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Thomas Leroy

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Reduced models and numerical methods for kinetic equations applied to photon transport

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par

Thomas Leroy

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Résumé

La modélisation d'expériences de fusion par confinement inertiel fait intervenir des équations cinétiques dont la discrétisation peut être très coûteuse. La recherche de modèles simplifiés permet de réduire la taille et donc la complexité de ces systèmes. La justification mathématique de ces modèles simplifiés devient alors un enjeu central.

Dans ce travail nous étudions plusieurs modèles réduits pour l'équation du transfert radiatif dans différents contextes, tant du point de vue théorique que du point de vue numérique. En particulier nous étudions l'équation du transfert radiatif relativiste dans le régime de diffusion hors équilibre, et nous montrons la convergence de la solution de cette équation vers la solution d'une équation de drift diffusion, dans laquelle les effets Doppler sont modélisés par un terme de transport en fréquence. Cette équation de transport est discrétisée par une nouvelle classe de schémas "bien équilibrés" (well-balanced), pour lesquels nous montrons que ces nouveaux schémas sont consistants lorsque la vitesse d'onde tend vers zéro, par opposition aux schémas de type Greenberg-Leroux [GL96]. Nous étudions également de nouveaux modèles réduits pour le scattering Compton (collision inélastique photon-électron). Une hiérarchie d'équations cinétiques non linéaires généralisant l'équation de Kompaneets [KOM57] pour des distributions anisotropes sont dérivées et leurs propriétés étudiées. Les modèles aux moments de type P_1 et M_1 sont dérivés à partir de l'une de ces équations, et nous montrons que la prise en compte de l'anisotropie du rayonnement peut modifier le phénomène de condensation de Bose expliqué par Caffisch et Levermore [CL86]. Ce manuscrit se termine avec les comptes rendus de deux projets. Le premier est une preuve technique de la convergence uniforme du schéma de Gosse-Toscani [GT02] sur maillages non structurés. Ce schéma est "asymptotic preserving", au sens où il préserve au niveau discret la limite de diffusion pour l'équation de la chaleur hyperbolique, et cette preuve de convergence uniforme sur maillage non structurés en 2D est originale. Le second concerne la dérivation d'un modèle cinétique pour le Bremsstrahlung électron-ion qui préserve la limite thermique.

Mots-clés: Equation de transfert, équations cinétiques, modèles réduits, schémas well-balanced et asymptotic preserving, scattering Compton, équation de Kompaneets, équation de Fokker-Planck, Bremsstrahlung électron-ion.

Abstract

The modeling of inertial confinement experiments involves kinetic equations whose discretization can become very costly. The research of reduced models allows to decrease the size and the complexity of these systems. The mathematical justification of such reduced models becomes an important issue.

In this work we study several reduced models for the transfer equation in several contexts, from the theoretical and numerical point of view. In particular we study the relativistic transfer equation in the non-equilibrium diffusion regime, and we prove the convergence of the solution

of this equation to the solution of a drift diffusion equation, in which the Doppler effects are modeled by a frequency transport term. This transport equation is discretized by a new class of well-balanced schemes, and we show that these schemes are consistent as the wave velocity tends to zero, by opposition to the Greenberg-Leroux type schemes [GL96]. We also study several original reduced models for the Compton scattering (inelastic electron-photon collision). A hierarchy of nonlinear kinetic equations generalizing the Kompaneets equation [KOM57] for anisotropic distributions are derived and their properties are studied. The M_1 and P_1 angular moments models are derived from one of these equations, and we show that the anisotropic part of a radiation beam can modify the Bose condensation phenomena observed by Caflisch and Levermore [CL86]. This work ends with the reports of two side projects. The first one is a technical proof of the uniform convergence of the Gosse-Toscani's [GT02] scheme on unstructured meshes. This scheme is asymptotic preserving, since it preserves at the discrete level the diffusion limit of the hyperbolic heat equation, and this proof on unstructured meshes in 2D is original. The second one is devoted to the derivation of a kinetic model for the electron-ion Bremsstrahlung that preserves the thermal limit.

Keywords: Transfer equation, kinetic equations, reduced models, well-balanced and asymptotic preserving schemes, Compton scattering, Kompaneets equation, Fokker-Planck equations, electron-ion Bremsstrahlung.

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Glossary

Physical quantities

- f_γ is the density distribution function of the photons,
- I is the specific intensity, defined in (1.1),
- T is the fluid temperature,
- $B(\nu, T)$ is the Planck function, defined in (1.2),
- $f_\mu(\nu)$ are the Bose-Einstein distributions, defined in (1.3),
- c is the speed of light,
- \mathbf{u} is the fluid velocity.
- σ_s is the scattering coefficient,
- σ_a is the emission absorption coefficient.

Notations

- LTE: local thermodynamic equilibrium,
- AP: asymptotic preserving,
- WB: well-balanced

Introduction

Introduction

This work focuses on the derivation of reduced models for kinetic equations applied to photon transport. This kind of equations arises in the modeling of laser transport, radiotherapy and more generally in astrophysics. These physical processes involve several interactions, such as emission absorption and scattering. The simulation of such interactions is a complicated issue. For instance the coupling between the light and the gas involves nonlinear operators, which increases the difficulty of the mathematical study. Moreover, in the case of inertial confinement fusion experiments, Rayleigh-Taylor instabilities appear, due to the variation of density between the successive layers of the sphere. The numerical study is complicated by the fact that very different time scales are involved, the velocity of the gas often being much smaller than the speed of light.

To fix the problem, we consider in this work a gas of photons interacting with a plasma (mixture of electrons and atomic nucleus with global electric neutrality) via several kind of interactions. From a mathematical point of view, these interactions are described by two Boltzmann equations: one for the distribution function of the photons and one for the distribution function of the electrons. In this work we do not consider the evolution of the atomic nucleus, assuming that they are motionless. The equation on the photons may be nonlinear if the induce effects (Pauli principle, quantum effects) are taken into account, leading to mathematical difficulties. The corresponding collision operators may be very complicated to approximate, which justifies the study of reduced models. Since in this work we are more interested in the description of the transport of light, we often deeply simplify the matter description by using only an equation describing the evolution of its macroscopic temperature.

An important issue in the derivation of reduced models is the preservation of the physical properties of the Boltzmann equation. Indeed, the situation that we want to model with mathematical tools satisfies several properties. For example the physical entropy of the system can not go down (non reversibility), the distribution function of the photons remains non negative... These properties have to be satisfied by the simpler model, as a necessary (but not necessarily sufficient) condition for the model to be valid.

Objectives of the study

The objectives of this work are multiple, but can all be explained in term of derivation and justification of reduced models, either theoretical or numerical. The main lines that we follow are: given a model, can we study its mathematical structure, and prove some theoretical results

(maximum principle, existence...)? When a new model is derived from a former one, does it preserves its physical properties (entropy, non negativity...)? Is it possible to justify rigorously, in a sense that has to be specified, the derivation of the model? We will come back to these questions for each model.

Main results

This work led to several original results. In particular the convergence result (theorem 2.8) of the solution of the relativistic transfer equation (2.1) to the solution of the drift diffusion system (2.11) in the non-equilibrium diffusion regime obtained in chapter 2 involves an innovating comparison principle (theorem 2.1). In chapter 3 is derived and studied a new class of well-balanced schemes for transport equations. Contrary to the Greenberg-Leroux type schemes [GL96], these schemes are proved to be uniformly convergent as the wave velocity tends to zero (lemma 3.6). Finally, in chapter 5 a multigroup angular moment model (frequency dependent P_1 model 5.10) is derived from an anisotropic kinetic model describing Compton scattering. In the homogeneous case, this model can be seen as an anisotropic extension of the Kompaneets equation. It is shown in lemma 5.2 that the anisotropic part of the radiation beam modifies the well-known long time behavior of the solution of the Kompaneets equation [CL86].

We now give some further details on the models we are about to study.

Chapter 1

In a first part (chapter 1) are recalled the basic concepts in radiation hydrodynamics. In particular we introduce the main physical quantities of interest, the Boltzmann equation for photons and the modeling of interactions with matter. Since they are widely used throughout this thesis, the main reduced models for the transfer equation are presented. In particular we explain the notion of equilibrium and non-equilibrium diffusion regimes and the angular moments methods (P_N , M_N).

Chapter 2

In chapter 2 is studied the influence of a gas of photons on a moving fluid, interacting via isotropic scattering and emission absorption. The relativistic transfer equation coupled to an equation describing the fluid temperature is studied, and we prove a maximum principle and an existence result. In a second part we study the non-equilibrium diffusion regime. Indeed, in physical literature [MWM99, WIN95], one can find a modeling of the Doppler effects by a frequency drift term $\nabla \cdot \mathbf{u}(\nu \partial_\nu \rho)$, where ρ , the zero-th order moment of the specific intensity I , satisfies a diffusion equation. We provide a justification of this equation, as the limit in the non-equilibrium diffusion regime of the relativistic transfer equation.

Chapter 3

In chapter 3 we design and study several costless numerical schemes for the (homogeneous in space) non-equilibrium diffusion regime obtained in chapter 2. Even if this equation is rather simple, it contains a lot of the involved physics. Indeed, the frequency drift term depends on the divergence of the fluid velocity, which may vanish, and the equation also depends on the absorption coefficient, which is known to be highly irregular with respect to the velocity.

Design a numerical scheme, well-balanced, consistent as $\nabla \cdot \mathbf{u}$ tends to 0 and costless even for these nonlinear absorption coefficient is thus an interesting issue. In a first step we study a scheme constructed according to the Greenberg-Leroux method [GL96]. Despite well-balanced, this scheme is not consistent with respect to the speed of the waves, which is a problem for our applications. We thus designed a new scheme, called spectrally well-balanced (SWB), which is proved to be uniformly convergent. Various numerical tests are performed to illustrate this from a numerical point of view.

Chapter 4

The chapter 4 deals with the derivation of a hierarchy of anisotropic kinetic models for the Compton scattering. This is a generalization of the derivation of the Kompaneets equation by M. Escobedo and co-authors [EMV03] for anisotropic distribution functions. It leads to nonlinear anisotropic Fokker-Planck type equations, which we study by proving theoretical results for one of them, such as a H-theorem and the conservation of the non negativity of the distribution function. In a second time this model is coupled to an equation describing the evolution of the electronic temperature, and the equilibrium and non-equilibrium diffusion regimes are studied.

Chapter 5

The chapter 5 deals with the derivation of angular moments methods for the coupled model derived in chapter 5. In particular we show that the P_1 model leads to an extension of the Kompaneets equation, perturbed by a Burgers type term on the first order moment. The difference between this P_1 model and the Kompaneets equation is studied, and we show that the introduction of an anisotropic part in a radiation beam may modify the stationary states. We also study the M_1 model. The introduction of the Compton scattering is shown to not modify the long time range in comparison with the classical isotropic scattering, characterized by the convergence of the radiation temperature to the electronic temperature, but modifies the convergence to this equilibrium. This grey M_1 model will be particularly studied since it is widely used in the context of our applications. Finally, an hybrid $P_1 - M_1$ model is derived.

Chapter 6

In chapter 6 is presented a side project concerning the electron ion Bremsstrahlung. This work has been initiated during the Cemracs 2014 (French summer school on plasma physics) with S. Brull, B. Dubroca and R. Ducloux and continued for several months after the summer school. This work includes the derivation of a kinetic model describing the electron-ion Bremsstrahlung, the derivation of the corresponding M_1 model and the construction of an analytic expression of the corresponding cross section. Several test cases are given in the context of the radiotherapy. The application of this work is the destruction of cancerous tumor by powerful electron beams.

Chapter 7

In chapter 7 is studied the Gosse-Toscani's scheme [GT02] for the hyperbolic heat equations, and its 2D extension [FRA12, JL96]. The hyperbolic heat equations corresponds to the P_1 model applied on the transfer equation in the case of no coupling with matter and isotropic scattering. In the non-equilibrium diffusion regime (rescaling of the equations with a small parameter), the solution of the hyperbolic heat equations converges to the solution of a diffusion equation. The Gosse-Toscani's scheme is asymptotic preserving (AP), which means that the limit of the scheme

as the parameter goes to zero is a consistent scheme with the diffusion equation. A proof of its uniform convergence (with respect to the small parameter) is provided, which is original for unstructured meshes, even in 1D. The proof in 1D on regular mesh was provided by L. Gosse and G. Toscani in [GT02].

Chapter 1

Reduced models for photon transport

1.1 Introduction

In this chapter are presented the basic concepts of radiation transport, the interaction between light and matter and the better known reduced models. The aim of this chapter is not to give an exhaustive list of all existing methods and models in radiative transfer, but to present an introduction on simple cases of reduced models and technics which are used in this work in more complicated situations.

This chapter is organized as follows. The section 2 is devoted to the presentation of the main models used to describe photons-electrons interactions. After presenting the physical quantities of interest, we introduce the Boltzmann equation for photons. The corresponding collision operator is given for classical interactions, such as scattering, emission absorption and Bremsstrahlung. We recall in particular the main simplified scattering (Compton, Thomson, isotropic...). After a brief introduction to the relativistic case, we introduce the gas equations by means of a hierarchy of models, from a Boltzmann equation for the distribution function of the electrons to a simple equation on the gas macroscopic temperature.

The section 3 is devoted to the presentation of the main reduced models in radiative transfer. We present two different methods. The first one consists to eliminate the angular dependence. We give the example of the diffusion approximations (equilibrium and non-equilibrium diffusion regimes), the angular moments methods and the discrete-ordinate methods. The second one consists to simplify the collision operator. We present in particular the case of the Compton scattering, whose collision operator can be approximated by the well-known nonlinear Kompaneets equation.

Finally, in section 4 are presented some numerical issues which will be studied in this work. In particular the notion of well-balanced scheme is recalled through the example of Greenberg-Leroux [GL96] type schemes, which are studied in chapter 3. We also discuss about asymptotic preserving (AP) schemes by presenting the Gosse-Toscani's scheme [GT02], which is studied in chapter 7.

1.2 The Boltzmann equation for photons

1.2.1 Physical quantities

As a starting point we present the main physical quantities that take place in the description of the radiation. We are interested in the description of the evolution of the density distribution function of the photons f_γ . It is defined such that the number of photons ΔN at time t , position \mathbf{x} and momentum \mathbf{p} in a volume $\Delta x \Delta \mathbf{p}$ is given by

$$\Delta N = f_\gamma(t, \mathbf{x}, \mathbf{p}) \Delta x \Delta \mathbf{p}.$$

With these notations, the total number of photons is

$$N(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\gamma d\mathbf{x} d\mathbf{p}.$$

Since the radiation momentum \mathbf{p} is related to the photons frequency ν and its direction $\boldsymbol{\Omega}$ by the relation

$$\mathbf{p} = \frac{h\nu}{c} \boldsymbol{\Omega},$$

where h is the Planck constant, for any density distribution function $f := f(\mathbf{p})$, the integral over momentum can be changed as

$$\int_{\mathbb{R}^3} f(\mathbf{p}) d\mathbf{p} = \frac{h}{c} \int_{\mathbb{R}^+} \int_{S^2} f(\nu, \boldsymbol{\Omega}) \nu^2 d\boldsymbol{\Omega} d\nu.$$

This change of variables will often be used along this work. Another physical important quantity is the specific intensity I , defined from the density distribution function by

$$I = \frac{2h}{c^2} \nu^3 f_\gamma. \quad (1.1)$$

The specific intensity is homogeneous to an energy. The attention of the reader is drawn to the fact that we adopt here the mathematical notations. Indeed in physical literature what we call the density distribution function f_γ is denoted n_γ , the photon occupation number. The notation n_γ is used in chapter 6 in which a kinetic model describing the electron-ion Bremsstrahlung is derived.

At the local thermodynamic equilibrium (ETL), the emission spectrum of the matter is given by the Planck function, also called the Black body radiation

$$B(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k_B T}} - 1}, \quad (1.2)$$

where h is once again the Planck constant, k_B is the Boltzmann constant and T is the matter temperature. This quantity is very important since when the radiation interacts with matter, this often leads to an equilibrium state defined by

$$\int_{\mathbb{R}^+} \int_{S^2} I d\boldsymbol{\Omega} d\nu = \int_{\mathbb{R}^+} \int_{S^2} B(\nu, T) d\boldsymbol{\Omega} d\nu = aT^4,$$

where $a = 4\sigma_R$, with σ_R the Stefan Boltzmann constant. This explains the fact that one often refers to a radiation temperature T_R , which tends to the matter temperature to the power four. The Planck function is also linked to the Bose-Einstein distributions f_μ , defined by

$$f_\mu(\nu) = \left[\exp\left(\frac{h}{k_B T}(\mu + \nu)\right) - 1 \right]^{-1}. \quad (1.3)$$

One thus has $B(\nu, T) = \frac{2h}{c^2}\nu^3 f_0(\nu)$. The Bose-Einstein distributions are the equilibrium states of the Boltzmann equation when the collision kernel describes Compton scattering. If emission absorption processes are taken into account, the set of equilibrium states reduces to $f_0(\nu)$.

Since in this work we are interested in the mathematical structure of the transfer equation, we work with dimensionless quantities. To this end we always take $h = k_B = 1$, and we consider the "normalized" specific intensity $I = \nu^3 f_\gamma$ and "normalized" Planck function

$$B(\nu, T) = \frac{\nu^3}{e^{\nu/T} - 1}.$$

1.2.2 The Boltzmann equation

From a general point of view, the equation describing the evolution of the density distribution function of the photons f_γ , which depends on the time $t \in [0, T^f]$, $0 < T^f < +\infty$, the position $\mathbf{x} \in \mathbb{R}_x^3$ of the photon, its frequency $\nu \in \mathbb{R}_\nu^+$ and its direction $\boldsymbol{\Omega} \in S^2$, interacting with electrons of density distribution function f_e is described by the Boltzmann equation for photons

$$\frac{1}{c}\partial_t f_\gamma + \boldsymbol{\Omega} \cdot \nabla_x f_\gamma = Q(f_\gamma, f_e). \quad (1.4)$$

The collision operator $Q(f_\gamma, f_e)$ describes the interactions between the photons and the electrons. These interactions usually involve scattering, emission absorption and Bremsstrahlung. The scattering describes the diffusion processes appearing during a collision. It might involve energy exchange (Compton scattering) or only a change of direction (Thomson or isotropic scattering). The emission absorption and the Bremsstrahlung processes describe the creation and absorption of photons. We thus decompose the collision operator $Q(f_\gamma, f_e)$ as the sum of three operators

$$Q(f_\gamma, f_e) = Q_{abs}(f_\gamma, f_e) + Q_{scat}(f_\gamma, f_e) + Q_{brem}(f_\gamma, f_e)$$

The next parts detail the expression of these operators for the most usual interactions.

Scattering

We introduce the expression of the collision operator for the scattering of photons by electrons. We start from the most general case, and through several levels of simplifications we explain the derivation of the Compton, Thomson and isotropic scattering.

We consider the collision between a photon of momentum \mathbf{p} with an electron of momentum \mathbf{p}_* , leading to another photon of momentum \mathbf{p}' and another electron of momentum \mathbf{p}'_* (figure 7.6). From a very general point of view, the collision operator $Q(f_\gamma, f_e)$, where f_e is the density distribution function of the electrons is given by [EMV03]

$$Q_{scat}(f_\gamma, f_e) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W(p, p_*, p', p'_*) \left(f'_\gamma f'_{e*} (1 + \tau f_\gamma) (1 + \tau' f_{e*}) - f_\gamma f_{e*} (1 + \tau f'_\gamma) (1 + \tau' f'_{e*}) \right) dp' dp_* dp'_*, \quad (1.5)$$

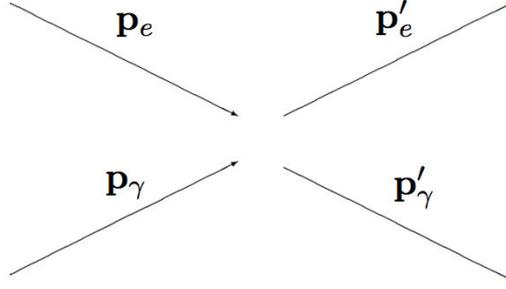


Figure 1.1: Diagram of photon-electron scattering

where $f_{e*} = f_e(\mathbf{p}_*)$, $f'_{e*} = f_e(\mathbf{p}'_*)$ and $f'_\gamma = f_\gamma(\mathbf{p}')$. The parameters $\tau, \tau' \in \{0, 1\}$ model the induce effects (quantum effects), whose value is 1 if they are taken into account and 0 otherwise. In this work we always take $\tau' = 0$, i.e. no induce effects for the electrons. The quantity $W(p, p_*, p', p'_*)$ is the probability of collision leading to the transition from the state (p, p_*) to the state (p', p'_*) , whose expressions are linked by the conservation of momentum and energy

$$\begin{cases} \mathbf{p} + \mathbf{p}_* = \mathbf{p}' + \mathbf{p}'_* \\ |\mathbf{p}| + \frac{|\mathbf{p}_*|^2}{2m} = |\mathbf{p}'| + \frac{|\mathbf{p}'_*|^2}{2m}, \end{cases} \quad (1.6)$$

where m is the mass of the electron at rest. We now present the expression of the collision operator $Q_{scat}(f_\gamma, f_e)$ in simplified models.

Compton scattering

In the case of the Compton scattering, the transition rate $W(p, p_*, p', p'_*)$ is given by [EMV03]

$$W(p, p_*, p', p'_*) = \frac{\sigma_s(\mathbf{x}, p, p')}{|p||p'|} \delta_\Sigma, \quad (1.7)$$

where δ_Σ is the manifold of 4-uplets (p, p_*, p', p'_*) satisfying (1.6) and $\sigma_s(\mathbf{x}, p, p')$ is given by the Klein-Nishina formula [EV55, POM73]. In this work we consider a simplified model describing the Compton scattering. This model is said to be simplified since in this description, we assume a simplified expression of the scattering kernel $\sigma_s(\mathbf{x}, p, p') = \bar{\sigma}_s(\mathbf{x})(1 + \cos^2 \theta)$, where $\cos \theta = \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}'$ is the cosine of the scattering angle.

From a physical point of view, the Compton scattering have been widely studied [FKM85, PL97, POM73]. On the other hand few mathematical results exist in the literature. We may cite the important papers of Escobedo and Mischler [EM01, EM99]. They considered the following Boltzmann equation modeling the Compton scattering

$$\nu^2 \partial_t f = \int_{\mathbb{R}_+} \left(f'(1 + f)B(\nu', \nu) - f(1 + f')B(\nu, \nu') \right) d\nu'. \quad (1.8)$$

This expression of the collision operator can be obtained from the general operator (1.5) by using the expression (1.7) of the transition rate, by assuming the isotropy of the distribution function of

the photons and that the electrons are at local thermodynamic equilibrium (ETL). This equation satisfies several physical properties. In particular it preserves the total number of photons $N(f) = \int_{\mathbb{R}^+} \nu^2 f d\nu$, it preserves the Boltzmann stationary states $f_\mu(\nu) = (e^{\mu+\nu} - 1)^{-1}$ and the (mathematical) entropy $H(f) = \int_{\mathbb{R}^+} \nu^2 (f \log f - (f+1) \log(f+1) + \nu f) d\nu$ decreases. Escobedo and Mischler studied the existence of solutions, and the long time range of these solutions (see also [LY77, LY78] for the Dirac masses formation). This result may be stated as follows [EM01, EM99]. Assume that the initial datum of equation (1.8) is of the form $F^{in} = \nu^2 f^{in} + \alpha^{in} \delta_0$, with $0 \leq f^{in} \leq f_0$ (f_0 defined in (1.3)) and $\alpha^{in} \geq 0$. Then, under physical assumptions on the kernel B ,

- There exists an entropy solution to equation (1.8) $F = \nu^2 f + \alpha \delta_0 \in C([0, T], \mathcal{E}_1)$, where \mathcal{E}_1 is the set of distributions $F \in (C_b(\mathbb{R}^+))'$ such that $F \geq 0$ and $\int_{\mathbb{R}^+} d(e^\nu F) < \infty$. Moreover, $0 \leq \alpha(t) \leq \alpha^{in}$, $0 \leq \nu^2 f(t, \cdot) \leq \nu^2 f^{in}$, $\forall t \geq 0$ and the total number of photons is preserved.
- Denoting $F \in C([0, \infty); (C_b(\mathbb{R}^+))')$ the solution of equation (1.8) and $m = \int_{\mathbb{R}^+} d(F^{in})$, they proved that

$$F(t, \cdot) \rightharpoonup \mathcal{B}_m, \quad \text{weakly } \star \text{ in } (C_b(\mathbb{R}^+))',$$

$$\lim_{t \rightarrow \infty} \|\nu^2 f(t, \cdot) - \nu^2 f_\mu\|_{L^1(\nu_0, \infty)} = 0, \quad \forall \nu_0 > 0,$$

where \mathcal{B}_m , the Bose Einstein condensate, is defined by

$$\mathcal{B}_m = \nu^2 f_\mu + \alpha \delta_0, \quad \text{with } \begin{cases} \alpha = 0 \text{ and } \mu \geq 0 \text{ such that } \int_{\mathbb{R}^+} \nu^2 f_\mu d\nu = m \text{ if } m \leq N_0, \\ \mu = 0 \text{ and } \alpha = m - N_0 \text{ if } m > N_0, \end{cases}$$

Moreover if $m \leq N_0$, we can take $\nu_0 = 0$.

In this work we are particularly interested in the description of anisotropic distribution functions. We thus need a more general expression of the collision operator for Compton scattering. Let us explain the main lines. We assume that the electrons are at thermodynamical equilibrium, and thus their distribution function is given by a Maxwellian of temperature T

$$f_e(p'_*) = e^{-\frac{|p'_*|^2}{2mT}} = e^{-c \frac{|p| - |p'|}{T}} e^{-\frac{|p_*|^2}{2mT}}$$

Under these assumptions, the integrals over the pre and post-collisions electrons momentum can be achieved in (1.5) and (1.7), and one finds [EMV03], taking into account the induce effects ($\tau = 1$),

$$Q_{scat}(f_\gamma, f_e) = Q_{scat}(f_\gamma) = \int_{\mathbb{R}^+} \int_{S^2} \frac{\nu'}{\nu} b(\mathbf{x}, \nu, \nu', \theta) e^{\frac{\nu'}{cT}} q(f_\gamma) d\nu' d\Omega', \quad (1.9)$$

where

$$\begin{cases} b(\mathbf{x}, \nu, \nu', \theta) = \sigma_s(\mathbf{x}) (1 + \cos^2 \theta) |\omega|^{-1} e^{-\frac{A^2 m c^2}{2T |\omega|^2}}, \\ A = \nu' - \nu + \frac{|\omega|^2}{2mc}, \\ \omega = \frac{1}{c} (\nu' \Omega' - \nu \Omega), \\ \cos \theta = \Omega \cdot \Omega', \end{cases} \quad (1.10)$$

and

$$q(f_\gamma) = e^{-\frac{\nu}{cT}} f_\gamma(\nu', \mathbf{\Omega}') (1 + \tau f_\gamma(\nu, \mathbf{\Omega})) - e^{-\frac{\nu'}{cT}} f_\gamma(\nu, \mathbf{\Omega}) (1 + \tau f_\gamma(\nu', \mathbf{\Omega}')).$$

This equation will be studied in details in chapter 4, and several Fokker-Planck equations will be derived from it. The Boltzmann equation (1.8) used in [EM01, EM99] can be obtained from (1.9) by assuming the isotropy of the distribution and by denoting

$$B(\nu, \nu') = \nu \nu' e^{\nu' - \nu} \int_{S^2} \int_{S^2} b(\mathbf{x}, \nu, \nu', \theta) d\mathbf{\Omega}' d\mathbf{\Omega}.$$

Simplified scattering

We present several simpler and more classical expression of the collision operator. Let us first assume that the electrons distribution function be constant, i.e. $f_e(p_*) = f_e(p'_*) = f_e$. One finds from the general collision operator (1.5) by denoting

$$\sigma_s(p, p') = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_e W(p, p_*, p', p'_*) dp_* dp'_*$$

and by taking $\tau = 0$ for simplicity the following expression of the collision operator

$$Q_{scat}(f_\gamma) = \int_{\mathbb{R}^3} \sigma_s(p, p') (f_\gamma(p') - f_\gamma(p)) dp'.$$

Since the momentum p of a photon is linked to its frequency ν and its direction $\mathbf{\Omega}$ by the (normalized) relation $p = \nu \mathbf{\Omega}$, this expression can also be written

$$Q_{scat}(f_\gamma) = \int_{\mathbb{R}^+} \int_{S^2} \sigma_s(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') (f_\gamma(\nu', \mathbf{\Omega}') - f_\gamma(\nu, \mathbf{\Omega})) d\mathbf{\Omega}' d\nu'.$$

Here we used the fact that the scattering coefficient σ_s depends only on the cosine of the scattering angle $\mathbf{\Omega} \cdot \mathbf{\Omega}'$. The simplest expression of the kernel $\sigma_s(p, p')$ is obtained when the scattering is coherent, i.e. there is no energy exchange during the collision. In this case, the scattering coefficient is given by $\sigma_s(\nu, \nu', \mathbf{\Omega}, \mathbf{\Omega}') = \bar{\sigma}_s(\mathbf{\Omega}, \mathbf{\Omega}') \delta_{\nu' = \nu}$, and the collision operator reduces to

$$Q_{scat}(f_\gamma) = \int_{S^2} \bar{\sigma}_s(\mathbf{\Omega}, \mathbf{\Omega}') (f_\gamma(\nu, \mathbf{\Omega}') - f_\gamma(\nu, \mathbf{\Omega})) d\mathbf{\Omega}'.$$

By taking $\bar{\sigma}_s(\mathbf{\Omega}, \mathbf{\Omega}') = 1 + (\mathbf{\Omega}, \mathbf{\Omega}')^2$, one finds the classical Thomson scattering [POM73]. If one takes $\bar{\sigma}_s(\mathbf{\Omega}, \mathbf{\Omega}') = 1$, one finds the classical isotropic scattering, which is often used in mathematical models [BGPS88, DOG01].

Emission absorption

In this part we describe the collision operator in the case of emission absorption. The process of emission can be separated in two different ways. First, an electron attached to an atomic nucleus can loss energy by emitting a photon, whose energy corresponds to the difference of potential energy between the two energy levels (Einstein principle). Inversely, an electron can increase its potential energy level by absorbing a photon. The second physical process leading to emission of photon is the electron-ion Bremsstrahlung. This phenomena is due to the Coulomb potential of an atomic nucleus, which slows down an ingoing electron which in turn emits a photon.

”Classical” emission absorption

Let us present here the simplest model of emission absorption. We consider that the matter is at local thermodynamic equilibrium (ETL), which means that the distribution function of the electrons is given by a Maxwellian of temperature T , where T is the macroscopic temperature of the matter. As explain in the introduction, the matter emits photons through the Planck function at temperature T . On the other hand the matter also absorbs photons, and the rate of absorption is given by the absorption coefficient σ_a . The emission absorption operator thus reads

$$Q_{abs}(f_\gamma, T) = \sigma_a(\nu, T)(B(\nu, T) - \nu^3 f_\gamma). \quad (1.11)$$

The absorption coefficient is in general highly nonlinear with respect to the frequency, and its behavior with respect to the electron temperature may be singular as $T \rightarrow 0$. The mathematical study of such models is thus a complicated issue. For example, in [BGPS88] the authors considered the Cramer’s opacity

$$\sigma_a(\nu, T) = \frac{1 - e^{-\frac{\nu}{T}}}{\nu^3 \sqrt{T}}.$$

In [GS08], the authors proved the existence of solutions in the grey case (no frequency dependence) by simply assuming that the absorption term satisfies

$$\lim_{T \rightarrow 0} \sigma_a(T) = +\infty, \quad \lim_{T \rightarrow +\infty} \sigma_a(T) = 0.$$

Due to this temperature dependence, these works were performed without the use of semigroups contraction. In this work, except in the chapter 5 and the study of the electron-ion Bremsstrahlung, we always assume that the absorption coefficient σ_a does not depend on the temperature T , but only on the frequency. In the case of the relativistic transfer equations (chapter 2), this allows us to prove the existence of solutions by using the semigroups theory [PAZ83].

Electron-ion Bremsstrahlung in kinetic transport theory

The electron-ion Bremsstrahlung effect describes the emission of a photon by an electron slowing down in presence of an electric field created by an atomic nucleus (figure 1.2). This photon

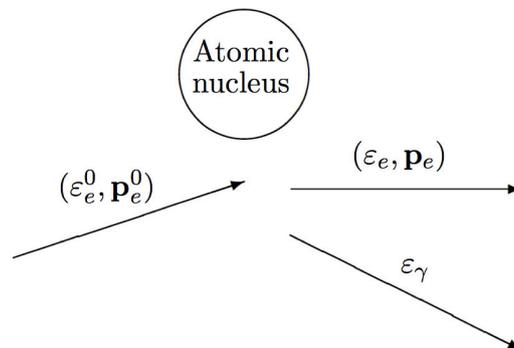


Figure 1.2: Diagram of Bremsstrahlung emission

might be absorbed in a second time by the environment. The conservation of energy yields

$\varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e \geq 0$, where ε_γ is the energy of the emitted photon, ε_e^0 (resp ε_e) is the energy of the pre (resp post) collision electron. One also defines the pre and post-collisional electron momentum \mathbf{p}_e^0 and \mathbf{p}_e . The Bremsstrahlung is often decomposed as direct and inverse Bremsstrahlung, the inverse Bremsstrahlung being seen as the absorption of a photon by an electron. The equations describing the electron-ion Bremsstrahlung are derived by considering all the possible transformation (Feynman' diagram, figure 1.3). In this diagram σ^B (resp σ^{IB}) represents the

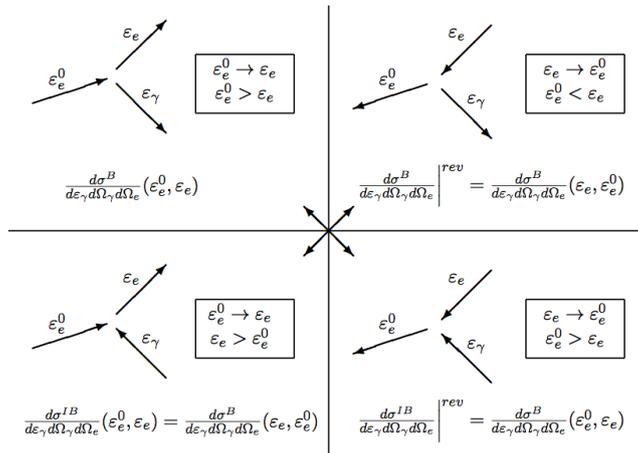


Figure 1.3: Feynman' diagram of electron-ion Bremsstrahlung

probability of direct Bremsstrahlung (resp inverse Bremsstrahlung). The derivation of these differential cross sections is performed in chapter 6, in which a kinetic model describing the electron-ion Bremsstrahlung is studied.

If one assumes that the electrons are at local thermodynamic equilibrium, and that the emitted photons are given by a Planck function of temperature T , then the Bremsstrahlung operator reduces to the classical emission absorption operator (1.11) (also referred as thermal Bremsstrahlung).

1.2.3 Relativistic case

In this part we introduce the relativistic transfer equation. We thus consider a gas of photons interacting with a moving fluid. The velocity of the fluid \mathbf{u} is assumed to satisfy $|\mathbf{u}|/c < 1$, which is physically relevant. Due to the velocity of the fluid, the physical quantities can be computed in several frames. In this work we consider two frames: the laboratory frame and the fluid frame, also called the co-mobile frame. The quantities computed in the co-mobile frame are denoted with a subscript 0. From a general point of view, the Boltzmann equation (1.4) in the relativistic case writes [MWM99, POM73]

$$\frac{1}{c} \partial_t I + \boldsymbol{\Omega} \cdot \nabla I = \Lambda^{-2} \left(Q(I, g) \right)_0, \quad (1.12)$$

where Λ is a relativistic coefficient

$$\Lambda = \frac{1 + \boldsymbol{\Omega} \cdot \mathbf{u}/c}{\sqrt{1 + |\mathbf{u}|^2/c^2}}.$$

In this equation g refers to the description of the fluid. According to the considered model, g can be the distribution function of the electrons in the most general case, or for example the fluid temperature in a simplified case. For ease of notations we introduce the Lorentz factor $\gamma = \sqrt{1 + |\mathbf{u}|^2/c^2}^{-1}$. Let us assume for simplicity that the interactions are described by classical isotropic scattering and emission absorption (with a fluid temperature T). One thus finds for the collision operator

$$\left(Q(I, T)\right)_0 = \sigma_s(\mathbf{x}) \left(\int_{S^2} I_0(\boldsymbol{\Omega}'_0) d\boldsymbol{\Omega}'_0 - I_0 \right) + \sigma_a(\nu_0) \left(B(\nu_0, T) - I_0 \right).$$

The relation that links the radiative intensity in the laboratory and in the co-mobile frames is given by

$$I = \Lambda^{-3} I_0$$

In the same way, the relation between the frequency in the laboratory and in the co-mobile frames is given by

$$\nu = \Lambda^{-1} \nu_0$$

The scattering operator can be computed in the laboratory frame [BD04, LMH99], which yields

$$\sigma_s(\mathbf{x}) \left(\int_{S^2} I_0(\boldsymbol{\Omega}'_0) d\boldsymbol{\Omega}'_0 - I_0 \right) = \sigma_s(x) \Lambda \left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I \right),$$

where $\Lambda' = \gamma(1 - \boldsymbol{\Omega}' \cdot \mathbf{u}/c)$ and $\nu' = \frac{\Lambda}{\Lambda'} \nu$. Here we see the influence of the fluid velocity to the frequency of the radiation, which is called the Doppler effects. Indeed the radiative intensity over the angular integral is computed at the frequency ν' , which goes to 0 with $|\mathbf{u}|/c$, but can be non negligible, depending once again on the value of $|\mathbf{u}|/c$.

The relativistic transfer equations has been studied by various authors. In particular the equilibrium and non equilibrium regimes have been studied in the grey case, in the context of a coupling with the Euler equations [BD04], and with a simpler coupling but with stronger results in [GLG05]. In chapter 2 we prove the strong convergence in the non equilibrium regime of the solution of the relativistic transfer equation (1.12) coupled to a fluid equation to the solution of a drift diffusion equation.

1.2.4 Coupling with the fluid equations

Let us now consider the equations describing the evolution of the fluid. In the same spirit than in the previous section, the evolution of the electrons distribution function f_e is governed by the following Boltzmann equation

$$\partial_t f_e + \mathbf{v}_e \cdot \nabla f_e = \bar{Q}(f_e, f_\gamma),$$

where the collision operator $\bar{Q}(f_e, f_\gamma)$ models the interaction between electrons and others particles (electrons, photons, atomic nucleus...). The collision operator has been widely studied in the case of the Coulomb correction. The case of the interactions with atomic nucleus (Bremsstrahlung) will be studied in chapter 6. In the case of the interaction with photons by scattering, one could ask that the total energy be conserved, which would leads to the relation

$$\int_{\mathbb{R}^3} \nu Q(f_\gamma, f_e) dp + \int_{\mathbb{R}^3} |\mathbf{v}_e|^2 \bar{Q}(f_e, f_\gamma) d\mathbf{v}_e = 0. \quad (1.13)$$

Euler equations

As in the classical theory of fluid equations, reduced models can be derived to simplify this equation. Let us assume for simplicity that this collision operator satisfies mass, momentum and energy conservation, i.e. $\bar{Q}(f_e, f_\gamma)(1, \mathbf{v}_e, |\mathbf{v}_e|^2)^T = 0$ (this is the case for example for the collision operator modeling the Coulomb interactions between electrons). We define the moments (with respect to the velocity) of the electrons distribution function f_e by

$$\begin{cases} \rho = \int_{\mathbb{R}^3} f_e d\mathbf{v}_e, \\ \rho \mathbf{u} = \frac{1}{\rho} \int_{\mathbb{R}^3} \mathbf{v}_e f_e d\mathbf{v}_e, \\ \rho E = \frac{1}{\rho} \int_{\mathbb{R}^3} \frac{|\mathbf{v}_e|^2}{2} f_e d\mathbf{v}_e, \end{cases}$$

and the pressure as $p = \int_{\mathbb{R}^3} (\mathbf{v}_e - \mathbf{u}) \otimes (\mathbf{v}_e - \mathbf{u}) f_e d\mathbf{v}_e$. Integrating the Boltzmann equation (1.13) with respect to the velocity \mathbf{v}_e yields

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$

which is the mass conservation equation. Multiplying equation (1.13) by \mathbf{v}_e and integrating with respect to the velocity gives

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \int_{\mathbb{R}^3} \mathbf{v}_e \otimes \mathbf{v}_e f_e d\mathbf{v}_e = 0.$$

Writing $\mathbf{v}_e \otimes \mathbf{v}_e = (\mathbf{v}_e - \mathbf{u}) \otimes (\mathbf{v}_e - \mathbf{u}) + \mathbf{u} \otimes \mathbf{u} + 2(\mathbf{v}_e - \mathbf{u}) \otimes \mathbf{u}$ and using the fact that $\int_{\mathbb{R}^3} (\mathbf{v}_e - \mathbf{u}) \otimes \mathbf{u} f_e d\mathbf{v}_e = 0$, one obtains

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p) = 0,$$

which is the momentum conservation equation. Now multiplying equation (1.13) by $\frac{1}{2}|\mathbf{v}_e|^2$ and integrating with respect to the velocity, one has

$$\partial_t (\rho E) + \nabla \cdot \int_{\mathbb{R}^3} \frac{|\mathbf{v}_e|^2}{2} \mathbf{v}_e f_e d\mathbf{v}_e = 0.$$

Once again, one remarks that

$$\int_{\mathbb{R}^3} \frac{|\mathbf{v}_e|^2}{2} \mathbf{v}_e f_e d\mathbf{v}_e = \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{v}_e - \mathbf{u}|^2 (\mathbf{v}_e - \mathbf{u}) f_e d\mathbf{v}_e + \frac{\mathbf{u}}{2} tr(P) + \rho \mathbf{u} \frac{|\mathbf{u}|^2}{2} + p \mathbf{u},$$

where $tr(P)$ is the trace of the matrix P . Since $|\mathbf{v}_e|^2 = |\mathbf{v}_e - \mathbf{u}|^2 + |\mathbf{u}|^2 + 2(\mathbf{u}, \mathbf{v}_e - \mathbf{u})$, one gets $\rho E = \frac{1}{2} tr(P) + \rho \frac{|\mathbf{u}|^2}{2}$, one finally obtains

$$\partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{u} + p \mathbf{u}) = -\frac{1}{2} \nabla \cdot \int_{\mathbb{R}^3} |\mathbf{v}_e - \mathbf{u}|^2 (\mathbf{v}_e - \mathbf{u}) f_e d\mathbf{v}_e.$$

Killing the term at the order 3 with respect to $\mathbf{v}_e - \mathbf{u}$ on the right hand side, one recover the classical Euler equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = 0, \\ \partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{u} + p \mathbf{u}) = 0. \end{cases}$$

Defining the internal energy as $\rho e = \frac{1}{2} \text{tr}(P) = \int_{\mathbb{R}^3} |\mathbf{v}_e - \mathbf{u}|^2 f_e d\mathbf{v}_e$, one can write the equation satisfied by ρe by subtracting to the equation on the total energy the momentum equation multiplied by \mathbf{u} , which yields

$$\partial_t(\rho e) + \nabla \cdot (\rho e \mathbf{u}) + p \nabla \cdot \mathbf{u} = 0.$$

Let us now consider the coupling between the Euler equations for the fluid with the Boltzmann equation (1.4) for the photons. The conservation of momentum and energy lead to the following system, known as the radiation hydrodynamic system [BD04]

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) = - \int_{\mathbb{R}^+} \int_{S^2} Q(f_\gamma, f_e) \nu^3 \boldsymbol{\Omega} d\nu d\Omega, \\ \partial_t(\rho E) + \nabla \cdot (\rho E \mathbf{u} + p \mathbf{u}) = -c \int_{\mathbb{R}^+} \int_{S^2} Q(f_\gamma, f_e) \nu^3 d\nu d\Omega, \\ \frac{1}{c} \partial_t f_\gamma + \boldsymbol{\Omega} \cdot \nabla f_\gamma = Q(f_\gamma, f_e), \end{cases} \quad (1.14)$$

where, with an abuse of notations, the dependence with respect to the fluid in the collision operator Q is denoted by f_e . This model can be seen as a first level of simplification, but for a lot of applications it remains too costly. Indeed, in most applications the velocity of the fluid \mathbf{u} is very small compared to speed of light c . From a numerical point of view, for each radiation time step will corresponds thousands of hydrodynamical steps. This explains why one often used a model describing only the evolution of a macroscopic quantity for the fluid, typically the internal energy or the fluid temperature, as explained in the next part.

Simplified models

These models are derived as follows [GLG05]. One first assumes that the fluid density ρ and the fluid velocity \mathbf{u} are given quantities. In a second step one writes the total energy of the fluid E as the sum of its internal energy e and its kinetic energy $|\mathbf{u}|^2/2$: $E = e + |\mathbf{u}|^2/2$. Subtracting to the equation on the total energy the equation on the momentum (second equation of (1.14)) multiplied by \mathbf{u} yields the following equation on the internal energy

$$\partial_t(\rho e) + \nabla \cdot (\rho e \mathbf{u}) + p \nabla \cdot \mathbf{u} = -c \int_{\mathbb{R}^+} \int_{S^2} \left(1 - \frac{\boldsymbol{\Omega} \cdot \mathbf{u}}{c}\right) Q(f_\gamma, f_e) \nu^3 d\nu d\Omega.$$

Denoting $\mathcal{E} = \rho e$ and taking the hydrodynamic pressure as $p = \Gamma \mathcal{E}$, where Γ is the adiabatic constant lead to

$$\partial_t \mathcal{E} + \nabla \cdot (\mathcal{E} \mathbf{u}) + \Gamma \mathcal{E} \nabla \cdot \mathbf{u} = -c \int_{\mathbb{R}^+} \int_{S^2} \left(1 - \frac{\boldsymbol{\Omega} \cdot \mathbf{u}}{c}\right) Q(f_\gamma, f_e) \nu^3 d\nu d\Omega.$$

This system will be considered in chapter 2 in a relativistic context. Finally, the simplest coupling of radiation and hydrodynamics is obtained by considering a static fluid, i.e. $\mathbf{u} = 0$. It yields

$$\partial_t \mathcal{E} = -c \int_{\mathbb{R}^+} \int_{S^2} Q(f_\gamma, f_e) \nu^3 d\nu d\Omega.$$

If one now consider the radiative transfer equation in the simple case of isotropic scattering and thermal Bremsstrahlung, one finds

$$\begin{cases} \frac{1}{c} \partial_t f_\gamma + \boldsymbol{\Omega} \cdot \nabla f_\gamma = \sigma_s \left(\int_{S^2} f_\gamma(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - f_\gamma \right) + \sigma_a(\nu)(f_0 - f_\gamma), \\ \partial_t \mathcal{E} = -c \int_{\mathbb{R}^+} \int_{S^2} \sigma_a(\nu)(B(\nu, T) - \nu^3 f_\gamma) d\nu d\Omega. \end{cases} \quad (1.15)$$

This model is considered for example in [DOG01, GP86, GS08]. A widely used simplification is to consider the grey case, i.e. when the absorption coefficient σ_a does not depend on the frequency ν . The integrability of the Planck function leads to

$$\int_{\mathbb{R}^+} B(\nu, T) d\nu = \int_{\mathbb{R}^+} \frac{\nu^3}{e^{\frac{\nu}{T}} - 1} d\nu = T^4$$

Multiplying the first equation of (1.15) by ν^3 and integrating it over the frequency $\nu \in \mathbb{R}$, one finds by defining $E = \int_{\mathbb{R}^+} f_\gamma \nu^3 d\nu$ the following grey model

$$\begin{cases} \frac{1}{c} \partial_t E + \mathbf{\Omega} \cdot \nabla E = \sigma_s \left(\int_{S^2} E(\mathbf{\Omega}') d\mathbf{\Omega}' - E \right) + \sigma_a (T^4 - E), \\ \partial_t \mathcal{E} = -c \sigma_a \left(T^4 - \int_{S^2} E d\Omega \right). \end{cases}$$

A simpler model can be obtained by assuming that the internal energy \mathcal{E} only depends on the electrons temperature T . Then

$$\partial_t (\mathcal{E}(T)) = \frac{\partial \mathcal{E}}{\partial T} \partial_t T$$

One often denotes $C_V = \frac{\partial \mathcal{E}}{\partial T}$. In this work we will often take $C_V = 1$, simplifying the resolution of the equation on the temperature T . One can obtain a linearized but unphysical model by assuming that $C_V = 4T^3$. Denoting $\Theta = T^4$, it leads to the linearized system

$$\begin{cases} \frac{1}{c} \partial_t E + \mathbf{\Omega} \cdot \nabla E = \sigma_s \left(\int_{S^2} E(\mathbf{\Omega}') d\mathbf{\Omega}' - E \right) + \sigma_a (\Theta - E), \\ \partial_t \Theta = -c \sigma_a \left(\Theta - \int_{S^2} E d\Omega \right). \end{cases}$$

1.3 Reduced models for the transfer equation

For many applications, the Boltzmann equation for photons (1.4) and (1.5) is too costly to be discretized directly. Indeed, on the one hand the phase space is of dimension 7 (time, space, angle and frequency). On the second hand the collision operator is an integral over \mathbb{R}^9 in the general case. This explains the motivation to derive reduced models. There is two main directions, which can be used in the same time. The first one is to remove the angular dependence, either by diffusions approximations, either by angular moments methods. The second way consists to simplify the source terms (collision operator). For instance in the case of Compton scattering, the Boltzmann equation can be approximate by the Kompaneets equation [COO71, KOM57]. In the next parts are explained the principles of these approximations.

1.3.1 Elimination of the angular dependence

There is two main classes of simplification of the angular dependence. The first one, the diffusion approximations, is explained in a first part. The second part is devoted to the famous angular moments methods.

Diffusions approximations

This part is devoted to the diffusions approximations. The basic concept of these approximations is to assume that some physical parameters are small compared to others. Rescaling the system with a small parameter and passing to the limit as this parameter goes to zero, one obtains a diffusion equation on the zero-th order moment of the distribution function. In the case of the radiative transfer, there are two main regimes. The equilibrium diffusion, in which the emission absorption is assumed to be large compared to the scattering, and the non-equilibrium diffusion regime in which the scattering is assumed to be large compared to the emission absorption.

To illustrate our subject, let us consider a simpler coupling between a fluid, described by an equation on its temperature, and a gas of photons, satisfying a Boltzmann equation, and interacting via isotropic scattering and emission absorption. The absorption coefficient σ_a is assumed to be constant (grey assumption) and we take $C_V = 1$

$$\begin{cases} \frac{1}{c} \partial_t I + \mathbf{\Omega} \cdot \nabla I = \sigma_s \left(\int_{S^2} I(\mathbf{\Omega}') d\mathbf{\Omega}' - I \right) + \sigma_a (B(\nu, T) - I), \\ \partial_t T = -c \sigma_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T) - I) d\nu d\Omega. \end{cases} \quad (1.16)$$

Dimension analysis

Let us follow [BD04, GLG05]. We consider typical time scale T and length scale L from which a typical velocity u_∞ is defined by $u_\infty = L/T$. We then define dimensionless time t_* and length \mathbf{x}_* such that

$$\begin{cases} t = T t_*, \\ \mathbf{x} = L \mathbf{x}_*. \end{cases}$$

Since the scattering (resp the absorption) coefficient depends on the inverse of a mean free path l_s (resp l_a), we define the new coefficients $\bar{\sigma}_s = \sigma_s/l_s$ and $\bar{\sigma}_a = \sigma_a/l_a$. Inserted in (1.16), it yields

$$\begin{cases} \frac{u_\infty}{c} \partial_{t_*} I + \mathbf{\Omega} \cdot \nabla_{\mathbf{x}_*} I = \frac{L}{l_s} \bar{\sigma}_s \left(\int_{S^2} I(\mathbf{\Omega}') d\mathbf{\Omega}' - I \right) + \frac{L}{l_a} \bar{\sigma}_a (B(\nu, T) - I), \\ \frac{u_\infty}{c} \partial_{t_*} T = -\frac{L}{l_a} \bar{\sigma}_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T) - I) d\nu d\Omega. \end{cases}$$

Denoting $\mathcal{C} = u_\infty/c$, $\mathcal{L}_s = L/l_s$, $\mathcal{L}_a = L/l_a$ and dropping the bars and the stars for ease of notations, one finds the following dimensionless system

$$\begin{cases} \frac{1}{\mathcal{C}} \partial_t I + \mathbf{\Omega} \cdot \nabla I = \mathcal{L}_s \sigma_s \left(\int_{S^2} I(\mathbf{\Omega}') d\mathbf{\Omega}' - I \right) + \mathcal{L}_a \sigma_a (B(\nu, T) - I), \\ \frac{1}{\mathcal{C}} \partial_t T = -\mathcal{L}_a \sigma_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T) - I) d\nu d\Omega. \end{cases}$$

The following paragraphs deal with the two limiting cases $1/\mathcal{C} \rightarrow 0$ and $\mathcal{L}_a \rightarrow \infty$ (equilibrium diffusion) on the one hand, and $1/\mathcal{C} \rightarrow 0$ and $\mathcal{L}_s \rightarrow \infty$ (non-equilibrium diffusion) on the second hand.

Equilibrium diffusion

In this part we study the so-called equilibrium diffusion regime, also called the Rosseland approximation [BGP87, BGPS88, DOG01, GLG05]. The main assumption is to assume that the speed of light is large compared to a typical velocity of the problem, and that the emission absorption is large compared to the scattering. One thus introduces a parameter $\varepsilon \in (0, 1]$, $\varepsilon \ll 1$ such that

$$\varepsilon = \frac{1}{\mathcal{L}} = \frac{1}{\mathcal{L}_a} = \mathcal{L}_s.$$

It yields the following system

$$\begin{cases} \partial_t I^\varepsilon + \frac{1}{\varepsilon} \boldsymbol{\Omega} \cdot \nabla I^\varepsilon = \sigma_s \left(\int_{S^2} I^\varepsilon(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I^\varepsilon \right) + \frac{\sigma_a}{\varepsilon^2} (B(\nu, T^\varepsilon) - I^\varepsilon), \\ \partial_t T^\varepsilon = -\frac{\sigma_a}{\varepsilon^2} \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T^\varepsilon) - I^\varepsilon) d\nu d\boldsymbol{\Omega}. \end{cases} \quad (1.17)$$

The following result deals with the limit, as $\varepsilon \rightarrow 0$, of the solution of (1.17).

Lemma 1.1 (Equilibrium diffusion). *A first order approximation of the system (1.17) is*

$$\partial_t (T_m + CT_m^4) - C \nabla \cdot \left(\frac{\nabla T_m^4}{3\sigma_a} \right) = 0,$$

where $C = \int_{\mathbb{R}^+} \nu^3 (e^\nu - 1)^{-1} d\nu$.

Proof. We perform a formal Hilbert (also called Chapman-Enskog) expansion of the unknown I^ε and T^ε , that is

$$\begin{cases} I^\varepsilon = I^0 + \varepsilon I^1 + \varepsilon^2 I^2 + \dots, \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

Injecting these expressions in the system (1.17), we study each term of same orders. At order $1/\varepsilon^2$, one finds

$$I^0 = (B(\nu, T^\varepsilon))^0.$$

A Taylor expansion of the Planck function $B(\nu, T^\varepsilon)$ formally shows that

$$B(\nu, T^\varepsilon) = B(\nu, T^0) + \varepsilon (T^1 + \varepsilon T^2 + \dots) \frac{\partial B}{\partial T}(\nu, T^0) + \dots$$

One thus obtains $(B(\nu, T^\varepsilon))^0 = B(\nu, T^0)$, which yields $I^0 = B(\nu, T^0)$, and thus I^0 does not depend on the angle, i.e. is isotropic. At the order $1/\varepsilon$, one has

$$\boldsymbol{\Omega} \cdot \nabla I^0 = \sigma_a \left((B(\nu, T^\varepsilon))^1 - I^1 \right).$$

The same arguments than for the zero-th order term of the Planck function yield $(B(\nu, T^\varepsilon))^1 = T^1 \frac{\partial B}{\partial T}(\nu, T^0)$. It yields

$$I^1 = -\frac{\boldsymbol{\Omega} \cdot \nabla I^0}{\sigma_a} + T^1 \frac{\partial B}{\partial T}(\nu, T^0).$$

Finally, at the order $1/\varepsilon^0$, one finds

$$\begin{cases} \partial_t I^0 + \mathbf{\Omega} \cdot \nabla I^1 = \sigma_s \left(\int_{S^2} I^0(\mathbf{\Omega}') d\mathbf{\Omega}' - I^0 \right) + \sigma_a ((B(\nu, T^\varepsilon))^2 - I^2), \\ \partial_t T^0 = -\sigma_a \int_{\mathbb{R}^+} \int_{S^2} ((B(\nu, T^\varepsilon))^2 - I^2) d\nu d\mathbf{\Omega}. \end{cases}$$

First, the scattering term vanishes since I^0 is isotropic. Using the previous equation of I^1 , integrating the first equation on $\mathbb{R}^+ \times S^2$ and adding the second equation, one writes

$$\partial_t \int_{\mathbb{R}^+} I^0 d\nu + \partial_t T^0 + \int_{\mathbb{R}^+} \int_{S^2} \mathbf{\Omega} \cdot \nabla \left(-\frac{\mathbf{\Omega} \cdot \nabla I^0}{\sigma_a} + T^1 \frac{\partial B}{\partial T}(\nu, T^0) \right) d\mathbf{\Omega} d\nu = 0.$$

One has $\int_{S^2} \mathbf{\Omega} \otimes \mathbf{\Omega} d\mathbf{\Omega} = \frac{1}{3}$. Moreover, the term depending on T^1 vanishes since $\int_{S^2} \mathbf{\Omega} d\mathbf{\Omega} = 0$. Finally, one has $\int_{\mathbb{R}^+} I^0 d\nu = \int_{\mathbb{R}^+} B(\nu, T^0) d\nu = C(T^0)^4$. Denoting $T_m = T^0$, one finds the announced equation, which is referred as the equilibrium diffusion. This name comes from the fact the radiation temperature equilibrates with the electrons temperature to the power four. \square

Non-equilibrium diffusion

In this part we study the so-called non-equilibrium diffusion regime, also called the Rosseland approximation [DOG01, GLG05]. The main assumption is to assume that the speed of light is large compared to a typical velocity of the problem, and that the scattering is large compared to the emission absorption. One thus introduce a parameter $\varepsilon \in (0, 1]$, $\varepsilon \ll 1$ such that

$$\varepsilon = \frac{1}{\mathcal{C}} = \frac{1}{\mathcal{L}_s} = \mathcal{L}_a.$$

It yields the following system

$$\begin{cases} \partial_t I^\varepsilon + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla I^\varepsilon = \frac{\sigma_s}{\varepsilon^2} \left(\int_{S^2} I^\varepsilon(\mathbf{\Omega}') d\mathbf{\Omega}' - I^\varepsilon \right) + \sigma_a (B(\nu, T^\varepsilon) - I^\varepsilon), \\ \partial_t T^\varepsilon = -\sigma_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T^\varepsilon) - I^\varepsilon) d\nu d\mathbf{\Omega}. \end{cases} \quad (1.18)$$

The following result deals with the limit, as $\varepsilon \rightarrow 0$, of the solution of (1.18).

Lemma 1.2 (Non-equilibrium diffusion). *A first order approximation of the system (1.18) is*

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\frac{\nabla I^0}{3\sigma_s} \right) = \sigma_a (B(\nu, T_m) - \rho), \\ \partial_t T_m = -\sigma_a \int_{\mathbb{R}^+} (B(\nu, T_m) - \rho) d\nu. \end{cases}$$

Proof. As we did for the equilibrium regime, we perform a formal Hilbert expansion of the unknown I^ε and T^ε , that is

$$\begin{cases} I^\varepsilon = I^0 + \varepsilon I^1 + \varepsilon^2 I^2 + \dots, \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

Injecting these expressions in the system (1.18), we study each terms of same orders. At order $1/\varepsilon^2$, one finds

$$I^0 = \int_{S^2} I^0(\mathbf{\Omega}') d\mathbf{\Omega}',$$

which means that I^0 is isotropic. At order $1/\varepsilon^1$, one has

$$\boldsymbol{\Omega} \cdot \nabla I^0 = \sigma_s \left(\int_{S^2} I^1(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I^1 \right),$$

which yields

$$I^1 = -\frac{\boldsymbol{\Omega} \cdot \nabla I^0}{\sigma_s} + \int_{S^2} I^1(\boldsymbol{\Omega}') d\boldsymbol{\Omega}'.$$

This expression of I^1 is kept despite it does not give an analytical value of I^1 . At the order $1/\varepsilon^0$, one has

$$\begin{cases} \partial_t I^0 + \boldsymbol{\Omega} \cdot \nabla I^1 = \sigma_s \left(\int_{S^2} I^2(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I^2 \right) + \sigma_a (B(\nu, T^0) - I^0), \\ \partial_t T^0 = -\sigma_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T^0) - I^0) d\nu d\boldsymbol{\Omega}. \end{cases}$$

First, since I is isotropic, the equation on the temperature reduces to

$$\partial_t T^0 = -\sigma_a \int_{\mathbb{R}^+} (B(\nu, T^0) - I^0) d\nu.$$

Integrating the photon transport equation over the angle and using the expression of I^1 , one finds

$$\partial_t I^0 + \int_{S^2} \boldsymbol{\Omega} \cdot \nabla \left(-\frac{\boldsymbol{\Omega} \cdot \nabla I^0}{\sigma_s} d\boldsymbol{\Omega} + \int_{S^2} I^1(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' \right) = \sigma_a (B(\nu, T^0) - I^0).$$

For the first integral, one has

$$\int_{S^2} \boldsymbol{\Omega} \cdot \nabla \left(\frac{\boldsymbol{\Omega} \cdot \nabla I^0}{\sigma_s} \right) d\boldsymbol{\Omega} = \nabla \cdot \left(\frac{1}{\sigma_s} \left(\int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} d\boldsymbol{\Omega} \right) \nabla I^0 \right) = \nabla \cdot \left(\frac{\nabla I^0}{3\sigma_s} \right)$$

Since the second integral vanishes, one finds by denoting $\rho = I^0$ and $T_m = T^0$ the announced system, known as the non-equilibrium diffusion regime. \square

Angular moments models

In this part we describe the angular moments methods. They are widely used in practice since they are known to be precise for a large range of applications. We focus on the two main models, which are the P_N and the M_N models. They are both constructed by taking the angular moments of the transfer equation. For simplicity we restrict ourself to the case $N = 1$ and we consider once again a simple radiative transfer system, given by

$$\begin{cases} \frac{1}{c} \partial_t I + \boldsymbol{\Omega} \cdot \nabla I = \sigma_s \left(\int_{S^2} I(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I \right) + \sigma_a (B(\nu, T) - I), \\ \partial_t T = -c\sigma_a \int_{\mathbb{R}^+} \int_{S^2} (B(\nu, T) - I) d\nu d\boldsymbol{\Omega}. \end{cases} \quad (1.19)$$

The two first angular moments of the specific intensity I are defined by

$$\begin{cases} U = \int_{S^2} I d\boldsymbol{\Omega}, \\ \mathbf{V} = \int_{S^2} \boldsymbol{\Omega} I d\boldsymbol{\Omega}. \end{cases}$$

As we will see, taking the two first moments of the radiative intensity in equation (1.19) will lead to a system of two equations for 3 unknowns. The difference between the P_1 and M_1 models comes from the choice of the closure, i.e. the expression of the second order moment in term of the zero-th and first order moments. Let us construct the equation satisfied by U and \mathbf{V} . First, integrating equation (1.19) over S^2 leads to

$$\frac{1}{c}\partial_t U + \nabla \cdot \mathbf{V} = \sigma_a(B(\nu, T) - U).$$

In the same way, multiplying equation (1.19) by $\boldsymbol{\Omega}$ and integrating equation (1.19) over S^2 lead to

$$\frac{1}{c}\partial_t \mathbf{V} + \nabla \cdot \mathbf{Q}_r = -\sigma_s \mathbf{V},$$

where $\mathbf{Q}_r = \int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \mathbf{I}_d I d\boldsymbol{\Omega}$, where \mathbf{I}_d stands as the identity matrix. The system composed on the two first angular moments is thus given by

$$\begin{cases} \frac{1}{c}\partial_t U + \nabla \cdot \mathbf{V} = \sigma_a(B(\nu, T) - U), \\ \frac{1}{c}\partial_t \mathbf{V} + \nabla \cdot \mathbf{Q}_r = -\sigma_s \mathbf{V}, \\ \partial_t T = -c\sigma_a \int_{\mathbb{R}^+} (B(\nu, T) - U) d\nu. \end{cases}$$

One thus needs a closure to close this system, i.e. a relation between the second order moment \mathbf{Q}_r and the two first moments U and \mathbf{V} . One can also consider the grey angular moments by defining the grey angular moments of the specific intensity I

$$\begin{cases} E = \int_{\mathbb{R}^+} \int_{S^2} I d\boldsymbol{\Omega} d\nu, \\ \mathbf{F} = \int_{\mathbb{R}^+} \int_{S^2} I \boldsymbol{\Omega} d\boldsymbol{\Omega} d\nu. \end{cases} \quad (1.20)$$

The corresponding system satisfied by the two first grey angular moments and the temperature is

$$\begin{cases} \frac{1}{c}\partial_t E + \nabla \cdot \mathbf{F} = \sigma_a(T^4 - E), \\ \frac{1}{c}\partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_r = -\sigma_s \mathbf{F}, \\ \partial_t T = -c\sigma_a(T^4 - E) d\nu, \end{cases}$$

where $\mathbf{P}_r = \int_{\mathbb{R}^+} \int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \mathbf{I}_d I d\boldsymbol{\Omega} d\nu$. In the two coming paragraphs we explicit the expression of the P_1 and M_1 models.

P_N models

The P_N models [BRU00, BRU02, HLM10] consists of assuming that the distribution I is polynomial with respect to the angle. For the P_1 model, it means that one assume the existence I^0 and \mathbf{I}^1 , which do not depend on $\boldsymbol{\Omega}$, such that

$$I(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) = I^0(t, \mathbf{x}, \nu) + \boldsymbol{\Omega} \cdot \mathbf{I}^1(t, \mathbf{x}, \nu).$$

The link between the two first moments U , \mathbf{V} and I^0 , \mathbf{I}^1 is trivially obtained, since

$$U = \int_{S^2} \left(I^0(t, \mathbf{x}, \nu) + \boldsymbol{\Omega} \cdot \mathbf{I}^1(t, \mathbf{x}, \nu) \right) d\boldsymbol{\Omega} = I^0,$$

and

$$\mathbf{V} = \int_{S^2} \left(I^0(t, \mathbf{x}, \nu) + \boldsymbol{\Omega} \cdot \mathbf{I}^1(t, \mathbf{x}, \nu) \right) \boldsymbol{\Omega} d\boldsymbol{\Omega} = \frac{1}{3} \mathbf{I}^1,$$

which yields $I(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) = U + 3\boldsymbol{\Omega} \cdot \mathbf{V}$. The second order moment (closure) can also easily be obtained by the relation

$$\mathbf{P}_r = \int_{S^2} \left(I^0(t, \mathbf{x}, \nu) + \boldsymbol{\Omega} \cdot \mathbf{I}^1(t, \mathbf{x}, \nu) \right) \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} d\boldsymbol{\Omega} = \frac{1}{3} \mathbf{I}_d I^0.$$

Finally, the P_1 model writes

$$\begin{cases} \frac{1}{c} \partial_t U + \nabla \cdot \mathbf{V} = \sigma_a (B(\nu, T) - U), \\ \frac{1}{c} \partial_t \mathbf{V} + \frac{1}{3} \nabla U = -\sigma_s \mathbf{V}, \\ \partial_t T = -c\sigma_a \int_{\mathbb{R}^+} (B(\nu, T) - U) d\nu. \end{cases} \quad (1.21)$$

If one takes $\sigma_a = 0$, one finds the well-known hyperbolic heat equations. The P_1 model will be considered in this work in the context of approximate Compton scattering (anisotropic Kompaneets equation).

The P_N models are in practice widely used since their polynomial dependence to $\boldsymbol{\Omega}$ allows an easy increasing in the number of moments. On the other hand, these models does not satisfy in general some physically relevant properties, such as the increase of the entropy and the preservation of the realizability domain, defined as the set of solutions $(U, \mathbf{V}) \in \mathbb{R}^4$ of (1.21) satisfying $U \geq 0$ and $|\mathbf{V}|/U < 1$.

Note also that it is possible to consider the grey P_1 model by integrating the previous P_1 model with respect to the frequency. It yields

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E), \\ \frac{1}{c} \partial_t \mathbf{F} + \frac{1}{3} \nabla E = -\sigma_s \mathbf{F}, \\ \partial_t T = -c\sigma_a (T^4 - E). \end{cases}$$

M_N models

The M_N models are another class of angular moments models [DF99, MBD15, MBD14]. As for the P_N models, the idea is to prescript the angular behavior of the radiative intensity. On the contrary to the P_N models, this behavior is nonlinear. The specific intensity is assumed to be given by

$$I(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) = \nu^3 \left[\exp \left(\frac{\nu}{a(t, \mathbf{x})} (1 + \mathbf{b}(t, \mathbf{x}) \cdot \boldsymbol{\Omega}) \right) - 1 \right]^{-1} \quad (1.22)$$

This expression is obtained by solving an entropy minimization problem under the constraint of solving the moments (1.20) [LEV84, DF99], a and \mathbf{b} being the Lagrange multipliers of the minimization problem. On the contrary to the P_1 model, the two cases frequency dependent / grey models are very different. Indeed, in the frequency dependent case the two first moments U and \mathbf{V} can not be analytically given in term of the Lagrange multipliers a and \mathbf{b} . The same problem arises for the closure, i.e. the second order moment \mathbf{Q}_r .

On the contrary, in the grey case the calculations can be achieved. This is due to the integrability with respect to the frequency of the specific intensity (1.22). Let us explain this by calculating the zero-th order moment E . By definition,

$$E = \int_{\mathbb{R}^+} \int_{S^2} \nu^3 \left[\exp \left(\frac{\nu}{a(t, \mathbf{x})} (1 + \mathbf{b}(t, \mathbf{x}) \cdot \boldsymbol{\Omega}) \right) - 1 \right]^{-1} d\boldsymbol{\Omega} d\nu$$

Removing the space and time dependence for ease of notations, one finds by performing the change of variables $\nu \mapsto \frac{\nu}{a}(1 + \mathbf{b} \cdot \boldsymbol{\Omega})$

$$E = a^4 \int_{S^2} \frac{d\boldsymbol{\Omega}}{(1 + \mathbf{b} \cdot \boldsymbol{\Omega})^4} \times \int_{\mathbb{R}^+} \frac{\nu^3}{e^\nu - 1} d\nu = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f d\boldsymbol{\Omega} d\nu = a^4 \frac{3 + \|\mathbf{b}\|^2}{3(1 - \|\mathbf{b}\|^2)^3}$$

For the first order moment \mathbf{F} , one finds $\mathbf{F} = -4a^4 \mathbf{b} / 3(1 - \|\mathbf{b}\|^2)^3$. Finally, the second order moment \mathbf{P}_r is given by [TUR05, BD04]

$$\mathbf{P}_r = \int_{\mathbb{R}^+ \times S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I d\nu d\boldsymbol{\Omega} = \left((1 - \chi) \mathbf{I}_d + (3\chi - 1) \frac{\mathbf{g} \otimes \mathbf{g}}{\|\mathbf{g}\|^2} \right) E,$$

where $\mathbf{g} = \mathbf{F}/E$ is the normalized flux and $\chi = (3 + 4\|\mathbf{g}\|^2) / (5 + 2\sqrt{4 - 3\|\mathbf{g}\|^2})$ is the Eddington factor [LEV84].

In the frequency dependent case, a generalization of this closure has been proposed by R. Turpault [TUR05, TUR12] in the context of a multigroup discretization. In this work we consider the grey M_1 model in the context of the Compton scattering, leading to nonlinear source terms.

Angular discretization: the S_N methods

In this part we make a brief introduction to the S_N methods [GJL99, JL91]. The idea is to discretize directly the transfer equation with respect to the angle. Let us explain it on a simple transfer equation without matter coupling

$$\frac{1}{c} \partial_t I + \boldsymbol{\Omega} \cdot \nabla I = \sigma_s \left(\int_{S^2} I(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I \right).$$

Let us define a discretization of the sphere S^2 by N vectors $\boldsymbol{\Omega}_i$, $i \in \{1, N\}$. One defines I_i as an approximation of $I(t, \mathbf{x}, \nu, \boldsymbol{\Omega}_i)$. The angular integral is approximated with a quadratic formula

$$\int_{S^2} I(\boldsymbol{\Omega}') d\boldsymbol{\Omega}' = \sum_{i=1}^N \omega_i I_i,$$

where the ω_i are the quadratic weights. One thus obtains

$$\frac{1}{c} \partial_t I_i + \boldsymbol{\Omega}_i \cdot \nabla I_i = \sigma_s \left(\sum_{j=1}^N \omega_j I_j - I_i \right), \quad 1 \leq i \leq N.$$

To obtain the correct diffusion equation in the equilibrium regime, the weights must satisfy [FRA12]

$$\sum_{i=1}^N \omega_i \boldsymbol{\Omega}_i \otimes \boldsymbol{\Omega}_i = \frac{1}{3} \mathbf{I}.$$

1.3.2 Source terms reductions: the case of the Compton scattering

Since it takes an important part of this thesis, we present the Kompaneets equation [KOM57]

$$\nu^2 \partial_t f = \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \frac{\partial}{\partial \nu} f + f(1+f) \right) \right],$$

where T is the macroscopic temperature of the electrons. The classical flux condition

$$\lim_{\nu \rightarrow 0} \nu^4 \left(T \frac{\partial}{\partial \nu} f + f(1+f) \right) = \lim_{\nu \rightarrow \infty} \nu^4 \left(T \frac{\partial}{\partial \nu} f + f(1+f) \right) = 0, \quad (1.23)$$

are considered. This equation approximates the Boltzmann equation (1.4)–(1.7) under the assumption of small energy exchange during the collision. Historically derived by A. S. Kompaneets [KOM57], this equation has been widely studied, from the numerical point of view [BC03, CL86, CC70] as well as from the mathematical point of view [EHV98, EM01, KAV02]. The Kompaneets equation preserves several physical properties of the Boltzmann equation, such as the conservation of the total number of photons, the Boltzmann equilibrium states, the positivity of the distribution function and the growth of the physical entropy. On the other hand, Escobedo and co-authors [EHV98] show that the flux condition (1.23) may be broken in finite time at $\nu \rightarrow 0$.

Let us explain the derivation of this equation performed by Escobedo and co-authors in [EMV03]. Starting from the Boltzmann equation (1.4), with the collision operator given by (1.9), one first assumes that the distribution function of the photons f is isotropic. Due to this assumption, the collision operator is now only an integral operator with respect to the outgoing photons $\nu' \in \mathbb{R}^+$. They assume that the energy of the ingoing and outgoing photons is small compared to the energy of the electron at rest mc^2 , i.e. $\nu \ll mc^2$ and $\nu' \ll mc^2$. They perform a Taylor expansion of $f(\nu')$ around the frequency ν . Inserted in the collision operator $Q(f_\gamma)$ (where we removed the dependence with respect to the electrons since they are assumed to be given), the collision operator now only depends on integrals of the kernel b (1.10) multiplied by power of $|\nu' - \nu|$. These integrals do not depend on the distribution function f . What is interesting in this derivation is that the computation of integrals is explicitly given.

Extensions of the Kompaneets equation to anisotropic distributions exist in the physical literature [BPR69, FKM85, PL97, POM73]. We propose a generalization of the Escobedo [EMV03] approach to anisotropic distributions.

1.4 Numerical issues

As explained in the introduction, the Boltzmann equation may be too costly to be discretized in its general form. Some work can be nevertheless found in this direction [BC03]. From a general point of view, we study two particular types of numerical schemes: the well-balanced schemes (part 1) and the asymptotic preserving schemes (part 2). These two kind of numerical

schemes can be of great interest in the case of the radiative transfer. Firstly, as explained in the introduction, the coupling between radiation and hydrodynamics involved very different velocities, the light often travelling much faster than the gas. One can thus expect the radiation to go quickly to a stationary state. The well-balanced schemes are thus interesting, since they will exactly capture this stationary state. Secondly we have seen that in radiative transfer issues, diffusion approximations can be used in different regimes. These diffusion equations are simpler to approximate, since the distribution function no longer depends on the photons direction Ω . It is thus interesting to design schemes that solve the kinetic equation in general, and only the diffusion equation when it is possible. This is the aim of the asymptotic preserving schemes.

1.4.1 Well-balanced schemes

The well-balanced schemes [JIN99, JIN10] are, by definition, schemes which preserve the stationary states, i.e. will exactly capture numerically the stationary states of the equation. They are justified in our case since as explained above, the coupling between the hydrodynamics and radiation involve different time scales. Thus, for each hydrodynamic time step, the radiation quickly goes to an equilibrium state, which is exactly captured by a well-balanced scheme. Let us present here the Greenberg-Leroux method [GL96] in a simple transport equation in 1D, with source and dumping terms

$$\begin{cases} \partial_t f + a \partial_\nu f = \sigma_a (B(\nu) - f), \\ f(0, \nu) = f^{in}(\nu), \end{cases} \quad (1.24)$$

where $B(\nu)$ is the Planck function (with a temperature $T = 1$), $a \in \mathbb{R}^*$ is a transport coefficient and σ_a is a non negative coefficient. The design of well-balanced schemes requires the knowledge of the stationary solution associated to (1.24). We thus consider the following problem

$$\begin{cases} a \partial_\nu \bar{f} = \sigma_a (B(\nu) - \bar{f}), \\ \bar{f}(\nu^*) = \bar{f}^*, \end{cases} \quad (1.25)$$

whose analytical solution is given by

$$\bar{f}(\nu; \bar{f}^*, \nu^*) = \bar{f}^* e^{-\frac{\sigma_a}{a}(\nu - \nu^*)} + \frac{\sigma_a}{a} \int_{\nu^*}^{\nu} B(s) e^{\frac{\sigma_a}{a}(s - \nu)} ds.$$

Considering a cartesian frequency mesh, the Greenberg-Leroux schemes that we study in this work are defined by

$$\begin{cases} \frac{f_i^{n+1} - f_i^n}{\Delta t} = -a \frac{f_i^n - \bar{f}(\nu_j; f_{i-1}^n; \nu_{i-1})}{\Delta \nu}, & a > 0, \\ \frac{f_i^{n+1} - f_i^n}{\Delta t} = -a \frac{\bar{f}(\nu_j; f_{i+1}^n; \nu_{i+1}) - f_i^n}{\Delta \nu}, & a < 0. \end{cases} \quad (1.26)$$

Despite being well-balanced, this kind of schemes are not consistent as the wave velocity a tends to zero. Such a scheme will be designed and studied in chapter 3 in the context of the non equilibrium regime for the relativistic transfer equation.

1.4.2 Asymptotic preserving schemes

The second wide class of costless numerical scheme is the asymptotic preserving schemes. Let us explain the main idea in the simple case of the grey P_1 model, the hyperbolic (uncoupled)

heat equations derived above (where we removed the $1/3$ for simplicity). Arguing like above, one can study the non-equilibrium diffusion regime for this system. Assuming that $\sigma_s \gg 1$ and $c \gg 1$, one writes the rescaled system

$$(P^\varepsilon) : \begin{cases} \partial_t E + \frac{1}{\varepsilon} \nabla \cdot \mathbf{F} = 0, \\ \partial_t F + \frac{1}{\varepsilon} \nabla E = -\frac{\sigma_s}{\varepsilon^2} \mathbf{F}. \end{cases} \quad (1.27)$$

One finds, in the regime $\varepsilon \rightarrow 0$, the following diffusion equation

$$(P^0) : \frac{1}{c} \partial_t E - \nabla \cdot \left(\frac{\nabla E}{\sigma_s} \right) = 0. \quad (1.28)$$

Numerically, this equation is much simpler to discretize, since the angular dependence has been removed and there is only one unknown. Thus, if a part of the physical domain satisfies the above assumptions, it would be interesting to solve only the diffusion equation. This is the idea of asymptotic preserving schemes. These are schemes defined for system (1.27), which depends on the parameter ε and denoted P_h^ε . The important point is that the limiting scheme P_h^0 , as $\varepsilon = 0$, is a scheme consistent with equation (1.28) (see figure 1.4). We illustrate this point with

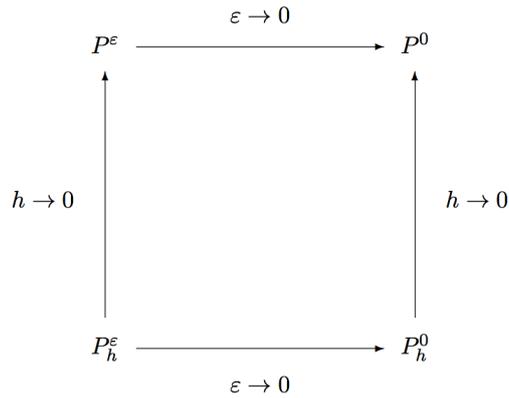


Figure 1.4: The AP diagram

the Gosse-Toscani' scheme [GT02], whose uniform convergence on unstructured meshes will be proved in chapter . In 1D and for a uniform mesh, this scheme is defined by

$$\begin{cases} \frac{d}{dt} E_j + M \frac{F_{j+1} - F_{j-1}}{2\varepsilon \Delta x} - M \frac{E_{j+1} + E_{j-1} - 2E_j}{2\varepsilon \Delta x} = 0 \\ \frac{d}{dt} F_j + M \frac{E_{j+1} - E_{j-1}}{2\varepsilon \Delta x} - M \frac{F_{j+1} + F_{j-1} - 2F_j}{2\varepsilon \Delta x} = -\frac{\sigma}{\varepsilon^2} M F_j, \end{cases} \quad (1.29)$$

where the coefficient M is defined by $M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x}$. On the one hand, if $\Delta x/\varepsilon \rightarrow 0$, one finds $M \rightarrow 1$ and it is an easy matter to check that this scheme is consistent with system (1.27). On the other hand, let us now assume that $\varepsilon/\Delta x \rightarrow 0$. We study the behavior of the scheme (1.29) by mean of the modify equation, i.e. we write the scheme as

$$\begin{cases} \partial_t E_j + \frac{M}{\varepsilon} (\partial_x F)_j - \Delta x \frac{M}{2\varepsilon} (\partial_{xx} E)_j = 0 \\ \partial_t F_j + \frac{M}{\varepsilon} (\partial_x E)_j - \Delta x \frac{M}{2\varepsilon} (\partial_{xx} F)_j = -\frac{\sigma}{\varepsilon^2} M F_j, \end{cases} \quad (1.30)$$

From the second equation of (1.30), one formally writes

$$F_j = -\frac{\varepsilon}{\sigma_s}(\partial_x E)_j + \mathcal{O}(\varepsilon),$$

which yields in the first equation of (1.30)

$$\partial_t E_j - M\left(\frac{1}{\sigma_s} + \frac{\Delta x}{2\varepsilon}\right)(\partial_{xx} E)_j = 0.$$

Here the importance of the coefficient M is highlighted. Indeed, since $\frac{1}{\sigma_s} + \frac{\Delta x}{2\varepsilon} = \frac{1}{\sigma_s M}$, one obtains

$$\partial_t E_j - \frac{1}{\sigma_s}(\partial_{xx} E)_j = 0.$$

A complete chapter of this document will be dedicated to the study of the Gosse-Toscani' scheme in 1 and 2D.

Chapter 2

Relativistic transfer equations: maximum principle and convergence to the non-equilibrium diffusion regime

This chapter is taken from a published article [LER15].

2.1 Introduction

We study some mathematical properties of a system describing the coupling between the relativistic transfer equation for photons and an equation describing the temperature of a fluid moving at the velocity \mathbf{u} . This kind of models was historically derived by physicists ([MWM99, POM73]) and some mathematical properties as existence, uniqueness and a maximum principle have been proved in the non relativistic case [GP86]. The system writes

$$\begin{cases} \frac{1}{c}\partial_t I + \boldsymbol{\Omega} \cdot \nabla_x I = Q_t & \text{in } [0, T^f] \times X \times \mathbb{R}_\nu^+ \times S^2, \\ \partial_t T + \nabla \cdot (T\mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u} = -c \int_{\nu, \Omega} \frac{\Lambda}{\gamma} Q_t d\nu d\Omega & \text{in } [0, T^f] \times X, \end{cases} \quad (2.1)$$

where $I = I(t, x, \nu, \boldsymbol{\Omega})$ is the radiative intensity, $T = T(t, x)$ the fluid temperature, $\mathbf{u} = \mathbf{u}(x)$ the fluid velocity, $t \in [0, T^f]$ for a given $0 < T^f < +\infty$ is the time, $x \in X$ is the position of the photons, where $X \subset \mathbb{R}^3$ is a bounded domain, $\nu \in \mathbb{R}_\nu^+$ is the frequency of the photons, $\boldsymbol{\Omega} \in S^2$ is the direction of the photons, c is the speed of light and $\Gamma > 0$ is the adiabatic constant. The operator $Q_t = Q_s + Q_a$ is composed of a scattering operator Q_s and an emission absorption operator Q_a , and describes the interactions between the light and the fluid. The Lorentz factor γ and the relativistic coefficient Λ are given by

$$\begin{cases} \gamma = \frac{1}{\sqrt{1 - |\mathbf{u}|^2/c^2}}, \\ \Lambda = \frac{1 - \boldsymbol{\Omega} \cdot \mathbf{u}/c}{\sqrt{1 - |\mathbf{u}|^2/c^2}}. \end{cases} \quad (2.2)$$

Due to the velocity of the fluid, the physical quantities defined previously can be computed in several frames. The derivation of the radiation hydrodynamics equations from the Lorentz

transformations and the relations between the different frames is a major issue, discussed in [POM73, MWM99]. In this work we consider two frames: a reference frame (typically the laboratory frame) and the fluid frame, in which the velocity of the fluid is zero. The fluid is considered as non relativistic (see [BD04] for a presentation of the Euler equations in the relativistic case), and thus only the measure of the physical quantities relative to the photons, that is the frequency ν , the direction $\boldsymbol{\Omega}$ and the radiative intensity I , are different according to the considered frame.

We adopt the following notations. The quantities measured in the moving frame are denoted with a subscript 0, while the notations without subscripts are relative to quantities measured in the reference frame. With these notations, the relation between the frequency ν of a photon measured in the reference frame and its frequency ν_0 measured in the moving frame is [POM73]

$$\nu_0 = \Lambda\nu. \quad (2.3)$$

In the same way, the relation between the direction $\boldsymbol{\Omega}$ of a photon in the reference frame and its direction $\boldsymbol{\Omega}_0$ in the moving frame is

$$\boldsymbol{\Omega}_0 = \frac{\nu}{\nu_0} \left(\boldsymbol{\Omega} - \frac{\gamma}{c} \mathbf{u} \left(1 - \frac{\boldsymbol{\Omega} \cdot \mathbf{u}}{c} \frac{\gamma}{\gamma + 1} \right) \right). \quad (2.4)$$

A fundamental property is the invariance by change of frame of the photons density distribution function $f = aI/\nu^3$, where a is a physical constant. This leads to a relation between the radiative intensity I measured in the reference frame and I_0 measured in the moving frame

$$I(\nu, \boldsymbol{\Omega}) = \Lambda^{-3} I_0(\nu_0, \boldsymbol{\Omega}_0). \quad (2.5)$$

As usual, the operator Q_t is the sum of a scattering operator Q_s and an emission absorption operator Q_a . The scattering operator models the diffusion phenomena between the light and the fluid. We assume that the scattering is coherent (no energy exchange) and isotropic (in the fluid frame). Under these assumptions, Q_s is defined by

$$Q_s = \sigma_s(x) \Lambda \left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I \right), \quad (2.6)$$

where, for ease of notations, we defined $\Lambda' = \gamma(1 - \boldsymbol{\Omega}' \cdot \mathbf{u}/c)$ and the measure $d\boldsymbol{\Omega}'$ such as $\int_{S^2} d\boldsymbol{\Omega}' = 1$. The frequency ν' quantifies the Doppler effects and is defined as $\nu' = \frac{\Lambda'}{\Lambda} \nu$. The coefficient σ_s is the scattering cross section and is assumed to depends only on the position x . We define $Q_{s,0}$ as the scattering operator measured in the moving frame

$$Q_{s,0} = \Lambda^2 Q_s = \sigma_s(x) \left(\int_{S_0^2} I_0 d\boldsymbol{\Omega}_0 - I_0 \right), \quad (2.7)$$

where, again, the measure $d\boldsymbol{\Omega}_0$ is such that $\int_{S_0^2} d\boldsymbol{\Omega}_0 = 1$. One recognizes in this expression the classical non relativistic scattering operator. Although the scattering is isotropic in the moving frame, i.e. $\int_{S_0^2} Q_{s,0} d\boldsymbol{\Omega}_0 = 0$, this is not true in the reference frame, due to the relativistic effects. The operator Q_a is the emission absorption operator. It is defined by

$$Q_a = \sigma_a(\Lambda\nu) \Lambda \left(\frac{B(\Lambda\nu, T)}{\Lambda^3} - I \right), \quad (2.8)$$

where $B(\nu, T) = \nu^3 (e^{\nu/T} - 1)^{-1}$ is the (normalized) Planck function. The coefficient σ_a is the emission absorption coefficient and is assumed to depend only on the frequency ν . The emission absorption measured in the moving frame is

$$Q_{a,0} = \Lambda^2 Q_a = \sigma_a(\nu_0) \left(B(\nu_0, T) - I_0 \right). \quad (2.9)$$

Once again, one recognizes in (2.9) the classical non relativistic emission absorption operator. It quantifies the gap between the radiation and the Planck function, which corresponds to the stationary states of the transfer equation in the non relativistic regime and at the local thermodynamical equilibrium (ETL). It shows in particular that the relativistic effects modify the stationary states of the transfer equation, from $I = B(\nu, T)$ in the non relativistic case to $I = \Lambda^{-3} B(\Lambda\nu, T)$ in the relativistic case.

The derivation of system (2.1) from the coupling between the Euler equations and the relativistic transfer equation is explained for example in [GLG05]: assuming a given fluid density ρ and a given velocity field \mathbf{u} , the equation of the internal energy is obtained by deducting to the equation on the total energy the equation on the kinetic energy. An important consequence of the absence of kinetic energy balance is that this system does not preserve the physical energy $\int I dx d\nu d\Omega + \int T dx$. Indeed, one has

$$\frac{d}{dt} \left(\int_{x,\nu,\Omega} I dx d\nu d\Omega + \int_x T dx \right) + \Gamma \int_x T \nabla \cdot \mathbf{u} dx = \int_{x,\nu,\Omega} (\mathbf{\Omega} \cdot \mathbf{u}) Q_t dx d\nu d\Omega.$$

The first remaining term comes from the hydrodynamic pressure. The second one corresponds to the variation of the kinetic energy of the fluid, which is not taken into account by assuming a given velocity. Due to this non conservation of the energy, the maximum principle proved by F. Golse and B. Perthame in [GP86] for the non relativistic transfer equation does not hold any more, and this leads to mathematical issues.

Despite a simple modelling of the hydrodynamic of the fluid, the system (2.1) can be used as a first approach to model the expansion of stellar atmospheres or inertial confinement fusion (ICF) experiments. In this last case, the shocks created by the bombardment of a small ball of gas struck by powerful photons beam eject a part of the gas. It is known that if the relativistic effects are not considered, a part of the physics, and in particular the radiative pressure, is not taken into account. This explains the importance of the relativistic correction terms, even for a non relativistic fluid. As a remark, the classical non relativistic transfer equations as studied in [DOG01, GP86] can be obtained from (2.1) by taking the limit $|\mathbf{u}|/c \rightarrow 0$.

In this chapter we prove two main results. In section 2, we prove a comparison principle for the system (2.1) by means of a suitable modification of the Golse and Perthame approach [GP86]. This result enables to write the system (2.1) as a Lipschitz perturbation of a linear transport equation, from which existence and uniqueness of a solution is obtained. Since our aim is to study the influence of a moving fluid on the radiative intensity, the coefficients σ_a , σ_s and \mathbf{u} in the system (2.1) will be taken as smooth as necessary. Several theoretical results for more realistic emission absorption coefficient can be found in the non relativistic case in [GP86, BGP87, BGPS88] (see also [YS14] in the context of radiation hydrodynamics). The second main result (section 3) is the proof of convergence with respect to a small parameter ε , formally equal to $|\mathbf{u}|/c$, of the solution of the system (2.1) to the solution of a drift diffusion system in the non-equilibrium

diffusion regime. This regime is obtained by assuming that the scattering is dominant in comparison with the emission absorption, and that the speed of light is large in comparison with the fluid velocity. After rescaling with a small parameter ε , it yields the following system

$$\begin{cases} \partial_t I^\varepsilon + \frac{1}{\varepsilon} \boldsymbol{\Omega} \cdot \nabla_x I^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon + Q_a^\varepsilon & \text{in } [0, T^f] \times X \times \mathbb{R}_\nu^+ \times S^2, \\ \partial_t T^\varepsilon + \nabla \cdot (T^\varepsilon \mathbf{u}) + \Gamma T^\varepsilon \nabla \cdot \mathbf{u} = - \int_{\nu, \Omega} \frac{\Lambda^\varepsilon}{\gamma^\varepsilon} \left(\frac{1}{\varepsilon^2} Q_s^\varepsilon + Q_a^\varepsilon \right) d\nu d\Omega & \text{in } [0, T^f] \times X, \end{cases} \quad (2.10)$$

with obvious notations for Q_s^ε and Q_a^ε , and where $\gamma^\varepsilon = \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}^{-1}$ and $\Lambda^\varepsilon = \gamma^\varepsilon (1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})$. The drift diffusion system writes

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s(x)} \right) + \nabla \cdot (\rho \mathbf{u}) = \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu \rho + \sigma_a(\nu) (B(\nu, T) - \rho) & \text{in } [0, T^f] \times X \times \mathbb{R}_\nu^+, \\ \partial_t T + \nabla \cdot (T \mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u} = - \int_\nu \sigma_a(\nu) (B(\nu, T) - \rho) d\nu & \text{in } [0, T^f] \times X, \end{cases} \quad (2.11)$$

where $\rho = \lim_{\varepsilon \rightarrow 0} \int_{S^2} I^\varepsilon d\Omega$ is the first angular moment of the radiative intensity. This equation has been formally derived by D. Mihalas and B. Weibel Mihalas in [MWM99]. The drift term $\frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu \rho$ that models the Doppler effects is also involved in an equation proposed by A. Winslow in [WIN95]. To our knowledge, the mathematical justification of the diffusion system (2.11) that is provided by means of a convergence result with respect to ε is original.

To obtain this convergence result some assumptions will be done on the regularity of the parameters. In particular the emission absorption coefficient will be assumed to belong to $L^2(\mathbb{R}_\nu^+)$, which has no physical meaning, but for technical reasons this assumption is necessary to obtain the convergence in L^2 . Indeed, the L^1 norm would be more natural since the quantities we are interested in, namely I and T , are homogeneous to energies. We make use of the L^2 norm since a priori estimates for the solution of the drift diffusion system (2.11) are much easily obtained in this norm. In return the proof of convergence is technical, which explains the division of this proof in several steps (section 3).

In the chapter we will use the following notations. The angular integral will be denoted $\langle \cdot \rangle$, i.e. $\langle f \rangle = \int_{S^2} f d\Omega$. The space variable x belongs to X , the frequency variable ν to \mathbb{R}_ν^+ and the time $t \in [0, T^f]$, for a given $0 < T^f < +\infty$. We denote by $\|\cdot\|_{L_{x,\nu,\Omega}^p}$ (respectively $\|\cdot\|_{L_{t,x}^p}$) the classical L^p norm on $X \times \mathbb{R}_\nu^+ \times S^2$ (respectively on $[0, T^f] \times X$), for a given $1 \leq p \leq +\infty$. We define the function sgn^+ as

$$\text{sgn}^+(f) = \begin{cases} 1 & f > 0, \\ 0 & f \leq 0, \end{cases} \quad (2.12)$$

and the positive part f^+ of a function f as $f^+ = f \text{sgn}^+(f)$. The measure in integrals will not be written, i.e. $\int_x \cdot = \int_X \cdot dx$, $\int_\nu \cdot = \int_{\mathbb{R}_\nu^+} \cdot d\nu$, $\int_\Omega \cdot = \int_{S^2} \cdot d\Omega$, ...

The chapter is organized as follows. In the next section we prove a comparison principle (theorem 2.1) for the relativistic transfer equation (2.1), from which the existence of solutions is obtained (lemma 2.2). The section 3 is devoted to the study of the non-equilibrium diffusion regime and is divided in two main parts. In the first part the drift diffusion system is obtained by means of formal Hilbert expansion of the solution of the transfer equations (2.1). In the second part

we prove a priori estimates for the drift diffusion system (2.11), such as a comparison principle (lemma 2.6) and several regularity results (lemma 2.7). We also introduce the main result, which is the theorem 2.8 of convergence of the solution of the relativistic transfer equations (2.10) to the solution of the drift diffusion system (2.11). The end of the second part deals with the proof of convergence, based on a reconstruction procedure, an original comparison principle and an appropriate weight in the lemma 2.12. The chapter ends with two appendices containing some technical results.

2.2 A comparison principle for the relativistic transfer equations

In this section we prove several results for the relativistic transfer equations (2.1). The main result is the following comparison principle (theorem 2.1), from which the existence result (lemma 2.2) follows easily using classical technics. To prove this comparison principle we need some assumptions

- (H1) Smoothness of the velocity field: $\mathbf{u} \in W^{1,\infty}([0, T^f] \times X)$. Moreover, $u^* = \|\mathbf{u}\|_{L_{t,x}^\infty}$ satisfies $u^* < c$, where c is the speed of light.
- (H2) Smoothness of the scattering coefficient: $\sigma_s \in W^{1,\infty}(X)$ and $\sigma_s > 0$.
- (H3) Smoothness of the emission absorption coefficient: $\sigma_a \in L^\infty(\mathbb{R}_\nu^+)$ and $\sigma_a \geq 0$.
- (H4) There exists two bounded and positive constants l_* and L_* which are respectively the infimum and the supremum of the temperature at the initial time: $l_* \leq T(t=0) \leq L_*$. Besides the radiative intensity at the initial time satisfies $B(\nu_0, l_*) \leq I_0(t=0) \leq B(\nu_0, L_*)$.
- (H5) We consider periodic boundary conditions.
- (H6) The velocity field is smooth $\mathbf{u} \in \mathcal{C}^1([0, T^f] \times X)$ and the emission absorption coefficient σ_a is integrable, i.e. $\sigma_a \in L^1(\mathbb{R}_\nu^+)$.

The assumption (H6) is used only to prove that the relativistic transfer system (2.1) is a Lipschitz perturbation of a \mathcal{C}^0 semigroup (lemma 2.2). In the forthcoming proofs, we will often use the following bounds, which easily come from the previous assumption on the velocity field, and where the constants $\Lambda_*, \Lambda^* \geq 0$ only depends on u^*

$$0 < \Lambda_* \leq \Lambda(t, x, \mathbf{\Omega}) \leq \Lambda^*, \quad \forall (t, x, \mathbf{\Omega}) \in [0, T^f] \times X \times S^2. \quad (2.13)$$

We introduce the first main result.

Theoreme 2.1 (Comparison principle). *We assume that hypotheses (H1)-(H5) are satisfied. Then, for all $(t, x, \nu) \in [0, T^f] \times X \times \mathbb{R}_\nu^+$, one has the a priori estimates $l(t) \leq T(t) \leq L(t)$ and $B(\nu_0, l(t)) \leq I_0(t) \leq B(\nu_0, L(t))$, where*

$$\begin{cases} l(t) = l_* \exp \left\{ -t \left[(\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}} + 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right) \right] \right\}, \\ L(t) = L_* \exp \left\{ t \left[(\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}} + 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right) \right] \right\}. \end{cases} \quad (2.14)$$

This result shows that although the system does not conserve the total energy, due to the absence of kinetic energy balance, energy does not blow up in finite time, and one can give lower and upper bounds on the radiative energy and on the fluid temperature. As a remark, the Golse and Perthame result [GP86] can be obtained by setting $\mathbf{u} = 0$.

