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Yunhui Hou

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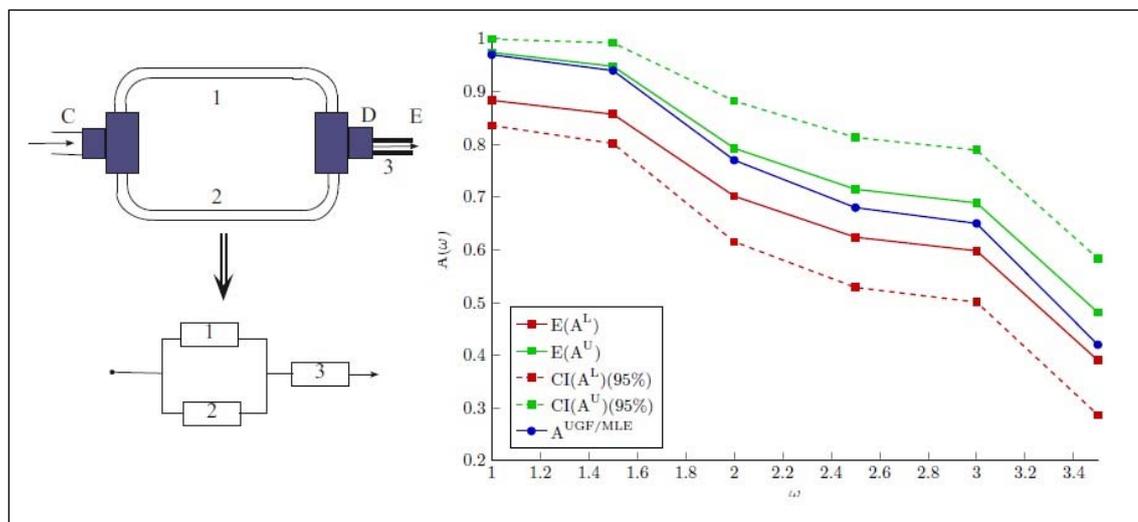
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Par Yunhui HOU

*Estimation of the probability and uncertainty of undesirable events in large-scale systems*

Thèse présentée  
pour l'obtention du grade  
de Docteur de l'UTC



Soutenue le 31 mars 2016

**Spécialité** : Technologies de l'Information et des Systèmes

D2272

# Estimation of the probability and uncertainty of undesirable events in large-scale systems

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# Chapter 1

## Introduction

### Research problem statement & objectives

The purpose of this thesis is to propose a reliability and uncertainty estimation method dealing with the existence of rare failure events and large-scale systems. This thesis started in Oct. 2012 under the supervision of Professor Nikolaos Limnios (LMAC) and Walter Schön (Heudiasyc) at Université de Technologie de Compiègne (UTC), France.

The purpose of reliability assessment is to predict the probability that a system is operating during a specified time interval. Over the last few years, the dependability and risk assessments community has recognized that there are different sources and/or types of uncertainties that play an important role in availability and risk evaluation [Winkler, 1996, Aven, 2011]. One of the major problems is how to properly estimate system dependability as well as the associated uncertainty with the existence of rare failure events. The rare failure events problem considered in this thesis is the condition that there are no component failures observed in historical data (or only a very small number of failures). This is what we call components with rare failure events. Then we have to raise the natural question: when a component has no failure, how can we consider its reliability parameters? Should we consider that its reliability is equal to one because the empirical estimator gives zero failure probability and zero variance? This is not reasonable, given that such highly reliable components may be critical to the system. Indeed, the rare failure events problem is mainly caused by the following situations:

- Small sample size: In many systems such as intelligent transportation systems, the sample size of some components is very small. For example, when a component under test is totally new there is no previous test data for reliability analysis; some of the components are very expensive or the test cannot be carried out frequently due to some cost considerations. Using Bayesian estimates, we should make some assumptions such as the choice of prior distributions. However, prior distributions may introduce more uncertainty in the reliability estimation results.
- Highly reliable component (system): For a highly reliable system or component, the frequency of failure is very low so that a large number of tests are needed to obtain the first failure observation, which is not possible due to limited test expense and time.

In our opinion, the associated problems on dependability assessment are caused by confusion among the following two types of uncertainties due to different conditions [Apostolakis, 1990, Helton and Burmaster, 1996] while presenting uncertainty measures:

- Aleatory uncertainty is caused by natural variability of random phenomena whose behavior can be different even under the same condition. It indicates an individual random phenomenon as well as its outcome does not have a pattern and is not predictable. This type of uncertainty has been studied for a long time using probability theory and is often quantified by dispersion measures such as variance.
- Epistemic uncertainty is caused by lack of information. This type of uncertainty has been represented under framework of uncertainty theories by set-value measures such as belief functions, possibility theory, imprecise probabilities, etc.

Our research objective is to build frameworks which can present the two types of uncertainties based on both probabilistic and uncertainty theory approaches to compare these methods and to find the proper application conditions for these methods.

The thesis also requires applying our methods on large-scale systems. Many articles have discussed similar problems but the definition of a large-scale system remains unclear. In this thesis, we study a large-scale system in terms of the complexity of its state probability calculation. It leads us to take into consideration two steps: model construction and reliability assessment. Comprehensive models such as fault trees, reliability block diagrams and Markov chains give simpler structure function representation and are easier to construct. However, the probabilistic assessment for these models needs complicated analysis and brings large amount of computation cost. Meanwhile, models like binary decision diagram require complicated building procedure while the system is decomposed and analyzed at the same time so that the cost of probabilistic assessment is reduced.

In our opinion, the complexity of both steps is impacted by the number of system components and also by the system structure:

- A large number of components is one of the necessary conditions for large-scale systems [Kolowrocki, 2014]. As the number of components increases, the number of possible component level state combination grows at least exponentially which increases the calculation needed to obtain system reliability function and corresponds to NP-difficult problem. For example, in a system composed of  $n$  items which have respectively  $m_i$  states for each component  $C_i$ , the number of all possible combinations is the product of all  $m_i$ . The exact minimal component number of the large-scale system definition indeed depends on the model in which the system is represented. In [Hwang et al., 1981], a system containing more than 10 binary components is defined as a large system which corresponds to  $2^{10}$  different states. For dynamic models like Markov process analysis [Sharma and Bazovsky, 1993], it means that we have a  $1024 \times 1024$  Markov kernel matrix and such systems can be defined as large-scale system. However, BDD have been applied on system with more  $s$ -independent components [NIKOLSKAIA, 2000]. Normally in these cases, a large-scale system requires more than 30 components.
- The system structure is also an important factor. Simple systems like parallel or series systems do not produce large calculation complexity even with a large component number. We have chosen the number of minimal cut-sets as measure for evaluating system structure complexity because minimal cut-sets are distinct from prime implicants and they have a great interest from both a computation complexity and practical viewpoint [Rauzy, 2001]. According to inclusion-exclusion theorem, among systems of the same component size, the ones having more cut-sets cause higher complexity. Dependent components increase the complexity too.

Finally, we have chosen component number and minimal cut-sets number as indicators to determine if the studied system is a large-scale system.

While uncertainty propagation is also considered, the complexity becomes more complicated. Uncertainty measures modeled by interval analysis [Zhang et al., 2002], belief functions [Aguirre et al., 2013], fuzzy number [Klir, 1997], etc. decline the minimal necessary condition of our definition. In this thesis, we apply the definition discussed previously in order to compare with other methods.

# Chapter 2

## Uncertainty theories

### 2.1 Random set theory

Random set theory is an extension of probability theory by studying set-valued variables rather than point-valued ones. The modern concept of random set was firstly mentioned in [Kolmogorov, 1950] (first published in 1933) even though confidence regions which are random sets had been used on statistics before that.

In this section, we begin with reviewing the definitions of closed random set on  $\mathbb{R}^d$ . Then we review the definitions of expectations and confidence intervals under random set theory framework.

#### 2.1.1 Basic definitions

Random sets are random variables whose values are sets. In other words, the theory of random sets includes the classical case of random variables as a special case.

Although the first random closed set is defined on subsets of a more general space by Matheron in [Matheron, 1975], we are just interested in random closed sets on  $\mathbb{R}^d$ .

Let  $\mathcal{F}$  denote the family of closed subsets of  $\mathbb{R}^d$  and  $\mathcal{K}$  the family of all compact subsets of  $\mathbb{R}^d$ . Similarly as the definition of random variable in probability theory, the random closed set is defined based on the idea that the range of a random set  $S$  is represented by the sets having common elements with  $S$ .

**Definition 1.** A closed random set on probability space  $(\Omega, \mathcal{A}, P)$  is a map  $S : \Omega \rightarrow \mathcal{F}$  if, for every compact set  $K \subset \mathbb{R}^d$ ,

$$\{\omega : S(\omega) \cap K \neq \emptyset\} \in \mathcal{A}.$$

The corresponding probability functional (hitting probability or capacity functional) of a random closed set  $S$ ,  $T : \mathcal{K} \rightarrow [0, 1]$ , is defined as

$$T(K) = \mathbb{P}\{S \cap K \neq \emptyset\} \quad \forall K \in \mathcal{K}.$$

**Theorem 1.** The capacity functional satisfies the following properties:

- $T(\emptyset) = 0$  and  $0 \leq T(K) \leq 1$  for every  $K \in \mathcal{K}$
- If  $K_n \downarrow K$ , i.e.

$$K_{n+1} \subseteq K_n, \quad K = \bigcap_{n \geq 1} K_n$$

then

$$T(K_n) \downarrow T(K).$$

- $T$  is monotone increasing

$$K_1 \subseteq K_2 \Rightarrow T(K_1) \leq T(K_2)$$

and for  $n \geq 2$ , and  $K_1, K_2, \dots, K_n \in \mathcal{K}$ ,

$$T\left(\bigcap_{i=1}^n K_i\right) \leq \sum_{\emptyset \neq I \subseteq \{1, 2, \dots, n\}} (-1)^{|I|+1} T\left(\bigcup_{i \in I} K_i\right)$$

**Theorem 2.**  $f(S)$  is a random closed set if  $S$  is a random closed set and the map  $f : \mathcal{F} \rightarrow \mathcal{F}$  is continuous or semi-continuous (and therefore measurable).

**Example 1.** Let us consider a random closed set  $S$  with a unique element on  $\mathbb{R}$ :

$$S = \{X\}$$

where  $X$  is a random variable on  $\mathbb{R}$ .  $S$  is a (singleton) random set. For all compact  $K \subset \mathbb{R}$ , the hitting probability of  $S$  is given by

$$T(K) = \mathbb{P}\{K \cap S \neq \emptyset\} = \mathbb{P}\{X \in K\}$$

which is exactly the probability of a random variable.

**Example 2.** Let us consider a random closed set  $S$  defined as

$$S = \{x \in \mathbb{R} | x \leq X\} = (-\infty, X]$$

where  $X$  is a random variable. For all compact  $K \subset \mathbb{R}$ , the hitting probability of  $S$  is given by

$$\begin{aligned} T(K) &= \mathbb{P}\{K \cap S \neq \emptyset\} \\ &= \mathbb{P}\{K \cap (-\infty, X] \neq \emptyset\} \\ &= \mathbb{P}\{\exists x \in K | x \leq X\} \end{aligned}$$

**Example 3.** Let  $X = (X_1, \dots, X_d)$  be a random variable on  $\mathbb{R}^d$ . Then a random closed set  $S$  on  $\mathbb{R}^d$  can be defined as

$$S = (-\infty, X_1] \times \dots \times (-\infty, X_d]$$

For all compact  $K \subset \mathbb{R}^d$ , the hitting probability of  $S$  is given by

$$\begin{aligned} T(K) &= \mathbb{P}\{K \cap S \neq \emptyset\} \\ &= \mathbb{P}\{K \cap (-\infty, X_1] \times \dots \times (-\infty, X_d] \neq \emptyset\} \\ &= \mathbb{P}\{\exists x = (x_1, \dots, x_d) \in K | x_1 \leq X_1, \dots, x_d \leq X_d\} \end{aligned}$$

### 2.1.2 Expectation of random sets

This subsection introduces the selection expectation (also called the Aumann expectation) which is the best and most used concept of expectation for random sets.

**Definition 2.** [Aumann, 1965] A random point  $\xi$  is said to be a selection of a random set  $S$  if  $\mathbb{P}(\xi \in S) = 1$ .

A random set can be approximated by all its selections. A random variable is called integrable if its expected value exists. The expectation of a random set is defined by grouping the expected value of all its integrable selections.

**Definition 3.** [Aumann, 1965] The expectation  $E(S)$  of an random set  $S$  on  $\mathbb{R}^d$  is the closure of the family of all expectations for its integrable selections, i.e.  $E(S) = \{E(\xi) | \xi \in \mathcal{T}(S)\}$ , where  $\mathcal{T}(S)$  is the set of all integrable selections of  $S$ .

**Example 4.** When considering  $S = \{X\}$  as defined in Example 1, there is only one selection for  $S$  given by  $\mathcal{T}(S) = \{X\}$ , so that  $E(S) = \{E(X)\}$ .

**Example 5.** When considering  $X$  as defined in Example 2. The selections are given by  $S = X - a$ , where  $a \in \mathbb{R}^+$  is a constant. We have

$$\mathcal{T}(S) = \{x | x = X - a, a \in \mathbb{R}^+\}.$$

Hence, the expectation of  $S$  is given by

$$E(S) = \{E(x) | x \in \mathcal{T}(S)\} = (-\infty, E(X)].$$

Then the expectation of the random closed set  $S'$  defined in Example 3 is given by

$$E(S') = (-\infty, E(X_1)] \times \dots \times (-\infty, E(X_d)].$$

### 2.1.3 Random intervals

**Definition 4.** [Gil, 1992] A random interval  $X$  of  $\mathbb{R}$  associated with the probability space  $(\Omega, \mathcal{A}, P)$  is a random set of  $\mathbb{R}$  associated with that probability space such that it may be characterized by means of a bi-dimensional random variable  $(X^L, X^U)$ . So that  $S(\omega) = [X^L(\omega), X^U(\omega)]$ , for all  $\omega \in \Omega$ , and it will be denoted by  $S = [X^L, X^U]$ .

Let  $f(x^L, x^U)$  be the joint probability density of  $X^L$  and  $X^U$ , the expectation of  $S = [X^L, X^U]$  is given by

$$E(S) = [E(X^L), E(X^U)] = \left[ \int_{-\infty}^{+\infty} t f^L(t) dt, \int_{-\infty}^{+\infty} t f^U(t) dt \right] \quad (2.1)$$

where  $f^L(t)$  and  $f^U(t)$  are respectively marginal pdf developed from the joint probability density function  $f(x^L, x^U)$  (see [Matheron, 1975]).

**Example 6.** Let  $U_1, U_2, \dots, U_n$  be i.i.d random variables following uniform distribution  $U(0, 1)$ . Let  $U_{(i)}$  the  $i$ th smallest value among  $U_1, U_2, \dots, U_n$ , i.e.

$$U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(n)}.$$

The random set

$$X = [U_{(k)}, U_{(k+1)}] \quad \forall k = 1, \dots, n-1$$

is a random interval. Since we have

$$U_{(k)} \sim \text{Beta}(k, n - k + 1) \quad \forall k = 1, \dots, n$$

the expectation of  $X = [U_{(k)}, U_{(k+1)}]$  is given by

$$E(X) = [E(U_{(k)}), E(U_{(k+1)})] = \left[ \frac{k}{n+1}, \frac{k+1}{n+1} \right].$$

### 2.1.4 Confidence interval

Consider a random variable  $X$  whose distribution depends on a parameter  $\theta = (\theta_1, \dots, \theta_m)$ ,  $\theta \in \Theta \subseteq \mathbb{R}^m$  where  $\Theta$  is the parameter space. Let  $\varphi(\theta) \in \mathbb{R}$  a parameter of the distribution of  $X$  which depends on  $\theta$ . Given a certain number of observations of  $X$ , the random set  $\mathcal{C} = [c_1, c_2] \subset \mathbb{R}$  which contains the true value parameter  $\varphi(\theta)$  is called a confidence set for  $\varphi(\theta)$  at level  $1 - \alpha$  ( $\alpha \in [0, 1]$ ) if for all possible values of  $\theta$ :  $\mathbb{P}\{\varphi(\theta) \in [c_1, c_2]\} = 1 - \alpha$ . Notice that here we use point estimate of  $\varphi(\theta)$  to construct the confidence interval.

Now we extend the use of confidence interval to apply on interval estimates of  $\varphi(\theta)$ .

**Definition 5** (Confidence interval for a random interval). Let a random interval  $[a^L, a^U]$  with known cumulative distribution functions  $F_L$  and  $F_U$ . The confidence interval of  $\varphi(\theta)$ ,  $\mathcal{C}' = [t^L, t^U]$  at level  $1 - \alpha$  is given by

$$[t^L, t^U] = [F_L^{-1}(\alpha/2), F_U^{-1}(1 - \alpha/2)] \quad (2.2)$$

where  $F_L^{-1}(\alpha)$  and  $F_U^{-1}(\alpha)$  are respectively the  $\alpha$ -quantile functions of  $F_L$  and  $F_U$  (see [Matheron, 1975]).

## 2.2 Imprecise probabilities

### 2.2.1 Introduction

On the same sample space and event space as probability theory,  $(\Omega, \mathcal{F})$ ,  $X$  is modeled not by a precise probability distribution but a set of probability distributions/measures  $M$  (normally closed and convex) [Seidenfeld and Wasserman, 1993] which is called representer in [Fraassen, 1990] (by terms of credal set normally closed convex) so that randomness (aleatory uncertainty) can be represented by (a single) probability distribution and epistemic uncertainty/doubts is represented by the range of the set of probability distributions.

The idea of set of probabilities is implemented as a class of sampling models in classical robustness and as a set of prior distributions in Bayesian robustness [Seidenfeld and Wasserman, 1993]. More generally

speaking, the set  $M$  can be summarized by its extreme points. The probability measure in  $M$  can be bounded by the upper probability envelope  $\underline{P}(A)$  and lower probability envelope  $\bar{P}(A)$  such that

$$\underline{P}(A) = \inf_{P \in M} \mathbb{P}(A) \quad (2.3)$$

$$\bar{P}(A) = \sup_{P \in M} \mathbb{P}(A) \quad (2.4)$$

There are several formal approaches/interpretations of this idea that introduces interval measures to present set of probabilities: Walley's approach [Smith, 1961, Walley, 1991] under the name of *imprecise probabilities* is based on a decision making situation and introduces more generally lower and upper expectations (previsions) where lower and upper probabilities can be seen as special type; Weichselberger's approach [Weichselberger, 2000, Weichselberger, 2001] extends the definition of Kolmogorov's axioms in precise probability theory and presents the envelopes by *interval probability*; the upper and lower probability envelopes can also be treated as examples of non-additive probabilities/measures [Denneberg, 1994] which we will discuss in the next section.

## 2.2.2 Upper and lower probabilities

In imprecise probability theory firstly introduced in [Smith, 1961, Walley, 1991], a *gamble*  $g(\omega)$  is defined as a decision that yields different outcomes (utilities) in different states of the world. In this section, a general presentation of upper and lower probabilities is given according to Walley's work *Statistical reasoning with imprecise probabilities* [Walley, 1991].

**Definition 6.** For  $\Omega$  the set of possible outcomes  $\omega$ , A *gamble*  $g$  is a bounded real-valued function on  $\Omega$ , i.e.

$$g : \Omega \rightarrow \mathbb{R} : \omega \rightarrow g(\omega)$$

Accepting a gamble means taking a decision/action in the face of uncertainty. If you were to accept gamble  $g$  and  $\omega$  turned out to be true then you would gain  $g(\omega)$  (so you would lose if  $g(\omega) < 0$ ) [Wilson and Moral, 1994]. Your set of desirable gambles  $\mathcal{D}$  contains the gambles that you accept. Buying a gamble  $g$  for a price  $\mu$  results in a new gamble  $g - \mu$ .

**Definition 7** ([Walley, 1991]). The lower prevision  $\underline{E}(g)$  of  $g$  is the supremum acceptable price for buying  $g$

$$\underline{E}(g) = \sup\{\mu : g - \mu \in \mathcal{D}\}$$

Selling a gamble  $g$  for a price  $\mu$  yields a new gamble  $\mu - g$ .

**Definition 8** ([Walley, 1991]). The upper prevision  $\bar{E}(g)$  of  $g$  is the infimum acceptable price for selling  $g$

$$\bar{E}(g) = \inf\{\mu : \mu - g \in \mathcal{D}\}$$

Consider a special gamble case  $I_A$  where  $A$  is an event  $A \in \Omega$  such that

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

This indicator function like gamble normalizes the possible gain/loss.

**Definition 9** ([Walley, 1991]). The upper and lower probability of the event  $A \subset \Omega$ ,  $\underline{P}(A)$  and  $\bar{P}(A)$ , are defined as the the upper and lower previsions of  $I_A$ , i.e.

$$\underline{P}(A) = \underline{E}(I_A) \quad \bar{P}(A) = \bar{E}(I_A)$$

In this way, prevision for gambles are transfered into probability measures. The corresponding desirable gamble set  $\mathcal{D}$  becomes a set of probabilities  $\mathcal{P}$  such that

$$\underline{P}(A) = \inf_{P \in \mathcal{P}} P(A) \quad \bar{P}(A) = \sup_{P \in \mathcal{P}} P(A)$$

The lower and upper probabilities,  $\underline{P}$  and  $\bar{P}$ , satisfy the following properties:

$$- 0 \leq \underline{P}(A) \leq \bar{P}(A) \leq 1;$$

- $\underline{P}(A) = 1 - \overline{P}(A^c)$ ;
- $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$  and  $\underline{P}(\Omega) = \overline{P}(\Omega) = 1$ ;
- If  $A \subset B$ , then  $\underline{P}(A) \leq \underline{P}(B)$  and  $\overline{P}(A) \leq \overline{P}(B)$ ;
- $\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$ ,  $A \cap B = \emptyset$ .
- $\overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B)$ ,  $A \cap B = \emptyset$ .
- $\underline{P}(A \cup B) \leq \underline{P}(A) + \overline{P}(B) \leq \overline{P}(A \cup B)$ ,  $A \cap B = \emptyset$

The epistemic uncertainty is described as the difference between the upper and lower probabilities:  $\Delta = \overline{P} - \underline{P}$ . When  $\underline{P}(A) = \overline{P}(A)$ , full knowledge on the random phenomenon is available so that the distribution set  $\mathcal{P}$  converges to a single point (precise) probability  $\mathbb{P}(A)$  [You and Tonon, 2012]. We can say then (precise) probability theory is a special case of imprecise probabilities.

Given only  $\underline{P}(A)$  and  $\overline{P}(A)$ , the exact set of probabilities bounded by them is given by

$$\mathcal{P}_{\underline{P}, \overline{P}} = \{P | \forall A \in \mathcal{A}, \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}$$

and normally

$$\mathcal{P} \subset \mathcal{P}_{\underline{P}, \overline{P}}.$$

The  $p$ -boxes model which represents the probability distribution by upper and lower bounds on cumulative probability functions  $[\underline{F}, \overline{F}]$ , can be seen as a special case imprecise probabilities where .

$$\mathcal{P}_{\underline{F}, \overline{F}} = \{P | \forall x \in \Omega, \underline{F}(x) \leq F(x) \leq \overline{F}(x)\}$$

**Example 7** ([Utkin and Kozine, 2010]). Another most used approach is the imprecise distributions for a certain distribution which is a set of reasonable priors (a set of distribution parameters  $\Theta$ ) which indicates the convergence speed of the whole set. Given the group of distribution the imprecise probabilities can also be presented as a group of parameters.

- Imprecise Dirichlet distribution [Utkin and Kozine, 2010]: a set of Dirichlet distribution  $Dir_i(s, t)$ ,  $\mathcal{P}$ , such that the hyper-parameters  $\Theta$  are  $t = (t_1, \dots, t_K)$  with  $t_i \in (0, 1), \forall i = 1, \dots, K$  the expected mean of the proportion of the  $i$  category and  $s$  (usually fixed and between one and two) is the influence of the prior distribution on posterior distribution, i.e.

$$\Theta = \{(s, t) : s > 0; \sum_{i=1}^n t_i = 1, t = (t_1, \dots, t_K) \in (0, 1)^K\}$$

Given  $n$  observations of a random variable following multinomial distribution with  $K$  categories where  $k_i$  is the number of observations of the  $i$ th category, then the estimation of the proportion of this category  $p_i$  is given by

$$\hat{p}_i = \frac{k_i + st_i}{s + n}$$

and varies freely between

$$\frac{k_i}{s + n} \leq \hat{p}_i \leq \frac{k_i + s}{s + n}$$

### Uncertainty combination

**Example 8** ([Kozine and Filimonov, 2000]). Consider two different lower and upper probabilities pairwise for the same random phenomenon  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Let  $\underline{P}_1(A), \overline{P}_1(A)$  and  $\underline{P}_2(A), \overline{P}_2(A)$  denote the upper and lower probabilities respectively based on  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , i.e.

$$\underline{P}_1(A) = \inf_{P \in \mathcal{P}_1} P(A) \quad \overline{P}_1(A) = \sup_{P \in \mathcal{P}_1} P(A)$$

$$\underline{P}_2(A) = \inf_{P \in \mathcal{P}_2} P(A) \quad \overline{P}_2(A) = \sup_{P \in \mathcal{P}_2} P(A)$$

the combination rules for two imprecise probabilities can be:

- Conjunction rule:

$$\begin{aligned}\underline{P}(A) &= \max\{\underline{P}_1(A), \underline{P}_2(A)\} \\ \overline{P}(A) &= \min\{\overline{P}_1(A), \overline{P}_2(A)\}\end{aligned}$$

- Unanimity rule:

$$\begin{aligned}\underline{P}(A) &= \min\{\underline{P}_1(A), \underline{P}_2(A)\} \\ \overline{P}(A) &= \max\{\overline{P}_1(A), \overline{P}_2(A)\}\end{aligned}$$

The conjunction rule narrows down the uncertainty range by taking the maximum of the lower probabilities and the minimum of the upper probabilities. In this case, both sources are trusted and it can be seen as an optimistic result. The unanimity rule on the contrary scarifies the precision to ensure the range of uncertainty, which means that there is doubt on both sources. Both combination rules are based on subjective hypothesis on information source reliability.

**Example 9** ([Kozine and Filimonov, 2000]). For events  $A_i, i = 1, \dots, n$ , we have

$$\underline{P}\left(\bigcap_{i=1}^n A_i\right) = \begin{cases} 0 & \text{if } \sum_{i=1}^n \underline{P}(A_i) \leq n-1 \\ \sum_{i=1}^n \underline{P}(A_i) - (n-1) & \text{if } \sum_{i=1}^n \underline{P}(A_i) \geq n-1 \end{cases} \quad (2.5)$$

$$\overline{P}\left(\bigcap_{i=1}^n A_i\right) = \min_{1 \leq i \leq n} \overline{P}(A_i) \quad (2.6)$$

$$\underline{P}\left(\bigcup_{i=1}^n A_i\right) = \max_{1 \leq i \leq n} \underline{P}(A_i) \quad (2.7)$$

$$\overline{P}\left(\bigcup_{i=1}^n A_i\right) = \begin{cases} \sum_{i=1}^n \overline{P}(A_i) & \text{if } \sum_{i=1}^n \overline{P}(A_i) \leq 1 \\ 1 & \text{if } \sum_{i=1}^n \overline{P}(A_i) > 1 \end{cases} \quad (2.8)$$

which can be applied on calculating reliability of series and parallel systems directly as well as path-cut system description.

### 2.2.3 Discussion

The upper and lower previsions are more general forms than upper and lower probabilities. The upper and lower probabilities are not additive measures which means they are not real "probability" according to probability theory definition. Uncertainty is presented by the interval between two envelopes. However, when we study the uncertainty propagation of outcome determined by a function of a set of others with or without dependence, the maximum or minimum calculation in combination rules (Example 8) seems to overestimate the degree of uncertainty at system level.

## 2.3 Theory of possibility

### 2.3.1 Fuzzy set theory

Fuzzy set theory is firstly introduced by Zadeh in [Zadeh, 1965] on extending the classical set theory (crisp set) where an element  $x \in \Omega$  can be either "in" or "not in" a certain set  $A \in \mathcal{A}$  (class of subsets of  $\Omega$ ). In this section we will use the definitions in [Zadeh, 1965] and in *Fuzzy Sets and Systems: Theory and Applications* by D. Dubois and H. Prade

Let  $\Omega$  be a space of points (objects), with a generic element of  $\Omega$  denoted by  $\omega$ . Thus,  $\Omega = \{\omega\}$ .

**Definition 10** (Fuzzy set). A fuzzy set (class)  $A$  in  $\Omega$  is characterized by a membership (characteristic) function  $\mu_A(\omega)$  which associates with each point in  $\Omega$  a real number in the interval  $[0, 1]$ , with the value of  $\mu_A(\omega)$  at  $\omega$  representing the "grade of membership" of  $\omega$  in  $A$ . The nearer the value of  $\mu_A(\omega)$  to unity, the higher the grade of membership of  $\omega$  in  $A$ .

$A$  is characterized by the set of pairs

$$A = \{(\omega, \mu_A(\omega)), \omega \in \Omega\}$$

As we have seen in the definition, comparing to classical set theory, fuzzy set theory has more space for uncertainty thanks to the membership function.

For  $\omega \in \Omega$

- $\mu_A(\omega) = 1$  signifies that  $\omega$  is a full member of  $A$ ;
- $\mu_A(\omega) = 0$  signifies that  $\omega$  is not a member of  $A$ ;
- $0 < \mu_A(\omega) < 1$  signifies that  $\omega$  is a fuzzy member of  $A$ ;

Indeed, when the membership function is binary, i.e.

$$\mu_A(\omega) : \Omega \rightarrow \{0, 1\}$$

it is equivalent to the indicator function in classical probability theory, i.e.

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

The *equality* of two fuzzy sets is defined by total equal membership functions, i.e.,  $A = B$  if and only if  $\mu_A(\omega) = \mu_B(\omega)$  for all  $\omega$  in  $\Omega$ .

The *containment* between two fuzzy sets is defined as follow:  $A \subset B$  if and only if  $\mu_A \leq \mu_B$ . The classical set operation *union*( $\cup$ ) and *intersection*( $\cap$ ) can be extended to fuzzy sets  $A$  and  $B$  in  $\Omega$  as follow

$$\forall \omega \in \Omega, \mu_{A \cup B}(\omega) = \max(\mu_A(\omega), \mu_B(\omega))$$

$$\forall \omega \in \Omega, \mu_{A \cap B}(\omega) = \min(\mu_A(\omega), \mu_B(\omega))$$

where  $\mu_{A \cup B}$  and  $\mu_{A \cap B}$  are respectively the membership functions of  $A \cup B$  and  $A \cap B$ . The *complement*  $\bar{A}$  of a fuzzy set  $A$  is defined by the membership function

$$\forall \omega \in \Omega, \mu_{\bar{A}}(\omega) = 1 - \mu_A(\omega)$$

We also have the following properties on set operations which are true for both crisp sets and fuzzy sets:

- Commutativity:  $A \cup B = B \cup A$ ;  $A \cap B = B \cap A$
- Associativity:  $A \cup (B \cup C) = (A \cup B) \cup C$ ,  $A \cap (B \cap C) = (A \cap B) \cap C$
- Idempotency:  $A \cup A = A$ ,  $A \cap A = A$
- Distributivity:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cap \emptyset = \emptyset$ ,  $A \cup \Omega = \Omega$
- Identity:  $A \cup \emptyset = A$ ,  $A \cap \Omega = A$
- Absorption:  $A \cup (A \cap B) = A$ ,  $A \cap (A \cup B) = A$
- De Morgan's laws:  $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$ ,  $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$
- Involution:  $\overline{\bar{A}} = A$
- Equivalence formula:  $(\bar{A} \cup B) \cap (A \cup \bar{B}) = (\bar{A} \cap \bar{B}) \cup (A \cap B)$
- Symmetrical difference formula:  $(\bar{A} \cap B) \cup (A \cap \bar{B}) = (\bar{A} \cup \bar{B}) \cap (A \cup B)$

The excluded-middle law no longer stays true for fuzzy sets, i.e.

$$A \cap \bar{A} \neq \emptyset, \quad A \cup \bar{A} \neq \Omega$$

**Definition 11.** A fuzzy set  $A$  is said to be *normalized* if and only if  $\exists \omega \in \Omega$  such that  $\mu_A(\omega) = 1$

**Definition 12.** The  $\alpha$ -cuts of a fuzzy set  $A$ ,  $A^\alpha$  is the fuzzy subset of  $A$  such that

$$A^\alpha = \{x : \mu(x) \geq \alpha\}$$

so that

$$A = \bigcup_{\alpha \in [0,1]} A^\alpha$$

and its membership function is given by

$$\mu_{A^\alpha}(x) = \alpha A(x)$$

**Definition 13.** A fuzzy set  $A$  is *convex* iff its  $\alpha$ -cuts are convex, i.e.

$$\mu_A(\lambda \omega_1 + (1 - \lambda) \omega_2) \geq \min(\mu_A(\omega_1), \mu_A(\omega_2)) \quad \forall \omega_1 \in \Omega, \forall \omega_2 \in \Omega, \forall \lambda \in [0, 1],$$

## Fuzzy numbers and fuzzy interval

**Definition 14** (Fuzzy number [Carlsson and Fullér, 2005]). A *fuzzy number*  $X$  is a fuzzy set of the real line  $\mathbb{R}$  with a normal, fuzzy convex and continuous membership function of bounded support.

A general fuzzy number membership function can be given as follow [Carlsson and Fullér, 2005]

$$\mu_X(\omega) = \begin{cases} L\left(\frac{a-\omega}{\alpha}\right) & \text{if } \omega \in [a-\alpha, a] \\ 1 & \text{if } \omega \in [a, b], \quad a \leq b \\ R\left(\frac{\omega-b}{\beta}\right) & \text{if } \omega \in [b, b+\beta] \\ 0 & \text{otherwise} \end{cases}$$

where  $[a, b]$  is the peak of  $A$  and  $a$  and  $b$  are the lower and upper modal values;  $L$  and  $R$  are shape functions:  $[0, 1] \rightarrow [0, 1]$  with  $L(0) = R(0) = 1$  and  $L(1) = R(1) = 0$  which are non-increasing, continuous mappings.

The fuzzy set is a fuzzy number if  $a = b$ ; otherwise, it is called a fuzzy interval. A value  $x$  is more likely to be included in the fuzzy set in this case as it is closer to the modal value or here the bounds of  $[a, b]$ . Among all possible membership functions, triangular/trapezoidal ones are widely used with both sides linearly increasing or decreasing.

**Example 10** (Triangular fuzzy number(TFN)).

$$\mu_X(x) = \begin{cases} \frac{x-c}{a-b}, & \text{if } x \in [b, a] \\ \frac{c-x}{c-a}, & \text{if } x \in [a, c] \\ 0, & \text{otherwise.} \end{cases}$$

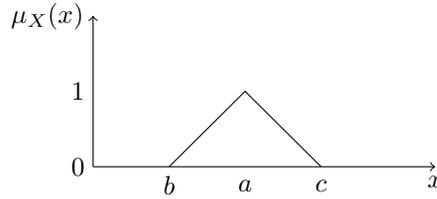


Figure 2.1: Triangular fuzzy number (TFN) membership function

We notice that the degree of uncertainty becomes higher as the size of the support space,  $[b, c]$ , increases.

**Example 11** (Trapezoidal fuzzy interval).

$$\mu_X(x) = \begin{cases} \frac{x-c}{a-c}, & \text{if } x \in [c, a] \\ 1, & \text{if } x \in [a, b] \\ \frac{d-x}{d-b}, & \text{if } x \in [b, d] \\ 0, & \text{otherwise.} \end{cases}$$

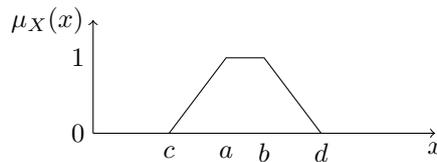


Figure 2.2: Trapezoidal fuzzy interval membership function

### 2.3.2 Theory of possibility

Possibility and necessity functions (theories) is firstly interpreted formally under fuzzy set framework by Zadeh [Zadeh, 1978]. The membership function defines gradual relationship between an element and a fuzzy set. The possibility function and the necessity function evaluate the relationship between subsets  $\Omega$  and the fuzzy set. The theory is then developed by Dubois and Prade [Dubois and Prade, 1988].

In this section we recall the basic notions in [Georgescu, 2012] and [Dubois and Prade, 1988].

Let  $\Omega$  be a non-empty set and  $2^\Omega$  its power set.

**Definition 15.** A *possibility measure* on  $\Omega$  is a function  $\Pi : 2^\Omega \rightarrow [0, 1]$  such that the following conditions are verified:

- $\Pi(\emptyset) = 0$
- $\Pi(\Omega) = 1$
- For any family  $\{A_i\}_{i \in I}$  of subsets of  $\Omega$ ,

$$\Pi\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \Pi(A_i)$$

**Definition 16.** A *possibility distribution* on  $\Omega$  is a function  $\mu : \Omega \rightarrow [0, 1]$  such that

$$\sup_{x \in \Omega} \mu(x) = 1$$

$\mu$  is said to be *normalized* if  $\mu(x) = 1$  for at least one  $x \in \Omega$ .

Under the fuzzy set framework the possibility distribution can also be seen as a membership function. Then if  $\Omega$  is finite, the possibility function can be defined as

$$\Pi(A) = \sup\{\mu(\omega) \mid \omega \in A\}, \quad \forall A \subseteq \Omega$$

The condition  $\Pi(A) = 1$  indicates “ $A$  is possible” which means that a subset of  $A$  is sure. However, there is no information about the location of the sure part inside  $A$ .

Then a conjugate measure is introduced as necessity.

**Definition 17.** A *necessity measure* on  $\Omega$  is a function  $N : 2^\Omega \rightarrow [0, 1]$  such that the following conditions are verified:

- $N(\emptyset) = 0$
- $N(\Omega) = 1$
- For any family  $\{A_i\}_{i \in I}$  of subsets of  $\Omega$ ,

$$N\left(\bigcup_{i \in I} A_i\right) = \inf_{i \in I} N(A_i)$$

$N(A) = 1$  means that  $A$  is sure or necessarily true. Thus, the possibility measure also can be defined as

$$\Pi(A) = 1 - N(\bar{A}), \quad A \subseteq \Omega$$

which means an event is *necessary* when its contrary is impossible. Using this relation necessity function can be constructed by possibility distribution as follows

$$N(A) = \inf\{1 - \mu(\omega) \mid \omega \notin A\}$$

**Proposition 1.** Let  $\Pi$  be a possibility measure and  $N$  a necessity measure on  $\Omega$ . For any  $A_1, A_2 \in 2^\Omega$ , if  $A_1 \subseteq A_2$ , then  $\Pi(A_1) \leq \Pi(A_2)$  and  $N(A_1) \leq N(A_2)$ .

The other properties are also verified

- $\Pi(A) \geq N(A), \quad \forall A \subseteq \Omega$
- $N(A) > 0 \Rightarrow \Pi(A) = 1$
- $\Pi(A) < 1 \Rightarrow N(A) = 0$
- $\Pi(A) + \Pi(\bar{A}) \geq 1, \quad \forall A \subseteq \Omega$
- $N(A) + N(\bar{A}) \leq 1, \quad \forall A \subseteq \Omega$

## 2.4 Theory of belief functions

### 2.4.1 Introduction

Theory of evidence is firstly introduced in [Dempster, 1968] as Dempster's upper and lower probabilities and in [Shafer, 1976] as Shafer's theory of evidence. The aim of this theory is to represent degree of belief with ignorance such as we know the truth lies on a domain  $A$  for a certain degree of belief but there is no information indicating where exactly the truth locates inside  $A$ . In order to achieve this goal, Dempster-Shafer theories extend the axiom of additivity in definition of probability theory while constructing basic probability numbers.

### 2.4.2 Dempster-Shafer (DS) theory of belief functions

In this section we adopt the basic concepts of belief functions formalized in *Mathematical Theory of Evidence* by Glenn Shafer [Shafer, 1976].

**Definition 18** (Frame of discernment). Given a question of interest, let  $\Omega$  be a finite set of possible answers to the question (*frame*), called a *frame of discernment*, and  $2^\Omega$  be the set of all subsets of  $\Omega$ :

$$2^\Omega = \{A | A \subseteq \Omega\}$$

**Definition 19** (Belief function). A real function over the subsets  $Bel : 2^\Omega \rightarrow [0, 1]$  is called a *belief function* if and only if it satisfies the following three axioms:

- Axiom 1:  $Bel(\emptyset) = 0$
- Axiom 2:  $Bel(\Omega) = 1$
- Axiom 3: For any whole number  $n$  and subsets  $A_1, A_2, \dots, A_n \subset \Omega$ ,

$$Bel\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{I \subset \{1,2,\dots,n\}; I \neq \emptyset} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right)$$

where  $|I| = \text{card}(I)$ .

**Definition 20.** A function  $m : 2^\Omega \rightarrow [0, 1]$  is called a *mass function* (also called *basic belief assignment* or *basic probability assignment*) if

- $m(\emptyset) = 0$
- $\sum_{A \subseteq \Omega} m(A) = 1$

An element  $A \in \Omega$  is called a *focal element* if  $m(A) > 0$ .

Belief function is not additive and is able to represent the degree of belief with ignorance. Thus, the mass function can not be divided into single elements of  $\Omega$  (limited divisibility) so that only focal elements have their mass function different than 0. Notice that probability measure is a special case of belief functions on limiting Axiom 3 to total equality, and a probability distribution over  $\Omega$  is a special case of mass function such that only singletons  $\{x\}$ ,  $x \in \Omega$  are focal elements.

However, if the mass function of empty set is not zero,  $m(\emptyset) > 0$ , the mass function is called *unnormalized*. In this case, the sample space is not properly defined due to lack of information which can be used to evaluate the used model of the random phenomenon. Generally, as the hypothesis are validated, we consider that  $m(\emptyset) = 0$  by default.

A mass function that has  $\Omega$  as unique focal element, i.e.  $m(\Omega) = 1$  is called *vacuous*, which under the closed world hypothesis, represents total ignorance.

Given mass functions  $m(A)$  the belief function is defined by

$$Bel(A) = \sum_{B \subseteq A} m(B)$$

**Definition 21** (Plausibility function). A *plausibility function*  $Pl(A)$  is defined as the sum of beliefs not committed to  $\bar{A}$

$$Pl(A) = 1 - Bel(\bar{A})$$

also as

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

For any  $A \in \Omega$ , we always have

$$Bel(A) \leq Pl(A)$$

Given a belief function or a plausibility function one may construct such a mass function for each  $A \in \Omega \setminus \{\emptyset\}$  using Möbius transformation

$$\begin{aligned} m(A) &= \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|A \setminus B|} Bel(B) \\ &= \sum_{B \subseteq A, B \neq \emptyset} (-1)^{|A \setminus B|} (1 - Pl(\bar{B})) \end{aligned}$$

where  $|A \setminus B|$  is the cardinality of  $A \setminus B$ .

