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Université Pierre et Marie Curie

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Lucas Kaufmann Sacchetto

Dynamique holomorphe, Théorie du Pluripotentiel et applications

---

dirigée par Tien-Cuong Dinh
et Elisha Falbel

Soutenue le 23 juin 2016 devant le jury composé de :

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Résumé

Cette thèse est consacrée à l'étude de quelques problèmes en dynamique holomorphe discrete et continue à l'aide de la Théorie du Pluripotentiel.

Le premier problème présenté concerne la description des paires d'endomorphismes holomorphes permutables du plan projectif complexe qui ne partagent pas une itérée. Nous nous intéressons au cas où les degrés des deux applications coïncident après un certain nombre d'itérations. Nous montrons que telles applications sont des exemples de Lattès ou bien des relèvements des exemples de Lattès unidimensionnels. Combiné avec un théorème de T.-C. Dinh et N. Sibony ce résultat complète la classification des paires permutables en dimension deux.

Ensuite, nous nous intéressons à la dynamique des laminations par variétés complexes. Nous montrons que, dans une variété kählérienne compacte, le carré de la classe de cohomologie d'un cycle feuilleté dirigé par une lamination transversalement Lipschitz est toujours zéro. Parmi les conséquences nous montrons que l'espace projectif complexe $\mathbb{P}^n$ n'admet pas de cycle feuilleté transversalement Lipschitz de dimension $q \leq \frac{n}{2}$. Cela généralise un résultat de J.E. Fornæss et N. Sibony.

Dans la dernière partie nous étudions les mesures de Monge-Ampère à potentiel höldérien. Nous montrons que ces mesures satisfont un analogue d'un théorème de H. Skoda concernant l'intégrabilité exponentielle d'une fonction plurisousharmonique en termes de ses nombres de Lelong. Ce résultat peut être vu comme une très forte compacité pour les fonctions plurisousharmoniques qui sont eux-mêmes un outil fondamental en dynamique holomorphe.

Mots-clés

Dynamique Holomorphe, Théorie du Pluripotentiel, endomorphismes holomorphes de $\mathbb{P}^k$, endomorphismes holomorphes permutables, laminations par variétés complexes, cycles feuilletés, mesures de Monge–Ampère.

Holomorphic Dynamics, Pluripotential Theory and applications

Abstract

This thesis is devoted to the study of some problems in discrete and continuous holomorphic dynamics with the tools of Pluripotential Theory.

The first problem we consider involves the description of commuting pairs of holomorphic endomorphisms of the complex projective plane that do not share an iterate. We consider the case when their degrees coincide after some number of iterations. We show that these maps are either Lattès maps or lifts of one-dimensional Lattès maps. Together with a theorem of T.-C. Dinh and N. Sibony this result completes the classification of commuting pairs in dimension two.

Later on, we turn our attention to the dynamics of laminations by complex manifolds. We show that, on a compact Kähler manifold, the square of the cohomology class of a foliated cycle directed by a transversally Lipschitz lamination is always zero. As a corollary we show that the complex projective space $\mathbb{P}^n$ do not carry any transversally Lipschitz foliated cycle of dimension $q \leq \frac{n}{2}$, generalizing a result by J.E. Fornæss and N. Sibony.

In the last part we study Monge-Ampère measures with Hölder continuous potential. We show that these measures satisfy an analogue of a theorem of H. Skoda concerning the exponential integrability of plurisubharmonic functions in terms of its Lelong numbers. This result can be viewed as a strong compactness property of plurisubharmonic functions, a class of functions of fundamental importance in holomorphic dynamics.

Keywords

Holomorphic Dynamics, Pluripotential Theory, holomorphic endomorphisms of $\mathbb{P}^k$, commuting holomorphic endomorphisms, laminations by complex manifolds, foliated cycles, Monge-Ampère measures.

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Introduction

Le domaine de la Dynamique Holomorphe est consacré à l'étude des système dynamiques dans une variété complexe définis à partir d'objets holomorphes. Son étude peut être divisée en deux parties : la dynamique discrète et la dynamique continue. Du côté discret on s'intéresse à l'itération d'une application holomorphe $f : X \to X$, où X est une variété complexe. Le cas où X est la sphère de Riemann est classique et fut étudié par Schröder, Lattès, Fatou, Julia, etc. au tournant du XX ^{ème} siècle et revisité activement à partir des années 1980. Pour les variétés de dimension supérieure le sujet est plus récent et fut introduit par des mathématiciens comme Bedford, Fornæss, Hubbard, Lyubich, Sibony, Smillie, entre autres. Du côté continu on s'intéresse aux équations différentielles complexes, aux champs des vecteurs holomorphes et, plus généralement, aux feuilletages et laminations holomorphes. Ce sujet remonte aux travaux fondateurs de Painlevé et Poincaré et continue à être activement étudié jusqu'à nos jours.

La Dynamique Holomorphe discrète et continue peuvent être, toutes les deux, étudiés à l'aide des outils de la Théorie du Pluripotentiel. Cette théorie fut introduite par Lelong et Oka dans les années 1940 et depuis lors elle s'est montré indispensable dans l'Analyse Complexe à plusieurs variables, une fois que quelques résultats fondamentaux de la théorie unidimensionnelle sont difficiles à utiliser en dimension supérieure ou sont même faux. Les objets principaux dans la théorie sont les fonctions plurisousharmoniques, les courants positifs fermés et l'opérateur de Monge–Ampère. La Théorie du Pluripotentiel est un outil très puissant en Géométrie et Dynamique Complexes et son développement et ses applications semblent loin d'être épuisés.

L'objectif de cette thèse est de présenter quelques problèmes en Dynamique Holomorphe discrète et continue et de montrer comment la Théorie du Pluripotentiel nous permet de les résoudre.

Le premier problème auquel nous nous intéressons est présenté dans le Chapitre 2 et concerne les endomorphismes holomorphes de l'espace projectif complexe $\mathbb{P}^k$. Une application holomorphe $f : \mathbb{P}^k \to \mathbb{P}^k$ est donnée en coordonnées homogènes par $f = [P_0 : \cdots : P_k]$, où les P_j sont des polynômes homogènes de même degré d_f et sans zéros communs dans $\mathbb{P}^k$. L'entier d_f est appelé degré algébrique de f.

Nous nous intéressons à la question suivante :

Problème A. Décrire les paires d'endomorphismes $f, g : \mathbb{P}^k \to \mathbb{P}^k$ avec $d_f, d_g \geq 2$ qui sont permutables

$$f \circ g = g \circ f \tag{0.0.1}$$

et qui ne partagent aucune itérée

$$f^n \neq g^m, \text{ pour tout } n, m \geq 1. \tag{$\star$}$$

Le cas unidimensionnel ($k = 1$) a été complètement classifié par Fatou, Julia et Ritt. Les seules examples sont les applications puissance, les polynômes de Tchebychev et les applications de Lattès : des endomorphismes de $\mathbb{P}^1$ induits par une application affine d'un tore complexe après le quotient par un groupe discret d'isométries, voir Théorème 2.1.2.

En dimension supérieure la question est plus délicate et une classification complète semble être difficile. Néanmoins, sous la condition supplémentaire sur les degrés de f et g :

$$d_f^n \neq d_g^m, \text{ pour tout } n, m \geq 1, \tag{0.0.2}$$

Dinh et Sibony ont montré que f et g sont de "type Lattès". Cela veut dire que dans un ouvert de Zariski de $\mathbb{P}^k$ les applications f et g sont obtenues à partir d'applications affines de $\mathbb{C}^k$ après un quotient par un groupe discret de transformations affines.

Une application de type Lattès comme ci-dessus est toujours critiquement finie, i.e., l'orbite de son ensemble critique est pré-périodique. Cette remarque nous permet d'obtenir un exemple de paire d'endomorphismes permutable de $\mathbb{P}^3$ de même degré qui n'est pas de type Lattès. En particulier la condition (0.0.2) est nécessaire pour la conclusion du théorème de Dinh et Sibony si la dimension est au moins 3.

Il est naturel donc de revenir au problème en dimension 2 et demander si la condition (0.0.2) est nécessaire dans ce cas. Le théorème suivant montre qu'elle peut être supprimée, ce qui complète la solution du Problème A en dimension 2, voir Théorème 2.1.9.