Proof. Since the arguments are the same, we only show the proof for the upper bound. This result is a suitable modification of the maximum principle proved by F. Golse and B. Perthame [GP86] in the non relativistic case. The method is based on varying bounds for which the point is to get the equations that define these bounds. In order to simplify the notations, we denote $B_{0,L} = B(\nu_0, L(t))$. The system (2.1) can be simplified. Indeed, using the invariance of the measure $\nu d\nu d\mathbf{\Omega}$ and the isotropy of the scattering operator in the fluid frame, one has

$$\int_{\nu,\Omega} \frac{\Lambda}{\gamma} Q_t = \int_{\nu,\Omega} \frac{1}{\Lambda\gamma} Q_{t,0} = \int_{\nu_0,\Omega_0} \frac{1}{\gamma} Q_{a,0}, \quad (2.15)$$

and thus system (2.1) reduces to

$$\begin{cases} \frac{1}{c} \partial_t I + \mathbf{\Omega} \cdot \nabla_x I = Q_t & \text{in } [0, T^f] \times X \times \mathbb{R}_\nu^+ \times S^2, \\ \partial_t T + \nabla \cdot (T\mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u} = -\frac{c}{\gamma} \int_{\nu_0,\Omega_0} Q_{a,0} & \text{in } [0, T^f] \times X. \end{cases} \quad (2.16)$$

Multiplying the first equation of (2.1) by $c \frac{\Lambda}{\gamma} \text{sgn}^+(I_0 - B_{0,L})$, integrating over $\mathbb{R}_\nu^+ \times S^2$ and reminding that $I(\nu, \mathbf{\Omega}) = \Lambda^{-3} I_0(\nu_0, \mathbf{\Omega}_0)$, one can deduce

$$\begin{aligned} & \int_{\nu,\Omega} \frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2 \gamma} \left(\partial_t I_0 + c \mathbf{\Omega} \cdot \nabla I_0 \right) \\ & + \int_{\nu,\Omega} \text{sgn}^+(I_0 - B_{0,L}) I_0 \frac{\Lambda}{\gamma} \left(\partial_t \Lambda^{-3} + c \mathbf{\Omega} \cdot \nabla \Lambda^{-3} \right) = c \int_{\nu,\Omega} \frac{\Lambda \text{sgn}^+(I_0 - B_{0,L})}{\gamma} Q_t. \end{aligned}$$

By expanding the derivatives of Λ^{-3} and by using the invariance of the measure $\nu d\nu d\mathbf{\Omega}$, one gets after algebraic manipulations

$$\begin{aligned} & \int_{\nu,\Omega} \left(\partial_t \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} + c \mathbf{\Omega} \cdot \nabla \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} \right) \\ & + \int_{\nu,\Omega} \frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2 \gamma} \left(\partial_t B_{0,L} + c \mathbf{\Omega} \cdot \nabla B_{0,L} \right) \\ & - \int_{\nu,\Omega} (I_0 - B_{0,L})^+ \left(\partial_t \frac{1}{\Lambda^2 \gamma} + c \mathbf{\Omega} \cdot \nabla \frac{1}{\Lambda^2 \gamma} \right) - 3 \int_{\nu,\Omega} \frac{(I_0 - B_{0,L})^+}{\Lambda^3 \gamma} \left(\partial_t \Lambda + c \mathbf{\Omega} \cdot \nabla \Lambda \right) \\ & - 3 \int_{\nu,\Omega} B_{0,L} \frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^3 \gamma} \left(\partial_t \Lambda + c \mathbf{\Omega} \cdot \nabla \Lambda \right) = \frac{c}{\gamma} \int_{\nu_0,\Omega_0} \text{sgn}^+(I_0 - B_{0,L}) Q_{t,0}. \end{aligned}$$

Regrouping the terms of the second line together, it yields after rearrangements

$$\begin{aligned} & \int_{\nu,\Omega} \left(\partial_t \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} + c \mathbf{\Omega} \cdot \nabla \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} \right) \\ & - \int_{\nu,\Omega} (I_0 - B_{0,L})^+ \Lambda^{-3} \left(\partial_t \frac{\Lambda}{\gamma} + c \mathbf{\Omega} \cdot \nabla \frac{\Lambda}{\gamma} \right) \\ & = \int_{\nu,\Omega} \frac{\text{sgn}^+(I_0 - B_{0,L})}{\Lambda^2 \gamma} \left\{ 3 B_{0,L} \left(\frac{\partial_t \Lambda}{\Lambda} + c \frac{\mathbf{\Omega} \cdot \nabla \Lambda}{\Lambda} \right) - \partial_t B_{0,L} - c \mathbf{\Omega} \cdot \nabla B_{0,L} \right\} \\ & + \frac{c}{\gamma} \int_{\nu_0,\Omega_0} \text{sgn}^+(I_0 - B_{0,L}) Q_{t,0}. \end{aligned}$$

We remind that $B(\nu_0, L(t)) = \frac{\nu_0^3}{e^{\nu_0/L(t)} - 1} = \frac{(\Lambda\nu)^3}{e^{\Lambda\nu/L(t)} - 1}$. It yields

$$\partial_t B(\nu_0, L(t)) = \frac{3\Lambda^2\nu^3\partial_t\Lambda}{e^{\nu_0/L(t)} - 1} - \frac{\nu L\partial_t\Lambda - \nu_0\partial_t L}{L^2} \frac{e^{\nu_0/L(t)}}{e^{\nu_0/L(t)} - 1} \frac{\nu_0^3}{e^{\nu_0/L(t)} - 1},$$

which can be written

$$\partial_t B(\nu_0, L(t)) = B(\nu_0, L(t)) \left(3\partial_t(\log \Lambda) - \frac{\nu_0}{L(t)} \frac{\partial_t(\log \Lambda) - \partial_t(\log L(t))}{1 - e^{-\nu_0/L(t)}} \right).$$

The same manipulations lead to

$$\nabla B(\nu_0, L(t)) = B(\nu_0, L(t)) \left(3\nabla(\log \Lambda) - \frac{\nu_0}{L(t)} \frac{\nabla(\log \Lambda)}{1 - e^{-\nu_0/L(t)}} \right).$$

The previous equation thus becomes

$$\begin{aligned} & \int_{\nu, \Omega} \partial_t \frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} + \int_{\nu, \Omega} c\mathbf{\Omega} \cdot \nabla \frac{(I_0 - B_{0,L})^+}{\Lambda^2\gamma} = \frac{c}{\gamma} \int_{\nu_0, \Omega_0} \operatorname{sgn}^+(I_0 - B_{0,L}) Q_{t,0} \\ & + \int_{\nu, \Omega} \frac{\operatorname{sgn}^+(I_0 - B_{0,L})}{L(t)\Lambda^2\gamma} \frac{\nu_0 B_{0,L}}{1 - e^{-\nu_0/L(t)}} \left(\partial_t(\log \Lambda) + c\mathbf{\Omega} \cdot \nabla(\log \Lambda) - \partial_t(\log L(t)) \right) \\ & + \int_{\nu, \Omega} (I_0 - B_{0,L})^+ \Lambda^{-3} \left(\partial_t \frac{\Lambda}{\gamma} + c\mathbf{\Omega} \cdot \nabla \frac{\Lambda}{\gamma} \right). \end{aligned} \quad (2.17)$$

The key of the proof relies on the fact that the second line of this equality is non positive. Indeed, the definition of Λ (2.2) yields

$$\partial_t \Lambda = \frac{\mathbf{\Omega} \cdot \partial_t \mathbf{u} / c}{\sqrt{1 - |\mathbf{u}|^2 / c^2}} + \left(1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c} \right) \frac{\mathbf{u} \cdot \partial_t \mathbf{u} / c^2}{\sqrt{1 - |\mathbf{u}|^2 / c^2}^3}.$$

Simple computations using the assumption (H1) on the velocity field and the estimate (2.13) lead to

$$\partial_t(\log \Lambda) \leq \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}} / c}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}} / c}{1 - (u^*/c)^2} \right).$$

The same manipulations for the space derivatives of Λ give us

$$\partial_t(\log \Lambda) + c\mathbf{\Omega} \cdot \nabla(\log \Lambda) \leq 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right).$$

Now using the definition of $L(t)$ (2.14), one has

$$\frac{\partial_t L(t)}{L(t)} = (\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}} + 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right).$$

In particular,

$$\frac{\partial_t L(t)}{L(t)} \geq 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right),$$

which yields

$$\partial_t(\log \Lambda) + c\mathbf{\Omega} \cdot \nabla(\log \Lambda) - \partial_t(\log L(t)) \leq 0.$$

We can write from equation (2.17)

$$\begin{aligned} \int_{\nu,\Omega} \partial_t \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} + \int_{\nu,\Omega} c \mathbf{\Omega} \cdot \nabla \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} &= \frac{c}{\gamma} \int_{\nu_0, \Omega_0} \text{sgn}^+(I_0 - B_{0,L}) Q_{t,0} \\ &+ \int_{\nu,\Omega} (I_0 - B_{0,L})^+ \Lambda^{-3} \left(\partial_t \frac{\Lambda}{\gamma} + c \mathbf{\Omega} \cdot \nabla \frac{\Lambda}{\gamma} \right). \end{aligned} \quad (2.18)$$

We now study the equation describing the evolution of the temperature. We make the same manipulations than for the equation of the radiative intensity I : multiplying it by $\text{sgn}^+(T - L(t))$ yields

$$\partial_t (T - L(t))^+ + \text{sgn}^+(T - L(t)) \left(\partial_t L(t) + \nabla \cdot (T \mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u} \right) = -\frac{c}{\gamma} \int_{\nu_0, \Omega_0} \text{sgn}^+(T - L(t)) Q_{a,0}.$$

In order to write the term in the brackets in terms of $(T - L(t))^+$ and $L(t)$, one remarks that $\nabla \cdot (T \mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u} = (T - L(t)) \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla (T - L(t)) + \Gamma (T - L(t)) \nabla \cdot \mathbf{u} + L(t) \nabla \cdot \mathbf{u} + \Gamma L(t) \nabla \cdot \mathbf{u}$, which yields

$$\begin{aligned} \text{sgn}^+(T - L(t)) (\nabla \cdot (T \mathbf{u}) + \Gamma T \nabla \cdot \mathbf{u}) &= \nabla \cdot ((T - L(t))^+ + \Gamma (T - L(t))^+) \nabla \cdot \mathbf{u} \\ &+ \text{sgn}^+(T - L(t)) (\Gamma + 1) L(t) \nabla \cdot \mathbf{u}. \end{aligned}$$

We thus have

$$\begin{aligned} \partial_t (T - L(t))^+ + \Gamma (T - L(t))^+ \nabla \cdot \mathbf{u} + \nabla \cdot ((T - L(t))^+ & \\ + \text{sgn}^+(T - L(t)) \left(\partial_t L(t) + (\Gamma + 1) \nabla \cdot \mathbf{u} L(t) \right) &= -\frac{c}{\gamma} \int_{\nu_0, \Omega_0} \text{sgn}^+(T - L(t)) Q_{a,0}. \end{aligned} \quad (2.19)$$

Once again, the key of the proof relies on the fact that $\partial_t L(t) + (\Gamma + 1) \nabla \cdot \mathbf{u} L(t)$ is non negative. Indeed, the definition of $L(t)$ (2.14) yields

$$\begin{aligned} \frac{\partial_t L(t)}{L(t)} &= (\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}} + 2 \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right), \\ &\geq (\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}}. \end{aligned}$$

We can write from equation (2.19)

$$\partial_t (T - L(t))^+ + \Gamma (T - L(t))^+ \nabla \cdot \mathbf{u} + \nabla \cdot ((T - L(t))^+ = -\frac{c}{\gamma} \int_{\nu_0, \Omega_0} \text{sgn}^+(T - L(t)) Q_{a,0}. \quad (2.20)$$

We now study the source terms involved in the inequality (2.18). By definition of the scattering operator (2.7), one has

$$\int_{\nu_0, \Omega_0} \text{sgn}^+(I_0 - B_{0,L}) Q_{s,0} = \sigma_s(x) \int_{\nu_0, \Omega_0} \text{sgn}^+(I_0 - B_{0,L}) \left(\int_{\Omega'_0} I_0(\vec{\Omega}'_0) - I_0 \right),$$

which we write

$$\begin{aligned} \int_{\nu_0, \Omega_0} \text{sgn}^+(I_0 - B_{0,L}) Q_{s,0} &= -\sigma_s(x) \int_{\nu_0} \left(\langle (I_0 - B_{0,L})^+ \rangle_0 \right. \\ &\quad \left. - \langle I_0 - B_{0,L} \rangle_0 \langle \text{sgn}^+(I_0 - B_{0,L}) \rangle_0 \right), \end{aligned}$$

where the notation $\langle \cdot \rangle_0$ means $\int_{S_0^2} \cdot d\mathbf{\Omega}_0$. Given a function $X \in L^1$ and denoting X^- its non positive part, one has $\langle X \rangle = \langle X^+ + X^- \rangle \leq \langle X^+ \rangle = \langle X \operatorname{sgn}^+(X) \rangle$. Multiplying this identity by $\langle \operatorname{sgn}^+(X) \rangle$ and using $\langle \operatorname{sgn}^+(X) \rangle \leq 1$, one obtains $\langle X \rangle \langle \operatorname{sgn}^+(X) \rangle \leq \langle X^+ \rangle$, which yields, using the positivity of the scattering coefficient (assumption (H2)), the identity $\int_{\nu_0, \Omega_0} \operatorname{sgn}^+(I_0 - B_{0,L}) Q_{s,0} \leq 0$. We now turn to the terms containing the emission absorption coefficients, both in equalities (2.18) and (2.20). By definition of the emission absorption operator (2.9), one has

$$\begin{aligned} & \int_{\nu_0, \Omega_0} (\operatorname{sgn}^+(I_0 - B_{0,L}) - \operatorname{sgn}^+(T - L(t))) Q_{a,0} \\ &= \int_{\nu_0, \Omega_0} \sigma_a(\nu_0) (B_{0,T} - I_0) \left(\operatorname{sgn}^+(I_0 - B_{0,L}) - \operatorname{sgn}^+(T - L(t)) \right). \end{aligned} \quad (2.21)$$

The proof that this term is non positive is done in [GP86,DOG01]. Actually, the equation (2.21) can be written

$$\begin{aligned} & \int_{\nu, \Omega} (\operatorname{sgn}^+(I_0 - B_{0,L}) - \operatorname{sgn}^+(T - L(t))) Q_{a,0} \\ &= - \int_{\nu_0, \Omega_0} \sigma_a(\nu_0) \left((I_0 - B_{0,L})^+ - (I_0 - B_{0,L}) \right) \operatorname{sgn}^+(T - L(t)) \\ & \quad - \int_{\nu_0, \Omega_0} \sigma_a(\nu_0) \left((T - L(t))^+ - (T - L(t)) \right) \operatorname{sgn}^+(I_0 - B_{0,L}), \end{aligned}$$

and thus this term is non positive since the function $T \mapsto B(\nu, T)$ is non decreasing and since the emission absorption coefficient σ_a is non negative (assumption (H3)). Adding the inequalities (2.18) and (2.20) and integrating over X , one gets, using all these results and the assumption (H5) on the boundary conditions

$$\begin{aligned} & \frac{d}{dt} \int_x \left(\int_{\nu, \Omega} \frac{(I_0 - B_{0,L})^+}{\Lambda^2 \gamma} + (T - L(t))^+ \right) \\ & \leq -\Gamma \int_x (T - L(t))^+ \nabla \cdot \mathbf{u} + \int_{x, \nu, \Omega} (I_0 - B_{0,L})^+ \Lambda^{-3} \left(\partial_t \frac{\Lambda}{\gamma} + c\mathbf{\Omega} \cdot \nabla \frac{\Lambda}{\gamma} \right), \\ & \leq \max \left(\left\| \frac{\partial_t (\Lambda \gamma^{-1}) + c\mathbf{\Omega} \cdot \nabla (\Lambda \gamma^{-1})}{\Lambda^3} \right\|_{L_x^\infty, \Omega}, \Gamma \|\nabla \cdot \mathbf{u}\|_{L_x^\infty} \right) \\ & \quad \times \int_x \left(\int_{\nu, \Omega} (I_0 - B_{0,L})^+ + (T - L(t))^+ \right). \end{aligned} \quad (2.22)$$

Integrating this inequality between 0 and t and using the positivity of the coefficients Λ and γ (estimate (2.13) and assumption (H1)) yield

$$\begin{aligned} & \min \left((\Lambda^*)^{-2} \inf_{t,x} \gamma^{-1}, 1 \right) \int_x \left(\int_{\nu, \Omega} (I_0 - B_{0,L})^+(t) + (T - L)^+(t) \right) \\ & \leq \int_x \left(\int_{\nu, \Omega} \frac{(I_0 - B_{0,L})^+(0)}{\Lambda^2 \gamma} + (T - L)^+(0) \right) \\ & \quad + \int_0^t C \int_x \left(\int_{\nu, \Omega} (I_0 - B_{0,L})^+ + (T - L)^+ \right). \end{aligned}$$

The Gronwall lemma gives

$$\begin{aligned} & \int_x \left(\int_{\nu, \Omega} (I_0 - B_{0,L})^+(t) + (T - L)^+(t) \right) \\ & \leq C_2 \int_x \left(\int_{\nu, \Omega} \frac{(I_0 - B_{0,L})^+(0)}{\Lambda^2 \gamma} + (T - L)^+(0) \right) e^{t \frac{\|c\|_{L_t^\infty}}{c_2}}, \end{aligned}$$

where $C_2 = \min((\Lambda^*)^{-2} \inf_{t,x} \gamma^{-1}, 1)^{-1}$. The assumption (H4) on the initial conditions thus yields the non positivity of the left hand side, which is the result of the claim. The proof of the minimum principle is similar. Writing the evolution equation satisfied by $\int_x \left(\int_{\nu, \Omega} \frac{(B_{0,L} - I_0)^+}{\Lambda^2 \gamma} + (l - T)^+ \right)$, the claim relies on the following inequalities satisfied by $l(t)$

$$\begin{cases} \frac{\partial_t l(t)}{l(t)} \leq -(\Gamma + 1) \|\mathbf{u}\|_{W_{t,x}^{1,\infty}}, \\ \frac{\partial_t l(t)}{l(t)} \leq -\frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{\Lambda_* \sqrt{1 - (u^*/c)^2}} \left(1 + \frac{2}{c} \frac{\|\mathbf{u}\|_{W_{t,x}^{1,\infty}}}{1 - (u^*/c)^2} \right). \end{cases}$$

□

To obtain existence of solutions of (2.1), one can use the semigroup theory for problem written as Lipschitz perturbation of semigroup operators. In the following lemma the notion of strong and classical solution is the one used by A. Pazy [PAZ83].

Lemma 2.2 (Existence of solutions). *Assume that hypotheses (H1)-(H6) are satisfied. Then system (2.1) has a unique solution $(I, T) \in \mathcal{C}^0([0, T^f]; L^p(X \times \mathbb{R}_\nu^+ \times S^2)) \times \mathcal{C}^0([0, T^f]; L^p(X))$. The solution is strong in the case $1 < p < +\infty$ and classical in the case $p = 1$.*

Proof. Let us rewrite the system (2.1) as an evolution system involving several operators, that is

$$\begin{cases} \partial_t(T, I) + Q(T, I) = Q_{lips}(T), \\ (T, I)(t = 0) = (T^{in}, I^{in}) \end{cases} \quad (2.23)$$

where $Q(T, I) = A(T, I) - Q_{lin}(T, I)$ and

$$\begin{cases} A(T, I) = \left(\mathbf{u} \cdot \nabla T; c \mathbf{\Omega} \cdot \nabla I \right), \\ Q_{lin}(T, I) = \left(-(1 + \Gamma) T \nabla \cdot \mathbf{u} + \frac{c}{\gamma} \int_{\nu, \Omega} \frac{\sigma_a(\nu_0)}{\Lambda} I_0; c Q_s - c \frac{\sigma_a(\nu_0)}{\Lambda^2} I_0 \right), \\ Q_{lips}(T) = c \left(- \int_{\nu, \Omega} \frac{\sigma_a(\nu_0)}{\gamma \Lambda^2} B(\nu_0, T); \frac{\sigma_a(\nu_0)}{\Lambda^2} B(\nu_0, T) \right), \end{cases}$$

where Q is the generator of the semigroup and Q_{lips} is the perturbation. The result of the claim is a consequence of the two following lemmas. In lemma 2.3 we prove that Q is the infinitesimal generator of a \mathcal{C}^0 semigroup on $L^p(X) \times L^p(X \times \mathbb{R}_\nu^+ \times S^2)$ and in lemma 2.4 we prove that Q_{lips} is a Lipschitz operator from $L^p(X) \times L^p(X \times \mathbb{R}_\nu^+ \times S^2)$ into itself. One then applies theorems 6.1.2 and 6.1.6 of [PAZ83] in the case $1 < p < +\infty$ and theorems 6.1.2 and 6.1.5 of [PAZ83] in the case $p = 1$. The difference comes from the fact that L^p is a reflexive Banach space only in the case $1 < p < +\infty$. □

Lemma 2.3. *Assume that hypotheses (H1)-(H6) are satisfied. Then, for all $1 \leq p < +\infty$, the operator $Q = A - Q_{lin}$ is the infinitesimal generator of a \mathcal{C}^0 semigroup on $L^p(X) \times L^p(X \times \mathbb{R}_\nu^+ \times S^2)$.*

Proof. It is known (see [DL83], chapter 21) that A is the infinitesimal generator of a \mathcal{C}^0 semigroup on $L^p(X) \times L^p(X \times \mathbb{R}_\nu^+ \times S^2)$. We need to prove that Q_{lin} is a linear continuous operator from $L^p(X) \times L^p(X \times \mathbb{R}_\nu^+ \times S^2)$ into itself. Using the inequality $\forall a, b \geq 0, (a+b)^p \leq 2^{p-1}(a^p + b^p)$, we just need to estimate each components of Q_{lin} in L^p . We start with the first component of Q_{lin} . For the first term, one easily obtains $\|(1 + \Gamma)T\nabla \cdot \mathbf{u}\|_{L_x^p} \leq (1 + \Gamma)\|\nabla \cdot \mathbf{u}\|_{L_x^\infty} \|T\|_{L_x^p}$. For the second term, the relation $I_0 = \Lambda^3 I$ and the estimate (2.13) give

$$\left\| \frac{c}{\gamma} \int_{\nu, \Omega} \frac{\sigma_a(\nu_0)}{\Lambda} I_0 \right\|_{L_x^p}^p \leq c^p (\Lambda^*)^{2p} \|\gamma^{-p}\|_{L_x^\infty} \int_x \left(\int_{\nu, \Omega} \sigma_a(\nu_0) I \right)^p.$$

By assumption (H1) on the velocity field, one has $\gamma^{-1} \leq 1$. Using the Hölder inequality, we get

$$\left\| \frac{c}{\gamma} \int_{\nu, \Omega} \frac{\sigma_a(\nu_0)}{\Lambda} I_0 \right\|_{L_x^p} \leq c \Lambda^{*2} \|\sigma_a\|_{L_{\nu'}^{\frac{p}{p-1}}} \|I\|_{L_{x, \nu, \Omega}^p}, \quad (2.24)$$

which is bounded thanks to the estimate (2.13) on Λ and to the assumptions (H3) and (H6) on the regularity of the emission absorption coefficient. The second component is a little more complicated. We remind that

$$Q_s = \sigma_s(x) \Lambda \left(\int_{\Omega'} \frac{\Lambda'}{\Lambda^3} I(\nu', \Omega') - I \right).$$

Thus, a Cauchy Schwarz inequality and the estimate (2.13) yield

$$\|Q_s\|_{L_{x, \nu, \Omega}^p}^p \leq \|\sigma_s\|_{L_x^\infty}^p (\Lambda^*)^p \left(\left(\frac{\Lambda^*}{\Lambda_3^*} \right)^p \int_{x, \nu, \Omega} I^p(\nu', \Omega') + \|I\|_{L_{x, \nu, \Omega}^p}^p \right).$$

Making the change of variable $\bar{\nu} = \frac{\Lambda}{\Lambda'} \nu$ in the integral, we find

$$\|Q_s\|_{L_{x, \nu, \Omega}^p}^p \leq \|\sigma_s\|_{L_x^\infty}^p (\Lambda^*)^p \left(\left(\frac{\Lambda^*}{\Lambda_3^*} \right)^{p+1} + 1 \right) \|I\|_{L_{x, \nu, \Omega}^p}^p,$$

which yields $\|Q_s\|_{L_{x, \nu, \Omega}^p} \leq C \|I\|_{L_{x, \nu, \Omega}^p}$. The second term of the second component is similar to (2.24). We deduce that if we denote $F = (T, I)$, there exists a constant $C \geq 0$ such that $\|Q_{lin}(F)\|_{L_x^p \times L_{x, \nu, \Omega}^p} \leq C \|F\|_{L_x^p \times L_{x, \nu, \Omega}^p}$. There just remains to apply the theorem 3.1.1 of [PAZ83] to conclude. \square

We study the operator Q_{lips} . We have the

Lemma 2.4. *Under hypotheses (H1)-(H6), there exists a constant $C \geq 0$ such that for all $T_1, T_2 \in (L^p(X) \cap L^\infty(X))^+$, the following estimate holds :*

$$\|Q_{lips}(T_1) - Q_{lips}(T_2)\|_{L_x^p \times L_{x, \nu, \Omega}^p} \leq C \|T_1 - T_2\|_{L_x^p}.$$

Proof. Studying the expression of $Q_{lips}(T_1) - Q_{lips}(T_2)$, the control of the second component relies on the control of $\|B(\nu_0, T_1) - B(\nu_0, T_2)\|_{L^p_{x,\nu,\Omega}}$. Using a Taylor expansion with integral remainder of the function $T \mapsto B(\nu_0, T)$, one gets

$$\begin{aligned} |B(\nu_0, T_2) - B(\nu_0, T_1)|^p &= \left| \int_{T_1}^{T_2} \partial_T B(\nu_0, s) ds \right|^p \\ &\leq |T_2 - T_1|^p \left| \frac{1}{|T_2 - T_1|} \int_{T_1}^{T_2} \partial_T B(\nu_0, s) ds \right|^p. \end{aligned}$$

The Jensen inequality yields

$$|B(\nu_0, T_2) - B(\nu_0, T_1)|^p \leq |T_2 - T_1|^p \frac{1}{|T_2 - T_1|} \int_{T_1}^{T_2} |\partial_T B(\nu_0, s)|^p ds. \quad (2.25)$$

By definition of the Planck function, one has $\partial_T B(\nu_0, s) = \frac{\nu_0^4 e^{\nu_0/s}}{s^2 (e^{\nu_0/s} - 1)^2}$. Integrating (2.25) on \mathbb{R}_ν^+ and making the change of variable $\nu \rightarrow \nu_0/s$ lead to

$$\int_\nu |B(\nu_0, T_2) - B(\nu_0, T_1)|^p d\nu \leq |T_2 - T_1|^p \frac{1}{\Lambda} \frac{1}{|T_2 - T_1|} \int_{T_1}^{T_2} s^{2p+1} ds \times \int_\nu \frac{\nu^{4p} e^{p\nu}}{(e^\nu - 1)^{2p}} d\nu.$$

Since there exists a constant C such that $\int_\nu \frac{\nu^{4p} e^{p\nu}}{(e^\nu - 1)^{2p}} d\nu \leq C$, this expression reduces to

$$\int_\nu |B(\nu_0, T_2) - B(\nu_0, T_1)|^p d\nu \leq |T_2 - T_1|^p \frac{C}{\Lambda(2p+2)} \frac{T_2^{2p+2} - T_1^{2p+2}}{|T_2 - T_1|}.$$

Using the formula $a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}$, one finds a constant C , which depends on $\max(T_1, T_2)^{2p+1}$, such that

$$\|B(\nu_0, T_2) - B(\nu_0, T_1)\|_{L^p_{x,\nu,\Omega}} \leq C \|T_2 - T_1\|_{L^p_x}.$$

For the first component of $Q_{lips}(T_1) - Q_{lips}(T_2)$, a Hölder inequality together with the change of variable $\nu \mapsto \Lambda\nu$ yield

$$\int_x \left| \int_{\nu,\Omega} \frac{c\sigma_a(\nu_0)}{\gamma\Lambda^2} (B(\nu_0, T_2) - B(\nu_0, T_1)) \right|^p \leq \frac{c^p}{\Lambda_*^{3p-1}} \|\sigma_a\|_{L_\nu^{\frac{p}{p-1}}}^p \|B(\nu_0, T_2) - B(\nu_0, T_1)\|_{L^p_{x,\nu,\Omega}}^p.$$

The previous result and the assumption (H5) on the emission absorption coefficient give the result. \square

2.3 Non-equilibrium diffusion regime

In this section we study the so called non-equilibrium diffusion regime. This regime has already been studied in the grey case with relativistic coefficients in [GLG05] and with non relativistic coefficients in [DOG01] (see also [BN14]). The idea is to assume that the speed of light c is very fast compared to the velocity field \mathbf{u} , i.e. $|\mathbf{u}|/c \ll 1$ and that the scattering coefficient is stiff compared to the emission absorption coefficient, i.e. $\sigma_s/\sigma_a \gg 1$. We thus introduce a coefficient ε , $0 < \varepsilon \leq 1$, formally equal to the ratio of a characteristic speed of the fluid by the velocity of light (a rigorous derivation of the equations can be found for example in [GLG05, BD04]).

Rescaling the emission absorption coefficient as $\sigma_a = \varepsilon \widehat{\sigma}_a$, the scattering coefficient as $\sigma_s = \widehat{\sigma}_s/\varepsilon$, where $\widehat{\sigma}_a, \widehat{\sigma}_s = \mathcal{O}(1)$ with respect to ε , and the speed of light as $c = 1/\varepsilon$ lead, after dropping the hats for ease of notations, to the following system

$$\begin{cases} \partial_t I^\varepsilon + \frac{1}{\varepsilon} \boldsymbol{\Omega} \cdot \nabla_x I^\varepsilon = \frac{Q_s^\varepsilon}{\varepsilon^2} + Q_a^\varepsilon, \\ \partial_t T^\varepsilon + \nabla \cdot (T^\varepsilon \mathbf{u}) + \Gamma T^\varepsilon \nabla \cdot \mathbf{u} = - \int_{\nu, \Omega} \frac{\Lambda^\varepsilon}{\gamma^\varepsilon} Q_a^\varepsilon, \end{cases} \quad (2.26)$$

where $\gamma^\varepsilon = (1 - \varepsilon^2 |\mathbf{u}|^2)^{-1/2}$ and $\Lambda^\varepsilon = \gamma^\varepsilon (1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})$. We introduce (ρ, \bar{T}) the solution of the following drift diffusion system:

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s(x)} \right) + \nabla \cdot (\rho \mathbf{u}) = \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu \rho + \sigma_a(\nu) (B(\nu, \bar{T}) - \rho), \\ \partial_t \bar{T} + \nabla \cdot (\bar{T} \mathbf{u}) + \Gamma \bar{T} \nabla \cdot \mathbf{u} = - \int_\nu \sigma_a(\nu) (B(\nu, \bar{T}) - \rho). \end{cases} \quad (2.27)$$

This section is devoted to the proof of convergence of the solution $(I^\varepsilon, T^\varepsilon)$ of the relativistic transfer equations (2.26) to the solution of the drift diffusion system (2.27) as $\varepsilon \rightarrow 0$. The proof will be done in several steps, following C. Dogbe [DOG01] or G. Allaire and F. Golse [AG12]. The idea is in a first time (part 2.3.1) to find formally the limit, as $\varepsilon \rightarrow 0$, of the solution $(I^\varepsilon, T^\varepsilon)$ using formal Hilbert expansions. In a second time (part 2.3.3), we prove the strong convergence in L^2 . To this end we prove a comparison principle and several a priori estimates for the solution of the drift diffusion system (2.27) (part 2.3.2). The end of this section deals with the proof of the convergence result, and uses a function, reconstructed from the truncated Hilbert expansion, solution of the system (2.26) with a remainder. Finally, we conclude by using a priori estimates on the solution of the drift diffusion system (2.27) and a stability result for the system (2.26).

As a remark, we do not consider the open problem of existence and uniqueness of a solution for the drift diffusion system (2.27). Indeed the equation on ρ is parabolic degenerate, since there is no frequency diffusion. A possible solution could be to introduce a small dissipation in frequency, apply the semi-group theory for the modified problem and then use similar a priori estimates than (2.47) to gain compactness. In this work we assume the existence of a solution for the system (2.27).

2.3.1 Formal asymptotic of the radiative transfer equations

In this part the drift diffusion system (2.27) will be obtained formally, using formal Hilbert expansions of the radiative transfer equations (2.26). Indeed, we prove the following lemma.

Lemma 2.5. *The drift diffusion system (2.27) is a first order approximation of the radiative transfer equations (2.26).*

Proof. The proof is divided in two steps. In a first one the scattering and emission absorption operators will be expanded in power of ε . In a second part the solution of the radiative transfer equations (2.26) will also be expanded, leading formally to the drift diffusion system (2.27).

First step: Study of the source terms

In order to simplify the next step, concerning the Hilbert expansion of the solution $(I^\varepsilon, T^\varepsilon)$ of the system (2.26), the scattering and the emission absorption operators are expanded in power

of ε . Since it is of order ε^{-2} , the study of the scattering operator will be more complicated, while the expansion of the emission absorption operator will be rather simple. We start with the scattering operator. Given a function $I : [0, T^f] \times X \times \mathbb{R}_\nu^+ \times S^2 \rightarrow \mathbb{R}^+$, it is defined by

$$Q_s^\varepsilon(I) = \sigma_s(x) \Lambda^\varepsilon \left(\int_{S^2} \frac{\Lambda^{\varepsilon'}}{(\Lambda^\varepsilon)^3} I(\nu^{\varepsilon'}, \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I \right),$$

where $\nu^{\varepsilon'} = (\Lambda^\varepsilon / \Lambda^{\varepsilon'}) \nu$. We expand it in power of ε : given $N \in \mathbb{N}$, we write

$$Q_s^\varepsilon(I) = \sum_{0 \leq i \leq N} \varepsilon^i Q_s^i(I) + \varepsilon^{N+1} \bar{Q}_{s,N}^\varepsilon(I).$$

Due to the Doppler shift, the radiative intensity I is computed at the frequency ν' , and a Taylor expansion with integral remainder will be performed. In order to simplify the notations we remove the dependence in ε in the coefficients $\Lambda^\varepsilon, \Lambda^{\varepsilon'}$ and $\nu^{\varepsilon'}$. Since it will be useful in the next part, expansions at order 2,1 and 0 with respect to ε of the scattering operator are performed.

Expansion of Q_s^ε at order 2

In this part the scattering operator is written as

$$Q_s^\varepsilon(I) = Q_s^0(I) + \varepsilon Q_s^1(I) + \varepsilon^2 Q_s^2(I) + \varepsilon^3 \bar{Q}_{s,2}^\varepsilon(I),$$

with

$$\left\{ \begin{array}{l} Q_s^0(I) = \sigma_s(x) \langle I \rangle - I, \\ Q_s^1(I) = \sigma_s(x) \left(\int_{S^2} \left(\nu \lambda_5 \partial_\nu I(\nu, \boldsymbol{\Omega}') + \lambda_3 I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' + \lambda_1 \sigma_s(x) \langle I \rangle - I \right), \\ Q_s^2(I) = \sigma_s(x) \int_{S^2} \left(\nu (\lambda_6 + \lambda_3 \lambda_5) \partial_\nu I(\nu, \boldsymbol{\Omega}') + \frac{\nu^2}{2} \lambda_5^2 \partial_\nu^2 I(\nu, \boldsymbol{\Omega}') + \lambda_4 I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_1 \int_{S^2} \left(\nu \lambda_5 \partial_\nu I(\nu, \boldsymbol{\Omega}') + \lambda_3 I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' + \sigma_s(x) \lambda_2 \langle I \rangle - I, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \bar{Q}_{s,2}^\varepsilon(I) = \sigma_s(x) \int_{S^2} \left(R_{I,2} + \lambda_3 \left(\nu \lambda_6 \partial_\nu I(\nu, \boldsymbol{\Omega}') + \frac{\nu^2}{2} \lambda_5^2 \partial_\nu^2 I(\nu, \boldsymbol{\Omega}') + \varepsilon R_{I,2} \right) \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \int_{S^2} \left(\lambda_4 \frac{I(\nu', \boldsymbol{\Omega}') - I(\nu, \boldsymbol{\Omega}')}{\varepsilon} + R_{\frac{\Lambda'}{\Lambda^3}, 2} I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_1 \int_{S^2} \left(\nu \lambda_6 \partial_\nu I(\nu, \boldsymbol{\Omega}') + \frac{\nu^2}{2} \lambda_5^2 \partial_\nu^2 I(\nu, \boldsymbol{\Omega}') + \varepsilon R_{I,2} \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_1 \int_{S^2} \left(\lambda_3 \frac{I(\nu', \boldsymbol{\Omega}') - I(\nu, \boldsymbol{\Omega}')}{\varepsilon} + (\lambda_4 + \varepsilon R_{\frac{\Lambda'}{\Lambda^3}, 2}) I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_2 \int_{S^2} \frac{I(\nu', \boldsymbol{\Omega}') - I(\nu, \boldsymbol{\Omega}')}{\varepsilon} d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_2 \int_{S^2} \left(\frac{\frac{\Lambda'}{\Lambda^3} - 1}{\varepsilon} \right) I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) R_{\Lambda, 2} \left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I(\nu, \boldsymbol{\Omega}) \right), \end{array} \right. \quad (2.28)$$

where all the coefficients involved in this system are given below. These coefficients come from two different contributions: a part of them comes from the expansion of the relativistic coefficients and the others come from the Taylor expansion of $I(\nu')$ around the frequency ν . We start with the expansion of the relativistic parameters involved in the expression of the scattering operator. Using a Taylor expansion with integral remainder, the coefficient Λ can be written as $\Lambda = 1 - \varepsilon \mathbf{\Omega} \cdot \mathbf{u} + \varepsilon^2 \frac{|\mathbf{u}|^2}{2} + \varepsilon^3 R_{\Lambda,2}$, which we write $\Lambda = 1 + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \varepsilon^3 R_{\Lambda,2}$, with $\lambda_1 = -\mathbf{\Omega} \cdot \mathbf{u}$, $\lambda_2 = \frac{|\mathbf{u}|^2}{2}$ and

$$R_{\Lambda,2} = \frac{1}{\varepsilon^3 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} \int_{1 - \varepsilon^2 |\mathbf{u}|^2}^1 \frac{1 - \varepsilon^2 |\mathbf{u}|^2 - s}{4} \frac{1}{s \sqrt{s}} ds + \frac{|\mathbf{u}|^2}{2 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} \left(\varepsilon \frac{|\mathbf{u}|^2}{2} - \mathbf{\Omega} \cdot \mathbf{u} \right).$$

In the same way, one has $\frac{\Lambda'}{\Lambda^3} = 1 + \varepsilon (3\mathbf{\Omega} - \mathbf{\Omega}') \cdot \mathbf{u} + \varepsilon^2 (3(\mathbf{\Omega}, \mathbf{u})(2\mathbf{\Omega} - \mathbf{\Omega}', \mathbf{u}) - |\mathbf{u}|^2) + \varepsilon^3 R_{\frac{\Lambda'}{\Lambda^3},2}$, which we write $\frac{\Lambda'}{\Lambda^3} = 1 + \lambda_3 \varepsilon + \lambda_4 \varepsilon^2 + \varepsilon^3 R_{\frac{\Lambda'}{\Lambda^3},2}$, with $\lambda_3 = (3\mathbf{\Omega} - \mathbf{\Omega}') \cdot \mathbf{u}$, $\lambda_4 = 3(\mathbf{\Omega}, \mathbf{u})(2\mathbf{\Omega} - \mathbf{\Omega}', \mathbf{u}) - |\mathbf{u}|^2$ and

$$R_{\frac{\Lambda'}{\Lambda^3},2} = \frac{1 - |\mathbf{u}|^2}{(1 - \varepsilon \mathbf{\Omega} \cdot \mathbf{u})^3} \left(3\mathbf{\Omega}' \cdot \mathbf{u} - 3\varepsilon (\mathbf{\Omega}, \mathbf{u})^2 + 2\varepsilon^2 (\mathbf{\Omega}, \mathbf{u})^3 \right. \\ \left. - 3(\mathbf{\Omega}, \mathbf{u})^2 (3\mathbf{\Omega} \cdot \mathbf{u} - 3\varepsilon (\mathbf{\Omega}, \mathbf{u})^2 + \varepsilon^2 (\mathbf{\Omega}, \mathbf{u})^3) \right) \\ - |\mathbf{u}|^2 \left((3\mathbf{\Omega} - \mathbf{\Omega}', \mathbf{u}) + 3\varepsilon \mathbf{\Omega} \cdot \mathbf{u} (2\mathbf{\Omega} - \mathbf{\Omega}', \mathbf{u}) \right).$$

Finally, we have $\frac{\Lambda}{\Lambda'} = 1 + \varepsilon (\mathbf{\Omega}' - \mathbf{\Omega}) \cdot \mathbf{u} + \varepsilon^2 \mathbf{\Omega}' \cdot \mathbf{u} (\mathbf{\Omega}' - \mathbf{\Omega}) \cdot \mathbf{u} + \varepsilon^3 R_{\frac{\Lambda}{\Lambda'},2}$, which we write $\frac{\Lambda}{\Lambda'} = 1 + \lambda_5 \varepsilon + \lambda_6 \varepsilon^2 + \varepsilon^3 R_{\frac{\Lambda}{\Lambda'},2}$, with $\lambda_5 = (\mathbf{\Omega}' - \mathbf{\Omega}) \cdot \mathbf{u}$, $\lambda_6 = \mathbf{\Omega}' \cdot \mathbf{u} (\mathbf{\Omega}' - \mathbf{\Omega}) \cdot \mathbf{u}$ and

$$R_{\frac{\Lambda}{\Lambda'},2} = \frac{(\mathbf{\Omega}', \mathbf{u})^2 (\mathbf{\Omega}' - \mathbf{\Omega}, \mathbf{u})}{1 - \varepsilon \mathbf{\Omega}' \cdot \mathbf{u}}.$$

We now expand the expression of $I(\nu')$ around the frequency ν . Using $\nu' - \nu = \left(\frac{\Lambda}{\Lambda'} - 1\right)\nu$, a Taylor expansion with integral remainder of I yields

$$I(\nu', \mathbf{\Omega}') = I(\nu, \mathbf{\Omega}') + \nu \lambda_5 \varepsilon \partial_\nu I(\nu, \mathbf{\Omega}') + \varepsilon^2 \left(\nu \lambda_6 \partial_\nu I(\nu, \mathbf{\Omega}') + \frac{\nu^2}{2} \lambda_5^2 \partial_\nu^2 I(\nu, \mathbf{\Omega}') \right) + \varepsilon^3 R_{I,2},$$

with

$$R_{I,2} = \nu R_{\frac{\Lambda}{\Lambda'},2} \partial_\nu I(\nu, \mathbf{\Omega}') + \frac{\nu^2}{2\varepsilon^3} \left(\left(\frac{\Lambda}{\Lambda'} - 1\right)^2 - (\varepsilon \lambda_5)^2 \right) \partial_\nu^2 I(\nu, \mathbf{\Omega}') \\ + \frac{1}{\varepsilon^3} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \mathbf{\Omega}') ds,$$

and this ends the definition of all the coefficients involved in the expansion at order 2 of the scattering operator.

Expansion of Q_s^ε at order 1

In this part the same expansion of the scattering operator is performed, but we stop at order 1. The method is the same. We write the scattering operator as

$$Q_s^\varepsilon(I) = Q_s^0(I) + \varepsilon Q_s^1(I) + \varepsilon^2 \bar{Q}_{s,1}(I),$$

with

$$\left\{ \begin{array}{l} Q_s^0(I) = \sigma_s(x)(\langle I \rangle - I), \\ Q_s^1(I) = \sigma_s(x) \left(\int_{S^2} \left(\nu \lambda_5 \partial_\nu I(\nu, \boldsymbol{\Omega}') + \lambda_3 I(\nu, \boldsymbol{\Omega}') \right) d\boldsymbol{\Omega}' + \sigma_s(x) \lambda_1 (\langle I \rangle - I) \right), \\ \bar{Q}_{s,1}^\varepsilon(I) = \sigma_s(x, \nu) \int_{S^2} \left(\lambda_3 \lambda_5 \nu \partial_\nu I(\nu, \boldsymbol{\Omega}') + R_{I,1} + R_{\frac{\Lambda'}{\Lambda^3},1} I(\nu) \right) d\boldsymbol{\Omega}' \\ \quad + \sigma_s(x) \lambda_1 \int_{S^2} \left(\frac{I(\nu') - I(\nu)}{\varepsilon} + (\lambda_3 + R_{\frac{\Lambda'}{\Lambda^3},1}) I(\nu') \right) d\boldsymbol{\Omega}' \\ \quad + R_{\Lambda,1} \left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I(\nu, \boldsymbol{\Omega}) \right). \end{array} \right. \quad (2.29)$$

Obviously the terms $Q_s^0(I)$ and $Q_s^1(I)$ are the same than for the expansion at order 2. The only difference with the previous expansions comes from the remainders of the expansions of the coefficients. Again, we have, $\Lambda = 1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u} + \varepsilon^2 R_{\Lambda,1}$, which we write $\Lambda = 1 + \lambda_1 \varepsilon + \varepsilon^2 R_{\Lambda,1}$, with

$$\left\{ \begin{array}{l} \lambda_1 = -\boldsymbol{\Omega} \cdot \mathbf{u}, \\ R_{\Lambda,1} = \frac{1 - \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}}{\varepsilon^2 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} (1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u}). \end{array} \right.$$

We also have $\frac{\Lambda'}{\Lambda^3} = 1 + \varepsilon(3\boldsymbol{\Omega} - \boldsymbol{\Omega}') \cdot \mathbf{u} + \varepsilon^2 R_{\frac{\Lambda'}{\Lambda^3},1}$, which we write $\frac{\Lambda'}{\Lambda^3} = 1 + \lambda_3 \varepsilon + \varepsilon^2 R_{\frac{\Lambda'}{\Lambda^3},1}$, with

$$\left\{ \begin{array}{l} \lambda_3 = (3\boldsymbol{\Omega} - \boldsymbol{\Omega}') \cdot \mathbf{u}, \\ R_{\frac{\Lambda'}{\Lambda^3},1} = \frac{(3\boldsymbol{\Omega} - \boldsymbol{\Omega}') \cdot \mathbf{u}}{\varepsilon} \frac{1 - (1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3}{(1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3} - \frac{((\boldsymbol{\Omega}, \mathbf{u})^2 - (\boldsymbol{\Omega}, \mathbf{u})^3)(1 - |\mathbf{u}|^2)}{(1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3}. \end{array} \right.$$

Finally, we have $\frac{\Lambda}{\Lambda'} = 1 + \varepsilon(\boldsymbol{\Omega}' - \boldsymbol{\Omega}) \cdot \mathbf{u} + \varepsilon^2 R_{\frac{\Lambda}{\Lambda'},1}$, which we write $\frac{\Lambda}{\Lambda'} = 1 + \lambda_5 \varepsilon + \varepsilon^2 R_{\frac{\Lambda}{\Lambda'},1}$, with

$$\left\{ \begin{array}{l} \lambda_5 = (\boldsymbol{\Omega}' - \boldsymbol{\Omega}) \cdot \mathbf{u}, \\ R_{\frac{\Lambda}{\Lambda'},1} = (\boldsymbol{\Omega}', \mathbf{u}) \frac{(\boldsymbol{\Omega}' - \boldsymbol{\Omega}, \mathbf{u})}{1 - \varepsilon \boldsymbol{\Omega}' \cdot \mathbf{u}}. \end{array} \right.$$

We now make a Taylor expansion, with respect to ν , of I

$$I(\nu', \boldsymbol{\Omega}') = I(\nu, \boldsymbol{\Omega}') + \nu \lambda_5 \varepsilon \partial_\nu I(\nu, \boldsymbol{\Omega}') + \varepsilon^2 R_{I,1},$$

with $R_{I,1} = \nu R_{\frac{\Lambda}{\Lambda'},1} \partial_\nu I(\nu, \boldsymbol{\Omega}') + \frac{1}{\varepsilon^2} \int_\nu^{\nu'} (\nu' - s) \partial_\nu^2 I(s, \boldsymbol{\Omega}') ds$.

Expansion of Q_s^ε at order 0

In this part we make the same development of the scattering operator but we stop at order 0. The method is the same. We write the scattering operator as

$$Q_s^\varepsilon(I) = Q_s^0(I) + \varepsilon \bar{Q}_{s,0}^\varepsilon(I).$$

with

$$\begin{aligned}\bar{Q}_{s,0}^\varepsilon(I) &= \sigma_s(x) \int_{S^2} (R_{I,0} + R_{\frac{\Lambda'}{\Lambda^3},0} I(\nu', \boldsymbol{\Omega}')) d\boldsymbol{\Omega}' \\ &+ R_{\Lambda,0} \left(\int_{S^2} \frac{\Lambda'}{\Lambda^3} I(\nu', \boldsymbol{\Omega}') d\boldsymbol{\Omega}' - I(\nu, \boldsymbol{\Omega}) \right).\end{aligned}\quad (2.30)$$

Obviously the term $Q_s^0(I)$ is the same than for the expansions at order 1 and 2. Once again, the only difference comes from the remainders. We have $\Lambda = 1 + \varepsilon R_{\Lambda,0}$, with

$$R_{\Lambda,0} = \frac{1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u} - \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}}{\varepsilon \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}}.$$

We also have $\frac{\Lambda'}{\Lambda^3} = 1 + \varepsilon R_{\frac{\Lambda'}{\Lambda^3},0}$, with

$$R_{\frac{\Lambda'}{\Lambda^3},0} = \frac{1 - \varepsilon^2 |\mathbf{u}|^2}{\varepsilon} \frac{1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u} - (1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3}{(1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3}.$$

Finally, we have $\frac{\Lambda}{\Lambda'} = 1 + \varepsilon R_{\frac{\Lambda}{\Lambda'},0}$, with $R_{\frac{\Lambda}{\Lambda'},0} = \frac{(\boldsymbol{\Omega}' - \boldsymbol{\Omega}, \mathbf{u})}{1 - \varepsilon \boldsymbol{\Omega}' \cdot \mathbf{u}}$. We make a Taylor expansion, with respect to ν , of I

$$I(\nu', \boldsymbol{\Omega}') = I(\nu, \boldsymbol{\Omega}') + \varepsilon R_{I,0},$$

with $R_{I,0} = \frac{1}{\varepsilon} \int_{\nu'}^{\nu} \partial_\nu I(s, \boldsymbol{\Omega}') ds$.

Expansion of the emission absorption operator

As for the scattering operator, a Hilbert expansion with exact residual term of the emission absorption operator is performed. The study is much simpler than for the scattering operator since the scaling is less severe. We recall here the definition of the emission absorption operator. Given two function $T : [0, T^f] \times X \rightarrow \mathbb{R}^+$ and $I : [0, T^f] \times X \times \mathbb{R}_\nu^+ \times S^2 \rightarrow \mathbb{R}^+$, it is defined by

$$Q_a^\varepsilon(I, T) = \frac{\sigma_a(\nu_0^\varepsilon)}{\Lambda \varepsilon^2} \left(B(\nu_0^\varepsilon, T) - (\Lambda^\varepsilon)^3 I \right).\quad (2.31)$$

Dropping the ε in the coefficients Λ^ε and ν_0^ε for ease of notations, we write

$$Q_a^\varepsilon(I, T) = Q_a^0(I, T) + \varepsilon \bar{Q}_a^\varepsilon(I, T),\quad (2.32)$$

with

$$\left\{ \begin{aligned} Q_a^0(I, T) &= \sigma_a(\nu) (B(\nu, T) - I), \\ \bar{Q}_a^\varepsilon(I, T) &= \frac{1 - \Lambda^2}{\varepsilon \Lambda^2} \sigma_a(\nu_0) (B(\nu_0, T) - I_0) + \frac{\sigma_a(\nu_0) - \sigma_a(\nu)}{\varepsilon} (B(\nu_0, T) - I) \\ &+ \sigma_a(\nu) \frac{B(\nu_0, T) - B(\nu, T)}{\varepsilon} + \sigma_a(\nu_0) \frac{I - I_0}{\varepsilon}. \end{aligned} \right.\quad (2.33)$$

Second step: Formal Hilbert expansion of the transport equation

In this part we find the formal asymptotic limit of the relativistic transfer equation in the non-equilibrium diffusion regime using a formal Hilbert expansion, that is

$$\left\{ \begin{aligned} I^\varepsilon &= I^0 + \varepsilon I^1 + \varepsilon^2 I^2 + \mathcal{O}(\varepsilon^3), \\ T^\varepsilon &= T^0 + \mathcal{O}(\varepsilon). \end{aligned} \right.\quad (2.34)$$

The limit will only be formal, in the sense that the remainders in (2.34) are not explicitly bounded in some norm. This will be performed in the next section. Recall that in the notations, the subscript 0 refers to quantities computed in the moving frame, while the power 0 refers to the first order term in the expansion in power of ε .

The choice of the scaling is driven by the fact that the temperature T is only involved in $\mathcal{O}(1)$ terms in (2.26), while I is involved in $\mathcal{O}(\varepsilon^{-2})$ terms. Since the scattering operator Q_s is linear, one has $Q_s^\varepsilon(I^\varepsilon) = Q_s^\varepsilon(I^0) + \varepsilon Q_s^\varepsilon(I^1) + \varepsilon^2 Q_s^\varepsilon(I^2) + \mathcal{O}(\varepsilon^3)$. We use the expansion at order 2 of the scattering operator for the zero-th order term I^0 , the expansion at order 1 for the first order term I^1 and the expansion at order 0 for the second order term I^2 . It yields

$$\begin{cases} Q_s^\varepsilon(I^0) = Q_s^0(I^0) + \varepsilon Q_s^1(I^0) + \varepsilon^2 Q_s^2(I^0) + \varepsilon^3 \bar{Q}_{s,2}^\varepsilon(I^0), \\ Q_s^\varepsilon(I^1) = Q_s^0(I^1) + \varepsilon Q_s^1(I^1) + \varepsilon^2 \bar{Q}_{s,1}^\varepsilon(I^1), \\ Q_s^\varepsilon(I^2) = Q_s^0(I^2) + \varepsilon \bar{Q}_{s,0}^\varepsilon(I^2). \end{cases}$$

The previous expansion (2.32) of the emission absorption operator yields $Q_a^\varepsilon(I^\varepsilon, T^\varepsilon) = Q_a^0(I^\varepsilon, T^\varepsilon) + \bar{Q}_a^\varepsilon(I^\varepsilon, T^\varepsilon)$. Moreover, the expansions (2.34) of the unknowns I^ε and T^ε and a Taylor expansion of the Planck function $B(\nu, T^\varepsilon)$ formally lead to $Q_a^0(I^\varepsilon, T^\varepsilon) = Q_a^0(I^0, T^0) + \mathcal{O}(\varepsilon)$. It yields, taking into account all the remainders $\bar{Q}_{s,i}$, $i = 0, 1, 2$ and \bar{Q}_a as $\mathcal{O}(\varepsilon)$ terms,

$$\begin{cases} \partial_t(I^0 + \varepsilon I^1 + \varepsilon^2 I^2) + \frac{1}{\varepsilon} \boldsymbol{\Omega} \cdot \nabla_x (I^0 + \varepsilon I^1 + \varepsilon^2 I^2) = \frac{Q_s^0(I^0)}{\varepsilon^2} + \frac{Q_s^1(I^0) + Q_s^0(I^1)}{\varepsilon} \\ \quad + Q_s^2(I^0) + Q_s^1(I^1) + Q_s^0(I^2) + Q_a^0(I^0, T^0) + \mathcal{O}(\varepsilon), \\ \partial_t I^0 + \nabla \cdot (T^0 \mathbf{u}) + \Gamma T^0 \nabla \cdot \mathbf{u} = - \int_{\nu, \Omega} Q_a^0(I^0, T^0) + \mathcal{O}(\varepsilon). \end{cases}$$

We now study all the terms with the same power of ε . In the forthcoming computations, the formula $\int_{\Omega} (\boldsymbol{\Omega} \cdot \mathbf{u})^2 = \frac{|\mathbf{u}|^2}{3}$ will be often used. First, in $\frac{1}{\varepsilon^2}$, one has $Q_s^0(I^0) = 0$, that is $\sigma_s(x)(I^0 - \int_{\Omega} I^0) = 0$, and thus I^0 is independent of the angular direction $\boldsymbol{\Omega}$. At the order $\frac{1}{\varepsilon}$, one has $\boldsymbol{\Omega} \cdot \nabla_x I^0 = Q_s^0(I^1) + Q_s^1(I^0)$, which yields

$$\boldsymbol{\Omega} \cdot \nabla_x I^0 = \sigma_s(x) \left(\int_{\Omega'} I^1(\boldsymbol{\Omega}') - I^1 \right) + \sigma_s(x) \left(\int_{\Omega'} \lambda_5 \nu \partial_\nu I^0 + \int_{\Omega'} \lambda_3 I^0 \right).$$

Using the relations $\int_{\Omega'} \lambda_5 = -\boldsymbol{\Omega} \cdot \mathbf{u}$ and $\int_{\Omega'} \lambda_3 = 3\boldsymbol{\Omega} \cdot \mathbf{u}$, one finds

$$\sigma_s(x) \left(I^1 - \int_{\Omega'} I^1(\boldsymbol{\Omega}') \right) = -\boldsymbol{\Omega} \cdot \nabla_x I^0 - \sigma_s(x) \boldsymbol{\Omega} \cdot \mathbf{u} \nu \partial_\nu I^0 + 3\sigma_s(x) \boldsymbol{\Omega} \cdot \mathbf{u} I^0 \quad (2.35)$$

This kind of equation is studied in [AG12], and we recall here the main lines. We introduce $\mathcal{H} : \phi \mapsto \int_{\Omega} \phi$, which is an Hilbert Schmidt operator, and we study the auxiliary equation

$$\begin{cases} (I_d - \mathcal{H})b_j(\boldsymbol{\Omega}) = \Omega_j, \\ \int_{\Omega} b_j = 0. \end{cases}$$

Since $\int_{\Omega} \Omega_j = (\int_{\Omega} \mathbf{\Omega})_j = 0$, the Fredholm theory gives the existence of a solution I_1 . In particular, it can be shown, see [AG12], that there exists a unique solution $b_j(\mathbf{\Omega}) \in \text{Ker}(I_d - \mathcal{K})^\perp$. The solutions of equation (2.35) are the functions of the form

$$I^1 = -\frac{1}{\sigma_s(x)} \mathbf{\Omega} \cdot \nabla_x I^0 - \mathbf{\Omega} \cdot \mathbf{u} \nu \partial_\nu I^0 + 3\mathbf{\Omega} \cdot \mathbf{u} I^0 + C^1(t, x, \nu), \quad (2.36)$$

where $C^1(t, x, \nu)$, constant in $\mathbf{\Omega}$, is an arbitrary solution of the homogeneous equation $(\mathcal{K} - I_d)C_1 = 0$. Finally, at the order 0, one has

$$\begin{cases} \partial_t I^0 + \mathbf{\Omega} \cdot \nabla_x I^1 = Q_s^0(I^2) + Q_s^1(I^1) + Q_s^2(I^0) + Q_a^0(I^0, T^0), \\ \partial_t T^0 + \nabla \cdot (T^0 \mathbf{u}) + \Gamma T^0 \nabla \cdot \mathbf{u} = - \int_\nu Q_a^0(I^0, T^0). \end{cases} \quad (2.37)$$

We have, using the definitions of the λ_i (part 2.3.1.0) and I^1 ,

$$\begin{aligned} Q_s^1(I^1) &= -\nu \int_{\Omega} (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla \partial_\nu I^0) - \sigma_s(x) \frac{|\mathbf{u}|^2}{3} (\nu \partial_\nu I^0 + \nu^2 \partial_\nu^2 I^0) + \sigma_s(x) |\mathbf{u}|^2 \nu \partial_\nu I^0 \\ &\quad + \int_{\Omega} (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla I^0) + \sigma_s(x) \frac{|\mathbf{u}|^2}{3} \nu \partial_\nu I^0 - \sigma_s(x) |\mathbf{u}|^2 I^0 - (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla I^0) \\ &\quad - \sigma_s(x) (\mathbf{\Omega}, \mathbf{u})^2 \nu \partial_\nu I^0 + 3\sigma_s(\mathbf{\Omega}, \mathbf{u})^2 I^0 + \sigma_s(x) (\mathbf{\Omega}, \mathbf{u})(3C^1 - \nu \partial_\nu C^1). \end{aligned}$$

In the same way, one has

$$\begin{aligned} Q_s^2(I^0) &= -2(\mathbf{\Omega}, \mathbf{u})^2 \sigma_s(x) \nu \partial_\nu I^0 + \sigma_s(x) \frac{\nu^2}{2} \partial_\nu^2 I^0 \left(\frac{|\mathbf{u}|^2}{3} + (\mathbf{\Omega}, \mathbf{u})^2 \right) \\ &\quad + \sigma_s(x) I^0 \left(3(\mathbf{\Omega}, \mathbf{u})^2 - |\mathbf{u}|^2 \right), \end{aligned}$$

and $Q_s^0(I^2) = \sigma_s(x) (\int_{\Omega} I^2 - I^2)$. We thus obtain

$$\left\{ \begin{aligned} \partial_t I^0 + \mathbf{\Omega} \cdot \nabla_x I^1 &= -\nu \int_{\Omega} (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla \partial_\nu I^0) + \sigma_s(x) \nu \partial_\nu I^0 \left(|\mathbf{u}|^2 - 3(\mathbf{\Omega}, \mathbf{u})^2 \right) \\ &\quad + \sigma_s(x) \nu^2 \partial_\nu^2 I^0 \left((\mathbf{\Omega}, \mathbf{u})^2 - \frac{|\mathbf{u}|^2}{3} \right) + \int_{\Omega} (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla I^0) - (\mathbf{\Omega}, \mathbf{u})(\mathbf{\Omega}, \nabla I^0) \\ &\quad + 2\sigma_s(x) I^0 \left(3(\mathbf{\Omega}, \mathbf{u})^2 - |\mathbf{u}|^2 \right) + Q_a^0(I^0, T^0) + \sigma_s(x) (\mathbf{\Omega}, \mathbf{u})(3C^1 - \nu \partial_\nu C^1) \\ &\quad + \sigma_s(x) \left(\int_{\Omega} I^2 - I^2 \right), \\ \partial_t T^0 + \nabla \cdot (T^0 \mathbf{u}) + \Gamma T^0 \nabla \cdot \mathbf{u} &= - \int_\nu Q_a^0(I^0, T^0). \end{aligned} \right. \quad (2.38)$$

The first equation can be rewritten $\sigma_s(x)(\mathcal{K} - I_d)I^2 = \partial_t I^0 - g$, with an obvious definition of the function g . Using once again the Fredholm theory, this equation has a solution if and only if the compatibility condition $\partial_t I^0 - g \in \text{Ker}(\mathcal{K} - I_d)^\perp$, i.e. $\int_{\Omega} (\partial_t I^0 - g) = \partial_t I^0 - \int_{\Omega} g = 0$ is satisfied. This gives us $\partial_t I^0 = \int_{\Omega} g$ and thus I^2 satisfy $\sigma_s(x)(\mathcal{K} - I_d)I_2 = \int_{\Omega} g - g$. Using the

same arguments than for the computation of I_1 , the solution of this equation is of the form

$$\begin{aligned}
I^2 &= \left((\boldsymbol{\Omega}, \mathbf{u})^2 - \frac{|\mathbf{u}|^2}{3} \right) \left(6I^0 - 3\nu\partial_\nu I^0 + \nu^2\partial_\nu^2 I^0 \right) \\
&+ \frac{1}{\sigma_s^2(x)} \sum_{i,j} \left(\boldsymbol{\Omega}_i \boldsymbol{\Omega}_j - \int_\Omega \boldsymbol{\Omega}_i \boldsymbol{\Omega}_j \right) \partial_{x_j} \partial_{x_i} I^0 - \frac{1}{\sigma_s(x)} (\boldsymbol{\Omega}, \nabla C^1) + C^2(t, x, \nu) \\
&+ \frac{1}{\sigma_s(x)} \sum_{i,j} \left(\boldsymbol{\Omega}_i \boldsymbol{\Omega}_j - \int_\Omega \boldsymbol{\Omega}_i \boldsymbol{\Omega}_j \right) \left((\nu\partial_\nu I^0 - 3I^0) \partial_{x_j} u_i + (\nu\partial_\nu \partial_{x_j} I^0 - 3\partial_{x_j} I^0) u_i \right) \\
&- \sum_{i,j} \left(\boldsymbol{\Omega}_i \boldsymbol{\Omega}_j - \int_\Omega \boldsymbol{\Omega}_i \boldsymbol{\Omega}_j \right) u_i \partial_{x_j} I^0 + (\boldsymbol{\Omega}, \mathbf{u}) (3C^1 - \nu\partial_\nu C^1),
\end{aligned} \tag{2.39}$$

where, $C^2(t, x, \nu)$ is an arbitrary solution of the homogeneous equation $(\mathcal{K} - I_d)C_2 = 0$. Setting $\rho = I^0$ and $\bar{T} = T^0$, integrating the first equation of (2.38) on S^2 and computing all the terms, we find

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s(x)} \right) + \nabla \cdot (\rho \mathbf{u}) = \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu \rho + \sigma_a(\nu)(B(\nu, \bar{T}) - \rho) + \mathcal{O}(\varepsilon), \\ \partial_t \bar{T} + \nabla \cdot (\bar{T} \mathbf{u}) + \Gamma \bar{T} \nabla \cdot \mathbf{u} = - \int_\nu \sigma_a(\nu)(B(\nu, \bar{T}) - \rho) + \mathcal{O}(\varepsilon). \end{cases}$$

Finally, dropping formally all the $\mathcal{O}(\varepsilon)$ terms, we find the following drift-diffusion equation

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s(x)} \right) + \nabla \cdot (\rho \mathbf{u}) = \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu \rho + \sigma_a(\nu)(B(\nu, \bar{T}) - \rho), \\ \partial_t \bar{T} + \nabla \cdot (\bar{T} \mathbf{u}) + \Gamma \bar{T} \nabla \cdot \mathbf{u} = - \int_\nu \sigma_a(\nu)(B(\nu, \bar{T}) - \rho), \end{cases} \tag{2.40}$$

which ends the proof of lemma 2.5. □

2.3.2 A priori estimates for the drift diffusion system

In this part are introduced several new assumptions, which are different from the assumptions H done in the previous section, due to the fact that we need additional regularity on the coefficients σ_a , σ_s and \mathbf{u} . We prove a priori estimates, such as a comparison principle and several regularity results for the solution of the drift diffusion system (2.27) (lemma 2.6). These results are important to prove the convergence result (theorem 2.8) in the next part.

We assume additional regularity on the parameters \mathbf{u} , σ_s and σ_a than in the previous section. This is summarized here, where the assumptions $(\overline{\text{H1}}) - (\overline{\text{H4}})$ are related to the regularity of the coefficients σ_a , σ_s , \mathbf{u} and on the parameter ε , and the assumption $(\overline{\text{H5}})$ (respectively the assumption $(\overline{\text{H6}})$) is related to the initial (respectively to the boundary) conditions of the solution of the drift diffusion system (2.27).

- $(\overline{\text{H1}})$ The velocity field satisfies $\mathbf{u} \in W^{4,\infty}([0, T^f] \times X)$.
- $(\overline{\text{H2}})$ The coefficients ε and $u^* = \|\mathbf{u}\|_{L^\infty}$ are such that there exists $\varepsilon^* < 1$ such that $u^* \varepsilon \leq \varepsilon^*$. It yields the positivity of $1 - \varepsilon^2 |\mathbf{u}|^2$ involved in the expression of the Lorentz coefficient γ^ε .

- $(\overline{H3})$ Smoothness of the scattering coefficient: $\sigma_s \in W^{3,\infty}(X)$ and $\sigma_s > 0$.
- $(\overline{H4})$ Smoothness of the emission absorption coefficient: $\sigma_a \in W^{3,\infty}(\mathbb{R}_\nu^+) \cap L^2(\mathbb{R}_\nu^+)$ and $\sigma_a \geq 0$.
- $(\overline{H5})$ The initial conditions $(\rho^{in}, \bar{T}^{in})$ of the drift diffusion system (2.27) are such that $\nu^k \partial_\nu^{k+q} \partial_{x_j}^l \rho^{in} \in L^2(X \times \mathbb{R}_\nu^+)$, $k, l, j \in (0, 1, 2, 3)$, $k \in \{0, 1\}$ and $\partial_{x_j}^k \bar{T}^{in} \in L^2(X)$, $k, j \in (0, 1, 2, 3)$. Moreover, there exists two bounded and positive constants \bar{T}_* and \bar{T}^* such that $\forall(x, \nu) \in X \times \mathbb{R}_\nu^+$, $\bar{T}_* \leq \bar{T}^{in}(x) \leq \bar{T}^*$ and $0 < B(\nu, \bar{T}_*) \leq \rho^{in}(x, \nu) \leq B(\nu, \bar{T}^*)$.
- $(\overline{H6})$ We consider periodic boundary conditions in space and we assume that the solution of the drift diffusion system (2.27) satisfies $\lim_{\nu \rightarrow 0} \nu^q \partial_\nu^q \partial_{x_1}^l \partial_{x_2}^m \partial_{x_3}^n \partial_t^p \rho = \lim_{\nu \rightarrow \infty} \nu^q \partial_\nu^q \partial_{x_1}^l \partial_{x_2}^m \partial_{x_3}^n \partial_t^p \rho = 0$, $q, l, m, n \in (0, 1, 2, 3)$, and $p \in \{0, 1\}$.

The assumption $(\overline{H2})$ yields in particular the positivity of the Lorentz factor γ^ε . Moreover, this assumption together with the assumption $(\overline{H1})$ lead to an equivalent of the estimates (2.13) for the relativistic coefficient Λ^ε

$$0 < \Lambda_* = 1 - \varepsilon^* \leq \Lambda^\varepsilon(t, x, \mathbf{\Omega}) \leq \Lambda^* = \frac{1 + \varepsilon^*}{\sqrt{1 - (\varepsilon^*)^2}}, \quad \forall(t, x, \mathbf{\Omega}) \in [0, T^f] \times X \times S^2, \quad (2.41)$$

where the notation Λ_* and Λ^* have been kept for simplicity. The regularity assumed on the emission absorption coefficient in assumption $(\overline{H4})$, which stipulates that the emission absorption coefficient satisfies $\sigma_a \in L^2(\mathbb{R}_\nu^+)$, on the initial conditions in assumption $(\overline{H5})$ and at the boundaries $\nu \rightarrow 0$ and $\nu \rightarrow \infty$ in assumption $(\overline{H6})$ are purely technical, in the sense that they are only used to prove that the solution (ρ, \bar{T}) and its derivatives belong to $L_t^\infty(L_{x,\nu}^2) \times L_t^\infty(L_x^2)$. They are due to the particular aspect of the proof, which uses a priori estimates (lemma 2.7). For example assumption $(\overline{H6})$ holds with $p = l = m = n = 0$ and $q \in (0, 1, 2, 3)$ for $\rho = B(\nu, \bar{T})$, the Planck function. We assume that the solution of the drift diffusion system (2.27) satisfies this condition.

The end of this section deals with a priori estimates and with our main result. First (lemma (2.6)), a comparison principle is proved for the solution of the drift diffusion system (2.27), using the same tools than for the radiative transfer equations (theorem 2.1) together with a trick to treat the diffusion term. Secondly (lemma 2.7), a regularity result is provided for the solution of the drift diffusion system (2.27), whose proof is postponed to the appendix.

Lemma 2.6 (Comparison Principle). *Assume that hypotheses $(\overline{H1})$ - $(\overline{H6})$ are satisfied. Then $\forall(t, x, \nu) \in [0, T^f] \times X \times \mathbb{R}_\nu^+$, the solution of the drift diffusion system (2.27) satisfies the a priori estimates $\bar{T}_*(t) \leq \bar{T}(t, x) \leq \bar{T}^*(t)$ and $B(\nu, \bar{T}_*(t)) \leq \rho(t, x, \nu) \leq B(\nu, \bar{T}^*(t))$, with*

$$\begin{cases} \bar{T}_*(t) = \bar{T}_* e^{-t(\Gamma + \frac{4}{3})\|\nabla \cdot \mathbf{u}\|_{L_{t,x}^\infty}}, \\ \bar{T}^*(t) = \bar{T}^* e^{t(\Gamma + \frac{4}{3})\|\nabla \cdot \mathbf{u}\|_{L_{t,x}^\infty}}, \end{cases} \quad (2.42)$$

where the constants \bar{T}_* and \bar{T}^* are defined in assumption $(\overline{H5})$.

Proof. Since the arguments are similar, we only show the proof for the upper bound. We denote $B_\nu^* = B(\nu, \bar{T}^*)$ for ease of notations. The proof is mainly the same than for the radiative transfer equations (2.1), except that we need to treat the second order derivative. To achieve this we use a method of Carrillo et al [CRS08], which is to introduce a function sgn_α^+ , where

$0 < \alpha \leq 1$ is a small parameter, defined as a non decreasing regularization of the sgn^+ function defined in (2.12).

Using algebraic manipulations, one can write from the first equation of the drift diffusion (2.27) the following equation

$$\begin{aligned} & \partial_t(\rho - B_\nu^*) - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s} \right) + \nabla \cdot \left((\rho - B_\nu^*) \mathbf{u} \right) + \left(\partial_t B_\nu^* + B_\nu^* \nabla \cdot \mathbf{u} - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu B_\nu^* \right) \\ & = \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu (\rho - B_\nu^*) + \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) \end{aligned} \quad (2.43)$$

The proof relies on the non negativity of the term $\partial_t B_\nu^* + B_\nu^* \nabla \cdot \mathbf{u} - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu B_\nu^*$. Indeed, the definition of the Planck function yields

$$\partial_t B_\nu^* + B_\nu^* \nabla \cdot \mathbf{u} - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu B_\nu^* = \frac{\nu}{\bar{T}^*} \frac{B_\nu^*}{1 - e^{-\nu/\bar{T}^*}} \left(\frac{\partial_t \bar{T}^*}{\bar{T}^*} + \frac{\nabla \cdot \mathbf{u}}{3} \right),$$

which is non negative by definition of \bar{T}^* (2.42). The equation (2.43) can be simplified to

$$\partial_t(\rho - B_\nu^*) - \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s} \right) + \nabla \cdot \left((\rho - B_\nu^*) \mathbf{u} \right) \leq \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu (\rho - B_\nu^*) + \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) \quad (2.44)$$

Multiplying this equation by $\text{sgn}_\alpha^+(\rho - B_\nu^*)$ and integrating over $\mathbb{R}_x^3 \times \mathbb{R}_\nu^+$ yield

$$\begin{aligned} & \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \left(\partial_t(\rho - B_\nu^*) + \nabla \cdot \left((\rho - B_\nu^*) \mathbf{u} \right) - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu (\rho - B_\nu^*) \right) \\ & - \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s} \right) \leq \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \sigma_a(\nu) (B(\nu, \bar{T}) - \rho). \end{aligned}$$

An integration by parts yields

$$- \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \nabla \cdot \left(\frac{\nabla \rho}{3\sigma_s} \right) = \int_{x,\nu} (\text{sgn}_\alpha^+)'(\rho - B_\nu^*) \frac{|\nabla \rho|^2}{3\sigma_s} \geq 0,$$

due to the non decreasing behavior of the sgn_α^+ function, to the fact that \bar{T}^* does not depend on the space variable x and since the scattering coefficient is positive by assumption $(\overline{\text{H3}})$. One thus obtains

$$\begin{aligned} & \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \left(\partial_t(\rho - B_\nu^*) + \nabla \cdot \left((\rho - B_\nu^*) \mathbf{u} \right) - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu (\rho - B_\nu^*) \right) \\ & \leq \int_{x,\nu} \text{sgn}_\alpha^+(\rho - B_\nu^*) \sigma_a(\nu) (B(\nu, \bar{T}) - \rho). \end{aligned}$$

We now pass to the limit as $\alpha \rightarrow 0$ in this inequality, where only the sgn_α^+ function depends on α . It yields

$$\begin{aligned} & \int_{x,\nu} \text{sgn}^+(\rho - B_\nu^*) \left(\partial_t(\rho - B_\nu^*) + \nabla \cdot \left((\rho - B_\nu^*) \mathbf{u} \right) - \frac{\nabla \cdot \mathbf{u}}{3} \nu \partial_\nu (\rho - B_\nu^*) \right) \\ & \leq \int_{x,\nu} \text{sgn}^+(\rho - B_\nu^*) \sigma_a(\nu) (B(\nu, \bar{T}) - \rho). \end{aligned} \quad (2.45)$$

As for the proof of the comparison principle for the transfer equations (theorem 2.1), we write an inequality satisfied by $\int_{x,\nu} (\rho - B_\nu^*)^+$. From the equation (2.45), one easily obtains by using integration by parts and the boundary conditions (assumption $(\overline{H6})$)

$$\frac{d}{dt} \int_{x,\nu} (\rho - B_\nu^*)^+ \leq - \int_{x,\nu} \frac{\nabla \cdot \mathbf{u}}{3} (\rho - B_\nu^*)^+ + \int_{x,\nu} \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) \operatorname{sgn}^+(\rho - B_\nu^*). \quad (2.46)$$

We keep this expression and study the equation satisfied by $(\bar{T} - \bar{T}^*)^+$, where \bar{T} is the solution of the drift diffusion equation (2.27). Using the same arguments than for the derivation of the equation (2.19) (proof of the comparison principle for the transfer equations (2.1)), one finds

$$\begin{aligned} & \partial_t (\bar{T} - \bar{T}^*)^+ + \nabla \cdot \left((\bar{T} - \bar{T}^*)^+ \mathbf{u} \right) + \Gamma (\bar{T} - \bar{T}^*)^+ \nabla \cdot \mathbf{u} + \operatorname{sgn}^+(\bar{T} - \bar{T}^*) \left(\partial_t \bar{T}^* + (\Gamma + 1) \bar{T}^* \nabla \cdot \mathbf{u} \right) \\ &= - \int_{\nu} \operatorname{sgn}^+(\bar{T} - \bar{T}^*) \sigma_a(\nu) (B(\nu, \bar{T}) - \rho). \end{aligned}$$

Once again, the key of the proof is that by definition of \bar{T}^* (2.42), one has $\partial_t \bar{T}^* + (\Gamma + 1) \bar{T}^* \nabla \cdot \mathbf{u} \geq 0$, which yields

$$\partial_t (\bar{T} - \bar{T}^*)^+ + \nabla \cdot \left((\bar{T} - \bar{T}^*)^+ \mathbf{u} \right) + \Gamma (\bar{T} - \bar{T}^*)^+ \nabla \cdot \mathbf{u} \leq - \int_{\nu} \operatorname{sgn}^+(\bar{T} - \bar{T}^*) \sigma_a(\nu) (B(\nu, \bar{T}) - \rho).$$

Adding this inequality integrated over X with the inequality (2.46), one gets by introducing $H(t) = \int_{x,\nu} (\rho - B(\nu, \bar{T}^*))^+ + \int_x (\bar{T} - \bar{T}^*)^+$ and by using the boundary conditions (assumption $(\overline{H6})$)

$$H'(t) \leq \left(\Gamma + \frac{1}{3} \right) \|\nabla \cdot \mathbf{u}\|_{L_x^\infty} H(t) + \int_{x,\nu} \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) \left(\operatorname{sgn}^+(\rho - B_\nu^*) - \operatorname{sgn}^+(\bar{T} - \bar{T}^*) \right).$$

Using the same reasoning than for the proof of the maximum principle for the relativistic transfer equation (see equation (2.21)), one has

$$\sigma_a(\nu) (B(\nu, \bar{T}) - \rho) \left(\operatorname{sgn}^+(\rho - B_\nu^*) - \operatorname{sgn}^+(\bar{T} - \bar{T}^*) \right) \leq 0,$$

and the Gronwall lemma gives the result. \square

We now turn to a regularity result for the solution of the drift diffusion system (2.27), which is needed for the proof of convergence in the next part. We have the

Lemma 2.7 (Regularity of the solution of the drift diffusion system). *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists a constant C such that the solution of the drift diffusion system (2.27) satisfies*

$$\begin{cases} \sum_{0 \leq p \leq 1} \sum_{0 \leq q, l, m, n \leq 3} \left\| \nu^q \partial_\nu^q \partial_{x_1}^l \partial_{x_2}^m \partial_{x_3}^n \partial_t^p \rho \right\|_{L_t^\infty(L_{x,\nu}^2)} \leq C, \\ \sum_{0 \leq p \leq 1} \sum_{0 \leq l, m, n \leq 3} \left\| \partial_{x_1}^l \partial_{x_2}^m \partial_{x_3}^n \partial_t^p \bar{T} \right\|_{L_t^\infty(L_x^2)} \leq C, \end{cases} \quad (2.47)$$

with the convention $\partial_\xi^0 = I_d$, $\xi = x, t$ or ν .

Proof. The proof of this lemma is long and tedious, and thus we postpone it to the appendix. It mainly uses the linearity of the equation on ρ and the comparison principle to treat the nonlinearity of the equation on \bar{T} . \square

2.3.3 A rigorous proof of convergence

This part is devoted to the proof of convergence of the solution $(I^\varepsilon, T^\varepsilon)$ of the relativistic transfer equations (2.26) to the solution of the drift diffusion system (2.27) as $\varepsilon \rightarrow 0$. Contrary to the previous section, in which the limit was obtained formally (lemma 2.5), the remainders of the source terms expansions are shown to be bounded in $L_t^\infty(L_{x,\nu}^2)$ (lemmas 2.9 and 2.10). The main result (theorem 2.8) deals with the proof of strong convergence in L^2 of the difference between the solution of the transfer equations (2.26) and the solution of the drift diffusion system (2.27).

The convergence result is

Theorem 2.8 (Convergence in the non-equilibrium diffusion regime). *Under assumption $(\overline{H1})$ - $(\overline{H6})$, there exists two constants C and C_1 , which do not depend on ε , such that the solution of the relativistic transfer system (2.26) and the solution of the drift diffusion system (2.27) satisfy the following estimate, where $0 \leq t \leq T^f$,*

$$\|I^\varepsilon(t) - \rho(t)\|_{L_{x,\nu,\Omega}^2} + \|T^\varepsilon(t) - \bar{T}(t)\|_{L_x^2} \leq C_1 \left(\|I^\varepsilon(0) - \rho(0)\|_{L_{x,\nu,\Omega}^2} + \|T^\varepsilon(0) - \bar{T}(0)\|_{L_x^2} + \varepsilon \right) e^{Ct}.$$

As a remark, the exponential in time in this result comes from the use of the Gronwall lemma. The end of the chapter is devoted to the proof of this convergence result. Since it is rather technical, it is divided in three steps. In a first one, the regularity needed on the solution of the drift diffusion system (2.27) to control the remainders of the expansions of the source term (part 2.3.1.0) is highlighted. In a second part, a function constructed from the truncated Hilbert expansion performed in the part 2.3.1.0, is proved to be solution of the radiative transfer equations (2.26) with a remainder, which is proved to be small with respect to ε . Finally, in a last part, the difference between this solution and the solution of the radiative transfer equations (2.26) is shown to tend to 0 with ε in L^2 . In particular a suitable weight is used to conclude the proof.

First step: control of the remainders of the expansions of the source term

In the last section, the scattering and emission absorption operators (see part 2.3.1.0) have been expanded in power of ε , but no attention was given to the remainders $\bar{Q}_{s,i}^\varepsilon$, for $i = 0, 1, 2$ defined respectively in section (2.30), (2.29) and (2.28) and to \bar{Q}_a^ε defined in (2.33). This is the purpose of this part. The following results are important, in the sense that they highlight the regularity needed on the solution of the drift diffusion system to control these remainders. The proofs of these results are postponed to the appendix.

Lemma 2.9. *Assume that J is a given function. Under assumptions $(\overline{H1})$ - $(\overline{H4})$, there exists a constant $C \geq 0$ which does not depend on ε , such that the following inequality holds*

$$\|\bar{Q}_{s,i}^\varepsilon(J)\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \sum_{k=0}^{i+1} \|\nu^k \partial_\nu^k J\|_{L_t^\infty(L_{x,\nu,\Omega}^2)}, \quad i = 0, 1, 2.$$

with the convention $\partial_\nu^0 = I_d$.

In the same way, the following lemma shows the regularity needed on given functions J and G to control the remainder \bar{Q}_a^ε of the emission absorption operator.

Lemma 2.10. *Assume that G and J are two given functions. Under assumptions $(\overline{H1})$ - $(\overline{H4})$, there exists a constant $C \geq 0$ which depends on the $L^\infty([0, T^J] \times X)$ norm of G , but does not depend on ε such that the following inequality holds*

$$\|\bar{Q}_a^\varepsilon(J, G)\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \left(\|G\|_{L_t^\infty(L_x^2)} + \|J\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} + \|\nu J\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

Second step: reconstruction of the solution

In this part is reconstructed a pair $(\hat{I}^\varepsilon, \hat{T})$ constructed from the formal Hilbert expansion, solution to (2.26) with a remainder, which is shown to be small with respect to ε in some norm. These functions are defined by

$$\begin{cases} \hat{I}^\varepsilon = \rho + \varepsilon\rho_1 + \varepsilon^2\rho_2, \\ \hat{T} = \bar{T}, \end{cases} \quad (2.48)$$

where ρ, ρ_1, ρ_2 and \bar{T} are constructed as follow: (ρ, \bar{T}) is the solution of the drift-diffusion equation (2.40),

$$\rho_1 = \frac{1}{\sigma_s(x)} \mathbf{\Omega} \cdot \nabla_x \rho - \mathbf{\Omega} \cdot \mathbf{u} \nu \partial_\nu \rho + 3\mathbf{\Omega} \cdot \mathbf{u} \rho, \quad (2.49)$$

and

$$\begin{aligned} \rho_2 = & \left((\mathbf{\Omega} \cdot \mathbf{u})^2 - \frac{|\mathbf{u}|^2}{3} \right) \left(6\rho - 3\nu \partial_\nu \rho + \nu^2 \partial_\nu^2 \rho \right) + \frac{1}{\sigma_s^2(x)} \sum_{i,j} (\mathbf{\Omega}_i \mathbf{\Omega}_j - \langle \mathbf{\Omega}_i \mathbf{\Omega}_j \rangle) \partial_{x_j} \partial_{x_i} \rho \\ & + \frac{1}{\sigma_s(x)} \sum_{i,j} (\mathbf{\Omega}_i \mathbf{\Omega}_j - \langle \mathbf{\Omega}_i \mathbf{\Omega}_j \rangle) \left((\nu \partial_\nu \rho - 3\rho) \partial_{x_j} \mathbf{u}_i + (\nu \partial_\nu \partial_{x_j} \rho - 3\partial_{x_j} \rho) \mathbf{u}_i \right) \\ & - \sum_{i,j} (\mathbf{\Omega}_i \mathbf{\Omega}_j - \langle \mathbf{\Omega}_i \mathbf{\Omega}_j \rangle) \mathbf{u}_i \partial_{x_j} \rho. \end{aligned} \quad (2.50)$$

Obviously ρ^1 and ρ^2 are related to the definition of I^1 (2.36) and I^2 (2.39). In this part is proved the following lemma, which shows that $(\hat{I}^\varepsilon, \hat{T})$ is solution of the radiative transfer equations (2.26) with remainders R^ε and S^ε .

Lemma 2.11. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, the pair $(\hat{I}^\varepsilon, \hat{T})$ previously constructed is solution of the following system*

$$\begin{cases} \partial_t \hat{I}^\varepsilon + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla_x \hat{I}^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon(\hat{I}^\varepsilon) + Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) + \varepsilon R^\varepsilon, \\ \partial_t \hat{T} + \nabla \cdot (\hat{T} \mathbf{u}) + \Gamma \hat{T} \nabla \cdot \mathbf{u} = - \int_{\nu, \Omega} \frac{\Lambda^\varepsilon}{\gamma^\varepsilon} Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) + \varepsilon S^\varepsilon, \end{cases} \quad (2.51)$$

where R^ε and S^ε are such that there exists a constant C which does not depend on ε such that $\|R^\varepsilon\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C$ and $\|S^\varepsilon\|_{L_t^\infty(L_x^2)} \leq C$.

Proof. Using the results of the previous section, it is easy to see that

$$\begin{cases} Q_s^0(\rho) = 0, \\ \mathbf{\Omega} \cdot \nabla \rho = Q_s^0(\rho_1) + Q_s^1(\rho), \\ \partial_t \rho + \mathbf{\Omega} \cdot \nabla \rho_1 = Q_s^0(\rho_2) + Q_s^1(\rho_1) + Q_s^2(\rho) + Q_a^0(\rho, \hat{T}). \end{cases}$$

We write $Q_s^\varepsilon(\hat{I}^\varepsilon)$ as

$$Q_s^\varepsilon(\hat{I}^\varepsilon) = Q_s^0(\rho) + \varepsilon(Q_s^0(\rho_1) + Q_s^1(\rho)) + \varepsilon^2(Q_s^0(\rho_2) + Q_s^1(\rho_1) + Q_s^2(\rho)) \\ + \varepsilon^3(\bar{Q}_{s,2}^\varepsilon(\rho) + \bar{Q}_{s,1}^\varepsilon(\rho_1) + \bar{Q}_{s,0}^\varepsilon(\rho_2)).$$

Using algebraic arguments, we obtain the system (2.51), where

$$\begin{cases} R^\varepsilon = \partial_t \rho_1 + \varepsilon \partial_t \rho_2 + \mathbf{\Omega} \cdot \nabla \rho_2 - (\bar{Q}_{s,2}^\varepsilon(\rho) + \bar{Q}_{s,1}^\varepsilon(\rho_1) + \bar{Q}_{s,0}^\varepsilon(\rho_2)) \\ \quad + \frac{Q_a^0(\rho, \hat{T}) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T})}{\varepsilon}, \\ S^\varepsilon = \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\frac{\Lambda}{\gamma} Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) - Q_a^0(\rho, \hat{T}) \right). \end{cases} \quad (2.52)$$

We now study the remainders R^ε and S^ε . First, using the lemma 2.9, one has

$$\left\| \bar{Q}_{s,2}^\varepsilon(\rho) + \bar{Q}_{s,1}^\varepsilon(\rho_1) + \bar{Q}_{s,0}^\varepsilon(\rho_2) \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \left(\sum_{k=0}^3 \|\nu^k \partial_\nu^k \rho\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right. \\ \left. + \sum_{k=0}^2 \|\nu^k \partial_\nu^k \rho_1\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} + \sum_{k=0}^1 \|\nu^k \partial_\nu^k \rho_2\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

The definitions of ρ_1 (2.49) and ρ_2 (2.50) yield, with another constant C ,

$$\|R^\varepsilon\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \left(\sum_{0 \leq p \leq 1} \sum_{0 \leq q \leq 3} \sum_{\substack{1 \leq i,j,k \leq 3 \\ 0 \leq l,m,n \leq 3}} \left\| \nu^q \partial_\nu^q \partial_{x_i}^l \partial_{x_j}^m \partial_{x_k}^n \partial_t^p \rho \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right. \\ \left. + \left\| \frac{Q_a^0(\rho, \hat{T}) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T})}{\varepsilon} \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

The first term is uniformly bounded thanks to the lemma 2.7. For the second one, one has

$$\frac{Q_a^0(\rho, \hat{T}) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T})}{\varepsilon} = -\bar{Q}_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) + \sigma_a(\nu)(\rho_1 + \varepsilon \rho_2),$$

where \bar{Q}_a^ε is defined in (2.33). Since a comparison principle has been provided, the lemma 2.10 can be applied to control the first term of the right member. Using the definition of ρ_1 (2.49), ρ_2 (2.50) and the lemma 2.7 to control the other ones, one easily gets

$$\left\| \frac{Q_a^0(\rho, \hat{T}) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T})}{\varepsilon} \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C,$$

where the constant C is uniform in ε . It yields, with another (uniform in ε) constant C ,

$$\|R^\varepsilon\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C. \quad (2.53)$$

We now look at the term S_ε . One can rewrite it as

$$\begin{aligned} S^\varepsilon &= \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\frac{\Lambda}{\gamma} - 1 \right) \left(\sigma_a(\nu_0) \left(B(\nu_0, \hat{T}) - \hat{I}^\varepsilon \right) \right) && := S_1^\varepsilon \\ &\quad - \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\frac{\Lambda}{\gamma} - 1 \right) \left(\sigma_a(\nu) \left(B(\nu, \hat{T}) - \rho \right) \right) && := S_2^\varepsilon \\ &\quad + \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\sigma_a(\nu_0) B(\nu_0, \hat{T}) - \sigma_a(\nu) B(\nu, \hat{T}) \right) && := S_3^\varepsilon \\ &\quad + \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\sigma_a(\nu) \rho - \sigma_a(\nu_0) \hat{I}^\varepsilon \right) && := S_4^\varepsilon \end{aligned}$$

We study each term separately. By definition, $\Lambda\gamma^{-1} - 1 = \varepsilon(\mathbf{\Omega}, \mathbf{u})$. The relation $(a + b)^2 \leq 2(a^2 + b^2)$ yields

$$\|S_1^\varepsilon\|_{L_x^2}^2 \leq 2\|\mathbf{u}\|_{L_x^\infty}^2 \left\{ \int_x \left(\int_\nu \sigma_a(\nu_0) B(\nu_0, \hat{T}) \right)^2 + \int_x \left(\int_{\nu, \Omega} \sigma_a(\nu_0) \hat{T}^\varepsilon \right)^2 \right\}.$$

A Cauchy Schwarz inequality in the second term gives

$$\|S_1^\varepsilon\|_{L_x^2}^2 \leq 2\|\mathbf{u}\|_{L_x^\infty}^2 \left\{ \|\sigma_a\|_{L_x^\infty}^2 \int_x \left(\int_\nu B(\nu_0, \hat{T}) \right)^2 + \|\sigma_a\|_{L_\nu^2}^2 \|\hat{T}^\varepsilon\|_{L_{x, \nu, \Omega}^2}^2 \right\},$$

which is bounded thanks to the assumption $(\overline{\text{H4}})$ on the emission absorption coefficient. The definition of the Planck function, together with the change of variable $\nu \mapsto \nu_0/\hat{T}$, yield $\int_\nu B(\nu_0, \hat{T}) = \hat{T}^4 \int_\nu \nu^3 (e^\nu - 1)^{-1} \leq C\hat{T}^4$. The comparison principle for the solution of the drift diffusion system (2.27) (lemma 2.6) yields in particular $\int_x \left(\int_\nu B(\nu_0, \hat{T}) \right)^2 \leq C\|\hat{T}\|_{L_x^2}$. Finally, the definition of \hat{T} in terms of ρ, ρ_1 and ρ_2 (2.48) and the regularity of the solution of the drift diffusion system (lemma 2.7) yield $\|S_1^\varepsilon\|_{L_x^2} \leq C$. The same arguments give $\|S_2^\varepsilon\|_{L_x^2} \leq C$. The control of the term S_3^ε is a little more technical. In order to use the fact that $\nu_0 - \nu$ is of order ε , we decompose it as

$$\begin{aligned} S_3^\varepsilon &= \frac{1}{\varepsilon} \int_{\nu, \Omega} \left(\left(\sigma_a(\nu_0) - \sigma_a(\nu) \right) B(\nu_0, \hat{T}) + \left(B(\nu_0, \hat{T}) - B(\nu, \hat{T}) \right) \sigma_a(\nu) \right) \\ &= S_{3,1}^\varepsilon + S_{3,2}^\varepsilon. \end{aligned}$$

The relations $\sigma_a(\nu_0) - \sigma_a(\nu) = \int_\nu^{\nu_0} \sigma_a'(s) ds$ and $\nu - \nu_0 = (1 - \Lambda)\nu$ yield

$$\|S_{3,1}^\varepsilon\|_{L_x^2}^2 \leq \|\sigma_a\|_{W^{1, \infty}}^2 \frac{\sup_{t, x} |1 - \Lambda|^2}{\varepsilon^2} \int_x \left(\int_\nu \nu B(\nu_0, \hat{T}) \right)^2.$$

The definition of the Planck function, together with the change of variable $\nu \mapsto \nu_0/\hat{T}$ give us $\int_\nu \nu B(\nu_0, \hat{T}) = \Lambda^{-2} \hat{T}^5 \int_\nu \nu^4 (e^\nu - 1)^{-1} \leq C\hat{T}^5$. Finally, the relation $|1 - \Lambda| \leq C\varepsilon$, the assumption $(\overline{\text{H4}})$ on the regularity of the emission absorption coefficient and the comparison principle (lemma 2.6) lead to $\|S_{3,1}^\varepsilon\|_{L_x^2}^2 \leq C$. For the term $S_{3,2}^\varepsilon$ the relation $B(\nu_0, \hat{T}) - B(\nu, \hat{T}) = \int_\nu^{\nu_0} \partial_\nu B(s, \hat{T}) ds$ gives

$$\|S_{3,2}^\varepsilon\|_{L_x^2}^2 = \frac{1}{\varepsilon^2} \int_x \left(\int_{\nu, \Omega} \int_{s=\nu}^{s=\nu_0} \sigma_a(\nu) \partial_\nu B(s, \hat{T}) \right)^2.$$

The Fubini's theorem yields $\int_\nu \int_{s=\nu}^{s=\Lambda\nu} \sigma_a(\nu) \partial_\nu B(s, \hat{T}) = \int_s \int_{\nu=s/\Lambda}^{\nu=s} \sigma_a(\nu) \partial_\nu B(s, \hat{T})$. One thus finds, using the estimate (2.41)

$$\|S_{3,2}^\varepsilon\|_{L_x^2}^2 \leq \frac{\sup_{t, x} |1 - \Lambda|^2}{\varepsilon^2} \frac{\|\sigma_a\|_{L_\nu^\infty}}{\Lambda_*} \int_x \left(\int_\nu \nu \partial_\nu B(\nu, \hat{T}) \right)^2.$$

The definition of the Planck function gives $\nu \partial_\nu B(\nu, \hat{T}) = (3 - (1 - e^{-\nu/\hat{T}})^{-1} \nu/\hat{T}) B(\nu, \hat{T})$. Once again, the change of variable $\nu \mapsto \nu_0/\hat{T}$ give us $\int_\nu \nu \partial_\nu B(\nu, \hat{T}) \leq C$, which yields, with another constant C , $\|S_{3,2}^\varepsilon\|_{L_x^2} \leq C$. The same arguments give $\|S_4^\varepsilon\|_{L_x^2} \leq C$, which concludes the proof. \square

Third step: end of the proof

In this part we end the proof of the theorem 2.8. The following lemma shows that for all $t \in [0, T^f]$, the function $\|I^\varepsilon(t) - \hat{I}^\varepsilon(t)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(t) - \hat{T}(t)\|_{L^2_x}$ tends to 0 with ε . The important point of the proof is the use of a well chosen weight to overcome the fact that the scattering operator is of order ε^{-2} .

