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# On goodness-of-fit tests with parametric hypotheses for some stochastic processes

Maroua Ben Abdeddaiem

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## Thèse de Doctorat

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## Tests d'ajustement pour des processus stochastiques dans le cas de l'hypothèse nulle paramétrique

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# Abstract

This work is devoted to the problem of the construction of several goodness-of-fit (GoF) tests in the case of some stochastic processes observed in continuous time. As models of observations, we take “small noise” and ergodic diffusion processes and an inhomogeneous Poisson process. Under the null hypothesis, we treat the case where each model depends on an unknown one-dimensional parameter and we consider the minimum distance estimator for this parameter. Our goal is to propose “asymptotically distribution free” (ADF) GoF tests of asymptotic size  $\alpha \in (0, 1)$  in the case of the parametric null hypotheses for the considered models. Indeed, we show that the limit of each studied statistic does not depend on the model and the unknown parameter. Therefore, the tests based on these statistics are ADF. The main purpose of this work is to construct a special linear transformation. In particular, we solve Fredholm equation of the second kind with degenerated kernel. Its solution gives us the desired linear transformation. Next, we show that the application of this transformation to the basic statistics allows us to introduce statistics with the same limit (the integral of the square of the Wiener process). The latter is “distribution free” because it does not depend on the models and the unknown parameter. Therefore, we construct the ADF GoF tests which are based on this linear transformation for the diffusion (“small noise” and ergodic) and inhomogeneous Poisson processes.

**key Words :** Goodness-of-fit tests, minimum distance estimator, asymptotically distribution free tests, stochastic processes, diffusion processes, inhomogeneous Poisson process.



# Résumé

Ce travail est consacré au problème de construction des tests d'ajustement dans le cas des processus stochastiques observés en temps continu. Comme modèles d'observations, nous considérons les processus de diffusion avec "petit bruit" et ergodique et le processus de Poisson non homogène. Sous l'hypothèse nulle, nous traitons le cas où chaque modèle dépend d'un paramètre inconnu unidimensionnel et nous proposons l'estimateur de distance minimale pour ce paramètre. Notre but est la construction des tests d'ajustement "asymptotically distribution free" (ADF) de niveau asymptotique  $\alpha \in (0, 1)$  dans le cas de cette hypothèse paramétrique pour les modèles traités. Nous montrons alors que la limite de chaque statistique étudiée ne dépend ni du modèle ni du paramètre inconnu. Les tests d'ajustement basés sur ces statistiques sont donc ADF. L'objectif principal de ce travail est la construction d'une transformation linéaire spéciale. En particulier, nous résolvons l'équation de Fredholm du second type avec le noyau dégénéré. Sa solution nous permet de construire la transformation linéaire désirée. Ensuite, nous montrons que l'application de cette transformation aux statistiques de base étudiées dans chaque modèle nous aide à introduire des statistiques ayant la même limite (l'intégrale du carré du processus de Wiener). Cette dernière est "distribution free" vu qu'elle ne dépend ni du modèle ni du paramètre inconnu. Par conséquent, nous proposons des tests d'ajustement ADF en se basant sur cette transformation linéaire pour les processus de diffusion avec "petit bruit" et ergodique et le processus de Poisson non homogène.

**Mots Clés :** Tests d'ajustement, estimateur de distance minimale, tests asymptotically distribution free, processus stochastiques, processus de diffusion, processus de Poisson non homogène.





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# Introduction

Les tests d'ajustement (goodness-of-fit) jouent un rôle important dans les statistiques théoriques et appliquées vu que leurs applications permettent de vérifier si le modèle mathématique proposé correspond bien aux observations. Ce problème a été introduit depuis 1928 par Cramér [10], von Mises [60], Kolmogorov [35] et Smirnov [57] et a été bien étudié par plusieurs statisticiens pendant les années 50. Nous citons par exemple les travaux de Darling [13], [15], Anderson & Darling [3] et Watson [61], etc.

Dans un premier temps, nous mentionnons le problème de construction des tests d'ajustement pour le modèle d'observations  $X^n = (X_1, \dots, X_n)$  des variables aléatoires indépendantes et identiquement distribuées (i.i.d.) admettant  $F(x)$  comme fonction de répartition continue. Notons que ce problème a été largement étudié par plusieurs auteurs par exemple dans les ouvrages de Durbin [18], Greenwood & Nikulin [24] et Lehmann & Romano [46]. Dans ces travaux, les tests proposés sont les tests de *Chi-deux*, *Kolmogorov-Smirnov*, *Cramér-von Mises* et *Anderson-Darling*.

Par la suite, nous rappelons le problème de construction du test de Cramér-von Mises de l'hypothèse de base simple  $\mathcal{H}_0$  contre l'alternative  $\mathcal{H}_1$

$$\mathcal{H}_0 : \quad F(x) = F_0(x), \quad \mathcal{H}_1 : \quad F(x) \neq F_0(x),$$

avec  $F_0(\cdot)$  une fonction de répartition continue et connue.

Notre but est de construire un test  $\Psi_n = \mathbb{I}_{\{\phi_n > c_\alpha\}}$  de niveau asymptotique  $\alpha \in (0, 1)$ , c'est à dire,

$$\lim_{n \rightarrow \infty} \mathbf{E}_0 \Psi_n = \alpha,$$

où  $\phi_n$  est la statistique de Cramér-von Mises,  $\Psi_n$  est la probabilité de rejeter

l'hypothèse  $\mathcal{H}_0$  et  $\mathbf{E}_0$  désigne l'espérance mathématique sous l'hypothèse  $\mathcal{H}_0$ .

Les tests d'ajustement admettant une statistique limite qui ne dépend pas du modèle, c'est à dire  $F_0(\cdot)$ , sont appelés "asymptotically distribution free" (ADF). Ainsi, le choix du seuil  $c_\alpha$ , solution de l'équation  $\mathbf{P}_0(\phi > c_\alpha) = \alpha$ , peut être fait une seule fois pour tous les problèmes admettant la même statistique limite  $\phi$ . Par conséquent,  $c_\alpha$  est le même pour toutes les fonctions  $F_0(\cdot)$ .

Rappelons que plusieurs tests d'ajustement ont été basés sur la propriété suivante : sous l'hypothèse  $\mathcal{H}_0$ , la fonction de répartition empirique normalisée

$$\sqrt{n} \left( \hat{F}_n(x) - F_0(x) \right)$$

converge en loi vers un pont Brownien  $B(F_0(x))$ ,  $x \in \mathbb{R}$ . La fonction de répartition empirique  $\hat{F}_n(\cdot)$  est définie comme suit :

$$\hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}_{\{X_j < x\}}.$$

Le processus stochastique  $B(s)$ ,  $0 \leq s \leq 1$  est un processus Gaussien avec  $\mathbf{E}B(s) = 0$  et  $\mathbf{E}B(s_1)B(s_2) = s_1 \wedge s_2 - s_1 s_2$ . Par conséquent, sous l'hypothèse  $\mathcal{H}_0$ , nous obtenons la convergence de la statistique de Cramér-von Mises

$$\begin{aligned} \phi_n &= n \int_{-\infty}^{\infty} \left( \hat{F}_n(x) - F_0(x) \right)^2 dF_0(x) \\ &\implies \int_{-\infty}^{\infty} B(F_0(x))^2 dF_0(x) = \int_0^1 B(s)^2 ds \equiv \phi, \end{aligned}$$

où nous utilisons le changement de variable  $s = F_0(x)$ . Ainsi, la statistique limite  $\phi$  ne dépend pas de  $F_0(\cdot)$  et donc le test de Cramér-von Mises  $\Psi_n = \mathbb{I}_{\{\phi_n > c_\alpha\}}$  est ADF et de niveau asymptotique  $\alpha \in (0, 1)$  (voir par exemple [46]).

Dans le cas de l'hypothèse de base paramétrique, nous allons voir par la suite que la situation va changer. Notons que ce problème a été développé dans plusieurs travaux par exemple Kac et al. [29], Babu & Rao [4], Khmaladze [31], Nikabadze [55] et Martynov [50]. Nous sommes intéressés par la construction d'un test  $\hat{\Psi}_n = \mathbb{I}_{\{\hat{\phi}_n > d_\alpha\}}$  basé sur la statistique de type Cramér-von Mises  $\hat{\phi}_n$  de l'hypothèse  $\mathcal{H}_0$

$$\mathcal{H}_0 \quad : \quad F(\cdot) \in \{F(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\}, \quad (1)$$

où  $a > 0$ ,  $b < \infty$ . Le paramètre  $\vartheta$  est inconnu unidimensionnel et  $F(\vartheta, x)$  est une fonction de répartition connue et continue qui dépend de  $\vartheta$ .

Une des méthodes proposées est de trouver un seuil  $d_\alpha = d_\alpha(\vartheta, F)$  tel que  $\mathbf{P}_\vartheta(\hat{\phi} > d_\alpha) = \alpha$ , où  $\hat{\phi}$  est la statistique limite de  $\hat{\phi}_n$ , vérifier que  $d_\alpha(\vartheta, F)$  est une fonction continue de  $\vartheta$  par les simulations numériques et finalement poser  $\bar{d}_\alpha = d_\alpha(\bar{\vartheta}_n, F)$ . Notons que  $\bar{\vartheta}_n$  est un estimateur quelconque de  $\vartheta$  qui peut être l'estimateur de maximum de vraisemblance (EMV). Ainsi, nous obtenons un test  $\bar{\Psi}_n = \mathbb{I}_{\{\hat{\phi}_n > \bar{d}_\alpha\}}$  de niveau asymptotique  $\alpha$ , c'est à dire,

$$\lim_{n \rightarrow \infty} \mathbf{E}_\vartheta \bar{\Psi}_n = \alpha, \quad \text{for all } \vartheta \in \Theta,$$

où  $\mathbf{E}_\vartheta$  désigne l'espérance mathématique sous l'hypothèse  $\mathcal{H}_0$ .

Par la suite, nous allons utiliser une autre méthode proposée par Khmaladze [30] pour construire le test ADF. Nous introduisons alors la statistique de type Cramér-von Mises

$$\hat{\phi}_n = \int_{-\infty}^{\infty} U_n(x)^2 dF(\hat{\vartheta}_n, x), \quad U_n(x) = \sqrt{n} \left( \hat{F}_n(x) - F(\hat{\vartheta}_n, x) \right),$$

où  $\hat{\vartheta}_n$  est l'EMV du paramètre inconnu  $\vartheta$ . Sous les conditions de régularités, nous mentionnons que l'EMV est consistant, asymptotiquement normale et admet en plus la représentation suivante :

$$\sqrt{n} \left( \hat{\vartheta}_n - \vartheta \right) = \frac{1}{\sqrt{n}I(\vartheta)} \sum_{j=1}^n \frac{\dot{f}(\vartheta, X_j)}{f(\vartheta, X_j)} + o(1), \quad I(\vartheta) = \int_{-\infty}^{\infty} \frac{\dot{f}(\vartheta, x)^2}{f(\vartheta, x)} dx.$$

Notons que  $I(\vartheta)$  et  $f(\vartheta, x)$  désignent l'information de Fisher et la densité, respectivement. Ci-dessus et par la suite le point représente la dérivation par rapport à  $\vartheta$ .

En appliquant la formule de Taylor, nous obtenons pour la statistique  $U_n(\cdot)$  la représentation suivante :

$$U_n(x) = B_n(x) - \sqrt{n} \left( \hat{\vartheta}_n - \vartheta \right) \dot{F}(\vartheta, x) + o(1),$$

où

$$B_n(x) = \sqrt{n} \left( \hat{F}_n(x) - F(\vartheta, x) \right). \quad (2)$$

D'une part, nous avons

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n \frac{\dot{f}(\vartheta, X_j)}{f(\vartheta, X_j)} = \sqrt{n} \int_{-\infty}^{\infty} \frac{\dot{f}(\vartheta, y)}{f(\vartheta, y)} d\hat{F}_n(y) = \int_{-\infty}^{\infty} \frac{\dot{f}(\vartheta, y)}{f(\vartheta, y)} dB_n(y).$$

D'autre part, nous écrivons la fonction  $\dot{F}(\vartheta, x)$  sous cette forme

$$\dot{F}(\vartheta, x) = \int_{-\infty}^x \dot{f}(\vartheta, y) dy = \int_{-\infty}^x \dot{l}(\vartheta, y) dF(\vartheta, y),$$

avec  $l(\vartheta, y) = \ln f(\vartheta, y)$ . Donc, nous obtenons

$$U_n(x) = B_n(x) - \int_{-\infty}^{\infty} \frac{\dot{l}(\vartheta, y)}{\sqrt{I(\vartheta)}} dB_n(y) \int_{-\infty}^x \frac{\dot{l}(\vartheta, y)}{\sqrt{I(\vartheta)}} dF(\vartheta, y) + o(1).$$

Ainsi, sous l'hypothèse  $\mathcal{H}_0$ , la convergence  $B_n(x) \Rightarrow B(F(\vartheta, x))$  nous donne

$$U_n(x) \Rightarrow B(F(\vartheta, x)) - \int_{-\infty}^{\infty} \frac{\dot{l}(\vartheta, y)}{\sqrt{I(\vartheta)}} dB(F(\vartheta, y)) \int_{-\infty}^x \frac{\dot{l}(\vartheta, y)}{\sqrt{I(\vartheta)}} dF(\vartheta, y) \equiv u(x).$$

Ensuite, les changements de variables  $s = F(\vartheta, y)$  et  $t = F(\vartheta, x)$  impliquent pour le processus limite  $u(\cdot)$  la représentation suivante :

$$U(t) = B(t) - \int_0^1 h(\vartheta, s) dB(s) \int_0^t h(\vartheta, s) ds, \quad (3)$$

avec

$$h(\vartheta, s) = \frac{\dot{l}(\vartheta, F_\vartheta^{-1}(s))}{\sqrt{I(\vartheta)}}, \quad \int_0^1 h(\vartheta, s)^2 ds = 1.$$

Notons que  $F_\vartheta^{-1}(s)$  est la fonction inverse de  $F(\vartheta, y)$ , c'est à dire  $s = F(\vartheta, y)$ .

Nous remarquons que la fonction  $h(\cdot, \cdot)$  apparaît deux fois dans l'expression de  $U(\cdot)$  vu l'utilisation de l'EMV.

Finalement, sous l'hypothèse  $\mathcal{H}_0$ , nous obtenons la convergence suivante :

$$\hat{\phi}_n = \int_{-\infty}^{\infty} U_n(x)^2 dF(\hat{\vartheta}_n, x) \Rightarrow \hat{\phi} \equiv \int_0^1 U(t)^2 dt.$$

Les détails de ce résultat et la représentation du processus limite  $U(\cdot)$  ont été introduits par exemple par Darling [13]. Nous mentionnons dans un certain sens que  $U(\cdot)$  est une limite universelle qui apparaît dans plusieurs problèmes de tests d'ajustement dans le cas de l'EMV [30], [34], [49] etc.

Nous remarquons que le test  $\hat{\Psi}_n = \mathbb{I}_{\{\hat{\phi}_n > d_\alpha\}}$  n'est pas ADF et le seuil  $d_\alpha$  défini comme solution de l'équation suivante :

$$\mathbf{P}_\vartheta \left( \hat{\phi} > d_\alpha \right) = \alpha$$

dépend fortement du modèle  $F(\cdot, \cdot)$  (connue) et du paramètre  $\vartheta$  (inconnu). Par conséquent, le choix du seuil  $d_\alpha = d_\alpha(F, \vartheta)$  est un problème complexe.

Une des méthodes développées pour résoudre ce problème a été introduite par Khmaladze [30]. Elle consiste à construire une transformation linéaire  $L[\cdot]$  pour le processus  $U(\cdot)$  défini par (3) telle que  $L[U](t) = W(t)$ , avec  $W(t), 0 \leq t \leq 1$  un processus de Wiener. Ainsi, sous l'hypothèse  $\mathcal{H}_0$ , nous avons la convergence suivante :

$$\begin{aligned} \hat{v}_n &= \int_{-\infty}^{\infty} L[U_n](x)^2 dF(\hat{\vartheta}_n, x) \\ &\implies \int_{-\infty}^{\infty} W(F(\vartheta, x))^2 dF(\vartheta, x) = \int_0^1 W(t)^2 dt \equiv \hat{v} \end{aligned}$$

et nous obtenons la statistique limite  $\hat{v}$  qui ne dépend ni de  $F(\cdot, \cdot)$  ni de  $\vartheta$ .

Par conséquent, le test

$$\hat{\Phi}_n = \mathbb{I}_{\{\hat{v}_n > k_\alpha\}}, \quad \mathbf{P}_\vartheta(\hat{v} > k_\alpha) = \alpha$$

est ADF et le seuil  $k_\alpha$  ne dépend ni de  $F(\cdot, \cdot)$  ni de  $\vartheta$ . En effet, le choix de  $k_\alpha$  ne pose plus de problème. Le test  $\hat{\Phi}_n$  est donc de niveau asymptotique  $\alpha$ . Plusieurs autres tests d'ajustement ADF ont été construits en se basant sur cette transformation linéaire par exemple dans [16], [17] et [32].

Ce problème existe aussi pour les processus stochastiques observés en temps continu et il a été largement traité par de nombreux auteurs. Par exemple, Dachian & Kutoyants ont étudié dans [12] des possibilités de construction des tests d'ajustement pour le processus de diffusion et le processus de Poisson non homogène dans le cas de l'hypothèse simple. Ce problème a été ensuite développé dans [42] et [43] dans le cas de l'hypothèse paramétrique. En particulier, Kutoyants a considéré le processus de diffusion avec "petit bruit" afin de construire des tests d'ajustement ADF dans le cas de l'hypothèse de base simple [40] et paramétrique [44]. Le même problème a été étudié dans [56] pour les cas régulier et singulier. Des résultats similaires pour les tests d'ajustement ADF ont été développés dans Kleptsyna & Kutoyants [34] et Kutoyants [39], Section 5.4 dans le cas du processus de diffusion ergodique. Plusieurs autres tests d'ajustement ont été construits dans le cas de ce processus, à savoir dans Gassem [23], Negri &

Nishiyama [51], [52], Negri & Zhou [54] et Zhou [62]. Notons que le cas général du processus de diffusion avec le paramètre de shift inconnu (unidimensionnel) a été traité dans [54]. En effet, les auteurs ont prouvé que la limite de la statistique de type Cramér-von Mises ne dépend pas du paramètre inconnu et par conséquent le test d'ajustement basé sur cette statistique est *asymptotically parameter free* (APF). Par ailleurs, les tests APF de type Kolmogorov-Smirnov ont été étudiés dans [62]. En outre, Kutoyants a proposé des tests d'ajustement APF de type Cramér-von Mises dans [41] pour le processus de diffusion ergodique y compris le processus d'Ornstein-Uhlenbeck.

Dans la littérature, de nombreux tests d'ajustement pour les processus de Poisson homogènes ont été proposés, par exemple, dans [47], [14] et [1]. Nous mentionnons dans [1] qu'un test a été introduit pour vérifier si un processus ponctuel est homogène. En outre, dans [5], les auteurs ont développé un test consistant basé sur la fonction génératrice empirique. Un autre test d'ajustement a été introduit dans [59], pour les distributions discrètes, basé sur une caractérisation par des distances moyennes. Par ailleurs, des tests d'ajustement basés sur les statistiques de Cramér-von Mises ont été proposés dans [58] pour la distribution de Poisson. De plus, une comparaison a été introduite pour montrer la puissance de ces tests. D'autres comparaisons entre le test de Pearson, le test régulier et le test de Kolmogorov-Smirnov modifié sont données dans [9].

Dans le cas du processus de Poisson non homogène et plus généralement le processus ponctuel, les tests d'ajustement ont été étudiés par de nombreux auteurs, par exemple, [2], [11], [25] et [49]. Dans [11], le cas de processus de Poisson non homogène avec le paramètre de shift (unidimensionnel) a été étudié et l'auteur a montré que la distribution limite de la statistique de type Cramér-von Mises est APF. Plus de problèmes de construction des tests d'ajustement dans le cas des observations en temps discret ont été largement étudiés par de nombreux auteurs, par exemple Klar [33] et Negri & Nishiyama [53] pour le processus de diffusion avec "petit bruit". Dans [52], des tests ADF ont été proposés dans le cas des observations en temps continu et discret pour les processus de diffusion ergodiques.



## Cas des observations i.i.d. avec un estimateur différent de l'EMV

Soulignons que dans de nombreux travaux, l'estimateur utilisé pour le paramètre inconnu était toujours l'EMV et ceci est important pour la construction de la transformation linéaire et du test ADF. Cependant, dans certains problèmes, il est impossible d'avoir une expression explicite pour l'EMV et donc il est parfois préférable d'utiliser d'autres estimateurs qui peuvent être facilement calculés. Par exemple, il pourrait être l'estimateur de distance minimale (EDM), l'estimateur de méthode des moments ou l'estimateur de "trajectory fitting" etc. Dans tous ces cas, l'expression des statistiques limites sera comme (3) (cas de l'EMV) mais avec deux fonctions différentes  $h(\cdot, \cdot)$  et  $g(\cdot, \cdot)$  (voir (6) ci-dessous).

En effet, nous considérons le modèle d'observations  $X^n = (X_1, \dots, X_n)$  i.i.d. et nous utilisons par exemple l'EDM pour la construction du test de l'hypothèse  $\mathcal{H}_0$  donné par (1). Nous introduisons la statistique de type Cramér-von-Mises

$$\phi_n^* = \int_{-\infty}^{+\infty} \xi_n(x)^2 dF(\vartheta_n^*, x), \quad \xi_n(x) = \sqrt{n} \left( \hat{F}_n(x) - F(\vartheta_n^*, x) \right), \quad (4)$$

où  $\vartheta_n^*$  est l'EDM qui est solution de l'équation de distance minimale

$$\int_{-\infty}^{+\infty} \left( \hat{F}_n(x) - F(\vartheta_n^*, x) \right) \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x) = 0.$$

Nous définissons  $u_n^* = \sqrt{n}(\vartheta_n^* - \vartheta)$ . Ainsi,  $\vartheta_n^* = \frac{u_n^*}{\sqrt{n}} + \vartheta$  et en utilisant la formule de Taylor l'équation précédente admet la représentation suivante :

$$\int_{-\infty}^{+\infty} \left( \hat{F}_n(x) - F(\vartheta, x) - \frac{u_n^*}{\sqrt{n}} \dot{F}(\tilde{\vartheta}, x) \right) \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x) = 0.$$

Par conséquent, la variable aléatoire  $u_n^*$  est représentée comme suit :

$$u_n^* = \left( \int_{-\infty}^{+\infty} \dot{F}(\tilde{\vartheta}, x) \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x) \right)^{-1} \int_{-\infty}^{+\infty} B_n(x) \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x), \quad (5)$$

où  $B_n(\cdot)$  est défini par (2). Par le théorème de Fubini, nous avons

$$\begin{aligned} & \int_{-\infty}^{+\infty} B_n(x) \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x) \\ &= \sqrt{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbb{1}_{\{y < x\}} \left[ d\hat{F}_n(y) - f(\vartheta, y) dy \right] \dot{F}(\vartheta_n^*, x) dF(\vartheta_n^*, x) \end{aligned}$$

$$\begin{aligned}
&= \sqrt{n} \int_{-\infty}^{+\infty} \int_y^{+\infty} \dot{F}(\vartheta_n^*, x) \, dF(\vartheta_n^*, x) \left[ d\hat{F}_n(y) - f(\vartheta, y) \, dy \right] \\
&= \sqrt{n} \int_{-\infty}^{+\infty} H(\vartheta_n^*, y) \left[ d\hat{F}_n(y) - f(\vartheta, y) \, dy \right],
\end{aligned}$$

avec  $H(\vartheta_n^*, y) = \int_y^{+\infty} \dot{F}(\vartheta_n^*, x) \, dF(\vartheta_n^*, x)$ . En utilisant l'expression suivante :

$$\int_{-\infty}^{+\infty} H(\vartheta_n^*, y) \, d\hat{F}_n(y) = \frac{1}{n} \sum_{j=1}^n H(\vartheta_n^*, X_j)$$

et le théorème central limite, nous obtenons

$$\begin{aligned}
&\int_{-\infty}^{+\infty} B_n(x) \dot{F}(\vartheta_n^*, x) \, dF(\vartheta_n^*, x) \\
&= \frac{1}{\sqrt{n}} \sum_{j=1}^n H(\vartheta_n^*, X_j) - \sqrt{n} \int_{-\infty}^{+\infty} H(\vartheta_n^*, y) f(\vartheta, y) \, dy \\
&= \frac{1}{\sqrt{n}} \sum_{j=1}^n (H(\vartheta_n^*, X_j) - \mathbf{E}_\vartheta H(\vartheta_n^*, X_j)) \implies \mathcal{N}(0, \sigma(\vartheta)^2),
\end{aligned}$$

où

$$\sigma(\vartheta)^2 = \mathbf{E}_\vartheta (H(\vartheta, X_1) - \mathbf{E}_\vartheta H(\vartheta, X_1))^2.$$

Ainsi, la variable aléatoire  $u_n^*$  converge en loi vers la loi normale

$$u_n^* \implies \mathcal{N}(0, d(\vartheta)^2), \quad d(\vartheta)^2 = \left( \mathbf{E}_\vartheta \dot{F}(\vartheta, X_1)^2 \right)^{-2} \sigma(\vartheta)^2.$$

De plus, grâce à la consistance de l'EDM, le processus  $u_n^*$  défini par (5) admet la représentation suivante :

$$u_n^* = \left( \int_{-\infty}^{+\infty} \dot{F}(\vartheta, x)^2 \, dF(\vartheta, x) \right)^{-1} \int_{-\infty}^{+\infty} H(\vartheta, y) \, dB_n(y) \, (1 + o(1)),$$

où  $B_n(\cdot)$  est défini par (2).

Maintenant, nous allons étudier le comportement asymptotique de la statistique de type Cramér-von-Mises définie par (4). En effet, la formule de Taylor et la convergence  $B_n(x) \implies B(F(\vartheta, x))$ , avec  $B(F(\vartheta, x))$  un pont Brownien, nous donnent la représentation suivante :

$$\begin{aligned}
\xi_n(x) &= B_n(x) - \sqrt{n}(\vartheta_n^* - \vartheta) \dot{F}(\tilde{\vartheta}, x) + o(1) \implies B(F(\vartheta, x)) \\
&\quad - \left( \int_{-\infty}^{+\infty} \dot{F}(\vartheta, y)^2 \, dF(\vartheta, y) \right)^{-1} \int_{-\infty}^{+\infty} H(\vartheta, y) \, dB(F(\vartheta, y)) \dot{F}(\vartheta, x).
\end{aligned}$$

Ensuite, si nous changeons les variables  $t = F(\vartheta, x)$  et  $s = F(\vartheta, y)$ , nous obtenons

$$\xi_n(x) \implies B(t) - J(\vartheta)^{-1} \int_0^1 H(\vartheta, F_\vartheta^{-1}(s)) dB(s) \int_0^t \frac{\dot{f}(\vartheta, F_\vartheta^{-1}(s))}{f(\vartheta, F_\vartheta^{-1}(s))} ds,$$

où  $B(t), 0 \leq t \leq 1$  est un pont Brownien,  $F_\vartheta^{-1}(s)$  est la fonction inverse de  $F(\vartheta, y)$  et

$$J(\vartheta) = \int_0^1 \dot{F}(\vartheta, F_\vartheta^{-1}(s))^2 ds.$$

Ainsi,

$$\xi_n(x) \implies B(t) - \int_0^1 g(\vartheta, s) dB(s) \int_0^t h(\vartheta, s) ds,$$

avec

$$g(\vartheta, s) = H(\vartheta, F_\vartheta^{-1}(s)) \left( \int_0^1 H(\vartheta, F_\vartheta^{-1}(v))^2 dv \right)^{-\frac{1}{2}}, \quad \int_0^1 g(\vartheta, s)^2 ds = 1$$

et

$$h(\vartheta, s) = J(\vartheta)^{-1} \frac{\dot{f}(\vartheta, F_\vartheta^{-1}(s))}{f(\vartheta, F_\vartheta^{-1}(s))} \left( \int_0^1 H(\vartheta, F_\vartheta^{-1}(v))^2 dv \right)^{\frac{1}{2}}.$$

Finalement, sous l'hypothèse  $\mathcal{H}_0$ , nous avons cette convergence

$$\phi_n^* \implies \phi^* \equiv \int_0^1 U(t)^2 dt,$$

où

$$U(t) = B(t) - \int_0^1 g(\vartheta, s) dB(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1. \quad (6)$$

Maintenant, nous pouvons remplacer le pont Brownien  $B(t), 0 \leq t \leq 1$  par  $B(t) = W(t) - W(1)t$ , où  $W(t)$  est un processus de Wiener. Ainsi, nous obtenons

$$\begin{aligned} U(t) &= W(t) - W(1)t \\ &\quad - \int_0^1 g(\vartheta, s) dW(s) \int_0^t h(\vartheta, s) ds + W(1) \int_0^1 g(\vartheta, s) ds \int_0^t h(\vartheta, s) ds \\ &= W(t) - \sum_{l=1}^3 \int_0^1 g_l(\vartheta, s) dW(s) \int_0^t h_l(\vartheta, s) ds, \end{aligned}$$

avec  $h_1(\vartheta, s) = g_1(\vartheta, s) = 1$ ,  $h_2(\vartheta, s) = h_3(\vartheta, s) = h(\vartheta, s)$ ,  $g_2(\vartheta, s) = g(\vartheta, s)$  et  $g_3(\vartheta, s) = -\int_0^1 g(\vartheta, s) ds$ , ce qui va augmenter la dimension des fonctions

$g(\vartheta, s) = (g_1(\vartheta, s), g_2(\vartheta, s), g_3(\vartheta, s))$  et  $h(\vartheta, s) = (h_1(\vartheta, s), h_2(\vartheta, s), h_3(\vartheta, s))$ . Alors, il est difficile de trouver une telle transformation linéaire pour ce processus limite, même dans le cas où le paramètre  $\vartheta$  est unidimensionnel. Ceci est probablement la raison pour laquelle ce problème n'a pas été considéré jusqu'à maintenant.

Ainsi, cette thèse est consacrée au problème de construction d'une transformation linéaire appliquée à un processus  $U(\cdot)$  admettant la représentation suivante :

$$U(t) = W(t) - \int_0^1 g(\vartheta, s) dW(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1, \quad (7)$$

où  $W(t), 0 \leq t \leq 1$  est un processus de Wiener. Notons que cette représentation peut être obtenue pour une large classe d'estimateurs. La différence principale avec le cas i.i.d. est dû au processus de Wiener et donc la possibilité de construire une transformation linéaire appliquée au processus limite (7). Par conséquent, nous allons voir dans ce travail que nous obtenons la même limite (7) en utilisant l'EDM dans le cas des processus de diffusion avec "petit bruit" et ergodique et dans le cas de processus de Poisson non homogène.

## Résultats principaux

Le but principal de ce travail est la construction des tests d'ajustement pour les processus de diffusion (avec "petit bruit" et ergodique) et le processus de Poisson non homogène observés en temps continu. Sous l'hypothèse nulle, nous considérons le cas où chaque modèle dépend d'un paramètre inconnu unidimensionnel estimé par l'EDM. Notre objectif est la construction des tests d'ajustement "asymptotically distribution free" dans le cas de cette hypothèse paramétrique pour les modèles traités. La problématique générale de cette thèse porte sur la construction d'une transformation linéaire spéciale. En particulier, elle est obtenue à l'aide de la solution de l'équation de Fredholm du second type avec le noyau dégénéré. Ensuite, nous montrons que l'application de cette transformation aux statistiques de base étudiées dans chaque modèle nous permet d'introduire des

statistiques ayant la même limite (l'intégrale du carré du processus de Wiener). Cette dernière est "distribution free" vu qu'elle ne dépend ni du modèle ni du paramètre inconnu.

En effet, nous construisons des tests d'ajustement ADF en se basant sur la transformation linéaire spéciale  $L[\cdot]$ . Cette méthode a été appliquée pour les processus de diffusion avec "petit bruit" et ergodique et pour le processus de Poisson non homogène observés en temps continu. La construction de  $L[\cdot]$  est basée sur la solution de l'équation de Fredholm du second type avec le noyau dégénéré (les étapes de cette construction sont dans Section 1.4). Pour ces trois modèles, nous montrons que les limites des statistiques de base correspondantes admettent la même représentation

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, r) dW(r) \int_0^\nu h(\vartheta, r) dr, \quad \int_0^1 g(\vartheta, r)^2 dr = 1, \quad (8)$$

où  $W(\nu), 0 \leq \nu \leq 1$  est un processus de Wiener et  $h(\cdot, \cdot)$  et  $g(\cdot, \cdot)$  sont des fonctions qui seront définies dans la suite. Ainsi, le problème se réduit à appliquer la transformation linéaire  $L[\cdot]$  au processus limite  $U(\cdot)$  telle que  $L[U](\nu) = w_\nu$ , où  $w_\nu, 0 \leq \nu \leq 1$  est un processus de Wiener. Par conséquent, l'application de cette transformation linéaire aux statistiques de base initiale nous permet de construire des tests d'ajustement ADF pour chaque modèle étudié.

Maintenant, nous considérons les trois modèles des processus stochastiques observés en temps continu : diffusion avec "petit bruit", diffusion ergodique et le processus de Poisson non homogène. Sous l'hypothèse de base  $\mathcal{H}_0$  (paramétrique), nous avons les processus suivants :

- **Processus de diffusion avec "petit bruit"**. Nous considérons le processus  $X^\varepsilon = (X_t, 0 \leq t \leq T)$ , avec

$$dX_t = S(\vartheta, X_t) dt + \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T,$$

où  $W_t$  est un processus de Wiener,  $S(\vartheta, x)$  est une fonction régulière et connue et  $\varepsilon \rightarrow 0$ .

- **Processus de Poisson non homogène**. Nous considérons  $X^n = (X_t, 0 \leq t \leq T = n\tau)$  le processus de Poisson non homogène et  $\tau$ -périodique, où

$\tau > 0$  est une période connue,  $n \rightarrow \infty$  et on a

$$\mathbf{E}_\vartheta X_t = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) \, ds.$$

La fonction d'intensité  $\lambda(\vartheta, s)$  est régulière, périodique et connue.

- **Processus de diffusion ergodique.** Nous considérons le processus  $X^T = (X_t, 0 \leq t \leq T)$ , avec

$$dX_t = S(\vartheta, X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

où  $S(\vartheta, x)$  et  $\sigma(x)$  sont deux fonctions régulières et connues et  $T \rightarrow \infty$ .

Le paramètre inconnu  $\vartheta \in \Theta = (a, b)$  est unidimensionnel et nous proposons l'EDM pour ce paramètre pour tous les processus définis précédemment. Le but principal est la construction des tests d'ajustement ADF de niveau asymptotique  $\alpha \in (0, 1)$  pour chaque modèle d'observations. Ces tests sont construits en deux étapes. En premier lieu, nous proposons des premières transformations pour les limites des statistiques de base respectivement pour chaque modèle. Ainsi, nous obtenons le processus  $U(\cdot)$  admettant la même représentation (8) pour les trois processus étudiés. En second lieu, nous montrons que les statistiques considérées ont la même limite :

- **Processus de diffusion avec “petit bruit”** lorsque  $\varepsilon \rightarrow 0$

$$\tilde{\Delta}_\varepsilon \Longrightarrow \int_0^1 U(\nu)^2 \, d\nu,$$

- **Processus de Poisson non homogène** lorsque  $T = n\tau \rightarrow \infty$

$$\Delta_n \Longrightarrow \int_0^1 U(\nu)^2 \, d\nu, \quad \delta_n \Longrightarrow \int_0^1 U(\nu)^2 \, d\nu,$$

- **Processus de diffusion ergodique** lorsque  $T \rightarrow \infty$

$$\Delta_T^* \Longrightarrow \int_0^1 U(\nu)^2 \, d\nu.$$

Nous rappelons que les fonctions  $h(\cdot, \cdot)$  et  $g(\cdot, \cdot)$  sont différentes pour chaque modèle étudié. La deuxième étape qui est générale pour les trois modèles s'agit d'appliquer une deuxième transformation linéaire  $L[\cdot]$  (1.26) au processus limite  $U(\cdot)$  et obtenir un processus de Wiener tel que

$$\int_0^1 L[U](\nu)^2 \, d\nu = \int_0^1 w_\nu^2 \, d\nu. \quad (9)$$

Cette propriété nous permet d'introduire les "versions empiriques" des statistiques admettant la même limite (9). Par conséquent, les tests d'ajustement basés sur ces statistiques sont ADF.

Nous abordons ensuite en détails le contenu des trois chapitres de cette thèse.

Dans Chapitre 1, nous traitons deux problèmes. Le premier est la construction d'une transformation linéaire spéciale. Tous les étapes de cette construction sont développées dans Section 1.4. Nous considérons ensuite le deuxième problème dans Section 1.5 : la construction de test d'ajustement dans le cas du processus de diffusion avec "petit bruit" observé en temps continu. Nous proposons un test basé sur la transformation linéaire définie dans Théorème 1.4.1 et nous montrons qu'il est ADF.

Supposons que nous observons un processus  $X^\varepsilon = (X_t, 0 \leq t \leq T)$  en temps continu, solution de l'équation différentielle stochastique suivante :

$$dX_t = S(X_t) dt + \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (10)$$

où  $W_t$ ,  $0 \leq t \leq T$  est un processus de Wiener,  $S(x)$  est une fonction inconnue et régulière et la valeur initiale  $x_0$  est déterministe. Nous construisons le test d'ajustement ADF de l'hypothèse nulle paramétrique

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

où  $\vartheta$  est un paramètre inconnu et unidimensionnel. Dans Section 1.2, nous proposons l'EDM pour ce paramètre

$$\vartheta_\varepsilon^* = \arg \min_{\vartheta \in \Theta} \|X - x(\vartheta)\|,$$

où  $\|\cdot\|$  est la norme  $L^2[0, T]$  définie par

$$\|X - x(\vartheta)\|^2 = \int_0^T (X_t - x_t(\vartheta))^2 dt.$$

Puis, nous étudions la statistique de base dans Section 1.3

$$u_\varepsilon(t) = \frac{X_t - x_t(\vartheta_\varepsilon^*)}{\varepsilon S(\vartheta_\varepsilon^*, X_t)}, \quad 0 \leq t \leq T,$$

où  $x_t(\vartheta)$  est la solution de l'équation (10) lorsque  $\varepsilon = 0$ . Notons que cette statistique a été proposée par Kutoyants [42] dans le cas de l'EMV. Nous montrons

qu'elle converge en loi sous  $\mathcal{H}_0$  et sous certaines conditions de régularités vers le processus  $u(t), 0 \leq t \leq T$  défini par (1.15). Ensuite, la transformation (1.16) de  $u(t)$  vers  $U\left(\frac{t}{T}\right), 0 \leq t \leq T$  nous donne la représentation (8). Nous construisons alors la transformation linéaire  $L[\cdot]$  dans Théorème 1.4.1 et nous prouvons que  $L[U](\nu) = w_\nu$ , où  $w_\nu, 0 \leq \nu \leq 1$  est un processus de Wiener.