Théorème B. *Soient f et g deux endomorphismes holomorphes permutables de $\mathbb{P}^2$ de degrés $d_f, d_g \geq 2$. Supposons que $f^n \neq g^m$ pour tout $n, m \geq 1$ et que $d_f^k = d_g^\ell$ pour certains $k, \ell \geq 1$. Alors f et g sont tous les deux d'un des types suivants :*

1. *Applications de Lattès permutables de $\mathbb{P}^2$.*
2. *Relèvements d'applications de Lattès permutables de $\mathbb{P}^1$: f et g sont des endomorphismes polynomiaux homogènes de $\mathbb{C}^2$ qui se prolongent holomorphiquement à $\mathbb{P}^2$ de façon que les applications induites sur la droite à l'infini sont des applications de Lattès de $\mathbb{P}^1$.*

La preuve est inspiré par les méthodes de Eremenko en dimension 1 et de Dinh–Sibony en dimension supérieure et consiste à donner une structure laminaire à la mesure d'équilibre μ commune de f et g. La mesure d'équilibre d'un endomorphisme est une

mesure de probabilité canonique invariante par la dynamique (voir Section 1.2.1 pour la définition). Cette mesure et son support ont en général une géométrie compliquée. Cependant, sous les hypothèses du Théorème B, nous avons que, dans un ouvert de $\mathbb{P}^2$, soit μ est équivalente à la mesure de Lebesgue, soit elle est équivalente à la mesure de Lebesgue dans une hypersurface strictement pseudoconvexe. D'après des travaux de Berteloot, Loeb, Dinh et Dupont, ces propriétés caractérisent, respectivement, les applications de Lattès et ses relèvements.

Dans la preuve du Théorème B il est nécessaire de comprendre en détail la géométrie du support de μ, l'ensemble de Julia d'ordre maximale. Dans ce cadre, nous obtenons des propriétés de ces ensembles qui ont un intérêt en soi. Nous renvoyons à la Section 1.2.1 pour une définition de J_1 et J_2.

Proposition C. *Soit $f : \mathbb{P}^2 \to \mathbb{P}^2$ un endomorphisme holomorphe de degré $d \geq 2$. Supposons que l'ensemble de Julia d'ordre maximale J_2 contienne un disque holomorphe passant par un point périodique répulsif de f. Alors $J_1 = J_2$.*

Dans le Chapitre 3 nous passons à la dynamique holomorphe continue et nous étudions l'intersection de cycles feuilletés dans les variétés complexes. Un cycle feuilleté est un courant positif fermé donné localement comme une moyenne de courants d'intégration le long de graphes holomorphes disjoints (voir Section 1.2.2 pour une définition précise). Ces graphes se recollent en une lamination par variétés complexes et l'existence d'un cycle feuilleté impose des restrictions sur la dynamique de cette lamination. Le résultat suivant (voir Théorème 3.1.1) montre que, sous une hypothèse sur la régularité de lamination, l'auto–intersection d'un cycle feuilleté est toujours zéro, ce qui peut être vu comme une obstruction à l'existence de tels objets. Cela généralise des résultats de Hurder–Mitsumatsu, Dujardin et Fornæss–Sibony.

Théorème D. *Soit X une variété kählérienne compacte de dimension n et soit T un cycle feuilleté diffus de dimension q dirigé par une lamination transversalement Lipschitz dans X. Si $\{T\} \in H^{n-q,n-q}(X)$ denote sa classe de cohomologie, alors*

$$\{T\} \smile \{T\} = 0 \quad \textit{dans} \quad H^{2n-2q,2n-2q}(X).$$

Comme conséquence nous pouvons démontrer la non-existence de cycles feuilletés transversalement Lipschitz dans certaines variétés. En particulier, nous avons le corollaire suivant, dont une version en dimension 2 et due à Fornæss et Sibony, voir [FS05].

Corollaire E. *Soit $\mathcal{L}$ une lamination transversalement Lipschitz de dimension $q \geq n/2$ dans $\mathbb{P}^n$. Alors il n'existe pas de cycle feuilleté diffus dirigé par $\mathcal{L}$. En particulier, si $\mathcal{L}$ n'a pas de feuilles compactes alors il n'existe pas de cycle feuilleté dirigé par $\mathcal{L}$.*

La preuve du Théorème D est une application de la théorie de densités de courants positifs fermés, introduite par Dinh et Sibony dans [DS14]. Cette notion généralise celle de

nombre de Lelong et permet de définir l'intersection de courants dans un cadre très général, notamment dans le cas d'intersections "non génériques" et de codimension supérieure à 1. Les points importants de la théorie de densités ici utilisés sont les faits que cette notion d'intersection est compatible avec le produit en cohomologie et que ces objets peuvent être calculés en coordonnées locales. Cela nous permet de déduire le Théorème D à partir de l'expression locale du cycle feuilleté dans une boîte à flot.

Dans le chapitre 4 nous changeons légèrement de sujet et nous étudions les mesures de Monge-Ampère à potentiel höldérien. Ce sont des mesures qui peuvent être écrites localement sous la forme

$$\mu = (\mathrm{dd}^c u)^n,$$

où u est une fonction plurisousharmonique Hölder continue. Ici, $\mathrm{dd}^c u = \frac{i}{\pi}\partial\bar{\partial}u$ est considéré dans le sens faible et sa puissance extérieure est définie selon la théorie de Bedford–Taylor (voir Section 1.1.3 pour plus de détails).

Ces mesures sont l'objet d'études récentes et apparaissent naturellement comme la mesure d'équilibre d'un endomorphisme holomorphe de $\mathbb{P}^k$ ainsi comme des solutions de certaines équations de Monge-Ampère complexe singulières.

Notre résultat principal dans cette direction est une version d'un théorème classique de Skoda, qui concerne l'intégrabilité exponentielle d'une fonction p.s.h..

Théorème F. *Soient $0 < \alpha \leq 1$ et u une fonction p.s.h. α-Hölder continue dans un domaine $\Omega \subset \mathbb{C}^n$ et $z \in \Omega$. Si φ est une fonction p.s.h. dans Ω et si $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$, alors il existe un voisinage $K \subset \Omega$ de z tel que l'integrale*

$$\int_K e^{-\varphi}(\mathrm{dd}^c u)^n$$

est finie.

Comme consequence nous obtenons des estimations sur la masse des ensembles de sous-niveau $\{\varphi < -N\}$ par rapport à la mesure $\mu = (\mathrm{dd}^c u)^n$. Ainsi comme dans le cas où μ est la mesure de Lebesgue, ces estimations peuvent être utiles dans les applications de la Théorie du Pluripotentiel à la Dynamique et à la Géométrie.

La preuve du Théorème F est inspirée par les méthodes de Dinh–Nguyen–Sibony dans [DNS10] et consiste à régulariser u et utiliser le fait que, d'après le théorème de Skoda, le résultat est valable si u est de classe $\mathcal{C}^2$.

Chapitre 1

Matériel préliminaire

Dans ce chapitre nous présentons quelques notions de base la Théorie du Pluripotentiel et de la Dynamique Holomorphe qui nous seront utiles dans les chapitres suivants. Les concepts ici présentés sont accompagnés de références et quelques exemples.

1.1 Théorie du Pluripotentiel

La Théorie du Pluripotentiel est l'analogue en dimension supérieure de la Théorie du Potential classique, consacré à l'étude de l'équation de Laplace et les fonctions sousharmoniques dans le plan complexe. Cette théorie fut introduite par Pierre Lelong et Kiyoshi Oka dans les années 1940 et s'est montrée depuis comme un outil indispensable dans l'Analyse Complexe et ses applications, notamment en Géométrie et Dynamique. Les objets centraux dans la Théorie du Pluripotentiel sont les fonctions plurisousharmoniques, un analogue des fonctions convexes dans le monde complexe, et les courants positifs fermés, des objets qui généralisent la notion d'ensemble analytique. Ces objets sont d'un côté assez souples, ce qui est utile en analyse, mais d'un autre côté ils possèdent aussi des propriétés remarquables de régularité et compacité.

1.1.1 Fonctions plurisousharmoniques

La classe des fonctions plurisousharmoniques a été introduite par Pierre Lelong et Kiyoshi Oka dans les années 1940. En contraste avec les fonctions holomorphes, qui sont assez rigides, Lelong appelle ces fonctions "les objets souples de l'analyse complexe". Les propositions 1.1.5 et 1.1.7 sont des exemples de cette souplesse.

Dans cette section nous introduisons la notion de fonction plurisousharmonique et énonçons ses propriétés basiques. Pour une étude complète et les démonstrations des résultats ici présentés nous renvoyons à [Kli91] et [Hör07].

Soit Ω un ouvert de l'espace euclidian complexe $\mathbb{C}^n$.