Lemma 2.12. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists two constants C and C_1 , which do not depend on ε such that $(I^\varepsilon - \hat{I}^\varepsilon, T^\varepsilon - \hat{T})$ satisfies the following estimate, where $0 \leq t \leq T^f$,*

$$\|I^\varepsilon(t) - \hat{I}^\varepsilon(t)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(t) - \hat{T}(t)\|_{L^2_x} \leq C_1 \left(\|I^\varepsilon(0) - \hat{I}^\varepsilon(0)\|_{L^2_{x,\nu,\Omega}} + \|T^\varepsilon(0) - \hat{T}(0)\|_{L^2_x} + \varepsilon \right) e^{Ct}.$$

Proof. We denote $E^\varepsilon = I^\varepsilon - \hat{I}^\varepsilon$ and $F^\varepsilon = T^\varepsilon - \hat{T}$. Using the lemma 2.11, the couple $(E^\varepsilon, F^\varepsilon)$ satisfies the following system :

$$\begin{cases} \partial_t E^\varepsilon + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla_x E^\varepsilon = \frac{1}{\varepsilon^2} Q_s^\varepsilon(E^\varepsilon) - \varepsilon R^\varepsilon + U^\varepsilon, \\ \partial_t F^\varepsilon + \nabla \cdot (F^\varepsilon \mathbf{u}) + \Gamma F^\varepsilon \nabla \cdot \mathbf{u} = -\varepsilon S^\varepsilon + V^\varepsilon, \end{cases} \quad (2.54)$$

with

$$\begin{cases} U^\varepsilon = Q_a^\varepsilon(I^\varepsilon, T^\varepsilon) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}), \\ V^\varepsilon = - \int_{\nu,\Omega} \frac{\Lambda}{\gamma} \left(Q_a^\varepsilon(I^\varepsilon, T^\varepsilon) - Q_a^\varepsilon(\hat{I}^\varepsilon, \hat{T}) \right). \end{cases}$$

Let us first prove that the $L^2_{x,\nu}$ norm of U^ε and the L^2_x norm of V^ε are controlled by the $L^2_{x,\nu}$ norm of E^ε and the L^2_x norm of F^ε . The definition of the emission absorption operator (2.31) yields

$$U^\varepsilon = \Lambda^{-2} \sigma_a(\nu_0) \left((B(\nu_0, T^\varepsilon) - B(\nu_0, \hat{T})) - \Lambda^3 (I^\varepsilon - \hat{I}^\varepsilon) \right)$$

The triangular inequality together with the estimate (2.41) yield

$$\|U^\varepsilon\|_{L^2_{x,\nu,\Omega}} \leq \Lambda_*^{-2} \max(1, \Lambda^{*3}) \|\sigma_a\|_{L^\infty} \left(\|B(\nu, T^\varepsilon) - B(\nu, \hat{T})\|_{L^2_{x,\nu}} + \|I^\varepsilon - \hat{I}^\varepsilon\|_{L^2_{x,\nu,\Omega}} \right).$$

The same arguments than in the proof of the lemma 2.4 give $\|B(\nu, T^\varepsilon) - B(\nu, \hat{T})\|_{L^2_{x,\nu}} \leq C \|T^\varepsilon - \hat{T}\|_{L^2_x}$, which yields

$$\|U^\varepsilon\|_{L^2_{x,\nu,\Omega}} \leq C \left(\|E^\varepsilon\|_{L^2_{x,\nu,\Omega}} + \|F^\varepsilon\|_{L^2_x} \right). \quad (2.55)$$

We turn to the control of V^ε . By definition of the operator Q_a^ε (2.8), one has

$$V^\varepsilon = \int_{\nu,\Omega} \frac{\Lambda^2}{\gamma} \sigma_a(\nu_0) \left(\frac{B(\nu_0, \hat{T}) - B(\nu_0, T^\varepsilon)}{\Lambda^3} + I^\varepsilon - \hat{I}^\varepsilon \right).$$

We treat these terms separately. For the first one, the estimate 2.41, the assumption $(\overline{H4})$ on the regularity of the emission absorption coefficient and the arguments used to control U^ε (2.55) give

$$\left\| \int_{\nu,\Omega} \frac{\Lambda^2}{\gamma} \sigma_a(\nu_0) \frac{B(\nu_0, \hat{T}) - B(\nu_0, T^\varepsilon)}{\Lambda^3} \right\|_{L^2_x} \leq (\Lambda^*)^{-1} C \|\hat{T} - T^\varepsilon\|_{L^2_x},$$

For the second one, a Cauchy Schwarz inequality yields

$$\left\| \int_{\nu, \Omega} \frac{\Lambda^2}{\gamma} \sigma_a(\nu_0) (I^\varepsilon - \hat{I}^\varepsilon) \right\|_{L_x^2} \leq C \|\sigma_a\|_{L_\nu^2} \|I^\varepsilon - \hat{I}^\varepsilon\|_{L_x^2}.$$

It finally yields, with another constant C

$$\|V^\varepsilon\|_{L_x^2} \leq C \left(\|T^\varepsilon - \hat{T}\|_{L_x^2} + \|I^\varepsilon - \hat{I}^\varepsilon\|_{L_{x,\nu,\Omega}^2} \right). \quad (2.56)$$

To obtain the proposed result, we need a stability result for the modified transfer system (2.54). The problematic term comes from the scattering operator, which is of order ε^{-2} . Since the scattering operator is isotropic in the moving frame, we multiply the first equation of (2.54) by the weight $\Lambda \gamma^{-1} E_0^\varepsilon$, with an obvious notation $E_0^\varepsilon = \Lambda^3 E^\varepsilon$ and we integrate it on $X \times \mathbb{R}_\nu^+ \times S^2$. This gives us

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 (E^\varepsilon)^2 &= \int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 E^\varepsilon (-\varepsilon R^\varepsilon + U^\varepsilon) \\ &+ \frac{1}{\varepsilon^2} \int_x \gamma^{-1} \sigma_s(x) \int_{\nu,\Omega} \frac{1}{\Lambda} E_0^\varepsilon (\langle E_0^\varepsilon \rangle_0 - E_0^\varepsilon) + \frac{1}{2} \int_{x,\nu,\Omega} (E^\varepsilon)^2 \left(\partial_t \gamma^{-1} \Lambda^4 + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla \gamma^{-1} \Lambda^4 \right), \end{aligned} \quad (2.57)$$

where we remind that $\langle E_0^\varepsilon \rangle = \int_{\Omega_0} E_0^\varepsilon$. Using the invariance of the measure $\nu d\nu d\mathbf{\Omega} = \nu_0 d\nu_0 d\mathbf{\Omega}_0$, one has

$$\int_{\nu,\Omega} \frac{1}{\Lambda} E_0^\varepsilon (\langle E_0^\varepsilon \rangle_0 - E_0^\varepsilon) = \int_{\nu_0} \left(\langle E_0^\varepsilon \rangle_0^2 - \langle E_0^{\varepsilon 2} \rangle_0 \right),$$

which is non positive thanks to a Cauchy-Schwarz inequality, and it shows the importance of the chosen weight. Adding equation (2.57) with the second equation of (2.54) multiplied by F^ε and integrated on \mathbb{R}_x^3 , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 (E^\varepsilon)^2 + \int_x (F^\varepsilon)^2 \right) &\leq \frac{1}{2} \int_{x,\nu,\Omega} (E^\varepsilon)^2 \left| \partial_t \gamma^{-1} \Lambda^4 + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla \gamma^{-1} \Lambda^4 \right| \\ &+ \int_x (F^\varepsilon)^2 |\nabla \cdot \mathbf{u}| \left(\frac{1}{2} + \Gamma \right) + \int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 |E^\varepsilon| |U^\varepsilon + \varepsilon R^\varepsilon| + \int_x |F^\varepsilon| |V^\varepsilon + \varepsilon S^\varepsilon|. \end{aligned}$$

We need to control the two last terms. Using the inequalities $ab \leq 1/2(a^2 + b^2)$ and $(a+b)^2 \leq 2(a^2 + b^2)$, one has

$$\int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 |E^\varepsilon| |U^\varepsilon + \varepsilon R^\varepsilon| \leq \Lambda^{*4} \left(\frac{1}{2} \|E^\varepsilon\|_{L_{x,\nu,\Omega}^2}^2 + \|U^\varepsilon\|_{L_{x,\nu,\Omega}^2}^2 + \varepsilon^2 \|R^\varepsilon\|_{L_{x,\nu,\Omega}^2}^2 \right).$$

Using the estimate (2.55), one finally finds

$$\int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 |E^\varepsilon| |U^\varepsilon + \varepsilon R^\varepsilon| \leq C \left(\|E^\varepsilon\|_{L_{x,\nu,\Omega}^2}^2 + \|F^\varepsilon\|_{L_{x,\nu}^2}^2 + \varepsilon^2 \|R^\varepsilon\|_{L_{x,\nu,\Omega}^2}^2 \right),$$

where the constant C is uniform in ε . One also has

$$\int_x |F^\varepsilon| |V^\varepsilon + \varepsilon S^\varepsilon| \leq \frac{1}{2} \|F^\varepsilon\|_{L_x^2}^2 + \|V^\varepsilon\|_{L_x^2}^2 + \varepsilon^2 \|S^\varepsilon\|_{L_x^2}^2,$$

which gives us, together with estimate (2.56)

$$\int_x |F^\varepsilon| |V^\varepsilon + \varepsilon S^\varepsilon| \leq C \left(\|F^\varepsilon\|_{L_x^2}^2 + \|E^\varepsilon\|_{L_{x,\nu}^2}^2 + \varepsilon^2 \right).$$

This gives us

$$\frac{1}{2} \frac{d}{dt} \left(\int_{x,\nu,\Omega} \gamma^{-1} \Lambda^4 (E^\varepsilon)^2 + \int_x (F^\varepsilon)^2 \right) \leq C \left(\int_{x,\nu,\Omega} (E^\varepsilon)^2 + \int_x (F^\varepsilon)^2 + \varepsilon^2 \right),$$

where the constant C is uniform in ε . Integrating this inequality between 0 and t and using the Gronwall lemma, one gets a constant C_1 such that

$$\|E^\varepsilon(t)\|_{L_{x,\nu,\Omega}^2} + \|F^\varepsilon(t)\|_{L_x^2} \leq C_1 \left(\|E^\varepsilon(0)\|_{L_{x,\nu,\Omega}^2} + \|F^\varepsilon(0)\|_{L_x^2} + \varepsilon \right) e^{2Ct},$$

where the constant C_1 depends (linearly) on the time t and on $\min(1, \inf_{t,x,\nu,\Omega} \gamma^{-1} \Lambda^4)^{-1}$. Using the estimate (2.41), and the assumption $(\overline{H2})$, which yields the positivity of the Lorentz factor γ , one sees that $\inf_{t,x,\nu,\Omega} \gamma^{-1} \Lambda^4 > 0$, and this concludes the proof. \square

2.4 Appendix

2.4.1 Appendix A: scattering and emission absorption operator's expansion

In this part we prove the lemma 2.9, which deals with the control of the remainders $\bar{Q}_{s,i}^\varepsilon$, $i = 0, 1, 2$ of the expansion of the scattering operator at order i defined respectively in section (2.30), (2.29) and (2.28), and the lemma 2.10, which deals with the control of the remainder \bar{Q}_a^ε of the expansion of the emission absorption operator defined in (2.33). These proofs need the following lemmas. The first one (lemma 2.13) provides integrability results for Planck type function, while the second (lemma 2.14) concerns the regularity of the coefficients λ_i (study of Q_s^ε in section 2.3.1.0) and the remainder of their expansions.

Lemma 2.13. *For any given function $T \in L^\infty(X) \cap L^2(X)$ and for all $\alpha \in \mathbb{N}$, there exists a constant $C \geq 0$ such that the following estimate holds $\|\nu^\alpha B(\nu, T)\|_{L_{x,\nu}^2} \leq C \|T\|_{L_x^\infty}^{\alpha + \frac{5}{2}} \|T\|_{L_x^2}$.*

Proof. We remind that $B(\nu, T) = \nu^3 (e^{\nu/T} - 1)^{-1}$. The change of variable $y \mapsto \nu/T$ leads to

$$\|\nu^\alpha B(\nu, T)\|_{L_{x,\nu}^2}^2 = \int_{x,\nu} \frac{\nu^{6+2\alpha}}{(e^{\nu/T} - 1)^2} d\nu dx = \int_x T^{7+2\alpha} \int_y \frac{y^{6+2\alpha}}{(e^y - 1)^2} dy dx.$$

If we denote $f(y) = y^{6+2\alpha} (e^y - 1)^{-2}$, one has $f(0) = 0$ and f has an exponential decay as $y \rightarrow +\infty$. Thus, there exists a bounded constant C such that $\|f\|_{L_y^1} = C$. It yields the existence of a constant C such that $\|\nu^\alpha B(\nu, T)\|_{L_{x,\nu}^2} \leq C \|T\|_{L_x^\infty}^{\alpha + \frac{5}{2}} \|T\|_{L_x^2}$, which is the announced result. \square

We prove the following lemma, which shows that under assumptions on the coefficients $\vec{u}, \sigma_s, \sigma_a$ and ε , the coefficients λ_i (study of Q_s^ε in section 2.3.1.0) are uniformly bounded in L^∞ .

Lemma 2.14. *Under assumptions $(\overline{H1})$ - $(\overline{H4})$, there exists a constant $C \geq 0$, which does not depend on ε , such that the following estimates holds*

$$\forall i \in [1, 6], \|\lambda_i\|_{L_{t,x,\Omega,\Omega'}^\infty} \leq C, \|R_{\frac{\Lambda}{\Lambda'}, 2}\|_{L_{t,x,\Omega,\Omega'}^\infty} \leq C, \|R_{\frac{\Lambda'}{\Lambda^3}, 2}\|_{L_{t,x,\Omega,\Omega'}^\infty} \leq C, \|R_{\Lambda, 2}\|_{L_{t,x,\Omega,\Omega'}^\infty} \leq C.$$

Proof. Using the expression of the λ_i , $i \in [1, 6]$, the first point is obvious. One has $R_{\frac{\Lambda}{\Lambda'}, 2} = \frac{(\boldsymbol{\Omega}', \mathbf{u})^2 (\boldsymbol{\Omega}' - \boldsymbol{\Omega}, \mathbf{u})}{1 - \varepsilon \boldsymbol{\Omega}' \cdot \mathbf{u}}$, and thus $R_{\frac{\Lambda}{\Lambda'}, 2} \in L_{t,x,\Omega,\Omega'}^\infty$ using assumptions $(\overline{H1})$ - $(\overline{H2})$ on \mathbf{u} and ε . Furthermore, on has

$$R_{\frac{\Lambda}{\Lambda'}, 2} = \frac{1 - |\mathbf{u}|^2}{(1 - \varepsilon \boldsymbol{\Omega} \cdot \mathbf{u})^3} \left(3\boldsymbol{\Omega}' \cdot \mathbf{u} - 3\varepsilon(\boldsymbol{\Omega}, \mathbf{u})^2 + 2\varepsilon^2(\boldsymbol{\Omega}, \mathbf{u})^3 - 3(\boldsymbol{\Omega}, \mathbf{u})^2(3\boldsymbol{\Omega} \cdot \mathbf{u} - 3\varepsilon(\boldsymbol{\Omega}, \mathbf{u})^2 + \varepsilon^2(\boldsymbol{\Omega}, \mathbf{u})^3) \right) - |\mathbf{u}|^2 \left((3\boldsymbol{\Omega} - \boldsymbol{\Omega}', \mathbf{u}) + 3\varepsilon \boldsymbol{\Omega} \cdot \mathbf{u} (2\boldsymbol{\Omega} - \boldsymbol{\Omega}', \mathbf{u}) \right)$$

And thus $R_{\frac{\Lambda}{\Lambda'}, 2} \in L_{t,x,\Omega,\Omega'}^\infty$ using assumptions $(\overline{H1})$ - $(\overline{H2})$ on \mathbf{u} and ε . We remind that

$$R_{\Lambda, 2} = \frac{1}{\varepsilon^3 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} \int_{1 - \varepsilon^2 |\mathbf{u}|^2}^1 \frac{1 - \varepsilon^2 |\mathbf{u}|^2 - s}{4} \frac{1}{s \sqrt{s}} ds + \frac{|\mathbf{u}|^2}{2 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} \left(\varepsilon \frac{|\mathbf{u}|^2}{2} - \boldsymbol{\Omega} \cdot \mathbf{u} \right)$$

We have $|R_{\Lambda, 2}| \leq \frac{\varepsilon |\mathbf{u}|^4}{4(1 - \varepsilon^2 |\mathbf{u}|^2)^2} + \frac{|\mathbf{u}|^2}{2 \sqrt{1 - \varepsilon^2 |\mathbf{u}|^2}} \left(\varepsilon \frac{|\mathbf{u}|^2}{2} - \boldsymbol{\Omega} \cdot \mathbf{u} \right)$, and thus one can see that it is bounded in $L_{t,x,\Omega,\Omega'}^\infty$ uniformly with respect to ε . \square

We now prove the lemma 2.9. Since the arguments are similar, the proof is provided only for the remainder $\bar{Q}_{s,2}$ of the expansion of the scattering operator at the order 2.

Proof of lemma 2.9. Studying the expression of $\bar{Q}_{s,2}^\varepsilon(I)$ (2.28), one can see that the only complicated terms come from $R_{I,2}$ and $I(\nu', \boldsymbol{\Omega}') - I(\nu, \boldsymbol{\Omega}')$. For the first one, one has

$$R_{I,2} = \nu R_{\frac{\Lambda}{\Lambda'}, 2} \partial_\nu I(\nu, \boldsymbol{\Omega}') + \frac{\nu^2}{2\varepsilon^3} \left(\left(\frac{\Lambda}{\Lambda'} - 1 \right)^2 - (\varepsilon \lambda_5)^2 \right) \partial_\nu^2 I(\nu, \boldsymbol{\Omega}') + \frac{1}{\varepsilon^3} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \boldsymbol{\Omega}') ds$$

First, one can check that $\left(\frac{\Lambda}{\Lambda'} - 1 \right)^2 - (\varepsilon \lambda_5)^2 \leq C\varepsilon^3$. We have to estimate $\int_{\Omega'} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I_\varepsilon(s, \boldsymbol{\Omega}') ds$ in $L_{x,\nu,\Omega}^2$. Using a Hölder inequality, the definition $\nu' = (\Lambda/\Lambda')\nu$ and the estimate $\Lambda/\Lambda' - 1 \leq C\varepsilon$,

$$\left| \int_{x,\nu,\Omega} \left| \int_{\Omega'} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \boldsymbol{\Omega}') ds \right|^2 \right| \leq \frac{\varepsilon^5}{4} \int_{x,\Omega,\Omega'} \int_\nu^{\nu'} \nu^5 |\partial_\nu^3 I(s, \boldsymbol{\Omega}')|^2 ds d\nu$$

Using Fubini's theorem, we get

$$\left| \int_{x,\nu,\Omega} \left| \int_{\Omega'} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \boldsymbol{\Omega}') ds \right|^2 \right| \leq \frac{\varepsilon^5}{4} \int_{x,\Omega,\Omega'} \int_s \int_{(\Lambda'/\Lambda)s}^s \nu^5 |\partial_\nu^3 I(s, \boldsymbol{\Omega}')|^2 d\nu ds$$

This gives us

$$\left| \int_{x,\nu,\Omega} \left| \int_{\Omega'} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \boldsymbol{\Omega}') ds \right|^2 \right| \leq C\varepsilon^6 \|\nu^3 \partial_\nu^3 I\|_{L_{x,\nu,\Omega}^2}^2$$

Thus,

$$\left\| \int_{\Omega'} \int_\nu^{\nu'} \frac{(\nu' - s)^2}{2} \partial_\nu^3 I(s, \boldsymbol{\Omega}') ds \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C\varepsilon^3 \|\nu^3 \partial_\nu^3 I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)}$$

Using the same arguments, one can see that

$$\left\| \int_{\Omega'} \frac{I(\nu', \boldsymbol{\Omega}') - I(\nu, \boldsymbol{\Omega}')}{\varepsilon} \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \|\nu \partial_\nu I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)}$$

Using these estimates, the assumption $(\overline{H3})$ on the scattering coefficient σ_s and the lemma 2.14, one easily finds the result \square

We now turn to the proof of lemma 2.10, which deals with the control of the remainder \bar{Q}_a^ε of the expansion of the emission absorption operator.

Proof of the lemma 2.10. We recall here the remainder \bar{Q}_a^ε

$$\left\{ \begin{array}{l} \bar{Q}_a^\varepsilon(I, T) = \frac{1 - \Lambda^2}{\varepsilon \Lambda^2} \sigma_a(\nu_0) (B(\nu_0, T) - I_0) + \frac{\sigma_a(\nu_0) - \sigma_a(\nu)}{\varepsilon} (B(\nu_0, T) - I) \\ \quad + \sigma_a(\nu) \frac{B(\nu_0, T) - B(\nu, T)}{\varepsilon} + \sigma_a(\nu_0) \frac{I - I_0}{\varepsilon}. \end{array} \right.$$

We start with the first term. One can see that $|1 - \Lambda^2| \leq C\varepsilon$, where C is bounded in $L^\infty([0, T^f] \times X \times S^2)$ uniformly in ε . One thus has

$$\left\| \frac{1 - \Lambda^2}{\varepsilon \Lambda^2} \sigma_a(\nu_0) (B(\nu_0, T) - I_0) \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq \frac{C}{\Lambda_*} \|\sigma_a\|_{L_\nu^\infty} \left(\|B(\nu_0, T)\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} + \|I_0\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

Using the change of variable $\nu \rightarrow \Lambda\nu$ and the relation $I_0 = \Lambda^3 I$, one finds a new constant C such that

$$\left\| \frac{1 - \Lambda^2}{\varepsilon \Lambda^2} \sigma_a(\nu_0) (B(\nu_0, T) - I_0) \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)}^2 \leq C \left(\|B(\nu, T)\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} + \|I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

Finally, the lemma 2.13 on the integrability of the Planck function (lemma 2.13) yield, with another constant C ,

$$\left\| \frac{1 - \Lambda^2}{\varepsilon \Lambda^2} \sigma_a(\nu_0) (B(\nu_0, T) - I_0) \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \left(\|T\|_{L_t^\infty(L_x^2)} + \|I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right).$$

We now look at the second component of \bar{Q}_a^ε . One has, using the relation $\sigma_a(\nu_0) - \sigma_a(\nu) = \int_\nu^{\nu_0} \sigma_a'(s) ds$ and a Cauchy-Schwarz inequality,

$$\int_\nu \left((\sigma_a(\nu_0) - \sigma_a(\nu)) (B(\nu_0, T) - I) \right)^2 \leq \|\partial_\nu \sigma_a\|_{L_\nu^\infty}^2 \int_\nu (B(\nu_0, T) - I)^2 (\nu - \nu_0)^2.$$

By definition $\nu_0 - \nu = \nu(\Lambda^\varepsilon - 1)$ and thus there exists a constant C such that $|\nu_0 - \nu| \leq C\varepsilon\nu$. This estimate, together with a Cauchy-Schwarz inequality and the lemma 2.13 give us, with another constant C ,

$$\left\| \frac{\sigma_a(\nu_0) - \sigma_a(\nu)}{\varepsilon} (B(\nu_0, T) - I) \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \|\partial_\nu \sigma_a\|_{L_\nu^\infty} \left(\|T\|_{L_t^\infty(L_x^2)} + \|\nu I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \right)$$

We turn to the third term of \bar{Q}_a^ε . One has, using a Taylor expansion, a Cauchy-Schwarz inequality and the Fubini's theorem,

$$\int_\nu (B(\nu_0, T) - B(\nu, T))^2 d\nu \leq \int_\nu \int_\nu^{\nu_0} |\nu_0 - \nu| |\partial_\nu B(s, T)|^2 ds d\nu \leq C\varepsilon \int_s |\partial_\nu B(s, T)|^2 \int_{\frac{s}{\Lambda}}^s \nu d\nu ds$$

As previously, one has $s(1 - \frac{1}{\Lambda}) \leq C\varepsilon s$ and thus

$$\left\| \sigma_a(\nu) \frac{B(\nu_0, T) - B(\nu, T)}{\varepsilon} \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C \|\sigma_a\|_{L_\nu^\infty} \|\nu \partial_\nu B\|_{L_t^\infty(L_{x,\nu,\Omega}^2)}$$

One has $\nu\partial_\nu B(\nu, T) = 3B(\nu, T) - \frac{\nu}{T}B(\nu, T)(1 - e^{-\nu/T})^{-1}$. Using the same kind of arguments than for the lemma 2.13, one finds $\|\nu\partial_\nu B\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C\|T\|_{L_t^\infty(L_x^2)}$. For the last term, the relations $I_0 = \Lambda^3 I$ and $1 - \Lambda^3 = C\varepsilon$ yield

$$\left\| \sigma_a(\nu_0) \frac{I - I_0}{\varepsilon} \right\|_{L_t^\infty(L_{x,\nu,\Omega}^2)} \leq C\|I\|_{L_t^\infty(L_{x,\nu,\Omega}^2)},$$

which ends the proof. \square

2.4.2 Appendix B: regularity of the solution of the drift diffusion system

In this part we prove the lemma 2.7, which deals with the regularity of the solution of the drift diffusion system. The proof mainly uses the linearity of the equation on ρ and the maximum principle to overcome the difficulties coming from the nonlinearity of the equation on T . The proof consists to estimate each terms of the sum in (2.47). Since the proof of the estimates of these terms is rather similar, we only show the development for some of them.

Lemma 2.15. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists a constant C_0 such that the solution (ρ, \bar{T}) of the drift diffusion system (2.40) satisfies the following estimate*

$$\|\rho(t)\|_{L_{x,\nu}^2} + \|\bar{T}(t)\|_{L_x^2} \leq e^{C_0 t}, \quad 0 \leq t \leq T^f.$$

Proof. Multiplying the first equation of (2.40) by ρ , integrating on $X \times \mathbb{R}_\nu^+$, multiplying the second by T , integrating on X and adding, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{x,\nu} \rho^2 + \int_x \bar{T}^2 \right) + \int_{x,\nu} \frac{|\nabla \rho|^2}{6\sigma_s(x)} + \int_{x,\nu} \frac{\rho^2}{2} \nabla \cdot \mathbf{u} + \left(\Gamma + \frac{1}{2} \right) \int_x \bar{T}^2 \nabla \cdot \mathbf{u} \\ = - \int_{x,\nu} \frac{\nabla \cdot \mathbf{u}}{3} \frac{\rho^2}{2} + \int_{x,\nu} \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) (\rho - \bar{T}). \end{aligned}$$

We study the last term. One has

$$\int_{x,\nu} \sigma_a(\nu) (B(\nu, \bar{T}) - \rho) (\rho - \bar{T}) = \int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \rho + \int_{x,\nu} \sigma_a(\nu) \bar{T} \rho - \int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \bar{T} - \int_{x,\nu} \sigma_a(\nu) \rho^2,$$

and we study successively each of those terms. For the first one, the Cauchy-Schwarz inequality yields

$$\int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \rho \leq \frac{1}{2} \|\sigma_a\|_{L_\nu^\infty} (\|B(\nu, \bar{T})\|_{L_{x,\nu}^2}^2 + \|\rho\|_{L_{x,\nu}^2}^2).$$

The relation $\|B(\nu, \bar{T})\|_{L_{x,\nu}^2} \leq C\|\bar{T}\|_{L_x^2}$ (lemma 2.13) and the assumption $(\overline{H4})$ on the regularity of the emission absorption coefficient finally give a constant C such that

$$\int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \rho \leq C(\|\bar{T}\|_{L_x^2}^2 + \|\rho\|_{L_{x,\nu}^2}^2).$$

The second one is a little more complicated. The inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ yields

$$\int_{x,\nu} \sigma_a(\nu) \bar{T} \rho \leq \frac{1}{2} \|\bar{T}\|_{L_x^2}^2 + \frac{1}{2} \int_x \left(\int_\nu \sigma_a(\nu) \rho \right)^2,$$

and a Cauchy Schwarz inequality gives

$$\int_{x,\nu} \sigma_a(\nu) \bar{T} \rho \leq \frac{1}{2} \|\bar{T}\|_{L_x^2}^2 + \frac{1}{2} \|\sigma_a\|_{L_\nu^2}^2 \|\rho\|_{L_{x,\nu}^2}^2. \quad (2.58)$$

For the third one, one has $\int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \bar{T} \leq \|\sigma_a\|_{L_\nu^\infty} \int_x \bar{T} \int_\nu B(\nu, \bar{T})$. The lemma 2.13 and the assumption $(\overline{H4})$ on the regularity of the emission absorption coefficient finally give a constant C such that $\int_{x,\nu} \sigma_a(\nu) B(\nu, \bar{T}) \bar{T} \leq C \|\bar{T}\|_{L_x^2}^2$. This gives us, using the assumption $(\overline{H3})$ on the positivity of the scattering coefficient and the assumption $(\overline{H1})$ on the regularity of the velocity field \mathbf{u} a constant C such that

$$\frac{1}{2} \frac{d}{dt} \left(\int_{x,\nu} \rho^2 + \int_x \bar{T}^2 \right) \leq C \left(\int_{x,\nu} \rho^2 + \int_x \bar{T}^2 \right).$$

The Gronwall lemma and the assumption $(\overline{H5})$ on the initial conditions give the expected result. \square

The following lemma deals with the control of $\nu \rho$ in $L_{x,\nu}^2$.

Lemma 2.16. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists a constant C such that the solution ρ of the first equation of the drift diffusion system (2.40) satisfies the following estimate*

$$\|\nu \rho(t)\|_{L_{x,\nu}^2} \leq C, \quad 0 \leq t \leq T^f.$$

Proof. Multiplying the first equation of (2.40) by $\nu^2 \rho$, integrating it on $X \times \mathbb{R}_\nu^+$, we get, denoting $h = \nu \rho$,

$$\frac{1}{2} \frac{d}{dt} \int_{x,\nu} h^2 + \int_{x,\nu} \frac{|\nabla h|^2}{6\sigma_s(x)} = - \int_{x,\nu} h^2 \frac{\nabla \cdot \mathbf{u}}{2} + \int_{x,\nu} \sigma_a(\nu h B(\nu, \bar{T}) - h^2)$$

We look at the last term. One has,

$$\int_{x,\nu} \sigma_a(\nu h B(\nu, \bar{T}) - h^2) \leq \frac{1}{2} \|\sigma_a\|_{L_\nu^\infty} \left(\|h\|_{L_{x,\nu}^2}^2 + \|\nu B(\nu, \bar{T})\|_{L_{x,\nu}^2}^2 \right).$$

Using the lemma 2.13 and the maximum principle (lemma 2.6), one has $\|\nu B(\nu, \bar{T})\|_{L_{x,\nu}^2}^2 \leq C \|\bar{T}\|_{L_x^2}^2$. This gives us, with another constants C_1 and C_2 depending on the L_x^2 norm of \bar{T} (lemma 2.15), $\frac{d}{dt} \int_{x,\nu} h^2 \leq C_1 \int_{x,\nu} h^2 + C_2$. The Gronwall lemma and the assumption $\overline{H5}$ on the initial conditions give the result. \square

We now turn to the estimate of $\nu \partial_\nu \rho$. One has the

Lemma 2.17. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists a constant C such that the solution ρ of the first equation of the drift diffusion system (2.40) satisfies the following estimate*

$$\|\nu \partial_\nu \rho(t)\|_{L_{x,\nu}^2} \leq C, \quad 0 \leq t \leq T^f.$$

Proof. We differentiate the equation of the drift diffusion system (2.40) with respect to ν , we multiply it by $\nu^2 \partial_\nu \rho$ and we integrate it on $X \times \mathbb{R}_\nu^+$. Denoting $h = \nu \partial_\nu \rho$, we get

$$\frac{1}{2} \frac{d}{dt} \int_{x,\nu} h^2 + \int_{x,\nu} \frac{|\nabla h|^2}{6\sigma_s(x)} = -\frac{2}{3} \int_{x,\nu} h^2 \nabla \cdot \mathbf{u} + \int_{x,\nu} \partial_\nu \sigma_a \left(\nu B(\nu, \bar{T}) - \nu \rho \right) h + \int_{x,\nu} \sigma_a \left(\nu \partial_\nu B(\nu, \bar{T}) h - h^2 \right).$$

Using a Cauchy-Schwarz inequality, the assumptions $(\overline{H1})$ on the velocity field, $(\overline{H3})$ on the positivity of the scattering coefficient, $(\overline{H4})$ on the regularity of the emission absorption coefficient and the lemma 2.16 on the control of $\nu\rho$ in $L^2_{x,\nu}$, one gets a constant C such that

$$\frac{1}{2} \frac{d}{dt} \|h(t)\|_{L^2_{x,\nu}} \leq C \left(\|h(t)\|_{L^2_{x,\nu}}^2 + \|\nu B(\nu, \bar{T})\|_{L^2_{x,\nu}} + \|\nu \partial_\nu B(\nu, \bar{T})\|_{L^2_{x,\nu}} \right).$$

Once again, the lemma (2.13) yields $\|\nu B(\nu, \bar{T})\|_{L^2_{x,\nu}}^2 \leq C \|\bar{T}\|_{L^2_x}^2$. Furthermore, one has $\nu \partial_\nu B(\nu, \bar{T}) = 3B(\nu, \bar{T}) - \frac{\nu}{\bar{T}}(1 - e^{-\nu/\bar{T}})^{-1} B(\nu, \bar{T})$. The same arguments than for the proof of the lemma (2.13) yields $\|\nu \partial_\nu B(\nu, \bar{T})\|_{L^2_{x,\nu}}^2 \leq C \|\bar{T}\|_{L^2_x}^2$. The Gronwall lemma and the assumption $(\overline{H5})$ on the initial conditions finally give the result. \square

Using exactly the same arguments, one can prove under assumptions $(\overline{H1})$ - $(\overline{H6})$, that $\forall t \in [0, T^f]$, $\nu^2 \partial_\nu^2 \rho(t)$ and $\nu^3 \partial_\nu^3 \rho(t)$ belong to $L^2(\mathbb{R}_x^3 \times \mathbb{R}_\nu^+)$. We now turn to the control of the space derivatives of the solution of the diffusion system (2.40). One has the

Lemma 2.18. *Under assumptions $(\overline{H1})$ - $(\overline{H6})$, there exists a constant C such that the solution (ρ, \bar{T}) of the drift diffusion system (2.40) satisfies the following estimate: $\|\nabla \rho(t)\|_{L^2_{x,\nu}} + \|\nabla \bar{T}(t)\|_{L^2_x} \leq C, 0 \leq t \leq T^f$.*

proof. Differentiating the first equation of (2.40) with respect to x_j , multiplying the obtained equation by $\partial_{x_j} \rho$ and integrating on $X \times \mathbb{R}_\nu^+$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x,\nu} |\partial_{x_j} \rho|^2 + \int_{x,\nu} \frac{|\nabla \partial_{x_j} \rho|^2}{6\sigma_s(x)} + \frac{1}{3} \int_{x,\nu} \nabla(\partial_{x_j} \rho) \cdot \nabla \rho \partial_{x_j} \sigma_s^{-1} + \int_{x,\nu} \partial_{x_j} \rho \nabla \cdot (\rho \partial_{x_j} \mathbf{u}) = \\ - \frac{2}{3} \int_{x,\nu} |\partial_{x_j} \rho|^2 \nabla \cdot \mathbf{u} + \frac{1}{3} \int_{x,\nu} \partial_{x_j} \rho \nu \partial_\nu \rho \nabla \cdot (\partial_{x_j} \mathbf{u}) + \int_{x,\nu} \sigma_a \partial_{x_j} \rho \partial_{x_j} \left((B(\nu, \bar{T}) - \rho) \right) \end{aligned}$$

One has $\nabla(\partial_{x_j} \rho) \cdot \nabla \rho = \frac{1}{2} \partial_{x_j} |\nabla \rho|^2$. Furthermore,

$$\partial_{x_j} \rho \nabla \cdot (\rho \partial_{x_j} \mathbf{u}) = \frac{1}{2} \partial_{x_j} \rho^2 \nabla \cdot \partial_{x_j} \mathbf{u} + \partial_{x_j} \rho \sum_i \partial_{x_i} \rho \partial_{x_i} \mathbf{u}_i$$

This gives us

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x,\nu} |\partial_{x_j} \rho|^2 + \int_{x,\nu} \frac{|\nabla \partial_{x_j} \rho|^2}{6\sigma_s(x)} + \frac{1}{6} \int_{x,\nu} \partial_{x_j} |\nabla \rho|^2 \partial_{x_j} \sigma_s^{-1} + \frac{1}{2} \int_{x,\nu} \partial_{x_j} \rho^2 \nabla \cdot \partial_{x_j} \mathbf{u} + \sum_i \int_{x,\nu} \partial_{x_j} \rho \partial_{x_i} \rho \partial_{x_i} \mathbf{u}_i \\ = - \frac{2}{3} \int_{x,\nu} |\partial_{x_j} \rho|^2 \nabla \cdot \mathbf{u} + \frac{1}{3} \int_{x,\nu} \partial_{x_j} \rho \nu \partial_\nu \rho \nabla \cdot (\partial_{x_j} \mathbf{u}) + \int_{x,\nu} \sigma_a \rho \partial_{x_j} \partial_{x_j} \left((B(\nu, \bar{T}) - \rho) \right) \end{aligned}$$

Making the sum of this equation for $j = 1, 2, 3$, using integration by parts, Cauchy Schwarz inequalities, the assumptions $(\overline{H1})$, $(\overline{H3})$ and $(\overline{H4})$ on the regularity of the coefficients \mathbf{u}, σ_s and σ_a and the lemmas 2.15 and 2.16 on the integrability of ρ and $\nu\rho$ in $L^2_{x,\nu}$, we get a constant C such that

$$\frac{1}{2} \frac{d}{dt} \int_{x,\nu} |\nabla \rho|^2 \leq C \left(1 + \int_{x,\nu} |\nabla \rho|^2 \right) + \sum_j \int_{x,\nu} \sigma_a \partial_{x_j} \rho \partial_{x_j} \left((B(\nu, \bar{T}) - \rho) \right)$$

We need to control the last term. One has

$$\sigma_a \partial_{x_j} \rho \partial_{x_j} \left((B(\nu, \bar{T}) - \rho) \right) = \sigma_a \left(\partial_{x_j} \rho \partial_{x_j} B(\nu, \bar{T}) - |\partial_{x_j} \rho|^2 \right)$$

Using the definition of $B(\nu, \bar{T})$, one has $\partial_{x_j} B(\nu, \bar{T}) = \partial_{x_j} \bar{T} \frac{\nu}{\bar{T}^2} \nu^4 e^{\nu/\bar{T}} (e^{\nu/\bar{T}} - 1)^{-2}$. Thus, one has

$$\sum_j \int_{x, \nu} \sigma_a \partial_{x_j} \rho \partial_{x_j} B(\nu, \bar{T}) \leq \frac{1}{2} \|\sigma_a\|_{L^\infty_\nu} \left(\int_{x, \nu} |\nabla \rho|^2 + \int_x \frac{|\nabla \bar{T}|^2}{\bar{T}^4} \int_\nu \frac{\nu^8 e^{2\nu/\bar{T}}}{(e^{\nu/\bar{T}} - 1)^4} \right).$$

The change of variable $y = \frac{\nu}{\bar{T}}$ leads to

$$\int_x \frac{|\nabla \bar{T}|^2}{\bar{T}^4} \int_\nu \frac{\nu^8 e^{2\nu/\bar{T}}}{(e^{\nu/\bar{T}} - 1)^4} = \int_x |\nabla \bar{T}|^2 \bar{T}^5 \int_{\mathbb{R}^+} \frac{y^8 e^{2y}}{(e^y - 1)^4}.$$

The maximum principle (lemma 2.6) together with the same idea than in the proof of the lemma 2.13 shows that

$$\int_x |\nabla \bar{T}|^2 \bar{T}^5 \int_y \frac{y^8 e^{2y}}{(e^y - 1)^4} \leq C \|\nabla \bar{T}\|_{L^2_x}^2.$$

Finally, one finds another constant C such that

$$\sum_j \int_{x, \nu} \partial_{x_j} \rho \partial_{x_j} \left(\sigma_a (B(\nu, \bar{T}) - \rho) \right) \leq C \left(\int_x |\nabla \bar{T}|^2 + \int_{x, \nu} |\nabla \rho|^2 \right).$$

This gives us, with another constants C_1 and C_2 ,

$$\frac{1}{2} \frac{d}{dt} \int_{x, \nu} |\nabla \rho|^2 \leq C_1 \left(\int_x |\nabla \bar{T}|^2 + \int_{x, \nu} |\nabla \rho|^2 \right) + C_2. \quad (2.59)$$

We turn to the equation on \bar{T} in (2.40). Differentiating this equation with respect to x_j , multiplying it by $\partial_{x_j} \bar{T}$, integrating on X and taking the sum for $j = 1, 2, 3$, we get

$$\frac{1}{2} \frac{d}{dt} \int_x |\nabla \bar{T}|^2 + \left(\frac{1}{2} + \Gamma \right) \int_x |\nabla \bar{T}|^2 + \sum_j \int_x (\partial_{x_j} T) \bar{T} \nabla \cdot (\partial_{x_j} \mathbf{u}) = - \sum_j \int_{x, \nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} \left((B(\nu, \bar{T}) - \rho) \right).$$

One has $\sum_j \int_{x, \nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} B = \sum_j \int_{x, \nu} \sigma_a |\partial_{x_j} \bar{T}|^2 \partial_T B$. Once again, the same arguments than for the lemma 2.13 on the integrability of the Planck function show that $\partial_T B(\nu, T) \in L^1_\nu$. One finds a constant C such that $\sum_j \int_{x, \nu} \partial_{x_j} \bar{T} \partial_{x_j} (\sigma_a B) \leq C \|\sigma_a\|_{L^\infty_\nu} \|\nabla \bar{T}\|_{L^2_x}^2$. Moreover, the Cauchy-Schwarz inequality together with the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ yield

$$\sum_j \int_{x, \nu} \sigma_a \partial_{x_j} \bar{T} \partial_{x_j} \rho \leq \frac{1}{2} \|\sigma_a\|_{L^2_\nu}^2 \|\nabla \bar{T}\|_{L^2_x}^2 + \frac{1}{4} \|\nabla \rho\|_{L^2_{x, \nu}}^2. \quad (2.60)$$

Finally, an integration by parts yields $\sum_j \int_x (\partial_{x_j} \bar{T}) \bar{T} \nabla \cdot (\partial_{x_j} \mathbf{u}) \leq \frac{1}{2} \|\mathbf{u}\|_{W_x^{2, \infty}} \|\nabla \bar{T}\|_{L^2_x}^2$. This gives us another constants C_1 and C_2 such that

$$\frac{1}{2} \frac{d}{dt} \int_x |\nabla \bar{T}|^2 \leq C_1 \left(\int_x |\nabla \bar{T}|^2 + \int_{x, \nu} |\nabla \rho|^2 \right) + C_2,$$

This result, together with the inequality (2.59), yields another constants C_1 and C_2 such that

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla \bar{T}\|_{L^2_x}^2 + \|\nabla \rho\|_{L^2_{x, \nu}}^2 \right) \leq C_1 \left(\|\nabla \bar{T}\|_{L^2_x}^2 + \|\nabla \rho\|_{L^2_{x, \nu}}^2 \right) + C_2.$$

The Gronwall lemma and the assumption $(\overline{\text{H5}})$ on the initial conditions give the result. \square

We do not give the proof of the remaining terms in (2.47) since it uses the same arguments.

Chapter 3

Well-balanced schemes for the non-equilibrium diffusion regime

This chapter is taken from two articles [LBD14, LER14].

3.1 Introduction

Our aim in this chapter is to design costless numerical schemes for the non-equilibrium diffusion regime (2.27) obtained in chapter 1 from the relativistic transfer equation. For simplicity, we only study the homogeneous case. The equation on the density distribution function of the photons writes

$$\partial_t \rho = \frac{\kappa}{3} \nu \partial_\nu \rho + \sigma(\nu)(B(\nu, T) - \rho), \quad (3.1)$$

Here $\rho = \rho(t, \nu)$ represents the density of photons and we denoted κ the divergence of the fluid velocity for simplicity. The coefficient σ is the absorption coefficient. It is known to be very irregular with respect to the frequency. The function B is the Planck's function, given by $B(\nu) = \nu^3 (e^{\nu/T} - 1)^{-1}$.

On the one hand, the model problem (3.1) is also representative of asymptotic preserving issues, due to the parameter κ . The limit system is

$$\partial_t \rho = \sigma(\nu)(B(\nu) - \rho) \quad (3.2)$$

Since κ represents the divergence of the fluid velocity, this parameter may vanish. It is important that the scheme be consistent with this limit equation as $\kappa \rightarrow 0$. On the other hand, the emission absorption coefficient is highly irregular, and plays an important role from the physical point of view. It is thus necessary that the scheme be accurate even for this kind of opacities. In a first approach a very fine mesh could be used, but it is not possible in general in photon transport. Indeed the distribution function of the photons is often rather peaked in frequency. In this context and due to the frequency shift, the classical upwind scheme is not accurate enough.

We propose in this chapter a new well-balanced scheme for the transport equation (3.1), which handle the consistency problem of the classical well-balanced scheme (3.14) as the velocity speed tends to 0. We prove that this scheme is uniformly convergent (with respect to the speed velocity κ , theorem 3.6). The long time behavior of the spectrally well-balanced scheme is study for several irregular opacities, including highly peaked emission absorption coefficient and the Cramer's

opacity. In particular we prove that the well-balanced property of the scheme is preserved, and we study numerically the consistency of the scheme as the velocity speed κ tends to 0 for these opacities. The case $\kappa < 0$ is study, and the numerical results agree with the theoretical result (lemma 3.6). Finally, the advantage of the SWB scheme with respect to the upwind scheme is shown in two test cases for different values of the coefficients κ and σ .

Numerical methods for the coupling between hydrodynamics and the transfer equations have been extensively studied (see for example [BD06,LMH99,TUR12]) and are still an active field of research. Indeed, the different time scales and the nonlinearity involved in this kind of coupled system make the design of costless numerical schemes a complicated issue. To treat the issue of the irregular and peaked behavior of the opacities in which we are interested, a method has been proposed in [RIP14]. Instead of discretizing the opacity σ on frequency bands, i.e. considering the set of $\{\sigma_i\}$, σ_i being constant on the frequency band $[\nu_{i-\frac{1}{2}}, \nu_{i+\frac{1}{2}}]$, the author considered a discretization of the value of the opacity, i.e. the sets $\{\alpha_{i\pm\frac{1}{2}}\}$ such that $\sigma(\nu) \in [\alpha_{i-\frac{1}{2}}, \alpha_{i+\frac{1}{2}}]$. It can be understood as the replacement of a Riemann integral by a Lebesgue integral. In this work we do not consider this treatment.

Since radiative phenomena are very fast compared to hydrodynamic ones, well-balanced schemes are a possible solution. As we explain in section 3, system (3.1) has stationary solutions, and our aim is to design numerical schemes which preserve these solutions. This could be interesting for kinetic equations for which the frequency discretization is known to be very costly.

A first approach consists to use Greenberg-Leroux [GL96] type schemes. These schemes are well-balanced, but we show in this chapter that they are not consistent in the limit regime $\kappa \rightarrow 0$. We propose a new scheme, called spectrally well-balanced (SWB) for which we prove a uniform (according to the parameter κ) convergence result. The proof of this results needs regularity assumptions on the absorption coefficient ($\sigma \in W^{2,\infty}(\mathbb{R}^+)$), but the numerical tests shows that these assumptions are purely technical, in the sense that $\sigma \in L^\infty(\mathbb{R}^+)$ is the only necessary assumption for the scheme to be stable.

The behavior of the spectrally well-balanced scheme is studied for several irregular opacities, and its numerical solution is compared with the Greenberg Leroux and upwind schemes. In particular numerical tests are performed with a coefficient σ constructed as the sum of a regular part and peaked functions, which seems to be realistic, and with the so called Cramer's opacity. The well-balanced property and the consistency as $\kappa \rightarrow 0^\pm$ of the spectrally well-balanced scheme are shown to be preserved for such opacities.

This chapter is organized as follows. In the first section, the transport equation for photons (3.1) is coupled to an equation describing the evolution of the fluid temperature. The aim of this section is to compute analytical solution for this coupled system. These analytical solutions could be used to validate some numerical schemes. In the remaining sections we focus only on the photon transport equation. The section 3 focuses on the transport equation (3.1). In particular we study the influence of the parameters κ and σ on the evolution of the L^1 norm of the solution. The beginning of the section 4 deals with the study of the stationary equation associated to (3.1). This is indeed important to construct well-balanced schemes. The section 4 is then divided in three steps. In a first one is constructed the Greenberg-Leroux type scheme. After explaining its derivation, we show its well-balanced property and explain why this scheme

is not consistent as the wave velocity κ tends to 0. The second step is devoted to the spectrally well-balanced scheme. We explain the derivation and show the L^2 stability and uniform (with respect to the parameter κ) convergence of this scheme. Finally, the third step deals with the upwind scheme, which will be used as a comparison point in the numerical tests. The last part (section 5) is devoted to the numerical comparison of the schemes. Several numerical tests are performed to highlight the uniform convergence of the spectrally well-balanced scheme, its well-balanced property and its advantages with respect to the up-wind scheme, all this with several more or less regular expressions of the opacity σ .

3.2 Analytical solutions

The aim of this section is to study the analytical solutions of a system composed of equation (3.1) for the photons and an equation describing the evolution of the internal energy of the fluid, in such a way that the total energy is preserved. The system writes

$$\begin{cases} \partial_t \rho = \frac{\kappa}{3} \nu \partial_\nu \rho + \sigma(\nu)(B(\nu, T) - \rho), \\ \frac{d}{dt} \mathcal{E}(t) = \int_{\mathbb{R}_+^\dagger} \left(\frac{\kappa}{3} + \sigma(\nu) \right) \rho d\nu - \int_{\mathbb{R}_+^\dagger} \sigma(\nu) B(\nu, T) d\nu. \end{cases}$$

In order to compute analytical solutions of the coupled system (3.1), we make two (unphysical) assumptions, which yields a linear equation on the macroscopic temperature of the fluid. These assumptions are first that the absorption coefficient σ is constant, and secondly that $\mathcal{E} = T^4$. Since $\int_{\mathbb{R}_+^\dagger} B(\nu, T) d\nu = aT^4$, where $a = \int_{\mathbb{R}_+^\dagger} \nu^3 (e^\nu - 1)^{-1} d\nu$, the fluid equation is now described by a linear equation on $\Theta = T^4$

$$\begin{cases} \partial_t \rho = \frac{\kappa}{3} \nu \partial_\nu \rho + \sigma(B(\nu, T) - \rho), \\ \frac{d}{dt} \Theta = \left(\frac{\kappa}{3} + \sigma \right) \int_{\mathbb{R}_+^\dagger} \rho d\nu - a\sigma\Theta, \end{cases} \quad (3.3)$$

where the notation T has been kept in the equation on the density distribution function of the photons for ease of notations. This part is devoted to the proof of the following lemma, which is concerned by the analytical solution of system (3.3).

Lemma 3.1 (Analytical solution). *Let us denote (ρ^{in}, Θ^{in}) the initial conditions associated to the system (3.3). The analytical solution is given by*

$$\begin{cases} \Theta(t) = \Theta^{in} \left(\frac{\sigma + \frac{\kappa}{3}}{(a+1)\sigma + \frac{\kappa}{3}} + \frac{a\sigma}{\sigma(a+1) + \frac{\kappa}{3}} e^{-((a+1)\sigma + \frac{\kappa}{3})t} \right) \\ \quad + \|\rho^{in}\|_{L^1} \left(\sigma + \frac{\kappa}{3} \right) \frac{1 - e^{-((a+1)\sigma + \frac{\kappa}{3})t}}{(a+1)\sigma + \frac{\kappa}{3}} \\ \rho(t, \nu) = \rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\sigma t} + \int_0^t \sigma B(\nu e^{\frac{\kappa}{3}(t-s)}, \Theta(s)^{\frac{1}{4}}) e^{-\frac{3}{\kappa}\sigma(t-s)} ds, \end{cases} \quad (3.4)$$

where we recall that $a = \int_{\mathbb{R}_+^\dagger} \nu^3 (e^\nu - 1)^{-1} d\nu$.

Proof. Instead of proving this result by inserted this expression in the system (3.3), it seems more interesting to show the method that leads to this solution. We start with the density distribution function of the photons ρ . We use the method of characteristic. The characteristic field is defined by:

$$\begin{cases} X'(t) = \frac{\kappa}{3}X(t), \\ X(0) = \nu \end{cases}$$

This gives us $X(t, \nu) = \nu e^{\frac{\kappa}{3}t}$. We make the change of variables $\nu \mapsto X(t, \nu)$ in the first equation of (3.3). We thus find:

$$\partial_t \left(\rho(t, X(t, \nu)) \right) = -\sigma \rho(t, X(t, \nu)) + \sigma B(X(t, \nu), T).$$

Solving this ODE, we find

$$\rho(t, X(t, \nu)) = \rho^{in}(\nu) e^{-\sigma t} + \int_0^t \sigma B(X(s, \nu), T(s)) e^{-\sigma(t-s)} ds.$$

Making the change of variables $y = \nu e^{-\frac{\kappa}{3}t}$ and then put $y = \nu$, we find the expression of the density distribution function of the photons for any temperature T

$$\rho(t, \nu) = \rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\sigma t} + \int_0^t \sigma_a B(\nu e^{\frac{\kappa}{3}(t-s)}, T(s)) e^{-\sigma(t-s)} ds,$$

which is the announced expression for ρ . Let us turn to the equation on the temperature in (3.4). Inserted the expression of ρ juste obtained, one has to solve the following equation

$$\frac{d}{dt} \Theta(t) = \left(\frac{\kappa}{3} + \sigma \right) \int_{\mathbb{R}_\nu^+} \left(\rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\sigma t} + \int_0^t \sigma B(\nu e^{\frac{\kappa}{3}(t-s)}, T(s)) e^{-\sigma(t-s)} ds \right) d\nu - a\sigma \Theta. \quad (3.5)$$

Let us study each term under the integral. First, one easily finds by a change of variables $\nu \rightarrow \nu e^{\frac{\kappa}{3}t}$

$$\int_{\mathbb{R}_\nu^+} \rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\sigma t} d\nu = e^{\sigma t} e^{\frac{\kappa}{3}t} \|\rho^{in}\|_{L^1}.$$

To treat the second one, the idea is to integrate first in frequency and to use the integrability of the Planck function, that is

$$\int_{\mathbb{R}_\nu^+} \int_0^t B(\nu e^{\frac{\kappa}{3}(t-s)}, T(s)) e^{-\sigma(t-s)} ds d\nu = \int_0^t e^{-\sigma(t-s)} \int_{\mathbb{R}_\nu^+} B(\nu e^{\frac{\kappa}{3}(t-s)}, T(s)) d\nu ds.$$

The change of variables $\nu \rightarrow \nu e^{\frac{\kappa}{3}(t-s)}$ yields

$$\int_{\mathbb{R}_\nu^+} B(\nu e^{\frac{\kappa}{3}(t-s)}, T(s)) d\nu = aT^4(s) e^{-\frac{\kappa}{3}(t-s)} = a\Theta(s) e^{-\frac{\kappa}{3}(t-s)}.$$

Finally, equation (3.5) becomes

$$\frac{d}{dt} \Theta(t) = \left(\frac{\kappa}{3} + \sigma \right) \left(\|\rho^{in}\|_{L^1} e^{-(\sigma + \frac{\kappa}{3})t} + a\sigma \int_0^t \Theta(s) e^{-(\sigma + \frac{\kappa}{3})(t-s)} ds \right) - a\sigma \Theta. \quad (3.6)$$

This is a linear integro-differential equation. One of the method to solve this kind of equation is the Laplace transform \mathcal{L} . It is define, for any given function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, by

$$\mathcal{L}(f)(y) = \int_{\mathbb{R}^+} e^{-yt} f(t) dt.$$

Let us construct the equation satisfied by $\mathcal{L}(\Theta)$. Multiplying equation (3.6) by e^{-yt} and integrating over \mathbb{R}^+ yields

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-yt} \partial_t \Theta(t) dt &= -\Theta^{in} + y\mathcal{L}(\Theta) \\ &= \left(\frac{\kappa}{3} + \sigma\right) \left(\|\rho^{in}\|_{L^1} \int_{\mathbb{R}^+} e^{-(y+\sigma+\frac{\kappa}{3})t} dt + a\sigma \int_{\mathbb{R}^+} e^{-yt} \int_0^t \Theta(s) e^{-(\sigma+\frac{\kappa}{3})(t-s)} ds dt \right) \\ &\quad - a\sigma\mathcal{L}(\Theta). \end{aligned}$$

We now study each integrals. For the first one, a direct computation yields

$$\|\rho^{in}\|_{L^1} \left(\frac{\kappa}{3} + \sigma\right) \int_{\mathbb{R}^+} e^{-(y+\sigma+\frac{\kappa}{3})t} dt = \|\rho^{in}\|_{L^1} \frac{\frac{\kappa}{3} + \sigma}{y + \frac{\kappa}{3} + \sigma}.$$

We use Fubini's theorem to invert the two integrals in the second integral.

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-(y+\sigma+\frac{\kappa}{3})t} \int_0^t \Theta(s) e^{(\sigma+\frac{\kappa}{3})s} ds dt &= \int_{\mathbb{R}^+} \Theta(s) e^{(\sigma+\frac{\kappa}{3})s} \int_s^{+\infty} e^{-(y+\sigma+\frac{\kappa}{3})t} dt ds \\ &= \frac{\mathcal{L}(\Theta)}{y + \frac{\kappa}{3} + \sigma}. \end{aligned}$$

One thus obtain the relation

$$-\Theta^{in} + y\mathcal{L}(\Theta) = \|\rho^{in}\|_{L^1} \frac{\frac{\kappa}{3} + \sigma}{y + \frac{\kappa}{3} + \sigma} + a\sigma \left(\frac{\kappa}{3} + \sigma\right) \frac{\mathcal{L}(\Theta)}{y + \frac{\kappa}{3} + \sigma} - a\sigma\mathcal{L}(\Theta),$$

which yields

$$\mathcal{L}(\Theta) = \frac{\Theta^{in} \left(y + \frac{\kappa}{3} + \sigma\right) + \|\rho^{in}\|_{L^1} \left(\frac{\kappa}{3} + \sigma\right)}{y \left(y + \frac{\kappa}{3} + \sigma(1+a)\right)}$$

In order to be able to find the inverse Laplace transform of this expression, it is necessary to write this term as a sum of terms for which this transformation is known. The relation $\mathcal{L}^{-1}\left(\frac{y+c}{(y+a)(y+b)}\right) = de^{-at} + (1-d)e^{-bt}$, $d = \frac{c-a}{b-a}$ $t \geq 0$ yields the announced result, and this concludes the proof. \square

3.3 Transport equation

In this section we focus on the frequency transport equation for photons. To this end we consider the following problem

$$\begin{cases} \partial_t \rho = \frac{\kappa}{3} \nu \partial_\nu \rho + \sigma(\nu)(B(\nu, T) - \rho), \\ \rho(t=0, \nu) = \rho^{in}(\nu), \end{cases} \quad (3.7)$$

where the temperature $T > 0$ is assumed to be given. This is a very simple transport equation which contains nevertheless several physical processes. Indeed the coefficient κ models the divergence of the fluid velocity, and σ_a represents the emission absorption rate. We aim here to understand the influence of these two parameters on the transport of photons. For equation (3.7), the analytical solutions given by the lemma (3.1) reduce to

$$\rho(t, \nu) = \rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}t}} \sigma(\tau) \tau^{-1} d\tau} + \int_0^t \sigma(\nu e^{\frac{\kappa}{3}s}) B(\nu e^{\frac{\kappa}{3}s}) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}s}} \sigma(\tau) \tau^{-1} d\tau} ds. \quad (3.8)$$

From this expression it is clear that for all $(t, \nu) \in [0, T^f] \times \mathbb{R}^+$ and for all $\kappa \in \mathbb{R}$, $\rho(t, \nu)$ is non negative. We assume for simplicity a constant opacity. Making the change of variables $y = \nu e^{\frac{\kappa}{3}s}$ in the integral, one gets

$$\rho(t, \nu) = \rho^{in}(\nu e^{\frac{\kappa}{3}t})e^{-\sigma t} + \frac{3\sigma}{\kappa} \int_{\nu}^{\nu e^{\frac{\kappa}{3}t}} \frac{B(y)}{y} e^{-\frac{3\sigma}{\kappa} \log(y/\nu)} dy. \quad (3.9)$$

We aim to study the time evolution of the L^1 norm of ρ , the L^1 norm being chosen for simplicity. One has the following result

Proposition 3.2. *The L^1 norm of the solution of the transport equation (3.7) satisfies*

- if $\frac{\kappa}{3} + \sigma > 0$, then $\lim_{t \rightarrow \infty} \|\rho(t)\|_{L^1} = \frac{\sigma}{\frac{\kappa}{3} + \sigma} \|B\|_{L^1}$,
- if $\frac{\kappa}{3} + \sigma < 0$, then the L^1 norm of ρ diverges exponentially with time,
- if $\frac{\kappa}{3} + \sigma = 0$, then the L^1 norm of ρ diverges linearly with time.

Proof. Since ρ is non negative, one gets from the expression (3.9)

$$\|\rho(t)\|_{L^1} = e^{-\sigma t} \int_{\mathbb{R}^+} \rho^{in}(\nu e^{\frac{\kappa}{3}t}) d\nu + \frac{3\sigma}{\kappa} \int_{\mathbb{R}^+} \int_{\nu}^{\nu e^{\frac{\kappa}{3}t}} \frac{B(y)}{y} e^{-\frac{3\sigma}{\kappa} \log(y/\nu)} dy d\nu,$$

which we write $\|\rho(t)\|_{L^1} = I_1(t) + I_2(t)$. Firstly, a change of variables $y = \nu e^{\frac{\kappa}{3}t}$ in the first integral yields

$$I_1(t) = e^{-(\sigma + \frac{\kappa}{3})t} \|\rho^{in}\|_{L^1}.$$

Secondly, we get by applying the Fubini's theorem on the second integral

$$I_2(t) = \frac{3\sigma}{\kappa} \int_{\mathbb{R}^+} \frac{B(y)}{y} e^{-\frac{3\sigma}{\kappa} \log(y)} \int_{ye^{-\frac{\kappa}{3}t}}^y e^{\frac{3\sigma}{\kappa} \log(\nu)} d\nu dy.$$

Integrating the integral with respect to ν and using several algebraic manipulations yield

$$I_2(t) = \sigma \frac{1 - e^{-(\frac{\kappa}{3} + \sigma)t}}{\frac{\kappa}{3} + \sigma} \int_{\mathbb{R}^+} B(y) dy.$$

We finally get

$$\|\rho(t)\|_{L^1} = e^{-(\sigma + \frac{\kappa}{3})t} \|\rho^{in}\|_{L^1} + \sigma \frac{1 - e^{-(\frac{\kappa}{3} + \sigma)t}}{\frac{\kappa}{3} + \sigma} \|B\|_{L^1}.$$

From this expression the expected result is easily obtained in the case $\frac{\kappa}{3} + \sigma \neq 0$. In the case $\frac{\kappa}{3} + \sigma = 0$, the result is obtained by using a Taylor expansion of e^{-X} near zero, and this concludes the proof. \square

This shows that the transport equation (3.7) might be more complicated to solve numerically in the case $\frac{\kappa}{3} + \sigma \leq 0$. For this reason and to simplify the problem we consider in this chapter mainly the case $\kappa > 0$.

3.4 Numerical schemes

For technical reasons, we restrict the frequency domain to $\mathbb{D} = [\varepsilon, \nu^*]$, for given $0 < \varepsilon < \nu^* < +\infty$. The parameter ε is introduced due to the fact that in this chapter we will consider opacities with singularity as $\nu \rightarrow 0$. Since we want to design well-balanced schemes, we are interested in the stationary solutions of (3.1). We thus solve the following Cauchy problem

$$\begin{cases} \frac{\kappa}{3} \nu \partial_\nu \rho + \sigma(\nu)(B(\nu) - \rho) = 0, \\ \rho(\hat{\nu}) = \hat{\rho}. \end{cases} \quad (3.10)$$

If the parameter κ is positive, this is a transport of the photons toward the frequency $\nu = \varepsilon$. On the other hand if the parameter κ is negative this is a transport of the photons toward the frequency $\nu = \nu^*$. The frequency $\hat{\nu}$ is thus defined by $\hat{\nu} = \varepsilon$ if $\kappa > 0$ and $\hat{\nu} = \nu^*$ if $\kappa < 0$. This is a simple O.D.E., and one can find the analytical solution, given by

$$\rho(\nu; \rho^*, \nu^*) = \rho^* e^{-\frac{3}{\kappa} \int_\nu^{\nu^*} \frac{\sigma(s)}{s} ds} + \frac{3}{\kappa} \int_\nu^{\nu^*} \frac{\sigma(s)B(s)}{s} e^{-\frac{3}{\kappa} \int_\nu^s \frac{\sigma(\tau)}{\tau} d\tau} ds. \quad (3.11)$$

Since one of the regimes we are interested in is the behavior of the scheme as $\kappa = 0$, we study the limit of this solution in this regime. Noting that $3\sigma(s)(s\kappa)^{-1} e^{-\frac{3}{\kappa} \int_\nu^s \frac{\sigma(\tau)}{\tau} d\tau} = -\frac{d}{ds} (e^{-\frac{3}{\kappa} \int_\nu^s \frac{\sigma(\tau)}{\tau} d\tau})$, one finds

$$\lim_{\kappa \rightarrow 0} \rho(\nu; \rho^*, \nu^*) = B(\nu). \quad (3.12)$$

For $1 \leq j \leq N$, we consider an irregular mesh defined by $(N + 1)$ points $0 < \varepsilon = \nu_{\frac{1}{2}} < \dots < \nu_{N+\frac{1}{2}} = \nu^*$. We define ν_j as the middle of the j -th frequency band, i.e. $\nu_j = (\nu_{j-\frac{1}{2}} + \nu_{j+\frac{1}{2}})/2$ and we denote $\Delta\nu_j$ its length. We also define the dual $(j + \frac{1}{2})$ -th frequency band as the cell $[\nu_j, \nu_{j+1}]$, which length is denoted $\Delta\nu_{j+\frac{1}{2}}$. We denote $h = \max_j \Delta\nu_j$. We assume that there exists a constant C such that $\forall j \in \{1, \dots, N\}$, $0 < Ch \leq \Delta\nu_j$.

In the following parts we adopt the following notations. We define $\rho_{ex}(t) = (\rho(t, \nu_j))_{1 \leq j \leq N}$ as the projection of the analytical solution of (3.1) over the mesh. For any vector $V \in \mathbb{R}^N$, we define the following discrete L^2 norm

$$\|V\|_{L_d^2} = \sqrt{\sum_{j=1}^N \Delta\nu_{j+\frac{1}{2}} V_j^2}. \quad (3.13)$$

In the next section we make some assumptions:

- (H1) The initial data satisfies $\rho^{in} \in H^2(\mathbb{D})$.
- (H2) The emission absorption coefficient satisfies $\sigma_a \in W^{2,\infty}(\mathbb{D})$. Moreover, there exists a constant $\sigma_* > 0$ such that $\forall \nu \in \mathbb{D}, \sigma(\nu) \geq \sigma_*$.

The assumption (H2) may be not satisfied by some physical opacities (Cramer's opacity, part 3.5.3), but for technical reasons is necessary to prove some theoretical results for the well-balanced schemes studied in the next parts.

3.4.1 A first class of well-balanced schemes

As presented in the introduction, we study a class of well-balanced schemes in the spirit of what was introduced by Greenberg-Leroux [GL96] (see [GOS13] for a recent state of the art on the topic). It consists to localized the source term at the interfaces and to use a Godunov method to construct a scheme for the resulting equation. For equation (3.1), it yields the following scheme, denoted as WB1 in the following

$$\begin{cases} \frac{d}{dt}\rho_j = \frac{\kappa}{3}\nu_j \frac{\rho(\nu_j; \rho_{j+1}; \nu_{j+1}) - \rho_j}{\Delta\nu_{j+\frac{1}{2}}}, & 1 \leq j \leq N-1, & \kappa > 0, \\ \frac{d}{dt}\rho_j = \frac{\kappa}{3}\nu_j \frac{\rho_j - \rho(\nu_j; \rho_{j-1}; \nu_{j-1})}{\Delta\nu_{j-\frac{1}{2}}}, & 2 \leq j \leq N, & \kappa < 0, \end{cases} \quad (3.14)$$

with the natural boundary conditions $\rho_N(t) = \rho(t, \nu_N)$ in the case $\kappa > 0$ and $\rho_1(t) = \rho(t, \nu_1)$ in the case $\kappa < 0$, where $\rho(t, \nu)$, defined in (3.31) is the analytical solution of equation (3.1). Considering a classical explicit Euler discretization of the time derivative, the CFL is given by

$$\Delta t \leq \max_j \frac{3\Delta\nu_j}{|\kappa|\nu_j}. \quad (3.15)$$

It is interesting to note that the CFL of this scheme does not depend on σ , but depends on the wave velocity κ . We now prove several properties for this scheme, and we start with the well-balanced property. In the following lemmas 3.3 and 3.4 we consider the case $\kappa > 0$.

Lemma 3.3 (Well-balanced property). *The WB1 scheme defined in (3.14) is well-balanced, i.e. preserves the stationary solutions.*

Proof. We consider a stationary state of equation (3.1). By definition (3.11), this solution may be written on the form $\rho(\nu) = \rho(\nu; \rho^*, \nu^*)$, and thus $\rho_{j+1} = \rho(\nu_{j+1}; \rho^*, \nu^*)$. The semigroup property yields

$$\rho(\nu_j; \rho_{j+1}; \nu_{j+1}) = \rho(\nu_j; \rho(\nu_{j+1}; \rho^*; \nu^*); \nu_{j+1}) = \rho(\nu_j; \rho^*; \nu^*) = \rho_j,$$

and thus $\frac{d}{dt}\rho_j = 0$, which is the expected result. \square

Let us now study the convergence of the WB1 scheme. The main point is that the WB1 scheme is convergent for any fixed and non zero κ , but is not convergent as κ tends to zero. For simplicity, we consider the case $\kappa > 0$. Without loss of generality we assume that $\rho_j(t=0) = \rho(\nu_j, t=0)$, where $\rho(\nu, t)$ is the solution of equation (3.1).

Lemma 3.4 (Convergence for $\kappa \neq 0$). *Assume that the parameter is fixed and positive. Under assumptions (H1)-(H2), there exists a constant $C := C(\|\sigma\|_{W^{1,\infty}}, \kappa^{-1}, \varepsilon^{-1}, \nu^*)$, where ε is the edge of the mesh, such that the solution $\rho_h = (\rho_j)_{1 \leq j \leq N}$ of the well balanced scheme (3.14) and the solution $\rho_{ex} = (\rho(\nu_j))_{1 \leq j \leq N}$ of the equation (3.1) satisfy the following estimate*

$$\|\rho_h(t) - \rho_{ex}(t)\|_{L^2} \leq Ch(\|B\|_{H^1} + \|\rho_{ex}(0)\|_{H^2}), \quad 0 < t < T.$$

Proof. The proof is decomposed in two steps. In a first time the WB1 scheme is written as the sum of two contributions: a part consistent with the equation (3.1), and a remainder which is proved to tends to zero with h . In a second time the result is proved by mean of a an error estimate between the solution of the WB1 scheme and the solution of the equation (3.1). To

obtain a consistent expression of the WB1 scheme, let us perform a Taylor expansion with integral remainder of the exponential formula contained in the expression of the stationary solution $\rho(\nu_j; \rho_{j+1}; \nu_{j+1})$, that is

$$e^{-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds} = 1 - \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds + \int_0^{-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds} \left(-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds - \gamma \right) e^\gamma d\gamma$$

Another Taylor expansion of the function $\nu \mapsto \sigma(\nu)/\nu$ around the frequency ν_j yields

$$e^{-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds} = 1 - \frac{3\sigma(\nu_j)}{\kappa\nu_j} \Delta\nu_{j+\frac{1}{2}} + R_1,$$

where

$$R_1 = -\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \int_{\nu_j}^s \left(\frac{\sigma(\gamma)}{\gamma} \right)' d\gamma ds + \int_0^{-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds} \left(-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds - \gamma \right) e^\gamma d\gamma. \quad (3.16)$$

In the same way, one writes

$$\begin{aligned} \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)B(s)}{s} e^{-\frac{3}{\kappa} \int_{\nu_j}^s \frac{\sigma(\tau)}{\tau} d\tau} ds &= \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)B(s)}{s} ds + \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)B(s)}{s} \left(1 - e^{-\frac{3}{\kappa} \int_{\nu_j}^s \frac{\sigma(\tau)}{\tau} d\tau} \right) ds \\ &= \frac{3\sigma(\nu_j)B(\nu_j)}{\kappa\nu_j} \Delta\nu_{j+\frac{1}{2}} + R_2 \end{aligned}$$

where

$$R_2 = \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \int_{\nu_j}^s \left(\frac{\sigma(\gamma)B(\gamma)}{\gamma} \right)' d\gamma ds + \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)B(s)}{s} \left(1 - e^{-\frac{3}{\kappa} \int_{\nu_j}^s \frac{\sigma(\tau)}{\tau} d\tau} \right) ds. \quad (3.17)$$

It is now possible to rewrite the WB1 scheme (3.14) as

$$\frac{d}{dt} \rho_j = \frac{\kappa}{3} \nu_j \frac{\rho_{j+1} - \rho_j}{\Delta\nu_{j+\frac{1}{2}}} + \sigma(\nu_j)(B(\nu_j) - \rho_{j+1}) + \frac{\kappa}{3} \nu_j \frac{\rho_{j+1}R_1 + R_2}{\Delta\nu_{j+\frac{1}{2}}}.$$

We keep this expression and we now study the solution of the equation (3.1). Evaluating this solution at the frequency ν_j yields $\partial_t \rho(\nu_j) = \frac{\kappa}{3} \nu_j \partial_\nu \rho(\nu_j) + \sigma(\nu_j)(B(\nu_j) - \rho(\nu_j))$. Using the expression $\partial_\nu \rho(\nu_j) = (\rho(\nu_{j+1}) - \rho(\nu_j)) \Delta\nu_{j+\frac{1}{2}}^{-1} + \Delta\nu_{j+\frac{1}{2}}^{-1} \int_{\nu_j}^{\nu_{j+1}} (s - \nu_{j+1}) \partial_\nu^2 \rho(s) ds$, one writes this equation as

$$\partial_t \rho(\nu_j) = \frac{\kappa}{3} \nu_j \frac{\rho(\nu_{j+1}) - \rho(\nu_j)}{\Delta\nu_{j+\frac{1}{2}}} + \sigma(\nu_j)(B(\nu_j) - \rho(\nu_j)) + R_3,$$

where

$$R_3 = \frac{\kappa}{3} \frac{\nu_j}{\Delta\nu_{j+\frac{1}{2}}} \int_{\nu_j}^{\nu_{j+1}} (s - \nu_{j+1}) \partial_\nu^2 \rho(s) ds \quad (3.18)$$

Denoting $e_j(t) = \rho_j(t) - \rho(\nu_j, t)$ and removing the time dependence for ease of notations, one finds

$$\frac{d}{dt} e_j = \frac{\kappa}{3} \nu_j \frac{e_{j+1} - e_j}{\Delta\nu_{j+\frac{1}{2}}} + \sigma(\nu_j)(\rho(\nu_j) - \rho_{j+1}) + \frac{\kappa}{3} \nu_j \frac{\rho_{j+1}R_1 + R_2}{\Delta\nu_{j+\frac{1}{2}}} - R_3$$

To obtain the equation satisfied by the discrete L^2 norm of e_j , we multiply this equation by $\Delta\nu_{j+\frac{1}{2}}e_j$ and we sum on all the cells. This leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e_h\|_{L_d^2}^2 &= \frac{1}{2} \frac{d}{dt} \sum_{j=0}^N \Delta\nu_{j+\frac{1}{2}} e_j^2 = \frac{\kappa}{3} \sum_{j=0}^N \nu_j (e_{j+1} - e_j) e_j + \sum_{j=0}^N \sigma(\nu_j) (\rho(\nu_j) - \rho_{j+1}) \Delta\nu_{j+\frac{1}{2}} e_j \\ &\quad + \sum_{j=0}^N \left(\frac{\kappa}{3} \nu_j \frac{\rho_{j+1} R_1 + R_2}{\Delta\nu_{j+\frac{1}{2}}} - R_3 \right) \Delta\nu_{j+\frac{1}{2}} e_j \end{aligned} \quad (3.19)$$

We study each of these terms. First, a reindexation of the sum yields

$$\frac{\kappa}{3} \sum_{j=0}^N \nu_j (e_{j+1} - e_j) e_j = \frac{\kappa}{3} \sum_{j=0}^N \nu_j e_{j+1} e_j - \frac{\kappa}{6} \sum_{j=0}^N \nu_j e_j^2 - \frac{\kappa}{6} \sum_{j=0}^N \nu_{j+1} e_{j+1}^2 - \frac{\kappa}{6} \nu_0 e_0^2 + \frac{\kappa}{6} \nu_{N+1} e_{N+1}^2.$$

The last term vanishes due to the boundary condition of the WB1 scheme. Rearranging some terms, one finds

$$\frac{\kappa}{3} \sum_{j=0}^N \nu_j (e_{j+1} - e_j) e_j = -\frac{\kappa}{6} \sum_{j=0}^N \nu_j (e_j - e_{j+1})^2 - \frac{\kappa}{6} \sum_{j=0}^N \Delta\nu_{j+\frac{1}{2}} e_{j+1}^2 - \frac{\kappa}{6} \nu_0 e_0^2 \leq 0$$

We now turn to the second term of the right member of (3.19). One can writes this term as

$$\sum_{j=0}^N \sigma(\nu_j) (\rho(\nu_j) - \rho_{j+1}) \Delta\nu_{j+\frac{1}{2}} e_j = \sum_{j=0}^N \sigma(\nu_j) (\rho(\nu_j) - \rho(\nu_{j+1})) \Delta\nu_{j+\frac{1}{2}} e_j - \sum_{j=0}^N \sigma(\nu_j) e_{j+1} \Delta\nu_{j+\frac{1}{2}} e_j$$

For the first term of the right member, a Cauchy-Schwarz inequality together with a Taylor expansion with integral remainder of the function $\nu \mapsto \rho(\nu)$ and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ yield

$$\sum_{j=0}^N \sigma(\nu_j) (\rho(\nu_j) - \rho(\nu_{j+1})) \Delta\nu_{j+\frac{1}{2}} e_j \leq \frac{1}{2} \|\sigma\|_{L^\infty} \left(\|e_h\|_{L_d^2}^2 + h^2 \|\partial_\nu \rho_{ex}\|_{L^2}^2 \right).$$

The second term is trivially controlled as

$$-\sum_{j=0}^N \sigma(\nu_j) e_{j+1} \Delta\nu_{j+\frac{1}{2}} e_j \leq C \|\sigma\|_{L^\infty} \|e_h\|_{L_d^2}^2,$$

where the constant C depends on the mesh. The proof now only relies on the control of the term depending on R_1, R_2 and R_3 in (3.19). Let us start with the term containing R_1 . One has, by definition of R_1 (3.16)

$$\frac{\kappa}{3} \nu_j R_1 = \frac{\kappa}{3} \nu_j \left(-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \int_{\nu_j}^s \left(\frac{\sigma(\gamma)}{\gamma} \right)' d\gamma ds + \int_0^{-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds} \left(-\frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds - \gamma \right) e^\gamma d\gamma \right),$$

A direct computation yields

$$\frac{\kappa}{3} \nu_j R_1 \leq \Delta\nu_{j+\frac{1}{2}}^2 \|\sigma\|_{W^{1,\infty}} \left[\left(1 + \frac{1}{\nu_j} \right) + \frac{3}{\kappa \nu_j} \|\sigma\|_{W^{1,\infty}} \right].$$

It yields

$$\begin{aligned} \sum_{j=0}^N \frac{\kappa}{3} \nu_j \rho_{j+1} R_1 e_j &= \sum_{j=0}^N \frac{\kappa}{3} \nu_j e_{j+1} R_1 e_j + \sum_{j=0}^N \frac{\kappa}{3} \nu_j \rho(\nu_{j+1}) R_1 e_j \\ &\leq C \left((1+h) \|e_h\|_{L_d^2}^2 + h^2 \|\rho_{ex}\|_{L_d^2}^2 \right), \end{aligned}$$

where $C := C(\|\sigma\|_{W^{1,\infty}}, \kappa^{-1}, \varepsilon^{-1})$, ε is edge of the mesh. We now turn to the term containing R_2 . One has, by definition of R_1 (3.17)

$$\frac{\kappa}{3} \nu_j R_2 = \nu_j \int_{\nu_j}^{\nu_{j+1}} \int_{\nu_j}^s \left(\frac{\sigma(\gamma) B(\gamma)}{\gamma} \right)' d\gamma ds + \nu_j \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s) B(s)}{s} \left(1 - e^{-\frac{3}{\kappa} \int_{\nu_j}^s \frac{\sigma(\tau)}{\tau} d\tau} \right) ds.$$

Using a Taylor expansion to treat the term $1 - e^{-\frac{3}{\kappa} \int_{\nu_j}^s \frac{\sigma(\tau)}{\tau} d\tau}$, a direct computation yields

$$\frac{\kappa}{3} \nu_j R_2 \leq \Delta \nu_{j+\frac{1}{2}}^{\frac{3}{2}} \|\sigma\|_{W^{1,\infty}} \|B\|_{H^1} \left(2 + \frac{1}{\nu_j} + \frac{3}{\kappa \nu_j} \|\sigma\|_{W^{1,\infty}} \right),$$

which leads to

$$\sum_{j=0}^N \frac{\kappa}{3} \nu_j R_2 e_j \leq C \left(\|e_h\|_{L_d^2}^2 + h^2 \|B\|_{H^1}^2 \right),$$

where, once again, $C := C(\|\sigma\|_{W^{1,\infty}}, \kappa^{-1}, \nu_j^{-1})$. Finally, there remains to control the term containing R_3 (3.18), for which one easily obtains

$$\sum_{j=0}^N R_3 \Delta \nu_{j+\frac{1}{2}} e_j \leq \frac{\kappa}{3} \nu^* \left(\|e_h\|_{L_d^2}^2 + h^2 \|\partial_\nu^2 \rho_{ex}\|_{L^2}^2 \right).$$

Using all these results in (3.19), one obtains

$$\frac{1}{2} \frac{d}{dt} \|e_h\|_{L_d^2}^2 \leq C \left((1+h) \|e_h\|_{L_d^2}^2 + h^2 (\|B\|_{H^1}^2 + \|\rho_{ex}\|_{H^2}^2) \right),$$

where $C := C(\|\sigma\|_{W^{1,\infty}}, \kappa^{-1}, \nu_j^{-1}, \nu^*)$. One easily gets from equation (3.1) the estimate $\|\rho_{ex}(t)\|_{H^2} \leq C \|\rho_{ex}(0)\|_{H^2}$. The expected result is obtained by applying the Gronwall lemma. \square

This result shows that the WB1 scheme (3.14) is convergent for any fixed and non zero κ . Indeed the constant C in this result depends on κ^{-1} , and it is not possible to remove this singularity. Indeed, taking into account the limit as κ tends to 0 of the analytical stationary solution (3.12), one finds for this scheme

$$\lim_{\kappa \rightarrow 0} \frac{d}{dt} \rho_j = 0,$$

which obviously is not a consistent discretization of the limit equation (3.2). We propose a new construction strategy to avoid this consistency problem.

3.4.2 Spectrally Well-Balanced Scheme

In this part we study and prove several properties for the following scheme, denoted as the spectrally well-balanced (SWB) scheme:

$$\begin{cases} \frac{d}{dt}\rho_j = \frac{\sigma(\nu_j)}{1 - M(\nu_{j+1}; \nu_j)} \left(\rho(\nu_j; \rho_{j+1}, \nu_{j+1}) - \rho_j \right), & 1 \leq j \leq N-1, \quad \kappa > 0, \\ \frac{d}{dt}\rho_j = \frac{\sigma(\nu_j)}{1 - M(\nu_{j-1}; \nu_j)} \left(\rho(\nu_j; \rho_{j-1}, \nu_{j-1}) - \rho_j \right), & 2 \leq j \leq N, \quad \kappa < 0, \end{cases} \quad (3.20)$$

where the boundary conditions are the same than for the WB1 scheme (3.14). Considering once again a classical explicit Euler discretization of the time derivative, the CFL is given by

$$\begin{cases} \Delta t \leq \max_j \frac{1 - M(\nu_{j+1}; \nu_j)}{\sigma(\nu_j)}, & \kappa > 0, \\ \Delta t \leq \max_j \frac{1 - M(\nu_{j-1}; \nu_j)}{\sigma(\nu_j)}, & \kappa < 0. \end{cases} \quad (3.21)$$

On the contrary to the previous scheme, the CFL of the SWB scheme strongly depends on the L^∞ norm of the opacity σ . Indeed, one has

$$\lim_{\sigma \rightarrow \infty} \Delta t = 0.$$

The scheme is built using the integrating factor. It is defined, for an arbitrary $\nu_0 \in \mathbb{D}$, by $M(\nu; \nu_0) = e^{-\frac{3}{\kappa} \int_{\nu_0}^{\nu} \frac{\sigma(s)}{s} ds}$. Multiplying equation (3.1) by $3\sigma(\nu)M(\nu; \nu_0)/\kappa\nu = -M'(\nu; \nu_0)$ yields

$$-M'(\nu; \nu_0)\partial_t \rho = \sigma(\nu) \left(\partial_\nu (M(\nu; \nu_0)\rho) + \frac{3\sigma(\nu)}{\kappa\nu} M(\nu; \nu_0)B(\nu) \right).$$

Integrating this equation between ν_j and ν_{j+1} and discretizing each term conveniently, one finds

$$-\left[M(\nu; \nu_0) \right]_{\nu_j}^{\nu_{j+1}} \frac{d}{dt} \rho_j = \sigma(\nu_j) \left(\left[M(\nu; \nu_0)\rho \right]_{\nu_j}^{\nu_{j+1}} + \int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_0)B(s) ds \right).$$

Taking $\rho(\nu_{j+1}) = \rho_{j+1}$, $\rho(\nu_j) = \rho_j$ and dividing this equation by $M(\nu_j; \nu_0)$ and $1 - M(\nu_{j+1}; \nu_j)$, one finds, using the relation $M(\nu; \nu_0)/M(s; \nu_0) = M(\nu; s)$,

$$\frac{d}{dt} \rho_j = \frac{\sigma(\nu_j)}{1 - M(\nu_{j+1}; \nu_j)} \left(M(\nu_{j+1}; \nu_j)\rho_{j+1} - \rho_j + \int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j)B(s) ds \right).$$

The definition of the stationary solution

$$\rho(\nu_j; \rho_{j+1}, \nu_{j+1}) = M(\nu_{j+1}; \nu_j)\rho_{j+1} + \int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j)B(s) ds$$

yields the SWB scheme (3.20). The same argument than for the WB1 scheme (3.14) shows that this scheme is well-balanced. We prove a uniform (in κ) convergence result for this scheme. We keep the same notations than in the previous part, that is $\rho_h = (\rho_j)_{1 \leq j \leq N}$ and $\rho_{ex} = (\rho(\nu_j))_{1 \leq j \leq N}$. We need the following stability result

Lemma 3.5 (L^2 Stability). *Assume that the parameter κ is non negative. Under the assumptions (H1)-(H2), the following estimate holds, where the constant C depends on all the parameters and the boundary condition but is uniform in $\kappa \in (0, \kappa^*]$*

$$\|\rho_h(t)\|_{L_d^2} \leq C \sqrt{1 + \|\rho_h(0)\|_{L_d^2}^2}, \quad 0 < t < T.$$

Proof. Since the proof is rather classical, we only develop the main ideas. We want to reveal in the SWB scheme (3.20) a consistent discretization of equation (3.1). Injecting the expression of the stationary solution (3.11), one can write it as

$$\frac{d}{dt} \rho_j = \frac{\kappa}{3} \nu_j \frac{\rho_{j+1} - \rho_j}{\Delta \nu_{j+\frac{1}{2}}} + \sigma(\nu_j)(B(\nu_j) - \rho_j) + \sigma(\nu_j)(\rho_{j+1} - \rho_j) R_{j,1} + R_{j,2}, \quad (3.22)$$

with

$$\begin{cases} R_{j,1} = \frac{1}{1 - M(\nu_{j+1}; \nu_j)} - \frac{\kappa \nu_j}{3\sigma(\nu_j)\Delta \nu_{j+\frac{1}{2}}} - 1, \\ R_{j,2} = \frac{\sigma(\nu_j)}{1 - M(\nu_{j+1}; \nu_j)} \int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j) (B(s) - B(\nu_j)) ds, \end{cases} \quad (3.23)$$

where we used, by definition of $M(s; \nu_j)$, $\int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j) ds = 1 - M(\nu_{j+1}; \nu_j)$. We introduce $B_h = (B(\nu_j))_{1 \leq j \leq N}$. Using the positivity of the coefficients κ and σ and the Cauchy-Schwarz inequality, one finds by multiplying equation (3.22) by $\Delta \nu_{j+\frac{1}{2}} \rho_j$ and adding on all the cells a positive constant C such that,

$$\frac{d}{dt} \|\rho_h\|_{L_d^2}^2 \leq C(1 + \max_j |R_{j,1}|) \|\rho_h\|_{L_d^2}^2 + \|\sigma\|_{L^\infty} \|B\|_{L_d^2}^2 + \|R_{j,2}\|_{L_d^2}^2 + \frac{\kappa}{6} \rho_{N+1}^2. \quad (3.24)$$

We thus need to control $R_{j,1}$ and the L_d^2 norm of $R_{j,2}$. Denoting $z_j = \frac{3}{\kappa} \int_{\nu_j}^{\nu_{j+1}} \frac{\sigma(s)}{s} ds$ and using the definition of $M(\nu_{j+1}; \nu_j)$, one can write $R_{j,1}$ as

$$R_{j,1} = \left(\frac{1}{1 - e^{-z_j}} - \frac{1}{z_j} \right) - 1 + \left(\frac{1}{z_j} - \frac{\kappa \nu_j}{3\sigma(\nu_j)\Delta \nu_{j+\frac{1}{2}}} \right).$$

For the first term one has $\frac{1}{1 - e^{-z_j}} - \frac{1}{z_j} \leq 1$. Using a Taylor expansion of the function $\nu \mapsto \sigma(\nu)\nu^{-1}$ at the frequency ν_j , one finds

$$\max_j |R_{j,1}| \leq C \frac{\kappa}{3} \frac{\|\sigma\|_{W^{1,\infty}}}{\sigma_*^2}, \quad (3.25)$$

where the constant C depends on the mesh but is independent of κ . We now turn to the term $R_{j,2}$. A Taylor expansion of the function $\nu \mapsto B(\nu)$ shows

$$|R_{j,2}| \leq \Delta \nu_{j+\frac{1}{2}} \|B'\|_{L^\infty} \frac{\sigma(\nu_j)}{1 - M(\nu_{j+1}; \nu_j)} \int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j) ds.$$

Using the relation $\int_{\nu_j}^{\nu_{j+1}} \frac{3\sigma(s)}{\kappa s} M(s; \nu_j) ds = 1 - M(\nu_{j+1}; \nu_j)$, one finds

$$\|R_{j,2}\|_{L_d^2} \leq h \sqrt{\nu^*} \|B'\|_{L^\infty} \|\sigma\|_{L^\infty}. \quad (3.26)$$

Using all these results in (3.24), one finds a constant C such that

$$\begin{aligned} \frac{d}{dt} \|\rho_h(t)\|_{L_d^2}^2 &\leq C \left(1 + \frac{\kappa}{3} \frac{\|\sigma\|_{W^{1,\infty}}}{\sigma_*^2} \right) \|\rho_h(t)\|_{L_d^2}^2 + \|\sigma\|_{L^\infty} \|B\|_{L_d^2}^2 + h^2 \nu^* \|B'\|_{L^\infty}^2 \|\sigma\|_{L^\infty}^2 \\ &\quad + \frac{\kappa}{6} \rho_{N+1}^2. \end{aligned}$$

The Gronwall lemma finally gives the result. \square

The key point was to prove a uniform estimate for the consistency errors $R_{j,1}$ and $R_{j,2}$. Actually, estimates (3.25) and (3.26) are no longer true for the WB1 scheme (3.14). We now turn to the uniform (in κ) convergence result of the scheme (3.20). Without loss of generality, we assume that $\forall j, \rho_j(0) = \rho(\nu_j, 0)$.

Lemma 3.6 (Uniform convergence). *Assume that the parameter κ is non negative. Under the assumptions (H1)-(H2), the numerical solution of the scheme (3.20) satisfies the following estimate, where the constant C is uniform in $\kappa \in (0, \kappa^*]$*

$$\|\rho_{ex}(t) - \rho_h(t)\|_{L_d^2} \leq Ch, \quad 0 < t < T.$$

Proof. Evaluating the solution of the P.D.E. (3.1) at the frequency ν_j and using a Taylor expansion of the function $\nu \mapsto \rho(t, \nu)$ with integral remainder, one has

$$\begin{aligned} \frac{d}{dt} \rho(t, \nu_j) &= \frac{\kappa}{3} \nu_j \left(\frac{\rho(t, \nu_{j+1}) - \rho(t, \nu_j)}{\Delta \nu_{j+\frac{1}{2}}} - \int_{\nu_j}^{\nu_{j+1}} \frac{\nu_{j+1} - s}{\Delta \nu_{j+\frac{1}{2}}} \partial_\nu^2 \rho(t, s) ds \right) \\ &\quad + \sigma(\nu_j) \left(B(\nu_j) - \rho(t, \nu_j) \right). \end{aligned} \quad (3.27)$$

We obtain an equation on the unknown $e_j(t) := \rho_j(t) - \rho(t, \nu_j)$ by deducting to the expression (3.22) of the SWB scheme this equation. Multiplying the obtained equation by $\Delta \nu_{j+\frac{1}{2}} e_j(t)$ and adding on all the cells, one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|e_h(t)\|_{L_d^2}^2 &= \sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) S_j(t) + \sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) \sigma(\nu_j) (\rho(\nu_{j+1}) - \rho(\nu_j)) R_{j,1} \\ &\quad + \sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) R_{j,2} + \sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) \int_{\nu_j}^{\nu_{j+1}} \frac{\nu_{j+1} - s}{\Delta \nu_{j+\frac{1}{2}}} \partial_\nu^2 \rho(t, s) ds, \end{aligned} \quad (3.28)$$

where $S_j(t) = \frac{\kappa}{3} \nu_j \frac{e_{j+1}(t) - e_j(t)}{\Delta \nu_{j+\frac{1}{2}}} + \sigma(\nu_j) (e_{j+1}(t) - e_j(t)) R_{j,1} - \sigma(\nu_j) e_j(t)$ and $R_{j,1}$ and $R_{j,2}$ are defined in (3.23). We control successively each of these terms. First, the term $S_j(t)$ has already been studied. Using the estimate (3.25), one has

$$\sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) S_j(t) \leq C \frac{\kappa}{3} \frac{\|\sigma\|_{W^{1,\infty}}^2}{\sigma_*^2} \|e_h(t)\|_{L_d^2}^2,$$

where the constant C depends on the mesh but is independent of κ . The term $R_{j,2}$ have also been controlled in the previous part. The estimate (3.26) and the inequality $ab \leq (a^2 + b^2)/2$ gives

$$\sum_j \Delta \nu_{j+\frac{1}{2}} e_j(t) R_{j,2} \leq \frac{1}{2} \|e_h(t)\|_{L_d^2}^2 + \frac{1}{2} h^2 \nu^* \|B'\|_{L^\infty}^2 \|\sigma\|_{L^\infty}^2.$$

Similar arguments associated to a Taylor expansion of the function $\nu \mapsto \rho(t, \nu)$ lead to

$$\sum_j \Delta\nu_{j+\frac{1}{2}} e_j(t) \sigma(\nu_j) (\rho(\nu_{j+1}) - \rho(\nu_j)) R_{j,1} \leq C \left(\|e_h(t)\|_{L_d^2}^2 + h^2 \|\partial_\nu \rho(t)\|_{L^2(\mathbb{D})}^2 \right),$$

where, once again, the constant C is independent of κ . In the same way, one has for the last term

$$\sum_j \Delta\nu_{j+\frac{1}{2}} e_j(t) \int_{\nu_j}^{\nu_{j+1}} \frac{\nu_{j+1} - s}{\Delta\nu_{j+\frac{1}{2}}} \partial_\nu^2 \rho(t, s) ds \leq \frac{1}{2} \left(\|e_h(t)\|_{L_d^2}^2 + h^2 \|\partial_{\nu\nu} \rho(t)\|_{L^2(\mathbb{D})}^2 \right).$$

As ρ is solution of a simple linear P.D.E., one easily controls its H^2 norm. Actually, using the regularity of σ (assumption (H2)), one finds a constant C such as $\|\partial_\nu \rho(t)\|_{L^2}^2 \leq C(1 + \|\partial_\nu \rho(0)\|_{L^2}^2)$ and $\|\partial_{\nu\nu} \rho(t)\|_{L^2}^2 \leq C(1 + \|\partial_{\nu\nu} \rho(0)\|_{L^2}^2)$. Using all these results in (3.28), one finds another constant C , once again uniform in κ , such that

$$\frac{d}{dt} \|e_h(t)\|_{L_d^2}^2 \leq C(\|e_h(t)\|_{L_d^2}^2 + h^2).$$

As before, the Gronwall lemma and the assumption on the initial data gives the announced result. \square

3.4.3 Upwind scheme

In all the forthcoming numerical results, the numerical solutions of the well-balanced schemes (3.14) and (3.20) are compared to the following classical upwind scheme

$$\begin{cases} \frac{d}{dt} \rho_j = \frac{\kappa}{3} \frac{\rho_{j+1} - \rho_j}{\Delta\nu_j} \nu_{j+\frac{1}{2}} + \sigma(\nu_j) (B(\nu_j) - \rho_j), & 1 \leq j \leq N-1, & \kappa > 0, \\ \frac{d}{dt} \rho_j = \frac{\kappa}{3} \frac{\rho_j - \rho_{j-1}}{\Delta\nu_j} \nu_{j-\frac{1}{2}} + \sigma(\nu_j) (B(\nu_j) - \rho_j), & 2 \leq j \leq N, & \kappa < 0, \end{cases} \quad (3.29)$$

where once again the boundary conditions are the same than for the WB1 scheme (3.14). Considering once again a classical explicit Euler discretization of the time derivative, the CFL is given by

$$\begin{cases} \Delta t \leq 3 \max_j \frac{\Delta\nu_j}{\kappa \nu_{j+\frac{1}{2}} + 3\Delta\nu_j \sigma(\nu_j)}, & \kappa > 0, \\ \Delta t \leq 3 \max_j \frac{\Delta\nu_j}{-\kappa \nu_{j-\frac{1}{2}} + 3\Delta\nu_j \sigma(\nu_j)}, & \kappa < 0. \end{cases} \quad (3.30)$$

As for the previous SWB scheme, the CFL of the upwind scheme strongly depends on the L^∞ norm of σ . Obviously the upwind scheme is consistent in both regimes $\Delta\nu_j \rightarrow 0$ and $\kappa \rightarrow 0^\pm$.

3.5 Numerical results

In this section numerical results are presented for the ‘‘classical’’ well-balanced scheme WB1 (3.14), the spectrally well-balanced scheme SWB (3.20) and the upwind scheme (3.29) in several configurations of the coefficients κ and σ . This section is divided in three parts, one for each considered opacities (constant, highly irregular and Cramer formula. For each opacities, the long time behavior and the consistency as $\kappa \rightarrow 0^\pm$ of the schemes of interest are studied. The main

point is that the SWB scheme exactly captures the numerical solution as time goes on and is consistent as $\kappa \rightarrow 0^\pm$. A last part is dedicated to the study of configurations of the parameters κ and σ in which the relative L^1 error of the upwind scheme is maximized, highlighting the interest of the SWB scheme.

Since this study involves a very simple model, the analytical solution of equation (3.1) can be computed, and its expression is given by

$$\rho(t, \nu) = \rho^{in}(\nu e^{\frac{\kappa}{3}t}) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}t}} \sigma(\tau) \tau^{-1} d\tau} + \int_0^t \sigma(\nu e^{\frac{\kappa}{3}s}) B(\nu e^{\frac{\kappa}{3}s}) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}s}} \sigma(\tau) \tau^{-1} d\tau} ds, \quad (3.31)$$

Since one of the regime of interest is $|\kappa| \ll 1$, it is numerically necessary to use a non singular form of this analytical solution. To this end one remarks that the expression under the integral can be integrated by parts. One finds

$$\begin{aligned} \rho(t, \nu) = & B(\nu) + \left(\rho^{in}(\nu e^{\frac{\kappa}{3}t}) - B(\nu e^{\frac{\kappa}{3}t}) \right) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}t}} \sigma(\tau) \tau^{-1} d\tau} \\ & + \frac{\kappa}{3} \nu \int_0^t e^{\frac{\kappa}{3}s} B'(\nu e^{\frac{\kappa}{3}s}) e^{-\frac{3}{\kappa} \int_\nu^{\nu e^{\frac{\kappa}{3}s}} \sigma(\tau) \tau^{-1} d\tau} ds, \end{aligned} \quad (3.32)$$

which is no longer singular in κ . The same procedure is applied to treat the singularity in the stationary solution (3.11). The following relative L^1 error is introduced, where the final time T^f will be defined for each test case

$$\frac{\|\rho_j(t) - \rho(\nu_j, t)\|}{\|\rho(\nu_j, T^f)\|}. \quad (3.33)$$

For all the numerical tests a random mesh composed of 50 cells is considered. The domain is $\mathbb{D} = [\varepsilon, 30]$, where ε varies between 0 and 10^{-1} according to the different opacities. The initial conditions are taken as $\rho^{in}(\nu) = B(\nu - 15)$ if $\kappa > 0$ and $\rho^{in}(\nu) = 0$ if $\kappa < 0$ and the boundary conditions are given by the analytical solution (3.32).

3.5.1 Constant σ

In this part the numerical tests are performed for the very simple case where σ is constant equal to 1. In figure 3.1 we displayed the relative L^1 error between the numerical solutions of the schemes and the analytical solution, with $N = 50$ cells and $\kappa = 1$. As expected, the SWB scheme and the WB1 scheme converge toward the analytical solution as time goes on. As we have seen previously, the WB1 scheme is not consistent in the regime $\kappa \rightarrow 0$. Figure 3.2 plots on the left side the evolution, as κ tends to 0 and at time $t = 2$, of the L^1 error between the solutions of the WB1, SWB and upwind schemes and the numerical solution of the following scheme, consistent with equation (3.2):

$$\frac{d}{dt} \rho_j = \sigma(\nu_j) (B(\nu_j) - \rho_j), \quad (3.34)$$

and confirms the theoretical study. On the right side we plotted the L^1 norm in a Log-Log scale between the analytical solution and the numerical solution of the schemes at time $t = 2$ and with $\kappa = 1$.

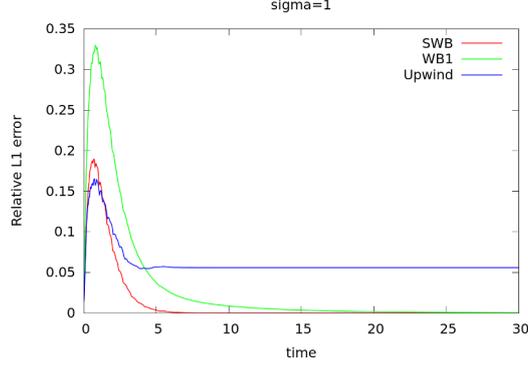


Figure 3.1: Evolution of the L^1 relative error between the analytical solution (3.31) and the numerical solutions of the WB1, SWB and upwind schemes, computed with $N=50$, $\kappa = 1$

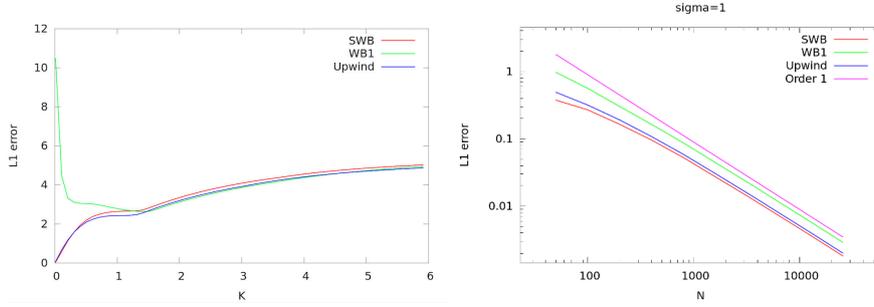


Figure 3.2: Left: L^1 error between the numerical solution of (3.34) and the numerical solutions of the WB1, SWB and upwind schemes versus K with $N=50$ and $t=2$. Right: L^1 error between the analytical solution (3.31) and the numerical solutions of the WB1, SWB and upwind schemes versus N in a Log-Log scale plan, with $t=2$ and $\kappa=1$.