Cela nous permet de construire un test d'ajustement ADF dans Théorème 1.5.2. En effet, nous obtenons la convergence suivante :

$$\Delta_\varepsilon = \frac{1}{T} \int_0^T L[U_\varepsilon](t)^2 dt \implies \int_0^1 w_\nu^2 d\nu,$$

où  $U_\varepsilon(\cdot)$  est le processus défini par (1.45) dans Section 1.5. Par conséquent, nous construisons le test ADF  $\psi_\varepsilon = \mathbb{I}_{\{\Delta_\varepsilon > c_\alpha\}}$  vu que la distribution limite ne dépend ni de  $S(\cdot)$  ni de  $\vartheta$ . Nous définissons  $c_\alpha$  comme le  $(1 - \alpha)$ -quantile de la statistique limite, c'est à dire, il est solution de l'équation suivante :

$$\mathbf{P} \left( \int_0^1 w_\nu^2 d\nu > c_\alpha \right) = \alpha, \quad \alpha \in (0, 1).$$

Nous mentionnons que le test  $\psi_\varepsilon$  est de niveau asymptotique  $\alpha$ , c'est à dire, nous avons lorsque  $\varepsilon \rightarrow 0$

$$\mathbf{E}_\vartheta \psi_\varepsilon = \alpha + o(1), \quad \text{pour tous } \vartheta \in \Theta,$$

où  $\mathbf{E}_\vartheta$  est l'espérance mathématique sous l'hypothèse nulle  $\mathcal{H}_0$ .

Finalement, dans Section 1.6, nous traitons le cas d'utilisation de l'EMV pour le paramètre inconnu. En effet, la représentation du processus limite est comme suit :

$$U(\nu) = W(\nu) - \int_0^\nu h(\vartheta, s) dW(s) \int_0^\nu h(\vartheta, s) ds, \quad \int_0^1 h(\vartheta, s)^2 ds = 1$$

et notre transformation (1.26) coïncide avec celle proposée dans [30].

Dans Chapitre 2, nous sommes intéressés par la construction de deux tests d'ajustement ADF basés sur les statistiques de type Cramér-von Mises et de "score-function". Cette construction est introduite en utilisant la transformation linéaire  $L[\cdot]$  définie par (1.26). En effet, nous observons un processus de Poisson non homogène périodique  $X^n = (X_t, 0 \leq t \leq T = n\tau)$  avec une période connue



$\tau > 0$ . La moyenne  $\Lambda(t)$  et la fonction d'intensité  $\lambda(t)$  vérifient les relations suivantes :

$$\mathbf{E}_{\vartheta} X_t = \Lambda(\vartheta, t), \quad \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) \, ds$$

sous l'hypothèse nulle paramétrique

$$\mathcal{H}_0 \quad : \quad \Lambda(\cdot) \in \{\Lambda(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

où  $a > 0$  et  $b < \infty$ . Nous introduisons dans Section 2.2 l'EDM

$$\vartheta_n^* = \arg \min_{\vartheta \in \Theta} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|,$$

où  $\hat{\Lambda}_n(\cdot)$  est la fonction de moyenne empirique définie par (2.2) et  $\|\cdot\|$  est la norme  $\mathcal{L}^2(0, \tau)$ , c'est à dire,

$$\left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|^2 = \int_0^\tau \left( \hat{\Lambda}_n(s) - \Lambda(\vartheta, s) \right)^2 \, ds.$$

Dans Section 2.3, nous considérons la statistique de type Cramér-von Mises

$$\Delta_n = \frac{n}{\Lambda(\vartheta_n^*, \tau)^2} \int_0^\tau \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_n^*, r) \right)^2 \lambda(\vartheta_n^*, r) \, dr,$$

Notons que dans cette section nous obtenons (sous l'hypothèse  $\mathcal{H}_0$ ) la convergence suivante :

$$\Delta_n \implies \int_0^1 U(\nu)^2 \, d\nu,$$

où le processus  $U(\cdot)$  est défini par (8). Ensuite, dans Section 2.4, nous construisons un autre test ADF pour le même modèle d'observations en se basant sur la statistique de “score-function”

$$u_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} [dX_j(v) - \lambda(\vartheta_n^*, v) \, dv].$$

Nous mentionnons que cette statistique a été introduite par Kutoyants [43]. Ainsi, nous montrons la convergence de  $u_n^*(\cdot)$  vers le processus  $u(\cdot)$  défini par (2.38). Par ailleurs, la transformation (2.39) nous donne la représentation (8) selon Lemme 2.4.5. Nous remarquons que dans les deux cas, c'est à dire, les statistiques de type Cramér-von Mises et “score-function”, nous obtenons le même processus limite (8) avec deux fonctions  $h(\cdot, \cdot)$  et  $g(\cdot, \cdot)$  différentes. Nous passons alors à

l'application de la transformation linéaire  $L[\cdot]$  au processus  $U(\cdot)$  (voir Théorème 1.4.1) tel que :

$$L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1.$$

La présentation ci-dessus suggère la construction des tests d'ajustement ADF en utilisant les “versions empiriques”  $U_n(\cdot, \cdot)$  et  $U_n^*(\cdot)$  de  $U(\cdot)$  (ces deux processus sont définis respectivement par (2.12) et (2.43)) comme suit : Nous introduisons les statistiques correspondantes aux statistiques de type Cramér-von Mises et “score-function”

$$\hat{\Delta}_n^* = \int_0^\tau \frac{L[U_n](r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) dr, \quad \hat{\delta}_n = \int_0^1 L[U_n^*](\nu)^2 d\nu.$$

Ensuite, nous montrons la convergence pour ces deux statistiques (sous l'hypothèse  $\mathcal{H}_0$ )

$$\hat{\Delta}_n^* \Longrightarrow \int_0^1 w_\nu^2 d\nu, \quad \hat{\delta}_n \Longrightarrow \int_0^1 w_\nu^2 d\nu$$

et par conséquent, les tests

$$\hat{\psi}_n^* = \mathbb{I}_{\{\hat{\Delta}_n^* > c_\alpha\}}, \quad \hat{\varphi}_n = \mathbb{I}_{\{\hat{\delta}_n > c_\alpha\}}, \quad \mathbf{P} \left( \int_0^1 w_\nu^2 d\nu > c_\alpha \right) = \alpha$$

sont ADF et de niveau asymptotique  $\alpha \in (0, 1)$ .

Le Chapitre 3 est consacré à la construction d'un test d'ajustement ADF dans le cas du processus de diffusion ergodique  $X^T = \{X_t, 0 \leq t \leq T\}$  observé en temps continu et solution de l'équation différentielle stochastique suivante :

$$dX_t = S(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

Soulignons que  $W_t$ ,  $0 \leq t \leq T$  est un processus de Wiener,  $S(x)$  est une fonction inconnue, le coefficient de diffusion  $\sigma(x)^2 > 0$  est une fonction connue et  $X_0$  est la valeur initiale de  $X_t$ . Supposons que l'hypothèse de base est paramétrique

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

c'est à dire, le processus  $X^T$  est solution de l'équation suivante :

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

où  $S(\vartheta, x)$  est une fonction régulière, connue et dépendante de  $\vartheta$ .

Nous proposons l'EDM  $\vartheta_T^*$  pour le paramètre inconnu  $\vartheta$  dans Section 3.3 :

$$\vartheta_T^* = \arg \inf_{\vartheta \in \Theta} \int_{\mathbb{R}} \left[ \hat{f}_T(x) - f(\vartheta, x) \right]^2 dx.$$

Notons que  $\hat{f}_T(x)$  est l'estimateur de temps local de la densité invariante  $f(\vartheta, x)$  (définis par (3.9) et (3.8), respectivement). Dans Section 3.4, nous considérons la statistique de type Cramér-von Mises

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x),$$

avec

$$\begin{aligned} \eta_T^*(\vartheta_T^*, x) &= \sqrt{T} \left( \hat{f}_T(x) - f(\vartheta_T^*, x) \right) \\ &\implies \eta(\vartheta, x) = 2 f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} dW(y) \\ &\quad - 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x). \end{aligned}$$

Une première transformation linéaire a été appliquée (voir Kutoyants [41])

$$L_1[\eta](x) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] = U(F(\vartheta, x))$$

pour le processus limite  $\eta(\cdot, \cdot)$ . Ainsi, nous obtenons dans Lemme 3.4.6 la représentation suivante :

$$U(t) = W(t) - \int_0^1 g(\vartheta, s) dW(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1,$$

où  $W(t), 0 \leq t \leq 1$  est un processus de Wiener. L'étape suivante s'agit d'introduire une seconde transformation linéaire  $L[\cdot]$  (1.26) comme dans les sections précédentes et obtenir l'égalité suivante :

$$L[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Par conséquent, nous avons la convergence

$$\delta_T^* = \int_{-\infty}^{\infty} L[\xi_T](x)^2 dF(\vartheta_T^*, x) \implies \int_0^1 w_t^2 dt,$$

où  $\xi_T(\cdot, \cdot)$  est le processus défini par (3.22) qui est asymptotiquement équivalent à  $L_1[\eta_T^*](\cdot)$ . Donc, le test  $\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}$  est de niveau asymptotique  $\alpha \in (0, 1)$

et ADF vu que la distribution limite de  $\delta_T^*$  ne dépend ni des fonctions  $S(\cdot, \cdot)$  et  $\sigma(\cdot)$  ni du paramètre inconnu  $\vartheta$ .

Dans les trois chapitres, nous avons considéré le problème de construction de quatre tests d'ajustement ADF de niveau asymptotique  $\alpha \in (0, 1)$ . Cette construction a été basée sur la transformation linéaire spéciale  $L[\cdot]$  obtenue à l'aide de la solution de l'équation de Fredholm du second type avec le noyau dégénéré. En effet, nous avons introduit les statistiques de base pour les processus de diffusion avec "petit bruit" et ergodique et le processus de Poisson non homogène. Après les premières transformations, nous avons montré que les limites de ces statistiques admettent la même représentation  $U(\cdot)$  (8). Nous pouvons donc dire que cette représentation est universelle. Nous avons prouvé aussi que  $L[U](\nu) = w_\nu$ , où  $w_\nu$  est un processus de Wiener. Par conséquent, nous en déduisons que l'application de  $L[\cdot]$  aux statistiques de base nous donne le processus de Wiener  $w_\nu$  comme une limite "distribution free". Ceci nous a permis de construire les tests d'ajustement ADF et de niveau asymptotique  $\alpha \in (0, 1)$ .

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# On GoF Test for Perturbed Dynamical Systems

## 1.1 Introduction

We consider the problem of the construction of an asymptotically distribution free test in the case of continuous time observations of a “small noise” diffusion process. Under the null hypothesis, we suppose that the trend coefficient depends on some unknown one-dimensional parameter. Therefore the basic (null) hypothesis is parametric.

Goodness-of-fit tests play an important role in theoretical and applied statistics. They allow to verify the correspondence between the proposed theoretical models and real data. In fact, these tests are widely used as mathematic models in many fields and studied very intensively in statistics since 1928 due to the works by Cramér (1928), von Mises (1931), Kolmogorov (1933) and Smirnov (1936) (see, e.g., [13] and [18]). Let us recall the well-known basic results for the i.i.d. model. Suppose that the null hypothesis is simple. Denoting the continuous distribution function under the null hypothesis by  $F_0(x)$ , we have to check if the i.i.d. observations  $X^n = (X_1, \dots, X_n)$  have this continuous distribution function. Many GoF tests are based on the following property : under the null hypothesis, the normalized empirical distribution function

$$\sqrt{n} \left( \hat{F}_n(x) - F_0(x) \right), \quad \hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}_{\{X_j < x\}}$$

converges in distribution to the Brownian bridge  $B(F_0(x))$ . In particular, for the Cramér-von Mises statistic we have (with the change of variable  $s = F_0(x)$ )

$$\phi_n = n \int_{-\infty}^{\infty} \left( \hat{F}_n(x) - F_0(x) \right)^2 dF_0(x) \implies \int_0^1 B(s)^2 ds \equiv \phi.$$

The limit  $\phi$  does not depend on  $F_0(\cdot)$  (distribution free). Therefore the Cramér-von Mises test

$$\Psi_n = \mathbb{I}_{\{\phi_n > c_\alpha\}}, \quad \mathbf{P} \{ \phi > c_\alpha \} = \alpha$$

is ADF and has the asymptotic ( $n \rightarrow \infty$ ) size  $\alpha \in (0, 1)$  (see, e.g., [46]).

In the case of the parametric null hypothesis

$$\mathcal{H}_0 : F_0(\cdot) \in \{F_0(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where  $\vartheta$  is some unknown one-dimensional parameter, the similar limit

$$U_n(x) = \sqrt{n} \left( \hat{F}_n(x) - F_0(\hat{\vartheta}_n, x) \right) \implies B(F_0(\vartheta, x)) - \zeta \dot{F}_0(\vartheta, x) \equiv u(x)$$

is no more distribution free and the choice of the threshold  $c_\alpha$  is much more complicated. Here  $\hat{\vartheta}_n$  is the maximum likelihood estimator (MLE) of the unknown parameter  $\vartheta$ ,  $\dot{F}_0(\vartheta, \cdot)$  is the derivative of  $F_0(\vartheta, \cdot)$  w.r.t.  $\vartheta$  and  $\zeta$  is a Gaussian variable. By the change of variables  $t = F_0(\vartheta, x)$  and  $s = F_0(\vartheta, y)$ , we have for  $u(\cdot)$  the representation (see, e.g., Darling [13])

$$U(t) = B(t) - \int_0^1 h(\vartheta, s) dB(s) \int_0^t h(\vartheta, s) ds. \quad (1.1)$$

Note that the function  $h(\cdot, \cdot)$  appears here twice because we used the MLE.

For the minimum distance estimator (MDE) using the same arguments as in the case of the MLE we obtain the limit

$$U(t) = B(t) - \int_0^1 g(\vartheta, s) dB(s) \int_0^t h(\vartheta, s) ds, \quad \int_0^1 g(\vartheta, s)^2 ds = 1 \quad (1.2)$$

with two different functions. In both cases, the corresponding tests are no more ADF and the choice of the threshold from the equation

$$\mathbf{P} \left\{ \int_0^1 U(s)^2 ds > c_\alpha \right\} = \alpha$$

can be a difficult problem.

There are several possibilities to solve this problem in the case of the limit (1.1). One of them is to find a linear transformation  $L[\cdot]$  of the random function  $U(\cdot)$ , such that  $L[U](t) = W(t)$ , where  $W(t), 0 \leq t \leq 1$  is some Wiener process. Then the statistic

$$\hat{v}_n = \int_{-\infty}^{\infty} L[U_n](x)^2 dF_0(\hat{\vartheta}_n, x) \implies \int_0^1 W(t)^2 dt \equiv \hat{v}$$

and we obtain once more the distribution free limit  $\hat{v}$ . Therefore the test

$$\hat{\Phi}_n = \mathbb{I}_{\{\hat{v}_n > d_\alpha\}}, \quad \mathbf{P}\{\hat{v} > d_\alpha\} = \alpha$$

is ADF. Such transformation was proposed by Khmaladze [30]. We have to emphasize that in [30] and in many other works (see, e.g., the paper Maglapheridze et al. [49]) the estimator used was always the MLE and this is important for the construction of this linear transformation. Many authors wrote that similar transformation can be obtained in the case of other estimators with limit (1.2), but as we know this work (construction of the linear transformation with other estimators) was not done.

The problem of GoF testing for the model of continuous time observations of “small noise” diffusion process, with a simple null hypothesis  $\Theta = \{\vartheta_0\}$ , was studied in [12] and [40]. Suppose that the observed diffusion process under hypothesis is

$$dX_t = S_0(X_t) dt + \varepsilon \sigma(X_t) dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (1.3)$$

with deterministic initial value  $x_0$ , known diffusion coefficient  $\varepsilon^2 \sigma(\cdot)^2 > 0$ , known smooth function  $S_0(\cdot)$  and  $W_t, 0 \leq t \leq 1$  is a Wiener process. Note that if the functions  $S_0(x)$  and  $\sigma(x)$  are Lipschitz w.r.t.  $x$ , then we have with probability 1

$$\sup_{0 \leq t \leq T} |X_t - x_t| \leq C\varepsilon \sup_{0 \leq t \leq T} |W_t|,$$

where  $C$  is some constant. Therefore the process  $X_t$  converges uniformly w.r.t.  $t \in [0, T]$  with probability 1 to  $x_t = x_t(\vartheta_0)$  (solution of the equation (1.3) as  $\varepsilon = 0$ ). We have as well

$$\sup_{0 \leq t \leq T} \mathbf{E} |X_t - x_t|^2 \leq C\varepsilon^2.$$

For the proof see [37], Lemma 1.13.

The GoF test was constructed on the basis of the normalized difference  $\varepsilon^{-1}(X_t - x_t)$  and the limit of this statistic is a Gaussian process. The latter can be transformed into a Wiener process  $w_s, 0 \leq s \leq 1$  as follows : Introduce the statistic

$$\delta_\varepsilon = \left[ \int_0^T \left( \frac{\sigma(x_t)}{S_0(x_t)} \right)^2 dt \right]^{-2} \int_0^T \left( \frac{X_t - x_t}{\varepsilon S_0(x_t)^2} \right)^2 \sigma(x_t)^2 dt.$$

The following convergence

$$\delta_\varepsilon \Longrightarrow \hat{\Delta} \equiv \int_0^1 w_s^2 ds$$

was proved and therefore the test  $\phi_\varepsilon = \mathbb{1}_{\{\delta_\varepsilon > c_\alpha\}}$  with  $\mathbf{P}(\hat{\Delta} > c_\alpha) = \alpha$  is ADF.

The case of the ADF GoF tests for “small noise” diffusion processes with parametric basic hypothesis was studied, for example, in [26], [56] and [42]– [43]. For the construction of the linear transformation and tests the estimator used was always the MLE (see Kutoyants [36]). Moreover, there are several GoF tests for the ergodic diffusion processes proposed, for example, in the works [51], [54] and [34].

Note that in some problems it is not possible to have an explicit expression for the MLE and therefore sometimes it is better to use other estimators, which can be easily calculated. For example, this can be the MDE, a method of moments estimator or a trajectory fitting estimator and so on. In all such cases, the limit expression for the underlying statistics will be like (1.2) but with two different functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  (see below).

In this Chapter, we are interested by two problems. We first deal with the problem of the construction of a special linear transformation. All steps of this construction are developed in Section 1.4. Then we consider the second problem in Section 1.5 : the construction of the ADF GoF test based on this linear transformation in the case of the diffusion process with “small noise” observed in continuous time. The main results are presented in Theorem 1.4.1 and Theorem 1.5.2, respectively. In particular, the linear transformation is defined in Theorem 1.4.1 and the ADF test is given below in Theorem 1.5.2.



In this work, we observe a continuous time process  $X^\varepsilon = (X_t, 0 \leq t \leq T)$ , which is the solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (1.4)$$

where  $W_t$ ,  $0 \leq t \leq T$  is a Wiener process,  $S(x)$  is some unknown smooth function and  $x_0$  is the deterministic initial value. We construct an ADF GoF test for the parametric null hypothesis

$$\mathcal{H}_0 : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where  $a > 0$ ,  $b < \infty$  and  $\vartheta$  is an unknown and one-dimensional parameter and we propose the MDE for this parameter

$$\vartheta_\varepsilon^* = \arg \min_{\vartheta \in \Theta} \|X - x(\vartheta)\|,$$

where  $\|\cdot\|$  is  $L^2[0, T]$  norm defined by

$$\|X - x(\vartheta)\|^2 = \int_0^T (X_t - x_t(\vartheta))^2 dt.$$

Here  $x_t = x_t(\vartheta)$  is the solution of the equation (1.4) as  $\varepsilon = 0$ .

We realize the following program. First, we show that the basic statistic

$$u_\varepsilon(t) = \frac{X_t - x_t(\vartheta_\varepsilon^*)}{\varepsilon S(\vartheta_\varepsilon^*, X_t)}, \quad 0 \leq t \leq T$$

converges to the random process  $u(t)$ ,  $0 \leq t \leq T$  (see (1.15) below). Note that this statistic was proposed by Kutoyants [42] in the case of the MLE. Then we transform  $u(t)$  in  $U\left(\frac{t}{T}\right)$ ,  $0 \leq t \leq T$  (see (1.16) below). Therefore, we obtain for  $U(\cdot)$  the following representation

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, r) dW(r) \int_0^\nu h(\vartheta, r) dr, \quad \int_0^1 g(\vartheta, r)^2 dr = 1. \quad (1.5)$$

We will give later the definitions of the functions  $g(\cdot, \cdot)$  (1.20) and  $h(\cdot, \cdot)$  (1.19).

The last step is to apply the special linear transformation  $L[\cdot]$  such that  $L[U](\nu) = w_\nu$ ,  $0 \leq \nu \leq 1$ , where  $w_\nu$  is a Wiener process (see Theorem 1.4.1).

The main contribution of this work is the form of this linear transformation.

Then we realize similar transformation (1.44) with the “empirical” process  $u_\varepsilon(\cdot)$ , apply the linear transformation  $L[\cdot]$  to  $U_\varepsilon(\cdot)$  given by (1.45) and obtain the convergence (under  $\mathcal{H}_0$ )

$$L[U_\varepsilon] \longrightarrow L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1.$$

Moreover, we show the convergence

$$\Delta_\varepsilon = \frac{1}{T} \int_0^T L[U_\varepsilon](t)^2 dt \implies \int_0^1 w_\nu^2 d\nu.$$

The limit distribution of  $\Delta_\varepsilon$  does not depend on  $S(\cdot, \cdot)$  and  $\vartheta$ . This allows us to construct the ADF test  $\psi_\varepsilon = \mathbb{I}_{\{\Delta_\varepsilon > c_\alpha\}}$  in Theorem 1.5.2. The threshold  $c_\alpha$  is solution of the equation

$$\mathbf{P} \left\{ \int_0^1 w_\nu^2 d\nu > c_\alpha \right\} = \alpha, \quad \alpha \in (0, 1).$$

Note that the test  $\psi_\varepsilon$  is of asymptotic size  $\alpha$ , i.e., we have as  $\varepsilon \rightarrow 0$

$$\mathbf{E}_\vartheta \psi_\varepsilon = \alpha + o(1), \quad \text{for all } \vartheta \in \Theta,$$

where  $\mathbf{E}_\vartheta$  is the mathematical expectation under the parametric hypothesis  $\mathcal{H}_0$ .

We shall mention that if in our problem we use the MLE of the unknown parameter, then the limit representation is

$$U(\nu) = W(\nu) - \int_0^\nu h(\vartheta, s) dW(s) \int_0^\nu h(\vartheta, s) ds, \quad \int_0^1 h(\vartheta, s)^2 ds = 1 \quad (1.6)$$

and our transformation coincides with the one proposed in [30]. We discuss this case in the Section 1.6.

We have to note that this linear transformation is rather cumbersome and the realization of the test can be a computationally difficult problem too. We suppose that the presented result is of theoretical interest and allows to “close the gap” in this field.

At the same time, we understand that this result is in some sense “negative” and says that if we have no MLE it is better to seek another GoF test, which is ADF. Note as well that in i.i.d. case, even if the estimated parameter is one-dimensional, the reduction of the equation with Brownian bridge (1.2) to the

equation (1.5), using the relation  $(B(t) = W(t) - t W(1))$ , leads to the corresponding Fredholm equation (see (1.28) below) with three-dimensional  $g(\cdot, \cdot)$  and  $h(\cdot, \cdot)$ . The expression for the solution of this equation and the form of the linear transformation become much more complicated. This is probably the reason why this problem was not considered till now. Our results thus should be understood as a constructive existence result for ADF tests based on the MDE.

## 1.2 Minimum distance estimator

Suppose that the continuous time observed process  $X^\varepsilon = (X_t, 0 \leq t \leq T)$  is the solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T, \quad (1.7)$$

where  $W_t$ ,  $0 \leq t \leq T$  is a Wiener process, the initial value  $x_0$  is deterministic and the trend coefficient  $S(x)$  is some unknown smooth function.

We consider the composite basic hypothesis

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where  $\vartheta$  is the one-dimensional (unknown) parameter, against alternative  $\mathcal{H}_1$  : not  $\mathcal{H}_0$ , i.e., the trend coefficient  $S(x)$  in the observed diffusion process (1.7) does not belong to the parametric family  $\{S(\vartheta, x), \vartheta \in \Theta\}$ . Therefore the process has the stochastic differential, under hypothesis  $\mathcal{H}_0$ ,

$$dX_t = S(\vartheta, X_t) dt + \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T. \quad (1.8)$$

Here  $S(\vartheta, x)$  is known smooth function.

We are interested in the properties of the test in the asymptotics of *small noise*, i.e., as  $\varepsilon \rightarrow 0$ . Below and in the sequel the dot means derivation w.r.t.  $\vartheta$ .

Let us introduce the regularity conditions.

$\mathcal{R}$ . *The function  $S(\vartheta, x)$  is strictly positive and has two continuous bounded derivatives with respect to  $\vartheta$  and  $x$ .*

In the presentation below we suppose that these conditions and the basic hypothesis  $\mathcal{H}_0$  are always fulfilled.

It is known that the solution  $X_t$  converges uniformly in  $t \in [0, T]$  to the solution  $x_t = x_t(\vartheta)$  of the ordinary differential equation

$$\frac{dx_t}{dt} = S(\vartheta, x_t), \quad x_0, \quad 0 \leq t \leq T,$$

where  $x_0$  is the same as in (1.7) (see [22] or [37], Lemma 1.13 for the proof).

Recall the properties of the maximum likelihood and minimum distance estimators of the parameter  $\vartheta$ . The likelihood ratio function in the case of observations (1.8) is

$$\ell(\vartheta, X^\varepsilon) = \frac{dP_\vartheta}{dP_0} = \exp \left\{ \int_0^T \frac{S(\vartheta, X_t)}{\varepsilon^2} dX_t - \int_0^T \frac{S(\vartheta, X_t)^2}{2\varepsilon^2} dt \right\}, \quad \vartheta \in \Theta,$$

where  $P_\vartheta$  and  $P_0$  are the measures induced respectively by the processes (1.8) and

$$dX_t = \varepsilon dW_t, \quad X_0 = x_0, \quad 0 \leq t \leq T,$$

(see [48] for more details).

The MLE  $\hat{\vartheta}_\varepsilon$  is solution of the equation

$$\sup_{\vartheta \in \Theta} \ell(\vartheta, X^\varepsilon) = \ell(\hat{\vartheta}_\varepsilon, X^\varepsilon).$$

This estimator is consistent and asymptotically normal (as  $\varepsilon \rightarrow 0$ )

$$\varepsilon^{-1}(\hat{\vartheta}_\varepsilon - \vartheta) \implies \mathcal{N}(0, I(\vartheta)^{-1}), \quad I(\vartheta) = \int_0^T \dot{S}(\vartheta, x_t)^2 dt.$$

Here and in the sequel  $x_t = x_t(\vartheta)$ . For the proof see Kutoyants [37].

Introduce the MDE

$$\vartheta_\varepsilon^* = \arg \min_{\vartheta \in \Theta} \|X - x(\vartheta)\|,$$

where  $\|\cdot\|$  is  $L^2[0, T]$  norm defined by

$$\|X - x(\vartheta)\|^2 = \int_0^T (X_t - x_t(\vartheta))^2 dt.$$

The properties of the MDE for this model were studied in [37]. The MDE satisfies the *minimum distance equation* (MDEq)

$$\int_0^T (X_t - x_t(\vartheta_\varepsilon^*)) \dot{x}_t(\vartheta_\varepsilon^*) dt = 0.$$

Let us put  $u_\varepsilon^* = \varepsilon^{-1} (\vartheta_\varepsilon^* - \vartheta)$ . Therefore the MDEq is as follows :

$$\int_0^T (X_t - x_t(\vartheta + \varepsilon u_\varepsilon^*)) \dot{x}_t(\vartheta_\varepsilon^*) dt = 0.$$

Then, by Taylor's formula, we can write

$$\int_0^T \left( X_t - x_t(\vartheta) - \varepsilon u_\varepsilon^* \dot{x}_t(\tilde{\vartheta}) \right) \dot{x}_t(\vartheta_\varepsilon^*) dt = 0,$$

where  $|\tilde{\vartheta} - \vartheta| \leq |\vartheta_\varepsilon^* - \vartheta|$  and for  $u_\varepsilon^*$  we obtain the following representation

$$u_\varepsilon^* = J(\vartheta)^{-1} \int_0^T \varepsilon^{-1} (X_t - x_t(\vartheta)) \dot{x}_t(\vartheta) dt + o(1), \quad (1.9)$$

where

$$J(\vartheta) = \int_0^T \dot{x}_t(\vartheta)^2 dt. \quad (1.10)$$

Let us consider the random process  $x_t^{(1)}$  defined as the derivative of  $X_t$  w.r.t.  $\varepsilon$  at  $\varepsilon = 0$ , i.e., we have  $\varepsilon^{-1} (X_t - x_t(\vartheta)) \rightarrow x_t^{(1)}$ . Here the random process  $x_t^{(1)}$  satisfies the linear equation

$$dx_t^{(1)} = S'(\vartheta, x_t) x_t^{(1)} dt + dW_t, \quad x_0^{(1)} = 0, \quad 0 \leq t \leq T.$$

Here and below the prime designates differentiation w.r.t. the variable  $x$ .

The process  $x_t^{(1)}$  has the representation

$$x_t^{(1)} = S(\vartheta, x_t) \int_0^t \frac{1}{S(\vartheta, x_s)} dW_s \quad (1.11)$$

(see, e.g., Section 3.3 in [37]). Due to the above representation of  $x_t^{(1)}$ , the equation (1.9) becomes

$$\begin{aligned} u_\varepsilon^* &= J(\vartheta)^{-1} \int_0^T x_t^{(1)} \dot{x}_t(\vartheta) dt + o(1) \\ &= J(\vartheta)^{-1} \int_0^T S(\vartheta, x_t) \int_0^t \frac{1}{S(\vartheta, x_s)} dW_s \dot{x}_t(\vartheta) dt + o(1). \end{aligned}$$

Here  $o(1)$  means the convergence in probability, i.e., for any  $\nu > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_\vartheta (|u_\varepsilon^* - \rho(\vartheta)| > \nu) = 0, \quad (1.12)$$

where

$$\rho(\vartheta) = J(\vartheta)^{-1} \int_0^T S(\vartheta, x_t) \int_0^t \frac{1}{S(\vartheta, x_s)} dW_s \dot{x}_t(\vartheta) dt.$$

Therefore, by Fubini's theorem,  $\vartheta_\varepsilon^*$  admits the following representation,

$$\varepsilon^{-1} (\vartheta_\varepsilon^* - \vartheta) = J(\vartheta)^{-1} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v + o(1). \quad (1.13)$$

Moreover, under conditions of regularity, the estimator  $\vartheta_\varepsilon^*$  is consistent and asymptotically normal (see Chapter 7 in [37])

$$\mathcal{L}_\vartheta\{\varepsilon^{-1}(\vartheta_\varepsilon^* - \vartheta)\} \implies \mathcal{L}\{\xi\} = \mathcal{N}(0, \sigma^2(\vartheta)),$$

where

$$\sigma^2(\vartheta) = J(\vartheta)^{-2} \int_0^T \frac{1}{S(\vartheta, x_v)^2} \left( \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds \right)^2 dv.$$

### 1.3 Basic statistic

Our goal is to find the GoF test, which is ADF, i.e., we seek the test statistic whose limit distribution, under hypothesis, does not depend on the underlying model given by the function  $S(\vartheta, x)$  and parameter  $\vartheta$ .

Introduce the statistic

$$\delta_\varepsilon^* = \int_0^T \left[ \frac{X_t - x_t(\vartheta_\varepsilon^*)}{\varepsilon S(\vartheta_\varepsilon^*, X_t)} \right]^2 dt.$$

To study it we need the behavior of the difference  $X_t - x_t(\vartheta_\varepsilon^*)$ , which can be described as follows :

$$\begin{aligned} \varepsilon^{-1} (X_t - x_t(\vartheta_\varepsilon^*)) &= \varepsilon^{-1} (X_t - x_t(\vartheta)) - \varepsilon^{-1} (\vartheta_\varepsilon^* - \vartheta) \dot{x}_t(\vartheta) + o(1) \\ &= x_t^{(1)}(\vartheta) - J(\vartheta)^{-1} \int_0^T x_s^{(1)} \dot{x}_s(\vartheta) ds \dot{x}_t(\vartheta) + o(1) \\ &= S(\vartheta, x_t) \int_0^t \frac{dW_v}{S(\vartheta, x_v)} \\ &\quad - J(\vartheta)^{-1} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v \dot{x}_t(\vartheta) + o(1), \end{aligned}$$

where the process  $x_t^{(1)}$  is defined by (1.11) and the derivative  $\dot{x}_t(\vartheta)$  w.r.t.  $\vartheta$  satisfies the equation

$$\frac{d\dot{x}_t(\vartheta)}{dt} = S'(\vartheta, x_t) \dot{x}_t(\vartheta) + \dot{S}(\vartheta, x_t), \quad \dot{x}_0(\vartheta) = 0.$$

Its solution is the function (it can be found in [42])

$$\dot{x}_t(\vartheta) = S(\vartheta, x_t) \int_0^t \frac{\dot{S}(\vartheta, x_v)}{S(\vartheta, x_v)} dv.$$

Here  $o(1)$  is the uniform convergence in probability w.r.t.  $t \in [0, T]$ , i.e., for any  $\nu > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_{\vartheta} \left( \sup_{t \in [0, T]} |\varepsilon^{-1} (X_t - x_t(\vartheta_{\varepsilon}^*)) - \mu_t(\vartheta)| > \nu \right) = 0, \quad (1.14)$$

where

$$\begin{aligned} \mu_t(\vartheta) &= S(\vartheta, x_t) \int_0^t \frac{dW_v}{S(\vartheta, x_v)} \\ &\quad - J(\vartheta)^{-1} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v \dot{x}_t(\vartheta). \end{aligned}$$

For the details see [37], Chapter 7.

Hence we have the uniform convergence w.r.t.  $t \in [0, T]$  (in probability)

$$\begin{aligned} u_{\varepsilon}(t) &= \frac{X_t - x_t(\vartheta_{\varepsilon}^*)}{\varepsilon S(\vartheta_{\varepsilon}^*, X_t)} \longrightarrow u(t) = \int_0^t \frac{dW_v}{S(\vartheta, x_v)} \\ &\quad - J(\vartheta)^{-1} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v \int_0^t \frac{\dot{S}(\vartheta, x_v)}{S(\vartheta, x_v)} dv \end{aligned} \quad (1.15)$$

and it can be shown that

$$\delta_{\varepsilon}^* \implies \int_0^T u(t)^2 dt,$$

(see in [42] the details in the similar problem where the MLE was used). Therefore the test based on this statistic is not ADF. Hence to obtain an ADF GoF test we introduce the Gaussian process

$$U \left( \frac{t}{T} \right) = \frac{1}{\sqrt{T}} \int_0^t S(\vartheta, x_s) du(s), \quad 0 \leq t \leq T, \quad (1.16)$$

where  $u(\cdot)$  is given by (1.15). Then, by the Itô formula,

$$d(S(\vartheta, x_s)u(s)) = dS(\vartheta, x_s) u(s) + du(s) S(\vartheta, x_s)$$

and using the equality

$$dS(\vartheta, x_s) = S'(\vartheta, x_s) S(\vartheta, x_s) ds, \quad (1.17)$$

we have

$$S(\vartheta, x_t) u(t) = \int_0^t S'(\vartheta, x_s) S(\vartheta, x_s) u(s) ds + \int_0^t S(\vartheta, x_s) du(s).$$

Therefore the process  $U(\cdot)$  given by (1.16) admits the following representation ( $0 \leq t \leq T$ )

$$U\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} S(\vartheta, x_t) u(t) - \frac{1}{\sqrt{T}} \int_0^t S'(\vartheta, x_s) S(\vartheta, x_s) u(s) ds. \quad (1.18)$$

Further we define two functions

$$h(\vartheta, r) = T \tilde{J}(\vartheta)^{-1} \dot{S}(\vartheta, x_{rT}(\vartheta)) C(\vartheta)^{1/2} \quad (1.19)$$

and

$$g(\vartheta, r) = S(\vartheta, x_{rT}(\vartheta))^{-1} \int_r^1 S(\vartheta, x_{zT}(\vartheta)) \dot{x}_{zT}(\vartheta) dz C(\vartheta)^{-1/2}. \quad (1.20)$$

Here  $\tilde{J}(\vartheta) = \int_0^1 \dot{x}_{vT}(\vartheta)^2 dv$  and

$$C(\vartheta) = \int_0^1 S(\vartheta, x_{vT}(\vartheta))^{-2} \left( \int_v^1 S(\vartheta, x_{zT}(\vartheta)) \dot{x}_{zT}(\vartheta) dz \right)^2 dv.$$

Observe that

$$\int_0^1 g(\vartheta, r)^2 dr = 1.$$

**Lemma 1.3.1.** *We have the equality*

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, r) dW(r) \int_0^\nu h(\vartheta, r) dr, \quad 0 \leq \nu \leq 1, \quad (1.21)$$

where  $W(\nu), 0 \leq \nu \leq 1$  is a Wiener process.

**Proof.** By the Itô formula, we have, for  $0 \leq t \leq T$ ,

$$\begin{aligned} dU\left(\frac{t}{T}\right) &= \frac{1}{\sqrt{T}} dS(\vartheta, x_t) u(t) \\ &\quad + \frac{1}{\sqrt{T}} S(\vartheta, x_t) du(t) - \frac{1}{\sqrt{T}} S'(\vartheta, x_t) S(\vartheta, x_t) u(t) dt. \end{aligned}$$

Then by (1.17) and using the representation (1.15) of the process  $u(\cdot)$ , we obtain

$$\begin{aligned} U\left(\frac{t}{T}\right) &= \frac{1}{\sqrt{T}} \int_0^t S(\vartheta, x_s) du(s) \\ &= \frac{W_t}{\sqrt{T}} - \frac{J(\vartheta)^{-1}}{\sqrt{T}} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v \int_0^t \dot{S}(\vartheta, x_v) dv. \end{aligned}$$



Hence, by the change of variables  $\nu = \frac{t}{T}$  and  $W(\nu) = T^{-1/2}W_{\nu T}$ ,  $0 \leq \nu \leq 1$ , we have

$$U(\nu) = W(\nu) - \frac{J(\vartheta)^{-1}}{\sqrt{T}} \int_0^T \frac{1}{S(\vartheta, x_v)} \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW_v \int_0^{\nu T} \dot{S}(\vartheta, x_v) dv.$$

Let us change the variables  $r = \frac{v}{T}$ ,  $z = \frac{s}{T}$ ,  $W(r) = T^{-1/2}W_{rT}$ ,  $0 \leq r \leq 1$ . Then we can write

$$\begin{aligned} U(\nu) &= W(\nu) - \tilde{J}(\vartheta)^{-1} \int_0^1 \frac{1}{S(\vartheta, x_{rT})} \int_{rT}^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds dW(r) \int_0^\nu \dot{S}(\vartheta, x_{rT}) dr \\ &= W(\nu) - \frac{T}{\tilde{J}(\vartheta)} \int_0^1 \frac{1}{S(\vartheta, x_{rT})} \int_r^1 S(\vartheta, x_{zT}) \dot{x}_{zT}(\vartheta) dz dW(r) \int_0^\nu \dot{S}(\vartheta, x_{rT}) dr \\ &= W(\nu) - \int_0^1 g(\vartheta, r) dW(r) \int_0^\nu h(\vartheta, r) dr. \end{aligned}$$

Therefore we obtain the representation (1.21) and this proves the Lemma 1.3.1.

