p_2}}$. Now, since $\gamma_1 - d(1/p_2 - 1/p_3)/2 + 1/2 - \beta$ is null, we have

$$\begin{aligned} \|t^{\gamma_1} \mathcal{B}f\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_3})} &\leq C \int_0^1 \frac{1}{|1-\tau|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}\tau\beta}} \|F(t\tau)\|_{L_t^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_2})} d\tau \\ &\leq C \int_0^1 \frac{1}{|1-\tau|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}\tau\beta+\frac{1}{2r}-\frac{\varepsilon}{2}}} d\tau \|F\|_{L^{\frac{2r}{1-\varepsilon r}}(0, T; L_x^{p_2})}, \end{aligned}$$

thanks to the Minkowski inequality. Thus (9.57) is true, since $\beta + 1/(2r) - \varepsilon/2 < 1$ and moreover $d(1/p_2 - 1/p_3)/2 + 1/2 = 2/3 - d/(6p) + 1/2 < 1 - 1/(2r) < 1$. Finally, observing that

$$\begin{aligned} \|t^{\gamma_2} \mathcal{B}f(t)\|_{L_x^{p_3}} &\leq C \int_0^t \frac{t^{\gamma_2}}{|t-s|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}}} \|f(s)\|_{L_x^{p_2}} ds \\ &\leq C \left(\int_0^t \left| \frac{t^{\gamma_2}}{|t-s|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}} s^\beta} \right|^{(2r)'} ds \right)^{1-\frac{1}{2r}} \|F\|_{L^{2r}(0, T; L_x^{p_2})}, \end{aligned}$$

we obtain

$$\|t^{\gamma_2} \mathcal{B}f(t)\|_{L_x^{p_3}} \leq C \left(\int_0^1 \left| \frac{1}{|1-\tau|^{\frac{d}{2}(\frac{1}{p_2}-\frac{1}{p_3})+\frac{1}{2}\tau\beta}} \right|^{(2r)'} d\tau \right)^{1-\frac{1}{2r}} \|F\|_{L^{2r}(0, T; L_x^{p_2})},$$

thanks to the change of variable $s = t\tau$ and since $(2r)' \{ \gamma_2 - d(1/p_2 - 1/p_3)/2 - 1/2 - \beta \} + 1$ is null. Hence (9.58) turns out from $\{d(1/p_2 - 1/p_3)/2 + 1/2\}(2r)' < 1$ and $\beta(2r)' < 1$. \square

We present the statement of a modified version of the Maximal Regularity Theorem, whose proof can be found in [60].

Theorem 9.5.4. *Let $T \in]0, \infty]$, $1 < \bar{r}, q < \infty$ and $\alpha \in (0, 1 - 1/\bar{r})$. Let the operator \mathcal{A} be defined as in Theorem 9.2.2. Suppose that $t^\alpha f(t)$ belongs to $L^{\bar{r}}(0, T; L_x^q)$. Then $t^\alpha \mathcal{A}f(t)$ belongs to*

$L^{\bar{r}}(0, T; L_x^q)$ and there exists $C > 0$ such that

$$\|t^\alpha \mathcal{A}f(t)\|_{L^{\bar{r}}(0, T; L_x^q)} \leq C \|t^\alpha f(t)\|_{L^{\bar{r}}(0, T; L_x^q)}.$$

As last part of these preliminaries, we have the following corollary, which will be useful in order to control the pressure Π .

Corollary 9.5.4.1. *Let $p \in (1, d)$, $\bar{r} \in (1, \infty)$ and $\alpha \in (0, 1 - 1/\bar{r})$. If $t^\alpha f$ belongs to $L^{\bar{r}}(0, T; L_x^p)$ then $t^\alpha \mathcal{B}f$ belongs to $L^{\bar{r}}(0, T; L_x^{p^*})$ and there exists a positive constant C (not dependent by f) such that*

$$\|t^\alpha \mathcal{B}f(t)\|_{L^{2\bar{r}}(0, T; L_x^{p^*})} \leq C \|t^\alpha f(t)\|_{L^{2\bar{r}}(0, T; L_x^p)}.$$

Proof. It is sufficient to observe that $\mathcal{B}f(t)$ reads as follows:

$$\mathcal{B}f(t) = -(\sqrt{-\Delta})^{-1} R \int_0^t \Delta e^{(t-s)\Delta} f(s) ds = -(\sqrt{-\Delta})^{-1} R \mathcal{A}f(t).$$

Recalling that R is a bounded operator from L_x^q to itself for any $q \in (1, \infty)$ and $(\sqrt{-\Delta})^{-1}$ from L_x^p into $L_x^{p^*}$, the lemma is a direct consequence of Theorem 9.5.4. \square

Proposition 9.5.5. *Let p, r, p_2, p_3 be as in Theorem 9.1.4. Suppose that $\bar{\theta}$ belongs to L_x^∞ and \bar{u} belongs to $\dot{B}_{p,r}^{d/p-1}$. If the smallness condition (9.7) holds, then there exists a global weak solution (θ, u, Π) of (9.12) such that it belongs to the functional framework defined by Theorem 9.1.4 and moreover it satisfies*

$$\begin{aligned} & \|t^\beta \nabla u^h\|_{L_t^{2r} L_x^{p_2}} + \|t^\alpha \nabla u^h\|_{L_t^{2r} L_x^{p^*}} + \|t^{\gamma_1} u^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u^h\|_{L_t^\infty L_x^{p_3}} \leq C_1 \eta, \\ & \|t^\beta \nabla u^d\|_{L_t^{2r} L_x^{p_2}} + \|t^\alpha \nabla u^d\|_{L_t^{2r} L_x^{p^*}} + \|t^{\gamma_1} u^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u^d\|_{L_t^\infty L_x^{p_3}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3 \quad (9.59) \\ & \|t^\alpha \Pi\|_{L_t^{2r} L_x^{p^*}} \leq C_4 \eta, \quad \|\theta\|_{L_{t,x}^\infty} \leq \|\bar{\theta}\|_{L_x^\infty}. \end{aligned}$$

for some positive constants C_1, C_2 and C_3 .

Proof. We proceed as in the proof of Proposition 9.3.2, considering the sequence of solutions for systems (9.14) and (9.15). We claim that such solutions belong to the same functional space defined in Theorem 9.1.4 and moreover that:

$$\begin{aligned} & \|t^\beta \nabla u_n^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_n^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \leq C_1 \eta, \\ & \|t^\beta \nabla u_n^d\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_n^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_n^d\|_{L_t^\infty L_x^{p_3}} \leq C_2 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C_3, \end{aligned} \quad (9.60)$$

for some suitable positive constants C_1, C_2 and C_3 , and for any positive integer n .

STEP 1: ESTIMATES NOT DEPENDENT ON ε AND μ . First, the maximal principle for parabolic equation implies that $\|\theta_n\|_{L_{t,x}^\infty}$ is bounded by $\|\bar{\theta}\|_{L_x^\infty}$. Now, we want to prove by induction that

$$\begin{aligned} & \|t^\beta \nabla u_n^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_n^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \leq \frac{C_1}{2} \tilde{\eta} \leq \frac{C_1}{2} \eta, \\ & \|t^\beta \nabla u_n^d\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_n^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_n^d\|_{L_t^\infty L_x^{p_3}} \leq \frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2}, \end{aligned} \quad (9.61)$$

for some positive constant C_1, C_2 and C_3 , where $\tilde{\eta}$ is defined by

$$\tilde{\eta} := (\|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \|\bar{\theta}\|_{L_x^\infty} + \|\nu - 1\|_\infty) \exp \left\{ \frac{C_r}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}^{2r} \right\} < \eta.$$

We begin with the horizontal component u_n^h . Let λ be a positive real number, and let $u_{n+1,\lambda}, \nabla u_{n+1,\lambda}$ and $\Pi_{n+1,\lambda}$ be defined by

$$(u_{n+1,\lambda}, \nabla u_{n+1,\lambda}, \Pi_{n+1,\lambda})(t) := h_{n,\lambda}(0, t)(u_{n+1}, \nabla u_{n+1}, \Pi_{n+1})(t), \quad (9.62)$$

where, for all $0 \leq s < t < \infty$,

$$h_{n,\lambda}(s, t) := \exp \left\{ -\lambda \int_s^t t^{2r\gamma_1} \|u_n^d(\tau)\|_{L_x^{p_3}}^{2r} d\tau - \lambda \int_s^t t^{2r\beta} \|\nabla u_n^d(\tau)\|_{L_x^{p_2}}^{2r} d\tau \right\}. \quad (9.63)$$

Decomposing $u_{n+1,\lambda}$ as in (9.23), $u_{n+1,\lambda} = u_L + F_{n+1,\lambda}^1 + F_{n+1,\lambda}^2 + F_{n+1,\lambda}^3$, the first estimate is given by Theorem 9.2.7 and Theorem 9.2.8:

$$\begin{aligned} \|t^\beta \nabla u_{L,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{L,\lambda}^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{L,\lambda}^h\|_{L_t^\infty L_x^{p_3}} &\leq \|t^\beta \nabla u_{L,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \\ &+ \|t^{\gamma_1} u_{L,\lambda}^h\|_{L_t^\infty L_x^{p_3}} + \|t^{\gamma_2} u_{L,\lambda}^h\|_{L_t^\infty L_x^{p_3}} \leq C \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}, \end{aligned} \quad (9.64)$$

for a positive constant C . Moreover, recalling the definition (9.16) of g_{n+1} , we get

$$\begin{aligned} \|t^\beta \nabla F_{n+1,\lambda}^{1,h}\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} F_{n+1,\lambda}^{1,h}\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} F_{n+1,\lambda}^{1,h}\|_{L_t^\infty L_x^{p_3}} &\leq C \left\{ \frac{1}{\lambda^{\frac{1}{2r}}} \|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \right. \\ &+ \left. \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_n^h\|_{L_t^{2r} L_x^{p_2}} + \frac{1}{\lambda^{\frac{1}{2r}}} \|t^{\gamma_1} u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_3}} \right\}. \end{aligned} \quad (9.65)$$

thanks to Lemma 9.5.1, Lemma 9.5.2, Lemma 9.7.3 and Lemma 9.7.4. Moreover,

$$\begin{aligned} \|t^{\gamma_1} F_{n+1,\lambda}^{2,h}\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} F_{n+1,\lambda}^{2,h}\|_{L_t^\infty L_x^{p_3}} + \|t^{\gamma_1} F_{n+1,\lambda}^{3,h}\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} F_{n+1,\lambda}^{3,h}\|_{L_t^\infty L_x^{p_3}} &\leq \\ \leq C \|t^\beta (\nu(\theta_{n+1}) - 1) D(u_n)\|_{L_t^{2r} L_x^{p_2}} \leq C \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}} \end{aligned} \quad (9.66)$$

by Lemma 9.5.2 and Lemma 9.5.3. Finally, Theorem 9.5.4 yields

$$\|t^\beta \nabla F_{n+1,\lambda}^{2,h}\|_{L_t^{2r} L_x^{p_2}} + \|t^\beta \nabla F_{n+1,\lambda}^{3,h}\|_{L_t^{2r} L_x^{p_2}} \leq C \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}}. \quad (9.67)$$

Summarizing (9.64), (9.65), (9.66) and (9.67), we deduce that

$$\begin{aligned} &\|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1,\lambda}^h\|_{L_t^\infty L_x^{p_3}} \leq \\ &\leq C \left\{ \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{1}{\lambda^{\frac{1}{2r}}} \|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_n^h\|_{L_t^{2r} L_x^{p_2}} + \right. \\ &\quad \left. + \frac{1}{\lambda^{\frac{1}{2r}}} \|t^{\gamma_1} u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_3}} + \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}} \right\}, \end{aligned} \quad (9.68)$$

for a suitable positive constant C . Setting $\lambda := (2C)^{2r}$, we can absorb the terms with index $n+1$ on the right-hand side by the left-hand side, hence there exists a positive constant \tilde{C} such that

$$\|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1,\lambda}^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1,\lambda}^h\|_{L_t^\infty L_x^{p_3}} \leq$$

$$\leq \tilde{C} \left\{ \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_1^2}{4} \tilde{\eta}^2 + \|\nu - 1\|_\infty \left(\frac{C_1}{2} \tilde{\eta} + \frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2} \right) \right\}.$$

Then we deduce that

$$\begin{aligned} & \|t^\beta \nabla u_{n+1}^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1}^h\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1}^h\|_{L_t^\infty L_x^{p_3}} \leq \\ & \leq \tilde{C} \sup_{t \in (0, \infty)} h_{n,\lambda}(0, t)^{-1} \left\{ \|\bar{u}^h\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_1^2}{4} \tilde{\eta}^2 + \|\nu - 1\|_\infty \left(\bar{C}_1 \tilde{\eta} + \frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2} \right) \right\} \\ & \leq \tilde{C} \exp \left\{ (2C)^{2r} \left(\frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2} \right)^{2r} \right\} \left\{ 1 + \left(\frac{C_1^2}{4} + \frac{C_1}{2} \right) \tilde{\eta} + \frac{C_2}{2} + \frac{C_3}{2} \right\} \tilde{\eta}. \end{aligned}$$