Définition 1.1.1. Une fonction $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ est dite *plurisousharmonique* (p.s.h. de façon abrégé) si φ est sémicontinue supérieurement, non identiquement égale à $-\infty$ sur une composante connexe de Ω et satisfait l'inégalité de la moyenne

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}\xi) d\theta, \tag{1.1.1}$$

pour tout $a \in \Omega$, $\xi \in \mathbb{C}^n$ et $r > 0$ suffisamment petit.

Autrement dit, une fonction est p.s.h. si sa restriction à toute droite complexe est une fonction sousharmonique ou identiquement égale à $-\infty$. L'espace des fonctions p.s.h. en Ω sera dénoté $\mathrm{PSH}(\Omega)$.

Exemple 1.1.2. Une fonction $\varphi : \Omega \to \mathbb{R}$ de classe $\mathcal{C}^2$ est p.s.h. si et seulement si l'hessienne complexe

$$(\mathrm{Hess}\ \varphi)(z) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}(z) \right)_{i,j=1,\ldots,n}$$

est définie semipositive pour tout $z \in \Omega$.

Exemple 1.1.3. Soit $h : \Omega \to \mathbb{C}$ une fonction holomorphe. Alors $\varphi(z) = \log |h(z)|$ est une fonction p.s.h.. Ces fonctions sont, en quelque sorte, les fonctions p.s.h. le plus singulières.

Le lemme ci-dessous et le fait que la plurisousharmonicité est une propriété locale nous permet de transférer cette notion aux fonctions définies sur une variété complexe.

Lemme 1.1.4 ([Kli91] Corollary 2.9.5)**.** *Si $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ est une fonction p.s.h. et $f : \Omega' \to \Omega$ est une application holomorphe alors $\varphi \circ f$ est p.s.h..*

Le résultat suivant nous permet de construire beaucoup d'exemples de fonctions p.s.h.. Pour une démonstration nous revoyons à [Dem], Chapitre I.

Proposition 1.1.5. *Soit $\chi : \mathbb{R} \cup \{-\infty\} \times \cdots \times \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ une fonction convexe et croissante en chaque variable dont la restriction à $\mathbb{R}^n$ est à valeurs réelles. Si $u_1, \ldots, u_n$ sont des fonctions p.s.h. alors $\chi(u_1, \ldots, u_n)$ est une fonction p.s.h..*

Exemple 1.1.6. Si $u_1, \ldots, u_n$ sont des fonctions p.s.h. alors $\max(u_1, \ldots, u_n)$ est une fonction p.s.h..

Proposition 1.1.7 ([Kli91], Thm 2.9.14)**.** *Soit $\Omega \subset \mathbb{C}^n$ un ouvert connexe. Alors pour toute suite décroissante de fonctions $u_j \in PSH(\Omega)$ la limite $u = \lim_{j \to +\infty} u_j$ et p.s.h. sur Ω ou identiquement égale à $-\infty$.*

Le principe du maximum est un outil fondamental. Sa démonstration est une consequence immédiate de l'inégalité de la moyenne.

Proposition 1.1.8 (Principe du maximum)**.** *Si une fonction p.s.h. définie dans un ouvert connexe admet un maximum alors elle est constante.*

Régularisation de fonctions p.s.h.

Dans les applications de la Théorie du Pluripotentiel en géométrie et dynamique nous trouvons souvent des fonctions p.s.h. qui ne sont pas lisses. Il est utile donc de les approcher par des fonctions p.s.h. lisses. Localement, ce problème peut être résolu à l'aide de la convolution alors que globalement le problème de régularisation est beaucoup plus délicat (voir par exemple [Dem92]). Le résultat suivant se trouve dans [Kli91], Theorem 2.9.2.

Théorème 1.1.9 (Régularisation locale)**.** *Soit $\Omega \subset \mathbb{C}^n$ un ouvert connexe et $u \in \mathrm{PSH}(\Omega)$. Soit χ une fonction lisse à support dans la boule unité avec $0 \leq \chi \leq 1$ et $\int_{\mathbb{C}^n} \chi(z)\mathrm{d}z = 1$. Posons $\chi_\varepsilon(z) = \varepsilon^{-2n}\chi(\varepsilon^{-1}z)$ et $\Omega_\varepsilon = \{z \in \Omega \,:\, dist(z, \partial\Omega) > \epsilon\}$.*

Alors la convolution $u_\varepsilon = u * \chi_\varepsilon$ est p.s.h. et lisse sur Ω_ε et décroit vers u lorsque $\epsilon \to 0$. De plus, si u est continue alors $u_\varepsilon \to u$ uniformément sur chaque compacte de Ω.

Propriétés de compacité des fonctions p.s.h.

Les fonctions p.s.h. possèdent des propriétés de compacité remarquables. Ces propriétés sont utiles en Analyse Complexe à plusieurs variables et ses applications en Dynamique et Géométrie. Cette compacité remplace par exemple le Théorème de Montel, qui n'est plus vrai en dimension $n \geq 2$.

Théorème 1.1.10 ([Hör07], p. 229)**.** *Soit u_j une suite de fonctions p.s.h. dans un ouvert connexe $\Omega \subset \mathbb{C}^n$. Si u_j est localement uniformément bornée supérieurment alors, soit $u_j \to -\infty$ uniformément sur chaque compact de Ω, soit il existe une sous-suite u_{j_k} qui converge dans $L^p_{loc}(\Omega)$ à une fonction p.s.h. pour tout $p \in [1, +\infty)$.*

Le nombre de Lelong d'une fonction p.s.h.

Un pôle d'une fonction p.s.h. est un point où cette fonction atteint la valeur $-\infty$. Le nombre de Lelong d'une fonction p.s.h. est un des premiers invariants attachés à l'un pôle et donne des informations sur le comportement de la fonction autour de ce point.

Définition 1.1.11. Soit $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ une fonction p.s.h.. Le nombre de Lelong de φ en $a \in \Omega$ est la limite

$$\nu(\varphi; a) = \liminf_{z \to a} \frac{\varphi(z)}{\log \|z - a\|} \cdot \tag{1.1.2}$$

Exemple 1.1.12. Si $\varphi(z) = \log|h|$ pour une fonction holomorphe h alors $\nu(\varphi; a) = \mathrm{ord}_a(h)$, l'ordre d'annulation de h en a.

Exemple 1.1.13. Si φ est localement bornée alors $\nu(\varphi; a) = 0$ pour tout $a \in \Omega$.

Le nombre de Lelong est lié à l'intégrabilité exponentielle de φ et au taux de décroissance du volume des ensembles de sous-niveau $\{\varphi < -M\}$ lorsque $M \to +\infty$. Plus précisément nous avons le théorème suivant, dû à H. Skoda.

Théorème 1.1.14 ([Sko72], [Kis00]). *Soit φ une fonction p.s.h. dans un ouvert $\Omega \subset \mathbb{C}^n$. Notons λ_{2n} la mesure de Lebesgue dans $\mathbb{C}^n$.*

1. *Si $\nu(\varphi; a) < 2$ alors l'intégrale*
$$\int e^{-\varphi}\, \mathrm{d}\lambda_{2n}$$
converge dans un voisinage de a.
2. *Soit $K \subset \Omega$ un compact. Alors pour tout $\gamma < 2/\sup_{z\in K} \nu(\varphi; z)$ il existe une constante $C_\gamma = C_\gamma(\varphi, \Omega, K)$ telle que*
$$\lambda_{2n}(K \cap \{\varphi \leq -M\}) \leq C_\gamma\, e^{-\gamma M}, \quad M \in \mathbb{R}.$$

Dans le Chapitre 4 nous allons donner une version de ce théorème où λ_{2n} est remplacée par une mesure singulière à potentiel höldérien.

1.1.2 Courants positifs fermés

Les courants positifs fermés sont des objets qui généralisent simultanément les sous-ensembles analytiques d'une variété complexe et les formes différentielles positives et fermées, comme par exemple les formes de Kähler. Ces courants sont très utiles par exemple dans l'étude de problèmes de nature globale en Géométrie et Dynamiques complexes. Pour une exposition complète sur ce sujet nous renvoyons à [Dem].

Soit M une variété différentielle de dimension m.

Définition 1.1.15. Un courant de degré p (ou p-courant) M est une forme linéaire continue T dans l'espace $\mathcal{D}^{m-p}(M)$ des formes différentielles de degré $m-p$ sur M à support compact. Le nombre $m-p$ est appelé la dimension de T.