3.5.2 Highly irregular σ

Many physical contexts imply highly irregular emission absorption coefficient σ , whose frequency dependence is sometimes regular and sometimes very peaked. The numerical study of such opacities is thus interesting. Numerically, the coefficient σ is constructed as the sum of a regular part and peaked functions (cf figure 1)

$$\sigma(\nu) = \frac{150}{1 + \nu} + \sum_{1 \leq j \leq 20} P_j, \quad (3.35)$$

where the peaked functions are distributed regularly on the mesh. In this part ε is taken equal to 0, so the domain is $[0, 30]$. The choice of σ (3.35) yields an exact numerical integration of all the integrals in the analytical solution (3.32). In the same way, the integrals in the expression of the stationary solutions (3.11) are numerically computed exactly. The first property to verify numerically is the well-balanced property of the SWB (and WB1) scheme. In figure 3.4 are displayed the time evolution of the relative L^1 error of the SWB, WB1 and upwind scheme for $\kappa = 1$ and $\kappa = -1$. As expected, both the SWB and the WB1 schemes exactly capture the stationary solution, but it is interesting to notice that the WB1 scheme is much slower than

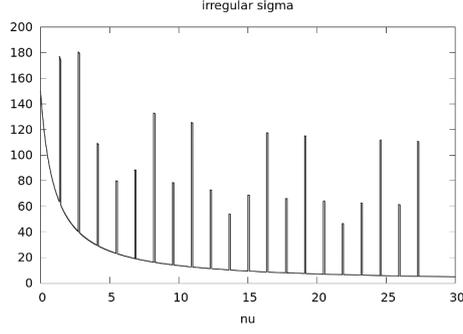


Figure 3.3: Irregular emission absorption coefficient

the SWB scheme. As explained in the previous section, the SWB scheme is uniformly (in κ)

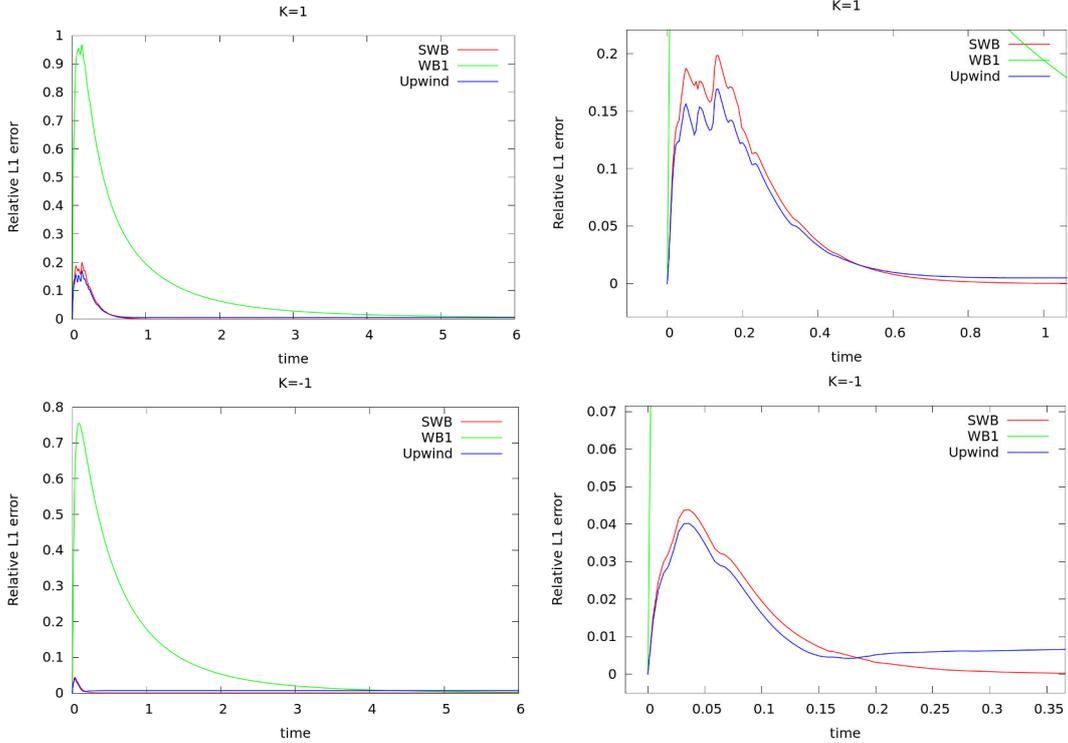


Figure 3.4: Relative L^1 error of the SWB, WB1 and upwind schemes versus time, $T^f = 6$. Top: $\kappa = 1$, bottom: $\kappa = -1$. Pictures on the right side are zoom of the pictures on the left side.

convergent, and this property has to be checked numerically for this choice of σ . Figure 3.5 displays the evolution, as $\kappa \rightarrow 0^\pm$, of the L^1 error between the solutions of the SWB, WB1 and the upwind schemes and the numerical solution of the following scheme, consistent with equation (3.1) as $\kappa = 0$:

$$\frac{d}{dt}\rho_j = \sigma(\nu_j)(B(\nu_j) - \rho_j). \quad (3.36)$$

Figure 3.6 displays, in a (Log,Log) scale, the L^1 error of the schemes with respect to the number

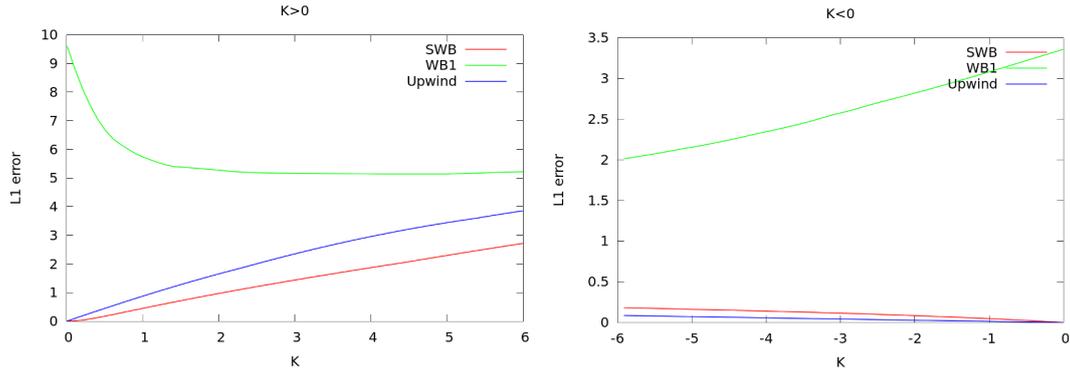


Figure 3.5: L^1 error between the numerical solution of (3.34) and the numerical solutions of the WB1, SWB and upwind schemes versus versus κ . Left: $\kappa > 0$, $t=0.1$. Right: $\kappa < 0$, $t=0.02$

of cells. It shows that as expected, all the schemes are of order 1.

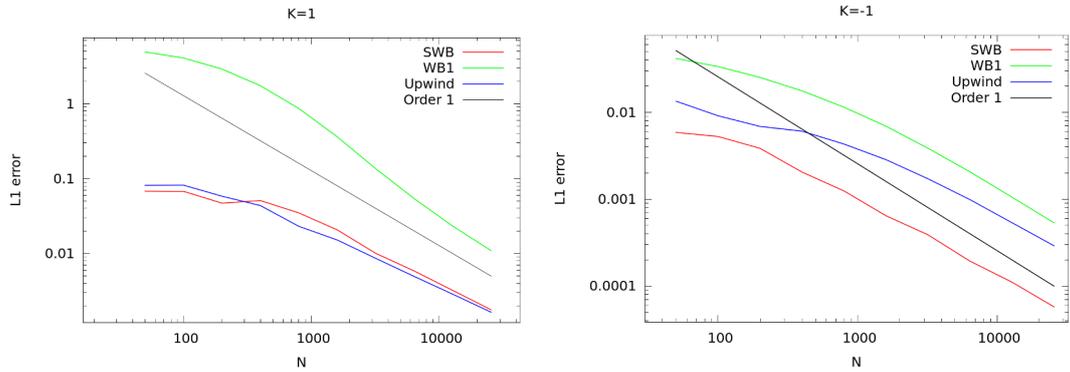


Figure 3.6: L^1 error of the SWB, WB1 and upwind schemes versus N , (Log,Log) scale. Left: $\kappa > 0$, $t=0.1$. Right: $\kappa < 0$, $t=0.02$

3.5.3 Cramer's opacity

In this part we study the behavior of the schemes with the Cramer's opacity:

$$\sigma(\nu) = \frac{1 - e^{-\nu}}{\nu^3}. \quad (3.37)$$

This opacity leads to $\sigma(\nu)B(\nu) = e^{-\nu}$, and has a singularity as $\nu \rightarrow 0$. For this coefficient the analytical value of the integrals in the expression of the solutions of (3.1) and (3.10) can no longer be computed analytically. Numerically, they are computed by classical five points Gaussian quadratures. In this part and due to the singularity of the opacity, ε is taken equal to 10^{-1} and thus the domain is $[10^{-1}; 30]$. The same study than for the previous opacity is performed, and as before the long time behavior of the numerical solutions of the schemes is studied. In figure 3.7 are displayed the time evolution of the relative L^1 error of the schemes. Once again and as expected, the SWB and WB1 schemes exactly capture the stationary solution, while the relative error of the upwind scheme is 18% for $\kappa = 1$ and 7% for $\kappa = -1$. As before,

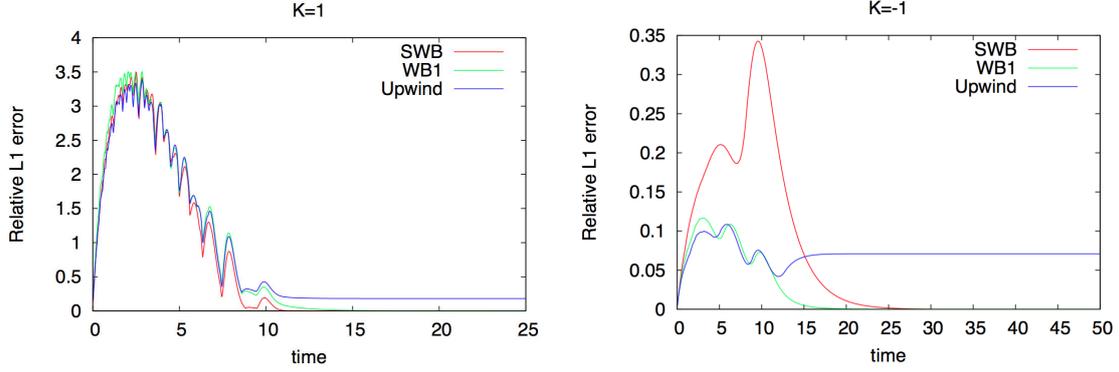


Figure 3.7: Relative L^1 error of the SWB, WB1 and upwind schemes versus time. Left: $\kappa = 1$, $T^f = 25$, right: $\kappa = -1$, $T^f = 50$

the behavior of the schemes for $\kappa \rightarrow 0^\pm$ is studied. Figure 3.8 displays the error between the schemes and the numerical solution of the scheme (3.36) as $\kappa \rightarrow 0^\pm$. It shows that the SWB scheme is consistent with equation (3.1), even for small coefficient κ .

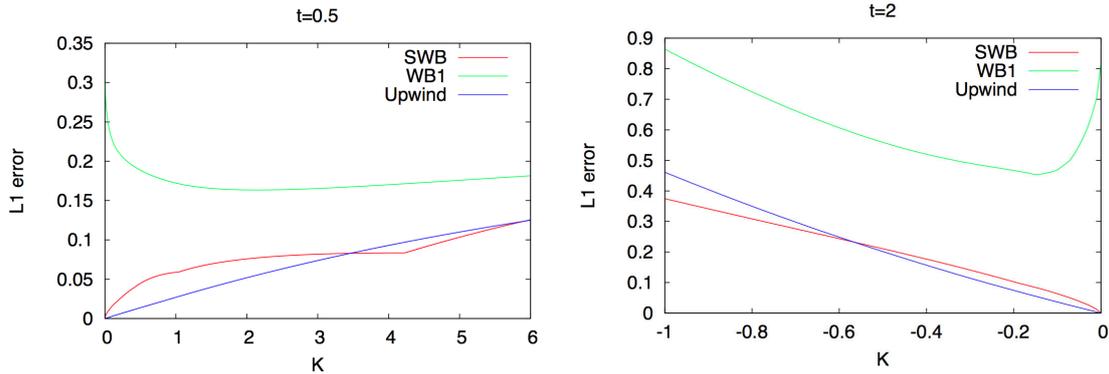


Figure 3.8: L^1 error of the SWB, WB1 and upwind schemes versus κ . Left: $\kappa > 0$, $t=0.5$. Right: $\kappa < 0$, $t=2$

3.5.4 SWB vs upwind

In this part the advantage of the SWB scheme compared with the upwind scheme is highlighted. Indeed, for certain values of κ and σ , the upwind L^1 error can become significant. In the previous part, for σ defined in (3.35) and $\kappa = \pm 1$, the upwind relative L^1 error was of order 1%. On the other hand we have seen that for the Cramer's opacity and for $\kappa = 1$, the relative L^1 error of the upwind scheme reaches 18%, which is non negligible. Two test cases are presented here in which the upwind relative L^1 error is significant. In both the case we take σ as a piecewise constant function. In the first following one, the discontinuity is placed at the middle of the mesh (figure 3.9). Figure 3.10 displays the relative L^1 error of the schemes as the time increases for several value of κ . The initialization is such that $\rho^{in}(\nu) = B(\nu - 15)$. It shows that while increasing the coefficient κ , the relative L^1 error of the upwind scheme also increases, and reach roughly 35%

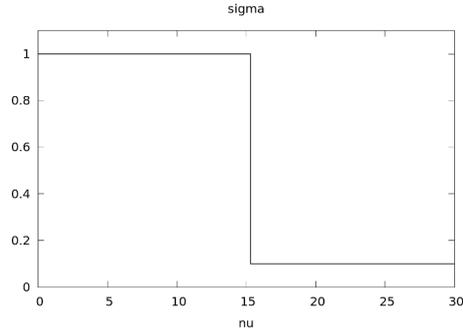


Figure 3.9: Piecewise constant emission absorption coefficient

in the case $\kappa = 10$. In figure 3.11 are displayed the corresponding solutions at $T^f = 30$. In the

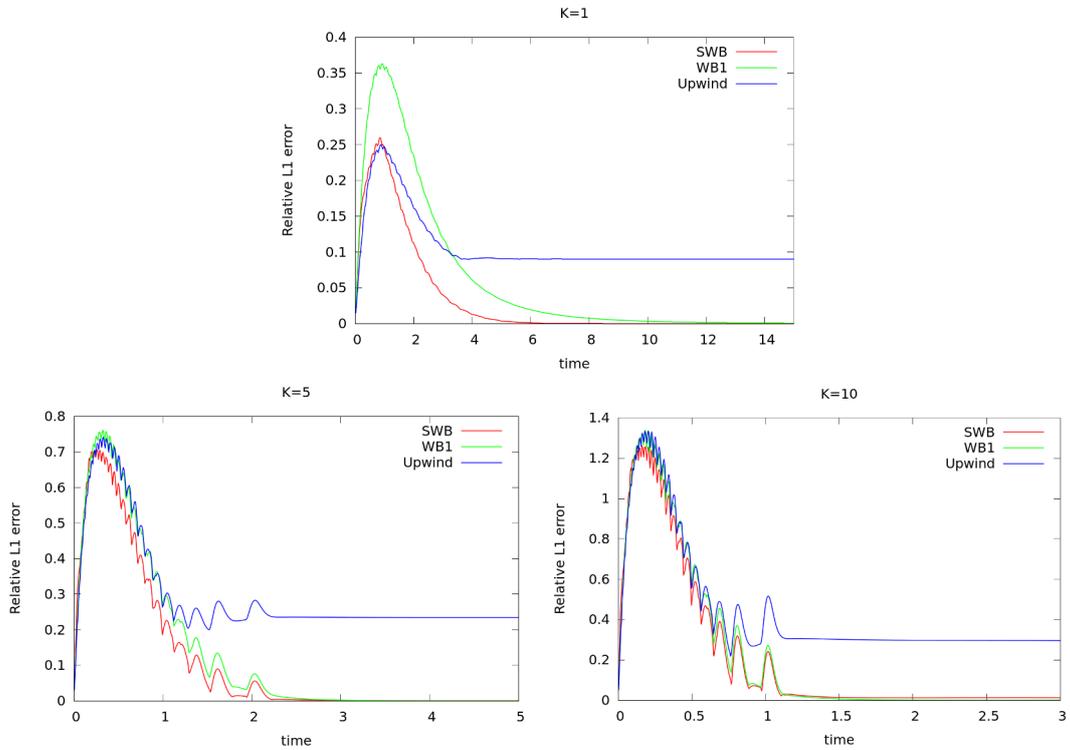


Figure 3.10: Relative L^1 error of the SWB, WB1 and upwind schemes versus time.

picture the curves of the analytical solution and the numerical solutions of the SWB and WB1 schemes are merged, due to the fact that the error of the SWB and WB1 schemes are close to the computer zero. The peak of the analytical solution is not well retranscribed by the upwind scheme. For the second test case the emission absorption coefficient is still taken as a piecewise constant function but we took a negative κ . The initial conditions are such that $\rho_j(0) = 0$. In figure 3.12 are displayed the evolution of the relative L^1 error as time goes on, for several values of κ . As previously, we see that the relative L^1 error of the upwind scheme increases with κ , and reaches roughly 20% for $\kappa = -10$. The corresponding solutions at $T^f = 30$ are displayed

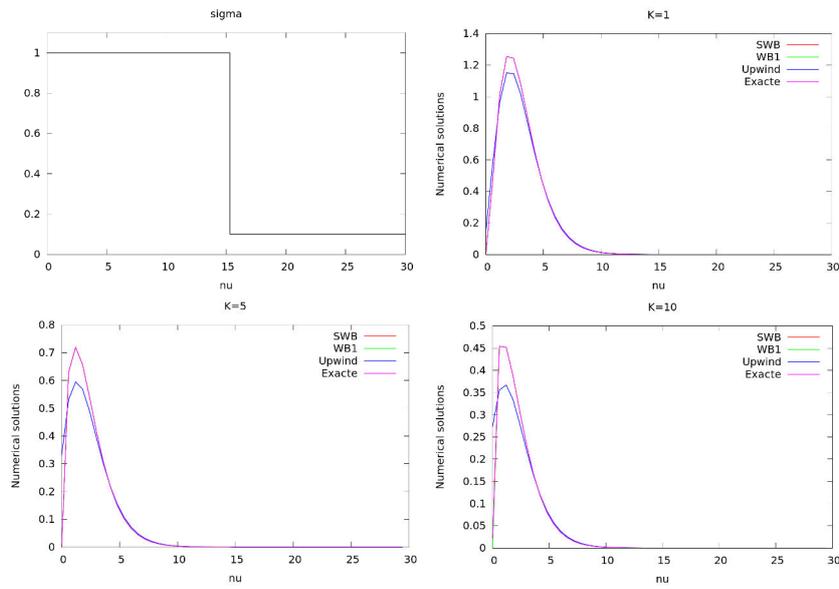


Figure 3.11: Corresponding numerical solutions of the SWB, WB1 and upwind schemes at $t=30$. The curves of the SWB and WB1 numerical solutions and the analytical solutions are merged.

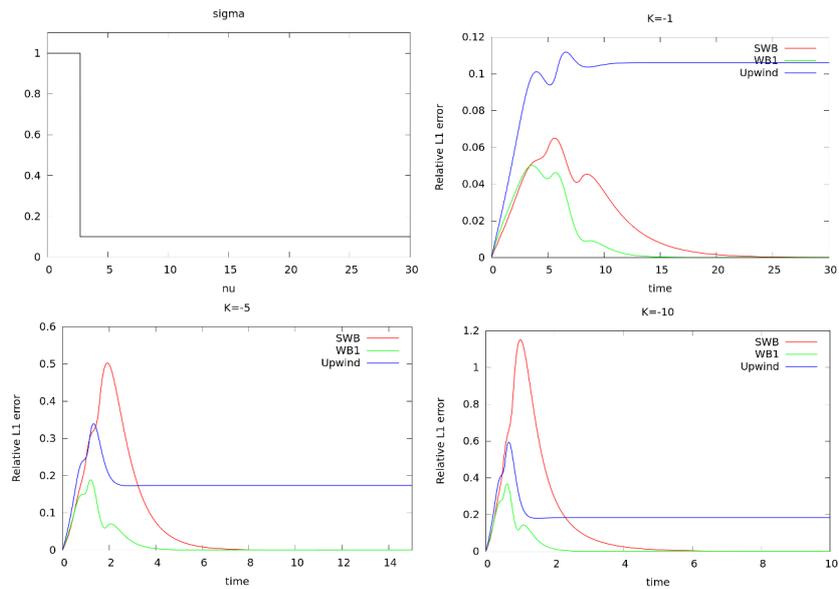


Figure 3.12: Relative L^1 error of the SWB, WB1 and upwind schemes versus time.

in figure 3.13. Once again, the curves of the analytical solution and the numerical solutions of the SWB and WB1 schemes curves are merged, and the upwind scheme does not retranscribe faithfully the behavior of the analytical solution.

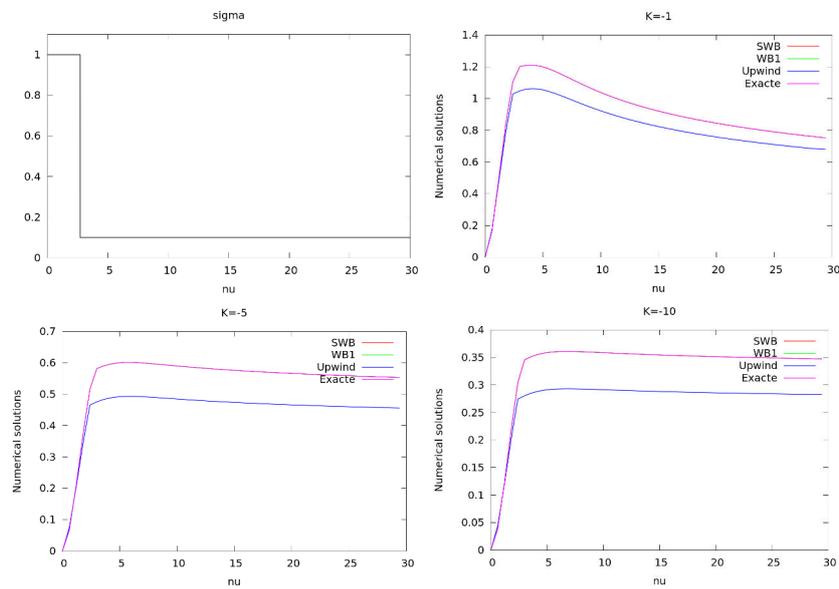


Figure 3.13: Corresponding numerical solutions of the SWB, WB1 and upwind schemes at $t=30$. The curves of the SWB and WB1 numerical solutions and the analytical solutions are merged.

Chapter 4

Compton scattering (I): anisotropic kinetic models

4.1 Introduction

The aim of this chapter is the study of a hierarchy of anisotropic kinetic models describing the Compton scattering. On the one hand, several kinetic models have been derived from the Boltzmann equation by physicists [BPR69, PL97, POM73, FKM85], leading to several anisotropic Fokker-Planck type equations. On the second hand, Escobedo et al [EMV03] recently derived from the Boltzmann equation for photons the Kompaneets equation [KOM57, EHV98], which is a nonlinear Fokker-Planck type equation, by assuming the isotropy of the distribution function and by using several original technics. The main goal of this chapter is to present a generalization of this approach for anisotropic distribution functions. This leads to Fokker-Planck equations, whose structure can be seen as anisotropic Kompaneets type equations.

The Compton scattering describes the change of energy and direction of a photon, of momentum \mathbf{p}_γ interacting with an electron, of momentum \mathbf{p}_e , leading to another photon and another electron of momentum \mathbf{p}'_γ and \mathbf{p}'_e respectively (figure 4.1). The Compton scattering between

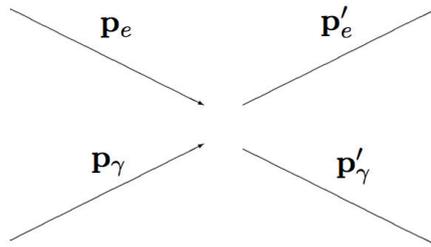


Figure 4.1: Photon-electron interaction by Compton scattering

photons and electrons can be described by a Boltzmann equation for the density distribution function of the photons. Since the induce effects (quantum effects) are taken into account for the photons, the collision operator is quadratic with respect to the distribution function. Assuming that the electrons are at thermodynamic equilibrium, their distribution function is given by a Maxwellian, and the equation describing the evolution of the density distribution of the photons

f , which depends on the time t , on the frequency of the photon $\nu \in \mathbb{R}^+$, on its space position $\mathbf{x} \in \mathbb{R}^3$ and on its direction $\boldsymbol{\Omega} \in S^2$, is given by [EMV03]

$$\frac{1}{c} \partial_t f + \boldsymbol{\Omega} \cdot \nabla f = \int_{\mathbb{R}^+ \times S^2} \frac{\nu'}{\nu} b(\mathbf{x}, \nu, \nu', \theta) e^{\frac{\nu'}{T}} q(f) d\nu' d\boldsymbol{\Omega}', \quad (4.1)$$

where

$$\begin{cases} b(\mathbf{x}, \nu, \nu', \theta) = \sigma_s(\mathbf{x}) (1 + \cos^2 \theta) |\omega|^{-1} e^{-\frac{A^2 m c^2}{2T |\omega|^2}}, \\ A = \frac{\nu'}{c} - \frac{\nu}{c} + \frac{|\omega|^2}{2mc}, \\ \omega = \frac{\nu'}{c} \boldsymbol{\Omega}' - \frac{\nu}{c} \boldsymbol{\Omega}, \\ \cos \theta = \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}', \end{cases} \quad (4.2)$$

and

$$q(f) = e^{-\frac{\nu'}{T}} f(\nu', \boldsymbol{\Omega}') (1 + f(\nu, \boldsymbol{\Omega})) - e^{-\frac{\nu}{T}} f(\nu, \boldsymbol{\Omega}) (1 + f(\nu', \boldsymbol{\Omega}')),$$

where the space dependence has been removed for ease of notations and where T is the temperature of the electrons, which is assumed positive and bounded. The parameter σ_s is a scattering coefficient and is assumed to depend only on the space variable \mathbf{x} . The parameter ω represents the transfer of impulsion and θ is the angle between the incoming and outgoing photon. The conservation of impulsion and energy during the collision is expressed through the following relations

$$\begin{cases} \frac{\nu}{c} \boldsymbol{\Omega} + \mathbf{p}_* = \frac{\nu'}{c} \boldsymbol{\Omega}' + \mathbf{p}'_*, \\ \nu + \frac{|\mathbf{p}_*|^2}{2m} = \nu' + \frac{|\mathbf{p}'_*|^2}{2m}, \end{cases} \quad (4.3)$$

in which \mathbf{p}_* (resp. \mathbf{p}'_*) is the impulsion of the electron before (resp. after) collision. This equation has important properties. Firstly, the kernel b satisfies the following relation, known as the detailed balance law

$$b(\mathbf{x}, \nu, \nu', \theta) e^{\nu/T} = b(\mathbf{x}, \nu', \nu, \theta) e^{\nu'/T}, \quad (4.4)$$

which can be easily verified from the definition of the kernel b (4.2). The stationary solutions of the Boltzmann equation (4.1) are given by the Bose-Einstein distributions, defined by

$$f_\mu(\nu) = \left(e^{(\mu+\nu)/T} - 1 \right)^{-1}, \quad (4.5)$$

where μ is a constant and non negative parameter. One remarks that $\nu^3 f_0(\nu) = B(\nu)$, where B is the Planck function and is the equilibrium state of the transfer equation under the assumption of local thermodynamic equilibrium (ETL) [MWM99]. Secondly, the Boltzmann equation preserves the total number of photons, whose proof directly comes from the detailed balance law (4.4). It is expressed by the following relation, where $N(f)$ is the total number of photons.

$$\frac{d}{dt} N(f) = \frac{d}{dt} \left[\int_{\mathbb{R}^3 \times \mathbb{R}^3} f d\mathbf{p} d\mathbf{x} \right] = 0. \quad (4.6)$$

We now turn to another important property of the Boltzmann equation, which is the H-theorem and express the non reversibility of the physical process [BC03, CRS08].

Theoreme 4.1 (H-Theorem). *Consider the (mathematical) entropy*

$$H(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(f \log f - (f+1) \log(f+1) + \frac{\nu}{T} f \right) d\mathbf{p} d\mathbf{x}. \quad (4.7)$$

Assume that the distribution of the photons f is non negative. Then the entropy H is monotone non increasing, i.e.

$$H'(t) \leq 0.$$

A simpler model than the Boltzmann equation (4.1) is the following Fokker-Planck type equation, named Kompaneets equation and historically derived by J. A. Kompaneets [KOM57] and widely studied in the literature [COO71, POM73, EHV98, EMV04, KAV02]

$$\partial_t f = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right]. \quad (4.8)$$

Such an equation satisfies a H-theorem, preserves the non negativity of the distribution function and preserves the total number of photons under flux conditions at $\nu = 0$ and $\nu = +\infty$ (see [EHV98] for a complete study). The main disadvantage of this equation is that by assumption, the distribution function is isotropic, which is not satisfied in the applications we have in mind.

The idea of the Fokker Planck approximation that we derive in this chapter from the Boltzmann equation (4.1) is to assume that the exchange of kinetic energy between the incoming and outgoing electrons during the Compton scattering is small compared to their temperature and that the energy of the incoming and outgoing photon is small compared to the energy of the electrons at rest, i.e. $\nu \ll mc^2$ and $\nu' \ll mc^2$. Under these assumptions, we show in this chapter that in this regime, the solution of the Boltzmann equation tends, at least formally, to the solution of the following anisotropic Kompaneets type equation

$$\begin{aligned} \frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f &= \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s}{16\pi mc^2} \nu^2 \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu f' (1+f) - \partial_\nu f (1+f') \right) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s}{16\pi mc^2} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f' + f'(1+f') \right) \right] d\mathbf{\Omega}'. \end{aligned} \quad (4.9)$$

In this equation one recognizes in the first term of the right hand side the classical Thomson scattering. The third term of the right hand side can be seen as an anisotropic Kompaneets type term. This equation can be simplified to get a simpler structure by evaluating all the terms of order $1/mc^2$ at the angle $\mathbf{\Omega}$ (isotropic assumption). It yields the following equation

$$\frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' + \sigma_s \frac{\nu^{-2}}{3mc^2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right]. \quad (4.10)$$

In this work we prove that this equation satisfies the same properties that the Boltzmann equation expressed above, namely a H-theorem, the conservation of the Boltzmann stationary states, the conservation of the total number of photons and the conservation of the non negativity of the distribution function. In a second time this equation is coupled to an equation describing

principle (lemma 4.6).

In this section we assume that the electron temperature T is positive and bounded, and does not depend on the other physical parameters. The attention of the reader is drawn on the relation that links the momentum of the photon \mathbf{p} and its direction $\boldsymbol{\Omega}$ and frequency ν . Since $\mathbf{p} = \frac{h\nu}{c}\boldsymbol{\Omega}$, one has for any density distribution function f

$$\int_{\mathbb{R}^3} f(\mathbf{p})d\mathbf{p} = \frac{h}{c} \int_{\mathbb{R}^+ \times S^2} f(\nu, \boldsymbol{\Omega})\nu^2 d\nu d\boldsymbol{\Omega}.$$

This relation will be used in this section to switch between the momentum on the one hand, and the direction and frequency on the other hand.

4.2.1 Derivation of anisotropic Kompaneets equations

We make the following assumptions

- (H1) The electron kinetic energy exchange during a collision is small compared to their temperature T , i.e.

$$\frac{1}{2m} \left| |\mathbf{p}'|^2 - |\mathbf{p}_*|^2 \right| \ll T.$$

- (H2) The energy of the incoming and outgoing photons is small compared to the energy of the electrons at rest, i.e. $\nu \ll mc^2$ and $\nu' \ll mc^2$.

From equation (4.3) the assumption (H1) is equivalent to the fact that the energy exchange between the pre and post-collision photons is small compared to the temperature of electrons T , i.e. $|\nu - \nu'| \ll T$. Let us introduce two parameters ε and $\bar{\varepsilon}$, $0 < \varepsilon, \bar{\varepsilon} < 1$ such that

$$\begin{cases} \varepsilon = \frac{1}{2mT} (|\mathbf{p}'|^2 - |\mathbf{p}_*|^2), \\ \bar{\varepsilon} = \frac{\nu}{mc^2}. \end{cases} \quad (4.12)$$

The assumption (H1) and (H2) are formally equivalent to $\varepsilon \ll 1$ and $\bar{\varepsilon} \ll 1$.

Lemma 4.2. *Under assumptions (H1)-(H2), a first order approximation of the Boltzmann equation (4.1) is*

$$\begin{aligned} \frac{1}{c} \partial_t f + \boldsymbol{\Omega} \cdot \nabla f &= \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\boldsymbol{\Omega}' \\ &+ \frac{\sigma_s}{16\pi mc^2} \nu^2 \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu f' (1 + f) - \partial_\nu f (1 + f') \right) d\boldsymbol{\Omega}' \\ &+ \frac{\sigma_s}{16\pi mc^2} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f' + f (1 + f') \right) \right] d\boldsymbol{\Omega}'. \end{aligned} \quad (4.13)$$

Proof. The proof is composed of two main steps. In a first one, the distribution functions $f(\nu', \boldsymbol{\Omega}')$ in the transfer term $q(f)$ of the Boltzmann equation (4.1) are expanded around the frequency $f(\nu, \boldsymbol{\Omega}')$ using formal Taylor expansion, justified by the assumption (H1). The integral over the frequency of all the outgoing photons $\nu' \in \mathbb{R}^+$ in the collision operator of the Boltzmann equation (4.1) will only relies on integrals of the kernel b multiplied by power of $\nu - \nu'$, which will be performed in a second step.

First step: Fokker-Planck expansion

In order to simplify the following study, we introduce the dimensionless frequency $\bar{\nu} = \frac{\nu}{T}$ and we rescale the time and the space in such a way that $\bar{t} = Tt$ and $\bar{\mathbf{x}} = T\mathbf{x}$. Denoting $g(\bar{\nu}) = f(\bar{\nu}T)$ and dropping the bars, equation (4.1) writes

$$\frac{1}{c}\partial_t g + \boldsymbol{\Omega} \cdot \nabla g = \int_{\mathbb{R}^+ \times S^2} \frac{\nu'}{\nu} b(T\nu, T\nu', \theta) q(g) e^{\nu'} d\nu' d\boldsymbol{\Omega}'. \quad (4.14)$$

This change of variables together with the assumption (H1) and the definition of ε yield $\nu' = \nu + T^{-1}\varepsilon$. We perform a formal Taylor expansion of the functions $g(\nu')$ around the frequency ν

$$g(\nu', \boldsymbol{\Omega}') = g(\nu, \boldsymbol{\Omega}') + (\nu' - \nu)\partial_\nu g(\nu, \boldsymbol{\Omega}') + \frac{1}{2}(\nu' - \nu)^2 \partial_\nu^2 g(\nu, \boldsymbol{\Omega}') + \mathcal{O}(\varepsilon^3).$$

In the same way, one writes

$$e^{-\nu'} = e^{-\nu} \left(1 + (\nu - \nu') + \frac{1}{2}(\nu' - \nu)^2 \right) + \mathcal{O}(\varepsilon^3).$$

It yields, omitting the dependence in ν in the notation for simplicity and denoting simply g' for $g(\boldsymbol{\Omega}')$,

$$q(g) = e^{-\nu} \left\{ g' - g + (\nu' - \nu) \left(\partial_\nu g' + g(1 + g') \right) + \frac{1}{2}(\nu' - \nu)^2 \left(\partial_\nu^2 g' + 2g\partial_\nu g' - g(1 + g') \right) \right\} + \mathcal{O}(\varepsilon^3).$$

An important point in the derivation of the model is to write this equation as

$$q(g) = e^{-\nu} \left\{ g' - g + \frac{1}{2}(\nu' - \nu)^2 \left(\partial_\nu g'(1 + g) - \partial_\nu g(1 + g') \right) + \frac{1}{2}(\nu' - \nu)^2 \partial_\nu \left(\partial_\nu g' + g(1 + g') \right) + \left((\nu' - \nu) - \frac{1}{2}(\nu' - \nu)^2 \right) \left(\partial_\nu g' + g(1 + g') \right) \right\} + \mathcal{O}(\varepsilon^3).$$

Introducing this expression of $q(g)$ in equation (4.14), one sees that the derivation of the model relies on the computation of the following integrals of the kernel b multiplied by power of $\nu - \nu'$

$$\begin{cases} I_1 = \int_{\mathbb{R}^+} b(T\nu, T\nu', \theta) \nu' e^{\nu'} d\nu', \\ I_2 = \frac{1}{2} \int_{\mathbb{R}^+} (\nu' - \nu)^2 b(T\nu, T\nu', \theta) \nu' e^{\nu'} d\nu', \\ I_3 = \int_{\mathbb{R}^+} \left((\nu' - \nu) - \frac{1}{2}(\nu' - \nu)^2 \right) b(T\nu, T\nu', \theta) \nu' e^{\nu'} d\nu'. \end{cases}$$

If moreover we introduce $R = \partial_\nu g' + g(1 + g')$, equation (4.14) writes

$$\begin{aligned} \frac{1}{c}\partial_t g + \boldsymbol{\Omega} \cdot \nabla g &= \int_{S^2} \frac{e^{-\nu}}{\nu} \left\{ (g' - g) I_1 + \left(\partial_\nu g'(1 + g) - \partial_\nu g(1 + g') \right) I_2 \right\} d\boldsymbol{\Omega}' \\ &+ \int_{S^2} \frac{e^{-\nu}}{\nu} \left(I_2 \partial_\nu R + I_3 R \right) d\boldsymbol{\Omega}' + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.15)$$

At this point, the closure of equation (4.15) relies on the computation of three unknowns, namely the integrals I_1 , I_2 and I_3 , but the integral I_3 can not be computed analytically due to the unsigned term $(\nu' - \nu)$. The difficulty is removed by the algebraic nature of the term $I_2\partial_\nu R + I_3R$

$$\frac{e^{-\nu}}{\nu} \left(I_2\partial_\nu R + I_3R \right) = \nu^{-2}d(\nu) \frac{\partial}{\partial\nu} \left[\alpha(\nu) \left(\partial_\nu g' + g(1 + g') \right) \right],$$

with

$$\begin{cases} \alpha(\nu) = \exp \left(\int_0^\nu I_3/I_2 ds \right), \\ d(\nu) = \nu e^{-\nu} I_2 \exp \left(- \int_0^\nu I_3/I_2 ds \right). \end{cases}$$

Equation (4.15) now writes

$$\begin{aligned} \frac{1}{c} \partial_t g + \mathbf{\Omega} \cdot \nabla g &= \int_{S^2} \frac{e^{-\nu}}{\nu} \left\{ (g' - g) I_1 + \left(\partial_\nu g' (1 + g) - \partial_\nu g (1 + g') \right) I_2 \right\} d\mathbf{\Omega}' \\ &+ \int_{S^2} \nu^{-2} d(\nu) \frac{\partial}{\partial\nu} \left[\alpha(\nu) \left(\partial_\nu g' + g(1 + g') \right) \right] d\mathbf{\Omega}' + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.16)$$

Second step: computation of the integrals

To ensure the conservation of the total number of photons $N(f) = \int_{x,\nu,\mathbf{\Omega}} f \nu^2$, being regardless on the contribution of the right member of the first line, it is sufficient to impose d constant. It yields in particular $\alpha = d^{-1} \nu e^{-\nu} I_2$, and the closure of equation (4.16) now only relies on the computation of I_1 and I_2 . To this end we use the assumption (H1) to simplify the kernel b . Indeed, the definition of ω yields (taking care that we performed the change of variables $\nu \rightarrow \nu/T$)

$$|\omega|^2 = \frac{\nu^2}{c^2} T^2 + \frac{\nu'^2}{c^2} T^2 - 2 \frac{\nu\nu'}{c^2} T^2 \cos \theta = 2 \frac{\nu^2}{c^2} T^2 (1 - \cos \theta) + \mathcal{O}(\varepsilon).$$

Let us introduce $\nu_0 = \nu - \frac{T}{mc^2} \nu^2 (1 - \cos \theta)$, from which $|\omega|^2$ can be written $|\omega|^2 = 2mT(\nu - \nu_0) + \mathcal{O}(\varepsilon)$. It yields in particular the following expression of A

$$A = \frac{\nu'T}{c} - \frac{\nu T}{c} + \frac{|\omega|^2}{2mc} = \frac{\nu T}{c} - \frac{\nu'T}{c} - \frac{T}{c} (\nu - \nu_0) + \mathcal{O}(\varepsilon) = \frac{T}{c} (\nu_0 - \nu') + \mathcal{O}(\varepsilon).$$

One formally writes the kernel b as

$$b(T\nu, T\nu') = \sigma_s (1 + \cos^2 \theta) \frac{c}{T} \frac{e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}}}{\nu \sqrt{2(1 - \cos \theta)}} + \mathcal{O}(\varepsilon).$$

We are now able to find an approximation of the integrals I_1 and I_2 . The integral I_2 has been computed by Escobedo et al [EMV03] in the following way. First, the previous expression of the kernel formally gives

$$\begin{aligned} I_2 &= \frac{1}{2} \int_{\mathbb{R}^+} (\nu' - \nu)^2 b(T\nu, T\nu') \nu' d\nu' \\ &= (1 + \cos^2 \theta) \frac{c\sigma_s}{2T\nu \sqrt{2(1 - \cos \theta)}} \int_{\mathbb{R}^+} (\nu' - \nu)^2 e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} \nu' d\nu' + \mathcal{O}(\varepsilon). \end{aligned}$$

The assumption (H2) and the definition of $\bar{\varepsilon}$ (4.12) yield $\nu_0 = \nu + \mathcal{O}(\bar{\varepsilon})$, and in particular

$$\begin{aligned} I_2 &= (1 + \cos^2 \theta) \frac{c\sigma_s}{2T\nu\sqrt{2(1 - \cos \theta)}} \int_{\mathbb{R}^+} (\nu' - \nu_0)^2 e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} \nu' d\nu' + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}) \\ &= c\sigma_s (1 + \cos^2 \theta) \frac{|\nu - \nu_0|^{\frac{3}{2}}}{2T\nu\sqrt{2(1 - \cos \theta)}} \int_{\mathbb{R}^+} \frac{1}{\sqrt{|\nu - \nu_0|}} \frac{(\nu' - \nu_0)^2}{|\nu - \nu_0|} e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} \nu' d\nu' + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}). \end{aligned}$$

Denoting $h(x) = x^2 e^{-x^2}$, we have an expression of the form $\frac{1}{\gamma} h(x/\gamma)$, with $\gamma = \sqrt{|\nu - \nu_0|}$. One thus has

$$\lim_{\nu \rightarrow \nu_0} \frac{1}{|\nu - \nu_0|} \frac{(\nu' - \nu_0)^2}{|\nu - \nu_0|} e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} = C_1 \delta_{\nu' = \nu_0},$$

where $C_1 = \int_{\mathbb{R}} r^2 e^{-r^2} dr$. This yields, together with the definition of ν_0 ,

$$I_2 = \sigma_s (1 + \cos^2 \theta) C_1 \frac{\sqrt{T}}{c^2 m^{\frac{3}{2}}} \frac{\nu^3}{2\sqrt{2}} (1 - \cos \theta) + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}),$$

and thus

$$\alpha = \sigma_s (1 + \cos^2 \theta) C_1 \frac{\sqrt{T}}{c^2 m^{\frac{3}{2}}} d^{-1} \frac{\nu^4}{2\sqrt{2}} (1 - \cos \theta) + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}).$$

There just remains to compute the second integral I_1 . Using the same idea, one has

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^+} b(T\nu, T\nu') \nu' e^{\nu'} d\nu' \\ &= (1 + \cos^2 \theta) \frac{c\sigma_s}{T\nu\sqrt{2(1 - \cos \theta)}} \int_{\mathbb{R}^+} e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} \nu' e^{\nu'} d\nu' + \mathcal{O}(\varepsilon) \\ &= c\sigma_s (1 + \cos^2 \theta) \frac{\sqrt{|\nu - \nu_0|}}{T\nu\sqrt{2(1 - \cos \theta)}} \int_{\mathbb{R}^+} \frac{1}{\sqrt{|\nu - \nu_0|}} e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} \nu' e^{\nu'} d\nu' + \mathcal{O}(\varepsilon). \end{aligned}$$

Again, one has

$$\lim_{\nu \rightarrow \nu_0} \frac{1}{\sqrt{|\nu - \nu_0|}} e^{-\frac{|\nu' - \nu_0|^2}{4|\nu - \nu_0|}} = C_2 \delta_{\nu' = \nu_0},$$

where $C_2 = \int_{\mathbb{R}} e^{-r^2} dr = 2C_1$. This yields

$$I_1 = c\sigma_s (1 + \cos^2 \theta) \frac{C_1}{\sqrt{mT}} \frac{2}{\sqrt{2}} \nu e^{\nu} + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}).$$

We thus obtain

$$\begin{aligned} \frac{1}{c} \partial_t g + \mathbf{\Omega} \cdot \nabla g &= \sigma_s c T^{\frac{3}{2}} C_1 \frac{2}{\sqrt{2m}} \int_{S^2} (1 + \cos^2 \theta) (g' - g) d\mathbf{\Omega}' + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}) \\ &\quad + \sigma_s c \frac{T^{\frac{5}{2}}}{c^2 m^{\frac{3}{2}}} C_1 \frac{\nu^2}{2\sqrt{2}} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu g' (1 + g) - \partial_\nu g (1 + g') \right) d\mathbf{\Omega}' \\ &\quad + \sigma_s c \frac{T^{\frac{5}{2}}}{c^2 m^{\frac{3}{2}}} C_1 \frac{1}{2\sqrt{2}} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(\partial_\nu g' + g(1 + g') \right) \right] d\mathbf{\Omega}'. \end{aligned}$$

Performing the change of time variable $t \rightarrow CT^{\frac{3}{2}} \frac{8\pi C_1}{\sqrt{2m}} t$, the model writes

$$\begin{aligned} \frac{1}{c} \partial_t g + \mathbf{\Omega} \cdot \nabla g &= \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (g' - g) d\mathbf{\Omega}' + \mathcal{O}(\varepsilon) + \mathcal{O}(\bar{\varepsilon}) \\ &+ \frac{\sigma_s T}{16\pi m c^2} \nu^2 \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu g' (1 + g) - \partial_\nu g (1 + g') \right) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s T}{16\pi m c^2} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(\partial_\nu g' + g (1 + g') \right) \right] d\mathbf{\Omega}'. \end{aligned} \quad (4.17)$$

We want to turn back to the equation on the distribution function f . We remind that $g(\nu) = f(\nu T)$. Making the change of variable $\bar{\nu} = \nu T$ yields, dropping the bars for ease of notations and all the $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\bar{\varepsilon})$ terms,

$$\begin{aligned} \frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f &= \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s}{16\pi m c^2} \nu^2 \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu f' (1 + f) - \partial_\nu f (1 + f') \right) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s}{16\pi m c^2} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f' + f (1 + f') \right) \right] d\mathbf{\Omega}'. \end{aligned}$$

which is the announced result. \square

The equation (4.13) is the sum of the classical Thomson scattering plus perturbations of order 1 with respect to ν/mc^2 . It is the same model (up to a renormalization of mc^2) than the one obtained in a different way in [BPR69, POM73, FKM85, PL97]. This can be seen by differentiating all the terms in the Kompaneets type term. This model preserves the total number of photons and the stationary states of the Boltzmann equation. Moreover, a generalization of the derivation of such models for more complicated differential cross section, and in particular for the Klein-Nishina cross section [EV55], could be obtained without any particular difficulties.

Let us prove the properties of this model announced above. The conservation of the stationary states of the Boltzmann equation is trivial. Indeed, the Bose-Einstein distributions are defined by $f_\mu(\nu) = (e^{\frac{\mu+\nu}{T}} - 1)^{-1}$. Since these distributions are isotropic, the first and second terms of the right member of (4.13) vanish, and so do the Kompaneets type term since these distributions satisfy $T \partial_\nu f + f(1 + f) = 0$. The conservation of the total number of photons is proved in the following

Lemma 4.3. *The equation (4.13) preserves the total number of photons, i.e.*

$$\frac{d}{dt} N(t) = \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} = 0.$$

Proof. Multiplying equation (4.13) by ν^2 and integrating over $\mathbb{R}^+ \times S^2$, one finds

$$\begin{aligned} \frac{d}{dt} N(t) &= \frac{\sigma_s}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+} \nu^2 \int_{S^2} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' d\mathbf{\Omega} d\nu d\mathbf{x} \\ &+ \frac{\sigma_s}{16\pi m c^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+} \nu^4 \int_{S^2} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) \left(\partial_\nu f' (1 + f) - \partial_\nu f (1 + f') \right) d\mathbf{\Omega}' d\mathbf{\Omega} d\nu d\mathbf{x}. \end{aligned} \quad (4.18)$$

Since $\cos \theta = \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}'$, a change of variable $\boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}'$ yields

$$\int_{S^2} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\boldsymbol{\Omega}' d\boldsymbol{\Omega} = \int_{S^2} \int_{S^2} f' \left((1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2) - (1 + (\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})^2) \right) d\boldsymbol{\Omega}' d\boldsymbol{\Omega} = 0.$$

The same manipulation in the second term of the right member of (4.18) gives the result. \square

4.2.2 A simplified model

Despite equation (4.13) is indeed interesting, since it gives a first description of the anisotropic part of the distribution function, we chose to simplify this equation to obtain a simpler model still containing anisotropic terms. The resulting model satisfies a H-theorem, what we do not succeed to prove for the equation (4.13). Indeed, the attempt to control the entropy defined in (4.7) for equation (4.13) leads to a sum of several terms, some of them being unsigned.

The simplification is the following: the terms of order 1 with respect to ν/mc^2 in (4.13) are assumed to be isotropic. Since $\frac{1}{16\pi} \int_{S^2} (1 + \cos^2 \theta) (1 - \cos \theta) d\boldsymbol{\Omega}' = \frac{1}{3}$, it yields the following model

$$\frac{1}{c} \partial_t f + \boldsymbol{\Omega} \cdot \nabla f = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\boldsymbol{\Omega}' + \sigma_s \frac{\nu^{-2}}{3mc^2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1 + f) \right) \right]. \quad (4.19)$$

This equation is the sum of the classical Thomson scattering plus a Kompaneets term. In this part we prove several theoretical results for this equation, such as the conservation of the non negativity of the distribution function (lemma 4.4) and a H-theorem (lemma 4.5). With these results, equation (4.19) owns all the properties of the Boltzmann equation expressed in the introduction, properties that we wanted to keep while deriving simplified models.

The end of this section deals with the proof of several mathematical properties of this model. The most important one is the proof of a H-theorem, which ensures the growth of the physical entropy of the model and thus the non reversibility of the process. We need to make the following assumptions

- $(\overline{H1})$ Initial conditions: the distribution function is non negative, i.e. $\forall(\mathbf{x}, \nu, \boldsymbol{\Omega}), f(0, \mathbf{x}, \nu, \boldsymbol{\Omega}) \geq 0$.
- $(\overline{H2})$ The scattering coefficient σ_s is non negative.

The equation (4.19) inherits of some of the properties of the equation (4.13). In particular it preserves the total number of photons and the stationary states of the Boltzmann equation (4.1). Since the H-theorem uses the logarithm of the solution of (4.19), one needs to prove that it remains non negative. This is done in the

Lemma 4.4 (Non negativity). *Under assumptions $(\overline{H1}) - (\overline{H2})$, equation (4.19) preserves the non negativity of the distribution function, i.e. $f(0, \mathbf{x}, \nu, \boldsymbol{\Omega}) \geq 0 \implies f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) \geq 0$, for all $0 < t < T$.*

Proof. We mainly follow the proof of Carrillo et al [CRS08], that is to introduce a regularized monotone increasing approximation sgn_ε of the sgn function and to define a regularized approximation $|f|_\varepsilon$ of the absolute value function $|f|$ via the primitive of $\text{sgn}_\varepsilon(f)$:

$$|f|_\varepsilon = \int_0^f \text{sgn}_\varepsilon(g) dg. \quad (4.20)$$

We also define a regularized approximation of the negative part of a function f as $f_\varepsilon^- = (|f|_\varepsilon - f)/2$. Multiplying equation (4.19) by $\nu^2 \text{sgn}_\varepsilon(f)$ and integrating over $\mathbb{R}_x^3 \times \mathbb{R}_\nu^+ \times S^2$, one has

$$\begin{aligned} \frac{1}{c} \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} |f|_\varepsilon \nu^2 d\nu d\Omega d\mathbf{x} &= \frac{\sigma_s}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \text{sgn}_\varepsilon(f) (1 + \cos^2 \theta) (f' - f) \nu^2 d\Omega' d\Omega d\nu d\mathbf{x} \\ &+ \frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \text{sgn}_\varepsilon(f) \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right] d\nu d\Omega d\mathbf{x}, \end{aligned}$$

since the transport term $\Omega \cdot \nabla f$ vanishes. The conservation of the total number of photons

$$\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f \nu^2 d\nu d\Omega d\mathbf{x} = 0 = \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (|f|_\varepsilon - 2f_\varepsilon^-) \nu^2 d\nu d\Omega d\mathbf{x}$$

leads in particular to

$$\begin{aligned} \frac{2}{c} \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f_\varepsilon^- \nu^2 d\nu d\Omega d\mathbf{x} &= \frac{\sigma_s}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \text{sgn}_\varepsilon(f) (1 + \cos^2 \theta) (f' - f) \nu^2 d\Omega' d\Omega d\nu d\mathbf{x} \\ &+ \frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \text{sgn}_\varepsilon(f) \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right] d\Omega d\nu d\mathbf{x}. \end{aligned}$$

Let us denote P_1 the first term of the right hand side and P_2 the second one. The invariance by change of variable $\Omega \rightarrow \Omega'$ of $\cos \theta$ yields

$$P_1 = -\frac{\sigma_s}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (1 + \cos^2 \theta) (f' - f) \left(\text{sgn}_\varepsilon(f') - \text{sgn}_\varepsilon(f) \right) \nu^2 d\Omega' d\Omega d\nu d\mathbf{x}.$$

The non decreasing monotonicity of the function $\text{sgn}_\varepsilon(f)$ thus yields $P_1 \leq 0$. For the term P_2 , one has, using an integration by parts

$$P_2 = -\frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \nu^4 \text{sgn}'_\varepsilon(f) \left(T |\partial_\nu f|^2 + f(1+f) \partial_\nu f \right) d\Omega d\nu d\mathbf{x}.$$

As pointed out by Carrillo et al, one has $\text{sgn}'_\varepsilon(f) f \partial_\nu f = \partial_\nu (f \text{sgn}_\varepsilon(f) - |f|_\varepsilon)$ and $\text{sgn}'_\varepsilon(f) f^2 \partial_\nu f = \partial_\nu (f^2 \text{sgn}_\varepsilon(f) - f|f|_\varepsilon)$. Passing to the limit as $\varepsilon \rightarrow 0$, it yields $P_2 \leq 0$ and finally

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f^-(t) \nu^2 d\nu d\mathbf{x} d\Omega \leq \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f^-(0) \nu^2 d\nu d\mathbf{x} d\Omega = 0,$$

which ends the proof. □

We now turn to the main result:

Theorem 4.5 (H-Theorem). *Assume that assumptions $(\overline{H1}) - (\overline{H2})$ are satisfied and consider the function $\phi(f) = f \log(f) - (f+1) \log(f+1) + \frac{\nu}{T} f$. The following inequality holds*

$$H'(t) = \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \phi(f) \nu^2 d\nu d\Omega d\mathbf{x} \leq 0.$$

The function ϕ is the sum of the (mathematical) entropy of the photons $f \log(f) - (f+1) \log(f+1)$ and a term $\frac{\nu}{T} f$ which is needed to prove the monotone non increasing behavior of H . In the next section, in which we couple equation (4.19) with an equation for the electrons, it will become clear that this term is in fact the entropy of the electrons.

Proof. Let us decompose equation (4.19) as

$$\frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f = P_{sym} + P_{komp},$$

where

$$\begin{cases} P_{sym} = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' \\ P_{komp} = \frac{\sigma_s}{3} \frac{\nu^{-2}}{mc^2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right]. \end{cases}$$

It yields, by definition of the entropy H

$$H'(t) = c \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\log \left(\frac{f}{f+1} \right) + \frac{\nu}{T} \right) \left(P_{sym} + P_{komp} - \mathbf{\Omega} \cdot \nabla f \right) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x}.$$

We thus have to estimate three terms. For the first one, one has by definition of P_{sym}

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} P_{sym} \phi'(f) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} = -\frac{\sigma_s}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (1 + \cos^2 \theta) (f' - f) \left(\log \left(1 + \frac{1}{f} \right) + \frac{\nu}{T} \right) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x}.$$

The same arguments than for the proof of lemma 4.3 yield

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (1 + \cos^2 \theta) (f' - f) \frac{\nu^3}{T} d\nu d\mathbf{\Omega} d\mathbf{x} = 0.$$

Using the invariance by change of variable $\mathbf{\Omega} \rightarrow \mathbf{\Omega}'$ of $\cos \theta = \mathbf{\Omega} \cdot \mathbf{\Omega}'$, one can write the remaining term as

$$\begin{aligned} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} P_{sym} \phi'(f) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} &= -\frac{\sigma_s}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (1 + \cos^2 \theta) f' f \left(\frac{1}{f'} - \frac{1}{f} \right) \\ &\quad \times \left\{ \log \left(1 + \frac{1}{f'} \right) - \log \left(1 + \frac{1}{f} \right) \right\} \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x}. \end{aligned}$$

The monotone increasing behavior of the function $X \mapsto \log(1 + X)$ and the non negativity of the distribution function (lemma 4.4) yield

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} P_{sym} \phi'(f) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} \leq 0.$$

We now turn to the Kompaneets type term P_{komp} . This term has already been studied for a slightly different Fokker-Planck equation in [CRS08] (see also [BC03]). One has, using an integration by parts,

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} P_{komp} \phi'(f) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} = -\frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\frac{\partial_\nu f}{f(f+1)} + \frac{1}{T} \right) \nu^4 \left(T \partial_\nu f + f(1+f) \right) d\nu d\mathbf{\Omega} d\mathbf{x}.$$

It thus yields

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} P_{komp} \phi'(f) \nu^2 d\nu d\mathbf{\Omega} d\mathbf{x} = -\frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \frac{T}{f(f+1)} \nu^4 \left(\partial_\nu f + \frac{f(1+f)}{T} \right)^2 d\nu d\mathbf{\Omega} d\mathbf{x}.$$

Once again, the non negativity of the distribution function gives $\int P_{komp} \phi'(f) \nu^2 d\nu d\Omega \leq 0$. There remains the term coming from the photon transport $\mathbf{\Omega} \cdot \nabla f$. Let us introduce P_{Trans} defined by

$$P_{Trans} = -c \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \mathbf{\Omega} \cdot \nabla f \left(\log \left(\frac{f}{f+1} \right) + \frac{\nu}{T} \right) \nu^2 d\nu d\Omega d\mathbf{x}.$$

One easily sees that $\int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \mathbf{\Omega} \cdot \nabla f \nu^3 d\nu d\Omega d\mathbf{x} = 0$. For the second one, an integration by parts gives

$$P_{Trans} = c \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f \mathbf{\Omega} \cdot \nabla \left[\log \left(\frac{f}{f+1} \right) \right] \nu^2 d\nu d\Omega d\mathbf{x}.$$

The relation $f \nabla (\log(f/(f+1))) = \nabla (\log(f+1))$ finally yields $P_{Trans} = 0$, which concludes the proof. \square

Finally, we prove a comparison result

Lemma 4.6 (Comparison principle). *Assume that f and g are two solutions of (4.19) that satisfied the assumptions $\overline{H1-H2}$ and $f(t=0) \geq g(t=0)$. Then for all $0 < t < T$, $f(t) \geq g(t)$.*

Proof. Once again, we follow [CRS08]. Since f and g are both solutions of (4.19), one can write the equation satisfies by $h = f - g$:

$$\frac{1}{c} \partial_t h + \mathbf{\Omega} \cdot \nabla h = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (h' - h) d\mathbf{\Omega}' + \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu h + h + h(f+g) \right) \right],$$

Multiplying this equation by $\text{sgn}_\varepsilon(h) \nu^2$ and using the conservation of the total number of photons, one gets, with the same notations than in the previous lemma (positivity, lemma 4.4),

$$\begin{aligned} \frac{1}{c} \frac{d}{dt} \int h_\varepsilon^- \nu^2 d\nu dx d\Omega &= \frac{\sigma_s}{4\pi} \int \text{sgn}_\varepsilon(h) \nu^2 (1 + \cos^2 \theta) (h' - h) d\mathbf{\Omega}' \\ &+ \frac{\sigma_s}{3mc^2} \int \text{sgn}_\varepsilon(h) \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu h + h + h(f+g) \right) \right]. \end{aligned}$$

The relation $\text{sgn}'_\varepsilon(h) h (f+g) \partial_\nu h = (f+g) \partial_\nu (h \text{sgn}_\varepsilon(h) - |h|_\varepsilon)$ together with the previous procedure (lemma 4.4) gives

$$\int h_\varepsilon^-(t) \nu^2 d\nu dx d\Omega \leq \int h_\varepsilon^-(0) \nu^2 d\nu dx d\Omega = 0$$

thanks to the initial conditions, and this gives the announced result. \square

4.3 Coupling electrons photons

In this section we couple the equation obtained in the previous section (equation (4.19)) with an equation describing the evolution of the temperature of the electrons. The system is the following

$$\left\{ \begin{aligned} \frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f &= \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' + \sigma_a(\nu) \left(f_0(\nu, T) - f \right) \\ &+ \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1+f) \right) \right], \\ C_V \partial_t T - \nabla \cdot (D \nabla T) &= -\frac{c}{4\pi} \int_{\mathbb{R}^+ \times S^2} \sigma_a(\nu) \left(B(\nu, T) - \nu^3 f \right) d\nu d\Omega \\ &+ \frac{1}{4\pi} \frac{\sigma_s}{3mc} \int_{\mathbb{R}^+ \times S^2} \nu^4 \left(T \partial_\nu f + f(1+f) \right) d\nu d\Omega, \end{aligned} \right. \quad (4.21)$$

where B is the planck function $B(\nu, T) = \nu^3(e^{\nu/T} - 1)^{-1}$, $f_0(\nu, T) = (e^{\nu/T} - 1)^{-1}$ is the Bose-Einstein distribution (4.5) for $\mu = 0$, C_V is a non negative constant coefficient, D is a non negative diffusion coefficient and σ_a is the emission absorption coefficient. It is important to notice that the coupling term $\sigma_a(\nu)(f_0(\nu, T) - f)$ does not change the stationary states but selected them, in the sense that not all the Bose-Einstein distributions are solutions but only the ones with $\mu = 0$. The choice of this coupling also leads in particular to

$$\frac{d}{dt} \left(\frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} f \nu^3 d\nu d\Omega d\mathbf{x} + C_V \int_{\mathbb{R}^3} T d\mathbf{x} \right) = 0, \quad (4.22)$$

which expressed the conservation of the total energy. Finally, the total number of photons is trivially preserved.

One of the aim of this section is to compare the system (4.21) with respect to the classical radiative transfer model in which the Kompaneets terms are cancelled, and in particular the convergence of the radiative temperature to the electron temperature. To this end we study the equilibrium and non-equilibrium diffusion regimes.

This section is organized as follows. A first part (part 4.3.1) is entirely dedicated to the proof of a H-theorem. This result is important to validate the model, in the sense that it ensures the non reversibility of the system, which is a basic physical principle, satisfied by the system (4.21). In a second part (part 4.3.2), we study the so-called equilibrium and non-equilibrium diffusion regimes for the system (4.21). We show that the Compton scattering does not change the equilibrium diffusion regime (lemma 4.8), since the obtained diffusion equation is the same than in the case of classical Thomson scattering, but that the Compton scattering changes the non-equilibrium diffusion regime (lemma 4.9). Indeed, in this regime, the obtained system is a diffusion system in which the Kompaneets terms $\frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} (\nu^4 (T \partial_\nu f + f(1+f)))$ remains.

4.3.1 H-Theorem

In this part we prove a H-theorem for the coupled system (4.21). What is interesting in this result is that it becomes clear that the term $\frac{\nu}{T} f$, needed in the case of the uncoupled anisotropic Kompaneets equation (4.19) to prove a H-theorem was in fact the entropy of the electrons.

Theoreme 4.7. *Consider the total physical entropy*

$$H(t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left((f+1) \log(f+1) - f \log f \right) \nu^2 d\nu d\Omega d\mathbf{x} + C_V \int_{\mathbb{R}^3} \log T d\mathbf{x}, \quad (4.23)$$

which is the sum of the entropy of the photons and the entropy of the electrons. Assume that the distribution of the photons f and the temperature of the electrons T are non negative functions. Then H satisfies $H'(t) \geq 0$.

Proof. Let us write the system (4.21) as

$$\begin{cases} \frac{1}{c} \partial_t f + \mathbf{\Omega} \cdot \nabla f = T_{aniso} + T_{komp} + T_{e,a} \\ C_V \partial_t T - \nabla \cdot (D \nabla T) = -\frac{c}{4\pi} \int_{\mathbb{R}^+ \times S^2} \left(T_{komp} + T_{e,a} \right) \nu^3 d\nu d\Omega, \end{cases} \quad (4.24)$$

where

$$\begin{cases} T_{aniso} = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\Omega', \\ T_{komp} = \sigma_a(\nu) (f_0(\nu, T) - f), \\ T_{e,a} = \frac{1}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1 + f) \right) \right]. \end{cases} \quad (4.25)$$

By definition of H (4.23), one has

$$H'(t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \log \left(\frac{f+1}{f} \right) \frac{\partial f}{\partial t} \nu^2 d\nu d\Omega d\mathbf{x} + C_V \int_{\mathbb{R}^3} \frac{1}{T} \frac{\partial T}{\partial t} d\mathbf{x} \quad (4.26)$$

Using the system (4.24), one writes equation (4.26) as $H'(t) = S_p + A + K + EA$, with

$$\begin{cases} S_p = -\frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \log \left(\frac{f+1}{f} \right) \nu^2 \boldsymbol{\Omega} \cdot \nabla f d\nu d\Omega d\mathbf{x} + \int_{\mathbb{R}^3} \nabla \cdot (D \nabla T) \frac{1}{T} d\mathbf{x}, \\ A = \frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \log \left(\frac{f+1}{f} \right) \nu^2 T_{aniso} d\nu d\Omega d\mathbf{x}, \\ K = \frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\log \left(\frac{f+1}{f} \right) - \frac{\nu}{T} \right) \nu^2 T_{Komp} d\nu d\Omega d\mathbf{x}, \\ EA = \frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\log \left(\frac{f+1}{f} \right) - \frac{\nu}{T} \right) \nu^2 T_{e,a} d\nu d\Omega d\mathbf{x}. \end{cases}$$

Let us prove that each of these terms are non negative. For the first one, the relation $\nabla f / (f+1) = \nabla \log(f+1)$ together with an integration by parts yield

$$E_{S_p} = -\frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \frac{\nabla f}{f+1} \cdot \boldsymbol{\Omega} \nu^2 d\nu d\Omega d\mathbf{x} + \int_{\mathbb{R}^3} D \frac{|\nabla T|^2}{T^2} d\mathbf{x} = \int_{\mathbb{R}^3} D \frac{|\nabla T|^2}{T^2} d\mathbf{x} \geq 0.$$

The second one, which corresponds to the anisotropic part, has already been signed in the previous part (taking care that here we consider the physical entropy). A symmetrization of this term leads to

$$A = \frac{c\sigma_s}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} (1 + \cos^2 \theta) f f' \left\{ \log \left(1 + \frac{1}{f} \right) - \log \left(1 + \frac{1}{f'} \right) \right\} \left(\frac{1}{f} - \frac{1}{f'} \right) \nu^2 d\nu d\Omega d\mathbf{x} \geq 0.$$

We now turn to the Kompaneets type term. The idea is still the same as for the uncoupled Kompaneets equation. By definition, one has

$$K = \frac{1}{4\pi} \frac{1}{3mc} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\log \left(\frac{f+1}{f} \right) - \frac{\nu}{T} \right) \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1 + f) \right) \right] d\nu d\Omega d\mathbf{x}.$$

An integration by parts yields

$$\begin{aligned} K &= \frac{1}{4\pi} \frac{1}{3mc} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\frac{\partial_\nu f}{f(f+1)} + \frac{1}{T} \right) \nu^4 \left(T \partial_\nu f + f(1 + f) \right) d\nu d\Omega d\mathbf{x} \\ &= \frac{1}{4\pi} \frac{1}{3mc} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \frac{(T \partial_\nu f + f(1 + f))^2}{T f(f+1)} d\nu d\Omega d\mathbf{x} \geq 0. \end{aligned}$$

There just remains to treat the emission absorption term EA. By definition, one has

$$EA = \frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(\log \left(\frac{f+1}{f} \right) - \frac{\nu}{T} \right) \nu^{-1} \sigma_a(\nu) (B(\nu, T) - I) d\nu d\Omega d\mathbf{x},$$

where by definition $I = \nu^3 f$. This term has already been studied in [BD04]. Introducing the monotone non decreasing function $g(x) = \log\left(\frac{x}{\nu^3}\left(\frac{x}{\nu^3} + 1\right)^{-1}\right)$, one has $-\frac{\nu}{T} = g(B(\nu, T))$. EA thus writes

$$EA = \frac{c}{4\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^+ \times S^2} \left(g(B(\nu, T)) - g(I) \right) \nu^{-1} \sigma_a(\nu) (B(\nu, T) - I) d\nu d\Omega d\mathbf{x} \geq 0,$$

and the proof is completed. \square

4.3.2 Diffusion approximations

In this part are studied the equilibrium [BGP87, BGPS88] and non-equilibrium diffusion regimes [BD04, GLG05]. As explained in chapter 1, the main idea of these approximations is to remove the angular dependence by assuming different scaling on the physical parameters. We assume in the whole part that σ_a does not depend on the frequency ν (grey assumption). We show in particular that the equilibrium diffusion regime for the system (4.21) is the same that in the case of classical isotropic scattering (no Kompaneets terms). On the contrary, the non-equilibrium diffusion regime exhibits new terms due to the Compton scattering.

Equilibrium diffusion

The equilibrium regime is a diffusion approximation of the previous coupled system (4.21), in which the emission absorption is assumed to be large compared to the scattering phenomena, i.e. $\sigma_s/\sigma_a \ll 1$ and the velocity of light c is very large. We thus introduce new variables

$$\begin{cases} c = \varepsilon^{-1} \bar{c}, \\ \sigma_a = \varepsilon^{-1} \bar{\sigma}_a, \\ \sigma_s = \varepsilon \bar{\sigma}_s, \end{cases}$$

in which ε is a small parameter $0 < \varepsilon < 1$. It yields, dropping the bars for ease of notations

$$\begin{cases} \frac{1}{c} \partial_t f^\varepsilon + \frac{1}{\varepsilon} \mathbf{\Omega} \cdot \nabla f^\varepsilon = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f'^\varepsilon - f^\varepsilon) d\mathbf{\Omega}' + \frac{\sigma_a}{\varepsilon^2} \left(f_0(\nu, T^\varepsilon) - f^\varepsilon \right) \\ \quad + \frac{\varepsilon^2 \sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T^\varepsilon \partial_\nu f^\varepsilon + f^\varepsilon (1 + f^\varepsilon) \right) \right], \\ C_V \partial_t T^\varepsilon - \nabla \cdot (D \nabla T^\varepsilon) = -\frac{c}{\varepsilon^2} \frac{\sigma_a}{4\pi} \int_{\mathbb{R}^+ \times S^2} \left(B(\nu, T^\varepsilon) - \nu^3 f^\varepsilon \right) d\nu d\mathbf{\Omega} \\ \quad + \frac{1}{4\pi} \frac{\varepsilon^2 \sigma_s}{3mc} \int_{\mathbb{R}^+ \times S^2} \nu^4 \left(T^\varepsilon \partial_\nu f^\varepsilon + f^\varepsilon (1 + f^\varepsilon) \right) d\nu d\mathbf{\Omega}. \end{cases} \quad (4.27)$$

Lemma 4.8 (Equilibrium diffusion regime). *A first order approximation of the system (4.27) in the equilibrium diffusion regime is*

$$\partial_t \left(\bar{C}_v T_m + T_m^4 \right) - \nabla \cdot \left(\bar{D} \nabla T_m + \frac{c}{3\sigma_a} \nabla T_m^4 \right) = 0, \quad (4.28)$$

where $\bar{C}_v = 15/\pi^4 C_V$ and $\bar{D} = 15/\pi^4 D$ and

$$T_m = \lim_{\varepsilon \rightarrow 0} T^\varepsilon = \lim_{\varepsilon \rightarrow 0} \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f^\varepsilon d\nu d\mathbf{\Omega}.$$

Proof. Let us expand the solution $(f^\varepsilon, T^\varepsilon)$ of system (4.27) in power of ε , that is

$$\begin{cases} f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

Take care to the fact that for the distribution function of the photons f the superscript refer to the Hilbert expansion while the subscript refer to the Bose-Einstein distributions. Since the Bose-Einstein distributions are nonlinear with respect to the temperature, one makes a Taylor expansion with respect to T :

$$f_0(\nu, T^\varepsilon) = f_0(\nu, T^0) + \varepsilon(T^1 + \varepsilon^2 T^2) \frac{\partial}{\partial T} f_0(\nu, T^0) + \dots \quad (4.29)$$

At the order ε^{-2} , one has

$$f^0 = f_0(\nu, T^0),$$

and thus the zero-th order term, equal to the Bose-Einstein distribution, does not depend on the angular $\mathbf{\Omega}$. If with classical notation we denote $E = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f d\nu d\mathbf{\Omega}$, one finds

$$E^0 = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f^0 d\nu d\mathbf{\Omega} = (T^0)^4, \quad (4.30)$$

where we used the relation $\int_\nu B(\nu, T) = \frac{\pi^4}{15} T^4$. At the order ε^{-1} , one finds $\mathbf{\Omega} \cdot \nabla f^0 = \sigma_a ((f_0(\nu, T^\varepsilon))^{1-f^1})$. The previous expansion (4.29) of the Bose-Einstein distributions thus yields

$$f^1 = -\frac{\mathbf{\Omega} \cdot \nabla f^0}{\sigma_a} + T^1 \frac{\partial}{\partial T} f_0(\nu, T^0). \quad (4.31)$$

Finally, at the order ε^0 , one finds

$$\begin{cases} \frac{1}{c} \partial_t f^0 + \mathbf{\Omega} \cdot \nabla f^1 = \sigma_a \left(T^2 \frac{\partial}{\partial T} f_0(\nu, T^0) - f^2 \right), \\ C_V \partial_t T^0 - \nabla \cdot (D \nabla T^0) = -\frac{c\sigma_a}{4\pi} \int_{\mathbb{R}^+ \times S^2} \nu^3 \left(T^2 \frac{\partial}{\partial T} f_0(\nu, T^0) - f^2 \right) d\nu d\mathbf{\Omega}. \end{cases}$$

Integrating the first equation on $\mathbb{R}^+ \times S^2$ multiplied by $15\nu^3/4\pi^5$ and multiplying the second equation by $15/\pi^4$, one finds by using the link between f_0 and E_0 (4.30) and the definition of f^1 (4.31)

$$\begin{cases} \frac{1}{c} \partial_t (T^0)^4 + \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 \mathbf{\Omega} \cdot \nabla \left(-\frac{\mathbf{\Omega} \cdot \nabla f^0}{\sigma_a} + T^1 \frac{\partial}{\partial T} f_0(\nu, T^0) \right) d\nu d\mathbf{\Omega} \\ \quad = \sigma_a \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 \left(T^2 \frac{\partial}{\partial T} f_0(\nu, T^0) - f^2 \right) d\nu d\mathbf{\Omega}, \\ \frac{15}{\pi^4} C_V \partial_t T^0 - \frac{15}{\pi^4} \nabla \cdot (D \nabla T^0) = -c\sigma_a \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 \left(T^2 \frac{\partial}{\partial T} f_0(\nu, T^0) - f^2 \right) d\nu d\mathbf{\Omega}. \end{cases}$$

Since $\int_{\mathbb{R}^+ \times S^2} \nu^3 \mathbf{\Omega} \cdot \nabla T^1 \frac{\partial}{\partial T} (f_0(\nu, T^0)) d\nu d\mathbf{\Omega} = 0$, due to the fact that f_0 does not depend on $\mathbf{\Omega}$, the announced result is obtained by adding the two equations. \square

This result shows that the Kompaneets term in equation (4.21) does not change the equilibrium regime, in the sense that the obtained equation is the same diffusion equation than in the case with no Compton scattering [BD04].

Non-equilibrium diffusion

We now turn to the study of the non-equilibrium diffusion regime. This regime is formally valid when the scattering coefficient σ_s is large compare to the emission absorption coefficient σ_a and when the speed of light c is large. The scaling is the following

$$\begin{cases} c = \varepsilon^{-1}\bar{c}, \\ \sigma_a = \varepsilon\bar{\sigma}_a, \\ \sigma_s = \varepsilon^{-1}\bar{\sigma}_s, \end{cases}$$

It yields the following system, in which we dropped the bars for ease of notations

$$\begin{cases} \frac{1}{c}\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{\Omega}\cdot\nabla f^\varepsilon = \frac{1}{\varepsilon^2}\frac{\sigma_s}{4\pi}\int_{S^2}(1+\cos^2\theta)(f'^\varepsilon - f^\varepsilon)d\mathbf{\Omega}' + \sigma_a\left(f_0(\nu, T^\varepsilon) - f^\varepsilon\right) \\ \quad + \frac{\sigma_s}{3mc^2}\nu^{-2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T^\varepsilon\partial_\nu f^\varepsilon + f^\varepsilon(1+f^\varepsilon)\right)\right], \\ C_V\partial_t T^\varepsilon - \nabla\cdot(D\nabla T^\varepsilon) = -\frac{c\sigma_a}{4\pi}\int_{\mathbb{R}^+\times S^2}\left(B(\nu, T^\varepsilon) - \nu^3 f^\varepsilon\right)d\nu d\mathbf{\Omega} \\ \quad + \frac{1}{4\pi}\frac{\sigma_s}{3mc}\int_{\mathbb{R}^+\times S^2}\nu^4\left(T^\varepsilon\partial_\nu f^\varepsilon + f^\varepsilon(1+f^\varepsilon)\right)d\nu d\mathbf{\Omega}. \end{cases} \quad (4.32)$$

We prove the

Lemma 4.9 (Non-equilibrium diffusion regime). *A first order approximation of the system (4.32) in the non-equilibrium diffusion regime is*

$$\begin{cases} \frac{1}{c}\partial_t f - \nabla\cdot\left(\frac{\nabla f}{4\sigma_s}\right) = \sigma_a(f_0 - f) + \frac{\sigma_s}{3mc^2}\nu^{-2}\frac{\partial}{\partial\nu}\left[\nu^4\left(T\partial_\nu f + f(1+f)\right)\right] \\ C_V\partial_t T - \nabla\cdot(D\nabla T) = -\frac{c\sigma_a}{4\pi}\int_{\mathbb{R}^+}\left(B(\nu, T) - \nu^3 f\right)d\nu + \frac{1}{4\pi}\frac{\sigma_s}{3mc}\int_{\mathbb{R}^+}\nu^4\left(T\partial_\nu f + f(1+f)\right)d\nu. \end{cases}$$

Proof. Once again, we performed a formal Hilbert expansion of the unknowns:

$$\begin{cases} f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

At the order ε^{-2} , one finds $f^0 = f'^0$, and thus f^0 is isotropic. At the order ε^{-1} , one finds

$$\mathbf{\Omega}\cdot\nabla f^0 = \frac{\sigma_s}{4\pi}\int_{S^2}(1+\cos^2\theta)(f'^1 - f^1)d\mathbf{\Omega}',$$

which yields

$$f^1\frac{1}{4\pi}\int_{S^2}(1+\cos^2\theta)d\mathbf{\Omega}' = \frac{1}{4\pi}\int_{S^2}(1+\cos^2\theta)f'^1d\mathbf{\Omega}' - \frac{\mathbf{\Omega}\cdot\nabla f^0}{\sigma_s}.$$

Since $\int_{S^2}(1+\cos^2\theta)d\mathbf{\Omega}' = \frac{4}{3}$, one finally finds

$$f^1 = -\frac{3}{4}\frac{\mathbf{\Omega}\cdot\nabla f^0}{\sigma_s} + \frac{3}{16\pi}\int_{S^2}(1+\cos^2\theta)f'^1d\mathbf{\Omega}'.$$

where $\alpha = 4mc^2$ stands for an absorption coefficient. This system is indeed interesting, since the classical Kompaneets equation (P_0 model, with $\mathbf{V} = 0$) is perturbed by a Burgers type term $\partial_\nu \nu^4(\mathbf{V}, \mathbf{V})$, where \mathbf{V} satisfies a Fokker-Planck type equation. Since several papers studied the long time behavior of the solution of the Kompaneets equation [CL86, EHV98, EMV04], it is interesting to study the influence of the anisotropic part of a radiation beam on the solution of the zero-th order moment U , in comparison with the solution of the Kompaneets equation. This is performed in the section 2. In particular we show (lemma 5.2) that in the limit case $\alpha = 0$, the long time range of the solution of the frequency dependent P_1 model exhibits new solutions in comparison with the Bose condensate result of Caffisch and Levermore [CL86] for the Kompaneets equation.

In this chapter we also study the M_1 model. For this model the distribution function is obtained by solving an entropy minimization problem [DF99, LEV84], and the obtained distribution has the form of a generalized Planck distribution. It yields a nonlinear closure, but the corresponding models are known to satisfy some relevant physical properties, such as the conservation of the positivity of the distribution function and a H-theorem.

For the coupled system (5.1), the grey M_1 model is given by

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \frac{4\sigma_s}{3mc^2} E \left(T - E^{\frac{1}{4}} Q_E(\|\mathbf{g}\|) \right) \\ \frac{1}{c} \partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_r = - \left[\frac{4}{3} \sigma_s + \sigma_a + \frac{4\sigma_s}{3mc^2} \left(E^{\frac{1}{4}} Q_F(\|\mathbf{g}\|) - T \right) \right] \mathbf{F}, \\ \partial_t T = c\sigma_a (E - T^4) + \frac{4\sigma_s}{3mc} E \left(E^{\frac{1}{4}} Q_E(\|\mathbf{g}\|) - T \right), \end{cases} \quad (5.4)$$

where \mathbf{P}_r , the second order moment of the distribution function f is given by $\mathbf{P}_r = \left(\frac{1 - \|\mathbf{b}\|^2}{3 + \|\mathbf{b}\|^2} \mathbf{I}_d + \frac{3 + \|\mathbf{b}\|^2}{4} \mathbf{g} \otimes \mathbf{g} \right) E$, with $\mathbf{g} = \mathbf{F}/E$. The operators Q_E and Q_F are defined by

$$\begin{cases} Q_E(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{5}{4}}}{4} \frac{4 + \|\mathbf{g}\|^2}{4 - \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}}, \\ Q_F(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{1}{4}}}{4} \frac{20 + \|\mathbf{g}\|^2}{4 + \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}}. \end{cases}$$

Despite nonlinear and rather complicated, it shows in particular that the Compton scattering can be simply used in numerical grey M_1 models by adding source terms. We show in this chapter that the Compton scattering does not change the equilibrium states, in the sense that the electron temperature T still equilibrates with the radiation temperature $T_R = T^4$, but can modify the transitional states, depending on the value of σ_s/mc^2 compared to σ_a . The equilibrium and non-equilibrium diffusion regimes are studied for this grey M_1 model.

This chapter is organized as follows. In section 2 is studied the frequency dependent P_1 model as presented in (5.3), and its solutions in the case of no coupling are compared to the solutions of the Kompaneets equation. In section 3 is derived and studied the grey M_1 model. In particular the equilibrium and non-equilibrium diffusion regimes are studied and the relaxation to the

electronic temperature are studied. Finally in a last part is studied an hybrid $P_1 - M_1$ model. Indeed, the study of the P_1 and M_1 models shows that according to the assumed expression of the distribution function, the construction of the frequency dependent and grey angular moment model is possible or not. It appears that it is possible to construct an expression for the distribution function f , constructed from a mix of the expression of M_1 and P_1 expressions of f , which allows to construct both the frequency dependent hybrid model (lemma 5.6), which is indeed exactly the P_1 frequency dependent model, and the grey hybrid model (lemma 5.7).

5.2 P_1 approximation

We derive here the so-called P_1 approximation [BRU00, BRU02, HLM10] for the system (5.1). The P_1 approximation consists of assuming that the distribution f writes

$$f(t, x, \nu, \mathbf{\Omega}) = f_0(t, x, \nu) + \mathbf{\Omega} \cdot \mathbf{f}_1(t, x, \nu), \quad (5.5)$$

i.e. is polynomial with respect to $\mathbf{\Omega}$. The main advantages of the P_1 approximation is that the moments equations are easily constructed. Moreover, due to the polynomial angular dependence of f , an increase of the order of the approximation is easily achieved. Moreover and as we will see in this part, even if the source terms are nonlinear (quadratic in our case), they can be expressed in terms in the moments. We introduce the two first angular moments of the distribution function f

$$\begin{cases} X(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \mathbf{\Omega}) d\mathbf{\Omega}, \\ \mathbf{Y}(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \mathbf{\Omega}) \mathbf{\Omega} d\mathbf{\Omega}, \end{cases} \quad (5.6)$$

The following lemma is concerned by the frequency dependent P_1 model

Lemma 5.1 (Frequency dependent P_1 model). *The frequency dependent P_1 model for the coupled system (5.1) writes*

$$\begin{cases} \frac{1}{c} \partial_t X + \nabla \cdot \mathbf{Y} = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + X^2 + 3(\mathbf{Y}, \mathbf{Y}) \right) \right] + \sigma_a (f_0 - X), \\ \frac{1}{c} \partial_t \mathbf{Y} + \frac{1}{3} \nabla X = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{Y} + \mathbf{Y} + 2X\mathbf{Y} \right) \right] - \alpha \mathbf{Y} - \sigma_a \mathbf{Y}, \\ \partial_t T = c \int_{\mathbb{R}^+} \nu^4 \left(T \partial_\nu X + X + X^2 + 3(\mathbf{Y}, \mathbf{Y}) \right) d\nu - \sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 X \right) d\nu, \end{cases} \quad (5.7)$$

where $\alpha = 4mc^2$.

Proof. The definition of X and \mathbf{Y} together with the property $\int \mathbf{\Omega} d\mathbf{\Omega} = 0$ yields

$$\begin{cases} f_0 = X, \\ \mathbf{f}_1 = 3\mathbf{Y}. \end{cases} \quad (5.8)$$

Integrating equation (5.1) over S^2 and dividing by 4π , we write the equation satisfied by X :

$$\frac{1}{c} \partial_t X + \nabla \cdot \mathbf{Y} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + \frac{1}{4\pi} \int_{S^2} f^2 d\mathbf{\Omega} \right) \right] + \sigma_a (f_0 - X).$$

Equations (5.5) and (5.8) gives

$$\frac{1}{4\pi} \int_{S^2} f^2 d\Omega = X^2 + 3(\mathbf{Y}, \mathbf{Y}),$$

and thus the equation on X is

$$\frac{1}{c} \partial_t X + \nabla \cdot \mathbf{Y} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + X^2 + 3(\mathbf{Y}, \mathbf{Y}) \right) \right] + \sigma_a (f_0 - X). \quad (5.9)$$

We now turn to the equation on \mathbf{Y} . Multiplying the first equation of (5.1) by Ω and integrating on S^2 , one has

$$\begin{aligned} \frac{1}{c} \partial_t \mathbf{Y} + \nabla \left(\frac{1}{4\pi} \int_{S^2} \Omega \otimes \Omega I_d f d\Omega \right) &= \frac{\sigma_s}{16\pi^2} \int_{S^2 \times S^2} (1 + \cos^2 \theta) \Omega (f' - f) d\Omega' d\Omega \\ &+ \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{Y} + \mathbf{Y} + \frac{1}{4\pi} \int_{S^2} \Omega f^2 d\Omega \right) \right] - \sigma_a \mathbf{Y}. \end{aligned}$$

Once again, equations (5.5) and (5.8) gives

$$\begin{cases} \nabla \left(\frac{1}{4\pi} \int_{S^2} \Omega \otimes \Omega I_d f d\Omega \right) = \frac{1}{3} \nabla X, \\ \frac{1}{16\pi^2} \int_{S^2 \times S^2} (1 + \cos^2 \theta) \Omega (f' - f) d\Omega' d\Omega = -\frac{4}{3} \mathbf{Y}, \\ \frac{1}{4\pi} \int_{S^2} \Omega f^2 d\Omega = 2X\mathbf{Y}. \end{cases}$$

The equation on the temperature is trivially obtained from the equation on X . Rescaling the time and the space as $t = \frac{3mc^2}{\sigma_s} \bar{t}$, $x = \frac{3mc^2}{\sigma_s} \bar{x}$ and $\frac{\sigma_s}{3mc^2} \bar{\sigma}_a$ lead to, dropping the bars for ease of notations,

$$\begin{cases} \frac{1}{c} \partial_t X + \nabla \cdot \mathbf{Y} = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + X^2 + 3(\mathbf{Y}, \mathbf{Y}) \right) \right] + \sigma_a (f_0 - X), \\ \frac{1}{c} \partial_t \mathbf{Y} + \frac{1}{3} \nabla X = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{Y} + \mathbf{Y} + 2X\mathbf{Y} \right) \right] - \alpha \mathbf{Y} - \sigma_a \mathbf{Y}, \\ \partial_t T = c \int_{\mathbb{R}^+} \nu^4 \left(T \partial_\nu X + X + X^2 + 3(\mathbf{Y}, \mathbf{Y}) \right) d\nu - \sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 X \right) d\nu, \end{cases} \quad (5.10)$$

where $\alpha = 4mc^2$, which is the announced result. \square

It is easy to see that this P_1 model preserves the stationary states of the Boltzmann equation, the Bose-Einstein distributions f_0 defined in (4.5). Indeed for such distributions one has $\mathbf{V} = \mathbf{0}$ and $U = f_0$, and these distributions satisfy $T \partial_\nu f_0 + f_0 + f_0^2 = 0$. The Bose-Einstein distributions are restricted to the case $\mu = 0$ due to the emission absorption term $\sigma_a (f_0 - f)$. From now on and until the end of this section, we take $c = 1$ in the transport term. This isotropic case $\mathbf{V} = \mathbf{0}$ can be seen as the P_0 model, and reduces to the Kompaneets equation in the case of no coupling

$$\partial_t f = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f(1 + f) \right) \right]. \quad (5.11)$$

We are interested in the comparison of the solution of this equation with the solution of the homogeneous P_1 model in the case of no coupling and in the 1D case. The system writes

$$\begin{cases} \partial_t X = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + X^2 + 3Y^2 \right) \right], \\ \partial_t Y = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu Y + Y + 2XY \right) \right] - \alpha Y \end{cases} \quad (5.12)$$

The end of this section deals with the comparison between the solution of the Kompaneets equation (P_0 model, equation (5.11)) and the solution of the homogeneous P_1 model (5.12). In a first time (part 5.2.1) are recalled the main result concerning the long time behavior of the Kompaneets equation. In a second time (part 5.2.2) are studied the difference between this two models. In particular we show that the anisotropic part of the radiation decreases the energy of the system, and that the stationary states in long time range are modified in the limiting case $\alpha = 0$ (lemma 5.2), which is an unexpected result.

It is interesting to note that despite we are able to construct the frequency dependent P_1 model, the grey P_1 model can not be constructed analytically. Indeed, considering the moments of $\nu^3 f$, integration by parts of equations (5.1) will lead to integrals of terms like $\nu^4 f$ and $\nu^4 f^2$, which we can not analytically compute in terms of the moments. This is the purpose of the section 4, in which an hybrid $P_1 - M_1$ model is constructed.

5.2.1 A review of the Kompaneets equation

In this part we recall the main results concerning the Kompaneets equation, which writes

$$(P_0) \quad \begin{cases} \partial_t f = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu f + f + f^2 \right) \right], \\ f(t=0) = f^{in}, \end{cases} \quad (5.13)$$

with the boundary conditions

$$\nu^4 \left(T \partial_\nu f + f + f^2 \right) = 0 \text{ at } \nu = 0 \text{ and } \nu = \infty. \quad (5.14)$$

In the following we denote $N(f_0)$ the number of photons associated to the Bose-Einstein distribution with $\mu = 0$, defined by

$$N(f_0) = \int_{\mathbb{R}^+} \frac{\nu^2}{e^{\nu/T} - 1} d\nu,$$

where T is the electronic temperature. More generally, we define for any function f its associated number of photons by

$$N(f) = \int_{\mathbb{R}^+} \int_{S^2} f \nu^2 d\nu d\Omega.$$

On the one hand Cagliostro and Levermore [CL86] studied and observed numerically the Bose condensation phenomena, recalled thereafter

- If $N(f^{in}) \leq N(f_0)$, then $\exists \mu \geq 0$ s. t. $\lim_{t \rightarrow \infty} \nu^2 f(t, \cdot) = \nu^2 f_\mu(\cdot)$.
 - Else if $N(f^{in}) > N(f_0)$, then $\lim_{t \rightarrow \infty} \nu^2 f(t, \cdot) = \nu^2 f_0(\cdot) + (N(f^{in}) - N(f_0)) \delta_0$.
- (5.15)

The parameter $\mu > 0$ in the proof is such that $N(f^{in}) = N(f_\mu)$. This result is referred to the Bose condensation phenomena; when the number of photons at the initial time is larger than a critical number, the excess of photons is concentrated at the origin. This is illustrated in picture 5.2. The Kompaneets equation is numerically solved for two different initial datum (picture 5.1): the first one satisfies $N(f^{in}) < N(f_0)$, the second one $N(f^{in}) > N(f_0)$. The curves represent

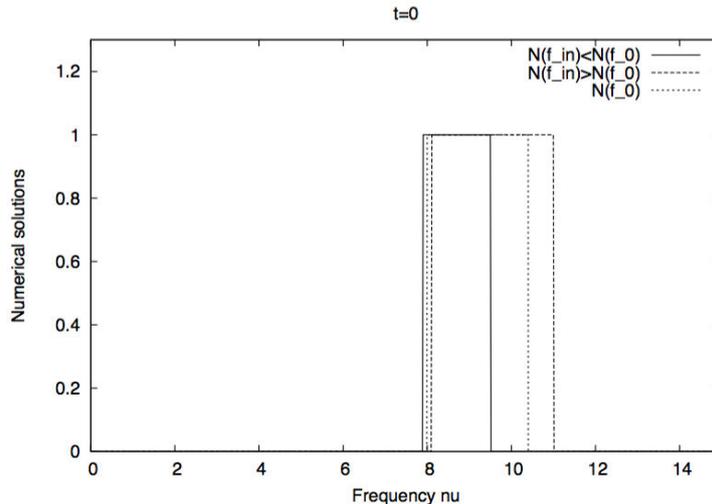


Figure 5.1: Initial conditions for the Kompaneets equation (5.11), written as $\nu^2 f$

the quantity ν^2 multiplied by the distribution function (and not the distribution function itself). This result was proved in [CL86] without discussions about the flux conditions (5.14).

On the other hand Escobedo and co-authors proved in [EHV98,EHV99] that there exists smooth nonnegative solutions of the Kompaneets equation, with arbitrarily small values of $N(f^{in})$, that may develop singularities near $\nu = 0$ in finite time, so that the flux condition (5.14) at $\nu = 0$ is lost. They also proved that if one replaces the flux condition at $\nu = 0$ (5.14) by an estimate of the form

$$0 \leq f(\nu, T) \leq \frac{C}{\nu^2}, \text{ as } \nu \rightarrow 0, C > 0,$$

then the corresponding modified problem has a unique solution for all times $t > T$.

The question of existence and uniqueness of global solutions is still an open problem. In this work our aim is to discuss from the numerical point of view the apparition of Bose condensation for new models, which permits to find back the issues discussed in [CL71] for these new models. Secondly, we aim to show that in certain cases the study of P_1 models reduces to two decoupled Kompaneets equations, which allows to extend all these previous results to our systems. We show in particular that the anisotropic part (first order moment \mathbf{Y}) changes the value of the threshold from which the Bose condensation appears.

5.2.2 P_0 - P_1 comparison

The goal of this part is to study the difference between the solution of the Kompaneets equation (5.11) and the solution of the multigroup P_1 model (5.12). Indeed, the question that we wish to

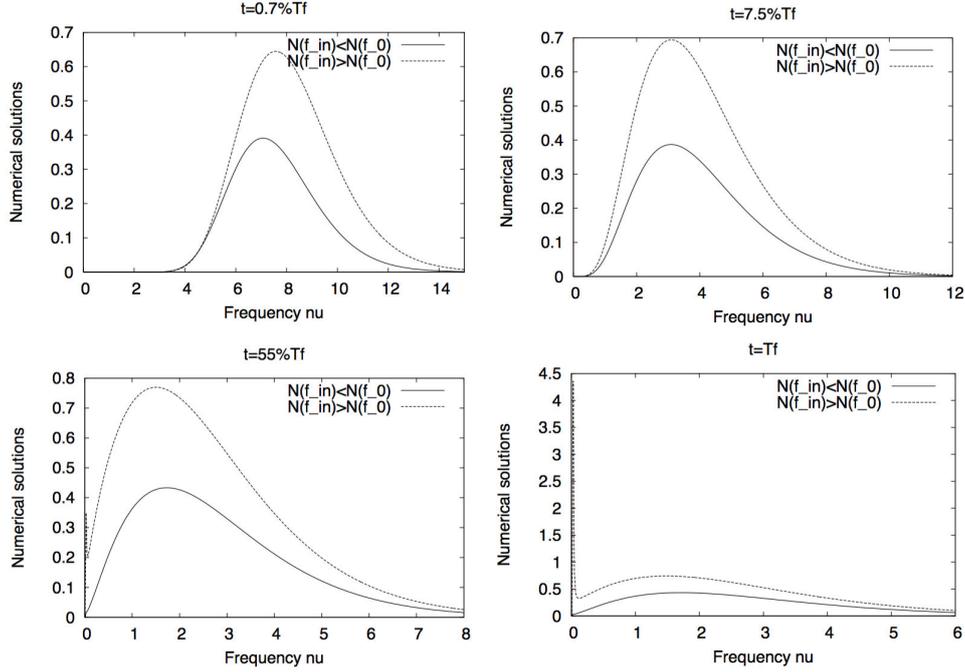


Figure 5.2: Numerical solutions of the Kompaneets equation (5.11) versus frequency ν . Convergence to a Planck function (full line) in the case $N(f^{in}) < N(f_0)$, and to a Planck function plus a Dirac function concentrated near $\nu = 0$ (dashed line) in the case $N(f^{in}) > N(f_0)$.

answer is how the anisotropic part of the radiation, modeled by the first moment \mathbf{Y} , modifies the zero-th order moment U . On the one hand, it is clear that if $\mathbf{Y}(t = 0) = 0$, one has for all $t \geq 0$ $X(t) = f(t)$, where f is the solution of the Kompaneets equation. Moreover, if $\alpha \gg 1$, one can expect \mathbf{Y} to tends quickly vers 0 and thus obtain $\lim_{t \rightarrow \infty} X(t) = f(t)$. One the other hand, the long time behavior of the solution of the P_1 model for $\alpha \approx 1$ or $\alpha \ll 1$ is not straightforward. This is the purpose of the forthcoming study. In particular we show in lemma 5.2 that in the limit case $\alpha = 0$ there exists solutions of the P_1 model such that the number of photons at the initial stage is lower than the critical number $N(f_0)$ and such that a Bose condensation nonetheless appears in long time range.

Let us emphasize some properties of the P_1 model. Firstly and as for the Kompaneets equation, the model (5.12) preserves the number of photons. Indeed, the number of photons is defined by

$$N(t) = \int_{\mathbb{R}^+ \times S^2} f(t, \nu, \mathbf{\Omega}) \nu^2 d\nu d\mathbf{\Omega}.$$

Since for the P_1 model $f = X + 3\mathbf{Y} \cdot \mathbf{\Omega}$, one easily obtains by using the P_1 model (5.12)

$$\frac{d}{dt} N(t) = \frac{d}{dt} \left[\int_{\mathbb{R}^+} X(t, \nu) \nu^2 d\nu \right] = 0,$$

assuming the correct flux conditions. What appears as a new property is the variation of the energy of the P_1 model in comparison with the P_0 model. The energy is defined by

$$E(t) = \int_{\mathbb{R}^+ \times S^2} f(t, \nu, \mathbf{\Omega}) \nu^3 d\nu d\mathbf{\Omega}.$$

Using the P_1 model (5.7) and an integration by parts, one gets

$$\frac{d}{dt}E(P_1) = \frac{d}{dt} \left[\int_{\mathbb{R}^+} X(t, \nu) \nu^3 d\nu \right] = - \int_{\mathbb{R}^+} \left(T \partial_\nu X + X + X^2 \right) \nu^4 d\nu - 3 \int_{\mathbb{R}^+} (\mathbf{Y}, \mathbf{Y}) \nu^4 d\nu.$$

This equation is recast as

$$\frac{d}{dt}E(P_1) = \frac{d}{dt}E(P_0) - 3 \int_{\mathbb{R}^+} (\mathbf{Y}, \mathbf{Y}) \nu^4 d\nu,$$

with an obvious definition of $E(P_0)$. This shows that the anisotropic part of the radiation does not modify the number of photons, but decreases the energy of the radiation.