Imposing C_1 big enough and $\tilde{\eta}$ small enough in order to have

$$\tilde{C} \exp \left\{ (2C)^{2r} \left(\frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2} \right)^{2r} \right\} \left\{ 1 + \left(\frac{C_1^2}{4} + \frac{C_1}{2} \right) \tilde{\eta} + \frac{C_2}{2} + \frac{C_3}{2} \right\} \leq \frac{C_1}{2} \tilde{\eta},$$

we finally deduce that the first inequality of (9.61) is true for any positive integer n . Now, let us handle the vertical component u_n^d . Proceeding as in the proof of (9.68), we obtain that the following inequality is satisfied:

$$\begin{aligned} & \|t^\beta \nabla u_{n+1}^d\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1}^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1}^d\|_{L_t^\infty L_x^{p_3}} \leq \\ & \leq C \left\{ \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \|t^\alpha g_{n+1}\|_{L_t^{2r} L_x^p} + \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}} \right\}, \end{aligned}$$

for a suitable positive constant C , where g_{n+1} is defined by (9.16). Recalling that $\alpha = \beta + \gamma_1$ and $1/p = 1/p_2 + 1/p_3$ we get

$$\begin{aligned} & \|t^\beta \nabla u_{n+1}^d\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1}^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1}^d\|_{L_t^\infty L_x^{p_3}} \leq C \left\{ \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \right. \\ & \quad + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_n^h\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_2} u_{n+1}^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_n^d\|_{L_t^{2r} L_x^{p_2}} + \\ & \quad \left. + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_{n+1}^d\|_{L_t^{2r} L_x^{p_2}} + \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}} \right\}, \end{aligned}$$

which yields that

$$\begin{aligned} & \|t^\beta \nabla u_{n+1}^d\|_{L_t^{2r} L_x^{p_2}} + \|t^{\gamma_1} u_{n+1}^d\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} u_{n+1}^d\|_{L_t^\infty L_x^{p_3}} \leq \\ & \leq C \left(1 + \frac{C_1 C_2}{4} \tilde{\eta} \right) \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + C \left(\frac{C_1 C_3}{4} + \frac{C_1^2}{4} \tilde{\eta} + \|\nu - 1\|_\infty \left(\frac{C_1}{2} + \frac{C_2}{2} \right) \right) \tilde{\eta}. \end{aligned}$$

Hence the second inequality of (9.61) is true for any positive integer n if we assume \bar{C}_2 big enough and η small enough in order to have

$$C \left(1 + \frac{C_1 C_2}{4} \tilde{\eta} \right) < \frac{C_2}{2} \quad \text{and} \quad C \left(\frac{C_1 C_3}{2} + \frac{C_1^2}{4} \eta + \eta \left(\frac{C_1}{2} + \frac{C_2}{2} \right) \right) \eta \leq \frac{C_3}{2}.$$

Proceeding again by induction, we claim that

$$\|t^\alpha \nabla u_n^h\|_{L_t^{2r} L_x^{p^*}} \leq \frac{C_1}{2} \eta \quad \text{and} \quad \|t^\alpha \nabla u_n^d\|_{L_t^{2r} L_x^{p^*}} \leq \frac{C_2}{2} \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}} + \frac{C_3}{2}, \quad (9.69)$$

for any positive integer n . First, we remark that ∇u_L can be rewritten in the following form $\nabla u_L = -(\sqrt{-\Delta})^{-1}R\Delta u_L$. Hence, recalling that $(\sqrt{-\Delta})^{-1}$ is a bounded operator from L_x^p into $L_x^{p^*}$ and R is a bounded operator from L_x^q into itself, for any $q \in (1, \infty)$, there exist two positive constants C and \tilde{C} such that

$$\|t^\alpha \nabla u_L\|_{L_t^{2r} L_x^{p^*}} \leq \tilde{C} \|t^\alpha \Delta u_L\|_{L_t^{2r} L_x^p} \leq C \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}, \quad (9.70)$$

thanks to Theorem 9.2.7. Moreover Theorem 9.5.4 and Corollary 9.5.4.1 imply

$$\|t^\alpha (\nabla F_{n+1}^2 + \nabla F_{n+1}^3)\|_{L_t^{2r} L_x^{p^*}} \leq C\eta \|t^\alpha \nabla u_n\|_{L_t^{2r} L_x^{p^*}}, \quad \|t^\alpha \nabla F_{n+1}^1\|_{L_t^{2r} L_x^{p^*}} \leq C \|t^\alpha g_{n+1}\|_{L_t^{2r} L_x^p} \leq C\eta.$$

Assuming η small enough we get that (9.69) is true for any $n \in \mathbb{N}$. Finally, recalling that Π_{n+1} is determined by

$$\Pi_{n+1} = (-\Delta)^{-1}R \cdot g_{n+1} - R \cdot R \cdot \{(\nu(\theta_{n+1}) - 1)\nabla u_n\},$$

we get

$$\|t^\alpha \Pi_{n+1}\|_{L_t^{2r} L_x^{p^*}} \leq C \left\{ \|t^\alpha g_{n+1}\|_{L_t^{2r} L_x^p} + \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{2r} L_x^{p_2}} \right\} \leq C_4 \eta,$$

for a suitable positive constant C_4 and for any positive integer n .

STEP 2: ε -DEPENDENT ESTIMATES. As second step, we establish some ε -dependent estimates which will be useful in order to show that $(\theta_n, u_n, \Pi_n)_{\mathbb{N}}$ is a Cauchy sequence in a suitable functional space. First, we claim that

$$\|t^{\gamma_1} u_{n,\lambda}\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla u_{n,\lambda}\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq \bar{C}_4 \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\varepsilon}}, \quad (9.71)$$

where $u_{n,\lambda}(t) = u_n(t)h(0,t)$, with h defined by (9.63). Recalling the characterization of the homogenous Besov spaces given by Theorem 9.2.7 and the embedding of Theorem 9.2.8, we get

$$\|t^{\gamma_1} u_L\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla u_L\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq C \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\varepsilon}}, \quad (9.72)$$

for a suitable $C > 0$. Furthermore, Lemma 9.7.3 and Lemma 9.7.4 yields

$$\begin{aligned} & \|t^{\gamma_1} F_{n+1,\lambda}^1\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla F_{n+1,\lambda}^1\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \\ & \leq \bar{C} \left\{ \frac{1}{\lambda^{\frac{1}{2r}}} \|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_{n,\lambda}^h\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \right. \\ & \quad \left. + \|t^{\gamma_2} u_{n+1}^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_{n,\lambda}^d\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} + \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} \|t^\beta \nabla u_{n+1,\lambda}^d\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \right\}, \end{aligned}$$

for a positive constant \bar{C} . Imposing $\lambda := (2\bar{C})^{2r}$, we deduce that

$$\begin{aligned} & \|t^{\gamma_1} F_{n+1,\lambda}^1\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla F_{n+1,\lambda}^1\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq \frac{1}{2} \|t^\beta \nabla u_{n+1,\lambda}^h\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} + \\ & \quad + \bar{C} C_1 \eta \|t^\beta \nabla u_{n,\lambda}^h\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} + \bar{C} C_1 \eta \|t^\beta \nabla u_{n,\lambda}^d\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} + \bar{C} C_1 \eta \|t^\beta \nabla u_{n+1,\lambda}^d\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}}. \end{aligned} \quad (9.73)$$

Moreover, Theorem 9.5.4 and Lemma 9.5.3 imply

$$\begin{aligned} & \|t^{\gamma_1} (F_{n+1,\lambda}^2 + F_{n+1,\lambda}^3)\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla (F_{n+1,\lambda}^2 + F_{n+1,\lambda}^3)\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq \\ & \leq \|t^{\gamma_1} (F_{n+1}^2 + F_{n+1}^3)\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^\beta \nabla (F_{n+1}^2 + F_{n+1}^3)\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \\ & \leq C \|\nu - 1\|_\infty \|t^\beta \nabla u_n\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq \tilde{C} \eta \|t^\beta \nabla u_{n,\lambda}\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}}, \end{aligned} \quad (9.74)$$

assuming C_r in the definition of η big enough. Summarizing (9.72), (9.73) and (9.74), there exists a positive constant C such that

$$\|t^{\gamma_1} u_{n+1, \lambda}\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^{\beta} \nabla u_{n+1, \lambda}\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq C \bar{C}_4 \eta \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\varepsilon}},$$

so that (9.71) is true for any positive integer n . Finally, multiplying both the left and right-hand sides of (9.71) by $\sup_{t \in \mathbb{R}} h^{-1}(0, t)$, we get

$$\|t^{\gamma_1} u_n\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_3}} + \|t^{\beta} \nabla u_n\|_{L_t^{\frac{2r}{1-\varepsilon r}} L_x^{p_2}} \leq C_5 \|\bar{u}\|_{\dot{B}_{p,r}^{\frac{d}{p}-1+\varepsilon}} \exp \left\{ C_6 \|\bar{u}^d\|_{\dot{B}_{p,r}^{\frac{d}{p}-1}}^{2r} \right\}, \quad (9.75)$$

for two suitable positive constants C_5 and C_6 .

STEP 3. μ -DEPENDENT ESTIMATES AND CONVERGENCE OF THE SERIES. We proceed as in the third step of Theorem 9.3.3, denoting $\delta u_n := u_{n+1} - u_n$, $\delta \nu_n := \nu(\theta_{n+1}) - \nu(\theta_n)$ and $\delta \theta_n := \theta_{n+1} - \theta_n$. We define

$$\delta U_{n, \lambda}(T) := \|t^{\gamma_1} \delta u_{n, \lambda}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta u_{n, \lambda}\|_{L^\infty(0, T; L_x^{p_3})} + \|t^{\beta} \nabla \delta u_{n, \lambda}\|_{L^{2r}(0, T; L_x^{p_2})},$$

where $\delta u_{n, \lambda}(t) := \delta u_n(t) h_{n, \lambda}(0, t)$. We claim that the series $\sum_{n \in \mathbb{N}} \delta U_n(T)$ is convergent. First, we split δu_n into $\delta u_{n, \lambda} = f_{n,1} + f_{n,2} + f_{n,3}$, where $f_{n,i}$ is defined by (9.45), for $i = 1, 2, 3$. We begin estimating $f_{n,1}$. Lemma 9.7.3 and Lemma 9.7.4 yield that

$$\begin{aligned} & \|t^{\gamma_1} \delta f_{n,1}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,1}\|_{L^\infty(0, T; L_x^{p_3})} + \|t^{\beta} \nabla \delta f_{n,1}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \\ & \leq C \left\{ \frac{1}{\lambda^{\frac{1}{2r}}} (\|t^{\beta} \partial_d \delta u_{n, \lambda}^h\|_{L^{2r}(0, T; L_x^{p_2})} + \|t^{\gamma_1} \delta u_{n, \lambda}^h\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\beta} \nabla^h \delta u_{n, \lambda}^h\|_{L^{2r}(0, T; L_x^{p_2})}) + \right. \\ & + \|t^{\gamma_2} \delta u_{n, \lambda}^d\|_{L^\infty(0, T; L_x^{p_3})} \|t^{\beta} \partial_d u_n^h\|_{L_t^2 L_x^{p_2}} + \|t^{\gamma_2} \delta u_{n-1, \lambda}^h\|_{L^\infty(0, T; L_x^{p_3})} \|t^{\beta} \nabla u_n^h\|_{L_t^2 L_x^{p_2}} + \\ & + \|t^{\gamma_2} u_{n-1}^h\|_{L_t^2 L_x^{p_3}} \|t^{\beta} \nabla \delta u_{n-1, \lambda}^h\|_{L^{2r}(0, T; L_x^{p_2})} + \|t^{\beta} \nabla^h \delta u_{n, \lambda}^d\|_{L^{2r}(0, T; L_x^{p_2})} \|t^{\gamma_2} u_n^h\|_{L_t^\infty L_x^{p_3}} + \\ & \left. + \|t^{\gamma_2} \delta u_{n-1, \lambda}^d\|_{L^\infty(0, T; L_x^{p_3})} \|t^{\beta} \nabla^h u_n^h\|_{L_t^2 L_x^{p_2}} \right\}. \end{aligned}$$

which yields,

$$\begin{aligned} & \|t^{\gamma_1} \delta f_{n,1}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,1}\|_{L^\infty(0, T; L_x^{p_3})} + \\ & + \|t^{\beta} \nabla \delta f_{n,1}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \frac{1}{4} (\delta U_{n, \lambda}(T) + \delta U_{n-1, \lambda}(T)), \end{aligned} \quad (9.76)$$

assuming η small enough. Now, we carry out the estimate of $f_{n,2}$. Lemma 9.5.1 and Theorem 9.5.4 imply

$$\begin{aligned} & \|t^{\gamma_1} \delta f_{n,2}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,2}\|_{L^\infty(0, T; L_x^{p_3})} + \|t^{\beta} \nabla \delta f_{n,2}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \\ & \leq C \|\nu - 1\|_\infty \|t^{\beta} \nabla \delta u_{n-1}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \tilde{C}_r \eta \|t^{\beta} \nabla \delta u_{n-1, \lambda}\|_{L^{2r}(0, T; L_x^{p_2})}, \end{aligned}$$

hence, we deduce that

$$\|t^{\gamma_1} \delta f_{n,2}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,2}\|_{L^\infty(0, T; L_x^{p_3})} + \|t^{\beta} \nabla \delta f_{n,2}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \tilde{C}_r \eta \delta U_{n-1, \lambda}(T). \quad (9.77)$$