Exemple 1.1.16. Une forme φ de degré p avec coefficients dans L^1_{loc} définit un courant de degré p par la formule
$$\langle \varphi, \alpha \rangle = \int_M \varphi \wedge \alpha, \quad \alpha \in \mathcal{D}^{m-p}(M).$$

Exemple 1.1.17. Soit $N \subset X$ une sous-variété lisse de M de codimension p avec ou sans bord. L'intégration le long de N définit un courant de degré p sur M, noté $[N]$. Plus précisément
$$\langle [N], \alpha \rangle = \int_M \alpha, \quad \alpha \in \mathcal{D}^{m-p}(M).$$

Nous pouvons étendre aux courants quelques opérations basiques définis pour les formes différentielles. Si T est un p-courant et φ est une k-forme lisse alors le produit extérieur $T \wedge \varphi$ est le $(p+k)$-courant défini par
$$\langle T \wedge \varphi, \alpha \rangle = \langle T, \varphi \wedge \alpha \rangle, \quad \alpha \in \mathcal{D}^{m-p-k}(M).$$

En général il n'est pas possible de définir le produit $T \wedge S$ de deux courant quelconques. Néanmoins, dans le cas des courants positifs fermés dans une variété complexe (définis ci-dessous), ce produit sera bien défini si T et S ne sont pas "très singuliers", voir Section 1.1.3.

Si T est un p-courant alors la dérivée extérieure $\mathrm{d}T$ est le $(p+1)$-courant défini par

$$\langle \mathrm{d}T, \alpha \rangle = (-1)^{p+1} \langle T, \mathrm{d}\alpha \rangle \quad \alpha \in \mathcal{D}^{m-p-1}(M).$$

D'après la formule d'intégration par parties si $T = \varphi$ est donné par une forme lisse alors la dérivée extérieure coïncide avec la dérivée extérieure usuelle. Un courant T est dit **fermé** si $\mathrm{d}T = 0$.

Soit maintenant X une variété complexe de dimension n. Une forme différentielle de bi-degré (p, q) sur X est une forme différentielle qui s'écrit en coordonnées locales comme

$$\varphi = \sum_{|I|=p, |J|=q} \varphi_{I,J} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J,$$

où les $\varphi_{I,J}$ sont des fonctions complexes lisses. On utilise ici la notation de multi-indice $\mathrm{d}z_I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$ pour $I = (i_1, \ldots, i_p)$ et $\mathrm{d}\overline{z}_J = \mathrm{d}\overline{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{j_q}$ pour $J = (j_1, \ldots, j_q)$.

Toute ℓ-forme sur X se décompose en une somme de formes de type (p, q) avec $p+q = \ell$. Par dualité nous avons ainsi une décomposition similaire pour les courants. Un courant sera de type (p, q) s'il s'annule dans toutes formes de bidegré $\neq (n-p, n-q)$.

Les opérateurs ∂ et $\bar{\partial}$ qui agissent sur les formes différentielles peuvent être prolongés, par dualité, aux courants. Nous pouvons définir ainsi l'opérateur $\mathrm{d}^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ de sorte que

$$\mathrm{d}\mathrm{d}^c = \frac{i}{\pi} \partial \bar{\partial}.$$

La notion de positivité pour les courants et les formes différentielles de type (p, p) a été introduite par Pierre Lelong, voir [Lel98].

Définition 1.1.18. Une $(p, p)-$forme φ est *positive* si en chaque point elle est une combinaison linéaire de formes du type

$$(i\, \alpha_1 \wedge \overline{\alpha}_1) \wedge \cdots \wedge (i\, \alpha_p \wedge \overline{\alpha}_p),$$

où les α_j sont de formes de type $(1, 0)$.

Un courant S de type (p, p) est *faiblement positif* si $S \wedge \varphi$ est une mesure positive pour toute $(n-p, n-p)-$forme différentielle positive. Un courant S de type (p, p) est **positif** si $S \wedge \varphi$ est une mesure positive pour toute $(n-p, n-p)-$forme différentielle faiblement positive.

Exemple 1.1.19. Une forme lisse de type $(1, 1)$, $\alpha = i \sum \alpha_{jk} dz_j \wedge d\overline{z}_k$ est positive si et

seulement si la forme hermitienne $\sum \alpha_{jk}\xi_j\overline{\xi_k}$ est défini semi-positive.

Soit ω une $(1,1)-$forme strictement positive sur X. On considère sur X la métrique hermitienne induite par ω. La mesure trace d'un $(p,p)-$courant positif par rapport à ω est la mesure positive $\sigma_T = T \wedge \omega^{n-p}$. La masse de T sur un compact K est la masse de σ_T sur K.

Une classe importante de courants positifs est donnée par le théorème suivant. Pour une démonstration, voir [Dem].

Théorème 1.1.20 (Lelong)**.** *Soit $A \subset X$ un sous-ensemble analytique de dimension pure $q = n - p$ et soit A_{reg} l'ensemble des points lisses de A. Alors la formule*

$$\langle [A], \alpha \rangle = \int_{A_{reg}} \alpha\,, \quad \alpha \in \mathcal{D}^{n-p,n-p}(X)$$

définit un courant positif fermé de type (p,p) sur X.

Positivité et fonctions p.s.h.

Le rapport entre les courants positifs et les fonctions p.s.h. définies auparavant est donné par les deux résultats suivants. Pour les démonstrations, consultez [Dem], Chapitre III.

Proposition 1.1.21. *Soit u une fonction p.s.h. dans une variété complexe. Alors $\mathrm{dd^c}u$ est un $(1,1)-$courant positif fermé. Réciproquement, si $\mathrm{dd^c}u$ est un courant positif alors u est égale presque partout à une fonction p.s.h. .*

Proposition 1.1.22 (Lemme de Poincaré)**.** *Soit T un courant positif fermé de type $(1,1)$ dans une variété complexe X. Alors pour tout point p de X il existe un voisinage Ω de p et une fonction p.s.h. u sur Ω telle que $T = \mathrm{dd^c}u$.*

Une fonction u comme dans la proposition ci-dessus est appelé un *potentiel local* du courant T. Deux potentiels locaux diffèrent par une fonction pluriharmonique, cet à dire, une fonction h telle que h et $-h$ sont tous les deux p.s.h. En particulier, h est harmonique, donc lisse. Cette remarque nous permet de donner un sens à des expressions du type "T est à potentiel local continu/borné", etc.

Exemple 1.1.23 (Formule de Lelong–Poincaré)**.** Soit V une hypersurface de $\mathbb{C}^n$ donnée par le lieu des zéros d'une fonction holomorphe f. La formule de Lelong-Poincaré dit que la fonction $u(z) = \log|f(z)|$ est un potentiel du courant d'intégration le long de V. Autrement dit, on a

$$\mathrm{dd^c} \log|f| = [V].$$

Cette formule découle du fait que la fonction $\frac{1}{2\pi}\log|z|$ définie sur $\mathbb{C}$ est la solution fondamentale de l'équation de Laplace, i.e. , $\frac{1}{2\pi}\Delta \log|z| = \delta_0$. Nous renvoyons au Chapitre III de [Dem] pour une démonstration et à [Hör07], p. 273 pour une discussion détaillée.

1.1.3 L'opérateur de Monge-Ampère

Ainsi comme le produit entre deux distributions n'est pas toujours bien défini, on ne peut pas donner un sens aux produit extérieur $T \wedge S$ entre deux courants arbitraires. Une partie importante de la Théorie du Pluripotentiel consiste à définir un tel produit pour une classe suffisamment grande de courants positifs fermés. Nous allons voir par exemple que, si $T = \mathrm{dd}^c u$ et $S = \mathrm{dd}^c v$ où u et v sont des fonctions p.s.h. bornées, alors leur produit $T \wedge S$ peut être défini.

Soit T un courant positif fermé de bi-degré (p, p) . Alors T est un courant d'ordre zéro. Cela veut dire que la fonctionnelle $T : \mathcal{D}^{n-p,n-p}(\Omega) \to \mathbb{C}$ se prolonge à l'espace de $(n-p, n-p)-$formes à coefficients continus et à support compact. Autrement dit, les coefficients de T sont des mesure de Radon (voir [Kli91], Proposition 3.3.4). Si u est une fonction p.s.h. bornée on peut définir alors le courant uT, dont les coefficients sont donnés par le produit de u par les coefficients de T. Cette observation nous permet de définir le produit extérieur entre $\mathrm{dd}^c u$ et T par la formule

$$\mathrm{dd}^c u \wedge T := \mathrm{dd}^c(uT). \tag{1.1.3}$$

Lorsque u est lisse et T est une forme différentielle, la formule d'intégration par parties nous dit que la définition ci-dessus coïncide avec le produit extérieur usuel.