We now turn to the main result of this part, which is concerned by the long time behavior of the solution of the P_1 model (5.12) in the case $\alpha = 0$. In the 1D case, the model writes

$$\begin{cases} \partial_t X = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + X^2 + 3Y^2 \right) \right], \\ \partial_t Y = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu Y + Y + 2XY \right) \right]. \end{cases} \quad (5.16)$$

We prove the following lemma, which shows that the anisotropic part of the radiation exhibits different stationary solutions in comparison with the Kompaneets equation in long time range. In the following $N(f_0)$ still refers to the number of photons of the Bose-Einstein distribution with $\mu = 0$. This comes from the very simple remark that one can find a new set of variables, namely $Z^\pm = X \pm \sqrt{3}Y$, which satisfies the Kompaneets equation.

Lemma 5.2 (Asymptotic behavior of the P_1 model in the case $\alpha = 0$). *There exists solutions of the P_1 model (5.16), whose total number of photons at the initial time is less than the critical number of photons $N(f_0)$, such that a condensation phenomena appears in long time range. For example if the initial conditions for the P_1 model (5.16) are chosen such that*

$$\begin{cases} N(X^{in}) = \frac{3}{4}N(f_0), \\ N(Y^{in}) = \frac{1}{2\sqrt{3}}N(f_0), \end{cases}$$

(which yields $N(f^{in}) = N(X^{in} + 3\Omega Y^{in}) = N(X^{in}) < N(f_0)$) then in long time range one has

$$\begin{cases} \lim_{t \rightarrow \infty} X(t, \cdot) = \nu^2 \frac{f_0 - f_\mu}{2} + \frac{N(f_0)}{8} \delta_0, \\ \lim_{t \rightarrow \infty} Y(t, \cdot) = \nu^2 \frac{f_0 - f_\mu}{2\sqrt{3}} + \frac{N(f_0)}{8\sqrt{3}} \delta_0, \end{cases}$$

although $N(X^{in}) < N(f_0)$, where the parameter μ is such that $N(f_\mu) = N(f_0)/4$.

Proof. We set $Z^\pm = X \pm \sqrt{3}Y$. Multiplying the second equation of (5.16) by $\sqrt{3}$ and adding and subtracting to the first equation, one sees that Z^\pm satisfies the Kompaneets equation

$$\partial_t Z^\pm = \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu Z^\pm + Z^\pm (1 + Z^\pm) \right) \right].$$

The initial conditions for Z^\pm are defined by $(Z^\pm)^{in} = X^{in} \pm \sqrt{3}Y^{in}$. Using the definition of X^{in} and Y^{in} , one gets $N((Z^+)^{in}) = \frac{5}{4}N(f_0) > N(f_0)$. In the same way one obtains $N((Z^-)^{in}) = \frac{1}{4}N(f_0) < N(f_0)$. One applies the result 5.14 on Z^+ and Z^- , and gets

$$\begin{cases} \lim_{t \rightarrow \infty} \nu^2 Z^+(t, \cdot) = \nu^2 f_0 + \frac{N(f_0)}{4} \delta_0, \\ \lim_{t \rightarrow \infty} \nu^2 Z^-(t, \cdot) = \nu^2 f_\mu, \quad \mu > 0 \text{ s. t. } N(f_\mu) = \frac{N(f_0)}{4}. \end{cases}$$

From the definition of Z^\pm , one has $X = (Z^+ + Z^-)/2$ and $Y = (Z^+ - Z^-)/2\sqrt{3}$, which yields

$$\begin{cases} \lim_{t \rightarrow \infty} X(t, \cdot) = \nu^2 \frac{f_0 - f_\mu}{2} + \frac{N(f_0)}{8} \delta_0, \\ \lim_{t \rightarrow \infty} Y(t, \cdot) = \nu^2 \frac{f_0 - f_\mu}{2\sqrt{3}} + \frac{N(f_0)}{8\sqrt{3}} \delta_0, \quad \mu > 0 \text{ s. t. } N(f_\mu) = \frac{N(f_0)}{4}, \end{cases}$$

which is the announced result. \square

Numerical illustration

We present here the numerical scheme (inspired from [LLPS84], see also [DWLM09]) designed for the frequency dependent P_1 model. Numerical results concerning the apparition of Dirac masses can be found in [ST95, ST97]. For a given $0 < \nu^*$, the frequency domain is $[0, \nu^*]$. For $1 \leq j \leq N$, we consider an irregular mesh defined by $(N + 1)$ points $0 = \nu_{\frac{1}{2}} < \dots < \nu_{N+\frac{1}{2}} = \nu^*$. We define ν_j as the middle of the j -th frequency band, i.e. $\nu_j = (\nu_{j-\frac{1}{2}} + \nu_{j+\frac{1}{2}})/2$ and we denote $\Delta\nu_j$ its length. We also define the dual $(j + \frac{1}{2})$ -th frequency band as the cell $[\nu_j, \nu_{j+1}]$, which length is denoted $\Delta\nu_{j+\frac{1}{2}}$. Since we consider the homogeneous (in space) case, we consider the 1D case, and thus the first moment \mathbf{Y} is a scalar.

Due to the term ν^{-2} in front of the right hand side of the P_1 model (5.12), it appears to be easier to work on the variables $U = \nu^2 X$ and $V = \nu^2 Y$. Using this set of variables, the frequency dependent P_1 model writes

$$\begin{cases} \partial_t U = \frac{\partial}{\partial \nu} \left[T \left(\partial_\nu (\nu^2 U) - 4\nu U \right) + U(\nu^2 + U) + 3V^2 \right], \\ \partial_t V = \frac{\partial}{\partial \nu} \left[T \left(\partial_\nu (\nu^2 V) - 4\nu V \right) + V + 2UV \right] - \alpha V. \end{cases} \quad (5.17)$$

We use a classical finite volume scheme with explicit Euler discretization of the time derivatives, defined by

$$\begin{cases} \frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta\nu_j}, \\ \frac{V_j^{n+1} - V_j^n}{\Delta t} = \frac{V_{j+\frac{1}{2}}^n - V_{j-\frac{1}{2}}^n}{\Delta\nu_j} - \alpha V_j^n, \end{cases} \quad (5.18)$$

where the fluxes are defined by

$$\begin{cases} U_{j+\frac{1}{2}}^n = T \left(\frac{\nu_{j+1}^2 U_{j+1}^n - \nu_j^2 U_j^n}{\Delta\nu_{j+\frac{1}{2}}} - 4\nu_j U_j^n \right) + U_{j+1}^n (\nu_{j+1}^2 + U_{j+1}^n) + 3(V_{j+1}^n, V_{j+1}^n), \\ V_{j+\frac{1}{2}}^n = T \left(\frac{\nu_{j+1}^2 V_{j+1}^n - \nu_j^2 V_j^n}{\Delta\nu_{j+\frac{1}{2}}} - 4\nu_j V_j^n \right) + V_{j+1}^n (\nu_{j+1}^2 + 2U_{j+1}^n). \end{cases} \quad (5.19)$$

Numerically, the conservation of the total number of photons is obtained by setting $U_{N+\frac{1}{2}}^n = U_1^n = 0$, for all n . To obtain the CFL condition, we write the system (5.17) as a drift diffusion system on the variable $\mathbf{W} = (U, V)$, that is

$$\partial_t \mathbf{W} = \nu^2 T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_\nu^2 \mathbf{W} + \underbrace{\begin{pmatrix} \nu^2 + 2U & 6V \\ 2V & \nu^2 + 2U \end{pmatrix}}_A \partial_\nu \mathbf{W} + \underbrace{\begin{pmatrix} 2(\nu - T) & 0 \\ 0 & 2(\nu - T) - \alpha \end{pmatrix}}_R \mathbf{W}. \quad (5.20)$$

Studying the eigenvalues of the matrix A and R , denoted respectively Λ_A^\pm and Λ_R^\pm , one easily finds

$$\begin{cases} \Lambda_A^\pm = \nu^2 + 2U \pm 2\sqrt{3}V, \\ \Lambda_R^+ = 2(\nu - T), \\ \Lambda_R^- = 2(\nu - T) - \alpha, \end{cases}$$

The stability of the scheme is obtained under the following CFL condition

$$\Delta t \sup_{1 \leq j \leq N} \left(\frac{T\nu_j^2}{\Delta\nu_j^2} + \frac{\max(|\Lambda_{A,j}^\pm|)}{\Delta\nu_j} + \max(|\Lambda_{R,j}^\pm|) \right) \leq 1,$$

with obvious notations of $\Lambda_{A,j}^\pm$ and $\Lambda_{R,j}^\pm$.

In the following, we compare the solution of the P_1 model with the solution of the P_0 model for several value of the absorption coefficient α . Each test cases are initialized with the same values, depicted in figure 5.3 and we take for the electron temperature $T = 1$. The mesh is composed

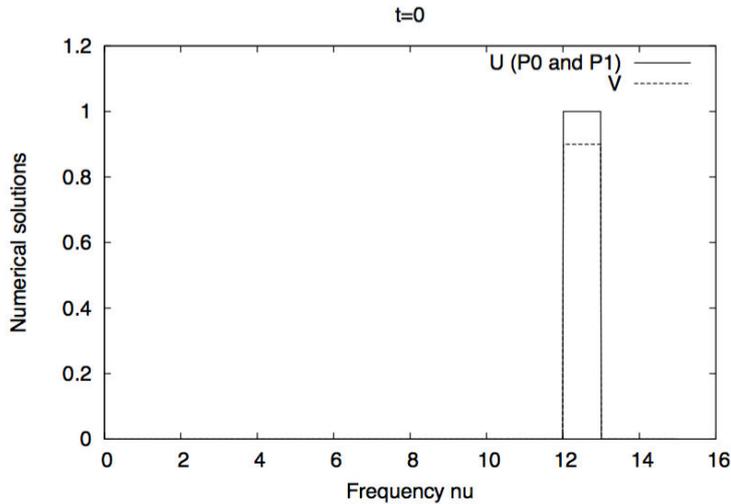


Figure 5.3: Initialization of the P_0 (Kompaneets) and P_1 model.

of 800 cells. With the choice of the variables (U, V) , the number of photons at the initial stage is simply $\int_{\mathbb{R}^+} U d\nu = 1 \leq N(f_0) \approx 2.4$. The long time behavior of the P_0 model is thus a regular Planck distribution, and our aim is to understand the influence of the first order moment V in the transitional and long time behavior of the zero-th order moment U .

Case $\alpha = 1$

In this part we solve the P_1 system with $\alpha = 1$ and we compare the zero-th order moment to the solution of the P_0 equation. The picture 5.4 displays the zero-th order moment U and the solution of the Kompaneets equation at different time. It shows that in the case $\alpha = 1$, the anisotropic part of the radiation does not sensibly modify the solution of the Kompaneets equation. Figure 5.5 displays the evolution of the first order moment V . The absorption term is

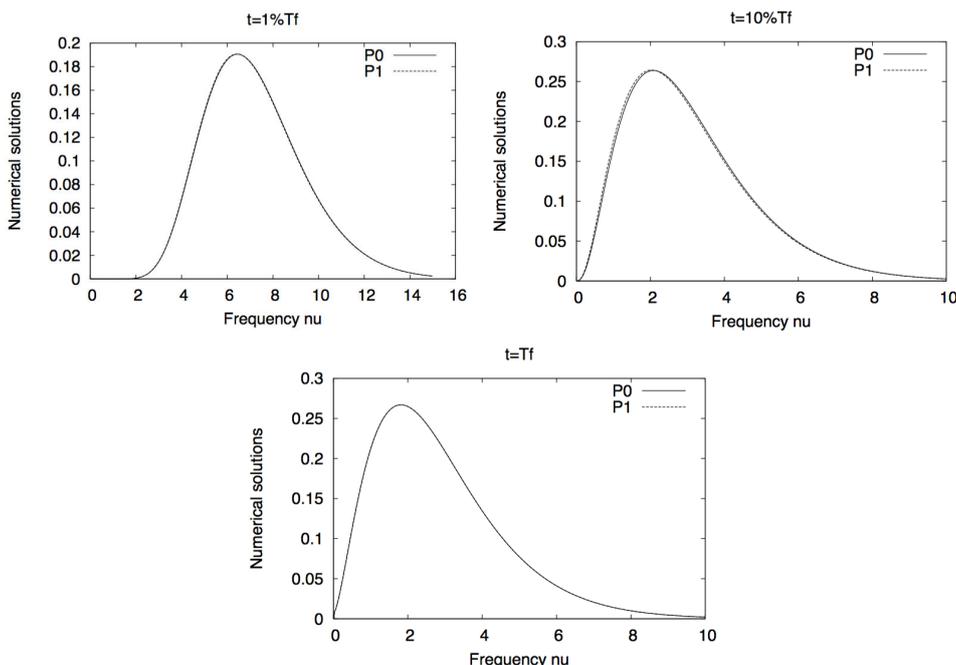


Figure 5.4: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 1$. The curves are almost merged, as expected due to the strong relaxation. The P_0 and P_1 models converge toward the same Planck distribution.

dominant in comparison to the Fokker-Planck term. Figure 5.6 presents the time evolution of the total energy of the P_0 and P_1 models. As expected (see equation (5.2.2)), the anisotropic part of the radiation decreases the energy, in comparison with the Kompaneets equation. Nevertheless, and since the first order moment V is small in comparison with the zero-th order moment, the variation in the total energy due to the anisotropic part is negligible.

Case $\alpha = 0.1$

In this part we perform the same study than in the previous part, but we take for the absorption coefficient $\alpha = 0.1$. The picture 5.7 displays the zero-th order moment U and the solution of the Kompaneets equation at different time. It shows in particular a different transitional regime. Contrary to the solution of the P_0 model, some photons are concentrated at the origin in a first time, a gain energy in a second time to equilibrate at the same Planck function than the solution of the Kompaneets equation (picture 5.7). This can be explained by studying the first

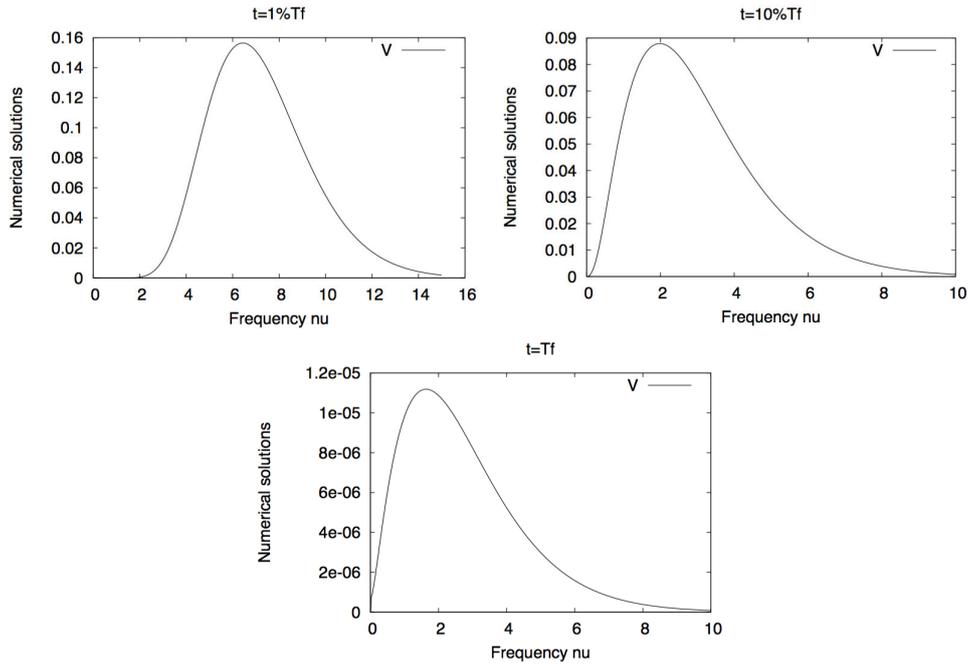


Figure 5.5: First order moment V versus frequency ν at different times, $\alpha = 1$. The first order moment has a Planck profile, decreasing with time due to the relaxation ($\alpha = 1$).

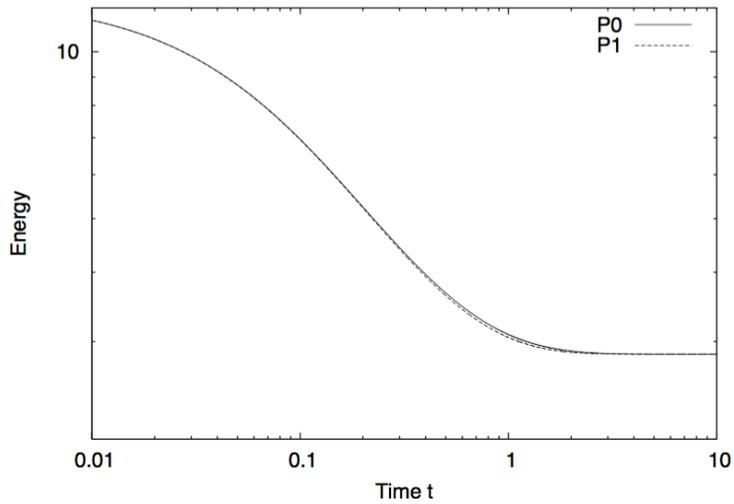


Figure 5.6: Evolution of the total energy versus time, $\alpha = 1$. Due to the strong relaxation, the anisotropic part does not sensibly modifies the energy, in comparison with the Kompaneets equation.

order moment V , whose time evolution is displayed in picture 5.8. Indeed, we see a competition between the thermalization term (Fokker-Planck term) and the absorption term $-\alpha V$. In long

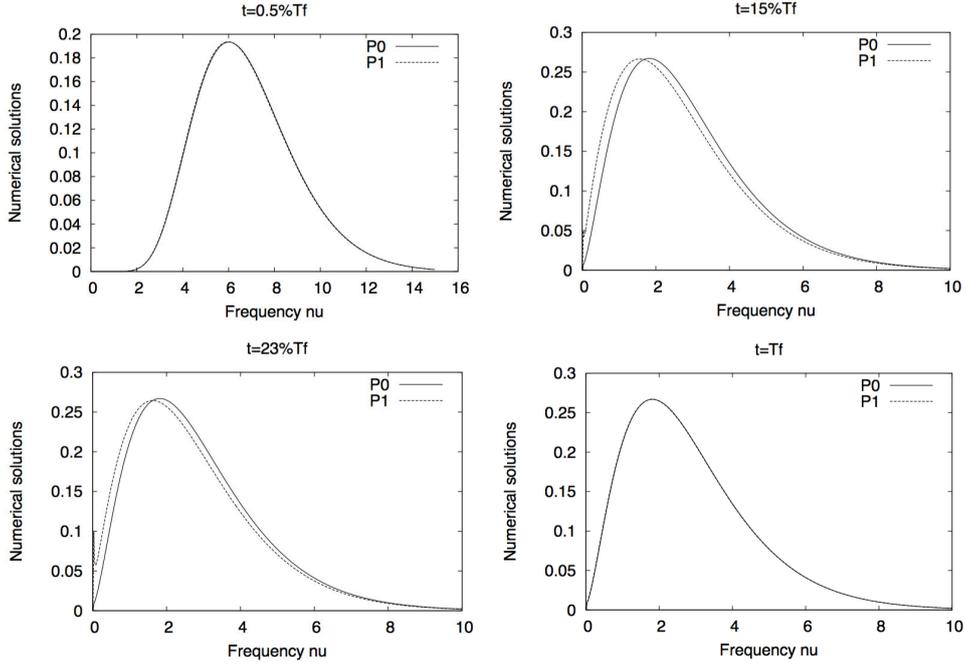


Figure 5.7: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 0.1$. A significant difference between the P_0 and P_1 models appears in a transitional regime. In long time, the P_0 and P_1 converge toward the same Planck distribution.

time, the absorption takes over for the thermalization, and one gets $\lim_{t \rightarrow \infty} V = 0$, and the P_1 model then reduces to the Kompaneets equation. This explains that one obtains $\lim_{t \rightarrow \infty} U = f$, where f is the solution of the Kompaneets equation. This transitional regime explains the significant difference in the time evolution of the total energy between the P_0 and P_1 model displayed in picture 5.9.

Case $\alpha = 0$

In this part we perform the same study than in the previous part, but we take for the absorption coefficient $\alpha = 0$. Since the initial conditions (figure 5.3) satisfy the properties of the lemma 5.2, we expect to observe numerically the conclusion of this lemma, that is the convergence in long time of the zero-th order moment U toward a Planck distribution plus a Dirac at the frequency $\nu = 0$. As for the previous numerical tests ($\alpha = 1$ and 0.1), we study the evolution, as time goes on, of the zero-th order moment U , compared to the solution of the Kompaneets equation, and the first order moment V .

Figure 5.10 displays the evolution of the solution of the Kompaneets equation (P_0) and the zero-th order moment of the P_1 model. It shows the expected result, that is the convergence of the zero-th order moment of the P_1 model toward a Dirac function plus a Planck distribution. In the same way, the figure 5.11 shows the convergence of the first order moment of the P_1 model toward a Dirac function plus a Planck function. Finally, the figure 5.12 shows the time evolution of the total energy for both the Kompaneets equation and the P_1 model. It shows in

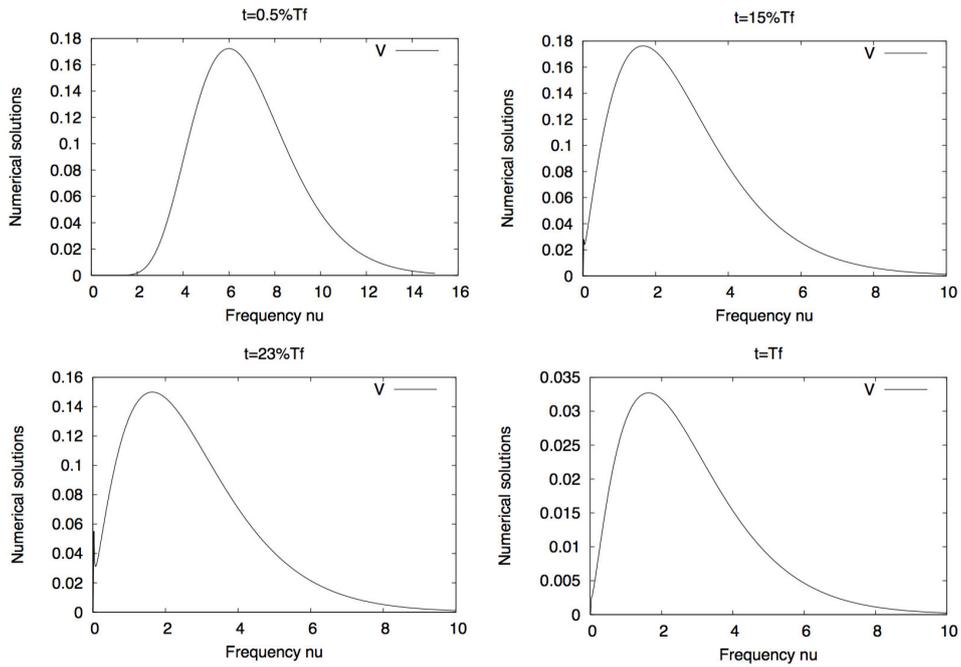


Figure 5.8: First order moment V versus frequency ν at different times, $\alpha = 0.1$. In short time range, there is a small concentration near $\nu = 0$. In long time the first order moment V tends to zero.

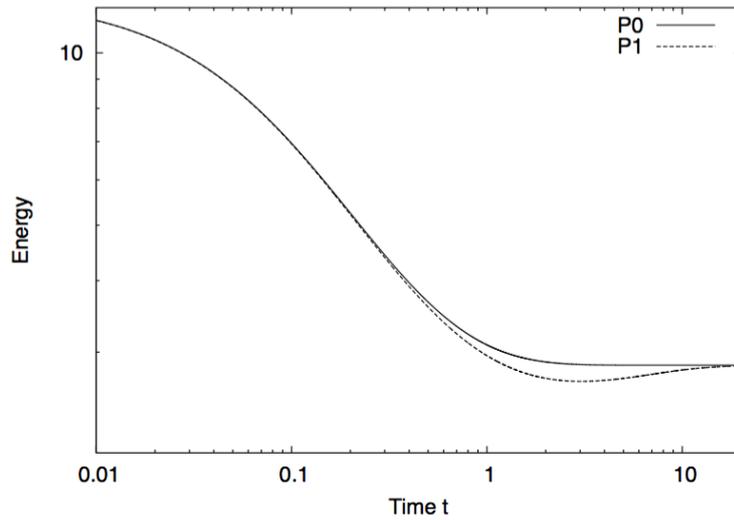


Figure 5.9: Evolution of the total energy versus time, $\alpha = 0.1$. The significant modification observed in the transitional regimes between the P_0 and P_1 models leads to a decrease of energy for the P_1 model.

particular that in this limit case $\alpha = 0$, the introduction of an anisotropic part in the radiation modified the stationary state in long time range.

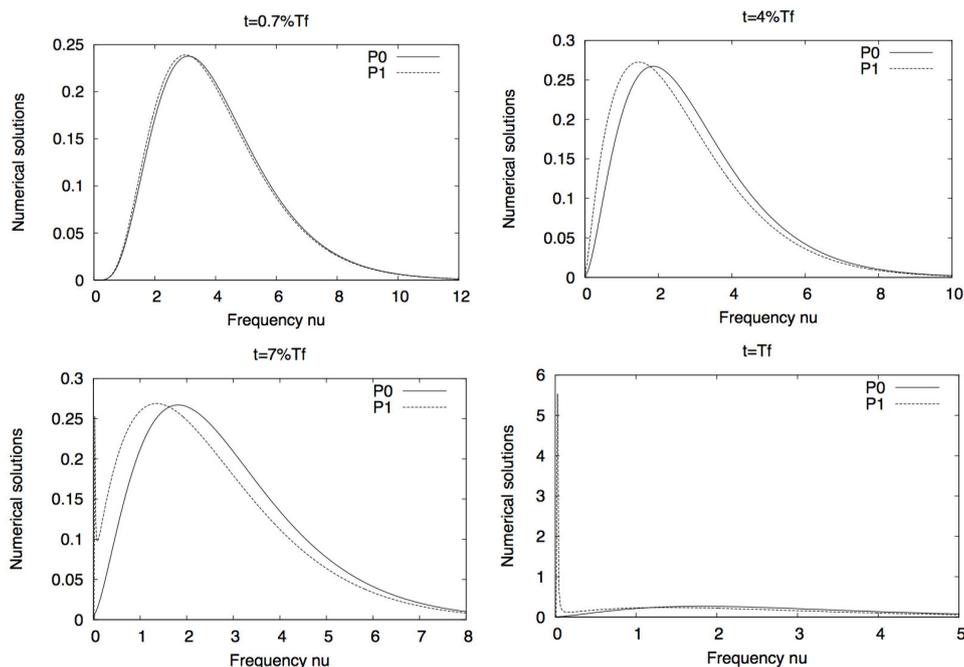


Figure 5.10: Zero-th order moments (P_0 and P_1) versus frequency ν at different times, $\alpha = 1$. The first order moment of the P_1 model converges to a different solution with respect to the Kompaneets equation. We observe that this solution is composed of the sum of a Planck type distribution and a concentration of photons near $\nu = 0$.

5.3 M_1 model

In this part we study the M_1 model for the system (5.1). The M_1 approximation consists of assuming that the distribution f writes

$$f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) = \left[\exp \left(\frac{\nu}{a(t, \mathbf{x})} \left(1 + \mathbf{b}(t, \mathbf{x}) \cdot \boldsymbol{\Omega} \right) \right) - 1 \right]^{-1}, \quad (5.21)$$

which is obtained by solving an entropy minimization problem under the constraints of preserving the moments [DF99, BD04]. The quantities a and \mathbf{b} are the Lagrange multipliers and must satisfy $a > 0$ and $|\mathbf{b}| < 1$ to preserve the non negativity of the distribution function. The distribution function f as defined by (5.21) can be seen as a generalized Planck distribution, in the sense that for $\mathbf{b} = 0$ and $a = T$, the electron temperature, one finds back the Planck distributions. The M_N methods are famous since they are known to preserve the non negativity of the distribution function and the growth of the (physical) entropy. The cost of these advantages is that since the distribution function is nonlinear with respect to $\boldsymbol{\Omega}$, an increase in the number of moments can be complicated to obtain, and in the particular case of our nonlinear source terms (Kompaneets terms), the expression of these terms in terms of the moments is not easily obtained. This is why the construction of the M_1 and more generally the M_N models mainly relies on the computations of integrals.

As for the P_1 approximation, we define the two first angular moments of the distribution function

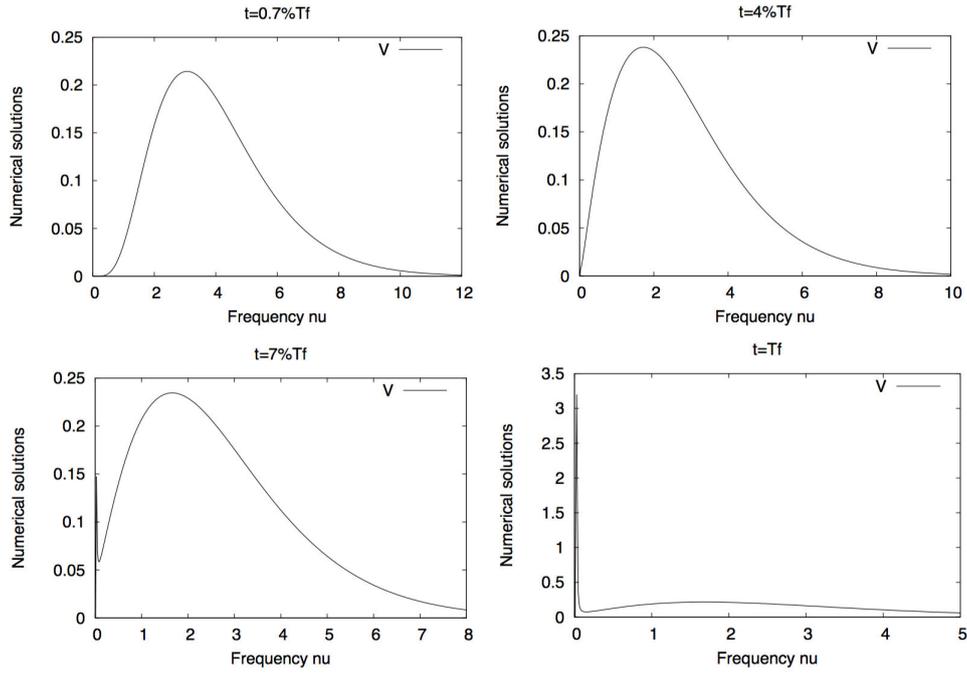


Figure 5.11: First order moment V versus frequency ν at different times, $\alpha = 0$. The first order moment converges toward the sum of a Planck type distribution and a concentration near $\nu = 0$.

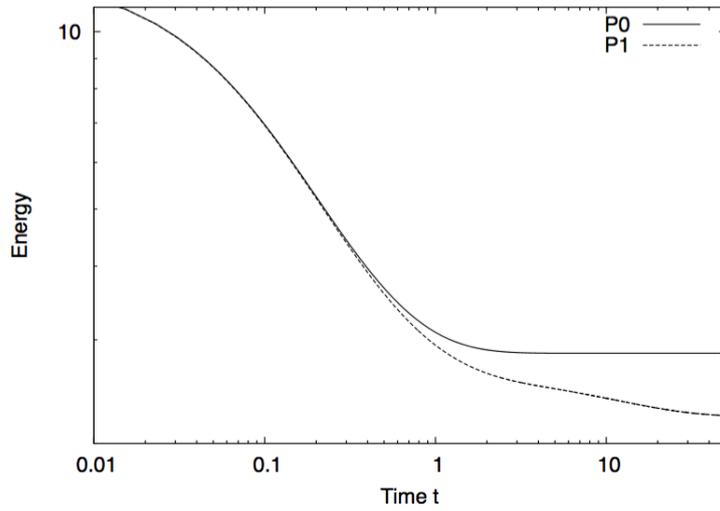


Figure 5.12: Evolution of the total energy versus time, $\alpha = 0$. There is a substantial decrease of the energy of the P_1 model in comparison with the Kompaneets equation, and the convergence of the numerical solution toward a different stationary solution.

f as

$$\begin{cases} X(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \Omega) d\Omega, \\ \mathbf{Y}(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \Omega) \Omega d\Omega. \end{cases} \quad (5.22)$$

Contrary to the P_1 approximation, in which the dependence with respect to the angle $\boldsymbol{\Omega}$ is linear, the nonlinearity of the distribution function f (5.21) prevents us to compute the quadratic terms in the model in terms of the two first moments X and \mathbf{Y} , i.e. to close the system. Indeed the same manipulations than for the P_1 model yield

$$\left\{ \begin{array}{l} \frac{1}{c} \partial_t X + \nabla \cdot \mathbf{Y} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu X + X + \frac{1}{4\pi} \int_{S^2} f^2 d\boldsymbol{\Omega} \right) \right] + \sigma_a (f_0 - X), \\ \frac{1}{c} \partial_t \mathbf{Y} + \nabla \cdot \left(\frac{1}{4\pi} \int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I_d f d\boldsymbol{\Omega} \right) = -\frac{4}{3} \sigma_s \mathbf{Y} - \sigma_a \mathbf{Y} \\ \quad + \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{Y} + \mathbf{Y} + \frac{1}{4\pi} \int_{S^2} f^2 \boldsymbol{\Omega} d\boldsymbol{\Omega} \right) \right], \\ C_V \partial_t T = -c\sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 X \right) d\nu + \frac{\sigma_s}{3mc} \int_{\mathbb{R}^+} \nu^4 \left(T \partial_\nu X + X + \frac{1}{4\pi} \int_{S^2} f^2 d\boldsymbol{\Omega} \right) d\nu. \end{array} \right. \quad (5.23)$$

The main problem here is that the integrals in (5.23) can not be formally computed in terms of the moments X and \mathbf{Y} . It could be achieved by numerical procedures (see [TUR05, TUR12] for a multigroup discretization of the second order moment), but this is not the purpose of this work. We show in this chapter that these computations can be achieved in the case of the grey M_1 .

5.3.1 Derivation of the grey M_1 model

We introduce the two first moments of $\nu^3 f$

$$\left\{ \begin{array}{l} E(t, \mathbf{x}) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) d\boldsymbol{\Omega} d\nu, \\ \mathbf{F}(t, \mathbf{x}) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f(t, x, \nu, \boldsymbol{\Omega}) \boldsymbol{\Omega} d\boldsymbol{\Omega} d\nu, \end{array} \right. \quad (5.24)$$

where we still assume that f is given by (5.21). The choice of taking the moments of $\nu^3 f$ is not insignificant. Indeed, $\nu^3 f$ is proportional to the specific intensity I and is thus a physical quantity. As a remark, one could take the moments on $\nu^N f$, with an arbitrary choice of $N \in \mathbb{N}^*$. Our choice is driven by the fact that we aim to study we exchange between E and the fluid temperature T , homogeneous to an energy. The following lemma is concerned by the grey M_1 model.

Lemma 5.3 (Grey M_1 model). *The grey M_1 model for the system (5.1) is given by*

$$\left\{ \begin{array}{l} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \frac{4\sigma_s}{3mc^2} \left(T - E^{\frac{1}{4}} Q_E(\|\mathbf{g}\|) \right) E \\ \frac{1}{c} \partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_r = -\left(\frac{4}{3} \sigma_s + \sigma_a \right) \mathbf{F} + \frac{4\sigma_s}{3mc^2} \left(T - E^{\frac{1}{4}} Q_F(\|\mathbf{g}\|) \right) \mathbf{F} \\ \bar{C}_V \partial_t T = -c\sigma_a (T^4 - E) - \frac{4\sigma_s}{3mc^2} \left(T - E^{\frac{1}{4}} Q_E(\|\mathbf{g}\|) \right) E, \end{array} \right. \quad (5.25)$$

where the operators Q_E and Q_F are defined by

$$\begin{cases} Q_E(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{5}{4}}}{4} \frac{4 + \|\mathbf{g}\|^2}{4 - \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}} \\ Q_F(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{1}{4}}}{4} \frac{20 + \|\mathbf{g}\|^2}{4 + \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}}, \end{cases}$$

and \mathbf{P}_r is the second order moment of f and is given by the Eddington formula [LEV84, DF99, BD04]

$$\mathbf{P}_r = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} f \nu^3 d\nu d\boldsymbol{\Omega} = \left((1 - \chi) \mathbf{I}_d + (3\chi - 1) \frac{\mathbf{g} \otimes \mathbf{g}}{\|\mathbf{g}\|^2} \right) E. \quad (5.26)$$

with $\mathbf{g} = \mathbf{F}/E$, $\mathbf{b} = -\frac{3}{\sqrt{4-3\|\mathbf{g}\|^2+2}}\mathbf{g}$, $\bar{C}_V = 15/\pi^4 C_V$ and $\chi = (3 + 4\|\mathbf{g}\|^2)/(5 + 2\sqrt{4 - 3\|\mathbf{g}\|^2})$ is the Eddington factor.

Proof. Multiplying the first equation of (5.1) by ν^3 and integrating over $\mathbb{R}^+ \times S^2$ yield, using an integration by parts

$$\frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \frac{4T}{3mc^2} \sigma_s E - \frac{5\sigma_s}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \frac{\nu^4}{mc^2} (f + f^2) d\nu d\boldsymbol{\Omega} \sigma_a (T^4 - E). \quad (5.27)$$

In the same way, multiplying the first equation of (5.1) by $\nu^3 \boldsymbol{\Omega}$ and integrating over $\mathbb{R}^+ \times S^2$ leads to

$$\begin{aligned} \frac{1}{c} \partial_t \mathbf{F} + \frac{15}{4\pi^5} \nabla \cdot \left(\int_{\mathbb{R}^+ \times S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} f \nu^3 d\nu d\boldsymbol{\Omega} \right) &= \frac{4}{3} \left(\frac{T}{mc^2} - 1 \right) \sigma_s \mathbf{F} - \sigma_a \mathbf{F} \\ &\quad - \frac{5\sigma_s}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \frac{\nu^4}{mc^2} \boldsymbol{\Omega} (f + f^2) d\nu d\boldsymbol{\Omega}. \end{aligned} \quad (5.28)$$

We thus need to determine five integrals in terms of the moments E and \mathbf{F} . Let us define

$$I_1 = \frac{5}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^4 f d\nu d\boldsymbol{\Omega}.$$

Using the definition of f (5.21) and the change of variable $y \mapsto (\nu/a)(1 + \mathbf{b} \cdot \boldsymbol{\Omega})$, one has

$$I_1 = \frac{5a^5}{4\pi^5} \int_{S^2} \frac{1}{(1 + \mathbf{b} \cdot \boldsymbol{\Omega})^5} d\boldsymbol{\Omega} \times \int_{\mathbb{R}^+} \frac{\nu^4}{e^\nu - 1} d\nu.$$

First, one has $\int_{\mathbb{R}^+} \frac{\nu^4}{e^\nu - 1} d\nu = 24\zeta(5)$, where ζ is the zeta Riemann function. The integral over S^2 can be computed by using the spherical coordinates $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$, where θ is chosen as the angle between \mathbf{b} and $\boldsymbol{\Omega}$. This frame is chosen such that $\mathbf{b} = (0, 0, \|\mathbf{b}\|)^T$. This gives, with the change of variables $\mu = \cos \theta$

$$I_1 = \zeta(5) \frac{60a^5}{\pi^4} \int_{-1}^1 \frac{1}{(1 + \mu \|\mathbf{b}\|)^5} d\mu = \zeta(5) \frac{120a^5}{\pi^4} \frac{1 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4}.$$

We then define

$$I_2 = \frac{5}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^4 f^2 d\nu d\boldsymbol{\Omega} = \frac{5a^5}{4\pi^5} \int_{S^2} \frac{1}{(1 + \mathbf{b} \cdot \boldsymbol{\Omega})^5} d\boldsymbol{\Omega} \times \int_{\mathbb{R}^+} \frac{\nu^4}{(e^\nu - 1)^2} d\nu.$$

Once again, one has $\int_{\mathbb{R}^+} \frac{\nu^4}{(e\nu-1)^2} d\nu = -24\zeta(5) + 4\pi^4/15$. Using the same technics than for the integral I_1 , one finds

$$I_2 = \left(-\frac{120}{\pi^4}\zeta(5) + \frac{4}{3} \right) a^5 \frac{1 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4}. \quad (5.29)$$

We finally finds

$$-\frac{5}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^4 (f + f^2) d\nu d\Omega = -a^5 \frac{4}{3} \frac{1 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4}.$$

We now turn to the integrals in the equation of \mathbf{F} . We define

$$\mathbf{I}_3 = \frac{5}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^4 \Omega f d\nu d\Omega.$$

Once again, one has

$$\mathbf{I}_3 = \frac{15 \times 24}{\pi^4} \zeta(5) \frac{a^5}{12\pi} \int_{S^2} \frac{\Omega}{(1 + \mathbf{b} \cdot \Omega)^5} d\nu d\Omega.$$

It yields

$$\mathbf{I}_3 = -\frac{140\zeta(5)}{3\pi^4} a^5 \frac{5 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4} \mathbf{b}. \quad (5.30)$$

In the same way, we define

$$\mathbf{I}_4 = \frac{5}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^4 \Omega f^2 d\nu d\Omega,$$

and finds

$$\mathbf{I}_4 = -\frac{5a^5}{3\pi^4} \left(-24\zeta(5) + \frac{4\pi^4}{15} \right) \frac{5 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4} \mathbf{b}. \quad (5.31)$$

It yields

$$-\frac{5}{4\pi^5} \int \nu^4 \Omega (f + f^2) d\nu d\Omega = \frac{4a^5}{9} \frac{5 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4} \mathbf{b}.$$

Finally, the second order moment \mathbf{P}_r can also be computed

$$\mathbf{P}_r = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \Omega \otimes \Omega f \nu^3 d\nu d\Omega = \left((1 - \chi) \mathbf{I}_d + (3\chi - 1) \frac{\mathbf{g} \otimes \mathbf{g}}{\|\mathbf{g}\|^2} \right) E,$$

where $\mathbf{g} = \mathbf{F}/E$ and $\chi = (3 + 4\|\mathbf{g}\|^2)/(5 + 2\sqrt{4 - 3\|\mathbf{g}\|^2})$ is the Eddington factor. The equation on the temperature T is trivially obtained by multiplying the last equation of (5.23) by $15/\pi^4$ since it involves the same terms as in the equation of the zero-th order moment E . We thus find the following grey M_1 model, where $\bar{C}_V = 15/\pi^4 C_V$

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \sigma_s \frac{T}{mc^2} \frac{4}{3} E - \frac{4\sigma_s a^5}{3mc^2} \frac{1 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4}, \\ \frac{1}{c} \partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_r = \frac{4}{3} \left(\frac{T}{mc^2} - 1 \right) \sigma_s \mathbf{F} - \sigma_a \mathbf{F} + \frac{4\sigma_s a^5}{9mc^2} \frac{5 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4} \mathbf{b}, \\ \bar{C}_V \partial_t T = -c\sigma_a (T^4 - E) - \frac{T}{mc} \frac{4}{3} \sigma_s E + \frac{4\sigma_s a^5}{3mc} \frac{1 + \|\mathbf{b}\|^2}{(1 - \|\mathbf{b}\|^2)^4}. \end{cases} \quad (5.32)$$

It remains to find the relation between (a, \mathbf{b}) and (E, \mathbf{F}) . The definition of E and \mathbf{F} (5.24), together with the expression of f (5.21) yield

$$\begin{cases} E(t, \mathbf{x}, \nu) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f d\Omega d\nu = a^4 \frac{3 + \|\mathbf{b}\|^2}{3(1 - \|\mathbf{b}\|^2)^3}, \\ \mathbf{F}(t, \mathbf{x}, \nu) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f \boldsymbol{\Omega} d\Omega d\nu = -4a^4 \frac{\mathbf{b}}{3(1 - \|\mathbf{b}\|^2)^3}. \end{cases} \quad (5.33)$$

Let us compute the expression of the coefficients a and \mathbf{b} in terms of E and \mathbf{F} . The relations (5.33) yield $\mathbf{g} = \mathbf{F}/E = -4\frac{\mathbf{b}}{3+\|\mathbf{b}\|^2}$. One thus has $\|\mathbf{g}\|^2(3 + \|\mathbf{b}\|^2)^2 = 16\|\mathbf{b}\|^2$. Solving this second order equation (with respect to $\|\mathbf{b}\|^2$), one finds $\|\mathbf{b}\|_{\pm}^2 = \|\mathbf{g}\|^{-2}(8 - 3\|\mathbf{g}\|^2 \pm 4\sqrt{4 - 3\|\mathbf{g}\|^2})$. Both these solutions are non negative, but only the one with the minus root is consistent with the fact that $\|\mathbf{b}\|$ has to be lesser than 1. It yields for the coefficients a and \mathbf{b}

$$\begin{cases} a = 2E^{\frac{1}{4}}(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{1}{4}} \left(\frac{1 - \|\mathbf{g}\|^2}{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2} \right)^{\frac{3}{4}}, \\ \mathbf{b} = -\frac{3}{\sqrt{4 - 3\|\mathbf{g}\|^2} + 2} \mathbf{g}. \end{cases} \quad (5.34)$$

Using these relations, we can write the source terms in (5.32) in terms of E and \mathbf{F} instead of a and \mathbf{b} , which yields the announced system. \square

5.3.2 Diffusion approximations

Since these regimes are of great interest in radiative transfer, we study the equilibrium and non-equilibrium diffusion regimes for the grey M_1 model.

Equilibrium diffusion

In this part we study the equilibrium diffusion regime for the M_1 grey model. Our aim is to check that we recover the same diffusion equation than in the lemma 4.8 for the kinetic coupled system (5.1), since in this regime the radiative temperature equilibrates to the electrons temperature to the power 4. For ease of notations we drop the bars in the notations of C_V . Once again we rescale the system (5.25) for $\sigma_s/\sigma_a \ll 1$ and $c \gg 1$, which yields

$$\begin{cases} \frac{1}{c} \partial_t E^\varepsilon + \frac{1}{\varepsilon} \nabla \cdot \mathbf{F}^\varepsilon = \varepsilon^2 \frac{4\sigma_s}{3mc^2} E^\varepsilon \left(T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_E^\varepsilon(\|g^\varepsilon\|) \right) + \frac{\sigma_a}{\varepsilon^2} ((T^\varepsilon)^4 - E^\varepsilon) \\ \frac{1}{c} \partial_t \mathbf{F}^\varepsilon + \frac{1}{\varepsilon} \nabla \cdot \mathbf{P}^\varepsilon_r = \varepsilon^2 \frac{4\sigma_s}{3} \left(\frac{T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_F^\varepsilon(\|g^\varepsilon\|)}{mc^2} - 1 \right) \mathbf{F}^\varepsilon - \frac{\sigma_a}{\varepsilon^2} \mathbf{F}^\varepsilon, \\ C_V \partial_t T^\varepsilon = -c \frac{\sigma_a}{\varepsilon^2} ((T^\varepsilon)^4 - E^\varepsilon) - \varepsilon^2 \frac{4\sigma_s}{3mc} E^\varepsilon \left(T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_E^\varepsilon(\|g^\varepsilon\|) \right) \end{cases} \quad (5.35)$$

We prove the

Lemma 5.4 (Equilibrium diffusion regime for the grey M_1 model). *A first order approximation of the grey M_1 model (5.35) in the equilibrium regime is*

$$\partial_t \left(C_V T_m + T_m^4 \right) - \nabla \cdot \left(\frac{\nabla T_m^4}{3\sigma_a} \right) = 0$$

Proof. As before, we perform a Hilbert expansion of the unknowns

$$\begin{cases} E^\varepsilon = E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \dots \\ \mathbf{F}^\varepsilon = \mathbf{F}^0 + \varepsilon \mathbf{F}^1 + \varepsilon^2 \mathbf{F}^2 + \dots \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

At the order ε^{-2} , one finds

$$\begin{cases} E^0 = (T^0)^4, \\ \mathbf{F}^0 = \mathbf{0}. \end{cases} \quad (5.36)$$

For the order ε^{-1} , one finds

$$\begin{cases} \nabla \cdot \mathbf{F}^0 = \sigma_a (4(T^0)^3 T^1 - E^1), \\ \nabla \mathbf{P}_r^0 = -\sigma_a \mathbf{F}^1. \end{cases}$$

The third order moment \mathbf{P}_r is defined by the Eddington factor (5.26) and one thus finds, using the relation between a and \mathbf{b} (5.33)

$$\mathbf{P}_r^0 = \frac{1}{3} E^0.$$

It yields in particular

$$\mathbf{F}^1 = -\frac{\nabla E^0}{3\sigma_a}.$$

Finally, one finds by denoting $T_m = T^0$ and by adding the equations at the order ε^0 satisfied by E^ε and T^ε

$$\partial_t \left(C_V T_m + T_m^4 \right) - \nabla \cdot \left(\frac{\nabla T_m^4}{3\sigma_a} \right) = 0,$$

which is exactly the diffusion equation obtained in the lemma 4.8 for the system (5.1) and which is the expected result. \square

Non-equilibrium diffusion regime

In this part we study the non-equilibrium diffusion regime of the grey M_1 moment model (5.25). Like in the beginning of this section, we perform the following scaling

$$\begin{cases} c = \varepsilon^{-1} \bar{c}, \\ \sigma_a = \varepsilon \bar{\sigma}_a, \\ \sigma_s = \varepsilon^{-1} \bar{\sigma}_s. \end{cases}$$

The moment model thus becomes

$$\begin{cases} \frac{1}{c}\partial_t E^\varepsilon + \frac{1}{\varepsilon}\nabla \cdot \mathbf{F}^\varepsilon = \sigma_a \left((T^\varepsilon)^4 - E^\varepsilon \right) + \frac{4\sigma_s}{3mc^2} \left(T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_E^\varepsilon(\|\mathbf{g}^\varepsilon\|) \right) E^\varepsilon, \\ \frac{1}{c}\partial_t \mathbf{F}^\varepsilon + \frac{1}{\varepsilon}\nabla \cdot \mathbf{P}_r^\varepsilon = -\frac{4\sigma_s}{3\varepsilon^2} \mathbf{F}^\varepsilon - \sigma_a \mathbf{F}^\varepsilon + \frac{4\sigma_s}{3mc^2} \left(T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_F^\varepsilon(\|\mathbf{g}^\varepsilon\|) \right) \mathbf{F}^\varepsilon, \\ C_V \partial_t T^\varepsilon = -c\sigma_a \left((T^\varepsilon)^4 - E^\varepsilon \right) - \frac{4\sigma_s}{3mc} \left(T^\varepsilon - (E^\varepsilon)^{\frac{1}{4}} Q_E^\varepsilon(\|\mathbf{g}^\varepsilon\|) \right) E^\varepsilon. \end{cases} \quad (5.37)$$

We prove the

Lemma 5.5 (Non-equilibrium diffusion regime for the grey M_1 model). *A first order approximation of the grey M_1 model (5.37) in the non-equilibrium diffusion regime is*

$$\begin{cases} \frac{1}{c}\partial_t E - \nabla \cdot \left(\frac{\nabla E}{4\sigma_s} \right) = \left(T - E^{\frac{1}{4}} \right) \left(\sigma_s \frac{4E}{3mc^2} + \sigma_a (T + E^{\frac{1}{4}}) (T^2 + E^{\frac{1}{2}}) \right), \\ C_V \partial_t T = - \left(T - E^{\frac{1}{4}} \right) \left(\sigma_s \frac{4E}{3mc} + c\sigma_a (T + E^{\frac{1}{4}}) (T^2 + E^{\frac{1}{2}}) \right). \end{cases} \quad (5.38)$$

One remarks that the obtained system can be seen as the classical non-equilibrium diffusion regime in the case of Thomson scattering perturbed by the term $\sigma_s \frac{4E}{3mc^2}$ coming from the Compton scattering. It is important to notice that the Compton scattering do not modify the equilibrium state $E = T^4$, but can modify the convergence rate to the equilibrium, depending to the value of $\sigma_s \frac{4E}{3mc}$ with respect to $c\sigma_a (T + E^{\frac{1}{4}}) (T^2 + E^{\frac{1}{2}})$.

Proof. We perform an Hilbert expansion of the unknowns $E, \mathbf{F}, \mathbf{P}_r^\varepsilon, \mathbf{g}^\varepsilon$ and T , that is

$$\begin{cases} E^\varepsilon = E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \dots \\ \mathbf{F}^\varepsilon = \mathbf{F}^0 + \varepsilon \mathbf{F}^1 + \varepsilon^2 \mathbf{F}^2 + \dots \\ \mathbf{P}_r^\varepsilon = \mathbf{P}_r^0 + \varepsilon \mathbf{P}_r^1 + \varepsilon^2 \mathbf{P}_r^2 + \dots \\ \mathbf{g}^\varepsilon = \mathbf{g}^0 + \varepsilon \mathbf{g}^1 + \varepsilon^2 \mathbf{g}^2 + \dots \\ T^\varepsilon = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots \end{cases}$$

Obviously the coefficients of the Hilbert expansion of \mathbf{P}_r^ε are directly linked to E^ε and \mathbf{F}^ε by the Eddington relation (5.26). In the same way, the coefficients of the Hilbert expansion of \mathbf{g}^ε are given by the coefficients of the Hilbert expansion of E^ε and \mathbf{F}^ε . At the order ε^{-2} , one finds $\mathbf{F}^0 = \mathbf{0}$, which yields in particular $\mathbf{g}^0 = \mathbf{0}$ and $\mathbf{b}^0 = \mathbf{0}$ (where \mathbf{b} is given in (5.34) in terms of E and \mathbf{F}). At the order ε^{-1} , one finds

$$\nabla \cdot \mathbf{P}_r^0 = -\sigma_s \frac{4}{3} \mathbf{F}^1.$$

Since $\mathbf{g}^0 = \mathbf{0}$ and $\mathbf{b}^0 = \mathbf{0}$, one finds by using the relation (5.26) $\nabla \cdot \mathbf{P}_r^0 = \frac{1}{3} E^0$. One thus finally finds $\mathbf{F}^1 = -\frac{1}{4\sigma_s} \nabla E^0$. At the order ε^0 , one has

$$\begin{cases} \frac{1}{c}\partial_t E^0 + \nabla \cdot \mathbf{F}^1 = \sigma_s \frac{4T^0}{3mc^2} E^0 + \sigma_a \left((T^0)^4 - E^0 \right) - \sigma_s \frac{4(E^0)^{\frac{5}{4}}}{3mc^2} \\ C_V \partial_t T^0 = -\sigma_s \frac{4T^0}{3mc} E^0 - c\sigma_a \left((T^0)^4 - E^0 \right) + \sigma_s \frac{4(E^0)^{\frac{5}{4}}}{3mc}. \end{cases}$$

It yields, denoting $E = E^0$ and $T = T^0$ and using the expression of \mathbf{F}^1

$$\begin{cases} \frac{1}{c}\partial_t E - \nabla \cdot \left(\frac{\nabla E}{4\sigma_s} \right) = \sigma_s \frac{4E}{3mc^2} \left(T - E^{\frac{1}{4}} \right) + \sigma_a (T^4 - E), \\ C_V \partial_t T = -\sigma_s \frac{4E}{3mc} \left(T - E^{\frac{1}{4}} \right) - c\sigma_a (T^4 - E). \end{cases}$$

Since $T^4 - E = (T - E^{\frac{1}{4}})(T + E^{\frac{1}{4}})(T^2 + E^{\frac{1}{2}})$, one finds the announced system. \square

5.3.3 Case of an isotropic distribution

Since the grey M_1 model (5.25) derived in the previous part is rather complicated, since it involves highly nonlinear source terms, it can be interesting to study some particular situations. In particular the case of an isotropic distribution, which often arises in long time range, can be considered. In this case, one has $\mathbf{F} = 0$ and the system reduces to

$$\begin{cases} \frac{1}{c}\partial_t E = \sigma_a (T^4 - E) + \frac{4\sigma_s}{3mc^2} (T - E^{\frac{1}{4}})E, \\ \bar{C}_V \partial_t T = -c\sigma_a (T^4 - E) - \frac{4\sigma_s}{3mc} (T - E^{\frac{1}{4}})E. \end{cases} \quad (5.39)$$

It is interesting to notice that this system corresponds to the non-equilibrium diffusion regime (5.38) in the homogeneous (in space) case.

5.3.4 Numerical illustration

In this part we study numerically the grey M_1 model (5.25). The main issue here is to understand how the introduction of the Kompaneets terms in the coupled model (5.1) may/do modify the long time behavior of the classical situation of isotropic scattering with emission absorption. Let us explain it by considering this classical system composed of isotropic scattering and a basic emission absorption operator

$$\begin{cases} \frac{1}{c}\partial_t f + \mathbf{\Omega} \cdot \nabla f = \frac{\sigma_s}{4\pi} \int_{S^2} (1 + \cos^2 \theta) (f' - f) d\mathbf{\Omega}' + \sigma_a \left(f_0(\nu, T) - f \right), \\ C_V \partial_t T = -\frac{c\sigma_a}{4\pi} \int_{\mathbb{R}^+ \times S^2} \left(B(\nu, T) - \nu^3 f \right) d\nu d\mathbf{\Omega}. \end{cases} \quad (5.40)$$

For this system, the grey M_1 model writes

$$\begin{cases} \frac{1}{c}\partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E), \\ \frac{1}{c}\partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_r = -\left(\frac{4}{3}\sigma_s + \sigma_a \right) \mathbf{F}, \\ \bar{C}_V \partial_t T = -c\sigma_a (T^4 - E), \end{cases} \quad (5.41)$$

and it is known that in long time range, the radiative temperature $T_R = E^4$ equilibrates with the electron temperature, i.e.

$$\lim_{t \rightarrow \infty} T_R = T.$$

On the other hand, denoting $\alpha = 1/3mc^2$, the grey M_1 model (5.25) writes

$$\begin{cases} \frac{1}{c}\partial_t E + \nabla \cdot \mathbf{F} = \sigma_a(T^4 - E) + 4\alpha\sigma_s \left(T - E^{\frac{1}{4}}Q_E(\|\mathbf{g}\|) \right) E \\ \frac{1}{c}\partial_t \mathbf{F} + \nabla \cdot \mathbf{P}_F = -\left(\frac{4}{3}\sigma_s + \sigma_a \right) \mathbf{F} + 4\alpha\sigma_s \left(1 - E^{\frac{1}{4}}Q_F(\|\mathbf{g}\|) \right) \mathbf{F} \\ \bar{C}_V \partial_t T = -c\sigma_a(T^4 - E) - 4\alpha\sigma_s \left(T - E^{\frac{1}{4}}Q_E(\|\mathbf{g}\|) \right) E, \end{cases} \quad (5.42)$$

where the operators Q_E and Q_F are defined by

$$\begin{cases} Q_E(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{5}{4}}}{4} \frac{4 + \|\mathbf{g}\|^2}{4 - \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}} \\ Q_F(\|\mathbf{g}\|) = \frac{(2 + \sqrt{4 - 3\|\mathbf{g}\|^2})^{\frac{1}{4}}}{4} \frac{20 + \|\mathbf{g}\|^2}{4 + \|\mathbf{g}\|^2 + 2\sqrt{4 - 3\|\mathbf{g}\|^2}} \left(\frac{2 + \sqrt{4 - 3\|\mathbf{g}\|^2} - \|\mathbf{g}\|^2}{1 - \|\mathbf{g}\|^2} \right)^{\frac{1}{4}}. \end{cases}$$

It is thus interesting to study the solution of this grey M_1 model with respect to α , and to compare its solution with the classical grey M_1 model (5.41).

To this end we restrict ourself to the 1D case and we consider a domain $\mathbb{D} = [0, 10]$. For $1 \leq j \leq N$, we consider an irregular mesh defined by $(N + 1)$ points $0 = x_{\frac{1}{2}} < \dots < x_{N+\frac{1}{2}} = x^*$. We define x_j as the middle of the j -th cell, i.e. $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and we denote Δx_j its length. We also define the dual $(j + \frac{1}{2})$ -th cell as the cell $[x_j, x_{j+1}]$, which length is denoted $\Delta x_{j+\frac{1}{2}}$. In all the numerical tests we consider periodic boundary conditions. The discretization of the grey M_1 model is performed by using a Roe type scheme [ACCD02] for the radiation transport (zero-th order moment E and first order moment F) and an explicit Euler discretization of the time derivatives. We consider a fully explicit discretization of the source terms. For simplicity the coefficients in the system (5.25) are taken as $C_V = \sigma_s = c = 1$, and the opacity σ_a changes according to the test case. The mesh is composed of 100 cells.

The section is organized as follows. In a first part we explain our procedure to get the stationary solution, either from an analytical or numerical expression. The end of the section presents two test cases to illustrate the influence of the Kompaneets terms in our grey M_1 model (5.42) in comparison with the classical grey M_1 model (5.41). In the first test case the initial conditions are tuned to obtained an analytical stationary solution. In the second test case, the stationary solution is obtained from a numerical procedure. The coefficient σ_a is chosen to increase the difference between the two models.

Stationary solutions

Since we are interested in the long time behavior of the solution of the grey M_1 model (5.25), and more precisely in the relaxation of the radiative temperature to the electron temperature, it is important to be able to compute the convergence rate of the solution. This can be achieved by using the energy conservation (5.2) satisfied by the system (5.1). For all $t \geq 0$, one has

$$\int_{\mathbb{D}} \left(E(t) + T(t) \right) dx = \int_{\mathbb{D}} \left(E^{in} + T^{in} \right) dx.$$

Since, as $t \rightarrow \infty$, the solution (E, T) converges to a stationary state satisfying $E^\infty = (T^\infty)^4$, one finds the following relation between the energy at initial time and the stationary solution

$$E^\infty + (E^\infty)^4 = \frac{1}{|\mathbb{D}|} \int_{\mathbb{D}} (E^{in} + T^{in}) dx. \quad (5.43)$$

There exists particular cases for which analytical solutions can be obtained. For example, if for all $x \in [0, x^*]$, $E^{in}(x) + T^{in}(x) = 2$, one easily finds $E^\infty = T^\infty = 1$. This class of initial conditions is used in the following first test case. In the second one, this equation is solved numerically with a small enough error.

First test case: piecewise constant initial conditions

In a first test case we consider piecewise constant initial conditions satisfying for all $x \in [0, x^*]$, $E^{in}(x) + T^{in}(x) = 2$. As explained at the end of the previous part, the stationary solutions in long time range can be computed and one finds $E^\infty = T^\infty = 1$. More precisely, we assume that the distribution at initial time is isotropic, i.e. $F^{in} = 0$, and the initial zero-th order moment and the initial electron temperature are displayed in picture 5.13. The opacity σ_a is

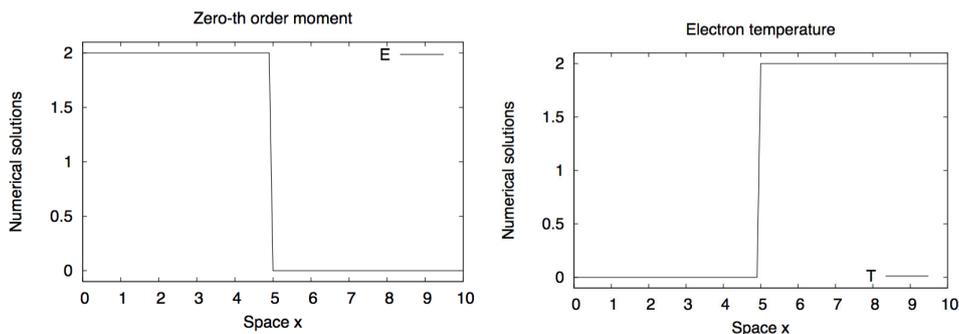


Figure 5.13: Initial conditions versus space. Left: zero-th order moment, right: electron temperature

taken equal to one. We are interested in the relaxation of the radiative temperature to the electron temperature, and more precisely to the influence of the Kompaneets terms with respect to the classical scattering case. We fix a final time $T^f = 200$ and we let the system move on for several values of the parameter α . Figure 5.14 shows the zero-th order moment (left), the electron temperature (right) and the first order moment (bottom) at time $t = 0.1$ for three values of the parameter α , namely $\alpha = 0, 0.5$ and 1 , the case $\alpha = 0$ being the solution of the classical system (5.41). It shows that the Kompaneets terms modify the transitional state. In particular remaining that the equilibrium state is defined by $E^\infty = T^\infty = 1$, we see that in the case $\alpha = 1$, the solution at time $t = 0.1$ is closer, in some sense, to the stationary state. The same solutions are displayed in picture 5.15 at time $t = 1$. The knowledge of the stationary states for given initial conditions allows to compute the time evolution of the numerical error between the numerical solution at time t and the stationary states. To this end we define the following L^1 error

$$\mathcal{E}(t) = \int_{\mathbb{D}} (|E(t, x) - E^\infty| + |T(t, x) - T^\infty|) dx. \quad (5.44)$$

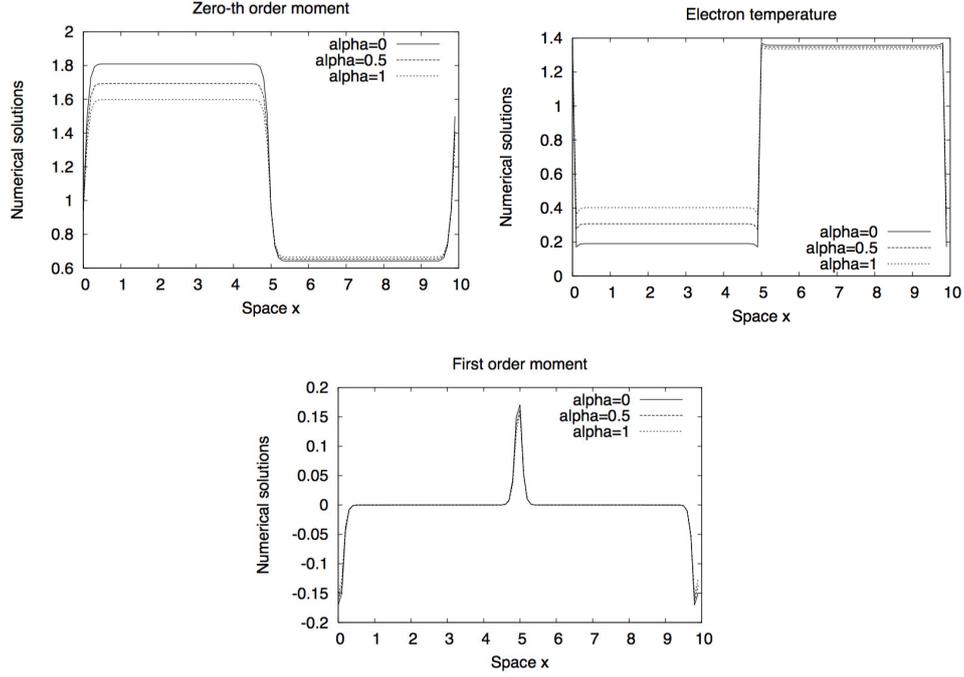


Figure 5.14: Numerical solutions of (5.42) versus space for several values of the parameter α at time $t = 0.1$. Top left: zero-th order moment, top right: electron temperature, bottom zero-th order moment. The Kompaneets terms modify the transitional regime with respect to the classical isotropic scattering.

Figure 5.16 displayed this numerical error with respect to the time for the three values of the parameter α in a log scaled plan. As observed in the previous pictures, it shows that in this test case the Kompaneets terms increase the rate of convergence toward equilibrium.

Second test case: crenel initial conditions

In this part we still aim to compare the solution of the grey M_1 model (5.42) and the solution of the classical grey M_1 model (5.41) for different values of the parameter α . To increase the difference between the two solutions, we consider an isotropic distribution and we take for the opacity $\sigma_a = 0.1$, σ_s being kept equal to one. In this case, the model reduces to (5.39) (case of an isotropic distribution). In particular the convergence speed is related only to $\sigma_a(T^4 - E)$ in the case of the classical grey M_1 model (no Kompaneets terms), while a competition between the parameters σ_a and σ_s drives this convergence rate in the other case.

The initial conditions are taken as follows (picture 5.17). The first order moment is taken as zero and the zero-th order moment and the electron temperature are taken as

$$E^{in}(x) = \begin{cases} 2 & \text{if } 3.25 \leq x \leq 3.75, \\ 10^{-14} & \text{elsewhere} \end{cases} \quad \text{and} \quad T^{in}(x) = \begin{cases} 2 & \text{if } 6.25 \leq x \leq 6.75, \\ 10^{-14} & \text{elsewhere} \end{cases}$$

For this initial conditions, the stationary solution is given by

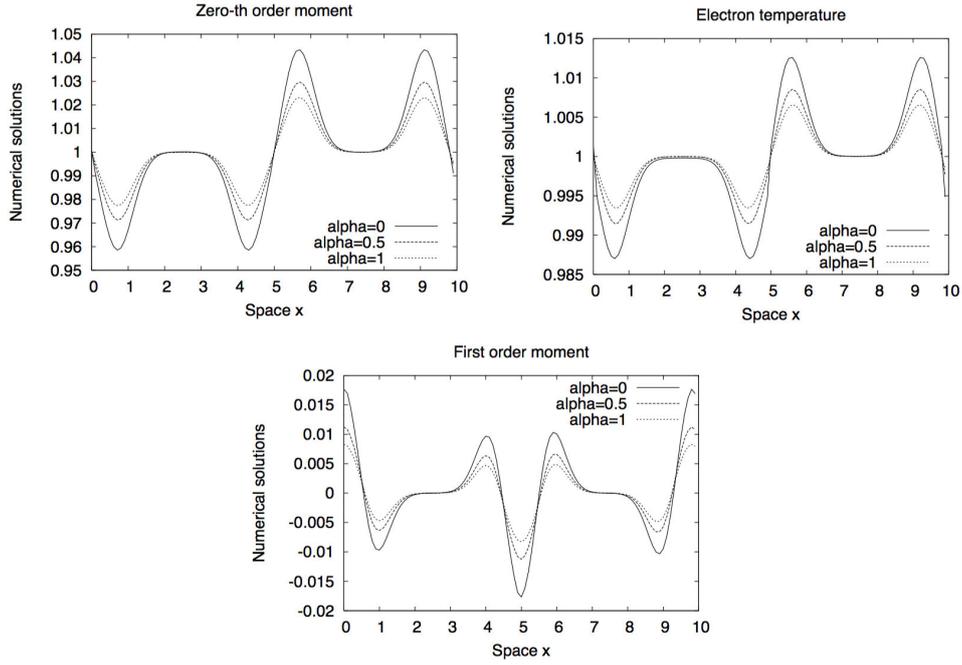


Figure 5.15: Numerical solutions of (5.42) versus space for several values of the parameter α at time $t = 1$. Top left: zero-th order moment, top right: electron temperature, bottom zero-th order moment.

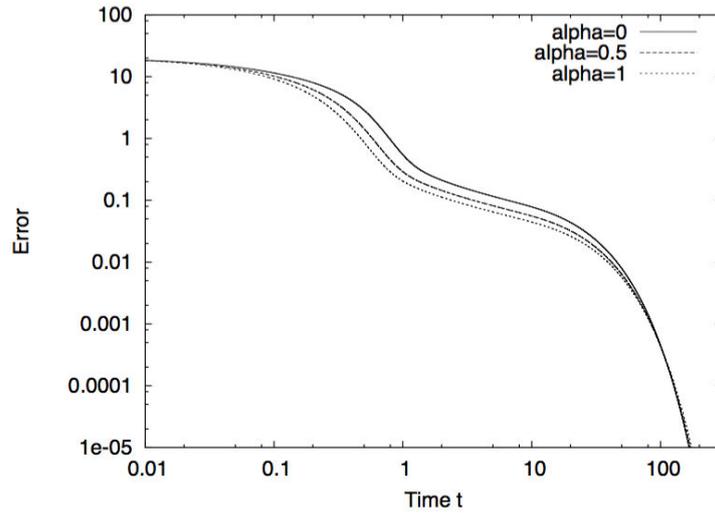


Figure 5.16: Numerical error $\mathcal{E}(t)$, defined in (5.44), versus time in a log scaled plan for several values of the parameter α . The Kompaneets terms increase the rate of convergence toward the equilibrium.

$$\begin{cases} E^\infty = 0.00340129681, \\ T^\infty = 0.24149666237. \end{cases} \quad (5.45)$$

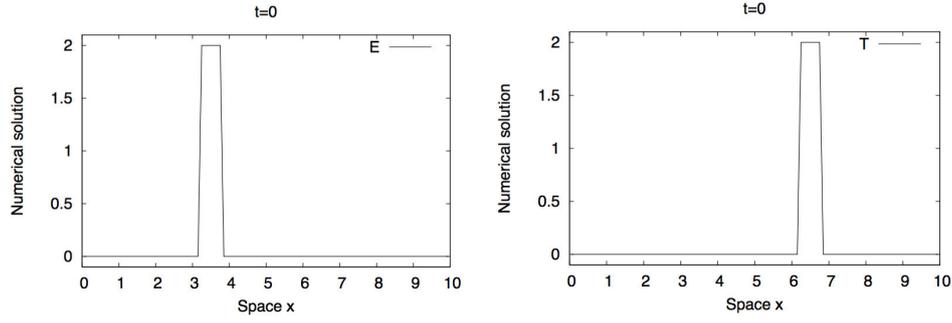


Figure 5.17: Initial conditions of the grey M_1 model (5.42) versus space. Left: zero-th order moment E , right: electron temperature T .

This solution, obtained as a numerical approximation of (5.43), is correct a order 10^{-10} . Once

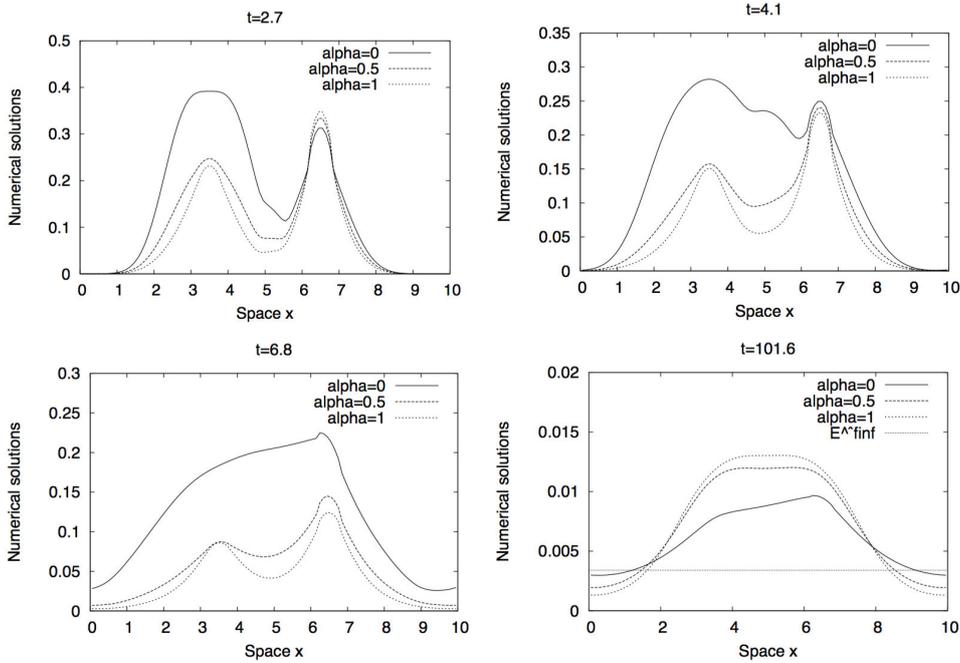


Figure 5.18: First order moment E versus space for several values of the parameter α at different times. The picture at the bottom right corner also displays the stationary solution E^∞ defined in (5.45). The Kompaneets terms in the grey M_1 model (5.42) bring a significant correction with respect to the classical case (model (5.41)).

again, we let the system move on for several values of the parameter α . Picture 5.18 displays for three values of the parameter α ($\alpha = 0, 0.5$ and 1) the first order moment E at several times. It shows that the junction of the two bumps generated by the particular choice of initial conditions takes a longer time when the Compton scattering (Kompaneets terms) is taken into account. The time evolution of the electron temperature Y is depicted on picture 5.19 for several values of the parameter α . Once again, some significant difference are observed between the case $\alpha = 0$

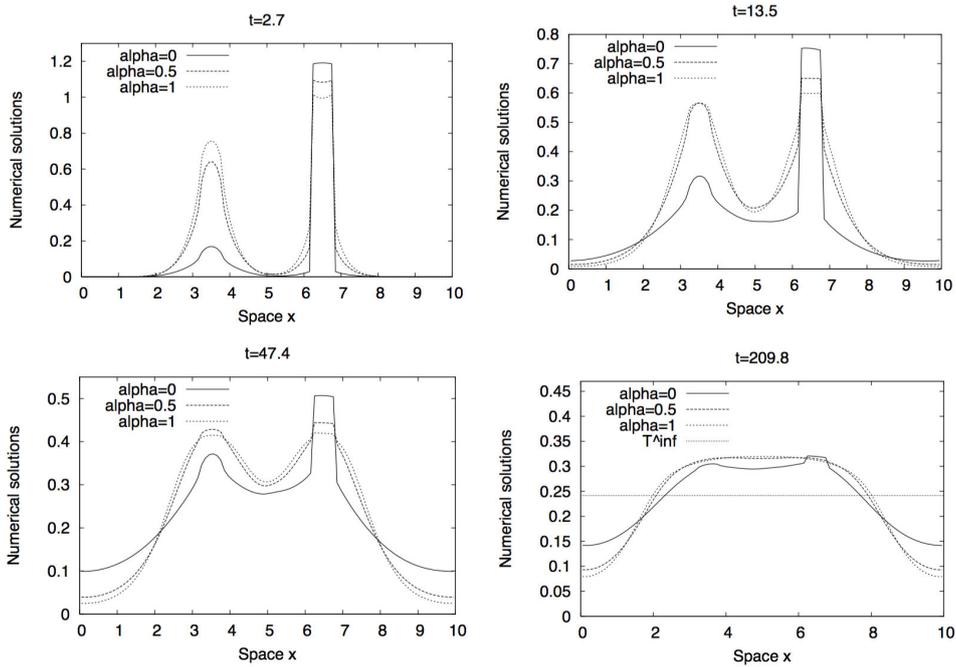


Figure 5.19: Electron temperature T versus space for several values of the parameter α at different times. The picture at the bottom right corner also displays the stationary solution T^∞ defined in (5.45).

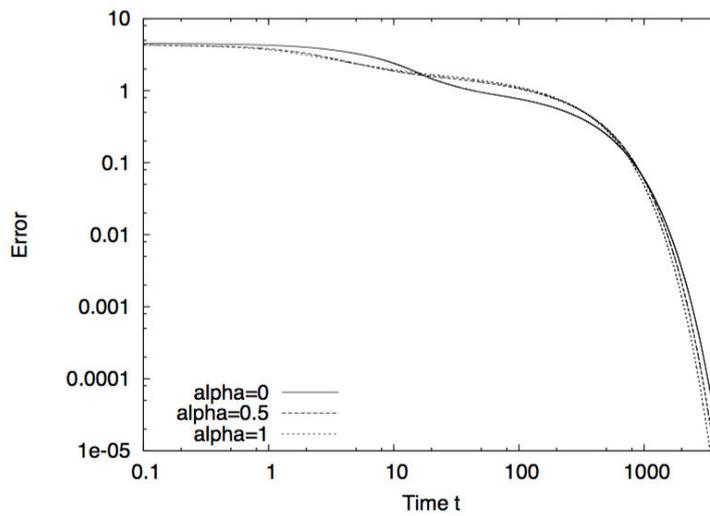


Figure 5.20: Numerical error $\mathcal{E}(t)$, defined in (5.44), versus time in a log scaled plan for several values of the parameter α .

(isotropic scattering) and $\alpha = 0.5$ or 1 (Compton scattering). In particular at time $t = 2.7$ the magnitude of the first bump of is five times greater in the case $\alpha = 1$ than is the case $\alpha = 0$. Once again, the important quantity is the time evolution of the numerical error $E(t)$ defined

in (5.44). Picture 5.20 displays the evolution of this numerical error. On the contrary to the previous test case, in which the Kompaneets terms increased the rate of convergence, the curves cross each other at two times. Picture 5.21 shows a zoom of this in the time region of interest.

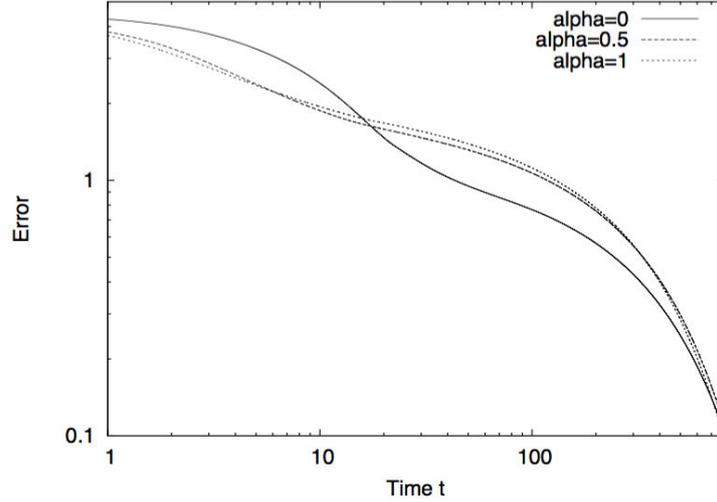


Figure 5.21: Numerical error $\mathcal{E}(t)$, defined in (5.44), versus time in a log scaled planj for several values of the parameter α .

5.4 Hybrid $P_1 - M_1$ angular moments model

From the preceding two parts, concerning the P_1 and M_1 models, one remarks that the choice of a particular expression for the distribution function, (5.5) in the case of the P_1 model and (5.21) in the case of the M_1 model, allows us to compute the frequency dependence model in the case of the P_1 model and the grey model in the case of the M_1 model. This comes from the fact that for the P_1 model the distribution function is assumed to be linear with respect to the angle Ω , and thus the frequency dependent model can be derived, even with the quadratic (with respect to f) terms. Since no particular behavior with respect to the frequency ν is assumed for the distribution function, the quadratic terms can not be analytically computed in terms of the moments in the grey case. On the other hand, in the case of the M_1 model, the dependence of the distribution function is nonlinear with respect to the angle Ω , and thus the frequency dependent model can not be computed. But since a particular behavior with respect to the frequency is assumed, the grey case can be derived.

From this discussion it appears that one can modify the assumed expression of the distribution function in the case of the P_1 model (expression (5.5)) by assuming a "M₁ kind" frequency dependence, i.e. assuming that the distribution function is given by an expression of the form

$$f(t, \mathbf{x}, \nu, \Omega) = \frac{1 + \mathbf{b}(t, \mathbf{x}) \cdot \Omega}{\exp\left(\frac{\nu}{a(t, \mathbf{x})}\right) - 1}. \quad (5.46)$$

As previously, we define

$$\begin{cases} U(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) d\boldsymbol{\Omega}, \\ \mathbf{V}(t, \mathbf{x}, \nu) = \frac{1}{4\pi} \int_{S^2} f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) \boldsymbol{\Omega} d\boldsymbol{\Omega}, \end{cases} \quad (5.47)$$

We prove the

Lemma 5.6 (Frequency dependent hybrid $P_1 - M_1$ model). *The frequency dependent hybrid model for system (5.1) is exactly the same than the P_1 frequency dependent model, i.e. is given by*

$$\begin{cases} \frac{1}{c} \partial_t U + \nabla \cdot \mathbf{V} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu U + U + U^2 + 3(\mathbf{V}, \mathbf{V}) \right) \right] + \sigma_a (f_0 - U), \\ \frac{1}{c} \partial_t \mathbf{V} + \frac{1}{3} \nabla U = -\frac{4}{3} \sigma_s \mathbf{V} + \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{V} + \mathbf{V} + 2U\mathbf{V} \right) \right] - \sigma_a \mathbf{F}, \\ C_V \partial_t T = -\sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 U \right) d\nu - \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu U + U + U^2 + 3(\mathbf{V}, \mathbf{V}) \right) \right]. \end{cases} \quad (5.48)$$

Proof. As before, we integrate the first equation of (5.1) over the two first angular moments, which yields

$$\begin{cases} \frac{1}{c} \partial_t U + \nabla \cdot \mathbf{V} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu U + U + \frac{1}{4\pi} \int_{S^2} f^2 d\boldsymbol{\Omega} \right) \right] + \sigma_a (f_0 - U), \\ \frac{1}{c} \partial_t \mathbf{V} + \nabla \left(\frac{1}{4\pi} \int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I_d f d\boldsymbol{\Omega} \right) = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{V} + \mathbf{V} + \frac{1}{4\pi} \int_{S^2} f^2 \boldsymbol{\Omega} d\boldsymbol{\Omega} \right) \right] \\ \quad - \left(\frac{4}{3} \sigma_s + \sigma_a \right) \mathbf{V}, \\ C_V \partial_t T = -\sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 U \right) d\nu + \frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^+} \nu^4 \left(T \partial_\nu U + U + \frac{1}{4\pi} \int_{S^2} f^2 d\boldsymbol{\Omega} \right) d\nu. \end{cases}$$

As before, we need to compute several integrals. Since the distribution function is assumed to be linear with respect to $\boldsymbol{\Omega}$, the computation of these integrals can be carried out and one finds

$$\begin{cases} \frac{1}{4\pi} \int_{S^2} f^2 d\boldsymbol{\Omega} = \frac{1}{4\pi} \int_{S^2} \left(\frac{1 + \mathbf{b} \cdot \boldsymbol{\Omega}}{e^{\nu/a} - 1} \right)^2 d\boldsymbol{\Omega} = \left(1 + \frac{\|\mathbf{b}\|^2}{3} \right) \left(e^{\nu/a} - 1 \right)^{-2} = U^2 \left(1 + \frac{\|\mathbf{b}\|^2}{3} \right), \\ \frac{1}{4\pi} \int_{S^2} f^2 \boldsymbol{\Omega} d\boldsymbol{\Omega} = \frac{1}{4\pi} \int_{S^2} \left(\frac{1 + \mathbf{b} \cdot \boldsymbol{\Omega}}{e^{\nu/a} - 1} \right)^2 \boldsymbol{\Omega} d\boldsymbol{\Omega} = \frac{2}{3} \left(e^{\nu/a} - 1 \right)^{-2} \mathbf{b} = 2U\mathbf{V}. \end{cases}$$

In the same way, the second order moment can be computed and one finds

$$\frac{1}{4\pi} \int_{S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I_d f d\boldsymbol{\Omega} = \frac{1}{4\pi} \int_{S^2} \frac{1 + \mathbf{b} \cdot \boldsymbol{\Omega}}{e^{\nu/a} - 1} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} d\boldsymbol{\Omega} = \frac{1}{3} \left(e^{\nu/a} - 1 \right)^{-1} = \frac{1}{3} E,$$

which is exactly the closure of the P_1 model. It yields

$$\begin{cases} \frac{1}{c} \partial_t U + \nabla \cdot \mathbf{V} = \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu U + U + U^2 \left(1 + \frac{\|\mathbf{b}\|^2}{3} \right) \right) \right] + \sigma_a (f_0 - U), \\ \frac{1}{c} \partial_t \mathbf{V} + \frac{1}{3} \nabla E = -\frac{4}{3} \sigma_s \mathbf{V} + \frac{\sigma_s}{3mc^2} \nu^{-2} \frac{\partial}{\partial \nu} \left[\nu^4 \left(T \partial_\nu \mathbf{V} + \mathbf{V} + 2U \mathbf{V} \right) \right] - \sigma_a \mathbf{V}, \\ C_V \partial_t T = -\sigma_a \int_{\mathbb{R}^+} \left(B(\nu, T) - \nu^3 U \right) d\nu + \frac{\sigma_s}{3mc^2} \int_{\mathbb{R}^+} \nu^4 \left(T \partial_\nu U + U + U^2 \left(1 + \frac{\|\mathbf{b}\|^2}{3} \right) \right) d\nu. \end{cases}$$

Since $\mathbf{b} = 3\mathbf{V}/U$, one finds exactly the P_1 frequency dependent model (5.7). \square

Since a "M₁ kind" frequency behavior is assumed for the distribution function, the grey model can be derived. Once again, we define

$$\begin{cases} E(t, \mathbf{x}) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f(t, \mathbf{x}, \nu, \boldsymbol{\Omega}) d\boldsymbol{\Omega} d\nu, \\ \mathbf{F}(t, \mathbf{x}) = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \nu^3 f(t, x, \nu, \boldsymbol{\Omega}) \boldsymbol{\Omega} d\boldsymbol{\Omega} d\nu, \end{cases}$$

We prove the following lemma.

Lemma 5.7 (Grey hybrid $P_1 - M_1$ model). *The grey hybrid $P_1 - M_1$ model for the system (5.1) is defined by*

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \frac{4T}{3mc^2} \sigma_s E - \frac{4\sigma_s}{3mc^2} E^{\frac{5}{4}} \left\{ 1 + 3\|\mathbf{g}\|^2 \left(1 - \frac{90}{\pi^4} \zeta(5) \right) \right\}, \\ \frac{1}{c} \partial_t \mathbf{F} + \frac{1}{3} \nabla E = \frac{4}{3} \left(\frac{T}{mc^2} - 1 \right) \sigma_s \mathbf{F} - \sigma_a \mathbf{F} - \frac{4\sigma_s}{3mc^2} E^{\frac{1}{4}} \left(2 - \frac{90}{\pi^4} \zeta(5) \right) \mathbf{F}, \\ C_V \partial_t T = -\sigma_a (T^4 - E) - \frac{4T}{3mc^2} \sigma_s E + \frac{4\sigma_s}{3mc^2} E^{\frac{5}{4}} \left\{ 1 + 3\|\mathbf{g}\|^2 \left(1 - \frac{90}{\pi^4} \zeta(5) \right) \right\}. \end{cases} \quad (5.49)$$

where $\mathbf{g} = \mathbf{F}/E$ and ζ is the zeta Riemann function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$.

It is remarkable to see that the structure of this model is the same that the grey M_1 model (5.25), only the nonlinear terms and the closure are different.

Proof. Taking the two first moments of system (5.1) and multiplying the temperature equation by $15\pi^4$, one writes by denoting once again $C_V = 15/\pi^4 C_V$ and $D = 15/\pi^4 D$

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \frac{4T}{3mc^2} \sigma_s E - \frac{5\sigma_s}{4\pi^5 mc^2} \int_{\mathbb{R}^+ \times S^2} \nu^4 (f + f^2) d\nu d\boldsymbol{\Omega}, \\ \frac{1}{c} \partial_t \mathbf{F} + \frac{15}{4\pi^5} \nabla \cdot \left(\int_{\mathbb{R}^+ \times S^2} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} f \nu^3 d\nu d\boldsymbol{\Omega} \right) = \frac{4}{3} \left(\frac{T}{mc^2} - 1 \right) \sigma_s \mathbf{F} - \sigma_a \mathbf{F} \\ \quad - \frac{5\sigma_s}{4\pi^5 mc^2} \int_{\mathbb{R}^+ \times S^2} \nu^4 \boldsymbol{\Omega} (f + f^2) d\nu d\boldsymbol{\Omega}, \\ C_V \partial_t T = -\sigma_a (T^4 - E) - \frac{4T}{3mc^2} \sigma_s E + \frac{5\sigma_s}{4\pi^5 mc^2} \int_{\mathbb{R}^+ \times S^2} \nu^4 (f + f^2) d\nu d\boldsymbol{\Omega} \end{cases}$$

As for the previous angular moments models, the proof mainly relies on the computation of integrals. The definition of f yields

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \nu^4 d\nu d\Omega = \frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} \left(1 + \frac{1 + \mathbf{b} \cdot \Omega}{e^{\nu/a} - 1}\right) \frac{\nu^4}{e^{\nu/a} - 1} (1 + \mathbf{b} \cdot \Omega) d\nu d\Omega,$$

which yields

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \nu^4 d\nu d\Omega = a^5 \left(1 + \frac{\|\mathbf{b}\|^2}{3}\right) \int_{\mathbb{R}^+} \frac{\nu^4}{(e^\nu - 1)^2} d\nu + a^5 \int_{\mathbb{R}^+} \frac{\nu^4}{e^\nu - 1} d\nu.$$

These integrals have already been computed during the derivation of the grey $M1$ model. Indeed, one finds $\int_{\mathbb{R}^+} \frac{\nu^4}{(e^\nu - 1)^2} d\nu = -24\zeta(5) + 4\pi^4/15$ and $\int_{\mathbb{R}^+} \frac{\nu^4}{e^\nu - 1} d\nu = 24\zeta(5)$. It gives us

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \nu^4 d\nu d\Omega = a^5 \left\{ \left(1 + \frac{\|\mathbf{b}\|^2}{3}\right) \left(-24\zeta(5) + \frac{4\pi^4}{15}\right) + 24\zeta(5) \right\},$$

which we write

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \nu^4 d\nu d\Omega = a^5 \left\{ \frac{4\pi^4}{15} + \frac{\|\mathbf{b}\|^2}{3} \left(-24\zeta(5) + \frac{4\pi^4}{15}\right) \right\},$$

At this point, one sees that one the differences with the grey M_1 model is the presence of the term $\frac{\|\mathbf{b}\|^2}{3}$. Let us compute the integrals in the equation of \mathbf{F} . One has, using the definition of f (5.46)

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \Omega \nu^4 d\nu d\Omega = \frac{a^5}{3} \left\{ \int_{\mathbb{R}^+} \frac{\nu^4}{e^\nu - 1} d\nu + 2 \int_{\mathbb{R}^+} \frac{\nu^4}{(e^\nu - 1)^2} d\nu \right\} \mathbf{b},$$

which yields

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \Omega \nu^4 d\nu d\Omega = \frac{a^5}{3} \left\{ 24\zeta(5) + 2 \left(-24\zeta(5) + \frac{4\pi^4}{15}\right) \right\} \mathbf{b}.$$

We write this expression

$$\frac{1}{4\pi} \int_{\mathbb{R}^+ \times S^2} (f + f^2) \Omega \nu^4 d\nu d\Omega = \frac{a^5}{3} \left(\frac{8\pi^4}{15} - 24\zeta(5) \right) \mathbf{b}.$$

In the same way, the two first moments E and \mathbf{F} can be computed in terms of a and \mathbf{b}

$$\begin{cases} E = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \frac{1 + \mathbf{b} \cdot \Omega}{e^{\nu/a} - 1} \nu^3 d\nu d\Omega = a^4 \frac{15}{\pi^4} \int_{\mathbb{R}^+} \frac{\nu^3}{e^\nu - 1} d\nu = a^4, \\ \mathbf{F} = \frac{15}{4\pi^5} \int_{\mathbb{R}^+ \times S^2} \frac{1 + \mathbf{b} \cdot \Omega}{e^{\nu/a} - 1} \Omega \nu^3 d\nu d\Omega = \frac{a^4}{3} \mathbf{b} \frac{15}{\pi^4} \int_{\mathbb{R}^+} \frac{\nu^3}{e^\nu - 1} d\nu = \frac{a^4}{3} \mathbf{b}, \end{cases}$$

which yields $a = E^{\frac{1}{4}}$ and $\mathbf{b} = 3 \frac{\mathbf{F}}{E} = 3\mathbf{g}$. One thus gets

$$\begin{cases} \frac{1}{c} \partial_t E + \nabla \cdot \mathbf{F} = \sigma_a (T^4 - E) + \frac{4T}{3mc^2} \sigma_s E - \frac{4\sigma_s}{3mc^2} E^{\frac{5}{4}} \left\{ 1 + 3\|\mathbf{g}\|^2 \left(1 - \frac{90}{\pi^4} \zeta(5)\right) \right\}, \\ \frac{1}{c} \partial_t \mathbf{F} + \frac{1}{3} \nabla E = \frac{4}{3} \left(\frac{T}{mc^2} - 1 \right) \sigma_s \mathbf{F} - \sigma_a \mathbf{F} - \frac{4\sigma_s}{3mc^2} E^{\frac{1}{4}} \left(2 - \frac{90}{\pi^4} \zeta(5) \right) \mathbf{F}, \\ C_V \partial_t T = -\sigma_a (T^4 - E) - \frac{4T}{3mc^2} \sigma_s E + \frac{4\sigma_s}{3mc^2} E^{\frac{5}{4}} \left\{ 1 + 3\|\mathbf{g}\|^2 \left(1 - \frac{90}{\pi^4} \zeta(5)\right) \right\}. \end{cases}$$

□

It is interesting to notice that this model has the same structure than the grey M_1 model (5.25) but the nonlinear sources terms are simpler. Obviously, the equilibrium regime of the grey hybrid $P_1 - M_1$ model (5.49) is the same than the equilibrium of the grey M_1 model (5.25) since the terms coming from the emission absorption are the same. Even if it also true for the non-equilibrium diffusion regime, it is not so obvious. This comes from the fact that in this regime, $\|\mathbf{g}\| \rightarrow 0$ and thus the terms which are different with the grey M_1 model vanish.

Chapter 6

A deterministic model for the electron-ion Bremsstrahlung: application to radiotherapy

This chapter is taken from a joint work with R. Ducloux (CEA), S. Brull (Bordeaux 1 University) and B. Dubroca (CEA) at the Cemracs 2014 (Coupomom project).

6.1 Introduction

The aim of this chapter is to derive and validate a deterministic model for the electron-ion Bremsstrahlung that allows a smooth transition between the thermal and non thermal limits. The Bremsstrahlung process arises when an electron gets close to an ion. Due to the Coulomb potential of the ion, the electron slows down and emits a photon (figure 6.1). The context of this work is the radiotherapy, and in particular the bombardment of cancerous tumor by fast electrons. As explained in the part 6.5.2, the slowing down of electrons is due both to

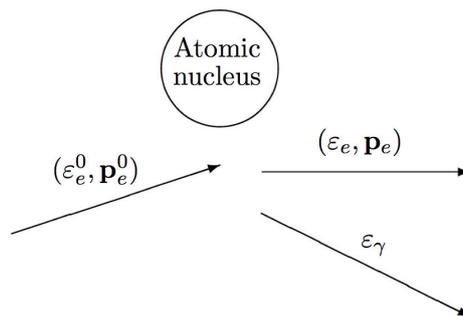


Figure 6.1: Schematic representation of electron-ion Bremsstrahlung

the electron-electron collisions and to the electron-ion Bremsstrahlung. This later contribution increases with the incoming electron's energy. For instance the electron-ion Bremsstrahlung slowing down becomes non negligible from 1 Mev, and dominates the electronic collision slowing down from roughly 60 Mev in the case of water. Since in the case of radiotherapy the energy range is $[0, 20]$ Mev, the electron-ion Bremsstrahlung brings a non negligible contribution to

the slowing down. This model could also be used in astrophysics or inertial confinement fusion (ICF), in which hard X-ray emission from Bremsstrahlung can bring significant modification in the photon spectrum [DOP12].

In the whole chapter we assume that there is no energy absorbed by the ion. It yields in particular the following energy conservation relation $\varepsilon_\gamma = h\nu = \varepsilon_e^0 - \varepsilon_e$, where h is the Planck constant, ν the photon's frequency, ε_e^0 (resp ε_e) is the energy of the incoming (resp outgoing electron). The momentum transferred to the ion, denoted \mathbf{q} , satisfies the relation $\mathbf{q} = \mathbf{p}_e^0 - \mathbf{p}_e - \mathbf{p}_\gamma$. In the following we denote Σ the 4-uplets of $(\mathbf{q}, \mathbf{p}_\gamma, \mathbf{p}_e, \mathbf{p}_e^0)$ that satisfy this conservation, i.e. Σ is defined by

$$\Sigma = \left\{ (\mathbf{q}, \mathbf{p}_\gamma, \mathbf{p}_e, \mathbf{p}_e^0) \in \mathbb{R}^{12} \mid \left\{ \begin{array}{l} \mathbf{q} = \mathbf{p}_e^0 - \mathbf{p}_e - \mathbf{p}_\gamma, \\ \varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e. \end{array} \right. \right\} \quad (6.1)$$

The probability of emission of a photon of energy ε_γ from an ingoing electron of energy ε_e^0 , leading to an electron of energy ε_e , is related to the quantity

$$\frac{d\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}(\varepsilon_e^0 \rightarrow \varepsilon_e). \quad (6.2)$$

This quantity is called the differential cross section, and its derivation will be discussed in section 6.4. The probability of the inverse transformation, called the inverse Bremsstrahlung, which is concerned by the absorption of a photon of energy ε_γ by an ingoing electron of energy ε_e^0 , leading to an electron of energy ε_e , is related to the quantity

$$\frac{d\sigma^{IB}}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}(\varepsilon_e^0 \rightarrow \varepsilon_e). \quad (6.3)$$

In practice, models and numerical codes require the angular integrated expression of these cross sections. In the case of plasmas, there is an additional dependance on the ionization degree and on the plasma temperature. Such dependance can be taken into account by analytical expressions [BET34, KM59]. This explains the extensive use of tabular values, as given for example in [SB85]. Despite precise, the numerical computation of the angular moments of the differential cross sections may take some unaffordable times.

This chapter is organized as follows. In section 2 we derive a kinetic model for the electron-ion Bremsstrahlung (system (6.16)). Since we consider the ions as fixed, we only consider an equation for the electrons and a equation for the photons. We prove that our model preserves the total energy (lemma 6.1) and the total number of electrons (lemma 6.2). We also explain how the thermal Bremsstrahlung can be obtained as the limit of our model in the regime of local thermodynamic equilibrium (LTE). Finally, we use the continuous slowing down approximation, developed by Pomraning [POM83, POM92], which is used to simplify the electrons' equations. This approximation can be seen as the grazing collisions approximation used in the case of electrons-electrons collisions (Boltzmann Landau equation [LP81]). In section 3 the angular M_1 model [DF99] is derived from our model, and we present the energy discretization. We also prove that the numerical scheme preserves the total energy (lemma 6.4). In section 4 we focus on analytical electron-ion Bremsstrahlung differential cross sections. Finally, in section 5 we present some numerical results. After a validation of the differential cross section from a comparison with tabulated values [SB85], we present a test case of dose deposition which arise in radiotherapy.