Now we deal with $f_{n,3}$. Thanks to Lemma 9.5.3 and Theorem 9.5.4, we have

$$\begin{aligned} & \|t^{\gamma_1} \delta f_{n,3}\|_{L^{2r}(0, T; L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,3}\|_{L^\infty(0, T; L_x^{p_3})} + \|t^{\beta} \nabla \delta f_{n,3}\|_{L^{2r}(0, T; L_x^{p_2})} \leq \\ & \leq \|t^{\beta} \delta \nu_n D(u_n)\|_{L^{2r}(0, T; L_x^{p_2})} \leq C \|\delta \nu_n\|_{L^{\frac{2}{\varepsilon}}(0, T; L_x^\infty)} \|t^{\beta} \nabla u_n\|_{L^{2r}(0, T; L_x^{p_2})} \leq \hat{C}_1(\bar{u}) \|\delta \theta_n\|_{L^{\frac{2}{\varepsilon}}(0, T; L_x^\infty)} \end{aligned} \quad (9.78)$$

where $\hat{C}_1(\bar{u})$ is a positive constant which depends on $\|\bar{u}\|_{\dot{B}_{p,r}^{d/p-1+\varepsilon}}$. Now, recalling that $\delta\theta_n$ is determined by (9.51), we get

$$\|\delta\theta_n(t)\|_{L_x^\infty} \leq \int_0^t \frac{s^{\gamma_1} \|\delta\theta_n(s)u_n(s)\|_{L_x^{p_3}}}{s^{\gamma_1} |\mu(t-s)|^{\frac{d}{2} \frac{1}{p_3} + \frac{1}{2}}} ds + \int_0^t \frac{s^{\gamma_1} \|\delta u_{n-1}(s)\theta_n(s)\|_{L_x^{p_3}}}{s^{\gamma_1} |\mu(t-s)|^{\frac{d}{2} \frac{1}{p_3} + \frac{1}{2}}} ds,$$

hence, defining $\alpha := (d/(2p_3) + 1/2)(2r)' < 1$, $\|\delta\theta_n(t)\|_{L_x^{2r}}$ is bounded by

$$2^{2r-1} \left(\int_0^t \frac{1}{s^{\gamma_1(2r)' |\mu(t-s)|^\alpha} ds \right)^{2r-1} \left\{ \int_0^t \|\delta\theta_n(s)\|_{L_x^{2r}}^{2r} s^{2r\gamma_1} \|u_n(s)\|_{L_x^{p_3}}^{2r} ds + \int_0^t \|\bar{\theta}\|_{L_x^{2r}}^{2r} s^{2r\gamma_1} \|\delta u_{n-1}(s)\|_{L_x^{p_3}}^{2r} ds \right\}.$$

Then, using the Gronwall inequality, we have

$$\|\delta\theta_n(t)\|_{L_x^{2r}}^{2r} \leq \hat{C}_2(t) \|\bar{\theta}\|_{L_x^{2r}}^{2r} \int_0^t s^{2r\gamma_1} \|\delta u_{n-1}(s)\|_{L_x^{p_3}}^{2r} ds \exp \left\{ \int_0^t s^{2r\gamma_1} \|u_n(s)\|_{L_x^{p_3}}^{2r} ds \right\},$$

which yields $\|\delta\theta_n(t)\|_{L_x^\infty} \leq \chi(t) \delta U_{n-1}(t)$, where χ is an increasing function. Hence, Recalling (9.78), we deduce that

$$\begin{aligned} & \|t^{\gamma_1} \delta f_{n,3}\|_{L^{2r}(0,T;L_x^{p_3})} + \|t^{\gamma_2} \delta f_{n,3}\|_{L^\infty(0,T;L_x^{p_3})} \\ & + \|t^\beta \nabla \delta f_{n,3}\|_{L^{2r}(0,T;L_x^{p_2})} \leq \hat{C}_1(\bar{u}) \chi(T) \|\delta U_{n-1}\|_{L^{\frac{4}{\varepsilon}}(0,T)}. \end{aligned}$$

Summarizing the last inequality with (9.76) and (9.77), we finally deduce that

$$\delta U_{n,\lambda}(T) \leq \left(\frac{1}{3} + \frac{4}{3} \tilde{C}_r \eta \right) \delta U_{n-1,\lambda}(T) + \frac{4}{3} \hat{C}_1(\bar{u}) \chi(T) \|\delta U_{n-1}\|_{L^{\frac{2}{\varepsilon}}(0,T)},$$

which is equivalent to (9.53). Thus we can conclude proceeding as in the last part of Theorem 9.3.3. \square

Now, we show that system (9.1) admits a weak solution, adding a tiny extra regularity to the initial data.

Theorem 9.5.6. *Let us assume that the hypotheses of Theorem 9.1.4 are fulfilled. Suppose that $\bar{\theta}$ belongs to $L_x^2 \cap L_x^\infty$ and \bar{u} belongs to $\dot{B}_{p,r}^{d/p-1} \cap \dot{B}_{p,r}^{d/p-1+\varepsilon}$ with $\varepsilon < \min\{1/(2r), 1 - 1/(2r), d/p - 1\}$. If the smallness condition (9.7) holds then there exists a global weak solution (θ, u, Π) of (9.12) which satisfies the properties of Theorem 9.1.4.*

Proof. By Proposition 9.5.5, there exists $(\theta_\mu, u_\mu, \Pi_\mu)$, solution of (9.12), such that $t^{\gamma_1} u_\mu$ belongs to $L_t^{2r} L_x^{p_3}$, $t^{\gamma_2} u_\mu$ belongs to $L_t^\infty L_x^{p_3}$, $t^\beta \nabla u_\mu$ lives in $L_t^{2r} L_x^{p_2}$, $t^\alpha \nabla u_\mu$ in $L_t^{2r} L_x^{p^*}$, θ_μ in $L_{t,x}^\infty$ and $t^\alpha \Pi_\mu$ in $L_t^{2r} L_x^{p^*}$. Then, thanks to inequalities (9.13), there exists (θ, u, Π) in the same space of $(\theta_\mu, u_\mu, \Pi_\mu)$, such that

$$\begin{aligned} t^{\gamma_1} u_{\mu_n} &\rightharpoonup t^{\gamma_1} u & w - L_t^{2r} L_x^{p_3}, & \quad t^{\gamma_2} u_{\mu_n} &\rightharpoonup t^{\gamma_2} u & w - L_t^\infty L_x^{p_3}, & \quad t^\beta \nabla u_{\mu_n} &\rightharpoonup t^\beta \nabla u & w - L_t^{2r} L_x^{p_2}, \\ t^\alpha \nabla u_{\mu_n} &\rightharpoonup t^\alpha \nabla u & w - L_t^{2r} L_x^{p^*}, & \quad \theta_{\mu_n} &\xrightarrow{*} \theta & w * - L_{t,x}^\infty, & \quad t^\alpha \Pi_{\mu_n} &\rightharpoonup t^\alpha \Pi & w - L_t^{2r} L_x^{p^*}, \end{aligned}$$

for a positive decreasing sequence $(\mu_n)_\mathbb{N}$ convergent to 0. We claim that (θ, u, Π) is a weak solution of (9.1). First, we show that u_{μ_n} strongly converges to u in $L^{\tau_3}(0, T; L_x^{p_3})$, up to a subsequence, with a suitable $\tau_3 > 1$. We proceed establishing that $\{u_\mu - u_L \mid \mu > 0\}$ is a compact

set in $C([0, T]; \dot{W}_x^{-1, p^*})$, for all $T > 0$. Applying $(\sqrt{-\Delta})^{-1}$ to the momentum equation of (9.12), we observe that $t^\alpha \partial_t (\sqrt{-\Delta})^{-1} u_\mu$ is uniformly bounded in $L^{2r}(0, T; L_x^{p^*})$. Hence, observing that $\alpha(2r)' < 1$, we get

$$\|\partial_t (\sqrt{-\Delta})^{-1} u_\mu\|_{L^1(0, T; L_x^{p^*})} \leq \frac{T^{1-\alpha(2r)'}}{1-\alpha(2r)'} \|t^\alpha \partial_t (\sqrt{-\Delta})^{-1} u_\mu\|_{L^{2r}(0, T; L_x^{p^*})}$$

Thus $\{(\sqrt{-\Delta})^{-1}(u_\mu - u_L) \mid \mu > 0\}$ is an equicontinuous and bounded family of $C([0, T], L_x^{p^*})$, namely it is a compact family. Then we can extract a subsequence (which we still denote by u_{μ_n}) such that $(\sqrt{-\Delta})^{-1}(u_{\mu_n} - u_L)$ strongly converges to $(\sqrt{-\Delta})^{-1}(u - u_L)$ in $L^\infty(0, T; L_x^{p^*})$, that is $u_{\mu_n} - u_L$ strongly converges to $u - u_L$ in $L^\infty(0, T; \dot{W}_x^{-1, p^*})$. Now, passing through the following real interpolation

$$\left[\dot{W}_x^{-1, p^*}, \dot{W}_x^{1, p^*} \right]_{\mu, 1} = \dot{B}_{p^*, 1}^{\frac{d}{p^*} - \frac{d}{p_3}} \hookrightarrow L_x^{p_3},$$

with $\mu := (d/p^* - d/p_3) + 1/2 < 1$ (see [11], Theorem 6.3.1 and [7], Theorem 2.39), we deduce that

$$\begin{aligned} \|u_{\mu_n} - u\|_{L^\tau(0, T; L_x^{p_3})} &\leq C \left\| \|u_{\mu_n} - u\|_{\dot{W}_x^{-1, p^*}}^{1-\mu} \|u_{\mu_n} - u\|_{\dot{W}_x^{1, p^*}}^\mu \right\|_{L^\tau(0, T)} \\ &\leq C \|u_{\mu_n} - u\|_{L^\infty(0, T; \dot{W}_x^{-1, p^*})}^{1-\mu} \|t^{-\alpha}\|_{L^{\frac{2r\tau}{2r-\tau}}(0, T)}^\mu \|t^\alpha \nabla(u_{\mu_n} - u)\|_{L^{2r}(0, T; L_x^{p^*})}^\mu, \end{aligned}$$

for all $T > 0$, where we have considered $\tau \in (1, 2r/(1+2\alpha r))$ so that $\alpha 2r\tau/(2r-\tau) < 1$. Moreover, we choose τ such that there exist τ_2 in $(1, 2r/(1+2\beta r))$ and τ_3 in $(1, 2r/(1+2\gamma_1 r))$ which fulfill $1/\tau_3 + 1/\tau_2 = 1/\tau_1$. Let us remark that the norms

$$\begin{aligned} \|u_{\mu_n}\|_{L^{\tau_3}(0, T; L_x^{p_3})} &\leq \|t^{\gamma_1}\|_{L^{\frac{2r\tau_3}{2r-\tau_3}}(0, T)} \|t^{\gamma_1} u_{\mu_n}\|_{L_t^{2r} L_x^{p_3}} < \infty, \\ \|\nabla u_{\mu_n}\|_{L^{\tau_2}(0, T; L_x^{p_2})} &\leq \|t^\beta\|_{L^{\frac{2r\tau_2}{2r-\tau_2}}(0, T)} \|t^\beta u_{\mu_n}\|_{L_t^{2r} L_x^{p_2}} < \infty, \end{aligned}$$

that is they are uniformly bounded in n . Now, we consider $\tau < \sigma < \tau_3$ strictly closed to τ_3 so that it still fulfills $1/\sigma + 1/\tau_2 > 1$. Then the following interpolation inequality

$$\|u_{\mu_n} - u\|_{L^\sigma(0, T; L_x^{p_3})} \leq \|u_{\mu_n} - u\|_{L^\tau(0, T; L_x^{p_3})}^{\frac{\tau_3 - \sigma}{\tau - \tau_3}} \|u_{\mu_n} - u\|_{L^{\tau_3}(0, T; L_x^{p_3})}^{\frac{\sigma - \tau_3}{\tau - \tau_3}},$$

which converges to 0 as n goes to ∞ , so that u_{μ_n} strongly converges to u in $L_{loc}^\sigma(\mathbb{R}_+; L_x^{p_3})$. This yields that $u_{\mu_n} \theta_{\mu_n}$ and $u_{\mu_n} \cdot \nabla u_{\mu_n}$ converge to $u \theta$ and $u \cdot \nabla u$, respectively, in the distributional sense. We deduce that θ is weak solution of

$$\partial_t \theta + \operatorname{div}(\theta u) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \quad \theta|_{t=0} = \bar{\theta} \quad \text{in } \mathbb{R}^d.$$

Arguing as in the proof of theorem 9.3.3, θ_{μ_n} converges almost everywhere to θ , up to a subsequence, so that $\nu(\theta_{\mu_n})$ strongly converges to $\nu(\theta)$ in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^d)$, for every $1 \leq m < \infty$, thanks to the Dominated Convergence Theorem. Then $\nu(\theta_{\mu_n})D(u_{\mu_n})$ converges to $\nu(\theta)D(u)$ in the distributional sense.