The belief function measures the degree of belief in a conservative point of view. On the contrary, the plausibility function gives the largest possible field of the studied set and the degree of belief in an optimistic point of view. Then the expectation of degree of belief for an event  $A$  is often expressed in form of  $[bel(A), pl(A)]$  where interval itself represents the uncertainty estimated. However, at decision level we still need a expectation with precision. From different propositions [Sudano, 2006], the pignistic transformation is the most used and gives a *pignistic probability*:

$$BetP(B) = \sum_{A \subseteq \Omega} \frac{|A \cap B|}{|A|} \frac{m^\Omega(A)}{(1 - m^\Omega(\emptyset))}$$

Consider that the focal sets of the mass function are consonant/nested, i.e.

$$A_1 \subset A_2 \subset \dots \subset A_{n_{focal}}$$

Then we have

$$pl(A \cap B) = \max(pl(A), pl(B)), \quad \forall A, B \subseteq \Omega$$

the plausibility function is a possibility function and the belief function is equivalent to the necessity function. Moreover, the possibility function can be seen as a contour function

$$\mu(t) = pl(\{x\}), \quad t \in \Omega$$

The theory of belief function generalizes theory of possibility and as result, the calculation of theory of belief function is more complex than the one of possibility theory.

### 2.4.3 Belief function on $\mathbb{R}$

In [Smets, 2005], the belief function is extended from finite frame of discernment to the extended set of real numbers  $\mathfrak{R} = [-\infty, +\infty]$  with all focal elements closed intervals on  $\mathbb{R}$ .

**Definition 22.** The set  $\mathfrak{R} = \mathbb{R} \cup \{-\infty, +\infty\}$  obtained by adjoining the two infinity elements to the set of real numbers  $\mathbb{R}$  is called the set of extended real numbers.

**Definition 23.** Suppose  $\alpha, \beta \in \mathfrak{R}$ ,  $\alpha < \beta$ . The  $\mathfrak{F}$  set is defined as the set of all intervals on  $\mathfrak{R}$  including the empty set and  $[-\infty, y]$ ,  $[x, \infty]$  and  $[-\infty, \infty]$ , i.e.

$$\mathfrak{F} = \{[x, y], (x, y), [x, y), (x, y) : x, y \in \mathfrak{R}\}$$

**Definition 24.** A basic belief density  $m$  is a non-negative function on  $\mathfrak{F}$  such that  $m(A) = 0$  if  $A$  is not a closed interval in  $\mathfrak{F}$  or  $\emptyset$  and

$$\kappa = \int_{x=-\infty}^{x=+\infty} \int_{y=-\infty}^{y=+\infty} m([x, y]) dy dx \leq 1$$

and  $m(\emptyset) = 1 - \kappa$ .

The mass function is transformed into probability density function(pdf), i.e.

$$f(a, b) = m([a, b]), \text{ if } a \leq b \tag{2.9}$$

$$= 0, \text{ otherwise} \tag{2.10}$$

The related functions (belief, plausibility, etc.) are defined on the Borel  $\sigma$ -algebra of closed intervals on  $\mathfrak{R}$ .

- Belief function:

$$Bel([a, b]) = \int_{x=a}^{x=b} \int_{y=x}^{y=b} m([x, y]) dy dx \quad Bel(\emptyset) = 0$$

- Plausibility function:

$$Pl([a, b]) = \int_{x=-\infty}^{x=b} \int_{y=\max(a, x)}^{y=+\infty} m([x, y]) dy dx \quad pl(\emptyset) = 0$$

#### 2.4.4 Uncertainty propagation: combination rules and operations

For evidence theory, the uncertainties from different random phenomena propagate as mass functions and are combined according to different combination rules and operations: vacuous extension, marginalization, conjunctive rule of combination and the Möbius Transform.

Let the mass function defined on Cartesian production space  $\Omega_i \times \Omega_j$ ,  $m^{\Omega_i \Omega_j} : 2^{\Omega_i} \times 2^{\Omega_j} \rightarrow [0, 1]$ .

**Definition 25.** The *marginalization operator* on  $\Omega_i$  transfers  $m^{\Omega_i \Omega_j}$  into a marginal mass function  $m^{\Omega_i \Omega_j \downarrow \Omega_i} : \mathcal{A}_i \rightarrow [0, 1]$  by

$$m^{\Omega_i \Omega_j \downarrow \Omega_i}(A) = \sum_{B \subseteq \Omega_i \times \Omega_j / Proj(B \downarrow \Omega_i) = A} m^{\Omega_i \Omega_j}(B)$$

for all  $A \subseteq \Omega_i$  or  $A \in \mathcal{A}_i$  where  $Proj(B \downarrow \Omega_i) = \{a \in \Omega_i | \exists b \in \Omega_j, (a, b) \in B\}$ .

**Definition 26.** *Vacuous extension* is the inverse operation which projects a mass function  $m^{\Omega_i} : \mathcal{A}_i \rightarrow [0, 1]$  into space  $\Omega_i \times \Omega_j$  as follow

$$m^{\Omega_i \uparrow \Omega_i \Omega_j}(A) = \begin{cases} m^{\Omega_i}(B) & \text{if } A = B \times \Omega_j \\ 0 & \text{otherwise} \end{cases}$$

Vacuous extension is the easiest way to extend the information including uncertainty of a single random component to a space combined of multiple random elements supposing that all components are independent and without any other assumption.

Combination rules combine mass functions defined on the same space and describing the same random phenomenon with different sources into a single mass function including all "useful" information. There are several different combination rules proposed with hypothesis on degree of belief on each source.

**Example 12.** Conjunctive rule of combination aggregates two mass functions  $m_1$  and  $m_2$  from distinct and reliable sources.

$$(m_1 \cap m_2)(A) = \sum_{C \cap B = A} m_1(B) m_2(C), \quad A, B, C \subseteq \Omega$$

**Example 13.** Dempster's rule of combination is a normalized alternative of conjunctive combination rule:

- $(m_1 \oplus m_2)(\emptyset) = 0$
- $(m_1 \oplus m_2)(A) = \frac{1}{\kappa} \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \neq \emptyset$

where  $\kappa = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) < 1$  quantifies the conflict between two mass functions.

**Example 14.** Disjunctive rule of combination is based on assumption that at least one of the two distinct information sources of mass functions  $m_1$  and  $m_2$  is reliable:

$$(m_1 \cup m_2)(A) = \sum_{C \cup B = A} m_1(B) m_2(C), \quad A, B, C \subseteq \Omega$$

The next step is to use the structure function  $\varphi : \Omega_1 \times \dots \times \Omega_n \rightarrow \Omega_s$  to obtain the mass functions at system level. In [Sallak et al., 2013] a configuration mass function is proposed which contains all possible combination between components and system  $m^{Config} : \Omega_1 \times \dots \times \Omega_n \times \Omega_s \rightarrow [0, 1]$ . This mass function is then combined with the mass function  $m^{\Omega_1 \dots \Omega_n}$  obtained by vacuous operation representing component level information by different combination rules. Finally, the system level mass function is calculated by marginalization operator.

## 2.5 Relation between uncertainty theories

As mentioned below, there are many different theories modeling uncertainties. There are a lot of similarities between them. In this section we are going to discuss the relation between these theories from different point of views. Finally, we also talk about how to combine information modeled by different representations.

**Element functions** In our opinion, starting with probability theory and set theory, the elementary functions proposed by other uncertainties theories are expanded from the same framework but with different constraints relaxed:

- **Imprecise probabilities:** Instead of using only one probability distribution to describe random behavior of a variable, imprecise probabilities propose a set of probability distributions. Each element of imprecise probability has all properties of probability distributions and gives different probability values for all events. Hence, the uncertainty is represented by the range of these probabilities.
- **Belief function:** Comparing with the ones defined in probability distributions (Probability mass function) which represents probability of singleton/atomic event,  $p_X(x) = \mathbb{P}(\{X = x\})$ , the mass of other events are given by additivity, the mass functions in belief function theory can be attributed to all events. Probability of unions of elementary events  $A$  and  $B$  can be given by

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

However, if the mass function value of such union of events is not not especially attribute, the mass function in belief theory gives 0 which indicates no certain/specific information is available for event  $A \cup B$ . On the contrary, if no information of a certain subset member is provided, the corresponding mass value should be attributed to this subset from a conservative point of view.

- **Fuzzy set and Possibility:** Fuzzy set theory extends the binary membership indicator to a function whose value varies from 0 to 1. Then the membership function can present how likely or probable a certain value is a member of the studied set.
- **Random set theory generalizes** the studied random outputs from a certain value to a set within similar probability space. In order to describe relation between two sets, the definition of event are extended to whether the sets have common members, which unlike total equality presents corresponding uncertainty.

**Measure functions** Next, we compare measures proposed by these theories. The lower and upper probabilities in imprecise probabilities are not probability measures as it is not sure that they come from the same distributions. Under the framework of measure theory, the membership function can also be defined as a monotonic measure on  $\Omega$  [Choquet, 1954, Sugeno, 1974]

**Definition 27** (Fuzzy measure [Sugeno, 1974]). Let  $\mathcal{A}$  be an  $\sigma$ -algebra on  $\Omega$ . A fuzzy measure is  $\mu : \mathcal{A} \rightarrow [0, 1]$  if it satisfies

- $\mu(\emptyset) = 0; \mu(\Omega) = 1$ .
- **Monotony:** if  $A, B \in \mathcal{A}$  and  $A \subseteq B$ , then  $\mu(A) \leq \mu(B)$

Probability, belief function, plausibility function, membership function, necessity and possibility functions, upper and lower probabilities, and random set probability are fuzzy measures [Garmendia, 2005] over the same measurable space  $(\Omega, \mathcal{A})$ . However, the mass function in belief function theory does not satisfy any of these above conditions so that it is not a fuzzy measure.

In a measurable space  $(\Omega, \mathcal{A})$ , theory of evidence introduces two principle fuzzy measures: belief function and plausibility function. Belief function  $Bel$  is super-additive such that

$$Bel(A \cup B) \geq Bel(A) + Bel(B)$$

Its dual function plausibility  $Pl$  is a sub-additive measure such that

$$Pl(A \cup B) \leq Pl(A) + Pl(B)$$

Uncertainty theory	Type of uncertainties	Difference comparing with probability theory	Measure
Probability theory	Aleatory		Probability
Imprecise probabilities	Aleatory; epistemic	A set of probability distributions	Upper and lower probabilities
Theory of belief functions	Epistemic	Need to specify mass of each subset	Belief; plausibility
Theory of possibility (based on fuzzy set)	Epistemic	Continue indicator function	Possibility; necessity
Random set theory	Aleatory; epistemic	Set-valued random variable	Capacity function

Table 2.1: Comparison of uncertainty theories

Since for any set  $I$ ,

$$\Pi\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \Pi(A_i)$$

$$N\left(\bigcap_{i \in I} A_i\right) = \inf_{i \in I} N(A_i)$$

possibility measure and necessity measure verify respectively super-additivity and sub-additivity. As in probability theory, we also have

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$$

in terms of additivity, the possibility measure is a plausibility function and the necessity and probability measures are special case of belief function.

The probability (capacity functional) measure in random set theory can be also seen as a fuzzy measure.

**Random set framework** Random set generalizes the probability theory. Under the random set framework, several widely used uncertainty theories can be treated as various interpretations of capacities [Nguyen, 2006]. The imprecise probability (upper and lower probabilities) can be seen as consonant(nested) random sets. The capacity functional is the plausibility function in term of theory of evidence with normalization. The containment functional of a random closed set is equivalent to the continuous belief function. The possibility and necessity function are also able to be interpreted with random set thanks to one-point-coverage function of random set which play the same role as membership function on fuzzy set theory.

## 2.6 Conclusion

In all the theories mentioned in previous sections, the notion of "set" is used to present epistemic uncertainty of a certain point value. The objective of this thesis is to propose a method to present both stochastic and epistemic uncertainties under the same framework. After comparing different uncertainty theories Table 2.1, imprecise probabilities and random set theory are able to present both types of uncertainties, and these two theories can be transferred between each other generally. In the following chapters, we use random set to present system dependability and other performance indicators and the associated uncertainties for the following reasons:

- Random set presents both stochastic and epistemic uncertainties. The notion of "set" is represented in a more comprehensible and direct way.
- Random set is a general form of random variable and can be seen as imprecise probabilities so that it is compatible with results obtained from classical probability theory imprecise probabilities.

## Chapter 3

# Reliability assessment on binary systems

### 3.1 Binary system modeling

In this section, we start to apply random set theory on system reliability assessment problem by studying the simplest component and system model where there are a full working state (good state) and a complete failure state (down state). Here, we use the notations and definitions from *Fault Trees* by N. Limnios [Limnios, 2007].

#### 3.1.1 Basic modeling

Consider a binary system composed of  $d$  binary components indexed by

$$\mathcal{C} = \{1, \dots, d\}.$$

The state of each component  $C_i$  ( $i = 1, \dots, d$ ) is represented by binary random variable  $x_i$  such that

$$x_i = \begin{cases} 1 & \text{if component } C_i \text{ is in good state} \\ 0 & \text{if component } C_i \text{ is in down state} \end{cases}$$

Let  $\mathbf{x} = (x_1, \dots, x_d) \in \{0, 1\}^d$  denote the vector describing the states of all components. Then the state of the system is determined by the states of all its components through a structure function

$$\varphi : \{0, 1\}^d \rightarrow \{0, 1\}$$

so that

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if the system is in good state} \\ 0 & \text{if the system is in down state} \end{cases}$$

Let binary variable  $x_s$  describe the system state, we have

$$x_s = \varphi(x_1, \dots, x_d)$$

Then the information concerning the system  $S$  can be represented by the couple

$$S = (\mathcal{C}, \varphi).$$

Let  $1_i$  and  $0_i$  ( $i \in \mathcal{C}$ ) respectively denote the conditions  $x_i = 1$  and  $x_i = 0$ , i.e.

$$(1_i, \mathbf{x}) = (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_d)$$

$$(0_i, \mathbf{x}) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$$

The structure function is composed of binary operations:

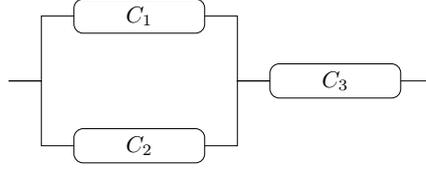


Figure 3.1: A system shown in RBD

- Addition ( $\dot{+}$ ): Let  $x_A$  and  $x_B$  denote respectively the binary indicator of “good state” events for component  $A$  and  $B$ . Consider that the system is in good state ( $x_S = 1$ ) if and only if  $A$  or  $B$  is in good state, the relation can be described by the set union operation  $\cup$  and binary operation  $\dot{+}$

$$S = A \cup B \Leftrightarrow x_S = x_A \dot{+} x_B = x_A + x_B - x_A x_B$$

The corresponding vector operation is defined as follows

$$\mathbf{x} \dot{+} \mathbf{y} = (x_1 \dot{+} y_1, \dots, x_d \dot{+} y_d)$$

- Multiplication ( $\cdot$ ): The system state is in good state if and only if both components  $A$  and  $B$  are in good state. The relation corresponds set event operation  $\cap$  and binary operation  $\cdot$

$$S = A \cap B \Leftrightarrow x_S = x_A \cdot x_B = x_A x_B$$

The corresponding vector operation is defined as follows

$$\mathbf{x} \cdot \mathbf{y} = (x_1 \cdot y_1, \dots, x_d \cdot y_d)$$

- Negation ( $\bar{x}$ ): Negation operation gives the complement of  $x$  such that

$$\bar{x} = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

The corresponding vector operation is defined as follows

$$\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_d)$$

### 3.1.2 Graphical representation of systems

#### Reliability Block Diagrams (RBDs)

Reliability Block Diagrams (RBDs) represent systems by a flow diagram:

- The flow cannot pass components and subsystems in down state.
- If there exists at least one path which allows the flow to pass from input to output, the system is in good state; otherwise the system is in down state.

For example, the structure function of the system shown in Figure 3.1 is given by

$$\varphi(\mathbf{x}) = (x_1 \dot{+} x_2) \cdot x_3 \quad (3.1)$$

In this thesis, we use RBD to present the simple systems for clarity.

#### Fault Trees (FTs)

Fault tree (FT) shows the relation between system failure or undesirable event and elementary failure event in tree structure:

- The root (the top node in a tree) denotes the system failure (or undesirable event).
- Each leaf (node with no children) represents an exact situation, for example, a component failure.
- Internal nodes (nodes with at least one child) are called gates which represent logical relationship between nodes, for example, logical AND/OR.

The system in Figure 3.1 can be also shown in fault tree shown (Figure 3.2) where the logical operation  $\dot{+}$  is equivalent to logical gate OR and  $\cdot$  to logical gate AND.

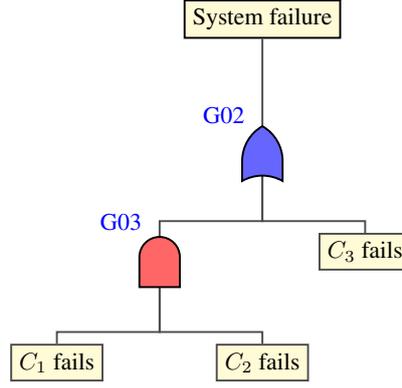


Figure 3.2: Fault tree (FT) of system shown in Figure 3.1

### Binary decision diagram (BDD)

Binary decision diagram (BDD) is a directed acyclic graph, where all paths through it start at the root node and end in one of the two terminal nodes denoting system states: system good state  $1_s$  and system down state  $0_s$  [Remenyte-Prescott and Andrews, 2008]. Except terminal nodes, all other nodes denote a component  $C_i$  from which there are two types of arrows: 0-arrow (dashed arrow) denotes the condition that  $C_i$  is in down state (event  $0_i$ ); 1-arrow (solid arrow) represents the condition that the component  $C_i$  is in good state (event  $1_i$ ). Each path from the root to a terminal node describes a compressed correspondence between component state condition (nodes and arrows) and system state (terminal nodes). All possible conditions which can determine the system state should be shown in BDD.

For example, Figure 3.3 illustrates the system shown in Figure 3.1. There are two paths ending with  $1_s$ :

$$(1_3, 1_2, 1_s), (1_3, 0_2, 1_1, 1_s)$$

They describe all possible conditions resulting in system good state ( $1_s$ ) so that we can rewrite the structure function by translating  $1_i$  as  $x_i$  and  $0_i$  as  $\bar{x}_i$  and then regrouping all paths using operator  $\dot{+}$ . The obtained structure function is given by

$$\varphi(\mathbf{x}) = x_3x_2 \dot{+} x_3\bar{x}_2x_1$$

which is equivalent to (3.1). With paths terminating in  $0_s$ :

$$(0_3, 0_s), (1_3, 0_2, 0_1, 0_s)$$

we can also rewrite the structure function  $\varphi$  as

$$\bar{\varphi}(\mathbf{x}) = \bar{x}_3 \dot{+} x_3\bar{x}_2\bar{x}_1 \quad \varphi(\mathbf{x}) = 1 - \bar{\varphi}(\mathbf{x})$$

### 3.1.3 Basic properties

In this section, we introduce some basic properties of binary systems.

**Definition 28** (Essential variable [Limnios, 2007]). A variable  $x_i$  is called to be essential if there exists a vector  $\mathbf{x}$  such that

$$\varphi(1_i, \mathbf{x}) \neq \varphi(0_i, \mathbf{x})$$

On the contrary, the variable  $x_i$  is said to be inessential.

**Definition 29** (Equal functions [Limnios, 2007]). Two functions  $\varphi_1(\mathbf{x})$  and  $\varphi_2(\mathbf{x})$  are said to be equal i.e.  $\varphi_1(\mathbf{x}) \equiv \varphi_2(\mathbf{x})$  if one is deduced from the other or vice visa by adjunction or elimination of the inessential variables.

**Definition 30** (Dual function [Limnios, 2007]). Given the function  $\varphi(\mathbf{x})$ , its dual function  $\bar{\varphi}(\mathbf{x})$  is defined as follows

$$\bar{\varphi}(\mathbf{x}) = 1 - \varphi(\mathbf{x})$$

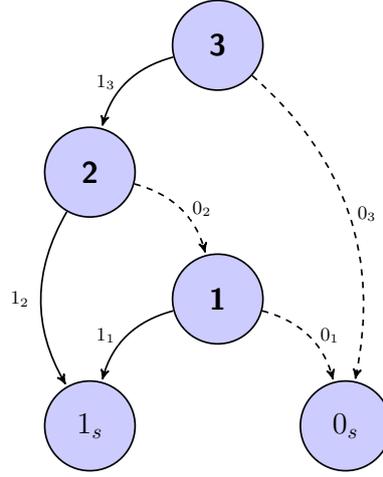


Figure 3.3: Binary decision diagram (BDD) of system shown in Figure 3.1

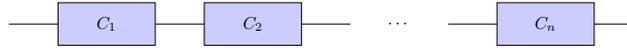


Figure 3.4: Serial system

**Definition 31** (Monotone structure function [Limnios, 2007]). The structure function  $\varphi(\mathbf{x})$  is said to be monotone with respect to the variable  $x_i$  if

$$\varphi(1_i, \mathbf{x}) \geq \varphi(0_i, \mathbf{x})$$

If a structure function  $\varphi(\mathbf{x})$  is monotone with respect to all the variables, then it will be called monotone.

**Definition 32** (Coherent structure function [Limnios, 2007]). A structure function is called coherent if the structure function is monotone and all its variables are essential.

The system is called coherent if its structure function is coherent.

**Example 15** (Elementary structure systems [Limnios, 2007]). - Serial system: A series system is in good state if and only if all its components are in good state simultaneously. All component failures can cause system failure so that its structure function is given by

$$\varphi(\mathbf{x}) = x_1 \cdot \dots \cdot x_d = \min(x_1, \dots, x_n) = \prod_{i=1}^d x_i$$

- Parallel system: On contrary to series systems, a parallel system is in good state if at least of its components is still in good state. The system level failure occurs only if all system components fail so that its structure function is given by

$$\varphi(\mathbf{x}) = x_1 \dot{+} \dots \dot{+} x_d = \max(x_1, \dots, x_d) = 1 - \prod_{i=1}^d (1 - x_i)$$

-  $k$ -out-of- $d$  system: A  $k$ -out-of- $d$  system is composed of  $d$  components. The system is in good state if at least  $k$  ( $1 \leq k \leq d$ ) components are in good state at the same time. The system fails if  $d - k + 1$  components are not in good state at the same time. Its system structure function is given by

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^d x_i \geq k \\ 0 & \text{otherwise} \end{cases}$$

When  $k = 1$ , the system becomes a parallel system; when  $k = d$ , it is a series system. The dual system of a  $k$ -out-of- $d$ :G system is a  $(d - k + 1)$ -out-of- $d$ :F system.

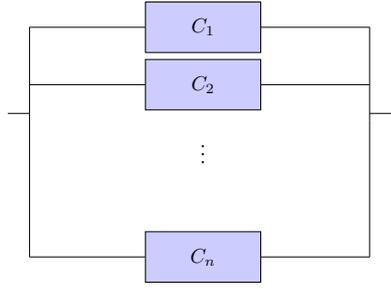


Figure 3.5: Parallel system

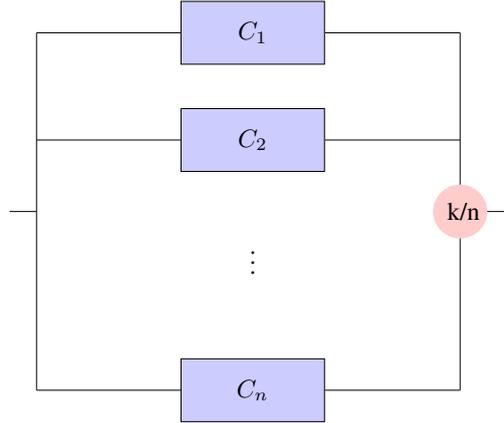


Figure 3.6:  $k$ -out-of- $d$  system

**Definition 33** (Complex system [Limnios, 2007]). A system is called a complex system or a system with complex structure if it cannot be directly decomposed into combination of elementary structures.

In order to describe the structure function of a complex system, the definition of path and cut sets is introduced.

The relation between component-level and system-level states can be simplified by listing only the combinations of good-state components resulting in system good state or the combinations of down-state components causing system failure (down state).

**Definition 34** (Path-set [Limnios, 2007]). A path-set is a sub-set of components whose simultaneous good functioning will assure the good working of the system, which is independent of the states of the other components. A path-set  $\mathcal{P}$  can be presented by component index subset of  $\mathcal{C} = \{1, \dots, d\}$  such that

$$\varphi(x) = 1$$

for every  $x = (x_1, \dots, x_d)$  given by

$$x_i = \begin{cases} 1 & \text{if } i \in \mathcal{P} \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

**Definition 35** (Cut-set [Limnios, 2007]). A cut set is defined as a sub-set of components whose simultaneous failure leads to the system failure, which is independent of the states of the other components. A cut-set  $\mathcal{K}$  can be described by a component index subset of  $\mathcal{C}$  such that

$$\varphi(x) = 0$$

for every  $x = (x_1, \dots, x_d)$  given by

$$x_i = \begin{cases} 0 & \text{if } i \in \mathcal{K} \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

Minimal path-sets and minimal cut-sets describe minimal requirements to obtain system level good or down states.

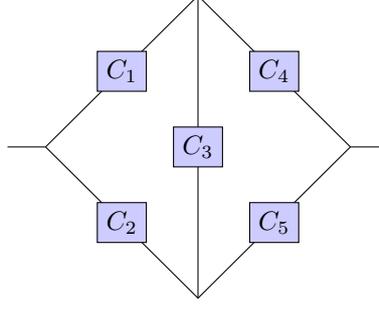


Figure 3.7: Bridge system

**Definition 36** (Minimal path-sets [Limnios, 2007]). A minimal path-set is a path-set which does not contain another path-set.

**Definition 37** (Minimal cut-sets [Limnios, 2007]). A minimal cut-set is a cut-set that does not contain another cut-set.

Let  $\mathcal{K}$  and  $\mathcal{P}$  denote respectively the minimal cut-sets composed of  $n_K$  minimal cuts  $\mathcal{K}_1, \dots, \mathcal{K}_{n_K}$  and minimal path-sets of  $n_P$  minimal paths  $\mathcal{P}_1, \dots, \mathcal{P}_{n_P}$ . The structure function can be written as a function of minimal cut-sets or path-sets, i.e.

$$\varphi(\mathbf{x}) = \prod_{i \in \mathcal{P}_1} x_i \dot{+} \dots \dot{+} \prod_{i \in \mathcal{P}_{n_P}} x_i \quad (3.2)$$

$$= 1 - \left( \prod_{i \in \mathcal{K}_1} \bar{x}_i \dot{+} \dots \dot{+} \prod_{i \in \mathcal{K}_{n_K}} \bar{x}_i \right) \quad (3.3)$$

**Example 16** (Bridge structure). Bridge structure (Figure 3.7) is one of the simplest complex system structures. It cannot be decomposed into series or parallel structures without replicated components. Its minimal cut-sets  $\mathcal{K}$  and path-sets  $\mathcal{P}$  are given by

$$\mathcal{K} = \{(1, 2), (4, 5), (1, 3, 5), (2, 3, 4)\}$$

$$\mathcal{P} = \{(1, 4), (2, 5), (1, 3, 5), (2, 3, 4)\}$$

Then using (3.2) and (3.3) its structure function becomes

$$\begin{aligned} \varphi(\mathbf{x}) &= x_1x_4 \dot{+} x_2x_5 \dot{+} x_1x_3x_5 \dot{+} x_2x_3x_4 \\ \varphi(\mathbf{x}) &= 1 - (\bar{x}_1\bar{x}_2 \dot{+} \bar{x}_4\bar{x}_5 \dot{+} \bar{x}_1\bar{x}_3\bar{x}_5 \dot{+} \bar{x}_2\bar{x}_3\bar{x}_4) \end{aligned}$$

## 3.2 Probabilistic modeling and exact methods

Consider a system  $S = (\mathcal{C}, \varphi)$  as described previously. Let  $X_i$  a binary random variable (r.v.) denote the state of the component  $C_i$  ( $i \in \mathcal{C}$ ), and  $\mathbf{X} = (X_1, \dots, X_d) \in \{0, 1\}^d$  denote the corresponding random vector as well as their complements  $\bar{X}_i = 1 - X_i$ .

The reliability of component  $C_i$  ( $i \in \mathcal{C}$ ),  $p_i$ , is defined as follows

$$p_i := \mathbb{P}[X_i = 1] = \mathbb{E}[X_i]$$

so that  $X_i$  follows Bernoulli distribution

$$X_i \sim \text{Bernoulli}(p_i)$$

We also define the component's failure probability  $q_i$  ( $i \in \mathcal{C}$ ) and the vector reliability (failure probability) of all system components  $\mathbf{p}$  ( $\mathbf{q}$ ) as follows

$$q_i := \mathbb{P}[\bar{X}_i = 1] = \mathbb{E}[\bar{X}_i] = \mathbb{P}[X_i = 0] = 1 - p_i$$

$$\begin{aligned}\mathbf{p} &= (p_1, \dots, p_d) \\ \mathbf{q} &= (q_1, \dots, q_d) = (1 - p_1, \dots, 1 - p_d) \\ \mathbf{p} + \mathbf{q} &= \mathbf{1} = (1, \dots, 1)\end{aligned}$$

Let  $X_s$  denote the binary random variable describing the system state  $x_s$  so that

$$X_s = \varphi(\mathbf{X})$$

The system reliability  $p_s$  (failure probability  $q_s$ ) is defined as

$$\begin{aligned}p_s &:= \mathbb{P}[X_s = 1] = \mathbb{E}[X_s] = \mathbb{P}[\varphi(\mathbf{X}) = 1] = \mathbb{E}[\varphi(\mathbf{X})] \\ q_s &:= \mathbb{P}[X_s = 0] = \mathbb{P}[\varphi(\mathbf{X}) = 0] = 1 - \mathbb{E}[\varphi(\mathbf{X})] = 1 - p_s\end{aligned}$$

Consider that  $X_s$  follows a Bernoulli distribution

$$X_s \sim \text{Bernoulli}(p_s)$$

so as the dual variable  $\bar{X}_s = 1 - X_s$

$$\bar{X}_s \sim \text{Bernoulli}(q_s)$$

The system reliability (failure probability) can be also treated as a function of component reliability  $\mathbf{p}$  (or failure probability  $\mathbf{q}$ )

$$R, Q : [0, 1]^d \longrightarrow [0, 1]$$

such that

$$\begin{aligned}p_s &= R(\mathbf{p}) = R(1 - \mathbf{q}) \\ q_s &= Q(\mathbf{p}) = Q(1 - \mathbf{q})\end{aligned}$$

Suppose that all system components are pairwise independent, we have

$$\begin{aligned}R(\mathbf{p}) &= \mathbb{E}[\varphi(\mathbf{X})] \\ &= \sum_{\mathbf{x} \in \{0,1\}^d} \varphi(\mathbf{x}) \mathbb{P}\{\mathbf{X} = \mathbf{x}\} \\ &= \sum_{\mathbf{x} \in \{0,1\}^d: \varphi(\mathbf{x})=1} \prod_{i \in \mathcal{C}} p_i^{x_i} (1 - p_i)^{1-x_i} \\ Q(\mathbf{p}) &= \mathbb{E}[1 - \varphi(\mathbf{X})] \\ &= \sum_{\mathbf{x} \in \{0,1\}^d} (1 - \varphi(\mathbf{x})) \mathbb{P}\{\mathbf{X} = \mathbf{x}\} \\ &= \sum_{\mathbf{x} \in \{0,1\}^d: \varphi(\mathbf{x})=0} \prod_{i \in \mathcal{C}} p_i^{x_i} (1 - p_i)^{1-x_i}\end{aligned}$$

**Example 17.** For series system composed of  $d$  independent components  $C_1, \dots, C_d$ , using the expression:

$$R(p) = \prod_{i=1}^d p_i$$

For parallel system composed of  $d$  independent components  $C_1, \dots, C_d$ , using the expression:

$$R(p) = 1 - \prod_{i=1}^d (1 - p_i)$$

### 3.2.1 Inclusion-exclusion method

Given  $\mathcal{K} = \{\mathcal{K}_1, \dots, \mathcal{K}_{n_K}\}$  and  $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_{n_P}\}$  the minimal cut-sets and minimal path-sets of the system  $(S, \varphi)$ , these two functions can be written as the union of down-state events in minimal cut-sets or good-state events in minimal path-sets

$$R(\mathbf{p}) = \mathbb{P}\{e_1^{\mathcal{P}} \cup \dots \cup e_{n_P}^{\mathcal{P}}\}$$

$$Q(\mathbf{p}) = \mathbb{P}\{e_1^{\mathcal{K}} \cup \dots \cup e_{n_K}^{\mathcal{K}}\}$$

where  $e_i^{\mathcal{P}}$  and  $e_i^{\mathcal{K}}$  denote respectively the following events

$$e_i^{\mathcal{P}} = \bigcap_{j \in \mathcal{P}_i} 1_j \quad 1 \leq i \leq n_P$$

$$e_i^{\mathcal{K}} = \bigcap_{j \in \mathcal{K}_i} 0_j \quad 1 \leq i \leq n_K$$

Using the inclusion-exclusion formula, we have the following development

$$\begin{aligned} R(\mathbf{p}) &= \mathbb{P}\left(\bigcup_{\mathcal{P}_i \in \mathcal{P}} e_i^{\mathcal{P}}\right) \\ &= \sum_{i=1}^{n_P} \mathbb{P}(e_i^{\mathcal{P}}) - \sum_{i=1}^{n_P-1} \sum_{j=i+1}^{n_P} \mathbb{P}(e_i^{\mathcal{P}} \cap e_j^{\mathcal{P}}) + \dots + (-1)^{n_P-1} P\left(\bigcap_{i=1}^{n_P} e_i^{\mathcal{P}}\right) \end{aligned}$$

$$\begin{aligned} Q(\mathbf{p}) &= \mathbb{P}\left(\bigcup_{\mathcal{K}_i \in \mathcal{K}} e_i^{\mathcal{K}}\right) \\ &= \sum_{i=1}^{n_K} \mathbb{P}(\mathcal{K}_i) - \sum_{i=1}^{n_K-1} \sum_{j=i+1}^{n_K} \mathbb{P}(e_i^{\mathcal{K}} \cap e_j^{\mathcal{K}}) + \dots + (-1)^{n_K-1} P\left(\bigcap_{i=1}^{n_K} e_i^{\mathcal{K}}\right) \end{aligned}$$

Supposing that all components are pairwise independent, we also have

$$R(\mathbf{p}) = \sum_{i=1}^{n_P} \prod_{k \in \mathcal{P}_i} p_k - \sum_{i=1}^{n_P-1} \sum_{j=i+1}^{n_P} \prod_{k \in \mathcal{P}_i \cap \mathcal{P}_j} p_k + \dots + (-1)^{n_P-1} \prod_{k \in (\bigcap_{i=1}^{n_P} \mathcal{P}_i)} p_k$$

$$Q(\mathbf{p}) = \sum_{i=1}^{n_K} \prod_{k \in \mathcal{K}_i} (1 - p_k) - \sum_{i=1}^{n_K-1} \sum_{j=i+1}^{n_K} \prod_{k \in \mathcal{K}_i \cap \mathcal{K}_j} (1 - p_k) + \dots + (-1)^{n_K-1} \prod_{k \in (\bigcap_{i=1}^{n_K} \mathcal{K}_i)} (1 - p_k)$$

or

$$Q(\mathbf{q}) = \sum_{i=1}^{n_K} \prod_{k \in \mathcal{K}_i} q_k - \sum_{i=1}^{n_K-1} \sum_{j=i+1}^{n_K} \prod_{k \in \mathcal{K}_i \cap \mathcal{K}_j} q_k + \dots + (-1)^{n_K-1} \prod_{k \in (\bigcap_{i=1}^{n_K} \mathcal{K}_i)} q_k$$

**Example 18** (Bridge system). According to its minimal cut/path sets, the reliability of the bridge system in Figure 3.7 is given by

$$R(\mathbf{p}) = p_1 p_4 + p_2 p_5 + p_1 p_3 p_5 + p_2 p_3 p_4 - (p_1 + p_2 + p_3 + p_4 + p_5)$$

or

$$Q(\mathbf{q}) = q_1 q_2 + q_4 q_5 + q_1 q_3 q_5 + q_2 q_3 q_4 - (q_1 + q_2 + q_5 + q_4 + q_3)$$

### 3.2.2 Disjunction-Conjunction method

Consider events  $E_1, \dots, E_n$ , for each  $i, j \in \{1, \dots, n\}$  and  $i \neq j$ , we have [Limnios, 2007]

$$E_i \cup E_j = E_i \cup (\bar{E}_i \cap E_j)$$

$$\overline{E_i \cap E_j} = \overline{E_i} \cup (E_i \cap \overline{E_j})$$

The disjunction and the negation of the conjunction of these events  $E_1, \dots, E_n$  become respectively written by

$$\begin{aligned} E_1 \cup E_2 \cup \dots \cup E_n &= E_1 \cup (\overline{E_1} \cap E_2) \cup \dots \cup (\overline{E_1} \cap \dots \cap \overline{E_{n-1}} \cap E_n) \\ \overline{E_1 \cap \dots \cap E_n} &= \overline{E_1} \cup (E_1 \cap \overline{E_2}) \cup \dots \cup (E_1 \cap \dots \cap E_{n-1} \cap \overline{E_n}) \end{aligned}$$

so that these events can be decomposed into disjunctive events. Similarly, the sequential binary operators  $\dot{+}$  and  $\cdot$  have different forms

$$\begin{aligned} x_1 \dot{+} x_2 \dot{+} \dots \dot{+} x_n &= x_1 + \overline{x_1}x_2 + \dots + \overline{x_1}\dots\overline{x_{n-1}}x_n \\ \overline{x_1 \cdot x_2 \dots \cdot x_n} &= \overline{x_1} + x_1\overline{x_2} + \dots + x_1x_2\dots x_{n-1}\overline{x_n} \end{aligned}$$

### 3.3 Asymptotic normality method using bootstrap method

#### 3.3.1 Introduction

To quantify uncertainty under probabilistic framework, confidence interval (CI) is proposed to be a presentation of uncertainty. It gives a range of possible values where the unknown parameter is included in a certain degree.

**Definition 38** (Confidence Interval). A level  $1 - \alpha$  confidence interval for an unknown parameter  $\theta \in \Theta$  is an random interval  $[L, U] \subset \Theta$  obtained from data such that

$$\mathbb{P}(\theta \in [L, U]) = 1 - \alpha$$

Variance is one of the most used probability measures which describe the dispersion due to randomness. The variance of an estimator signifies the quantity of uncertainty of the estimation. Regarding the system as a single component, its estimator is obtained by propagating the component reliability estimation through the reliability function  $R(\mathbf{p})$  but not directly from observations. The variance of the system reliability estimation should be then calculated similarly. In this section, we are going to propose a method based on asymptotic normality hypothesis to calculate the variance of estimator and give the confidence interval for system reliability with independent components.

#### 3.3.2 Hypothesis

Supposing that for each component  $C_i$  ( $i = 1, \dots, d$ ), we have  $n_i$  observations of  $X_i$ . The sample size  $n_i$  can be described as  $n_i := n_i(n)$ ,  $n \in \mathbb{N}^*$ , such that, as  $n \rightarrow \infty$ , we have

$$n_i(n) \rightarrow \infty \tag{3.4}$$

$$\frac{n_i(n)}{n} \rightarrow m_i \tag{3.5}$$

where  $m_i$  is a constant which indicates the relative sample size of the component  $C_i$ . Of course,  $m_i = 1$ , if all components' sample sizes are the same.

We consider that random variable  $X_i$  follows Bernoulli distribution, i.e.

$$X_i \sim \text{Bernoulli}(p_i)$$

$$E(X_i) = p_i$$

$$\text{Var}(X_i) = p_i(1 - p_i)$$

The first hypothesis is that the reliability estimate for each component  $C_i$  in the system,  $\hat{p}_{i,n_i}$ , is strongly consistent, i.e., as  $n_i \rightarrow \infty$ ,

$$\hat{p}_{i,n_i} \xrightarrow{a.s.} p_i \quad i = 1, \dots, d. \tag{3.6}$$

The other hypothesis is that  $\hat{p}_{i,n_i}$  satisfies asymptotic normality, i.e.

$$\sqrt{n_i}(\hat{p}_{i,n_i} - p_i) \xrightarrow{D} N(0, \sigma_i^2) \quad n_i \rightarrow \infty \quad i = 1, \dots, d. \tag{3.7}$$

where  $\sigma_i^2$  is the variance of  $X_i$ .

### 3.3.3 Estimation

The “plug-in” estimator of system reliability is given by

$$\hat{p}_{s,n} = R(\hat{\mathbf{p}}_n)$$

where

$$\hat{\mathbf{p}}_n = (\hat{p}_{1,n_1}, \dots, \hat{p}_{1,n_d}).$$

**Proposition 1.**  $\hat{p}_{s,n}$  has the following properties:

- Convergence

$$\hat{p}_{s,n} \xrightarrow{a.s.} p_s \quad n \rightarrow \infty \quad (3.8)$$

- Asymptotic normality

$$\sqrt{n}(\hat{p}_{s,n} - p_s) \xrightarrow{D} N(0, \sigma_s^2) \quad n \rightarrow \infty \quad (3.9)$$

where

$$\sigma_s^2 = \sum_{i=1}^d [R(\mathbf{p}, 1_i) - R(\mathbf{p}, 0_i)]^2 \frac{\sigma_i^2}{m_i} \quad (3.10)$$

and  $R(\mathbf{p}, 1_i)$  and  $R(\mathbf{p}, 0_i)$  are the conditional system reliability given the state of the component  $c_i$ , i.e.

$$\begin{aligned} R(\mathbf{p}, 1_i) &= \mathbb{P}(X_s = 1 | X_i = 1) \\ &= R(p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_d) \end{aligned}$$

and

$$\begin{aligned} R(\mathbf{p}, 0_i) &= \mathbb{P}(X_s = 1 | X_i = 0) \\ &= R(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_d). \end{aligned}$$

*Proof.* Since  $X_i$  are independent, Shannon relation gives

$$R(p) = p_i R(\mathbf{p}, 1_i) + (1 - p_i) R(\mathbf{p}, 0_i) \quad (3.11)$$

The partial derive of the function  $R(\mathbf{p})$  can be written by

$$\frac{\partial R}{\partial p_i}(\mathbf{p}) = R(\mathbf{p}, 1_i) - R(\mathbf{p}, 0_i) \quad (3.12)$$

We can also say that  $R(\mathbf{p})$  is once differentiable and continuous on  $(0, 1)$ .