We end this section by introducing the main physical quantities of interest used in this chapter. Firstly, we chose to compute the population of electrons from their angular flux, which is defined from their distribution function f_e by the relation

$$\psi_e(t, x, \varepsilon_e, \mathbf{\Omega}_e) = v_e(\varepsilon_e) f_e(t, x, \varepsilon_e, \mathbf{\Omega}_e), \quad (6.4)$$

where v_e is the absolute value of the electrons' velocity. We also define the photon angular flux by the relation

$$\psi_\gamma(t, x, \varepsilon_\gamma, \mathbf{\Omega}_\gamma) = \frac{I_\nu(t, x, \varepsilon_\gamma, \mathbf{\Omega}_\gamma)}{h\varepsilon_\gamma}, \quad (6.5)$$

where I_ν is the radiative intensity, h is the Planck constant and $\varepsilon_\gamma = h\nu$ is the photons' energy. The photon occupation number n_γ is defined by

$$n_\gamma(t, x, \varepsilon_\gamma, \mathbf{\Omega}_\gamma) = \frac{c^2 h^2 I_\nu(t, x, \varepsilon_\gamma, \mathbf{\Omega}_\gamma)}{2 \varepsilon_\gamma^3}. \quad (6.6)$$

The attention of the reader is drawn to the fact that the occupation number defined here is the same quantity that the density distribution function f_γ defined in the chapter 1. We define the number of electrons per unit volume by

$$\rho_e(t, x) = \int_0^\infty d\varepsilon_e \int_{S^2} d\mathbf{\Omega}_e f_e(t, x, \varepsilon_e, \mathbf{\Omega}_e).$$

The number of electrons at time t is thus defined by

$$N_e(t) = \int_{\mathbb{R}^3} dx \rho_e(t, x) = \int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\mathbf{\Omega}_e f_e(t, x, \varepsilon_e, \mathbf{\Omega}_e). \quad (6.7)$$

In this chapter the impulsion (resp the energy) is expressed in mc (resp in mc^2) unit, i.e we define $\mathbf{p} := \mathbf{p}/mc$ and $\varepsilon := \varepsilon/mc^2$.

6.2 Kinetic models for electron-ion Bremsstrahlung

In this section we present our kinetic model for electron-ion Bremsstrahlung (system (6.16)). This model is constructed from the master equations, satisfies an energy conservation result (lemma 6.1) and preserves the total number of electrons (lemma 6.2). In a second step, we apply the continuous slowing down approximation on the electrons equation. This approximation makes use of the peak of the differential cross section at low energy exchanges to expand the function evaluated at $(\varepsilon_e^0, \mathbf{\Omega}_e^0)$ around $(\varepsilon_e, \mathbf{\Omega}_e)$, obtaining a drift diffusion equation in energy and angle [POM83, POM92].

6.2.1 Model derivation from the master equations

The aim of this section is to derive a model from the master equations. The derivation of this model is performed by considering all the possible transformations produced by the electron-ion Bremsstrahlung (Feynman's diagram, picture 6.2). There is two possible ways. One can consider photon emission (resp. absorption). Either an incoming electron can lose (resp. gain) energy by emitting (resp. absorbing) a photon. In addition to the differential cross sections $\frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}(\varepsilon_e^0 \rightarrow \varepsilon_e)$ and $\frac{d\sigma^{IB}}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}(\varepsilon_e^0 \rightarrow \varepsilon_e)$ defined respectively in (6.2) and (6.3), we define

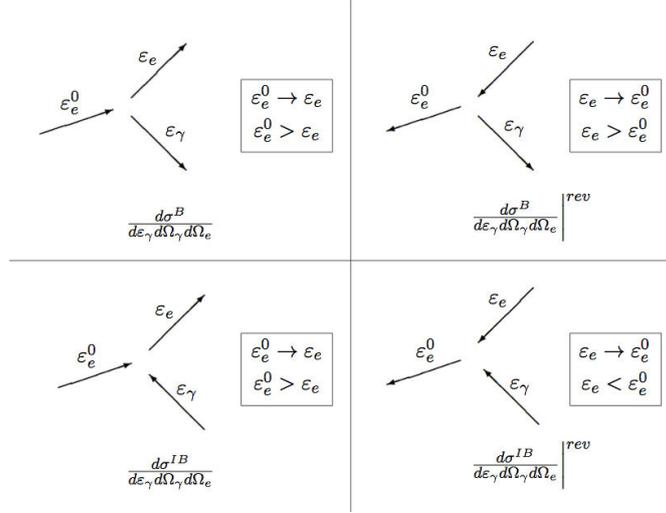


Figure 6.2: Feynman's diagram for electron-ion Bremsstrahlung

the probability for the reverse Bremsstrahlung (top right in the Feynman diagram figure 6.2) and reverse inverse Bremsstrahlung (bottom right in the Feynman diagram figure 6.2), whose quantities refer respectively to the probability of emission of a photon by an electron of energy ε_e and the absorption of a photon by an electron of energy ε_e . The conservation of the total energy for each separated process (direct and inverse Bremsstrahlung) implies a link between all these probabilities, namely

$$\begin{cases} \left. \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e} \right|^{rev} (\varepsilon_e \rightarrow \varepsilon_e^0) = \frac{d\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e} (\varepsilon_e^0 \rightarrow \varepsilon_e), \\ \left. \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e} (\varepsilon_e^0 \rightarrow \varepsilon_e) = \frac{d\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e} \right|^{rev} (\varepsilon_e \rightarrow \varepsilon_e^0). \end{cases} \quad (6.8)$$

Nevertheless, we keep the entire notations in the next parts since it highlights the origin of each terms.

We consider a Boltzmann equation for the electrons and the photons. Since we are only interested in the electron-photon interactions, we do not consider the electron-electron interactions. Moreover, we assume that the ions are fixed. The derivation of the electron equation is performed first (part 6.2.1.0), and the derivation of the photon equation comes in a second step (part 6.2.1.0).

Electrons equation

The Boltzmann equation for the electrons is composed of a classical transport term and a collision operator. As shown in the Feynman's diagram (picture 6.2), this collision operator involves two gain terms and two loss terms, coming from direct and inverse Bremsstrahlung. One thus obtains the general Boltzmann equation

$$\frac{1}{v_e} \partial_t \psi_e(\varepsilon_e, \mathbf{\Omega}_e) + \mathbf{\Omega}_e \cdot \nabla \psi_e(\varepsilon_e, \mathbf{\Omega}_e) = Q_e^B - Q_e^{B|rev} + Q_e^{IB} - Q_e^{IB|rev}, \quad (6.9)$$

where we removed the time and space dependence in the angular flux for ease of notations. The collision operators Q_e^B , $Q_e^{B|rev}$, Q_e^{IB} and $Q_e^{IB|rev}$ are obtained as follows. The collision operator for direct Bremsstrahlung Q_e^B is obtained by integrating over all the photons energy ε_γ and direction $\mathbf{\Omega}_\gamma$, over all the ingoing electrons energy ε_e^0 and direction $\mathbf{\Omega}_e^0$ and over all the ion impulsion \mathbf{q} , the probability for direct Bremsstrahlung multiplied by the electrons angular flux distribution ψ_e , the ionic density ρ_i and a Dirac function δ_Σ that ensures the conservation of energy and impulsion (whose set is defined in (6.1))

$$Q_e^B = \rho_i(t, x) \int_{\mathbb{R}^3} d^3\mathbf{q} \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \\ \times \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} (\varepsilon_e^0 \rightarrow \varepsilon_e) \delta_\Sigma.$$

The integral over the ingoing electrons is performed on the interval $[\varepsilon_e, \infty]$ (and not $[0, \infty]$) since the ingoing electron must have a higher energy than the outgoing electron ($\varepsilon_e^0 = \varepsilon_e + \varepsilon_\gamma > \varepsilon_e$). The factor $1 + n_\gamma$, where n_γ is the photons occupation number defined in (6.6), models the induced effects (quantum effects). Since the integration of the Dirac function over the impulsion \mathbf{p} and over the photons energy ε_γ vanishes, one obtains

$$Q_e^B = \rho_i(t, x) \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} (\varepsilon_e^0 \rightarrow \varepsilon_e). \quad (6.10)$$

In the same way, the Bremsstrahlung operator $Q_e^{B|rev}$ for the reverse path is obtained by integrating over all the photons energy ε_γ and direction $\mathbf{\Omega}_\gamma$, over all the outgoing electrons energy ε_e^0 and direction $\mathbf{\Omega}_e^0$ and over all the ion impulsion \mathbf{q} , the transition probability multiplied by the electron angular flux distribution ψ_e and by the Dirac function δ_Σ

$$Q_e^{B|rev} = \rho_i(t, x) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \int_{\mathbb{R}^3} d^3\mathbf{q} \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \\ \times \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \Big|^{rev} (\varepsilon_e^0 \rightarrow \varepsilon_e) \delta_\Sigma.$$

The integral over the outgoing electrons is performed on the interval $[0, \varepsilon_e]$ since the outgoing electron (in this case ε_e^0) must have a lower energy than the ingoing electron. Once again, this expression can be simplified to get

$$Q_e^{B|rev} = \rho_i(t, x) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \Big|^{rev} (\varepsilon_e^0 \rightarrow \varepsilon_e). \quad (6.11)$$

Using the same reasoning, one obtains for the inverse Q_e^{IB} and reverse inverse $Q_e^{IB|rev}$ Bremsstrahlung operators the following expressions

$$\left\{ \begin{array}{l} Q_e^{IB} = \rho_i(t, x) \int_{S^2} d\mathbf{\Omega}_\gamma n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \frac{v_e \mathcal{D}(\varepsilon_e)}{v_e^0 \mathcal{D}(\varepsilon_e^0)} \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} (\varepsilon_e^0 \rightarrow \varepsilon_e), \\ Q_e^{IB|rev} = \rho_i(t, x) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \int_{S^2} d\mathbf{\Omega}_\gamma n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e \mathcal{D}(\varepsilon_e)} \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \Big|^{rev} (\varepsilon_e^0 \rightarrow \varepsilon_e). \end{array} \right. \quad (6.12)$$

The process of inverse Bremsstrahlung involves the absorption of photons, and the corresponding collision operators must be directly proportional to the photons population. This explains the

coefficient n_γ (and not $1 + n_\gamma$) in the expression of the inverse Bremsstrahlung operators. In these expressions v_e is the electrons' velocity and \mathcal{D} , defined by

$$\mathcal{D}(\varepsilon_e) = \sqrt{\frac{\varepsilon_e}{m_e c^2} \left(\frac{\varepsilon_e}{m_e c^2} + 2 \right) \left(\frac{\varepsilon_e}{m_e c^2} + 1 \right)},$$

is the density of state. The coefficient $\frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)}$ is introduced to ensure the correct thermal regime at the local thermodynamic equilibrium (LTE), in which case the electron angular flux is given by $\psi_e(\varepsilon_e) = v_e \frac{\rho_e}{m_e c^2} \frac{\beta e^{-\beta}}{K_2(\beta)} \mathcal{D}(\varepsilon_e) e^{-\varepsilon_e/k_B T}$, where T is the electrons' temperature at equilibrium, k_B the Boltzmann constant (see the part 6.2.3), K_2 is the modified Bessel function of the first kind and $\beta = m_e c^2 / k_B T$.

Photon equation

As shown in the Feynman diagram (figure 6.2), the collision operator in the photons' equation is composed of a gain term coming from direct Bremsstrahlung Q_γ^B and a loss term coming from inverse Bremsstrahlung Q_γ^{IB} . We thus obtain the general Boltzmann equation

$$\frac{1}{c} \partial_t \psi_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) + \mathbf{\Omega}_\gamma \cdot \nabla \psi_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) = Q_\gamma^B - Q_\gamma^{IB}. \quad (6.13)$$

In the same way than for the electron equation, the collision operator for direct Bremsstrahlung Q_γ^B is obtained by integrating over all the energy and direction of the ingoing and outgoing electron and over all the ionic impulsion \mathbf{q} the probability of direct Bremsstrahlung multiplied by the electron's angular flux distribution ψ_e , the ionic density ρ_i and the Dirac function δ_Σ

$$Q_\gamma^B = \rho_i(t, x) [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_{\mathbb{R}^3} d^3 \mathbf{q} \int_0^\infty d\varepsilon_e \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \psi_e(\varepsilon_e^o, \mathbf{\Omega}_e^o) \int_{S^2} d\mathbf{\Omega}_e \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}(\varepsilon_e^o \rightarrow \varepsilon_e) \delta_\Sigma.$$

The integral over all the ingoing electrons ε_e^o is performed on the interval $[\varepsilon_e, \infty]$ (and not $[0, \infty]$) since the energy of the ingoing electron must be higher than the energy of the outgoing electron. Since the integration of the Dirac function over the ion impulsion \mathbf{q} and over the outgoing electrons is equal to 1, one obtains the following expression

$$Q_\gamma^B = \rho_i(t, x) [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \psi_e(\varepsilon_e^o, \mathbf{\Omega}_e^o) \int_{S^2} d\mathbf{\Omega}_e \frac{d\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}(\varepsilon_e^o \rightarrow \varepsilon_e). \quad (6.14)$$

The same reasoning is performed to obtain the expression of the inverse Bremsstrahlung operator Q_γ^{IB} . The differences come from the integration domain on the ingoing and outgoing electrons, from the induced effects and from the introduction of the factor $\frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)}$. Indeed, in this case the outgoing electron has a higher energy ε_e^o than the ingoing electron. The integration domain is thus $\varepsilon_e^o \in [\varepsilon_e, \infty]$. The induced effects should here be taken into account with a factor n_γ (and not $1 + n_\gamma$). One obtains

$$Q_\gamma^{IB} = \rho_i(t, x) n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_{\mathbb{R}^3} d^3 \mathbf{q} \int_0^\infty d\varepsilon_e \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \int_{S^2} d\mathbf{\Omega}_e^o \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \Big|^{rev} (\varepsilon_e^o \rightarrow \varepsilon_e) \delta_\Sigma,$$

which, as previously, simplified as

$$Q_\gamma^{IB} = \rho_i(t, x) n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \int_{S^2} d\mathbf{\Omega}_e^o \frac{d\sigma^{IB}}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \Big|^{rev} (\varepsilon_e^o \rightarrow \varepsilon_e). \quad (6.15)$$

General model

Finally, using (6.9), (6.10), (6.11), and (6.12) for the electrons and (6.13), (6.14) and (6.15) for the photons and the link between the different differential cross sections (6.8), one obtains the following general model for electron-ion Bremsstrahlung

$$\left\{ \begin{array}{l} \frac{1}{v_e} \partial_t \psi_e(\varepsilon_e, \mathbf{\Omega}_e) + \mathbf{\Omega}_e \cdot \nabla \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \\ = \rho_i(t, x) \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_{\varepsilon_e}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \psi_e(\varepsilon_e^o, \mathbf{\Omega}_e^o) \frac{d\sigma^B(\varepsilon_e^o \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \\ - \rho_i(t, x) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \int_{S^2} d\mathbf{\Omega}_\gamma [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_0^{\varepsilon_e} d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^o)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \\ + \rho_i(t, x) \int_{S^2} d\mathbf{\Omega}_\gamma n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_0^{\varepsilon_e} d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \psi_e(\varepsilon_e^o, \mathbf{\Omega}_e^o) \frac{v_e \mathcal{D}(\varepsilon_e)}{v_e^o \mathcal{D}(\varepsilon_e^o)} \frac{d\sigma^{IB}(\varepsilon_e \rightarrow \varepsilon_e^o)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \\ - \rho_i(t, x) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \int_{S^2} d\mathbf{\Omega}_\gamma n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_{\varepsilon_e}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \frac{d\sigma^{IB}(\varepsilon_e^o \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \\ \frac{1}{c} \partial_t \psi_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) + \mathbf{\Omega}_\gamma \cdot \nabla \psi_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \\ = \rho_i(t, x) [1 + n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma)] \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e^o \psi_e(\varepsilon_e^o, \mathbf{\Omega}_e^o) \int_{S^2} d\mathbf{\Omega}_e \frac{d\sigma^B(\varepsilon_e^o \rightarrow \varepsilon_e^o - \varepsilon_\gamma)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} \\ - \rho_i(t, x) n_\nu(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\mathbf{\Omega}_e \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \int_{S^2} d\mathbf{\Omega}_e^o \frac{d\sigma^{IB}(\varepsilon_e^o \rightarrow \varepsilon_e^o - \varepsilon_\gamma)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}. \end{array} \right. \quad (6.16)$$

For simplicity we only kept the dependence with respect to the ingoing and outgoing electrons in the expressions of the differential cross sections.

6.2.2 Properties of the model

In this part are summarized the properties of the model (6.16), which are the conservation of the total energy (lemma 6.1) and the conservation of the total number of electrons (lemma 6.2). The conservation of the total energy is given in the following lemma.

Lemma 6.1 (Energy conservation). *Assuming that the photons' and electrons' angular fluxes ψ_e and ψ_γ vanish at the limit $|x| \rightarrow \infty$ and that the differential cross sections satisfy the relation (6.8), the model for direct and inverse electron-ion Bremsstrahlung defined in (6.16) preserves the total energy, i.e.*

$$\frac{d}{dt} E(t) = \frac{d}{dt} \left[\int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\mathbf{\Omega}_e \frac{\varepsilon_e}{v_e} \psi_e + \int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\mathbf{\Omega}_\gamma \frac{\varepsilon_\gamma}{c} \psi_\gamma \right] = 0.$$

Proof. By definition of the model, the variation of the total energy is given by

$$\begin{aligned} \frac{d}{dt}E(t) = \int_{\mathbb{R}^3} dx \left[\int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e \varepsilon_e \left(Q_e^B - Q_e^{B|rev} + Q_e^{IB} - Q_e^{IB|rev} \right) \right. \\ \left. + \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\Omega_\gamma \varepsilon_\gamma \left(Q_\gamma^B + Q_\gamma^{IB} \right) \right]. \end{aligned}$$

Since both the direct and inverse Bremsstrahlung processes preserve the energy, it is sufficient to prove that

$$\begin{cases} \partial_t E^B(t, x) = \int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e \varepsilon_e (Q_e^B - Q_e^{B|rev}) + \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\Omega_\gamma \varepsilon_\gamma Q_\gamma^B = 0, \\ \partial_t E^{IB}(t, x) = \int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e \varepsilon_e (Q_e^{IB} - Q_e^{IB|rev}) + \int_0^\infty d\varepsilon_\gamma \int_{S^2} d\Omega_\gamma \varepsilon_\gamma Q_\gamma^{IB} = 0. \end{cases}$$

We start with the direct Bremsstrahlung. Using the definition of Q_e^B , $Q_e^{B|rev}$ and Q_γ^B defined in (6.10), (6.11) and (6.14), one has

$$\partial_t E^B(t, x) = \rho_i(t, x) \left(E_{e,1}^B + E_{e,2}^B + E_\gamma^B \right),$$

where

$$\begin{cases} E_{e,1}^B = \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^6} d\Omega [1 + n_\gamma(\varepsilon_e^0 - \varepsilon_e, \Omega_\gamma)] \varepsilon_e \psi_e(\varepsilon_e^0, \Omega_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ E_{e,2}^B = - \int_0^\infty d\varepsilon_e \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e - \varepsilon_e^0, \Omega_\gamma)] \varepsilon_e \psi_e(\varepsilon_e, \Omega_e) \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ E_\gamma^B = \int_0^\infty d\varepsilon_\gamma \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_\gamma, \Omega_\gamma)] \varepsilon_\gamma \psi_e(\varepsilon_e^0, \Omega_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \end{cases}$$

where we used the notation $d\Omega = d\Omega_\gamma d\Omega_e^0 d\Omega_e$ for ease of notations. We need to invert the two energy integrals in the expression of $E_{e,2}^B$. The Fubini theorem yields

$$E_{e,2}^B = - \int_0^\infty d\varepsilon_e^0 \int_{\varepsilon_e^0}^\infty d\varepsilon_e \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e - \varepsilon_e^0, \Omega_\gamma)] \varepsilon_e \psi_e(\varepsilon_e, \Omega_e) \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}.$$

Now inverting the notation $(\varepsilon_e, \Omega_e) \leftrightarrow (\varepsilon_e^0, \Omega_e^0)$, one finds by adding with the expression of $E_{e,2}^B$

$$E_{e,1}^B + E_{e,2}^B = \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^6} d\Omega [1 + n_\gamma(\varepsilon_e^0 - \varepsilon_e, \Omega_\gamma)] (\varepsilon_e - \varepsilon_e^0) \psi_e(\varepsilon_e^0, \Omega_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}.$$

Using once again the Fubini's theorem to invert the two integrals, one finds

$$E_{e,1}^B + E_{e,2}^B = \int_0^\infty d\varepsilon_e^0 \int_0^{\varepsilon_e^0} d\varepsilon_e \int_{S^6} d\Omega [1 + n_\gamma(\varepsilon_e^0 - \varepsilon_e, \Omega_\gamma)] (\varepsilon_e - \varepsilon_e^0) \psi_e(\varepsilon_e^0, \Omega_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}.$$

Making the change of variables $\varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e$, one gets

$$E_{e,1}^B + E_{e,2}^B = - \int_0^\infty d\varepsilon_e^0 \int_0^{\varepsilon_e^0} d\varepsilon_\gamma \int_{S^6} d\Omega [1 + n_\gamma(\varepsilon_\gamma, \Omega_\gamma)] \varepsilon_\gamma \psi_e(\varepsilon_e^0, \Omega_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}.$$

Finally, the Fubini's theorem yields $E_{e,1}^B + E_{e,2}^B = E_\gamma^B$, which is the result of the claim. We turn to the terms containing the inverse Bremsstrahlung. Once again, one writes

$$\partial_t E^{IB}(t, x) = \rho_i(t, x) \left(E_{e,1}^{IB} + E_{e,2}^{IB} + E_\gamma^{IB} \right),$$

where

$$\left\{ \begin{array}{l} E_{e,1}^{IB} = \int_0^\infty d\varepsilon_e \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\Omega n_\gamma(\varepsilon_e - \varepsilon_e^0, \mathbf{\Omega}_\gamma) \varepsilon_e \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \frac{v_e \mathcal{D}(\varepsilon_e)}{v_e^0 \mathcal{D}(\varepsilon_e^0)} \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \\ E_{e,2}^{IB} = - \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega n_\gamma(\varepsilon_e^0 - \varepsilon_e, \mathbf{\Omega}_\gamma) \varepsilon_e \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e \mathcal{D}(\varepsilon_e)} \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \\ E_\gamma^{IB} = - \int_0^\infty d\varepsilon_\gamma \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega n_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \varepsilon_\gamma \psi_e(\varepsilon_e^0 - \varepsilon_\gamma, \mathbf{\Omega}_e) \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e(\varepsilon_e^0 - \varepsilon_\gamma) \mathcal{D}(\varepsilon_e^0 - \varepsilon_\gamma)} \\ \quad \times \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}. \end{array} \right.$$

As for the direct Bremsstrahlung, the point is to associate the integrals $E_{e,1}^{IB}$ and $E_{e,2}^{IB}$, and to obtain the expression of the integral E_γ^{IB} after manipulations. Using the same arguments than for the direct Bremsstrahlung, the Fubini's theorem and the change of variables $\varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e$ lead to the following expression for $E_{e,1}^{IB}$

$$E_{e,1}^{IB} = \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega n_\gamma(\varepsilon_e^0 - \varepsilon_e, \mathbf{\Omega}_\gamma) \varepsilon_e^0 \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e \mathcal{D}(\varepsilon_e)} \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}.$$

Adding with $E_{e,2}^{IB}$, one gets

$$E_{e,1}^{IB} + E_{e,2}^{IB} = \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega n_\gamma(\varepsilon_e^0 - \varepsilon_e, \mathbf{\Omega}_\gamma) (\varepsilon_e^0 - \varepsilon_e) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e \mathcal{D}(\varepsilon_e)} \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}.$$

Using once again the Fubini's theorem and introducing $\varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e$, one writes this expression as

$$E_{e,1}^{IB} + E_{e,2}^{IB} = \int_0^\infty d\varepsilon_e^0 \int_0^{\varepsilon_e^0} d\varepsilon_\gamma \int_{S^2} d\Omega n_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \varepsilon_\gamma \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{v_e^0 \mathcal{D}(\varepsilon_e^0)}{v_e(\varepsilon_e^0 - \varepsilon_\gamma) \mathcal{D}(\varepsilon_e^0 - \varepsilon_\gamma)} \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}.$$

The expression of E_γ^{IB} is obtained by using the Fubini's theorem, which concludes the proof. \square

Another important property of the model is that it preserves the number of electrons. This is demonstrated in the following lemma.

Lemma 6.2 (Conservation of the electron number). *Assuming that the photon and electron angular fluxes ψ_e and ψ_γ vanish at the limit $|x| \rightarrow \infty$, the model for direct and inverse electron Bremsstrahlung defined in (6.16) preserves the number of electrons defined in (6.7), i.e.*

$$\frac{d}{dt} N_e(t) = \frac{d}{dt} \left[\int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\mathbf{\Omega}_e \frac{1}{v_e} \psi_e \right] = 0.$$

Proof. By definition of the model, the variation of the number of electrons is given by

$$\frac{d}{dt}N_e(t) = \left[\int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e \left(Q_e^B - Q_e^{B|rev} + Q_e^{IB} - Q_e^{IB|rev} \right) \right] = 0.$$

As for the energy conservation, each direct and inverse Bremsstrahlung processes preserve the number of electrons. It means that it is sufficient to prove that both the terms in the parenthesis vanish, i.e.

$$\begin{cases} \frac{d}{dt}N_{e,1}(t) = \int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e (Q_e^B - Q_e^{B|rev}) = 0, \\ \frac{d}{dt}N_{e,2}(t) = \int_{\mathbb{R}^3} dx \int_0^\infty d\varepsilon_e \int_{S^2} d\Omega_e (Q_e^{IB} - Q_e^{IB|rev}) = 0. \end{cases} \quad (6.17)$$

We start with the first term $N_{e,1}$. The definition of Q_e^B and $Q_e^{B|rev}$, defined in (6.10) and (6.11), yield

$$\begin{aligned} \frac{d}{dt}N_{e,1}(t) &= \int_{\mathbb{R}^3} dx \rho_i(t, x) \int_0^\infty d\varepsilon_e \int_{\varepsilon_e}^\infty d\varepsilon_e^0 \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e^0 - \varepsilon_e, \mathbf{\Omega}_\gamma)] \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \\ &\quad - \int_{\mathbb{R}^3} dx \rho_i(t, x) \int_0^\infty d\varepsilon_e \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e - \varepsilon_e^0, \mathbf{\Omega}_\gamma)] \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \end{aligned}$$

where we used once again the notation $d\mathbf{\Omega} = d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e^0 d\mathbf{\Omega}_e$ for ease of notations. Using the Fubini's theorem, the second integral can be written

$$\begin{aligned} &\int_{\mathbb{R}^3} dx \rho_i(t, x) \int_0^\infty d\varepsilon_e \int_0^{\varepsilon_e} d\varepsilon_e^0 \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e - \varepsilon_e^0, \mathbf{\Omega}_\gamma)] \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \\ &= \int_{\mathbb{R}^3} dx \rho_i(t, x) \int_0^\infty d\varepsilon_e^0 \int_{\varepsilon_e^0}^\infty d\varepsilon_e \int_{S^2} d\Omega [1 + n_\gamma(\varepsilon_e - \varepsilon_e^0, \mathbf{\Omega}_\gamma)] \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^0)}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}, \end{aligned}$$

and the first equation in (6.17) is obtained by making the change of variables $(\varepsilon_e, \mathbf{\Omega}_e) \leftrightarrow (\varepsilon_e^0, \mathbf{\Omega}_e^0)$. The second equation is obtained by using the same arguments, which concludes the proof. \square

6.2.3 Link with thermal Bremsstrahlung

In this part our aim is to find back from our model (6.16) the general expression of thermal Bremsstrahlung introduced in chapter 1 and used in chapter 2 in a relativistic context. This expression is given by

$$Q^{BT} = \sigma_a(\nu, T) \left(f(\nu, T) - n_\gamma(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) \right), \quad (6.18)$$

where f is the Planck function at temperature T , whose expression is given by $f(\nu, T) = (e^{h\nu/k_B T} - 1)^{-1}$, where k_B is the Boltzmann constant, n_γ is the photon occupation number and σ_a is the absorption coefficient. This simplified expression of the Bremsstrahlung is known to be valid at or close to the local thermodynamic equilibrium (LTE). We thus assume that the electrons are given by a Maxwell distribution at temperature T

$$\psi_e(t, x, \varepsilon_e) = C v_e \mathcal{D}(\varepsilon_e) e^{-\varepsilon_e/k_B T},$$

where $C = \frac{\rho_e}{mc^2} \frac{\beta e^{-\beta}}{K_2(\beta)}$. Using this expression in the direct and inverse collision operator for photons defined in (6.16), one gets the simplified expression

$$\begin{cases} Q_\gamma^B = C \rho_i(t, x) [1 + n_\gamma(\varepsilon_\gamma)] \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \int_{S^2} d\Omega_e^0 \int_{S^2} d\Omega_e v_e^0 \mathcal{D}(\varepsilon_e^0) e^{-\varepsilon_e^0/k_B T} \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ Q_\gamma^{IB} = -C \rho_i(t, x) n_\gamma(\varepsilon_\gamma) \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \int_{S^2} d\Omega_e^0 \int_{S^2} d\Omega_e e^{-(\varepsilon_e^0 - \varepsilon_\gamma)/k_B T} v_e^0 \mathcal{D}(\varepsilon_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}. \end{cases}$$

Now assuming in the emission term Q_γ^B that the photons are given by a Planck function $f(\nu, T)$, which is justified by the fact that close to the equilibrium, the emission spectrum is a Planck distribution, one gets the relation $1 + n_\gamma(\varepsilon_\gamma) = f(\nu, T) e^{h\nu/k_B T}$. Defining the absorption coefficient by

$$\sigma_a(\nu, T) = C \rho_i(t, x) \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \int_{S^2} d\Omega_e^0 \int_{S^2} d\Omega_e e^{-(\varepsilon_e^0 - \varepsilon_\gamma)/k_B T} v_e^0 \mathcal{D}(\varepsilon_e^0) \frac{d\sigma^B(\varepsilon_e^0 \rightarrow \varepsilon_e^0 - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e},$$

one obtains the expression of classical thermal Bremsstrahlung, as defined in (6.18).

6.2.4 CSD approximation

In this part we use the well known continuous slowing down (CSD) approximation to simplify the electrons' equation. The idea of this approximation, developed by Pomraning in [POM83, POM92], is to use the fact that the differential cross section is peaked at the small energy exchanges (this point concerning the differential cross section will be discussed in section 6.4). One thus has $\varepsilon_\gamma = \varepsilon_e^0 - \varepsilon_e \ll 1$. Expanding the function of $(\varepsilon_e^0, \Omega_e^0)$ around $(\varepsilon_e, \Omega_e)$ in the two gain terms of the electrons' collision operators (6.10) and (6.12), one obtains by stopping the expansion at the order 1 with respect to $|\varepsilon_e^0 - \varepsilon_e|$,

$$\begin{aligned} \frac{1}{v_e} \partial_t \psi_e(\varepsilon_e, \Omega_e) + \Omega_e \cdot \nabla \psi_e(\varepsilon_e, \Omega_e) &= \frac{\partial}{\partial \varepsilon_e} \left[\left(\mathcal{S}_B(\varepsilon_e) + \mathcal{S}_{IB}(\varepsilon_e) \right) \psi_e(\varepsilon_e, \Omega_e) \right] \\ &+ \left(\mathcal{L}_B(\varepsilon_e) + \mathcal{L}_{IB}(\varepsilon_e) \right) \nabla_{\Omega_e} \cdot \left[(\mathbf{I} - \Omega_e \otimes \Omega_e) \cdot \nabla_{\Omega_e} \right] \psi_e(\varepsilon_e, \Omega_e) \end{aligned} \quad (6.19)$$

The important point in this approximation is that although the differential cross section $d\sigma^B/d\varepsilon_\gamma d\Omega_\gamma d\Omega_e$ is singular at the limit $\varepsilon_e^0 \rightarrow \varepsilon_e$, $|\varepsilon_e^0 - \varepsilon_e| d\sigma^B/d\varepsilon_\gamma d\Omega_\gamma d\Omega_e$ is not singular. All the coefficients (defined below) in the energy drift terms of this equation involve power of $|\varepsilon_e^0 - \varepsilon_e|$ multiplying the differential cross section. The angular diffusion operators involve terms of the form $(1 - \Omega_e \cdot \Omega_e^0) d\sigma^B/d\varepsilon_\gamma d\Omega_\gamma d\Omega_e$. This term is not singular neither since in this approximation $(1 - \Omega_e \cdot \Omega_e^0)$ is of the same order than $|\varepsilon_e^0 - \varepsilon_e|$ (small deviation angle).

The operators \mathcal{S}_B and \mathcal{S}_{IB} are the stopping power coefficients. The one for the direct Bremsstrahlung is non negative and corresponds to a slowing down of the electrons. On contrary, the one for the inverse Bremsstrahlung is non positive and corresponds to an acceleration of the electrons. These operators are defined by

$$\begin{cases} \mathcal{S}_B(\varepsilon_e) = \rho_i(t, x) [1 + n_\nu(t, x)] \int_0^{\varepsilon_e} d\varepsilon_e^o(\varepsilon_e - \varepsilon_e^o) \int_{S^2} d\Omega_e^o \int_{S^2} d\Omega_\gamma \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^o)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ \mathcal{S}_{IB}(\varepsilon_e) = \rho_i(t, x) n_\nu(t, x) \int_{\varepsilon_e}^{\infty} d\varepsilon_e^o(\varepsilon_e^o - \varepsilon_e) \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \int_{S^2} d\Omega_e^o \int_{S^2} d\Omega_\gamma \frac{d\sigma^B(\varepsilon_e^o \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e} \end{cases} \quad (6.20)$$

Finally, the operator defined by the third line of (6.19) is an angular diffusion operator. The coefficients \mathcal{L}^B and \mathcal{L}^{IB} are defined by

$$\begin{cases} \mathcal{L}_B(\varepsilon_e) = \frac{\rho_i(t, x)}{2} [1 + n_\nu(t, x)] \int_0^{\varepsilon_e} d\varepsilon_e^o \int_{S^2} d\Omega_e^o (1 - \Omega_e \cdot \Omega_e^o) \int_{S^2} d\Omega_\gamma \frac{d\sigma^B(\varepsilon_e \rightarrow \varepsilon_e^o)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ \mathcal{L}_{IB}(\varepsilon_e) = \frac{\rho_i(t, x)}{2} n_\nu(t, x) \int_{\varepsilon_e}^\infty d\varepsilon_e^o \frac{v_e^o \mathcal{D}(\varepsilon_e^o)}{v_e \mathcal{D}(\varepsilon_e)} \int_{S^2} d\Omega_e^o (1 - \Omega_e \cdot \Omega_e^o) \int_{S^2} d\Omega_\gamma \frac{d\sigma^B(\varepsilon_e^o \rightarrow \varepsilon_e)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}. \end{cases} \quad (6.21)$$

6.3 M_1 angular moment model and numerical scheme

The aim of this section is to derive an M_1 angular moment model and a numerical scheme from our kinetic model. We use the continuous slowing down approximation derived in the previous section for the electrons and the same equation for the photons (second equation of (6.16)). Since we now focus only on radiotherapy application, we consider the stationary case and we do not consider the induced effects. This mean that we do not consider inverse Bremsstrahlung and that the coefficient $1 + n_\gamma$ in the direct Bremsstrahlung operators are replaced by 1. This is justified since in the considered energy range, the induced effects can be neglected. The model thus reduces to

$$\begin{cases} \Omega_\gamma \cdot \nabla \psi_\gamma(\varepsilon_\gamma, \Omega_\gamma) = \rho_i(t, x) \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^o \int_{S^2} d\Omega_e^o \psi_e(\varepsilon_e^o, \Omega_e^o) \int_{S^2} d\Omega_e \frac{d\sigma^B(\varepsilon_e^o \rightarrow \varepsilon_e^o - \varepsilon_\gamma)}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}, \\ \Omega_e \cdot \nabla \psi_e(\varepsilon_e, \Omega_e) = \frac{\partial}{\partial \varepsilon_e} \left[\bar{\mathcal{S}}_B(\varepsilon_e) \psi_e(\varepsilon_e, \Omega_e) \right] \\ \quad + \bar{\mathcal{L}}_B(\varepsilon_e) \nabla_{\Omega_e} \cdot \left[(\mathbf{I} - \Omega_e \otimes \Omega_e) \cdot \nabla_{\Omega_e} \right] \psi_e(\varepsilon_e, \Omega_e), \end{cases} \quad (6.22)$$

where the operators $\bar{\mathcal{S}}_B$ and $\bar{\mathcal{L}}_B$ are defined by

$$\begin{cases} \bar{\mathcal{S}}_B = \rho_i(t, x) \int_0^{\varepsilon_e} d\varepsilon_e^o (\varepsilon_e - \varepsilon_e^o) \int_{S^2} d\Omega_e^o K_e(\varepsilon_e, \varepsilon_e^o, \Omega_e \cdot \Omega_e^o) \\ \bar{\mathcal{L}}_B = \frac{\rho_i(t, x)}{2} \int_0^{\varepsilon_e} d\varepsilon_e^o \int_{S^2} d\Omega_e^o (1 - \Omega_e \cdot \Omega_e^o) K_e(\varepsilon_e, \varepsilon_e^o, \Omega_e \cdot \Omega_e^o). \end{cases} \quad (6.23)$$

The end of this section is organized as follows. In a first part the M_1 approximation for the system (6.22) is derived. In a second part its energy discretization is presented. We prove that the resulting scheme preserves the total energy (lemma 6.4).

For the sake of clarity we introduce K_e (resp K_γ and $K_{\gamma,2}$) defined as the integral over the all the photons angle Ω_γ (resp over the ingoing and outgoing electrons angle Ω_e and Ω_e^o).

$$\begin{cases} K_e(\varepsilon_e^o, \varepsilon_e, \Omega_e \cdot \Omega_e^o) = \int_{S^2} d\Omega_\gamma \frac{d\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}(\varepsilon_e^o \rightarrow \varepsilon_e), \\ K_\gamma(\varepsilon_e^o, \varepsilon_e, \Omega_\gamma \cdot \Omega_e^o) = \int_{S^2} d\Omega_e \frac{d\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}(\varepsilon_e^o \rightarrow \varepsilon_e). \end{cases} \quad (6.24)$$

This important point is that the resulting expression depends only on a scattering angle.

6.3.1 M_1 angular moment model

In this section we derive the M_1 angular moment model [DF99] for the previous model (6.22). This model is widely used in practice since it is known to preserve the realizability domain, defined as the set of $\{\psi_\alpha^{(i)}, i = 0, 1\}$ such that $\psi_\alpha^{(0)} > 0$ and $|\psi_\alpha^{(1)}| < \psi_\alpha^{(0)}$. The compensation is that the closure (the expression of the second order moment $\psi_\alpha^{(2)}$ in terms of the previous ones) is nonlinear. This last point is not detailed in this part since it has been studied in chapter 5.

We define the angular moments as

$$\psi_\alpha^{(i)}(\varepsilon) = \frac{1}{4\pi} \int_{S^2} d\Omega_\alpha \psi_\alpha(\varepsilon) \Omega_\alpha^i,$$

with $\alpha = e, \gamma$ and $i \in \{0, 1, 2\}$. The M_1 model is given in the following lemma.

Lemma 6.3 (M_1 model). *Under the assumption that the induce effects are isotropic, the angular M_1 model for the system (6.22) writes*

$$\begin{cases} \nabla \psi_\gamma^{(1)} = \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \psi_e^{(0)}(\varepsilon_e^0) \sigma_{\gamma,1}(\varepsilon_e^0, \varepsilon_\gamma), \\ \nabla \psi_\gamma^{(2)} = \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \psi_e^{(1)}(\varepsilon_e^0) \sigma_{\gamma,2}(\varepsilon_e^0, \varepsilon_\gamma), \\ \nabla \psi_e^{(1)} = \frac{\partial}{\partial \varepsilon_e} \left[\psi_e^{(0)}(\varepsilon_e) \bar{\mathcal{S}}_B(\varepsilon_e) \right], \\ \nabla \psi_e^{(2)} = \frac{\partial}{\partial \varepsilon_e} \left[\psi_e^{(1)}(\varepsilon_e) \bar{\mathcal{S}}_B(\varepsilon_e) \right] - 2\bar{\mathcal{L}}_B(\varepsilon_e) \psi_e^{(1)}(\varepsilon_e), \end{cases} \quad (6.25)$$

where

$$\begin{cases} \sigma_{\gamma,1}(\varepsilon_e^0, \varepsilon_\gamma) = 2\pi \rho_i(t, x) \int_{-1}^1 d\mu K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu), \\ \sigma_{\gamma,2}(\varepsilon_e^0, \varepsilon_\gamma) = 2\pi \rho_i(t, x) \int_{-1}^1 \mu K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu). \end{cases}$$

Proof. We start with the photons' equation, the electrons's equation being treated in a second step.

Moments equations for the photons

Integrating the photons equation (6.22) over S^2 , one finds

$$\nabla \cdot \psi_\gamma^{(1)}(\varepsilon_\gamma) = \int_{S^2} d\Omega_\gamma Q_\gamma^B.$$

We need to compute these two integrals. Using the definition of the direct Bremsstrahlung operator \bar{Q}_γ^B (6.23), one has

$$\int_{S^2} d\Omega_\gamma Q_\gamma^B = \rho_i(t, x) \int_{\varepsilon_\gamma}^{\infty} d\varepsilon_e^0 \int_{S^2} d\Omega_e^0 \psi_e(\varepsilon_e^0, \Omega_e^0) \int_{S^2} d\Omega_e K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \Omega_e \cdot \Omega_e^0).$$

The point is to remark that $\int_{S^2} d\mathbf{\Omega}_e K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^0)$ does not depend on $\mathbf{\Omega}_e^0$. Indeed, we fix $\mathbf{\Omega}_e^0$ and we choose a polar coordinate system such that $\mathbf{\Omega}_e = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T$ and $\mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^0 = \cos \theta$, i.e. θ is the angle between $\mathbf{\Omega}_e$ and $\mathbf{\Omega}_e^0$. It yields, denoting $\mu = \cos \theta$,

$$\begin{aligned} \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \int_{S^2} d\mathbf{\Omega}_e K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^0) &= 2\pi \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \int_{-1}^1 d\mu K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu), \\ &= \psi_e^{(0)}(\varepsilon_e^0) 2\pi \int_{-1}^1 d\mu K_e(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu). \end{aligned}$$

For the equation on the second order moment, one gets by integrating over S^2 the first equation of (6.22) multiplied by $\mathbf{\Omega}_\gamma$

$$\nabla \cdot \psi_\gamma^{(2)}(\varepsilon_\gamma, \mathbf{\Omega}_\gamma) = \int_{S^2} d\mathbf{\Omega}_\gamma \mathbf{\Omega}_\gamma Q_\gamma^B.$$

The definition of the direct Bremsstrahlung operator \bar{Q}_γ^B (6.23) yields

$$\nabla \cdot \psi_\gamma^{(2)}(\varepsilon_\gamma) = \rho_i(t, x) \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^0 \int_{S^2} d\mathbf{\Omega}_e^0 \psi_e(\varepsilon_e^0, \mathbf{\Omega}_e^0) \int_{S^2} d\mathbf{\Omega}_\gamma \mathbf{\Omega}_\gamma K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mathbf{\Omega}_e^0 \cdot \mathbf{\Omega}_\gamma).$$

Using once again polar coordinates, one finds

$$\int_{S^2} d\mathbf{\Omega}_\gamma \mathbf{\Omega}_\gamma K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mathbf{\Omega}_e^0 \cdot \mathbf{\Omega}_\gamma) = 2\pi \begin{pmatrix} 0 \\ 0 \\ \int_{-1}^1 d\mu \mu K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu) \end{pmatrix} = 2\pi \mathbf{\Omega}_e^0 \int_{-1}^1 d\mu \mu K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu).$$

One thus obtains

$$\nabla \cdot \psi_\gamma^{(2)}(\varepsilon_\gamma) = 2\pi \rho_i(t, x) \int_{\varepsilon_\gamma}^\infty d\varepsilon_e^0 \psi_e^{(1)}(\varepsilon_e^0) \int_{-1}^1 \mu K_\gamma(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma, \mu).$$

Moments equations for the electrons

Integrating the electrons equation (6.22) over S^2 , one finds

$$\nabla \psi_e^{(1)}(\varepsilon_e, \mathbf{\Omega}_e) = \int_{S^2} d\mathbf{\Omega}_e Q_e^{SP} + \int_{S^2} d\mathbf{\Omega}_e Q_e^{AD},$$

where the stopping power Q_e^{SP} and the angular diffusion Q_e^{AD} operators are defined by

$$\begin{cases} Q_e^{SP} = \frac{\partial}{\partial \varepsilon_e} \left[\bar{\mathcal{S}}_B(\varepsilon_e) \psi_e(\varepsilon_e, \mathbf{\Omega}_e) \right] \\ Q_e^{AD} = \bar{\mathcal{L}}_B(\varepsilon_e) \nabla_{\mathbf{\Omega}_e} \cdot \left[(\mathbf{I} - \mathbf{\Omega}_e \otimes \mathbf{\Omega}_e) \cdot \nabla_{\mathbf{\Omega}_e} \right] \psi_e(\varepsilon_e, \mathbf{\Omega}_e). \end{cases} \quad (6.26)$$

Using the definition of $\bar{\mathcal{S}}_B$ (6.23), one finds

$$\bar{\mathcal{S}}_B = 2\pi \rho_i(t, x) \int_0^{\varepsilon_e} d\varepsilon_e^0 (\varepsilon_e - \varepsilon_e^0) \int_{-1}^1 d\mu K_e(\varepsilon_e, \varepsilon_e^0, \mu),$$

which yields

$$\int_{S^2} d\mathbf{\Omega}_e Q_e^{SP} = 2\pi \rho_i(t, x) \frac{\partial}{\partial \varepsilon_e} \left[\psi_e^{(0)}(\varepsilon_e) \int_0^{\varepsilon_e} d\varepsilon_e' (\varepsilon_e - \varepsilon_e') \int_{-1}^1 d\mu K_e(\varepsilon_e, \varepsilon_e', \mu) \right],$$

We now turn to the zero-th order moment of the angular diffusion operator. One has by definition of Q_e^{AD} and since the diffusion coefficient \mathcal{L}_B does not depend on the electrons' angle $\mathbf{\Omega}_e$

$$\int_{S^2} d\mathbf{\Omega}_e Q_e^{AD} = \bar{\mathcal{L}}_B(\varepsilon_e) \int_{S^2} d\mathbf{\Omega}_e \nabla_{\mathbf{\Omega}_e} \cdot \left[(\mathbf{I} - \mathbf{\Omega}_e \otimes \mathbf{\Omega}_e) \cdot \nabla_{\mathbf{\Omega}_e} \right] \psi_e(\varepsilon_e, \mathbf{\Omega}_e).$$

We use once again polar coordinates: we introduce $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$ such that $\mathbf{\Omega}_e = (\sqrt{1 - \mu^2} \cos \phi, \sqrt{1 - \mu^2} \sin \phi, \mu)^T$. It yields

$$\begin{aligned} \int_{S^2} d\mathbf{\Omega}_e Q_e^{AD} &= \bar{\mathcal{L}}_B(\varepsilon_e) \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \left[\frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial}{\partial \mu} \right) + \frac{1}{1 - \mu^2} \frac{\partial^2}{\partial^2 \phi} \right] \psi_e(\varepsilon_e, \mu, \cos \phi, \sin \phi), \\ &= \bar{\mathcal{L}}_B(\varepsilon_e) \left\{ \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial}{\partial \mu} \right) \psi_e(\varepsilon_e, \mu, \cos \phi, \sin \phi) \right. \\ &\quad \left. + \int_{-1}^1 d\mu \frac{1}{1 - \mu^2} \int_0^{2\pi} d\phi \frac{\partial^2}{\partial^2 \phi} \psi_e(\varepsilon_e, \mu, \cos \phi, \sin \phi) \right\}, \end{aligned}$$

and an integration by parts for each of these integrals yields $\int_{S^2} d\mathbf{\Omega}_e Q_e^{AD} = 0$. We turn to the equation on the second order moment $\psi_e^{(2)}$. Multiplying the second equation of (6.22) by $\mathbf{\Omega}_e$ and integrating over S^2 , one finds

$$\nabla \psi_e^{(2)}(\varepsilon_e, \mathbf{\Omega}_e) = \int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e Q_e^{SP} + \int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e Q_e^{AD}.$$

For the first integral, one easily finds by using the definition of Q_e^{SP} (6.26)

$$\int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e Q_e^{SP} = \frac{\partial}{\partial \varepsilon_e} \left[\bar{\mathcal{S}}_B(\varepsilon_e) \psi_e^{(1)}(\varepsilon_e) \right].$$

The second integral is a little more complicated. By definition of Q_e^{AD} (6.26), one has

$$\int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e Q_e^{AD} = \bar{\mathcal{L}}_B(\varepsilon_e) \int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e \nabla_{\mathbf{\Omega}_e} \cdot \left[(\mathbf{I} - \mathbf{\Omega}_e \otimes \mathbf{\Omega}_e) \cdot \nabla_{\mathbf{\Omega}_e} \right] \psi_e(\varepsilon_e, \mathbf{\Omega}_e).$$

Using once again polar coordinates, one finds $\int_{S^2} d\mathbf{\Omega}_e \mathbf{\Omega}_e Q_e^{AD} = D_\phi + D_\mu$, with

$$\begin{cases} D_\phi = \bar{\mathcal{L}}_B(\varepsilon_e) \int_{-1}^1 d\mu \frac{1}{1 - \mu^2} \int_0^{2\pi} d\phi \left(\frac{\sqrt{1 - \mu^2} \cos \phi}{\sqrt{1 - \mu^2} \sin \phi} \right) \frac{\partial^2}{\partial^2 \phi} \psi_e \\ D_\mu = \bar{\mathcal{L}}_B(\varepsilon_e) \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \left(\frac{\sqrt{1 - \mu^2} \cos \phi}{\sqrt{1 - \mu^2} \sin \phi} \right) \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial}{\partial \mu} \right) \psi_e. \end{cases}$$

We treat the term D_ϕ first. Two successive integration by parts on the variable ϕ leads to

$$D_\phi = -\bar{\mathcal{L}}_B(\varepsilon_e) \int_{-1}^1 d\mu \frac{1}{1 - \mu^2} \int_0^{2\pi} d\phi \left(\frac{\sqrt{1 - \mu^2} \cos \phi}{\sqrt{1 - \mu^2} \sin \phi} \right) \psi_e.$$

We keep this expression and we turn to the computation of D_μ . An integration by parts yields

$$D_\mu = \bar{\mathcal{L}}_B(\varepsilon_e) \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \begin{pmatrix} -\mu\sqrt{1-\mu^2} \cos \phi \\ -\mu\sqrt{1-\mu^2} \sin \phi \\ (1-\mu^2) \end{pmatrix} \frac{\partial}{\partial \mu} \psi_e.$$

A second integration by parts gives

$$D_\mu = -\bar{\mathcal{L}}_B(\varepsilon_e) \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \begin{pmatrix} \frac{1-2\mu^2}{\sqrt{1-\mu^2}} \cos \phi \\ \frac{1-2\mu^2}{\sqrt{1-\mu^2}} \sin \phi \\ 2\mu \end{pmatrix} \psi_e.$$

Adding the expression of D_ϕ , one gets

$$D_\phi + D_\mu = -2\bar{\mathcal{L}}_B(\varepsilon_e) \int_0^{2\pi} d\phi \int_{-1}^1 d\mu \begin{pmatrix} \frac{1}{\sqrt{1-\mu^2}} \cos \phi \\ \frac{1}{\sqrt{1-\mu^2}} \sin \phi \\ 0 \end{pmatrix} \psi_e = -2\bar{\mathcal{L}}_B(\varepsilon_e) \psi_e^{(1)}(\varepsilon_e),$$

which concludes the proof. \square

6.3.2 Numerical scheme

In this part we detail the numerical scheme used in the numerical tests in section 6.5. We are mostly interested here in the energy discretization. We do not detail the space discretization, which is performed in practice with an Aregba Natallini scheme [AN00]. We thus consider only the semi-discrete scheme. We consider a supremum (resp infimum) in energy ε_{max} (resp ε_{min}) and a cartesian energy discretization mesh defined by N points ε_j , $j \in \{1, N\}$ such that $\varepsilon_j = \varepsilon_{max} - j\Delta\varepsilon$, where the mesh step is such that $\varepsilon_N = \varepsilon_{max} - N\Delta\varepsilon = \varepsilon_{min}$. In the following we set $\varepsilon_{min} = 0$.

Définition 6.1 (Numerical scheme). *The semi-discrete numerical scheme writes, for $i = 1, 2$,*

$$\begin{cases} \nabla \psi_{\gamma,j}^{(i)} = \sum_{k=1}^j \Delta\varepsilon \sigma_{\gamma,i}(\varepsilon_k, \varepsilon_j) \psi_{e,k}^{(i)}, \\ \nabla \psi_{e,j}^{(i)} = \sum_{k=1}^j \Delta\varepsilon \sigma_e(\varepsilon_k, \varepsilon_j) \psi_{e,k}^{(i)} - \delta_{i2} \bar{\mathcal{L}}_B(\varepsilon_j) \psi_e^{(1)}(\varepsilon_j) \end{cases} \quad 1 \leq j \leq N, \quad (6.27)$$

where

$$\sigma_e(\varepsilon_k, \varepsilon_j) = \begin{cases} -\bar{\mathcal{S}}_B(\varepsilon_j)/\Delta\varepsilon^2 & \text{if } k = j, \\ \bar{\mathcal{S}}_B(\varepsilon_{j-1})/\Delta\varepsilon^2 & \text{if } k = j-1, \\ 0 & \text{elsewhere.} \end{cases}$$

The discretization of the stopping power operator $\bar{\mathcal{S}}_B$ is performed through a first order approximation

$$\bar{\mathcal{S}}_B(\varepsilon_j) = \rho_i(t, x) \sum_{k=j+1}^N \Delta\varepsilon(\varepsilon_j - \varepsilon_k) \int_{S^2} d\Omega_e^o K_e(\varepsilon_j, \varepsilon_k, \Omega_e \cdot \Omega_e^o) \quad 1 \leq j \leq N-1, \quad (6.28)$$

$$\bar{\mathcal{S}}_B(\varepsilon_N) = 0.$$

The main point here is to understand that the proposed scheme is a consistent discretization of the M_1 model (6.25). The discretization of the photons equation is obvious, since it simply uses a first order approximation of the energy integral. For the electrons' equations, the definition of σ_e yields

$$\nabla \psi_{e,j}^{(i)} = \frac{\bar{\mathcal{S}}_B(\varepsilon_{j-1})\psi_{e,j-1}^{(i)} - \bar{\mathcal{S}}_B(\varepsilon_j)\psi_{e,j}^{(i)}}{\Delta\varepsilon}.$$

Reminding that we consider a decreasing energy discretization mesh ($\varepsilon_i \leq \varepsilon_j$ for $i \geq j$), we obtain a consistent discretization of the energy drift term $\partial/\partial\varepsilon(\bar{\mathcal{S}}_B\psi_e)$. The second important point is the discretization of the stopping power operator $\bar{\mathcal{S}}_B$. Indeed there is at least two choices of first order approximation, consisting to use either the left or the right value in each cell. The left value gives the conservation of energy, which is not true if we use a discretization from the right value. This is proved in the following lemma.

Lemma 6.4 (Energy conservation). *Define the energy $E(t, x)$ as the sum of the photons' energy $E_\gamma(t, x)$ and the electrons' energy $E_e(t, x)$, where*

$$\begin{cases} E_\gamma(t, x) = \sum_{j=1}^N \varepsilon_j \Delta\varepsilon \nabla \psi_\gamma^{(1)}, \\ E_e(t, x) = \sum_{j=1}^N \varepsilon_j \Delta\varepsilon \nabla \psi_e^{(1)}. \end{cases}$$

Then, under the boundary conditions $\psi_\alpha^{(0)}(\varepsilon_0) = \psi_\alpha^{(0)}(\varepsilon_N) = 0$, for $\alpha = e, \gamma$, the scheme (6.27) preserves the total energy, i.e.

$$\partial_t E(t, x) = 0.$$

Proof. Using the discrete equation for the electrons (6.27), one has

$$\partial_t E_e(t, x) = \sum_{j=1}^N \Delta\varepsilon^2 \sum_{k=1}^j \varepsilon_j \sigma_e(\varepsilon_k, \varepsilon_j) \psi_{e,k}^{(0)},$$

and the definition of σ_e yields

$$\partial_t E_e(t, x) = \sum_{j=1}^N \varepsilon_j \left(\bar{\mathcal{S}}_B(\varepsilon_{j-1})\psi_{e,j-1}^{(0)} - \bar{\mathcal{S}}_B(\varepsilon_j)\psi_{e,j}^{(0)} \right).$$

Rearranging the terms to get a sum on $\bar{\mathcal{S}}_B(\varepsilon_j)\psi_{e,j}^{(0)}$, one finds

$$\partial_t E_e(t, x) = - \sum_{j=1}^N \Delta\varepsilon \bar{\mathcal{S}}_B(\varepsilon_j)\psi_{e,j}^{(0)} + \varepsilon_1 \bar{\mathcal{S}}_B(\varepsilon_0)\psi_{e,0}^{(0)} - \varepsilon_{N+1} \bar{\mathcal{S}}_B(\varepsilon_N)\psi_{e,N}^{(0)}.$$

Due to the definition of $\bar{\mathcal{S}}_B$ at the edge of the mesh ε_N (6.28) and the boundary conditions, the two last terms vanish. Now using the definition of $\bar{\mathcal{S}}_B$, one gets

$$\partial_t E_e(t, x) = -\rho_i(t, x) \Delta\varepsilon^2 \sum_{j=1}^N \sum_{k=j+1}^N (\varepsilon_j - \varepsilon_k) \int_{S^2} d\Omega_e^o K_e(\varepsilon_j, \varepsilon_k, \Omega_e \cdot \Omega_e^o) \psi_{e,j}^{(0)}. \quad (6.29)$$

Since the mesh is cartesian, the energy $\varepsilon_j - \varepsilon_k$ corresponds to a grid point, say ε_l . From the definition of the mesh, the indice l is defined by $l = N + j - k$. Making this change of indices in (6.29), one finds

$$\partial_t E_e(t, x) = -\rho_i(t, x) \Delta \varepsilon^2 \sum_{j=1}^N \sum_{l=j}^{N-1} \varepsilon_l \int_{S^2} d\Omega_e^o K_e(\varepsilon_j, \varepsilon_j - \varepsilon_l, \mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^o) \psi_{e,j}^{(0)}.$$

Using a discrete version of the Fubini's theorem, one has $\sum_{j=1}^N \sum_{l=j}^{N-1} f(\varepsilon_j, \varepsilon_l) = \sum_{l=1}^{N-1} \sum_{j=1}^l f(\varepsilon_j, \varepsilon_l)$. It yields

$$\partial_t E_e(t, x) = -\rho_i(t, x) \sum_{l=1}^{N-1} \Delta \varepsilon \varepsilon_l \sum_{j=1}^l \Delta \varepsilon \int_{S^2} d\Omega_e^o K_e(\varepsilon_j, \varepsilon_j - \varepsilon_l, \mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^o) \psi_{e,j}^{(0)}.$$

Finally, and since $\psi_{e,N}^{(0)} = 0$, one finds

$$\partial_t E_e(t, x) = -\rho_i(t, x) \underbrace{\sum_{l=1}^N \Delta \varepsilon \varepsilon_l \sum_{j=1}^l \Delta \varepsilon \int_{S^2} d\Omega_e^o K_e(\varepsilon_j, \varepsilon_j - \varepsilon_l, \mathbf{\Omega}_e \cdot \mathbf{\Omega}_e^o) \psi_{e,j}^{(0)}}_{\nabla \psi_{\gamma,j}^{(0)}},$$

which concludes the proof. □

6.4 Differential cross sections

This part is devoted to the derivation of a differential cross section for electron-ion Bremsstrahlung. In a first part (part 6.4.1) we give the general expression that can be found in physical literature both in the relativistic and non relativistic regimes. It is important to notice that the relativistic case is not a simplification of the non relativistic case. Indeed, the relativistic differential cross section (6.30) is derived from the Dirac equation, while the non relativistic differential cross section (6.31) is derived from the Schrödinger equation. In a second part (part 6.4.2), we explain the way the screening effects are taken into account in the differential cross section. These screening effects model the interaction between the ingoing electron and the electrons of the ion. Finally, in part 6.4.3 we give the analytical differential cross section that is used in the numerical tests in section 6.5.

6.4.1 General expression

The inspection of available theoretical works on the Bremsstrahlung differential cross sections in energy and angles, shows that one cannot expect more than qualitative behavior from analytical formulae [LKPT76, SB85, TSE89]. A quantitative assessment should result from partial-wave calculations. Having mentioned that point, we nevertheless use analytical formulae [TPL79, TSE89, HAU08] that are legitimate in some regimes, and fail in reasonable proportion with respect to what is considered the best available data set [SB85]. As a matter of fact, this increases our knowledge and confidence while crossing data sets, and enables us to validate our numerical schemes at a lower cost.

As presented in the introduction, we denote by $\frac{d^4\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e}$ the Bremsstrahlung fourth differential cross section with respect to the photon energy and solid angle $d\mathbf{\Omega} \equiv \sin\theta^o d\theta^o \sin\theta d\theta d\phi$. In the Born approximation (plane waves), we obtain the expression [KM59]

$$\begin{aligned} \frac{d^4\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} &= \alpha_f Z^2 \left(\frac{r_0}{2\pi}\right)^2 \frac{(1-F(q))^2}{q^4} \frac{1}{|\varepsilon_e^o - \varepsilon_e|} \frac{p}{p^o} \\ &\times \left[\frac{p^2 \sin^2 \theta}{(\varepsilon_e - p \cos \theta)^2} (4\varepsilon_e^{o2} - q^2) + \frac{p^{o2} \sin^2 \theta^o}{(\varepsilon_e^o - p^o \cos \theta^o)^2} (4\varepsilon_e^2 - q^2) \right. \\ &\quad - \frac{2p^o p \sin \theta^o \sin \theta \cos \phi}{(\varepsilon_e^o - p^o \cos \theta^o)(\varepsilon_e - p \cos \theta)} (4\varepsilon_e^o \varepsilon_e - q^2) \\ &\quad \left. + \frac{2k^2 (p^{o2} \sin^2 \theta^o + p^2 \sin^2 \theta - 2p^o p \sin \theta^o \sin \theta \cos \phi)}{(\varepsilon_e^o - p^o \cos \theta^o)(\varepsilon_e - p \cos \theta)} \right], \end{aligned} \quad (6.30)$$

$$q^2 = p^{o2} + p^2 + k^2 - 2p^o p \mathbf{\Omega}_e^o \cdot \mathbf{\Omega}_e - 2p^o k \mathbf{\Omega}_e^o \cdot \mathbf{\Omega}_\gamma + 2pk \mathbf{\Omega}_e \cdot \mathbf{\Omega}_\gamma$$

where $\cos\theta^o = \mathbf{\Omega}_e^o \cdot \mathbf{\Omega}_\gamma$ (resp. $\cos\theta = \mathbf{\Omega}_e \cdot \mathbf{\Omega}_\gamma$) is the pitch angle between the emitted photon momentum and the ingoing (resp. outgoing) electron momentum. ϕ is the angle between the plane (\mathbf{p}, \mathbf{k}) and $(\mathbf{p}^o, \mathbf{k})$. $\mathbf{q} = \mathbf{p}^o - \mathbf{p} + \mathbf{k}$ is the momentum transferred to the ion.

In the non relativistic limit, the differential cross section (6.30) reduces to [KM59, BMO70]

$$\begin{aligned} \frac{d^4\sigma^B}{d\varepsilon_\gamma d\mathbf{\Omega}_\gamma d\mathbf{\Omega}_e} &= \alpha_f Z^2 \left(\frac{r_0}{2\pi}\right)^2 \frac{(1-F(q))^2}{q^4} \frac{1}{|\varepsilon_e^o - \varepsilon_e|} \frac{p}{p^o} \left(p^2 \sin^2 \theta + p^{o2} \sin^2 \theta^o \right. \\ &\quad \left. - 2p^o p \sin \theta^o \sin \theta \cos \phi \right), \end{aligned} \quad (6.31)$$

$$q^2 = p^{o2} + p^2 - 2p^o p (\cos\theta^o \cos\theta + \sin\theta^o \sin\theta \cos\phi).$$

6.4.2 Analytical screening: plasma effects and partial ionization treatment in the cross sections

Again, a quantitative evaluation of screening effects requires partial-wave calculations. An analytical evaluation can be obtained with the introduction of a form factor $F(q)$ in the differential cross sections. It is related to the Fourier transform of the electron-ion potential, $\mathcal{V}(q)$, with the Poisson equation

$$-\frac{\alpha_f Z}{2\pi^2} \frac{1-F(q)}{q^2} = \mathcal{V}(q) \equiv \frac{1}{(2\pi)^3} \int_{S^2} V(r) \exp(i\mathbf{q} \cdot \mathbf{r}) d\mathbf{r}.$$

The form factor accounts for electric field screening, whether Thomas-Fermi or Debye, in solids or plasmas. A combination of at least two Yukawa potentials is then required for the interpolation. The following potential can be chosen, for instance [ROZ79]

$$V(r) = -\alpha_f Z \left[\left(1 - \frac{Z^*}{Z}\right) \frac{1}{r} \exp(-r/l_{TF}) + \frac{Z^*}{Z} \frac{1}{r} \exp(-r/\lambda_D) \right], \quad (6.32)$$

where r is the radius vector from a center, l_{TF} the Thomas-Fermi length, and λ_D the Debye length. Z^* is the effective atomic number for free plasma electrons. In this Section, the distances are expressed in units of the Compton wavelength [KM59]. The momentum transfer to the ion,

q , is expressed in $m_e c$ units, and the potential is expressed in $m_e c^2$ units. When Fourier transformed, the potential becomes

$$\mathcal{V}(q) = -\frac{\alpha_f Z}{2\pi^2} \left[\left(1 - \frac{Z^*}{Z}\right) \frac{l_{TF}^2}{l_{TF}^2 q^2 + 1} + \frac{Z^*}{Z} \frac{l_D^2}{l_D^2 q^2 + 1} \right]$$

The injection of the potential (6.32) in the form factor of the cross sections introduces new nonlinearities, because the cross sections are proportional to the square of the Fourier transformed potential. A brute force method would require the computation of all the new differential cross sections with this potential. However, this is not always straightforward. Moreover, the interpolation of the potential (6.32) comes from a linearization (valid at relatively high temperature and low density), so a possible path to circumvent the difficulty is proposed in [LOB15], where an effective screening length, l^* , is employed. It encompasses the two limit regimes, where references can be obtained, while preserving the analytical expressions of the cross sections available in the literature.

6.4.3 Analytical expression

In the following we denote by $\varepsilon_\gamma \frac{d\sigma^B}{d\varepsilon_\gamma}$ the differential cross section integrated over momentum \mathbf{q} and angles Ω_γ and Ω_e , i.e.

$$\varepsilon_\gamma \frac{d\sigma^B}{d\varepsilon_\gamma} = \int_{S^2} d\Omega_e \int_{S^2} d\Omega_\gamma \int_{\mathbb{R}^3} d\mathbf{q}^3 \varepsilon_\gamma \frac{d^4\sigma^B}{d\varepsilon_\gamma d\Omega_\gamma d\Omega_e}.$$

The method used to compute the momentum integrals is detailed in [BET34]. One obtains different expressions according to the relativistic (derived from the Dirac equation) or non relativistic (derived from the Schrödinger equation) regimes. The expression given below are valid under the Born (plane wave) approximation. This approximation is valid if $2\pi Z\alpha_f \ll \beta_e^0$ and $2\pi Z\alpha_f \ll \beta_e$, where $\beta_e := \beta(\varepsilon_e) = 1 - (1 + \varepsilon_e/mc^2)^{-2}$, Z is the atomic number and $\alpha_f = 1/137$ is the fine structure constant.

Non relativistic case

In the non relativistic regime, the Schrödinger equation can be used to derive the Bremsstrahlung cross section. In [TSE89], a partial-wave analysis is performed to obtain the differential cross section in electron recoil angle and energy.

In the Born approximation, analytical Bremsstrahlung coefficients can be expressed from the integration of equation (6.31). The integration procedure from [BMO70] is applied and gives

$$\varepsilon_\gamma \frac{d\sigma_{NR}^B}{d\varepsilon_\gamma}(\varepsilon_e^o, \varepsilon_e) = \frac{16}{3} \frac{r_o^2 \alpha_f Z^2}{p_e^{o2}} \frac{1}{2} \left[\ln \left(\frac{k_+^2 l^{*2} + 1}{k_-^2 l^{*2} + 1} \right) + \frac{1}{k_+^2 l^{*2} + 1} - \frac{1}{k_-^2 l^{*2} + 1} \right] \quad (6.33)$$

In this expression $k_\pm = p_e^0 \pm p_e$, where p_e is the ingoing electron momentum, and is defined by

$$p_e = mc \sqrt{\frac{\varepsilon_e}{mc^2} \left(\frac{\varepsilon_e}{mc^2} + 2 \right)}.$$

The constant α_f is the fine structure constant, and is defined by $\alpha_f = 1/137$, Z is the atomic number, r_o is the electron' radius and l^* is the screening length. As explained in the part 6.4.2, it can be either the Thomas-Fermi length l_{TF} (solid), the Deby length l_D (plasma), or an interpolation of these two length.

Relativistic case

Even in the Born approximation, the analytical computation of the Bremsstrahlung coefficients in the relativistic regime is a challenge. For instance, in [BOR72], a unique analytical integration is made, which is far from the three integrations required to reach an analytical formulation. Thus, it seems that we are compelled to introduce new assumptions, and adopt those of Bethe [BET34], considering high ingoing and outgoing electron energies compared to the rest energy. In this limit very simple analytical expressions can be obtained.

In the relativistic regime, the significant values of the cross section are restricted to the small scattering angles between photons and electrons [DBM54, HOU48]. In the case of a Thomas-Fermi ion model, two limiting cases may be distinguished, depending on the quantity [KM59]

$$\gamma = \frac{100\varepsilon_\gamma}{E_e^o E_e Z^{1/3}}, \quad (6.34)$$

where $E_e = 1 + \varepsilon_e$. These are labelled I and II, and correspond to "high" and "small" energy regions (though both relativistic).

- limiting case I: $\gamma = 0$ (high energy)
Coulomb corrections to the Born approximation need to be included in case I only, while screening effects can be neglected. The corrected cross sections are available in [DBM54, BET34].
- limiting case II: $1 \ll \gamma$ (small energy)
In the case II, the Born approximation is valid, but screening effects must be taken into account.

The photon energy differential cross section, with arbitrary screening, is expressed by formula (50) in [BET34]. It is valid for the cases I and II, and intermediate regimes as well:

$$\varepsilon_\gamma \frac{d\sigma_R^B}{d\varepsilon_\gamma}(\varepsilon_\gamma, \varepsilon_e^o) = 4r_0^2 \alpha_f Z^2 \left[\left(\left[1 + \frac{E_e^2}{E_e^{o2}} \right] (I_1 + f(Z)) - \frac{2}{3} \frac{E_e}{E_e^o} (I_2 + f(Z)) \right) \right], \quad (6.35)$$

where $a(Z) = Ze^2/\hbar c = \alpha_f Z$ and $f(Z) = \sum_{\nu=1}^{\infty} \frac{a^2(Z)}{\nu(\nu^2 + a^2(Z))}$ is the Coulomb correction that becomes non-negligible in the ultra-relativistic regime (domain I) [DBM54]. This correction can be evaluated, for $a(Z) < 2/3$, as [DBM54]

$$f(Z) \simeq a^2 \left[(1 + a^2)^{-1} + 0.20206 - 0.0369a^2 + 0.0083a^4 - 0.002a^6 \right].$$

In the general case of an arbitrary atomic form factor F , we have

$$\begin{cases} I_1 = \int_{\delta}^1 (q - \delta)^2 (1 - F(q))^2 \frac{dq}{q^3} + 1, \\ I_2 = \int_{\delta}^1 (q^3 - 6\delta^2 q \ln \frac{q}{\delta} + 3\delta^2 q - 4\delta^3) (1 - F(q))^2 \frac{dq}{q^4} + \frac{5}{6}, \end{cases} \quad (6.36)$$

where $\delta = p_e^o - p_e - k$ and $\delta \simeq \varepsilon_\gamma / 2E_e^o E_e$ if $E_e^o, E_e \gg 1$. Following [LOB15], analytical expressions for I_1 and I_2 can be obtained if the nucleus is considered as a punctual charge, $\rho_n(r) = Ze\delta(r)$.

In this case, the nucleus form factor is $F_n = \frac{1}{Z} \int_{\omega} 4\pi\rho_n \exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r} \simeq 1$. Then using the Fourier transformed Poisson equation [HAU08], together with the screened Yukawa potential screened at a effective length l^* , we obtain

$$\left\{ \begin{array}{l} 1 - F_e = \frac{l^{*2}q^2}{l^{*2}q^2 + 1}, \\ I_1 = l^*\delta \arctan(l^*\delta) + \frac{1}{2} \ln\left(\frac{1+l^{*2}}{1+l^{*2}\delta^2}\right) - l^*\delta \arctan(l^*) - \frac{l^{*2}(1-\delta)^2}{2(1+l^{*2})} + 1, \\ I_2 = 2l^{*3}\delta^3 \arctan(l^*\delta) + \frac{1}{2}(1+3l^{*2}\delta^2) \ln\left(\frac{1+l^{*2}}{1+l^{*2}\delta^2}\right) + \frac{3l^{*4}\delta^2}{1+l^{*2}} \ln\delta - 2l^{*3}\delta^3 \arctan(l^*) \\ + \frac{1}{2} \frac{l^{*2}(\delta-1)(\delta+1-4l^{*2}\delta^2)}{1+l^{*2}} + \frac{5}{6}. \end{array} \right.$$

In this expression l^* is the screening length. As explained in the part 6.4.2, it can be either the Thomas-Fermi length l_{TF} (solid), the Deby length l_D (plasma), or an interpolation of these two length.

Composite formula

We define our differential cross section as the maximum of the relativistic and non relativistic expression, i.e.

$$\frac{d\sigma^B}{d\varepsilon_\gamma}(\varepsilon_e^o, \varepsilon_e) = \max\left(\frac{d\sigma_{NR}^B}{d\varepsilon_\gamma}(\varepsilon_e^o, \varepsilon_e), \frac{d\sigma_R^B}{d\varepsilon_\gamma}(\varepsilon_e^o, \varepsilon_e)\right). \quad (6.37)$$

6.4.4 Cross section validation

The aim of this part is to validate qualitatively the analytical expression (6.37) obtained in the previous section. The strategy consists to compare this expression to tabulated values obtained

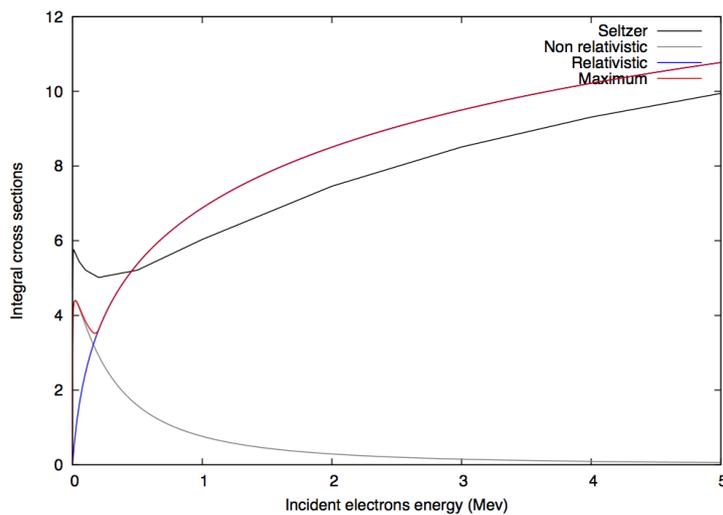


Figure 6.3: Integral of the cross sections versus incident electrons' energy

by Seltzer [SB85]. To this end we consider the dimensionless integrated in photons' energy cross section, normalized as

$$\phi(\varepsilon_e^0) = \frac{\alpha_f r_0^2 Z^2}{1 + \varepsilon_e^0} \int_0^{\varepsilon_e^0} d\varepsilon_\gamma \varepsilon_\gamma \frac{d\sigma}{d\varepsilon_\gamma}(\varepsilon_e^0, \varepsilon_e^0 - \varepsilon_\gamma).$$

Figure 6.3 displays, for $Z = 6$ (carbon), the function ϕ versus the incident electrons' energy for the non relativistic expression (6.33) (black curve), the relativistic expression (6.35) (blue curve) and the maximum of the two expressions (6.37) (red curve). We also displayed the Seltzer values (black curve). It shows a good agreement in both the non relativistic range $\varepsilon_e^0 \in [0 : 500\text{Kev}]$ and in the ultra relativistic case $\varepsilon_e^0 \geq 2\text{Mev}$. The intermediate zone is rather unknown. Never-

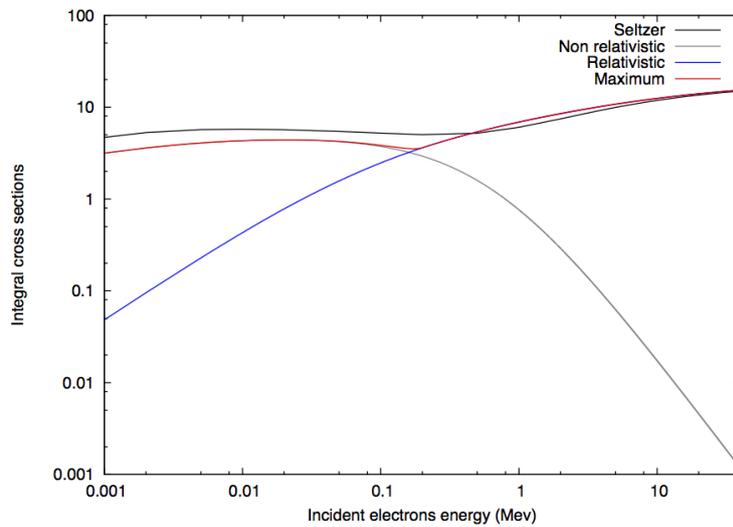


Figure 6.4: Integral of the cross sections versus incident electrons' energy; (log-log) plan

theless, the maximum of the relativistic and non relativistic curves remains qualitatively correct. The figure 6.4 displays the same quantities in a (log-log) plan.

The previous validation has also been performed in the case $Z = 10$ (for example water). From now on we consider our cross section as validated, keeping in mind that the derivation, and in particular the Born approximation, is only valid for small atoms.

6.5 Numerical tests

This section is devoted to numerical tests. The M_1 model (6.27) is implemented in a radiotherapy code, that aims to model an electrons beam. Several kind of interactions take place together with the Bremsstrahlung. In particular the electrons-electrons interactions also imply a stopping power coefficient \mathcal{S}_Q . It is thus important to make a comparison between these two coefficients \mathcal{S}_Q and \mathcal{S}_B defined in (6.28) for the Bremsstrahlung. This is performed in a first part 6.5.1. The second part 6.5.2 presents a test case of dose deposition for several values of the electrons beam. It computes the energy deposited by the fast electrons to the thermal electrons.

6.5.1 Electron-electron vs Bremsstrahlung slowing down

As explain above, the Bremsstrahlung is one of the considered effects. Indeed, the electrons act to each other by collisions. The model used has the same structure than for electron-ion Bremsstrahlung. The resulting model for electrons writes, neglecting the angular diffusion terms

$$\nabla\psi_e = \frac{\partial}{\partial\varepsilon} \left((\mathcal{S}_Q + \bar{\mathcal{S}}_B)\psi_e \right),$$

where $\mathcal{S}_Q \geq 0$ is a slowing down operator. The (normalized) dose \mathcal{D} is defined by

$$\mathcal{D}(x) = \frac{1}{\mathcal{D}_{max}} \int_0^{\varepsilon_{max}} \mathcal{S}_Q(\varepsilon)\psi_e^{(0)}(\varepsilon)g(\varepsilon)d\varepsilon,$$

where, for a given $\varepsilon_{seuil} > 0$, $g(\varepsilon) = \varepsilon$ if $\varepsilon \leq \varepsilon_{seuil}$ and $g(\varepsilon) = \varepsilon_{seuil}$ elsewhere and \mathcal{D}_{max} is the maximum of \mathcal{D} on the space domain.

In particular the stopping power due to Bremsstrahlung $\bar{\mathcal{S}}_B$ does not acts directly on the computation of the dose, but can of course modify the electrons' zero-th order distribution $\psi_e^{(0)}$, which in turns can modify the dose. It is thus interesting to compare the values of the slowing down due to electron-electron collisions, and the slowing down due to the electron-ion Bremsstrahlung. We recall here the expression of the electron-ion Bremsstrahlung coefficient

$$\bar{\mathcal{S}}_B = \rho_i(t, x) \int_0^{\varepsilon_e} \varepsilon_\gamma \frac{d\sigma^B(\varepsilon_e^o, \varepsilon_e^o - \varepsilon_\gamma)}{d\varepsilon_\gamma} d\varepsilon_e^o.$$

Figure 6.5 displays in a (log-log) plan the electron-electron (black curve) and Bremsstrahlung (red curve) stopping power versus the incident energy. The sum $\mathcal{S}_Q + \bar{\mathcal{S}}_B$ is also displayed (blue curve). It shows that in the range of energy $\varepsilon \in [0, 0.1MeV]$, the Bremsstrahlung stopping power

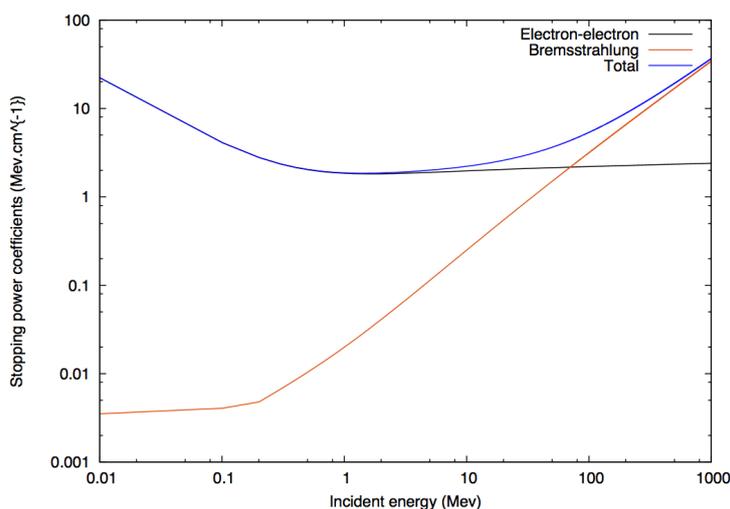


Figure 6.5: Stopping power coefficients ($MeV.cm^{-1}$) versus energy (MeV); (log-log) plan

is negligible in comparison with the electron-electron stopping power. On the other hand, in the range of energy $\varepsilon \geq 1MeV$, the Bremsstrahlung stopping power start to be comparable to the

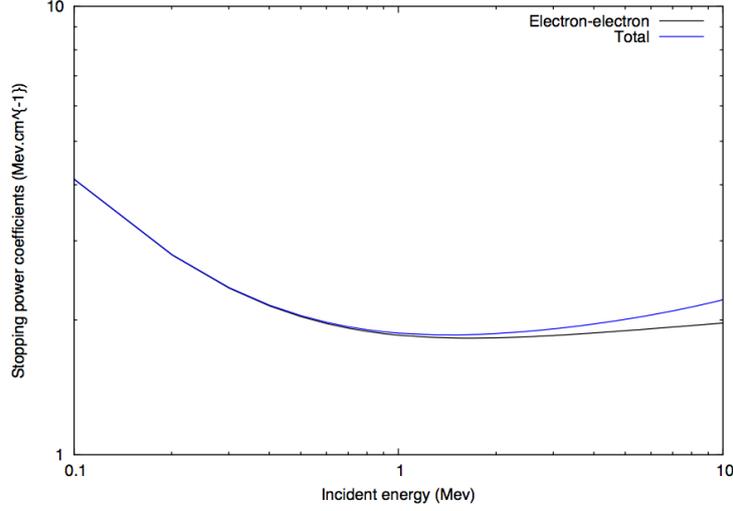


Figure 6.6: Electron-electron and total stopping power coefficients ($Mev.cm^{-1}$) versus energy (Mev); (log-log) plan

electron-electron stopping power, and become preponderant. Figure 6.6 presents a zoom in on this range of energy, plotting only the electron-electron and total stopping power coefficients, which highlights this phenomena. This will help the comprehension of the variation in the dose deposition observed when the Bremsstrahlung is activated, according to the energy of the electrons beam.

6.5.2 Test-case: dose deposition

The test case that is presented in this part is concerned by the deposition of energy by fast electrons to the electrons of the medium (thermal electrons). In this context, the important quantity is the dose \mathcal{D} , whose expression is recalled here

$$\mathcal{D}(x) = \frac{1}{\mathcal{D}_{max}} \int_0^{\varepsilon_{max}} \mathcal{S}_Q(\varepsilon) \psi_\varepsilon^{(0)}(\varepsilon) g(\varepsilon) d\varepsilon,$$

where, for a given $\varepsilon_{seuil} > 0$, $g(\varepsilon) = \varepsilon$ if $\varepsilon \leq \varepsilon_{seuil}$ and $g(\varepsilon) = \varepsilon_{seuil}$ elsewhere and \mathcal{D}_{max} is the maximum of \mathcal{D} on the space domain.

For the test cases, we consider a cartesian energy discretization mesh defined on the interval $[10^{-15}, \varepsilon + 2]Mev$, where ε is the energy of the incident electron beam, discretized with 1000 cells. The space discretization is performed with a cartesian mesh of 100 cells on an interval $[0, x_{max}]$, where x_{max} depends on the energy value of the incident electron beam.

We consider a source of electron beam at the left edge of the space domain ($x = 0$). The domain is composed of water ($Z = 10$) and we compute the dose deposition with and without Bremsstrahlung for several values of the incident electron beam energy, from 0.1 Mev to 20 Mev. According to the previous part 6.5.1, we expect that the bremsstrahlung does not modify the dose deposition in the case $\varepsilon = 0.1$ Mev. On the other hand, we expect a slight slowing down in the case $\varepsilon \in [1, 5]$ Mev and a substantial modification in the case $\varepsilon > 10$ Mev. Figure 6.7

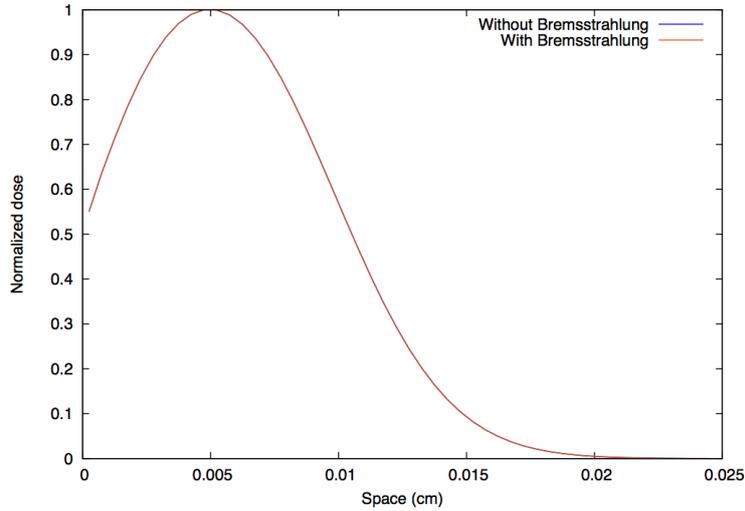


Figure 6.7: Normalized dose \mathcal{D} with (blue) and without (red) Bremsstrahlung versus space (cm) for an electron beam of 0.1 Mev

displays the dose curves with and without Bremsstrahlung in the case of an incident electron beam of $\varepsilon = 0.1$ Mev. As expected, the two curves are merged. In this case the slowing down due to the Bremsstrahlung is negligible in comparison with the electron-electron stopping power.

We now consider incident electron beams of 1 and 5 Mev. Considering figure 6.6, in this energy range the Bremsstrahlung slightly modifies the total slowing down of the electrons. We thus

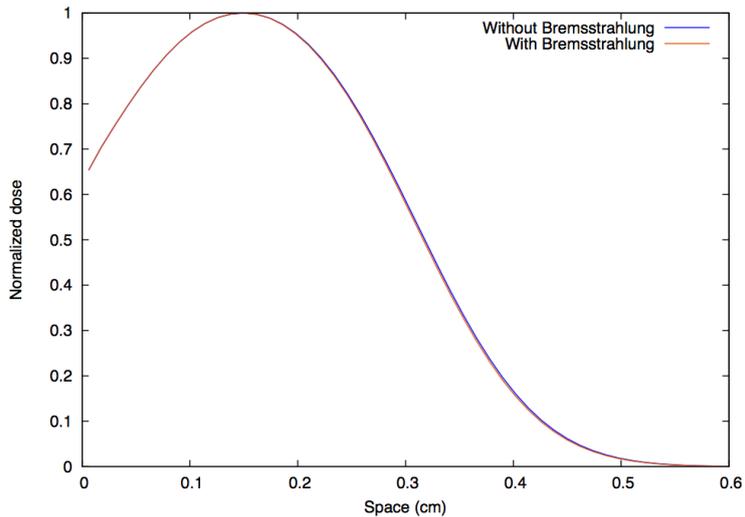


Figure 6.8: Normalized dose \mathcal{D} with (blue) and without (red) Bremsstrahlung versus space (cm) for an electron beam of 1 Mev

expect the curves of dose deposition with Bremsstrahlung to be more "on the left" of the domain, since the electron beams start at the left edge of the space domain, and this should be more

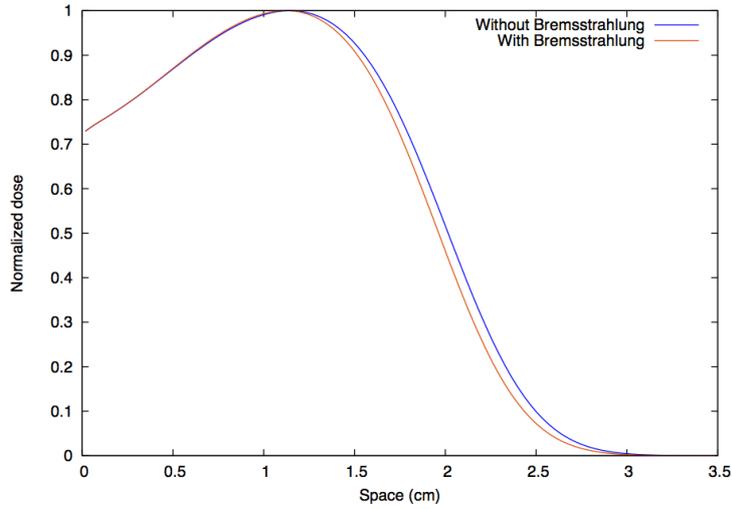


Figure 6.9: Normalized dose \mathcal{D} with (blue) and without (red) Bremsstrahlung versus space (cm) for an electron beam of 5 Mev

significant for the 5 Mev beam. Figure 6.8 (respectively figure 6.9) shows the curves of dose deposition with and without Bremsstrahlung for an incident electron beam of 1 Mev (respectively 5 Mev). The expected behavior is observed, and we see in particular that the Bremsstrahlung introduces non negligible modification in the spatial distribution of the dose.

We conclude this chapter by studying the dose in the case of high energy incident electron beam, namely $\varepsilon \in [10, 20]$ Mev. From the previous numerical results and the previous part 6.5.1, and in particular from figure 6.6, it is clear that the Bremsstrahlung will considerably

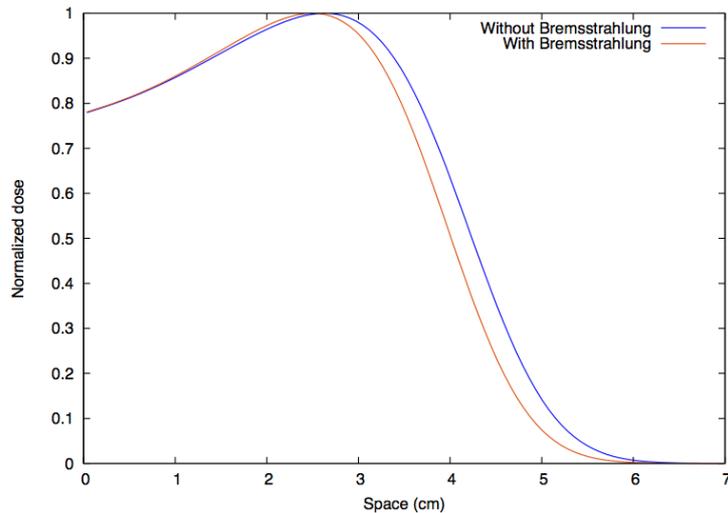


Figure 6.10: Normalized dose \mathcal{D} with (blue) and without (red) Bremsstrahlung versus space (cm) for an electron beam of 10 Mev

slow down the electron beam. Figure 6.10 (respectively figure 6.11) shows the curves of dose deposition with and without Bremsstrahlung for an incident electron beam of 10 Mev (respectively 20 Mev). The figure 6.11 for an incident electron beam of 20 Mev is particularly interesting.

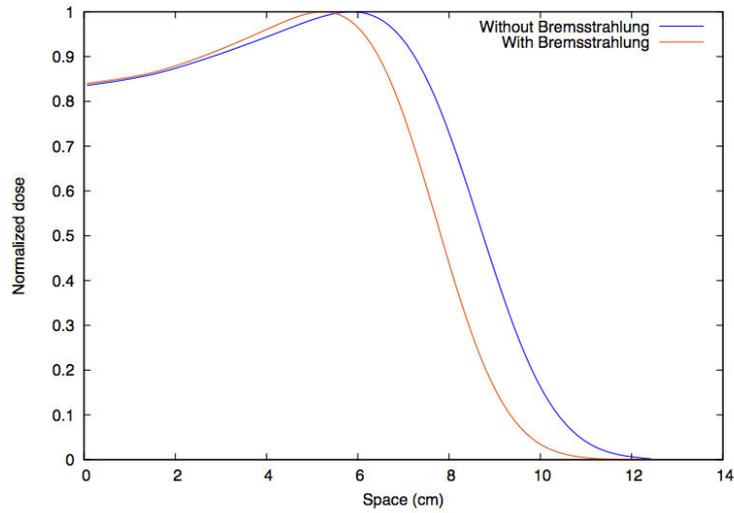


Figure 6.11: Normalized dose \mathcal{D} with (blue) and without (red) Bremsstrahlung versus space (cm) for an electron beam of 20 Mev

Indeed, in addition to modify the slope of the dose curve, the introduction of the bremsstrahlung also modifies the peak of the dose, which is now closer to the electrons' source.

Chapter 7

Proof of uniform convergence for a cell-centered AP discretization of the hyperbolic heat equation on general meshes

This work is taken from a submitted article [BDFL14]

7.1 Introduction

We address the convergence analysis on unstructured meshes of diffusion asymptotic preserving schemes for the discretization of a problem with a stiff parameter denoted as $0 < \varepsilon \leq 1$. The model problem considered in this work is the hyperbolic heat equation in the domain $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^n$

$$P^\varepsilon : \quad \begin{cases} \partial_t p^\varepsilon + \frac{1}{\varepsilon} \operatorname{div}(\mathbf{u}^\varepsilon) = 0, & p^\varepsilon \in \mathbb{R}, \\ \partial_t \mathbf{u}^\varepsilon + \frac{1}{\varepsilon} \nabla p^\varepsilon = -\frac{\sigma}{\varepsilon^2} \mathbf{u}^\varepsilon, & \mathbf{u}^\varepsilon \in \mathbb{R}^n \end{cases} \quad (7.1)$$

discretized with first order finite volume schemes. This problem is representative of many transport problem such as transfer and neutron transport, for which the small parameter ε is the ratio of two very different sound velocities and σ is the absorption or the opacity. For simplicity both ε and $\sigma > 0$ are kept constant in space in this study. The system (7.1) can also be introduced as a specific linearization of a pressure-velocity system of partial differential equations in the acoustic regime. In this work we will need the following well known energy estimates concerning the solution \mathbf{V}^ε of the Cauchy problem for the partial differential equation (7.1).

Proposition 7.1. *If $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$, then*

$$\|\mathbf{V}^\varepsilon\|_{H^p(\Omega)} \leq \|\mathbf{V}^\varepsilon(0)\|_{H^p(\Omega)} \quad (7.2)$$

and moreover

$$\frac{\sigma}{\varepsilon^2} \|\mathbf{u}^\varepsilon\|_{L^2([0,T];H^p(\Omega))}^2 \leq \|\mathbf{V}^\varepsilon(0)\|_{H^p(\Omega)}^2. \quad (7.3)$$

We will consider well prepared data in the sense that: $p^\varepsilon(t = 0)$ is independent of ε and is sufficiently smooth; the initial velocity satisfies the equality in the second equation of (7.1) at leading order. It writes

$$p^\varepsilon(t = 0) = p_0 \text{ and } \mathbf{u}^\varepsilon(0) = -\frac{\varepsilon}{\sigma} \nabla p_0. \quad (7.4)$$

For such well prepared data, it can be easily shown that the formal limit of P^ε for small ε is

$$P^0 : \quad \partial_t p - \frac{1}{\sigma} \Delta p = 0. \quad (7.5)$$

Remarque 7.1. *We do not consider the regime $\sigma \rightarrow 0$, since it introduces a singularity both in the initial data of the hyperbolic heat equation and in the limit parabolic equation.*

7.1.1 Precision of AP discretizations

Before addressing the main difficulty of this work which is the discretization on unstructured meshes, we briefly recall the now well known notion of an asymptotic preserving technique [JIN99]- [JIN10] which is illustrated in the figure 7.1. For the simplicity of the presentation, we will consider mainly semi-discrete numerical methods, this is why the time step does not show up in the graphic. The parameter h designs a numerical method with characteristic length $h \leq 1$: that is we assume a numerical method P_h^ε for the discretization of P^ε .

Définition 7.1 (Uniform AP). *If P_h^ε is consistent with P^ε uniformly with respect to ε , then we say that the scheme P_h^ε is uniformly AP (uniformly asymptotic preserving).*

However the design of such methods and the numerical proof of this property is difficult. This is why it has been proposed in [JIN99] to rely on the simpler necessary condition, where the limit as $\varepsilon \rightarrow 0$ of P_h^ε is called the limit diffusion scheme P_h^0 .

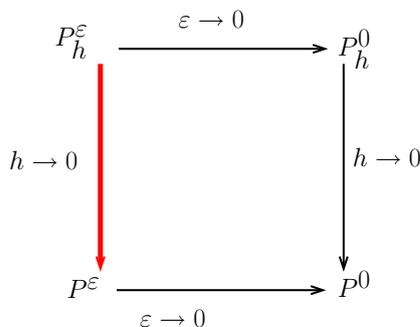


Figure 7.1: The AP (asymptotic diagram) diagram

Définition 7.2 (AP). *If P_h^0 is consistent with the limit model P^0 , then we say that the scheme P_h^ε is AP (asymptotic preserving).*

This property is simpler to analyze than the uniform AP. It explains why it has been very fruitful in the past. In 1D, many AP schemes have been designed for some PDE and physical problems: S. Jin, C. D. Levermore [JL96] or L. Gosse, G. Toscani [GT02] for the hyperbolic heat equation, M. Lemou, L. Mieussens, N. Crouseilles [LM08]- [CL11]- [CR13] for some kinetic equations, L. Gosse [GOS11], C. Buet and co-workers [BCLM02] or S. Jin and C. D. Levermore

[JL91] for S_N equations and C. Berthon, R. Turpault [BCT07]- [BLT13]- [BT11] for generic systems and a non linear radiative transfer model. Recently some asymptotic preserving schemes for linear systems and non linear radiative transfer model have been designed in 2D [BDF12-1]- [BDF12-2]- [BDF11]. Other application to non linear hyperbolic systems of conservation laws with stiff diffusive relaxation is to be found in [NP00]. Relaxation systems are treated in [FR13]. More general situation for transport and discrete velocity systems are in [JPT00, JPT98]. However for this type of schemes it is difficult to obtain convergence estimates due to the competition between the two parameters ε and h . To our knowledge this type of proof are only given for uniform grids [BDF12-1] (consistence and stability, Lax theorem), [GT02] (L^1 and BV estimates), [LM10] (L^2 estimates). The goal of this work is to prove the uniform AP property on unstructured grids.

To this end we adapt a strategy developed in [GJL99] in a slightly different context. It relies on the derivation of a priori estimates attached to the AP diagram in figure 7.1. To have a more global perspective on this strategy, let us assume some natural abstract a priori estimates for a given norm which is in our work based on $\|f\| = \|f\|_{L^2([0,T] \times \Omega)}$ or $\|f\| = \|f\|_{L^\infty([0,T]; L^2(\Omega))}$ where $T > 0$ is a given final time, $\Omega = \mathbb{R}$, in 1D or $\Omega = [0, 1]^2$ with periodic boundary conditions in 2D. We assume five constants $a, b, c, d, e > 0$ and four additional constants $C_\downarrow, C^\rightarrow, C_\leftarrow, C_\downarrow > 0 > 0$ such that the error attached to the branches of the AP diagram can be bounded like

$$\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}} \leq C_\downarrow \varepsilon^{-b} h^c, \quad (7.6)$$

$$\|P_h^\varepsilon - P_h^0\| \leq C^\rightarrow \varepsilon^e. \quad (7.7)$$

$$\|P_h^0 - P^0\| \leq C_\downarrow h^d, \quad (7.8)$$

$$\|P^\varepsilon - P^0\| \leq C_\leftarrow \varepsilon^a, \quad (7.9)$$

The first inequality is the naive error bound which typically blows up for small ε . The second inequality for $\|P_h^\varepsilon - P_h^0\|$ is assumed to have a form similar to the last one which expresses that P^0 is the limit of P^ε . The third inequality corresponds to the usual AP property.