Summarizing all the previous considerations we finally conclude that (θ, u, Π) is a weak solution of (9.1) and it satisfies (9.59). \square

9.6 Weak solutions: the general case

In this section we present the proof of Theorem (9.1.4). We proceed similarly as in the proof of Theorem 9.1.3, approximating our initial data by

$$\bar{\theta}_n := \chi_n \sum_{|j| \leq n} \dot{\Delta}_j \bar{\theta} \quad \text{and} \quad \bar{u}_n := \sum_{|j| \leq n} \dot{\Delta}_j \bar{u}, \quad \text{for every } n \in \mathbb{N},$$

where $\chi_n \leq 1$ is a cut-off function which has support on the ball $B(0, n) \subset \mathbb{R}^d$ and $\chi_n \equiv 1$ in $B(0, n/2)$. Then $\bar{\theta}_n \in L_x^\infty \cap L_x^2$ and $\bar{u} \in \dot{B}_{p,r}^{d/p} \cap \dot{B}_{p,r}^{d/p-1+\varepsilon}$, with $\varepsilon < \min\{1/(2r), 1 - 1/r, 2(d/p - 2 + 1/r)\}$. Then, by Theorem 9.5.6, there exists (θ_n, u_n, Π_n) weak solution of

$$\begin{cases} \partial_t \theta_n + \operatorname{div}(\theta_n u_n) = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u_n + u_n \cdot \nabla u_n - \operatorname{div}(\nu(\theta_n)D(u_n)) + \nabla \Pi_n = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u_n = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ (\theta_n, u_n)_{t=0} = (\bar{\theta}_n, \bar{u}_n) & \mathbb{R}^d, \end{cases}$$

which belongs to the functional space defined in Theorem 9.1.4 and it fulfills the inequalities (9.9), uniformly in $n \in \mathbb{N}$. Then there exists a subsequence (which we still denote by $(\theta_n, u_n, \Pi_n)_{\mathbb{N}}$) and an element (θ, u, Π) in the same space of (θ_n, u_n, Π_n) , such that

$$\begin{aligned} t^{\gamma_1} u_n &\rightharpoonup t^{\gamma_1} u & w - L_t^{2r} L_x^{p_3}, & \quad t^{\gamma_2} u_n &\rightharpoonup t^{\gamma_2} u & w - L_t^\infty L_x^{p_3}, & \quad t^\beta \nabla u_n &\rightharpoonup t^\beta \nabla u & w - L_t^{2r} L_x^{p_2}, \\ t^\alpha \nabla u_n &\rightharpoonup t^\alpha \nabla u & w - L_t^{2r} L_x^{p_*}, & \quad \theta_{\varepsilon_n} &\xrightarrow{*} \theta & w * -L_{t,x}^\infty, & \quad t^\alpha \Pi_n &\rightharpoonup t^\alpha \Pi & w - L_t^{2r} L_x^{p_*}. \end{aligned}$$

In order to complete the proof, we claim that (θ, u, Π) is weak solution of (9.1). We first rewrite $u_n = t^{-\gamma_1} t^{\gamma_1} u_n$, $\nabla u = t^{-\beta} t^\beta \nabla u$ and $\Pi_n = t^{-\alpha} t^\alpha \Pi_n$, so that the Hölder inequality guarantees that u_n , ∇u_n and Π_n are uniformly bounded in $L^{\tau_3}(0, T; L_x^{p_3})$, $L^{\tau_2}(0, T; L_x^{p_2})$ and $L^{\tau_1}(0, T; L_x^{p_*})$ respectively, with $T \in (0, \infty)$ and

$$\tau_1 \in \left(1, \frac{2r}{1 + 2\alpha r}\right), \quad \tau_2 \in \left(1, \frac{2r}{1 + 2\beta r}\right), \quad \tau_3 \in \left(1, \frac{2r}{1 + 2\gamma_1 r}\right), \quad \text{such that} \quad \frac{1}{\tau_1} = \frac{1}{\tau_2} + \frac{1}{\tau_3}.$$

The same properties are preserved by (θ, u, Π) . Moreover, arguing as in Theorem 9.5.6, u_n strongly converges to u in $L_{loc}^\sigma(\mathbb{R}_+; L_x^{p_3})$, with $\sigma \in (\tau_1, \tau_3)$ strictly closed to τ_3 so that $1/\sigma + 1/\tau_2 > 1$. This yields that $u_n \cdot \nabla u_n$ and $u_n \theta_n$ converge to $u \cdot \nabla u$ and $u \theta$ respectively, in the distributional sense. Moreover, proceeding as in theorem 9.3.3, θ_n converges almost everywhere to θ , up to a subsequence, so that $\nu(\theta_n)$ strongly converges to $\nu(\theta)$ in $L_{loc}^m(\mathbb{R}_+ \times \mathbb{R}^d)$, for every $1 \leq m < \infty$, thanks to the Dominated Convergence Theorem. Then $\nu(\theta_n)D(u_n)$ converges to $\nu(\theta)D(u)$ in the distributional sense and this allows us to conclude that (θ, u, Π) is weak solution of (9.1). Finally, passing through the limit as n goes to ∞ , (θ, u, Π) still fulfills inequalities (9.9) and this concludes the proof of the Theorem.

9.7 Inequalities

In this section we improve Lemma 9.2.5 and Lemma 9.2.6 for a particular choice of the function f and also with a perturbation of the operators, which is dependent on a parameter $\lambda > 0$. This Lemmas are useful for the Theorem of section 3, more precisely during the proof of the inequalities, since, for an opportune choice of λ , they permit to “absorb” some uncontrolled terms. Here the

statements and the proofs.

Lemma 9.7.1. *Let $1 < r < \infty$ and $q_1, q_2 \in (1, \infty]$ such that $1/q = 1/q_1 + 1/q_2 \in ((2r - 1)/dr, 1)$. Let $v \in L_t^{2r} L_x^{q_1}$ and for all $\lambda > 0$ let $h = h_\lambda$ be defined by*

$$h(s, t) := \exp \left\{ -\lambda \int_s^t \|v\|_{L_x^{q_1}}^{2r} ds \right\},$$

for all $0 \leq s \leq t < \infty$ and consider \mathcal{C}_λ , the operator defined by

$$\mathcal{C}_\lambda(f)(t) := \int_0^t h(s, t) e^{(t-s)\Delta} f(s) ds.$$

Then there exists a positive constant C_r , such that

$$\|\mathcal{C}_\lambda(v\omega)\|_{L_t^{2r} L_x^{q_3}} \leq C_r \frac{1}{\lambda^{\frac{1}{4r}}} \|v\|_{L_t^{2r} L_x^{q_1}}^{\frac{1}{2}} \|\omega\|_{L_t^{2r} L_x^{q_2}},$$

where q_3 is defined by $1/q_3 = 1/q - (2r - 1)/dr$.

Proof. Notice that

$$\begin{aligned} \left\| \int_0^t h(s, t) K(t-s) * v\omega(s) ds \right\|_{L_x^{q_3}} &\leq \int_0^t h(s, t) \|K(t-s) * v\omega(s)\|_{L_x^{q_3}} ds \\ &\leq \int_0^t h(s, t) \|K(t-s)\|_{L_x^{\tilde{q}}} \|v\omega(s)\|_{L_x^q} ds, \end{aligned}$$

where $1/\tilde{q}' = 1 - 1/\tilde{q} = 1/q - 1/q_3 = (2r - 1)/(dr)$. By Remark 9.2.3 and Holder inequality, we obtain

$$\begin{aligned} \|\mathcal{C}(v\omega)(t)\|_{L_x^q} &\leq \int_0^t h(s, t) \|v(s)\|_{L^{p_1}}^{\frac{1}{2}} \frac{1}{|t-s|^{\frac{2r-1}{2r}}} \|v(s)\|_{L^{p_1}}^{\frac{1}{2}} \|\omega(s)\|_{L_x^{p_2}} ds \\ &\leq \left(\int_0^t h(s, t)^{4r} \|v(s)\|_{L_x^{p_1}}^{2r} ds \right)^{\frac{1}{4r}} \left(\int_{\mathbb{R}_+} \frac{(\|v(s)\|_{L^{q_1}}^{\frac{1}{2}} \|\omega(s)\|_{L_x^{q_2}})^{\frac{4r}{4r-1}}}{|t-s|^{\frac{2r-1}{2r} \frac{4r}{4r-1}}} ds \right)^{1-\frac{1}{4r}}. \end{aligned} \quad (9.79)$$

Since

$$g := \left(\|v(\cdot)\|_{L^{q_1}}^{\frac{1}{2}} \|\omega(\cdot)\|_{L_x^{q_2}} \right)^{\frac{4r}{4r-1}} \in L_t^{\frac{4r-1}{3}},$$

by Hardy-Littlewood-Sobolev inequality,

$$|\cdot|^{-\frac{4r-2}{4r-1}} * g \in L_t^{\frac{4r-1}{2}},$$

and then

$$\left(|\cdot|^{-\frac{4r-2}{4r-1}} * g \right)^{1-\frac{1}{4r}} \in L_t^{2r}.$$

Moreover there exists $C >$ such that

$$\begin{aligned} \left\| \left(|\cdot|^{-\frac{4r-2}{4r-1}} * g \right)^{1-\frac{1}{4r}} \right\|_{L_t^{2r}} &= \left\| |\cdot|^{-\frac{4r-2}{4r-1}} * g \right\|_{L_t^{\frac{4r-1}{2}}}^{1-\frac{1}{4r}} \\ &\leq C \|g\|_{L_t^{\frac{4r-1}{3}}}^{1-\frac{1}{4r}} \leq \left(\int_{\mathbb{R}_+} (\|v(t)\|_{L_x^{q_1}}^{\frac{1}{2}} \|\omega(t)\|_{L_x^{q_2}})^{\frac{4}{3}r} dt \right)^{\frac{3}{4r}} \end{aligned}$$

$$\leq C \|v\|_{L_x^{q_1}}^{\frac{1}{2}} \|v\|_{L_t^{4r}} \|\omega\|_{L_t^{2r} L_x^{q_2}} \leq C \|v\|_{L_t^{2r} L_x^{q_1}}^{\frac{1}{2}} \|\omega\|_{L_t^{2r} L_x^{q_2}}.$$

Observing that

$$\left(\int_0^t h(s, t)^{4r} \|v(s)\|_{L_x^{q_1}}^{2r} ds \right)^{\frac{1}{4r}} \leq \left(\frac{1}{4r\lambda} \right)^{\frac{1}{4r}},$$

the Lemma is proved. \square

Lemma 9.7.2. *Let $1 < r < \infty$, $q_1 \in [1, \frac{dr}{r-1}]$ and $v \in L_t^{2r} L_x^{q_1}$. For all $\lambda > 0$ let $h = h_\lambda$ be defined as in Lemma 9.7.1 and let \mathcal{B}_λ the operator defined by*

$$\mathcal{B}_\lambda(f)(t) := \int_0^t h(s, t) \nabla e^{(t-s)\Delta} f(s) ds.$$

For all $q_2 \in [q_1', \infty]$, there exists a positive constant C_r , such that

$$\|\mathcal{B}_\lambda(v\omega)\|_{L_t^{2r} L_x^q} \leq C_r \frac{1}{\lambda^{\frac{1}{4r}}} \|v\|_{L_t^{2r} L_x^{p_1}}^{\frac{1}{2}} \|\omega\|_{L_t^{2r} L_x^{p_2}},$$

where q is defined by $1/q := 1/q_1 + 1/q_2 - (r-1)/dr$.

Lemma 9.7.3. *Let $r \in (1, \infty)$, $p_1 \in (d/2, d)$, $p_3 > dr/(r-1)$ and p_2 be given by $1/p_1 + 1/p_2 = 1/p_3$. Let $t_1^\gamma v \in L_t^{2r} L_x^{p_3}$ and $t^\beta \omega \in L_t^{2r} L_x^{p_2}$. Defining*

$$h_\lambda(s, t) := \exp \left\{ -\lambda \int_s^t \tau^{2r\gamma_1} \|v(\tau)\|_{L_x^{p_3}}^{2r} d\tau - \lambda \int_s^t \tau^{2r\beta} \|\omega(\tau)\|_{L_x^{p_2}}^{2r} d\tau \right\},$$

where λ is a positive constant, there exists a positive constant C_r such that

$$\|t^{\beta_1} \mathcal{B}(v\omega)_\lambda(t)\|_{L_t^{2r} L_x^{p_2}} \leq \frac{C_r}{\lambda^{\frac{1}{2r}}} \|t^\beta \omega\|_{L_t^{2r} L_x^{p_2}}, \tag{9.80}$$

$$\|t^{\beta_1} \mathcal{B}(v\omega)_\lambda(t)\|_{L_t^{2r} L_x^{p_2}} \leq \frac{C_r}{\lambda^{\frac{1}{2r}}} \|t^{\gamma_1} v\|_{L_t^{2r} L_x^{p_3}}. \tag{9.81}$$

Proof. Remark 9.2.3 yields that there exists a positive constant C such that

$$\begin{aligned} t^\beta \|\mathcal{B}(v\omega)_\lambda(t)\|_{L_x^{p_2}} &\leq C \int_0^t \frac{t^{\beta_1}}{|t-s|^{\frac{d}{2p_3} + \frac{1}{2} s \alpha_2}} h_\lambda(s, t) s^{\gamma_1} \|v(s)\|_{L_x^{p_3}} s^{\beta_1} \|\omega_\lambda(s)\|_{L_x^{p_2}} ds \\ &\leq C \left(\int_0^t h_\lambda(s, t)^{2r} s^{2r\gamma_1} \|v(s)\|_{L_x^{p_3}}^{2r} ds \right)^{\frac{1}{2r}} \left(\int_0^t \left| \frac{t^{\beta_1}}{|t-s|^{\frac{d}{2p_3} + \frac{1}{2} s \alpha_2}} F(s) \right|^{(2r)'} ds \right)^{\frac{1}{(2r)'}}. \end{aligned} \tag{9.82}$$