Using the convergence of the “empirical version”  $U_\varepsilon(\cdot)$  to  $U(\cdot)$  (see proof of Theorem 1.5.2 below) and due to the continuous mapping theorem, it can be shown that we have the convergence

$$\tilde{\Delta}_\varepsilon = \frac{1}{T^2} \int_0^T \left( \int_0^t S(\vartheta_\varepsilon^*, X_s) du_\varepsilon(s) \right)^2 dt \implies \int_0^1 U(\nu)^2 d\nu.$$

We remark that the test based on this statistic is not ADF. Hence we have to find the transformation  $L[U](\cdot)$  into the Wiener process such that

$$\int_0^1 L[U](\nu)^2 d\nu = \int_0^1 w_\nu^2 d\nu, \quad 0 \leq \nu \leq 1,$$

where  $w_\nu$ ,  $0 \leq \nu \leq 1$  is a Wiener process. This property allows us to present the “empirical version” of the test statistic with the same limit. Therefore the test based on this statistic is ADF.

## 1.4 Linear transformation

Now the problem is to find such transformation  $L[\cdot]$  of  $U(\cdot)$  (see (1.21)) using the MDE that  $L[U](\nu) = w_\nu$ . Recall that for the limit process (1.1) such linear

transformation and the corresponding ADF test were proposed by Khmaladze [30]. Another (direct) proof of this result was recently obtained by Kleptsyna and Kutoyants [34]. Note that in these works the estimator used was always the MLE and in our work it is the MDE. The limit processes in these two cases are quite different. That is why we have to present here a special modification of the proof given in [34]. Our proof follows the main steps of the work [34]. Specifically, we have to solve Fredholm equation of the second kind with degenerated kernel. The solution of it gives us the desired linear transformation.

Denote

$$\begin{aligned} I_1 &= \int_0^r g(\vartheta, q)^2 dq, \quad I_2 = \int_0^r h(\vartheta, q) g(\vartheta, q) dq, \quad I_3 = \int_0^r h(\vartheta, q) dq, \\ I_4 &= \int_0^r h(\vartheta, q)^2 dq, \quad I_5 = \int_0^r g(\vartheta, q) dq. \end{aligned} \quad (1.22)$$

Below we omit  $\vartheta$  and  $r$  for simplicity and put  $g = g(\vartheta, r)$  and  $h = h(\vartheta, r)$ . Introduce the functions

$$\begin{aligned} \varphi_1(r) &= g - h - 3I_2g + I_5hg + I_3g^2 + 2I_2h - 2I_2I_3g^2 + I_1I_2^2h + I_4I_5g^2 - I_2^3g \\ &\quad - I_2I_4g + 3I_2^2g + I_2I_5h^2 - 2I_2I_5hg - 2I_1I_2h + I_2^2I_5hg + I_1^2I_3h^2 - I_4h \\ &\quad + 2I_1I_4h - I_1I_4g + I_1I_2I_4g + I_1I_4I_5hg - I_1^2I_4h + I_1h + 2I_2I_3hg \\ &\quad - I_2I_4I_5g^2 - I_5h^2 + 2I_1I_3hg - 2I_1I_2I_3hg - 2I_1I_3h^2 - I_2^2h + I_3h^2 \\ &\quad - 2I_3hg - I_4I_5hg - I_1I_2I_5h^2 + I_2^2I_3g^2 + I_1I_5h^2 + I_4g, \end{aligned} \quad (1.23)$$

$$\begin{aligned} \varphi_2(r) &= 1 + I_5h - 3I_2I_5h + I_1I_3h + I_3g - 3I_2I_3g + I_4I_5g - I_3h - I_1I_4^2I_5g \\ &\quad + 3I_2^2I_3g - 2I_2I_4I_5g + 2I_2I_3h - I_2^3I_5h + I_1I_2^2I_3h - I_2^3I_3g + 3I_2^2I_5h \\ &\quad + I_4I_5h - I_2I_4I_5h + 2I_1I_3I_4h + I_3I_4g - I_2I_3I_4g + I_4^2 + 2I_2^2I_4 + I_2^4 \\ &\quad + I_1I_2I_4I_5h - I_1^2I_3I_4h - I_1I_3I_4g + I_1I_2I_3I_4g - I_2^2I_3h - 2I_1I_2^2I_4 \\ &\quad - 2I_1I_4^2 + I_1^2I_4^2 + 2I_4 - 2I_1I_4 - 4I_2I_4 + 4I_1I_2I_4 - 4I_2 + I_4^2I_5g \\ &\quad + I_2^2I_4I_5g - 2I_1I_2I_3h - I_3I_4h + 6I_2^2 - 4I_2^3 - I_1I_4I_5h \end{aligned} \quad (1.24)$$

and

$$\psi_2(r) = h + I_3hg - 2I_2I_4g + I_5h^2 - 3I_2h - 2I_2I_3hg - I_1I_2I_3h^2 + I_4g + 3I_2^2h$$

$$\begin{aligned}
 & + I_2^2 I_4 g + I_2^2 I_5 h^2 - I_2^3 h - I_3 I_4 h g + I_4^2 g + I_4 h - I_2 I_4 h - 2 I_2 I_5 h^2 \\
 & - I_1 I_4 h + I_1 I_2 I_4 h + I_1 I_3 h^2 + I_3 I_4 g^2 + I_2 I_3 h^2 - I_3 h^2 + I_1 I_3 I_4 h g \\
 & - I_2 I_3 I_4 g^2 - 2 I_2 I_4 I_5 h g + 2 I_4 I_5 h g + I_2^2 I_3 h g - I_1 I_4^2 g + I_4^2 I_5 g^2. \quad (1.25)
 \end{aligned}$$

The following Theorem is the main result of this work.

**Theorem 1.4.1.** *Suppose that  $h(q)$  and  $g(q)$  are continuous functions such that  $\int_0^1 g(q)^2 dq = 1$  and  $\varphi_2(r)$  is a strictly positive function on  $[0, 1)$ . Then the equality*

$$L[U](\nu) = U(\nu) + \int_0^\nu \int_0^r \frac{\varphi_1(r) h(q) + \varphi_2(r) g(q)}{\varphi_2(r)} dU(q) dr = w_\nu \quad (1.26)$$

holds. Here  $w_\nu$ ,  $0 \leq \nu \leq 1$  is a Wiener process.

**Proof.** The proof will be done in several steps.

**Step 1 :** Introduce a Gaussian process

$$M_t = \int_0^t q(t, s) dU(s), \quad 0 \leq t \leq 1,$$

where the function  $q(t, s)$  is chosen as solution of Fredholm equation described in the next step. Observe that

$$M_t = \int_0^t q(t, u) dW(u) - \int_0^1 g(u) dW(u) \int_0^t q(t, u) h(u) du.$$

**Step 2 :** For the correlation function of  $M_t$

$$R(t, s) = \mathbf{E} [M_t M_s], \quad t > s,$$

we have

$$\begin{aligned}
 \mathbf{E} [M_t M_s] &= \mathbf{E} \left[ \int_0^t q(t, u) dW(u) - \int_0^1 g(u) dW(u) \int_0^t q(t, u) h(u) du \right] \\
 &\quad \left[ \int_0^s q(s, v) dW(v) - \int_0^1 g(v) dW(v) \int_0^s q(s, v) h(v) dv \right] \\
 &= \int_0^s q(t, u) q(s, u) du - \int_0^t q(t, u) g(u) du \int_0^s q(s, v) h(v) dv \\
 &\quad - \int_0^s q(s, v) g(v) dv \int_0^t q(t, u) h(u) du \\
 &\quad + \underbrace{\int_0^1 g(v)^2 dv}_{=1} \int_0^t q(t, u) h(u) du \int_0^s q(s, v) h(v) dv
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^s q(s, u) \left[ q(t, u) - \int_0^t q(t, v) g(v) dv h(u) \right. \\
 &\quad \left. - \int_0^t q(t, v) h(v) dv g(u) + \int_0^t q(t, v) h(v) dv h(u) \right] du \\
 &= \int_0^s q(s, u) \left[ q(t, u) - \int_0^t q(t, v) [g(v) h(u) \right. \\
 &\quad \left. + h(v) g(u) - h(v) h(u)] dv \right] du.
 \end{aligned}$$

Denote the kernel

$$K(u, v) = g(v) h(u) + h(v) g(u) - h(v) h(u).$$

Then

$$\mathbf{E} [M_t M_s] = \int_0^s q(s, u) \left[ q(t, u) - \int_0^t q(t, v) K(u, v) dv \right] du. \quad (1.27)$$

Therefore if we take  $q(t, s)$  such that it solves the Fredholm equation of the second kind ( $t$  is fixed)

$$q(t, s) - \int_0^t q(t, v) K(s, v) dv = 1, \quad s \in [0, t], \quad (1.28)$$

then (1.27) becomes

$$\mathbf{E} [M_t M_s] = \mathbf{E} [M_s^2] = \int_0^s q(s, u) du. \quad (1.29)$$

**Step 3 :** The solution  $q(t, s)$  can be found as follows : We have

$$q(t, s) = 1 + \int_0^t q(t, v) K(s, v) dv.$$

Denote

$$A(t) = \int_0^t q(t, v) h(v) dv$$

and

$$B(t) = \int_0^t q(t, v) g(v) dv.$$

Then  $q(t, s)$  has the representation

$$q(t, s) = 1 + B(t) h(s) + A(t) (g(s) - h(s)), \quad (1.30)$$

where the function  $A(t)$  itself is solution of the following equation (after multiplying (1.30) by  $h(s)$  and integrating)

$$\begin{aligned} \int_0^t h(s) ds &= A(t) - B(t) \int_0^t h(s)^2 ds \\ &\quad - A(t) \int_0^t h(s) g(s) ds + A(t) \int_0^t h(s)^2 ds. \end{aligned} \quad (1.31)$$

The function  $B(t)$  is solution of the following equation (after multiplying (1.30) by  $g(s)$  and integrating)

$$\begin{aligned} \int_0^t g(s) ds &= B(t) - B(t) \int_0^t h(s) g(s) ds \\ &\quad - A(t) \int_0^t g(s)^2 ds + A(t) \int_0^t h(s) g(s) ds. \end{aligned} \quad (1.32)$$

Using the notation (1.22), we can write (1.31)–(1.32) as follows :

$$A(t) - B(t) I_4 - A(t) I_2 + A(t) I_4 = I_3 \quad (1.33)$$

and

$$B(t) - B(t) I_2 - A(t) I_1 + A(t) I_2 = I_5. \quad (1.34)$$

Further, we have to find the expressions of  $A(t)$  and  $B(t)$ . Therefore we obtain from (1.34)

$$B(t) = \frac{I_5 + A(t) (I_1 - I_2)}{(1 - I_2)}. \quad (1.35)$$

Then we insert (1.35) in (1.33) and obtain

$$A(t) = \frac{I_3 (1 - I_2) + I_4 I_5}{(1 - I_2)^2 + I_4 I_6}, \quad B(t) = \frac{I_5 (1 + I_4 - I_2) + I_3 (I_1 - I_2)}{(1 - I_2)^2 + I_4 I_6},$$

where

$$I_6 = \int_t^1 g(s)^2 ds.$$

Therefore the solution  $q(t, s)$  of (1.28) is

$$q(t, s) = 1 + \frac{I_5 (1 + I_4 - I_2) + I_3 (I_1 - I_2)}{(1 - I_2)^2 + I_4 I_6} h(s) + \frac{I_3 (1 - I_2) + I_4 I_5}{(1 - I_2)^2 + I_4 I_6} (g(s) - h(s)).$$

The final expression of  $q(t, s)$  is

$$q(t, s) = 1 + \frac{\int_0^t g(s) ds \left( 1 + \int_0^t h(s)^2 ds - \int_0^t h(s) g(s) ds \right) h(s)}{\left( 1 - \int_0^t h(s) g(s) ds \right)^2 + \int_0^t h(s)^2 ds \int_t^1 g(s)^2 ds}$$

$$\begin{aligned}
 & + \frac{\int_0^t h(s) ds \left( \int_0^t g(s)^2 ds - \int_0^t h(s)g(s) ds \right) h(s)}{\left( 1 - \int_0^t h(s)g(s) ds \right)^2 + \int_0^t h(s)^2 ds \int_t^1 g(s)^2 ds} \\
 & + \frac{\int_0^t h(s) ds \left( 1 - \int_0^t h(s)g(s) ds \right) + \int_0^t h(s)^2 ds \int_0^t g(s) ds}{\left( 1 - \int_0^t h(s)g(s) ds \right)^2 + \int_0^t h(s)^2 ds \int_t^1 g(s)^2 ds} (g(s) - h(s)).
 \end{aligned}$$

**Step 4 :** To show that  $M_t$  is martingale we need the following Lemma.

**Lemma 1.4.2.** *We have the following equality*

$$\int_0^t q(t, s) ds = \int_0^t q(s, s)^2 ds, \quad 0 \leq t \leq 1. \quad (1.36)$$

**Proof.** Show that

$$\frac{d}{dt} \int_0^t q(t, s) ds = \frac{d}{dt} \int_0^t q(s, s)^2 ds = q(t, t)^2.$$

Denote

$$\begin{aligned}
 C(t) &= 1 - \int_0^t h(s) g(s) ds, \\
 D(t) &= \int_0^t g(s) ds \left( 1 + \int_0^t h(s)^2 ds - \int_0^t h(s) g(s) ds \right) \\
 &\quad + \int_0^t h(s) ds \left( \int_0^t g(s)^2 ds - \int_0^t h(s) g(s) ds \right), \\
 K(t) &= C(t)^2 + \int_0^t h(s)^2 ds \int_t^1 g(s)^2 ds
 \end{aligned}$$

and

$$N(t) = C(t) \int_0^t h(s) ds + \int_0^t h(s)^2 ds \int_0^t g(s) ds.$$

Then  $q(t, s)$  has the following expression

$$q(t, s) = 1 + \frac{D(t)}{K(t)} h(s) + \frac{N(t)}{K(t)} (g(s) - h(s)).$$

Hence

$$\begin{aligned}
 \frac{d}{dt} \int_0^t q(t, s) ds &= 1 + \frac{D(t) h(t) + N(t) (g(t) - h(t))}{K(t)} \\
 &\quad + \frac{\left( D'(t) K(t) - D(t) K'(t) \right)}{K(t)^2} \int_0^t h(s) ds \\
 &\quad + \frac{\left( N'(t) K(t) - N(t) K'(t) \right)}{K(t)^2} \int_0^t (g(s) - h(s)) ds.
 \end{aligned} \quad (1.37)$$

Then we obtain the equalities

$$\begin{aligned} C(t) &= 1 - I_2, \\ D(t) &= I_5 (1 + I_4 - I_2) + I_3 (I_1 - I_2), \\ K(t) &= (1 - I_2)^2 + I_4 (1 - I_1), \\ N(t) &= I_3 (1 - I_2) + I_4 I_5. \end{aligned}$$

The derivatives of these functions w.r.t.  $t$  have such expressions

$$\begin{aligned} C'(t) &= -h(t) g(t), \\ D'(t) &= g(t) + I_4 g(t) - I_2 g(t) + I_5 h(t)^2 - I_5 h(t) g(t) \\ &\quad + I_1 h(t) - I_2 h(t) + I_3 g(t)^2 - I_3 h(t) g(t), \\ K'(t) &= -2h(t) g(t) + 2I_2 h(t)g(t) + h(t)^2 - I_1 h(t)^2 - I_4 g(t)^2, \\ N'(t) &= -I_3 h(t) g(t) + I_4 g(t) + I_5 h(t)^2 + h(t) - I_2 h(t). \end{aligned}$$

Thus, (1.37) has the following representation

$$\frac{d}{dt} \int_0^t q(t, s) ds = 1 + \frac{\Phi_1(t) + \Phi_2(t) + \Phi_3(t)}{K(t)^2}, \quad (1.38)$$

where

$$\begin{aligned} \Phi_1(t) &= (D(t) h(t) + N(t) (g(t) - h(t)))K(t), \\ \Phi_2(t) &= (D'(t) K(t) - D(t) K'(t))I_3 \end{aligned}$$

and

$$\Phi_3(t) = (N'(t) K(t) - N(t) K'(t)) (I_5 - I_3).$$

Then returning to initial notation we obtain the expressions

$$\begin{aligned} \Phi_1(t) &= 3I_2^2 I_5 h - 3I_2 I_5 h + I_1 I_3 h - 3I_2 I_3 g + I_1 I_2^2 I_3 h - I_2^3 I_5 h + I_3 g - I_2^3 I_3 g \\ &\quad + I_4 I_5 g - I_1 I_3 I_4 g - I_3 h - I_1^2 I_3 I_4 h + 2I_2 I_3 h + I_5 h - I_1 I_4^2 I_5 g - I_2 I_3 I_4 g \\ &\quad - 2I_1 I_2 I_3 h + 3I_2^2 I_3 g - 2I_2 I_4 I_5 g + I_4 I_5 h - I_2 I_4 I_5 h + I_1 I_2 I_3 I_4 g + I_4^2 I_5 g \\ &\quad + I_2^2 I_4 I_5 g - I_2^2 I_3 h + 2I_1 I_3 I_4 h - I_3 I_4 h - I_1 I_4 I_5 h + I_3 I_4 g + I_1 I_2 I_4 I_5 h, \\ \Phi_2(t) &= I_2 I_3^2 h^2 + 2I_3 I_4 g - 3I_2 I_3 g + I_3 I_5 h g + I_3 I_4^2 I_5 g^2 + I_3^2 I_4 g^2 + I_1 I_3 h \\ &\quad - 3I_2 I_3 I_4 g + 3I_2^2 I_3 g - I_2 I_3 I_5 h^2 - 2I_2 I_3 I_5 h g + I_3^2 g^2 - I_1 I_3^2 h^2 \end{aligned}$$

$$\begin{aligned}
& + I_2^2 I_3 I_4 g - I_2^3 I_3 g + I_2^2 I_3 I_5 h^2 + I_2^2 I_3 I_5 h g + I_1 I_2^2 I_3 h - 2 I_1 I_2 I_3 h \\
& - I_3^2 h g - 2 I_2 I_3^2 g^2 + I_2^2 I_3^2 g^2 + 2 I_2^2 I_3 h - 2 I_2 I_3 I_4 I_5 h g - I_1^2 I_3 I_4 h \\
& + I_2^2 I_3^2 h g + I_3 I_4^2 g + I_3 I_4 I_5 h g - 2 I_1 I_2 I_3^2 h g + I_1^2 I_3^2 h^2 - I_1 I_2 I_3^2 h^2 \\
& - I_1 I_3 I_4 g - I_1 I_3 I_4^2 g + I_1 I_2 I_3 I_4 g + I_1 I_3 I_4 I_5 h g - I_1 I_2 I_3 I_5 h^2 + I_3 g \\
& + I_1 I_3^2 I_4 h g + I_1 I_3 I_5 h^2 + I_3 I_4 I_5 g^2 + I_1 I_3 I_4 h - I_2 I_3 h - I_2 I_3^2 I_4 g^2 \\
& - I_2 I_3 I_4 I_5 g^2 + 2 I_1 I_3^2 h g - I_3^2 I_4 h g + I_1 I_2 I_3 I_4 h - I_2 I_3 I_4 h - I_2^3 I_3 h
\end{aligned}$$

and

$$\begin{aligned}
\Phi_3(t) = & I_3 I_5 h g + I_2 I_3 I_4 h - I_3 I_5 h^2 - 3 I_2 I_5 h - I_3 I_4^2 g - I_2^2 I_3^2 h g \\
& - 2 I_2 I_5^2 h^2 + 3 I_2^2 I_5 h + I_2^2 I_3 I_5 h g - I_2^2 I_3 I_5 h^2 + I_2^2 I_5^2 h^2 - I_3 I_4^2 I_5 g^2 \\
& + I_4^2 I_5 g - 3 I_2^2 I_3 h - I_2 I_4 I_5 h + I_1 I_3 I_4 I_5 h g - I_3^2 h g - I_1 I_4^2 I_5 g \\
& + I_2 I_3^2 I_4 g^2 + I_1 I_3 I_5 h^2 + I_3 I_4 I_5 g^2 + I_2 I_3 I_5 h^2 + I_5 h - 2 I_2 I_3 I_5 h g \\
& + 2 I_4 I_5^2 h g - 2 I_2 I_4 I_5^2 h g + I_4^2 I_5^2 g^2 + I_2^2 I_4 I_5 g - I_3 h + I_1 I_2 I_4 I_5 h \\
& + 3 I_2 I_3 h + 2 I_2 I_3^2 h g + 2 I_2 I_3 I_4 g + 2 I_2 I_3 I_5 h^2 + I_5^2 h^2 - I_1 I_4 I_5 h \\
& - I_2^2 I_3 I_4 g + I_2^3 I_3 h + I_3^2 I_4 h g + I_4 I_5 h - I_2^3 I_5 h + I_3^2 h^2 + I_4 I_5 g \\
& - I_1 I_3^2 I_4 h g + I_1 I_2 I_3^2 h^2 + I_1 I_3 I_4^2 g + I_1 I_3 I_4 h - I_3 I_4 h - I_3 I_4 g \\
& - I_3^2 I_4 g^2 - I_2 I_3^2 h^2 - I_3 I_5 h^2 - I_1 I_2 I_3 I_4 h - I_2 I_3 I_4 I_5 g^2 - I_1 I_3^2 h^2 \\
& - 2 I_3 I_4 I_5 h g + 2 I_2 I_3 I_4 I_5 h g - 2 I_2 I_4 I_5 g - I_1 I_2 I_3 I_5 h^2 - I_3 I_4 I_5 h g.
\end{aligned}$$

Denote

$$\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t),$$

then we can write

$$\begin{aligned}
\Phi(t) = & h(t) \left( 2 I_5 - 6 I_2 I_5 + 2 I_1 I_3 - 2 I_3 + 6 I_2^2 I_5 - 4 I_1 I_2 I_3 + 4 I_2 I_3 + 4 I_1 I_3 I_4 \right. \\
& - 2 I_2^3 I_5 + 2 I_1 I_2^2 I_3 + 2 I_4 I_5 - 2 I_2 I_4 I_5 - 2 I_3 I_4 + 2 I_1 I_2 I_4 I_5 - 2 I_1^2 I_3 I_4 \\
& \left. - 2 I_1 I_4 I_5 - 2 I_2^2 I_3 \right) + g(t) \left( 2 I_3 - 2 I_1 I_3 I_4 + 2 I_1 I_2 I_3 I_4 - 6 I_2 I_3 + 2 I_4 I_5 \right. \\
& - 2 I_1 I_4^2 I_5 + 6 I_2^2 I_3 - 4 I_2 I_4 I_5 - 2 I_2^3 I_3 + 2 I_2^2 I_4 I_5 + 2 I_3 I_4 + 2 I_4^2 I_5 \\
& \left. - 2 I_2 I_3 I_4 \right) + h(t) g(t) \left( 2 I_3 I_5 - 2 I_3^2 - 4 I_2 I_3 I_5 - 2 I_3 I_4 I_5 + 2 I_1 I_3 I_4 I_5 \right. \\
& \left. + 2 I_2 I_3^2 + 2 I_4 I_5^2 + 2 I_1 I_3^2 - 2 I_1 I_2 I_3^2 - 2 I_2 I_4 I_5^2 + 2 I_2^2 I_3 I_5 \right) + h(t)^2
\end{aligned}$$



$$\begin{aligned} & \left( 2I_1I_3I_5 - 2I_1I_2I_3I_5 - 2I_2I_5^2 + I_2^2I_5^2 - 2I_3I_5 + 2I_2I_3I_5 - 2I_1I_3^2 + I_3^2 \right. \\ & \left. + I_5^2 + I_1^2I_3^2 \right) + g(t)^2 \left( I_3^2 - 2I_2I_3^2 + 2I_3I_4I_5 - 2I_2I_3I_4I_5 + I_4^2I_5^2 + I_2^2I_3^2 \right). \end{aligned}$$

Finally, (1.38) can be written as follows :

$$\frac{d}{dt} \int_0^t q(t, s) ds = 1 + \frac{\Phi(t)}{K(t)^2}. \quad (1.39)$$

Now the expression of  $q(t, t)^2$  is

$$\begin{aligned} q(t, t)^2 &= \left( 1 + \frac{D(t)}{K(t)} h(t) + \frac{N(t)}{K(t)} (g(t) - h(t)) \right)^2 \\ &= 1 + \frac{2 h(t) K(t) (D(t) - N(t)) + 2 g(t) K(t) N(t)}{K(t)^2} \\ &\quad + \frac{h(t)^2 (D(t) - N(t))^2 + g(t)^2 N(t)^2}{K(t)^2} \\ &\quad + \frac{2 h(t) g(t) N(t) (D(t) - N(t))}{K(t)^2}. \end{aligned}$$

Denote

$$\begin{aligned} M(t) &= 2 K(t) (D(t) - N(t)), \\ Q(t) &= 2 K(t) N(t), \\ L(t) &= (D(t) - N(t))^2, \\ E(t) &= N(t)^2, \\ H(t) &= 2 N(t) (D(t) - N(t)). \end{aligned}$$

Therefore the final expression for  $q(t, t)^2$  is

$$\begin{aligned} q(t, t)^2 &= 1 + \frac{h(t) M(t) + g(t) Q(t) + h(t)^2 L(t)}{K(t)^2} \\ &\quad + \frac{g(t)^2 E(t) + h(t) g(t) H(t)}{K(t)^2}, \end{aligned} \quad (1.40)$$

where

$$\begin{aligned} M(t) &= 2I_1I_3 - 6I_2I_5 - 2I_3 + 6I_2^2I_5 - 4I_1I_2I_3 + 4I_2I_3 - 2I_2^3I_5 + 2I_4I_5 \\ &\quad + 2I_1I_2^2I_3 - 2I_1^2I_3I_4 - 2I_2^2I_3 - 2I_2I_4I_5 + 2I_5 - 2I_3I_4 - 2I_1I_4I_5 \\ &\quad + 2I_1I_2I_4I_5 + 4I_1I_3I_4, \\ Q(t) &= 2I_3I_4 - 6I_2I_3 + 6I_2^2I_3 - 2I_2^3I_3 + 2I_4I_5 - 4I_2I_4I_5 + 2I_2^2I_4I_5 \end{aligned}$$

$$\begin{aligned}
 & + 2I_3 - 2I_2I_3I_4 - 2I_1I_3I_4 + 2I_1I_2I_3I_4 + 2I_4^2I_5 - 2I_1I_4^2I_5, \\
 L(t) = & I_5^2 + I_2^2I_5^2 - 2I_2I_5^2 + I_1^2I_3^2 + I_3^2 - 2I_1I_3^2 + 2I_1I_3I_5 - 2I_3I_5 \\
 & + 2I_2I_3I_5 - 2I_1I_2I_3I_5, \\
 E(t) = & I_3^2 - 2I_2I_3^2 + I_2^2I_3^2 + I_4^2I_5^2 + 2I_3I_4I_5 - 2I_2I_3I_4I_5, \\
 H(t) = & 2I_3I_5 - 4I_2I_3I_5 - 2I_3I_4I_5 + 2I_1I_3^2 + 2I_2I_3^2 + 2I_2^2I_3I_5 \\
 & - 2I_2I_4I_5^2 + 2I_1I_3I_4I_5 - 2I_3^2 - 2I_1I_2I_3^2 + 2I_4I_5^2.
 \end{aligned}$$

The comparison of these expressions with (1.39)–(1.40) shows that the Lemma is proved.

**Step 5 :** In the next step, we need the following Lemma to show that the linear transformation is a Wiener process.

**Lemma 1.4.3.** *If the Gaussian process  $M_s$  satisfies (1.29) and we have relation (1.36), then*

$$z(t) = \int_0^t q(s, s)^{-1} dM_s$$

*is a Wiener process.*

**Proof.** The proof can be found, e.g., in [34], Lemma 2.

Hence

$$M_t = \int_0^t q(s, s) dw_s, \quad 0 \leq t \leq 1$$

is a Gaussian martingale, where  $w_s, 0 \leq s \leq 1$  is a Wiener process.

Therefore we have the equality

$$w_t = \int_0^t q(s, s)^{-1} dM_s = U(t) + \int_0^t q(s, s)^{-1} \int_0^s q'_s(s, v) dU(v) ds,$$

where  $w_t, 0 \leq t \leq 1$  is a Wiener process (by Lemma 1.4.3).

Now we have to calculate the right side of the above expression. The derivative  $q'_t(t, s)$  w.r.t.  $t$  can be written as follows :

$$q'_t(t, s) = \frac{(\psi_1(t) - \psi_2(t)) h(s) + \psi_2(t) g(s)}{K(t)^2},$$

where

$$\psi_1(t) = D'(t) K(t) - D(t) K'(t)$$

and

$$\psi_2(t) = N'(t) K(t) - N(t) K'(t).$$

Returning to the initial notation we obtain the following expression :

$$\begin{aligned} \psi_1(t) = & g + 2I_4g - 3I_2g + I_5hg + I_1h - I_2h + I_1I_4I_5hg + I_4^2g + 3I_2^2g \\ & - I_2I_5h^2 - 2I_2I_5hg - 2I_1I_2h + 2I_2^2h + I_2^2I_3g^2 - I_1I_2I_3h^2 - I_2^3g \\ & + I_2^2I_5hg + I_1I_2^2h - I_2^3h - 2I_2I_4I_5hg + I_1^2I_3h^2 - I_1I_2I_5h^2 - I_3hg \\ & - I_2I_4h + I_3I_4g^2 - I_3I_4hg - I_1I_4g - I_1I_4^2g + I_1I_2I_4g - 2I_2I_3g^2 \\ & + I_1I_2I_4h + I_1I_5h^2 + I_4I_5g^2 + I_4^2g + I_2^2I_3hg - I_1I_3h^2 + I_4^2I_5g^2 \\ & - I_2I_4I_5g^2 + 2I_1I_3hg - 2I_1I_2I_3hg + I_2^2I_4g - I_1^2I_4h + I_2I_3h^2 \\ & - I_2I_3I_4g^2 - 3I_2I_4g + I_2^2I_5h^2 + I_1I_4h + I_1I_3I_4hg + I_4I_5hg + I_3g^2. \end{aligned}$$

The function  $\psi_2(t)$  is defined by (1.25). Hence we obtain

$$\frac{q'_t(t, s)}{q(t, t)} = \frac{(\psi_1(t) - \psi_2(t)) h(s) + \psi_2(t) g(s)}{K(t)^2 + \Phi_1(t)}.$$

Then if we put

$$\varphi_1(t) = \psi_1(t) - \psi_2(t)$$

and

$$\varphi_2(t) = K(t)^2 + \Phi_1(t),$$

then this implies that

$$\frac{q'_t(t, s)}{q(t, t)} = \frac{\varphi_1(t) h(s) + \psi_2(t) g(s)}{\varphi_2(t)},$$

with  $\varphi_1(t)$  and  $\varphi_2(t)$  are defined by (1.23)–(1.24). Finally, we obtain the expression

$$w_t = \int_0^t q(s, s)^{-1} dM_s = U(t) + \int_0^t \int_0^s \frac{\varphi_1(s) h(v) + \psi_2(s) g(v)}{\varphi_2(s)} dU(v) ds.$$

This is the linear transformation  $L[U](\cdot)$  of the process  $U(\cdot)$  into the Wiener process  $w_t$  and this proves the Theorem 1.4.1.

**Remark.** Let us present a sufficient condition for  $\varphi_2(t) > 0$ .

$\mathcal{R}_0$ . Suppose that  $h(t)$  and  $g(t)$  are continuous strictly positive functions such that  $g(t) > h(t)$  and

$$\int_0^t h(s) g(s) ds < 1, \quad \int_0^1 g(s)^2 ds = 1, \quad 0 \leq t < 1,$$

then  $\varphi_2(t)$  defined by (1.24) is strictly positive function on  $[0, 1)$ .

Now we will verify that if the condition  $\mathcal{R}_0$  is satisfied, then  $\varphi_2(t)$  is strictly positive function. Remind that  $\varphi_2(t)$  has the following expression :

$$\varphi_2(t) = K(t)^2 + \Phi_1(t), \quad 0 \leq t < 1,$$

where

$$\Phi_1(t) = (D(t) h(t) + N(t) (g(t) - h(t)))K(t),$$

with

$$\begin{aligned} D(t) = & \int_0^t g(s) ds \left( 1 + \int_0^t h(s)^2 ds - \int_0^t h(s) g(s) ds \right) \\ & + \int_0^t h(s) ds \left( \int_0^t g(s)^2 ds - \int_0^t h(s) g(s) ds \right), \end{aligned}$$

$$N(t) = C(t) \int_0^t h(s) ds + \int_0^t h(s)^2 ds \int_0^t g(s) ds$$

and

$$K(t) = C(t)^2 + \int_0^t h(s)^2 ds \int_t^1 g(s)^2 ds.$$

Here

$$C(t) = 1 - \int_0^t h(s) g(s) ds.$$

We shall mention that it is sufficient to check that  $\Phi_1(t) > 0$  to obtain  $\varphi_2(t) > 0$ . Recall that  $K(t)$  is strictly positive function. Then due to the following condition

$$\int_0^t h(s) g(s) ds < 1, \tag{1.41}$$

we have  $C(t) > 0$ . Consequently, we obtain  $N(t) > 0$  by the conditions

$$h(t) > 0 \quad \text{and} \quad g(t) > 0. \tag{1.42}$$

Finally, we see that  $D(t) > 0$  by the conditions  $\int_0^1 g(s)^2 ds = 1$  and (1.41)–(1.42).

We conclude that we have  $\varphi_2(t) > 0$  if we suppose that  $g(t) > h(t)$ .

## 1.5 Test

Our objective is to test the composite parametric hypothesis  $\mathcal{H}_0$  and to do this we will propose a statistic based on the MDE  $\vartheta_\varepsilon^*$ . Recall that the starting statistic

$$u_\varepsilon(t) = \frac{X_t - x_t(\vartheta_\varepsilon^*)}{\varepsilon S(\vartheta_\varepsilon^*, X_t)} \quad (1.43)$$

converges to the random function

$$u(t) = \int_0^t \frac{dW_s}{S(\vartheta, x_s)} - \int_0^T \int_v^T \frac{S(\vartheta, x_s) \dot{x}_s(\vartheta)}{J(\vartheta)S(\vartheta, x_v)} ds dW_v \int_0^t \frac{\dot{S}(\vartheta, x_s)}{S(\vartheta, x_s)} ds.$$

By the Itô formula, the linear transformation

$$U\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} \int_0^t S(\vartheta, x_s) du(s)$$

has the following representation

$$U\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} S(\vartheta, x_t) u(t) - \frac{1}{\sqrt{T}} \int_0^t S'(\vartheta, x_s) S(\vartheta, x_s) u(s) ds$$

which leads to the random function

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, r) dW(r) \int_0^\nu h(\vartheta, r) dr, \quad 0 \leq \nu \leq 1.$$

The last step is to apply the transformation  $L[\cdot]$  from Theorem 1.4.1 and to obtain the Wiener process  $L[U](\nu) = w_\nu$ . Now we have to realize the similar transformations with the “empirical” process  $u_\varepsilon(\cdot)$  given by (1.43), i.e., we (formally) calculate

$$U_\varepsilon\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} \int_0^t S(\vartheta_\varepsilon^*, X_s) du_\varepsilon(s). \quad (1.44)$$

Then we apply the transformation  $L[\cdot]$  to the process  $U_\varepsilon(\cdot)$  and we show that this statistic converges in distribution to the Wiener process  $w_\nu, 0 \leq \nu \leq 1$ . Therefore the test  $\psi_\varepsilon = \mathbb{I}_{\{\Delta_\varepsilon > c_\alpha\}}$  with

$$\Delta_\varepsilon = \frac{1}{T} \int_0^T L[U_\varepsilon](t)^2 dt \implies \int_0^1 w_\nu^2 d\nu$$

will be ADF in view of the limit distribution of  $\Delta_\varepsilon$  which does not depend on  $S(\cdot, \cdot)$  and  $\vartheta$ .

Let us realize this program. We have for the process (1.44) the following representation

$$U_\varepsilon \left( \frac{t}{T} \right) = \frac{1}{\sqrt{T}} S(\vartheta_\varepsilon^*, X_t) u_\varepsilon(t) - \frac{1}{\sqrt{T}} \int_0^t S'(\vartheta_\varepsilon^*, X_s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) ds. \quad (1.45)$$

Introduce the functions

$$\hat{h}(\vartheta, v) = \frac{1}{\sqrt{T}} J(\vartheta)^{-1} \dot{S}(\vartheta, x_v), \quad J(\vartheta) = \int_0^T \dot{x}_s(\vartheta)^2 ds \quad (1.46)$$

and

$$\hat{g}(\vartheta, v) = S(\vartheta, x_v)^{-1} I(\vartheta, v), \quad I(\vartheta, v) = \int_v^T S(\vartheta, x_s) \dot{x}_s(\vartheta) ds \quad (1.47)$$

and their “empirical versions”, respectively

$$h_\varepsilon(\vartheta_\varepsilon^*, v) = \frac{1}{\sqrt{T}} J_\varepsilon(\vartheta_\varepsilon^*)^{-1} \dot{S}(\vartheta_\varepsilon^*, X_v), \quad g_\varepsilon(\vartheta_\varepsilon^*, v) = S(\vartheta_\varepsilon^*, X_v)^{-1} I_\varepsilon(\vartheta_\varepsilon^*, v).$$

Here

$$J_\varepsilon(\vartheta_\varepsilon^*) = \int_0^T \dot{x}_s(\vartheta_\varepsilon^*)^2 ds, \quad I_\varepsilon(\vartheta_\varepsilon^*, v) = \int_v^T S(\vartheta_\varepsilon^*, X_s) \dot{x}_s(\vartheta_\varepsilon^*) ds.$$

Note that in the functions  $\hat{h}(\cdot, \cdot)$  and  $\hat{g}(\cdot, \cdot)$  we omit the normalizing constants in the expressions of the functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  defined by (1.19)–(1.20), for simplicity of exposition, because the structure of the used statistic is such that we can do this without changing the limit distribution of the statistic.