Proposition 7.2. *Assume that all these inequalities are at hand and that $d \geq c$ and $e \geq a$. Then the uniform AP holds with a rate at least $O\left(h^{\frac{ac}{a+b}}\right)$.*

Proof. The triangular inequality writes

$$\|P_h^\varepsilon - P^\varepsilon\| \leq \min\left(\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}, \|P_h^\varepsilon - P_h^0\| + \|P_h^0 - P^0\| + \|P^\varepsilon - P^0\|\right)$$

which yields, using $\min(x, y + z) \leq \min(x, y) + \min(x, z)$, $d \geq c$ and $e \geq a$,

$$\|P_h^\varepsilon - P^\varepsilon\| \leq C \left(\min\left(\varepsilon^{-b} h^c, \varepsilon^e\right) + h^d + \min\left(\varepsilon^{-b} h^c, \varepsilon^a\right) \right) \leq C \left(2 \min\left(\varepsilon^{-b} h^c, \varepsilon^a\right) + h^d \right) \quad (7.10)$$

with $C = \max(C_\downarrow, C^\rightarrow, C_\downarrow, C_\leftarrow)$. We define a threshold value $\varepsilon_{\text{thresh}}$ by $\varepsilon_{\text{thresh}}^{-b} h^c = \varepsilon_{\text{thresh}}^a$. So either $\varepsilon \leq \varepsilon_{\text{thresh}}$ so that

$$\min\left(\varepsilon^{-b} h^c, \varepsilon^a\right) \leq \varepsilon_{\text{thresh}}^a = h^{\frac{ac}{a+b}},$$

or $\varepsilon \geq \varepsilon_{\text{thresh}}$ and the same bound is obtained by taking the other term as the minimum. And since $d \geq c$, one gets the abstract bound $\|P_h^\varepsilon - P^\varepsilon\| \leq 3Ch^{\frac{ac}{a+b}}$ which ends the proof. \square

7.1.2 Organization of the proof

The structure of these inequalities explains our strategy: that is we prove separately each of these inequalities (7.9-7.7) with care, so that the inequalities $d \geq c$ and $e \geq a$ are true. This part of the proof relies on specific hyperbolic and parabolic numerical methods. Even if it is technical, the first three inequalities (7.9-7.8) do not yield additional difficulties with respect to the state of the art. The proof of inequality (7.7) is provided in 1D, and can be probably be generalized straightforwardly on cartesian meshes in 2D and 3D. On the other hand our researches on proving (7.7) for $\|P_h^\varepsilon - P_h^0\|$ show a fundamental obstruction in dimension greater than one on unstructured meshes which was not expected initially. Since the main difficulty is related to P_h^0 , it motivates the definition of a new diffusion scheme. To this end we remark that another diffusion scheme is naturally defined from P_h^ε by killing the derivative $\partial_t v_h$ in the discrete version of the second equation of (7.1). Killing at the continuous level the $\partial_t v$ is absolutely equivalent to taking the formal limit $\varepsilon \rightarrow 0^+$. But at the discrete level, it appears that it generates a new family of diffusion schemes, where both parameters h and ε are present. We call them Diffusion Asymptotic schemes, DA_h^ε . By construction $P_h^0 = \lim_{\varepsilon \rightarrow 0} DA_h^\varepsilon$. This is summarized in figure 7.2. Finally since the scheme DA_h^ε is still an accurate discretization of P^0 , our proof of the uniform AP property is based on the new AP diagram displayed in figure 7.3.

$$P_h^\varepsilon \xrightarrow{\partial_t v_h = 0} DA_h^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} P_h^0$$

Figure 7.2: Definition of the diffusion asymptotic scheme DA_h^ε .

Our main theorem 7.26 in dimension 2 is based on this structure and it may be stated as follows: **The so-called JL-(b) scheme defined in [BDF12-1] for the discretization of the hyperbolic heat equation (7.1) (the scheme is cell-centered with nodal based fluxes) is uniformly AP on unstructured meshes, with a rate of convergence at least $O(h^{\frac{1}{4}})$ for sufficiently smooth initial data.** This is an improvement with respect to [BDF12-1] where only AP was proven. To our knowledge this is the first time that such a result is obtained on general unstructured multidimensional meshes. More precisely the convergence estimate can be written as

$$\mathbf{error} \leq C \min \left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \max \left(1, \sqrt{\frac{\varepsilon}{h}} \right) + h + \varepsilon \right)$$

where the first argument in the min function comes from the hyperbolic analysis and the second argument comes from the parabolic analysis. Some natural regularity assumptions are nevertheless imposed on the mesh in the hypothesis 7.3, this is not very restrictive. For example meshes with angles greater than 90 degrees are allowed. If the mesh is made with triangles, the hypothesis is fulfilled if all angles are greater than 12 degrees, see [BDF12-1]. It is interesting to notice that the rate of uniform convergence is $O(h^{\frac{1}{3}})$ in dimension one. The difference essentially comes from the estimate of the reconstruction of the initial velocity which is needed to rewrite a diffusion scheme as a non homogeneous hyperbolic scheme: it is much simpler in dimension one (see equation (7.23)) than in dimension two (see proposition (7.23)). In this work we considered mainly semi-discrete numerical schemes, since it simplifies a lot the notations and allow to focus on the main difficulties, but the final estimates of convergence can be generalized to fully discrete schemes, using the a priori estimates developed in [DES04]. For explicit schemes, these estimates add a term proportional to the square root of the maximal time step allowed by

the CFL condition. Since our problem is an hyperbolic+relaxation problem, with a limit which is parabolic, this additional term can be computed and is of the order between h (for purely hyperbolic) to h^2 (for purely parabolic). We refer to [BDF12-1] for the detail of CFL condition in 1D and 2D. Concerning the implicit fully discrete version of the semi-discrete scheme which is unconditionally stable and well adapted to the test problem analyzed at the end of this work, the same kind of error terms can be analyzed. We will obtain the following result in dimension two.

Theorem 7.1. *With some usual regularity assumptions on the mesh, the error between our cell-centered finite volume corner-based-flux implicit discretization $\mathbf{P}_{h,\Delta t}^\varepsilon$ and the exact solution is*

$$\|\mathbf{V}_h^\varepsilon(t_n) - \mathbf{V}^\varepsilon(t_n)\|_{L^2(\Omega)} \leq C \left(h^{\frac{1}{4}} + \Delta t^{\frac{1}{2}} \right) \|p_0\|_{H^4(\Omega)}, \quad t_n = n\Delta t \leq T.$$

The constant is independent of h , ε and Δt and behaves less than $T^{\frac{3}{2}}$ for large T .

The proof is an easy add-on on the space estimate $O(h^{\frac{1}{4}})$ of theorem 7.26, by means of an abstract method [DES04] which gives a general bound $O(\Delta t^{\frac{1}{2}})$ of the difference between the semi-discrete scheme and the implicit Euler scheme. This will be explained at the end of this work. The rate of convergence is confirmed by the numerical results of section 7.5, which show an even better rate of convergence.

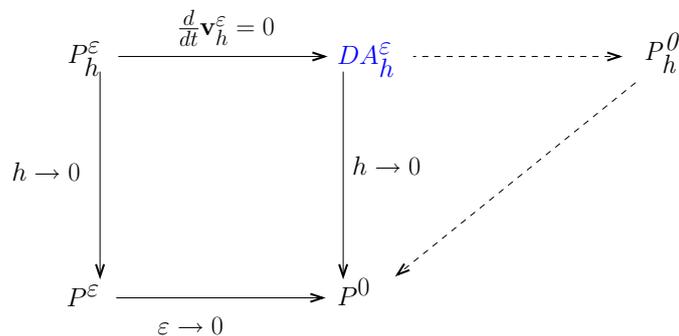


Figure 7.3: The new AP diagram, where the previous branch is still displayed in dashed lines.

We think that some of our results can have an interest for the development and use of such methods in research or industrial codes with complex non linear physics on unstructured meshes. Indeed for such codes cell-centered Finite Volume schemes are a natural solution in terms of data structure. The point is the following: the scheme studied in this work is the only cell-centered one we know in 2D to compute the solutions of problems which admit diffusion limits in certain regimes and for which it is possible to prove the AP property. Since the structure of this cell-centered scheme is nodal based, it strongly questions the ability of standard Finite Volume methods with edge-based fluxes to recover asymptotic diffusion regimes. As demonstrated in this work, nodal based Finite Volume techniques do not suffer from this drawback. For linear wave equation the nodal scheme can be understood as some 1D Riemann problem written in some direction around each node, so can be interpreted as an approximation of the 2D Riemann problem [GOS14].

7.1.3 Organization of the work

This work is organized as follows. Section 7.2 is dedicated to the discretization of the model problem in dimension one on irregular grids. The convergence is proved in theorem 7.11 with order $h^{\frac{1}{3}}$ in the L^2 space-time norm. In the next section, the nodal solvers for the hyperbolic equation are defined, and the various a priori estimates proved. The main theorem of uniform AP for the JL-(b) scheme with a rate $O(h^{\frac{1}{4}})$ is proved at the end of the section. Section 7.5 provides numerical results that sustain the fact that the convergence order depends on the relative value of ε and h , and so is mixed hyperbolic/parabolic. Our final remarks will be gathered in a conclusion. All our results and numerical methods in 2D can be generalized in 3D provided a convenient definition of the nodal corner vector is used as in [DES10].

7.2 Analysis in 1D

The model problem in dimension one writes

$$P^\varepsilon : \quad \begin{cases} \partial_t p^\varepsilon + \frac{1}{\varepsilon} \partial_x u^\varepsilon = 0, \\ \partial_t u^\varepsilon + \frac{1}{\varepsilon} \partial_x p^\varepsilon = -\frac{\sigma}{\varepsilon^2} u^\varepsilon. \end{cases} \quad (7.11)$$

As stressed already in (7.4), we consider well-prepared data $p^\varepsilon(t=0) = p_0$ and $u_0^\varepsilon = -\frac{\varepsilon}{\sigma} \partial_x p_0$. The equations (7.11) admit the formal diffusion limit when ε tends to 0:

$$P^0 : \quad \partial_t p - \frac{1}{\sigma} \partial_{xx} p = 0. \quad (7.12)$$

A useful variable will be the scaled gradient

$$v = -\frac{1}{\sigma} \partial_x p. \quad (7.13)$$

7.2.1 Notations

We denote $x_{j+1/2}$ the nodes, the cells j are the intervals $[x_{j-1/2}, x_{j+1/2}]$, thus $\Delta x_j = x_{j+1/2} - x_{j-1/2}$, x_j is the center of the cell j that is $x_j = \frac{1}{2}(x_{j+1/2} + x_{j-1/2})$, and $\Delta x_{j+1/2} = x_{j+1} - x_j = \frac{1}{2}(\Delta x_{j+1} + \Delta x_j)$. Natural assumptions on the mesh are summarized below:

Hypothèse 7.3 (Regularity of the mesh in 1D and constant $C_{\mathcal{M}}$). *We consider that there exists a universal constant $0 < C_{\mathcal{M}} \leq 1$ independent of the mesh size $h = \sup_{j \in \mathbb{Z}} \Delta x_j$ which controls the mesh from below*

$$C_{\mathcal{M}} h \leq \Delta x_j \leq h \quad \forall j \in \mathbb{Z}. \quad (7.14)$$

The semi-discrete JL(b) scheme, derived in [BDF12-1] in 2D, can also be written in 1D on irregular meshes as

$$P_h^\varepsilon : \quad \begin{cases} \frac{d}{dt} p_j^\varepsilon + \frac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = 0, \\ \frac{d}{dt} u_j^\varepsilon + \frac{p_{j+\frac{1}{2}}^\varepsilon - p_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = -\frac{\sigma}{\varepsilon^2} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2}, \end{cases} \quad (7.15)$$

with the fluxes $p_{j+\frac{1}{2}}^\varepsilon$ and $u_{j+\frac{1}{2}}^\varepsilon$ are the solutions of the well-posed linear system

$$j \in \mathbb{Z} : \quad \begin{cases} p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \frac{\sigma \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}^\varepsilon = p_j^\varepsilon + u_j^\varepsilon, \\ -p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \frac{\sigma \Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}^\varepsilon = -p_{j+1}^\varepsilon + u_{j+1}^\varepsilon. \end{cases} \quad (7.16)$$

This scheme is the same as the Gosse-Toscani scheme¹. Other equivalent forms of P_h^ε can be obtained by various manipulations, as in (7.29). We use another formulation of the Gosse-Toscani obtained using the Jin-Levemore scheme [JL96] and a discretization of the source term which uses the fluxes. Contrary to the Gosse-Toscani scheme which uses Riemann problem, this formulation based an elementary algebraic computation is easier to write in 2D on unstructured meshes (the design is detailed in [BDF12-1]). The natural pointwise initialization is chosen

$$p_j^\varepsilon(0) = p_0(x_j) \text{ and } u_j^\varepsilon(0) = -\frac{\varepsilon}{\sigma} \partial_x p_0(x_j) \text{ for all } j \in \mathbb{Z}. \quad (7.17)$$

When ε tends to 0, the scheme P_h^ε admits the diffusion limit scheme P_h^0

$$P_h^0 : \quad \Delta x_j \frac{d}{dt} p_j - \frac{1}{\sigma} \left(\frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} - \frac{p_j - p_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right) = 0 \quad (7.18)$$

with the pointwise initialization

$$p_j(0) = p_0(x_j) \text{ for all } j \in \mathbb{Z}. \quad (7.19)$$

Other quantities are the reconstructed gradient

$$\begin{cases} v_{j+\frac{1}{2}} = -\frac{1}{\sigma} \frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}}, \\ v_j = \frac{v_{j+\frac{1}{2}} + v_{j-\frac{1}{2}}}{2}. \end{cases} \quad (7.20)$$

We denote by $\mathbf{V}^\varepsilon(t, x) = (p^\varepsilon(x, t), u^\varepsilon(x, t))$ the solution of the hyperbolic heat equations P^ε . We reconstruct similar quantities from the diffusion scheme: it yields $\mathbf{W}^\varepsilon(t, x) = (p(x, t), \varepsilon v(x, t))$ which is the solution of the diffusion limit (7.12)-(7.13). The indicatrix function of the interval $(x_{j-1/2}, x_{j+1/2})$ is denoted as $1_j(x) = 1$ if $x \in (x_{j-1/2}, x_{j+1/2})$ and $1_j(x) = 0$ in the other case. With this notation we note $\mathbf{V}_h^\varepsilon(t, x) = \left(\sum_{j \in \mathbb{Z}} p_j^\varepsilon(t) 1_j(x), \sum_{j \in \mathbb{Z}} u_j^\varepsilon(t) 1_j(x) \right)$ the solution

¹A long and tedious computation shows that the scheme is strictly equivalent to the Gosse-Toscani's scheme, described in [GT02] but only for uniform meshes, which writes in terms of $w^\varepsilon, v^\varepsilon = p^\varepsilon \pm u^\varepsilon$

$$\begin{cases} \frac{dw_j}{dt} + \frac{M_{j-\frac{1}{2}}}{\varepsilon} \frac{w_j^\varepsilon - w_{j-1}^\varepsilon}{\Delta x_j} = \frac{1}{\varepsilon \Delta x_j} (1 - M_{j-\frac{1}{2}}) (v_j^\varepsilon - w_j^\varepsilon) = M_{j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} \frac{\sigma}{2\varepsilon^2} (v_j^\varepsilon - w_j^\varepsilon), \\ \frac{dv_j^\varepsilon}{dt} - \frac{M_{j+\frac{1}{2}}}{\varepsilon} \frac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\Delta x_j} = \frac{1}{\varepsilon \Delta x_j} (1 - M_{j+\frac{1}{2}}) (w_j^\varepsilon - v_j^\varepsilon) = M_{j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} \frac{\sigma}{2\varepsilon^2} (w_j^\varepsilon - v_j^\varepsilon) \end{cases}$$

with $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{\sigma \Delta x_{j+\frac{1}{2}} + 2\varepsilon}$ and $\Delta x_{j+\frac{1}{2}} = \frac{\Delta x_j + \Delta x_{j+1}}{2}$. By writing

$$\begin{cases} M_{j-\frac{1}{2}} (w_{j-1}^\varepsilon - w_j^\varepsilon) = M_{j-\frac{1}{2}} w_{j-1} - M_{j+\frac{1}{2}} w_j + (M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}) w_j^\varepsilon \\ M_{j+\frac{1}{2}} (v_{j+1}^\varepsilon - v_j^\varepsilon) = M_{j+\frac{1}{2}} v_{j+1}^\varepsilon - M_{j-\frac{1}{2}} v_j^\varepsilon - (M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}) v_j^\varepsilon \end{cases}$$

then in terms of p^ε an u^ε we have evidently

$$\begin{cases} \frac{dp_j^\varepsilon}{dt} + \frac{1}{\varepsilon} \frac{M_{j+\frac{1}{2}} u_{j+\frac{1}{2}}^\varepsilon - M_{j-\frac{1}{2}} u_{j-\frac{1}{2}}^\varepsilon}{\Delta x_j} = 0, \\ \frac{du_j^\varepsilon}{dt} + \frac{1}{\varepsilon} \frac{M_{j+\frac{1}{2}} p_{j+\frac{1}{2}}^\varepsilon - M_{j-\frac{1}{2}} p_{j-\frac{1}{2}}^\varepsilon}{\Delta x_j} = -\frac{1}{2} \left(M_{j+\frac{1}{2}} \frac{\Delta x_{j+\frac{1}{2}}}{\Delta x_j} \frac{\sigma}{\varepsilon^2} + M_{j-\frac{1}{2}} \frac{\Delta x_{j-\frac{1}{2}}}{\Delta x_j} \frac{\sigma}{\varepsilon^2} \right) u_j^\varepsilon + \frac{M_{j+\frac{1}{2}} - M_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} p_j^\varepsilon \end{cases}$$

with the fluxes given by $p_{j+\frac{1}{2}}^\varepsilon = \frac{p_j^\varepsilon + p_{j+1}^\varepsilon}{2} + \frac{u_j^\varepsilon - u_{j+1}^\varepsilon}{2}$ and $u_{j+\frac{1}{2}}^\varepsilon = \frac{u_j^\varepsilon + u_{j+1}^\varepsilon}{2} + \frac{p_j^\varepsilon - p_{j+1}^\varepsilon}{2}$.

of the JL-(b) scheme P_h^ε . Finally we note $\mathbf{W}_h^\varepsilon(t, x) = \left(\sum_{j \in \mathbb{Z}} p_j(t) 1_j(x), \varepsilon \sum_{j \in \mathbb{Z}} v_j(t) 1_j(x) \right)$ the solution of the diffusion scheme P_h^0 (7.18)-(7.20).

For simplicity we choose a final time $T > 0$. All error estimates will be given for $t \leq T$, either in the norm $\|f(t)\|_{L^\infty([0, T]; L^2(\mathbb{R}))}$, or mostly in the norm $\|f\|_{L^2([0, T] \times \mathbb{R})}$.

Hypothèse 7.4 (Regularity of the initial data and constant C_A). *We consider that there exists a universal constant $C_A > 0$ which controls all kind of approximations/interpolations/projections on the mesh of exact functions. We will write for example the error estimates at the initial time under the form*

$$\|\mathbf{V}_h^\varepsilon(0) - \mathbf{V}^\varepsilon(0)\|_{L^2(\mathbb{R})} \leq C_A h \|p_0\|_{H^2(\mathbb{R})} \quad (7.21)$$

and

$$\|\mathbf{W}_h^\varepsilon(0) - \mathbf{W}^\varepsilon(0)\|_{L^2(\mathbb{R})} \leq C_A h \|p_0\|_{H^2(\mathbb{R})}. \quad (7.22)$$

The second inequality in the hypothesis can be related to the sharper inequality

$$\left\| \sum_j (u_j^\varepsilon(0) - \varepsilon v_j(0)) 1_j \right\|_{L^2(\mathbb{R})} \leq C_A h \varepsilon \|p_0\|_{H^2(\mathbb{R})}. \quad (7.23)$$

The other technical constants used to bound the errors of the left, top, right and bottom branches of the AP diagram 7.1 will be denoted as $\downarrow C$, C^\leftarrow , C_\downarrow and C_{\leftarrow} .

7.2.2 Study of $\|P^\varepsilon - P^0\|$

In this section we prove a natural error estimate [GJL99] between the solution of the hyperbolic heat equations (7.11) and the solution of the diffusion limit equation (7.12).

Lemma 7.5. *One has the estimate*

$$\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0, T] \times \mathbb{R})} \leq C_{\leftarrow} \varepsilon \|p_0\|_{H^3(\mathbb{R})}, \quad C_{\leftarrow} = \frac{T^{\frac{3}{2}}}{\sigma^2}. \quad (7.24)$$

Proof. We redefine $v = -\frac{\varepsilon}{\sigma} \partial_x p$ with p the diffusion solution of (7.12) and introduce R^ε such that the solution of the diffusion equation satisfies

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon} \partial_x v = 0, \\ \partial_t v + \frac{1}{\varepsilon} \partial_x p + \frac{\sigma}{\varepsilon^2} v = R^\varepsilon \end{cases} \quad (7.25)$$

where $R^\varepsilon = \partial_t v = -\frac{\varepsilon}{\sigma} \partial_{tx} p = -\frac{\varepsilon}{\sigma^2} \partial_{xxx} p$. Note that $\|R^\varepsilon(t)\|_{L^2(\mathbb{R})} \leq \|R^\varepsilon(0)\|_{L^2(\mathbb{R})} \leq \frac{\varepsilon}{\sigma^2} \|p_0\|_{H^3(\mathbb{R})}$. Denoting $e^\varepsilon = p - p^\varepsilon$, $f^\varepsilon = v - v^\varepsilon$, we make the difference between the systems (7.11) et (7.25)

$$\begin{cases} \partial_t e^\varepsilon + \frac{1}{\varepsilon} \partial_x f^\varepsilon = 0, \\ \partial_t f^\varepsilon + \frac{1}{\varepsilon} \partial_x e^\varepsilon + \frac{\sigma}{\varepsilon^2} f^\varepsilon = R^\varepsilon. \end{cases} \quad (7.26)$$

Since data are well-prepared, one has $e^\varepsilon(0) = f^\varepsilon(0) = 0$. Consider $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2(\mathbb{R})}^2 = \|e^\varepsilon\|_{L^2(\mathbb{R})}^2 + \|f^\varepsilon\|_{L^2(\mathbb{R})}^2$. Adding the first equation of (7.26) multiplied by e^ε and the second multiplied by f^ε and integrating on \mathbb{R} , we find out that: $\frac{1}{2} \frac{d}{dt} \|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} R^\varepsilon f^\varepsilon dx \leq \|R^\varepsilon\|_{L^2(\mathbb{R})} \|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2(\mathbb{R})}$. One gets a bound of $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}))}$ by integration between 0 and T. Finally $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0, T] \times \mathbb{R})} \leq \sqrt{T} \|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0, T]; L^2(\mathbb{R}))}$ which ends the proof. \square

7.2.3 Stability estimates for P_h^ε and P_h^0

The estimates (7.27-7.28) and (7.31) characterize the dissipation rate of both schemes.

Proposition 7.6. *The scheme P_h^ε is stable in L^2 norm. Moreover,*

$$\sqrt{\int_0^T \left(\sum \Delta x_{j+\frac{1}{2}} (u_{j+\frac{1}{2}}^\varepsilon)^2 \right) dt} \leq \frac{\varepsilon}{\sqrt{\sigma}} \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})} \quad (7.27)$$

and

$$\sqrt{\int_0^T \left(\sum_{j \in \mathbb{Z}} (u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2 + \sum_{j \in \mathbb{Z}} (u_{j-\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2 \right) dt} \leq \sqrt{\varepsilon} \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}. \quad (7.28)$$

Remarque 7.2. *The strategy of the proof of many estimates in this work consists in analyzing the balance between the dissipation of the fluxes and the physical dissipation (all source terms like $-\frac{\sigma}{\varepsilon^2}u$) on the one hand, and some truncation errors on the other hand. This is why it is convenient to reformulate P_h^ε so that the pressure fluxes $p_{j+\frac{1}{2}}^\varepsilon$ and $p_{j-\frac{1}{2}}^\varepsilon$ are eliminated in the second equation of (7.15). This elimination is technically convenient since all dissipation terms are expressed using the same variable, namely u . It will simplify a lot the comparisons between all kinds of dissipation terms and other errors terms.*

Proof. According to the above remark we obtain the formulation (7.29) which is equivalent to P_h^ε

$$\begin{cases} \Delta x_j \frac{d}{dt} p_j^\varepsilon + \frac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} = 0, \\ \Delta x_j \frac{d}{dt} u_j^\varepsilon - \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} + \frac{2}{\varepsilon} u_j^\varepsilon = 0, \\ \left(2 + \frac{\sigma \Delta x_{j+\frac{1}{2}}}{\varepsilon} \right) u_{j+\frac{1}{2}}^\varepsilon = p_j^\varepsilon - p_{j+1}^\varepsilon + u_j^\varepsilon + u_{j+1}^\varepsilon. \end{cases} \quad (7.29)$$

Consider now the discrete quadratic energy $E(t) = \frac{1}{2} \sum_j \Delta x_j ((p_j^\varepsilon)^2 + (u_j^\varepsilon)^2)$. Multiplying the first equation of (7.29) by p_j^ε and the second equation by u_j^ε and adding on all the cells, one finds

$$E'(t) = - \sum_{j \in \mathbb{Z}} \frac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} p_j^\varepsilon + \sum_{j \in \mathbb{Z}} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} u_j^\varepsilon - \frac{2}{\varepsilon} \sum_j (u_j^\varepsilon)^2.$$

Since $\sum_j (u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon) p_j^\varepsilon = \sum_j u_{j+\frac{1}{2}}^\varepsilon (p_j^\varepsilon - p_{j+1}^\varepsilon)$, one has by using the third equation of (7.29) and rearranging the terms

$$E'(t) + \sum_{j \in \mathbb{Z}} \frac{(u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} + \sum_{j \in \mathbb{Z}} \frac{(u_{j-\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} + \frac{\sigma}{\varepsilon^2} \sum_{j \in \mathbb{Z}} \Delta x_j \frac{(u_{j+\frac{1}{2}}^\varepsilon)^2 + (u_{j-\frac{1}{2}}^\varepsilon)^2}{2} = 0. \quad (7.30)$$

Integrating (7.30) between 0 and t , one finds $E(t) \leq E(0)$, that is the L^2 stability of P_h^ε . The estimate (7.27) comes from $\Delta x_{j+\frac{1}{2}} = \frac{1}{2}(\Delta x_j + \Delta x_{j+1})$. The estimate (7.28) is directly deduced from (7.30). \square

Some similar bounds hold for the quantities related to the diffusion scheme (7.18). First, multiplying the diffusion scheme by p_j and adding on all the cells, one has the L^2 stability in the sense

$$\frac{1}{2} \frac{d}{dt} \sum_j \Delta x_j p_j^2 = -\frac{1}{\sigma} \sum_j \frac{(p_{j+1} - p_j)^2}{\Delta x_{j+\frac{1}{2}}}.$$

Thus the following estimate holds for the function $\bar{v}_h = \left(v_{j+\frac{1}{2}}\right)_j$ defined by (7.20)

$$\|\bar{v}_h\|_{L^2([0,T] \times \mathbb{R})} = \sqrt{\int_0^T \sum_j \Delta x_{j+\frac{1}{2}} (v_{j+\frac{1}{2}})^2} \leq \sqrt{\frac{\sigma}{2}} \|p_h(0)\|_{L^2(\mathbb{R})}, \quad C > 0. \quad (7.31)$$

7.2.4 Study of $\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}$

In this section we prove the convergence of P_h^ε to P^ε . We still denote $V^\varepsilon(t) = (p^\varepsilon, u^\varepsilon)$.

Lemma 7.7. *There exists a constant $\downarrow C > 0$ independent of $h, \varepsilon, C_{\mathcal{M}}$, with at most a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\downarrow C}{\sqrt{C_{\mathcal{M}}}} \sqrt{\frac{h}{\varepsilon}} \|p_0\|_{H^2(\mathbb{R})}. \quad (7.32)$$

Proof. We use the method introduced by C. Mazeran [MAZ07] in his PhD thesis. It starts with an estimate for the time derivative of $\mathcal{E} = \frac{1}{2} \|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2(\mathbb{R})}^2$. For the sake of simplicity, q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$ for any quantity q . One has

$$\begin{aligned} \mathcal{E}'(t) &= \underbrace{\frac{1}{2} \int_{\mathbb{R}} ((p_h^\varepsilon)^2 + (u_h^\varepsilon)^2)' dx}_{D_1} + \underbrace{\frac{1}{2} \int_{\mathbb{R}} ((p^\varepsilon)^2 + (u^\varepsilon)^2)' dx}_{D_2} \\ &+ \underbrace{\int_{\mathbb{R}} (-(p_h^\varepsilon)' p^\varepsilon - (u_h^\varepsilon)' u^\varepsilon) dx}_{D_3} + \underbrace{\int_{\mathbb{R}} (-p_h^\varepsilon (p^\varepsilon)' - u_h^\varepsilon (u^\varepsilon)') dx}_{D_4} \end{aligned}$$

We will successively estimate each of those terms, the fundamental idea being that $D_1 \leq 0$ and $D_2 \leq 0$ are used to control spurious contributions in D_3 and D_4 . First D_1 corresponds to the entropy production of the scheme. Thanks to (7.30), one has

$$D_1 = -\frac{1}{\varepsilon} \sum_{j \in \mathbb{Z}} (u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2 - \frac{1}{\varepsilon} \sum_{j \in \mathbb{Z}} (u_{j-\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2} \sum_{j \in \mathbb{Z}} \Delta x_j \frac{(u_{j+\frac{1}{2}}^\varepsilon)^2 + (u_{j-\frac{1}{2}}^\varepsilon)^2}{2} \leq 0.$$

One also directly obtains

$$D_2 = -\sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u^\varepsilon)^2 dx \right) \leq 0.$$

For D_4 , one gets directly

$$D_4 = \sum_{j \in \mathbb{Z}} p_j^\varepsilon \frac{u^\varepsilon(x_{j+\frac{1}{2}}) - u^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon} + \sum_{j \in \mathbb{Z}} u_j^\varepsilon \frac{p^\varepsilon(x_{j+\frac{1}{2}}) - p^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon} + \sum_{j \in \mathbb{Z}} \frac{\sigma}{\varepsilon^2} u_j^\varepsilon \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx$$

In this method the third term D_3 is more complicated to study

$$D_3 = \sum_{j \in \mathbb{Z}} \frac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p^\varepsilon(x) dx \right) + \sum_{j \in \mathbb{Z}} \frac{p_{j+\frac{1}{2}}^\varepsilon - p_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) \\ + \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right).$$

It is decomposed in several pieces. We add and subtract in each fluxes the value of the unknowns in the cell. We also add and subtract to the two first integrals the value of the unknowns on the edge. Denoting by $\delta_j^\pm(g) = \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} g(x) dx - g(x_{j \pm \frac{1}{2}})$, one gets after rearrangements

$$D_3 = \sum_{j \in \mathbb{Z}} \frac{u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon}{\varepsilon} \delta_j^+(p^\varepsilon) + \sum_{j \in \mathbb{Z}} \frac{u_j^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} \delta_j^-(p^\varepsilon) \\ + \sum_{j \in \mathbb{Z}} \frac{p_{j+\frac{1}{2}}^\varepsilon - p_j^\varepsilon}{\varepsilon} \delta_j^+(u^\varepsilon) + \sum_{j \in \mathbb{Z}} \frac{p_j^\varepsilon - p_{j-\frac{1}{2}}^\varepsilon}{\varepsilon} \delta_j^-(u^\varepsilon) \\ - \sum_{j \in \mathbb{Z}} \frac{u^\varepsilon(x_{j+\frac{1}{2}}) - u^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon} p_j^\varepsilon - \sum_{j \in \mathbb{Z}} \frac{p^\varepsilon(x_{j+\frac{1}{2}}) - p^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon} u_j^\varepsilon \\ + \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)$$

Using the fluxes' definition (7.16), one can eliminate the pressure fluxes. With a Young's inequality $ab \leq \alpha a^2 + \frac{1}{4\alpha} b^2$ where $\alpha > 0$, one gets

$$\sum_{j \in \mathbb{Z}} \frac{p_{j+\frac{1}{2}}^\varepsilon - p_j^\varepsilon}{\varepsilon} \delta_j^+(u^\varepsilon) = \sum_{j \in \mathbb{Z}} \frac{1}{\varepsilon} (u_j^\varepsilon - u_{j+\frac{1}{2}}^\varepsilon) \delta_j^+(u^\varepsilon) - \frac{\sigma}{2\varepsilon^2} \sum_{j \in \mathbb{Z}} \Delta x_j u_{j+\frac{1}{2}}^\varepsilon \delta_j^+(u^\varepsilon) \\ \leq \alpha \sum_{j \in \mathbb{Z}} \frac{(u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} + \left(\frac{1}{4\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) \sum_{j \in \mathbb{Z}} \delta_j^+(u^\varepsilon)^2 + \frac{\sigma}{8\varepsilon^2} \sum_{j \in \mathbb{Z}} \Delta x_j^2 \left(u_{j+\frac{1}{2}}^\varepsilon \right)^2.$$

Using this expression in D_3 and using again Young's inequality, one gets for arbitrary $\alpha > 0$

$$D_3 \leq \alpha \sum_{j \in \mathbb{Z}} \frac{(u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} + \alpha \sum_{j \in \mathbb{Z}} \frac{(u_{j-\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} \\ + \sum_{j \in \mathbb{Z}} \left(\left(\frac{1}{2\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) \left(\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2 \right) + \frac{\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2}{2\varepsilon\alpha} \right) \\ + \sum_{j \in \mathbb{Z}} \frac{1}{8\varepsilon} \sigma \Delta x_j^2 \frac{(u_{j-\frac{1}{2}}^\varepsilon)^2 + (u_{j+\frac{1}{2}}^\varepsilon)^2}{\varepsilon} + \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) \\ - \sum_{j \in \mathbb{Z}} u_j^\varepsilon \frac{p^\varepsilon(x_{j+\frac{1}{2}}) - p^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon} - \sum_{j \in \mathbb{Z}} p_j^\varepsilon \frac{u^\varepsilon(x_{j+\frac{1}{2}}) - u^\varepsilon(x_{j-\frac{1}{2}})}{\varepsilon}.$$

We now sum all bounds contributing to $\mathcal{E}'(t)$ and we get:

$$\begin{aligned}
\mathcal{E}'(t) &\leq (-1 + \alpha) \sum_{j \in \mathbb{Z}} \frac{(u_{j+\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2 + (u_{j-\frac{1}{2}}^\varepsilon - u_j^\varepsilon)^2}{\varepsilon} \\
&+ \sum_{j \in \mathbb{Z}} \left(\left(\frac{1}{2\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) (\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2) + \frac{\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2}{2\varepsilon\alpha} \right) \\
&+ \sum_{j \in \mathbb{Z}} \frac{1}{8\varepsilon} \sigma \Delta x_j^2 \frac{(u_{j-\frac{1}{2}}^\varepsilon)^2 + (u_{j+\frac{1}{2}}^\varepsilon)^2}{\varepsilon} \\
&+ \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) - \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{2\varepsilon} \frac{(u_{j-\frac{1}{2}}^\varepsilon)^2 + (u_{j+\frac{1}{2}}^\varepsilon)^2}{\varepsilon} \\
&+ \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} u_j^\varepsilon \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) - \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \left(\frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u^\varepsilon)^2(x) dx \right).
\end{aligned}$$

We now examine the sum of all terms in the two last lines of the RHS of the above inequality, which we denote S . One finds

$$\begin{aligned}
S &= - \sum_{j \in \mathbb{Z}} \Delta x_j \frac{\sigma}{2\varepsilon^2} \left[\left(u_{j-\frac{1}{2}}^\varepsilon - \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2 + \left(u_{j+\frac{1}{2}}^\varepsilon - \frac{1}{\Delta x_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2 \right] \\
&+ \frac{\sigma}{2\varepsilon^2} \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) \left(u_j^\varepsilon - u_{j+\frac{1}{2}}^\varepsilon + u_j^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon \right) \\
&\leq \frac{\sigma}{2\varepsilon^2} \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right) \left(u_j^\varepsilon - u_{j+\frac{1}{2}}^\varepsilon + u_j^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon \right).
\end{aligned}$$

Using another Young's inequality, one has for all $\hat{\alpha} > 0$

$$S \leq \frac{\sigma^2}{8\hat{\alpha}\varepsilon^3} \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2 + \hat{\alpha} \sum_{j \in \mathbb{Z}} \frac{(u_j^\varepsilon - u_{j+\frac{1}{2}}^\varepsilon)^2 + (u_j^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon)^2}{\varepsilon}.$$

For example by choosing $\alpha = \frac{1}{2}$ and $\hat{\alpha} = \frac{1}{2}$, and coming back to $\mathcal{E}'(t)$ we get

$$\mathcal{E}'(t) \leq \sum_{j \in \mathbb{Z}} \left(\left(\frac{1}{\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) (\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2) + \frac{1}{\varepsilon} (\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2) \right) \quad (7.33)$$

$$+ \sum_{j \in \mathbb{Z}} \frac{1}{8\varepsilon} \sigma \Delta x_j^2 \frac{(u_{j-\frac{1}{2}}^\varepsilon)^2 + (u_{j+\frac{1}{2}}^\varepsilon)^2}{\varepsilon} + \frac{\sigma^2}{4\varepsilon^3} \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2. \quad (7.34)$$

To estimate the contributions on the first line we use the following fact: for any quantity q , one can use $q(x_{j-\frac{1}{2}}) = q(x) + \int_x^{x_{j-\frac{1}{2}}} \frac{d}{ds} q(s) ds$ and integrate this expression in the cell Δx_j ; we get $\sum_{j \in \mathbb{Z}} \delta_j^\pm(q)^2 \leq h \|q\|_{H^1(\mathbb{R})}^2$. Therefore the first terms on the right hand side of (7.33) can be estimated as

$$\left(\frac{1}{\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) \int_0^t \sum_{j \in \mathbb{Z}} (\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2) dt \leq 2h \left(\frac{1}{\varepsilon} + \frac{\sigma}{2\varepsilon^2} \right) \|u^\varepsilon\|_{L^2([0,t];H^1(\mathbb{R}))}^2.$$

Since $\|\mathbf{u}^\varepsilon\|_{L^2([0,t];H^1(\mathbb{R}))}^2 \leq t\|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}^2$ and also $\frac{\sigma}{\varepsilon^2}\|\mathbf{u}^\varepsilon\|_{L^2([0,t];H^1(\mathbb{R}))}^2 \leq \|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}^2$ by (7.2) and (7.3), one gets that

$$\left(\frac{1}{\varepsilon} + \frac{\sigma}{2\varepsilon^2}\right) \int_0^t \sum_{j \in \mathbb{Z}} \left(\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2\right) dt \leq 2h \left(\frac{t}{\varepsilon} + \frac{1}{2}\right) \|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}^2. \quad (7.35)$$

A similar and simpler formula for the next terms is

$$\frac{1}{\varepsilon} \int_0^t \sum_{j \in \mathbb{Z}} \left(\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2\right) \leq 2\frac{ht}{\varepsilon} \|p^\varepsilon\|_{H^1(\mathbb{R})}^2 \leq 2\frac{ht}{\varepsilon} \|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}^2. \quad (7.36)$$

Next, using the assumption (7.3) on the mesh and the estimate (7.27), one controls the next term by

$$\int_0^T \sum_{j \in \mathbb{Z}} \frac{1}{8\varepsilon} \sigma \Delta x_j^2 \frac{(u_{j-\frac{1}{2}}^\varepsilon)^2 + (u_{j+\frac{1}{2}}^\varepsilon)^2}{\varepsilon} \leq \frac{h}{4C_{\mathcal{M}}} \|\mathbf{V}^\varepsilon(0)\|_{L^2(\mathbb{R})}^2. \quad (7.37)$$

Finally the last term in (7.34) can be bounded as

$$\frac{\sigma^2}{4\varepsilon^3} \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2 \leq \frac{\sigma^2}{4\varepsilon^3} h \|u^\varepsilon\|_{L^2(\mathbb{R})}^2$$

so that

$$\frac{\sigma^2}{4\varepsilon^3} \int_0^t \sum_{j \in \mathbb{Z}} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x) dx \right)^2 \leq \frac{\sigma^2}{4\varepsilon^3} h \|u^\varepsilon\|_{L^2([0,t] \times \mathbb{R})}^2 \leq \frac{\sigma}{4\varepsilon} h \|\mathbf{V}^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 \quad (7.38)$$

by means of the energy identity. We note that

$$\|\mathbf{V}^\varepsilon(0)\|_{H^p(\mathbb{R})} \leq (1 + \varepsilon/\sigma) \|p_0\|_{H^{p+1}(\mathbb{R})} \leq (1 + 1/\sigma) \|p_0\|_{H^{p+1}(\mathbb{R})} \quad \forall p \in \mathbb{N}. \quad (7.39)$$

So using (7.35-7.38) we obtain for all time $t \leq T$

$$\begin{aligned} \mathcal{E}(t) &\leq \mathcal{E}(0) + \left(\frac{t}{\varepsilon} + \frac{1}{2} + \frac{t}{\varepsilon} + \frac{1}{4C_{\mathcal{M}}} + \frac{\sigma}{4\varepsilon}\right) h \|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}^2 \\ &\leq (1 + 1/\sigma) \left(C_{\mathcal{A}}^2 h + \frac{2t}{\varepsilon} + \frac{1}{2} + \frac{1}{4C_{\mathcal{M}}} + \frac{\sigma}{4\varepsilon}\right) h \|p_0\|_{H^2(\mathbb{R})}^2 \end{aligned}$$

where the initialization stage is estimated using (7.21). One obtains after integration

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T] \times \mathbb{R})} \leq \sqrt{T} \left(\sqrt{1 + 1/\sigma} \times \sqrt{C_{\mathcal{A}}^2 h \varepsilon + 2T + \frac{\varepsilon}{2} + \frac{\varepsilon}{4C_{\mathcal{M}}} + \frac{\sigma}{4}} \right) \sqrt{\frac{h}{\varepsilon}} \|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}.$$

The constant in parentheses is $\sqrt{T} \sqrt{1 + 1/\sigma} \sqrt{C_{\mathcal{A}}^2 h \varepsilon + 2TC_{\mathcal{M}} + \frac{\varepsilon}{2} C_{\mathcal{M}} + \frac{\varepsilon}{4} + \frac{\sigma}{4} C_{\mathcal{M}} / \sqrt{C_{\mathcal{M}}}} \leq \frac{\downarrow C}{\sqrt{C_{\mathcal{M}}}}$ with

$$\downarrow C = \sqrt{T} \sqrt{1 + 1/\sigma} \times \sqrt{C_{\mathcal{A}}^2 + 2T + \frac{1}{2} + \frac{1}{4} + \frac{\sigma}{4}}.$$

The proof is ended. □

7.2.5 Study of $\|P_h^0 - P^0\|$

We first recall a fundamental error estimate [EGH00] for the diffusion limit scheme (7.18-7.19).

Lemma 7.8. *There exists a constant $C_\downarrow > 0$ independent of $h, \varepsilon, C_{\mathcal{M}}$, with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0;T] \times \mathbb{R})} \leq \frac{C_\downarrow}{\sqrt{C_{\mathcal{M}}}} h \|p_0\|_{H^2(\mathbb{R})}. \quad (7.40)$$

Proof. We use a method that one can find in Eymard-Gallouet-Herbin [EGH00]. It is based on a notion of consistency for finite volumes schemes. We set

$$s_j = \partial_{xx} p(x_j) - \frac{\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_{j-\frac{1}{2}})}{\Delta x_j} \text{ and } r_{j+\frac{1}{2}} = \partial_x p(x_{j+\frac{1}{2}}) - \frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}},$$

so that one has the identity

$$\frac{d}{dt} p(x_j) - \frac{1}{\sigma \Delta x_j} \left(\frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}} - \frac{p(x_j) - p(x_{j-1})}{\Delta x_{j-\frac{1}{2}}} \right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma \Delta x_j}.$$

We next introduce the difference $e_j = p(x_j) - p_j$ which satisfies

$$\frac{d}{dt} e_j - \frac{1}{\sigma \Delta x_j} \left(\frac{e_{j+1} - e_j}{\Delta x_{j+\frac{1}{2}}} - \frac{e_j - e_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma \Delta x_j}$$

with $e_j(0) = 0$ for all j . By multiplying this equation by e_j and denoting by $\|e_h\|_{L^2(\mathbb{R})}^2 = \sum_j \Delta x_j e_j^2$, one finds that

$$\frac{1}{2} \frac{d}{dt} \|e_h\|_{L^2(\mathbb{R})}^2 + \frac{1}{\sigma} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} = \frac{1}{\sigma} \sum_j \Delta x_j s_j e_j + \frac{1}{\sigma} \sum_j r_{j+\frac{1}{2}} (e_j - e_{j+1}).$$

The Cauchy-Schwarz inequality yields

$$\sum_j r_{j+\frac{1}{2}} (e_j - e_{j+1}) \leq \frac{1}{2} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} + \frac{1}{2} \sum_j \Delta x_{j+\frac{1}{2}} r_{j+\frac{1}{2}}^2.$$

One finds out with natural notations

$$\frac{1}{2} \frac{d}{dt} \|e_h\|_{L^2(\mathbb{R})}^2 + \frac{1}{2\sigma} \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} \leq \frac{1}{\sigma} \|s_h\|_{L^2(\mathbb{R})} \|e_h\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|r_h\|_{L^2(\mathbb{R})}^2. \quad (7.41)$$

Using the definitions of the truncation error s_h , one easily obtains by using classical arguments $\|s_h\|_{L^2([0;T] \times \mathbb{R})} \leq \sqrt{2}h \|\partial_{xxx} p\|_{L^2([0;T] \times \mathbb{R})}$: since p satisfies the diffusion equation (7.12), one gets $\|\partial_{xxx} p\|_{L^2([0;T] \times \mathbb{R})} \leq \sqrt{\sigma/2} \|\partial_{xx} p_0\|_{L^2(\mathbb{R})}$; one gets $\|s_h\|_{L^2([0;T] \times \mathbb{R})} \leq \sqrt{\sigma}h \|\partial_{xx} p_0\|_{L^2(\mathbb{R})}$. The same manipulations on the second truncation error r_h yield

$$\|s_h\|_{L^2([0;T] \times \mathbb{R})} + \|r_h\|_{L^2([0;T] \times \mathbb{R})} \leq \sqrt{\sigma}h \|p_0\|_{H^2(\mathbb{R})}. \quad (7.42)$$

One gets the bound from (7.41)

$$\|e_h\|_{L^2(\mathbb{R})}^2(t) \leq e_h\|_{L^2(\mathbb{R})}^2(0) + \int_0^t \frac{1}{\sigma} \|s_h\|_{L^2(\mathbb{R})} \|e_h\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|r_h\|_{L^2([0;T] \times \mathbb{R})}^2.$$

The use of the lemma (7.34), which is a corollary of the Bihari's inequality, gives us:

$$\|e_h\|_{L^2([0,T]\times\Omega)}^2 \leq \frac{1}{2}T \left(2\sqrt{\|e_h\|_{L^2(\Omega)}^2(0) + \frac{1}{\sigma}\|r_h\|_{L^2([0,T]\times\Omega)}^2} + \frac{1}{\sigma}\sqrt{T}\|s_h\|_{L^2([0,T]\times\Omega)} \right)^2. \quad (7.43)$$

The initial value is bounded using (7.22) and taking into account (7.42) we obtain

$$\|e_h\|_{L^2([0,T]\times\Omega)} \leq \sqrt{\frac{T}{2}} \left(2\sqrt{C_{\mathcal{A}} + \frac{1}{\sqrt{\sigma}}} + \sqrt{\frac{T}{\sigma}} \right) h\|p_0\| = B h\|p_0\|. \quad (7.44)$$

We also deduce from (7.41)

$$\begin{aligned} \int_0^T \sum_j \frac{(e_{j+1} - e_j)^2}{\Delta x_{j+\frac{1}{2}}} &\leq (\|s_h\|_{L^2([0,T]\times\mathbb{R})} + \|r_h\|_{L^2([0,T]\times\mathbb{R})})^2 + \|e_h\|_{L^2([0,T]\times\mathbb{R})}^2 \\ &\leq (\sigma + B^2) h^2 \|p_0\|_{H^2(\mathbb{R})}^2 \end{aligned}$$

The other term that we must bound in (7.40) is $f_h = \varepsilon(v(x_j) - v_j)1_j(x) = -\varepsilon\left(\frac{\partial_x p(x_j)}{\sigma} + v_j\right)1_j(x)$ with v_j defined in (7.20). It yields

$$\|f_h\|_{L^2([0,T]\times\mathbb{R})} = \frac{\varepsilon}{2\sigma} \left(\int_0^T \sum_j \Delta x_j \left| \frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} - \partial_x p(x_j) + \frac{p_j - p_{j-1}}{\Delta x_{j-\frac{1}{2}}} - \partial_x p(x_j) \right|^2 \right)^{\frac{1}{2}} \quad (7.45)$$

where the definition of e_j yields $\frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} - \partial_x p(x_j) = \left(\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_j)\right) + \left(\frac{e_{j+1} - e_j}{\Delta x_{j+\frac{1}{2}}}\right) - r_{j+\frac{1}{2}}$. One gets from the triangular inequality

$$\begin{aligned} \left(\int_0^T \sum_j \Delta x_{j+\frac{1}{2}} \left(\frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} - \partial_x p(x_j) \right)^2 \right)^{\frac{1}{2}} &\leq \left(\int_0^T \sum_j \Delta x_{j+\frac{1}{2}} \left(\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_j) \right)^2 \right)^{\frac{1}{2}} \\ &\quad + [\sigma + B^2]^{\frac{1}{2}} h\|p_0\|_{H^2(\mathbb{R})} + \|r_h\|_{L^2([0,T]\times\mathbb{R})}. \end{aligned}$$

Since $\left(\int_0^T \sum_j \Delta x_{j+\frac{1}{2}} \left(\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_j)\right)^2\right)^{\frac{1}{2}} \leq h\sqrt{\frac{\sigma}{2}}\|p_0\|_{H^1(\mathbb{R})}$ and the estimate (7.42) holds, one gets

$$\left(\int_0^T \sum_j \Delta x_{j+\frac{1}{2}} \left(\frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} - \partial_x p(x_j) \right)^2 \right)^{\frac{1}{2}} \leq \left([\sigma + B^2]^{\frac{1}{2}} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma} \right) h\|p_0\|_{H^2(\mathbb{R})}. \quad (7.46)$$

Taking into account that the weight Δx_j (7.45) is different from the weight $\Delta x_{j+\frac{1}{2}}$ in (7.46), one gets

$$\|f_h\|_{L^2([0,T]\times\mathbb{R})} \leq \frac{\varepsilon}{\sqrt{C_{\mathcal{M}}}} \left([\sigma + B^2]^{\frac{1}{2}} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma} \right) h\|p_0\|_{H^2(\mathbb{R})} \leq \frac{1}{\sqrt{C_{\mathcal{M}}}} \left([\sigma + B^2]^{\frac{1}{2}} + \sqrt{\frac{\sigma}{2}} + \sqrt{\sigma} \right) h\|p_0\|_{H^2(\mathbb{R})} \quad (7.47)$$

since $\varepsilon \leq 1$. Finally, the difference $\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0;T] \times \mathbb{R})}^2 = \|e_h\|_{L^2([0;T] \times \mathbb{R})}^2 + \|f_h\|_{L^2([0;T] \times \mathbb{R})}^2$ is bounded using (7.44) and (7.47)

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0;T] \times \mathbb{R})} \leq \frac{C_\downarrow}{\sqrt{C_{\mathcal{M}}}} h \|p_0\|_{H^2(\mathbb{R})}$$

and the constant C_\downarrow with the definition of B by (7.44), has, at most, a linear growth in time. It ends the proof. \square

7.2.6 Study of $\|P_h^\varepsilon - P_h^0\|$

In this section we prove an error estimate between the solution of the scheme (7.29) and the solution of the diffusion scheme (7.18). It is necessary to use some comparison estimates between the initial data of P_h^ε and P_h^0 .

Lemma 7.9. *There exists a constant $C^\rightarrow > 0$ independent of $h, \varepsilon, C_{\mathcal{M}}$ and growth as $T^{\frac{3}{2}}$ for large T such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0;T] \times \mathbb{R})} \leq \frac{C^\rightarrow}{C_{\mathcal{M}}} \varepsilon \|p_0\|_{H^2(\mathbb{R})}. \quad (7.48)$$

Proof. For practical reasons we use the formulation (7.29) of the hyperbolic scheme which is equivalent to (7.15-7.16) and we reformulate the diffusion scheme (7.18-7.20) as

$$\left\{ \begin{array}{l} \Delta x_j \frac{d}{dt} p_j + \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\varepsilon} = 0, \\ \Delta x_j \frac{d}{dt} u_j - \frac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{\varepsilon} + \frac{2}{\varepsilon} u_j = \Delta x_j R_j, \\ p_j - p_{j+1} + u_j + u_{j+1} = 2u_{j+\frac{1}{2}} + \sigma \Delta x_{j+\frac{1}{2}} \frac{u_{j+\frac{1}{2}}}{\varepsilon} + \Delta x_{j+\frac{1}{2}} S_{j+\frac{1}{2}}, \\ u_{j+\frac{1}{2}} = -\frac{\varepsilon p_{j+1} - p_j}{\sigma \Delta x_{j+\frac{1}{2}}}, \\ u_j = \frac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{2}, \end{array} \right. \quad (7.49)$$

where the error terms are R_j and $S_{j+\frac{1}{2}}$. A simple computation using the last two identities in (7.49) yields

$$R_j = \frac{d}{dt} u_j \text{ and } S_{j+\frac{1}{2}} = \frac{1}{\Delta x_{j+\frac{1}{2}}} \left(u_j + u_{j+1} - 2u_{j+\frac{1}{2}} \right).$$

One has from the triangular inequality applied to $u_j = \frac{u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}}{2}$

$$\begin{aligned} \left\| \frac{d}{dt} u_h \right\|_{L^2([0;T] \times \mathbb{R})} &= \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_j \left| \frac{d}{dt} u_j \right|^2} \\ &\leq \frac{1}{\sqrt{C_{\mathcal{M}}}} \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} u_{j+\frac{1}{2}} \right|^2} = \sqrt{\frac{\varepsilon}{C_{\mathcal{M}}}} \sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} v_{j+\frac{1}{2}} \right|^2}. \end{aligned}$$

Since the scheme is invariant with respect to the time variable, one can apply (7.31) to the derivative with respect to time. It yields

$$\sqrt{\int_0^T \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} \left| \frac{d}{dt} v_{j+\frac{1}{2}} \right|^2} \leq \sqrt{\frac{\sigma}{2}} \left\| \frac{d}{dt} p_h(0) \right\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2\sigma}} \|p_0\|_{H^2(\mathbb{R})}$$

where the last inequality is from the well preparedness of the initial data, as detailed in proposition 7.10. So one has the bound

$$\|R\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon}{\sqrt{2\sigma C_{\mathcal{M}}}} \|p_0\|_{H^2(\mathbb{R})}. \quad (7.50)$$

Using the definitions of u_j (7.49), $S_{j+\frac{1}{2}}$ can be written in terms of $\frac{d}{dt} p_j$ and $\frac{d}{dt} p_{j+1}$

$$S_{j+\frac{1}{2}} = \frac{\varepsilon}{2} \left(\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}} \frac{d}{dt} p_j - \frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}} \frac{d}{dt} p_{j+1} \right).$$

Using the technical proposition 7.10, one finds out that $S = (S_{j+\frac{1}{2}})_{j \in \mathbb{Z}}$ satisfies

$$\|S\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon}{C_{\mathcal{M}}} \left\| \frac{d}{dt} p \right\|_{L^2([0,T] \times \mathbb{R})} \leq \frac{\varepsilon \sqrt{T}}{\sigma C_{\mathcal{M}}} \|p_0\|_{H^2(\mathbb{R})}. \quad (7.51)$$

We now introduce the differences

$$e_j = p_j - p_j^\varepsilon, \quad f_j = u_j - u_j^\varepsilon \text{ and } f_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}^\varepsilon. \quad (7.52)$$

Let us look at the difference between the scheme (7.29) and (7.49). We get

$$\begin{cases} \Delta x_j \frac{d}{dt} e_j + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\varepsilon} = 0, \\ \Delta x_j \frac{d}{dt} f_j - \frac{f_{j+\frac{1}{2}} + f_{j-\frac{1}{2}}}{\varepsilon} + \frac{2}{\varepsilon} f_j = \Delta x_j R_j, \\ e_j - e_{j+1} + f_j + f_{j+1} - 2f_{j+\frac{1}{2}} - \sigma \Delta x_{j+\frac{1}{2}} \frac{f_{j+\frac{1}{2}}}{\varepsilon} = \Delta x_{j+\frac{1}{2}} S_{j+\frac{1}{2}}. \end{cases}$$

We use the notation $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 = \sum_j \Delta x_j (e_j^2 + f_j^2)$. Using the same kind of proof than for the L^2 stability of proposition 7.6, one gets that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \sum_j \Delta x_j R_j f_j - \sum_j \Delta x_{j+\frac{1}{2}} \frac{f_{j+\frac{1}{2}}}{\varepsilon} S_{j+\frac{1}{2}} - \frac{\sigma}{\varepsilon^2} \sum_{j \in \mathbb{Z}} \Delta x_{j+\frac{1}{2}} f_{j+\frac{1}{2}}^2.$$

Using a Young's inequality on the second term of the right side of this inequality, one finds out that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \sum_j \Delta x_j R_j f_j + \frac{1}{4\sigma} \sum_j \Delta x_{j+\frac{1}{2}} S_{j+\frac{1}{2}}^2. \quad (7.53)$$

Using the Cauchy-Schwarz inequality, we have

$$\frac{d}{dt} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \leq \|R\|_{L^2(\mathbb{R})} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|S\|_{L^2(\mathbb{R})}^2.$$

Integrating in time on $[0, t]$

$$\|\mathbf{V}_h^\varepsilon(t) - \mathbf{W}_h^\varepsilon(t)\|_{L^2(\mathbb{R})}^2 \leq \left(\int_0^t \|\mathbf{V}_h^\varepsilon(t) - \mathbf{W}_h^\varepsilon(t)\|_{L^2(\mathbb{R})} \|R\|_{L^2(\mathbb{R})} + \frac{1}{2\sigma} \|S\|_{L^2([0, T] \times \mathbb{R})}^2 + \|\mathbf{V}_h^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 \right).$$

Another use of the Bihari's inequality, lemma (7.34), yields

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0, T] \times \mathbb{R})}^2 \leq \frac{1}{2} T \left(2 \sqrt{\|\mathbf{V}_h^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 + \int_0^T \frac{1}{2\sigma} \|S\|_{L^2([0, T] \times \mathbb{R})}^2 + \sqrt{T} \|R\|_{L^2([0, T] \times \mathbb{R})}^2} \right)^2$$

Finally, using the previous estimates on R , S , the well-preparedness of the data (7.23) one gets

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0, T] \times \mathbb{R})}^2 \leq \frac{1}{2} T \left(2 \sqrt{(C_{\mathcal{A}h\varepsilon})^2 + \frac{\varepsilon^2 T^2}{4\sigma^3 C_{\mathcal{M}}^2}} + \sqrt{T} \frac{\varepsilon^2}{2C_{\mathcal{M}}} \right)^2 \|p_0\|_{H^2(\mathbb{R})}^2$$

The proof is ended. \square

Proposition 7.10 (Technical result). *The bound $\sqrt{\sum_j \Delta x_j (\frac{d}{dt} p_j)^2(t)} \leq \sigma^{-1} \|p_0\|_{H^2(\mathbb{R})}$ holds at any time.*

Proof. By linearity of the diffusion scheme, $z_h = \frac{d}{dt} p_h$ is solution of P_h^0 :

$$\Delta x_j \frac{d}{dt} z_j - \frac{1}{\sigma} \left(\frac{z_{j+1} - z_j}{\Delta x_{j+\frac{1}{2}}} - \frac{z_j - z_{j-1}}{\Delta x_{j-\frac{1}{2}}} \right) = 0,$$

with initial condition

$$z_j(0) = \frac{d}{dt} p_0(x_j) = \frac{1}{\Delta x_j \sigma} \left(\frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac{1}{2}}} - \frac{p_0(x_j) - p_0(x_{j-1})}{\Delta x_{j-\frac{1}{2}}} \right). \quad (7.54)$$

One gets from a Taylor expansion with integral residue that

$$\left| \frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j) \right| \leq \int_{x_j}^{x_{j+1}} |\partial_{xx} p_0(y)| dy.$$

Similarly one has the bound $\left| \frac{p_0(x_j) - p_0(x_{j-1})}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j) \right| \leq \int_{x_{j-1}}^{x_j} |\partial_{xx} p_0(y)| dy$. Therefore $|z_j(0)| \leq \frac{1}{\Delta x_j \sigma} \int_{x_{j-1}}^{x_{j+1}} |\partial_{xx} p_0(y)| dy$ from which the bound $\sqrt{\sum_j \Delta x_j z_j^2(0)} \leq \sigma^{-1} \|p_0\|_{H^2(\mathbb{R})}$ is deduced. Since the scheme P_h^0 is stable in L^2 , this bound is true at any time. Considering (7.54) the discrete second derivative attached to P_h^0 is bounded at any time, which ends the proof of the claim. \square

7.2.7 End of the proof of uniform AP property

Theoreme 7.11. *Assuming a sufficiently smooth well prepared initial data, the scheme P_h^ε converges to P^ε at order at least $\frac{1}{3}$ in $L^2([0, T] \times \mathbb{R})$, uniformly with respect to ε*

Proof. All the previous estimates show that (7.9-7.8) are true with $a = 1$, $b = c = \frac{1}{2}$ and $d = 1$. More specifically, estimates (7.32), (7.48), (7.40) and (7.24) shows that

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T] \times \mathbb{R})} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, h + 2\varepsilon\right) \|p_0\|_{H^3(\mathbb{R})}$$

where

$$C = \max\left[\frac{\downarrow C}{\sqrt{C_M}}, \frac{C_{\rightarrow}}{C_M}, \frac{C_{\downarrow}}{\sqrt{C_M}}, C_{\leftarrow}\right]$$

and behaves less than $T^{\frac{3}{2}}$ for large T . Using the general method described at the beginning of this work in proposition 7.2, one obtains the convergence estimate $\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T] \times \mathbb{R})} \leq C(T)h^q$ with the order of convergence $q = \frac{ac}{a+b} = \frac{1}{3}$. □

7.3 The 2D case

In this section we prove the uniform convergence of the solution of the diffusion AP scheme introduced in [BDF12-1] to the solution of the hyperbolic heat equation. The structure of our proof is globally the same as in the previous section. However two major difficulties will be treated: a) the first one consists in the adaptation to our problem of a combination of specific finite volumes techniques for hyperbolic and parabolic equations; b) the second one is to derive new bounds for the scheme $\mathbf{DA}_h^\varepsilon$.

The model problem is the hyperbolic heat equation in the domain $\Omega =]0, 1[^2$ with periodic boundary conditions and well-prepared data

$$\mathbf{P}^\varepsilon : \begin{cases} \partial_t p^\varepsilon + \frac{1}{\varepsilon} \operatorname{div}(\mathbf{u}^\varepsilon) = 0, \\ \partial_t \mathbf{u}^\varepsilon + \frac{1}{\varepsilon} \nabla p^\varepsilon = -\frac{\sigma}{\varepsilon^2} \mathbf{u}^\varepsilon, \\ p^\varepsilon(t=0) = p_0, \mathbf{u}^\varepsilon(t=0) = \mathbf{u}_0^\varepsilon = -\frac{\varepsilon}{\sigma} \nabla p_0. \end{cases}$$

When ε tends to zero, this problem admits the following diffusion limit

$$\mathbf{P}^0 : \quad \partial_t p - \frac{1}{\sigma} \operatorname{div}(\nabla p) = 0, \quad p(t=0) = p_0.$$

The rescaled gradient is $\mathbf{v} = -\frac{1}{\sigma} \nabla p$. We will admit the following proposition, the proof of which can be easily obtained by a method similar to the one of proposition 7.5.

Proposition 7.12. *The error between the two solutions can be upper bounded by*

$$\|p^\varepsilon - p\|_{L^\infty([0,T]; H^n(\Omega))} + \|\mathbf{v}\|_{L^\infty([0,T]; H^n(\Omega))} \leq \frac{T}{\sigma^2} \varepsilon \|p_0\|_{H^{3+n}(\Omega)}, \quad n \in \mathbb{N}. \quad (7.55)$$

Proof. The structure of the proof in the $L^\infty([0,T]; L^2(\Omega))$ norm is the same as the one of proposition 7.5. Since the coefficients of the problem are constant, similar bounds are obtained at any order of derivation which proves the estimate for any $n > 0$. □

7.3.1 Definition of \mathbf{P}_h^ε

Let us consider an unstructured mesh in dimension 2. The mesh is defined by a finite number of vertices \mathbf{x}_r and cells Ω_j . We denote \mathbf{x}_j a point chosen arbitrarily inside Ω_j . For simplicity we will call this point the center of the cell. By convention the vertices are listed counter-clockwise $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_{r+1}$ with coordinates $\mathbf{x}_r = (x_r, y_r)$. We note $l_{jr} \mathbf{n}_{jr}$ the vector as follows

$$l_{jr} = \frac{1}{2} \text{dist}(\mathbf{x}_{r-1}, \mathbf{x}_{r+1}) \text{ and } \mathbf{n}_{jr} = \frac{1}{2l_{jr}} (\mathbf{x}_{r+1} - \mathbf{x}_{r-1})^\perp. \quad (7.56)$$

This notion of a corner vector can be rigorously introduced also in any dimension using the definition [DES10]. The scalar product of two vectors is denoted as (\mathbf{x}, \mathbf{y}) .

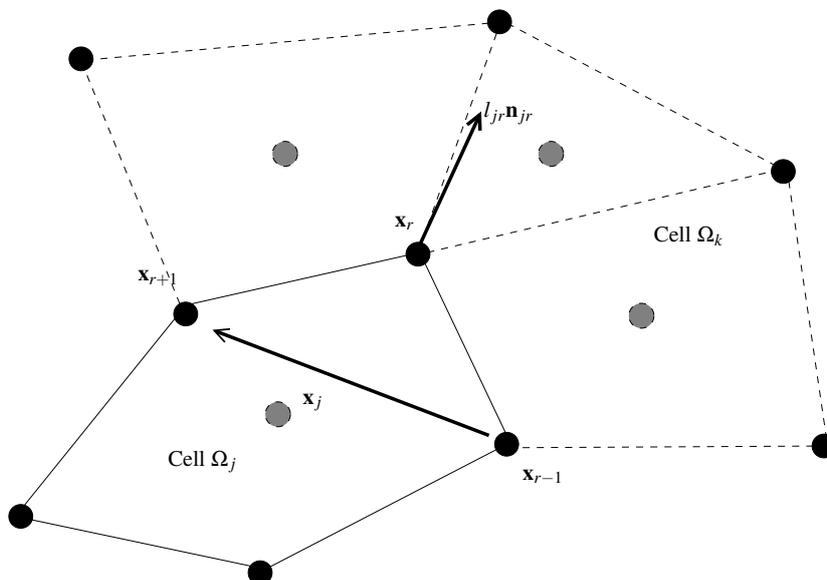


Figure 7.4: Notation for node formulation. The corner length l_{jr} and the corner normal \mathbf{n}_{jr} are defined in equation (7.56). The point \mathbf{x}_j is an arbitrary point inside the cell, typically the centroid of the cell or an averaged of the corners.

The numerical approximation of the problem \mathbf{P}^ε that we study is the JL-(b) scheme defined in [BDF12-1]

$$\mathbf{P}_h^\varepsilon : \begin{cases} |\Omega_j| \frac{d}{dt} p_j^\varepsilon + \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ |\Omega_j| \frac{d}{dt} \mathbf{u}_j^\varepsilon + \frac{1}{\varepsilon} \sum_r l_{jr} \mathbf{n}_{jr} p_{jr}^\varepsilon = -\frac{\sigma}{\varepsilon^2} \sum_r \hat{\beta}_{jr} \mathbf{u}_r^\varepsilon, \end{cases} \quad (7.57)$$

with for simplicity point wise initial data $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$ and $\mathbf{u}_j^\varepsilon(0) = -\varepsilon \sigma^{-1} \nabla p_0(\mathbf{x}_j)$. The fluxes are defined by the so-called corner problem

$$\begin{cases} p_{jr}^\varepsilon - p_j^\varepsilon = (\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \frac{\sigma}{\varepsilon} (\mathbf{x}_r - \mathbf{x}_j, \mathbf{u}_r^\varepsilon), \\ \sum_j l_{jr} p_{jr}^\varepsilon \mathbf{n}_{jr} = 0. \end{cases} \quad (7.58)$$

This corner problem has been introduced in [BDF12-1] as a multidimensional version of the 1D

Jin-Levermore technique [JL96]. Its solution is provided by the solution of the linear system

$$\left(\sum_j \hat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon} \hat{\beta}_{jr} \right) \mathbf{u}_r^\varepsilon = \sum_j l_{jr} p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \hat{\alpha}_{jr} \mathbf{u}_j^\varepsilon, \quad (7.59)$$

where the geometry of the mesh is used to define the matrices $\hat{\alpha}_{jr}$ and $\hat{\beta}_{jr}$

$$\hat{\alpha}_{jr} = l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}, \text{ and } \hat{\beta}_{jr} = l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \quad (7.60)$$

We will use the notations $A_j = \sum_r \hat{\alpha}_{jr}$, $A_r = \sum_j \hat{\alpha}_{jr}$ and $B_r = \sum_j \hat{\beta}_{jr}$. By comparison with the scheme P_h^ε in dimension one, one sees that the multi-dimensional scheme (7.57-7.60) is more tricky than the 1D scheme (7.15-7.16).

Starting from (7.57) and taking into account of the definitions of the fluxes (7.58) and also the identity $\sum_r l_{jr} \mathbf{n}_{jr} = 0$, the scheme \mathbf{P}_h^ε can also be rewritten as

$$\mathbf{P}_h^\varepsilon : \begin{cases} |\Omega_j| \frac{d}{dt} p_j^\varepsilon + \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ |\Omega_j| \frac{d}{dt} \mathbf{u}_j^\varepsilon + \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon) \mathbf{n}_{jr} = 0 \end{cases} \quad (7.61)$$

When $\varepsilon \rightarrow 0$ the scheme \mathbf{P}_h^ε , see (7.57) or (7.61), admits the limit diffusion scheme \mathbf{P}_h^0

$$\mathbf{P}_h^0 : \begin{cases} |\Omega_j| \frac{d}{dt} p_j + \sum_r l_{jr} (\mathbf{v}_r, \mathbf{n}_{jr}) = 0, \\ \mathbf{v}_r = \frac{1}{\sigma} B_r^{-1} \sum_j l_{jr} p_j \mathbf{n}_{jr}, \end{cases} \quad (7.62)$$

with $B_r = \sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$. We define additionally \mathbf{v}_j by a kind of mean

$$\left(\sum_r \hat{\alpha}_{jr} \right) \mathbf{v}_j = \sum_r \hat{\alpha}_{jr} \mathbf{v}_r.$$

This is well defined since the matrix $\sum_r \hat{\alpha}_{jr}$ is symmetric positive by definition of the $\hat{\alpha}_{jr}$.

7.3.2 Definition of $\mathbf{DA}_h^\varepsilon$

We define now that is call thereafter the "diffusion approximation" scheme. We just neglect the time derivative in the second equation, that we make $\partial_t \mathbf{u}_j^\varepsilon = 0$ for (7.61). It leads to the scheme

$$\mathbf{DA}_h^\varepsilon : \begin{cases} |\Omega_j| \frac{d}{dt} p_j^\varepsilon + \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon) \mathbf{n}_{jr} = 0 \\ \left(\sum_j \hat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon} \hat{\beta}_{jr} \right) \mathbf{u}_r^\varepsilon = \sum_j l_{jr} p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \hat{\alpha}_{jr} \mathbf{u}_j^\varepsilon \end{cases} \quad (7.63)$$

This scheme depends of two parameters, the size of the mesh h and the small parameter ε . We notice that $\mathbf{DA}_h^\varepsilon \neq \mathbf{P}_h^0$ for $\varepsilon > 0$, and that $\lim_{\varepsilon \rightarrow 0^+} \mathbf{DA}_h^\varepsilon = \mathbf{P}_h^0$. The point wise initial data for (7.63) is $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$. There is no need of initial data for $(u_j^\varepsilon(0))$, which will be obtained as a function of $(p_j^\varepsilon(0))$ by solving a linear system.

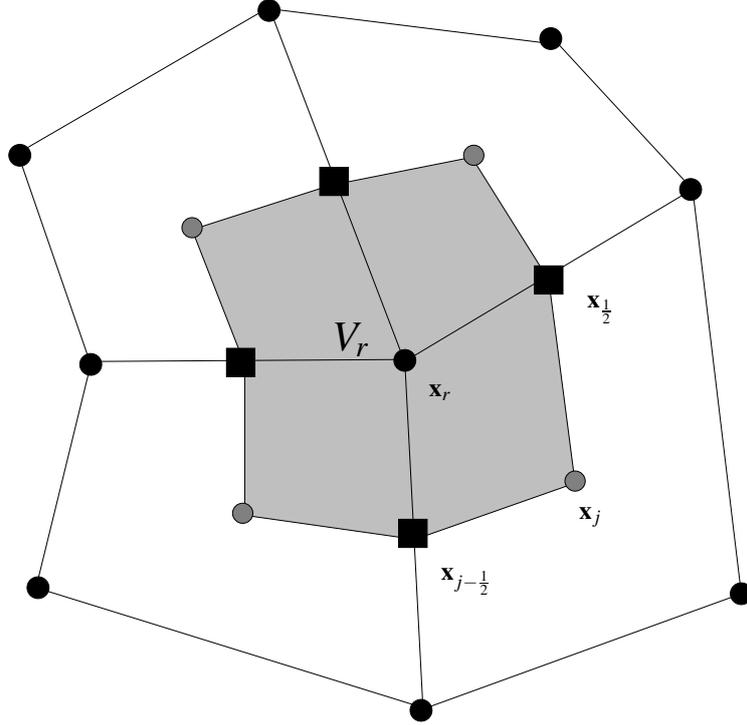


Figure 7.5: Definition of the control volume V_r around vertex \mathbf{x}_r . The control volume around the vertex \mathbf{x}_r is defined by the closed loop that joins the center of the cells (\mathbf{x}_j 's) and the middle of the edges ($\mathbf{x}_{j+\frac{1}{2}}$'s).

7.3.3 Mesh assumptions

The characteristic length of the mesh is $h = \max_j (\text{diam}(\Omega_j))$, so that

$$\begin{cases} l_{jr} \leq h, & \forall j, r, \\ |\Omega_j| \leq h^2, & \forall j. \end{cases} \quad (7.64)$$

The control volume V_r around the vertex \mathbf{x}_r is defined by the closed loop $\dots, \mathbf{x}_{j-\frac{1}{2}}, \mathbf{x}_j, \mathbf{x}_{j+\frac{1}{2}}, \dots$. Here the \mathbf{x}_j 's are the center of the cells, and the $\mathbf{x}_{j+\frac{1}{2}}$'s are the middle of the edges around the vertices \mathbf{x}_r . A typical example is depicted in figure 7.5.

Additional geometrical assumptions are always necessary in dimension greater than one to guarantee some minimal regularity of the mesh. We make the usual assumptions listed below from 1 to 3. The last items are more specific.

Hypothèse 7.13. *Our geometrical assumptions will be the following*

1. *The numbers of cells which share a node r is bounded independently of h , which means there exists $P \in \mathbb{N}$ independent of h such that*

$$\sum_j \delta_{jr} \leq P. \quad (7.65)$$

For example, for a structured mesh of quadrangular cells $P = 4$.

2. For each cell of the mesh, the number of edges is bounded independently of h , or equivalently the numbers of vertices for a cell is bounded independently of h .
3. The mesh is regular in the sense that there exists a universal constant $C_{\mathcal{M}} > 0$ such that the inverse inequalities hold:

$$C_{\mathcal{M}}h \leq l_{jr}, \quad \forall j, r \quad \text{uniformly with respect to } h \quad (7.66)$$

where \mathbf{x}_r is a vertex of the cell Ω_j , and

$$C_{\mathcal{M}}h^2 \leq |\Omega_j|, \quad \forall j \quad \text{uniformly with respect to } h. \quad (7.67)$$

and

$$C_{\mathcal{M}}h^2 \leq |V_r| \leq Ph^2, \quad \forall r \quad \text{uniformly with respect to } h. \quad (7.68)$$

We recall that V_r is the volume control (centered on \mathbf{x}_r) and Ω_j is the cell j . The inequality $|V_r| \leq Ph^2$ is immediate to check on the figure 7.5.

4. A consequence of the items 1-3 is that there exists a constant $\alpha > 0$ such that

$$(A_j \mathbf{u}, \mathbf{u}) \geq \alpha h(\mathbf{u}, \mathbf{u}), \quad A_j = \sum_r \hat{\alpha}_{jr}. \quad (7.69)$$

It can be proved with a geometrical identity that we borrow from [DES10] (proposition 8).

5. The matrix $B_r = \sum_j \hat{\beta}_{jr}$ is positive in the sense that

$$(B_r \mathbf{u}, \mathbf{u}) = (B_r^s \mathbf{u}, \mathbf{u}) \geq \alpha |V_r|(\mathbf{u}, \mathbf{u}), \quad (7.70)$$

where $B_r^s = \frac{1}{2}(B_r + B_r^t)$ is the symmetric part of B_r , and α is the same constant as in (7.69). Square meshes satisfy (7.70). This assumption is however not trivial to check in the general case. We point out [BDF12-1] where sufficient conditions such that (7.70) is satisfied can be found; in particular it is shown that triangular meshes with all angles greater than 12 degrees satisfy it.

7.3.4 Norms and error measurements

The quadratic norms below are usual integral norms. It yields for any cell centered quantity $f = (f_j)_{j \in \text{Cells}}$: $\|f\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j| |f_j|^2}$. For vertex based quantity $g = (g_r)_{r \in \text{Vertices}}$, we use $\|g\|_{L^2(\Omega)} = \sqrt{\sum_r |V_r| |g_r|^2}$: it is more a convention. Useful quantities are

- $\mathbf{V}_h^\varepsilon(t, \mathbf{x}) = \left(\sum_{j \in \text{Cells}} p_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}), \sum_{j \in \text{Cells}} \mathbf{u}_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}) \right)$ which is the solution of \mathbf{P}_h^ε .
- $\mathbf{V}^\varepsilon(t, \mathbf{x}) = (p^\varepsilon, \mathbf{u}^\varepsilon)(t, \mathbf{x})$ which is the solution of \mathbf{P}^ε ,
- $\mathbf{W}_h^\varepsilon(t, \mathbf{x}) = \left(\sum_{j \in \text{Cells}} p_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}), \sum_{j \in \text{Cells}} \mathbf{u}_j^\varepsilon(t) 1_{\Omega_j}(\mathbf{x}) \right)$ which is the solution of $\mathbf{DA}_h^\varepsilon$. Notice that an abuse of notations is made with the solution of \mathbf{P}_h^ε .
- $\mathbf{W}^\varepsilon(t, \mathbf{x}) = (p, -\frac{\varepsilon}{\sigma} \nabla p)(t, \mathbf{x})$ which is the solution of \mathbf{P}^0 .

As in dimension one, the differences between these quantities are characterized at the initial time with a universal constant $C_{\mathcal{A}} > 0$ which indicates it can be related to the approximation/interpolation/projection of a smooth function on the mesh. We will use for example some bounds that can be obtained as by-product or corollary of the first technical inequality below.

$$\begin{cases} \|\mathbf{V}^\varepsilon(0) - \mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)} \leq C_{\mathcal{A}}h\|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)} \leq C_{\mathcal{A}}h\|\mathbf{V}^1(0)\|_{H^2(\Omega)}, \\ \|\mathbf{W}^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\Omega)} \leq C_{\mathcal{A}}h\|\mathbf{W}^\varepsilon(0)\|_{H^2(\Omega)}. \end{cases} \quad (7.71)$$

We will need additional technical estimates for the corner based Finite Volume scheme \mathbf{P}_h^ε . These technical estimates can be formulated as follows. Let f be a regular function. We define $\delta_{j,r}(f) = \frac{1}{|\Omega_j|} \int_{\Omega_j} f d\mathbf{x} - f(\mathbf{x}_r)$ which is the interpolation error term that compares mean value in a cell Ω_j and point values at a vertex \mathbf{x}_r of the same cell. Let $\Gamma_{j,r} = [\mathbf{x}_r, \mathbf{x}_{r+1}]$ be the edge oriented toward the outside of the cell j , with length $|\Gamma_{j,r}|$. We define also $\tilde{\delta}_{j,r}(h) = \frac{1}{|\Gamma_{j,r}|} \int_{\Gamma_{j,r}} h ds - \frac{h(\mathbf{x}_r) + h(\mathbf{x}_{r+1})}{2}$ which is another interpolation error contribution that compares the mean value and the mid sum, on the edge.

Proposition 7.14. *One has the technical inequalities*

$$|\delta_{jr}(f)| \leq C_{\mathcal{A}}\|f\|_{H^2(\Omega_j)} \quad (7.72)$$

and

$$|\tilde{\delta}_{jr}(f)| \leq C_{\mathcal{A}}h\|f\|_{H^3(\Omega_j)} \quad (7.73)$$

Proof. These non optimal inequalities are consequences of classical approximations results. We will not prove them. However one can notice that the scaling is correct. That if a function f has its third derivatives bounded in $L^\infty(\Omega_j)$, then $\|f\|_{H^2(\Omega_j)} = O(h)$ because the problem is 2D: this is compatible with the fact that δ_{jr} is a first order difference. Similarly $h\|f\|_{H^3(\Omega_j)} = O(h^2)$ is compatible with the fact that $\tilde{\delta}_{jr}$ is a second order difference. An alternative proof is by assuming that f is in $H^p(\Omega)$ for a sufficiently large p . Then by the Sobolev embeddings, all derivatives up to fourth order are in L^∞ which is enough to prove that (7.72) is a first order interpolation error term, and that (7.73) is a second order interpolation error term. In this case it also explains very simply why the constant $C_{\mathcal{A}}$ is independent of the mesh size. \square

The first technical inequality is actually true for any points in the cell. So it allows to compare the mean value and the point value in the cell. This is why it yields (7.71) after summation over all cells and redefinition of $C_{\mathcal{A}}$.

As in dimension one, we will use constants $C_\downarrow, C^\leftarrow, C_\downarrow$ and C_\leftarrow in the errors bounds for the four branches of the new AP diagram. The important point is that these constants are independent of h and ε . They have of course some dependence with respect to other parameters such as the constant of the mesh $C_{\mathcal{M}}$ for example, but we will not keep track of these dependence in order to simplify the notations. Nevertheless the interested reader can compare with the same estimates in dimension one where the dependence with respect to the mesh constant is indicated. A first result is the inequality (7.55) which yields the basic estimate for the lower branch of the AP diagram. It can be formalized as follows.

Lemma 7.15. *One has the estimate*

$$\|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T] \times \Omega)} \leq C_\leftarrow \varepsilon \|p_0\|_{H^4(\Omega)} \quad (7.74)$$

where the constant C_\leftarrow is independent of h and ε , with a growth in time less than $T^{\frac{3}{2}}$ by comparison with (7.55).

7.3.5 Study of $\|\mathbf{P}_h^\varepsilon - \mathbf{P}^\varepsilon\|_{\text{naive}}$

In this part, we exploit the hyperbolic nature of both \mathbf{P}^ε and \mathbf{P}_h^ε to obtain the main bound. As one will see below, the convergence estimate (7.75) is not trivial. It indicates that, for a problem with $O(\varepsilon^{-2})$ terms, a scheme converges, with h , with at rate $O(\varepsilon^{-\frac{1}{2}})$ with respect to ε .

Lemma 7.16 (Naive estimate). *There exists a constant $\downarrow C$ independent of h and ε , with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^\infty([0,T];L^2(\Omega))} \leq \downarrow C \sqrt{\frac{h}{\varepsilon}} \|p_0\|_{H^4(\Omega)}. \quad (7.75)$$

The norm is slightly stronger than the $L^2([0,T] \times \Omega)$ needed to complete the proof.

Stability

We first prove the L^2 stability of the scheme P_h^ε defined in (7.57,7.58).

Proposition 7.17 (Stability). *The semi-discrete general JL-(b) scheme defined by (7.57,7.58) is stable in the L^2 norm in the sense that $\frac{d}{dt}\|\mathbf{V}_h^\varepsilon(t)\| \leq 0$. Moreover we have the bounds*

$$\frac{\sigma}{\varepsilon^2} \|\mathbf{u}_r^\varepsilon\|_{L^2([0,T] \times \Omega)} \leq \frac{1}{\alpha} \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)}, \quad (7.76)$$

$$\int_0^T \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 dt \leq \varepsilon \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)}^2. \quad (7.77)$$

Proof. We define the functions p_h^ε and \mathbf{u}_h^ε by $p_h^\varepsilon = p_j$ and $\mathbf{u}_h^\varepsilon = \mathbf{u}_j$ on Ω_j . We set for convenience $E(t) = \|\mathbf{V}_h^\varepsilon(t)\|^2$. One has

$$E'(t) = \frac{1}{2} \int_\Omega \frac{d}{dt} (|p_h^\varepsilon|^2 + (\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon)) = \int_\Omega p_h^\varepsilon \frac{d}{dt} p_h^\varepsilon + (\mathbf{u}_h^\varepsilon, \frac{d}{dt} \mathbf{u}_h^\varepsilon) = \sum_j |\Omega_j| p_j^\varepsilon \frac{d}{dt} p_j^\varepsilon + (\mathbf{u}_j^\varepsilon, \frac{d}{dt} \mathbf{u}_j^\varepsilon).$$

Using the definition of scheme

$$E'(t) = -\frac{1}{\varepsilon} \sum_j \sum_r l_{jr} p_j^\varepsilon (\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \frac{1}{\varepsilon} \sum_j \sum_r (l_{jr} p_{j,r}^\varepsilon \mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon^2} \sum_j \sum_r (\hat{\beta}_{jr} \mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \quad (7.78)$$

Using (7.58) we expand the second term of the previous equation

$$\sum_j \sum_r (l_{jr} p_{j,r}^\varepsilon \mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) = \sum_j \sum_r l_{jr} p_j^\varepsilon (\mathbf{u}_j^\varepsilon, \mathbf{n}_{jr}) + \sum_j \sum_r (\hat{\alpha}_{jr} (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon} \sum_j \sum_r (\hat{\beta}_{jr} \mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \quad (7.79)$$

Since $\sum_r l_{jr} \mathbf{n}_{jr} = 0$ the first term of (7.79) is zero. Summing on r the second equation of (7.58) and permuting the sums, we show that $0 = \sum_j \sum_r l_{jr} p_{jr} (\mathbf{u}_r, \mathbf{n}_{jr})$ which yields that

$$0 = \sum_j \sum_r l_{jr} p_j^\varepsilon (\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \sum_j \sum_r ((\hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \hat{\beta}_{jr}) \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j \sum_r (\hat{\alpha}_{jr} \mathbf{u}_j^\varepsilon, \mathbf{u}_r^\varepsilon). \quad (7.80)$$

Plugging (7.79) and (7.80) in (7.78) and permuting the sums in $E'(t)$ gives

$$E'(t) = -\frac{1}{\varepsilon} \sum_j \sum_r (\hat{\alpha}_{jr} (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \frac{\sigma}{\varepsilon^2} \sum_r \sum_j (\hat{\beta}_{jr} \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$$

which gives

$$E'(t) + \frac{1}{\varepsilon} \sum_r \sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = 0. \quad (7.81)$$

By geometrical assumption (7.70) we have $E'(t) \leq 0$, that is the L^2 stability, and by integrating this equality on $[0, T]$ we obtain

$$E(T) + \int_0^T \frac{1}{\varepsilon} \sum_r \sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \int_0^T \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = E(0)$$

Using again the geometrical assumption (7.70) for the terms $(B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$ we have

$$E(T) + \int_0^T \frac{1}{\varepsilon} \sum_r \sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \alpha \int_0^T \frac{\sigma}{\varepsilon^2} \sum_r |V_r| |\mathbf{u}_r^\varepsilon|^2 \leq E(0)$$

which gives (7.76) and (7.77). The proof is ended. \square

Main estimate

Our goal now is to prove the lemma 7.16 as the consequence of propositions 7.18 to 7.32. This part is the more technical one of the paper, but is essential to be able to use the general strategy of proposition 7.2 with convenient exponents. Like in 1D, we use the method introduced by Mazeran [MAZ07] and decompose the proof in several steps. We introduce $\mathcal{E}(t) = \frac{1}{2} \|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2(\Omega)}^2$. As for the 1D proof and for the sake of simplicity, for any quantity q , q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$.

Proposition 7.18. *One has the formula*

$$\mathcal{E}'(t) = -\frac{1}{\varepsilon} \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + E_1 + E_2 + E_3 \quad (7.82)$$

where

$$\begin{aligned} E_1 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) \delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} \mathbf{n}_{j,r} (p_{jr}^\varepsilon - p_j^\varepsilon), \delta_{j,r}(\mathbf{u}^\varepsilon) \right), \\ E_2 &= \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \right) \\ E_3 &= \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_j \int_{\Omega_j} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x} - \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon). \end{aligned}$$

Proof. We first consider the time derivative

$$\begin{aligned} \mathcal{E}'(t) &= \underbrace{\int_{\Omega} (p_h^\varepsilon (p_h^\varepsilon)' + (\mathbf{u}_h^\varepsilon, (\mathbf{u}_h^\varepsilon)')) d\mathbf{x}}_{D_1} + \underbrace{\int_{\Omega} (p^\varepsilon (p^\varepsilon)' + (\mathbf{u}^\varepsilon, (\mathbf{u}^\varepsilon)')) d\mathbf{x}}_{D_2} \\ &\quad + \underbrace{\int_{\Omega} (-(p_h^\varepsilon)' p^\varepsilon - ((\mathbf{u}_h^\varepsilon)', \mathbf{u}^\varepsilon)) d\mathbf{x}}_{D_3} + \underbrace{\int_{\Omega} (-p_h^\varepsilon (p^\varepsilon)' - (\mathbf{u}_h^\varepsilon, (\mathbf{u}^\varepsilon)')) d\mathbf{x}}_{D_4}. \end{aligned}$$

One has thanks to (7.81)

$$D_1 = -\frac{1}{\varepsilon} \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon).$$

One also directly has

$$D_2 = -\frac{\sigma}{\varepsilon^2} \int_{\Omega} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x} = -\frac{\sigma}{\varepsilon^2} \sum_j \int_{\Omega_j} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x}.$$

Then, using the definition (7.57,7.58) of the scheme we have

$$\begin{aligned} D_3 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} \mathbf{u}_r^\varepsilon, \mathbf{n}_{j,r} \right) \frac{1}{|\Omega_j|} \int_{\Omega_j} p^\varepsilon d\mathbf{x} \\ &\quad + \frac{1}{\varepsilon} \sum_j \left(\sum_r l_{jr} \mathbf{n}_{jr} p_{j,r}^\varepsilon + \frac{\sigma}{\varepsilon} \sum_r \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \end{aligned}$$

Since $\sum_r l_{jr} \mathbf{n}_{jr} = 0$, we can write

$$\begin{aligned} D_3 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) \frac{1}{|\Omega_j|} \int_{\Omega_j} p^\varepsilon d\mathbf{x} \\ &\quad + \frac{1}{\varepsilon} \sum_j \left(\sum_r l_{jr} \mathbf{n}_{jr} (p_{j,r}^\varepsilon - p_j^\varepsilon), \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \\ &\quad + \frac{\sigma}{\varepsilon^2} \left(\sum_r \sum_j \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right). \end{aligned}$$

One gets

$$\begin{aligned} D_3 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) \delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{jr} \mathbf{n}_{jr} (p_{j,r}^\varepsilon - p_j^\varepsilon), \delta_{j,r}(\mathbf{u}^\varepsilon) \right) \\ &\quad + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) p^\varepsilon(\mathbf{x}_r) + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{jr} \mathbf{n}_{jr} (p_{j,r}^\varepsilon - p_j^\varepsilon), \mathbf{u}^\varepsilon(\mathbf{x}_r) \right) \\ &\quad + \frac{\sigma}{\varepsilon^2} \left(\sum_r \sum_j \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right). \end{aligned}$$

We have the identities $\sum_{j,r} l_{jr} \mathbf{n}_{jr} = 0$ and $\sum_j l_{jr} \mathbf{n}_{jr} p_{j,r}^\varepsilon = 0$ by definition (7.58). Therefore one can simplify the third and fourth term in the previous expression and get

$$\begin{aligned} D_3 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) \delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{jr} \mathbf{n}_{jr} (p_{j,r}^\varepsilon - p_j^\varepsilon), \delta_{j,r}(\mathbf{u}^\varepsilon) \right) \\ &\quad - \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} \mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r} \right) p^\varepsilon(\mathbf{x}_r) - \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} p_j^\varepsilon \mathbf{n}_{j,r}, \mathbf{u}^\varepsilon(\mathbf{x}_r) \right) \\ &\quad + \frac{\sigma}{\varepsilon^2} \left(\sum_r \sum_j \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right). \end{aligned}$$

We now look at D_4 . By definition, one has

$$D_4 = \frac{1}{\varepsilon} \sum_j p_j^\varepsilon \sum_r \int_{\Gamma_{j,r}} (\mathbf{u}^\varepsilon, \tilde{\mathbf{n}}_{j,r}) d\sigma + \frac{1}{\varepsilon} \sum_j \left(\mathbf{u}_j^\varepsilon, \left(\sum_r \int_{\Gamma_{j,r}} p^\varepsilon \tilde{\mathbf{n}}_{j,r} d\sigma + \frac{\sigma}{\varepsilon} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \right)$$

where $\tilde{\mathbf{n}}_{j,r}$ is the normal to the edge $\Gamma_{j,r} = [\mathbf{x}_r, \mathbf{x}_{r+1}]$ oriented toward the outside of the cell j . This expression needs an important manipulation which is to approximate the integral on edges by corner values. This necessary manipulation is one of the ideas that was introduced in [MAZ07] in order to proceed to the numerical analysis of such corner based finite volume schemes. This is why interpolation terms $\tilde{\delta}_{j,r}(h) = \frac{1}{|\Gamma_{j,r}|} \int_{\Gamma_{j,r}} h - \frac{h(\mathbf{x}_r) + h(\mathbf{x}_{r+1})}{2}$ are introduced. One gets after an algebraic manipulation

$$\begin{aligned} D_4 &= \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\tilde{\mathbf{n}}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon) \right) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon \right) \\ &+ \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\tilde{\mathbf{n}}_{j,r}, \frac{\mathbf{u}^\varepsilon(\mathbf{x}_r) + \mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2} \right) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r} \frac{p^\varepsilon(\mathbf{x}_r) + p^\varepsilon(\mathbf{x}_{r+1})}{2} \right) \end{aligned}$$

By definition (7.56), $\mathbf{n}_{jr} l_{jr} = \frac{\tilde{n}_{j,r} |\Gamma_{j,r}| + \tilde{n}_{j,r-1} |\Gamma_{j,r-1}|}{2}$, so one can see that

$$\sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\tilde{\mathbf{n}}_{j,r}, \frac{\mathbf{u}^\varepsilon(\mathbf{x}_r) + \mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2} \right) = \sum_j \sum_r l_{jr} p_j^\varepsilon (\mathbf{n}_{jr}, \mathbf{u}^\varepsilon(\mathbf{x}_r)).$$

It yields a slightly simpler expression

$$\begin{aligned} D_4 &= \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\tilde{\mathbf{n}}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon) \right) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon \right) \\ &+ \frac{1}{\varepsilon} \sum_j \sum_r l_{jr} p_j^\varepsilon (\mathbf{n}_{jr}, \mathbf{u}^\varepsilon(\mathbf{x}_r)) + \frac{1}{\varepsilon} \sum_j \sum_r l_{jr} p^\varepsilon(\mathbf{x}_r) (\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) \end{aligned}$$

One can now compute the sum $D_3 + D_4$

$$\begin{aligned} D_3 + D_4 &= \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \right) \tilde{\delta}_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon} \sum_j \sum_r \left(l_{j,r} \mathbf{n}_{j,r} (p_{j,r}^\varepsilon - p_j^\varepsilon), \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon) \right) \\ &+ \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon \left(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon) \right) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \right) \\ &+ \frac{\sigma}{\varepsilon^2} \left(\sum_r \sum_j \hat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right). \end{aligned}$$

One finally gets after rearrangement the final result (7.82) for $\mathcal{E}'(t) = D_1 + D_2 + D_3 + D_4$. \square

The proof of the dissipative identity relies on a careful and technical evaluation of E_1 , E_2 and E_3 . Using the damping of the first term in (7.82), it is sufficient to obtain the desired result. We refer the reader to the appendix for all details.

7.3.6 Study of $\|\mathbf{DA}_h^\varepsilon - \mathbf{P}^0\|$

This main result in this section is the following.

Lemma 7.19. *There exists a constant C_\downarrow independent of h and ε , with a growth in time less than $T^{\frac{3}{2}}$ such that one has the estimate*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0,T] \times \Omega)} \leq C_\downarrow (h + \varepsilon) \|p_0\|_{H^4(\Omega)}. \quad (7.83)$$

This result, which is merely a consequence of (7.96) and (7.97) in proposition 7.22, will be obtained after studying in details the well-posedness, stability and consistency of the new diffusion asymptotic scheme rewritten after a convenient rescaling. The proof is provided just after the proof of the proposition. Additional technical results will be derived at the end of the section.

Rescaling of the equations

We rescale the semi-discrete diffuse asymptotic scheme $\mathbf{DA}_h^\varepsilon$ (7.63) wherein for convenience we made the following change of unknowns

$$\bar{\mathbf{u}}_r^\varepsilon = \frac{\mathbf{u}_r^\varepsilon}{\varepsilon} \text{ and } \bar{\mathbf{u}}_j^\varepsilon = \frac{\mathbf{u}_j^\varepsilon}{\varepsilon}. \quad (7.84)$$

In order to keep a simple notation we dropped the superscript ε and the bars. Thus the scheme (7.63) is now written as:

$$\begin{cases} |\Omega_j| \frac{d}{dt} p_j + \sum_r (l_{jr} \mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r - \mathbf{u}_j) \mathbf{n}_{jr} = 0 \\ \left(\varepsilon \sum_j \hat{\alpha}_{jr} + \sigma B_r \right) \mathbf{u}_r = \sum_j l_{jr} p_j \mathbf{n}_{jr} + \varepsilon \sum_j \hat{\alpha}_{jr} \mathbf{u}_j \end{cases} \quad (7.85)$$

Remarque 7.3. *If we set $\varepsilon = 0$ we naturally recover the limit diffusion scheme (7.62).*

This way of writing the system is much better to help the intuition, since it can be naturally interpreted as the discretization of a diffusion equation.

Well-posedness

What we mean about well-posedness is the following: if we are able to write the last two relations of (7.85) as a non singular linear system with the \mathbf{u}_r 's and \mathbf{u}_j 's as unknowns, then we have a unique solution in terms of the p_j 's. This notion is the relevant one for numerical discretization.

Let us denote $Y = (\{\mathbf{u}_j\}, \{\mathbf{u}_r\})$ the vector of unknowns. We can write the last two relations of (7.85) as $MY = b$ where M is a $(J + R)^2$ square matrix, J is the number of cells and R . One can observe that unless $\varepsilon = 0$, M is not a blockwise triangular matrix. One has

$$(MY, Y) = \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2$$

Assume $(MY, Y) = 0$: in this case the geometrical assumption (7.70) implies that all the \mathbf{u}_r are null and therefore it remains to study $\sum_j \sum_r l_{jr} (\mathbf{u}_j, \mathbf{n}_{jr})^2 = 0$ that is $\sum_j (\mathbf{u}_j, C_j \mathbf{u}_j) = 0$ where $C_j = \sum_r l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$. Since the C_j are all invertible unless the mesh is degenerate, all the \mathbf{u}_j are null: we have proved the invertibility of the matrix M and thus the scheme (7.85) exists and is uniquely defined.

Stability

We note $E(t) = \frac{1}{2} \sum_j |\Omega_j| p_j^2$. The initial data is $p_h(0) = (p_j(0))_{j \in \text{Cells}}$.

Proposition 7.20. *Under the geometrical assumption (7.70), the diffusion approximation scheme (7.85) is stable in the L^2 norm, in the sense that $E'(t) \leq 0$. One has*

$$\|\mathbf{u}_r\|_{L^2([0,T] \times \Omega)} \leq \frac{1}{\alpha} \|p_h(0)\|_{L^2(\Omega)} \quad (7.86)$$

and

$$\varepsilon \int_{[0,T]} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j - \mathbf{u}_r))^2 \leq \|p_h(0)\|_{L^2(\Omega)}. \quad (7.87)$$

Proof. One has

$$E'(t) = \sum_j |\Omega_j| p_j \frac{d}{dt} p_j = - \sum_j p_j \sum_r (l_{jr} \mathbf{u}_r, \mathbf{n}_{jr}) = \sum_r \left(\mathbf{u}_r, \sum_j l_{jr} \mathbf{n}_{jr} p_j \right).$$

With the last equation of (7.85), one finds $E'(t) = - \sum_r \left(\mathbf{u}_r, \left(\varepsilon \sum_j \hat{\alpha}_{jr} + \sigma B_r \right) \mathbf{u}_r - \varepsilon \sum_j \hat{\alpha}_{jr} \mathbf{u}_j \right)$.