Hence, raising to the power of $(2r)'$ both the left-hand and the right-hand sides, we get

$$\begin{aligned} t^{(2r)'\beta_1} \|\mathcal{B}(v\omega)_\lambda(t)\|_{L_x^{p_2}}^{(2r)'} &\lesssim \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^t \left| \frac{t^{\beta_1}}{|t-s|^{\frac{d}{2p_3} + \frac{1}{2} s \alpha_2}} s^{\beta_1} \|\omega(s)\|_{L_x^{p_2}} \right|^{(2r)'} ds \\ &\lesssim \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^1 \left| \frac{t^{\beta_1 - \alpha_2 - \frac{N}{2p_3} - \frac{1}{2}}}{|1-\tau|^{\frac{d}{2p_3} + \frac{1}{2} \tau \alpha_2}} F(t\tau) \right|^{(2r)'} t d\tau, \end{aligned}$$

where $F(s) = s^\beta \|\omega_\lambda(s)\|_{L_x^{p_2}}$. Observing that $\beta - \alpha_2 - N/(2p_3) - 1/2 = 1/(2r) - 1 = -1/(2r)'$, we

get

$$t^{(2r)'\beta_1} \|\mathcal{B}(v\omega)_\lambda(t)\|_{L_x^{p_2}}^{(2r)'} \lesssim \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^1 \left| \frac{1}{|1-\tau|^{\frac{d}{2p_3} + \frac{1}{2}} \tau^{\alpha_2}} F(t\tau) \right|^{(2r)'} d\tau, \quad (9.83)$$

Hence, applying the $L_t^{(2r)/(2r)'}$ -norm to both the left and right-hand sides,

$$\begin{aligned} \|t^{\beta_1} \mathcal{B}(v\omega)_\lambda(t)\|_{L_t^{2r} L_x^{p_2}}^{(2r)'} &\lesssim \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^1 \left| \frac{1}{|1-\tau|^{\frac{d}{2p_3} + \frac{1}{2}} \tau^{\alpha_2}} \right|^{(2r)'} \left(\int_0^\infty F(t\tau)^{2r} d\tau \right)^{\frac{1}{2r-1}} dt \\ &\lesssim \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^1 \left| \frac{1}{|1-\tau|^{\frac{d}{2p_3} + \frac{1}{2}} \tau^{\alpha_1}} \right|^{(2r)'} d\tau \|t^\beta \omega_\lambda\|_{L_t^{2r} L_x^{p_2}}^{(2r)'} \end{aligned}$$

thanks to Minkowski inequality. Since $\alpha_1(2r)' < 1$ and $(d/(2p_3) + 1/2)(2r)' < 1$ we finally obtain (9.80). Now, defining $F(t) := s^{\gamma_1} \|v_\lambda(s)\|_{L_x^{p_3}}$, we also have

$$\begin{aligned} &t^\beta \|\mathcal{B}(v\omega)_\lambda(t)\|_{L_x^{p_2}} \\ &\leq C \left(\int_0^t h_\lambda(s, t)^{2r} s^{2r\beta} \|\omega(s)\|_{L_x^{p_2}} ds \right)^{\frac{1}{2r}} \left(\int_0^t \left| \frac{t^\beta}{|t-s|^{\frac{d}{2p_3} + \frac{1}{2}} s^{\alpha_2}} F(s) \right|^{(2r)'} ds \right)^{\frac{1}{(2r)'}}, \end{aligned}$$

which is equivalent to (9.82). Thus, arguing as for proving (9.80), we also obtain (9.81). \square

Lemma 9.7.4. *Let $r \in (2, \infty)$, $p_1 \in (dr/(2r-2), N)$ and $p_3 \geq Nr/(r-2)$ such that $1/p_1 + 1/p_2 = 1/p_3$. Let h_λ, v and ω be defined as in the previous Lemma. Then there exists $C_r > 0$ such that*

$$\begin{aligned} \|t^{\gamma_1} \mathcal{C}(v\omega)_\lambda(t)\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} \mathcal{C}(v\omega)_\lambda(t)\|_{L_t^\infty L_x^{p_3}} &\leq \frac{C_r}{\lambda^{\frac{1}{2r}}} \|t^{\beta_1} \omega_\lambda\|_{L_t^{2r} L_x^{p_2}}, \\ \|t^{\gamma_1} \mathcal{C}(v\omega)_\lambda(t)\|_{L_t^{2r} L_x^{p_3}} + \|t^{\gamma_2} \mathcal{C}(v\omega)_\lambda(t)\|_{L_t^\infty L_x^{p_3}} &\leq \frac{C_r}{\lambda^{\frac{1}{2r}}} \|t^{\beta_1} v_\lambda\|_{L_t^{2r} L_x^{p_3}}. \end{aligned} \quad (9.84)$$

Proof. We control the $L_t^{2r} L_x^{p_3}$ norm arguing as in previous proof. Indeed we have

$$t^{(2r)'\gamma_1} \|\mathcal{C}(v\omega)_\lambda(t)\|_{L_x^{p_3}}^{(2r)'} \leq C \frac{1}{\lambda^{\frac{(2r)'}{2r}}} \int_0^1 \left| \frac{1}{|1-\tau|^{\frac{d}{2p_2}} \tau^{\alpha_2}} F(t\tau) \right|^{(2r)'} d\tau,$$

where $F(s) = s^\beta \|\omega_\lambda\|_{L_x^{p_2}}$ or $F(s) = s^{\gamma_1} \|v_\lambda\|_{L_x^{p_3}}$ instead of (9.83). Let us take in consideration the $L_t^\infty L_x^{p_3}$ norm. With a direct computation we get

$$\begin{aligned} \|t^{\gamma_2} \mathcal{C}(t)\|_{L_x^{p_3}} &\leq C \left(\int_0^t \left| \frac{t^{\gamma_2}}{|t-s|^{\frac{N}{2p_2}} s^{\alpha_2}} \right|^{r'} ds \right)^{\frac{1}{r'}} \left(\int_0^t h(s, t)^r s^{r\gamma_1} \|v(s)\|_{L_x^{p_3}}^r s^{r\beta_1} \|\omega(s)\|_{L_x^{p_2}}^r ds \right)^{\frac{1}{r}} \\ &\leq C \left(\int_0^1 \left| \frac{t^{\gamma_2 - \alpha_2 - \frac{N}{2p_2}}}{|1-\tau|^{\frac{N}{2p_2}} \tau^{\alpha_2}} \right|^{r'} t d\tau \right)^{\frac{1}{r'}} \left(\int_0^t h(s, t)^r s^{r\gamma_1} \|v(s)\|_{L_x^{p_3}}^r s^{r\beta_1} \|\omega(s)\|_{L_x^{p_2}}^r ds \right)^{\frac{1}{r}} \end{aligned}$$

Thus, observing that $\gamma_2 - \alpha_2 - d/(2p_2) + 1/r' = 0$, $dr'/(2p_2) < 1$ and $\alpha_2 r' < 1$, we conclude that

$$\begin{aligned} \|t^{\gamma_2} \mathcal{C}(t)\|_{L_x^{p_3}} &\leq \bar{C}_r \left(\int_0^t h_\lambda(s, t)^{2r} s^{2r\gamma_1} \|v(s)\|_{L_x^{p_3}} ds \right)^{\frac{1}{2r}} \|t^\beta \omega_\lambda\|_{L_t^{2r} L_x^{p_2}} \quad \text{and} \\ \|t^{\gamma_2} \mathcal{C}(t)\|_{L_x^{p_3}} &\leq \bar{C}_r \left(\int_0^t h_\lambda(s, t)^{2r} s^{2r\beta} \|\omega(s)\|_{L_x^{p_2}} ds \right)^{\frac{1}{2r}} \|t^{\gamma_1} \omega_\lambda\|_{L_t^{2r} L_x^{p_3}}, \end{aligned}$$

for a suitable positive constant \tilde{C}_r , which finally yields (9.84). \square

Theorem 9.7.5. *Let $r \in (1, \infty)$, $p \in (1, dr/(2r - 1))$ and $\bar{u} \in \dot{B}_{p,r}^{d/p-1}$. Let us suppose that*

$$f_1 \in (L_t^r L_x^{\frac{dr}{3r-2}})^d \cap (L_t^r L_x^{\check{p}})^d, \quad f_2 \in (L_t^{2r} L_x^{\frac{dr}{2r-1}})^{d \times d} \cap (L_t^r L_x^{\frac{dr}{2(r-1)}})^{d \times d},$$

Let v belongs to $L_t^{2r} L_x^{dr/(r-1)}$ with $\nabla v \in L_t^{2r} L_x^{\frac{dr}{2r-1}}$. Then system

$$\begin{cases} \partial_t u^h + v \partial_d u^h - \Delta u^h + \nabla^h \Pi = f_1^h + \operatorname{div} f_2^h & \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t u^d + \nabla^h v \cdot u^h - v \operatorname{div}^h u^h - \Delta u^d + \partial_d \Pi = f_1^d + \operatorname{div} f_2^d & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ u|_{t=0} = \bar{u} & \mathbb{R}^d, \end{cases} \quad (9.85)$$

admits a weak solution (u, Π) , such that u belongs to $L_t^{2r} L_x^{\frac{dr}{r-1}}$ with ∇u in $L_t^{2r} L_x^{\frac{dr}{2r-1}}$ and Π in $L_t^r L_x^{\frac{dr}{2(r-1)}}$.

Proof. For all u in $L_t^{2r} L_x^{dr/(r-1)}$ with $\nabla u \in L_t^{2r} L_x^{dr/(2r-1)}$, let $g(u)$ be defined by

$$g(u) := (-v \partial_d u^h, -\nabla^h v \cdot u^h + v \operatorname{div}^h u^h) \in L_t^r L_x^{\frac{dr}{3r-2}}. \quad (9.86)$$

Then, the momentum equations of (9.85) reads as follows:

$$\partial_t u - \Delta u + \nabla \Pi = g(u) + f_1 + \operatorname{div} f_2 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \quad (9.87)$$

We want to prove the existence of a weak solution for this system, using the Fixed-Point Theorem. We define the functional space Y_r by

$$Y_r := \left\{ u \in L_t^{2r} L_x^{\frac{dr}{r-1}} \quad \text{such that} \quad \nabla u \in L_t^{2r} L_x^{\frac{dr}{2r-1}} \right\},$$

then, fixing a positive constant λ , we consider the norm $\|\cdot\|_\lambda$ on Y_r , defined by

$$\|u\|_\lambda := \|u(t) h_\lambda(0, t)\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} + \|\nabla u(t) h_\lambda(0, t)\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}},$$

where, for all $0 \leq s \leq t \leq \infty$,

$$h_\lambda(s, t) := \exp \left\{ -\lambda \left(\int_s^t \|v(\tau)\|_{L_x^{\frac{dr}{r-1}}}^{2r} + \int_s^t \|\nabla v(\tau)\|_{L_x^{\frac{dr}{r-1}}}^{2r} + \int_s^t \|\nabla v(\tau)\|_{L_x^q}^{2r} \right) \right\} \leq 1. \quad (9.88)$$

Let Ψ be the operator from Y_r to itself, such that, for all $\omega \in Y_r$, $\Psi(\omega)$ is the velocity of the weak solution of

$$\begin{cases} \partial_t u - \Delta u + \nabla \Pi = g(\omega) + f_1 + \operatorname{div} f_2 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} u = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ u|_{t=0} = \bar{u} & \mathbb{R}^d. \end{cases}$$

Let us prove that, for a good choice of λ , Ψ is a contraction on Y_r . First of all, for all $\omega_1, \omega_2 \in Y_r$,

the difference $\delta\Psi := \Psi(\omega_1) - \Psi(\omega_2)$ is the velocity field of the weak solution of

$$\begin{cases} \partial_t \delta\Psi - \Delta \delta\Psi + \nabla \Pi = g(\delta\omega) & \mathbb{R}_+ \times \mathbb{R}^d, \\ \operatorname{div} \delta\Psi = 0 & \mathbb{R}_+ \times \mathbb{R}^d, \\ \delta\Psi|_{t=0} = 0 & \mathbb{R}^d, \end{cases}$$

where $\delta\omega := \omega_1 - \omega_2$. Since the Mild formulation yields

$$\delta\Psi(t) = \int_0^t e^{(t-s)\Delta} \mathbb{P}g(\delta\omega)(s) ds,$$

then, by the definition (9.86) of g , Lemma 9.7.1 and Lemma 9.7.2 the following inequality is fulfilled:

$$\|\delta\Psi\|_\lambda \leq \frac{C}{\lambda^{4r}} \left\{ \|v\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}}^{\frac{1}{2}} \|\delta\nabla\omega(t)h(0,t)\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}} + \|\nabla v\|_{L_t^{2r} L_x^{\frac{dr}{2r-1}}}^{\frac{1}{2}} \|\delta\omega(t)h(0,t)\|_{L_t^{2r} L_x^{\frac{dr}{r-1}}} \right\}.$$

Imposing $\lambda > 0$ big enough we finally obtain $\|\delta\Psi\|_\lambda \leq \|\delta\omega\|_\lambda/2$, namely Ψ is a contraction on Y_r . Then, by the Fixed-Point Theorem, there exists a function u in Y_r such that, u is the velocity field of the weak solution (u, Π) of (9.87). Let us remark that ∇u belongs also to $L_t^r L_x^{dr/(2r-2)}$. Indeed ∇u is formulated by

$$\begin{aligned} \nabla u(t) := & e^{t\Delta} \nabla \bar{u} + \int_0^t \nabla e^{(t-s)\Delta} \mathbb{P}(f_1(s) + g(u)(s)) ds + \\ & - \int_0^t \Delta e^{(t-s)\Delta} RRR \cdot R \cdot f_2(s) ds - \int_0^t \Delta e^{(t-s)\Delta} R \cdot R \cdot f_2(s) ds, \end{aligned}$$

then the result holds thanks to Corollary 9.2.7, Lemma 9.2.4 and Theorem 9.2.2. Finally, recalling that Π is determined by

$$\Pi := -(-\Delta)^{-\frac{1}{2}} R \cdot (f_1 + g(u)) - R \cdot R \cdot f_2,$$

we deduce that Π belongs to $L_t^r L_x^{dr/(2r-2)}$, by Corollary 9.2.1.1. \square