Then denote the “empirical versions”

$$I_{1,\varepsilon} = C_1(T) \int_0^s g_\varepsilon(\vartheta_\varepsilon^*, v)^2 dv, \quad I_{2,\varepsilon} = C_2(T) \int_0^s h_\varepsilon(\vartheta_\varepsilon^*, v) g_\varepsilon(\vartheta_\varepsilon^*, v) dv,$$

$$I_{3,\varepsilon} = C_3(T) \int_0^s h_\varepsilon(\vartheta_\varepsilon^*, v) dv, \quad I_{4,\varepsilon} = C_4(T) \int_0^s h_\varepsilon(\vartheta_\varepsilon^*, v)^2 dv$$

and

$$I_{5,\varepsilon} = C_5(T) \int_0^s g_\varepsilon(\vartheta_\varepsilon^*, v) dv$$

of the integrals

$$\hat{I}_1 = C_1(T) \int_0^s \hat{g}(\vartheta, v)^2 dv, \quad \hat{I}_2 = C_2(T) \int_0^s \hat{h}(\vartheta, v) \hat{g}(\vartheta, v) dv,$$

$$\hat{I}_3 = C_3(T) \int_0^s \hat{h}(\vartheta, v) \, dv, \quad \hat{I}_4 = C_4(T) \int_0^s \hat{h}(\vartheta, v)^2 \, dv$$

and

$$\hat{I}_5 = C_5(T) \int_0^s \hat{g}(\vartheta, v) \, dv,$$

where  $C_1(T) = \frac{1}{T^3}$ ,  $C_2(T) = \sqrt{T}$ ,  $C_3(T) = T\sqrt{T}$ ,  $C_4(T) = T^4$  and  $C_5(T) = \frac{1}{T^2}$ . This allows us to introduce the “empirical versions”  $\varphi_{1,\varepsilon}(\cdot)$ ,  $\varphi_{2,\varepsilon}(\cdot)$  and  $\psi_{2,\varepsilon}(\cdot)$  of  $\hat{\varphi}_1(\cdot)$ ,  $\hat{\varphi}_2(\cdot)$  and  $\hat{\psi}_2(\cdot)$  defined respectively by (1.23)–(1.25), where we replace the functions  $g$  by  $\hat{g} = \frac{1}{T} \hat{g}(\vartheta, s)$  and  $h$  by  $\hat{h} = T^2\sqrt{T} \hat{h}(\vartheta, s)$ ,

$$\begin{aligned} \varphi_{1,\varepsilon}(s) &= g_\varepsilon - h_\varepsilon + I_{4,\varepsilon}g_\varepsilon - 3I_{2,\varepsilon}g_\varepsilon + I_{5,\varepsilon}h_\varepsilong_\varepsilon + I_{1,\varepsilon}h_\varepsilon + 2I_{2,\varepsilon}h_\varepsilon + I_{2,\varepsilon}^2I_{3,\varepsilon}g_\varepsilon^2 \\ &\quad - 2I_{2,\varepsilon}I_{3,\varepsilon}g_\varepsilon^2 - I_{5,\varepsilon}h_\varepsilon^2 + I_{3,\varepsilon}g_\varepsilon^2 - I_{2,\varepsilon}I_{4,\varepsilon}g_\varepsilon + 3I_{2,\varepsilon}^2g_\varepsilon - 2I_{3,\varepsilon}h_\varepsilong_\varepsilon \\ &\quad - 2I_{2,\varepsilon}I_{5,\varepsilon}h_\varepsilong_\varepsilon - I_{2,\varepsilon}^2g_\varepsilon + I_{2,\varepsilon}^2I_{5,\varepsilon}h_\varepsilong_\varepsilon + I_{1,\varepsilon}I_{2,\varepsilon}^2h_\varepsilon - I_{2,\varepsilon}^2h_\varepsilon \\ &\quad + 2I_{1,\varepsilon}I_{4,\varepsilon}h_\varepsilon - I_{1,\varepsilon}I_{4,\varepsilon}g_\varepsilon + I_{1,\varepsilon}^2I_{3,\varepsilon}h_\varepsilon^2 + I_{1,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilong_\varepsilon + I_{2,\varepsilon}I_{5,\varepsilon}h_\varepsilon^2 \\ &\quad + I_{1,\varepsilon}I_{5,\varepsilon}h_\varepsilon^2 + I_{4,\varepsilon}I_{5,\varepsilon}g_\varepsilon^2 - I_{1,\varepsilon}I_{2,\varepsilon}I_{5,\varepsilon}h_\varepsilon^2 - I_{4,\varepsilon}h_\varepsilon - I_{2,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}g_\varepsilon^2 \\ &\quad + 2I_{1,\varepsilon}I_{3,\varepsilon}h_\varepsilong_\varepsilon - 2I_{1,\varepsilon}I_{2,\varepsilon}I_{3,\varepsilon}h_\varepsilong_\varepsilon - 2I_{1,\varepsilon}I_{3,\varepsilon}h_\varepsilon^2 - 2I_{1,\varepsilon}I_{2,\varepsilon}h_\varepsilon \\ &\quad + I_{1,\varepsilon}I_{2,\varepsilon}I_{4,\varepsilon}g_\varepsilon + 2I_{2,\varepsilon}I_{3,\varepsilon}h_\varepsilong_\varepsilon - I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilong_\varepsilon + I_{3,\varepsilon}h_\varepsilon^2 - I_{1,\varepsilon}^2I_{4,\varepsilon}h_\varepsilon, \\ \varphi_{2,\varepsilon}(s) &= 1 - 2I_{1,\varepsilon}I_{4,\varepsilon}^2 - 3I_{2,\varepsilon}I_{5,\varepsilon}h_\varepsilon + I_{1,\varepsilon}I_{3,\varepsilon}h_\varepsilon + I_{3,\varepsilon}g_\varepsilon - 3I_{2,\varepsilon}I_{3,\varepsilon}g_\varepsilon \\ &\quad + I_{4,\varepsilon}I_{5,\varepsilon}g_\varepsilon - I_{3,\varepsilon}h_\varepsilon + 3I_{2,\varepsilon}^2I_{5,\varepsilon}h_\varepsilon - 2I_{1,\varepsilon}I_{2,\varepsilon}I_{3,\varepsilon}h_\varepsilon - 2I_{1,\varepsilon}I_{2,\varepsilon}^2I_{4,\varepsilon} \\ &\quad - 2I_{2,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}g_\varepsilon + 2I_{2,\varepsilon}I_{3,\varepsilon}h_\varepsilon - I_{2,\varepsilon}^3I_{5,\varepsilon}h_\varepsilon + I_{1,\varepsilon}I_{2,\varepsilon}^2I_{3,\varepsilon}h_\varepsilon - 2I_{1,\varepsilon}I_{4,\varepsilon} \\ &\quad - I_{2,\varepsilon}^3I_{3,\varepsilon}g_\varepsilon + I_{4,\varepsilon}^2 + I_{2,\varepsilon}^2I_{4,\varepsilon}I_{5,\varepsilon}g_\varepsilon + I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilon + I_{4,\varepsilon}^2I_{5,\varepsilon}g_\varepsilon \\ &\quad - I_{2,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilon + 2I_{1,\varepsilon}I_{3,\varepsilon}I_{4,\varepsilon}h_\varepsilon + I_{3,\varepsilon}I_{4,\varepsilon}g_\varepsilon + 3I_{2,\varepsilon}^2I_{3,\varepsilon}g_\varepsilon \\ &\quad + 2I_{2,\varepsilon}^2I_{4,\varepsilon} + 2I_{4,\varepsilon} - I_{3,\varepsilon}I_{4,\varepsilon}h_\varepsilon + 6I_{2,\varepsilon}^2 + I_{1,\varepsilon}I_{2,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilon \\ &\quad - I_{1,\varepsilon}^2I_{3,\varepsilon}I_{4,\varepsilon}h_\varepsilon - I_{1,\varepsilon}I_{3,\varepsilon}I_{4,\varepsilon}g_\varepsilon + I_{5,\varepsilon}h_\varepsilon + I_{1,\varepsilon}I_{2,\varepsilon}I_{3,\varepsilon}I_{4,\varepsilon}g_\varepsilon \\ &\quad - I_{1,\varepsilon}I_{4,\varepsilon}^2I_{5,\varepsilon}g_\varepsilon - I_{1,\varepsilon}I_{4,\varepsilon}I_{5,\varepsilon}h_\varepsilon + 4I_{1,\varepsilon}I_{2,\varepsilon}I_{4,\varepsilon} - I_{2,\varepsilon}I_{3,\varepsilon}I_{4,\varepsilon}g_\varepsilon \\ &\quad - I_{2,\varepsilon}^2I_{3,\varepsilon}h_\varepsilon + I_{2,\varepsilon}^4 - 4I_{2,\varepsilon}^3 - 4I_{2,\varepsilon} - 4I_{2,\varepsilon}I_{4,\varepsilon} + I_{1,\varepsilon}^2I_{4,\varepsilon}^2 \end{aligned} \tag{1.48}$$

and

$$\begin{aligned} \psi_{2,\varepsilon}(s) &= h_\varepsilon + I_{3,\varepsilon}h_\varepsilong_\varepsilon + I_{4,\varepsilon}g_\varepsilon + I_{5,\varepsilon}h_\varepsilon^2 - 3I_{2,\varepsilon}h_\varepsilon - I_{1,\varepsilon}I_{2,\varepsilon}I_{3,\varepsilon}h_\varepsilon^2 \\ &\quad - 2I_{2,\varepsilon}I_{4,\varepsilon}g_\varepsilon + I_{4,\varepsilon}h_\varepsilon - 2I_{2,\varepsilon}I_{5,\varepsilon}h_\varepsilon^2 + 3I_{2,\varepsilon}^2h_\varepsilon + I_{2,\varepsilon}^2I_{4,\varepsilon}g_\varepsilon \end{aligned}$$

$$\begin{aligned}
& + I_{2,\varepsilon}^2 I_{5,\varepsilon} h_\varepsilon^2 - I_{2,\varepsilon}^3 h_\varepsilon - I_{3,\varepsilon} I_{4,\varepsilon} h_\varepsilon g_\varepsilon + I_{4,\varepsilon}^2 g_\varepsilon - I_{2,\varepsilon} I_{4,\varepsilon} h_\varepsilon \\
& + I_{1,\varepsilon} I_{3,\varepsilon} I_{4,\varepsilon} h_\varepsilon g_\varepsilon - I_{1,\varepsilon} I_{4,\varepsilon}^2 g_\varepsilon - I_{1,\varepsilon} I_{4,\varepsilon} h_\varepsilon + I_{1,\varepsilon} I_{2,\varepsilon} I_{4,\varepsilon} h_\varepsilon \\
& - I_{3,\varepsilon} h_\varepsilon^2 + I_{1,\varepsilon} I_{3,\varepsilon} h_\varepsilon^2 + I_{3,\varepsilon} I_{4,\varepsilon} g_\varepsilon^2 + I_{2,\varepsilon} I_{3,\varepsilon} h_\varepsilon^2 - 2I_{2,\varepsilon} I_{3,\varepsilon} h_\varepsilon g_\varepsilon \\
& - 2I_{2,\varepsilon} I_{4,\varepsilon} I_{5,\varepsilon} h_\varepsilon g_\varepsilon + 2I_{4,\varepsilon} I_{5,\varepsilon} h_\varepsilon g_\varepsilon - I_{2,\varepsilon} I_{3,\varepsilon} I_{4,\varepsilon} g_\varepsilon^2 + I_{2,\varepsilon}^2 I_{3,\varepsilon} h_\varepsilon g_\varepsilon \\
& + I_{4,\varepsilon}^2 I_{5,\varepsilon} g_\varepsilon^2.
\end{aligned}$$

Here  $g_\varepsilon = g_\varepsilon(\vartheta_\varepsilon^*, s)$ ,  $h_\varepsilon = h_\varepsilon(\vartheta_\varepsilon^*, s)$  and  $I_{1,\varepsilon}$ ,  $I_{2,\varepsilon}$ ,  $I_{3,\varepsilon}$ ,  $I_{4,\varepsilon}$ ,  $I_{5,\varepsilon}$  are the ‘‘empirical versions’’ of  $\hat{I}_1$ ,  $\hat{I}_2$ ,  $\hat{I}_3$ ,  $\hat{I}_4$ ,  $\hat{I}_5$ , respectively.

In the construction of the test we introduce one condition else.

$\mathcal{R}_4$ . We suppose that  $\varphi_2(r)$ ,  $r \in [0, 1)$  defined by (1.24) is strictly positive function.

We have the uniform convergence in probability w.r.t.  $s \in [0, T]$  ( $\varepsilon \rightarrow 0$ )

$$\sup_{s \in [0, T]} |\varphi_{2,\varepsilon}(s) - \hat{\varphi}_2(s)| \rightarrow 0.$$

This convergence we obtain due to the consistency of the estimator and the smoothness of the functions  $g_\varepsilon(\cdot, \cdot)$  and  $h_\varepsilon(\cdot, \cdot)$ .

Therefore we can introduce the function

$$\varphi_{2,\varepsilon}^+(s) = \begin{cases} \varphi_{2,\varepsilon}(s)^{-1}, & \text{if } \varphi_{2,\varepsilon}(s) > 0, \\ 0, & \text{else,} \end{cases}$$

which asymptotically coincides with  $\hat{\varphi}_2(s)^{-1}$  and therefore the limit distribution does not change.

Hence we first consider (formally) the statistic

$$\begin{aligned}
W_\varepsilon(t) &= U_\varepsilon\left(\frac{t}{T}\right) \\
&+ \frac{1}{T} \int_0^t \int_0^s \varphi_{2,\varepsilon}^+(s) [\lambda_1(T) \varphi_{1,\varepsilon}(s) h_\varepsilon(\vartheta_\varepsilon^*, q) + \lambda_2(T) \psi_{2,\varepsilon}(s) g_\varepsilon(\vartheta_\varepsilon^*, q)] dU_\varepsilon\left(\frac{q}{T}\right) ds,
\end{aligned} \tag{1.49}$$

where  $\lambda_1(T) = T^2 \sqrt{T}$ ,  $\lambda_2(T) = \frac{1}{T}$  and  $U_\varepsilon(\cdot)$  is defined by (1.45). If we prove that

$$W_\varepsilon(t) \rightarrow L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1$$

then the test based on this statistic will be ADF. The main technical problem in carrying out this program is to define the stochastic integrals

$$K_\varepsilon(\vartheta_\varepsilon^*, s) = \int_0^s h_\varepsilon(\vartheta_\varepsilon^*, q) dU_\varepsilon\left(\frac{q}{T}\right) \tag{1.50}$$



and

$$L_\varepsilon(\vartheta_\varepsilon^*, s) = \int_0^s g_\varepsilon(\vartheta_\varepsilon^*, q) dU_\varepsilon\left(\frac{q}{T}\right). \quad (1.51)$$

Unfortunately we can not calculate them as they are written now, because the integrand contains the MDE  $\vartheta_\varepsilon^*$  and this estimator depends on the whole trajectory  $X^\varepsilon = (X_t, 0 \leq t \leq T)$ . Therefore the corresponding stochastic integrals  $K_\varepsilon(\vartheta_\varepsilon^*, s)$  and  $L_\varepsilon(\vartheta_\varepsilon^*, s)$  are not well defined.

To avoid this problem we use an approach based on the application of the Itô formula, i.e., we replace the corresponding stochastic integrals by the ordinary ones. We shall mention that this approach was applied in the similar problem in [44]. Introduce the statistic

$$K(\vartheta, s) = \int_0^s \hat{h}(\vartheta, q) dU\left(\frac{q}{T}\right) = \frac{1}{\sqrt{T}} \int_0^s \hat{h}(\vartheta, q) S(\vartheta, x_q) du(q), \quad (1.52)$$

where the process  $U(\cdot)$  and the function  $\hat{h}(\cdot, \cdot)$  are defined by (1.16) and (1.46), respectively. Indeed the Itô formula gives us the following representation

$$\begin{aligned} d\left(\hat{h}(\vartheta, q)S(\vartheta, x_q)u(q)\right) &= \left(\hat{h}'(\vartheta, q)S(\vartheta, x_q) + \hat{h}(\vartheta, q)S'(\vartheta, x_q)S(\vartheta, x_q)\right)u(q) dq \\ &\quad + \hat{h}(\vartheta, q)S(\vartheta, x_q) du(q). \end{aligned}$$

Here  $\hat{h}'(\vartheta, q)$  is the derivative of  $\hat{h}(\vartheta, q)$  w.r.t.  $q$ , given by the following expression

$$\hat{h}'(\vartheta, q) = \frac{1}{\sqrt{T}} J(\vartheta)^{-1} \dot{S}'(\vartheta, x_q(\vartheta)) S(\vartheta, x_q(\vartheta)). \quad (1.53)$$

Therefore the statistic  $K(\vartheta, \cdot)$  defined by (1.52) can be written as follows :

$$\begin{aligned} K(\vartheta, s) &= \frac{1}{\sqrt{T}} \hat{h}(\vartheta, s) S(\vartheta, x_s) u(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s \left(\hat{h}'(\vartheta, q) S(\vartheta, x_q) + \hat{h}(\vartheta, q) S'(\vartheta, x_q) S(\vartheta, x_q)\right) u(q) dq. \end{aligned}$$

Hence we obtain for the process (1.50) the following representation

$$\begin{aligned} K_\varepsilon(\vartheta_\varepsilon^*, s) &= \frac{1}{\sqrt{T}} h_\varepsilon(\vartheta_\varepsilon^*, s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s (h'_\varepsilon(\vartheta_\varepsilon^*, q)S(\vartheta_\varepsilon^*, X_q) + h_\varepsilon(\vartheta_\varepsilon^*, q)S'(\vartheta_\varepsilon^*, X_q)S(\vartheta_\varepsilon^*, X_q)) u_\varepsilon(q) dq \end{aligned} \quad (1.54)$$

and the integral is now well defined. Here

$$h'_\varepsilon(\vartheta_\varepsilon^*, q) = \frac{1}{\sqrt{T}} J_\varepsilon(\vartheta_\varepsilon^*)^{-1} \dot{S}'(\vartheta_\varepsilon^*, X_q) S(\vartheta_\varepsilon^*, X_q).$$

Similarly, the process (1.51) can be written as follows :

$$\begin{aligned} L_\varepsilon(\vartheta_\varepsilon^*, s) &= \frac{1}{\sqrt{T}} g_\varepsilon(\vartheta_\varepsilon^*, s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) \\ &- \frac{1}{\sqrt{T}} \int_0^s (g'_\varepsilon(\vartheta_\varepsilon^*, q) S(\vartheta_\varepsilon^*, X_q) + g_\varepsilon(\vartheta_\varepsilon^*, q) S'(\vartheta_\varepsilon^*, X_q) S(\vartheta_\varepsilon^*, X_q)) u_\varepsilon(q) dq, \end{aligned} \quad (1.55)$$

where

$$g'_\varepsilon(\vartheta_\varepsilon^*, q) = -S(\vartheta_\varepsilon^*, X_q)^{-1} S'(\vartheta_\varepsilon^*, X_q) \int_q^T S(\vartheta_\varepsilon^*, X_s) \dot{x}_s(\vartheta_\varepsilon^*) ds - \dot{x}_q(\vartheta_\varepsilon^*)$$

is the “empirical version” of

$$\hat{g}'(\vartheta, q) = -S(\vartheta, x_q(\vartheta))^{-1} S'(\vartheta, x_q(\vartheta)) \int_q^T S(\vartheta, x_s(\vartheta)) \dot{x}_s(\vartheta) ds - \dot{x}_q(\vartheta). \quad (1.56)$$

Then the formal expression (1.49) for  $W_\varepsilon(t)$  can be replaced by

$$\begin{aligned} \tilde{W}_\varepsilon(t) &= U_\varepsilon\left(\frac{t}{T}\right) \\ &+ \frac{1}{T} \int_0^t \varphi_{2,\varepsilon}^+(s) [\lambda_1(T) \varphi_{1,\varepsilon}(s) K_\varepsilon(\vartheta_\varepsilon^*, s) + \lambda_2(T) \psi_{2,\varepsilon}(s) L_\varepsilon(\vartheta_\varepsilon^*, s)] ds, \end{aligned} \quad (1.57)$$

where  $\lambda_1(T) = T^2 \sqrt{T}$ ,  $\lambda_2(T) = \frac{1}{T}$  and the processes  $U_\varepsilon(\cdot)$ ,  $K_\varepsilon(\vartheta_\varepsilon^*, \cdot)$  and  $L_\varepsilon(\vartheta_\varepsilon^*, \cdot)$  admit the representations (1.45), (1.54) and (1.55), respectively.

The test is given in the following Theorem.

**Theorem 1.5.2.** *Suppose that the conditions of regularity  $\mathcal{R}$  and  $\mathcal{R}_4$  are fulfilled, then the test*

$$\psi_\varepsilon = \mathbb{I}_{\{\Delta_\varepsilon > c_\alpha\}}, \quad \mathbf{P}\left(\hat{\Delta} > c_\alpha\right) = \alpha,$$

with

$$\Delta_\varepsilon = \frac{1}{T} \int_0^T \tilde{W}_\varepsilon(t)^2 dt, \quad \hat{\Delta} \equiv \int_0^1 w_\nu^2 d\nu$$

is ADF and of asymptotic size  $\alpha \in (0, 1)$ .

**Proof.** Under the hypothesis  $\mathcal{H}_0$ , we have to show that the convergence

$$\Delta_\varepsilon \implies \hat{\Delta} \quad (1.58)$$

holds.

Recall that the process  $U_\varepsilon(\cdot)$  has the following representation

$$U_\varepsilon\left(\frac{t}{T}\right) = \frac{1}{\sqrt{T}} S(\vartheta_\varepsilon^*, X_t) u_\varepsilon(t) - \frac{1}{\sqrt{T}} \int_0^t S'(\vartheta_\varepsilon^*, X_s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) ds.$$

We shall mention that we have already the convergence in probability uniformly w.r.t.  $s \in [0, T]$  (as  $\varepsilon \rightarrow 0$ )

$$\sup_{s \in [0, T]} |X_s - x_s(\vartheta)| \longrightarrow 0, \quad \sup_{s \in [0, T]} |u_\varepsilon(s) - u(s)| \longrightarrow 0.$$

Further, we can write

$$\begin{aligned} |S(\vartheta_\varepsilon^*, X_s) - S(\vartheta, x_s)| &\leq |S(\vartheta_\varepsilon^*, X_s) - S(\vartheta, X_s)| + |S(\vartheta, X_s) - S(\vartheta, x_s)| \\ &\leq |\vartheta_\varepsilon^* - \vartheta| |\dot{S}(\tilde{\vartheta}, X_s)| + |X_s - x_s| |S'(\vartheta, \tilde{X}_s)|. \end{aligned}$$

Here  $|\tilde{\vartheta} - \vartheta| \leq |\vartheta_\varepsilon^* - \vartheta|$  and

$$|\tilde{X}_s - X_s| \leq |x_s(\vartheta_\varepsilon^*) - X_s| \leq |x_s(\vartheta_\varepsilon^*) - x_s(\vartheta)| + |x_s(\vartheta) - X_s| \rightarrow 0.$$

This convergence is uniform w.r.t.  $s \in [0, T]$ .

Then we know that the functions  $\dot{S}(\vartheta, x)$  and  $S'(\vartheta, x)$  are bounded by the regularity conditions  $\mathcal{R}$ , the process  $X_s$  converges uniformly w.r.t.  $s \in [0, T]$  to  $x_s(\vartheta)$  and due to the consistency of the estimator  $\vartheta_\varepsilon^*$ , we obtain the uniform convergence w.r.t.  $s \in [0, T]$  (in probability)

$$\sup_{s \in [0, T]} |S(\vartheta_\varepsilon^*, X_s) - S(\vartheta, x_s)| \longrightarrow 0.$$

Moreover, similar arguments give the uniform convergence w.r.t.  $s \in [0, T]$  (in probability)

$$\sup_{s \in [0, T]} |S'(\vartheta_\varepsilon^*, X_s) - S'(\vartheta, x_s)| \longrightarrow 0.$$

Therefore, we obtain the uniform convergence w.r.t.  $t \in [0, T]$  (in probability)

$$\begin{aligned} U_\varepsilon\left(\frac{t}{T}\right) &\longrightarrow \frac{1}{\sqrt{T}} S(\vartheta, x_t) u(t) - \frac{1}{\sqrt{T}} \int_0^t S'(\vartheta, x_s) S(\vartheta, x_s) u(s) ds \\ &= \frac{1}{\sqrt{T}} \int_0^t S(\vartheta, x_s) du(s) = U\left(\frac{t}{T}\right). \end{aligned}$$

Now we have to show that  $K_\varepsilon(\vartheta_\varepsilon^*, s) \longrightarrow K(\vartheta, s)$ , where

$$\begin{aligned} K_\varepsilon(\vartheta_\varepsilon^*, s) &= \frac{1}{\sqrt{T}} h_\varepsilon(\vartheta_\varepsilon^*, s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s (h'_\varepsilon(\vartheta_\varepsilon^*, q) S(\vartheta_\varepsilon^*, X_q) + h_\varepsilon(\vartheta_\varepsilon^*, q) S'(\vartheta_\varepsilon^*, X_q) S(\vartheta_\varepsilon^*, X_q)) u_\varepsilon(q) dq \end{aligned}$$

and

$$\begin{aligned} K(\vartheta, s) &= \frac{1}{\sqrt{T}} \hat{h}(\vartheta, s) S(\vartheta, x_s) u(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s \left( \hat{h}'(\vartheta, q) S(\vartheta, x_q) + \hat{h}(\vartheta, q) S'(\vartheta, x_q) S(\vartheta, x_q) \right) u(q) dq, \end{aligned}$$

where  $\hat{h}(\cdot, \cdot)$  and  $\hat{h}'(\cdot, \cdot)$  are defined by (1.46) and (1.53), respectively.

We have

$$\begin{aligned} \left| \dot{S}(\vartheta_\varepsilon^*, X_s) - \dot{S}(\vartheta, x_s) \right| &\leq \left| \dot{S}(\vartheta_\varepsilon^*, X_s) - \dot{S}(\vartheta, X_s) \right| + \left| \dot{S}(\vartheta, X_s) - \dot{S}(\vartheta, x_s) \right| \\ &\leq |\vartheta_\varepsilon^* - \vartheta| |\ddot{S}(\tilde{\vartheta}, X_s)| + |X_s - x_s| |\dot{S}'(\vartheta, \tilde{X}_s)| \\ &\leq \tilde{C}_1 |\vartheta_\varepsilon^* - \vartheta| + \tilde{C}_2 |X_s - x_s|. \end{aligned}$$

Here  $\tilde{C}_1$  and  $\tilde{C}_2$  are some constants. Due to the regularity conditions  $\mathcal{R}$ , the second derivative  $\ddot{S}(\vartheta, x)$  w.r.t.  $\vartheta$  and  $\dot{S}'(\vartheta, x)$  are bounded functions. Therefore we obtain the convergence in probability

$$\sup_{s \in [0, T]} \left| \dot{S}(\vartheta_\varepsilon^*, X_s) - \dot{S}(\vartheta, x_s) \right| \longrightarrow 0.$$

Further, we have (as  $\varepsilon \rightarrow 0$ )

$$J_\varepsilon(\vartheta_\varepsilon^*) - J(\vartheta) = \int_0^T \dot{x}_s(\vartheta_\varepsilon^*)^2 ds - \int_0^T \dot{x}_s(\vartheta)^2 ds \longrightarrow 0.$$

Hence we prove the convergence in probability

$$\sup_{s \in [0, T]} \left| h_\varepsilon(\vartheta_\varepsilon^*, s) - \hat{h}(\vartheta, s) \right| \longrightarrow 0.$$

In the same manner, it is shown that we have the convergence in probability

$$\sup_{s \in [0, T]} \left| h'_\varepsilon(\vartheta_\varepsilon^*, s) - \hat{h}'(\vartheta, s) \right| \longrightarrow 0.$$

Therefore, by the uniform convergence of  $u_\varepsilon(s)$  to  $u(s)$  w.r.t.  $s \in [0, T]$ , we obtain the convergence in probability

$$K_\varepsilon(\vartheta_\varepsilon^*, s) \longrightarrow K(\vartheta, s).$$

Now we have to show that  $L_\varepsilon(\vartheta_\varepsilon^*, s) \longrightarrow L(\vartheta, s)$ , where

$$\begin{aligned} L_\varepsilon(\vartheta_\varepsilon^*, s) &= \frac{1}{\sqrt{T}} g_\varepsilon(\vartheta_\varepsilon^*, s) S(\vartheta_\varepsilon^*, X_s) u_\varepsilon(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s (g'_\varepsilon(\vartheta_\varepsilon^*, q) S(\vartheta_\varepsilon^*, X_q) + g_\varepsilon(\vartheta_\varepsilon^*, q) S'(\vartheta_\varepsilon^*, X_q) S(\vartheta_\varepsilon^*, X_q)) u_\varepsilon(q) dq \end{aligned}$$

and

$$\begin{aligned} L(\vartheta, s) &= \frac{1}{\sqrt{T}} \hat{g}(\vartheta, s) S(\vartheta, x_s) u(s) \\ &\quad - \frac{1}{\sqrt{T}} \int_0^s (\hat{g}'(\vartheta, q) S(\vartheta, x_q) + \hat{g}(\vartheta, q) S'(\vartheta, x_q) S(\vartheta, x_q)) u(q) dq, \end{aligned}$$

where  $\hat{g}(\cdot, \cdot)$  and  $\hat{g}'(\cdot, \cdot)$  are defined by (1.47) and (1.56), respectively.

Observe that for  $s \in [0, T]$ , we have

$$\begin{aligned} |I_\varepsilon(\vartheta_\varepsilon^*, s) - I(\vartheta, s)| &\leq \int_s^T |S(\vartheta_\varepsilon^*, X_v) (\dot{x}_v(\vartheta_\varepsilon^*) - \dot{x}_v(\vartheta))| dv \\ &\quad + \int_s^T |\dot{x}_v(\vartheta) (S(\vartheta_\varepsilon^*, X_v) - S(\vartheta, x_v(\vartheta)))| dv \\ &\leq \int_s^T |S(\vartheta_\varepsilon^*, X_v)| |\dot{x}_v(\vartheta_\varepsilon^*) - \dot{x}_v(\vartheta)| dv \\ &\quad + |\vartheta_\varepsilon^* - \vartheta| \int_s^T |\dot{x}_v(\vartheta)| |\dot{S}(\tilde{\vartheta}, X_v)| dv \\ &\quad + \int_s^T |\dot{x}_v(\vartheta)| |X_v - x_v(\vartheta)| |S'(\vartheta, \tilde{X}_v)| dv \\ &\leq \left( \tilde{C}_3 |\vartheta_\varepsilon^* - \vartheta| + \tilde{C}_4 |\vartheta_\varepsilon^* - \vartheta| + \tilde{C}_5 \sup_{s \in [0, T]} |X_s - x_s(\vartheta)| \right) (T - s), \end{aligned}$$

where  $\tilde{C}_3$ ,  $\tilde{C}_4$  and  $\tilde{C}_5$  are constants. The estimator  $\vartheta_\varepsilon^*$  is consistent, the process  $X_s$  converges uniformly w.r.t.  $s \in [0, T]$  to  $x_s(\vartheta)$  and the derivatives are bounded due to the conditions of regularity  $\mathcal{R}$ . Therefore we obtain the convergence in probability uniformly w.r.t.  $s \in [0, T]$

$$\sup_{s \in [0, T]} |I_\varepsilon(\vartheta_\varepsilon^*, s) - I(\vartheta, s)| \longrightarrow 0.$$

Moreover, we proved already the uniform convergence w.r.t.  $s \in [0, T]$  of  $u_\varepsilon(s)$  to  $u(s)$  and  $S(\vartheta_\varepsilon^*, X_s)$  to  $S(\vartheta, x_s)$ . Hence, we have the convergence in probability

$$\sup_{s \in [0, T]} |g_\varepsilon(\vartheta_\varepsilon^*, s) - \hat{g}(\vartheta, s)| \longrightarrow 0.$$

Similarly, by the regularity conditions  $\mathcal{R}$ , it is shown that (in probability)

$$\sup_{s \in [0, T]} |g'_\varepsilon(\vartheta_\varepsilon^*, s) - \hat{g}'(\vartheta, s)| \longrightarrow 0.$$

Therefore we obtain the convergence in probability

$$L_\varepsilon(\vartheta_\varepsilon^*, s) \longrightarrow L(\vartheta, s).$$

Due to the regularity conditions  $\mathcal{R}$  and the consistency of the estimator  $\vartheta_\varepsilon^*$ , similar arguments give the convergence in probability uniformly w.r.t.  $s \in [0, T]$

$$\sup_{s \in [0, T]} |\psi_{2,\varepsilon}(s) - \hat{\psi}_2(s)| \longrightarrow 0$$

and

$$\sup_{s \in [0, T]} |\varphi_{1,\varepsilon}(s) - \hat{\varphi}_1(s)| \longrightarrow 0, \quad \sup_{s \in [0, T]} |\varphi_{2,\varepsilon}(s) - \hat{\varphi}_2(s)| \longrightarrow 0.$$

Finally, the convergence mentioned in (1.58) is proved and using this result the test  $\psi_\varepsilon$  is ADF and of asymptotic size  $\alpha \in (0, 1)$ , which completes the proof.

## 1.6 The case of MLE

This case was studied in [30]–[34]. The authors proposed a linear transformation, which yields the convergence of the test statistic to the integral of Wiener process. Therefore they showed that the test based on this statistic is ADF. To obtain the linear transformation mentioned in [34], we put  $h(\vartheta, r) = g(\vartheta, r)$  in (1.22) and we obtain

$$I_1 = I_2 = I_4 = \int_0^r h(\vartheta, q)^2 dq, \quad I_3 = I_5 = \int_0^r h(\vartheta, q) dq.$$

Therefore we can write

$$\begin{aligned} \varphi_1(r) = & h - h + I_1 h - 3I_1 h + I_3 h^2 + I_3 h^2 + 2I_1 h - I_1^2 h - 2I_1 I_3 h^2 - I_1^2 I_3 h^2 \\ & - I_1 h + I_1^2 I_3 h^2 - I_1^2 h + 3I_1^2 h + I_1 I_3 h^2 - 2I_1 I_3 h^2 - 2I_1^2 h + 2I_1 I_3 h^2 \\ & + I_1^2 I_3 h^2 + I_1^3 h + I_3 h^2 + I_1 h + 2I_1^2 h - I_1^2 h + I_1^3 h + I_1^2 I_3 h^2 - 2I_1^2 I_3 h^2 \\ & - I_1^3 h + I_1 I_3 h^2 + I_1 I_3 h^2 + 2I_1 I_3 h^2 - I_1^3 h - I_1^2 I_3 h^2 - I_3 h^2 - 2I_1 I_3 h^2 \\ & + I_1^2 I_3 h^2 - I_1 I_3 h^2 - 2I_3 h^2 \end{aligned}$$

$$= 0,$$

$$\begin{aligned} \varphi_2(r) &= 1 + I_3h - 3I_1I_3h + I_1I_3h + I_3h - 3I_1I_3h + I_1I_3h - I_3h + 6I_1^2 - 2I_1^4 \\ &\quad + 3I_1^2I_3h - 2I_1^2I_3h + 2I_1I_3h - I_1^3I_3h + I_1^3I_3h - I_1^3I_3h + I_1^2 - 4I_1 \\ &\quad + I_1I_3h - I_1^2I_3h + 2I_1^2I_3h + I_1I_3h - I_1^2I_3h + 2I_1^3 + I_1^2I_3h - 4I_1^3 \\ &\quad + I_1^3I_3h - I_1^3I_3h - I_1^2I_3h + I_1^3I_3h - I_1^2I_3h + 4I_1^3 + 3I_1^2I_3h - I_1^3I_3h \\ &\quad + I_1^4 - 2I_1^3 + I_1^4 + 2I_1 - 2I_1^2 + I_1^3I_3h - 2I_1^2I_3h - I_1I_3h - I_1^2I_3h - 4I_1^2 \\ &= 1 + I_3h - I_1I_3h + I_1^2 - 2I_1 \\ &= (1 - I_1)(1 + I_3h - I_1) \end{aligned}$$

and

$$\begin{aligned} \psi_2(r) &= h + I_3h^2 + I_1h + I_3h^2 - 3I_1h - 2I_1I_3h^2 - 2I_1^2h - 2I_1I_3h^2 - I_1^3h \\ &\quad + I_1^3h + I_1^2I_3h^2 - I_1^3h - I_1I_3h^2 + I_1^2h + I_1h - I_1^2h - I_3h^2 + 3I_1^2h \\ &\quad - I_1^2h + I_1^3h + I_1I_3h^2 + I_1I_3h^2 + I_1I_3h^2 - I_1^2I_3h^2 + 2I_1^2I_3h^2 \\ &\quad - I_1^2I_3h^2 - 2I_1^2I_3h^2 + 2I_1I_3h^2 + I_1^2I_3h^2 \\ &= h(1 + I_3h - I_1), \end{aligned}$$

where  $h = h(\vartheta, r)$ . Hence the linear transformation (1.26) will have the following expression :

$$L[U](\nu) = U(\nu) + \int_0^\nu \int_0^r h(r) \mathbb{N}(r)^{-1} h(q) dU(q) dr = w_\nu, \quad (1.59)$$

where  $\mathbb{N}(r) = \int_r^1 h(q)^2 dq$  and

$$U(\nu) = W(\nu) - \int_0^1 h(r) dW(r) \int_0^\nu h(r) dr,$$

with  $W(\nu)$  and  $w_\nu, 0 \leq \nu \leq 1$  are some standard Wiener processes.

The transformation  $L[\cdot]$  given by (1.59) of the limit process  $U(\cdot)$  coincides with the one proposed by Khmaladze [30].





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# On GoF Test for Inhomogeneous Poisson Process

## 2.1 Introduction

This chapter is dedicated to the problem of the construction of two ADF GoF tests based on the empirical and score-function basic statistics. Suppose that we observe a periodic inhomogeneous Poisson process  $X^n = (X_t, 0 \leq t \leq T = n\tau)$  with known period  $\tau > 0$ . The mean  $\Lambda(t)$  and intensity functions  $\lambda(t)$  satisfy the relations

$$\mathbf{E}X_t = \Lambda(t), \quad \Lambda(t) = \int_0^t \lambda(s) \, ds.$$

Our aim is to construct two ADF GoF tests for the parametric null hypothesis

$$\mathcal{H}_0 \quad : \quad \Lambda(\cdot) \in \{\Lambda(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\}, \quad (2.1)$$

where  $a > 0$  and  $b < \infty$ . These tests are based on the special linear transformation  $L[\cdot]$  (1.26) applied to the empirical and score-function basic statistics.