Par recurrence nos pouvons définir le produit $\mathrm{dd}^c u_1 \wedge \ldots \wedge \mathrm{dd}^c u_k$ lorsque que les u_j, $1 \leq j \leq k \leq n$ sont des fonctions p.s.h. localement bornées. L'opérateur

$$u \longmapsto (\mathrm{dd}^c u)^n$$

est appelé *l'opérateur de Monge–Ampère.*

La formule (1.1.3) étant assez formelle, le résultat suivant est parfois utile dans le calcul du produit extérieur. Combiné avec la régularisation locale (Théorème 1.1.9) il nous permet de calculer le produit $\mathrm{dd}^c u_1 \wedge \ldots \wedge \mathrm{dd}^c u_k$ à l'aide du produit extérieur standard de formes différentielles.

Théorème 1.1.24 ([Kli91], Thm 3.4.3)**.** *Soit Ω un ouvert de $\mathbb{C}^n$, $k \leq n$ et $u_1^j, \ldots, u_k^j$, $j \in \mathbb{N}$ des suites des fonctions p.s.h. qui décroissent ponctuellement vers $u_1, \ldots, u_k \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$. Alors*

$$\lim_{j\to\infty} \mathrm{dd}^c u_1^j \wedge \ldots \wedge \mathrm{dd}^c u_k^j = \mathrm{dd}^c u_1 \wedge \ldots \wedge \mathrm{dd}^c u_k.$$

Remarque 1.1.25. Le théorème ci-dessus montre que l'opérateur de Monge–Ampère est continu par rapport aux suites décroissantes de fonctions p.s.h. . Cependant, l'opérateur de Monge–Ampère est discontinu dans un sens assez fort. On peut démontrer par exemple que toute fonction p.s.h. localement bornée définie dans un ouvert pseudoconvexe de $\mathbb{C}^n$

est une limite dans L^1_{loc} de fonctions p.s.h. continues u_j telles que $(\mathrm{dd}^c u_j)^n = 0$, voir [Kli91], Section 3.8.

1.2 Dynamique Holomorphe

Un système dynamique holomorphe dans une variété complexe X peut être de deux natures différentes :

1. Un système discret, donné par une application holomorphe ou méromorphe $f : X \to X$ et ses itérées $f^n = f \circ \cdots \circ f : X \to X$.
2. Un système continu, donné par un champs de vecteurs holomorphe ou plus généralement un feuilletage holomorphe sur X.

Dans les deux cas on s'intéresse à des questions similaires : comprendre le comportement asymptotique du système, ses objets invariants, les phénomènes chaotiques, etc.

1.2.1 Endomorphismes holomorphes de $\mathbb{P}^k$

L'espace projectif complexe $\mathbb{P}^k$ est une compactification naturelle de l'espace euclidien complexe $\mathbb{C}^k$, où on ajoute un point à l'infini pour chaque direction complexe passant par l'origine. L'espace projectif possède une grande quantité d'endomorphismes holomorphes et il sert également comme une compactification dynamique de certains endomorphismes polynomiaux de $\mathbb{C}^k$. Les endomorphismes holomorphes de $\mathbb{P}^k$ sont donnés par polynômes à plusieurs variable, ce qui nous permet de produire beaucoup d'exemples. Pour une exposition plus complète sur la dynamique des endomorphismes de $\mathbb{P}^k$ le lecteur pourra consulter [DS10].

L'espace projectif complexe peut être défini comme

$$\mathbb{P}^k = (\mathbb{C}^{k+1} - \{0\})/\mathbb{C}^*,$$

où $\mathbb{C}^*$ agit sur $\mathbb{C}^{k+1} - \{0\}$ par multiplication coordonnée à coordonnée.

La classe d'un point $\mathbf{z} = (z_0, \ldots, z_k)$ est dénoté par $[\mathbf{z}] = [z_0 : \cdots : z_k]$ et les z_j sont appelés coordonnées homogènes du point $p = [\mathbf{z}] \in \mathbb{P}^k$.

L'espace euclidian $\mathbb{C}^k$ se plonge naturelement dans $\mathbb{P}^k$ via l'application

$$(z_1, \ldots, z_k) \mapsto [1 : z_1 : z_2 : \ldots : z_k].$$

Le complémentaire $\mathbb{P}^k \setminus \mathbb{C}^k = \{z_0 = 0\}$ est appelé hyperplan à l'infini et il est dénoté par H_∞. Nous avons donc $\mathbb{P}^k = \mathbb{C}^k \cup H_\infty$ et $H_\infty \simeq \mathbb{P}^{k-1}$.

Un **endomorphisme holomorphe** de $\mathbb{P}^k$ est une application holomorphe $f : \mathbb{P}^k \to \mathbb{P}^k$.

En coordonnées homogènes f s'écrit comme

$$f = [P_0 : \ldots : P_k],$$

où les P_j sont des polynômes homogènes en $(z_0, \ldots, z_k)$ de même degré d et sans zéros communs dans $\mathbb{C}^{k+1} - \{0\}$, voir [GH94]. Le nombre d est appelé degré algébrique de f.

Nous nous intéresserons à l'étude dynamique de f, c'est à dire, nous allons considérer f et ses itérées $f^n = f \circ \cdots \circ f : \mathbb{P}^k \to \mathbb{P}^k$, $n \geq 1$ comme un système dynamique discret dans $\mathbb{P}^k$. La dynamique des applications linéaires étant simple à décrire, nous allons toujours supposer que $d \geq 2$.

Proposition 1.2.1 (cf. [DS10])**.** *Soit f un endomorphisme holomorphe de degré d de $\mathbb{P}^k$.*

1. *Le degré algébrique de f^n est égal à d^n.*
2. *Pour tout $a \in \mathbb{P}^k$ la fibre $f^{-1}(a)$ contient d^k points comptés avec multiplicité. De plus f est une application ouverte et définit un revêtement ramifié de degré d^k.*
3. *Le nombre de points fixes de f comptés avec multiplicité est $\frac{d^{k+1}-1}{d-1}$. En particulier le nombre de points périodiques d'ordre n comptés avec multiplicité est $d^{nk} + o(d^{kn})$.*

La dichotomie Fatou-Julia

Lorsque $k = 1$ les endomorphismes $f : \mathbb{P}^1 \to \mathbb{P}^1$ sont donnés par fractions rationnelles. L'etude dynamique de telles applications est un domaine classique et beaucoup développé. Nous renvoyons à [Mil06a] pour une exposition complète du sujet.

Ainsi comme dans la théorie unidimensionnelle l'espace $\mathbb{P}^k$ peut être divisé en deux parties selon la stabilité de la dynamique de f par rapport aux conditions initiales.

Définition 1.2.2. L'ensemble de Fatou $\mathcal{F}$ de f est le plus grand ouvert de $\mathbb{P}^k$ ou la famille $\{f^n\}_{n\in\mathbb{N}}$ est normale. Autrement dit, un point p appartient à $\mathcal{F}$ s'il existe un voisinage U de p tel que la suite $\{f^n|_U\}_{n\in\mathbb{N}}$ admet une sous-suite qui converge localement uniformément vers une application holomorphe. L'ensemble de Julia de f est le complémentaire $J = \mathbb{P}^k \setminus \mathcal{F}$.

Moralement, dans l'ensemble de Fatou la dynamique de f n'est pas sensible au conditions initiales. En revanche, dans l'ensemble de Julia la dynamique est "chaotique" dans le sens où des petites perturbations des conditions initiales peuvent mener à des comportements dynamiques complètement différents.

Exemple 1.2.3. L'endomorphisme polynomial $f : (z_1, z_2, \ldots, z_k) \mapsto (z_1^d, z_2^d, \ldots, z_k^d)$ de $\mathbb{C}^k$ se prolonge en un endomorphisme holomorphe $f : \mathbb{P}^k \to \mathbb{P}^k$ par la formule $f : [z_0 : z_2 : \cdots : z_k] \mapsto [z_0^d : z_2^d : \ldots : z_k^d]$.

L'ensemble de Fatou est l'union des ouverts $\mathcal{F}_i$, $i = 0, \ldots, k$, où

$$\mathcal{F}_i = \{p = [\mathbf{z}] \in \mathbb{P}^k \ : \ |z_j| < |z_i| \text{ pour tout } i \neq j\}.$$

Notons que pour tout $p \in \mathcal{F}_i$ la suite $f^n(p)$ converge vers le point fixe $p_i = [\mathbf{e}_i] = [0 : \cdots : 1 : \cdots : 0]$ lorsque $n \to \infty$. Autrement dit, $\mathcal{F}_i$ est le bassin d'attraction de p_i.