Remark 9.7.6. *If we add a small extra regularity on \bar{u} in Theorem 9.7.5 assuming \bar{u} in $\dot{B}_{p,r}^{d/p-1+\varepsilon}$, with $\varepsilon < \min\{1/(2r), 1 - 1/r, 2(d/p - 2 + 1/r)\}$, the weak solution (u, Π) fulfills also*

$$u \in L_t^{2r} L_x^{\frac{2dr}{(2-\varepsilon)r-2}} \cap L_t^{\frac{4r}{2-\varepsilon}} L_x^{\frac{dr}{r-1}} \quad \text{with} \quad \nabla u \in L_t^{2r} L_x^{\frac{2dr}{(4-\varepsilon)r-2}} \cap L_t^{\frac{4r}{2-\varepsilon}} L_x^{\frac{dr}{2r-1}}.$$

Part III

Appendix

Chapter 10

Appendix

10.1 Relation between the Oseen-Frank energy density and the Gibbs free energy density

In section 3.2.2 we showed the explicit formula of the Oseen-Frank energy density, that is

$$w_F(d, \nabla d) = \frac{k_{11}}{2}(\operatorname{div} d)^2 + \frac{k_{22}}{2}(d \cdot \operatorname{rot} d)^2 + \frac{k_{33}}{2}|d \wedge \operatorname{rot} d|^2 + \frac{k_{22} + k_{24}}{2}\{\operatorname{tr}\{\nabla d^2\} - (\operatorname{div} d)^2\}. \quad (10.1)$$

We justified this expression as a consequence of the Gibbs free energy density

$$w_F = k_1(s_1 + s_2) + k_2(t_1 + t_2) + \frac{k_{11}}{2}(s_1 + s_2)^2 + \frac{k_{22}}{2}(t_1 + t_2)^2 + \frac{k_{33}}{2}(b_1^2 + b_2^2) + k_{12}(s_1 + s_2)(t_1 + t_2) - (k_{22} + k_{24})(s_1 s_2 + t_1 t_2), \quad (10.2)$$

evolving the six components of curvature (3.8). Now, we want to give an exhaustive proof of the equivalence between (10.1) and (10.2). First, let us recall that in section 3.2.2 we introduced a local right-handed Cartesian coordinate system $y := (y_1, y_2, y_3)$, centred at a point ζ , as depicted in figure Figure 3.2. We also imposed y_3 parallel to $d(\zeta)$ and we denoted by \tilde{d} the director d under the new coordinates. From (3.7) and (3.8) we get

$$\begin{aligned} \operatorname{div}_y \tilde{d}(0) &= \frac{\partial \tilde{d}_1}{\partial y_1}(0) + \frac{\partial \tilde{d}_2}{\partial y_2}(0) = s_1 + s_2, \\ \tilde{d}(0) \cdot \operatorname{rot}_y \tilde{d}(0) &= \frac{\partial \tilde{d}_2}{\partial y_1}(0) - \frac{\partial \tilde{d}_1}{\partial y_2}(0) = -(t_1 + t_2), \\ |\tilde{d}(0) \wedge \operatorname{rot}_y \tilde{d}(0)|^2 &= \left(\frac{\partial \tilde{d}_1}{\partial y_3}(0)\right)^2 + \left(\frac{\partial \tilde{d}_2}{\partial y_3}(0)\right)^2 = b_1^2 + b_2^2, \end{aligned} \quad (10.3)$$

with also

$$[\operatorname{tr}\{(\nabla_y \tilde{d})^2\} - (\operatorname{div}_y \tilde{d})^2](0) = \left[2 \frac{\partial d_1}{\partial y_2} \frac{\partial d_2}{\partial y_1} - 2 \frac{\partial d_1}{\partial y_1} \frac{\partial d_2}{\partial y_2}\right](0) = -2(t_1 t_2 + s_1 s_2). \quad (10.4)$$

Thus (10.1) is satisfied replacing d by \tilde{d} and deriving under the new coordinates (y_1, y_2, y_3) . Now, we claim that the four terms (10.3) and (10.4) does not depend on the considered change of variables, namely

$$\begin{aligned} \operatorname{div}_y \tilde{d}(0) &= \operatorname{div}_x d(\zeta), & \tilde{d}(0) \cdot \operatorname{rot}_y \tilde{d}(0) &= d(\zeta) \cdot \operatorname{rot}_x d(\zeta), \\ |\tilde{d}(0) \wedge \operatorname{rot}_y \tilde{d}(0)|^2 &= |d(\zeta) \wedge \operatorname{rot}_x d(\zeta)|^2, & \operatorname{tr}\{(\nabla_y \tilde{d}(0))^2\} &= \operatorname{tr}\{(\nabla_x d(\zeta))^2\}. \end{aligned} \quad (10.5)$$

The first equality turns out from

$$\begin{aligned} \operatorname{div}_y \tilde{d}(0) &= \sum_{i=1}^3 \frac{\partial \tilde{d}_i}{\partial y_i}(0) = \sum_{i,j,k}^3 R_{ij} \frac{\partial}{\partial y_i} \left[d_j(\zeta + {}^t R y) \right](0) \\ &= \sum_{i,j,k}^3 R_{ij} {}^t R_{ki} \frac{\partial d_j}{\partial x_k}(\zeta) = \sum_{j,k=1}^3 \delta_{jk} \frac{\partial d_j}{\partial x_k}(\zeta) = \operatorname{div}_x d(\zeta), \end{aligned}$$

while the second equality is achieved by

$$\begin{aligned} \tilde{d}(0) \cdot \operatorname{rot}_y \tilde{d}(0) &= \sum_{r,k,l=1}^3 \left(\sum_{s,i,j=1}^3 \epsilon_{sij} R_{sr} R_{ik} {}^t R_{lj} \right) d_r(\zeta) \frac{\partial d_k}{\partial x_l}(\zeta) \\ &= \sum_{r,k,l=1}^3 \epsilon_{rkl} d_r(\zeta) \frac{\partial d_k}{\partial x_l}(\zeta) = d(\zeta) \cdot \operatorname{rot}_x d(\zeta), \end{aligned}$$

where we have used the following identity

$$\sum_{s,i,j=1}^3 \epsilon_{sij} R_{sr} R_{ik} R_{jl} = \epsilon_{rkl}.$$

Now, in order to prove the third equality, we make use of the constraint $|d| = 1$ through

$$d \wedge \operatorname{rot}_x d = \nabla_x (|d|^2) - d \cdot \nabla_x d = -d \cdot \nabla_x d.$$

The same identity is satisfied for \tilde{d} in the y -coordinates, hence the third identity is equivalent to

$$\begin{aligned} |\tilde{d}(0) \cdot \nabla_y \tilde{d}(0)|^2 &= \sum_{i=1}^3 \left| \sum_{j=1}^3 \tilde{d}_j(0) \frac{\partial \tilde{d}_i}{\partial y_j}(0) \right|^2 = \sum_{i=1}^3 \left| \sum_{j,s,l,k=1}^3 R_{js} d_s(\zeta) R_{ik} {}^t R_{lj} \frac{\partial d_k}{\partial x_l}(\zeta) \right|^2 \\ &= \sum_{i=1}^3 \left| \sum_{s,l,k=1}^3 R_{ik} \delta_{ls} d_s(\zeta) \frac{\partial d_k}{\partial x_l}(\zeta) \right|^2 = \sum_{i=1}^3 \left| \sum_{l,k=1}^3 R_{ik} d_l(\zeta) \frac{\partial d_k}{\partial x_l}(\zeta) \right|^2 \\ &= \sum_{i=1}^3 |(R \{d(\zeta) \cdot \nabla d(\zeta)\})_i|^2 = |R \{d(\zeta) \cdot \nabla d(\zeta)\}|^2 = |d(\zeta) \cdot \nabla d(\zeta)|^2. \end{aligned}$$

Lastly, we achieve the fourth identity as follows:

$$\begin{aligned} \operatorname{tr}\{(\nabla_y \tilde{d}(0))^2\} &= \sum_{i,j=1}^3 \frac{\partial \tilde{d}_i}{\partial y_j}(0) \frac{\partial \tilde{d}_j}{\partial y_i}(0) = \sum_{i,j,k,l,r,s=1}^3 R_{ij} {}^t R_{lj} \frac{\partial d_k}{\partial x_l}(\zeta) R_{jr} {}^t R_{si} \frac{\partial d_r}{\partial x_s}(\zeta) \\ &= \sum_{k,l,r,s=1}^3 \delta_{sk} \delta_{lr} \frac{\partial d_k}{\partial x_l}(\zeta) \frac{\partial d_r}{\partial x_s}(\zeta) = \sum_{k,l=1}^3 \delta_{sk} \delta_{lr} \frac{\partial d_k}{\partial x_l}(\zeta) \frac{\partial d_l}{\partial x_k}(\zeta) = \operatorname{tr}\{(\nabla_x d(\zeta))^2\}. \end{aligned}$$

Thus, summarizing the above considerations, we can finally conclude that (10.1) and (10.2) stand for the same energy density.

10.2 The Oseen-Frank identity

In this section we prove a useful tensor identity for the Oseen-Frank energy density w_F (3.9). It reads as follows:

$$\mathcal{I}_s := \epsilon_{isl} \left(d_l \frac{\partial w_F}{\partial d_i} + \frac{\partial w_F}{\partial d_{i,j}} d_{l,j} + \frac{\partial w_F}{\partial d_{j,i}} d_{j,l} \right) = 0, \quad (10.6)$$

for any $s = 1, 2, 3$. First, let us remark that from the identity

$$d \wedge \text{rot } d = \nabla(d \cdot d) - d \cdot \nabla d = -d \cdot \nabla d,$$

w_F can be expressed in summation form as follows:

$$\begin{aligned} w_F(d, \nabla d) &= \\ &= \frac{k_{11}}{2} (d_{i,i})^2 + \frac{k_{22}}{2} (d_i \epsilon_{ijk} n_{k,j})^2 + \frac{k_{33}}{2} d_j d_{i,j} d_k d_{i,k} + \frac{k_{22} + k_{24}}{2} (d_{i,j} d_{j,i} - (d_{i,i})^2) \\ &= \frac{k_{11} - k_{22} - k_{24}}{2} (d_{i,i})^2 + \frac{k_{22}}{2} \left(d_{k,j} d_{k,j} - d_{k,j} d_{j,k} - d_i d_j d_{k,j} d_{k,i} \right) + \\ &\quad + \frac{k_{33}}{2} d_j d_{i,j} d_k d_{i,k} + \frac{k_{22} + k_{24}}{2} d_{i,j} d_{j,i} \\ &= \frac{k_{11} - k_{22} - k_{24}}{2} (d_{i,i})^2 + \frac{k_{22}}{2} d_{i,j} d_{i,j} + \frac{k_{24}}{2} d_{i,j} d_{j,i} + \frac{k_{33} - k_{22}}{2} d_i d_j d_{k,j} d_{k,i}. \end{aligned} \quad (10.7)$$

Thus, we can compute each derivative of w_F , obtaining

$$\begin{aligned} \frac{\partial w_F}{\partial d_i} &= (k_{33} - k_{22}) d_j d_{k,j} d_{k,i}, \\ \frac{\partial w_F}{\partial d_{i,j}} &= (k_{11} - k_{22} - k_{24}) \delta_{ij} d_{k,k} + k_{22} d_{i,j} + k_{24} d_{j,i} + (k_{33} - k_{22}) d_k d_j d_{i,k}. \end{aligned}$$

Hence, applying this result to (10.6) yields that

$$\begin{aligned} \mathcal{I}_s &= \epsilon_{isl} \left\{ (k_{33} - k_{22}) d_j d_{k,j} d_{k,i} d_l + [(k_{11} - k_{22} - k_{24}) \delta_{ij} d_{k,k} d_{l,j} + k_{22} d_{i,j} + k_{24} d_{j,i} + \right. \\ &\quad \left. (k_{33} - k_{22}) d_k d_j d_{i,k}] d_{l,j} + [(k_{11} - k_{22} - k_{24}) \delta_{ji} d_{k,k} d_{l,j} + k_{22} d_{j,i} + k_{24} d_{i,j} + \right. \\ &\quad \left. (k_{33} - k_{22}) d_k d_i d_{j,k}] d_{j,l} \right\} = (k_{33} - k_{22}) \underbrace{\epsilon_{isl} [d_j d_{k,j} d_{k,i} d_l + d_k d_j d_{i,k} d_{l,j} + d_i d_k d_{j,k} d_{j,l}]}_{T_{il}^1} + \\ &\quad \underbrace{(k_{11} - k_{22} - k_{24}) d_{k,k} \epsilon_{isl} [d_{l,i} + d_{i,l}]}_{T_{il}^2} + k_{22} \underbrace{\epsilon_{isl} [d_{i,j} d_{l,j} + d_{j,i} d_{j,l}]}_{T_{il}^3} + k_{24} \underbrace{\epsilon_{isl} [d_{j,i} d_{l,j} + d_{i,j} d_{j,l}]}_{T_{il}^4}. \end{aligned}$$

Each tensor T_{il}^k , for $k = 1, 2, 3, 4$, is symmetric on (i, l) , thus we deduce that $\epsilon_{isl} T_{il}^k$ is null, for any $s = 1, 2, 3$, which yields (10.6).