- Convergence

From (3.6) we have

$$\hat{\mathbf{p}}_n \xrightarrow{a.s.} \mathbf{p} \quad (3.13)$$

Then, as  $R$  is continuous, applying Continuous Mapping Theorem (CMT) on (3.13), we obtain the following result:

$$R(\hat{\mathbf{p}}_n) \xrightarrow{a.s.} R(\mathbf{p})$$

and

$$\hat{p}_{s,n} \xrightarrow{a.s.} p_s$$

- Asymptotic normality

From (3.4) and (3.5), we have, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \sqrt{n_i(n)}(\hat{p}_{i,n_i} - p_i) &\xrightarrow{D} N(0, \sigma_i^2) \\ \frac{\sqrt{n_i(n)}}{\sqrt{m_i}}(\hat{p}_{i,n_i} - p_i) &\xrightarrow{D} N(0, \frac{\sigma_i^2}{m_i}) \\ \sqrt{n}(\hat{p}_{i,n_i} - p_i) &\xrightarrow{D} N(0, \frac{\sigma_i^2}{m_i}) \end{aligned}$$

As  $\{X_i\}$  are independent and (3.7),

$$\sqrt{n}(\hat{\mathbf{p}}_n - \mathbf{p}) \xrightarrow{D} N(0, \Sigma)$$

where

$$\Sigma = \begin{pmatrix} \frac{\sigma_1^2}{m_1} & & 0 \\ & \ddots & \\ 0 & & \frac{\sigma_d^2}{m_d} \end{pmatrix}$$

Using delta method, we have, as  $n \rightarrow \infty$ ,

$$\sqrt{n}(R(\hat{\mathbf{p}}_n) - R(\mathbf{p})) \xrightarrow{D} N(0, \sigma_s^2)$$

and

$$\sqrt{n}(\hat{p}_{s,n} - p_s) \xrightarrow{D} N(0, \sigma_s^2)$$

with

$$\begin{aligned} \sigma_s^2 &= \nabla R^\top \Sigma \nabla R \\ &= \sum_{i=1}^d [R(\mathbf{p}, 1_i) - R(\mathbf{p}, 0_i)]^2 \frac{\sigma_i^2}{m_i} \end{aligned}$$

where the gradient of  $R(\mathbf{p})$  is given by

$$\nabla R = \left( \frac{\partial R}{\partial p_1}, \dots, \frac{\partial R}{\partial p_d} \right)^\top$$

□

The variance of the system reliability can be estimated by

$$\hat{\sigma}_{s,n}^2 = \sum_{i=1}^d [R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i)]^2 \frac{\hat{\sigma}_{i,n_i}^2}{m_i} \quad (3.14)$$

We also have, for large  $n$ ,

$$\text{Var}(\hat{p}_{s;n}) \approx \frac{\sigma_{s;n}^2}{n}$$

which suggests the estimator

$$\widehat{\text{Var}}(\hat{p}_{s;n}) = \frac{\hat{\sigma}_{s;n}^2}{n}$$

As  $\hat{p}_{i,n_i}$  is a sample mean of  $n_i$  observations on the random variable  $X_i$ ,  $\sigma_i^2/m_i$  indicates the variance of  $\hat{p}_{i,n_i}$ , which is equal to  $\sigma_i^2/n_i$ . From (3.14), we observe directly how reliability uncertainty propagates from single component to system level. The term  $R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i)$ , obtained by partial differentiation of the system reliability with respect to  $p_i$ , is called Birnbaum importance measure [Birnbaum, 1969] [van der Borst and Schoonakker, 2001], which evaluates the influence of component  $C_i$  on system reliability from a structural and also a probabilistic point of view.

The confidence interval of our method is given by

$$[L_\alpha, U_\alpha] = \left[ \hat{p}_{s,n} - \frac{\hat{\sigma}_{s,n} z_{1-\alpha/2}}{\sqrt{n}}, \hat{p}_{s,n} + \frac{\hat{\sigma}_{s,n} z_{1-\alpha/2}}{\sqrt{n}} \right]$$

The width of our confidence interval then becomes

$$U_\alpha - L_\alpha = \frac{2\hat{\sigma}_{s,n} z_{1-\alpha/2}}{\sqrt{n}} \quad (3.15)$$

The width only depends on  $n$  since  $\alpha$  is constant and we assume that  $\hat{\sigma}_{s,n}$  converges to  $\sigma_s$  as  $n \rightarrow \infty$ .

**Example 19** (Application on Series/Parallel System Structures). Our method was tested on several standard structures: series system, parallel system, series-parallel system and parallel-series system. For a parallel system with  $d$  components, we have from (3.12)

$$R(\mathbf{p}) = 1 - \prod_{j=1}^d (1 - p_j)$$

$$\frac{\partial R}{\partial p_i}(\mathbf{p}) = \prod_{j \neq i} (1 - p_j) = \frac{1 - p_s}{1 - p_i} \quad (3.16)$$

$$\sigma_s^2 = \sum_{i=1}^d \left( \frac{1 - p_s}{1 - p_i} \right)^2 \frac{\sigma_i^2}{m_i} \quad (3.17)$$

Based on (3.17), we get the following estimator

$$\widehat{Var}(\hat{p}_{s;n}) = \sum_{i=1}^d \left[ \frac{1 - \hat{p}_{s;n}}{1 - \hat{p}_{i;n_i}} \right]^2 \frac{\hat{\sigma}_{i;n_i}^2}{n_i} \quad (3.18)$$

For a series system with  $d$  components, we also have from (3.12)

$$R(\mathbf{p}) = \prod_{j=1}^d p_j$$

$$\frac{\partial R}{\partial p_i}(\mathbf{p}) = \prod_{j \neq i} p_j = \frac{p_s}{p_i} \quad (3.19)$$

$$\sigma_s^2 = \sum_{i=1}^d \left( \frac{p_s}{p_i} \right)^2 \frac{\sigma_i^2}{m_i} \quad (3.20)$$

Then, according to (3.20), we get the following estimator

$$\widehat{Var}(\hat{p}_{s;n}) = \sum_{i=1}^d \left[ \frac{\hat{p}_{s;n}}{\hat{p}_{i;n_i}} \right]^2 \frac{\hat{\sigma}_{i;n_i}^2}{n_i} \quad (3.21)$$

In a series-parallel system composed of  $d_s$  parallel subsystems, where each subsystem  $g$  is composed of  $d_g$  components, and  $c_{g,i}$  is the  $i$ th component in the  $g$ th subsystem with sample size  $n_{g,i}$ , we have from (3.12) (3.16) and (3.19)

$$R(\mathbf{p}) = \prod_{j=1}^{d_s} \left[ 1 - \prod_{k=1}^{d_j} (1 - p_{j,k}) \right]$$

$$\frac{\partial R}{\partial p_{g,i}}(\mathbf{p}) = \frac{p_s(1 - R_g)}{R_g(1 - p_{g,i})}$$

where  $R_j = 1 - \prod_{k=1}^{d_j} (1 - p_{j,k})$ , and

$$\sigma_s^2 = \sum_{j=1}^{d_s} \sum_{k=1}^{d_j} \left[ \frac{p_s(1 - R_j)}{R_j(1 - p_{j,k})} \right]^2 \frac{\sigma_{j,k}^2}{m_{j,k}}. \quad (3.22)$$

$$\widehat{Var}(\hat{p}_{s;n}) = \sum_{j=1}^{d_s} \sum_{k=1}^{d_j} \left[ \frac{\hat{p}_{s;n}(1 - \hat{R}_j)}{\hat{R}_j(1 - \hat{p}_{j,k})} \right]^2 \frac{\hat{\sigma}_{j,k,n_{j,k}}^2}{n_{j,k}}. \quad (3.23)$$

In a parallel-series system with  $d_s$  series subsystems in parallel, each subsystem  $g$  is composed of  $d_g$  components, and  $c_{g,i}$  is the  $i$ th component in the  $g$ th subsystem with sample size  $n_{g,i}$ . From (3.12) (3.16) and (3.19), we have

$$R(\mathbf{p}) = \prod_{j=1}^{d_s} \left[ 1 - \prod_{k=1}^{d_j} p_{j,k} \right]$$

$$\frac{\partial R}{\partial p_{g,i}}(\mathbf{p}) = \frac{(1 - p_s)R_g}{(1 - R_g)p_{g,i}}$$

where  $R_g = \prod_{j=1}^{d_g} p_{g,j}$ , and

$$\sigma_s^2 = \sum_{j=1}^{d_s} \sum_{k=1}^{d_j} \left[ \frac{(1 - p_s)R_j}{(1 - R_j)p_{j,k}} \right]^2 \frac{\sigma_{j,k}^2}{m_{j,k}}. \quad (3.24)$$

$$\widehat{Var}(\hat{p}_{s;n}) = \sum_{j=1}^{d_s} \sum_{k=1}^{d_j} \left[ \frac{(1 - \hat{p}_{s;n})\hat{R}_j}{(1 - \hat{R}_j)\hat{p}_{j,k}} \right]^2 \frac{\hat{\sigma}_{j,k,n_{j,k}}^2}{n_{j,k}}. \quad (3.25)$$

Then in case of large system, since Birnbaum factor is the difference of system reliability in case that a certain component is surely in good or bad state, we can use similar algorithm as calculating system reliability to estimate our confidence interval (CI) (Algorithm 1).

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**Algorithm 1** Estimate Birnbaum importance factor and system reliability with Monte Carlo simulation/Bootstrap technique

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**Require:** Component reliability estimate  $\hat{p}_{1,n_1}, \dots, \hat{p}_{d,n_d}$ ; number of simulation trials  $N$

Initialize

$$R_p = 0 \quad R_0 = (0, \dots, 0) \quad R_1 = (0, \dots, 0)$$

**for**  $j = 1$  to  $N$  **do**

**for**  $i = 1$  to  $d$  **do**

    Randomly sample  $X_i$  following *Bernoulli*( $\hat{p}_{i,n_i}$ )

**end for**

**for**  $i = 1$  to  $d$  **do**

    Count the simulation results:

$$R_p = R_p + \varphi(X_1, \dots, X_i, \dots, X_d)$$

$$R_0(i) = R_0(i) + \varphi(X_1, \dots, X_i = 0, \dots, X_d)$$

$$R_1(i) = R_1(i) + \varphi(X_1, \dots, X_i = 1, \dots, X_d)$$

**end for**

**end for**

Aggregate the sampled values of  $R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i)$  by average:

$$R(\hat{\mathbf{p}}_n) = \frac{R_p}{N}$$

**for**  $i = 1$  to  $d$  **do**

$$R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i) = \frac{R_1(i) - R_0(i)}{N}$$

**end for**

$$Var(\hat{p}_s) = \sum_{i=1}^d (R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i))^2 \frac{\hat{p}_i(1 - \hat{p}_i)}{n_i}$$

$$CI: [R(\hat{\mathbf{p}}_n) - z_{\alpha/2} \sqrt{Var(\hat{p}_s)}, R(\hat{\mathbf{p}}_n) + z_{\alpha/2} \sqrt{Var(\hat{p}_s)}]$$


---

### 3.3.4 Cases of small sample size or rare event

When the sample sizes are not large enough or the component has high reliability (the failure event is rare), sometimes there is not failure state sample observed so that the consistency of the maximum likelihood estimator which gives 1 as reliability is doubted.

In order to solve problems with small sample size ( $n_i < 30$  or  $k_i = n_i$ ), assumptive information should be added to the estimation procedure. Some of these assumptions simply increase the failure number or sample size and/or add failure sample to the observation sample pool. Bayesian method assumes that the parameter  $p$  follows a certain prior distribution. It is introduced as the subjective uncertainty on the studied quantity before evidence (observations) are brought in. There two kinds of prior distributions: informative priors and uninformative priors. Informative priors express the subjective knowledge on uncertainty. Uninformative priors give only general information about the quantity. The posterior probability distribution is the conditional distribution of the studied quantity with uncertainty given the data.

$$p(p|x_i) = \mathbb{P}(p = p | X_{i,1} = x_{i,1}, \dots, X_{i,n_i} = x_{i,n_i}) = \frac{p(x|p)p(p)}{p(x)}$$

where  $p(x) = \mathbb{P}(X_{i,1} = x_{i,1}, \dots, X_{i,n_i} = x_{i,n_i})$  is supposed to be constant.

Then the estimator of  $p$  and its variance are respectively given the expected value and the variance of  $p$  given evidence  $x$

$$\hat{p}_n = E(p|x)$$

$$Var(\hat{p}_n) = Var(p|x)$$

with normalized posterior probability distribution  $f(p|x)$

$$f(p|x) = \frac{p(x|p)p(p)}{\int_{-\infty}^{+\infty} p(x|p)p(p)dp}$$

In [Leemis, 2006], six different assumptions of prior distributions are considered to build Bayesian estimator of  $p$ , when there is not failure observed, i.e.  $k = n$ :

**Assumption 1.** The observation reflects perfectly the reliability of the studied component so that we believe the fact that there is not any failure and  $X \sim Bernoulli(\hat{p})$

$$\hat{p} = \frac{k}{n} = 1$$

**Assumption 2.** We assume that there is always a failure in the  $n$  observation so that

$$\hat{p} = \frac{n-1}{n}$$

This assumption changes the value of one of observations.

**Assumption 3.** One more failure sample is assumed to be added so that the sample size is increased and

$$\hat{p} = \frac{n}{n+1}$$

**Assumption 4.** Assume that  $p$  follows uniform prior on  $[0, 1]$ , i.e.

$$p \sim U(0, 1)$$

which is equivalent to beta distribution with  $\alpha = 1$  and  $\beta = 1$

$$p \sim Beta(1, 1)$$

Then the posterior distribution becomes

$$p \sim Beta(n+1, 1)$$

and the estimation is given by

$$\hat{p} = \frac{n+1}{n+2}$$

**Assumption 5.** The Clopper-Pearson (CP) exact lower bound at confidence level  $1 - \alpha$  satisfies [Agresti and Coull, 1998]

$$\sum_{i=k}^n \binom{n}{i} p_L^i (1-p_L)^{n-i} = \alpha$$

In this assumption, the prior information is that the probability that  $p < p_L$  is equal to  $\alpha$ , i.e.

$$\mathbb{P}(p \leq p_L) = \alpha$$

With beta prior distribution  $Beta(\alpha, \beta)$ , we have

$$\int_0^{p_L} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = \alpha$$

Let  $\alpha = 1$ , then the prior beta distribution is given by

$$p \sim Beta(1, \frac{\log(1-\alpha)}{\log(1-\alpha^{1/n})})$$

Then the posterior distribution becomes

$$p \sim \text{Beta}(n + 1, \frac{\log(1 - \alpha)}{\log(1 - \alpha^{1/n})})$$

so as the estimation

$$\hat{p} = \frac{n + 1}{n + 1 + \frac{\log(1 - \alpha)}{\log(1 - \alpha^{1/n})}}$$

**Assumption 6.** Jeffreys prior is another non-informative prior based on the assumption that the prior should be proportional to the square root of the determinant of the Fisher information, i.e.

$$p(\vec{\theta}) \propto \sqrt{\det \mathcal{I}(\vec{\theta})}.$$

In case of Bernoulli distribution  $X$ , we obtain

$$p \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$$

so that the posterior distribution and the estimation become

$$p \sim \text{Beta}(n + \frac{1}{2}, \frac{1}{2})$$

$$\hat{p} = \frac{n + \frac{1}{2}}{n + 1}$$

Let  $k_i$  be the number of good-state observations from  $n_i$  samples of component  $C_i$ . In case of rare failure event, no failure is observed so that  $k_i = n_i$  and our empirical variance estimation becomes zero. Bayesian methods are proposed and use Beta priors  $\text{Beta}(\alpha, \beta)$  with parameters fixed as in Assumption 1 - 6 mentioned previously. Then the posterior distribution of the component reliability  $p_i$  follows beta distribution

$$p_i \sim \text{Beta}(k_i + \alpha, n_i - k_i + \beta).$$

such that

$$E(p_i) = \frac{k_i + \alpha}{n_i + \alpha + \beta}$$

The obtained estimate

$$\hat{p}_{i,n_i} = \frac{k_i + \alpha}{n_i + \alpha + \beta},$$

is equivalent to the assumption that  $\alpha + \beta$  imaginary samples are added to the initial sample pool. Then the variance of this estimate is given by

$$\text{Var}(\hat{p}_{i,n_i}) = \frac{\hat{p}_{i,n_i}(1 - \hat{p}_{i,n_i})}{n_i + \alpha + \beta}$$

which also fits the central limit theorem (CLT)

$$\sqrt{n_i + \alpha + \beta}(\hat{p}_{i,n_i} - E(\hat{p}_i)) \xrightarrow{d} N(0, \text{Var}(\hat{p}_{i,n_i}))$$

Hence, we have Algorithm 2, an alternative version of Algorithm 1 which can be applied on cases with rare event/failure.

## 3.4 Random set approach

### 3.4.1 Reliability assessment of components

Consider the observation pool  $O$  containing  $n$  independent and identically distributed (i.i.d.) samples of a binary random variable  $X$

$$O = \{x_1, x_2, \dots, x_n\}$$

---

**Algorithm 2** Estimate Birnbaum importance factor and system reliability with Monte Carlo simulation/Bootstrap technique with  $Beta(\alpha, \beta)$  prior probability

---

**Require:** Component reliability estimate  $\hat{p}_{1,n_1}, \dots, \hat{p}_{d,n_d}$ ; number of simulation trials  $N$

Initialize  $R_{\mathbf{p}} = 0$      $R_0 = (0, \dots, 0)$      $R_1 = (0, \dots, 0)$

**for**  $j = 1$  to  $N$  **do**

**for**  $i = 1$  to  $d$  **do**

**if**  $k_i \neq n_i$  **then**

$$\hat{p}_{i,n_i} = \frac{k_i}{n_i} \quad \text{Var}(\hat{p}_i) = \frac{\hat{p}_i(1 - \hat{p}_i)}{n_i}$$

**else**

$$\hat{p}_{i,n_i} = \frac{k_i + \alpha}{n_i + \alpha + \beta} \quad \text{Var}(\hat{p}_{i,n_i}) = \frac{\hat{p}_{i,n_i}(1 - \hat{p}_{i,n_i})}{n_i + \alpha + \beta}$$

**end if**

    Randomly sample  $X_i$  following  $Bernoulli(\hat{p}_{i,n_i})$

**end for**

**for**  $i = 1$  to  $d$  **do**

  Count the simulation results:

$$R_{\mathbf{p}} = R_{\mathbf{p}} + \varphi(X_1, \dots, X_i, \dots, X_d)$$

$$R_0(i) = R_0(i) + \varphi(X_1, \dots, X_i = 0, \dots, X_d) \quad R_1(i) = R_1(i) + \varphi(X_1, \dots, X_i = 1, \dots, X_d)$$

**end for**

**end for**

Aggregate the sampled values of  $R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i)$  by average:

$$R(\hat{\mathbf{p}}_n) = \frac{R_{\mathbf{p}}}{N}$$

**for**  $i = 1$  to  $d$  **do**

$$R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i) = \frac{R_1(i) - R_0(i)}{N}$$

**end for**

$$\text{Var}(\hat{p}_s) = \sum_{i=1}^d (R(\hat{\mathbf{p}}_n, 1_i) - R(\hat{\mathbf{p}}_n, 0_i))^2 \text{Var}(\hat{p}_i)$$

$$\text{CI: } [R(\hat{\mathbf{p}}_n) - z_{\alpha/2} \sqrt{\text{Var}(\hat{p}_s)}, R(\hat{\mathbf{p}}_n) + z_{\alpha/2} \sqrt{\text{Var}(\hat{p}_s)}]$$


---

where  $x_i$  is the  $i$ th observation of  $X$ . The reliability of component  $C$  given these observations  $O$  can be represented by a random variable  $P$  such that

$$P = \mathbb{P}\{X = 1|O\}.$$

In this section, a random closed set  $S$  on  $[0, 1]$  represents  $P$  which is selection of  $S$ , i.e.

$$\mathbb{P}(P \in S) = 1$$

without any additional assumption. The confidence interval of  $S$  is calculated as well.

### Expected values

Let  $k$  samples be the number of observations in good state in  $O$ . According to the theorem introduced in [David and Nagaraja, 2003], and more detailed in [Aguirre et al., 2013], these  $n$  samples of  $X$  following Bernoulli distribution with

$$\mathbb{P}(X = 1) = p$$

can be generated as follows:

- Introduce  $n$  independent pivotal variables following uniform distribution  $U(0, 1)$

$$U_1, U_2, \dots, U_n \sim U(0, 1)$$

- Transform the pivotal variables to observation variables of  $X$  by

$$x_i = \begin{cases} 1 & \text{if } U_i \leq p \\ 0 & \text{if } U_i > p \end{cases}$$

Let  $P^L = U_{(k)}$  and  $P^U = U_{(k+1)}$  denote respectively the  $k$  and  $k + 1$  order statistics  $U_{(k)}$  and  $U_{(k+1)}$  of  $U_1, U_2, \dots, U_n$ . Then given the fact that there are  $k$  good-state observations obtained, there must be exactly  $k$  pivotal variables smaller or equal to  $p$  so that

$$U_{(k)} \leq_{a.s.} P \leq_{a.s.} U_{(k+1)}$$

The random closed set  $S$  can be given by

$$S = [P^L, P^U] \subseteq [0, 1]$$

such that

$$\mathbb{P}(P \in S) = 1. \tag{3.26}$$

Since [David and Nagaraja, 2003]

$$P^L = U_{(k)} \sim \text{Beta}(k, n + 1 - k) \tag{3.27}$$

$$P^U = U_{(k+1)} \sim \text{Beta}(k + 1, n - k) \tag{3.28}$$

the expected value of  $S$  is given by

$$E(S) = [E(P^L), E(P^U)] = \left[ \frac{k}{n+1}, \frac{k+1}{n+1} \right] \tag{3.29}$$

The epistemic uncertainty with which  $P$  is associated due to size-limited available observations is then quantified by the length of  $S$ , i.e.

$$|E(P^U) - E(P^L)| = \frac{1}{n+1}.$$

As the sample size  $n$  converges to infinity ( $n \rightarrow \infty$ ), the value converges to 0.

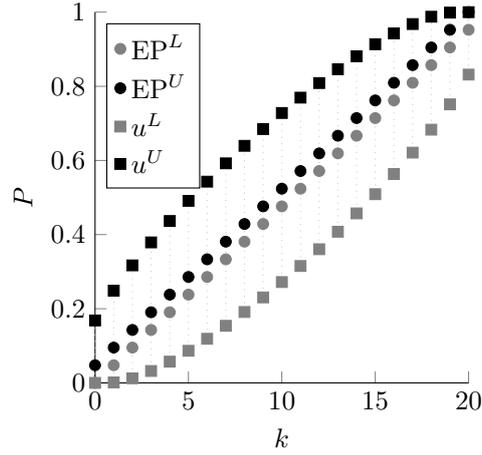


Figure 3.8: Expected values  $[EP^L, EP^U]$  and confidence intervals of the component reliability  $[u^L, u^U]$  ( $1 - \alpha = 0.95, n = 20, k = 0 : 20$ )

### Confidence intervals

Using (2.2), the confidence interval at level  $1 - \alpha$  for  $P$  is given by

$$[u^L, u^U] = [I_{\alpha/2}^{-1}(k, n + 1 - k), I_{1-\alpha/2}^{-1}(k + 1, n - k)] \quad (3.30)$$

where  $I_x(a, b)$  is the regularized incomplete beta function. The confidence interval describes the most probably range of  $P$  and it takes both aleatory and epistemic uncertainty into account.

Since

$$I_{\alpha/2}^{-1}(k, n + 1 - k) < I_{\alpha/2}^{-1}(k + 1, n - k) < I_{1-\alpha/2}^{-1}(k + 1, n - k)$$

the difference between  $u^U$  and  $u^L$  becomes smaller but does not converge to 0 as  $n$  converges to infinity, which corresponds the fact that aleatory uncertainty is not reducible,

**Example 20.** Let  $n = 20$  observations be available. Using (3.29) and (3.30), we computed the expected values and confidence intervals of the component reliability  $P$  on function of  $k$  (cf. Figure 3.8). The value  $|E(P^U) - E(P^L)|$  is constant as epistemic uncertainty only depends on the amount of information contained in  $O$  represented by the number of observations  $n$ . However,  $u^U - u^L$  has smaller value when  $k$  is near 0 or 1 because the aleatory variation of  $P$  is smaller when  $k$  is near 0 or 1.

### Estimation of system reliability

The common method to estimate system reliability is to

- estimate reliability and its uncertainty of each component directly from the corresponding component-level observations with probability theory and/or uncertainty theory
- propagate the quantity of reliability and uncertainty to system level through the system reliability function  $R(p_1, \dots, p_d)$  or the structure function  $\varphi(x_1, \dots, x_d)$ .

However, in case of large-scale systems where both the reliability and uncertainty propagation from component level to system level is very difficult to calculate, the complexity of the method is at-least exponential.

Since mathematical tools to estimate single component's reliability and the associated uncertainty are well developed, if the system is treated as a single component, this problem will be much simpler. Then, the problem becomes how to obtain the system-level observations which are almost impossible to be obtained directly by tests in reality so that system-level pseudo-observation construction is proposed to estimate system reliability.

### 3.4.2 Pseudo observation construction with resampling procedure

Let  $O_1, \dots, O_d$  be respectively the observation pools of the random variables  $X_1, \dots, X_d$  representing states of all system components, and  $n_i$  the number of observations in  $O_i$ , i.e.

$$O_i = \{x_{i,1}, \dots, x_{i,n_i}\} \quad n_i = |O_i|$$

where  $x_{i,1}, \dots, x_{i,n_i}$  are all i.i.d. observations ( $1 \leq i \leq d$ ) and

$$x_{i,j} \sim \text{Bernoulli}(p_i) \quad 1 \leq j \leq n_i$$

The basic idea is that since the system is composed of these  $d$  components and samples of each component are available. It is possible

- to draw one sample from each component's observation pool,
- to construct a system observation using these component observations,
- to obtain the system state corresponding to these picked components through structure function.

This procedure is repeated to build multiples pseudo-systems. With system-level observations, the studied system can be treated as a single component. Then, the system reliability can be estimated using random set model. Since these system observations are not exactly observed from real systems, we call them pseudo-observations.

Let  $X_S$  be the system state determined by  $X_1, \dots, X_d$  through structure function, i.e.

$$X_s = \varphi(X_1, \dots, X_d)$$

and  $p_s$  denote the system reliability given all components observation pools, i.e.

$$p_s = \mathbb{P}\{X_s = 1 | O_1, \dots, O_d\}$$

With the pseudo-observation construction procedure mentioned above, the random pseudo-observation pools  $O_S$  is able to present information in  $O_1, \dots, O_d$  such that

$$p_s = \mathbb{P}\{X_s = 1 | O_S\} \approx \mathbb{P}\{X_s = 1 | O_1, \dots, O_d\}$$

Consider that  $O_s$  contains  $n_s$  observations where  $k_s$  good-state samples are observed, i.e.

$$k_s = \sum_{x \in O_S} x$$

On applying exactly the same reliability estimation method for one component, the random set  $S$  becomes

$$S = [P_s^L, P_s^U]$$

such that

$$\mathbb{P}\{p_s \in [P_s^L, P_s^U]\} = 1$$

where

$$\begin{aligned} P_s^L &\sim \text{Beta}(k_s, n_s + 1 - k_s) \\ P_s^U &\sim \text{Beta}(k_s + 1, n_s - k_s) \end{aligned}$$

The expected value of  $S$  is then given by

$$\begin{aligned} E(S) &= [E(P_s^L), E(P_s^U)] \\ &= \left[ \frac{k_s}{n_s + 1}, \frac{k_s + 1}{n_s + 1} \right] \end{aligned} \tag{3.31}$$

Hence, the confidence interval at level  $1 - \alpha$  of the system reliability  $p_s$  is given by

$$[u^L, u^U] = [I_{\alpha/2}^{-1}(k_s, n_s + 1 - k_s), I_{1-\alpha/2}^{-1}(k_s + 1, n_s - k_s)] \tag{3.32}$$

Notice that in this case  $k_s$  is also a discrete random variable  $k_s \in \{0, 1, 2, \dots, n_s\}$ . Then the expected value of  $S$  and the cumulative distribution function (cdf) of  $P^L$  and  $P^U$  given  $O_S$  become

$$\begin{aligned}
E(S) &= [E(P^L), E(P^U)] \\
&= [E(E(P^L|k_s)), E(E(P^U|k_s))] \\
&= \left[ \sum_{k=0}^{n_s} \mathbb{P}(k_s = k) E(P^L|k_s = k), \sum_{k=0}^{n_s} \mathbb{P}(k_s = k) E(P^U|k_s = k) \right] \\
&= \left[ \sum_{k=0}^{n_s} \mathbb{P}(k_s = k) \frac{k}{n_s + 1}, \sum_{k=0}^{n_s} \mathbb{P}(k_s = k) \frac{k+1}{n_s + 1} \right] \\
&= \left[ \frac{\sum_{k=0}^{n_s} \mathbb{P}(k_s = k) k}{n_s + 1}, \frac{\sum_{k=0}^{n_s} \mathbb{P}(k_s = k) (k+1)}{n_s + 1} \right] \\
&= \left[ \frac{E(k_s)}{n_s + 1}, \frac{E(k_s) + 1}{n_s + 1} \right] \tag{3.33}
\end{aligned}$$

$$\begin{aligned}
F_L(x) &= \mathbb{P}(P^L \leq x) \\
&= \sum_{k=0}^{n_s} \mathbb{P}(P^L \leq x, k_s = k) \\
&= \sum_{k=0}^{n_s} \mathbb{P}(P^L \leq x | k_s = k) \mathbb{P}(k_s = k) \\
&= \mathbb{P}(k_s = 0) + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \mathbb{P}(k_s = k) \tag{3.34}
\end{aligned}$$

$$\begin{aligned}
F_U(x) &= \mathbb{P}(P^U \leq x) \\
&= \sum_{k=0}^{n_s} \mathbb{P}(P^U \leq x, k_s = k) \\
&= \sum_{k=0}^{n_s} \mathbb{P}(P^U \leq x | k_s = k) \mathbb{P}(k_s = k) \\
&= \mathbf{1}_{\{x=1\}}(x) \mathbb{P}(k_s = n_s) + \sum_{k=0}^{n_s-1} I_x(k+1, n_s - k) \mathbb{P}(k_s = k) \tag{3.35}
\end{aligned}$$

where  $I_x(\alpha, \beta)$  is the regularized incomplete beta function and

$$\begin{aligned}
\mathbb{P}(P^L \leq x, k_s = 0) &= \mathbb{P}(P^L \leq x | k_s = 0) \mathbb{P}(k_s = 0) \\
&= \mathbf{1}_{[0,1]}(x) \mathbb{P}(k_s = 0) \\
&= \mathbb{P}(k_s = 0)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(P^U \leq x, k_s = n_s) &= \mathbb{P}(P^U \leq x | k_s = n_s) \mathbb{P}(k_s = n_s) \\
&= \mathbf{1}_{\{x=1\}}(x) \mathbb{P}(k_s = n_s)
\end{aligned}$$

as

$$\begin{aligned}
0 \leq p_s \leq U_{(1)} & \quad k_s = 0 \\
U_{(n_s)} \leq p_s \leq 1 & \quad k_s = n_s.
\end{aligned}$$

Then on system level,  $k_s$  follows binomial distribution

$$k_s \sim \text{Binomial}(n_s, p_s)$$

so that

$$\mathbb{P}(k_s = k) = \binom{n_s}{k} p_s^k (1 - p_s)^{n_s - k}, \quad \forall k \in \{0, \dots, n_s\}$$

Hence,  $F_L(x)$  and  $F_U(x)$  become

$$F_L(x) = (1 - p_s)^{n_s} + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \binom{n_s}{k} p_s^k (1 - p_s)^{n_s - k} \quad (3.36)$$

$$F_U(x) = \mathbf{1}_{\{x=1\}}(x) p_s^{n_s} + \sum_{k=0}^{n_s-1} I_x(k+1, n_s - k) \binom{n_s}{k} p_s^k (1 - p_s)^{n_s - k} \quad (3.37)$$

Finally, the confidence interval at level  $1 - \alpha$  of  $S$  is given by

$$[u^L, u^U] = [F_L^{-1}(\alpha/2), F_U^{-1}(1 - \alpha/2)] \quad (3.38)$$

In case of highly reliable system where  $p_s$  is close to 1, the variance of  $k_s$  is small and close to  $E(k_s)$  so that the cdf of  $F^L(x)$  and  $F^U(x)$  and the confidence interval  $[u^L, u^U]$  can be approximated as follows

$$F_L(x) \approx \mathbb{P}(P^L \leq x | E(k_s)) \quad (3.39)$$

$$F_U(x) \approx \mathbb{P}(P^U \leq x | E(k_s)) \quad (3.40)$$

$$[u^L, u^U] \approx [I_{\alpha/2}^{-1}(E(k_s), n_s + 1 - E(k_s)), I_{1-\alpha/2}^{-1}(E(k_s), n_s - E(k_s))] \quad (3.41)$$

on replacing  $k_s$  by  $E(k_s)$ . As shown in Figure 3.9, the exact method gives larger CI than approximation method. For numerical application, Monte Carlo simulations with  $N$  iterations is proposed to construct pseudo observations and to obtain  $\hat{k}_s$  and  $\hat{p}_{s,k}$ , the estimates of  $E(k_s)$  and  $\mathbb{P}(k_s = k)$ , as follows

$$\hat{k}_s = \frac{\sum_{j=1}^N k_{s,j}}{N} \quad (3.42)$$

$$\hat{p}_{s,k} = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j})}{N} \quad (3.43)$$

where  $k_{s,j}$  denotes the  $k_s$  value obtained by the  $j$ th iteration, and  $\mathbf{1}_k(x)$  is an indicator function giving 1 if  $x = k$ , 0 otherwise.

Then the estimate of  $p_s$  is calculated on function of  $\hat{k}_s$ , i.e.

$$\hat{p}_s = \frac{\hat{k}_s}{n_s}$$

For small systems, the estimates of  $E(k_s)$  and  $p_s$  can also be given by

$$\hat{p}_s = R(\hat{p}_1, \dots, \hat{p}_d)$$

$$\hat{k}_s = R(\hat{p}_1, \dots, \hat{p}_d) n_s$$

where  $\hat{p}_i$  is an estimator of  $p_i$ .

Finally, the estimators of  $E(s) = [E(P^L), E(P^U)]$  becomes

$$[\hat{p}^L, \hat{p}^U] = \left[ \frac{\hat{k}_s}{n+1}, \frac{\hat{k}_s + 1}{n+1} \right] \quad (3.44)$$

Thus, the confidence interval can be obtained by both approximation and empirical methods, i.e.

$$[\hat{u}^L, \hat{u}^U] \approx [I_{\alpha/2}^{-1}(\hat{k}_s, n_s + 1 - \hat{k}_s), I_{1-\alpha/2}^{-1}(\hat{k}_s + 1, n_s - \hat{k}_s)] \quad (3.45)$$

$$[\hat{u}^L, \hat{u}^U] = [\hat{F}_L^{-1}(\alpha/2), \hat{F}_U^{-1}(1 - \alpha/2)] \quad (3.46)$$

where

$$\hat{F}_L(x) = (1 - \hat{p}_s)^{n_s} + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \binom{n_s}{k} \hat{p}_s^k (1 - \hat{p}_s)^{n_s - k} \quad (3.47)$$

$$\hat{F}_U(x) = \mathbf{1}_{\{x=1\}}(x) \hat{p}_s^{n_s} + \sum_{k=0}^{n_s-1} I_x(k+1, n_s - k) \binom{n_s}{k} \hat{p}_s^k (1 - \hat{p}_s)^{n_s - k} \quad (3.48)$$

Here, several things need to be discussed:

- The number of pseudo-observations to be constructed ( $n_s$ );
- The resampling procedure to assure that the resampled observations are independent and identically distributed (i.i.d.).

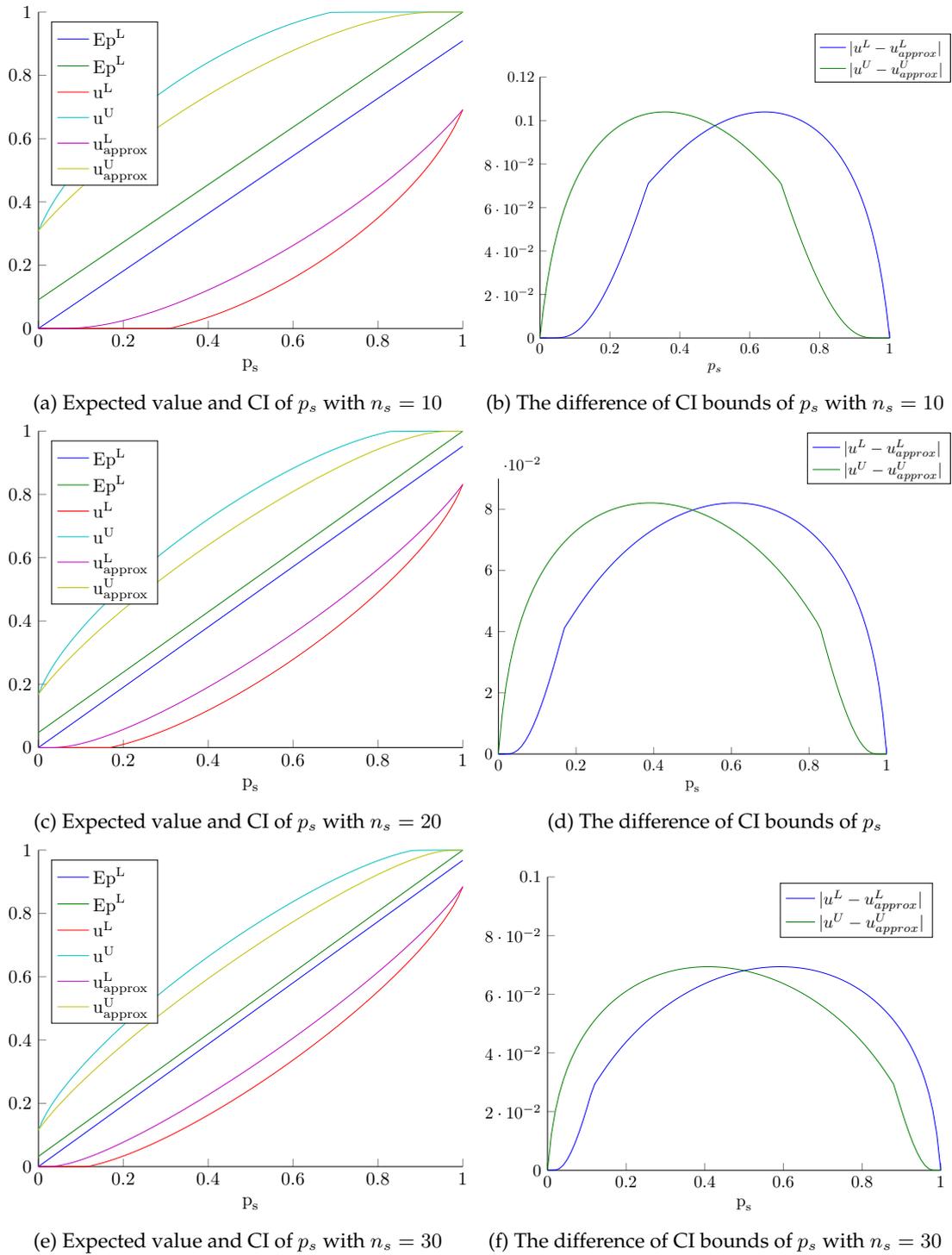


Figure 3.9: Results given by exact method (3.38) and approximation method (3.41) as a function of  $\hat{p}_s$  with  $\alpha = 0.05$

**Proposal 1** Following our basic idea,  $n_s$  system samples must be available simultaneously which means all component observations used are different ones. So the number of system samples is obviously limited by the smallest component sample size

$$n_s \leq \min_i(n_i)$$

Like cross validation, after being shuffled, the observation data set  $O_i$  is splitted into two sets:

- $\tilde{O}_i$  which contains the first  $n_s$  observations drawn from  $O_S$  and is used to construct pseudo systems;
- $\tilde{O}_i^-$  which is not used to construct pseudo systems.

$$\begin{aligned}\tilde{O}_i &= \{x_{i,1}, \dots, x_{i,n_s}\} & |\tilde{O}_i| &= n_s \\ \tilde{O}_i^- &= \{x_{i,n_s+1}, \dots, x_{i,n_i}\} & |\tilde{O}_i^-| &= n_i - n_s\end{aligned}$$

The system state pseudo-observation pool  $\tilde{O}_S$  is obtained as follows

$$\tilde{O}_S = \{x_{s,1} = \varphi(x_{1,1}, \dots, x_{d,1}), \dots, x_{s,n_s} = \varphi(x_{1,n_s}, \dots, x_{d,n_s})\}$$

However, since  $n_s$  is the minimal sample size among all component observation pools, no component sample in each observation pool  $O_i (1 \leq i \leq d)$  is used more than once while  $\tilde{O}_S$  is constructed. Meanwhile, the fact that there are observations unused in each iteration indicates that information provided in  $\tilde{O}_S$  leads to overestimating the system reliability uncertainty because the system level uncertainty is indeed determined by component level uncertainty and in most of uncertainty theories sample size is negatively related to uncertainty quantity. In order to fully use available data,  $n_s$  is extended to

$$\min_i n_i \leq n_s \leq \max_i n_i$$

Replication of observations is allowed for observation pools smaller than  $n_s$ . Like bootstrap resampling,  $n_s$  observations samples are randomly drawn from  $O_i (|O_i| < n_s)$  with replacement which ensures all observations to be independent and uniformly drawn. However, some component observations may be used repeatedly to build different pseudo system samples, these system level observations can not be available at the same time. Since the number of repeated samples picked is limited when the difference in sample size is not very large, in my opinion, this method is close to the experiment developed directly from our basic idea and gives i.i.d. samples.

In summary,  $n_s$  is fixed between  $\min(n_1, \dots, n_d)$  and  $\max(n_1, \dots, n_d)$ . For component with sample size smaller than  $n_s$ , component samples are re-sampled with replacement; while for those with sample size equal to or larger than  $n_s$ , component level re-sampling is without replacement.

**Proposal 2** A method using the idea of systematic sampling is also proposed: each observation pool  $O_i$  is firstly shuffled so that a sequence of i.i.d. observations is built; then starting with a randomly chosen point  $a$ ,  $n_s$  observations are sequentially drawn  $x_{i,a}, \dots, x_{i,a+n_s-1}$ ; if the end of the sequence is reached, drawing is continued by starting with the beginning of the observation sequence  $X_1, X_2, \dots, X_d$ .

**Proposal 3** Bootstrap technique is used to practice the experiment that one sample from each component's observation pool is randomly picked; and a system is built with these components; then the system state is tested and noted as a system level observation; after the experiment the system is decomposed and the components are returned to their observation pools respectively. Then  $n_s$  i.i.d. pseudo system observations are obtained by repeating this experiment.

Let  $I$  be the set of the component indices selected in our experiments

$$I = \{(j_{1,1}, \dots, j_{d,1}), \dots, (j_{1,n_s}, \dots, j_{d,n_s}) \in \{1, \dots, n_1\} \times \dots \times \{1, \dots, n_d\}\}$$

Then the pseudo system observation pools becomes

$$\tilde{O}_S = \{\varphi(X_{1,j_{1,1}}, \dots, X_{d,j_{d,1}}), \dots, \varphi(X_{1,j_{1,n_s}}, \dots, X_{d,j_{d,n_s}})\}$$

Proposal	Re-sample method	Number of pseudo-systems	Results iid
Proposal 1	Randomly pick component observations without replacement	$\min n_i$	No
Proposal 2	Randomly pick component observations without replacement	between $\min$ and $\max n_i$	Yes
Proposal 3	Randomly pick component observations with replacement	unlimited	Yes

Table 3.1: Comparison of proposals

**Discussion** Proposal 3 uses bootstrap techniques and gives i.i.d. samples. Here, the observations are treated as in case of a small finite population.