In the literature on the statistical analysis of point processes numerous GoF tests for homogeneous Poisson processes were proposed, e.g., in [47], [14] and [1]. We shall mention that in [1] a test procedure was introduced for testing whether a point process is homogeneous : First whether the process is a mixed Poisson process and then whether the process is a renewal process. Further, in [5], the authors developed a test based on the empirical generating function which is consistent. Another GoF test was given in [59] for discrete distributions based on

a characterization by mean distances. Moreover, GoF tests based on the Cramér-von Mises statistics were proposed in [58] for the Poisson distribution. Power comparisons showed that these statistics give good overall tests of fit. Other comparisons between the Pearson, smooth tests and a modified Kolmogorov-Smirnov test are given in [4].

In the case of inhomogeneous Poisson process and more general point processes, the GoF tests for parametric hypothesis were studied by many authors, e.g., [25], [2], [49], [11] and [42]–[43]. Note that the estimator used in all these works is always the MLE and the proposed tests are ADF. In [42], the author proposed two linear transformations of the limit statistics, which allow the construction of the ADF tests. However, in [43], it was shown that a special change of time transformed the limit score-function processes into the Brownian bridge and the ADF tests were constructed due to this property. The case of inhomogeneous Poisson process with shift parameter (one dimensional case) was studied in [11]. This work showed that the limit distribution of the Cramér-von Mises statistic does not depend on the unknown (shift) parameter and therefore the statistic is *asymptotically parameter free*. In the case of the simple null hypothesis, several results concerning the construction of the Cramér-von Mises and Kolmogorov-Smirnov type goodness-of-fit tests for continuous time processes were presented in [12]. As models of observations, the authors take a stochastic differential equation with small noise, ergodic diffusion process, Poisson process and self-exciting point processes.

Let us review some of the related results for the simple null hypothesis, i.e.,  $\Theta = \{\vartheta_0\}$  (see, e.g., [45] for more details)

$$\mathcal{H}_0 \quad : \quad \Lambda(t) = \Lambda(\vartheta_0, t).$$

Introduce the Cramér-von Mises type statistic

$$\eta_n^* = \frac{n}{\Lambda(\vartheta_0, \tau)^2} \int_0^\tau \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_0, r) \right)^2 \lambda(\vartheta_0, r) \, dr.$$

Here  $\hat{\Lambda}_n(\cdot)$  is the empirical mean function on one period constructed as follows :

$$\hat{\Lambda}_n(r) = \frac{1}{n} \sum_{j=1}^n [X_{\tau(j-1)+r} - X_{\tau(j-1)}] = \frac{1}{n} \sum_{j=1}^n X_j(r), \quad 0 \leq r \leq \tau. \quad (2.2)$$

The following convergence (under  $\mathcal{H}_0$ )

$$\eta_n^* \Longrightarrow \hat{\Delta} = \int_0^1 w_\nu^2 d\nu$$

holds, where  $w_\nu, 0 \leq \nu \leq 1$  is a Wiener process. It is clear that the limit statistic  $\hat{\Delta}$  does not depend on the model  $\Lambda(\cdot, \cdot)$ . Therefore, the test  $\varphi_n^* = \mathbb{I}_{\{\eta_n^* > c_\alpha\}}$  with  $\mathbf{P}(\hat{\Delta} > c_\alpha) = \alpha, \alpha \in (0, 1)$  is ADF.

Let us recall what happens in the case of the parametric null hypothesis (2.1). The details concerning this result can be found, e.g., in [42]–[45].

Introduce the MLE  $\hat{\vartheta}_n$  of the unknown parameter  $\vartheta$  constructed by the observations of a periodic Poisson process  $X^n$ . We suppose that  $\hat{\vartheta}_n$  is consistent and asymptotically normal (as  $n \rightarrow +\infty$ ) under regularity conditions (see [38] for more details)

$$\sqrt{n}(\hat{\vartheta}_n - \vartheta) \Longrightarrow \mathcal{N}(0, I(\vartheta)^{-1}), \quad I(\vartheta) = \int_0^\tau \frac{\dot{\lambda}(\vartheta, s)^2}{\lambda(\vartheta, s)} ds. \quad (2.3)$$

The GoF test is based on the Cramér-von Mises type statistic

$$\tilde{V}_n = \int_0^\tau \frac{\tilde{U}_n(\hat{\vartheta}_n, r)^2}{\Lambda(\hat{\vartheta}_n, \tau)} \lambda(\hat{\vartheta}_n, r) dr,$$

where

$$\tilde{U}_n(\hat{\vartheta}_n, r) = \frac{\sqrt{n}(\hat{\Lambda}_n(r) - \Lambda(\hat{\vartheta}_n, r))}{\sqrt{\Lambda(\hat{\vartheta}_n, \tau)}}. \quad (2.4)$$

The convergence follows immediately from the central limit theorem

$$\sqrt{n}(\hat{\Lambda}_n(r) - \Lambda(\vartheta, r)) \Longrightarrow W(\Lambda(\vartheta, r)) \sim \mathcal{N}(0, \Lambda(\vartheta, r)),$$

where  $W(\cdot)$  is a Wiener process. Therefore we have the convergence, under the hypothesis (2.1), (see, e.g., [42] for details)

$$\tilde{V}_n \Longrightarrow \int_0^1 \tilde{U}(\nu)^2 d\nu \equiv \tilde{V},$$

where

$$\tilde{U}(\nu) = w(\nu) - \int_0^1 h(\vartheta, s) dw(s) \int_0^\nu h(\vartheta, s) ds, \quad 0 \leq \nu \leq 1 \quad (2.5)$$

and

$$h(\vartheta, s) = \frac{\dot{\lambda}(\vartheta, v(s))}{\lambda(\vartheta, v(s))} \sqrt{\frac{\Lambda(\vartheta, \tau)}{I(\vartheta)}}, \quad \int_0^1 h(\vartheta, s)^2 ds = 1.$$

Here  $w(\nu), 0 \leq \nu \leq 1$  is a Wiener process and  $v(s)$  is the inverse function of  $s = \Lambda(\vartheta, v)\Lambda(\vartheta, \tau)^{-1}$ . It is easy to see that the limit process  $\tilde{V}$  depends strongly on the model  $\Lambda(\cdot, \cdot)$  and the unknown parameter  $\vartheta$ . Hence the corresponding test  $\tilde{\varphi}_n = \mathbb{I}_{\{\tilde{V}_n > c_\alpha\}}$  is not ADF and the threshold  $c_\alpha$  defined as solution of the equation

$$\mathbf{P}(\tilde{V} > c_\alpha) = \alpha, \quad \alpha \in (0, 1)$$

depends on  $\Lambda(\cdot, \cdot)$  (known) and  $\vartheta$  (unknown), i.e.,  $c_\alpha = c_\alpha(\Lambda, \vartheta)$ . One way to find the threshold is using numerical simulations to calculate the function  $\{c_\alpha(\Lambda, \vartheta), \vartheta \in \Theta\}$  w.r.t.  $\vartheta$ . Then we choose the threshold  $\bar{c}_\alpha = c_\alpha(\Lambda, \bar{\vartheta}_n)$ , where  $\bar{\vartheta}_n$  is some consistent estimator of  $\vartheta$  (say, MLE). Indeed it can be shown that, for the test  $\bar{\varphi}_n = \mathbb{I}_{\{\tilde{V}_n > \bar{c}_\alpha\}}$ , we have

$$\lim_{n \rightarrow \infty} \mathbf{E}_\vartheta \bar{\varphi}_n = \alpha \quad \text{for all } \vartheta \in \Theta = (a, b),$$

i.e., the test  $\bar{\varphi}_n$  is of asymptotic size  $\alpha$ .

There is another possibility to obtain an ADF test. It is done by introducing a linear transformation  $L[\cdot]$  of the random function  $\tilde{U}(\cdot)$  defined by (2.5) such that  $L[\tilde{U}](\nu) = w_\nu$ , where  $w_\nu, 0 \leq \nu \leq 1$  is a Wiener process. Consequently, we obtain the convergence

$$\tilde{\delta}_n = \int_0^\tau \frac{L[\tilde{U}_n](r)^2}{\Lambda(\hat{\vartheta}_n, \tau)} \lambda(\hat{\vartheta}_n, r) dr \implies \int_0^1 w_\nu^2 d\nu \equiv \hat{\Delta},$$

where  $\tilde{U}_n(\cdot)$  given by (2.4) is the ‘‘empirical version’’ of  $\tilde{U}(\cdot)$ . Finally, we have the distribution free limit  $\hat{\Delta}$ , the test

$$\tilde{\psi}_n = \mathbb{I}_{\{\tilde{\delta}_n > d_\alpha\}}, \quad \mathbf{P}\{\hat{\Delta} > d_\alpha\} = \alpha$$

is ADF and of asymptotic size  $\alpha \in (0, 1)$ . For more details we refer the reader to [49].

In this chapter, we are going to construct two ADF GoF tests, for the empirical and score-function basic statistics, based on the linear transformation

defined in Theorem (1.4.1). The parametric null hypothesis is given by (2.1) and in Section 2.2, we propose for the unknown parameter  $\vartheta$  the MDE

$$\vartheta_n^* = \arg \min_{\vartheta \in \Theta} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|,$$

where  $\hat{\Lambda}_n(\cdot)$  is defined by (2.2) and  $\|\cdot\|$  is  $\mathcal{L}^2(0, \tau)$ -norm, i.e.,

$$\left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|^2 = \int_0^\tau \left( \hat{\Lambda}_n(s) - \Lambda(\vartheta, s) \right)^2 ds.$$

In Section 2.3, we begin by introducing the Cramér-von Mises type statistic

$$\Delta_n = \frac{n}{\Lambda(\vartheta_n^*, \tau)^2} \int_0^\tau \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_n^*, r) \right)^2 \lambda(\vartheta_n^*, r) dr.$$

Under the hypothesis  $\mathcal{H}_0$ , we first show in Lemma 2.3.4 that we have the convergence

$$\Delta_n \implies \int_0^1 U(\nu)^2 d\nu,$$

where

$$U(\nu) = w(\nu) - \int_0^1 g(\vartheta, s) dw(s) \int_0^\nu h(\vartheta, s) ds, \quad \int_0^1 g(\vartheta, s)^2 ds = 1. \quad (2.6)$$

Then, in Section 2.4, we introduce for the same model of observations the corresponding score-function statistic (formally) (can be found in [43])

$$u_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} [dX_j(v) - \lambda(\vartheta_n^*, v) dv].$$

We say “formally” because the MDE  $\vartheta_n^*$  depends on the whole trajectory  $X^n$  and the corresponding stochastic integral

$$\int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} dX_j(v)$$

is not well defined. The correct definition will be given later using an approach which was applied in the similar problem in Kutoyants [43]. The score-function statistic  $u_n^*(\cdot)$  converges to the random function  $u(\cdot)$  defined by (2.38) below. Hence, the transformation (2.39) leads to the Gaussian process  $U(\cdot)$  given by (2.42) (see Lemma 2.4.5 below).

We shall mention that in both cases, i.e., the empirical and score-function basic statistics, we obtain the same limit process  $U(\cdot)$  with different functions

$h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$ . Now to obtain the ADF limit processes we apply the linear transformation  $L[\cdot]$  defined by (1.26) to the process  $U(\cdot)$  such that

$$L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1.$$

The above presentation suggests the construction of tests by introducing the “empirical versions”  $U_n(\cdot, \cdot)$  and  $U_n^*(\cdot)$  of  $U(\cdot)$  (given by (2.12) and (2.43), respectively below). Then, we apply the transformation  $L[\cdot]$  to both statistics  $U_n(\cdot, \cdot)$  and  $U_n^*(\cdot)$  and we show the convergence to the Wiener process (under  $\mathcal{H}_0$ )

$$L[U_n](r) \Longrightarrow L[U](\nu) = w_\nu, \quad L[U_n^*](\nu) \Longrightarrow L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1.$$

Introduce the corresponding statistics for the empirical and score-function basic statistics, respectively

$$\hat{\Delta}_n^* = \int_0^\tau \frac{L[U_n](r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) \, dr, \quad \hat{\delta}_n = \int_0^1 L[U_n^*](\nu)^2 \, d\nu.$$

Indeed, we will show that for both statistics we have the convergence (under  $\mathcal{H}_0$ )

$$\hat{\Delta}_n^* \Longrightarrow \int_0^1 w_\nu^2 \, d\nu, \quad \hat{\delta}_n \Longrightarrow \int_0^1 w_\nu^2 \, d\nu.$$

Therefore the tests

$$\hat{\psi}_n^* = \mathbb{I}_{\{\hat{\Delta}_n^* > c_\alpha\}}, \quad \hat{\varphi}_n = \mathbb{I}_{\{\hat{\delta}_n > c_\alpha\}},$$

with the threshold  $c_\alpha$  satisfying the equation

$$\mathbf{P} \left( \int_0^1 w_\nu^2 \, d\nu > c_\alpha \right) = \alpha,$$

will be of asymptotic size  $\alpha \in (0, 1)$ , i.e., they belong to the class

$$\mathcal{K}_\alpha = \left\{ \phi_n : \lim_{n \rightarrow \infty} \mathbf{E}_\vartheta \phi_n = \alpha, \forall \vartheta \in \Theta = (a, b) \right\}$$

and will be ADF. We propose these tests in Theorem 2.3.3 and 2.4.4 below.

## 2.2 Minimum distance estimator

Suppose that we have continuous time observations of a periodic Poisson process

$$X^n = (X_t, 0 \leq t \leq T = n\tau) \tag{2.7}$$

of known period  $\tau > 0$ . We have for  $0 \leq s < t$  and  $k = 0, 1, 2, \dots$

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\Lambda(s, t)^k}{k!} e^{-\Lambda(s, t)}, \quad \Lambda(s, t) = \int_s^t \lambda(v) \, dv.$$

We see that  $X_t - X_s$  is a Poisson random variable with mean  $\Lambda(s, t)$ . In particular, the mean and intensity functions of  $X_t$  are given respectively by  $\Lambda(t) = \Lambda(0, t)$  and  $\lambda(t + k\tau) = \lambda(t)$ .

Our goal is to construct GoF tests for the parametric null hypothesis

$$\mathcal{H}_0 \quad : \quad \Lambda(\cdot) \in \{\Lambda(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where  $\Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, v) \, dv$  is a known smooth function. We introduce the MDE for the unknown parameter  $\vartheta$

$$\vartheta_n^* = \arg \min_{\vartheta \in \Theta} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|,$$

where  $\hat{\Lambda}_n(\cdot)$  is defined by (2.2) and  $\|\cdot\|$  is  $\mathcal{L}^2(0, \tau)$ -norm, i.e.,

$$\left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|^2 = \int_0^\tau \left( \hat{\Lambda}_n(s) - \Lambda(\vartheta, s) \right)^2 \, ds.$$

Introduce the regularity conditions  $\mathcal{R}$ .

$\mathcal{R}_0$ . *The intensity function  $\lambda(t)$ ,  $0 \leq t \leq T$  of the observed process  $X^n$  belongs to the parametric family*

$$\{(\lambda(\vartheta, t), 0 \leq t \leq T), \vartheta \in \Theta = (a, b)\},$$

*i.e., there exists  $\vartheta_0 \in \Theta$  such that  $\lambda(t) = \lambda(\vartheta_0, t)$  for all  $t \in [0, T]$ .*

$\mathcal{R}_1$ . *The intensity function  $\lambda(\vartheta, t)$  is strictly positive, has two continuous bounded derivatives w.r.t.  $\vartheta$  and the derivatives  $\dot{\lambda}(\vartheta, t)$  and  $\ddot{\lambda}(\vartheta, t)$  are also  $\tau$ -periodic.*

$\mathcal{R}_2$ . *The Fisher information (2.3) is strictly positive.*

$\mathcal{R}_3$ . *The identifiability condition is as follows : for any  $\nu > 0$  and  $\vartheta_0 \in \Theta$*

$$g(\nu) \equiv \inf_{\vartheta: |\vartheta - \vartheta_0| > \nu} \int_0^\tau [\Lambda(\vartheta, t) - \Lambda(\vartheta_0, t)]^2 \, dt > 0.$$

We shall mention that conditions of this type appear in almost all statistical problems of parametric estimation for regular case (see, e.g., in [45]). In the

presentation below, we suppose that these conditions and the null hypothesis  $\mathcal{H}_0$  are always fulfilled.

The properties and the asymptotic ( $n \rightarrow \infty$ ) behavior of the MDE for this model were studied under regularity conditions  $\mathcal{R}$  in [45]. The MDE  $\vartheta_n^*$  satisfies the *minimum distance equation* (MDEq) (can be found in [45])

$$\int_0^\tau \left( \hat{\Lambda}_n(s) - \Lambda(\vartheta_n^*, s) \right) \dot{\Lambda}(\vartheta_n^*, s) \, ds = 0. \quad (2.8)$$

Denote  $u_n^{**} = \sqrt{n}(\vartheta_n^* - \vartheta)$ . By Taylor's formula,  $\Lambda(\vartheta_n^*, s) - \Lambda(\vartheta, s) = \frac{u_n^{**}}{\sqrt{n}} \dot{\Lambda}(\tilde{\vartheta}, s)$ , the equation (2.8) becomes

$$\int_0^\tau \left( \hat{\Lambda}_n(s) - \Lambda(\vartheta, s) - \frac{u_n^{**}}{\sqrt{n}} \dot{\Lambda}(\tilde{\vartheta}, s) \right) \dot{\Lambda}(\vartheta_n^*, s) \, ds = 0,$$

where  $|\tilde{\vartheta} - \vartheta| \leq |\vartheta_n^* - \vartheta|$ . Indeed after replacing  $\hat{\Lambda}_n(\cdot)$  by (2.2), we obtain

$$\begin{aligned} u_n^{**} &= \left( \int_0^\tau \dot{\Lambda}(\vartheta_n^*, s) \dot{\Lambda}(\tilde{\vartheta}, s) \, ds \right)^{-1} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau (X_j(s) - \Lambda(\vartheta, s)) \dot{\Lambda}(\vartheta_n^*, s) \, ds \\ &= \left( \int_0^\tau \dot{\Lambda}(\vartheta_n^*, s) \dot{\Lambda}(\tilde{\vartheta}, s) \, ds \right)^{-1} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau \int_0^s d\pi_j(v) \dot{\Lambda}(\vartheta_n^*, s) \, ds. \end{aligned}$$

Here we define the centered Poisson process by  $\pi_j(v) = X_j(v) - \Lambda(\vartheta, v)$ . Using Fubini's theorem, the random variable  $u_n^{**}$  admits the following representation :

$$u_n^{**} = \left( \int_0^\tau \dot{\Lambda}(\vartheta_n^*, s) \dot{\Lambda}(\tilde{\vartheta}, s) \, ds \right)^{-1} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta_n^*, s) \, ds \, d\pi_j(v).$$

By the central limit theorem for stochastic integrals (see, e.g., [38]), we have the convergence in distribution

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta_n^*, s) \, ds \, d\pi_j(v) \implies \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, s) \, ds \sqrt{\lambda(\vartheta, v)} \, dW_v,$$

where  $W_v, 0 \leq v \leq \tau$  is some Wiener process. Therefore we obtain

$$\sqrt{n}(\vartheta_n^* - \vartheta) \implies J(\vartheta)^{-1} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, s) \, ds \sqrt{\lambda(\vartheta, v)} \, dW_v, \quad (2.9)$$

where

$$J(\vartheta) = \int_0^\tau \dot{\Lambda}(\vartheta, s)^2 \, ds. \quad (2.10)$$



Hence, under the regularity conditions  $\mathcal{R}$ , we have the asymptotic normality

$$\mathcal{L}_{\vartheta}\{\sqrt{n}(\vartheta_n^* - \vartheta)\} \implies \mathcal{L}\{\xi\} = \mathcal{N}(0, \sigma(\vartheta)^2).$$

Here the variance limit is as follows :

$$\sigma(\vartheta)^2 = J(\vartheta)^{-2} \int_0^\tau \left( \int_v^\tau \dot{\Lambda}(\vartheta, s) ds \right)^2 \lambda(\vartheta, v) dv. \quad (2.11)$$

Our aim is to construct GoF tests, which are ADF, i.e., we seek the tests statistics whose limits distribution under hypothesis  $\mathcal{H}_0$  do not depend on the underlying model given by the function  $\Lambda(\cdot, \cdot)$  and parameter  $\vartheta$ . Indeed, we apply the special linear transformation (1.26) to the statistic  $U(\cdot)$  (2.6) in order to obtain the ADF GoF tests. We shall mention that we will treat the problem of the construction of two tests. The first one is based on the empirical basic statistic (Section 2.3) and the second one on the score-function statistic (Section 2.4).

## 2.3 Empirical basic statistic

We begin by constructing the ADF GoF test based on the empirical basic statistic. Consider the Cramér-von Mises type statistic

$$\Delta_n = \int_0^\tau \frac{U_n(\vartheta_n^*, r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) dr.$$

Here

$$U_n(\vartheta_n^*, r) = \frac{\zeta_n(\vartheta_n^*, r)}{\sqrt{\Lambda(\vartheta_n^*, \tau)}}, \quad \zeta_n(\vartheta_n^*, r) = \sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_n^*, r) \right), \quad (2.12)$$

where  $\vartheta_n^*$  is the MDE and  $\hat{\Lambda}_n(\cdot)$  is defined by (2.2).

Below it is shown that the limit statistic of  $\Delta_n$  depends strongly on the model and the parameter  $\vartheta$ . Therefore, we introduce the linear transformation (1.26) in order to obtain the ADF limit statistic.

Define the functions

$$g(\vartheta, s) = C_1(\vartheta)^{-\frac{1}{2}} \int_{v(s)}^\tau \dot{\Lambda}(\vartheta, z) dz \quad (2.13)$$

and

$$h(\vartheta, s) = C_1(\vartheta)^{\frac{1}{2}} J(\vartheta)^{-1} \Lambda(\vartheta, \tau) \frac{\dot{\lambda}(\vartheta, v(s))}{\lambda(\vartheta, v(s))}, \quad (2.14)$$

where  $v(s)$  is the inverse function of  $s = \Lambda(\vartheta, v) \Lambda(\vartheta, \tau)^{-1}$ ,  $J(\vartheta)$  is given by (2.10) and

$$\int_0^1 g(\vartheta, s)^2 ds = 1, \quad C_1(\vartheta) = \int_0^1 \left( \int_{v(s)}^{\tau} \dot{\Lambda}(\vartheta, z) dz \right)^2 ds.$$

Let us denote the Gaussian process

$$U(\nu) = w(\nu) - \int_0^1 g(\vartheta, s) dw(s) \int_0^\nu h(\vartheta, s) ds, \quad (2.15)$$

where  $w(\nu), 0 \leq \nu \leq 1$  is a Wiener process. We have the following result.

**Lemma 2.3.4.** *Let the regularity conditions  $\mathcal{R}$  be fulfilled, then the convergence*

$$\Delta_n \implies \Delta = \int_0^1 U(\nu)^2 d\nu \quad (2.16)$$

holds, under  $\mathcal{H}_0$ .

**Proof.** Let us first show that (under  $\mathcal{H}_0$ )

$$U_n(\vartheta_n^*, r) \implies U(\nu), \quad 0 \leq \nu \leq 1. \quad (2.17)$$

We prove this as follows. By Taylor's formula, the process  $\zeta_n(\cdot, \cdot)$  defined by (2.12) has the following representation

$$\begin{aligned} \zeta_n(\vartheta_n^*, r) &= \sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_n^*, r) \right) \\ &= \sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta, r) \right) - \sqrt{n} (\vartheta_n^* - \vartheta) \dot{\Lambda}(\tilde{\vartheta}, r) + o(1). \end{aligned}$$

The central limit theorem yields the convergence in distribution

$$\sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta, r) \right) \implies W(\Lambda(\vartheta, r)) \sim \mathcal{N}(0, \Lambda(\vartheta, r)),$$

where  $W(\cdot)$  is a Wiener process.

Under the regularity conditions  $\mathcal{R}$ , we have the convergence

$$\sqrt{n}(\vartheta_n^* - \vartheta) \implies J(\vartheta)^{-1} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) dz dW(\Lambda(\vartheta, v)) \sim \mathcal{N}(0, \sigma(\vartheta)^2), \quad (2.18)$$

where  $J(\vartheta)$  and  $\sigma(\vartheta)^2$  are given by (2.10)–(2.11), respectively. Consequently, under  $\mathcal{H}_0$ , we get

$$\begin{aligned} U_n(\vartheta_n^*, r) &= \frac{\zeta_n(\vartheta_n^*, r)}{\sqrt{\Lambda(\vartheta_n^*, \tau)}} \Longrightarrow \frac{W(\Lambda(\vartheta, r))}{\sqrt{\Lambda(\vartheta, \tau)}} \\ &\quad - J(\vartheta)^{-1} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) \, dz \, dW(\Lambda(\vartheta, v)) \frac{\dot{\Lambda}(\vartheta, r)}{\sqrt{\Lambda(\vartheta, \tau)}} \equiv u(r). \end{aligned} \quad (2.19)$$

Further, we can write

$$\dot{\Lambda}(\vartheta, r) = \int_0^r \dot{\lambda}(\vartheta, v) \, dv = \int_0^r \frac{\dot{\lambda}(\vartheta, v)}{\lambda(\vartheta, v)} \, d\Lambda(\vartheta, v).$$

Hence the limit process  $u(\cdot)$  defined by (2.19) admits the following representation

$$\begin{aligned} u(r) &\equiv \frac{W(\Lambda(\vartheta, r))}{\sqrt{\Lambda(\vartheta, \tau)}} \\ &\quad - J(\vartheta)^{-1} \Lambda(\vartheta, \tau) \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) \, dz \frac{dW(\Lambda(\vartheta, v))}{\sqrt{\Lambda(\vartheta, \tau)}} \int_0^r \frac{\dot{\lambda}(\vartheta, v)}{\lambda(\vartheta, v)} \frac{d\Lambda(\vartheta, v)}{\Lambda(\vartheta, \tau)}. \end{aligned} \quad (2.20)$$

Let us change the variables

$$s = \frac{\Lambda(\vartheta, v)}{\Lambda(\vartheta, \tau)}, \quad \nu = \frac{\Lambda(\vartheta, r)}{\Lambda(\vartheta, \tau)}, \quad w(\nu) = \frac{W(\Lambda(\vartheta, r))}{\sqrt{\Lambda(\vartheta, \tau)}}, \quad w(s) = \frac{W(\Lambda(\vartheta, v))}{\sqrt{\Lambda(\vartheta, \tau)}}. \quad (2.21)$$

Then,

$$\begin{aligned} U_n(\vartheta_n^*, r) &\Longrightarrow w(\nu) \\ &\quad - J(\vartheta)^{-1} \Lambda(\vartheta, \tau) \int_0^1 \int_{v(s)}^\tau \dot{\Lambda}(\vartheta, z) \, dz \, dw(s) \int_0^\nu \frac{\dot{\lambda}(\vartheta, v(s))}{\lambda(\vartheta, v(s))} \, ds \\ &= w(\nu) - \int_0^1 g(\vartheta, s) \, dw(s) \int_0^\nu h(\vartheta, s) \, ds = U(\nu), \end{aligned}$$

where  $v(s)$  is the inverse function of  $s = \Lambda(\vartheta, v) \Lambda(\vartheta, \tau)^{-1}$ ,  $g(\cdot, \cdot)$  and  $h(\cdot, \cdot)$  are defined by (2.13) and (2.14), respectively. We finally obtain the convergence (2.17) under  $\mathcal{H}_0$ .

Let  $Z_n(\cdot, \cdot)$  and  $Z(\cdot)$  be given by

$$Z_n(\vartheta_n^*, r) = U_n(\vartheta_n^*, r)^2, \quad Z(\nu) = U(\nu)^2.$$

Now in order to get the final result (2.16), we will prove the convergence of the integrals, i.e., we have to verify the following properties : (see [27], Theorem A1.22)

1. For any  $r_1, \dots, r_k$  and  $\nu_1, \dots, \nu_k$ , the convergence of the vectors

$$(Z_n(\vartheta_n^*, r_1), \dots, Z_n(\vartheta_n^*, r_k)) \implies (Z(\nu_1), \dots, Z(\nu_k)) \quad (2.22)$$

holds.

2. For  $r_i \in [0, \tau]$ ,  $i = 1, 2$ ,

$$\mathbf{E}_\vartheta |Z_n(\vartheta_n^*, r_1) - Z_n(\vartheta_n^*, r_2)| \leq C |r_1 - r_2|^{\frac{1}{2}}, \quad (2.23)$$

where  $C > 0$  is some constant.

3. For any  $\kappa > 0$ ,

$$\sup_{n, \vartheta} \mathbf{E}_\vartheta Z_n(\vartheta_n^*, r) < \kappa. \quad (2.24)$$

By using the same arguments to prove (2.17), for any real numbers  $\nu_1, \dots, \nu_k$ ,  $r_1, \dots, r_k$  and  $\nu_1, \dots, \nu_k$ , we obtain the convergence of the vectors

$$\sum_{l=1}^k \nu_l U_n(\vartheta_n^*, r_l) \implies \sum_{l=1}^k \nu_l U(\nu_l).$$

Therefore, by continuous mapping theorem, we have the convergence of finite-dimensional distributions

$$(Z_n(\vartheta_n^*, r_1), \dots, Z_n(\vartheta_n^*, r_k)) \implies (Z(\nu_1), \dots, Z(\nu_k)).$$

Moreover, using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} & \left( \mathbf{E}_\vartheta |Z_n(\vartheta_n^*, r_1) - Z_n(\vartheta_n^*, r_2)| \right)^2 \\ &= \left( \mathbf{E}_\vartheta |U_n(\vartheta_n^*, r_1) - U_n(\vartheta_n^*, r_2)| |U_n(\vartheta_n^*, r_1) + U_n(\vartheta_n^*, r_2)| \right)^2 \\ &\leq \mathbf{E}_\vartheta |U_n(\vartheta_n^*, r_1) - U_n(\vartheta_n^*, r_2)|^2 \mathbf{E}_\vartheta |U_n(\vartheta_n^*, r_1) + U_n(\vartheta_n^*, r_2)|^2. \end{aligned}$$

For the first mathematical expectation, by applying Taylor's formula, we get

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \sqrt{n} \left( \hat{\Lambda}_n(r_1) - \Lambda(\vartheta_n^*, r_1) \right) - \sqrt{n} \left( \hat{\Lambda}_n(r_2) - \Lambda(\vartheta_n^*, r_2) \right) \right|^2 \\ &= \frac{1}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_{r_2}^{r_1} [dX_j(r) - \lambda(\vartheta_n^*, r) dr] \right|^2 \\ &\leq \frac{2}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_{r_2}^{r_1} [dX_j(r) - \lambda(\vartheta, r) dr] \right|^2 + \frac{2}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_{r_2}^{r_1} [\lambda(\vartheta, r) - \lambda(\vartheta_n^*, r)] dr \right|^2 \end{aligned}$$

$$\begin{aligned} &\leq \frac{2}{n} \sum_{j=1}^n \mathbf{E}_\vartheta \left| \int_{r_2}^{r_1} d\pi_j(r) \right|^2 + \frac{2}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_{r_2}^{r_1} (\vartheta - \vartheta_n^*) \dot{\lambda}(\tilde{\vartheta}, r) dr \right|^2 \\ &\leq 2 \int_{r_2}^{r_1} \lambda(\vartheta, r) dr + 2 n \mathbf{E}_\vartheta (\vartheta_n^* - \vartheta)^2 \left( \int_{r_2}^{r_1} \dot{\lambda}(\tilde{\vartheta}, r) dr \right)^2. \end{aligned}$$

By the regularity condition  $\mathcal{R}_1$ ,

$$\mathbf{E}_\vartheta |U_n(\vartheta_n^*, r_1) - U_n(\vartheta_n^*, r_2)|^2 \leq C |r_1 - r_2|.$$

Similarly, for the second mathematical expectation, we have for  $r_i \in [0, \tau], i = 1, 2$

$$\begin{aligned} &\mathbf{E}_\vartheta \left| \sqrt{n} \left( \hat{\Lambda}_n(r_1) - \Lambda(\vartheta_n^*, r_1) \right) + \sqrt{n} \left( \hat{\Lambda}_n(r_2) - \Lambda(\vartheta_n^*, r_2) \right) \right|^2 \\ &\leq \frac{2}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_1} [dX_j(r) - \lambda(\vartheta_n^*, r) dr] \right|^2 + \frac{2}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_2} [dX_j(r) - \lambda(\vartheta_n^*, r) dr] \right|^2 \\ &\leq \frac{4}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_1} [dX_j(r) - \lambda(\vartheta, r) dr] \right|^2 + \frac{4}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_2} [dX_j(r) - \lambda(\vartheta, r) dr] \right|^2 \\ &\quad + \frac{4}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_1} (\vartheta - \vartheta_n^*) \dot{\lambda}(\tilde{\vartheta}, r) dr \right|^2 + \frac{4}{n} \mathbf{E}_\vartheta \left| \sum_{j=1}^n \int_0^{r_2} (\vartheta - \vartheta_n^*) \dot{\lambda}(\tilde{\vartheta}, r) dr \right|^2 \\ &\leq 4 \int_0^{r_1} \lambda(\vartheta, r) dr + 4 \int_0^{r_2} \lambda(\vartheta, r) dr + 4 n \mathbf{E}_\vartheta (\vartheta_n^* - \vartheta)^2 \left( \int_0^{r_1} \dot{\lambda}(\tilde{\vartheta}, r) dr \right)^2 \\ &\quad + 4 n \mathbf{E}_\vartheta (\vartheta_n^* - \vartheta)^2 \left( \int_0^{r_2} \dot{\lambda}(\tilde{\vartheta}, r) dr \right)^2. \end{aligned}$$

Hence, by regularity condition  $\mathcal{R}_1$ ,

$$\mathbf{E}_\vartheta |U_n(\vartheta_n^*, r_1) + U_n(\vartheta_n^*, r_2)|^2 \leq C.$$

From what has already been proved, it follows that the estimate (2.23) holds.

Now to check (2.24), we can write

$$\begin{aligned} &\mathbf{E}_\vartheta \left| \sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta_n^*, r) \right) \right|^2 \\ &\leq 2 \mathbf{E}_\vartheta \left| \sqrt{n} \left( \hat{\Lambda}_n(r) - \Lambda(\vartheta, r) \right) \right|^2 + 2 \mathbf{E}_\vartheta \left| \sqrt{n} \left( \Lambda(\vartheta, r) - \Lambda(\vartheta_n^*, r) \right) \right|^2 \\ &\leq 2 \mathbf{E}_\vartheta (X_1(r) - \Lambda(\vartheta, r))^2 + 2 n \mathbf{E}_\vartheta (\vartheta_n^* - \vartheta)^2 \dot{\Lambda}(\tilde{\vartheta}, r)^2 \\ &\leq 2 \Lambda(\vartheta, r) + C \dot{\Lambda}(\tilde{\vartheta}, r)^2. \end{aligned}$$

We thus get (2.24) by the regularity condition  $\mathcal{R}_1$ .

Finally, the convergence of finite-dimensional distributions (2.22), the estimate (2.23) and the condition (2.24) allow us to verify the convergence (2.16) and the Lemma is proved.

It is obvious that the limit

$$\Delta = \int_0^1 U(\nu)^2 d\nu,$$

where  $U(\cdot)$  is defined by (2.15), depends on the model  $\Lambda(\cdot, \cdot)$  and the unknown parameter  $\vartheta$ . By applying the special linear transformation  $L[\cdot]$  (1.26) to the studied process  $U(\cdot)$ , we obtain

$$\int_0^1 L[U](\nu)^2 d\nu = \int_0^1 w_\nu^2 d\nu. \quad (2.25)$$

In view of the above property, we will introduce the “empirical version” of the test statistic with the same limit (2.25). Indeed we will show that

$$\hat{\Delta}_n^* = \int_0^\tau \frac{L[U_n](r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) dr \implies \int_0^1 w_\nu^2 d\nu$$

and therefore the test  $\hat{\psi}_n^* = \mathbb{I}_{\{\hat{\Delta}_n^* > c_\alpha\}}$  will be ADF.

Define the functions

$$\tilde{g}(\vartheta, v) = \tilde{C}_1(\vartheta)^{-\frac{1}{2}} \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \quad (2.26)$$

and

$$\tilde{h}(\vartheta, v) = \tilde{C}_1(\vartheta)^{\frac{1}{2}} J(\vartheta)^{-1} \Lambda(\vartheta, \tau) \frac{\dot{\lambda}(\vartheta, v)}{\lambda(\vartheta, v)}. \quad (2.27)$$

Here  $J(\vartheta)$  is defined by (2.10) and

$$\tilde{C}_1(\vartheta) = \int_0^\tau \left( \int_q^\tau \dot{\Lambda}(\vartheta, z) dz \right)^2 \frac{d\Lambda(\vartheta, q)}{\Lambda(\vartheta, \tau)}.$$

Then denote

$$\tilde{I}_1 = \frac{1}{\Lambda(\vartheta, \tau)} \int_0^s \tilde{g}(\vartheta, v)^2 d\Lambda(\vartheta, v), \quad \tilde{I}_2 = \frac{1}{\Lambda(\vartheta, \tau)} \int_0^s \tilde{h}(\vartheta, v) \tilde{g}(\vartheta, v) d\Lambda(\vartheta, v),$$

$$\tilde{I}_3 = \frac{1}{\Lambda(\vartheta, \tau)} \int_0^s \tilde{h}(\vartheta, v) d\Lambda(\vartheta, v), \quad \tilde{I}_4 = \frac{1}{\Lambda(\vartheta, \tau)} \int_0^s \tilde{h}(\vartheta, v)^2 d\Lambda(\vartheta, v)$$

and put

$$\tilde{I}_5 = \frac{1}{\Lambda(\vartheta, \tau)} \int_0^s \tilde{g}(\vartheta, v) d\Lambda(\vartheta, v).$$

This allows us to introduce the functions  $\tilde{\varphi}_1(\cdot, \cdot)$ ,  $\tilde{\varphi}_2(\cdot, \cdot)$  and  $\tilde{\psi}_2(\cdot, \cdot)$  given by (1.23), (1.24) and (1.25), respectively, where instead of  $g$  and  $h$ , we use  $\tilde{g} = \tilde{g}(\vartheta, s)$  and  $\tilde{h} = \tilde{h}(\vartheta, s)$ .

In the sequel we replace the unknown parameter  $\vartheta$  by the MDE  $\vartheta_n^*$  in  $\tilde{g}(\cdot, \cdot)$ ,  $\tilde{h}(\cdot, \cdot)$ ,  $\tilde{\varphi}_1(\cdot, \cdot)$ ,  $\tilde{\varphi}_2(\cdot, \cdot)$  and  $\tilde{\psi}_2(\cdot, \cdot)$  and we define the “empirical versions”  $\tilde{g}_n(\cdot, \cdot)$ ,  $\tilde{h}_n(\cdot, \cdot)$ ,  $\tilde{\varphi}_{1,n}(\cdot, \cdot)$ ,  $\tilde{\varphi}_{2,n}(\cdot, \cdot)$  and  $\tilde{\psi}_{2,n}(\cdot, \cdot)$ , respectively.