L'ensemble de Julia est l'union de J_{ij} pour $i \neq j$ où

$$J_{ij} = \{p = [\mathbf{z}] \in \mathbb{P}^k \ : \ |z_i| = |z_j| \text{ et } |z_\ell| \leq |z_i| \text{ pour tout } \ell\}.$$

Exemple 1.2.4 (Exemples de Lattès)**.** Une classe importante d'endomorphismes holomorphes de $\mathbb{P}^k$ et celle des exemples de Lattès. Un endomorphisme holomorphe $f : \mathbb{P}^k \to \mathbb{P}^k$ de degré $d \geq 2$ est un *exemple de Lattès* s'il existe un tore complexe $\mathbb{T}^k$ de dimension k, une application affine $\Lambda_f : \mathbb{T}^k \to \mathbb{T}^k$ de partie linéaire $\sqrt{d}\,U$ avec U unitaire et un revêtement fini $\varphi : \mathbb{T}^k \to \mathbb{P}^k$ tels que le diagramme

$$\begin{tikzcd} \mathbb{T}^k \arrow[r, "\Lambda_f"] \arrow[d, "\varphi"'] & \mathbb{T}^k \arrow[d, "\varphi"] \\ \mathbb{P}^k \arrow[r, "f"] & \mathbb{P}^k \end{tikzcd}$$

commute. De plus il existe un groupe $\mathcal{A}$ d'isometries affines de $\mathbb{T}^k$ agissant transitivement sur les fibres de φ. Ici, on considère la métrique plate sur $\mathbb{T}^k$ donnée par la forme kählérienne standard $\omega_0 = i \sum \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$.

Remarque 1.2.5. Dans la plupart des cas l'ensemble de Julia est très compliqué du point de vue géométrique/topologique (voir Figure 1.1). Néanmoins il existe des exemples exceptionnels où ces ensembles sont plus réguliers que d'habitude. Il se peut par exemple que J soit, dans un ouvert, une sous-variété réelle lisse, comme dans l'exemple 1.2.3 ou bien que J soit tout $\mathbb{P}^k$ comme dans l'exemple 1.2.4. D'autres exemples de ce phénomène apparaissent lorsque un endomorphisme possède un centralisateur non trivial (voir Chapitre 2) ou lorsqu'il preserve un feuilletage holomorphe (voir [FP11]).

En dimension 1, la dichotomie Fatou-Julia donne beaucoup d'informations sur la dynamique de f. Un théorème célèbre de Sullivan dit que toute composante de Fatou (i.e., une composante connexe de $\mathcal{F}$) est pré-périodique. Ce résultat nous permet de réduire l'étude de la dynamique dans l'ensemble Fatou au cas de composantes invariantes, ce qui aboutit à une classification complète des composantes de Fatou (voir [Mil06a]). Sur l'ensemble de Julia, d'après les travaux de Brolin [Bro65], Freire–Lopez–Mañé [FLM83] et Lyubich [Lju83], la dynamique peut être décrite à l'aide d'une mesure invariante canonique portée par J, ce qui fait intervenir la Théorie Ergodique.

En dimension $k \geq 2$ la situation est plus complexe. Tout d'abord on ne peut pas espérer avoir une classification des composantes de Fatou une fois que, par exemple, il existe des composantes de Fatou errantes (i.e. non pré-périodiques), comme montre des exemples récents d'Astorg, Buff, Dujardin, Peter et Raissy [Ast+16]. Concernant la dynamique sur

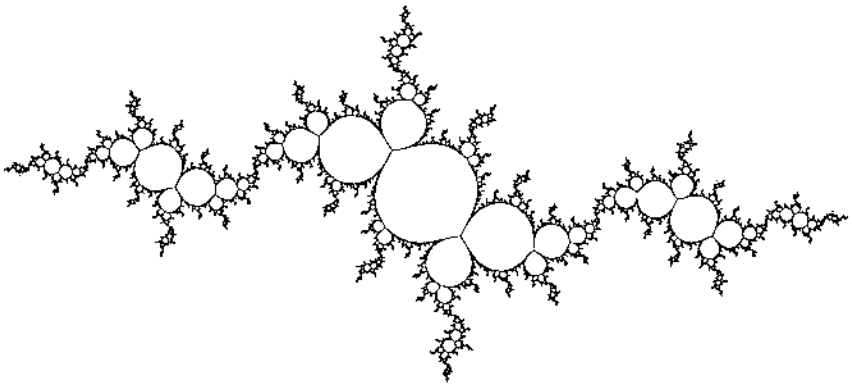

FIGURE 1.1 – Ensemble de Julia de l'application $z \mapsto z^2 + \frac{e^{\frac{2\pi i}{3}}}{4} - 1$ (réf. : Wikipédia).

l'ensemble de Julia, la situation est aussi plus délicate puisque la mesure de probabilité canoniquement associée à f est, en général, portée par un ensemble strictement contenu dans J, voir (1.2.1) ci-dessous.

Le courant de Green et la mesure d'équilibre

Le courant de Green est un objet essentiel pour la description de la dynamique globale d'un endomorphisme holomorphe. Nous verrons qu'à l'aide de ce courant nous pouvons construire une mesure invariante canonique pour le système dynamique $(\mathbb{P}^k, f)$ et tirer des informations comme la distribution de points périodiques, la distribution des pré-images des points et de sous-variétés génériques, la dimension d'Hausdorff de l'ensemble de Julia, etc.

Théorème 1.2.6 ([FS95], [Sib99], [DS10], Thm. 1.16)**.** *Soit f un endomorphisme holomorphe de $\mathbb{P}^k$ de degré algébrique d. Soit ω la forme de Fubini-Study de $\mathbb{P}^k$ normalisé de sorte que $\int_{\mathbb{P}^k} \omega^k = 1$. Alors la suite $d^{-n}(f^n)^*\omega$ converge vers un courant positif fermé T de type $(1,1)$. Le courant T est totalement invariant $f^*T = d \cdot T$. De plus, le courant T est l'unique courant à potentiels locaux bornés et de masse 1 tel que $f^*T = dT$.*

Définition 1.2.7. Le courant T est appelé le **courant de Green** de f.

Proposition 1.2.8 ([DS10], Prop. 1.18)**.** *Soit f un endomorphisme holomorphe de $\mathbb{P}^k$ et soit T son courant de Green. Alors les potentiels locaux de T sont des fonctions p.s.h. de classe $\mathcal{C}^\alpha$ pour tout α tel que $0 < \alpha < \min(1, \log d / \log d_\infty)$ où $d_\infty = \lim_{n\to\infty} \|Df^n\|_\infty^{1/n}$.*

En particulier, nous voyons que les potentiels locaux de T sont bornés. D'après la discussion de la Section 1.1.3 nous pouvons définir les produits

$$T^\ell = T \wedge \cdots \wedge T.$$

Ces courants sont des courants positifs fermés de type (ℓ, ℓ). Lorsque $\ell = k$ on obtient une mesure positive $\mu = T^k$, appelée **mesure de Green** ou mesure d'équilibre de f.

L'ensemble $J_\ell = \operatorname{supp} T^\ell$ est appelé *l'ensemble de Julia d'odre* ℓ de f. Nous avons donc une suite d'inclusions

$$\operatorname{supp} \mu = J_k \subset J_{k-1} \subset \cdots \subset J_1. \tag{1.2.1}$$

L'ensemble Julia usuel (i.e., le complémentaire de l'ensemble de Fatou) est égale à J_1.

Théorème 1.2.9 ([Ued94],[FS95], [Sib99],[DS10], Thm 1.22)**.** *Soit f un endomorphisme holomorphe de $\mathbb{P}^k$. Alors son ensemble de Fatou $\mathcal{F} = \mathbb{P}^k \setminus J_1$ est hyperbolique au sens de Kobayashi. De plus, l'ensemble de Julia d'ordre ℓ est connexe pour tout $\ell \leq k/2$.*

Rappelons que l'image directe de d'un courant S par f est définie par dualité $\langle f_* S, \alpha \rangle := \langle S, f^* \alpha \rangle$, où α est une $(k-\ell, k-\ell)$-forme lisse. L'image réciproque d'un courant de type $(1,1)$ peut être définie localement : si $S = \mathrm{dd}^c u$ dans un ouvert U de $\mathbb{P}^k$ alors $f^* S := \mathrm{dd}^c (u \circ f)$ dans $f^{-1}(U)$. Nous pouvons ainsi définir $f^* S^\ell := (f^* S)^\ell$.