Someone will then ask why we cannot use Proposal 3 with a large  $n_s$  in order to apply more precise estimation. The problem is that the bootstrap method needs large amount of samples to establish reliable results. However, in our case, we suppose that the available observations are relatively small so that uncertainty theory must be applied and then the size of our pseudo-observation pools should be limited. In Proposal 1 and Proposal 2 for component  $C_i$  with  $n_s \leq n_i$ , the component samples are randomly drawn without replacement. The number of good-state samples of  $C_i$ ,  $k_{s,i}$  follows hyper-geometric distribution with parameters  $(n_i, k_i, n_s)$  if  $n_s \leq n_i$ .

$$\mathbb{P}(k_{s,i} = k) = h(x; n_i, n_s, k_i) = \frac{\binom{k_i}{k} \binom{n_i - k_i}{n_s - k}}{\binom{n_i}{n_s}}$$

It is not a binomial experiment which requires the condition that the probability of good-state drawn is constant for every trial. The  $i$ th selected observations shares equal likelihood of being in any trial so that  $\forall m, m' \in \{1, \dots, n_s\}$

$$\mathbb{P}(X_{i,m} = 1) = \frac{k_i}{n_i}$$

$$\text{Cov}(X_{i,m}, X_{i,m'}) = -\frac{\frac{k_i}{n_i} (1 - \frac{k_i}{n_i})}{n_i - 1}$$

The expected value of each system pseudo observation is the same as Proposal 3

$$\mathbb{P}(\varphi(X_{1,m}, \dots, X_{d,m}) = 1) = R\left(\frac{k_1}{n_1}, \dots, \frac{k_d}{n_d}\right)$$

$$E(\varphi(X_{1,m}, \dots, X_{d,m})) = \mathbb{P}(\varphi(X_{1,m}, \dots, X_{d,m}) = 1)$$

However, these observations are not i.i.d. but interchangeable, which violates the condition mentioned in the beginning of our random set application. Since the covariance is negative, the variances of  $X_{s,j}$  ( $1 \leq j \leq n_s$ ) and  $k_s$  given by Proposal 1 and 2 are much smaller than i.i.d. ones. As  $n_s$  grows, the results obtained by Proposal 1 approach the ones of Proposal 3 because more and more component observations are resampled with replacement.

In summary, pseudo-observations provided by Proposal 1 and 2 do not fit our basic assumption on binomial distribution so that only Proposal 3 can be applied. Proposal 1 and Proposal 2 give the same expected value of  $k_s$  and  $k_s$  estimate with much smaller variance. In order to just estimate  $E(k_s)$ , these two proposals can not only give the same results as Proposal 3 but also accelerate the Monte Carlo simulation. Finally Algorithms 3-6 are built according to these proposals and our approach.

### Estimation of system reliability

In order to compare the complexity of the proposed methods with classical probabilistic assessment (PA) methods, we need to focus on the estimation of  $k_s$ . Algorithm 5 with MLE uses classical assessment function  $R(\mathbf{p})$  to estimate  $k_s$ . This function  $R(\mathbf{p})$  requires minimal cut-sets whose building algorithms as well as the inclusion-exclusion calculation have at least exponential complexity. In case of large scale systems, without truncation on cut-sets length or ignorance of some components, it is very difficult. Then Algorithm 3, 4 and 6 using Monte Carlo simulation are given as the function  $\varphi$  is easier to build.

---

**Algorithm 3** Estimate  $k_s$  with Monte Carlo simulation (Proposal 2)

---

**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$

**for**  $j = 1$  to  $N$  **do**

    Shuffle  $O_1, \dots, O_d$

    Randomly select a starting point

$$(a_1, \dots, a_d) \in \{1, \dots, n_1\} \times \dots \times \{1, \dots, n_d\}$$

**for**  $m = 1$  to  $n_s$  **do**

**for**  $i = 1$  to  $d$  **do**

$$b_i = \text{mod}(a_i, n_i) + 1$$

$$a_i = a_i + 1$$

**end for**

        Construct pseudo system lifetime observations:

$$x_{s,m} = \varphi(x_{1,b_1}, \dots, x_{d,b_d})$$

**end for**

    Calculate  $k_{s,j}$ :

$$k_{s,j} = \sum_{i=1}^{n_s} x_{s,i}$$

**end for**

Aggregate the sampled values of  $k_s$  by average:

$$\hat{k}_s = \frac{\sum_{j=1}^N k_{s,j}}{N} \quad \hat{p}_s = \frac{\hat{k}_s}{n_s}$$

$$\hat{p}_{s,k} = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j})}{N} \quad k = 0, \dots, n_s$$

---

---

**Algorithm 4** Estimate  $k_s$  with Monte Carlo simulation (Proposal 1)

---

**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$ ;

**for**  $j = 1$  to  $N$  **do**

    Shuffle  $O_1, \dots, O_d$

**for**  $i = 1$  to  $d$  **do**

**if**  $n_i < n_s$  **then**

            Select randomly with replacement  $n_s$  samples from  $O_i$  :

$$\tilde{O}_i : x_{i,1}, \dots, x_{i,n_s}$$

**else**

            Select randomly without replacement  $n_s$  samples from  $O_i$  :

$$\tilde{O}_i : x_{i,1}, \dots, x_{i,n_s}$$

**end if**

**end for**

**for**  $m = 1$  to  $n_s$  **do**

        Construct pseudo system state observations from  $\tilde{O}_1, \dots, \tilde{O}_d$ :

$$x_{s,m} = \varphi(x_{1,m}, \dots, x_{d,m})$$

**end for**

    Count the number of systems in operational state:

$$k_{s,j} = \sum_{i=1}^{n_s} x_{s,i}$$

**end for**

Aggregate the sampled value of  $k_s$  by average:

$$\hat{k}_s = \frac{\sum_{j=1}^N k_{s,j}}{N} \quad \hat{p}_s = \frac{\hat{k}_s}{n_s}$$

$$\hat{p}_{s,k} = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j})}{N} \quad k = 0, \dots, n_s$$

---

---

**Algorithm 5** Estimate  $k_s$  with MLE (Proposal 3)

---

**Require:** Component observations  $O_1, \dots, O_d$ ;  $n_s$ ;

**for**  $i = 1$  to  $d$  **do**

    Estimate component reliability:

$$\hat{p}_i = \frac{\sum_{j=1}^{n_i} x_{i,j}}{n_i}$$

**end for**

Estimate the value of  $k_s$ :

$$\hat{k}_s = R(\hat{p}_1, \dots, \hat{p}_d)n_s \quad \hat{p}_s = R(\hat{p}_1, \dots, \hat{p}_d)$$

$$\hat{p}_{s,k} = \binom{n_s}{k} \hat{p}_s^k (1 - \hat{p}_s)^{n_s - k} \quad k = 0, \dots, n_s$$

---

---

**Algorithm 6** Estimate  $k_s$  with Monte Carlo simulation (Proposal 3)
 

---

**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$

**for**  $j = 1$  to  $N$  **do**

**for**  $m = 1$  to  $n_s$  **do**

    Randomly draw  $x_{1,m}, \dots, x_{d,m}$  respectively from  $O_1, \dots, O_d$  with replacement

    Construct pseudo system lifetime observations:

$$x_{s,m} = \varphi(x_{1,m}, \dots, x_{d,m})$$

**end for**

  Estimate  $k_{s,j}$ :

$$k_{s,j} = \sum_{i=1}^{n_s} x_{s,i}$$

**end for**

Aggregate the sampled values of  $k_s$  by average:

$$\hat{k}_s = \frac{\sum_{j=1}^N k_{s,j}}{N} \quad \hat{p}_s = \frac{\hat{k}_s}{n_s}$$

$$\hat{p}_{s,k} = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j})}{N} \quad k = 0, \dots, n_s$$


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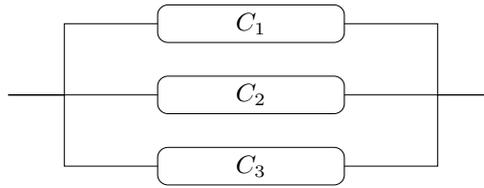


Figure 3.10: S1: Parallel system

For example, for a fault tree composed of  $N_G$  gates and  $d$  components, the complexity of its structure function  $\varphi$  is in order of  $O(N_G \cdot d)$ .

Probability methods providing system level confidence interval estimate the variance of system availability which requires either Birnbaum importance factor (BIF) of each components or system block diagram decomposition [Coit, 1997], and then give the CI bounds on different assumptions on the distribution of system reliability. Notice that the calculation of BIF is at least as complex as  $R(\mathbf{p})$  and not all systems can be decomposed into parallel/series structures without duplicated components. Avoiding the NP-difficult minimal cut set (MCS) building procedure, our method with Monte Carlo simulation reduces efficiently the execution cost (time and complexity) to estimate the system reliability and its confidence interval.

**Example 21.** Consider a parallel system S1 (cf. Figure 3.10), with observation results presented in Table 3.2. The estimation of  $k_s$  is given by Algorithm 3, 4, 5 and 6.

Using Algorithm 5, we obtained

$$\hat{p}_s = (1 - (1 - \hat{p}_1)(1 - \hat{p}_2)(1 - \hat{p}_3)) = 1.000000$$

In system S1, we have  $k_s = n_s$  for all  $n_s$  by all four algorithms because there is no failure observations

$O_i$	$k_i$	$n_i$	$\hat{p}_i = \frac{k_i}{n_i}$
1	20	20	1
2	28	30	0.9333
3	34	35	0.9714

Table 3.2: Component observations of  $C_1, C_2$  and  $C_3$

$n_s$	$\hat{u}^L(3.38)$	$\hat{u}^U(3.38)$	$\hat{u}^L(3.41)$	$\hat{u}^U(3.41)$
20	0.8316	1	0.8316	1
25	0.8628	1	0.8628	1
30	0.8843	1	0.8843	1
35	0.9000	1	0.9000	1

Table 3.3: S1: Confidence interval with  $\alpha = 0.05$  given by (3.38) and (3.41)

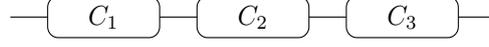


Figure 3.11: S2: Serial system

of component  $C_1$ . The expected value of  $S$  is given by

$$E(S) = \left[ \frac{n_s}{n_s + 1}, \frac{n_s + 1}{n_s + 1} \right]$$

In this case, the confidence intervals obtained by (3.38) and (3.41) are the same because  $E(k_s) = k_s = n_s$ .

**Example 22.** Consider a series system (cf. Figure 3.11) with observation results presented in Table 3.2. The estimation of  $k_s$  was obtained by Algorithm 3, 4, 5 and 6. According to Algorithm 5, we obtained

$$\hat{p}_s = R(\hat{\mathbf{p}}) = \hat{p}_1 \hat{p}_2 \hat{p}_3 = 0.906667$$

It is observed that the distribution of  $k_s$  in Proposal 3 has the form of a binomial distribution. The value of  $k_s$  obtained by Proposal 2 is more concentrated and the one given by Proposal 1 is between these two proposals and becomes more and more similar to the one obtained using Proposal 3 as  $n_s$  grows. The cdf of  $k_s$  obtained by (3.38) and (3.41) with parameters given by Algorithm 6 (Proposal 3) were also compared.

There is a large difference between the two cdf because the  $\hat{p}_s$  is not close to 1 and the variance of  $k_s$  is very large which does not correspond the condition of our approximation.

**Example 23.** Consider a system (cf. Figure 3.14) with observation results presented in Table 3.2. The estimation of  $k_s$  was given by Algorithm 3, 4, 5 and 6. According to Algorithm 5, we obtained

$$\hat{p}_s = R(\hat{\mathbf{p}}) = ((1 - \hat{p}_1)\hat{p}_2\hat{p}_3 + \hat{p}_1(1 - \hat{p}_2)\hat{p}_3 + \hat{p}_1\hat{p}_2\hat{p}_3) = 0.933333$$

$$\hat{p}_s = R(\hat{\mathbf{p}}) = ((1 - \hat{p}_1)(1 - \hat{p}_2)\hat{p}_3 + (1 - \hat{p}_1)\hat{p}_2\hat{p}_3 + \hat{p}_1(1 - \hat{p}_2)\hat{p}_3 + \hat{p}_1\hat{p}_2(1 - \hat{p}_3) + \hat{p}_1\hat{p}_2\hat{p}_3) = 0.998095$$

The distribution of  $k_s$  of Proposal 3 has the form of a binomial distribution in S3 (cf. Figure 3.15).

It is observed that the results obtained by Proposal 1, Proposal 2 and Proposal 3 are similar even as  $n_s$  grows from 20 to 35 because the system has high reliability in S4 (Figure 3.16).

The empirical cdf obtained by (3.38) and (3.41) with  $k_s$  given by Algorithm 6 (Proposal 3) are also compared (Figure 3.17 and 3.18).

Since  $\hat{p}_s$  becomes closer to 1, the variance of  $k_s$  decreases. The cdf obtained by (3.38) and (3.41) become closer to each other and our approximation stands.

### 3.4.3 Reliability assessment with unknown data part

#### Reliability assessment of components with unknown data part

Consider that besides observation pool  $O$  of a random variable  $X$  with precise value, there are also  $n'$  observations of  $X$

$$O' = \{x'_1, x'_2, \dots, x'_{n'}\}$$

where the value of  $x'_i$  is unknown due to censored time of observation or other problems.

More observations equal to 0 there are in  $O'$ , the more underestimated the probability  $P = \mathbb{P}\{X = 1|O, O'\}$  is; the more observations equal to 1 there are in  $O'$ , the more overestimated the probability  $P = \mathbb{P}\{X = 1|O, O'\}$  is.

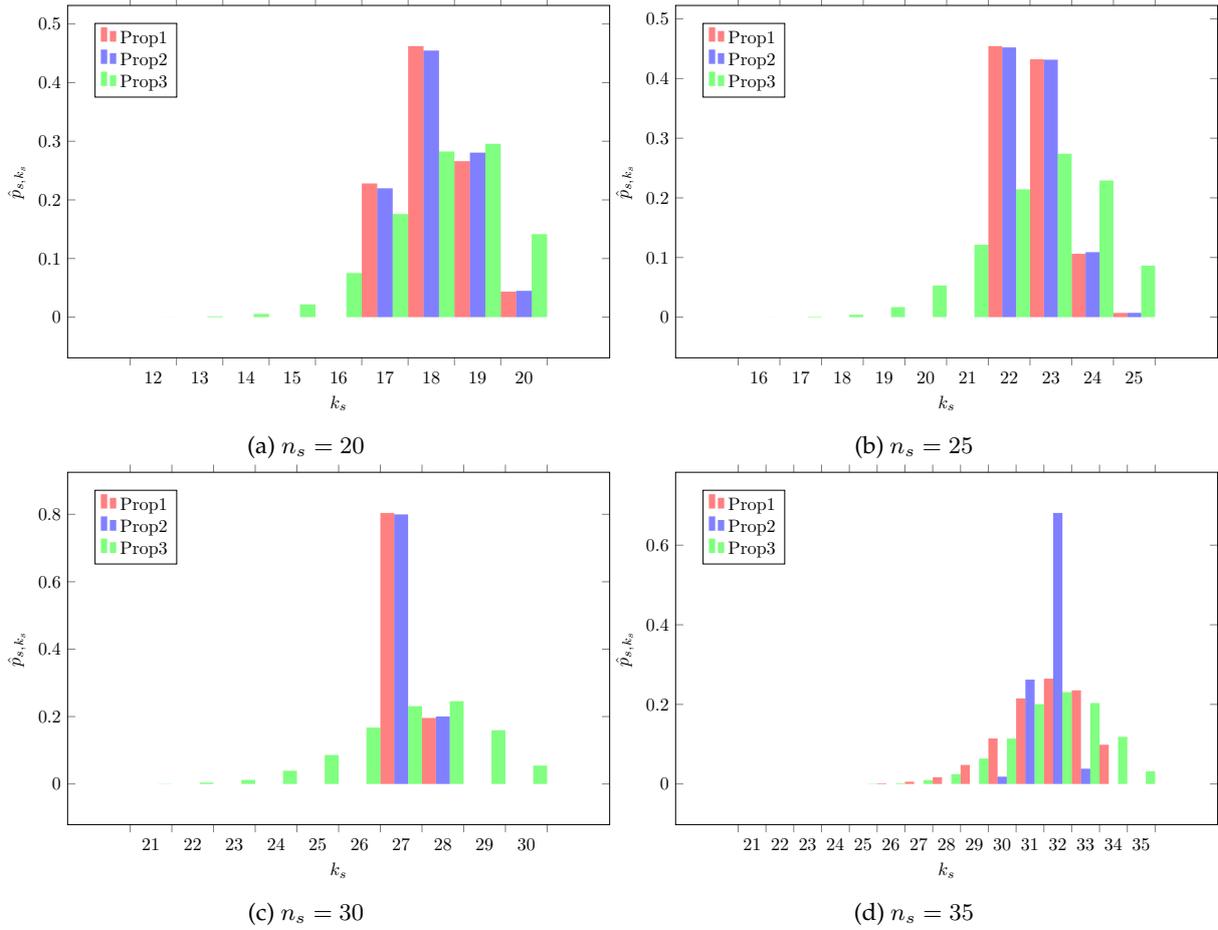


Figure 3.12: S2: Distribution of  $k_s$  values obtained by different proposals with  $N = 10000$

Algorithm	Proposal	$\hat{k}_s$	$var(\hat{k}_s)$	$\hat{p}_s$
$n_s = 20$				
3	2	18.1506	0.657385	0.90753
4	1	18.125	0.65264	0.90625
5	3	18.133333	-	0.90666665
6	3	18.1569	1.639646	0.907845
$n_s = 25$				
3	2	22.6712	0.481939	0.906848
4	1	22.666	0.477492	0.90664
5	3	22.666667	-	0.90666668
6	3	22.6923	2.063427	0.907692
$n_s = 30$				
3	2	27.2002	0.160136	0.906673333
4	1	27.1959	0.157539	0.90653
5	3	27.2	-	0.906666667
6	3	27.1977	2.522467	0.90659
$n_s = 35$				
3	2	31.7394	0.306318	0.90684
4	1	31.7388	2.140789	0.906822857
5	3	31.733333	-	0.906666657
6	3	31.7505	2.885338	0.907157143

Table 3.4: S2: Results obtained by different algorithms with  $N = 10000$

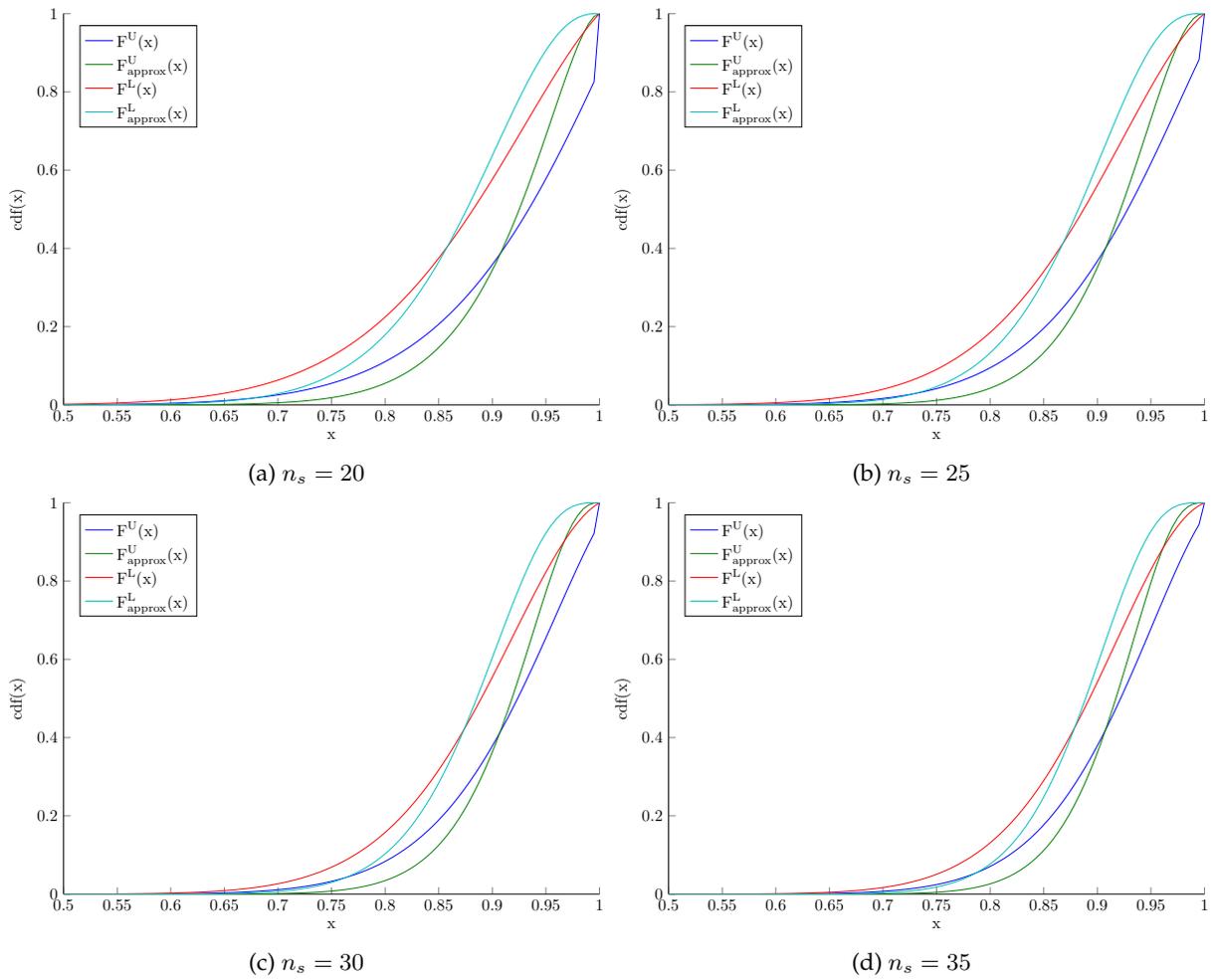


Figure 3.13: S2: cdf obtained by (3.38) and (3.41) with parameters given by Algorithm 6 (Proposal 3)  $N = 10000$

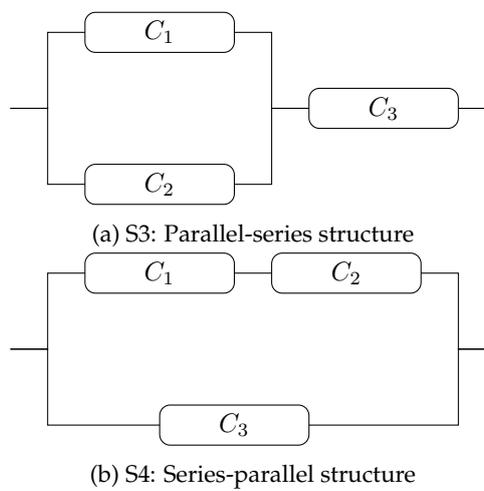


Figure 3.14: Parallel-series and series-parallel systems composed of three components

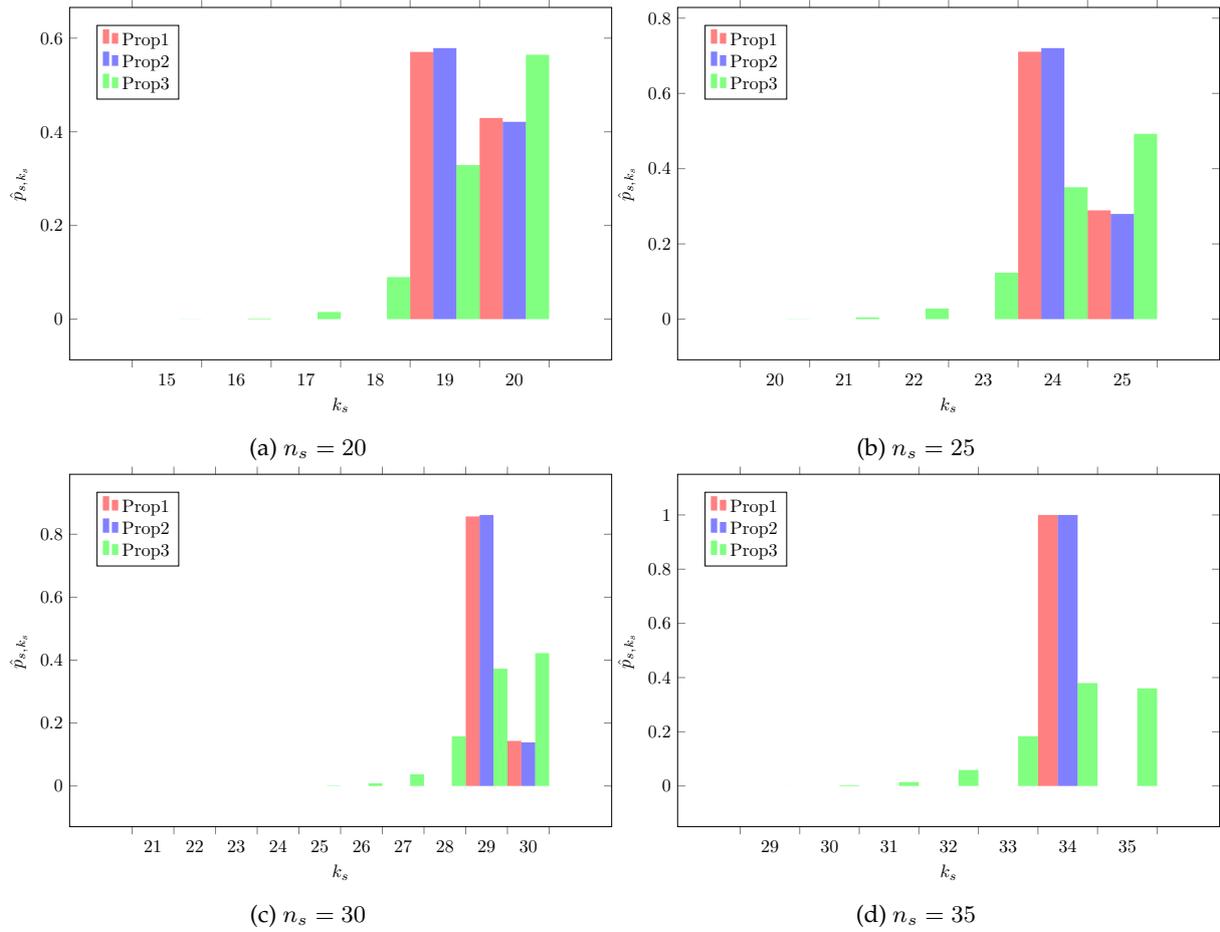


Figure 3.15: S3: Distribution of  $k_s$  values obtained by different proposals with  $N = 10000$

Algorithm	Proposal	$\hat{k}_s$	$var(\hat{k}_s)$	$\hat{p}_s$
$n_s = 20$				
3	2	19.4213	0.243831	0.971065
4	1	19.4295	0.245054	0.971475
5	3	18.666667	-	0.93333335
6	3	19.4391	0.538745	0.971955
$n_s = 25$				
3	2	24.2796	0.201444	0.971184
4	1	24.2892	0.205584	0.971568
5	3	23.333333	-	0.93333332
6	3	24.2964	0.691616	0.971856
$n_s = 30$				
3	2	29.1386	0.119402	0.971286667
4	1	29.1433	0.122777	0.971443333
5	3	28	-	0.933333333
6	3	29.1591	0.802067	0.97197
$n_s = 35$				
3	2	34	0	0.971428571
4	1	34	0	0.971428571
5	3	32.666667	-	0.933333343
6	3	34.0024	0.96249	0.971497143

Table 3.5: S3: Results obtained by different algorithms with  $N = 10000$

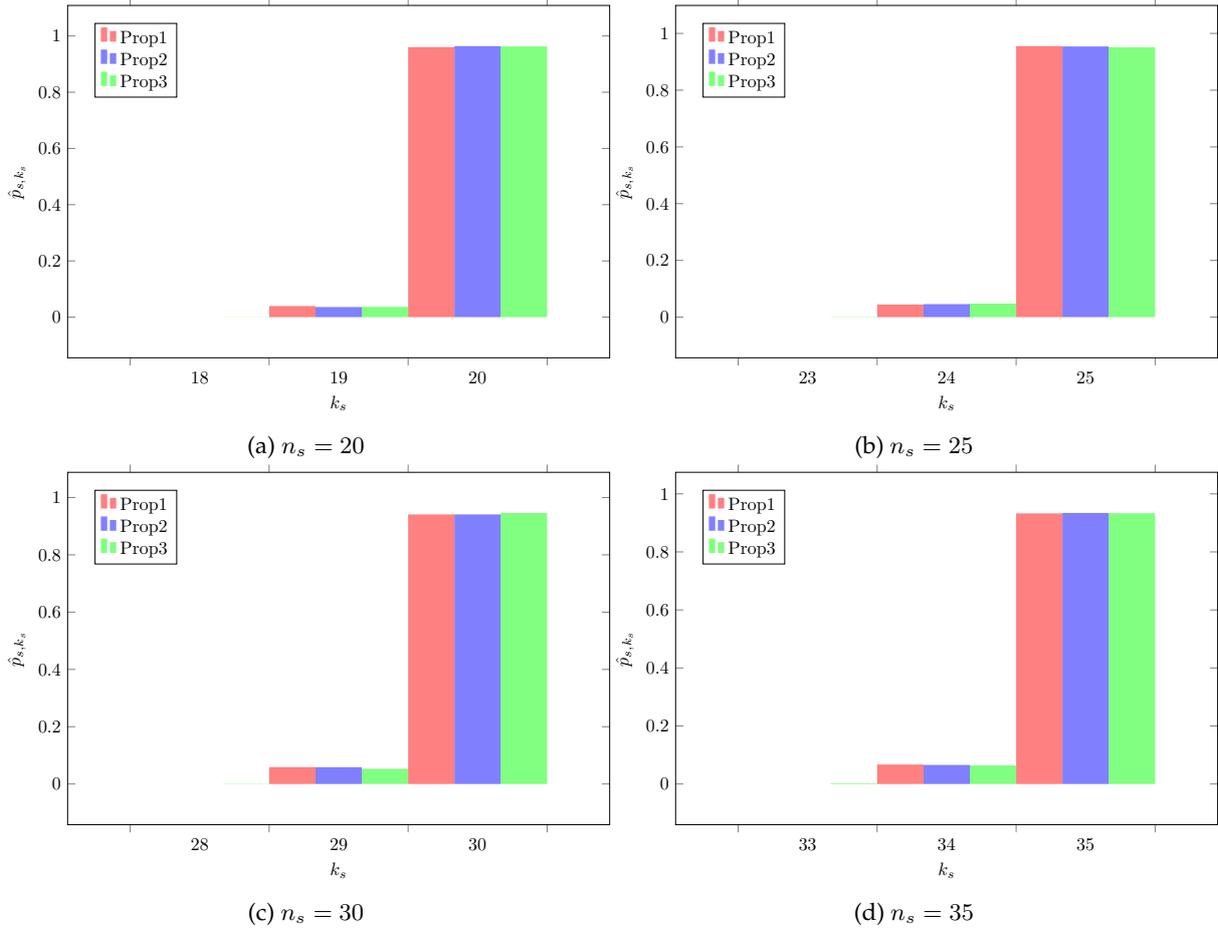


Figure 3.16: S4: Distribution of  $k_s$  values obtained by different proposals with  $N = 10000$

Algorithm	Proposal	$\hat{k}_s$	$var(\hat{k}_s)$	$\hat{p}_s$
$n_s = 20$				
3	2	19.964	0.034707	0.9982
4	1	19.9604	0.038036	0.99802
5	3	19.961905	-	0.99809525
6	3	19.962	0.03796	0.9981
$n_s = 25$				
3	2	24.9541	0.043798	0.998164
4	1	24.9553	0.042706	0.998212
5	3	24.952381	-	0.99809524
6	3	24.95	0.049705	0.998
$n_s = 30$				
3	2	29.9412	0.055348	0.99804
4	1	29.941	0.055525	0.998033333
5	3	29.942857	-	0.998095233
6	3	29.9452	0.053602	0.998173333
$n_s = 35$				
3	2	34.9343	0.06139	0.998122857
4	1	34.9328	0.06269	0.99808
5	3	34.933333	-	0.998095229
6	3	34.9308	0.069618	0.998022857

Table 3.6: S4: Results obtained by different algorithms with  $N = 10000$

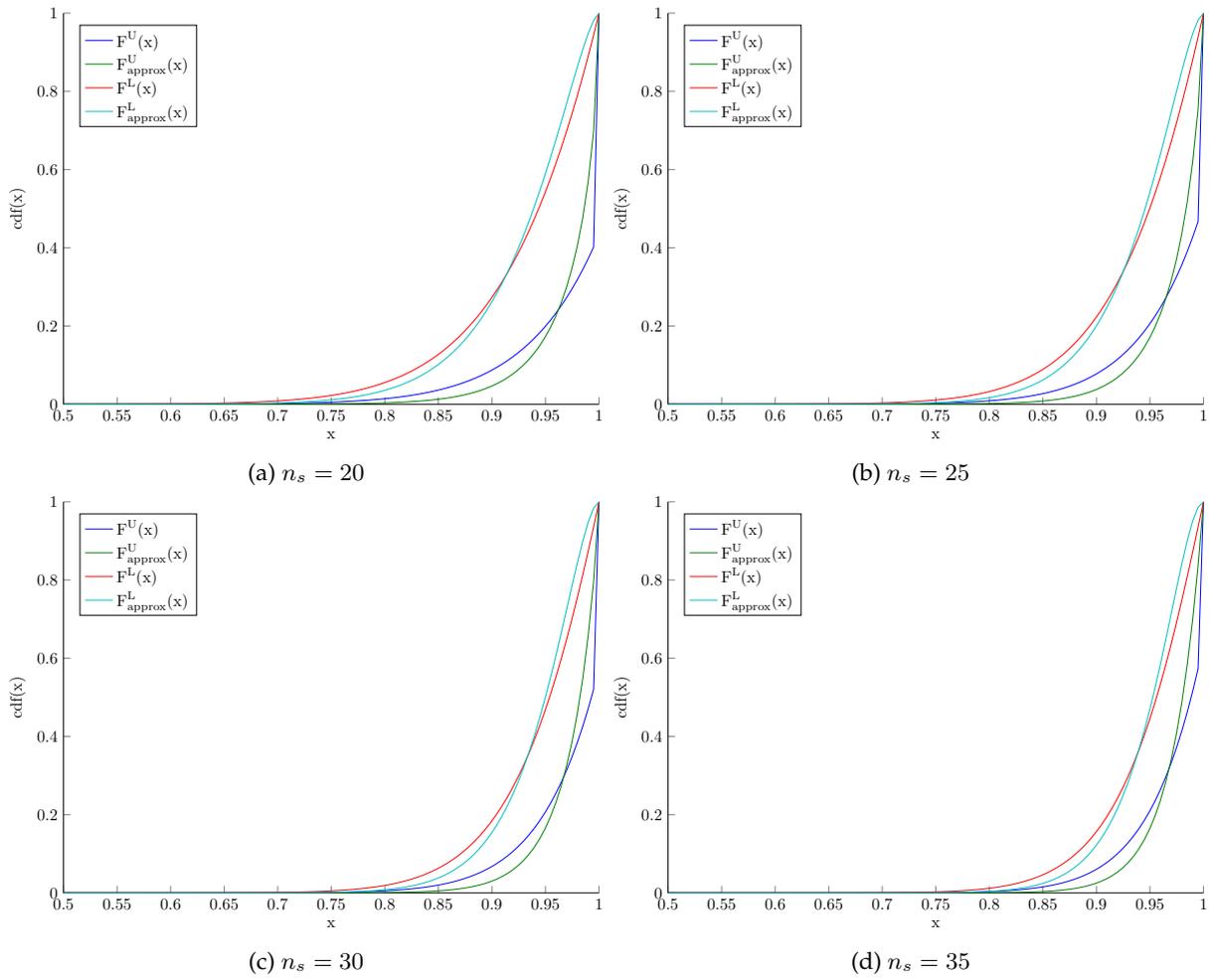


Figure 3.17: S3: cdf obtained by (3.38) and (3.41) with parameters given by Algorithm 6 (Proposal 3) with  $N = 10000$

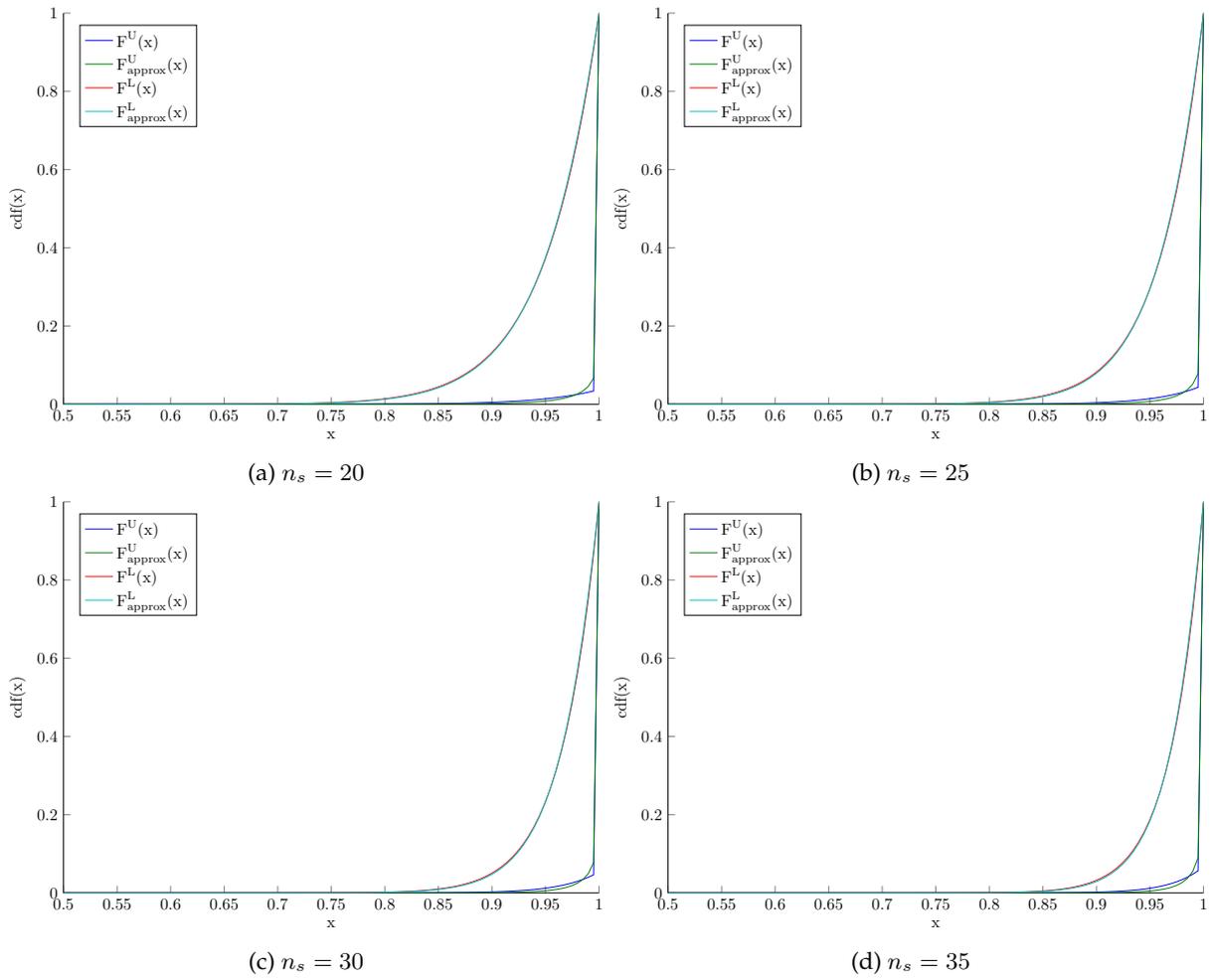


Figure 3.18: S4: cdf obtained by (3.38) and (3.41) with parameters given by Algorithm 6 (Proposal 3) with  $N = 10000$

Let

$$O^L = \{x_1, x_2, \dots, x_n, \underbrace{x_{n+1} = 0, \dots, x_{n+n'} = 0}_{n'}\} \quad (3.49)$$

$$O^U = \{x_1, x_2, \dots, x_n, \underbrace{x_{n+1} = 1, \dots, x_{n+n'} = 1}_{n'}\} \quad (3.50)$$

be the observation pools containing  $n + n'$  observations obtained by merging  $O$  and  $O'$  in the most underestimated and overestimated ways, and  $k^L$  and  $k^U$  be respectively the number of observations equal to 1 in  $O^L$  and  $O^U$  so that

$$k^L = k \quad k^U = k + n'.$$

Then

$$\mathbb{P}\{X = 1|O^L\} \leq \mathbb{P}\{X = 1|O, O'\} \leq \mathbb{P}\{X = 1|O^U\}$$

Without any assumption, the random closed set  $S$  representing  $P$  such that  $P$  is in  $S$  almost surely, i.e.

$$\mathbb{P}\{P \in S\} = 1$$

can be seen as a random closed set containing almost surely  $\mathbb{P}\{X = 1|O^L\}$  and  $\mathbb{P}\{X = 1|O^U\}$ , i.e.

$$\mathbb{P}\{\mathbb{P}\{X = 1|O^L\} \in S; \mathbb{P}\{X = 1|O^U\} \in S\} = 1$$

According to our results in the previous section, let  $P^L$  be the lower random variable bound of  $\mathbb{P}\{X = 1|O^L\}$  and  $P^U$  be the upper random variable bound of  $\mathbb{P}\{X = 1|O^U\}$  so that

$$P^L \sim \text{Beta}(k, n + n' + 1 - k) \quad (3.51)$$

$$P^U \sim \text{Beta}(k + 1 + n', n - k) \quad (3.52)$$

Then we have

$$\begin{aligned} P^L &\leq \mathbb{P}\{X = 1|O^L\} \leq \mathbb{P}\{X = 1|O, O'\} \\ \mathbb{P}\{X = 1|O, O'\} &\leq \mathbb{P}\{X = 1|O^U\} \leq P^U \end{aligned}$$

The random interval  $[P^L, P^U]$  satisfies the condition of  $S$ , so we can use this random interval to represent  $P$ , i.e.

$$S = [P^L, P^U]$$

The expected value of  $S$  is given by

$$E(S) = [E(P^L), E(P^U)] = \left[ \frac{k}{n + n' + 1}, \frac{k + n' + 1}{n + n' + 1} \right] \quad (3.53)$$

The confidence interval at level  $1 - \alpha$  for  $\mathbb{P}\{X = 1|O, O'\}$  is given by

$$[u^L, u^U] = [I_{\alpha/2}^{-1}(k^L, n + n' + 1 - k^L), I_{1-\alpha/2}^{-1}(k^U + 1, n + n' - k^U)] \quad (3.54)$$

$$= [I_{\alpha/2}^{-1}(k, n + 1 + n' - k), I_{1-\alpha/2}^{-1}(k + n' + 1, n - k)] \quad (3.55)$$

### 3.4.4 System reliability assessment with unknown data part

Using the same approach that system level pseudo observations are constructed based on component observations and structure function in previous section,  $n_s$  system pseudo-observations are built according to (3.49) or (3.50) where  $k_s^L$  ( $k_s^U$ ) is the number of pseudo observations in good state. Since  $k_s^L, k_s^U \in \{0, 1, 2, \dots, n_s\}$  are also random variables, we have

$$\begin{aligned} E(S) &= [E(P^L), E(P^U)] \\ &= [E(E(P^L|k_s^L)), E(E(P^U|k_s^U))] \\ &= \left[ \sum_{k=0}^{n_s} \mathbb{P}(k_s^L = k) E(P^L|k_s^L = k), \sum_{k=0}^{n_s} \mathbb{P}(k_s^U = k) E(P^U|k_s^U = k) \right] \\ &= \left[ \sum_{k=0}^{n_s} \mathbb{P}(k_s^L = k) \frac{k}{n_s + 1}, \sum_{k=0}^{n_s} \mathbb{P}(k_s^U = k) \frac{k + 1}{n_s + 1} \right] \\ &= \left[ \frac{\sum_{k=0}^{n_s} \mathbb{P}(k_s^L = k) k}{n_s + 1}, \frac{\sum_{k=0}^{n_s} \mathbb{P}(k_s^U = k) (k + 1)}{n_s + 1} \right] \\ &= \left[ \frac{E(k_s^L)}{n_s + 1}, \frac{E(k_s^U) + 1}{n_s + 1} \right] \end{aligned} \quad (3.56)$$

where

$$n_s \in \left[ \min_{1 \leq i \leq d} (n_i + n'_i), \max_{1 \leq i \leq d} (n_i + n'_i) \right]$$

The cdf of  $P^L$  and  $P^U$  are given by

$$\begin{aligned} F_L(x) &= \mathbb{P}(P^L \leq x) \\ &= \sum_{k=0}^{n_s} \mathbb{P}(P^L \leq x, k_s^L = k) \\ &= \sum_{k=0}^{n_s} \mathbb{P}(P^L \leq x | k_s^L = k) \mathbb{P}(k_s^L = k) \\ &= \mathbb{P}(k_s^L = 0) + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \mathbb{P}(k_s^L = k) \end{aligned} \quad (3.57)$$

$$\begin{aligned} F_U(x) &= \mathbb{P}(P^U \leq x) \\ &= \sum_{k=0}^{n_s} \mathbb{P}(P^U \leq x, k_s^U = k) \\ &= \sum_{k=0}^{n_s} \mathbb{P}(P^U \leq x | k_s^U = k) \mathbb{P}(k_s^U = k) \\ &= \mathbf{1}_{\{x=1\}}(x) \mathbb{P}(k_s^U = n_s) + \sum_{k=0}^{n_s-1} I_x(k+1, n_s - k) \mathbb{P}(k_s^U = k) \end{aligned} \quad (3.58)$$

where  $I_x(\alpha, \beta)$  is the regularized incomplete beta function.