The consistency of the estimator  $\vartheta_n^*$  and the regularity condition  $\mathcal{R}_1$  give us the uniform convergence ( $n \rightarrow \infty$ )

$$\sup_{s \in [0, \tau]} |\tilde{\varphi}_{2,n}(\vartheta_n^*, s) - \tilde{\varphi}_2(\vartheta, s)| \longrightarrow 0.$$

Hence we consider the function

$$\tilde{\varphi}_{2,n}^+(\vartheta_n^*, s) = \begin{cases} \tilde{\varphi}_{2,n}(\vartheta_n^*, s)^{-1}, & \text{if } \tilde{\varphi}_{2,n}(\vartheta_n^*, s) > 0, \\ 0, & \text{else.} \end{cases} \quad (2.28)$$

We shall mention that this function coincides asymptotically with  $\tilde{\varphi}_2(\vartheta, s)^{-1}$  and therefore the limit distribution does not changed.

Let us introduce (formally) the following statistic

$$\begin{aligned} \hat{W}_n(r) = U_n(\vartheta_n^*, r) + \frac{1}{\Lambda(\vartheta_n^*, \tau)} \int_0^r \int_0^s \tilde{\varphi}_{2,n}^+(\vartheta_n^*, s) \left[ \tilde{\varphi}_{1,n}(\vartheta_n^*, s) \tilde{h}_n(\vartheta_n^*, q) \right. \\ \left. + \tilde{\psi}_{2,n}(\vartheta_n^*, s) \tilde{g}_n(\vartheta_n^*, q) \right] dU_n(\vartheta_n^*, q) \lambda(\vartheta_n^*, s) ds. \end{aligned} \quad (2.29)$$

Here the process  $U_n(\cdot, \cdot)$  is defined by (2.12).

In order to get the ADF GoF test, it will be necessary to prove the convergence

$$\hat{W}_n(r) \Longrightarrow L[U](\nu) = w_\nu.$$

Thus the test based on  $\hat{W}_n(\cdot)$  will be ADF. It is easily seen that the process

$$A_n(\vartheta_n^*, s) = \int_0^s \tilde{g}_n(\vartheta_n^*, q) dU_n(\vartheta_n^*, q)$$

has the following representation

$$\begin{aligned} A_n(\vartheta_n^*, s) = \frac{1}{\sqrt{n} \sqrt{\Lambda(\vartheta_n^*, \tau)}} \sum_{j=1}^n \int_0^s \tilde{g}_n(\vartheta_n^*, q) dX_j(q) \\ - \frac{\sqrt{n}}{\sqrt{\Lambda(\vartheta_n^*, \tau)}} \int_0^s \tilde{g}_n(\vartheta_n^*, q) \lambda(\vartheta_n^*, q) dq. \end{aligned} \quad (2.30)$$

It is clear that we can not calculate this process directly, because in the first integral the integrand contains the MDE  $\vartheta_n^*$ , which depends on the whole trajectory  $X^n = (X_t, 0 \leq t \leq T = n\tau)$ . Therefore the corresponding stochastic integral

$$\int_0^s \tilde{g}_n(\vartheta_n^*, q) dX_j(q) \quad (2.31)$$

is not well defined. Indeed, to avoid this problem, we use an approach, which is based on some *preliminary consistent estimator*  $\bar{\vartheta}_N$ . Note that this approach was applied in the similar problem in [43]. Hence we can consider two different estimators in the calculation of the process  $A_n(\cdot, \cdot)$  as follows : For the stochastic integral (2.31), it is sufficient for  $\bar{\vartheta}_N$  to be consistent and for the second ordinary integral

$$\int_0^s \tilde{g}_n(\vartheta_n^*, q) \lambda(\vartheta_n^*, q) dq$$

we need the asymptotic normality of  $\vartheta_n^*$ .

Let us introduce the *preliminary estimator* constructed by the first  $N = [\sqrt{n}]$  observations  $X^N = (X_1, \dots, X_N)$

$$\bar{\vartheta}_N = \arg \min_{\vartheta \in \Theta} \int_0^\tau [\hat{\Lambda}_N(t) - \Lambda(\vartheta, t)]^2 dt. \quad (2.32)$$

Here  $N$  is the entire part of  $\sqrt{n}$  and  $\hat{\Lambda}_N(\cdot)$  is the empirical mean function

$$\hat{\Lambda}_N(t) = \frac{1}{N} \sum_{j=1}^N X_j(t),$$

where  $X_j(t) = X_{\tau(j-1)+t} - X_{\tau(j-1)}$ ,  $t \in [0, \tau]$ . It can be shown that this estimator is consistent and asymptotically normal under regularity conditions  $\mathcal{R}$  (see [45] for more details).

Consequently, we modify the statistic  $A_n(\cdot, \cdot)$  given by (2.30) as follows :

$$\begin{aligned} \hat{A}_n(\bar{\vartheta}_N, s) &= \frac{1}{\sqrt{n}\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \sum_{j=N+1}^n \int_0^s \tilde{g}_n(\bar{\vartheta}_N, q) dX_j(q) \\ &\quad - \frac{\sqrt{n}}{\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \int_0^s \tilde{g}_n(\bar{\vartheta}_N, q) \lambda(\vartheta_n^*, q) dq. \end{aligned} \quad (2.33)$$

Now the *preliminary estimator*  $\bar{\vartheta}_N$  and the processes  $(X_j(\cdot), j = N + 1, \dots, n)$  are independent and the stochastic integral w.r.t. the Poisson process

$$\int_0^s \tilde{g}_n(\bar{\vartheta}_N, q) dX_j(q)$$



is well defined. We shall mention that this stochastic integral is called Stieltjes integral and its definition and properties can be found in Liptser and Shirayev [48], Section 18.4.

In the same manner, we can see that

$$\begin{aligned} \hat{B}_n(\bar{\vartheta}_N, s) &= \frac{1}{\sqrt{n}\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \sum_{j=N+1}^n \int_0^s \tilde{h}_n(\bar{\vartheta}_N, q) dX_j(q) \\ &\quad - \frac{\sqrt{n}}{\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \int_0^s \tilde{h}_n(\bar{\vartheta}_N, q) \lambda(\vartheta_n^*, q) dq \end{aligned} \quad (2.34)$$

is the modified version of the process

$$B_n(\vartheta_n^*, s) = \int_0^s \tilde{h}_n(\vartheta_n^*, q) dU_n(\vartheta_n^*, q).$$

Therefore, we introduce

$$\hat{\Delta}_n^* = \int_0^\tau \frac{\hat{W}_n^*(r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) dr, \quad \hat{\Delta} \equiv \int_0^1 w_\nu^2 d\nu,$$

where  $w_\nu, 0 \leq \nu \leq 1$  is a Wiener process and the formal expression (2.29) is replaced by the following statistic

$$\begin{aligned} \hat{W}_n^*(r) &= U_n(\vartheta_n^*, r) + \frac{1}{\Lambda(\bar{\vartheta}_N, \tau)} \int_0^r \tilde{\varphi}_{2,n}^+(\bar{\vartheta}_N, s) \left[ \tilde{\varphi}_{1,n}(\bar{\vartheta}_N, s) \hat{B}_n(\bar{\vartheta}_N, s) \right. \\ &\quad \left. + \tilde{\psi}_{2,n}(\bar{\vartheta}_N, s) \hat{A}_n(\bar{\vartheta}_N, s) \right] \lambda(\bar{\vartheta}_N, s) ds. \end{aligned}$$

Here the processes  $U_n(\cdot, \cdot)$ ,  $\hat{A}_n(\cdot, \cdot)$  and  $\hat{B}_n(\cdot, \cdot)$  are given by (2.12), (2.33) and (2.34), respectively. The first test is given in the following Theorem.

**Theorem 2.3.3.** *Suppose that the conditions of regularity  $\mathcal{R}$  and  $\mathcal{R}_4$  are fulfilled, then the test*

$$\hat{\psi}_n^* = \mathbb{I}_{\{\hat{\Delta}_n^* > c_\alpha\}}, \quad \mathbf{P}(\hat{\Delta} > c_\alpha) = \alpha$$

is ADF and of asymptotic size  $\alpha \in (0, 1)$ .

**Proof.** The main idea of the proof is to show the convergence

$$\hat{\Delta}_n^* \Longrightarrow \hat{\Delta} \quad (2.35)$$

under hypothesis  $\mathcal{H}_0$ .

For the first term of  $\hat{W}_n^*(\cdot)$ , we already proved the convergence

$$U_n(\vartheta_n^*, r) \implies U(\nu), \quad 0 \leq \nu \leq 1,$$

where  $U(\cdot)$  is defined by (2.15) (see proof of Lemma 2.3.4).

We now apply Taylor's formula and the regularity condition  $\mathcal{R}_1$ , to obtain

$$\left| \int_v^\tau \dot{\Lambda}(\bar{\vartheta}_N, z) dz - \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \right| \leq |\bar{\vartheta}_N - \vartheta| \int_v^\tau |\ddot{\Lambda}(\tilde{\vartheta}, z)| dz \leq C |\bar{\vartheta}_N - \vartheta|,$$

where  $C$  is some positive constant. Therefore, the consistency of the *preliminary estimator*  $\bar{\vartheta}_N$  gives the uniform convergence (as  $n \rightarrow \infty$ ) w.r.t.  $v \in [0, \tau]$

$$\sup_{v \in [0, \tau]} |\tilde{g}_n(\bar{\vartheta}_N, v) - \tilde{g}(\vartheta, v)| \rightarrow 0.$$

By using the same arguments, we have the uniform convergence w.r.t.  $v \in [0, \tau]$

$$\begin{aligned} \sup_{v \in [0, \tau]} |\tilde{h}_n(\bar{\vartheta}_N, v) - \tilde{h}(\vartheta, v)| &\rightarrow 0, & \sup_{v \in [0, \tau]} |J_n(\bar{\vartheta}_N) - J(\vartheta)| &\rightarrow 0, \\ \sup_{v \in [0, \tau]} |\tilde{\psi}_{2,n}(\bar{\vartheta}_N, v) - \tilde{\psi}_2(\vartheta, v)| &\rightarrow 0, & \sup_{v \in [0, \tau]} |\tilde{\varphi}_{1,n}(\bar{\vartheta}_N, v) - \tilde{\varphi}_1(\vartheta, v)| &\rightarrow 0. \end{aligned}$$

Then, we proceed to show that  $\hat{A}_n(\bar{\vartheta}_N, s) \implies \hat{A}(\vartheta, s)$ , where  $\hat{A}_n(\cdot, \cdot)$  is defined by (2.33) and

$$\begin{aligned} \hat{A}(\vartheta, s) &= \int_0^s \tilde{g}(\vartheta, q) du(q) = \int_0^s \tilde{g}(\vartheta, q) \frac{dW(\Lambda(\vartheta, q))}{\sqrt{\Lambda(\vartheta, \tau)}} \\ &\quad - \frac{\Lambda(\vartheta, \tau)}{J(\vartheta)} \int_0^\tau \int_q^\tau \dot{\Lambda}(\vartheta, z) dz \frac{dW(\Lambda(\vartheta, q))}{\sqrt{\Lambda(\vartheta, \tau)}} \int_0^s \tilde{g}(\vartheta, q) \frac{\dot{\lambda}(\vartheta, q)}{\lambda(\vartheta, q)} \frac{d\Lambda(\vartheta, q)}{\Lambda(\vartheta, \tau)}. \end{aligned}$$

Here the process  $u(\cdot)$  is defined by (2.20) and the function  $\tilde{g}(\vartheta, q)$  given by (2.26) is continuous w.r.t.  $q$ .

By Taylor's formula, the process (2.33) has the following representation

$$\begin{aligned} \hat{A}_n(\bar{\vartheta}_N, s) &= \frac{1}{\sqrt{n} \sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \sum_{j=N+1}^n \int_0^s \tilde{g}_n(\bar{\vartheta}_N, q) d\pi_j(q) \\ &\quad - \sqrt{n}(\vartheta_n^* - \vartheta) \frac{1}{\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \int_0^s \tilde{g}_n(\bar{\vartheta}_N, q) \dot{\lambda}(\tilde{\vartheta}, q) dq, \end{aligned}$$

where  $d\pi_j(q) = dX_j(q) - \lambda(\vartheta, q) dq$ . Using the central limit theorem for stochastic integrals and the representation given by (2.18), we obtain directly the convergence in distribution

$$\hat{A}_n(\bar{\vartheta}_N, s) \implies \hat{A}(\vartheta, s).$$

Now by the change of variables

$$v = \frac{\Lambda(\vartheta, q)}{\Lambda(\vartheta, \tau)}, \quad r = \frac{\Lambda(\vartheta, s)}{\Lambda(\vartheta, \tau)}, \quad w(v) = \frac{W(\Lambda(\vartheta, q))}{\sqrt{\Lambda(\vartheta, \tau)}}, \quad (2.36)$$

it is obvious that

$$\hat{A}(\vartheta, s) \equiv A(\vartheta, r),$$

where

$$A(\vartheta, r) = \int_0^r g(\vartheta, v) \, dw(v) - \int_0^1 g(\vartheta, v) \, dw(v) \int_0^r g(\vartheta, v) \, h(\vartheta, v) \, dv.$$

Here the functions  $g(\cdot, \cdot)$  and  $h(\cdot, \cdot)$  are defined by (2.13) and (2.14), respectively and  $w(\cdot)$  is some Wiener process.

Further, by the same arguments, we show that  $\hat{B}_n(\bar{\vartheta}_N, s) \implies \hat{B}(\vartheta, s)$ , where  $\hat{B}_n(\cdot, \cdot)$  is given by (2.34) and

$$\begin{aligned} \hat{B}(\vartheta, s) &= \int_0^s \tilde{h}(\vartheta, q) \, du(q) = \int_0^s \tilde{h}(\vartheta, q) \frac{dW(\Lambda(\vartheta, q))}{\sqrt{\Lambda(\vartheta, \tau)}} \\ &\quad - \frac{\Lambda(\vartheta, \tau)}{J(\vartheta)} \int_0^\tau \int_q^\tau \dot{\Lambda}(\vartheta, z) \, dz \frac{dW(\Lambda(\vartheta, q))}{\sqrt{\Lambda(\vartheta, \tau)}} \int_0^s \tilde{h}(\vartheta, q) \frac{\dot{\lambda}(\vartheta, q)}{\lambda(\vartheta, q)} \frac{d\Lambda(\vartheta, q)}{\Lambda(\vartheta, \tau)}. \end{aligned}$$

Note that the function  $\tilde{h}(\vartheta, q)$  defined by (2.27) is continuous w.r.t.  $q$ .

Using Taylor's formula, it is easily seen that

$$\begin{aligned} \hat{B}_n(\bar{\vartheta}_N, s) &= \frac{1}{\sqrt{n}\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \sum_{j=N+1}^n \int_0^s \tilde{h}_n(\bar{\vartheta}_N, q) \, d\pi_j(q) \\ &\quad - \sqrt{n}(\vartheta_n^* - \vartheta) \frac{1}{\sqrt{\Lambda(\bar{\vartheta}_N, \tau)}} \int_0^s \tilde{h}_n(\bar{\vartheta}_N, q) \, \dot{\lambda}(\bar{\vartheta}, q) \, dq. \end{aligned}$$

Moreover, we have by the change of variables (2.36)

$$\hat{B}(\vartheta, s) \equiv B(\vartheta, r),$$

where

$$B(\vartheta, r) = \int_0^r h(\vartheta, v) \, dw(v) - \int_0^1 g(\vartheta, v) \, dw(v) \int_0^r h(\vartheta, v)^2 \, dv.$$

Finally, the convergence in distribution (2.35) holds, and the Theorem is proved.

## 2.4 Score-function basic statistic

Our objective is to construct the ADF GoF test under hypothesis  $\mathcal{H}_0$ . To do this, we introduce (formally) the score-function process (can be found in [43])

$$u_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} [dX_j(v) - \lambda(\vartheta_n^*, v) dv], \quad (2.37)$$

which converges to  $u(\cdot)$  defined below by (2.38).

Then the transformation for the limit process  $u(\cdot)$

$$U(\nu) = \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta, v)}}{\dot{\lambda}(\vartheta, v)} du(v), \quad 0 \leq \nu \leq 1$$

has the following representation

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, s) dW(s) \int_0^\nu h(\vartheta, s) ds.$$

The last step is to apply the transformation  $L[\cdot]$  from Theorem 1.4.1 and to obtain the Wiener process

$$L[U](\nu) = w_\nu, \quad 0 \leq \nu \leq 1.$$

Below we realize this program with the empirical process  $u_n^*(\cdot)$ , apply the linear transformation  $L[\cdot]$  and obtain the convergence

$$\hat{\delta}_n = \int_0^1 L[U_n^*](\nu)^2 d\nu \implies \int_0^1 w_\nu^2 d\nu.$$

Therefore the test  $\hat{\varphi}_n = \mathbb{I}_{\{\hat{\delta}_n > c_\alpha\}}$  will be ADF in view of the distribution free limit statistic.

It is obvious that the process (2.37) has the following representation (formally)

$$u_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} dX_j(v) - \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \dot{\lambda}(\vartheta_n^*, v) dv.$$

Here we shall mention that the problem is about the definition of the stochastic integral w.r.t. the Poisson process  $(X_j(\cdot), j = 1, \dots, n)$

$$\int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} dX_j(v)$$

because the integrand depends on the future, i.e., the MDE  $\vartheta_n^*$  depends on the whole trajectory  $X^n$ . The correct definition of this integral will be given later. The task is now to find (as well formally) the limit of the process  $u_n^*(\cdot)$ .

Using Taylor's formula, the formal expansion of  $u_n^*(\cdot)$  provides us the following expressions :

$$\begin{aligned} u_n^*(t) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} [dX_j(v) - \lambda(\vartheta, v) dv - (\lambda(\vartheta_n^*, v) - \lambda(\vartheta, v)) dv] \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} d\pi_j(v) - \sqrt{n}(\vartheta_n^* - \vartheta) \int_0^t \frac{\dot{\lambda}(\vartheta_n^*, v)}{\lambda(\vartheta_n^*, v)} \dot{\lambda}(\tilde{\vartheta}, v) dv + o(1). \end{aligned}$$

The central limit theorem for stochastic integrals implies the convergence in distribution

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \frac{\dot{\lambda}(\vartheta, v)}{\lambda(\vartheta, v)} d\pi_j(v) \Longrightarrow \int_0^t \frac{\dot{\lambda}(\vartheta, v)}{\sqrt{\lambda(\vartheta, v)}} dW_v,$$

where  $W_v$  is some Wiener process. Then, by the representation (2.9), we obtain the convergence in distribution

$$\begin{aligned} u_n^*(t) \Longrightarrow u(t) &\equiv \int_0^t \frac{\dot{\lambda}(\vartheta, v)}{\sqrt{\lambda(\vartheta, v)}} dW_v \\ &- J(\vartheta)^{-1} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, s) ds \sqrt{\lambda(\vartheta, v)} dW_v \int_0^t \frac{\dot{\lambda}(\vartheta, v)^2}{\lambda(\vartheta, v)} dv. \end{aligned} \quad (2.38)$$

It is easily seen that the limit process  $u(\cdot)$  depends on  $\Lambda(\cdot, \cdot)$  and the unknown parameter  $\vartheta$ . Therefore the test constructed on the base of this statistic is not ADF. Indeed to obtain the ADF GoF test we propose a transformation for the process (2.38) as follows : We introduce the Gaussian process

$$U(\nu) = \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta, v)}}{\dot{\lambda}(\vartheta, v)} du(v), \quad 0 \leq \nu \leq 1 \quad (2.39)$$

and the functions

$$h(\vartheta, s) = C_2(\vartheta)^{\frac{1}{2}} \tilde{J}(\vartheta)^{-1} \frac{\dot{\lambda}(\vartheta, s\tau)}{\sqrt{\lambda(\vartheta, s\tau)}} \quad (2.40)$$

and

$$g(\vartheta, s) = C_2(\vartheta)^{-\frac{1}{2}} \int_{s\tau}^\tau \dot{\Lambda}(\vartheta, r) dr \sqrt{\lambda(\vartheta, s\tau)}. \quad (2.41)$$

Here

$$\tilde{J}(\vartheta) = \int_0^1 \lambda(\vartheta, s\tau) \, ds, \quad \int_0^1 g(\vartheta, s)^2 \, ds = 1$$

and

$$C_2(\vartheta) = \int_0^1 \lambda(\vartheta, r\tau) \left( \int_{r\tau}^\tau \dot{\lambda}(\vartheta, z) \, dz \right)^2 \, dr.$$

**Lemma 2.4.5.** *We have the equality*

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, s) \, dW(s) \int_0^\nu h(\vartheta, s) \, ds, \quad 0 \leq \nu \leq 1, \quad (2.42)$$

where  $W(\nu)$ ,  $0 \leq \nu \leq 1$  is a Wiener process.

**Proof.** Let us first show that

$$\begin{aligned} U(\nu) &= \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta, v)}}{\dot{\lambda}(\vartheta, v)} \, du(v) = \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} dW_v \\ &\quad - \frac{J(\vartheta)^{-1}}{\sqrt{\tau}} \int_0^\tau \int_v^\tau \dot{\lambda}(\vartheta, z) \, dz \sqrt{\lambda(\vartheta, v)} \, dW_v \int_0^{\nu\tau} \frac{\dot{\lambda}(\vartheta, v)}{\sqrt{\lambda(\vartheta, v)}} \, dv. \end{aligned}$$

By the change of variables  $s = \frac{v}{\tau}$  and  $W(\nu) = \tau^{-1/2}W_{\nu\tau}$ ,  $0 \leq \nu \leq 1$ , we can write

$$\begin{aligned} U(\nu) &= \frac{1}{\sqrt{\tau}} \int_0^\nu dW_{s\tau} \\ &\quad - \frac{\tilde{J}(\vartheta)^{-1}}{\sqrt{\tau}} \int_0^1 \int_{s\tau}^\tau \dot{\lambda}(\vartheta, z) \, dz \sqrt{\lambda(\vartheta, s\tau)} \, dW_{s\tau} \int_0^\nu \frac{\dot{\lambda}(\vartheta, s\tau)}{\sqrt{\lambda(\vartheta, s\tau)}} \, ds \\ &= W(\nu) - \int_0^1 g(\vartheta, s) \, dW(s) \int_0^\nu h(\vartheta, s) \, ds. \end{aligned}$$

Therefore, the process  $U(\cdot)$  has the expression (2.42) and the proof of the Lemma is completed. We shall mention that we obtain once more the same representation of the limit process given by (2.15) in Section 2.3.

Using the convergence of the “empirical version”  $U_n^*(\cdot)$  (defined below by (2.43)) to  $U(\cdot)$  (see proof of Theorem 2.4.4 below) and the continuous mapping theorem, we show that the following result

$$\delta_n = \frac{1}{\tau} \int_0^1 \left( \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta_n^*, v)}}{\dot{\lambda}(\vartheta_n^*, v)} \, du_n^*(v) \right)^2 \, d\nu \implies \delta \equiv \int_0^1 U(\nu)^2 \, d\nu$$

is satisfied. One can see that the test constructed on the base of this statistic

$$\psi_n^* = \mathbb{I}_{\{\delta_n > c_\alpha\}}, \quad \mathbf{P}(\delta > c_\alpha) = \alpha, \quad \alpha \in (0, 1)$$

is not ADF. To avoid this problem we will apply the linear transformation  $L[\cdot]$  defined by (1.26) to the process (2.42) to get the Wiener process, i.e,

$$\int_0^1 L[U](\nu)^2 d\nu = \int_0^1 w_\nu^2 d\nu, \quad 0 \leq \nu \leq 1.$$

Our next goal is to construct the ADF GoF test which is based on this property using the “empirical version” of the test statistic with the same above limit.

Indeed we calculate (formally)

$$U_n^*(\nu) = \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta_n^*, v)}}{\dot{\lambda}(\vartheta_n^*, v)} du_n^*(v), \quad 0 \leq \nu \leq 1, \quad (2.43)$$

where the process  $u_n^*(\cdot)$  is given by (2.37). Further, we apply the transformation  $L[\cdot]$  to the process  $U_n^*(\cdot)$  and we show that this statistic converges to the Wiener process

$$L[U_n^*](\nu) \implies w_\nu, \quad 0 \leq \nu \leq 1.$$

Therefore, the test  $\hat{\varphi}_n = \mathbb{1}_{\{\hat{\delta}_n > c_\alpha\}}$  with (under hypothesis)

$$\hat{\delta}_n = \int_0^1 L[U_n^*](\nu)^2 d\nu \implies \int_0^1 w_\nu^2 d\nu$$

will be ADF because the limit distribution does not depend on  $\Lambda(\cdot, \cdot)$  and  $\vartheta$ .

To do this let us proceed as follows : We have for the statistic  $U_n^*(\cdot)$  (formally)

$$\begin{aligned} U_n^*(\nu) &= \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta_n^*, v)}}{\dot{\lambda}(\vartheta_n^*, v)} du_n^*(v) \\ &= \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\sqrt{\lambda(\vartheta_n^*, v)}}{\dot{\lambda}(\vartheta_n^*, v)} d \left[ \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^v \frac{\dot{\lambda}(\vartheta_n^*, s)}{\lambda(\vartheta_n^*, s)} [dX_j(s) - \lambda(\vartheta_n^*, s) ds] \right] \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\vartheta_n^*, v)}} dX_j(v) - \frac{\sqrt{n}}{\sqrt{\tau}} \int_0^{\nu\tau} \sqrt{\lambda(\vartheta_n^*, v)} dv. \end{aligned}$$

Remark that the corresponding stochastic integral

$$\int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\vartheta_n^*, v)}} dX_j(v)$$

is not well defined. To solve this difficulty we introduce the *preliminary estimator*  $\bar{\vartheta}_N$  given by (2.32) and constructed by the observations  $X^N = (X_1, \dots, X_N)$ .

Hence, we consider the statistic

$$\hat{U}_n(\nu) = \frac{1}{\sqrt{n}} \sum_{j=N+1}^n \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} dX_j(v) - \frac{\sqrt{n}}{\sqrt{\tau}} \int_0^{\nu\tau} \sqrt{\lambda(\bar{\vartheta}_N, v)} dv \quad (2.44)$$

and the stochastic integral is well defined.

Below we insert the *preliminary estimator*  $\bar{\vartheta}_N$  in the functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  defined by (2.40) and (2.41) in order to obtain the “empirical versions”  $h_n(\cdot, \cdot)$  and  $g_n(\cdot, \cdot)$ , respectively :

$$h_n(\bar{\vartheta}_N, s) = C_2 (\bar{\vartheta}_N)^{\frac{1}{2}} \tilde{J}_n(\bar{\vartheta}_N)^{-1} \frac{\dot{\lambda}(\bar{\vartheta}_N, s\tau)}{\sqrt{\lambda(\bar{\vartheta}_N, s\tau)}}$$

and

$$g_n(\bar{\vartheta}_N, s) = C_2 (\bar{\vartheta}_N)^{-\frac{1}{2}} \int_{s\tau}^{\tau} \dot{\lambda}(\bar{\vartheta}_N, r) dr \sqrt{\lambda(\bar{\vartheta}_N, s\tau)}.$$

Denote

$$\begin{aligned} I_{1,n} &= \int_0^r g_n(\bar{\vartheta}_N, q)^2 dq, & I_{2,n} &= \int_0^r h_n(\bar{\vartheta}_N, q) g_n(\bar{\vartheta}_N, q) dq, \\ I_{3,n} &= \int_0^r h_n(\bar{\vartheta}_N, q) dq, & I_{4,n} &= \int_0^r h_n(\bar{\vartheta}_N, q)^2 dq \end{aligned}$$

and put

$$I_{5,n} = \int_0^r g_n(\bar{\vartheta}_N, q) dq$$

the “empirical versions” of (1.22). This allows us to introduce the “empirical versions”  $\varphi_{1,n}(\cdot, \cdot)$ ,  $\varphi_{2,n}(\cdot, \cdot)$  and  $\psi_{2,n}(\cdot, \cdot)$  of the functions  $\varphi_1(\cdot, \cdot)$ ,  $\varphi_2(\cdot, \cdot)$  and  $\psi_2(\cdot, \cdot)$  given by (1.23), (1.24) and (1.25), respectively.

Due to the consistency of the *preliminary estimator*  $\bar{\vartheta}_N$  and the regularity condition  $\mathcal{R}_1$ , we have the uniform convergence (as  $n \rightarrow \infty$ )

$$\sup_{r \in [0,1]} |\varphi_{2,n}(\bar{\vartheta}_N, r) - \varphi_2(\vartheta, r)| \longrightarrow 0.$$

We can introduce the function

$$\varphi_{2,n}^+(\bar{\vartheta}_N, r) = \begin{cases} \varphi_{2,n}(\bar{\vartheta}_N, r)^{-1}, & \text{if } \varphi_{2,n}(\bar{\vartheta}_N, r) > 0, \\ 0, & \text{else,} \end{cases} \quad (2.45)$$

which asymptotically coincides with  $\varphi_2(\vartheta, r)^{-1}$  and therefore the limit distribution does not change.

Namely, as a suitable statistic to base the testing procedure on we will take

$$\hat{\delta}_n = \int_0^1 W_n(\nu)^2 d\nu, \quad \hat{\Delta} \equiv \int_0^1 w_\nu^2 d\nu,$$



where  $w_\nu, 0 \leq \nu \leq 1$  is a Wiener process and

$$W_n(\nu) = \hat{U}_n(\nu) + \int_0^\nu \int_0^r \hat{\varphi}_{2,n}^+(\bar{\vartheta}_N, r) \left[ \hat{\varphi}_{1,n}(\bar{\vartheta}_N, r) h_n(\bar{\vartheta}_N, q) + \hat{\psi}_{2,n}(\bar{\vartheta}_N, r) g_n(\bar{\vartheta}_N, q) \right] d\hat{U}_n(q) dr.$$

Here  $\hat{U}_n(\cdot)$  is defined by (2.44). The second test is given in the following Theorem.

**Theorem 2.4.4.** *Suppose that the conditions of regularity  $\mathcal{R}$  and  $\mathcal{R}_4$  are fulfilled, then the test*

$$\hat{\varphi}_n = \mathbb{I}_{\{\hat{\delta}_n > c_\alpha\}}, \quad \mathbf{P}(\hat{\Delta} > c_\alpha) = \alpha$$

is ADF and of asymptotic size  $\alpha \in (0, 1)$ .

**Proof.** The main idea of the proof is to verify the convergence

$$\hat{\delta}_n \implies \hat{\Delta} \tag{2.46}$$

under hypothesis  $\mathcal{H}_0$ .

Note that due to the regularity condition  $\mathcal{R}_1$  and the consistency of the preliminary estimator  $\bar{\vartheta}_N$ , we have the uniform convergence (as  $n \rightarrow \infty$ ) w.r.t.  $s \in [0, 1]$

$$\sup_{s \in [0,1]} |g_n(\bar{\vartheta}_N, s) - g(\vartheta, s)| \longrightarrow 0, \quad \sup_{s \in [0,1]} |h_n(\bar{\vartheta}_N, s) - h(\vartheta, s)| \longrightarrow 0,$$

$$\sup_{s \in [0,1]} |\tilde{J}_n(\bar{\vartheta}_N) - \tilde{J}(\vartheta)| \rightarrow 0, \quad \sup_{s \in [0,1]} |\psi_{2,n}(\bar{\vartheta}_N, s) - \psi_2(\vartheta, s)| \rightarrow 0$$

and

$$\sup_{s \in [0,1]} |\varphi_{1,n}(\bar{\vartheta}_N, s) - \varphi_1(\vartheta, s)| \rightarrow 0.$$

Moreover, the process  $\hat{U}_n(\cdot)$  given by (2.44) can be written as follows :

$$\begin{aligned} \hat{U}_n(\nu) &= \frac{1}{\sqrt{n}} \sum_{j=N+1}^n \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} [dX_j(v) - \lambda(\vartheta, v) dv] \\ &\quad - \frac{\sqrt{n}}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{[\lambda(\vartheta_n^*, v) - \lambda(\vartheta, v)]}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} dv \\ &= \frac{1}{\sqrt{n}} \sum_{j=N+1}^n \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} d\pi_j(v) \\ &\quad - \sqrt{n}(\vartheta_n^* - \vartheta) \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} \frac{\dot{\lambda}(\bar{\vartheta}, v)}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} dv + o(1), \end{aligned} \tag{2.47}$$

where  $\pi_j(v) = X_j(v) - \Lambda(\vartheta, v)$  is the centered Poisson process. By the central limit theorem for stochastic integrals, we first obtain the convergence in distribution

$$\frac{1}{\sqrt{n}} \sum_{j=N+1}^n \int_0^{\nu\tau} \frac{1}{\sqrt{\lambda(\bar{\vartheta}_N, v)}} d\pi_j(v) \implies \int_0^{\nu\tau} dW_v.$$

Then using the representation (2.9), the change of variables  $s = \frac{v}{\tau}$  and  $W(\nu) = \tau^{-1/2}W_{\nu\tau}$ ,  $0 \leq \nu \leq 1$ , we get the second convergence

$$\begin{aligned} \hat{U}_n(\nu) &\implies \frac{1}{\sqrt{\tau}} \int_0^{\nu\tau} dW_v \\ &\quad - \frac{J(\vartheta)^{-1}}{\sqrt{\tau}} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, v)} dW_v \int_0^{\nu\tau} \frac{\dot{\lambda}(\vartheta, v)}{\sqrt{\lambda(\vartheta, v)}} dv \\ &= W(\nu) - \int_0^1 g(\vartheta, s) dW(s) \int_0^\nu h(\vartheta, s) ds \equiv U(\nu), \end{aligned}$$

where  $W_v$ ,  $0 \leq v \leq \tau$  and  $W(\nu)$ ,  $0 \leq \nu \leq 1$  are some Wiener processes.

Now we have to prove the convergence

$$K_n(\bar{\vartheta}_N, r) = \int_0^r g_n(\bar{\vartheta}_N, q) d\hat{U}_n(q) \implies K(\vartheta, r),$$

where

$$K(\vartheta, r) = \int_0^r g(\vartheta, s) dW(s) - \int_0^1 g(\vartheta, s) dW(s) \int_0^r g(\vartheta, s) h(\vartheta, s) ds.$$

$h(\vartheta, q)$  and  $g(\vartheta, q)$  defined by (2.40)–(2.41) are continuous functions w.r.t.  $q$ .

By (2.47), it is easily seen that  $K_n(\cdot, \cdot)$  can be written as follows :

$$\begin{aligned} K_n(\bar{\vartheta}_N, r) &= \frac{C_2(\bar{\vartheta}_N)^{-\frac{1}{2}}}{\sqrt{\tau}\sqrt{n}} \sum_{j=N+1}^n \int_0^r \int_{q\tau}^\tau \dot{\Lambda}(\bar{\vartheta}_N, z) dz d\pi_j(q\tau) \\ &\quad - \sqrt{n}(\vartheta_n^* - \vartheta) C_2(\bar{\vartheta}_N)^{-\frac{1}{2}} \sqrt{\tau} \int_0^r \int_{q\tau}^\tau \dot{\Lambda}(\bar{\vartheta}_N, z) dz \dot{\lambda}(\bar{\vartheta}, q\tau) dq \\ &= \frac{C_2(\bar{\vartheta}_N)^{-\frac{1}{2}}}{\sqrt{\tau}\sqrt{n}} \sum_{j=N+1}^n \int_0^{r\tau} \int_v^\tau \dot{\Lambda}(\bar{\vartheta}_N, z) dz d\pi_j(v) \\ &\quad - \sqrt{n}(\vartheta_n^* - \vartheta) \frac{C_2(\bar{\vartheta}_N)^{-\frac{1}{2}}}{\sqrt{\tau}} \int_0^{r\tau} \int_v^\tau \dot{\Lambda}(\bar{\vartheta}_N, z) dz \dot{\lambda}(\bar{\vartheta}, v) dv, \end{aligned}$$

where we use the Taylor formula and the change of variable  $q = \frac{v}{\tau}$ . By the central limit theorem for stochastic integrals and the representation (2.9), we obtain

$$K_n(\bar{\vartheta}_N, r) \implies \frac{C_2(\vartheta)^{-\frac{1}{2}}}{\sqrt{\tau}} \int_0^{r\tau} \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, v)} dW_v$$

$$- \frac{C_2(\vartheta)^{-\frac{1}{2}}}{\sqrt{\tau}J(\vartheta)} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, v)} dW_v \int_0^{r\tau} \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \dot{\lambda}(\vartheta, v) dv.$$

Let us change the variables  $s = \frac{v}{\tau}$  and  $W(s) = \tau^{-\frac{1}{2}}W_{s\tau}$ . Therefore, we have the convergence

$$\begin{aligned} K_n(\bar{\vartheta}_N, r) &\implies C_2(\vartheta)^{-\frac{1}{2}} \int_0^r \int_{s\tau}^{\tau} \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, s\tau)} dW(s) \\ &- \frac{C_2(\vartheta)^{-\frac{1}{2}}}{\tilde{J}(\vartheta)} \int_0^1 \int_{s\tau}^\tau \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, s\tau)} dW(s) \int_0^r \int_{s\tau}^\tau \dot{\Lambda}(\vartheta, z) dz \dot{\lambda}(\vartheta, s\tau) ds \\ &= K(\vartheta, r). \end{aligned}$$

Similarly, we have to show that

$$L_n(\bar{\vartheta}_N, r) = \int_0^r h_n(\bar{\vartheta}_N, q) d\hat{U}_n(q) \implies L(\vartheta, r),$$

where

$$L(\vartheta, r) = \int_0^r h(\vartheta, s) dW(s) - \int_0^1 g(\vartheta, s) dW(s) \int_0^r h(\vartheta, s)^2 ds.$$

Using the expression (2.47), Taylor's formula and the change of variable  $q = \frac{v}{\tau}$ , the process  $L_n(\cdot, \cdot)$  has the following representation

$$\begin{aligned} L_n(\bar{\vartheta}_N, r) &= \frac{C_2(\bar{\vartheta}_N)^{\frac{1}{2}} \tilde{J}_n(\bar{\vartheta}_N)^{-1}}{\sqrt{\tau}\sqrt{n}} \sum_{j=N+1}^n \int_0^r \frac{\dot{\lambda}(\bar{\vartheta}_N, q\tau)}{\lambda(\bar{\vartheta}_N, q\tau)} d\pi_j(q\tau) \\ &- \sqrt{n}(\vartheta_n^* - \vartheta) C_2(\bar{\vartheta}_N)^{\frac{1}{2}} \sqrt{\tau} \tilde{J}_n(\bar{\vartheta}_N)^{-1} \int_0^r \frac{\dot{\lambda}(\bar{\vartheta}, q\tau) \dot{\lambda}(\bar{\vartheta}_N, q\tau)}{\lambda(\bar{\vartheta}_N, q\tau)} dq \\ &= \frac{C_2(\bar{\vartheta}_N)^{\frac{1}{2}} \tilde{J}_n(\bar{\vartheta}_N)^{-1}}{\sqrt{\tau}\sqrt{n}} \sum_{j=N+1}^n \int_0^{r\tau} \frac{\dot{\lambda}(\bar{\vartheta}_N, v)}{\lambda(\bar{\vartheta}_N, v)} d\pi_j(v) \\ &- \sqrt{n}(\vartheta_n^* - \vartheta) \frac{C_2(\bar{\vartheta}_N)^{\frac{1}{2}} \tilde{J}_n(\bar{\vartheta}_N)^{-1}}{\sqrt{\tau}} \int_0^{r\tau} \frac{\dot{\lambda}(\bar{\vartheta}, v) \dot{\lambda}(\bar{\vartheta}_N, v)}{\lambda(\bar{\vartheta}_N, v)} dv. \end{aligned}$$

Further, by the central limit theorem for stochastic integrals and the representation (2.9), we obtain

$$\begin{aligned} L_n(\bar{\vartheta}_N, r) &\implies \frac{C_2(\vartheta)^{\frac{1}{2}} \tilde{J}(\vartheta)^{-1}}{\sqrt{\tau}} \int_0^{r\tau} \frac{\dot{\lambda}(\vartheta, v)}{\sqrt{\lambda(\vartheta, v)}} dW_v \\ &- \frac{C_2(\vartheta)^{\frac{1}{2}} \tilde{J}(\vartheta)^{-1}}{\sqrt{\tau}J(\vartheta)} \int_0^\tau \int_v^\tau \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, v)} dW_v \int_0^{r\tau} \frac{\dot{\lambda}(\vartheta, v)^2}{\lambda(\vartheta, v)} dv. \end{aligned}$$

Therefore using the change of variables  $s = \frac{v}{\tau}$  and  $W(s) = \tau^{-1/2}W_{s\tau}$ ,  $0 \leq s \leq 1$ , we can write

$$\begin{aligned} L_n(\bar{\vartheta}_N, r) &\implies C_2(\vartheta)^{\frac{1}{2}} \tilde{J}(\vartheta)^{-1} \int_0^r \frac{\dot{\lambda}(\vartheta, s\tau)}{\sqrt{\lambda(\vartheta, s\tau)}} dW(s) \\ &\quad - C_2(\vartheta)^{\frac{1}{2}} \tilde{J}(\vartheta)^{-2} \int_0^1 \int_{s\tau}^{\tau} \dot{\Lambda}(\vartheta, z) dz \sqrt{\lambda(\vartheta, s\tau)} dW(s) \int_0^r \frac{\dot{\lambda}(\vartheta, s\tau)^2}{\lambda(\vartheta, s\tau)} ds \\ &= L(\vartheta, r). \end{aligned}$$

Finally, the convergence (2.46) is proved and using this result the test  $\hat{\varphi}_n$  is ADF.