We expand the right hand side $E'(t) = - \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_r \left(\mathbf{u}_r, \sum_j \hat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j) \right)$. Permuting the sums in the second term of the right hand side, we show that

$$E'(t) = - \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r (\mathbf{u}_r, \hat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j)). \quad (7.88)$$

Using the definition of the \mathbf{u}_j , second line of (7.85), one has

$$\sum_j \left(\mathbf{u}_j, \sum_r \hat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j) \right) = 0. \quad (7.89)$$

Combining (7.89) $\times \varepsilon$ with (7.88) and using the definition of the matrices $\hat{\alpha}_{jr}$ one has finally

$$E'(t) = - \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2.$$

By the geometrical assumption (7.70) we have $E'(t) \leq 0$, that is the L^2 stability. By integrating this equality on $[0, T]$ we obtain

$$E(T) + \int_0^T \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \int_0^T \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2 = E(0)$$

Using again the geometrical assumption (7.70) for the terms $(B_r \mathbf{u}_r, \mathbf{u}_r)$ we have

$$E(T) + \alpha \int_0^T \sum_r |V_r| \|\mathbf{u}_r\|^2 + \int_0^T \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2 \leq E(0)$$

which gives (7.86) and (7.87). □

Consistency

For convenience we set

$$\bar{p}_j = p(\mathbf{x}_j, t) \quad \bar{\mathbf{u}}_j = -\frac{1}{\sigma} \nabla p(\mathbf{x}_j, t) \quad \bar{\mathbf{u}}_r = -\frac{1}{\sigma} \nabla p(\mathbf{x}_r, t) \quad (7.90)$$

where $p(x, t)$ is the solution of the diffusion equation. We define three consistency errors by inserting these quantities into the three equations of (7.85) which are also rescaled by a factor $\frac{1}{|\Omega_j|}$, $\frac{1}{h}$ and $\frac{1}{|V_r|}$. It yields

$$\begin{cases} a_j = \frac{d}{dt} \bar{p}_j + \frac{1}{|\Omega_j|} \sum_r (l_{jr} \bar{\mathbf{u}}_r, \mathbf{n}_{jr}), \\ \mathbf{b}_j = \frac{1}{h} \sum_r l_{jr} (\mathbf{n}_{jr}, \bar{\mathbf{u}}_r - \bar{\mathbf{u}}_j) \mathbf{n}_{jr}, \\ \mathbf{c}_r = \frac{1}{|V_r|} \left(\sigma B_r \bar{\mathbf{u}}_r - \sum_j l_{jr} \bar{p}_j \mathbf{n}_{jr} + \varepsilon \sum_j \hat{\alpha}_{jr} (\bar{\mathbf{u}}_r - \bar{\mathbf{u}}_j) \right). \end{cases}$$

Proposition 7.21. *There exists a constant $C_c > 0$ independent on h and ε such that the following estimates hold*

$$\|a_h\|_{L^\infty([0,T];L^2(\Omega))} \leq C_c h \|p_0\|_{H^4(\Omega)}, \quad (7.91)$$

$$\|\mathbf{b}_h\|_{L^\infty([0,T];L^2(\Omega))} \leq C_c h \|p_0\|_{H^3(\Omega)}, \quad (7.92)$$

and

$$\|\mathbf{c}_h\|_{L^\infty([0,T];L^2(\Omega))} \leq C_c (h + \varepsilon) \|p_0\|_{H^3(\Omega)}. \quad (7.93)$$

Proof. The proof uses the inequalities of proposition 7.14. For example one has

$$a_j = \frac{1}{\sigma} \left(\underbrace{\Delta p(\mathbf{x}_j, t) - \frac{\int_{\Omega_j} \Delta p(\mathbf{x}, t) dx}{|\Omega_j|}}_{=d_j^1} \right) + \frac{1}{\sigma |\Omega_j|} \left(\underbrace{\int_{\partial\Omega_j} \partial_n p d\tau - \sum_r l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t))}_{=d_j^2} \right).$$

The first term is $|d_j^1| \leq C \|p\|_{H^4(\Omega_j)}$ by virtue of the first inequality of the proposition (7.14) with \mathbf{x}_r changed into \mathbf{x}_j . The second term d_j^2 can be rearranged. Indeed by definition of $l_{jr} \mathbf{n}_{jr}$ one has

$$\sum_r l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t)) = \sum_k \int_{\partial\Omega_{jk}} \left(\frac{\nabla p(x_{jk}^+) + \nabla p(x_{jk}^-)}{2}, \mathbf{n}_j \right) d\tau$$

where $n_j = \tilde{n}_{j,r}$ defined in the previous part and the nodes x_{jk}^+ and x_{jk}^- are the end of the edge $\partial\Omega_{jk} = \Omega_j \cap \Omega_k$, with the relation $\partial\Omega_j = \bigcup \partial\Omega_{jk}$. Therefore

$$d_j^2 = \sum_k \int_{\partial\Omega_{jk}} \left(\nabla p - \frac{\nabla p(x_{jk}^+) + \nabla p(x_{jk}^-)}{2}, \mathbf{n}_j \right) d\tau.$$

The second inequality of the proposition 7.14 yields $|d_j^2| \leq C_{\mathcal{A}} h^2 \|p\|_{H^4(\Omega_j)}$. Therefore one can write $a_j \leq \tilde{C} \|p_0\|_{H^4(\Omega_j)}$ where the constant is uniform with respect to j . It yields

$$\|a_h\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j| a_j^2} \leq \sqrt{\sum_j |\Omega_j| C^2 \|p\|_{H^4(\Omega_j)}^2} \leq Ch \|p\|_{H^4(\Omega)} \leq Ch \|p_0\|_{H^4(\Omega)}. \quad (7.94)$$

Since it is true at any time t , it yields the first bound (7.91). The second inequality can be obtained with the same argument. Consider the decomposition

$$\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j) = \left(\nabla p(\mathbf{x}_r) - \frac{\int_{\Omega_j} \nabla p(\mathbf{x}) dv}{|\Omega_j|} \right) - \left(\nabla p(\mathbf{x}_j) - \frac{\int_{\Omega_j} \nabla p(\mathbf{x}) dv}{|\Omega_j|} \right).$$

Each parenthesis can be estimated with the first inequality of proposition 7.14. The rest of the proof of the second bound (7.92) is immediate since l_{jr} is neutralized by the $\frac{1}{h}$. The third bound is analyzed as follows. We write $\mathbf{c}_r = \mathbf{c}_r^a + \mathbf{c}_r^b$ with

$$\begin{aligned} \mathbf{c}_r^a &= \frac{1}{|V_r|} \left(\left(\sigma \sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right) \left(-\frac{\nabla p(\mathbf{x}_r, t)}{\sigma} \right) - \sum_j l_{jr} \mathbf{n}_{jr} p(\mathbf{x}_j, t) \right) \\ &= \frac{1}{|V_r|} \sum_j \left((\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) \right) l_{jr} \mathbf{n}_{jr} \\ &= \frac{1}{|V_r|} \sum_j \left((\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) + p(\mathbf{x}_r, t) \right) l_{jr} \mathbf{n}_{jr} \end{aligned}$$

and $\mathbf{c}_r^b = \frac{\varepsilon}{\sigma |V_r|} \left(\sum_j l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} (\nabla p(\mathbf{x}_j, t) - \nabla p(\mathbf{x}_r, t)) \right)$. The first interpolation of proposition 7.14 can be used to evaluate the difference of point values in \mathbf{c}_r^b . It yields $|\mathbf{c}_r^b| \leq C \frac{\varepsilon}{h} \|p\|_{H^3(\Omega_j)}$. Concerning \mathbf{c}_r^a we notice that

$$(\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) + p(\mathbf{x}_r, t) = \left(\frac{1}{|\mathbf{x}_r - \mathbf{x}_j|} \int_{\mathbf{x}_j}^{\mathbf{x}_r} \nabla p(\mathbf{x}) d\tau - \nabla p(\mathbf{x}_r), \mathbf{x}_r - \mathbf{x}_j \right)$$

where the integral is along the chord between \mathbf{x}_j and \mathbf{x}_r . The first term in the scalar product is the comparison between a mean value and a point value. So it can be estimated as in proposition 7.14. It yields similarly

$$\left| \left(\frac{1}{|\mathbf{x}_r - \mathbf{x}_j|} \int_{\mathbf{x}_j}^{\mathbf{x}_r} \nabla p(\mathbf{x}) d\tau - \nabla p(\mathbf{x}_r), \mathbf{x}_r - \mathbf{x}_j \right) \right| \leq Ch \|p\|_{H^3(\Omega_j)}. \quad (7.95)$$

Thus $|\mathbf{c}_r^a| \leq Ch \|p\|_{H^3(\Omega_j)}$. After summation of the \mathbf{c}_r^a s and \mathbf{c}_r^b s, one gets the last inequality of the claim. The constant C_c is the maximum of the three constants that show up in the three inequalities. \square

Convergence

We study the numerical error between the solution of the diffusion asymptotic scheme written as (7.85) and the point values of the exact solution (7.90). Let us define three error variables

$$e_j = p_j - \bar{p}_j, \quad \mathbf{f}_r = \mathbf{u}_r - \bar{\mathbf{u}}_r \quad \text{and} \quad \mathbf{g}_j = \mathbf{u}_j - \bar{\mathbf{u}}_j$$

Proposition 7.22. *There exists constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ independent of h and ε , bounded for any time T and growing at most as $T^{\frac{3}{2}}$, such that*

$$\|e_h\|_{L^\infty([0,T];L^2(\Omega))} \leq C_1(h + \varepsilon)\|p_0\|_{H^4(\Omega)}, \quad (7.96)$$

$$\|\mathbf{f}_h\|_{L^2([0,T]\times\Omega)} \leq C_2(h + \varepsilon)\|p_0\|_{H^4(\Omega)}, \quad (7.97)$$

and

$$\|\mathbf{g}_h\|_{L^2([0,T]\times\Omega)} \leq C_3(h + \varepsilon)\sqrt{\left(1 + \frac{h}{\varepsilon}\right)}\|p_0\|_{H^4(\Omega)}. \quad (7.98)$$

Moreover

$$\sqrt{\varepsilon \int_0^T \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2} \leq C_4(h + \varepsilon)\|p_0\|_{H^4(\Omega)}. \quad (7.99)$$

Proof. By construction

$$\begin{cases} |\Omega_j| e'_j + \sum_r (l_{jr} \mathbf{f}_r, \mathbf{n}_{jr}) = -|\Omega_j| a_j \\ \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_r - \mathbf{f}_j) \mathbf{n}_{jr} = -h \mathbf{b}_j, \\ \left(\varepsilon \sum_j \hat{\alpha}_{jr} + \sigma B_r \right) \mathbf{f}_r - \sum_j l_{jr} e_j \mathbf{n}_{jr} - \varepsilon \sum_j \hat{\alpha}_{jr} \mathbf{f}_j = -|V_r| \mathbf{c}_r. \end{cases}$$

The errors are measured with $E(t) = \frac{1}{2}\|e_h\|_{L^2(\Omega)}^2$, $F(t) = \|\mathbf{f}_h\|_{L^2([0,t]\times\Omega)}^2 = \int_0^t \sum_r |V_r| |\mathbf{f}_r|^2$ and $\|\mathbf{g}_h\|_{L^2([0,t]\times\Omega)}^2 = \int_0^t \sum_j |\Omega_j| |\mathbf{f}_j|^2$. By proceeding as for the results of stability one has the identity

$$\begin{aligned} E'(t) &= \sum_j |\Omega_j| e_j e'_j = \sum_j e_j \left(- \left(\sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_r) \right) - |\Omega_j| a_j \right) \\ &= - \sum_r \sum_j (l_{jr} \mathbf{n}_{jr} e_j, \mathbf{f}_r) - \sum_j |\Omega_j| e_j a_j \\ &= - \sum_r \left(\mathbf{f}_r, \left(\varepsilon \sum_j \hat{\alpha}_{jr} + \sigma B_r \right) \mathbf{f}_r - \varepsilon \sum_j \hat{\alpha}_{jr} \mathbf{f}_j \right) - \sum_j |\Omega_j| e_j a_j - \sum_r |V_r| (\mathbf{c}_r, \mathbf{f}_r) \\ &= - \sum_r (\sigma B_r \mathbf{f}_r, \mathbf{f}_r) - \varepsilon \sum_r \left(\mathbf{f}_r, \sum_j \hat{\alpha}_{jr} (\mathbf{f}_r - \mathbf{f}_j) \right) - \sum_j |\Omega_j| e_j a_j - \sum_r |V_r| (\mathbf{c}_r, \mathbf{f}_r) \\ &= - \sum_r (\sigma B_r \mathbf{f}_r, \mathbf{f}_r) - \varepsilon \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 - \sum_j |\Omega_j| e_j a_j - \sum_r |V_r| (\mathbf{c}_r, \mathbf{f}_r) + \varepsilon \sum_j h (\mathbf{b}_j, \mathbf{f}_j). \end{aligned}$$

Using Young's inequality and assumptions (7.67) and (7.70), one gets

$$\begin{aligned} E'(t) &\leq -\alpha\sigma\|\mathbf{f}_h\|_{L^2(\Omega)}^2 - \varepsilon \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 + \sqrt{2E(t)}\|a_h\|_{L^2(\Omega)} \\ &\quad + \left(\frac{\mu}{2}\|\mathbf{f}_h\|_{L^2(\Omega)}^2 + \frac{1}{2\mu}\|\mathbf{c}_h\|_{L^2(\Omega)}^2 \right) + \frac{\varepsilon}{2hC_{\mathcal{M}}} \left(\eta\|\mathbf{g}_h\|_{L^2(\Omega)}^2 + \frac{1}{\eta}\|\mathbf{b}_h\|_{L^2(\Omega)}^2 \right) \end{aligned} \quad (7.100)$$

where $\mu, \eta > 0$ are two arbitrary coefficients that will be specified later. Now using (7.69) and (7.64), we have

$$|\Omega_j| |\mathbf{f}_j|^2 \leq \frac{h}{\alpha} \sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{f}_j)^2. \quad (7.101)$$

Therefore

$$|\Omega_j| |\mathbf{f}_j|^2 \leq \frac{h}{\alpha} \left(2 \sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + 2 \sum_r l_{jr} |\mathbf{f}_r|^2 \right),$$

which yields, using (7.68) and (7.65)

$$\|\mathbf{g}_h\|_{L^2(\Omega)}^2 \leq \frac{2h}{\alpha} \sum_{jr} l_{jr}(\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + \frac{2P}{\alpha C_{\mathcal{M}}} \|\mathbf{f}_h\|_{L^2(\Omega)}^2. \quad (7.102)$$

So from (7.100) we obtain

$$\begin{aligned} E'(t) &\leq \sqrt{2E(t)} \|a_h\|_{L^2(\Omega)} + \frac{1}{2\mu} \|\mathbf{c}_h\|_{L^2(\Omega)}^2 + \left(\frac{\mu}{2} + \frac{\varepsilon P \eta}{h C_{\mathcal{M}}^2 \alpha} - \sigma \alpha \right) \|\mathbf{f}_h\|_{L^2(\Omega)}^2 \\ &\quad + \left(\frac{\eta}{C_{\mathcal{M}} \alpha} - 1 \right) \varepsilon \sum_j \sum_r l_{jr}(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 + \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} \|\mathbf{b}_h\|^2, \quad \forall \mu, \eta > 0. \end{aligned}$$

Let us choose the free coefficients μ and η so that

$$\frac{\mu}{2} + \frac{\varepsilon P \eta}{h C_{\mathcal{M}}^2 \alpha} - \sigma \alpha \leq -\frac{\sigma \alpha}{4} \quad \text{and} \quad \frac{\eta}{C_{\mathcal{M}} \alpha} - 1 \leq -\frac{1}{2}.$$

Let us choose first $\mu = \frac{\alpha \sigma}{2}$. The two inequalities reduce to

$$\frac{\varepsilon P \eta}{h C_{\mathcal{M}}^2 \alpha} \leq \frac{\sigma \alpha}{2} \quad \text{and} \quad \frac{\eta}{C_{\mathcal{M}} \alpha} \leq \frac{1}{2}.$$

A natural solution is $\eta = \frac{C_{\mathcal{M}} \alpha}{2} \min(1, \frac{\alpha \sigma h C_{\mathcal{M}}}{\varepsilon P})$. So

$$\begin{aligned} E'(t) &\leq \sqrt{2E(t)} \|a_h\|_{L^2(\Omega)} - \frac{\alpha \sigma}{4} F'(t) - \frac{\varepsilon}{2} \sum_j \sum_r l_{jr}(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 \\ &\quad + \frac{1}{\alpha \sigma} \|\mathbf{c}_h\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} \|\mathbf{b}_h\|_{L^2(\Omega)}^2. \end{aligned}$$

By the consistency estimates (7.91-7.92-7.93), one has

$$\begin{aligned} &\frac{1}{2} \|a_h\|_{L^2(\Omega)}^2 + \frac{1}{\alpha \sigma} \|\mathbf{c}_h\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} \|\mathbf{b}_h\|_{L^2(\Omega)}^2 \\ &\leq C_c^2 \left(\frac{1}{2} h^2 + \frac{1}{\alpha \sigma} (h + \varepsilon)^2 + \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} h^2 \right) \|p\|_{L^\infty([0, T]; H^4(\Omega))}^2 \\ &\leq C_c^2 \left(\frac{1}{2} h^2 + \frac{1}{\alpha \sigma} (h + \varepsilon)^2 + \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} h^2 \right) \|p_0\|_{H^4(\Omega)}^2. \end{aligned}$$

The last term in the parenthesis is

$$\begin{aligned} \frac{\varepsilon}{2h C_{\mathcal{M}} \eta} h^2 &= \frac{1}{C_{\mathcal{M}}^2 \alpha} \varepsilon h \max(1, \varepsilon P / (\alpha \sigma h C_{\mathcal{M}})) \\ &\leq \frac{1}{C_{\mathcal{M}}^2 \alpha} \varepsilon h (1 + \varepsilon P / (\alpha \sigma h C_{\mathcal{M}})) = \frac{1}{C_{\mathcal{M}}^2 \alpha} \varepsilon h + \frac{P}{C_{\mathcal{M}}^3 \alpha^2 \sigma} \varepsilon^2. \end{aligned}$$

So there exists a constant C_e independent of h and ε such that

$$E'(t) \leq \sqrt{2E(t)} - \frac{\alpha\sigma}{4}F'(t) - \frac{\varepsilon}{2} \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 + C_e(h + \varepsilon)^2 \|p_0\|_{H^4(\Omega)}^2. \quad (7.103)$$

Integrating (7.103), we find that for any for $t \leq T$

$$E(t) + \frac{\alpha\sigma}{4}F(t) + \frac{\varepsilon}{2} \int_0^t \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 \leq E(0) + \int_0^t \sqrt{2E(s)} \|a_h\|_{L^2(\Omega)} ds + tC_e(h + \varepsilon)^2 \|p_0\|_{H^4(\Omega)}^2. \quad (7.104)$$

that is

$$\begin{aligned} E(t) + \frac{\alpha\sigma}{4}F(t) + \frac{\varepsilon}{2} \int_0^t \sum_j \sum_r l_{jr} (\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr})^2 \\ \leq E(0) + \int_0^t \sqrt{2E(s)} \sqrt{C_e}(h + \varepsilon) ds + TC_e(h + \varepsilon)^2 \|p_0\|_{H^4(\Omega)}^2. \end{aligned} \quad (7.105)$$

With another use of the Bihari's inequality, lemma (7.34), we obtain

$$\int_0^T E(t) \leq \frac{1}{2}T \left(2\sqrt{E(0) + TC_e(h + \varepsilon)^2 \|p_0\|_{H^4(\Omega)}^2} + T\sqrt{2C_e}(h + \varepsilon) \|p_0\|_{H^4(\Omega)} \right)^2 \quad (7.106)$$

By construction $E(0) \leq C_A^2 h^2 \|p_0\|_{H^2(\Omega)}^2$, which comes from inequality (7.71) which compares mean value and point value. $E(t) \leq C_1^2(t)(h + \varepsilon)^2 \|p_0\|_{H^4(\Omega)}^2$, where the constant C_1 is bounded for any T and growing as $T^{\frac{3}{2}}$. It gives (7.96), and one easily obtains (7.97) and (7.99) from (7.105) and the constants C_2 and C_3 are bounded for any time T and behave as a linear polynomial in T . Integrating (7.102) and using the estimates (7.97) and (7.99), one gets

$$\int_0^T \|\mathbf{g}\|_{L^2(\Omega)}^2 \leq \frac{2}{\alpha}h \int_0^T \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + \frac{2P}{\alpha C_M} \|\mathbf{f}\|_{L^2([0,T] \times \Omega)}^2 \leq C_4^2(h + \varepsilon)^2 \left(1 + \frac{h}{\varepsilon}\right) \|p_0\|_{H^4(\Omega)},$$

where the constant C_4 is uniform in h and ε and is bounded for any T with, at most, and behave as a linear polynomial in T . The proof is finished. \square

Proof of lemma 7.19. The norm of the estimate in the lemma 7.19 can be bounded from the sum of (7.96) and (7.98). However one must rescale back (7.98) since it corresponds to the scaled variable (7.84). This is why (7.98) must be multiplied by ε . It eliminates the $\varepsilon^{-\frac{1}{2}}$ divergence in (7.98). The constant $C_\downarrow \max(C_1, C_2)$ is bounded for any time T and behaves as $T^{\frac{3}{2}}$ for large T since it is the case for C_1 . and ends the proof. \square

Technical estimates

These technical estimates are needed in the next section. These results compare two different velocities at the initialization stage: on the one hand the velocity computed as the solution of the linear system made of the two last equations of (7.85), on the other hand the exact point wise velocity.

Proposition 7.23. *There exists a constant C independent of h and ε such that*

$$\left\| \left(\mathbf{u}_r + \frac{1}{\sigma} \nabla p(\mathbf{x}_r) \right) (t=0) \right\|_{L^2(\Omega)} \leq C \sqrt{h \max(h, \varepsilon)} \|p_0\|_{H^3(\Omega)} \quad (7.107)$$

and

$$\left\| \left(\mathbf{u}_j + \frac{1}{\sigma} \nabla p(\mathbf{x}_j) \right) (t=0) \right\|_{L^2(\Omega)} \leq C \sqrt{\frac{h}{\varepsilon}} \max(h, \varepsilon) \|p_0\|_{H^3(\Omega)}. \quad (7.108)$$

Proof. Let us write $\mathbf{q}_r = \mathbf{u}_r + \frac{1}{\sigma} \nabla p(\mathbf{x}_r)$ and $\mathbf{s}_j = \mathbf{u}_j + \frac{1}{\sigma} \nabla p(\mathbf{x}_j)$. These quantities are solution of the system

$$\begin{cases} \left(\varepsilon \sum_j l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} + \sigma B_r \right) \mathbf{q}_r - \varepsilon \sum_j l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \mathbf{s}_j = \mathbf{d}_r^1 + \mathbf{d}_r^2, & \forall r, \\ -\varepsilon \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r) \mathbf{n}_{jr} + \varepsilon \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{s}_j) \mathbf{n}_{jr} = \mathbf{d}_j, & \forall j, \end{cases}$$

where the right hand sides are

$$\begin{aligned} \mathbf{d}_r^1 &= \sum_j l_{jr} p(\mathbf{x}_j) \mathbf{n}_{jr} + \sum_j l_{jr} (\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r)) \mathbf{n}_{jr}, \\ \mathbf{d}_r^2 &= \varepsilon \sum_j l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)) \mathbf{n}_{jr} \end{aligned}$$

and

$$\mathbf{d}_j = -\varepsilon \sum_r l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)).$$

The right hand side \mathbf{d}_r^1 can be interpreted as a consistency error. Indeed it can be rewritten as

$$\mathbf{d}_r^1 = \sum_j [p(\mathbf{x}_j) - p(\mathbf{x}_r) + (\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r))] l_{jr} \mathbf{n}_{jr},$$

one obtains from (7.95) the bound $|\mathbf{d}_r^1| \leq \sum_{j \text{ neighboring } r} [\tilde{C}h \|p\|_{H^3(\Omega_j)}] h$. It yields after summation

$$\|\mathbf{d}^1\|_{L^2(\Omega)} \leq Ch^3 \|p\|_{H^3(\Omega)}, \quad C = \tilde{C}P. \quad (7.109)$$

Taking the scalar product of the first line by \mathbf{q}_r and of the second line by \mathbf{s}_j , one gets the identity

$$\begin{aligned} & \sigma \sum_r (B_r \mathbf{q}_r, \mathbf{q}_r) + \varepsilon \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2 \\ &= \sum_r (\mathbf{d}_r^1, \mathbf{q}_r) + \varepsilon \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j) (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)) \end{aligned}$$

where \mathbf{d}^1 shows up explicitly. A Young's inequality yields

$$\sigma \sum_r (B_r \mathbf{q}_r, \mathbf{q}_r) + \frac{\varepsilon}{2} \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2 \leq \sum_r (\mathbf{d}_r^1, \mathbf{q}_r) + \frac{\varepsilon}{2} \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j))^2. \quad (7.110)$$

The first term in the right hand side is

$$\sum_r (\mathbf{d}_r^1, \mathbf{q}_r) = \sum_r h^2 \left(\frac{1}{h^2} \mathbf{d}_r^1, \mathbf{q}_r \right) \leq C \frac{1}{h^2} \|\mathbf{d}^1\|_{L^2(\Omega)} \|\mathbf{q}\|_{L^2(\Omega)} \leq Ch \|p\|_{H^3(\Omega)} \|\mathbf{q}\|_{L^2(\Omega)}.$$

A similar reasoning as for (7.109), which one more time can be viewed as a consequence of the first technical inequality of proposition (7.14), is

$$\sum_{jr} l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j))^2 \leq Ch \|p\|_{H^3(\Omega)}^2.$$

So (7.110) implies (after redefinition of the constants)

$$\|\mathbf{q}\|_{L^2(\Omega)}^2 \leq C \left(\|\mathbf{q}\|_{L^2(\Omega)} \|p\|_{H^3(\Omega)} + \varepsilon \|p\|_{H^3(\Omega)}^2 \right) h.$$

It means that $z = \frac{\|\mathbf{q}\|_{L^2(\Omega)}}{\|p\|_{H^3(\Omega)}}$ is below the maximal root of the polynomial $p(z) = z^2 - Chz - C\varepsilon h$, that is for some constant $K > 0$ $z \leq x^+ = \frac{Ch + \sqrt{C^2 h^2 + 4C\varepsilon h}}{2} \leq K \sqrt{\max(h^2, h\varepsilon)}$. Noticing that $\|p\|_{H^3(\Omega)} \leq \|p_0\|_{H^3(\Omega)}$, It finishes the proof of the first inequality (7.107). Concerning the second inequality, we start from (7.101) to show that

$$\begin{aligned} \|\mathbf{s}\|_{L^2(\Omega)}^2 &= \sum_j |\Omega_j| |\mathbf{s}_j|^2 \leq \frac{h}{\alpha} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{s}_j)^2 \\ &\leq 2 \frac{h}{\alpha} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r)^2 + 2 \frac{h}{\alpha} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2 \\ &\leq 2 \frac{1}{C_M \alpha} \sum_r |V_r| |\mathbf{q}_r|^2 + 2 \frac{h}{\alpha} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2 \\ &\leq 2 \frac{1}{C_M \alpha} \|\mathbf{q}\|_{L^2(\Omega)}^2 + 2 \frac{h}{\alpha} \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2. \end{aligned}$$

The first term is natural bounded bounded using (7.110). The crux of the estimate is the second term which is naturally bounded by (7.107)

$$\sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j)^2 \leq \frac{2}{\varepsilon} \left(K \sqrt{\max(h^2, h\varepsilon)} h + C\varepsilon h \right) \|p_0\|_{H^3(\Omega)}^2 \leq D(h + \varepsilon) \frac{h}{\varepsilon} \|p_0\|_{H^3(\Omega)}^2, \quad D > 0.$$

We obtain therefore

$$\|\mathbf{s}\|_{L^2(\Omega)}^2 \leq C \left(\max(h^2, h\varepsilon) + h(h + \varepsilon) \frac{h}{\varepsilon} \right) \|p_0\|_{H^3(\Omega)}^2, \quad C > 0.$$

The numbers h and ε can be considered less than 1. There are two cases: Either $h < \varepsilon$ so $\|\mathbf{s}\|_{L^2(\Omega)}^2 \leq \tilde{C} h \varepsilon \|p_0\|_{H^3(\Omega)}^2$ for another constant \tilde{C} ; or $\varepsilon \leq h$, so $\|\mathbf{s}\|_{L^2(\Omega)}^2 \leq \tilde{C} \frac{h^3}{\varepsilon} \|p_0\|_{H^3(\Omega)}^2$ for another constant \tilde{C} . So we can writes $\|\mathbf{s}\|_{L^2(\Omega)} \leq C \sqrt{\frac{h}{\varepsilon}} \max(h, \varepsilon)$ for a certain constant $C > 0$ independent of h and ε . The proof of (7.108) is ended. \square

Proposition 7.24. *There exists a constant C independent of h and ε such that*

$$\left\| \left(\frac{d}{dt} \mathbf{u}_r \right) \right\|_{L^2([0, T] \times \Omega)} \leq C \max \left(1, \sqrt{\frac{\varepsilon}{h}} \right) \|p_0\|_{H^3(\Omega)}. \quad (7.111)$$

Proof. The proof is essentially a consequence of the previous proposition. Let us denote the time derivative of any f as $\dot{f} = \partial_t f$. By linearity of the system (7.85), one has

$$\begin{cases} |\Omega_j| \frac{d}{dt} \tilde{p}_j + \sum_r (l_{jr} \tilde{\mathbf{u}}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr} (\mathbf{n}_{jr}, \tilde{\mathbf{u}}_r - \tilde{\mathbf{u}}_j) \mathbf{n}_{jr} = 0 \\ \left(\varepsilon \sum_j \hat{\alpha}_{jr} + \sigma B_r \right) \tilde{\mathbf{u}}_r = \sum_j l_{jr} \tilde{p}_j \mathbf{n}_{jr} + \varepsilon \sum_j \hat{\alpha}_{jr} \tilde{\mathbf{u}}_j \end{cases}$$

The L^2 stability property yields

$$\|\tilde{p}_h\|_{L^\infty([0,T];L^2(\Omega))}^2 + \int_0^T \sum_r (B_r \tilde{\mathbf{u}}_r, \tilde{\mathbf{u}}_r) dt \leq \|\tilde{p}_h(0)\|_{L^2(\Omega)}^2 \quad (7.112)$$

where this last quantity can be estimated with the first equation of (7.85): the square of the norm in (7.111) is also bounded by the same quantity. It remains to bound $\|\tilde{p}_h(0)\|_{L^2(\Omega)}$. Using again the notation $\mathbf{q}_r = \mathbf{u}_r + \frac{1}{\sigma} \nabla p(\mathbf{x}_r)$, we consider at time $t = 0$ the relation

$$\tilde{p}_j = \frac{d}{dt} p_j = -\frac{1}{|\Omega_j|} \sum_r l_{jr} (\mathbf{u}_r, \mathbf{n}_{jr}) = \frac{1}{\sigma} \underbrace{\frac{1}{|\Omega_j|} \sum_r l_{jr} (\nabla p(\mathbf{x}_r), \mathbf{n}_{jr})}_{=v_j^1} - \frac{1}{|\Omega_j|} \underbrace{\sum_r l_{jr} (\mathbf{q}_r, \mathbf{n}_{jr})}_{=v_j^2}.$$

One has $v_j^1 = \frac{1}{|\Omega_j|} \sum_r l_{jr} (\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j), \mathbf{n}_{jr})$. Using techniques which have been used many times in this paper, one has $|v_j^1| \leq C \frac{1}{h} \|p\|_{H^3(\Omega_j)}$, which turns into

$$\|v^1\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j| (v_j^1)^2} \leq C \|p\|_{H^3(\Omega)} \leq C \|p_0\|_{H^3(\Omega)}, \quad C > 0.$$

The other term is naturally bounded by the norm of \mathbf{q} , that is $\|v^2\|_{L^2(\Omega)} \leq \frac{P}{C_M h} \|\mathbf{q}\|_{L^2(\Omega)}$, P the maximal number of vertices per cell. Going back to (7.107), one obtains

$$\|v^2\|_{L^2(\Omega)} \leq C \frac{1}{h} \sqrt{h \max(h, \varepsilon)} \|\mathbf{p}_0\|_{H^3(\Omega)} \leq C \sqrt{\max(1, \varepsilon/h)} \|\mathbf{p}_0\|_{H^3(\Omega)}. \quad (7.113)$$

The sum $\|v^1\|_{L^2(\Omega)} + \|v^2\|_{L^2(\Omega)}$ yields the bound for $\tilde{p}_h(0)$ that was looked for. The estimate is dominated by the worst term which is the right hand side of (7.113). Plugging in (7.112), the proof is finished. \square

7.3.7 Study of $\|\mathbf{P}_h^\varepsilon - \mathbf{DA}_h^\varepsilon\|$

In this section we estimate the difference between the hyperbolic scheme \mathbf{P}_h^ε and the diffusion asymptotic scheme $\mathbf{DA}_h^\varepsilon$. Since the discrete of the discrete equations are very similar, this proof is simple. This is where we get the clear benefit of the introduction of the new diffusion asymptotic scheme.

Lemma 7.25. *There exists a constant C^\rightarrow independant of h and ε , with a linear growth in time, such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T] \times \Omega)} \leq C^\rightarrow \left(h^2 + \varepsilon \max \left(1, \sqrt{\frac{\varepsilon}{h}} \right) \right) \|p_0\|_{H^3(\Omega)}. \quad (7.114)$$

Proof. We introduce $\mathbf{R}_j = \frac{d}{dt} \mathbf{u}_j$ such that the solution \mathbf{V}_h of the diffusion scheme (7.63) satisfies

$$\begin{cases} |\Omega_j| \frac{d}{dt} p_j + \frac{1}{\varepsilon} \sum_r (l_{jr} \mathbf{n}_{jr}, \mathbf{u}_r) = 0, \\ |\Omega_j| \frac{d}{dt} \mathbf{u}_j + \frac{1}{\varepsilon} \sum_r (l_{jr} p_j \mathbf{n}_{jr} + \hat{\alpha}_{jr} (\mathbf{u}_j - \mathbf{u}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \frac{\sigma}{\varepsilon} B_r \right) \mathbf{u}_r - \sum_j l_{jr} p_j \mathbf{n}_{jr} - \sum_j \hat{\alpha}_{jr} \mathbf{u}_j = 0. \end{cases} \quad (7.115)$$

By definition $\|\mathbf{R}\|_{L^2(\Omega)} = \|\frac{d}{dt} \mathbf{u}_j\|_{L^2(\Omega)}$. Using the second line of (7.63), one has $\mathbf{u}_j = A_j^{-1} \sum_r \hat{\alpha}_{jr} \mathbf{u}_r$ and thus $\|\frac{d}{dt} \mathbf{u}_j\|_{L^2(\Omega)} \leq C \|\frac{d}{dt} \mathbf{u}_r\|_{L^2(\Omega)}$. Using (7.111) (and taking care that rescaling (7.84) by a factor ε was systematically used in the previous section), one gets for a smooth initial data

$$\|\mathbf{R}\|_{L^2([0,T] \times \Omega)} \leq C\varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) \|p_0\|_{H^3(\Omega)}.$$

We denote by $e_j = p_j - p_j^\varepsilon$, $\mathbf{f}_j = \mathbf{u}_j - \mathbf{u}_j^\varepsilon$ and $\mathbf{f}_r = \mathbf{u}_r - \mathbf{u}_r^\varepsilon$. One finds, making the difference between the schemes (7.115) and (7.57):

$$\begin{cases} |\Omega_j| \frac{d}{dt} e_j + \frac{1}{\varepsilon} \sum_r (l_{jr} \mathbf{n}_{jr}, \mathbf{f}_r) = 0, \\ |\Omega_j| \frac{d}{dt} \mathbf{f}_j + \frac{1}{\varepsilon} \sum_r (l_{jr} e_j \mathbf{n}_{jr} + \hat{\alpha}_{jr} (\mathbf{f}_j - \mathbf{f}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \frac{\sigma}{\varepsilon} B_r \right) \mathbf{f}_r - \sum_j l_{jr} e_j \mathbf{n}_{jr} - \sum_j \hat{\alpha}_{jr} \mathbf{f}_j = 0. \end{cases}$$

We are going to write an inequality satisfied by $E(t) = \|e(t)\|_{L^2(\Omega)}^2 + \|\mathbf{f}(t)\|_{L^2(\Omega)}^2$, knowing that $e(0) = 0$. Using the same kind of proof than for the L^2 stability of the JL-(b) scheme (proposition 7.17), one can show that

$$\frac{1}{2} \frac{d}{dt} E(t) \leq \sum_j |\Omega_j| (\mathbf{R}_j, \mathbf{f}_j) \leq \|\mathbf{f}\|_{L^2(\Omega)} \|\mathbf{R}\|_{L^2(\Omega)} \leq \sqrt{E(t)} \|\mathbf{R}\|_{L^2(\Omega)}.$$

By integration, one has for $t \leq T$

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \sqrt{T} \|\mathbf{R}\|_{L^2([0,T] \times \Omega)} = \|\mathbf{f}(0)\|_{L^2(\Omega)} + \sqrt{T} \|\mathbf{R}\|_{L^2([0,T] \times \Omega)}.$$

One has $\|\mathbf{f}(0)\|_{L^2(\Omega)} \leq C\sqrt{h\varepsilon} \max(h, \varepsilon) \|p_0\|_{H^3(\Omega)}$ by virtue of (7.108) (taking care that there is a rescaling (7.84) by ε). We simplify a little $\|\mathbf{f}(0)\|_{L^2(\Omega)} \leq C(h^2 + \varepsilon^2) \|p_0\|_{H^3(\Omega)}$, so

$$\sqrt{E(t)} \leq C \left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) \sqrt{T} \right) \|p_0\|_{H^3(\Omega)}, \quad C > 0.$$

Since $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T] \times \Omega)} = \sqrt{\int_0^T E(t) dt}$, the proof is ended with

$$C^\rightarrow = CT \left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) \right) \|p_0\|_{H^3(\Omega)}$$

□

7.3.8 Space estimate for uniform AP property in 2D

We have the following result of uniform convergence for a mesh satisfying the geometrical assumptions (7.13).

Theorem 7.26 (Space estimate). *There exists a constant C_0 independent of h and ε , increasing at most as $T^{\frac{3}{2}}$, such that the following estimate holds*

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T] \times \Omega)} \leq Ch^{\frac{1}{4}} \|p_0\|_{H^4(\Omega)}.$$

Proof. The proof is a slight adaptation of our initial proposition 7.2, where we use the norm $\|\cdot\| = \|\cdot\|_{L^2([0,T] \times \Omega)}$. From the triangular inequality applied to the AP diagram, one has

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| \leq \min(\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{\text{naive}}, \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\| + \|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\| + \|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\|).$$

All these norms are estimated with (7.75), (7.114), (7.83) and (7.74). Therefore one can write

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, \left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right) + (h + \varepsilon) + \varepsilon\right) \|p_0\|_{H^4(\Omega)}, \quad C > 0,$$

where

$$C = \max\left[\frac{\downarrow C}{\sqrt{C_M}}, \frac{C \rightarrow}{C_M}, \frac{C \downarrow}{\sqrt{C_M}}, C \leftarrow\right]$$

and behaves as $T^{\frac{3}{2}}$ for large T .

The parenthesis is

$$\begin{aligned} \mathcal{Z} &= \min\left(\sqrt{\frac{h}{\varepsilon}}, \left(h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)\right) + (h + \varepsilon) + \varepsilon\right) \\ &\leq \min\left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + 2h + 2\varepsilon\right) \leq \min\left(\sqrt{\frac{h}{\varepsilon}}, 3\varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + 2h\right). \end{aligned}$$

As in proposition 7.2, a threshold value is obtained by equating the more singular terms, that is $\sqrt{\frac{h}{\varepsilon_{\text{thresh}}}} = \varepsilon_{\text{thresh}} \sqrt{\frac{\varepsilon_{\text{thresh}}}{h}}$, with solution $\varepsilon_{\text{thresh}} = \sqrt{h}$. Two case occur. The first case is $\varepsilon \geq \varepsilon_{\text{thresh}}$. Then the first term in \mathcal{Z} shows that $\mathcal{Z} \leq \sqrt{\frac{h}{\varepsilon_{\text{thresh}}}} = h^{\frac{1}{4}}$. The second case is $\varepsilon \leq \varepsilon_{\text{thresh}}$. Then the second term in \mathcal{Z} shows that $\mathcal{Z} \leq 3\varepsilon_{\text{thresh}} \sqrt{\frac{\varepsilon_{\text{thresh}}}{h}} + 2h = 3h^{\frac{1}{4}} + 2h \leq 5h^{\frac{1}{4}}$. In both case $\mathcal{Z} \leq Ch^{\frac{1}{4}}$. The proof is ended. \square

7.4 Implicit discretization and proof of theorem 7.1

We explain hereafter how to compare the implicit scheme and the semi-discrete scheme, in a way that produces immediately abstract error bounds. This technique comes from [DES04] where applications to the numerical analysis of explicit schemes was the main goal. In what follows we concentrate on implicit Euler discretization for two reasons. First reason is that the theory is a little simpler to explain than for the explicit scheme, for which the interested reader may nevertheless refer to the cited work. The very simple proof that is provided is new. Second reason is that implicit discretization is somehow necessary to take into the account the intrinsic stiffness of the problem. In particular the numerical tests have been performed with the implicit method. With the explicit method the CFL condition is so restrictive that it makes impossible the convergence study. The proof is a consequence of the abstract estimate (7.120) with the technical estimate (7.126) for the initial data.

7.4.1 An abstract estimate

The idea is to compare the solution $U_h(t)$ of a semi-discrete scheme

$$U_h(t) = A_h U_h(t), \quad U_h(0) = U_h^{\text{ini}} \quad (7.116)$$

with the solution of the corresponding implicit Euler scheme with time step Δt

$$\frac{U_h^{n+1} - U_h^n}{\Delta t} = A_h U_h^{n+1}, \quad U_h^0 = U_h^{\text{ini}} \quad (7.117)$$

The operator depends on an abstract parameter h : with symbolic notation, this abstract parameter is $h \leftarrow (h, \varepsilon)$ in the case of our problem \mathbf{P}_h^ε . The question is to bound the difference of these two uniformly with respect to Δt and uniformly with respect to the abstract parameter h .

We assume a natural L^2 norm denoted as $\|\cdot\|$ with the associated scalar product. For simplicity we also assume that A_h is dissipative in the sense that

$$(U_h, A_h U_h) \leq 0 \quad \text{for all } U_h \text{ in the appropriate discrete space.}$$

Taking the scalar product of (7.117) with U_h^{n+1} , one deduces that $\|U_h^{n+1}\| \leq \|U_h^n\|$ for all U_h^n . Assuming the discrete space is finite (this is always true for discrete methods in a compact domain), one gets the unconditional stability estimate

$$\|(I_h - \Delta t A_h)^{-1}\| \leq 1 \quad \forall \Delta t > 0 \quad (7.118)$$

where I_h is the discrete identity operator and the norm is the induced one for operators. Note that (7.118) ultimately shows that the matrix $I_h - \Delta t A_h$ is non singular. So the matrix of the problem can be assembled and inverted on a computer.

Let us define for convenience $V_h^n = U_h(n\Delta t)$ so that the semi-discrete scheme can be rewritten as

$$\begin{aligned} \frac{V_h^{n+1} - V_h^n}{\Delta t} - A_h V_h^{n+1} &= \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} U_h'(s) ds - A_h U_h((n+1)\Delta t) \\ &= \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} A_h U_h(s) ds - A_h U_h((n+1)\Delta t) = \Delta t A_h s_h^{n+1} \end{aligned}$$

where the residual is $s_h^{n+1} = \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \frac{U_h(s) - U_h((n+1)\Delta t)}{\Delta t} ds$. We notice that

$$\|s_h^n\| \leq \sup_{0 \leq s \leq T} \|U_h'(s)\| \leq \|A_h U_h^{\text{ini}}\|, \quad n\Delta t \leq T. \quad (7.119)$$

Therefore this special residual is uniformly bounded provided $\|A_h U_h^{\text{ini}}\|$ is uniformly bounded. This is actually true: it comes from the fact that $W_h(t) = U_h'(t)$ is solution of $W_h'(t) = A_h W_h(t)$ and $W_h(0) = A_h U_h^{\text{ini}}$. So the strong L^2 stability of the semi-discrete scheme due to (7.121) yields the bound (7.119).

Proposition 7.27 (Time estimate). *Let $T > 0$ be a final time. Then there exists a constant C independent of h , ε and Δt , proportional to \sqrt{T} , such that*

$$\|U_h^n - U_h(n\Delta t)\| \leq C\sqrt{\Delta t} \|A_h U_h^{\text{ini}}\|, \quad n\Delta t \leq T. \quad (7.120)$$

Proof. The initial data is the same $V_h^0 = U_h^{\text{ini}}$. Let us define the error $E_h^n = V_h^n - U_h^n$ which is solution of

$$\frac{E_h^{n+1} - E_h^n}{\Delta t} = A_h E_h^{n+1} + \Delta t A_h s_h^{n+1}, \quad E_h^0 = 0. \quad (7.121)$$

It yields $(I_h - \Delta t A_h) E_h^{n+1} = E_h^n + \Delta t^2 A_h s_h^{n+1}$, that is $E_h^{n+1} = (I_h - \Delta t A_h)^{-1} E_h^n + \Delta t (I_h - \Delta t A_h)^{-1} \Delta t A_h s_h^{n+1}$. We obtain the representation formula (discrete Duhamel's formula)

$$E_h^n = \Delta t \sum_{p=0}^{n-1} [(I_h - \Delta t A_h)^{-1}]^{n-1-p} (I_h - \Delta t A_h)^{-1} \Delta t A_h s_h^{p+1}. \quad (7.122)$$

Let us define the operator $T_h = (I_h - \Delta t A_h)^{-1}$ which is bounded $\|T_h\| \leq 1$. One has the formula $T_h - I_h = (I_h - \Delta t A_h)^{-1} \Delta t A_h$ and the formula $(I_h - \frac{\Delta t}{2} A_h)^{-1} \frac{I_h + T_h}{2} = (I_h - \Delta t A_h)^{-1}$. Plugging in the discrete Duhamel's formula, one obtains another representation

$$E_h^n = \Delta t \sum_{p=0}^{n-1} \left[(I_h - \frac{\Delta t}{2} A_h)^{-(n-1-p)} \right] \left[\left(\frac{I_h + T_h}{2} \right)^{n-1-p} (T_h - I_h) \right] s_h^{p+1}. \quad (7.123)$$

The first operator in brackets is bounded by 1 due to the stability (7.118). On the other hand it is an easy exercise in number theory to show that for $q \geq 0$

$$\left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) = \frac{1}{2^q} \sum_r \left(\binom{q}{r-1} - \binom{q}{r} \right) T_h^r$$

where the binomial coefficients are $\binom{q}{r} = \frac{q!}{r!(q-r)!}$ for $0 \leq r \leq q$, otherwise zero. Therefore

$$\left\| \left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) \right\| \leq \frac{1}{2^q} \sum_r \left| \binom{q}{r-1} - \binom{q}{r} \right| \leq \frac{1}{2^q} 2 \binom{q}{r_*}$$

where the last inequality is from a telescoping reasoning and r_* is one of the closest entire number to $q/2$, that is $|\frac{q}{2} - r_*| \leq 1$. But there exists a universal constant, denoted K , such that $\frac{1}{2^{q-1}} \binom{q}{r_*} \leq \frac{K}{\sqrt{q+1}}$. Therefore $\left\| \left(\frac{I_h + T_h}{2} \right)^q (T_h - I_h) \right\| \leq K/\sqrt{q+1}$. Using this universal estimate in (7.123) and the estimate on s_h^n , we obtain $\|E_h^n\| \leq \Delta t \sum_{p=0}^{n-1} \frac{K}{\sqrt{n-1-p+1}} \|A_h U_h^{\text{ini}}\| = \Delta t \sum_{p=1}^n \frac{K}{\sqrt{p}} \|A_h U_h^{\text{ini}}\|$. A basic bound shows that $\sum_{p=1}^n \frac{1}{\sqrt{p}} \leq \tilde{K} \sqrt{n}$. Therefore

$$\|E_h^n\| \leq \Delta t K \tilde{K} \sqrt{n} \|A_h U_h^{\text{ini}}\| \leq (K \tilde{K} \sqrt{T}) \sqrt{\Delta t} \|A_h U_h^{\text{ini}}\|, \quad n \Delta t \leq T.$$

The proof is ended. □

To finish the proof of the theorem 7.1, it is now necessary and sufficient to show that $\|\frac{d}{dt} U_h(0)\| = \|A_h U_h^{\text{ini}}\|$ is bounded independently of h for the initial data of \mathbf{P}_h^ε . This is the purpose of the next section.

7.4.2 Technical estimates

To prove the uniform on the initial data, we will use in a slightly different manner the estimates for the initial data that have been obtained for the diffusion approximation scheme $\mathbf{DA}_h^\varepsilon$. However we will need an additional assumption of the mesh

$$(A_r \mathbf{u}, \mathbf{u}) \geq \alpha h(\mathbf{u}, \mathbf{u}), \quad A_r = \sum_j \hat{\alpha}_{jr}. \quad (7.124)$$

This assumption is not restrictive so we do not comment on it. The following technical estimates show two things. First it explains in what sense the corner velocity is a good approximation of the gradient at the corner at initial stage. Second it provides in (7.126) a control of the time derivative at time $t = 0$ uniformly with respect to h and ε , it immediately shows the boundedness of the abstract quantity $A_h U_h^{\text{ini}}$ in (7.119). So it is possible to apply the above proposition and the main theorem is proved. We now turn to the proof the technical estimates.

Proposition 7.28. *There exists a constant C independent of h and ε such that the initial data of \mathbf{P}_h^ε satisfies*

$$\left\| \mathbf{u}_r^\varepsilon(0) + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right\|_{L^2(\Omega)} \leq Ch\varepsilon \|p_0\|_{H^3(\Omega)}. \quad (7.125)$$

Proof. The corner problem (7.59) that defines $\mathbf{u}_r = \mathbf{u}_r^\varepsilon(0)$ at initial time is rewritten as

$$\left(A_r + \frac{\sigma}{\varepsilon} B_r \right) \mathbf{u}_r = \sum_j l_{jr} p_0(\mathbf{x}_j) \mathbf{n}_{jr} - \frac{\varepsilon}{\sigma} \sum_j \hat{\alpha}_{jr} \nabla p_0(\mathbf{x}_j).$$

Let us defined $\mathbf{d}_r^1 = \sum_j l_{jr} (p_0(\mathbf{x}_j) - p_0(\mathbf{x}_r) - (\mathbf{x}_j - \mathbf{x}_r, \nabla p_0(\mathbf{x}_r))) \mathbf{n}_{jr}$, already defined and bounded in (7.109). So elimination of $p(\mathbf{x}_j)$ and simplification with $\sum_j l_{jr} \mathbf{n}_{jr} p(\mathbf{x}_r) = 0$ yield

$$\begin{aligned} \left(A_r + \frac{\sigma}{\varepsilon} B_r \right) \mathbf{u}_r &= \sum_j l_{jr} (\mathbf{x}_r - \mathbf{x}_j, \nabla p_0(\mathbf{x}_r)) \mathbf{n}_{jr} + \mathbf{d}_r^1 \\ &\quad - \frac{\varepsilon}{\sigma} \sum_j \hat{\alpha}_{jr} \nabla p_0(\mathbf{x}_r) + \frac{\varepsilon}{\sigma} \sum_j \hat{\alpha}_{jr} (\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)). \end{aligned}$$

that is with the definition of the matrices

$$\left(A_r + \frac{\sigma}{\varepsilon} B_r \right) \left(\mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right) = \mathbf{d}_r^1 + \frac{\varepsilon}{\sigma} \sum_j \hat{\alpha}_{jr} (\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)).$$

The coercivity (7.124) of the matrices A_r and B_r yields

$$\alpha \left(h + \frac{\sigma h^2}{\varepsilon} \right) \left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq |\mathbf{d}_r^1| + \frac{\varepsilon}{\sigma} \sum_j \hat{\alpha}_{jr} |\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)|.$$

With estimate of \mathbf{d}_r^1 (7.109), estimate of the difference $\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j)$, it yields

$$\alpha \left(h + \frac{\sigma h^2}{\varepsilon} \right) \left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq C(h^2 + \frac{\varepsilon}{\sigma} h) \|p\|_{\Omega_j},$$

with a constant uniform with respect to h , ε and the index of the cell j . That is

$$\left| \mathbf{u}_r + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r) \right| \leq \frac{C}{\alpha} \varepsilon \|p\|_{\Omega_j}.$$

After squaring and summation with respect to j , it yields the result. \square

Proposition 7.29. *There exists a constant $C > 0$ which do not depend on h and ε such that the initial data of \mathbf{P}_h^ε satisfies*

$$\left\| \frac{d}{dt} \mathbf{V}_h^\varepsilon \right\|_{L^2(\Omega)} \leq C \|p_0\|_{H^3(\Omega)}. \quad (7.126)$$

Proof. The \mathbf{P}_h^ε scheme (7.57) or (7.61) can be rewritten

$$\mathbf{P}_h^\varepsilon : \quad \begin{cases} \frac{d}{dt} p_j^\varepsilon &= -\frac{1}{\varepsilon|\Omega_j|} \sum_r l_{jr} (\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon, \mathbf{n}_{jr}) \\ \frac{d}{dt} \mathbf{u}_j^\varepsilon &= -\frac{1}{\varepsilon|\Omega_j|} \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon) \mathbf{n}_{jr}. \end{cases} \quad (7.127)$$

At time $t = 0$ one has $\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon = (\mathbf{u}_r^\varepsilon + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r)) + \frac{\varepsilon}{\sigma} (\nabla p_0(\mathbf{x}_j) - \nabla p_0(\mathbf{x}_r))$: the first term can be estimated by (7.125) and the second one as usual. Therefore there exists constants such that

$$\left\| \frac{1}{\varepsilon|\Omega_j|} \sum_r l_{jr} (\mathbf{u}_r^\varepsilon + \frac{\varepsilon}{\sigma} \nabla p_0(\mathbf{x}_r), \mathbf{n}_{jr}) \right\|_{L^2(\Omega)} \leq C \frac{h}{C_{\mathcal{M}} h^2} P h \varepsilon \|p_0\|_{H^3(\Omega)} \leq \widehat{C} \|p_0\|_{H^3(\Omega)}.$$

In a similar way

$$\frac{1}{\sigma|\Omega_j|} \left| \sum_r l_{jr} (\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j), \mathbf{n}_{jr}) \right| \leq \frac{h}{\sigma C_{\mathcal{M}} h^2} \widetilde{C} \|p_0\|_{H^3(\Omega_j)} \leq \frac{\overline{C}}{h} \|p_0\|_{H^3(\Omega_j)}.$$

Therefore

$$\left\| \frac{1}{\sigma|\Omega_j|} \sum_r l_{jr} (\nabla p_0(\mathbf{x}_r) - \nabla p_0(\mathbf{x}_j), \mathbf{n}_{jr}) \right\|_{L^2(\Omega)} \leq \overline{C} \|p_0\|_{H^3(\Omega)}.$$

It shows that $\left\| \frac{d}{dt} p^\varepsilon(0) \right\|_{L^2(\Omega)} \leq C \|p_0\|_{H^3(\Omega_j)}$. Considering (7.127), a similar result for $\frac{d}{dt} \mathbf{u}^\varepsilon(0)$. It shows $\left\| \frac{d}{dt} \mathbf{V}_h^\varepsilon(0) \right\|_{L^2(\Omega)} \leq C \|p_0\|_{H^3(\Omega)}$. The proof is ended. \square

7.5 Numerical illustration

To illustrate the theory and have a more quantitative version of the error estimates studied in this work, we consider the academic square $\Omega = [0, 1]^2$ and discretize the hyperbolic heat equation of a mesh made with random quads. A random quad mesh is made of quads where the vertices are moved randomly around their initial position, by a factor between 10% and 30%. We use the fully implicit time discretization version of the 2D scheme detailed in this work. The solution of the linear systems is computed via an iterative GMRES algorithm, which converges smoothly in our numerical experiments. The reference analytical solution used in our tests is designed by separation of variables. A solution of (7.1) is

$$p = f + \frac{\varepsilon^2}{\sigma} \partial_t f \quad \text{and} \quad \mathbf{u} = -\frac{\varepsilon}{\sigma} \nabla f,$$

with f solution of

$$\partial_t f + \frac{\varepsilon^2}{\sigma} \partial_t^2 f - \frac{1}{\sigma} \Delta f = 0. \quad (7.128)$$

Figure 7.6: The error is plotted in log scale versus the number of cells per direction for the test problem described in section 7.5. Each curve corresponds to a value of $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$, plus a reference line for order one. One sees that the order of convergence is an increasing function of τ .

We propose to construct a solution for a subset of small ε to validate the uniform convergence. Firstly we consider that the solution is a periodic solution on the square $[0, 2] \times [0, \frac{2}{L}]$. For this we use the separation of the variables. We consider the following function

$$f(t, x, y) = \alpha(t) \cos(L\pi x) \cos(L\pi y).$$

and we propose to find the function $\alpha(t)$ such that $f(t, x, y) = \alpha(t) \cos(L\pi x) \cos(L\pi y)$ is a periodic solution of (7.128). The function α is determined as the solution of

$$\alpha'(t) + \frac{\varepsilon^2}{\sigma} \alpha''(t) + \frac{2L^2\pi^2}{\sigma} \alpha(t) = 0$$

with $\alpha'(0) = 0$ and $\alpha(0) = 1$. For small ε , which is the case we are interested in, the solution is computed as follows. First determine

$$\lambda_1 = -\frac{\sigma \left(\sqrt{1 - \frac{\varepsilon^2}{\sigma^2} 8L^2\pi^2} + 1 \right)}{2\varepsilon^2} \quad \text{and} \quad \lambda_2 = \frac{\sigma \left(\sqrt{1 - \frac{\varepsilon^2}{\sigma^2} 8L^2\pi^2} - 1 \right)}{2\varepsilon^2}.$$

Then

$$\alpha(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t}$$

from which $p(t)$ and $\mathbf{u}(t)$ are easily recovered.

We decide that an exact relation is enforced between ε and $h = \frac{1}{N}$, so that the error can be expressed as a function of h solely. The relation between ε and h writes $\varepsilon = 0.01(40h)^\tau$ for $\tau \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$. The error between the exact solution and the numerical solution is computed numerically in function of $h = \frac{1}{N}$, for different values of τ , and the results of some of these numerical experiments is displayed in figure 7.6. The results correspond to the time $T = 0.02$ using the time step $\Delta t = 0.2h^2$.

As predicted by the theory, the scheme is uniformly AP and the error behavior is a continuous function of γ between the hyperbolic and parabolic limits. However the results are much better, in the sense the order is greater than the theoretical prediction since the order is approximatively 1 for $\gamma = 0$ (hyperbolic limit) and 2 for $\gamma = 2$ (parabolic regime). We can find a closed result on the second order convergence for the parabolic regime in the paper [ABN12] (1D linear problem). The reason is probably that the theory is based on worst case estimates, as it is often the case for the numerical analysis of finite volume schemes [EGH00].

7.6 Conclusion

The proof that was given of the uniform AP property is quite technical. It relies on specific hyperbolic and parabolic estimates for linear nodal finite volume schemes on general meshes. We observe that the multidimensional case yields an additional contribution in the error that ultimately slightly degrades the convergence rate. It is an open problem to determine if these inequalities are optimal. The numerical results indicate that it is probably not the case.

7.7 Appendix A: detailed proof of the naive estimate (7.75)

Our aim is to now examine each term in the right hand side of the dissipative identity (7.82). Its first term is already non positive.

Proposition 7.30. *Let $\gamma > 0$ be a number whose precise value will be determined further. There exists a constant $C_1(\gamma)$ which depends on γ such that one has the bound for the second term of the dissipative identity (7.82)*

$$\int_0^T E_1(t) dt \leq \frac{\gamma}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + C_1(\gamma) \frac{h}{\varepsilon \sqrt{C_{\mathcal{M}}}} \|\mathbf{V}^\varepsilon(0)\|_{H^1(\Omega)}^2. \quad (7.129)$$

Proof. We use a Young's inequality $ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2$, with some positive constant γ which will be defined later, for the second term and the definition of the fluxes (7.58) for the third term: we get

$$\begin{aligned} E_1 &\leq \frac{\gamma}{2\varepsilon} \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon} \sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \\ &\quad + \sum_j \sum_r l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)(\mathbf{n}_{j,r}, \delta_{j,r}(\mathbf{u}^\varepsilon)) - \frac{1}{\varepsilon} \sum_j \sum_r \left(\frac{\sigma}{\varepsilon} \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \delta_{j,r}(\mathbf{u}^\varepsilon) \right) \end{aligned}$$

Another use of Young's inequality with the same coefficient γ for the third term yields

$$\begin{aligned} E_1 &\leq \frac{\gamma}{\varepsilon} \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon} \sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \\ &\quad + \frac{1}{2\gamma\varepsilon} \sum_j \sum_r l_{j,r} |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 - \frac{1}{\varepsilon} \sum_j \sum_r \left(\frac{\sigma}{\varepsilon} \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \delta_{j,r}(\mathbf{u}^\varepsilon) \right) \\ &\leq \frac{\gamma}{\varepsilon} \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon} \sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \\ &\quad + \frac{h}{2\gamma\varepsilon} \sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 - \frac{1}{\varepsilon} \sum_j \sum_r \left(\frac{\sigma}{\varepsilon} \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \delta_{j,r}(\mathbf{u}^\varepsilon) \right). \end{aligned}$$

We now look at the last term of this inequality $W = -\frac{1}{\varepsilon} \sum_j \sum_r \left(\frac{\sigma}{\varepsilon} \widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \delta_{j,r}(\mathbf{u}^\varepsilon) \right)$. By definition (7.60) of $\widehat{\beta}_{j,r}$, one has $|\widehat{\beta}_{j,r}| \leq h^2$. Therefore

$$\begin{aligned} |W| &\leq \frac{\sigma h^2}{\varepsilon^2} \left(\sum_j \sum_r |\mathbf{u}_r^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\sigma h^2}{\varepsilon^2} \sqrt{P} \left(\sum_r |\mathbf{u}_r^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 \right)^{\frac{1}{2}} \leq \frac{\sigma h}{\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \left(\sum_r |V_r| |\mathbf{u}_r^\varepsilon|^2 \right)^{\frac{1}{2}} \left(\sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \left(\sum_r |V_r| (\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 \right). \end{aligned}$$

It yields

$$\begin{aligned}
E_1 &\leq \frac{\gamma}{\varepsilon} \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{h}{2\gamma\varepsilon} \sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \\
&\quad + \left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \right) \sum_j \sum_r |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \sum_r |V_r|(\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon).
\end{aligned} \tag{7.130}$$

Using the first interpolation result of proposition 7.14 and the assumption (7.65), one has

$$\sum_j \sum_r \delta_{j,r}(p^\varepsilon)^2 \leq PC_{\mathcal{A}}^2 \|p^\varepsilon\|_{H^2(\Omega)}^2 \quad \text{and} \quad \sum_j \sum_r |\delta_{j,r}\mathbf{u}^\varepsilon|^2 \leq PC_{\mathcal{A}}^2 \|\mathbf{u}^\varepsilon\|_{H^2(\Omega)}^2.$$

So we obtain

$$\begin{aligned}
\int_0^T E_1 dt &\leq \frac{\gamma}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 dt + \frac{PC_{\mathcal{A}}^2 h}{2\gamma\varepsilon} \|p^\varepsilon\|_{L^2([0,T];H^2(\Omega))}^2 \\
&\quad + PC_{\mathcal{A}}^2 \left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \right) \|\mathbf{u}^\varepsilon\|_{L^2([0,T];H^2(\Omega))}^2 + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \|\mathbf{u}_r^\varepsilon\|_{L^2([0,T]\times\Omega)}^2.
\end{aligned}$$

Using energy estimate (7.2) for the the second term of the rhs of the above inequality, (7.3) for the third term and (7.76) for the last term, one gets finally

$$\begin{aligned}
\int_0^T E_1(t) dt &\leq \frac{\gamma}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \\
&\quad + \left(T \frac{PC_{\mathcal{A}}^2 h}{2\gamma\varepsilon} + PC_{\mathcal{A}}^2 \left(\frac{h}{2\gamma\varepsilon} + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \right) \frac{\varepsilon^2}{\sigma} + \frac{\sigma h}{2\varepsilon^2} \sqrt{\frac{P}{C_{\mathcal{M}}}} \frac{\varepsilon^2}{\sigma\alpha} \right) \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}^2.
\end{aligned} \tag{7.131}$$

After a convenient definition of the constant $C_1(\gamma)$, it ends the proof. \square

Proposition 7.31. *There exists a constant C_2 such that the third term in the dissipative identity (7.82) can be bounded as*

$$\int_0^T E_2(t) dt \leq C_2 \frac{h}{\varepsilon C_{\mathcal{M}}} \|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)}^2. \tag{7.132}$$

Proof. We decompose E_2 in (7.82) in two terms. Making use of the second set of inequalities of the proposition 7.14 and the assumptions (7.64) and (7.65), the first one can be bounded as

$$|A| = \left| \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)) \right| \leq \frac{C_{\mathcal{A}} P}{\varepsilon} h^2 \sum_j |p_j^\varepsilon| \|\mathbf{V}^\varepsilon(t)\|_{H^3(\Omega_j)}.$$

Using the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, it yields $|A| \leq \frac{C_{\mathcal{A}} P}{2\varepsilon} h^3 \sum_j |p_j^\varepsilon|^2 + \frac{C_{\mathcal{A}} P}{2\varepsilon} h \sum_j \|\mathbf{V}^\varepsilon(t)\|_{H^3(\Omega_j)}^2$. The assumption (7.67) yields

$$|A| \leq \frac{C_{\mathcal{A}} P}{2\varepsilon} \frac{h}{C_{\mathcal{M}}} \|\mathbf{V}_h^\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{C_{\mathcal{A}} P}{2\varepsilon} h \|\mathbf{V}^\varepsilon(t)\|_{H^3(\Omega)}^2.$$

The L^2 stability (7.76) of the scheme \mathbf{P}_h^ε shows that

$$\|\mathbf{V}_h^\varepsilon(t)\|_{L^2(\Omega)} \leq \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\Omega)} \leq \|\mathbf{V}^\varepsilon(0)\|_{L^2(\Omega)} + \|\mathbf{V}_h^\varepsilon(0) - \mathbf{V}^\varepsilon(0)\|_{L^2(\Omega)} \leq (1 + C_{\mathcal{A}h}) \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}$$

where the last inequality comes from the initialization stage (7.71). With the basic energy estimate (7.2), and since h is bounded, we obtain

$$\int_0^T |A| dt \leq T \left(\frac{C_{\mathcal{A}P}}{2\varepsilon} \frac{h}{C_{\mathcal{M}}} (1 + C_{\mathcal{A}h}) + \frac{C_{\mathcal{A}P}}{2\varepsilon} h \right) \|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)}^2.$$

The second contribution in E_2 is $B = \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \left(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \right)$. Almost the same calculations show the bound

$$\int_0^T |B| dt \leq T \left(\frac{C_{\mathcal{A}P}}{2\varepsilon} \frac{h}{C_{\mathcal{M}}} (1 + C_{\mathcal{A}h}) + \frac{C_{\mathcal{A}P}}{2\varepsilon} h \right) \|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)}^2.$$

Summing the two contributions, it concludes the proof after a convenient definition of C_2 . \square

Proposition 7.32. *Let $\hat{\gamma} > 0$ be a number which precise value will be determined further. There exists a constant $C_3(\hat{\gamma})$ which depends on $\hat{\gamma}$ such that one has the bound for the last term of the dissipative identity (7.82)*

$$\int_0^T E_3(t) dt \leq \frac{\hat{\gamma}\sigma}{2\varepsilon} \int_0^T \sum_r \sum_j l_{jr} \left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right)^2 dt + C_3(\hat{\gamma}) \frac{h}{\varepsilon C_{\mathcal{M}}} \|\mathbf{V}^\varepsilon(0)\|_{H^1(\Omega)}^2. \quad (7.133)$$

Proof. The definition of E_3 in (7.82) is

$$\begin{aligned} E_3 &= \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon \right) \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_j \int_{\Omega_j} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x} - \frac{\sigma}{\varepsilon^2} \sum_j \sum_r (\hat{\beta}_j \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon). \end{aligned}$$

Using the Cauchy-Schwarz inequality on the third term $\int (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon)$, one gets

$$\begin{aligned} E_3 &\leq \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) + \frac{\sigma}{\varepsilon^2} \sum_j \left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon \right) \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_j \frac{1}{|\Omega_j|} \left(\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right)^2 - \frac{\sigma}{\varepsilon^2} \sum_j \sum_r (\hat{\beta}_j \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon), \end{aligned}$$

which can be written

$$E_3 \leq -\frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) - \frac{\sigma}{\varepsilon^2} \sum_j \left(\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon \right)$$

that is

$$\begin{aligned} E_3 &\leq -\frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right), \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\hat{\beta}_{j,r} \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_j \left(\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon \right). \end{aligned}$$

One has, using the geometric identity $\sum_r \widehat{\beta}_{jr} = |\Omega_j| I_d$ which can be found in [BDF12-1, DES10],

$$\begin{aligned} & \sum_r \sum_j \left(\widehat{\beta}_{j,r} \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) + \sum_j \left(\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon \right) \\ &= \sum_r \sum_j \left(\widehat{\beta}_{j,r} \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right). \end{aligned}$$

We thus get after simplification

$$\begin{aligned} E_3 &\leq -\frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\widehat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right), \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \Bigg| := S_1 \\ &\quad - \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\widehat{\beta}_{j,r} \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right) \Bigg| := S_2 \end{aligned} \quad (7.134)$$

We add and subtract at each average on the cell the nodal value. We recall the notation $\delta_{j,r}(\mathbf{u}^\varepsilon) = \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}^\varepsilon(\mathbf{x}_r)$. We get for the term under the first sum in (7.134)

$$\begin{aligned} & \left(\widehat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right), \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \\ &= \left(\widehat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right) - \left(\widehat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right), \delta_{j,r}(\mathbf{u}^\varepsilon) \right) \\ &\quad - \left(\widehat{\beta}_{j,r} \delta_{j,r}(\mathbf{u}^\varepsilon), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right) + \left(\widehat{\beta}_{j,r} \delta_{j,r}(\mathbf{u}^\varepsilon), \delta_{j,r}(\mathbf{u}^\varepsilon) \right). \end{aligned} \quad (7.135)$$

The first of these quantities is purely nodal: one has after summation

$$\begin{aligned} & \sum_j \sum_r \left(\widehat{\beta}_{j,r} \left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right) \\ &= \sum_r \left(B_r \left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right), \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r) \right) \geq \alpha \sum_r |V_r| |\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)|^2 \end{aligned} \quad (7.136)$$

with the help of (7.70). The second and third term in the identity (7.135) can be bounded by a Young's inequality with a convenient constant $C = \frac{C_{\mathcal{M}} \alpha}{2P}$ so that all terms containing $\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)$ are controlled by (7.136). So we obtain concerning S_1 defined in (7.134)

$$S_1 \leq \left(1 + \frac{2P}{C_{\mathcal{M}} \alpha} \right) \frac{h^2 \sigma}{\varepsilon^2} \sum_r \sum_j |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2.$$

Using the first interpolation result stressed in proposition 7.14, one has in dimension two $|\delta_{j,r}(\mathbf{u}^\varepsilon)| \leq C_{\mathcal{A}} \|\mathbf{u}^\varepsilon(t)\|_{H^2(\Omega_j)}$. So, taking into account energy estimate (7.3) we have for the first term

$$\int_0^T S_1 dt \leq C_{\mathcal{A}}^2 P \left(1 + \frac{2P}{C_{\mathcal{M}} \alpha} \right) h^2 \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}^2.$$

We now consider the second term called S_2 in (7.134)

$$S_2 = -\frac{\sigma}{\varepsilon^2} \sum_r \sum_j \left(\widehat{\beta}_{j,r} \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right).$$

Using $(\vec{a} \otimes \vec{b}, \vec{c}, \vec{d}) = (\vec{b}, \vec{c})(\vec{a}, \vec{d})$, one has

$$S_2 = -\frac{\sigma}{\varepsilon^2} \sum_r \sum_j l_{jr} \left((\mathbf{x}_r - \mathbf{x}_j), \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right) \left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right)$$

Using the Young's inequality $ab \leq \frac{\hat{\gamma}\varepsilon}{2} a^2 + \frac{1}{2\hat{\gamma}\varepsilon} b^2$, we get

$$\int_0^T S_2 dt \leq \frac{\hat{\gamma}\sigma}{2\varepsilon} \int_0^T \sum_r \sum_j l_{jr} \left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right)^2 dt + \int_0^T \frac{\sigma}{2\hat{\gamma}\varepsilon^3} \sum_r \sum_j l_{jr} \left((\mathbf{x}_r - \mathbf{x}_j), \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \right)^2 dt$$

Using one more time the energy estimate (7.3) the second term in the right hand side of the above inequality is bounded by $\frac{Ph}{2C_{\mathcal{M}}\hat{\gamma}\varepsilon} \|\mathbf{V}^\varepsilon(0)\|_{L^2(\Omega)}^2$. Thus

$$\int_0^T E_3(t) dt \leq \frac{\hat{\gamma}\sigma}{2\varepsilon} \int_0^T \sum_r \sum_j l_{jr} \left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right)^2 dt + P \left\{ C_{\mathcal{A}}^2 \left(1 + \frac{2P}{C_{\mathcal{M}}\alpha} \right) h^2 + \frac{h}{2C_{\mathcal{M}}\hat{\gamma}\varepsilon} \right\} \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}^2,$$

which is the expected result after convenient redefinition of the constant in front of the last term. \square

End of the proof of the naive estimate of proposition (7.16). One gets

$$\mathcal{E}(T) \leq \mathcal{E}(0) - \frac{1}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \int_0^T E_1(t) dt + \int_0^T E_2(t) dt + \int_0^T E_3(t) dt$$

where integrals are estimated in (7.129), (7.132) and (7.133). Using equation (7.71), one finds

$$\begin{aligned} \mathcal{E}(T) &\leq C_0^2 h^2 \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}^2 \\ &\quad - \frac{1}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \\ &\quad + \frac{\gamma}{\varepsilon} \int_0^T \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + C_1(\gamma) \frac{h}{\varepsilon\sqrt{C_{\mathcal{M}}}} \|\mathbf{V}^\varepsilon(0)\|_{H^1(\Omega)}^2 \\ &\quad + C_2 \frac{h}{\varepsilon C_{\mathcal{M}}} \|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)}^2 \\ &\quad + \frac{\hat{\gamma}\sigma}{2\varepsilon} \int_0^T \sum_r \sum_j l_{jr} \left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon \right)^2 dt + C_3(\hat{\gamma}) \frac{h}{\varepsilon C_{\mathcal{M}}} \|\mathbf{V}^\varepsilon(0)\|_{H^2(\Omega)}^2. \end{aligned}$$

This estimate is fundamental, since it shows the competition between different kind of error terms and the dissipation of the fluxes. Choosing by example $\hat{\gamma} < \frac{1}{\sigma}$ and $\gamma < \frac{1}{2}$, all terms like $\int_0^T \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2$ vanish. All other terms can put together as $\mathcal{E}(T) \leq \frac{1}{2} \frac{C}{\varepsilon} h \|p_0\|_{H^4(\Omega)}^2$, where the constant $\downarrow C$ has, as in 1D, has at most a linear growth in time. It ends the proof of the naive estimate. \square

7.8 Appendix B: Bihari's inequality and application

We recall a nonlinear generalization of the Gronwall-Bellman inequality known as Bihari's inequality

Lemma 7.33. *If*

$$y(t) \leq a + \int_0^t b(s)g(y(s))ds, \quad (7.137)$$

with a non negative constante, $b(t)$ a positive function and g a positive non decreasing function then, noting by $G(x)$ an antiderivative of $1/g(x)$, one has

$$y(t) \leq G^{-1} \left(G(a) + \int_0^t b(s)ds \right). \quad (7.138)$$

The proof is trivial by setting $Z = a + \int_0^t b(s)g(y(s))ds$ and and verifying that $Z' \leq bg(Z)$, see [BI56]. In our work $g(x) = \sqrt{x}$. Moreover a , $b(s)^2$ and y are square of $L^2(\Omega)$ norms. More precisely in our convergence's proofs one ends to inequality of the type

$$\|Y\|_{L^2(\Omega)}^2(t) \leq \|Y\|_{L^2(\Omega)}^2(0) + \int_0^t \|A\|_{L^2(\Omega)}^2 + \int_0^t \|B\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)}ds. \quad (7.139)$$

for Y , A and B functions of $L^2(\Omega)$. Thus for all $t \leq T$

$$\|Y\|_{L^2(\Omega)}^2(t) \leq \|Y\|_{L^2(\Omega)}^2(0) + \int_0^T \|A\|_{L^2(\Omega)}^2 + \int_0^t \|B\|_{L^2(\Omega)}\|Y\|_{L^2(\Omega)}ds, \quad (7.140)$$

Using the Bihari's inequality (7.138) and the Cauchy-Schwarz inequality one obtain for all $t \leq T$,

$$\|Y\|_{L^2(\Omega)}^2(t) \leq \frac{1}{2} \left(2\sqrt{\|Y\|_{L^2(\Omega)}^2(0) + \|A\|_{L^2([0,T] \times \Omega)}^2} + \sqrt{t} \sqrt{\int_0^t \|B\|_{L^2([0,T] \times \Omega)}^2 ds} \right)^2, \quad (7.141)$$

and majorizing t by T in the right-hand side

$$\|Y\|_{L^2(\Omega)}^2(t) \leq \frac{1}{2} \left(2\sqrt{\|Y\|_{L^2(\Omega)}^2(0) + \|A\|_{L^2([0,T] \times \Omega)}^2} + \sqrt{T}\|B\|_{L^2([0,T] \times \Omega)} \right)^2. \quad (7.142)$$

Integrating in time that gives

$$\|Y\|_{L^2([0,T] \times \Omega)}^2 \leq \frac{1}{2}T \left(2\sqrt{\|Y\|_{L^2(\Omega)}^2(0) + \|A\|_{L^2([0,T] \times \Omega)}^2} + \sqrt{T}\|B\|_{L^2([0,T] \times \Omega)} \right)^2 \quad (7.143)$$

We can summarize these calculations by the lemma

Lemma 7.34. *If Y , A and B are functions of $L^2(\Omega)$ satisfying (7.139) then*

$$\|Y\|_{L^2([0,T] \times \Omega)} \leq \sqrt{\frac{T}{2}} \left(2\sqrt{\|Y\|_{L^2(\Omega)}^2(0) + \|A\|_{L^2([0,T] \times \Omega)}^2} + \sqrt{T}\|B\|_{L^2([0,T] \times \Omega)} \right) \quad (7.144)$$

If $\|A\|_{L^2(\Omega)} \leq C$ et $\|B\|_{L^2(\Omega)} \leq C$, with C constant then the right-hand side behaves as $T^{\frac{3}{2}}$ for large time. If $\|A\|_{L^2(\Omega)} \leq C$ or $\int_0^T \|A\|_{L^2(\Omega)} \leq C$ and $\int_0^T \|B\|_{L^2(\Omega)} \leq C$, then the right-hand side behaves now as T for large time.

Conclusion and prospects

We conclude this study with a summary of the major results and the possible prospects of this work.

First, the proof of convergence of the solution of the relativistic transfer equation to the solution of a drift diffusion equation given in chapter 2 is original. In particular it involves a comparison principle (theorem 2.1) that generalizes the maximum principle of Golse and Perthame [GP86] for a moving fluid. The proof of this convergence result assumes that the opacity σ_a is regular (belongs to $W^{3,\infty}(\mathbb{R}_\nu^+) \cap L^2(\mathbb{R}_\nu^+)$), which is rather unphysical and is the limit of this result. An improvement could be obtained by proving the result using compactness arguments.

In chapter 3 has been derived a new class of well-balanced schemes for transport equations, which are consistent as the wave velocity tends to zero. The uniform convergence of this scheme in the context of the relativistic transfer equations has been provided, both from the theoretical as from the numerical point of view. These new schemes may be generalized to construct well-balanced schemes for more general transport equations. A work in this direction can be found in [BD14] for the Friedrichs systems.

In chapter 4 several original anisotropic Fokker-Planck equations have been derived to model the Compton scattering, generalizing the Kompaneets equation in the case of anisotropic distributions. A rigorous mathematical justification of these equations, and in particular a H-theorem for the equation (4.13) would improve this study. Several angular moments models, and in particular the M_1 and P_1 models have been derived from a simple anisotropic Kompaneets type equation, leading to original models. The M_1 model, derived in the grey case, could be extended in the multigroup case by using the results of Turpault [TUR05, TUR12].

In chapter 5 a prospective study has been performed concerning the electron-ion Bremsstrahlung. An energy conservative kinetic model and the corresponding M_1 model have been derived, and a numerical scheme implemented. The numerical results show a good agreement with stochastic tests. The numerical tests have been performed in the case of cold matter, and a reflection concerning inertial confinement fusion simulations is in progress.

Finally, the chapter 6 presents a proof of uniform convergence of the Gosse-Toscani's scheme for the hyperbolic heat equations on unstructured meshes in 2D. This result is original, and involves a new diffusion scheme, called DA (diffusion approximation) scheme. Since the numerical convergence rate ($h = 1$) is better than the theoretical convergence rate ($h = 1/4$), one could expect an improvement of this theoretical rate.

Bibliography

- [ABN12] D. Aregba, M. Briani, R. Natalini, *Time Asymptotic High Order Schemes for Dissipative BGK Hyperbolic Systems*, arXiv:1207.6279.
- [AG12] G. Allaire, F. Golse, *Transport et diffusion* Cours de l'école Polytechnique, chapitre 4, 2012.
- [ACCD02] E. Audit, P. Charrier, J. P. Chièze, B. Dubroca *A radiation-hydrodynamics scheme valid from the transport to the diffusion limit*, arXiv, 2002.
- [AN00] D. Aregba-Driollet, R. Natalini *Discrete kinetic schemes for multidimensional systems of conservation laws*, Siam Journal on Numerical Analysis 37 (6), 1973-2004, 2000.
- [BC03] C. Buet, S. Cordier *Numerical method for the Compton scattering operator* Series on Advances in Mathematics for Applied Science 63, Chap. 9, 2003.
- [BCLM02] C. Buet, S. Cordier, B. Lucquin-Desreux, S. Mancini, *Diffusion limit of the lorentz model: asymptotic preserving schemes*. ESAIM: M2AN vol. 36, 4, pp 631-655, 2002.
- [BCT07] C. Berthon, P. Charrier, R. Turpault *An HLLC scheme to solve the M^1 model of radiative transfer in two dimensions* J. Scie. Comput., 31, no. 3, pp 347-389, 2007.
- [BD04] C. Buet, B. Després, *Asymptotic Analysis of Fluid Models for the Coupling of radiation and Hydrodynamics* JQSRT 85, 385-418, 2004.
- [BD06] C. Buet, B. Després *Asymptotic preserving and positive schemes for radiation hydrodynamics*, Journal of Computational Physics 215, 717-740, 2006.
- [BD14] C. Buet, B. Després *The structure of well-balanced schemes for Friedrichs systems with linear relaxation*, HAL 2014.
- [BDF11] C. Buet, B. Després, E. Franck, *Asymptotic preserving finite volumes discretization for non-linear moment model on unstructured meshes*, Finite Volumes for Complex Applications VI Problems, Perspectives, Springer Proceedings in Mathematics Volume 4, pp 467-474, 2011.
- [BDF12-1] C. Buet, B. Després, E. Franck *Design of asymptotic preserving schemes for the hyperbolic heat equation on unstructured meshes* Numerish Mathematik, Volume 122, Issue 2, pp 227-278, 2012.
- [BDF12-2] C. Buet, B. Després, E. Franck *An asymptotic preserving scheme with the maximum principle for the M_1 model on distorted meshes*, C.R. Acad. Sci., Paris, Sér. I, Math., Vol 350, 11-12, 633-638, 2012.

- [BDFL14] C. Buet, B. Després, E. Franck, T. Leroy *Proof of uniform convergence for a cell-centered AP discretization of the hyperbolic heat equation on general meshes*, Hal 2014, submitted.
- [BET34] H. A. Bethe *The influence of screening on the creation and stopping of electrons*, Mathematical Proceedings of the Cambridge Philosophical Society 30 (4), 524-539, 1934.
- [BI56] I. Bihari. *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Mathematica Hungarica, pp 81-94,1956.
- [BGP87] C. Bardos, F. Golse, B. Perthame, *The Rosseland Approximation for the Radiative Transfer Equations* Communications on Pure and Applied Mathematics 1987.
- [BGPS88] C. Bardos, F. Golse, B. Perthame, R. Sentis, *The Nonaccretive Radiative Transfer Equations: Existence of Solutions and Rosseland Approximation* Journal of Functional Analysis 1988.
- [BLT13] C. Berthon, P. LeFloch, R. Turpault *Late-time relaxation limits of nonlinear hyperbolic systems. A general framework*, Math. of Comput., 82 , pp. 831-860, 2013.
- [BMO70] E. Borie., L. C. Maximon, H. Olsen *Molecular Coherence Effects in Radiation Processes: Bremsstrahlung*, Physical Review A 2 (4), 1970.
- [BN14] B. Ducomet, S. Nečasová, *Diffusion limits in a model of radiative flow*, Annali Dell'Universita di Ferrara, 2014.
- [BRE10] H. Brezis, *Fonctional Analysis, Sobolev Spaces and Partial Differential Equations* Springer, 2010.
- [BOR72] E. Borie *Screening and second Born corrections to Bremsstrahlung at intermediate energies*, Il Nuovo Cimento 11 A (4), 1972.
- [BPR69] J. P. Babuel-Peyrissac, G. Rouvillois *Diffusion compton dans un gaz d'électrons maxwelliens* Le Journal de Physique 30 (in french), pp 301, 1969.
- [BRU00] T. A. Brunner *Riemann solvers for time-dependant transport based on the maximum entropy and spherical harmonics closures*, PhD thesis, 2000.
- [BRU02] T. A. Brunner *Forms of Approximate Radiation Transport*, Sandia Report, 2002.
- [BT11] C. Berthon, R. Turpault *Asymptotic preserving HLL schemes*, Numerical Methods for Partial Differential Equations 27 (6), 1396-1422, 2011.
- [CC70] J.S. Chang, G. Cooper *A practical scheme for Fokker-Planck equations*, J. Comp. Physics 6, pp 1-16, 1970.
- [CL71] M. G. Crandall, T. M. Liggett, *Generation of Semi-Groups of Nonlinear Transformations on General Banach Spaces* The John Hopkins university Press 1971.
- [CL86] R. E. Caflisch, C. D. Levermore *Equilibrium for radiation in a homogeneous plasma*, Phys. Fluids 29 (3), 1986.
- [CL11] N. Crouseilles, M. Lemou *An asymptotic preserving scheme based on a micro-macro decomposition for collisional Vlasov equations: diffusion and high-field scaling limits*, Kinetic and related models 4, 2 pp 441-477, 2011.

- [COO71] G. Cooper *Compton Fokker-Planck equation for hot plasmas*, Physical Review D 3 (10), 1971.
- [CR13] N. Crouseilles, M. Roger *A dynamic multi-scale model for transient radiative transfer calculations* Journal of Quantitative Spectroscopy and Radiative Transfer 116, 110-121, 2013.
- [CRS08] J.A. Carrillo, J. Rosado, F. Salvarani, *1D nonlinear Fokker Planck equations for fermions and bosons* Applied Mathematics Letters 21 (2008) 148-154.
- [DBM54] H. Davies, H. A. Bethe, L. C. Maximon *Theory of Bremsstrahlung and Pair production. II. Integral Cross section for Pair Production*, Physical Review 93 (4), 1954.
- [DES04] B. Després, *Lax theorem and finite volume schemes*, Math. Comput 73 (247), 1203-1234, 2004.
- [DES10] B. Després, *Weak consistency of the cell-centered Lagrangian GLACE scheme on general meshes in any dimension*, Comput. Methods Appl. Mech. Engrg. 199, 2669-2679, 2010.
- [DF99] B. Dubroca, J. L. Feugeas, *Theoretical and numerical study on a moment closure hierarchy for the radiative transfer equation*. C. R. Acad. Sci Paris, t. 329, Série 1, pp. 915-920, 1999.
- [DL83] R. Dautray, J.L. Lions, *Analyse mathématique et calcul numérique* Masson, 1985.
- [DOG01] C. Dogbe, *The Radiative Transfer Equations: Diffusion Approximation Under AcCRETIVENESS and Compactness Assumptions* Computers & Mathematics with Applications 42, 783-791, 2001.
- [DOP12] T. Doppner et al *Direct measurements of energetic electrons coupling to an imploding low-adiabat inertial confinement fusion capsule*, Physical Review Letters 108, 135006, 2012.
- [DWLM09] J. D. Densmore, J. S. Warsa, R. B. Lowrie, J. E. Morel *Stability analysis of implicit time discretizations for the Compton-scattering Fokker-Planck equation*, Journal of computational Physics 228, 5933-5960, 2009.
- [EGH00] R. Eymard, T. Gallouet, R. Herbin, *The finite volume method*, Handbook for Numerical Analysis, Ph. Ciarlet J.L. Lions eds, North Holland, 715-1022, 2000.
- [EM01] M. Escobedo, S. Mischler *On a quantum Boltzmann equation for a gas of photons*, J. Math. Pures Appl. 80 (5), pp 471-515, 2001.
- [EM99] M. Escobedo, S. Mischler *Équation de Boltzmann quantique homogène : existence et comportement asymptotique*, Cr. Acad. Sci. Paris 329 (Série 1), 593-598, 1999.
- [EMV03] M. Escobedo, S. Mischler, M.A. Valle *Homogeneous Boltzmann equation in quantum relativistic kinetic theory* Electronic Journal of Differential Equations, Monograph 04, 2003.
- [EHV98] M. Escobedo, M. A. Herrero, J. J. L. Velazquez *A nonlinear Fokker-Planck equation modelling the approach to thermal equilibrium in a homogeneous plasma*, Transactions of the American Mathematical Society 350 (10), pp 3837-3901, 1998.

- [EHV99] M. Escobedo, M. A. Herrero, J. J. L. Velazquez *Radiation dynamics in homogeneous plasma*, Physica D 126 (3-4), 236-260, 1999.
- [EMV04] M. Escobedo, S. Mischler, J. J. L. Velazquez *Asymptotic description of Dirac mass formation in kinetic equations for quantum particles*, J. Differential equations 202, 208-230, 2004.
- [EV55] R. D. Evans *The Atomic Nucleus* Tata McGraw-Hill Publishing Company Ltd, 1955.
- [FKM85] J. Fukue, S. Kato, R. Matsumoto *Radiation hydrodynamics in a moving plasma with compton scattering* Publ. Astron. Soc. Japan 37, pp 383-397, 1985.
- [FRA12] E. Franck *Construction et analyse numérique de schémas asymptotic preserving sur maillages non structurés. Application au transport linéaire et aux systèmes de Friedrichs*, PhD thesis, hal-00735956 (in french), 2012.
- [FR13] F. Filbet, A. Rambaud, *Analysis of an asymptotic preserving scheme for relaxation systems*, ESAIM M2AN, Vol. 47 pp. 609-633 (2013)
- [GJL99] F. Golse, S. Jin, C. D. Levermore *The convergence of numerical transfer schemes in diffusive regimes I: discrete-ordinate method*, SIAM J. Numer. Anal. 36, 1333-1369, 1999.
- [GL96] J. M. Greenberg, A. Y. Leroux *A well-balanced scheme for the numerical processing of source terms in hyperbolic equations*, Journal on Numerical Analysis 33 (1), 1-16, 1996.
- [GLG05] P. Godillon-Lafitte, T. Goudon, *A coupled model for radiative transfer: doppler effects, equilibrium, and non-equilibrium diffusion asymptotics* SIAM, 2005.
- [GOS11] L. Gosse *Transient radiative transfer in the grey case: Well-balanced and asymptotic-preserving schemes built on Cases's elementary solutions* Journal of Quantitative Spectroscopy and Radiative Transfer 112, pp 1995-2012, 2011.
- [GOS13] L. Gosse *Computing qualitatively correct approximations of balance laws*, Springer Series, ISBN 978-88-470-2892-0, 2013.
- [GOS14] L. Gosse, *A two-dimensional version of the Godunov scheme for scalar balance laws*. SIAM J. Numer. Anal. 52, 626-652, 2014.
- [GP86] F. Golse, B. Perthame, *Generalized solutions of the radiative transfer equations in a singular case* Communications in Mathematical Physics, 1986.
- [GS08] F. Golse, F. Salvarani *The Rosseland limit for radiative transfer in gray matter*, hal-00268799, 2008.
- [GT02] L. Gosse, G. Toscani, *An asymptotic preserving well-balanced scheme for the hyperbolic heat equations* C. R. Acad. Sci. Paris, Ser I 334, 337-342, 2002.
- [HAU08] E. Haug *Bremsstrahlung cross-section with screening and Coulomb corrections at high energies*, Rad. Phys. Chem. 77, 207-214, 2008.
- [HLM10] C. D. Hauck, R. B. Lowrie, R. G. McClarren *Methods for diffusive relaxation in P_n equations*, Numerical methods for relaxation systems and balance equations, Quaderni di Matematica International Series, 2010.

- [HOU48] P. V. C. Hough *The Angular Distribution of Pair-Produced Electrons and Bremsstrahlung*, Physical Review, 74 (1), 1948.
- [JIN99] S. Jin *Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations*, SIAM J. Sci. Comp. 21, 441-454, 1999.
- [JIN10] S. Jin *Asymptotic-preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review*, Lecture Notes for Summer School on $\epsilon^{\frac{1}{2}}$ Methods and Models of Kinetic Theory $\epsilon^{\frac{1}{2}}$ (M and MKT), Porto Ecole (Grosseto, Italy), 2010.
- [JL91] S. Jin, C. D. Levermore *The discrete-ordinate method in diffusive regimes*, Transport Theory and Statistical Physics 20 (5), 413-439, 1991.
- [JL96] S. Jin, C. D. Levermore *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms*, J. Comp. Physics 126, 449-467, 1996.
- [JPT98] S. Jin, L. Pareschi and G. Toscani, *Diffusive Relaxation Schemes for Discrete- Velocity Kinetic Equations*, SIAM J. Num. Anal. 35 (1998) 2405-2439
- [JPT00] S. Jin; L. Pareschi and G. Toscani, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM J. Numer. Anal. 38 (2000), 913-936
- [KAV02] O. Kaviani *Chapter 21 Remarks on the kompaneets equation, a simplified model of the Fokker-Planck equation*, Studies in Mathematics and its Applications 31, 469-487, 2002.
- [KM59] H. W. Koch, J. W. Motz *Bremsstrahlung cross-section formulas and related data*, Reviews on Modern Physics 31 (4), 920-956, 1959.
- [KOM57] A. S. Kompaneets *The establishment of thermal equilibrium between quanta and electrons*, Soviet Physics JETP 4 (5), 1957.
- [LBD14] T. Leroy, C. Buet, B. Després *A well-balanced scheme for a transport equation with varying velocity arising in relativistic transfer equation*, Springer Proceedings in Mathematics & Statistics 78, 891-899, 2014.
- [LER14] T. Leroy *A well-balanced scheme for a transport equation with varying velocity and irregular opacities*, ESAIM: Proceedings and Surveys 50, 133-143, 2015.
- [LER15] T. Leroy *Relativistic transfer equations: comparison principle and convergence to the non-equilibrium regime*, Kinetic and Related Models 8 (4), 725-763, 2015.
- [LEV84] C. D. Levermore, *Relating Eddington Factors to flux limiters*. J. Quant. Spectrosc. Radiat. Transfer Vol 31, No 2, pp 149-160, 1984.
- [LKPT76] C.M. Lee, L. Kissel, R. H. Pratt, H. K. Tseng *Electron Bremsstrahlung spectrum, 1-500 keV*, Physical Review A, 12 (5), 1976.
- [LLPS84] E. W. Larsen, C. D. Levermore, G. C. Pomraning, J. G. Sanderson *Discretization methods for one-dimensional Fokker-Planck operators*, Journal of Computational Physics 61, 359-390, 1985.
- [LM08] M. Lemou, L. Mieussens *A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit*. SIAM J. Sci. COMPUT. Vol. 31, 1, pp 334-368, 2008.

- [LM10] J. G. Liu, L. Mieussens *Analysis of an Asymptotic Preserving Scheme for Linear Kinetic Equations in the Diffusion Limit* SIAM J. Numer. Anal, 48(4), pp 1474-1491, 2010.
- [LMH99] R. B. Lowrie, J. E. Morel, J. A. Hittinger *The coupling of radiation and hydrodynamics*, The Astrophysical Journal 521, 432-450, 1999.
- [LOB15] M. Lobet PhD thesis, 2014.
- [LP81] E.M. Lifchitz and L.P. Pitaevskii *Physical Kinetics ? Course in theoretical physics*, 10, Pergamon, Oxford, 1981.
- [LY77] E. Levich, V. Yakhot *Time evolution of a Bose system passing through the critical point*, Physical Review B 15 (1), 243-251, 1977.
- [LY78] E. Levich, V. Yakhot *Time development of coherent and superfluid properties in the course of a λ -transition*, Journal of Physics A 11 (11), 2237-2254, 1978.
- [MAZ07] C. Mazeran *Sur la structure mathématique et l'approximation numérique de l'hydrodynamique lagrangienne bidimensionnelle*, PhD, University Bordeaux 1, 2007.
- [MBD14] J. Mallet, S. Brull, B. Dubroca *An Entropic Scheme for an Angular Momentum Model for the Classical Fokker-Planck-Landau Equation of Electrons*, Commun. Comput. Phys. 15, 422-450, 2014.
- [MBD15] J. Mallet, S. Brull, B. Dubroca *General moment system for plasma physics based on minimum entropy principle*, Kinetic and Related Models 8 (3), 533-558, 2015.
- [MWM99] D. Mihalas, B. Weibel Mihalas, *Foundations of Radiation Hydrodynamics* Dover Publications 1999.
- [NP00] G. Naldi and L. Pareschi, *Numerical schemes for hyperbolic systems of conservation laws with stiff diffusive relaxation*, SIAM J. Numer. Anal., 37 (2000), pp. 1246-1270.
- [PAZ83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations* Springer-Verlag, 1983.
- [PL97] D. Psaltis, F. K. Lamb *Compton scattering by static and moving media. I. The transfer equation and its moments* The Astrophysical Journal 488, pp 881-894, 1997.
- [POM73] G. C. Pomraning *The Equations of Radiation Hydrodynamics* Dover 1973.
- [POM83] G. C. Pomraning, *Flux-Limited Diffusion and Fokker-Planck Equations*, Nuclear Science and Engineering 85, 116-126, 1983.
- [POM92] G. C. Pomraning, *The Fokker-Planck operator as an asymptotic limit*, Mathematical Models and Methods in Applied Sciences 2 (1), 21-36, 1992.
- [RIP14] J. R. Ripoll *Some contributions to radiative transfer and plasma modeling*, HDR, 2014.
- [ROZ79] B.F. Rozsnyai *Computation of free-free gaunt factors and conductive opacities in hot matter*, J. Quant. Spectrosc. Radiat. Transfer 22, 337-343, 1979.
- [SAM58] D. H. Sampson *The opacity at high temperatures due to Compton scattering* The Astrophysical Journal 129, pp 734, 1958.

- [SB85] S. M. Seltzer, M. J. Berger *Bremsstrahlung spectra from electron interactions with screened atomic nuclei and orbital electrons*, Nuclear Instruments and Methods in Physics Research, B12, 95-134, 1985.
- [ST95] D. V. Semikoz, I. I. Tkachev *Kinetics of Bose condensation*, Physical review Letters 74 (16), 3093-3097, 1995.
- [ST97] D. V. Semikoz, I. I. Tkachev *Condensation of bosons in the kinetic regime*, Physical review D 55 (2), 489-502, 1997.
- [TPL79] H. K. Tseng, R. H. Pratt, C. M. Lee *Electron bremsstrahlung angular distribution in the 1-500 keV energy range*. Physical Review A, 19 (1), 188-195, 1979.
- [TSE89] H. K. Tseng *Non relativistic calculation of electron Bremsstrahlung angular distributions of recoiled electrons*, Physical Review A, 39 (3), 1012-1015, 1989.
- [TUR05] R. Turpault *A consistent multigroup model for the radiative transfer and its underlying mean opacities*. J. Quant. Spectrosc. Radiat. Transfer 94, pp 357-371, 2005.
- [TUR12] R. Turpault *Properties and frequential hybridisation of the multigroup M1 model for radiative transfer*, Non Linear Analysis: Real World Applications 11, 2514-2528, 2012.
- [WIN95] A. M. Winslow, *Multifrequency-Grey Method for Radiation Diffusion with Compton Scattering* JCP 117 2 (1995) 262-273.
- [YS14] Y. Li, S. Zhu, *Existence results and blow-up criterion of compressible radiation hydrodynamic equations* arXiv:1407.7830, 2014.