The number of good-state components  $k_s^L$  and  $k_s^U$  follow binomial distribution, i.e.

$$k_s^L \sim \text{Binomial}(n_s, p_s^L)$$

$$k_s^U \sim \text{Binomial}(n_s, p_s^U)$$

so that

$$\mathbb{P}(k_s^L = k) = \binom{n_s}{k} (p_s^L)^k (1 - p_s^L)^{n_s - k}$$

$$\mathbb{P}(k_s^U = k) = \binom{n_s}{k} (p_s^U)^k (1 - p_s^U)^{n_s - k}$$

The cdf of  $P^L$  and  $P^U$  become

$$F_L(x) = (1 - p_s^L)^{n_s} + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \binom{n_s}{k} (p_s^L)^k (1 - p_s^L)^{n_s - k} \quad (3.59)$$

$$(3.60)$$

The confidence interval at level  $1 - \alpha$  of  $S$  is then given by

$$[u^L, u^U] = [F_L^{-1}(\alpha/2), F_U^{-1}(1 - \alpha/2)] \quad (3.61)$$

In case of highly reliable systems, on replacing  $k_s^L(k_s^U)$  by the expected values, the cdf of  $P^L$  and  $P^U$  can also be approximated by

$$F_L(x) \approx \mathbb{P}(P^L \leq x | E(k_s^L)) \quad (3.62)$$

$$F_U(x) \approx \mathbb{P}(P^U \leq x | E(k_s^U)) \quad (3.63)$$

so as the confidence interval  $[u^L, u^U]$

$$[u^L, u^U] \approx [I_{\alpha/2}^{-1}(E(k_s^L), n_s + 1 - E(k_s^L)), I_{1-\alpha/2}^{-1}(E(k_s^U), n_s - E(k_s^U))] \quad (3.64)$$

Algorithms 7, 8, 9, 10 are then proposed to apply random set theory on cases with unknown data part.

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**Algorithm 7** Estimate  $k_s$  with Monte Carlo simulation based in data with unknown part (Proposal 1)
 

---

**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$

Composite the upper and lower observation pools of component  $C_i$  with  $O_i = \{X_{i,1}, \dots, X_{i,n_i}\}$

$$O_i^L = \{x_{i,1}^L, x_{i,2}^L, \dots, x_{i,n_i}^L, \underbrace{x_{i,n_i+1}^L = 0, \dots, x_{i,n_i+n'_i}^L = 0}_{n'_i}\}$$

$$O_i^U = \{x_{i,1}^U, x_{i,2}^U, \dots, x_{i,n_i}^U, \underbrace{x_{i,n_i+1}^U = 1, \dots, x_{i,n_i+n'_i}^U = 1}_{n'_i}\}$$

**for**  $j = 1$  to  $N$  **do**

Shuffle  $O_1^L, \dots, O_d^L$  and  $O_1^U, \dots, O_d^U$

Randomly select starting points

$$(a_1^L, \dots, a_d^L) \in \{1, \dots, n_1 + n'_1\} \times \dots \times \{1, \dots, n_d + n'_d\}$$

$$(a_1^U, \dots, a_d^U) \in \{1, \dots, n_1 + n'_1\} \times \dots \times \{1, \dots, n_d + n'_d\}$$

**for**  $m = 1$  to  $n_s$  **do**

**for**  $i = 1$  to  $d$  **do**

$$b_i^L = \text{mod}(a_i^L, n_i + n'_i) + 1; b_i^U = \text{mod}(a_i^U, n_i + n'_i) + 1$$

$$a_i^L = a_i^L + 1; a_i^U = a_i^U + 1$$

**end for**

Construct pseudo system lifetime observations:

$$x_{s,m}^L = \varphi(x_{1,b_1}^L, \dots, x_{d,b_d}^L)$$

$$x_{s,m}^U = \varphi(x_{1,b_1}^U, \dots, x_{d,b_d}^U)$$

**end for**

Estimate  $k_{s,j}^L$  and  $k_{s,j}^U$  :

$$k_{s,j}^L = \sum_{i=1}^{n_s} x_{s,i}^L \quad k_{s,j}^U = \sum_{i=1}^{n_s} x_{s,i}^U$$

**end for**

Aggregate the sampled values of  $k_s^L$  and  $k_s^U$  by average:

$$\hat{k}_s^L = \frac{\sum_{j=1}^N k_{s,j}^L}{N} \quad \hat{k}_s^U = \frac{\sum_{j=1}^N k_{s,j}^U}{N}$$

$$\hat{p}_s^L = \frac{\hat{k}_s^L}{n_s} \quad \hat{p}_s^U = \frac{\hat{k}_s^U}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}^L = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^L)}{N} \quad \hat{p}_{s,k}^U = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^U)}{N}$$


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**Algorithm 8** Estimate  $k_s$  with Monte Carlo simulation based in data with unknown part (Proposal 2)
 

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**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$

Composite the upper and lower observation pools of component  $C_i$  with  $O_i = \{x_{i,1}, \dots, x_{i,n_i}\}$

$$O_i^L = \{x_{i,1}^L, x_{i,2}^L, \dots, x_{i,n_i}^L, \underbrace{x_{i,n_i+1}^L = 0, \dots, x_{i,n_i+n'_i}^L = 0}_{n'_i}\}$$

$$O_i^U = \{x_{i,1}^U, x_{i,2}^U, \dots, x_{i,n_i}^U, \underbrace{x_{i,n_i+1}^U = 1, \dots, x_{i,n_i+n'_i}^U = 1}_{n'_i}\}$$

**for**  $j = 1$  to  $N$  **do**

Shuffle  $O_1^L, \dots, O_d^L$  and  $O_1^U, \dots, O_d^U$

**for**  $m = 1$  to  $n_s$  **do**

**for**  $i = 1$  to  $d$  **do**

**if**  $n_i + n'_i < n_s$  **then**

Select randomly with replacement  $n_s$  samples from  $O_i^L$  and  $O_i^U$  respectively:

$$\{x_{i,1}^L, \dots, x_{i,n_s}^L\} \quad \{x_{i,1}^U, \dots, x_{i,n_s}^U\}$$

**else**

Select randomly without replacement  $n_s$  samples from  $O_i^L$  and  $O_i^U$  respectively:

$$\{x_{i,1}^L, \dots, x_{i,n_s}^L\} \quad \{x_{i,1}^U, \dots, x_{i,n_s}^U\}$$

**end if**

**end for**

Construct upper and lower pseudo system lifetime observations:

$$x_{s,m}^L = \varphi(x_{1,m}^L, \dots, x_{d,m}^L) \quad x_{s,m}^U = \varphi(x_{1,m}^U, \dots, x_{d,m}^U)$$

**end for**

Calculate the upper and lower  $k_{s,j}$  estimates:

$$k_{s,j}^L = \sum_{i=1}^{n_s} x_{s,i}^L \quad k_{s,j}^U = \sum_{i=1}^{n_s} x_{s,i}^U$$

**end for**

Aggregate the sampled value of  $k_s^L$  and  $k_s^U$  by average:

$$\hat{k}_s^L = \frac{\sum_{j=1}^N k_{s,j}^L}{N} \quad \hat{k}_s^U = \frac{\sum_{j=1}^N k_{s,j}^U}{N}$$

$$\hat{p}_s^L = \frac{\hat{k}_s^L}{n_s} \quad \hat{p}_s^U = \frac{\hat{k}_s^U}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}^L = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^L)}{N} \quad \hat{p}_{s,k}^U = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^U)}{N}$$


---

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**Algorithm 9** Estimate  $k_s$  with MLE based in data with unknown part

---

**Require:** Component observations  $O_1, \dots, O_d$  and  $O'_1, \dots, O'_d$

**for**  $i = 1$  to  $d$  **do**

Estimate component reliability:

$$\hat{p}_i^L = \frac{\sum_{j=1}^{n_i} x_{i,j}}{n_i + n'_i} \quad \hat{p}_i^U = \frac{\sum_{j=1}^{n_i} x_{i,j} + n'_i}{n_i + n'_i}$$

**end for**

Calculate the probability of system failure according to components' unavailability and Estimate the value of  $k_s$ :

$$\begin{aligned} \hat{k}_s^L &= n_s R(p_1^L, \dots, p_d^L) & \hat{k}_s^U &= n_s R(p_1^U, \dots, p_d^U) \\ \hat{p}_s^L &= R(\hat{p}_1^L, \dots, \hat{p}_d^L) & \hat{p}_s^U &= R(\hat{p}_1^U, \dots, \hat{p}_d^U) \end{aligned}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}^L = \binom{n_s}{k} (\hat{p}_s^L)^k (1 - \hat{p}_s^L)^{n_s - k} \quad \hat{p}_{s,k}^U = \binom{n_s}{k} (\hat{p}_s^U)^k (1 - \hat{p}_s^U)^{n_s - k}$$

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**Algorithm 10** Estimate  $k_s$  with Monte Carlo simulation based in data with unknown part

---

**Require:** Component observations  $O_1, \dots, O_d$ ; number of simulation trials  $N$

Composite the upper and lower observation pools of component  $C_i$  with  $O_i = \{x_{i,1}, \dots, x_{i,n_i}\}$

$$O_i^L = \{x_{i,1}^L, x_{i,2}^L, \dots, x_{i,n_i}^L, \underbrace{x_{i,n_i+1}^L = 0, \dots, x_{i,n_i+n'_i}^L = 0}_{n'_i}\}$$

$$O_i^U = \{x_{i,1}^U, x_{i,2}^U, \dots, x_{i,n_i}^U, \underbrace{x_{i,n_i+1}^U = 1, \dots, x_{i,n_i+n'_i}^U = 1}_{n'_i}\}$$

**for**  $j = 1$  to  $N$  **do**

Shuffle the observation pools:  $O_i^U$  and  $O_i^L$  ( $1 \leq i \leq d$ )

**for**  $m = 1$  to  $n_s$  **do**

Randomly draw  $x_{1,m}^L, \dots, x_{d,m}^L$  and  $x_{1,m}^U, \dots, x_{d,m}^U$  respectively from  $O_1^L, \dots, O_d^L$  and  $O_1^U, \dots, O_d^U$

Construct upper and lower pseudo system state observations:

$$x_{s,m}^L = \varphi(x_{1,m}^L, \dots, x_{d,m}^L) \quad x_{s,m}^U = \varphi(x_{1,m}^U, \dots, x_{d,m}^U)$$

**end for**

Calculate the upper and lower  $k_{s,j}$  estimates:

$$k_{s,j}^L = \sum_{i=1}^{n_s} x_{s,i}^L \quad k_{s,j}^U = \sum_{i=1}^{n_s} x_{s,i}^U$$

**end for**

Aggregate the sampled value of  $k_s$  by average:

$$\hat{k}_s^L = \frac{\sum_{j=1}^N k_{s,j}^L}{N} \quad \hat{k}_s^U = \frac{\sum_{j=1}^N k_{s,j}^U}{N}$$

$$\hat{p}_s^L = \frac{\hat{k}_s^L}{n_s} \quad \hat{p}_s^U = \frac{\hat{k}_s^U}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}^L = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^L)}{N} \quad \hat{p}_{s,k}^U = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}^U)}{N}$$

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### 3.4.5 Estimation of system reliability with time-dependent observations

#### Reliability assessment of components

**Survival model with binary states** Let  $T$ , a non-negative continuous random variable, denote the “waiting time until an occurrence of event” where an “event” represents a change of states on the studied object such as a component failure, a patient’s death, etc.

Let  $X(t)$ , a time-dependent random variable, denote the state of a binary component or system

$$X(t) = \begin{cases} 1 & \text{if the component or system is in good state at time } t \\ 0 & \text{if the component or system is in down state at time } t \end{cases}$$

A *failure* is defined as the event that state changes from 1 to 0 and a *repair* is defined as the event that the state changes from 0 to 1. A component is *repairable* if repairs are possible. The availability  $A(t)$  of a component is defined as the probability that the component is in good state at time  $t$ , i.e.

$$A(t) = \mathbb{P}(X(t) = 1) = E[X(t)] \quad t > 0$$

The *reliability*  $R(t)$  is defined as the probability that the component is in good state for all instants before time  $t$  or the failure does not occur before time  $t$ , i.e.

$$R(t) = \mathbb{P}\{X(\mu) = 1, \forall \mu \in [0, t]\}$$

The *maintainability*  $M(t)$  is the probability that the system will be repaired within time  $t$  when failure occurs, i.e.

$$M(t) = \mathbb{P}\{X(t) = 1 | X(0) = 0\}$$

Supposing that the initial state is always state 1 (in good state), then the following properties are available

$$R(0) = A(0) = 1$$

In case of unrepairable systems, the relation between  $T$  and  $X(t)$  are given by

$$X(t) = \begin{cases} 1 & \text{if } T > t \\ 0 & \text{if } T \leq t \end{cases}$$

The *survival function*  $S(t)$  is the complement of cumulative distribution function (cdf) of  $T$ , i.e.

$$F(t) = \mathbb{P}\{T \leq t\}$$

$$S(t) = \mathbb{P}\{T > t\} = 1 - F(t)$$

so that it is equivalent to the reliability function  $R(t)$ .

If the probability density function (pdf) of  $T$ ,  $f(t)$ , exists, we have

$$S(t) = \int_t^{\infty} f(\mu) d\mu$$

The expected value of  $T$  is

$$E(T) = \int_0^{\infty} t f(t) dt = \int_0^{\infty} S(t) dt$$

In this section, we are going to explain how to obtain system (un)availability estimates from component sample data by constructing pseudo system observations and with random set model. The following key assumptions are taken into account:

- System and components are allowed to take only two possible states: either good state, or down state.
- Component failures are independent: State (failure) of one component does not impact the other components.
- The structure function is coherent. That is, improvement of component states cannot damage the system.
- The components are not repairable so that the system reliability and availability are the same.

**Reliability assessment of components using random set** Consider an observation pool  $O$  which contains  $n$  independent and identically distributed (i.i.d.) samples of  $T$  for a unrepairable component  $C$ , i.e.

$$O = \{t_1, t_2, \dots, t_n\}$$

where  $t_i$  is the  $i$ th observation.

In this section, a random closed set  $S(t)$  as well as a confidence interval is introduced to represent the component reliability given observations  $O$  without any additional assumption. Let  $x_i(t)$  denote the state of the  $i$ th observation at time  $t$ ,  $i = 1, \dots, n$ , i.e.

$$x_i(t) = \begin{cases} 1 & \text{if } t_i > t \\ 0 & \text{if } t_i \leq t \end{cases}$$

With fixed instant  $t$ , the observation pool  $O(t)$  of  $X(t)$  is the same as  $O$  in previous sections

$$O(t) = \{x_1(t), x_2(t), \dots, x_n(t)\}$$

Let  $P(t)$  a random variable denote the availability  $A(t)$  given  $O(t)$ , i.e.

$$P(t) = \mathbb{P}\{X(t) = 1|O(t)\}$$

According to (3.27) and (3.26),  $P(t)$  can be represented by a random closed set (random interval)

$$S(t) = [P^L(t), P^U(t)] \subset [0, 1]$$

such that  $P(t)$  is selection of  $S(t)$ , i.e.

$$\mathbb{P}(P(t) \in S(t)) = 1$$

Both bounds of  $S(t)$  follow beta distribution

$$\begin{aligned} P^L(t) &\sim \text{Beta}(k(t), n - k(t) + 1) \\ P^U(t) &\sim \text{Beta}(k(t) + 1, n - k(t)) \end{aligned}$$

where  $k(t)$  is the number of observations in good state in  $O(t)$ .

The expectation of  $S(t)$  is then given by

$$E[S(t)] = [E(P^L(t)), E(P^U(t))] = \left[ \frac{k(t)}{n+1}, \frac{k(t)+1}{n+1} \right] = \left[ \frac{\sum_{i=1}^n \mathbf{1}_{(t, \infty)}(t_i)}{n+1}, \frac{\mathbf{1}_{(t, \infty)}(t_i) + 1}{n+1} \right]$$

where  $\mathbf{1}_{(t, \infty)}(x)$  is an indicator function giving 1 if  $x > t$ , 0 otherwise.

Hence, the confidence interval of  $S(t)$  at level  $1 - \alpha$  becomes

$$[u^L(t), u^U(t)] = [I_{\alpha/2}^{-1}(k(t), n+1-k(t)), I_{1-\alpha/2}^{-1}(k(t)+1, n-k(t))]$$

### Estimation of system reliability

Consider a system composed of  $d$  independent components. Let  $T_s$  and  $T_1, \dots, T_d$  denote respectively the times to failure of the system and all components, and  $X_s(t)$  and  $X_1(t), \dots, X_d(t)$  denote the states of system and all components at time  $t$ .

The system state at time  $t$  is given by

$$X_s(t) = \varphi(X_1(t), \dots, X_d(t))$$

Besides  $\varphi : \{0, 1\}^d \rightarrow \{0, 1\}$ , there exists another structure function giving system lifetime

$$T_s = \varphi_T(T_1, \dots, T_d) \tag{3.65}$$

so that

$$X_s(t) = \begin{cases} 1 & \text{if } T_s = \varphi_T(T_1, \dots, T_d) > t \\ 0 & \text{if } T_s = \varphi_T(T_1, \dots, T_d) \leq t \end{cases} \tag{3.66}$$

The advantage of applying our method here is that the probability distribution at system level is very complicated if we propagate the reliability distribution of each component using the structure function. For example, if all components follows exponential distribution, only a series system has exponential distributed reliability. Our method uses directly empirical distribution functions which always exist and can present all possible distributions.

**Estimation of system failure probability/distribution** Let

$$O_{T_i} = \{t_{1,i}, \dots, t_{n_i,i} \in \mathbb{R}^+\}$$

be an observation pool which contains  $n_i$  i.i.d lifetime observations of component  $C_i$ . Instead of using failure probabilities calculated from each component observation pool and then propagating them to system level, our proposed idea to calculate the system reliability  $p_s(t)$  is to build directly pseudo-observations of  $T_s$

$$O_{T_s} = \{t_{s,1}, \dots, t_{s,n_s}\}.$$

The same procedure as in binary static systems introduced previously is carried out:

- Randomly draw one observation with or without from each component observation pool  $O_{T_i}$
- Build a pseudo system composed of these observation combinations
- Obtain a pseudo sample of  $T_s$  through function  $\varphi_T$ .

The number of system pseudo-observations in good state in  $O_{T_s}$  at time  $t$ ,  $k_s(t)$ , follows binomial distribution

$$k_s(t) \sim \text{Binomial}(n_s, p_s(t))$$

where  $p_s(t)$  denotes the system reliability.

With Monte Carlo simulation, the pseudo-observation pool  $O_{T_s}$  are supposed to represent all information provided by  $O_{T_1}, \dots, O_{T_d}$  so that

$$P_s(t) = \mathbb{P}\{X_s(t) = 1 | O_{T_1}, \dots, O_{T_d}\} \approx \mathbb{P}\{X_s(t) = 1 | O_{T_s}\}$$

The random set  $S(t)$  representing the system reliability such that

$$\mathbb{P}\{P_s(t) \in S(t)\} = 1$$

is a random interval

$$S(t) = [P^L(t), P^U(t)]$$

where  $P^L(t)$  and  $P^U(t)$  are random variables following distributions with cdf given by

$$F_L(x, t) = (1 - p_s(t))^{n_s} + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \binom{n_s}{k} (p_s(t))^k (1 - p_s(t))^{n_s - k} \quad (3.67)$$

$$F_U(x, t) = \mathbf{1}_{\{x=1\}}(x) (p_s(t))^{n_s} + \sum_{k=0}^{n_s-1} I_x(k+1, n_s - k) \binom{n_s}{k} (p_s(t))^k (1 - p_s(t))^{n_s - k} \quad (3.68)$$

In case of highly reliable systems, the distributions of  $P^L(t)$  and  $P^U(t)$  can be approximated by Beta distribution

$$P^L(t) \sim \text{Beta}(E(k_s(t)), n_s + 1 - E(k_s(t)))$$

$$P^U(t) \sim \text{Beta}(E(k_s(t)) + 1, n_s - E(k_s(t)))$$

The expected value of  $S(t) = [P^L(t), P^U(t)]$  is then given by

$$E(S(t)) = [E(P_s^L(t)), E(P_s^U(t))] = \left[ \frac{E(k_s(t))}{n_s + 1}, \frac{E(k_s(t)) + 1}{n_s + 1} \right] \quad (3.69)$$

The confidence interval at level  $1 - \alpha$  of  $P_s(t)$  is given by

$$[u^L(t), u^U(t)] = [F_{L_{\alpha/2}}^{-1}(t), F_{U_{\alpha/2}}^{-1}(t)] \quad (3.70)$$

where  $F_{L_{\alpha/2}}^{-1}(t)$  and  $F_{U_{\alpha/2}}^{-1}(t)$  are the inverse function of (3.67) and (3.68)

It can also be approximated by

$$[u^L(t), u^U(t)] = [I_{\alpha/2}^{-1}(E(k_s^L(t)), n_s + 1 - E(k_s^L(t))), I_{1-\alpha/2}^{-1}(E(k_s^U(t)) + 1, n_s - E(k_s^U(t)))] \quad (3.71)$$

The proposed estimators of  $E(k_s(t))$ ,  $p_s(t)$  and  $\mathbb{P}(k_s = k)$  are shown in Algorithm 11, 12, 13 and 14.

---

**Algorithm 11** Estimate  $k_s(t)$  with MLE (Proposal 3)

---

**Require:** Component observations  $O_{T_1}, \dots, O_{T_d}$ ; instant  $t$ ; sample size  $n_s$

**for**  $i = 1$  to  $d$  **do**

Estimate component availability at instant  $t$ :

$$\hat{p}_i(t) = \frac{\sum_{j=1}^{n_i} \mathbf{1}_{(t,\infty)}(t_{i,j})}{n_i}$$

**end for**

Calculate the system reliability according to components' reliability:

$$\hat{p}_s(t) = R(\hat{p}_1(t), \dots, \hat{p}_d(t))$$

Estimate the value of  $k_s(t)$ :

$$\hat{k}_s(t) = \hat{p}_s(t)n_s$$

---

---

**Algorithm 12** Estimate  $k_s(t)$  with Monte Carlo re-sampling/simulation (Proposal 1)

---

**Require:** Component observations  $O_{T_1}, \dots, O_{T_d}$ ; number of simulation trials  $N$ ; instant  $t$

**for**  $j = 1$  to  $N$  **do**

Shuffle  $O_{T_1}, \dots, O_{T_d}$

**for**  $i = 1$  to  $d$  **do**

**if**  $n_i < n_s$  **then**

Select randomly with replacement  $n_s$  samples from  $O_{T_i}$  :

$$\tilde{O}_{T_i} : t_{i,1}, \dots, t_{i,n_s}$$

**else**

Select randomly without replacement  $n_s$  samples from  $O_{T_i}$  :

$$\tilde{O}_{T_i} : t_{i,1}, \dots, t_{i,n_s}$$

**end if**

**end for**

**for**  $m = 1$  to  $n_s$  **do**

Construct pseudo system lifetime observations from  $\tilde{O}_1, \dots, \tilde{O}_d$ :

$$t_{s,m} = \varphi_T(t_{1,m}, \dots, t_{d,m})$$

**end for**

Count the number of systems in good state:

$$k_{s,j}(t) = \sum_{i=1}^{n_s} \mathbf{1}_{(t,\infty)}(t_{s,i})$$

**end for**

Aggregate the sampled value of  $k_s(t)$  by average:

$$\hat{k}_s(t) = \frac{\sum_{j=1}^N k_{s,j}(t)}{N} \quad \hat{p}_s(t) = \frac{\hat{k}_s(t)}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}(t) = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}(t))}{N}$$

---

---

**Algorithm 13** Estimate  $k_s(t)$  with Monte Carlo re-sampling/simulation (Proposal 2)

---

**Require:** Component observations  $O_{T_1}, \dots, O_{T_d}$ ; number of simulation trials  $N$ ; instant  $t$

**for**  $j = 1$  to  $N$  **do**

Shuffle  $O_{T_1}, \dots, O_{T_d}$

Randomly select a starting point

$$(a_1, \dots, a_d) \in \{1, \dots, n_1\} \times \dots \times \{1, \dots, n_d\}$$

**for**  $m = 1$  to  $n_s$  **do**

**for**  $i = 1$  to  $d$  **do**

$$b_i = \text{mod}(a_i, n_i) + 1$$

$$a_i = a_i + 1$$

**end for**

Construct pseudo system lifetime observations:

$$t_{s,m} = \varphi_T(t_{1,b_1}, \dots, t_{d,b_d})$$

**end for**

Calculate  $k_{s,j}(t)$ :

$$k_{s,j}(t) = \sum_{i=1}^{n_s} \mathbf{1}_{(t,\infty)}(t_{s,i})$$

**end for**

Aggregate the sampled value of  $k_{s,t}$  by average:

$$\hat{k}_s(t) = \frac{\sum_{j=1}^N k_{s,j}(t)}{N} \quad \hat{p}_s(t) = \frac{\hat{k}_s(t)}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}(t) = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}(t))}{N}$$

---

---

**Algorithm 14** Estimate  $k_s(t)$  with Monte Carlo re-sampling/simulation (Proposal 3)

---

**Require:** Component observations  $O_{T_1}, \dots, O_{T_d}$ ; number of simulation trials  $N$ ; instant  $t$

**for**  $j = 1$  to  $N$  **do**

**for**  $m = 1$  to  $n_s$  **do**

Randomly draw  $t_{1,m}, \dots, t_{d,m}$  respectively from  $O_{T_1}, \dots, O_{T_d}$  with replacement

Construct pseudo system lifetime observations:

$$t_{s,m} = \varphi_T(t_{1,m}, \dots, t_{d,m})$$

**end for**

Estimate  $k_{s,j}(t)$ :

$$k_{s,j}(t) = \sum_{i=1}^{n_s} \mathbf{1}_{(t,\infty)}(t_{s,i})$$

**end for**

Aggregate the sampled value of  $k_s(t)$  by average:

$$\hat{k}_s(t) = \frac{\sum_{j=1}^N k_{s,j}(t)}{N} \quad \hat{p}_s(t) = \frac{\hat{k}_s(t)}{n_s}$$

For  $k = 0, \dots, n_s$ ,

$$\hat{p}_{s,k}(t) = \frac{\sum_{j=1}^N \mathbf{1}_k(k_{s,j}(t))}{N}$$

---

$C_i$	$n_i$	$O_{T_i}$
1	5	78 73 36 55 28
2	4	30 21 56 30
3	3	60 84 56

Table 3.7: Parallel/series systems: Observations of three system components

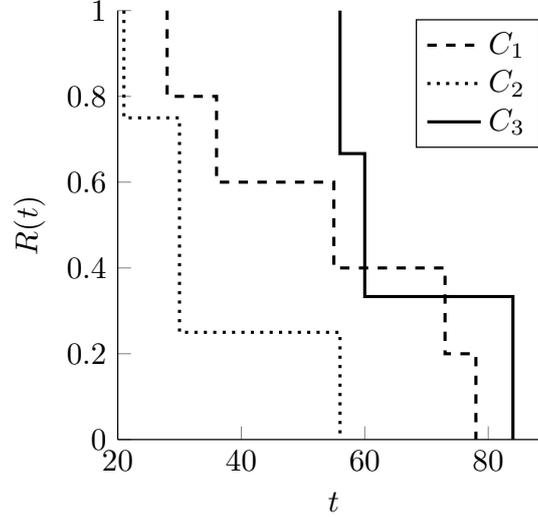


Figure 3.19: Parallel/series systems: Empirical reliability for three system components:  $C_1$ ,  $C_2$  and  $C_3$

**Example 24** (Parallel/series systems). Consider a system composed of three independent components,  $C_1$ ,  $C_2$  and  $C_3$ , whose lifetime observations are given in Table 3.7 (cf. Figure 3.19) with different possible structures (cf. Figure 3.10, 3.11 and 3.14) using Algorithm 11 with  $n_s = 4$  and  $1 - \alpha = 0.95$ .

A system fails when the last working component in one of the minimal cut-set fails. In the obtained stair-step graphs (Figure 3.20), the instants where estimation and confidence interval bound values drop correspond to one of component lifetime observations in  $O_{T_i}$  (Table 3.7): 21, 28, 30, 30, 36, 55, 56, 60, 73, 78, 84.

For S1, the empirical reliability estimate  $\hat{P}_{MLE}(t)$  obtained by probabilistic structure function  $R(\hat{p}_1(t), \dots, \hat{p}_d(t))$  with  $\hat{p}_i(t)$ , the MLE estimator of  $p_i(t)$ , is always bounded by our upper and lower estimates.

The reliability of parallel systems mostly depends on the most reliable component which is  $C_3$  in our case. As shown in Figure 3.20a, the reliability does not start decreasing until the instant  $t = 56$ , the first failure time of component  $C_3$ . For series systems, once one of the components fails, the system fails. The reliability depends on the least reliable component which is  $C_2$  in our case. The curves stop decreasing on function of time after the instant  $t = 56$  where the last sample of  $C_2$  fails (Figure 3.20b). All components in the two previous examples have equal importance. The magnitude of each decrease is only proportional to the number of samples corresponding to this instant and the weight of such samples in their observation pool ( $1/n_i$ ). The two following systems S3 and S4 combine parallel and series structures so that the instants where system reliability estimates start or stop decreasing can be traced. Meanwhile, the component importance becomes different comparing between components and varies on function of time and component reliability. In S3, the magnitude of decreases caused by samples of  $C_1$  are different on function of time. The shape of obtained curves are similar to the one of  $C_1$  reliability. According to all available observations  $C_3$  is much more reliable than parallel structure composed of the other two components, which indicates that  $C_1$  is critical to the system reliability. In S4, the shape of all estimates especially  $\hat{P}_{MLE}(t)$  is close to the one of  $C_3$  reliability. The series part according to available observations fails much faster than  $C_3$  which is on parallel. In this specific case,  $C_3$  is critical to the system reliability, which confirms what is observed from our results.

**Case with censored component observation pools** Consider that there are  $n'_i$  samples  $O'_{T_i} = \{t'_{i,1}, \dots, t'_{i,n'_i}\}$  censored with maximal observation time  $t_i^{max}$  of component  $C_i$ . The only information available is

$$t'_{i,j} \in [t_i^{max}, +\infty), \quad j = 1, \dots, n'_i, \quad i = 1, \dots, d$$

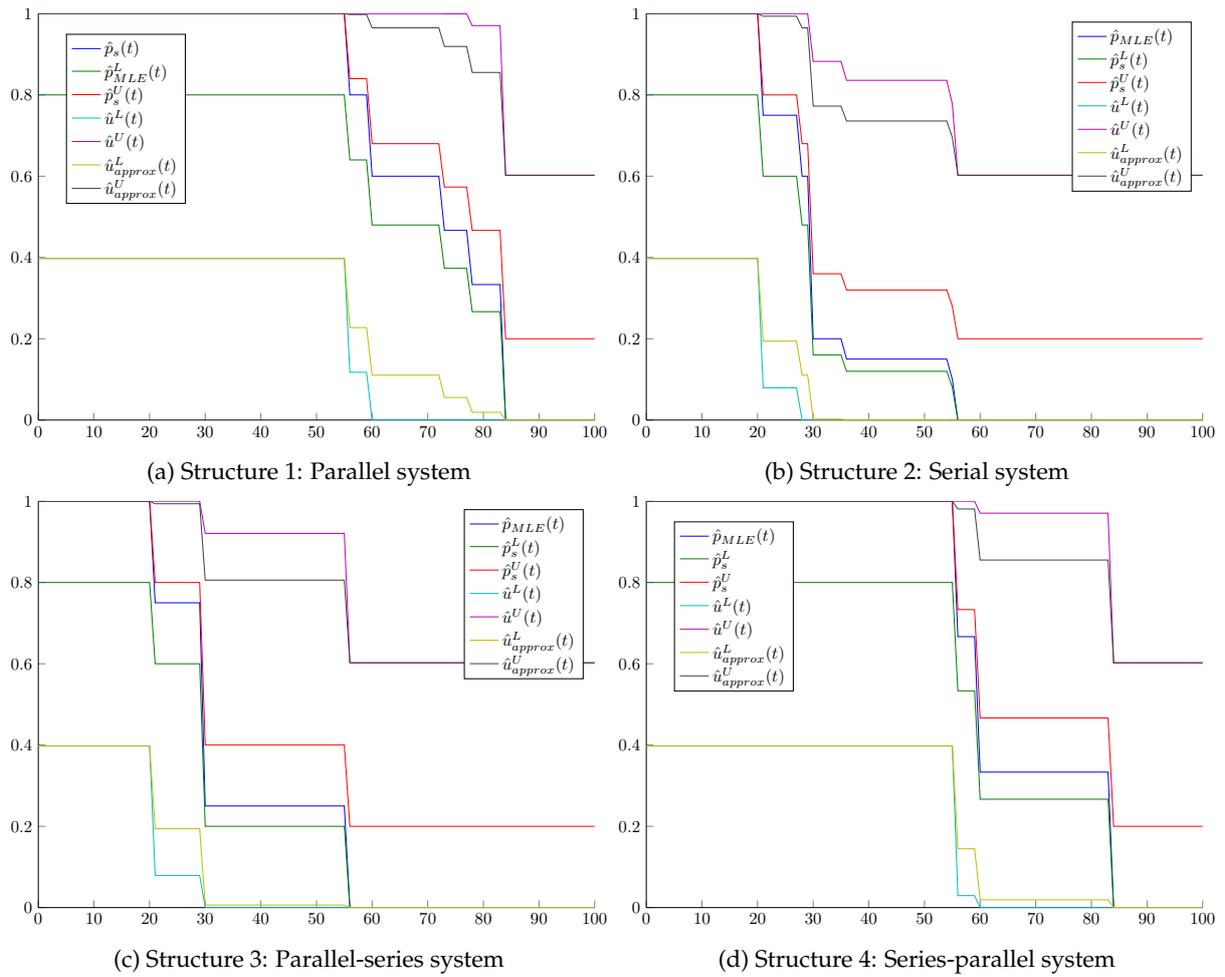


Figure 3.20: Parallel/series systems: System reliability  $R_s(t)$  with  $n_s = 4$

In order to merge the precise observations and censored ones, the upper and lower observation pools are build without additional assumption as follows

$$O_{T_i}^L = \{t_{i,1}, \dots, t_{i,n_i}, \underbrace{t_i^{max}, \dots, t_i^{max}}_{n'_i}\}$$

$$O_{T_i}^U = \{t_{i,1}, \dots, t_{i,n_i}, \underbrace{+\infty, \dots, +\infty}_{n'_i}\}$$

System reliability  $p_s(t)$  can be presented in form of a random interval  $S(t) = [P_s^L(t), P_s^U(t)]$  where  $P_s^L(t)$  is the lower random variable bound of  $\mathbb{P}\{T_s > t | O_{T_1}^L, \dots, O_{T_d}^L\}$  and  $P_s^U$  is the upper random variable bound of  $\mathbb{P}\{T_s > t | O_{T_1}^U, \dots, O_{T_d}^U\}$  so that the cdf of  $P_s^L(t)$  and  $P_s^U(t)$  are given by

$$F_L(x, t) = (1 - p_s^L(t))^{n_s} + \sum_{k=1}^{n_s} I_x(k, n_s + 1 - k) \binom{n_s}{k} (p_s^L(t))^k (1 - p_s^L(t))^{n_s - k} \quad (3.72)$$

$$F_U(x, t) = \mathbf{1}_{\{1\}}(x) (p_s^U(t))^{n_s} + \sum_{k=0}^{n_s - 1} I_x(k + 1, n_s - k) \binom{n_s}{k} (p_s^U(t))^k (1 - p_s^U(t))^{n_s - k} \quad (3.73)$$

where  $p_s^L(t)$  and  $p_s^U(t)$  are the empirical system reliability observed before instant  $t$  in pseudo observations  $O_{T_s}^L$  and  $O_{T_s}^U$  which are deviated respectively from  $O_{T_1}^L, \dots, O_{T_d}^L$  and  $O_{T_1}^U, \dots, O_{T_d}^U$ . Both  $O_{T_s}^L$  and  $O_{T_s}^U$  should be constructed by random re-sampling with Proposal 1, 2 and 3 respectively. The algorithms estimating  $\hat{k}_{s,t}^L$  and  $\hat{k}_{s,t}^U$  are given in Algorithms 5 and 6.

Using (2.2), the confidence interval at level  $1 - \alpha$  for  $P_s(t)$  is given by

$$[u^L(t), u^U(t)] = [F_{L_{\alpha/2}}^{-1}(t), F_{U_{\alpha/2}}^{-1}(t),]$$

where  $F_{L_{\alpha/2}}^{-1}(t)$  and  $F_{U_{\alpha/2}}^{-1}(t)$  are the inverse function of (3.72) and (3.73). It can also be approximated by

$$[u^L(t), u^U(t)] = [I_{\alpha/2}^{-1}(E(k_s^L(t)), n_s + 1 - E(k_s^L(t))), I_{1-\alpha/2}^{-1}(E(k_s^U(t)) + 1, n_s - E(k_s^U(t)))]$$

when the system is highly reliable.

### 3.5 Comparison of two approaches on binary systems

In this section, the two previous approaches, the asymptotic normality (AN) approach and random set (RS) approach, are applied on simple elementary structures and large system fault trees and compared.

#### 3.5.1 Comparison on elementary structures

##### Comparing method

For analyzing quality of interval estimates, two indices are introduced: average coverage percentage (ACP) to evaluate the confidence interval accuracy and length of confidence interval which characterizes how conservative and precise it is.

In statistics, coverage probability of a confidence interval is the proportion of the time that the interval contains the true value of interest [Dodge, 2006]. While constructing a confidence interval (CI), we try to obtain coverage probability close to the confidence level  $1 - \alpha$ , i.e.

$$\mathbb{P}_\theta(\theta \in CI) \approx 1 - \alpha$$

Average coverage probability (ACP) is the empirical estimate of the actual confidence level  $\mathbb{P}_\theta(\theta \in CI)$ . If it is greater than  $1 - \alpha$ , the confidence interval tends to be conservative; if ACP is smaller than  $1 - \alpha$ , the CI should be considered less accurate. In order to obtain ACP, for each component  $C_i$ ,  $n_i$  samples were generated according to Table 3.8 for each iteration. After repeating this procedure for  $n_{acp} = 10000$  times, the calculated ACP corresponds to the empirical probability that the actual value  $p_s$  is bounded inside the obtained confidence intervals.

For confidence interval estimation of binomial proportion  $p$ , erratic and chaotic behaviors of ACP are observed as well as oscillation phenomenon on ACP with fixed sample size as the proportion parameter

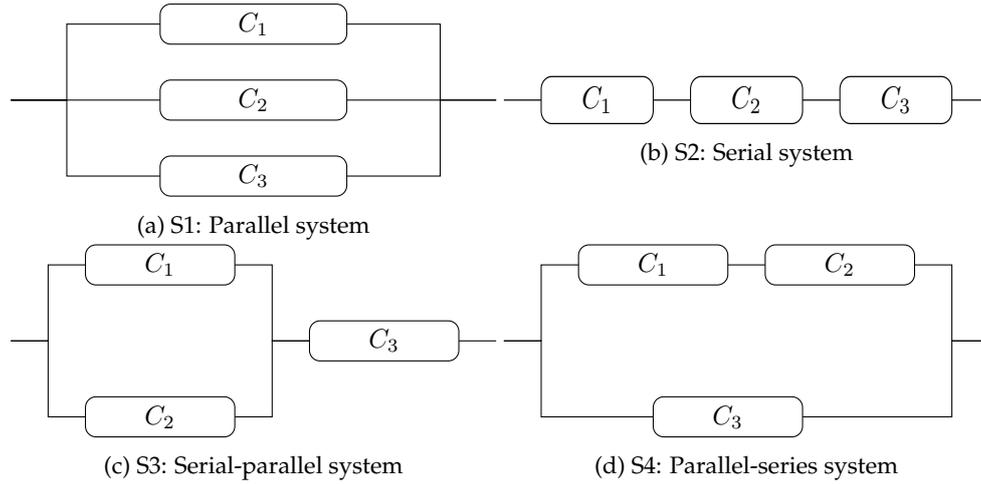


Figure 3.21: System structures of three-component system

Component	Sample size ( $n_i$ )	Reliability Configuration 1	Reliability Configuration 2
1	20	0.98	0.85
2	30	0.93	0.70
3	35	0.95	0.80

Table 3.8: Parameters for ACP test: Component Reliability

$p$  increases from 0 to 1 [Brown et al., 2001]. It is also widely recognized that unsatisfactory coverage appears for  $p$  close to 0 or 1. The CI results of our asymptotic normality method are based on indirectly calculated variance of system reliability estimate. And the ones given by random set approach have both bounds with beta distribution whose parameters are given by pseudo system level observations. Therefore, even though both CI are not classical estimates for binomial proportion, we can still suppose that the ACP behavior of both methods should be impacted by the value of system reliability and more directly components' reliability so that two different component reliability configurations were introduced (Table 3.8) in this test.

Algorithm 5 based on random set (RS) approach and Algorithm 2 using asymptotic normality (AN) approach are used to conduct this test.

In general, RS methods treat system as single component by constructing pseudo samples; AN methods are based on a bottom-up scheme where the component reliability estimate and the corresponding variance are propagated from single component level to system level using reliability function.

RS methods provide conservative results such that the ACP are always higher than the actual confidence  $1 - \alpha = 0.95$  because the upper and lower bounds variables are supposed to satisfy the following condition

$$\mathbb{P}(P^L \leq p \leq P^U) = 1$$

The CI lengths are also much larger than the results using AN methods with assumption. The lengths of interval estimates and CIs are impacted by two factors: the system reliability and the number of pseudo system level component which is chosen according to component sample sizes and presents the quantity of available information and its uncertainty.