# On GoF Test for Ergodic Diffusion Process

## 3.1 Introduction

This chapter is devoted to the problem of the construction of the GoF test for the parametric basic hypothesis in the case of the ergodic diffusion process. We propose an ADF test, which is based on two linear transformations applied to the normalized deviation of the empirical density.

Remind what happens in the case of the simple basic hypothesis (see [41] for more details) :

$\mathcal{H}_0$  : the observed diffusion process satisfies the stochastic differential

$$dX_t = S_0(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where  $W_t$  is a Wiener process,  $S_0(\cdot)$  is some known function and  $X_0$  is the initial value of  $X_t$ .

In this statement, the diffusion process is supposed to be ergodic with the density of invariant law  $f_{S_0}(x)$ . We denote by  $F_{S_0}(x)$  the corresponding distribution function. The Cramér-von Mises type statistic based on the *empirical density*  $\hat{f}_T(x)$  (will be defined by (3.9) below) given by

$$\hat{\delta}_T = \int_{-\infty}^{\infty} [\zeta_T(x)]^2 dF_{S_0}(x), \quad \zeta_T(x) = \sqrt{T} \left( \hat{f}_T(x) - f_{S_0}(x) \right)$$

admits the following limit (under the hypothesis  $\mathcal{H}_0$ )

$$\hat{\delta}(S_0) = \int_{-\infty}^{\infty} \zeta(S_0, x)^2 dF_{S_0}(x).$$

Here

$$\zeta(S_0, x) = 2 f_{S_0}(x) \int_{-\infty}^{\infty} \frac{F_{S_0}(y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f_{S_0}(y)}} dW(y),$$

where  $W(\cdot)$  is double-sided Wiener process and the limit process  $\hat{\delta}(S_0)$  depends on the model (the proof can be found in [39]). To avoid this problem, a linear transformation  $L_1[\zeta_T](\cdot)$  of the random function  $\zeta_T(\cdot)$  was proposed (see [41] for more details) such that

$$\tilde{\delta}_T = \int_{-\infty}^{\infty} L_1[\zeta_T](x)^2 dF_{S_0}(x) \implies \int_0^1 w_s^2 ds, \quad (3.1)$$

where  $w_s, 0 \leq s \leq 1$  is a Wiener process. Then, after transformation, the test statistic is as follows :

$$\tilde{\delta}_T^* = \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S_0(X_s) ds] \right]^2 dF_{S_0}(x)$$

with the same limit (3.1). Note that this statistic was introduced in the similar problem to construct the Kolmogorov-Smirnov type ADF test in [51]. Therefore, the test

$$\phi_T^* = \mathbb{I}_{\{\tilde{\delta}_T^* > c_\alpha\}}, \quad \mathbf{P} \left( \int_0^1 w_s^2 ds > c_\alpha \right) = \alpha$$

is ADF under the simple basic hypothesis  $\mathcal{H}_0$ .

This problem was also studied in [34] in the case of the parametric null hypothesis  $\mathcal{H}_0$  that

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T, \quad \vartheta \in \Theta = (a, b),$$

i.e., the trend coefficient  $S(\vartheta, x)$  is some known function which depends on the unknown parameter  $\vartheta$ . Let us denote by  $\hat{f}_T(x)$  the *local time estimator (empirical density)* of the invariant density  $f(\vartheta, x)$  which will be defined by (3.9) and (3.8) below, respectively.

The statistic is given by

$$\hat{\Delta}_T = \int_{-\infty}^{\infty} \hat{\mu}_T(\hat{\vartheta}_T, x)^2 dF(\hat{\vartheta}_T, x), \quad \hat{\mu}_T(\hat{\vartheta}_T, x) = \sqrt{T} \left( \hat{f}_T(x) - f(\hat{\vartheta}_T, x) \right),$$

where  $\hat{\vartheta}_T$  is the MLE of the parameter  $\vartheta$ . Unfortunately, the immediate use of the test  $\hat{\varphi}_T = \mathbb{I}_{\{\hat{\Delta}_T > d_\alpha\}}$  leads to the same problem as in the i.i.d. case.

Indeed, the limit distribution of this statistic (as  $T \rightarrow \infty$ ) under  $\mathcal{H}_0$  depends on the model, i.e,  $S(\cdot, \cdot)$ ,  $\sigma(\cdot)$  and the unknown parameter  $\vartheta$ . To solve this problem, a first linear transformation  $L_1[\hat{\mu}_T](\cdot)$  of the statistic  $\hat{\mu}_T(\cdot, \cdot)$  was proposed in [34] and [41]. Moreover, it was shown that this linear transformation is asymptotically equivalent to the statistic

$$\hat{\xi}_T(\hat{\vartheta}_T, x) = \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} \left[ dX_s - S(\hat{\vartheta}_T, X_s) ds \right]$$

with the same limit (as  $T \rightarrow \infty$ )

$$U(t) = W(t) - \int_0^1 h(\vartheta, s) dW(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1. \quad (3.2)$$

Here  $W(s)$ ,  $0 \leq s \leq 1$  is a Wiener process. The limit process  $U(\cdot)$  is not distribution free. Hence, the ADF test was based on a second linear transformation  $L_2[\cdot]$  (can be found, e.g., in [30] and [34]) of the statistic  $\hat{\xi}_T(\cdot, \cdot)$  such that  $L_2[U](t) = w_t$  and the convergence

$$\int_{-\infty}^{\infty} L_2 \left[ \hat{\xi}_T(\hat{\vartheta}_T, \cdot) \right] (x)^2 dF(\hat{\vartheta}_T, x) \implies \int_0^1 w_t^2 dt$$

was proved. Here  $w_t$ ,  $0 \leq t \leq 1$  is a Wiener process.

There are several GoF tests for the continuous time diffusion and inhomogeneous Poisson processes proposed, for example, in the works [56], [44], [42] and [11]. See also [12] where some GoF tests for diffusion and inhomogeneous Poisson processes with simple basic hypothesis were proposed. It was shown that these tests are ADF. The Kolmogorov-Smirnov statistic for ergodic diffusion process was studied, e.g., in [20] and [21]. In [39], Section 5.4, the author discusses some possibilities of the construction of the Kolmogorov-Smirnov and the Cramér-von Mises tests when the ergodic diffusion process is observed on continuous time. The same problem was considered in [23] but for the hypothesis with sign-type trend coefficient. In this case, two Cramér-von Mises GoF tests based on empirical distribution function and empirical density function were studied. More about GoF tests for the ergodic diffusion process can be found,

e.g., in Kleptsyna & Kutoyants [34] for the parametric hypotheses and Negri & Nishiyama [51] for the simple hypotheses.

We shall mention that the general case of ergodic diffusion process with unknown shift (one dimensional) parameter was studied in [54]. The authors showed that the limit distribution of the Cramér-von Mises type statistic does not depend on the unknown parameter and therefore is *asymptotically parameter free* (APF). Moreover, the APF Kolmogorov-Smirnov type tests were studied in [62]. Similar results for the APF tests of the Cramér-von Mises type have been obtained by Kutoyants [41]. The author studied the case of the composite basic hypothesis with a parametric class of diffusion processes including the Ornstein-Uhlenbeck and simple switching processes.

More problems of the construction of the GoF tests for the discrete time observations were widely studied by many authors, e.g., Negri & Nishiyama [53] in the case of the small diffusion process. In [52], the proposed ADF GoF tests were based on continuous time observations, on discrete time observations and on the so-called tick time sampled data for ergodic diffusion processes.

We have to emphasize that in [34] and in many other works, e.g., [49] the estimator used was always the MLE and this is important for the construction of the linear transformation. In all such cases, the limit expression for the underlying statistics was always like (3.2). However, in some problems the MLE is not given in an explicit form and therefore sometimes it is better to use other estimators like the MDE. Later, we will get the limit process (3.3) with two different functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  in the case of the MDE.

The main contribution of this chapter is to construct an ADF test based on the linear transformation (1.26) when the basic model is a continuous time ergodic diffusion process  $X^T = \{X_t, 0 \leq t \leq T\}$  solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

Here  $W_t$ ,  $0 \leq t \leq T$  is a Wiener process. The trend coefficient  $S(\cdot)$  is an unknown function, the diffusion coefficient  $\sigma(x)^2 > 0$  is known and  $X_0$  is the initial value



of  $X_t$ . We consider the parametric null hypothesis

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

i.e, the process  $X^T$  is solution of the following equation :

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where  $S(\vartheta, x)$  is known smooth function. Suppose that the stochastic process  $X^T$  is *recurrent positive*, i.e., it has ergodic properties in Section 3.2. In Section 3.3, we propose for the unknown parameter  $\vartheta$  the MDE

$$\vartheta_T^* = \arg \inf_{\vartheta \in \Theta} \int_{\mathbb{R}} \left[ \hat{f}_T(x) - f(\vartheta, x) \right]^2 dx.$$

We consider the Cramér-von Mises type statistic

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x)$$

and under hypothesis  $\mathcal{H}_0$ , we have the convergence

$$\begin{aligned} \eta_T^*(\vartheta_T^*, x) &= \sqrt{T} \left( \hat{f}_T(x) - f(\vartheta_T^*, x) \right) \\ &\implies \eta(\vartheta, x) = 2 f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} dW(y) \\ &\quad - 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x). \end{aligned}$$

A first linear transformation (can be found in [41])

$$L_1[\eta](x) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] = U(F(\vartheta, x))$$

will be applied for the limit process  $\eta(\cdot, \cdot)$ . Then we show that the above transformation admits the following representation

$$U(t) = w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1, \quad (3.3)$$

where  $w(t), 0 \leq t \leq 1$  is some Wiener process. The next step is to introduce the second linear transformation  $L_2[\cdot]$  (1.26) and to obtain

$$L_2[U](t) = w_t, \quad 0 \leq t \leq 1. \quad (3.4)$$

Now, we have to realize the similar transformation with the process  $\xi_T(\cdot, \cdot)$  and then we show the convergence in distribution to the Wiener process

$$L_2[\xi_T](x) \implies L_2[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Note that the process  $\xi_T(\cdot, \cdot)$  defined by (3.22) below is asymptotically equivalent to  $L_1[\eta_T^*](\cdot)$ . Hence, we have

$$\delta_T^* = \int_{-\infty}^{\infty} L_2[\xi_T](x)^2 dF(\vartheta_T^*, x) \implies \int_0^1 w_t^2 dt.$$

Therefore, the test  $\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}$  will be ADF because the limit distribution of  $\delta_T^*$  does not depend on  $S(\cdot, \cdot)$ ,  $\sigma(\cdot)$  and  $\vartheta$ .

## 3.2 Preliminaries

Suppose that we have continuous time observations  $X^T = \{X_t, 0 \leq t \leq T\}$  of a diffusion process, which is solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T, \quad (3.5)$$

where  $W_t$ ,  $0 \leq t \leq T$  is a Wiener process. The trend coefficient  $S(\cdot)$  is an unknown function, the diffusion coefficient  $\sigma(x)^2 > 0$  is known and  $X_0$  is the initial value of  $X_t$ .

We consider the problem of testing the parametric (composite) hypothesis

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where  $0 < a < b < \infty$ , i.e., the process  $X^T$  is solution of the equation

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T. \quad (3.6)$$

Here  $S(\vartheta, x)$  is known smooth function depending on the unknown parameter  $\vartheta$ .

We suppose that the functions  $S(\cdot)$  and  $\sigma(\cdot)$  of the observed diffusion process satisfy the conditions  $\mathcal{ES}$  (existence of the solution),  $\mathcal{A}_0$  and  $\mathcal{RP}$  (recurrent positive)(ergodicity) :

$\mathcal{ES}$ . The function  $S(\cdot, \cdot)$  is locally bounded, the function  $\sigma(\cdot)^2$  is continuous and positive and for some  $C > 0$ , the condition

$$xS(\vartheta, x) + \sigma(x)^2 \leq C(1 + x^2)$$

holds.

By this condition, the stochastic differential equation (3.5) has a unique weak solution for all  $\vartheta \in \Theta$  (see for instance [19]).

Denote by  $\mathcal{P}$  the class of locally bounded functions with polynomial majorants ( $p > 0$ )

$$\mathcal{P} = \{h(\cdot) : |h(y)| \leq C(1 + |y|^p)\}.$$

Let us introduce the following two conditions (see [39] for more details) :

$\mathcal{A}_0$ . The functions  $S(\cdot, \cdot)$ ,  $\sigma(\cdot)^{\pm 1} \in \mathcal{P}$  and for all  $\vartheta$

$$\overline{\lim}_{|y| \rightarrow \infty} \sup_{\vartheta \in \Theta} \operatorname{sgn}(y) \frac{S(\vartheta, y)}{\sigma(y)^2} < 0.$$

$\mathcal{RP}$ . Note that if  $S(\cdot, \cdot)$  and  $\sigma(\cdot)$  satisfy  $\mathcal{A}_0$ , then we have

$$V(\vartheta, x) = \int_0^x \exp \left\{ -2 \int_0^y \frac{S(\vartheta, z)}{\sigma(z)^2} dz \right\} dy \longrightarrow \pm \infty, \quad \text{as } x \rightarrow \pm \infty$$

and

$$G(\vartheta) = \int_{-\infty}^{+\infty} \sigma(x)^2 \exp \left\{ 2 \int_0^x \frac{S(\vartheta, y)}{\sigma(y)^2} dy \right\} dx < \infty. \quad (3.7)$$

If both conditions are fulfilled then the stochastic process  $X^T$  is *recurrent positive*, i.e., it has ergodic properties. Then there exists an invariant distribution with the density function (see, e.g., [39], Theorem 1.16)

$$f(\vartheta, x) = \frac{1}{G(\vartheta) \sigma(x)^2} \exp \left\{ 2 \int_0^x \frac{S(\vartheta, y)}{\sigma(y)^2} dy \right\}, \quad x \in \mathbb{R}, \quad (3.8)$$

where  $G(\vartheta)$  defined by (3.7) is the normalizing constant. We denote by  $F(\vartheta, x)$  the corresponding distribution function.

Introduce the regularity conditions  $\mathcal{R}$ . (can be found in Kutoyants [39])

$\mathcal{R}_1$ . The invariant density function  $f(\vartheta, x)$  is differentiable w.r.t.  $\vartheta$  for all  $x \in \mathbb{R}$  and the derivative  $\dot{f}(\vartheta, x) \in \mathcal{P}$  is uniformly continuous in the following sense : for any  $\vartheta_0 \in \Theta$  and  $\nu > 0$

$$\lim_{\nu \rightarrow 0} \sup_{\vartheta_0 \in \Theta} \sup_{|\vartheta - \vartheta_0| \leq \nu} \left\| \dot{f}(\vartheta, \cdot) - \dot{f}(\vartheta_0, \cdot) \right\| = 0.$$

$\mathcal{R}_2$ . The identifiability condition is

$$\inf_{|\vartheta - \vartheta_0| > \nu} \|f(\vartheta, \cdot) - f(\vartheta_0, \cdot)\| > 0.$$

$\mathcal{R}_3$ . The function  $J(\vartheta) = \left\| \dot{f}(\vartheta, \cdot) \right\|^2$  is positive uniformly in  $\vartheta$ . Here  $\|\cdot\|$  is the  $\mathcal{L}^2$ -norm :

$$\left\| \dot{f}(\vartheta, \cdot) \right\|^2 = \int_{\mathbb{R}} \dot{f}(\vartheta, x)^2 dx.$$

In the presentation below we suppose that these conditions and the null hypothesis  $\mathcal{H}_0$  are always fulfilled.

### 3.3 Minimum distance estimator

The MDE can be constructed with the help of the invariant density function (3.8) and the empirical density  $\hat{f}_T(x)$  defined by the relation

$$\hat{f}_T(x) = \frac{2\Lambda_T(x)}{\sigma(x)^2 T}, \quad (3.9)$$

where  $\Lambda_T(x)$  is the local time of the observed diffusion process (see, e.g., [39] Chapter 1, Section 1.1.3 for the definition).

The MDE is defined as a solution of the problem

$$\left\| \hat{f}_T(\cdot) - f(\vartheta_T^*, \cdot) \right\| = \inf_{\vartheta \in \Theta} \left\| \hat{f}_T(\cdot) - f(\vartheta, \cdot) \right\|, \quad (3.10)$$

where  $\vartheta \in \Theta = (a, b)$  is the one-dimensional (unknown) parameter. Note that the properties and the asymptotic ( $T \rightarrow \infty$ ) behavior of the MDE for this model were studied under regularity conditions  $\mathcal{R}$  in [39].

The MDE  $\vartheta_T^*$  can be written as well as follows :

$$\vartheta_T^* = \arg \inf_{\vartheta \in \Theta} \int_{\mathbb{R}} \left[ \hat{f}_T(x) - f(\vartheta, x) \right]^2 dx \quad (3.11)$$

and it satisfies the minimum distance equation (MDEq) (can be found in [39])

$$\int_{\mathbb{R}} \left[ \hat{f}_T(x) - f(\vartheta_T^*, x) \right] \dot{f}(\vartheta_T^*, x) dx = 0.$$

Let us put  $u_T^* = \sqrt{T}(\vartheta_T^* - \vartheta)$ . Then, by Taylor's formula,  $f(\vartheta_T^*, x) = f(\vartheta, x) + (\vartheta_T^* - \vartheta) \dot{f}(\tilde{\vartheta}, x)$ , where  $|\tilde{\vartheta} - \vartheta| \leq |\vartheta_T^* - \vartheta|$ , the MDEq becomes as follows :

$$\int_{\mathbb{R}} \left[ \sqrt{T} \left[ \hat{f}_T(x) - f(\vartheta, x) \right] - u_T^* \dot{f}(\tilde{\vartheta}, x) \right] \dot{f}(\vartheta_T^*, x) dx = 0.$$

Using the consistency of the MDE, we can write

$$u_T^* = \left[ \int_{\mathbb{R}} \dot{f}(\vartheta, x)^2 dx \right]^{-1} \int_{\mathbb{R}} \sqrt{T} \left[ \hat{f}_T(x) - f(\vartheta, x) \right] \dot{f}(\vartheta, x) dx + o(1).$$

Remind that we have the convergence

$$\sqrt{T} \left[ \hat{f}_T(x) - f(\vartheta, x) \right] \Longrightarrow 2 f(\vartheta, x) \int_{-\infty}^{\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} dW(y), \quad (3.12)$$

where  $W(\cdot)$  is double-sided Wiener process. For the proof see [39].

Therefore, by Fubini's theorem,  $\vartheta_T^*$  admits the following representation

$$\sqrt{T}(\vartheta_T^* - \vartheta) \Longrightarrow 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y). \quad (3.13)$$

Here

$$J(\vartheta) = \int_{\mathbb{R}} \dot{f}(\vartheta, x)^2 dx. \quad (3.14)$$

Moreover, under regularity conditions  $\mathcal{R}$  and  $\mathcal{A}_0$ , the MDE  $\vartheta_T^*$  is consistent and asymptotically normal (for more details see Chapter 2, Section 2.2 in [39])

$$\mathcal{L}_{\vartheta}\{\sqrt{T}(\vartheta_T^* - \vartheta)\} \Longrightarrow \mathcal{L}\{\xi\} = \mathcal{N}(0, \sigma^2(\vartheta)),$$

where

$$\sigma^2(\vartheta) = J(\vartheta)^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\vartheta, x) \dot{f}(\vartheta, x) D_{\vartheta}(x, y) f(\vartheta, y) \dot{f}(\vartheta, y) dx dy.$$

Here

$$D_{\vartheta}(x, y) = 4 \mathbf{E}_{\vartheta} \left[ \frac{[F(\vartheta, \xi) - \mathbb{I}_{\{\xi>x\}}][F(\vartheta, \xi) - \mathbb{I}_{\{\xi>y\}}]}{\sigma(\xi)^2 f(\vartheta, \xi)^2} \right].$$

### 3.4 Cramér-von Mises type statistic

We are going to study the ADF GoF test based on the Cramér-von Mises type statistic

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x),$$

where

$$\eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left( \hat{f}_T(x) - f(\vartheta_T^*, x) \right). \quad (3.15)$$

Here  $\vartheta_T^*$  is the MDE defined by (3.11). Introduce the Gaussian function

$$\begin{aligned} \eta(\vartheta, x) &= 2 f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} dW(y) \\ &- 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x). \end{aligned} \quad (3.16)$$

It can be shown that we have this convergence (see Theorem 3.4.5 below)

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x) \implies \delta \equiv \int_{-\infty}^{\infty} \eta(\vartheta, x)^2 dF(\vartheta, x).$$

Remark that the limit statistic  $\delta$  depends strongly on the model, i.e.,  $f(\cdot, \cdot)$  and the unknown parameter  $\vartheta$ . Therefore the test based on this statistic is not ADF. To avoid this difficulty we propose a first transformation for  $\eta(\cdot, \cdot)$  given as follows :

$$U(F(\vartheta, x)) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right]. \quad (3.17)$$

Remind that the above transformation was introduced in the similar problem in [41]. Further, let us define the functions

$$h(\vartheta, s) = 2 \tilde{J}(\vartheta)^{-1} C(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, F^{-1}(\vartheta, s))}{\sigma(F^{-1}(\vartheta, s))} \quad (3.18)$$

and

$$g(\vartheta, s) = C(\vartheta)^{-\frac{1}{2}} \int_0^1 \frac{[s - \mathbb{I}_{\{s>t\}}]}{a(\vartheta, s) b(\vartheta, s)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt, \quad (3.19)$$

where  $a(\vartheta, s) = \sigma(F^{-1}(\vartheta, s))$ ,  $b(\vartheta, s) = f(\vartheta, F^{-1}(\vartheta, s))$ ,

$$C(\vartheta) = \int_0^1 \left[ \int_0^1 \frac{[v - \mathbb{I}_{\{v>t\}}]}{a(\vartheta, v) b(\vartheta, v)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt \right]^2 dv \quad (3.20)$$

and

$$\tilde{J}(\vartheta) = \int_0^1 \frac{\dot{f}(\vartheta, F^{-1}(\vartheta, s))^2}{f(\vartheta, F^{-1}(\vartheta, s))} ds, \quad \int_0^1 g(\vartheta, s)^2 ds = 1.$$

Here  $F^{-1}(\vartheta, s)$  is the inverse function of  $F(\vartheta, y)$ , i.e.,  $y$  is solution of the equation  $s = F(\vartheta, y)$ .

Then, we show that the transformation (3.17) has the following representation

$$U(t) = w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1, \quad (3.21)$$

where  $w(s), 0 \leq s \leq 1$  is some Wiener process.

Moreover, we show that the transformation (3.17) of  $\eta_T^*(\cdot, \cdot)$  defined by (3.15) gives us a statistic which is asymptotically equivalent to

$$\xi_T(\vartheta_T^*, x) = \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S(\vartheta_T^*, X_s) ds], \quad (3.22)$$

i.e., both statistics have the same limit process (3.21). We shall mention that the statistic (3.22) was introduced in [34] where the MLE was used for the unknown parameter. Therefore, our ADF test will be based on the statistic  $\xi_T(\cdot, \cdot)$  for simplicity of calculus.

The last step is to apply the second transformation  $L_2[\cdot]$  to the limit process (3.21) (from Theorem 1.4.1) and to obtain the Wiener process

$$L_2[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Then we can construct the ADF GoF test which is based on this transformation. Below we realize this program. We have the following result.

**Theorem 3.4.5.** *Let the conditions  $\mathcal{R}$ ,  $\mathcal{ES}$  and  $\mathcal{A}_0$  be fulfilled, then*

$$\eta_T^*(\vartheta_T^*, x) \implies \eta(\vartheta, x), \quad \xi_T(\vartheta_T^*, x) \implies U(F(\vartheta, x)) \quad (3.23)$$

and

$$U(F(\vartheta, x)) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right].$$

**Proof.** The statistic  $\eta_T^*(\cdot, \cdot)$  defined by (3.15) has the following representation

$$\eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left( \hat{f}_T(x) - f(\vartheta, x) \right) - \sqrt{T} (\vartheta_T^* - \vartheta) \dot{f}(\vartheta, x) + o(1).$$

Remind that the normalized difference  $\sqrt{T}(\hat{f}_T(x) - f(\vartheta, x))$  has the asymptotic behavior (3.12) and the MDE  $\vartheta_T^*$  admits the representation (3.13). Therefore, the following convergence (under hypothesis  $\mathcal{H}_0$ )

$$\begin{aligned} \eta_T^*(\vartheta_T^*, x) \implies \eta(\vartheta, x) &= 2f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y > x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} dW(y) \\ &- 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y > x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x) \end{aligned}$$

holds. Here

$$\dot{f}(\vartheta, x) = 2 f(\vartheta, x) \left( -\frac{\dot{G}(\vartheta)}{2 G(\vartheta)} + \int_0^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)^2} dy \right).$$

Below, we show that the transformation (3.17) admits the representation (3.24).

**Lemma 3.4.6.** *We have the equality*

$$U(t) = w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds, \quad (3.24)$$

where  $w(t), 0 \leq t \leq 1$  is a Wiener process.

**Proof.** We have following (3.17) :

$$\begin{aligned} U(F(\vartheta, x)) &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] \\ &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \int_{-\infty}^{\infty} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &\quad - 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y). \end{aligned}$$

Further,

$$\begin{aligned} &\int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \int_{-\infty}^{\infty} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \int_{-\infty}^y \frac{F(\vartheta, z)}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &\quad + \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[ \int_y^{\infty} \frac{F(\vartheta, z) - 1}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &= \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} U(F(\vartheta, x)) &= \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) \\ &\quad - 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y). \end{aligned}$$

Let us change the variables  $t = F(\vartheta, x)$ ,  $s = F(\vartheta, y)$ ,  $a(\vartheta, s) = \sigma(F^{-1}(\vartheta, s))$ ,  $b(\vartheta, s) = f(\vartheta, F^{-1}(\vartheta, s))$  and

$$w(s) = \int_{-\infty}^{F^{-1}(\vartheta, s)} \sqrt{f(\vartheta, y)} dW(y),$$



where  $w(s), 0 \leq s \leq 1$  is a Wiener process. Hence, we can write

$$\begin{aligned} U(t) &= w(t) \\ &- 2 \tilde{J}(\vartheta)^{-1} \int_0^1 \int_0^1 \frac{[s - \mathbb{1}_{\{s>t\}}]}{a(\vartheta, s)b(\vartheta, s)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt dw(s) \int_0^t \frac{\dot{S}(\vartheta, F^{-1}(\vartheta, s))}{\sigma(F^{-1}(\vartheta, s))} ds \\ &= w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds. \end{aligned}$$

Therefore, the process  $U(\cdot)$  has the representation (3.24) and the Lemma is proved.

Now, we show that the linear transformation (3.17) of  $\eta_T^*(\cdot, \cdot)$  gives us a statistic which is asymptotically equivalent to (3.22). Indeed, we have

$$\begin{aligned} \xi_T(\vartheta_T^*, x) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{1}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S(\vartheta, X_s) ds] \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{1}_{\{X_s < x\}}}{\sigma(X_s)} [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\ &= \frac{1}{\sqrt{T}} \int_0^T \mathbb{1}_{\{X_s < x\}} dW_s \\ &\quad - \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\mathbb{1}_{\{X_s < x\}} \dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} ds + o(1). \end{aligned}$$

Using central limit theorem, the first stochastic integral is asymptotically normal

$$\frac{1}{\sqrt{T}} \int_0^T \mathbb{1}_{\{X_s < x\}} dW_s \implies \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) \sim \mathcal{N}(0, F(\vartheta, x))$$

and by the law of large numbers, we have for the second integral (as  $T \rightarrow \infty$ )

$$\frac{1}{T} \int_0^T \frac{\mathbb{1}_{\{X_s < x\}} \dot{S}(\vartheta, X_s)}{\sigma(X_s)} ds \longrightarrow \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y).$$

Therefore, using (3.13) and the above convergence results, one can write

$$\begin{aligned} \xi_T(\vartheta_T^*, x) &\implies \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) \\ &- 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{1}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y) \\ &= U(F(\vartheta, x)). \end{aligned}$$

Hence, we have the asymptotic equivalence between the linear transformation (3.17) of  $\eta_T^*(\cdot, \cdot)$  and the statistic (3.22). Finally, we obtain the convergence (3.23), which proves the theorem.

Due to Theorem 3.4.5, one can write

$$\Delta = \int_{-\infty}^{\infty} \left( \int_{-\infty}^x \sigma(y) f(\vartheta, y) \, d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] \right)^2 dF(\vartheta, x) = \int_0^1 U(t)^2 dt.$$

It is easily seen that

$$\Delta_T = \int_{-\infty}^{\infty} \left( \int_{-\infty}^x \sigma(y) f(\vartheta_T^*, y) \, d \left[ \frac{\eta_T^*(\vartheta_T^*, y)}{2f(\vartheta_T^*, y)} \right] \right)^2 dF(\vartheta_T^*, x)$$

and the statistic

$$\Delta_T^* = \int_{-\infty}^{\infty} \xi_T(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x)$$

have the same limit process  $\Delta$ . Then, it is obvious to see that the test constructed on the base of this statistic

$$\psi_T = \mathbb{I}_{\{\Delta_T^* > c_\alpha\}}, \quad \mathbf{P}(\Delta > c_\alpha) = \alpha, \quad \alpha \in (0, 1)$$

is not ADF. So we introduce the linear transformation  $L_2[\cdot]$  defined by (1.26) such that

$$\int_0^1 L_2[U](t)^2 dt = \int_0^1 w_t^2 dt, \quad 0 \leq t \leq 1.$$

Here the process  $U(\cdot)$  is defined by (3.24). Finally, we can construct the ADF GoF test which is based on this transformation using the “empirical version” of the test statistic with the same limit.

### 3.5 Test

In this Section, our purpose is to test the parametric hypothesis  $\mathcal{H}_0$ . Thus, we will propose a statistic based on the MDE  $\vartheta_T^*$ . Recall that the starting statistic

$$\eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left( \hat{f}_T(x) - f(\vartheta_T^*, x) \right) \tag{3.25}$$

converges to the random function  $\eta(\vartheta, x)$  defined by (3.16).

Then the linear transformation

$$L_1[\eta](x) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) \, d \left[ \frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right]$$

leads to the random function  $U(\cdot)$  defined by (3.24) which is the same limit of the process (3.22).

To finish, we apply the transformation  $L_2[\cdot]$  from Theorem 1.4.1 such that

$$L_2[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Now, we have to realize the similar transformation with the process  $\xi_T(\cdot, \cdot)$  given by (3.22) and then we show that this statistic converges in distribution to the Wiener process

$$L_2[\xi_T](x) \Longrightarrow w_t, \quad 0 \leq t \leq 1.$$

Therefore the test  $\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}$  with

$$\delta_T^* = \int_{-\infty}^{\infty} L_2[\xi_T](x)^2 dF(\vartheta_T^*, x) \Longrightarrow \int_0^1 w_t^2 dt$$

will be ADF because the limit distribution of  $\delta_T^*$  does not depend on  $S(\cdot, \cdot)$ ,  $\sigma(\cdot)$  and  $\vartheta$ .

Let us realize this program. Denote the functions

$$\hat{h}(\vartheta, z) = 2 J(\vartheta)^{-1} \widehat{C}(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, z)}{\sigma(z)} \quad (3.26)$$

and

$$\hat{g}(\vartheta, z) = \widehat{C}(\vartheta)^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy. \quad (3.27)$$

Here  $J(\vartheta)$  is defined by (3.14) and

$$\widehat{C}(\vartheta) = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \right]^2 f(\vartheta, z) dz. \quad (3.28)$$

Introduce

$$\hat{I}_1 = \int_{-\infty}^y \hat{g}(\vartheta, z)^2 f(\vartheta, z) dz, \quad \hat{I}_2 = \int_{-\infty}^y \hat{h}(\vartheta, z) \hat{g}(\vartheta, z) f(\vartheta, z) dz,$$

$$\hat{I}_3 = \int_{-\infty}^y \hat{h}(\vartheta, z) f(\vartheta, z) dz, \quad \hat{I}_4 = \int_{-\infty}^y \hat{h}(\vartheta, z)^2 f(\vartheta, z) dz$$

and put

$$\hat{I}_5 = \int_{-\infty}^y \hat{g}(\vartheta, z) f(\vartheta, z) dz.$$

Indeed, we define the functions  $\hat{\varphi}_1(\cdot, \cdot)$ ,  $\hat{\varphi}_2(\cdot, \cdot)$  and  $\hat{\psi}_2(\cdot, \cdot)$  given by (1.23), (1.24) and (1.25), respectively, where we replace  $g(\cdot, \cdot)$  and  $h(\cdot, \cdot)$  by  $\hat{g} = \hat{g}(\vartheta, y)$  and  $\hat{h} = \hat{h}(\vartheta, y)$ . Then we introduce

$$w_{F(\vartheta, x)} = U(F(\vartheta, x)) \tag{3.29}$$

$$+ \int_{-\infty}^x \int_{-\infty}^y \hat{\varphi}_2(\vartheta, y)^{-1} [\hat{\varphi}_1(\vartheta, y) \hat{h}(\vartheta, z) + \hat{\psi}_2(\vartheta, y) \hat{g}(\vartheta, z)] dU(F(\vartheta, z)) f(\vartheta, y) dy.$$

To construct the test, we have to replace  $U(F(\vartheta, x))$  by  $\xi_T(\vartheta_T^*, x)$  defined by (3.22) and the functions  $\hat{h}(\cdot, \cdot)$ ,  $\hat{g}(\cdot, \cdot)$ ,  $\hat{\varphi}_1(\cdot, \cdot)$ ,  $\hat{\varphi}_2(\cdot, \cdot)$  and  $\hat{\psi}_2(\cdot, \cdot)$  by their “empirical versions” based only on the observations. In fact, we insert the estimator  $\vartheta_T^*$  given by (3.11) instead of the unknown parameter  $\vartheta$  in the expressions of the above functions.

In the construction of the test, we need to introduce an additional condition.

$\mathcal{R}_5$ . The function  $S(\vartheta, x)$  has two continuous derivatives w.r.t.  $\vartheta$  such that

$$\dot{S}(\vartheta, x), \ddot{S}(\vartheta, x) \in \mathcal{P}$$

and the functions  $\dot{S}(\vartheta, x)$  and  $\sigma(x)$  have continuous derivatives w.r.t.  $x$  such that

$$\dot{S}'(\vartheta, x), \sigma'(x) \in \mathcal{P}.$$

Moreover, due to the consistency of the MDE, we have the convergence (as  $T \rightarrow \infty$ )

$$\hat{\varphi}_2(\vartheta_T^*, y) - \hat{\varphi}_2(\vartheta, y) \longrightarrow 0.$$

Hence, we can introduce the function

$$\hat{\varphi}_2^+(\vartheta_T^*, y) = \begin{cases} \hat{\varphi}_2(\vartheta_T^*, y)^{-1}, & \text{if } \hat{\varphi}_2(\vartheta_T^*, y) > 0, \\ 0, & \text{else,} \end{cases}$$

which asymptotically coincides with  $\hat{\varphi}_2(\vartheta, y)^{-1}$  and therefore the limit distribution does not changed.

We construct the ADF GoF test based on the following statistic :

$$W_T^*(x) = \xi_T(\vartheta_T^*, x) + \int_{-\infty}^x \int_{-\infty}^y \hat{\varphi}_2^+(\vartheta_T^*, y) \left[ \hat{\varphi}_1(\vartheta_T^*, y) \hat{h}(\vartheta_T^*, z) \right. \tag{3.30}$$

$$\left. + \hat{\psi}_2(\vartheta_T^*, y) \hat{g}(\vartheta_T^*, z) \right] d\xi_T(\vartheta_T^*, z) dF(\vartheta_T^*, y).$$

Then we have to show that

$$W_T^*(x) \implies L_2[U](t) = w_t.$$

The main technical problem in the realization of this program is the definition of the following stochastic integrals :

$$N(\vartheta_T^*, y) = \int_{-\infty}^y \hat{h}(\vartheta_T^*, z) \, d\xi_T(\vartheta_T^*, z) \quad (3.31)$$

and

$$M(\vartheta_T^*, y) = \int_{-\infty}^y \hat{g}(\vartheta_T^*, z) \, d\xi_T(\vartheta_T^*, z). \quad (3.32)$$

Remark that we cannot compute these integrals directly because the integrands contain the MDE  $\vartheta_T^*$  which depends on the whole trajectory  $X^T = \{X_t, 0 \leq t \leq T\}$ . Therefore, the corresponding stochastic integrals (3.31) and (3.32) are not well defined.