À patrir de la définition de T et du fait que $f_* f^* = d^k \operatorname{id}$ on vérifie que les courants T^ℓ sont totalement invariants, c'est à dire

$$f^* T^\ell = d^\ell \cdot T^\ell \quad \text{et} \quad f_* T^\ell = d^{k-\ell} \cdot T^\ell.$$

La mesure de Green est une mesure canoniquement attachée à la dynamique de f sur $\mathbb{P}^k$. Le théorème suivant montre que μ possède des bonnes propriétés ergodiques et elle décrit la distribution des points dynamiquement intéressants, comme les point périodiques répulsifs et les pré-images d'un point générique. Pour une définition des termes 'ergodique', 'mélangeante', entropie', etc. nous renvoyons à [KH95].

Théorème 1.2.10 (cf. [DS10])**.** *Soit f un endomorphisme holomorphe de degré d de $\mathbb{P}^k$ et μ sa mesure de Green.*

1. *(Équidistribution des points périodiques) Soit P_n l'ensemble des points périodiques répulsifs d'ordre n de f dans le support de μ. Alors*

$$\lim_{n \to \infty} \frac{1}{d^{nk}} \sum_{x \in P_n} \delta_x \longrightarrow \mu.$$

2. *(Équidistribution des pré-images) Il existe un sous-ensemble algébrique $\mathcal{E}$ de $\mathbb{P}^k$ qui est totalement invariant $f^{-1}(\mathcal{E}) = \mathcal{E} = f(\mathcal{E})$ tel que pour tout $a \notin \mathcal{E}$ on a*

$$\lim_{n \to \infty} \frac{1}{d^{nk}} f^* \delta_a = \lim_{n \to \infty} \frac{1}{d^{nk}} \sum_{f^n(x) = a} \delta_x = \mu.$$

L'ensemble $\mathcal{E}$ est maximal dans le sens suivant : si E est un sous-ensemble algébrique propre de $\mathbb{P}^k$ et $f^{-n}(E) \subset E$ pour un certain $n \geq 1$, alors $E \subset \mathcal{E}$.

3. *La mesure μ est l'unique mesure invariante d'entropie maximale* $\log d^k$.

4. *La mesure μ est mélangeante, en particulier μ est ergodique.*

Remarque 1.2.11. Le point (2) du théorème précédent se généralise au cas de pré-images d'hypersurfaces de $\mathbb{P}^k$. Dans ce cas, c'est le courant de Green qui décrit l'équidis-tribution. Plus précisément, si $H \subset \mathbb{P}^k$ est une hypersurface générique de degré s alors la suite de courants $d^{-n}(f^n)^*[H]$ converge vers sT à une vitesse exponentielle (voir [DS10], Theorem 1.64 et [Taf11]). Pour les sous-variétés $V \subset \mathbb{P}^k$ de degré s et codimension p avec $1 < p < k$ il est conjecturé que $d^{-np}(f^n)^*[V]$ converge vers sT^p pour V générique. Le résultat est connu pour un endomorphisme générique, voir [DS10], Theorem 1.70.

Dans quelques exemples, le courant de Green et la mesure de Green peuvent être calculés explicitement. Nous allons revenir à ces exemples dans le Chapitre 2.

Exemple 1.2.12. Soit $f : \mathbb{P}^k \to \mathbb{P}^k$ donné par $[z_0 : z_1 : \cdots : z_k] \mapsto [z_0^d : z_1^d : \ldots : z_k^d]$. Soit $U_0 = \{z_0 \neq 0\}$ et $\zeta_j = z_j/z_0$ les coordonnés affines dans U_0. Alors

$$T|_{U_0} = \mathrm{dd}^c \max\left(0, \log|\zeta_1|, \ldots, \log|\zeta_k|\right).$$

La mesure $\mu = T^k$ est la mesure de Haar sur le tore $n-$dimensionnel

$$J_k = \{\zeta \in U_0 : |\zeta_1| = \cdots = |\zeta_k| = 1\}.$$

Exemple 1.2.13. Soit $f : \mathbb{P}^k \to \mathbb{P}^k$ une application de Lattès et soient $\mathbb{T}^k, \varphi, \Lambda_f, \mathcal{A}$ et ω_0 comme dans l'Exemple 1.2.4. Comme $\mathcal{A}$ preserve la forme ω_0 elle descend à un courant S sur $\mathbb{P}^k$. Autrement dit, il existe un courant S positif fermé de type $(1,1)$ sur $\mathbb{P}^k$ tel que $\varphi^* S = \omega_0$.

Le courant de Green de f est égal à un multiple du courant S. En effet, si u est un potentiel local de S alors $u \circ \varphi$ est un potentiel local de ω_0, qui est lisse. Cela montre que S est à potentiel local borné. En outre, nous avons que

$$\varphi^* f^* S = \Lambda_f^* \varphi^* S = \Lambda_f^* \omega_0 = d \cdot \omega_0 = d \cdot \varphi^* S = \varphi^*(d \cdot S)$$

et comme φ est surjective cela implique que $f^* S = dS$. Nous avons donc, à partir du Théorème 1.2.6, que le courant de Green de f est un multiple de S.

En particulier, nous voyons que le courant de Green de f est lisse en dehors des valeurs critiques de φ (car φ est localement inversible sur cet ensemble). On peut montrer que les applications de Lattès sont les seules exemples d'endomorphisme holomoprhe tel que son courant de Green est lisse et strictement positif dans un ouvert de $\mathbb{P}^k$ (voir [BL01]). Ce résultat nous sera utile dans le Chapitre 2.

1.2.2 Laminations par variétés complexes

Soit Σ un espace métrique compacte. Une *lamination* $\mathcal{L}$ de Σ par des variétés complexes de dimension q consiste d'un atlas d'homéomophismes $\varphi_i : U_i \to \mathbb{D}^q \times K_i$, où $\mathbb{D}^q$ est le polydisque unité dans $\mathbb{C}^q$ et K_i est un espace topologique de sorte que les changements des coordonnées $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ soient de la forme

$$\varphi_{ij}(z,t) = (f_{ij}(z,t), \gamma_{ij}(t)), \tag{1.2.2}$$

avec f_{ij} holomorphe dans la première variable.

Les ouverts U_i sont appelés des *boîtes à flot* et les ensembles $\Pi_t = \varphi_i^{-1}(\mathbb{D}^q \times \{t\})$ sont appelés les *plaques* de $\mathcal{L}$ dans U_i. Une *feuille* de la lamination est un sous-ensemble connexe minimal $L \subset \Sigma$ avec la propriété que si L rencontre une plaque Π alors $\Pi \subset L$.

Si Σ est un sous-ensemble d'une variété complexe compacte X de dimensions n nous disons que $\mathcal{L}$ est une *lamination plongée dans* X si les transversales K_i sont contenus dans $\mathbb{D}^{n-q}$ et les cartes φ_i^{-1} se prolongent en homéomorphismes entre $\mathbb{D}^q \times \mathbb{D}^{n-q}$ et un ouvert de X qui sont holomorphes dans la première variable.

Dans les exemples ci-dessous, X est une variété complexe compacte de dimension n.

Exemple 1.2.14. Soit V une sous-variété complexe de X. Alors V porte une lamination triviale où les transversales K_i sont des singletons.

Exemple 1.2.15 (Feuilletages holomorphes non-singuliers)**.** Si $\Sigma = X$, les transversales K_i sont des polydisques de dimension $n-q$ et les cartes $\varphi_i : U_i \to \mathbb{D}^q \times \mathbb{D}^{n-q}$ sont des biholomorphismes satisfaisant (1.2.2) alors $\mathcal{L}$ est ce qu'on appelle un *feuilletage holomorphe non-singulier* sur X.

L'existence d'un feuilletage holomorphe non-singulier est assez rare. Voici quelques exemples et non-exemples.

1. S'il existe une variété complexe compacte Y de dimension $m < n$ et une submersion holomorphe $\pi : X \to Y$ alors X admet un feuilletage non-singulier dont les feuilles sont les fibres de π.
2. Si X est un tore complexe et $\pi : \mathbb{C}^n \to X$ denote la projection canonique alors le feuilletage de $\mathbb{C}^n$ donné par des sous-espaces affines de dimension q parallèles à un sous-espace fixé E induit via π un feuilletage holomorphe non-singulier $\mathcal{F}$ sur X. Selon la position de E par rapport au réseau définissant X les feuilles peuvent avoir des comportements variés. Il se peux par exemple que les feuilles $\mathcal{F}$ soient toutes compactes (données par des sous-tores de dimension q), ou encore que toutes les feuilles soient non-compactes et denses dans X. Il est possible aussi que les feuilles s'accumulent sur un sous-tore réel de codimension 1.
3. L'espace projectif complexe n'admet pas de feuilletage holomorphe non-singulier de codimension 1, voir [Bru15].