With more specific information (prior distributions), AN methods gave different results: AN methods using typical MLE had always the lowest ACP and smallest CI length as the variance of estimated components' reliability was small; amid the proposed assumptions, Assumptions 2-4 showed the best performance. Indeed, with zero failure observed in some cases, these assumptions decrease the reliability estimate and enlarge its variance estimate. As the variance estimates for each component are the same for all these four systems, the results also showed the role played by Birnbaum importance factor (BIF) determined by component reliability and system structure in calculating system level variance using AN approach.

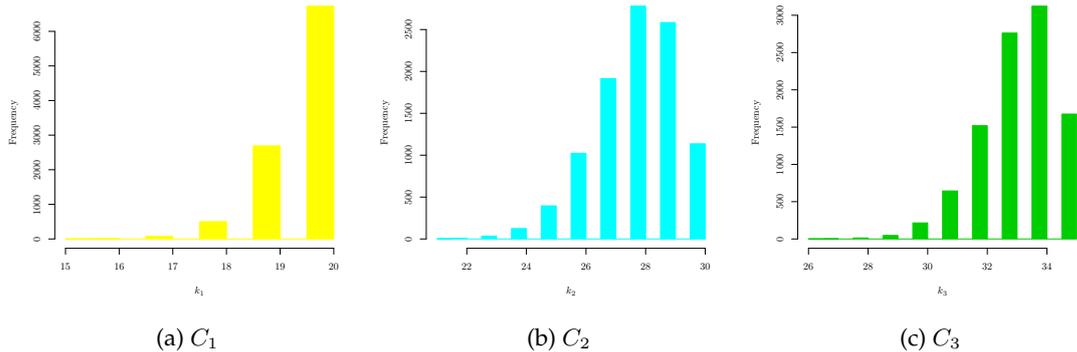


Figure 3.22: Distribution of generated  $k_i$  value for each component  $C_i$  in Configuration 1

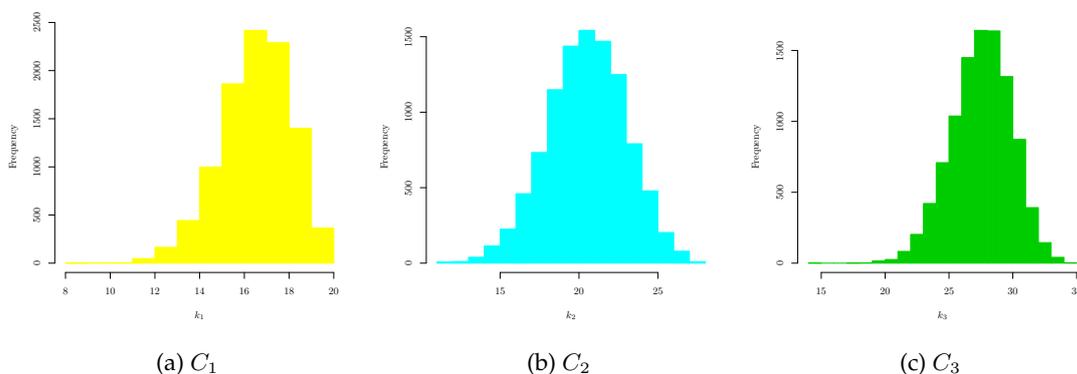


Figure 3.23: Distribution of generated  $k_i$  value for each component  $C_i$  in Configuration 2

### Systems composed of highly reliable components

In Configuration 1, we had high component reliabilities so that the condition  $n_i = k_i$  where prior assumptions are more likely needed to estimate  $p_i$  and  $Var(p_i)$  especially for Component 1 (Figure 3.22). The results obtained by AN methods (Assumptions 2-4) have high ACP values which shows that the prior distributions assumptions fit our system and data conditions.

In case of S2, the AN results have larger average CI length comparing with the ones in case of S1. The single component in series system has more influence on system reliability than in parallel system so that its BIF increases so as the system reliability variance estimate and CI length.

The evolution of CI length between different system structures composed of the same components is more sensitive for AN method than for RS method.

$n_s$	ACP CI	Average CI Length
20	1	0.1685644
25	1	0.1373198
30	1	0.1158404
35	1	0.1001713

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.2476	0.0002478518
2	1	0.0007779568
3	1	0.0007515397
4	1	0.0007275831
5	0.4402	0.0002888466
6	0.9267	0.0005256422

(b) Asymptotic normality (AN) method

Table 3.9: ACP test Results on S1 Configuration 1,  $n_{acp} = 10000$ ,  $p_s = 0.99993$

$n_s$	ACP CI	Average CI Length
20	0.9972	0.3258477
25	0.9928	0.2890787
30	0.9728	0.2620569
35	0.9666	0.2411718

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.9015	0.2238406
2	0.9833	0.2693917
3	0.9833	0.2662084
4	0.9833	0.2633749
5	0.9231	0.2253339
6	0.9890	0.2476414

(b) Asymptotic normality (AN) method

Table 3.10: ACP test Results on S2 Configuration 1,  $n_{acp} = 10000$ ,  $p_s=0.86583$

$n_s$	ACP CI	Average CI Length
20	0.9982	0.2435901
25	0.9936	0.2113461
30	0.9935	0.1884120
35	0.9910	0.1711320

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.8299	0.1132770
2	0.9971	0.1292080
3	0.9971	0.1286954
4	0.9971	0.1282165
5	0.8300	0.1145025
6	0.9977	0.1226694

(b) Asymptotic normality (AN) method

Table 3.11: ACP test Results on S3 Configuration 1,  $n_{acp} = 10000$ , the theoretical value of system reliability  $p_s = 0.94867$

$n_s$	ACP CI	Average CI Length
20	1	0.1762370
25	1	0.1451650
30	1	0.1237884
35	1	0.1081813

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.7162	0.01191240
2	1	0.01811855
3	1	0.01782051
4	1	0.01754955
5	0.7162	0.01219797
6	0.93	0.01503417

(b) Asymptotic normality (AN) method

Table 3.12: ACP test Results on S4 Configuration 1,  $n_{acp} = 10000$ ,  $p_s=0.99557$

$n_s$	ACP CI	Average CI Length
20	1	0.1841664
25	1	0.1532332
30	1	0.1319220
35	1	0.1163369

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.8309	0.02073847
2	0.8479	0.02108163
3	0.8462	0.02106560
4	0.8453	0.02105100
5	0.8309	0.02077456
6	0.8342	0.02094286

(b) Asymptotic normality (AN) method

Table 3.13: ACP test Results on S1 Configuration 2,  $n_{acp} = 10000$ ,  $p_s = 0.991$ 

$n_s$	ACP CI	Average CI Length
20	0.9969	0.4501559
25	0.9909	0.4038223
30	0.9827	0.3690131
35	0.9703	0.3416780

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.9354	0.3212212
2	0.9407	0.3213518
3	0.9402	0.3213107
4	0.9403	0.3212761
5	0.9357	0.3212241
6	0.9384	0.3212717

(b) Asymptotic normality (AN) method

Table 3.14: ACP test Results on S2 Configuration 2,  $n_{acp} = 10000$ ,  $p_s = 0.476$ 

### Systems composed of non-highly reliable components

In Configuration 2, components reliabilities are not close to 1. It is more likely to use directly MLE in AN method which fits our observation of similar results between different assumptions (Figure 3.23). The ACP value of RS results still stayed stable near 1 and was much higher than the confidence level. The CI has large average length. An method results give lower ACP than  $1 - \alpha$ . It indicates that the system level reliability variance was underestimated.

### Conclusion

In conclusion, RS methods give always conservative results while AN method gave precise results but not always accurate. In case of highly reliable systems, all results are very close to 1 and AN method gave smaller and more accurate CI; RS methods give larger CI with ACP equal to 1. However, in case of less reliable systems, the accuracy of AN results dropped while RS method keep conservative with high ACP. Given the knowledge that studied system is highly reliable, we propose AN method; if such information is not available or conservative results are demanded, RS method should be a better choice.

## 3.5.2 Application on large systems

### Benchmark

In order to illustrate that our proposed approach can be applied efficiently to reliability studies of large systems, we used fault trees BAOBAB1 and BAOBAB2 (which are constituted of real-life fault trees using

$n_s$	ACP CI	Average CI Length
20	0.9985	0.3923593
25	0.9947	0.3507160
30	0.9867	0.3196582
35	0.9775	0.2953970

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.9277	0.2630549
2	0.9289	0.2631137
3	0.9285	0.2631030
4	0.9285	0.2630940
5	0.9277	0.2630562
6	0.9277	0.2630802

(b) Asymptotic normality (AN) method

Table 3.15: ACP test Results on S3 Configuration 2,  $n_{acp} = 10000$ ,  $p_s = 0.764$

$n_s$	ACP CI	Average CI Length
20	0.9999	0.2815794
25	0.9991	0.2479342
30	0.9981	0.2235574
35	0.9962	0.2049100

(a) Random set (RS) method

Assumption	ACP CI	Average CI Length
1	0.9093	0.1267270
2	0.9134	0.1270949
3	0.9127	0.1270722
4	0.9124	0.1270520
5	0.9093	0.1267378
6	0.9108	0.1269044

(b) Asymptotic normality (AN) method

Table 3.16: ACP test Results on S4 Configuration 2,  $n_{acp} = 10000$ ,  $p_s=0.919$ 

Fault tree	BAOBAB1	BAOBAB2
Component number	61	32
Number of minimal cutsets (max length=4)	72	395
Number of minimal cutsets (max length=5)	472	1025
Number of minimal cutsets (max length=6)	2684	4805
Number of minimal cutsets (max length=7)	17432	4805
Number of minimal cutsets (max length=8)	25892	4805
Total number of minimal cutsets	46188	4805

Table 3.17: Complexity information about studied fault trees

various sources) from Aralia benchmark<sup>1</sup>. The coherent fault trees BAOBAB1 and BAOBAB2 contain respectively 61 and 32 components and both have more than 4000 minimal cutsets (shown in Table 3.17) such that the calculation of the system reliability using probabilistic assessment by propagation from component level is very expensive. Both systems are complex so that it is not possible to decompose them into series/parallel subsystems without repeated components.

### Real-time system application

Suppose that the lifetime observations of the component  $C_i$  are

$$O_{T_i} = \{T_{1,i}, T_{2,i}, \dots, T_{n_i,i}\}$$

Our normality asymptotic method uses Kaplan-Meier estimator to estimate the component reliability at time  $t$ ,  $p_i(t)$ , i.e.

$$\hat{p}_i(t) = \prod_{w_{j,i} < t} \frac{b_{j,i} - d_{j,i}}{b_{j,i}}$$

where  $b_{j,i}$  is the number of observations in good state just before the time  $T_{j,i}$  and  $d_{j,i}$  the number of failures observed at time  $T_{j,i}$ .

The variance of  $\hat{p}_i(t)$  is given as

$$\widehat{\text{Var}}(\hat{p}_i(t)) = \hat{p}_i(t)^2 \sum_{w_{j,i} \leq t} \frac{d_{j,i}}{b_{j,i}(b_{j,i} - d_{j,i})}$$

$$\hat{p}_s(t) = R(\hat{p}_1(t), \dots, \hat{p}_d(t))$$

$$\widehat{\text{Var}}(\hat{p}_s(t)) = \sum_{i=1}^d (R(\hat{p}_1(t), \dots, \hat{p}_i(t) = 1, \dots, \hat{p}_d(t)) - R(\hat{p}_1(t), \dots, \hat{p}_i(t) = 0, \dots, \hat{p}_d(t)))^2 \widehat{\text{Var}}(\hat{p}_i(t))$$

<sup>1</sup>Fault trees distributed by Antoine Rauzy which can be found at: <http://www.itu.dk/research/cla/externals/clib/Aralia.zip>

$C_i$	$n_i$	$O_{W_i}$	$C_i$	$n_i$	$O_{W_i}$	$C_i$	$n_i$	$O_{W_i}$
1	2	100,120	23	3	35,42,2	45	3	54,66,46
2	1	20	24	3	200+,200+,200+	46	4	3,30,7,145
3	3	200+,200+,200+	25	4	93,111,117,23	47	2	18,10
4	5	24,12,33,134,25	26	3	200+,200+,200+	48	2	68,46
5	4	4,28,3,39	27	2	73,35	49	3	124,17,9
6	5	20,26,41,30,24	28	4	138,5,48,30	50	2	160,34
7	3	200+,200+,200+	29	2	73,94	51	4	38,9,67,12
8	2	200+,200+	30	2	200+,200+	52	1	227
9	3	88,42,8	31	4	41,78,15,57	53	4	31,62,155,144
10	2	33,25	32	4	26,152,1,250	54	3	1,9,2
11	5	16,16,19,51,3	33	2	145,6	55	3	200+,200+,200+
12	3	117,89,32	34	1	61	56	3	200+,200+,200+
13	2	100,150	35	2	7,59	57	4	3,31,4,5
14	1	123	36	2	100,90	58	3	200+,200+,200+
15	2	132,164	37	2	25,91	59	1	26
16	4	39,46,151,68	38	1	107	60	4	27,19,16,24
17	2	184,253	39	3	9,5,4	61	5	24,1,14,14,53
18	3	12,39,29	40	3	24,170,61			
19	2	81,15	41	5	1,4,13,12,27			
20	3	31,84,38	42	1	60			
21	5	125,25,25,9,74	43	4	143,5,35,79			
22	5	102,60,105,28,6	44	3	42,8,9			

Table 3.18: BAOBAB1: Component  $C_i, i = 1, \dots, d = 61$

$C_i$	$n_i$	$O_{W_i}$	$C_i$	$n_i$	$O_{W_i}$	$C_i$	$n_i$	$O_{W_i}$
1	1	223	12	1	58	23	2	67,19
2	2	28,253	13	2	69,76	24	1	16
3	3	2,23,20	14	1	122	25	2	3,15
4	4	20,28,18,99	15	3	105,638,1462	26	5	29,8,10,28,75
5	5	67,93,71,90,114	16	5	95,70,136,160,97	27	3	200+,200+,200+
6	5	92,98,74,60	17	5	78,122,133,52,34	28	2	14,17
7	5	206,35,115,123,27	18	1	19	29	5	6,32,5,23,2
8	4	42,126,54,69	19	1	70	30	3	200+,200+,200+
9	4	338,375,128,1823	20	2	100+,100+	31	3	10,1,3
10	5	190,104,105,42,69	21	4	17,3,9,1	32	2	200+,200+
11	2	208,161	22	1	54			

Table 3.19: BAOBAB2: Component  $C_i, i = 1, \dots, d = 32$

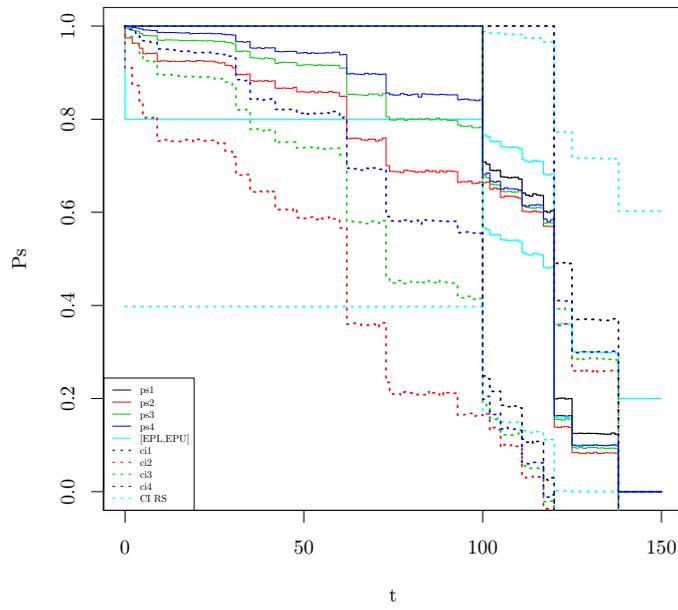


Figure 3.24: BAOBAB1: reliability estimates and their CIs obtained by RS and AN methods (ps1-4,ci1-4) ( $n_s = 4$ , Assumption 1-4)

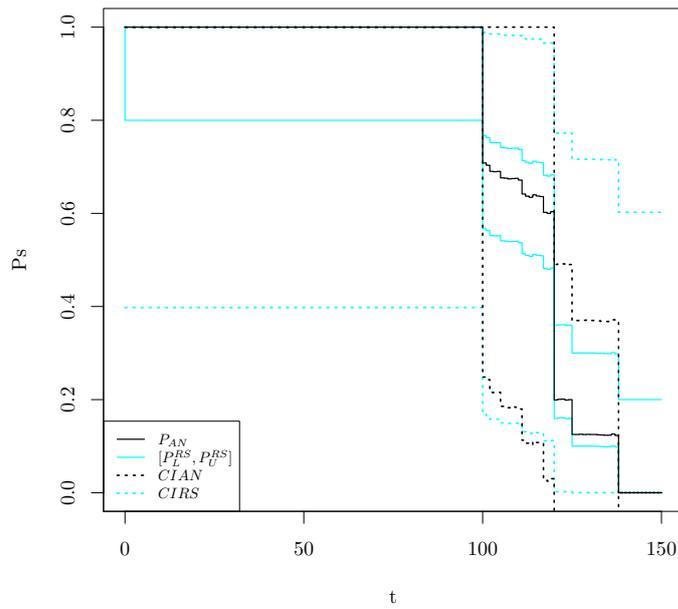


Figure 3.25: BAOBAB1: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 1)

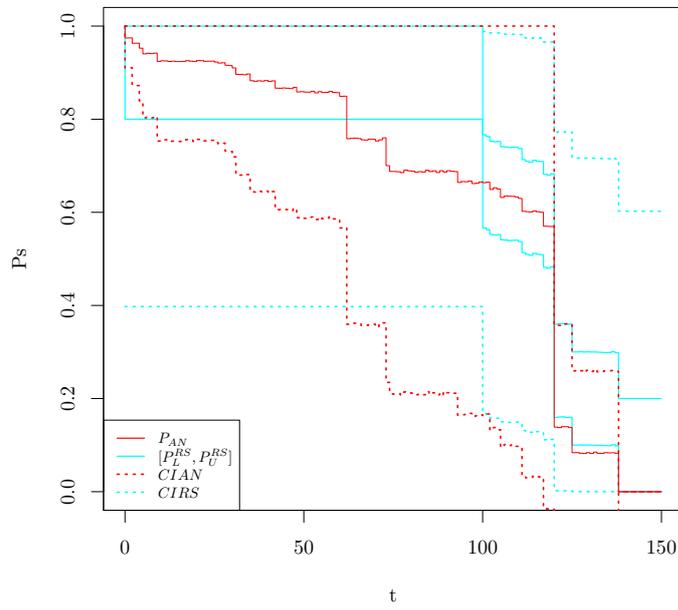


Figure 3.26: BAOBAB1: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 2)

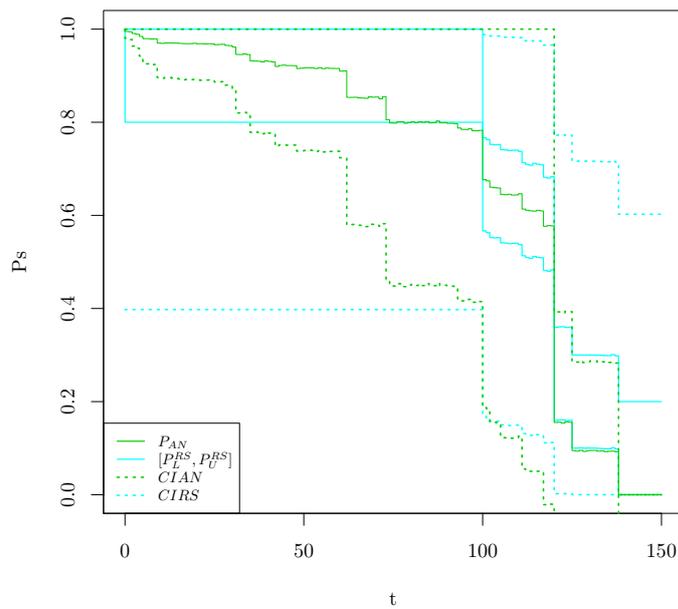


Figure 3.27: BAOBAB1: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 3)

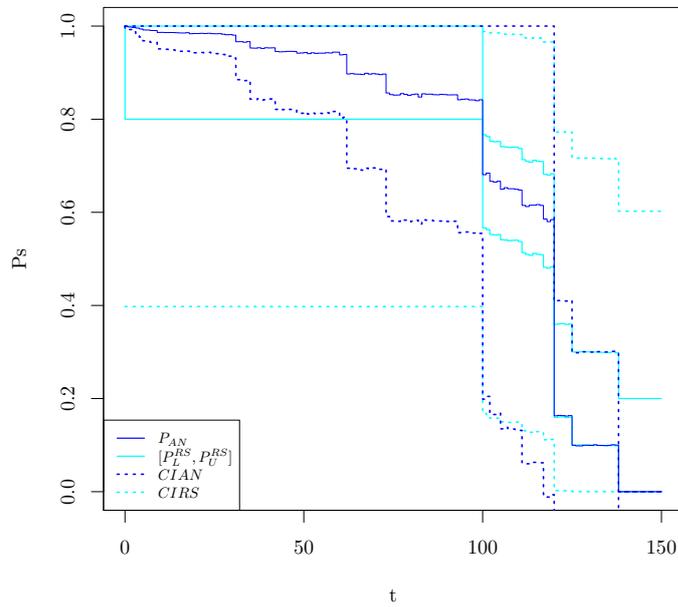


Figure 3.28: BAOBAB1: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 4)

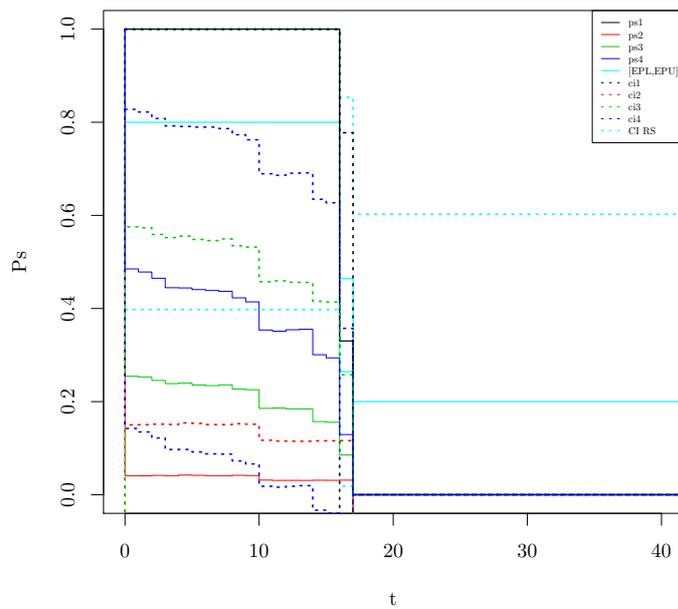


Figure 3.29: BAOBAB2: reliability estimates and their CIs obtained by RS and AN methods ( $ps1-4, ci1-4$ ) ( $n_s = 4$ , Assumption 1-4)

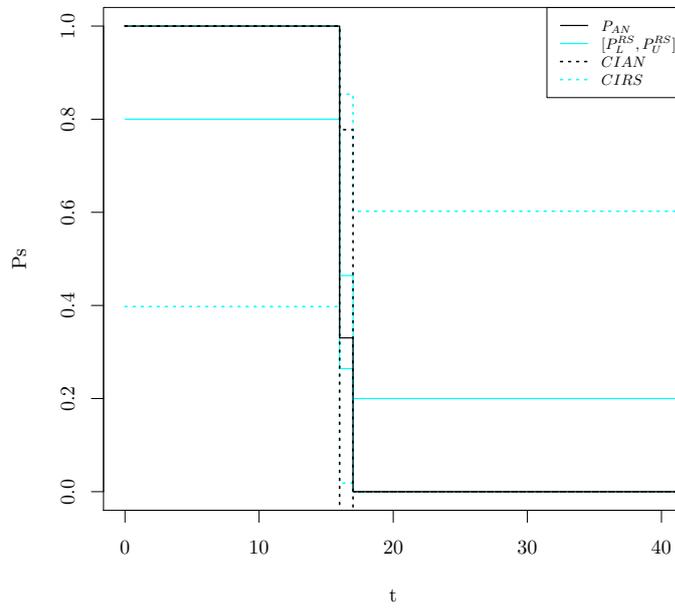


Figure 3.30: BAOBAB2: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 1)

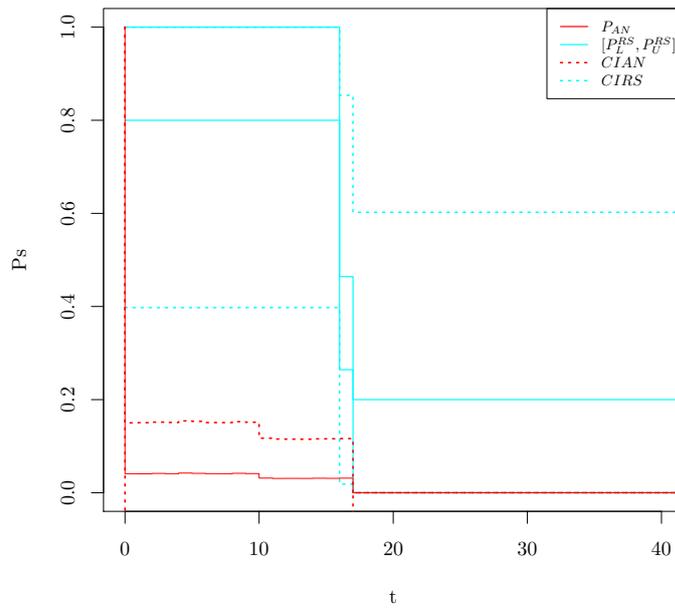


Figure 3.31: BAOBAB2: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 2)

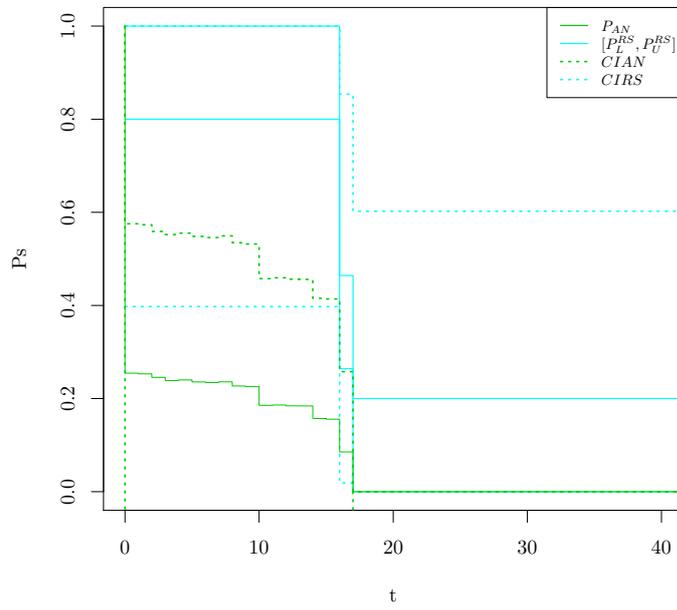


Figure 3.32: BAOBAB2: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 3)

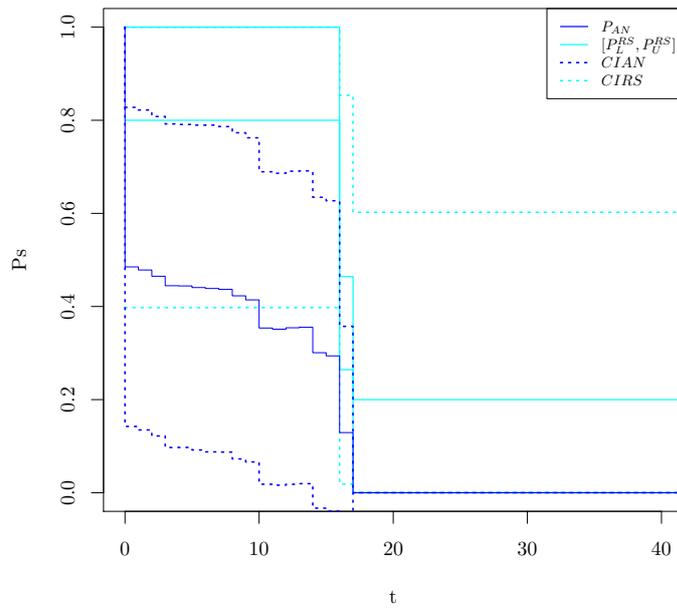


Figure 3.33: BAOBAB2: reliability estimates and their CIs obtained by RS and AN methods ( $n_s = 4$ , Assumption 4)

We obtained reliability estimated value and confidence interval at level and  $1-\alpha = 0.95$  using Algorithm 2 and Algorithm 7 with  $N = 10^5$  and  $n_s = 4$  as well as reference method with 100000 simulations for both fault trees shown in Figures 3.24 - 3.28 and Figures 3.29 - 3.33.

AN results start to drop from the beginning of our observation and gives lower system reliability expected value because at the early age of observation all components have zero failure, i.e.  $k_i = n_i$ , so that Assumption 2-4 gave reduced  $p_i$  estimates, which, in our case, a small  $n_i$  can be very different from the facts. For example, with  $n_i = k_i = 2$ , we obtain respectively  $p_i = 1/2, p_i = 2/3$  and  $p_i = 3/4$  using Assumption 2-4.

Similar to simple structures, the RS results start to drop near instant  $t = 100$  for BAOBAB2 and  $t = 18$  for BAOBAB1 where the first pseudo system sample failure probably appears. The instants where the RS results values decrease correspond to one of component lifetime observations in  $O_{W_i}$  as well as AN results. The results obtained by AN method with Assumption 1 are always bounded by the RS interval estimate for both fault trees.

## Chapter 4

# Reliability assessment on multi-state systems

### 4.1 Introduction

Besides binary model which gives components and systems only two states: perfect operational and complete failure, a component or a system can be evaluated more precisely by multiple states according to its different levels of performance. Such system model is called multi-state system (MSS) and was introduced in [Barlow and Wu, 1978, El-Newehi et al., 1978].

The structure function of a MSS is extended from binary system model. It takes multi-valued input component level states and gives multi-valued system state [Pourret et al., 1999, Ushakov, 1994]. It can be applied with Monte Carlo simulation technique for MSS reliability assessment [Ramirez-Marquez and Coit, 2005, Zio et al., 2007] or with Markov method for dynamic MSS performance evaluation [Xue and Yang, 1995, Lisnianski, 2007].

#### 4.1.1 MSS performance indices

In a system or component of  $m_S$  states, the performance output of each state  $i$  is evaluated by a performance rate  $g_i$ . Let variable  $G \in \{g_1, \dots, g_{m_S}\}$  denote the system performance rate and  $X$  a random variable represent the corresponding system or component state (index). Let variables  $G_s$  and  $G_1, \dots, G_d$  denote respectively the performance rate of a system with  $m_S$  states and  $d$  system components with  $m_i$  states for each. Random variables  $X_s \in E_S = \{1, \dots, m_S\}$  and  $X_i \in E_i = \{1, \dots, m_i\}$ ,  $i = 1, \dots, d$ , as the corresponding state so that

$$G_s = g_{X_s} \quad G_i = g_{X_i}.$$

Then, the structure function of performance rate  $\varphi_G$  becomes a  $\mathbb{R}^d \rightarrow \mathbb{R}$  function

$$G_s = \varphi_G(G_1, \dots, G_d).$$

In practice, we also propose the structure function of state index as in case of binary systems such that

$$X_s = \varphi(X_1, \dots, X_d).$$

The performance are evaluated by several linear MSS reliability indices on function of demand  $\omega$  which indicates the minimal performance rate that should be satisfied to identify the studied object as "in functioning or good state".

- MSS availability: All states can be grouped into a working state set

$$U(\omega) = \{i : g_i - \omega \geq 0, i = 1, \dots, m_S\} \quad (4.1)$$

and a failure state set

$$D(\omega) = \{i : g_i - \omega < 0, i = 1, \dots, m_S\} \quad (4.2)$$

The MSS availability for an arbitrary demand constant  $\omega$  is the probability that the system is in working states, i.e.

$$A(\omega) = \mathbb{P}\{X \in U(\omega)\} = \sum_{i=1}^{m_S} p_i \mathbf{1}_{\{g_i - \omega \geq 0\}}$$

where  $p_i, i \in E_S$  denotes the probability of each state, i.e.

$$p_i = \mathbb{P}(X = i)$$

- MSS expected output performance: The MSS expected output performance  $E_d$  is defined as

$$E_d = \sum_{i=1}^{m_S} p_i G_i \quad (4.3)$$

- MSS expected performance deficiency: The MSS expected performance deficiency  $E_u(\omega)$  for an arbitrary demand constant  $\omega$  is defined as

$$E_u(\omega) = \sum_{i=1}^{m_S} p_i \max(\omega - G_i, 0)$$

#### 4.1.2 Universal generating function (UGF)

The universal generating function (UGF) was firstly introduced in [Ushakov, 1986] and then developed by [Lisnianski and Levitin, 2003]. In mathematics, a *generating function* (also called *generating series*) is a formal power series in variable(s) whose coefficients encode information about a sequence of numbers  $a_n$  indexed by the natural numbers. The ordinary generation function of a sequence  $a_n$  is defined by

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$$

In probability theory and statistics, the *moment generating function* (mgf) of a random variable  $G$  is defined as

$$m_G(t) := E[e^{tG}], \quad t \in \mathbb{R},$$

where the expectation of  $e^{tG}$  exists.

Consider that  $G$  takes discrete values from set  $E_G = \{g_1, \dots, g_m\}$  and

$$p_i = \mathbb{P}(G = g_i)$$

the *probability generating function* (pgf) of  $G$  is defined as

$$m_G(t) = E(e^{tG}) = \sum_{i=1}^m e^{tg_i} p_i$$

By replacing  $e^t$  by a variable  $z$ , we obtain the *z-transform function* of  $G$

$$w_G(z) = E(z^G) = \sum_{i=1}^m z^{g_i} p_i$$

Consider that random variables  $G_1, \dots, G_d$  denote performance rates of independent system components such that

$$G_i \in E_{G_i} = \{g_{i1}, \dots, g_{im_i}\} \quad p_{ij_i} = \mathbb{P}(G_i = g_{ij_i}),$$

the z-transform function of  $G_i$  is given by

$$u_i(z) = \sum_{j_i=1}^{m_i} z^{g_{ij_i}} p_{ij_i}.$$

The system *universal generating function* (UGF) is then defined as a function of z-transforms of component random variables  $u_1(z), \dots, u_d(z)$ , i.e.

$$\begin{aligned} U(z) &= \otimes_{\varphi}(u_1(z), \dots, u_d(z)) \\ &= \sum_{i_1=0}^{m_1} z^{g_{1i_1}} p_{1i_1} \otimes_{\varphi} \dots \otimes_{\varphi} \sum_{i_d=0}^{m_d} z^{g_{di_d}} p_{di_d} \\ &= \sum_{i_1=1}^{m_1} \dots \sum_{i_d=1}^{m_d} \prod_{j=1}^d p_{ji_j} z^{\varphi(g_{1i_1}, \dots, g_{di_d})} \end{aligned}$$

where  $\otimes_{\varphi}$  is an operator describing the system (or subsystem) level performance in terms of performance rate of all its components.

## 4.2 Random set approach

In this section, we firstly introduce how to estimate performance indices of a multi-state component using random set theory; then we give the propagation method to obtain system-level performance indices using random set model with resampling procedure; finally, a case study is illustrated to demonstrate the effectiveness of our approach and compares results obtained by our method with the results given by universal generating function (UGF) method accompanied with maximum likelihood estimators (MLE).

### 4.2.1 Estimation of availability of multi-state components

#### General case

Consider a component  $C$  with  $m$  states and that the random state variable  $X \in E = \{1, \dots, m\}$  follows categorical distribution parameterized by

$$\mathbb{P}(X = i | X \geq i - 1), \quad 1 \leq i \leq m - 1.$$

Then  $N$  observations of  $X$  are constructed

$$O = \{x_1, \dots, x_N\}$$

using stick-breaking construction scheme as follows [Connor and Mosimann, 1969]:

- Initialize the number of samples to be generated in  $O$ :

$$n = N$$

- For each state  $i = 1 : m - 1$

- Generate  $n$  pivotal samples

$$z_1^i, \dots, z_n^i \sim U[0, 1]$$

- Count  $n_i$  the number of samples in category  $i$ :

$$n_i = \sum_{j=1}^{j=n} \mathbf{1}_{[0, \mathbb{P}(X=i|X \geq i-1)]}(z_j^i)$$

- Add  $n_i$  observations in category  $i$  to the observation pool  $O$ :

$$x_{N-n+1} = \dots = x_{N-n+n_i} = i$$

- Calculate the number of samples not affected yet:

$$n = n - n_i$$

- Count the number of samples in category  $m$  and complete the observation pool  $O$ :

$$n_m = n, \quad x_{N-n+1} = \dots = x_N = m$$

Let  $\theta_i$  denote the distribution parameter given by

$$\theta_i = \mathbb{P}(X = i | X \geq i - 1; O)$$

Then we have

$$\underline{\theta}_i = z_{(n_i)}^i \leq \theta_i \leq z_{(n_i+1)}^i = \bar{\theta}_i \quad 1 \leq i \leq m - 1$$

where  $\underline{\theta}_i = z_{(n_i)}^i$  ( $\bar{\theta}_i = z_{(n_i+1)}^i$ ) is the  $n_i$ th ( $(n_i + 1)$ th) order statistic for the  $i$ th sub-sampling so that

$$\underline{\theta}_i \sim \text{Beta}(n_i, \sum_{j=i+1}^m n_j + 1) \quad (4.4)$$

$$\bar{\theta}_i \sim \text{Beta}(n_i + 1, \sum_{j=i+1}^m n_j) \quad (4.5)$$

Thus the probability or proportion of each category  $i$  given observations  $O$ ,

$$P_i = \mathbb{P}(X = i|O) \quad 1 \leq i \leq m$$

can be written by

$$P_i = \begin{cases} \theta_1 & \text{if } i = 1 \\ \theta_i \prod_{j=1}^{i-1} (1 - \theta_j) & \text{if } i = 2, \dots, m-1 \\ \prod_{j=1}^{m-1} (1 - \theta_j) & \text{if } i = m \end{cases}$$

Let  $F : [0, 1]^m \rightarrow \mathbb{R}$  be a linear function and  $Y$  a random variable given by

$$Y = F(p) = \sum_{i=1}^m c_i P_i \quad (4.6)$$

where  $c_1, \dots, c_m$  are distinct constant coefficients and  $0 < c_1 < c_2 < \dots < c_m$ . Then  $Y$  can be represented by a random interval  $S_Y = [\underline{Y}, \bar{Y}]$  such that

$$\mathbb{P}\{Y \in [\underline{Y}, \bar{Y}]\} = 1$$

where

$$\underline{Y} = \sum_{i=1}^m \underline{P}_i c_i \quad \bar{Y} = \sum_{i=1}^m \bar{P}_i c_i \quad (4.7)$$

are two random variables;  $(\underline{P}_1, \dots, \underline{P}_m)$  and  $(\bar{P}_1, \dots, \bar{P}_m)$  are random vectors in  $[0, 1]^m$  with the constraint  $\sum_{j=1}^m \bar{P}_j = \sum_{j=1}^m \underline{P}_j = 1$ .

To obtain  $(\bar{P}_1, \dots, \bar{P}_m)$  describing the condition in which  $Y$  is maximized, we just need to maximize the  $P_m$  whose coefficient is the maximum by setting

$$\bar{P}_i = \begin{cases} \theta_1 & \text{if } i = 1 \\ \theta_i \prod_{j=1}^{i-1} (1 - \theta_j) & \text{if } i = 2, \dots, m-1 \\ \prod_{j=1}^{m-1} (1 - \theta_j) & \text{if } i = m \end{cases}$$

Using (4.5) and results in [Albert and Denis, 2012], the random vector  $(\bar{P}_1, \dots, \bar{P}_m)$  follows a Dirichlet distribution

$$(\bar{P}_1, \dots, \bar{P}_m) \sim \text{Dirichlet}(\bar{k}_1, \dots, \bar{k}_m) \quad (4.8)$$

where

$$\bar{k}_i = \begin{cases} n_i & \text{if } i \neq m \\ n_i + 1 & \text{if } i = m \end{cases} \quad (4.9)$$

Similarly, to obtain  $(\underline{P}_1, \dots, \underline{P}_m)$  describing the condition in which  $Y$  is minimized we need to maximize the  $P_1$  while minimizing the others by setting

$$\underline{P}_i = \begin{cases} \bar{\theta}_1 & \text{if } i = 1 \\ \bar{\theta}_i (1 - \bar{\theta}_1) \prod_{j=2}^{i-1} (1 - \bar{\theta}_j) & \text{if } i = 2, \dots, m-1 \\ (1 - \bar{\theta}_1) \prod_{j=2}^{m-1} (1 - \bar{\theta}_j) & \text{if } i = m \end{cases}$$

Then we obtain the distribution of  $(\underline{P}_1, \dots, \underline{P}_m)$

$$(\underline{P}_1, \dots, \underline{P}_m) \sim \text{Dirichlet}(\underline{k}_1, \dots, \underline{k}_m) \quad (4.10)$$

with

$$\underline{k}_i = \begin{cases} n_i & \text{if } i \neq 1 \\ n_i + 1 & \text{if } i = 1 \end{cases} \quad (4.11)$$

The estimation of  $S_Y$  becomes

$$E(S_Y) = \left[ \sum_{i=1}^m \frac{\underline{k}_i}{N+1} c_i, \sum_{i=1}^m \frac{\bar{k}_i}{N+1} c_i \right] \quad (4.12)$$

The corresponding confidence interval (CI) can be obtained by simulating the Dirichlet distributions. In case that different categories have the same coefficient in (4.6), they should be merged them firstly by adding the probabilities and the number of observations.

### MSS reliability indices calculation

In a system of  $m_S$  states, the performance output of each state  $i$  is evaluated by a performance rate  $G_i$ . Suppose that there is not two system states having the same performance rate. Let  $X \in E_S = \{1, \dots, m_S\}$  a random variable representing system state. Given  $N$  observations of  $X$ ,  $O = \{x_1, x_2, \dots, x_N\}$ , we apply our method on several linear MSS reliability indices on function of demand  $\omega$ .