To avoid this problem we replace the corresponding stochastic integrals by the ordinary ones using the Itô formula. We shall mention that this approach was introduced in the similar problem in [34] and [44].

Let us suppose that  $\hat{h}(\cdot, \cdot)$  and  $\hat{g}(\cdot, \cdot)$  are piece-wise continuous functions and consider the calculation of the integrals

$$\int_a^b \hat{h}(\vartheta, z) \, d\xi_T(\vartheta, z), \quad \int_a^b \hat{g}(\vartheta, z) \, d\xi_T(\vartheta, z).$$

For any partition  $a = z_1 < z_2 < \dots < z_m = b$ , we can write

$$\begin{aligned} \sum_{k=1}^{m-1} \hat{h}(\vartheta, \tilde{z}_k) [\xi_T(\vartheta, z_{k+1}) - \xi_T(\vartheta, z_k)] &= \frac{1}{\sqrt{T}} \int_0^T \sum_{k=1}^{m-1} \frac{\hat{h}(\vartheta, \tilde{z}_k)}{\sigma(X_s)} \mathbb{I}_{\{z_k \leq X_s < z_{k+1}\}} \, dX_s \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \sum_{k=1}^{m-1} \frac{\hat{h}(\vartheta, \tilde{z}_k)}{\sigma(X_s)} \mathbb{I}_{\{z_k \leq X_s < z_{k+1}\}} S(\vartheta, X_s) \, ds. \end{aligned}$$

Therefore as  $\max |z_{k+1} - z_k| \rightarrow 0$ , we obtain the limit

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{k=1}^{m-1} \hat{h}(\vartheta, \tilde{z}_k) [\xi_T(\vartheta, z_{k+1}) - \xi_T(\vartheta, z_k)] &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{a \leq X_s < b\}} \, dX_s \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{a \leq X_s < b\}} S(\vartheta, X_s) \, ds \end{aligned}$$

and for  $a = -\infty$  and  $b = y$  (our case), we have the equality

$$\begin{aligned} \int_{-\infty}^y \hat{h}(\vartheta, z) \, d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} \, dX_s \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} S(\vartheta, X_s) \, ds. \end{aligned}$$

By using the same argument, we obtain

$$\begin{aligned} \int_{-\infty}^y \hat{g}(\vartheta, z) \, d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{g}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} \, dX_s \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{g}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} S(\vartheta, X_s) \, ds. \end{aligned} \tag{3.33}$$

Now, we introduce the function

$$F(\vartheta, y, x) = \int_{X_0}^x R(\vartheta, y, z) \, dz,$$

where

$$R(\vartheta, y, z) = 2 J(\vartheta)^{-1} \widehat{C}(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, z)}{\sigma(z)^2} \mathbb{I}_{\{z < y\}}.$$

Here  $\widehat{C}(\vartheta)$  and  $J(\vartheta)$  are defined by (3.28) and (3.14), respectively. By the Itô formula,

$$dF(\vartheta, y, X_s) = F'_x(\vartheta, y, X_s) \, dX_s - \frac{1}{2} F''_{xx}(\vartheta, y, X_s) \sigma(X_s)^2 \, ds,$$

we can write

$$\int_0^T R(\vartheta, y, X_s) \, dX_s = \int_{X_0}^{X_T} R(\vartheta, y, z) \, dz - \frac{1}{2} \int_0^T R'_x(\vartheta, y, X_s) \sigma(X_s)^2 \, ds.$$

Indeed, we obtain

$$\begin{aligned} \int_{-\infty}^y \hat{h}(\vartheta, z) \, d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta, y, z) \, dz \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] \, ds. \end{aligned}$$

Hence, we have no more stochastic integral and we can substitute the MDE  $\vartheta_T^*$  given by (3.11). Now, the process (3.31) is well defined and has the following expression

$$\begin{aligned} N(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) \, dz \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] \, ds. \end{aligned} \tag{3.34}$$

Using once more the Itô formula, the representation (3.33) becomes as follows :

$$\begin{aligned} \int_{-\infty}^y \hat{g}(\vartheta, z) \, d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta, y, z) \, dz \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] \, ds. \end{aligned}$$

Here

$$Q(\vartheta, y, z) = \frac{\widehat{C}(\vartheta)^{-\frac{1}{2}}}{\sigma(z)} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) \, dy \mathbb{I}_{\{z < y\}},$$

where  $\widehat{C}(\vartheta)$  is given by (3.28). Therefore, the process (3.32) is well defined and has the following representation

$$\begin{aligned} M(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) \, dz \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] \, ds. \end{aligned} \quad (3.35)$$

Now, using the representations (3.34) and (3.35), the statistic (3.30) becomes as follows :

$$\begin{aligned} W_T^{**}(x) &= \xi_T(\vartheta_T^*, x) + \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) \, dz \right. \\ &\quad \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] \, ds \right] \, dF(\vartheta_T^*, y) \\ &\quad + \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) \, dz \right. \\ &\quad \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] \, ds \right] \, dF(\vartheta_T^*, y). \end{aligned} \quad (3.36)$$

Let us put

$$\delta_T^* = \int_{-\infty}^{\infty} W_T^{**}(x)^2 \, dF(\vartheta_T^*, x).$$

The main result of this chapter is given in the following Theorem.

**Theorem 3.5.6.** *Suppose that the conditions  $\mathcal{ES}$ ,  $\mathcal{A}_0$ ,  $\mathcal{R}$ ,  $\mathcal{R}_4$  and  $\mathcal{R}_5$  are fulfilled, then the test*

$$\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}, \quad \mathbf{P} \left( \int_0^1 w_t^2 \, dt > c_\alpha \right) = \alpha$$

*is ADF and of asymptotic size  $\alpha \in (0, 1)$ .*

**Proof.** It is enough to verify the convergence

$$\delta_T^* \implies \int_{-\infty}^{\infty} w_{F(\vartheta, x)}^2 dF(\vartheta_T^*, x) \equiv \delta^* \quad (3.37)$$

under hypothesis  $\mathcal{H}_0$ . Here  $w_{F(\vartheta, x)}$  is given by (3.29).

Note that due to the consistency of the MDE  $\vartheta_T^*$ , we have the convergence (as  $T \rightarrow \infty$ )

$$\begin{aligned} \hat{h}(\vartheta_T^*, z) &\longrightarrow \hat{h}(\vartheta, z) & \hat{g}(\vartheta_T^*, z) &\longrightarrow \hat{g}(\vartheta, z), \\ J(\vartheta_T^*) &\longrightarrow J(\vartheta), & \hat{\psi}_2(\vartheta_T^*, z) &\longrightarrow \hat{\psi}_2(\vartheta, z) \end{aligned}$$

and

$$\hat{\varphi}_1(\vartheta_T^*, z) \longrightarrow \hat{\varphi}_1(\vartheta, z), \quad \hat{\varphi}_2(\vartheta_T^*, z) \longrightarrow \hat{\varphi}_2(\vartheta, z).$$

Remind that we proved the convergence (in Theorem 3.4.5)

$$\xi_T(\vartheta_T^*, x) \implies U(F(\vartheta, x)).$$

Then we have to show that

$$N(\vartheta_T^*, y) \implies N(\vartheta, y), \quad (3.38)$$

where  $N(\vartheta_T^*, y)$  is defined by (3.34) and

$$\begin{aligned} N(\vartheta, y) &= \int_{-\infty}^y \hat{h}(\vartheta, z) dU(F(\vartheta, z)) \\ &= J(\vartheta)^{-1} \hat{C}(\vartheta)^{\frac{1}{2}} \int_{-\infty}^y \frac{\dot{S}(\vartheta, z)}{\sigma(z)} \sqrt{f(\vartheta, z)} dW(z) \\ &\quad - \frac{4\hat{C}(\vartheta)^{\frac{1}{2}}}{J(\vartheta)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{1}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \int_{-\infty}^y \frac{\dot{S}(\vartheta, z)^2}{\sigma(z)^2} f(\vartheta, z) dz. \end{aligned} \quad (3.39)$$

Here the function  $\hat{h}(\vartheta, z)$  is defined by (3.26). The process  $N(\vartheta_T^*, y)$  has the following representation

$$\begin{aligned} N(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\ &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \end{aligned}$$



$$\begin{aligned} & - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\ & - \frac{1}{\sqrt{T}} \int_0^T R(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds = I_T^*(\vartheta_T^*, y) - K_T^*(\vartheta_T^*, y). \end{aligned}$$

Combining Itô's formula and the central limit theorem for stochastic integrals, we obtain

$$\begin{aligned} I_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \\ & - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\ &= \frac{1}{\sqrt{T}} \int_0^T R(\vartheta, y, X_s) \sigma(X_s) dW_s + o(1) \\ &\implies \int_{-\infty}^{\infty} R(\vartheta, y, z) \sigma(z) \sqrt{f(\vartheta, z)} dW(z), \end{aligned}$$

where  $W(\cdot)$  is double-sided Wiener process. Further, by Taylor's formula, the law of large numbers and using the representation (3.13), we can write

$$\begin{aligned} K_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_0^T R(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\ &= \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T R(\vartheta_T^*, y, X_s) \dot{S}(\tilde{\vartheta}, X_s) ds \\ &\implies 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\ &\quad \int_{-\infty}^{\infty} R(\vartheta, y, z) \dot{S}(\vartheta, z) f(\vartheta, z) dz. \end{aligned}$$

Therefore, we obtain  $N(\vartheta_T^*, y) \implies N(\vartheta, y)$ , where  $N(\vartheta, y)$  is given by (3.39).

In the same manner, we show that we have the convergence

$$M(\vartheta_T^*, y) \implies M(\vartheta, y), \quad (3.40)$$

where  $M(\vartheta_T^*, y)$  is defined by (3.35) and

$$\begin{aligned} M(\vartheta, y) &= \int_{-\infty}^y \hat{g}(\vartheta, z) dU(F(\vartheta, z)) \\ &= \hat{C}(\vartheta)^{-\frac{1}{2}} \int_{-\infty}^y \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z) f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \sqrt{f(\vartheta, z)} dW(z) \\ & - 4 J(\vartheta)^{-1} \hat{C}(\vartheta)^{-\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\ & \int_{-\infty}^y \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z)^2 f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \dot{S}(\vartheta, z) f(\vartheta, z) dz. \end{aligned} \quad (3.41)$$

Here the function  $\hat{g}(\vartheta, z)$  is defined by (3.27). For the process  $M(\vartheta_T^*, y)$ , we have

$$\begin{aligned}
 M(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) \, dz \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) \, dz \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] \, ds \\
 &= \tilde{I}_T^*(\vartheta_T^*, y) - \tilde{K}_T^*(\vartheta_T^*, y).
 \end{aligned}$$

For the first integral, due to the Itô formula and the central limit theorem for stochastic integrals, we can write

$$\begin{aligned}
 \tilde{I}_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) \, dz \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &= \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta, y, X_s) \sigma(X_s) \, dW_s + o(1) \\
 &\implies \int_{-\infty}^{\infty} Q(\vartheta, y, z) \sigma(z) \sqrt{f(\vartheta, z)} \, dW(z).
 \end{aligned}$$

Further, we have for the second integral

$$\begin{aligned}
 \tilde{K}_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] \, ds \\
 &= \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T Q(\vartheta_T^*, y, X_s) \dot{S}(\tilde{\vartheta}, X_s) \, ds \\
 &\implies 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{1}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) \, dy \, dW(z) \\
 &\quad \int_{-\infty}^{\infty} Q(\vartheta, y, z) \dot{S}(\vartheta, z) f(\vartheta, z) \, dz.
 \end{aligned}$$

Therefore  $M(\vartheta_T^*, y) \implies M(\vartheta, y)$ , where  $M(\vartheta, y)$  is defined by (3.41). Finally, (3.23), (3.38) and (3.40) give us the convergence

$$W_T^{**}(x) \implies w_{F(\vartheta, x)}.$$

Now, to obtain (3.37), we will prove the convergence of the integrals by checking the following three conditions : (see [27] and [34])

1. For any  $x_1, \dots, x_k$

$$(W_T^{**}(x_1), \dots, W_T^{**}(x_k)) \implies (w_{F(\vartheta, x_1)}, \dots, w_{F(\vartheta, x_k)}),$$

where  $W_T^{**}(x)$  and  $w_{F(\vartheta, x)}$  are defined by (3.36) and (3.29), respectively.

2. For  $|x_i| < L, i = 1, 2$

$$\mathbf{E}_\vartheta |W_T^{**}(x_1) - W_T^{**}(x_2)|^2 \leq C |x_1 - x_2|^{\frac{1}{2}}, \quad (3.42)$$

where  $C$  and  $L$  are some positive constants.

3. For any  $\kappa > 0$  there exist  $L > 0$  such that

$$\int_{|x|>L} \mathbf{E}_\vartheta W_T^{**}(x)^2 f(\vartheta_T^*, x) dx < \kappa. \quad (3.43)$$

The first convergence of finite-dimensional distributions follows from (3.23), (3.38) and (3.40) for any  $x_1, \dots, x_k$ .

Then, in order to obtain the estimate (3.42), we write

$$\begin{aligned} \mathbf{E}_\vartheta |W_T^{**}(x_1) - W_T^{**}(x_2)|^2 &\leq 3 \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 \\ &+ 3 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\ &- \left. \left. \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ &+ 3 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\ &- \left. \left. \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2. \end{aligned}$$

Further, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 &\leq 2 \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \mathbb{I}_{\{x_2 < X_s < x_1\}} dW_s \right|^2 \\ &+ 2 \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{I}_{\{x_2 < X_s < x_1\}} ds \right|^2 \\ &\leq 2 \int_{x_2}^{x_1} f(\vartheta, y) dy + 2 \left( \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^4 \right)^{\frac{1}{2}} \left( \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \frac{\dot{S}(\tilde{\vartheta}, y)}{\sigma(y)} \hat{f}_T(y) dy \right|^4 \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq 2 \int_{x_2}^{x_1} f(\vartheta, y) dy + C \left( \int_{x_2}^{x_1} P(y) f(\vartheta, y) dy \right)^{\frac{1}{2}}.$$

Here for any measurable function  $l(\cdot)$ , we use the following relation (can be found in [39], Section 1.1.3)

$$\frac{1}{T} \int_0^T l(X_t) dt = \int_{-\infty}^{\infty} l(x) \hat{f}_T(x) dx,$$

where  $\hat{f}_T(x)$  is the *local time estimator* given by (3.9) of the invariant density  $f(\vartheta, x)$ .

Remind that by the conditions  $\mathcal{A}_0$  and  $\mathcal{R}_5$ , the functions  $\dot{S}(\vartheta, \cdot)$  and  $\sigma(\cdot)^{\pm 1}$  have polynomial majorants  $P(y)$ . In addition, the invariant density has exponentially decreasing tails by condition  $\mathcal{A}_0$ , i.e., there exist the constants  $C_1 > 0$  and  $\lambda > 0$  such that

$$f(\vartheta, x) \leq C_1 e^{-\lambda|x|}.$$

Therefore,

$$\mathbf{E}_{\vartheta} |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 \leq C |x_1 - x_2| + C |x_1 - x_2|^{\frac{1}{2}} \leq C |x_1 - x_2|^{\frac{1}{2}}.$$

Then, we have

$$\begin{aligned} & \mathbf{E}_{\vartheta} \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \leq 2 \mathbf{E}_{\vartheta} \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \quad + 2 \mathbf{E}_{\vartheta} \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \sqrt{T} (\vartheta_T^* - \vartheta) \left[ \frac{1}{T} \int_{X_0}^{X_T} \dot{R}(\tilde{\vartheta}, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{T} \int_0^T \left[ \dot{R}(\tilde{\vartheta}, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} \dot{R}'_x(\tilde{\vartheta}, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \leq 2 \mathbf{E}_{\vartheta} \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \end{aligned}$$

$$+ 2\mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T \dot{R}(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 + o(1).$$

Indeed, here we used the Itô formula

$$\begin{aligned} & \int_{X_0}^{X_T} R(\vartheta, y, z) dz - \int_0^T \left[ R(\vartheta, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] ds \\ &= \int_0^T R(\vartheta, y, X_s) dW_s \end{aligned}$$

and

$$\begin{aligned} & \int_{X_0}^{X_T} \dot{R}(\tilde{\vartheta}, y, z) dz - \int_0^T \left[ \dot{R}(\tilde{\vartheta}, y, X_s) S(\vartheta, X_s) + \frac{1}{2} \dot{R}'_x(\tilde{\vartheta}, y, X_s) \sigma(X_s)^2 \right] ds \\ &= \int_0^T \dot{R}(\tilde{\vartheta}, y, X_s) dW_s. \end{aligned}$$

Additionally, by Fubini's theorem and Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \leq \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \int_{x_2}^{x_1} R(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \\ & \leq \mathbf{E}_\vartheta \frac{1}{T} \int_0^T \left| \int_{x_2}^{x_1} R(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^2 ds \\ & \leq \mathbf{E}_\vartheta \int_{-\infty}^{\infty} \left| \int_{x_2}^{x_1} R(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^2 \hat{f}_T(z) dz \\ & \leq [F(\vartheta, x_1) - F(\vartheta, x_2)] \\ & \quad \int_{x_2}^{x_1} \int_{-\infty}^{\infty} [R(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y)]^2 f(\vartheta, z) dz dF(\vartheta, y). \end{aligned}$$

Therefore, using the conditions  $\mathcal{A}_0$  and  $\mathcal{R}_5$ , we obtain

$$\mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \leq C |x_1 - x_2|.$$

Then, we have

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T \dot{R}(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \leq \mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^2 \left| \frac{1}{T} \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \\
 &\leq \left( \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^4 \right)^{\frac{1}{2}} \\
 &\quad \frac{1}{T^2} \left( \mathbf{E}_\vartheta \left| \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^4 \right)^{\frac{1}{2}} \\
 &\leq \frac{C}{T^2} \left( T \mathbf{E}_\vartheta \int_0^T \left| \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^4 ds \right)^{\frac{1}{2}} \\
 &\leq \frac{C}{T} \left( \mathbf{E}_\vartheta \int_{-\infty}^{\infty} \left| \int_{x_2}^{x_1} \dot{R}(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^4 \hat{f}_T(z) dz \right)^{\frac{1}{2}} \\
 &\leq \frac{C}{T} \left( [F(\vartheta, x_1) - F(\vartheta, x_2)]^3 \right. \\
 &\quad \left. \int_{x_2}^{x_1} \int_{-\infty}^{\infty} \left| \dot{R}(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \right|^4 f(\vartheta, z) dz dF(\vartheta, y) \right)^{\frac{1}{2}} \leq \frac{C}{T}.
 \end{aligned}$$

Similarly, we obtain the following estimate

$$\begin{aligned}
 &\mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
 &\quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
 &\leq C |x_1 - x_2|.
 \end{aligned}$$

Now, we have to check the condition (3.43). Indeed,

$$\begin{aligned}
 &\mathbf{E}_\vartheta |W_T^{**}(x)|^2 \leq 3 \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x)|^2 \\
 &\quad + 3 \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\
 &\quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
 &\quad + 3 \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
 &\quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T \left[ Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \right|^2.
 \end{aligned}$$

We can write

$$\begin{aligned}
\mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x)|^2 &\leq 2 \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \mathbb{I}_{\{X_s < x\}} dW_s \right|^2 \\
&+ 2 \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < x\}} ds \right|^2 \\
&\leq 2 F(\vartheta, x) + 2 \left( \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^4 \right)^{\frac{1}{2}} \left( \mathbf{E}_\vartheta \left| \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < x\}} ds \right|^4 \right)^{\frac{1}{2}} \\
&\leq 2 F(\vartheta, x) + C \left( \int_{-\infty}^x P(y) f(\vartheta, y) dy \right)^{\frac{1}{2}} \leq C.
\end{aligned}$$

Moreover, using the same representation as above, we have

$$\begin{aligned}
\mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\
\left. \left. - \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \leq C
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[ \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
\left. \left. - \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
\leq C.
\end{aligned}$$

Finally, the convergence (3.37) is proved and the test  $\psi_T^*$  is ADF.





# Conclusion

The main purpose of this work is the construction of the goodness-of-fit (GoF) tests for the parametric null hypotheses in the case of some stochastic processes observed in continuous time. In particular, we consider three models of observations : diffusion process with “small noise”  $X^\varepsilon = (X_t, 0 \leq t \leq T)$  ( $\varepsilon \rightarrow 0$ ), ergodic diffusion process  $X^T = \{X_t, 0 \leq t \leq T\}$  ( $T \rightarrow \infty$ ) and inhomogeneous  $\tau$ -periodic Poisson process  $X^n = (X_t, 0 \leq t \leq T = n\tau)$  ( $n \rightarrow \infty$ ). Under the basic hypotheses, we assume that the models depend on some unknown one-dimensional parameter.

For all these processes, we construct “asymptotically distribution free” (ADF) GoF tests  $\psi_\varepsilon$ ,  $\hat{\psi}_n^*$ ,  $\hat{\varphi}_n$  and  $\psi_T^*$  of asymptotic size  $\alpha \in (0, 1)$  based on the special linear transformation  $L[\cdot]$ . The form of this transformation is the main contribution of this work. It is based on the solution of the Fredholm equation of the second kind with degenerated kernel.

The tests are constructed in two steps. First, we introduce the basic statistics  $u_\varepsilon(\cdot)$ ,  $u_n^*(\cdot)$  and  $\eta_T^*(\cdot, \cdot)$  and we use the minimum distance estimator for the unknown parameter. Then we show that these statistics converge to the processes  $(u(t), 0 \leq t \leq T)$ ,  $(u(t), 0 \leq t \leq \tau)$  and  $(\eta(\vartheta, x), x \in \mathbb{R})$ , respectively. Then, we propose the (first) transformations of these limits which lead to the same Gaussian process ( $0 \leq \nu \leq 1$ )

$$U(\nu) = W(\nu) - \int_0^1 g(\vartheta, s) dW(s) \int_0^\nu h(\vartheta, s) ds, \quad \int_0^1 g(\vartheta, s)^2 ds = 1.$$

In addition, we consider in Section 2.3 the empirical basic statistic  $U_n(\cdot, \cdot)$  which admits the same limit  $U(\cdot)$  and this shows the universality of this limit process. Note that the functions  $h(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  in different models are different.

The second step (general for all models) is to apply the (second) linear transformation  $L[\cdot]$  to the process  $U(\cdot)$  such that  $L[U](\nu) = w_\nu$ , where  $w_\nu, 0 \leq \nu \leq 1$  is a Wiener process. Based on the above property, we introduce the “empirical versions”  $U_\varepsilon(\cdot), U_n(\cdot, \cdot), U_n^*(\cdot)$  and  $\xi_T(\cdot, \cdot)$  of the process  $U(\cdot)$  and we have the convergence of the corresponding statistics to the same limit :

$$\Delta_\varepsilon = \frac{1}{T} \int_0^T L[U_\varepsilon](t)^2 dt \implies \hat{\Delta}, \quad \hat{\Delta}_n^* = \int_0^\tau \frac{L[U_n](r)^2}{\Lambda(\vartheta_n^*, \tau)} \lambda(\vartheta_n^*, r) dr \implies \hat{\Delta},$$

$$\hat{\delta}_n = \int_0^1 L[U_n^*](\nu)^2 d\nu \implies \hat{\Delta}, \quad \delta_T^* = \int_{-\infty}^\infty L[\xi_T](x)^2 dF(\vartheta_T^*, x) \implies \hat{\Delta} \equiv \int_0^1 w_\nu^2 d\nu.$$

Therefore we propose the ADF GoF tests of asymptotic size  $\alpha \in (0, 1)$  for each studied model

$$\psi_\varepsilon = \mathbb{I}_{\{\Delta_\varepsilon > c_\alpha\}}, \quad \hat{\psi}_n^* = \mathbb{I}_{\{\hat{\Delta}_n^* > c_\alpha\}}, \quad \hat{\varphi}_n = \mathbb{I}_{\{\hat{\delta}_n > c_\alpha\}}, \quad \psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}},$$

where,  $\mathbf{P}(\hat{\Delta} > c_\alpha) = \alpha$ .

We emphasize that the founded linear transformation in this work is rather cumbersome and its application in real problems can be not so easy. At the same time, we understand that this result is in some sense “negative” and says that if we have no MLE it is better to seek another GoF tests, which are ADF. Nevertheless this study has to be done to cover the gap which exists in this well-known statistical problem.

So the further work will be concentrated on doing numerical simulations. Our starting point is proving that the proposed test  $\hat{\psi}_n^*$  in the case of the inhomogeneous Poisson process is of asymptotic size  $\alpha$  but we do not obtain the desired result till now. Due to the universality of the special linear transformation, we shall mention that it can be applied to many other statistical models to construct ADF GoF tests, e.g., nonlinear time series. This is an interesting problem but not yet been treated. In addition, as we mentioned before, the linear transformation is constructed in the case of other (non MLE) estimators and in our work we used the minimum distance estimator to construct the ADF GoF tests. It will be important to construct ADF GoF tests with other estimators.

# Bibliography

- [1] Albrecht, P. (1982) Testing the goodness of fit of a mixed Poisson process. *Insurance Math. Econom.* 1, 27-33.
- [2] Andersen, P. K., Borgan, O., Gill, R. D. and Keiding, N. (1993) *Statistical Models Based on Counting Processes*. Springer, New York.
- [3] Anderson, T. W. and Darling, D. A. (1952) Asymptotic theory of certain “goodness of fit” criteria based on stochastic processes. *Ann. Math. Statist.* 23(2), 193-212.
- [4] Babu, G. J. and Rao, C. R. (2004) Goodness-of-fit tests when parameters are estimated. *Sankhya :The Indian Journal of Statistics.* 66(1), 63-74.
- [5] Baringhaus, L. and Henze, N. (1992) A goodness of fit test for the Poisson distribution based on the empirical generating function. *Statist. Probab. Lett.* 13(4), 269-274.
- [6] Ben Abdeddaiem, M. (2015a) On goodness-of-fit tests with parametric hypotheses for inhomogeneous Poisson process using a minimum distance estimator. Submitted.
- [7] Ben Abdeddaiem, M. (2015b) On ADF goodness-of-fit test with parametric hypotheses for ergodic diffusion process using a minimum distance estimator. Submitted.
- [8] Ben Abdeddaiem, M. (2016) On goodness-of-fit tests for parametric hypotheses in perturbed dynamical systems using a minimum distance estimator. *Stat. Inference Stoch. Process.* 1-29, DOI 10.1007/s11203-016-9132-6.
- [9] Best, D. J. and Rayner, J. C. W. (1999) Goodness of fit for the Poisson distribution. *Statist. Probab. Lett.* 44(3), 259-265.

- 
- [10] Cramér, H. (1928) On the composition of elementary errors. *Scand. Actuar. J.* 11(1), 141-180.
- [11] Dabye, A. S. (2013) On the Cramér-von Mises test with parametric hypothesis for Poisson processes. *Stat. Inference Stoch. Process.* 16(1), 1-13.
- [12] Dachian, S. and Kutoyants, Yu. A. (2008) On the goodness-of-fit tests for some continuous time processes. *Statistical Models and Methods for Biomedical and Technical Systems* (F. Vonta, M. Nikulin, N. Limnios and C. Huber-Carol, eds.). Birkhäuser, Boston, 385-403.
- [13] Darling, D. A. (1955) The Cramér-Smirnov test in the parametric case. *Ann. Math. Statist.* 26(1), 1-20.
- [14] Davies, R. B. (1977) Testing the hypothesis that a point process is Poisson. *Adv. Appl. Prob.* 9(4), 724-746.
- [15] Darling, D. A. (1957) The Kolmogorov-Smirnov, Cramér-von Mises tests. *Ann. Math. Statist.* 28(4), 823-838.
- [16] Delgado, M. A., Hidalgo, J. and Velasco, C. (2005) Distribution free goodness-of-fit tests for linear processes. *Ann. Statist.* Vol. 33(6), 2568-2609.
- [17] Dette, H. and Hetzler, B. (2009) Khmaladze transformation of integrated variance processes with applications to goodness-of-fit testing. *Math. Methods Statist.* 18(2), 97-116.
- [18] Durbin, J. (1973) *Distribution Theory for Tests Based on the Sample Distribution Function*. SIAM, Philadelphia.
- [19] Durrett, R. (1996) *Stochastic Calculus : A Practical Introduction*. CRC Press, Boca Raton.
- [20] Fournie, E. (1992) Un test de type Kolmogorov-Smirnov pour processus de diffusion ergodiques. Rapport de Recherche, 1696, INRIA, Sophia-Antipolis.
- [21] Fournie, E. and Kutoyants, Yu. A. (1993) Estimateur de la distance minimale pour des processus de diffusion ergodiques. Rapport de Recherche, 1952, INRIA, Sophia-Antipolis.
- [22] Freidlin, M. I. and Wentzell, A. D. (1984) *Random Perturbations of Dynamical Systems*. Springer, New York.

- 
- [23] Gasseem, A. (2010) Goodness-of-fit test for switching diffusion. *Stat. Inference Stoch. Process.* 13(2), 97-123.
- [24] Greenwood, P. E. and Nikulin, M. (1996) *A Guide to Chi-Squared Testing*. John Wiley and Sons, New York.
- [25] Hjort, N. L. (1990) Goodness of fit tests in models for life history data based on cumulative hazard rates. *Ann. Statist.* 18(3), 1221-1258.
- [26] Iacus, S. and Kutoyants, Yu. A. (2001) Semiparametric hypotheses testing for dynamical systems with small noise. *Math. Methods Statist.* 10(1), 105-120.
- [27] Ibragimov, I. A. and Khasminskii, R. Z. (1981) *Statistical Estimation*. Springer, New York.
- [28] Justel, A., Peña, D. and Zamar, R. (1997) A multivariate Kolmogorov-Smirnov test of goodness of fit. *Statist. Probab. Lett.* 35(3), 251-259.
- [29] Kac, M., Kiefer, J. and Wolfowitz, J. (1955) On tests of normality and other tests of goodness of fit based on distance methods. *Ann. Math. Statist.* 26(2), 189-211.
- [30] Khmaladze, E. V. (1981) Martingale approach in the theory of goodness-of-fit tests. *Theory Probab. Appl.* 26(2), 240-257.
- [31] Khmaladze, E. V. (1993) Goodness-of-fit problem and scanning innovation martingales. *Ann. Statist.* 21(2), 798-829.
- [32] Khmaladze, E. V. and Koul, H. L. (2004) Martingale transforms goodness-of-fit tests in regression models. *Ann. Statist.* 32(3), 995-1034.
- [33] Klar, B. (1999) Goodness-of-fit tests for discrete models based on the integrated distribution function. *Metrika.* 49(1), 53-69.
- [34] Kleptsyna, M. and Kutoyants, Yu. A. (2014) On asymptotically distribution free tests with parametric hypothesis for ergodic diffusion processes. *Stat. Inference Stoch. Process.* 17(3), 295-319.
- [35] Kolmogorov, A. N. (1933) Sulla determinazione empirica di una legge di distribuzione. *Giornale dell'Istituto Italiano degli Attuari.* 4, 83-91.

- [36] Kutoyants, Yu. A. (1978) Estimation of a parameter of a diffusion process. *Theory Probab. Appl.* 23(3), 641-649.
- [37] Kutoyants, Yu. A. (1994) *Identification of Dynamical Systems with Small Noise*. Kluwer, Dordrecht.
- [38] Kutoyants, Yu. A. (1998) *Statistical Inference for Spatial Poisson Processes*. Lect. Notes Statist. 134, Springer, New York.
- [39] Kutoyants, Yu. A. (2004) *Statistical Inference for Ergodic Diffusion Processes*. Springer, London.
- [40] Kutoyants, Yu. A. (2011) On goodness-of-fit tests for perturbed dynamical systems. *J. Statist. Plann. Inference.* 141(5), 1655-1666.
- [41] Kutoyants, Yu. A. (2014a) On asymptotic distribution of parameter free tests for ergodic diffusion processes. *Stat. Inference Stoch. Process.* 17(2), 139-161.
- [42] Kutoyants, Yu. A. (2014b) On ADF goodness-of-fit tests for stochastic processes. *New Perspectives on Stochastic Modeling and Data Analysis* (J. R. Bozeman., V. Girardin and C. H. Skiadas, eds.). ISAST, Athens, 3-18.
- [43] Kutoyants, Yu. A. (2014c) On score-functions and goodness-of-fit tests for stochastic processes. (arXiv :1403.7715) To appear in *Math. Methods Statist.*
- [44] Kutoyants, Yu. A. (2015) On ADF goodness-of-fit tests for perturbed dynamical systems. *Bernoulli.* 21(4), 2430-2456.
- [45] Kutoyants, Yu. A. (2016) *Introduction to Statistics of Poisson Processes*. To appear.
- [46] Lehmann, E. L. and Romano, J. P. (2005) *Testing Statistical Hypotheses*. (3rd ed.) Springer, New York.
- [47] Lewis, P. A. W. (1965) Some results on tests for Poisson processes. *Biometrika.* 52(1-2), 67-77.
- [48] Liptser, R. and Shirayev, A. N. (2005) *Statistics of Random Processes II Applications*. (2-nd ed.) Springer, New York.

- [49] Maglapheridze, N., Tsigroshvili, Z. P. and van Pul, M. (1998) Goodness-of-fit tests for parametric hypotheses on the distribution of point processes. *Math. Methods. Statist.* 7, 60-77.
- [50] Martynov, G. V. (1992) Statistical tests based on empirical processes and related questions. *J. Soviet. Math.* 61(4), 2195-2271.
- [51] Negri, I. and Nishiyama, Y. (2009) Goodness of fit test for ergodic diffusion processes. *Ann. Inst. Statist. Math.* 61(4), 919-928.
- [52] Negri, I. and Nishiyama, Y. (2010) Review on goodness of fit tests for ergodic diffusion processes by different sampling schemes. *Economic Notes.* 39(1-2), 91-106.
- [53] Negri, I. and Nishiyama, Y. (2011) Goodness of fit test for small diffusions by discrete time observations. *Ann. Inst. Statist. Math.* 63(2), 211-225.
- [54] Negri, I. and Zhou, L. (2014) On goodness-of-fit testing for ergodic diffusion process with shift parameter. *Stat. Inference Stoch. Process.* 17(1), 51-73.
- [55] Nikabadze, A. M. (1984) On a method for constructing goodness-of-fit tests for parametric hypotheses in  $\mathbb{R}^{m*}$ . *Theory Probab. Appl.* 32(3), 539-544.
- [56] Rabhi, A. (2009) On the goodness-of-fit testing of composite hypothesis for dynamical systems with small noise. *Ann. I. S. U. P.* 53(2-3), 31-48.
- [57] Smirnov, N. V. (1936) Sur la distribution de  $\omega^2$ . *C. R. Acad. Sci. Paris.* 202, 449-452.
- [58] Spinelli, J. J. and Stephens, M. A. (1997) Cramér-von Mises tests of fit for the Poisson distribution. *Canad. J. Statist.* 25(2), 257-268.
- [59] Székely, G. J. and Rizzo, M. L. (2004) Mean distance test of Poisson distribution. *Statist. Probab. Lett.* 67(3), 241-247.
- [60] von Mises, R. (1931) *Wahrscheinlichkeitsrechnung*. Deuticke, Leipzig and Wien.
- [61] Watson, G. S. (1959) Some recent results in chi-square goodness-of-fit tests. *International Biometric Society.* 15(3), 440-468.
- [62] Zhou, L. (2014) On asymptotically parameter free test for ergodic diffusion process. *Ann. I. S. U. P.* 58(1-2), 37-56.

# Thèse de Doctorat

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## Tests d'ajustement pour des processus stochastiques dans le cas de l'hypothèse nulle paramétrique

On goodness-of-fit tests with parametric hypotheses for some stochastic processes

### Résumé

Ce travail est consacré au problème de construction des tests d'ajustement dans le cas des processus stochastiques observés en temps continu. Comme modèles d'observations, nous considérons les processus de diffusion avec « petit bruit » et ergodique et le processus de Poisson non homogène. Sous l'hypothèse nulle, nous traitons le cas où chaque modèle dépend d'un paramètre inconnu unidimensionnel et nous proposons l'estimateur de distance minimale pour ce paramètre. Notre but est la construction des tests d'ajustement « asymptotically distribution free » (ADF) de niveau asymptotique  $\alpha \in (0,1)$  dans le cas de cette hypothèse paramétrique pour les modèles traités. Nous montrons alors que la limite de chaque statistique étudiée ne dépend ni du modèle ni du paramètre inconnu. Les tests d'ajustement basés sur ces statistiques sont donc ADF.

L'objectif principal de ce travail est la construction d'une transformation linéaire spéciale. En particulier, nous résolvons l'équation de Fredholm du second type avec le noyau dégénéré. Sa solution nous permet de construire la transformation linéaire désirée. Ensuite, nous montrons que l'application de cette transformation aux statistiques de base étudiées dans chaque modèle nous aide à introduire des statistiques ayant la même limite (l'intégrale du carré du processus de Wiener). Cette dernière est « distribution free » vu qu'elle ne dépend ni du modèle ni du paramètre inconnu. Par conséquent, nous proposons des tests d'ajustement ADF en se basant sur cette transformation linéaire pour les processus de diffusion avec « petit bruit » et ergodique et le processus de Poisson non homogène.

### Mots Clés

Tests d'ajustement, estimateur de distance minimale, tests asymptotically distribution free, processus stochastiques, processus de diffusion, processus de Poisson non homogène.

### Abstract

This work is devoted to the problem of the construction of several goodness of-fit (GoF) tests in the case of some stochastic processes observed in continuous time. As models of observations, we take “small noise” and ergodic diffusion processes and an inhomogeneous Poisson process. Under the null hypothesis, we treat the case where each model depends on an unknown one-dimensional parameter and we consider the minimum distance estimator for this parameter. Our goal is to propose “asymptotically distribution free” (ADF) GoF tests of asymptotic size  $\alpha \in (0,1)$  in the case of the parametric null hypotheses for the considered models. Indeed, we show that the limit of each studied statistic does not depend on the model and the unknown parameter. Therefore, the tests based on these statistics are ADF.

The main purpose of this work is to construct a special linear transformation. In particular, we solve Fredholm equation of the second kind with degenerated kernel. Its solution gives us the desired linear transformation. Next, we show that the application of this transformation to the basic statistics allows us to introduce statistics with the same limit (the integral of the square of the Wiener process). The latter is “distribution free” because it does not depend on the models and the unknown parameter. Therefore, we construct the ADF GoF tests which are based on this linear transformation for the diffusion (“small noise” and ergodic) and inhomogeneous Poisson processes.

### Key Words

Goodness-of-fit tests, minimum distance estimator, asymptotically distribution free tests, stochastic processes, diffusion process, and inhomogeneous Poisson process.