10.3 Some specifics about the Ericksen-Leslie system

In section 3.2.4 we presented the general Ericksen-Leslie system, deriving its formulation from the balance of the linear momentum (3.2), the balance of angular momentum (3.12), the conservation of mass (3.1) and finally the work postulate (3.16). In this section we give some more specifics for the interested reader.

First, recalling that the tensor formulations of the surface force σ_i and the surface moment l_i

are $\sigma_{ij}\nu_j$ and $l_{ij}\nu_j$, with ν the normal to the boundary ∂U , then the work postulate (3.16) becomes

$$\int_U \rho(F \cdot u + K \cdot w) dx + \underbrace{\int_{\partial U} (\sigma \cdot u + l \cdot w) d\nu}_{\mathcal{I}} = \underbrace{\frac{D}{Dt} \int_U \left(\frac{1}{2} \rho |u|^2 + w_F \right) dx}_{\mathcal{II}} + \int_U \mathcal{D} dx. \quad (10.8)$$

Moreover, thanks to the Green's theorem, we can develop \mathcal{I} into

$$\begin{aligned} \mathcal{I} &= \int_{\partial U} (\sigma_{ij}u_i + l_{ij}w_i)\nu_j = \int_U (\sigma_{ij}u_i + l_{ij}w_i)_{,j} dx \\ &= \int_U (\sigma_{ij,j}u_i + \sigma_{ij}u_{i,j} + l_{ij,j}w_i + l_{ij}w_{i,j}) dx, \end{aligned}$$

hence, recalling (3.4) and (3.14), we achieve

$$\mathcal{I} = \int_U \left(\rho \dot{u}_i u_i - \rho F_i u_i + \sigma_{ij} u_{i,j} - \rho K_i w_i - \epsilon_{ijk} \sigma_{jk} w_i + l_{ij} w_{i,j} \right) dx,$$

while the Reynolds' transport theorem allows to express \mathcal{II} as follows

$$\mathcal{II} = \int_U \left(\frac{D}{Dt} \left[\rho \frac{|u|^2}{2} \right] + \dot{w}_F \right) dx = \int_U \left(\dot{\rho} \frac{|u|^2}{2} + \rho \dot{u} \cdot u + \dot{w}_F \right) dx = \int_U \left(\rho \dot{u} \cdot u + \dot{w}_F \right) dx.$$

Substituting the terms \mathcal{I} and \mathcal{II} and from the arbitrariness of the domain U we can write (10.8) in point form as follows

$$\sigma_{ij} u_{i,j} + l_{ij} w_{i,j} - \epsilon_{ijk} \sigma_{jk} w_i = \dot{w}_F + \mathcal{D}. \quad (10.9)$$

Now, we analyse the material derivative of the Oseen-Frank energy density \dot{w}_F . First, we recall that from the explicit formula (3.9), w_F depends only on d and ∇d . Thus, its material derivative fulfils

$$\dot{w}_F = \frac{\partial w_F}{\partial d_i} \dot{d}_i + \frac{\partial w_F}{\partial d_{i,j}} \frac{D}{Dt} [d_{i,j}].$$

The definition of material derivative implies

$$\frac{D}{Dt} [d_{i,j}] = \partial_t d_{i,j} + u_k \partial_{k,j}^2 d_i = \partial_j (\partial_t d_i + u_k \partial_k d_i) - u_{k,j} d_{i,k} = (\dot{d}_i)_{,j} - u_{k,j} d_{i,k},$$

so that, \dot{w}_F is determined by

$$\begin{aligned} \dot{w}_F &= \frac{\partial w_F}{\partial d_i} \dot{d}_i + \frac{\partial w_F}{\partial d_{i,j}} ((\dot{d}_i)_{,j} - u_{k,j} d_{i,k}) \\ &= \frac{\partial w_F}{\partial d_i} \epsilon_{isl} w_s d_l + \frac{\partial w_F}{\partial d_{i,j}} (\epsilon_{isl} w_s d_l)_{,j} - \frac{\partial w_F}{\partial d_{i,j}} u_{k,j} d_{i,k} \\ &= \frac{\partial w_F}{\partial d_i} \epsilon_{isl} w_s d_l + \frac{\partial w_F}{\partial d_{i,j}} \epsilon_{isl} w_{s,j} d_l + \frac{\partial w_F}{\partial d_{i,j}} \epsilon_{isl} w_s d_{l,j} - \frac{\partial w_F}{\partial d_{i,j}} u_{k,j} d_{i,k} \\ &= \epsilon_{isl} \left[w_s \left(\frac{\partial w_F}{\partial d_i} d_l + \frac{\partial w_F}{\partial d_{i,j}} d_{l,j} \right) + \frac{\partial w_F}{\partial d_{i,j}} w_{s,j} d_l \right] - \frac{\partial w_F}{\partial d_{i,j}} u_{k,j} d_{i,k}. \end{aligned}$$

Making use of the following identity concerning the material derivative \dot{w}_F

$$\mathcal{I}_s := \epsilon_{isl} \left(d_l \frac{\partial w_F}{\partial d_i} + \frac{\partial w_F}{\partial d_{i,j}} d_{l,j} + \frac{\partial w_F}{\partial d_{j,i}} d_{j,l} \right) = 0, \quad (10.10)$$

for any $s = 1, 2, 3$ (see the appendix for more details), we achieve that \dot{w}_F can be expressed as follows

$$\dot{w}_F = \epsilon_{isl} \left[\frac{\partial w_F}{\partial d_{i,j}} w_{s,j} d_l - w_s \frac{\partial w_F}{\partial d_{j,i}} d_{j,l} \right] - \frac{\partial w_F}{\partial d_{i,j}} u_{k,j} d_{i,k}.$$

Then, the balance law (10.9) becomes

$$\sigma_{ij} u_{i,j} + l_{ij} w_{i,j} - \epsilon_{ijk} \sigma_{jk} w_i = \epsilon_{isl} \left[\frac{\partial w_F}{\partial d_{i,j}} w_{s,j} d_l - w_s \frac{\partial w_F}{\partial d_{j,i}} d_{j,l} \right] - \frac{\partial w_F}{\partial d_{i,j}} u_{k,j} d_{i,k} + \mathcal{D},$$

namely

$$u_{i,j} \left(\sigma_{ij} + \frac{\partial w_F}{\partial d_{k,j}} d_{k,i} \right) + w_{i,j} \left(l_{ij} - \epsilon_{isl} d_s \frac{\partial w_F}{\partial d_{l,j}} \right) - w_i \epsilon_{isl} \left(\sigma_{sl} - \frac{\partial w_F}{\partial d_{k,l}} d_{k,s} \right) = \mathcal{D}. \quad (10.11)$$

As exposed by Leslie in [67], the rate of dissipation \mathcal{D} is necessarily positive, thus the sign arbitrariness of w_i , $w_{i,j}$ and $u_{i,j}$ leads to the conclusion that the stress tensor σ_{ij} and the couple stress l_{ij} can be expressed as follows:

$$\sigma_{ij} = -p \delta_{ij} - \frac{\partial w_F}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij}, \quad (10.12)$$

$$l_{ij} = \epsilon_{isl} d_s \frac{\partial w_F}{\partial d_{l,j}} + \tilde{l}_{ij}. \quad (10.13)$$

Here p is an arbitrary pressure which arises from the incompressibility condition, while $\tilde{\sigma}$ and \tilde{l} are dynamic contribution. The tensor $\tilde{\sigma}_{ij}$ is known as the *Leslie stress tensor* or the *viscous stress*. Moreover (10.11) reduces on a constriction for the dynamic contributions, namely

$$\tilde{\sigma}_{ij} u_{i,j} + \tilde{l}_{i,j} w_{i,j} + w_i \epsilon_{ijk} \tilde{\sigma}_{kj} = \mathcal{D} \geq 0. \quad (10.14)$$

The Ericksen-Leslie equations

In order to proceed further, we have to analyse the dynamic contributions $\tilde{\sigma}_{ij}$ and \tilde{l}_{ij} to the total stress tensor σ_{ij} and the couple stress tensor l_{ij} . This contribution is going to be expressed as relations between the stresses and the dynamic of the material, the so called *constitutive relations*.

First, we assume that the dynamic contributions can be formulated as functions of the director d , the gradient of the velocity field ∇u and the local angular velocity of the director w . Then inequality (10.14) presents a linear term on \tilde{l}_{ij} , since there is no dependence on the gradient ∇w . Assuming sign-arbitrariness of $w_{i,j}$, we achieve that there is no dynamic contribution to the couple stress tensor, that is

$$\tilde{l}_{ij} = 0.$$

Thus the dynamic contribution reduces to the viscous stress $\tilde{\sigma}_{ij}$ and inequality (10.14) becomes

$$\tilde{\sigma}_{ij} u_{i,j} + w_i \epsilon_{ijk} \tilde{\sigma}_{kj} = \mathcal{D} \geq 0. \quad (10.15)$$

Now, we also assume the dynamic contributions to be invariant under a rigid body motion, which leads (see for instance [107]) the viscous stress $\tilde{\sigma}_{i,j}$ to be dependent on d , w and the rate of strain tensor A , given by (3.15). Further developments allow us to formulate the viscous stress $\tilde{\sigma}_{ij}$ in its most widely adopted and well known form

$$\tilde{\sigma}_{ij} = \alpha_1 A_{lk} d_k d_l d_i d_j + \alpha_2 d_i N_j + \alpha_3 N_i d_j + \alpha_4 A_{ij} + \alpha_5 d_j A_{ik} d_k + \alpha_6 d_i A_{jk} d_k. \quad (10.16)$$

Here, the coefficients $\alpha_1, \dots, \alpha_6$ are known as the *Leslie viscosity coefficients*. Inequality (10.15) yields the following restriction (see [107], section 4.2.3):

$$\begin{aligned}\gamma_1 &:= \alpha_3 - \alpha_2 \geq 0, \\ \alpha_4 &\geq 0, \\ 2\alpha_4 + \alpha_5 + \alpha_6 &\geq 0, \\ 2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 &\geq 0, \\ 4\gamma_1(2\alpha_4 + \alpha_5 + \alpha_6) &\geq (\alpha_2 + \alpha_3 + \gamma_2)^2.\end{aligned}$$

We are now able to perform the dynamic equations, coming back to the balance laws of linear momentum (3.4) and angular momentum (3.14). Replacing the total stress tensor σ_{ij} (10.12) and the couple stress tensor (10.13), the dynamic equations become

$$\rho(u_{i,t} + u_k u_{i,k}) + \nabla p = \rho F_i + \left[\frac{\partial w_F}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij} \right]_{,j}, \quad (10.17)$$

$$\rho K_i + \epsilon_{isl} \left[\tilde{\sigma}_{sl} + \left(d_s \frac{\partial w_F}{\partial d_{l,j}} \right)_{,j} - \frac{\partial w_F}{\partial d_{l,j}} d_{k,l} \right] = 0. \quad (10.18)$$

Now, let us remark that

$$\begin{aligned}\epsilon_{isl} \tilde{\sigma}_{sl} &= \underbrace{\alpha_1 \epsilon_{isl} A_{rk} d_k d_r d_s d_l}_{=0} + \alpha_2 \epsilon_{isl} d_s N_l - \alpha_3 \epsilon_{isl} N_l d_s + \underbrace{\alpha_4 \epsilon_{isl} A_{sl}}_{=0} + \\ &\quad + \alpha_5 \epsilon_{isl} d_l A_{sk} d_k - \alpha_6 \epsilon_{isl} d_l A_{sk} d_k,\end{aligned}$$

hence, denoting with $\gamma_1 := \alpha_3 - \alpha_2$, $\gamma_2 := \alpha_6 - \alpha_5$ and defining $\tilde{g} = -\gamma_1 N - \gamma_2 A n$, we get

$$\epsilon_{isl} \tilde{\sigma}_{sl} = \epsilon_{isl} d_s \tilde{g}_l = (d \wedge \tilde{g})_i.$$

Finally, recalling also the identity (10.10), the angular momentum equation (10.18) assumes the following form

$$\rho K_i + \epsilon_{isl} d_s \left[\tilde{g}_l + \left(\frac{\partial w_F}{\partial d_{l,j}} \right)_{,j} - \frac{\partial w_F}{\partial d_l} \right] = 0. \quad (10.19)$$

If we assume that the external body force K can be expressed as $\rho K = d \wedge G$, with G the so-called *generalized body force*, then (10.18) becomes

$$\epsilon_{isl} d_s \left[G_l + \tilde{g}_l + \left(\frac{\partial w_F}{\partial d_{l,j}} \right)_{,j} - \frac{\partial w_F}{\partial d_l} \right] = 0,$$

or in a no-index form

$$d \wedge \left[G + \tilde{g} + \operatorname{div} \left\{ \frac{\partial w_F}{\partial \nabla d} \right\} - \frac{\partial w_F}{\partial d} \right] = 0.$$

As suggested by Ericksen [41], it is natural assuming F and G to be produced by gravitational or electromagnetic fields, however in this thesis we are going to ignore these contributions. Summarizing all the previous considerations, we can finally perform the well-known Ericksen-Leslie

equations, which reads as follows

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) = \operatorname{div} \sigma, \\ \operatorname{div} u = 0, \\ d \wedge (\tilde{g} + h) = 0, \\ |d|^2 = 1 \end{cases} \quad (10.20)$$

where the total stress tensor σ is given by

$$\sigma_{ij} = -p\delta_{ij} - \frac{\partial w_F}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij}$$

with the viscous stress $\tilde{\sigma}$ determined by (3.17), and also denoting by h the *molecular field*

$$h = \operatorname{div} \left\{ \frac{\partial w_F}{\partial \nabla d} \right\} - \frac{\partial w_F}{\partial d}.$$

10.4 Uniaxial behaviour of the Qian-Sheng system

In this section we show the main correlation between the Ericksen-Leslie and the Qian-Sheng theories. The Ericksen-Leslie system is the most widespread model for the hydrodynamics of *uniaxial* nematic materials, while the Qian-Sheng one makes use of the de Gennes order tensor that also describes the evolution of *biaxial* nematic liquid crystals. Nevertheless, the Qian-Sheng model should be seen as an extension of the Ericksen-Leslie one: we are going to prove that if we restrict the order tensor to be uniaxial, as in (10.22), then the two theories coincide.