**MSS availability** All states can be grouped into a working state set

$$U(\omega) = \{i : G_i - \omega \geq 0\} \quad (4.13)$$

and a failure state set

$$D(\omega) = \{i : G_i - \omega < 0\} \quad (4.14)$$

Given the observations  $O$ , the MSS availability for the arbitrary demand constant  $\omega$  is the probability that the system is in working states, i.e.

$$A(\omega) = \mathbb{P}\{X \in U(\omega) | O\} = \sum_{i=1}^{m_S} P_i \mathbf{1}_{\{G_i - \omega \geq 0\}}$$

With (4.7) (4.8) (4.9) (4.10) and (4.11), it can be represented by the random set  $S_A(\omega) = [A^L(\omega), A^U(\omega)]$  with

$$A^L(\omega) \sim \text{Beta}(n_{U(\omega)}, N - n_{U(\omega)} + 1)$$

$$A^U(\omega) \sim \text{Beta}(n_{U(\omega)} + 1, N - n_{U(\omega)})$$

where  $n_{U(\omega)}$  is the number of observations in working states. The expected value of  $S_A(\omega)$  is given by

$$E(S_A(\omega)) = \left[ \frac{n_{U(\omega)}}{N+1}, \frac{n_{U(\omega)} + 1}{N+1} \right]$$

**MSS expected output performance** The MSS expected output performance  $E_d$  is defined as

$$E_d = \sum_{i=1}^{m_S} P_i G_i \quad (4.15)$$

Let  $i_{max}^d$  and  $i_{min}^d$  be the states with the maximum and minimum performance rate, i.e.

$$G_{i_{max}^d} = \max_{1 \leq i \leq m_S} G_i \quad G_{i_{min}^d} = \min_{1 \leq i \leq m_S} G_i$$

Using the procedure described in the previous section, the random interval describing  $E_d$  is given by

$$S_d = [\underline{E}_d, \bar{E}_d]$$

where

$$\underline{E}_d = \sum_{i=1}^{m_S} \underline{P}_i^d G_i \quad \bar{E}_d = \sum_{i=1}^{m_S} \bar{P}_i^d G_i$$

are two random variables on function of the random vectors

$$(\underline{P}_1^d, \dots, \underline{P}_m^d) \sim \text{Dir}(\underline{k}_1^d, \dots, \underline{k}_m^d)$$

$$(\bar{P}_1^d, \dots, \bar{P}_m^d) \sim \text{Dir}(\bar{k}_1^d, \dots, \bar{k}_m^d)$$

where

$$\underline{k}_i^d = \begin{cases} n_i & \text{if } i \neq i_{min}^d \\ n_i + 1 & \text{if } i = i_{min}^d \end{cases}$$

$$\bar{k}_i^d = \begin{cases} n_i & \text{if } i \neq i_{max}^d \\ n_i + 1 & \text{if } i = i_{max}^d \end{cases}$$

The estimation of  $S_d$  is given by

$$E(S_d) = \left[ \sum_{i=1}^{m_S} \frac{\underline{k}_i^d}{N+1} G_i, \sum_{i=1}^{m_S} \frac{\bar{k}_i^d}{N+1} G_i \right]$$

**MSS performance deficiency** The MSS performance deficiency  $E_u(\omega)$  for the arbitrary demand constant  $\omega$  is defined as

$$E_u(\omega) = \sum_{i=1}^{m_S} P_i \max(\omega - G_i, 0)$$

After grouping the states with the same value of the coefficient  $c_i = \max(\omega - G_i, 0)$ , we obtain  $m'$  categories  $(1, \dots, m')$  whose coefficients are  $(c'_1, \dots, c'_{m'})$  and  $c'_1 < c'_2 < \dots < c'_{m'}$ . Then the random interval describing  $E_u$  is given by

$$S_u = [\underline{E}_u, \bar{E}_u]$$

where

$$\underline{E}_u = \sum_{i=1}^{m'} \underline{P}_i^u c'_i \quad \bar{E}_u = \sum_{i=1}^{m'} \bar{P}_i^u c'_i$$

are two random variables where

$$\begin{aligned} (\underline{P}_1^u, \dots, \underline{P}_{m'}^u) &\sim Dir(\underline{k}_1^u, \dots, \underline{k}_{m'}^u) \\ (\bar{P}_1^u, \dots, \bar{P}_{m'}^u) &\sim Dir(\bar{k}_1^u, \dots, \bar{k}_{m'}^u) \end{aligned}$$

with

$$\begin{aligned} \underline{k}_i^u &= \begin{cases} n'_i & \text{if } i \neq 1 \\ n'_i + 1 & \text{if } i = 1 \end{cases} \\ \bar{k}_i^u &= \begin{cases} n'_i & \text{if } i \neq m' \\ n'_i + 1 & \text{if } i = m' \end{cases} \end{aligned}$$

and

$$n'_i = \sum_{x \in O} \mathbf{1}_{\{\max(\omega - G_x, 0) = c'_i\}}(x)$$

the number of observation for each category  $i$  after merging.

The estimation of  $S_u$  is given by

$$E(S_u) = \left[ \sum_{i=1}^{m'} \frac{\underline{k}_i^u}{N+1} c'_i, \sum_{i=1}^{m'} \frac{\bar{k}_i^u}{N+1} c'_i \right]$$

## 4.2.2 Estimation of system availability

Resampling method is used to simulate a condition that  $n_S$  systems are constructed by the available component samples and the system-level pseudo-observations of each system-level samples are obtained.

Consider a system composed of  $d$  components. Let  $O_j$  ( $1 \leq j \leq d$ ) be the observation pool containing independent and identically distributed (i.i.d.) state samples of component  $C_j$ . To construct a system-level pseudo-observation  $y_i$  ( $1 \leq i \leq n_S$ ), for each component  $C_j$ , a sample  $x_i^j$  of  $C_j$  is resampled from  $O_j$  with replacement.

Then the observation pool  $O_S$  containing  $n_S$  system-level pseudo-observations is given by

$$O_S = \{y_i = \varphi(x_i^1, \dots, x_i^d), 1 \leq i \leq n_S\}$$

where  $\varphi$  is the structure function of the studied system.

In order to ensure the quantity of uncertainty, the sample size  $n_S$  is bounded by

$$\min_{1 \leq i \leq d} |O_i| \leq n_S \leq \max_{1 \leq i \leq d} |O_i|$$

The pseudo-observations resampled with replacement can be approximately regarded as i.i.d. samples so that the vector samples  $(x_i^1, \dots, x_i^d)$ , as well as  $y_i$  which are obtained through the structure function  $\varphi$  on function of  $(x_i^1, \dots, x_i^d)$ , are i.i.d. observations.

Then, the system performance indices can be obtained using method mentioned in the previous section on regarding the system as a single component given  $n_S$  observations in  $O_S$ .

However, the resampling is also random, in case of categorical distributed  $X$ , the observation number of each category  $i$  can be estimated by Monte Carlo simulation where the number of executions  $N_{sim}$  conducts convergent results.

System state $i$	1	2	3	4	5	6	7
Performance $G_i$	0	1	1.5	2	2.5	3	3.5

Table 4.1: Correspondence table of system states and performance

$X_1$	$X_2$	$X_3$	$X_S = \varphi(X_1, X_2, X_3)$	$G$
1	1	1	1	0
1	1	2	1	0
1	2	1	1	0
1	2	2	3	1.5
1	3	1	1	0
1	3	2	4	2
2	1	1	1	0
2	1	2	2	1
2	2	1	1	0
2	2	2	5	2.5
2	3	1	1	0
2	3	2	6	3
3	1	1	1	0
3	1	2	3	1.5
3	2	1	1	0
3	2	2	6	3
3	3	1	1	0
3	3	2	7	3.5

Table 4.2: Case study: Truth table

### 4.2.3 Case study: oil transmission system

In this section, we apply our method on an oil transmission system composed of three pipes in [Ding and Lisnianski, 2008](See Figure 4.1) given component observation data which we generated shown in Table 4.3.

In the studied system, the oil flow is transmitted from point C to point E. The performance for each pipe  $C_i$  in its state  $j$  is measured by their transmission capacity (tons per minute)  $g_j^i$ . Pipe 1  $C_1$  and Pipe 2  $C_2$  have three states: a total failure state 1 where the capacity falls to zero ( $g_1^1 = g_1^2 = 0$ ); a failure state 2 where the capacity is 1 ton/min for  $C_1$  and 1.5 tons/min for  $C_2$  ( $g_2^1 = 1, g_2^2 = 1.5$ ); an operational state 3 where the capacity is 1.5 ton/min for  $C_1$  and 2 tons/min for  $C_2$  ( $g_3^1 = 1.5, g_3^2 = 2$ ). Pipe 3  $C_3$  is binary: state 1 indicates a total failure with zero capacity ( $g_1^3 = 0$ ); state 2 indicates an operational state with a capacity of 4 tons/min ( $g_2^3 = 4$ ).

The system output performance  $G_i$  is defined as the maximum flow that can be transmitted from C to E: the total flow between points C and D through the parallel Pipe 1 and Pipe 2 is equal to the sum of the flows in the two pipes. The flow from point D to point E is limited by the transmission capacity of Pipe 3. This flow cannot be greater than the flow between points C and D. Therefore, the flow between points C and E (the system performance rate) is given by

$$G_s = \min(g_i^1 + g_j^2, g_k^3), \quad i, j \in \{1, 2, 3\}, k \in \{1, 2\} \quad (4.16)$$

where  $s = 1, \dots, m_S = 7$  is the index of system states determined according to the correspondence table (Table 4.1).

The structure function  $\varphi$  is given by combining (4.16) and the correspondence between system state index and the performance rate  $G_i$  (shown in Table 4.1).

#### Reference method

The proposed reference method used on MSS reliability assessment is the universal generating function (UGF) method used by Ding and Lisianski [Ding and Lisnianski, 2008] with maximum likelihood estimators (MLE).

Let  $p_j^i$  the corresponding probabilities that the component  $C_i$  is in state  $j$ , the  $u$ -function for  $C_i$ ,  $u_i(z)$ , is

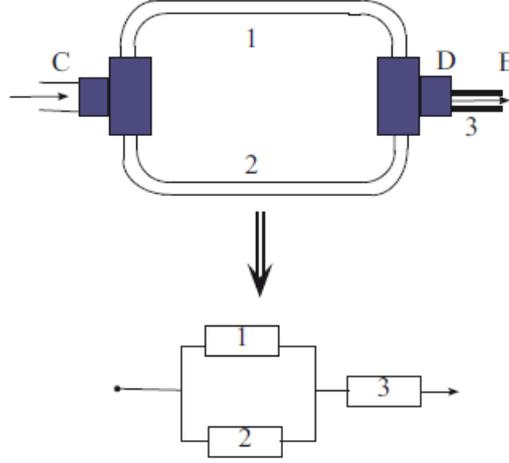


Figure 4.1: Oil transmission system [Ding and Lisnianski, 2008]

Component $C_i$	$n_1^i$	$n_2^i$	$n_3^i$
$C_1$	3	3	14
$C_2$	3	3	9
$C_3$	0	10	-

Table 4.3: Observation data

defined as

$$u_i(z) = \sum_{j=1}^{m_i} p_j^i z^{g_j^i}$$

The  $u$ -function of the oil transmission system is

$$U(z) = \sum_{k=1}^{m_3} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} p_i^1 p_j^2 p_k^3 z^{f_g(g_i^1, g_j^2, g_k^3)} \quad (4.17)$$

where  $f_g(x_1, x_2, x_3) = \min(x_1 + x_2, x_3)$ .

The corresponding performance indices are given by

$$A(\omega) = \sum_{k=1}^{m_3} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} p_i^1 p_j^2 p_k^3 \mathbf{1}(f_g(g_i^1, g_j^2, g_k^3) - \omega \geq 0) \quad (4.18)$$

$$E_d = \sum_{k=1}^{m_3} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} p_i^1 p_j^2 p_k^3 f_g(g_i^1, g_j^2, g_k^3) \quad (4.19)$$

$$E_u(\omega) = \sum_{k=1}^{m_3} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} p_i^1 p_j^2 p_k^3 \max(\omega - f_g(g_i^1, g_j^2, g_k^3), 0) \quad (4.20)$$

where  $m_1 = m_2 = 3$  and  $m_3 = 2$  are the numbers of states for Pipe 1, 2 and 3 respectively. The estimation of the three indices can be calculated on replacing the probabilities  $p_j^i$  by its MLE estimators

$$\hat{p}_j^i = \frac{n_j^i}{\sum_{k=1}^{m_i} n_k^i}$$

where  $n_j^i$  is the number of observations in state  $j$  for component  $C_i$ .

## Results and discussion

We set the pseudo-observation size  $n_S = 10$  which is equal to the minimal size among three components' observations. The value of system-level  $n_i$  (the number of pseudo-observations in state  $i$  in  $O_S$ ) is

$E(E_d^L)$	$E(E_d^U)$	$CI(E_d^L)$ (95%)	$CI(E_d^U)$ (95%)	$E_d^{UGF/MLE}$
2.4691	2.7873	1.7219	3.2219	2.7000

Table 4.4: MSS expected output performance  $E_d$

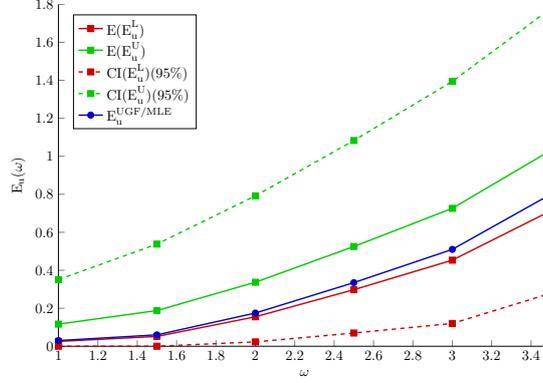


Figure 4.3: MSS performance deficiency  $E_u(\omega)$  for the arbitrary demand constant  $\omega$

estimated by Monte-Carlo simulation with  $N_{sim} = 4.10^4$  executions and the confidence intervals of performance indices at level  $1 - \alpha = 0.95$  are also obtained by Monte Carlo simulation of the corresponding Beta or Dirichlet distribution.

We obtained the results using both methods for different demands  $\omega = \{1, 1.5, 2, 2.5, 3, 3.5\}$  shown in Figure 4.2, Table 4.4 and Figure 4.3.

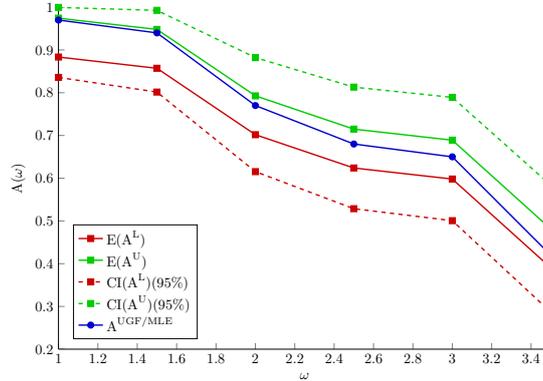


Figure 4.2: MSS availability  $A(\omega)$  for the arbitrary demand constant  $\omega$

We observed that the results given by UGF method with MLE estimators lies inside the estimations of our random intervals. Since the observed number of state 1 of Pipe 3 is zero because of the limited observation size, for MLE estimators the estimated probability  $p_1^3$  is zero so that the system is in state 1 only if Pipe 1 and Pipe 2 are all in state 1, which can not reflect the all situations and causes overestimation of  $A(\omega)$  and  $E_d$  as well as underestimation of  $E_u(\omega)$ . The random interval estimation is conducted without prior distribution information. Our method provides larger range estimations and confidence intervals compatible with unobserved events and limited component-level sample sizes.

## 4.3 Stochastic modeling of multi-state systems

### 4.3.1 Introduction

Markov (renewal) or semi-Markov process gives a general and more flexible presentation for processes having Markov properties for modeling real life problems in finance [Janssen and Manca, 2007], engineering [Grabski, 2007, N. Limnios, 2001], biology [V.S. Barbu, 2008], etc.

Continuous time Markov process are widely used to describe system dependability in many studies [Osaki and Nakagawa, 1976, Osaki, 1985, Lindqvist, 1987, Gnedenko et al., 1995]. In this approach, all state sojourn time follows exponential distribution which means that the passage rates are homogeneous so that it is easy to be implemented. However, the exponential hypothesis does not always correspond the fact in real world systems whose failure time and/or repair time are not always exponentially distributed.

Semi-Markov process (SMP) provides a more precise model to evaluate system reliability. The passage rates do not have to be homogeneous. In [N. Limnios, 2001, Malefaki et al., 2014], system dependability measure of repairable systems is modeled by a continuous time semi-Markov process (CTSMP). One of the disadvantage of SMP is that a system composed of semi-Markov subsystems does not always fit Markov property.

However, an important drawback in these applications is the difficulty of obtaining tractable solution in application with a large enough number of states. Several methods were proposed in the past using algebraic and complementary variables approaches [Cox, 2008, Limnios, 2011, Limnios, 2014, N. Limnios, 2001] to obtain transition functions of the semi-Markov process.

In section 4.3.3, we study the existence and uniqueness solution of a Markov renewal equation (MRE). We also propose a method solving Markov renewal equations based on an iterative scheme inspired by n-fold convolution. Theoretical and numerical applications on system reliability and availability calculation as well as a case study on system availability and reliability assessment are also provided in the following sections.

In section 4.3.4, the objective of this chapter is to apply the state merging and splitting scheme on discrete state space continuous time semi-Markov describing highly reliably (repairable) system evaluation in order to reduce state space and to approximately estimate the system reliability and availability.

The state merging and splitting scheme gives a method that transforms semi-Markov subsystems whose state space can be divided into groups where jumps occur much more frequently between states inside the group, than states between groups, into single states with homogeneous passage rate, instead of just replacing non-exponential passage time distribution by exponential distribution with the same mean passage rate. As a system (process) composed of Markov subsystems still keeps its Markov property, we can easily simplify the space state by grouping some of states. The grouping procedure is based on average theorem with ergodic assumptions. It is very useful for system reliability (availability) estimation because in repairable systems composed of highly reliable systems, we often observe that jumps happen at most of time inside groups of states; the jumps between the group(s) of system working state and the one(s) of system failure states are rather rare. We just need to understand the long term process behavior.

## 4.3.2 Semi-Markov model on MSS evaluation: State of the art

### Stochastic multistate components/systems

The stochastic behavior of a multi-state component or a system described in previous chapter is studied by observing the object changes of states during a period of time, for example, failure or repair of the component.

Consider that the studied object is put into observation at time instant  $t = 0$ , the state of the component at instant  $t \geq 0$  is denoted by  $Z_t \in E$  where the state space discrete finite/countable set  $E = \{1, \dots, m\}$  contains  $m$  different states and  $\mathcal{E}$  is the power set of  $E$ .

Let a chain  $\{S_i, i \in \mathbb{N}\}$  such that

$$0 = S_0 < S_1 < \dots < S_i < \dots,$$

be the time of the  $i$ th jump (state change of  $Z_t$ ). Then the (sojourn/passage) time between the  $(n - 1)$ th and  $n$ th jumps  $\{X_n, n \in \mathbb{N}^+\}$  is given by

$$X_n = S_n - S_{n-1}$$

The number of jumps during the time period  $(0, t]$ ,  $N(t)$ , is given by

$$N(t) = \sup\{n : S_n \leq t\} \quad t > 0$$

Hence,  $(J_n, S_n), n \in \mathbb{N}$  is called the embedded process of  $Z_t$ , so that the embedded chain of  $Z_t$ ,  $(J_n, n \in \mathbb{N})$  is defined as

$$J_n = Z_{S_n}$$

and

$$Z_t = J_{N(t)}$$

## Semi-Markov process

**Definition 39.** A process  $Z_t$  is called a semi-Markov process if the embedded process  $(J_n, S_n), n \in \mathbb{N}$  of the stochastic process  $Z_t, t \in \mathbb{R}^+$ , satisfies

$$\begin{aligned} & \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_n, J_{n-1}, \dots, J_0, S_n, S_{n-1}, \dots, S_0) \\ & = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_n) \end{aligned}$$

The cumulative semi-Markov (matrix) kernel associated with  $Z_t$  is defined as

$$\mathbf{Q}(t) = (Q_{ij}(t), i, j \in E, t \in \mathbb{R}^+)$$

$$Q_{ij}(t) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq t | J_n = i) = \int_0^t q_{ij}(s) ds$$

where

$$\mathbf{q}(t) = (q_{ij}(t), i, j \in E, t \in \mathbb{R}^+)$$

is the transition kernel of the process which satisfies the following properties:

- $0 \leq q_{ij}(t), i, j \in E, t \geq 0$
- $\int_0^\infty q_{ij}(t) dt \leq 1, i, j \in E$

The embedded chain  $(J_n)$  is a Markov chain whose Markov transition probability is given by

$$\mathbf{P} = (p_{ij}, i, j \in E)$$

where

$$\begin{aligned} p_{ij} &= \mathbb{P}(J_{n+1} = j | J_n = i) \\ &= Q_{ij}(\infty) \end{aligned}$$

We consider that jumps are all triggered by events. Let  $e_{ij}$  the event drives a jump from state  $i$  to state  $j$  whose waiting time is represented by a random variable  $T_{ij}$  whose cumulative distribution is denoted by  $F_{ij}(t)$ .  $F_{ij}(t)$  can be also considered as the conditional cumulative distribution function of sojourn time in state  $i$  given the next visited state  $j$ , i.e.

$$F_{ij}(t) := \mathbb{P}(S_{n+1} - S_n \leq t | J_n = i, J_{n+1} = j) = \begin{cases} \frac{Q_{ij}(t)}{p_{ij}}, & \text{if } p_{ij} \neq 0, \\ 1_{\{t=\infty\}}, & \text{if } p_{ij} = 0 \end{cases}$$

Suppose the current state  $\{Z_t = i\}$ , the next state is the state  $k$  whose corresponding event  $e_{ik}$  is first event occurs among all possible next events  $\{e_{ij}, j \in E \setminus \{i\}\}$  so that the sojourn time in state  $i$  is given by

$$T_i = \min_{j \in E \setminus \{i\}} T_{ij}$$

Hence, the kernel probability  $Q_{ik}(t)$  is the probability that  $T_{ik} \leq t$  given that  $T_{ik}$  is less than the other possible jumping sojourn times, i.e.

$$\begin{aligned} Q_{ik}(t) &= \mathbb{P}(T_{ik} \leq t \text{ and } T_{ik} < T_{ij}, \forall j \in E \setminus \{i, k\}) \\ &= \int_0^t F_{ik}(du) \prod_{j \in E \setminus \{i, k\}} (1 - F_{ij}(u)) \end{aligned}$$

$$q_{ik}(t) = f_{ik}(t) \prod_{j \in E \setminus \{i, k\}} (1 - F_{ij}(t))$$

The probability that the system leaves state  $i$  by time  $t$ ,  $H_i(t)$ , is given by

$$\begin{aligned} H_i(t) &:= \mathbb{P}(S_{n+1} - S_n \leq t | J_n = i) \\ &= \mathbb{P}(T_{ij} \leq t, j \in E \setminus \{i\}) \\ &= \sum_{j \in E \setminus \{i\}} Q_{ij}(t) \\ &= \int_0^t h_i(s) ds \\ &= \sum_{j \in E} \int_0^t q_{ij}(s) ds \end{aligned}$$

where

$$h_i(t) := \sum_{j \in E} q_{ij}(t)$$

The survival function of state  $i$ ,  $\bar{H}_i(t)$  is then defined as

$$\bar{H}_i(t) := \mathbb{P}(S_{n+1} - S_n > t | J_n = i) = 1 - H_i(t)$$

Hence, we can define the mean sojourn time in state  $i$  by

$$m_i := \mathbb{E}(S_1 | J_0 = i) = \int_0^\infty \bar{H}_i(t) dt$$

If  $J_n$  is irreducible with stationary distribution  $\rho = (\rho_i, i \in E)$ , the mean sojourn time of  $Z_t$  is defined by

$$m := \sum_{i \in E} \rho_i m_i$$

The stationary distribution of  $Z_t$  is defined by

$$\pi_i := \lim_{t \rightarrow \infty} P_{ji}(t), \quad i, j \in E$$

where

$$P_{ji}(t) = \mathbf{P}(Z_t = j | Z_0 = i).$$

**Application on MSS evaluation** The performance of each state is quantified by its performance rate  $G_i$ ,  $i \in E$ . Supposing that it is demanded that the performance rate should be not less than a constant value  $\omega$  (called demand of MSS). The system is considered in functional state if its performance rate is not less than the demand. Then using (4.13) the set of all functional states with constant demand  $\omega$  is given by

$$U = U(\omega) = \{i \in E | G_i \geq \omega\}$$

Define  $p_i(t)$  as the probability that the system is in a certain state  $i$  at instant  $t$ . Consider a system start at time  $t = 0$ . The system reliability is given by

$$\begin{aligned} R(t) &= \mathbb{P}(Z_s \in U, \forall s \in [0, t]) \\ &= \sum_{i \in E} \alpha_i R_i(t) \end{aligned}$$

where  $\alpha_i = \mathbb{P}(Z_0 = i)$  is the initial state probability for state  $i$ , and  $R_i(t)$  is the system reliability (conditional reliability) given the initial state  $i$ , i.e.

$$R_i(t) = \mathbb{P}(Z_s \in U, \forall s \in [0, t] | Z_0 = i), i \in U$$

Similarly, the system availability is defined as the multi-state component/system's availability  $A(t)$  is given by

$$A(t) = \mathbb{P}(Z_t \in U) = \sum_{i \in U} p_i(t) = \sum_{i \in E} \alpha_i A_i(t)$$

where  $A_i(t)$  is the conditional system availability given the initial state  $i$ :

$$A_i(t) = \mathbb{P}(Z_t \in U | X_0 = i), i \in E$$

Let  $\mathbf{P}(t) = (P_{ij}(t))$  where

$$P_{ij}(t) = \mathbb{P}(Z_t = j | Z_0 = i),$$

the instantaneous probability of each state, be the solution of the Markov renewal equation (MRE)

$$\mathbf{P}(t) = \mathbf{1} - \mathbf{H}(t) + \mathbf{Q} * \mathbf{P}(t)$$

where  $\mathbf{H}(t) = \text{diag}(H_1(t), \dots, H_m(t))$  is a diagonal matrix.

We have the instantaneous probability  $p_i(t)$  given by

$$p_i(t) = \sum_{j \in E} \alpha_j p_{ji}(t)$$

System state $Z_t^s$	Component state $(Z_t^1, Z_t^2)$	Description
1	(1,1)	Both components are in good state.
2	(1,0)	$C_1$ is in good state; $C_2$ is in failure state.
3	(0,1)	$C_1$ is in failure state; $C_2$ is in good state.
4	(0,0)	Both components are in failure state.

where  $p_{ji}(t)$  denotes the conditional probability

$$p_{ji}(t) = \mathbb{P}(Z_t = i | Z_0 = j) = \begin{cases} 1 - H_j(t) + \sum_{k \in U} \int_0^t Q_{jk}(ds) p_j(t-s), & j = i \\ \sum_{k \in U} \int_0^t Q_{jk}(ds) p_{kj}(t-s), & j \neq i \end{cases}$$

Then the conditional reliability and the conditional availability are given by

$$R_i(t) = \begin{cases} 1 - H_i(t) + \sum_{j \in U(\omega)} \int_0^t Q_{ij}(ds) R_j(t-s), & i \in U(\omega) \\ \sum_{j \in U(\omega)} \int_0^t Q_{ij}(ds) R_j(t-s), & i \in D(\omega) \end{cases}$$

$$A_i(t) = \mathbb{P}(X_t \in U | X_0 = i), i \in E \\ = \begin{cases} 1 - H_i(t) + \sum_{j \in E} \int_0^t Q_{ij}(ds) A_j(t-s), & i \in U(\omega) \\ \sum_{j \in E} \int_0^t Q_{ij}(ds) A_j(t-s), & i \in D(\omega) \end{cases}$$

The value of conditional probabilities can be approximated as follows

$$p_{ji}(t) \approx \begin{cases} 1 - H_i(t) + \sum_{j \in E} \sum_{l=1}^n p_{ji}(t-x_l)(Q_{ij}(x_l) - Q_{ij}(x_{l-1})), & i = j \\ \sum_{k \in E} \sum_{l=1}^n p_{ki}(t-x_l)(Q_{ik}(x_l) - Q_{ik}(x_{l-1})), & i \neq j \end{cases}$$

$$R_i(t) \approx \begin{cases} 1 - H_i(t) + \sum_{j \in U} \sum_{l=1}^k R_j(t-x_l)(Q_{ij}(x_l) - Q_{ij}(x_{l-1})), & i \in U \\ \sum_{k \in U} \sum_{l=1}^n R_k(t-x_l)(Q_{ik}(x_l) - Q_{ik}(x_{l-1})), & i \in D \end{cases}$$

$$A_i(t) \approx \begin{cases} 1 - H_i(t) + \sum_{j \in E} \sum_{l=1}^k A_j(t-x_l)(Q_{ij}(x_l) - Q_{ij}(x_{l-1})), & i \in U \\ \sum_{k \in E} \sum_{l=1}^n A_k(t-x_l)(Q_{ik}(x_l) - Q_{ik}(x_{l-1})), & i \in D \end{cases}$$

where  $0 = x_0 < x_1 < \dots < x_n = t$ .

### State space specification

Supposing that we have a system composed of  $d$  independent components  $C_1, \dots, C_d$ . For each component  $C_i$ , its state is represented by  $Z_t^i \in E_i = \{1, \dots, m_i\}$  such that

$$\mathbb{P}(Z_t^i = j) = p_j^i(t)$$

One of the most used specification for system level is to start (bottom-up) from component level and list all possible combinations of

$$Z_t = (Z_t^1, \dots, Z_t^d)$$

so that the system state number  $|E|$  is the product of all component state numbers, i.e.

$$E = \{1, \dots, m_1\} \times \dots \times \{1, \dots, m_d\}$$

$$|E| = \prod_{i=1}^d m_i \geq m_S$$

where  $m_S$  is the number of system level states and will exponentially increase when the system has more components.

**Example 25.** A system composed of two binary components  $C_1$  and  $C_2$  can be presented by  $Z_t = \varphi(Z_t^1, Z_t^2)$  The state space  $E = \{1, 2, 3, 4\}$  where the space size is equal to  $2^d$  in case of system composed of  $d$  binary components.

If the system is a parallel system, the sets of good and failure system states are respectively given by  $U = \{1, 2, 3\}$  and  $D = \{4\}$ ; similarly, if the system is a series system, we have  $U = \{1\}$  and  $D = \{2, 3, 4\}$ .

Another approach is that we use the reliability function or UGF to calculate the instantaneous probability of each system state  $p_i^s(t)$  on function of component state probabilities obtained by semi-Markov process.

$$p_i^s(t) = \mathbb{P}(Z_t^s = i) = \sum_{\mathbf{x}=(x_1, \dots, x_d) \in E} \mathbf{1}_{\{\varphi(\mathbf{x})=i\}} \prod_{j=1}^d p_{x_j}^j(t)$$

Here instead of create state space on combination of component states, we only build model for each single component.

We do not merge different states as in many examples because firstly we are treating component with multiple states; secondly in order to evaluate indices such as system performance expectation and availability on function of demand it is complicated to identify two conditions with exactly the same impact on system level.

The advantage of using semi-Markov process modeling is that it allows us to adapt more complicated situations, for example, component with different maintenance strategies and repair results.

### 4.3.3 Solution of Markov renewal equation

#### Markov renewal function and n-fold convolution

Since  $Z_t$  is regular, which means that the number of jumps in any finite time interval is finite almost surely, we suppose that  $Z_t$  is continuous on the right having left limits in any point of time  $t > 0$ .

Let  $\phi(i, t)$ ,  $i \in E$ ,  $t \geq 0$ , be a real-valued measurable function, then the convolution of  $\phi$  by  $Q$  is defined as follows

$$Q * \phi(i, t) := \sum_{k \in E} \int_0^t Q_{ik}(ds) \phi(k, t - s). \quad (4.21)$$

It is easy to prove the following fundamental equality

$$Q_{ij}^{(n)}(t) = P_i(J_n = j, S_n \leq t). \quad (4.22)$$

where  $P_i(\cdot)$  means  $\mathbb{P}(\cdot | J_0 = i)$  and  $Q_{ij}^{(n)}(t)$ ,  $i, j \in E$ , is the  $n$ -fold convolution of  $Q$  by itself, i.e.

$$Q_{ij}^{(n)}(t) = \begin{cases} \sum_{k \in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t - s) & n \geq 2 \\ Q_{ij}(t) & n = 1 \\ \delta_{ij} \mathbf{1}_{\{t \geq 0\}} & n = 0, \end{cases}$$

with  $\delta_{ij} = 1$ , if  $i = j$  and  $\delta_{ij} = 0$ , if  $i \neq j$ .

Then the Markov renewal function  $\psi_{ij}(t)$ ,  $i, j \in E, t \geq 0$ , can be defined by

$$\psi_{ij}(t) := \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \quad (4.23)$$

Let us write the Markov renewal function (4.23) in matrix form

$$\psi(t) = (I(t) - Q(t))^{(-1)} = \sum_{n=0}^{\infty} Q^{(n)}(t). \quad (4.24)$$

This can also be written as

$$\psi(t) = I(t) + Q * \psi(t), \quad (4.25)$$

where  $I(t) = I$  (the identity matrix), if  $t \geq 0$  and  $I(t) = 0$ , if  $t < 0$ .

**Markov renewal equation** Equation (4.25) is a special case of what is called Markov renewal equation (MRE) which is generally defined as

$$U(t) = V(t) + Q * U(t), \quad (4.26)$$

where  $U(t) = (U_{ij}(t))_{i,j \in E}$ ,  $V(t) = (V_{ij}(t))_{i,j \in E}$  are matrix-valued measurable functions, with  $U_{ij}(t) = V_{ij}(t) = 0$  for  $t < 0$ . The function  $V(t)$  is a given matrix-valued function and  $U(t)$  is an unknown matrix-valued function.

### MRE solution given by iterative method

**Assumption 7.** Let  $\mathbb{B}$  be the space of all locally bounded, on  $\mathbb{R}^+$ , matrix functions  $U(t)$ , i.e.,  $\|U(t)\| = \sup_{i,j} |U_{i,j}(t)|$  is bounded on sets  $[0, \xi]$ , for every  $\xi \in \mathbb{R}^+$ .

**Proposition 2.** ([Pyke, 1961, N. Limnios, 2001]). The transition function  $P(t) = (P_{ij}(t), i, j \in E)$  satisfies the following MRE

$$P(t) = \bar{H}(t) + Q * P(t), \quad (4.27)$$

which, under Assumption 7, has the unique solution

$$P(t) = \psi * \bar{H}(t), \quad (4.28)$$

Here  $\bar{H}(t) = I(t) - H(t)$  with  $H(t) = \text{diag}(H_i(t))$  is a diagonal matrix.

Let  $(P_{ij}^n(t), n \in \mathbb{N})$  a sequence of functions given by

$$P_{ij}^n(t) = \begin{cases} \delta_{ij} \bar{H}_i(t) & \text{if } n = 0 \\ \delta_{ij} \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) P_{kj}^{n-1}(t-s) & \text{if } n \geq 1. \end{cases} \quad (4.29)$$

where  $\bar{H}_i(t) := 1 - H_i(t)$ .

**Proposition 3.** For any fixed  $t \geq 0$  and  $i, j \in E$ , the limit

$$\lim_{n \rightarrow \infty} P_{ij}^n(t)$$

exists and it is the smallest solution of the MRE (4.27) that we denote by  $\tilde{P}_{ij}(t)$ .

*Proof.* From equation (4.29) we get

$$\begin{aligned} \sum_{j \in E} P_{ij}^n(t) &= \delta_{ij} \bar{H}_i(t) + \sum_{j \in E} \sum_{k \in E} \int_0^t Q_{ik}(ds) P_{kj}^{n-1}(t-s) \\ &\leq \delta_{ij} \bar{H}_i(t) + \int_0^t H_i(ds) \\ &= 1. \end{aligned}$$

On the other hand, we have  $P_{ij}^n(t) \geq 0$  for any  $n \geq 0, i, j \in E$  and  $t \geq 0$ .

Define

$$D_{ij}^n(t) := \begin{cases} P_{ij}^0(t) & \text{if } n = 0 \\ P_{ij}^n(t) - P_{ij}^{n-1}(t) & \text{if } n \geq 1. \end{cases} \quad (4.30)$$

Then we get

$$D_{ij}^n(t) = \sum_{k \in E} \int_0^t Q_{ik}(ds) D_{kj}^{n-1}(t-s), \quad n \geq 1. \quad (4.31)$$

Since  $D_{ij}^0(t) \geq 0$  we have  $D_{ij}^n(t) \geq 0$  for all  $n \geq 1$  and then  $P_{ij}^n(t) \geq P_{ij}^{n-1}(t)$ .

Finally, from the above inequality and the inequality  $0 \leq P_{ij}^n(t) \leq 1$  we get the desired result that the limit  $\tilde{P}_{ij}(t)$  exists.

Let us now prove that  $\tilde{P}_{ij}(t)$  is the smallest solution of the MRE (4.27). The fact that  $\tilde{P}_{ij}(t)$  is a solution of the MRE is obtained directly by considering limits in both sides of equation (4.29).

Now let us consider another solution of the MRE, say  $P_{ij}^\#(t)$ . Then we have  $P_{ij}^\#(t) \geq \delta_{ij} \bar{H}_i(t) =: P_{ij}^0(t)$  and, suppose that  $P_{ij}^\#(t) \geq P_{ij}^{n-1}(t)$ , we get

$$\begin{aligned} P_{ij}^n(t) &= \delta_{ij} \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) P_{kj}^{n-1}(t-s) \\ &\leq \delta_{ij} \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) P_{kj}^\#(t-s) \\ &= P_{ij}^\#(t), \end{aligned}$$

and passing to limit when  $n \rightarrow \infty$ , the proof is achieved.  $\square$

Moreover, the probabilistic meaning of  $P_{ij}^n(t)$  can be given by

$$P_{ij}^n(t) = P_i(Z_t = j, S_{n+1} > t) \quad (4.32)$$

It indicates that for a fixed instant  $t$ , the convergence speed of the sequence  $P_{ij}^n(t)$  depends on distribution of the jump number  $N(t)$  or  $H_i(t)$ , the sojourn distribution of each state  $i$ .

As the time  $t$  increases, the minimal  $n$  needed to obtain a precise value of  $P_{ij}^n(t)$  using our method increases too.

*Proof.* Let us start with  $n = 0$

$$P_{ij}^0(t) = \delta_{ij} \bar{H}_i(t) = P_i(Z_t = j, S_1 > t)$$

If for  $n \in \mathbb{N}$  the proposition stands, then at  $n + 1$ , we have

$$\begin{aligned} P_{ij}^{n+1}(t) &= \delta_{ij} \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) P_{kj}^n(t-s), \quad n \geq 1 \\ &= P_i(Z_t = j, S_1 > t) \\ &+ \sum_{k \in E} \int_0^t \mathbb{P}(J_1 = k, S_1 \in ds | J_0 = i) \mathbb{P}(Z_{t-s} = j, S_{n+2} - S_1 > t-s | J_1 = k, S_1 = s) \\ &= P_i(Z_t = j, S_1 > t) + \sum_{k \in E} \int_0^t P_i(Z_t = j, S_1 \in ds, S_{n+2} > t, J_1 = k) \\ &= P_i(Z_t = j, S_1 > t) + P_i(Z_t = j, S_1 \leq t < S_{n+2}) \\ &= P_i(Z_t = j, S_{n+2} > t) \end{aligned}$$

The hypothesis also stands. Finally, we prove equality (4.32) for all  $n \in \mathbb{N}$ , and we also have

$$D_{ij}^n(t) = P_i(Z_t = j, N(t) = n)$$

□

### Application on system reliability assessment

Besides transition functions, Proposal 3 is also valuable for other probability functions under the same framework. In this section, the method proposed here is applied on system survival or reliability functions and availability assessment.

Let us consider the survival or reliability function in the case where a subset of down states is given, i.e., say  $D \subset E$ , and the lifetime of the system  $T$  is defined by  $T := \inf\{t \geq 0 : Z_t \in D\}$ . The survival function is then  $R_i(t) := P_i(T > t)$ ,  $i \in E \setminus D$ . This function satisfy the following MRE (see, e.g., [N. Limnios, 2001])

$$R_i(t) = \bar{H}_i(t) + \sum_{k \in E \setminus D} \int_0^t Q_{ik}(ds) R_k(t-s).$$

In that case we can use also the above iterative scheme, i.e.,

$$R_i^n(t) = \begin{cases} \bar{H}_i(t) & \text{if } n = 0 \\ \bar{H}_i(t) + \sum_{k \in E \setminus D} \int_0^t Q_{ik}(ds) R_k^{n-1}(t-s) & \text{if } n \geq 1. \end{cases}$$

Similarly, the availability is defined by  $A_i(t) := P_i(Z_t \in E \setminus D)$  and satisfy the following MRE

$$A_i(t) = \mathbf{1}_{\{E \setminus D\}}(i) \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) A_k(t-s).$$

with the corresponding iterative scheme

$$A_i^n(t) = \begin{cases} \mathbf{1}_{\{E \setminus D\}}(i) \bar{H}_i(t) & \text{if } n = 0 \\ \mathbf{1}_{\{E \setminus D\}}(i) \bar{H}_i(t) + \sum_{k \in E} \int_0^t Q_{ik}(ds) A_k^{n-1}(t-s) & \text{if } n \geq 1. \end{cases}$$

where  $\mathbf{1}_{E \setminus D}(i)$  is the indicator function gives 1 if  $i \in E \setminus D$ , 0 otherwise.

In both cases, the limits  $\tilde{R}_i(t) := \lim_{n \rightarrow \infty} R_i^n(t)$  and  $\tilde{A}_i(t) := \lim_{n \rightarrow \infty} A_i^n(t)$  give the smallest solutions of the above corresponding MREs. It can be also applied directly on Markov renewal equation of semi-Markov chains [V.S. Barbu, 2008].

State	Description
1	Machine fails
2	Machine waits for new task
3	Operation task T1
4	Operation task T2
5	Operation task T3

Table 4.5: States and their description

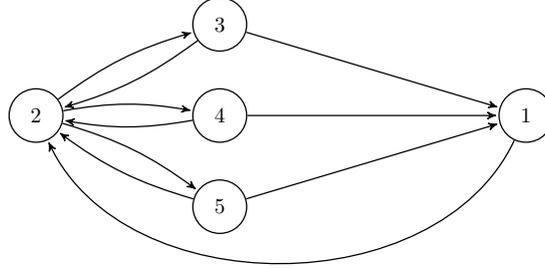


Figure 4.4: The semi-Markov process of the case study

### Case study: Multiple task machine

In this section, we apply our method discussed in the previous section on a specific case study: a multi-task machine.

**System description** Consider a multi-task machine which performs three different tasks T1, T2 and T3 whose arrival times follow exponential distributions with parameter  $\lambda = 2$ , density function  $f_{exp}(x; \lambda)$  and cumulative distribution function  $F_{exp}(x; \lambda)$ . The arrival proportion is 0.45, 0.35 and 0.20 for Task T1, T2 and T3 respectively. The time to complete of each task follows lognormal distribution with different parameters:

- Task T1:  $\mu_1 = 1.3550, \sigma_1 = 0.25$ ;
- Task T2:  $\mu_2 = 1.4728, \sigma_2 = 0.25$ ;
- Task T3:  $\mu_3 = 1.5782, \sigma_3 = 0.25$ .

whose distribution function  $F_{logn}(x; \mu_i, \sigma_i)$  and density function  $f_{logn}(x; \mu_i, \sigma_i)$ .

We assume that at the end of each operation, the machine is maintained and failures happen only during the operations with a failure time following Weibull distribution with scale parameter  $a = 56.4190$  and shape parameter  $b = 2$ : for  $x \geq 0$  the cumulative distribution function and the density function are respectively given by  $F_{wbl}(x; a, b)$  and  $f_{wbl}(x; a, b)$ .

$$\begin{aligned}
 F_{wbl}(x; a, b) &= 1 - e^{-(x/a)^b} \\
 f_{wbl}(x; a, b) &= \frac{b}{a} \left(\frac{x}{a}\right)^{b-1} e^{-(x/a)^b}
 \end{aligned}$$

Once the machine fails, it takes 20 hours to repair it which indicates the repair time follows a Dirac distribution with parameter  $c = 20$  whose cumulative distribution function is a unit step function.

$$H(x; 20) = \begin{cases} 1 & \text{if } x \geq 20 \\ 0 & \text{if } x < 20. \end{cases}$$

**Semi-Markov modeling** According to the description, the system can be presented by a five-state semi-Markov process (Figure 4.4) where each state corresponds a specific system state (Table 4.5) and the semi-Markov kernel  $(Q_{ij}(t))$  are shown in Table 4.6 using method mentioned in [V.S. Korolyuk, 1966]. (The conditional reliability and availability are shown in Figure 4.8 and Figure 4.7).