We begin our proof showing that the bulk energy $\psi_B(Q)$ (3.25) is null when the order tensor Q assumes an uniaxial form.

Lemma 10.4.1. *Let Q be an uniaxial order tensor*

$$Q = s \left(d \otimes d - \frac{\operatorname{Id}}{3} \right), \quad \text{such that} \quad s = \frac{b \pm \sqrt{b^2 - 24ac}}{4c} \quad (10.21)$$

where d is a unit vector in $\mathbb{S}^2 \subset \mathbb{R}^3$ and a , b and c are the constants appearing in the bulk energy density ψ_B (3.25). Then

$$\mathcal{L} \frac{\partial \psi_B(Q)}{\partial Q} = aQ - b \left(Q^2 - \operatorname{tr}\{Q^2\} \frac{\operatorname{Id}}{3} \right) + cQ \operatorname{tr}\{Q^2\} = 0.$$

Proof. As Q assumes the uniaxial form (10.21), we have that

$$Q^2 = s^2 \left[(d \otimes d)(d \otimes d) - \frac{2}{3} d \otimes d + \frac{\operatorname{Id}}{9} \right] = \frac{s^2}{3} \left(d \otimes d + \frac{\operatorname{Id}}{3} \right) \quad \text{and} \quad \operatorname{tr}\{Q^2\} = \frac{2s^2}{3}.$$

Thus the projection through \mathcal{L} of the variational derivative of $\psi_B(Q)$ assumes the following form:

$$\begin{aligned} \mathcal{L} \frac{\partial \psi_B(Q)}{\partial Q} &= as \left(d \otimes d - \frac{\operatorname{Id}}{3} \right) - b \left[\frac{s^2}{3} \left(d \otimes d + \frac{\operatorname{Id}}{3} \right) - \frac{2s^2}{9} \operatorname{Id} \right] + \frac{2cs^3}{3} \left(d \otimes d - \frac{\operatorname{Id}}{3} \right) \\ &= \left(a - \frac{bs}{3} + \frac{2cs^2}{3} \right) s \left(d \otimes d - \frac{\operatorname{Id}}{3} \right) = \frac{1}{3} (3a - bs + 2cs^2) Q \end{aligned}$$

Since the values of s in (10.21) are the zeros of $3a - bs + 2cs^2$, then the result follows. \square

We are now ready to prove our main result, which reads as follows:

Proposition 10.4.2. *Let (u, Q) be a smooth solution of the Qian-Sheng system (3.37) with inertial density $J = 0$. Let us assume that the order tensor $Q(t, x)$ is uniaxial at every point x and for any time t ,*

$$Q(t, x) = s \left(d(t, x) \otimes d(t, x) - \frac{Id}{3} \right). \quad (10.22)$$

where $d(t, x)$ is a smooth function which returns value into the sphere \mathbb{S}^2 in \mathbb{R}^3 and with the constant s given by

$$s = \frac{b - \sqrt{b^2 - 24ac}}{4c}.$$

Then the couple (u, d) is solution of the general Ericksen-Leslie system (3.18), for suitable value of the Leslie viscosity coefficients.

Proof. We begin analysing the uniaxiality of Q (10.22) together with the order tensor equation

$$\mu_1 (\dot{Q} - [\Omega, Q]) - \Delta Q = -\mathcal{L} \frac{\partial \psi_B(Q)}{\partial Q} + \mu_2 A.$$

We claim that these lead to the angular momentum equation of the Ericksen-Leslie system

$$d \wedge (-\gamma_1 \mathcal{N} - \gamma_2 Ad + h) = 0, \quad (10.23)$$

for a suitable value of coefficients γ_1 and γ_2 , where we recall that \mathcal{N} and h stand for the co-rotational time flux of d and the molecular field respectively, whose formulas are given by

$$\mathcal{N}_i = \dot{d}_i - (\Omega d)_i = \dot{d}_i - \frac{u_{i,j} - u_{j,i}}{2} d_j, \quad \text{and} \quad h_i = \left(\frac{\partial w_F(d, \nabla d)}{\partial d_{i,j}} \right)_{,j} - \frac{\partial w_F(d, \nabla d)}{\partial d_i}.$$

For the sake of simplicity, in what follows we consider the simplest form of the Oseen Frank energy density, that is

$$w_F(d, \nabla d) = w_F(\nabla d) = \frac{|\nabla d|^2}{2},$$

so that the elastic field h in (10.23) reads as $h = \Delta d$. In this form, the density w_F is of the same shape with respect to the elastic energy density (3.29) for the de Gennes tensor Q , i.e.

$$\psi_e(Q) = \frac{|\nabla Q|^2}{2}.$$

We leave to the interested reader the task of formulating analogous results when the general case w_F and ψ_e given by (3.9) and (3.26) respectively, occurs.

Lemma 10.4.1 and the uniaxiality of Q (10.21) yield the order tensor equation to read as

$$\mu_1 (\dot{Q} - [\Omega, Q]) - \Delta Q = \mu_2 A, \quad (10.24)$$

that is

$$\mu_1 s^2 (\dot{d} \otimes d + d \otimes \dot{d} - [\Omega, (d \otimes d)]) - s^2 \Delta (d \otimes d) = \mu_2 A.$$

Now, let us remark that for any $i, j \in \{1, 2, 3\}$

$$\begin{aligned} [\Omega, (d \otimes d)]_{ij} &= (\Omega(d \otimes d) - (d \otimes d)\Omega)_{ij} \\ &= (\Omega_{ik} d_k) d_j - d_i (d_k \Omega_{kj}) = (d \otimes (\Omega d) + (\Omega d) \otimes d)_{ij}, \end{aligned}$$

hence

$$\dot{Q} - [\Omega, Q] = s (\dot{d} \otimes d + d \otimes \dot{d} - [\Omega, (d \otimes d)]) = s \mathcal{N} \otimes d + s d \otimes \mathcal{N}. \quad (10.25)$$

Now

$$\Delta (d \otimes d)_{ij} = \Delta d_i d_j + d_i \Delta d_j + 2d_{i,k} d_{j,k},$$

So that (10.24) in index form reads as

$$\mu_1 s^2 (\mathcal{N}_i d_j + d_i \mathcal{N}_j) - s^2 (\Delta d_i d_j + d_i \Delta d_j + 2d_{i,k} d_{j,k}) = \mu_2 A_{ij}.$$

Multiplying both the left-hand and right-hand sides by d_i and taking the sum in i , we get

$$\mu_1 s^2 (\mathcal{N}_i d_i d_j + \mathcal{N}_j) - s^2 (\Delta d_i d_i d_j + \Delta d_j + 2 \underbrace{d_{i,k} d_i}_{=0} d_{j,k}) = \mu_2 A_{ij} d_i,$$

where we have used $d_{i,k} d_i = \partial_k |d|^2 / 2 = 0$. Thus (10.24) becomes

$$\mu_1 s^2 \mathcal{N} - s^2 \Delta d - \mu_2 A d = -\alpha d$$

where $\alpha = (\mu_1 s^2 \mathcal{N} - s^2 \Delta d) \cdot d$. Taking the wedge product between the above equation and d , we finally obtain

$$(\mu_1 s^2 \mathcal{N} - s^2 \Delta d - \mu_2 A d) \wedge d = 0,$$

that is the angular momentum equation (10.23), imposing $\gamma_1 = \mu_1$ and $\gamma_2 = -\mu_2 / s^2$.

Now, we handle the linear momentum equation for the velocity field u , that is

$$\begin{aligned} u_t + u \cdot \nabla u - \frac{\beta_4}{2} \Delta u + \nabla p = \operatorname{div} \{ -\nabla Q \odot \nabla Q + \beta_1 Q \operatorname{tr}\{QA\} + \beta_5 A Q + \beta_6 Q A \} + \\ + \operatorname{div} \{ \frac{\mu_2}{2} (\dot{Q} - [\Omega, Q]) + \mu_1 [Q, \dot{Q} - [\Omega, Q]] \}. \end{aligned}$$

and we claim that under the uniaxial condition (10.22) it corresponds to the linear momentum equation of the Ericksen-Leslie system (3.18), i.e.

$$u_t + u \cdot \nabla u = \operatorname{div} \sigma.$$

We begin analysing each stress tensor on the right-hand side, first

$$\begin{aligned} (\nabla Q \odot \nabla Q)_{ij} &= s^2 (\nabla(d \otimes d) \odot \nabla(d \otimes d))_{ij} = s^2 (d_{\alpha} d_{\beta})_{,i} (d_{\alpha} d_{\beta})_{,j} \\ &= s^2 d_{\alpha,i} d_{\beta} d_{\alpha,j} d_{\beta} + s^2 d_{\alpha} d_{\beta,i} d_{\alpha,j} d_{\beta} + s^2 d_{\alpha,i} d_{\beta} d_{\alpha,j} d_{\beta} + s^2 d_{\alpha,i} d_{\beta} d_{\alpha} d_{\beta,j} \\ &= 2s^2 d_{\alpha,i} d_{\alpha,j} = s^2 (\nabla d \odot \nabla d)_{ij}, \end{aligned}$$

where we have used $d_{\alpha,i} d_{\alpha} = d_{\beta,i} d_{\beta} = 0$. Thus, recalling that the total stress tensor σ (10.12) is given by

$$\sigma_{ij} = -p \delta_{ij} - \frac{\partial w_F}{\partial d_{k,j}} d_{k,i} + \tilde{\sigma}_{ij} = -p \delta_{ij} - (\nabla d \odot \nabla d)_{ij} + \tilde{\sigma}_{ij},$$

it remains to identify the Leslie viscous stress $\tilde{\sigma}$ (10.16), given by

$$\tilde{\sigma}_{ij} = \alpha_1 A_{lk} d_k d_l d_i d_j + \alpha_2 d_i \mathcal{N}_j + \alpha_3 \mathcal{N}_i d_j + \alpha_4 A_{ij} + \alpha_5 d_j A_{ik} d_k + \alpha_6 d_i A_{jk} d_k. \quad (10.26)$$

First, we observe that

$$\beta_1 Q_{ij} \operatorname{tr}\{QA\} = \beta_1 s^2 (d_i d_j - \frac{\delta_{ij}}{3}) A_{lk} d_l d_k,$$

$$\beta_5 (AQ)_{ij} + \beta_6 (QA)_{ij} = \beta_5 s A_{ik} d_k d_j + \beta_6 s A_{jk} d_k d_i + \frac{\beta_5 + \beta_6}{3} s A_{ij},$$

moreover, recalling (10.25),

$$\begin{aligned} \frac{\mu_2}{2} (\dot{Q} - [\Omega, Q])_{ij} &= \frac{\mu_2}{2} s (d_i \mathcal{N}_j + \mathcal{N}_i d_j), \\ \mu_1 [Q, \dot{Q} - [\Omega, Q]]_{ij} &= \mu_1 s^2 [d \otimes d, d \otimes \mathcal{N} + \mathcal{N} \otimes d]_{ij} \\ &= \mu_1 s^2 (d_i \mathcal{N}_j + d_i d_k \mathcal{N}_k d_j - d_i \mathcal{N}_k d_k d_j - \mathcal{N}_i d_j) = \mu_1 s^2 (d_i \mathcal{N}_j - \mathcal{N}_i d_j), \end{aligned}$$

where we have used $d \cdot \mathcal{N} = d \cdot (\dot{d} - \Omega d) = 0$. Hence, summarizing the previous considerations, imposing

$$\begin{aligned} \alpha_1 &:= \beta_1 s^2, & \alpha_2 &:= \mu_1 s^2 + \frac{\mu_2}{2} s, & \alpha_3 &:= -\mu_1 s^2 + \frac{\mu_2}{2} s, \\ \alpha_4 &:= \beta_4 + \frac{\beta_5 + \beta_6}{3} s, & \alpha_5 &:= \beta_5 s, & \alpha_6 &:= \beta_6 s. \end{aligned}$$

and considering the new pressure $\tilde{p} = p + \mu_2 s A d \cdot d/3$, u is solution of

$$u_t + u \cdot \nabla u = \operatorname{div} \sigma,$$

which concludes the proof of the Proposition. □

Remark 10.4.3. *Vice versa, one can show that if (u, d) is solution for the Ericksen-Leslie system (3.18), then the couple (u, Q) , with Q defined by (10.22), is solution of the Qian-Sheng equations. The proof is equivalent to the one of the above proposition, and it is left to the interested reader.*

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