$Q_{ij}(t)$	
$Q_{12}(t)$	$H(t; 20)$
$Q_{23}(t)$	$0.45F_{exp}(t; \lambda)$
$Q_{24}(t)$	$0.35F_{exp}(t; \lambda)$
$Q_{25}(t)$	$0.20F_{exp}(t; \lambda)$
$Q_{31}(t)$	$\int_0^t (1 - F_{logn}(s; \mu_1, \sigma_1)) f_{wbl}(s; a, b) ds$
$Q_{32}(t)$	$\int_0^t f_{logn}(s; \mu_1, \sigma_1) (1 - F_{wbl}(s; a, b)) ds$
$Q_{41}(t)$	$\int_0^t (1 - F_{logn}(s; \mu_2, \sigma_2)) f_{wbl}(s; a, b) ds$
$Q_{42}(t)$	$\int_0^t f_{logn}(s; \mu_2, \sigma_2) (1 - F_{wbl}(s; a, b)) ds$
$Q_{51}(t)$	$\int_0^t (1 - F_{logn}(s; \mu_3, \sigma_3)) f_{wbl}(s; a, b) ds$
$Q_{52}(t)$	$\int_0^t f_{logn}(s; \mu_3, \sigma_3) (1 - F_{wbl}(s; a, b)) ds$

Table 4.6:  $Q_{ij}(t)$  for passage between states (for those not mentioned  $Q_{ij}(t) = 0$ )

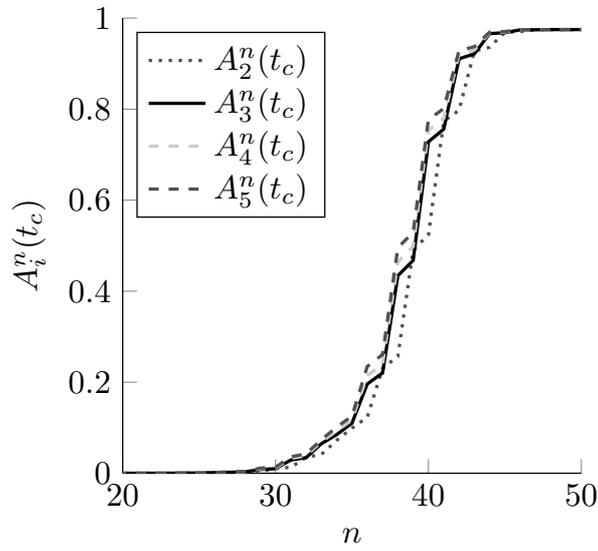


Figure 4.5: Plot of  $A_i^n(t_c)$ ,  $t_c = 100$  on function of  $n$

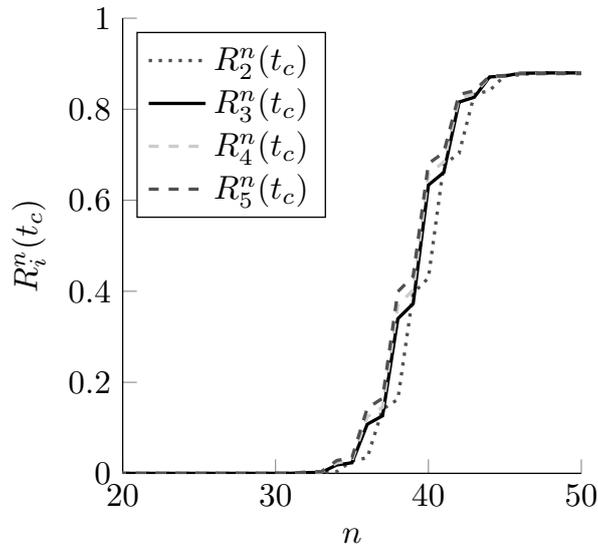


Figure 4.6: Plot of  $R_i^n(t_c)$ ,  $t_c = 100$  on function of  $n$

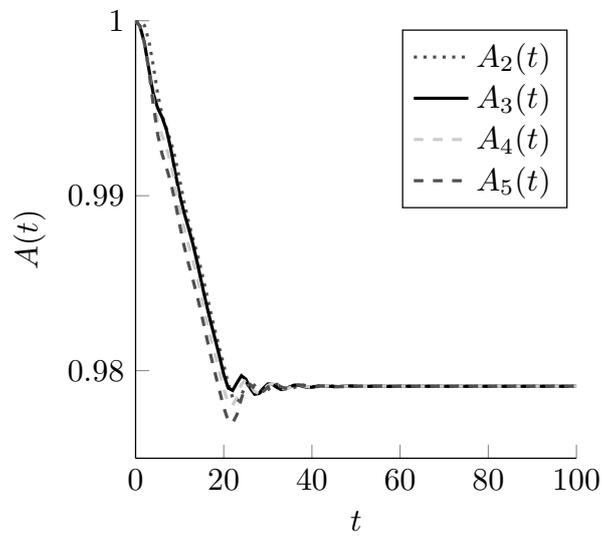


Figure 4.7: Plot of  $A_i(t)$ ,  $t = 0 : 100$  on function of  $t$

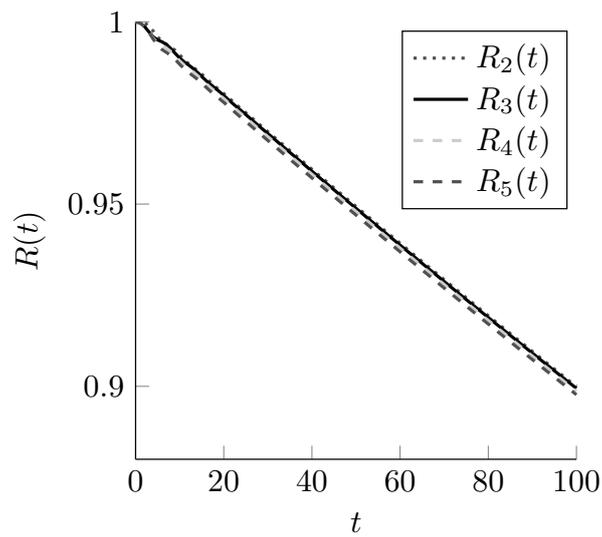


Figure 4.8: Plot of  $R_i(t)$ ,  $t = 0 : 100$  on function of  $t$

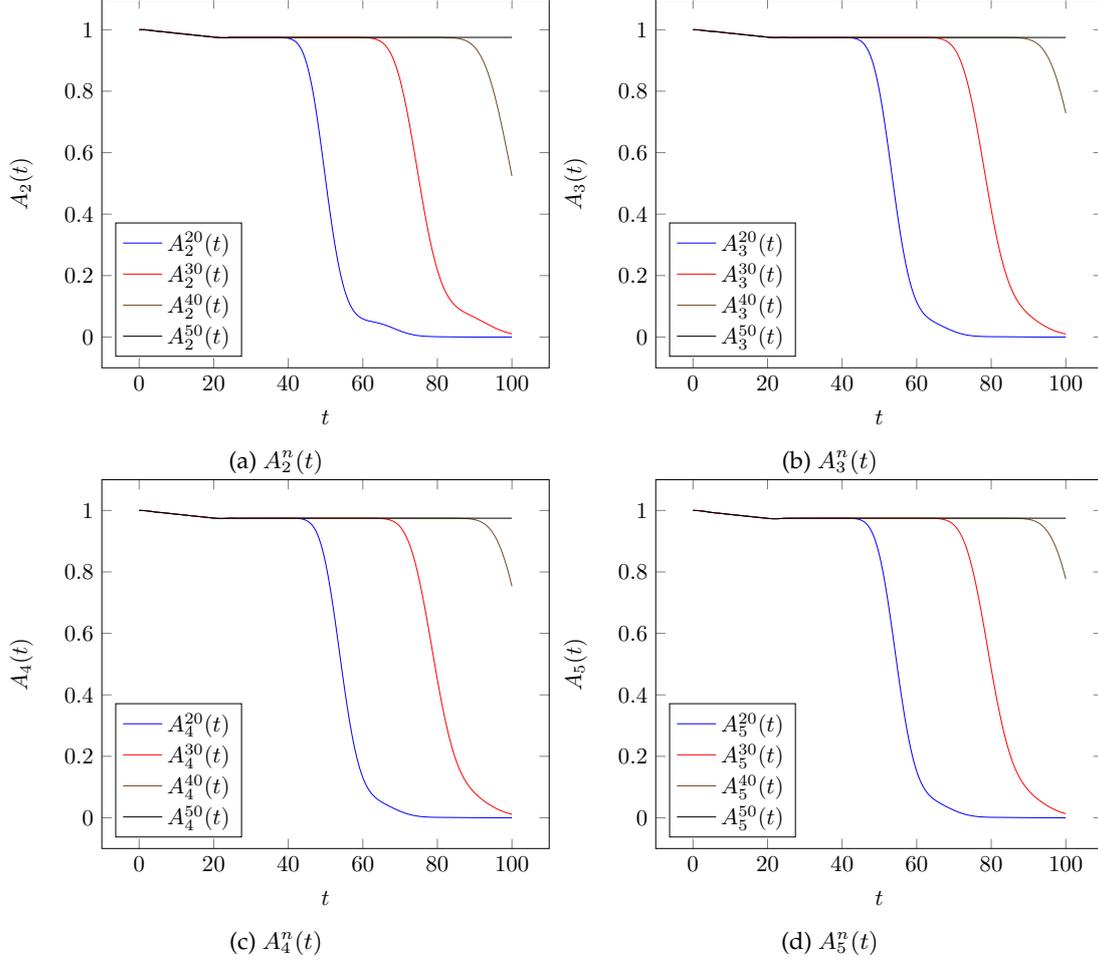


Figure 4.9: Plot of  $A_i^n(t)$ ,  $t = 0 : 100$  on function of  $t$  for different number of iterations

Using our proposed method, the obtained conditional reliability  $R_i^n(t_c)$  and availability  $A_i^n(t_c)$  at instant  $t_c = 100$  on function of iteration number  $n$  (cf. Figures 4.6, 4.5).

The curves of states 2,3,4,5 are slightly different because the sojourn time distributions  $H_i(t)$  of this four states are similar. Except the constant sojourn time of state 1, it is observed that the obtained system reliability and availability estimates have the same order as their expected sojourn times. The convergence rates of all curves are almost the same as they share the same jumping count process  $N(t)$ . Figure 4.9 also shows the same convergence and verifies the probability meaning of  $A_i^n(t)$  corresponding to (4.32), i.e.

$$A_i^n(t) = P_i(Z_t \in E \setminus D, S_{n+1} > t | Z_0 = i)$$

## Conclusion

In this section, we studied the existent and unique solution for Markov renewal equation and proposed an iterative method for solving MRE. We proved the existence and uniqueness of the obtained solution. The advantage of this method is that there is little limitation on state space which can be extended to infinite and its simplicity. The convergence speed of our method depends on the jump count process which is easier to be estimated. We also applied it on calculating system reliability and availability.

### 4.3.4 Reliability approximation by asymptotic merging state space with perturbation

#### State merging scheme

In this section, state merging scheme is introduced and Semi-Markov system with discrete space defined in book section 4.2.2 in [Koroliuk and Limnios, 2005] and in [Koroliuk and Limnios, 2010] is applied on

system reliability assessment.

For  $i \in E, y \subseteq E, t \geq 0$ , we define the kernel and transition functions of semi-Markov process  $X_t$  as follows

$$Q(i, y, t) = \sum_{j \in y} Q_{i,j}(t)$$

$$P(i, y) = \sum_{j \in y} p_{i,j}$$

so that we have

$$Q(i, y, t) = P(i, y)H_i(t)$$

Supposing the  $X_t \in E^0$  where there is one absorbing state 0, we then separate state 0 from  $E^0$  by defining a non-absorption state space  $E$  such that

$$E^0 = E \cup \{0\}$$

The non-absorbing state space is composed of  $N$  disjoint subset  $E_k, 1 \leq k \leq N$ , i.e.

$$E = \bigcup_{k \in \hat{E}} E_k$$

$$E_k \subset E$$

$$E_i \cap E_j = \emptyset, \quad i \neq j$$

where  $\hat{E} = \{1, \dots, N\}$

Given a function  $v : E^0 \rightarrow \hat{E}^0$  such that

$$v(x) := \begin{cases} k & \text{if } x \in E_k \\ 0 & \text{if } x = 0 \end{cases}$$

where  $\hat{E}^0 = \{0\} \cup \hat{E} = \{0, 1, \dots, N\}$ , we aggregate states of  $X_t$  into  $N + 1$  groups and the new aggregated process is given by

$$v(X_t) \in \hat{E}^0$$

**Theorem 3** ([Koroliuk and Limnios, 2005] [Koroliuk and Limnios, 2010]). Under the following assumptions

- Assumption MA1: The transition kernel of the embedded Markov chain  $X_n, n \geq 0$  has the following representation

$$P^\varepsilon(i, y) = P^0(i, y) + \varepsilon P^1(i, y) \quad (4.33)$$

The stochastic kernel  $P(x, y)$  is coordinated with the split phase space as follows

$$P^0(i, E_k) = 1_k(x) := \begin{cases} 1 & \text{if } x \in E_k \\ 0 & \text{if } x \notin E_k \end{cases} \quad (4.34)$$

The split phase kernel  $P^0$  also defines a supporting Markov chain  $X_n^0$  on  $E$  which is uniformly ergodic in every class  $E_k$  with stationary distributions  $\rho_k(x), k \in \hat{E}$ .

The perturbing signed kernel  $P^1$  satisfies the conservative condition

$$\sum_{j \in E} P^1(i, j) = 0$$

because  $\sum_{j \in E} P^\varepsilon(i, j) = \sum_{j \in E} P^0(i, j) = 1$ .

- Assumption MA2: The perturbing kernel  $P^1(i, j)$  satisfies the following absorption condition. There exists at least one  $k \in \hat{E}$ , such that the absorption probability from  $k$  is positive, that is

$$p_{k0} := - \sum_{x \in E_k} \rho_k(x) P^1(x, E) > 0$$

the weak convergence

$$v(Z(t/\varepsilon)) \Rightarrow \hat{Z}(t), \quad \varepsilon \rightarrow 0$$

takes place.

The limit Markov process  $\hat{X}(t), t \geq 0$ , on the merging phase space  $\hat{E}^0$  is defined by the generating matrix

$$\begin{aligned} \hat{Q} &= (\hat{q}_{kr}; 0 \leq k, r \leq N) \\ \hat{q}_{kr} &= q_k p_{kr} \quad p_{kr} = \sum_{x \in E_k} \rho_k(x) P_1(x, E_r) \quad k \neq r \in \hat{E} \\ q_k &= \frac{1}{m_k} \quad m_k = \sum_{x \in E_k} \rho_k(x) m(x) \quad m(x) := \int_0^\infty \bar{H}_x(t) dt \end{aligned}$$

Let the component be binary,  $E$  denote all working states, 0 is the only failure state so that we have  $N = 1$ , then the sojourn time in working state  $T$  follows the exponential distribution

$$\mathbb{P}(T > t) = \exp(-\Lambda \varepsilon t)$$

where

$$\begin{aligned} \Lambda &= qp \quad q = \sum_{x \in E} \pi(x) q(x) \quad q(x) := 1/m(x) \\ p &= \sum_{x \in E} \rho(x) p(x) \quad p(x) := -P^1(x, E) = -\sum_{y \in E} P^1(x, y) \end{aligned}$$

Hence the component reliability is given by

$$R(t) = \mathbb{P}(T > t) = \exp(-\Lambda \varepsilon t) \quad (4.35)$$

We can also estimate the system availability according to the following relation

$$A(t) = R(t) + \int_0^t A(t-u) q(u) du$$

where  $q(u)$  is the density function of repair time. In addition, the probability of a state  $i$  in  $E$  can be written as

$$p_i(t) = \pi(i) A(t) \quad i \in E$$

where  $\pi(i)$  is the stationary distribution of each state  $i$  in  $E$ .

### Application on reliability estimation

Consider that we have observed a path (or several independent sample paths) of the studied semi-Markov process in an interval of time  $[0, t_{max}]$ , and the observation is given by

$$\mathcal{H} = \{\mathcal{H}^m | \mathcal{H}^m = \{J_0^m, J_1^m, \dots, J_{N^m}^m, Z_0^m, Z_1^m, \dots, Z_{N^m}^m\}, m = \{1, \dots, n_{\mathcal{H}}\}\}$$

The empirical estimator of the semi-Markov kernel is defined as

$$\hat{Q}_{ij}(t) := \hat{p}_{ij} \hat{F}_{ij}(t)$$

where

$$\hat{p}_{ij} := \frac{N_{ij}}{N_i} \quad (4.36)$$

$$\hat{F}_{ij}(t) := \sum_{m=1}^{n_{\mathcal{H}}} \sum_{k=1}^{N^k(t_{max}^m)} \mathbf{1}_{\{J_{k-1}^m=i, J_k^m=j, Z_k^m \leq t\}} / N_i \quad (4.37)$$

and  $N_{ij}$  denotes the jump count from state  $i$  to state  $j$  during the period of time  $[0, t_{max}]$  and  $N_i(t_{max})$  denotes the number of visits to state  $i$

$$N_i = \sum_{j \in E} N_{ij}$$

$$N_{ij} = \sum_{m=1}^{n\mathcal{H}} N_{ij}^m = \sum_{m=1}^{n\mathcal{H}} \sum_{k=1}^{N^k} \mathbf{1}_{\{J_{k-1}^m=i, J_k^m=j\}}$$

In case that for a certain jump from one state  $i$  to an absorbing state  $j$  we observe  $N_{ij} = 0$  or  $N_{ij}$  is too small to build model for  $Q_{ij}(t)$ , the estimated transition probability becomes  $\hat{p}_{ij} = 0$ . However, the system configuration gives  $\hat{p}_{ij} > 0$  so that we have a similar condition like rare event discussed previously.

In order to solve this problem of lack of information, our proposition is that the system is represented by a series of semi-Markov process on function of the only parameter  $\varepsilon$  under the state merging scheme in Section 4.3.4.

For a transition probability  $p_{ij}$  with the knowledge that its value is strictly positive and  $N_{ij}$  is small, using our random set approach,  $p_{ij}$  is bounded by a random interval

$$\mathbb{P}(\underline{p}_{ij} \leq p_{ij} \leq \bar{p}_{ij}) = 1$$

where

$$\underline{p}_{ij} \sim \text{Beta}(N_{ij}, N_i + 1 - N_{ij})$$

$$\bar{p}_{ij} \sim \text{Beta}(N_{ij} + 1, N_i - N_{ij})$$

so that

$$\mathbb{E}(p_{ij}) \in \left[ \frac{N_{ij}}{N_i + 1}, \frac{N_{ij} + 1}{N_i + 1} \right] = [\underline{n}_{ij}\varepsilon, \bar{n}_{ij}\varepsilon]$$

with  $\underline{n}_{ij} = \frac{N_{ij}}{(N_i + 1)\varepsilon}$  and  $\bar{n}_{ij} = \frac{N_{ij} + 1}{(N_i + 1)\varepsilon}$ .

It is also possible to apply Bayesian estimator with beta distribution prior  $\text{beta}(\alpha, \beta)$  (Assumptions in Section 3.3.4). The posteriori distribution of the estimator follows then a beta distribution

$$\hat{p}_{ij} \sim \text{Beta}(N_{ij} + \alpha, N_i + \beta)$$

The expected value of  $p_{ij}$  is given by

$$\mathbb{E}(\hat{p}_{ij}) = \frac{N_{ij} + \alpha}{N_i + \alpha + \beta} = \hat{n}_{ij}\varepsilon$$

where  $\hat{n}_{ij} = \frac{N_{ij} + \alpha}{(N_i + \alpha + \beta)\varepsilon}$ .

Hence, the perturbing kernel  $P^1$  can be written as

$$p_{ij}^1 = n_{ij}$$

where  $n_{ij} \in [\underline{n}_{ij}, \bar{n}_{ij}]$  or  $n_{ij} = \hat{n}_{ij}$  for both approaches.

Finally, the system is modeled by a series of semi-Markov process with kernel

$$\mathbf{Q}(t) = \mathbf{P} * \mathbf{H}(t) \quad \mathbf{P} = \mathbf{P}^0 + \varepsilon \mathbf{P}^1$$

on function of  $\varepsilon$  where  $*$  denotes a matrix multiplication such that

$$Q_{ij}(t) = P_{ij} H_i(t).$$

The lower bound of system reliability can be then estimated as the reliability of the systems with following perturbing transition probability such that with random set method

$$\hat{P}_{RS}^1(i, j) = \bar{n}_{ij}$$

or with Bayesian method

$$\hat{P}_B^1(i, j) = \hat{n}_{ij}$$

for  $i, j \in E^0$ .

According to Theorem 3, with  $\varepsilon \rightarrow 0$  we have

$$\hat{R}_B(t) = \exp(-\hat{\Lambda}_B \varepsilon t)$$

$$\hat{R}_{RS}(t) = \exp(-\hat{\Lambda}_B \varepsilon t)$$

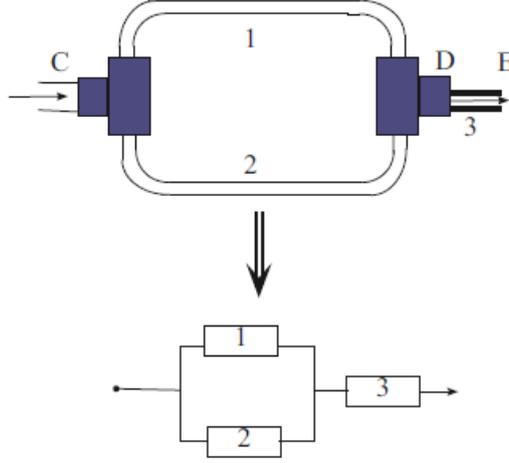


Figure 4.10: Oil transmission system [Ding and Lisnianski, 2008]

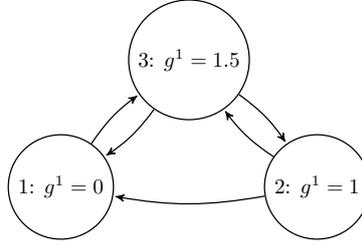


Figure 4.11: States of Component 1

where

$$\begin{aligned} \hat{\Lambda}_B &= \hat{q}\hat{p}_B & \hat{\Lambda}_{RS} &= \hat{q}\hat{p}_{RS} & \hat{q} &= \sum_{x \in E} \hat{\pi}(x)\hat{q}(x) & \hat{q}(x) &= 1/\hat{m}(x) \\ \hat{p}_B &= \sum_{x \in E} \hat{\rho}(x)\hat{p}_B(x) & \hat{p}_B(x) &= -\hat{P}_B^1(x, E) = -\sum_{y \in E} \hat{P}_B^1(x, y) \\ \hat{p}_{RS} &= \sum_{x \in E} \hat{\rho}(x)\hat{p}_{RS}(x) & \hat{p}_{RS}(x) &= -\hat{P}_{RS}^1(x, E) = -\sum_{y \in E} \hat{P}_{RS}^1(x, y) \end{aligned}$$

and  $\hat{\pi}(x)$  is the stationary distribution of  $X^0(t)$  and  $\hat{m}(x)$  is the average of sojourn time in state  $x$  and  $\hat{\rho}(x)$  is the stationary distribution of the Markov chain defined in  $E$ .

### Numerical Application: oil transport system

In this example, we take the same system as the case study in Section 4.2.3 with more details on each component:

**Component 1 and Component 2** Component 1 and Component 2 have three states: component starts with a full working state (state 3) then jumps to degradation state (state 2) or to complete failure state (state 1). In degradation state, the component will be maintained in order to return to state 3. During the maintenance, there is a small chance that the component totally fails. Once it completely fails, it takes a period of time to finish repair. After the repair, the tube functions as a new one. The state passage observation summaries of these two components are shown in Table 4.7 and 4.8. Here we have already enough information to model the distribution functions of sojourn time  $F_{23}$  and  $F_{32}$ . There is only one observation of jumping time from state 2 to state 1 and no observation of jump from state 3 to state 1. However, after analyzing the transition probabilities  $p_{31}$  and  $p_{21}$  are small but not zero.

Then we introduce the representation in (4.33) and (4.34). For the jumps with number of observations far more enough to build precise empirical distribution function and the visited state is absorbing state,

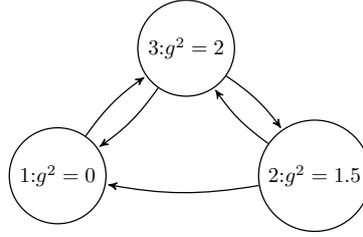


Figure 4.12: States of Component 2

$i \setminus j$	1	2	3
1	-	0	1
2	1	-	499
3	0	499	-

Table 4.7: Observation of Component 1,  $N_{ij}$ , in Figure 4.11

we can estimate this probabilities as

$$\mathbb{E}(p_{ij}) \in \left[ \frac{N_{ij}}{N_i + 1}, \frac{N_{ij} + 1}{N_i + 1} \right] = [\underline{n}_{ij}\varepsilon, \bar{n}_{ij}\varepsilon] \quad \mathbb{E}(\hat{p}_{ij}) = \frac{N_{ij} + 0.5}{N_i + 1}$$

where the Bayesian estimate has prior distribution  $Beta(0.5, 0.5)$ .

The perturbing transition probability with random set approach for components  $C_1$  and  $C_2$  is then given by

$$\hat{P}_{RS}^1 = \begin{pmatrix} 0 & 0 & 0 \\ \bar{n}_{12} & 0 & -\bar{n}_{12} \\ \bar{n}_{13} & -\bar{n}_{13} & 0 \end{pmatrix}$$

The perturbing transition probability with Bayesian approach becomes

$$\hat{P}_B^1 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{N_{21}+0.5}{N_2+1} & 0 & -\frac{N_{21}+0.5}{N_2+1} \\ \frac{N_{31}+0.5}{N_3+1} & -\frac{N_{31}+0.5}{N_3+1} & 0 \end{pmatrix}$$

where  $\varepsilon$  is a small value.

Given the knowledge that the repair time follows uniform distribution between 20 and 25 for component  $C_1$  and between 15 and 20 for component  $C_2$ , we have obtained the system availability using the two approaches (Figure 4.13, Table 4.9, Figure 4.14, Table 4.10).

**Component 3** Component  $C_3$  is a binary repairable component. For component  $C_3$ , we observed 6 failures during a period of  $t = 2287.717$ . Suppose that we know its time to failure follows an exponential distribution and the repair time follows uniform distribution from 40 to 45, we use the method in [Aguirre et al., 2013] to estimate the upper bound of failure rate so that

$$\hat{\lambda}_{RS} = \frac{N_{21} + 1}{t} = 0.003059818$$

or MLE

$$\hat{\lambda}_{MLE} = \frac{N_{21}}{t} = 0.002622701$$

We obtained the following results shown in Figure 4.16.

$i \setminus j$	1	2	3
1	-	0	5
2	2	-	744
3	3	746	-

Table 4.8: Observation of Component 2,  $N_{ij}$ , in Figure 4.12

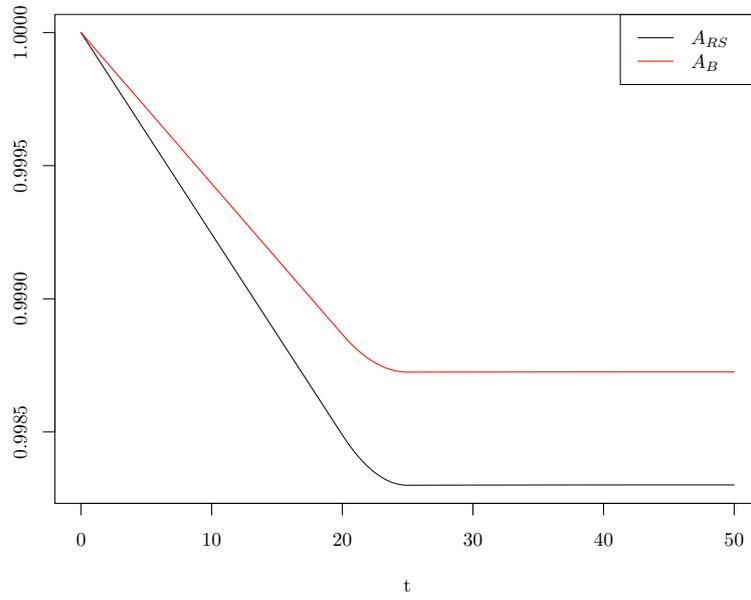


Figure 4.13: Lower bound of Component  $C_1$  availability

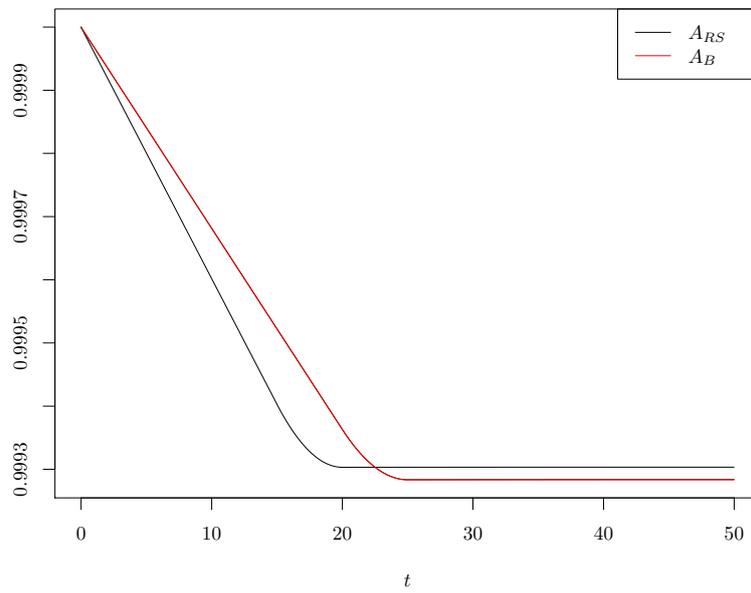


Figure 4.14: Lower bound of Component  $C_2$  availability

$x$	$P_{RS}(x)$	$P_B(x)$
$E$	0.9983011	0.9987253
1	0.0016989	0.0012747
2	0.8770656	0.8774383
3	0.1237341	0.1237867

Table 4.9: State probability of Component  $C_1$

$x$	$P_{RS}(x)$	$P_B(x)$
$E$	0.999303362749677	0.999283633296796
1	0.000696637250323207	0.000716366703203741
2	0.885952572516417	0.885935080971622
3	0.120202532053764	0.120200158870324

Table 4.10: State probability of Component  $C_2$

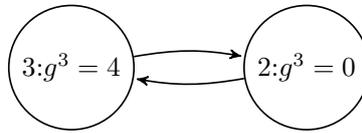


Figure 4.15: States of Component 3

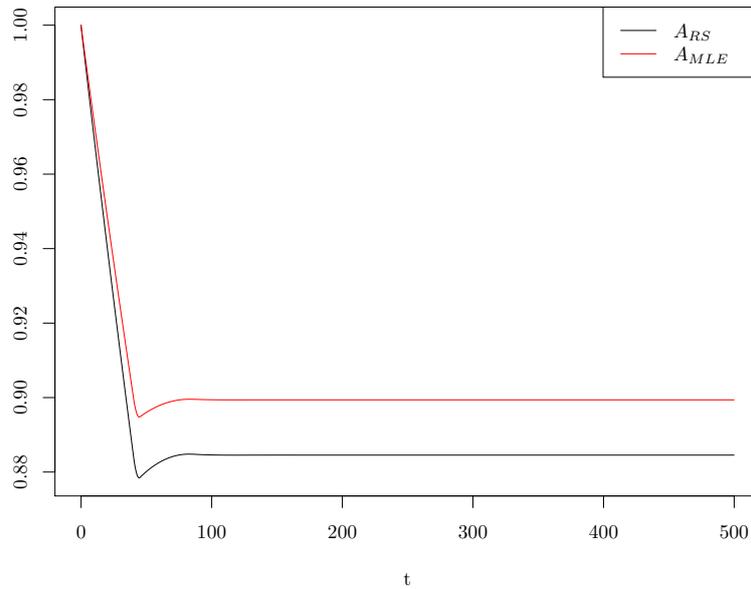


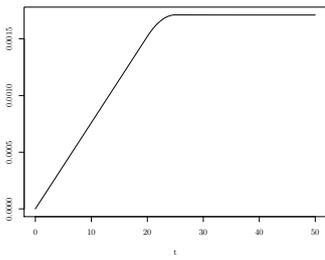
Figure 4.16: Lower bound of Component  $C_3$  availability

System state	State probability
1	0.0017099716
2	0.0013068991
3	0.0014912730
4	0.0001843738
5	0.6746774707
6	0.1903634026
7	0.0134279810

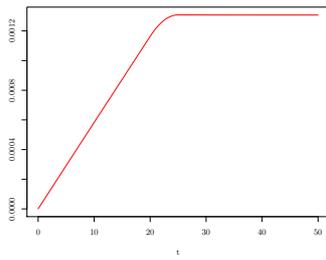
Table 4.11: Oil transition system: Stationary distribution of system state probabilities with lower bounds of component availability

Using the system structure function and correspondence between system state and system output (performance), we obtained the stationary output estimate  $g_s = 2.308694$  and system probability of each state shown in Table 4.11 and Figure 4.17.

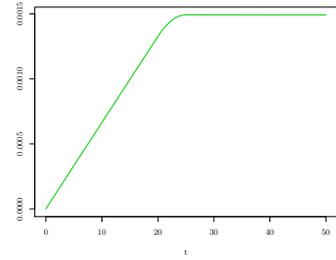
We notice that it is available only for long-term estimation. The initial state distribution is not taken into account because in the long term instantaneous state probabilities converge to stationary distribution.



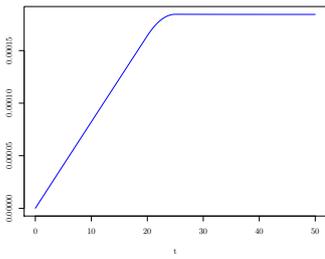
(a)  $i = 1$



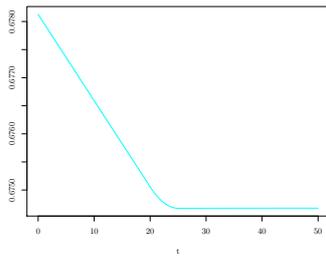
(b)  $i = 2$



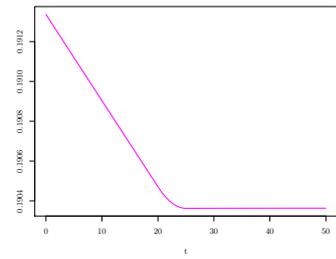
(c)  $i = 3$



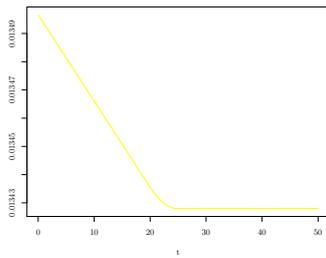
(d)  $i = 4$



(e)  $i = 5$



(f)  $i = 6$



(g)  $i = 7$

Figure 4.17: System state probabilities  $p_s^i$

# Chapter 5

## Conclusions and perspectives

### 5.1 Conclusions

**Availability of binary systems** After reviewing different uncertainty theories which can be applied for system reliability calculations, we firstly studied the simplest reliability model: binary components and systems.

Starting with probability theory, the most studied theory among uncertainty theories, we proposed a new method to estimate uncertainty under probabilistic approach which means measures of dispersion. As mentioned previously, probability theory describes appropriately aleatory uncertainty but the epistemic uncertainty is presented by different prior distribution based on assumptions. Normality asymptotic hypothesis on component reliability estimates allow us to propagate normality properties from component level to system level via reliability function. Not only system reliability can be calculated using plug-in estimator but the corresponding estimation variance can be also estimated, which makes it possible to build a confidence interval (CI) of the system reliability presenting uncertainty by a range of possible values where the unknown parameter is included for a certain degree. In order to solve the problems bought by rare failure events, Bayesian assumptions are used when there is no failure observation in available data of a certain component. In case of large-scale system, we proposed a Monte-Carlo method to estimate the system level reliability and propagate the variance measure through structure function.

We also proposed random set theory to model system reliability [Matheron, 1975]. Random set theory is a mathematical theory which can handle both aleatory and epistemic uncertainties under the same framework. It is an extension of probability theory to set-valued rather than point-value mappings. Using this approach, we managed to present system reliability by a random interval, a special case of random set [Gil, 1992]. We also defined its confidence interval which can be compared with the one of point value variable. The common method estimates system reliability is to firstly estimate reliability and uncertainty of each component directly from the corresponding component-level observations with probability theory and/or an uncertainty theory and then propagate the reliability and uncertainty to system level through the system reliability function or the structure function. However, the studied systems are large-scale systems where both the reliability and uncertainty propagation from components to system is very difficult to calculate because of the at-least exponential complexity. Since the mathematical tools to estimate single component's reliability and the associated uncertainty are well developed, if the system can be regarded as a single component, the problem will be much more simplified. Then the problem becomes how to obtain the system-level observations which is almost impossible to be obtained directly by test in reality so that we proposed to construct system-level pseudo-observations using bootstrap technique.

Then confidence intervals given by asymptotic normality (AN) method and random set (RS) method were evaluated with respect to average coverage percentage (ACP) and their interval width on simple elementary structures with different component reliability settings. ACP is the empirical estimator of the confidence level which presents the accuracy of confidence interval; the interval width describes the precision of obtained confidence interval. We observed that RS method always gives conservative results while AN method gives precise results but not always accurate. In case of highly reliable systems, all results are very close to 1 and AN method gave smaller and more accurate CI; RS method gave larger CI with ACP equal to 1. However, in case of lower reliable systems, the accuracy of AN results dropped sharply; and RS method kept conservative with high ACP. On conclusion, given the knowledge

that studied system is highly reliable, we propose AN method; if such information is not available or conservative results are demanded, we propose RS method. We applied both methods on large-scale system fault trees as well and obtained similar results.

**Availability of multi-state systems** In the second part of our work, we extended our proposed random set and semi-Markov approaches on multi-state components and systems in order to deal with rare failure events.

With random set the system state probabilities and system performance indices can be presented in form of random set bounded by random variables following Dirichlet distribution or Beta distribution. We also applied our method on an oil transmission system and compared the obtained results with the ones given by universal generating function method (UGF). Similar algorithms are also given for large-scale system cases.

In addition, we studied dependability estimation of repairable dynamic multi-state systems which means this time the observation time  $t$  is no more fixed. The system in our study is modeled by semi-Markov process. Markov (renewal) or semi-Markov process gives a general and more flexible presentation for processes having Markov properties. Semi-Markov process (SMP) provides a more precise model to evaluate system reliability [N. Limnios, 2001, Malefaki et al., 2014]. The passage rates do not have to be homogeneous as in case of Markov process modeling. System dependability measure of repairable systems is modeled by a continuous time semi-Markov process (CTSMP). An important drawback in these applications is the difficulty of obtaining traceable solution in application with a large enough number of states. In this thesis, we firstly studied the existence and uniqueness solution of a Markov renewal equation (MRE). A method solving Markov renewal equations based on an iterative scheme inspired by  $n$ -fold convolution was then proposed. Theoretical and numerical applications on system reliability and availability calculation as well as a case study on system availability and reliability assessment are also provided.

We applied the state merging and splitting scheme on discrete state space continuous time semi-Markov describing highly reliably (repairable) system evaluation in order to reduce state space and to approximately estimate the system reliability and availability. The state merging and splitting scheme [Koroliuk and Limnios, 2005, Koroliuk and Limnios, 2010] gives a method that transforms semi-Markov subsystems whose state space can be divided into groups where jumps occur much more frequently between states inside the group, than states between groups, into single states with homogeneous passage rate, instead of just replacing non-exponential passage time distribution by exponential distribution with the same mean passage rate. As a system (process) composed of Markov subsystems still keeps its Markov property, we can easily simplify the space state by grouping some of states. The grouping procedure is based on average theorem with ergodic assumptions. It is very useful for system reliability (availability) estimation because in reparable systems composed of highly reliable systems, we often observe that jumps happen at most of time inside groups of states; the jumps between the group(s) of system working state and the one(s) of system failure states are rather rare. We just need to understand the long term process behavior.

During the thesis, we proposed methods and algorithms to estimate system reliability and its uncertainty with both probabilistic and random set theory approaches via confidence interval. Random set method presents both aleatory and epistemic uncertainties in a flexible way and gives rather conservative result which corresponds to our demand of treating rare failure observation conditions. We also associated the two methods with semi-Markov process while possible state passages were barely observed. Pseudo-observation construction and bootstrap method are widely used to avoid exponentially increasing computation cost in case of large-scale systems.

## 5.2 Perspectives

Our future work will focus on

- application of pseudo-observation construction on other uncertainty theories
- application of our random set theory method on multi-state system performance evaluation with uncertain demand and other conditions
- adaptation of the state merging scheme on large-scale systems

Uncertainty theories give various opinions about the origin and the presentation of uncertainty. On applying these approaches on system reliability evaluation, one of the main disadvantages is that the direct calculation of uncertainty propagation from component level to system level through structure function or reliability function brings computational cost growing exponentially with the size of the studied system.

In our research, we introduced a resampling method in order to reduce calculation complexity when random set theory is applied. The proposed method constructs pseudo system samples and treats a system as a single component and then studies the system reliability or other performance measures using random set theory. The uncertainty propagation from component level to system level has then been taken into account during the pseudo system construction. Since methods calculating the reliability of one component/system are well developed using uncertainty theories such as belief function, imprecise probabilities, given size  $n_s$  pseudo system samples among which there are  $k_s$  good-state observations, we can apply imprecise probabilities, belief functions, fuzzy set theory or other theories using similar algorithms under the same framework.

We have already studied random set model for evaluating binary and multi-state systems performance with constant demand (constant availability state set). The next task is to study the possibility of applying the model on complicated cases such as systems with uncertain demand and uncertain system structures and conditions.

Uncertain demand is not presented by a constant value, but is modeled by a range or an uncertainty model such as interval value, membership function, mass function and probability distribution. In my opinion, uncertainty system demand can be combined with the system level performance rate model. For example, a system with  $m$  states and the demand can be both modeled by random set based on Dirichlet distribution, all possible the system possible conditions can be presented in a  $m \times m$  space and then projected to  $\mathbb{R}$  as another random set.

The system structure contains uncertain parts sometimes due to lack of information. The result of structure function can be described by a set. Then the system reliability estimation  $R(\hat{\mathbf{p}})$  would be represented by a random set where  $\hat{\mathbf{p}}$  denotes the component reliability estimated using probability theory estimators and follows asymptotically a normal distribution. Finally, it is possible to give a set-valued estimation and a corresponding confidence interval of the system reliability.

Since our random set results converges to probability theory results as more observations (information) are available, it is possible to build algorithms adapted to incremental historical system data and study the importance impact of each component or subsystem in terms of uncertainty measure.

Both of our estimation methods have been applied on semi-Markov system reliability assessment with the state merging scheme in order to solve problem of rare failure events. System time to failure in long term is described by a series of exponential distributions with parameter  $\varepsilon$ . Indeed, it is also possible to bring more uncertainty theories under this framework.

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