Exemple 1.2.16. Si $\mathcal{F}$ est un feuilletage non-singulier de X alors tout compact donné par une réunion de feuilles de $\mathcal{F}$ et laminé. Si L est une feuille de $\mathcal{F}$, l'adhérence $\overline{L}$ est un compacte laminé de X.

Exemple 1.2.17 (Laminations par surfaces de Riemann et Hypersurfaces Levi-plates)**.** Une lamination par des variétés complexes de dimension 1 est appelée une *lamination par surfaces de Riemann*. Ces objets appairassent naturellement dans des problèmes de Géométrie et Dynamique Complexe. Le lecteur pourra consulter [Ghy99] pour une introduction au sujet. Un exemple important de ce type de lamination est donné par les hypersurfaces Levi-plates

Soit X une surface complexe compacte. Une hypersurface réelle compacte lisse $M \subset X$ est dite *Levi-plate* si dans des coordonnées holomorphes locales $(z, w) \in \mathbb{D}^2$, M est donnée par $\{\text{Im } w = 0\}$, voir par exemple [LFP15]. Dans ces coordonnées, les ensembles $\{w = c\}$, $c \in \mathbb{R}$ définissent les plaques d'une lamination par surfaces de Riemann sur M. Cet exemple se généralise facilement en dimension supérieure. Les hypersurfaces Levi-plates sont donc des exemples de compactes laminées. La question de l'existence de tels hypersurfaces est très délicate (voir par exemple [LN99] et [Siu00]).

Mesures transverses, mesures harmoniques et cycles feuillétés

Soit $\mathcal{L}$ une lamination de dimension q plongée dans une variété complexe X. Fixons un atlas d'homéomorphismes (U_i, φ_i) dont les changements de cartes sont comme dans (1.2.2).

Définition 1.2.18. Une **mesure transverse invariante** μ est la donnée d'une mesure positive μ_i sur chaque transverse $K_i \subset \mathbb{D}^{n-q}$ satisfaisant la condition d'invariance

$$\mu_i(E) = \mu_j(\gamma_{ij}(E)) \tag{1.2.3}$$

pour tout borélien E contenu dans le domaine de définition de γ_{ij}.

Une mesure transverse invariante nous permet de définir un courant positif fermé de bi-dimension (q, q) sur U_i de la façon suivante. Si α est une $(q, q)-$forme à support compact dans U_i, alors

$$\langle T, \alpha \rangle = \int_{K_i} \left(\int_{\mathbb{D}^q \times \{t\}} (\varphi_i)_* \alpha \right) \mathrm{d}\mu_i(t). \tag{1.2.4}$$

Autrement dit, T est donné par une moyenne des courants d'intégration le long des plaques. Notons que le pullback $(\varphi_i^{-1})^* \alpha = (\varphi_i)_* \alpha$ n'est pas bien défini, car φ_i n'est pas lisse. Néanmoins, l'intégrale ci-dessus est bien définie car φ_i est différentiable le long des plaques.

La relation d'invariance (1.2.3) entraîne que les formules locales (1.2.4) se recollent en un courant défini globalement sur X.

Définition 1.2.19. Un **cycle feuilleté** subordonné à $\mathcal{L}$ est un courant de bi-dimension (q, q) sur X donné localement par l'expression (1.2.4) pour une mesure invariante μ.

Exemple 1.2.20. Supposons que $\mathcal{L}$ admette une feuille compacte L. Alors le courant d'intégration $T = [L]$ est un cycle feuilleté subordonné à $\mathcal{L}$. Dans ce cas, les mesures transverses μ_i sont des masses de Dirac. D'autres exemples seront donnés dans le Chapitre 3.

Il s'avère que beaucoup de laminations n'admettent pas de cycles feuilletés appart ceux qui sont donnés par intégration le long de feuilles compactes. Ceci est vrai par exemple pour des laminations par surfaces de Riemann transversalement Lipschitz plongées dans $\mathbb{P}^2$, voir [FS05]. Dans le Chapitre 3 nous généralisons ce résultat pour des laminations de dimension supérieure.

Même si l'existence de cycles feuilletés (ou de façon équivalente des mesures transverse invariantes) est assez rare, toute lamination par surfaces de Riemann admet ce qu'on appelle un **courant harmonique invariante**, voir [Ghy99],[BS02] et [FS08]. Ce sont des courants positifs T subordonnés à $\mathcal{L}$ vérifiant $\mathrm{dd}^c T = 0$. Dans une boîte à flot (U_i, φ_i) nous avons une expression locale du type

$$\langle T, \alpha \rangle = \int_{K_i} \left(\int_{\mathbb{D}^q \times \{t\}} h_i^t \, (\varphi_i)_* \alpha \right) \mathrm{d}\mu_i(t),$$

où les h_i^t sont des fonctions harmoniques positives sur $\mathbb{D}^q \times \{t\}$ et les μ_i sont des mesures transverses invariantes comme dans (1.2.3).

Pour les laminations de dimension supérieure, même l'existence de courants harmoniques invariantes n'est pas assurée. Dans [FSW11] les auteurs donnent un exemple de lamination de dimension 2 dans $\mathbb{P}^2 \times \mathbb{P}^2$ sans aucun courant harmonique invariant.

Laminations avec singularités

Pour compenser la manque d'exemples de laminations dans les variétés compactes nous pouvons élargir cette notion et considérer des laminations avec singularités : ce sont des laminations de $Y = X \setminus E$, où E est l'ensemble singulier supposé "petit". Le cas des feuilletages holomorphes est peut-être le plus étudié, voir par exemple [Bru15]. Concernant les laminations, la plupart des références étudient les cas des laminations singulières par surfaces de Riemann. Nous renvoyons à [FS08] et ses références. Pour les laminations par surfaces de Riemann, beaucoup de résultats se généralisent aux cas singulier. Par exemple, nous pouvons toujours définir les cycles feuilletés et les courants harmoniques dans ce contexte . Leur étude est cependant beaucoup plus compliquée et demande des outils nouveaux.

Chapitre 2

Endomorphismes Permutables de $\mathbb{P}^2$

In this chapter we study commuting pairs of endomorphisms of $\mathbb{P}^2$ that do not share an iterate. The one-dimensional case of this problem is classical and it traces back to some of the creators of the iteration theory of rational functions, such as Fatou, Julia and Ritt. In this case, these pairs are completely classified. In higher dimensions the problem was first considered by T.-C.Dinh and N. Sibony, and they gave a structure theorem for this maps under the assumption that their degrees remain distinct after iteration. Here we study the two dimensional case without this additional assumption, completing the classification, see Theorem 2.1.9. The results in this chapter were originally presented in the paper [Kau15].

2.1 Introduction and main result

The study of functional equations is at the origin of the early developments of the iteration theory of polynomials and rational functions, carried out by Fatou, Julia, Ritt and others. Among these equations, the commutation relation

$$f \circ g = g \circ f \tag{2.1.1}$$

is one of the simplest and most commonly encountered in mathematics and its applications. It can also be thought as a sort “integrability condition" for the dynamical system defined by f (see [Ves87]).

The problem considered by Fatou, Julia and Ritt is the following.

Problem 2.1.1. Describe all pairs of holomorphic maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree at least

two satisfying (2.1.1) and not sharing an iterate, that is, satisfying

$$f^n \neq g^m, \text{ for all } n, m \geq 1. \tag{$\star$}$$

The solution was already obtained by the above mentioned authors (see [Fat24], [Jul22], [Rit23]) and a new proof was given by Eremenko in [Ere89] with the use of a modern language.

Theorem 2.1.2 (Fatou, Julia, Ritt, Eremenko)**.** *Let $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ be holomorphic maps of degree $d_f, d_g \geq 2$ satisfying (2.1.1) and ($\star$). Then, after conjugation we have one of the following possibilities*

1. *$f(z) = z^{\pm d_f}$ and $g(z) = \lambda z^{\pm d_g}$, where λ is a $(d_f - 1)$-th root of unity.*
2. *$f(z) = \pm T_{d_f}(z)$ and $g(z) = \pm T_{d_g}(z)$, where T_d denotes the Chebyshev polynomial of degree d, defined by*
$$T_d(\cos z) = \cos(d \cdot z).$$
3. *f and g are commuting Lattès maps of $\mathbb{P}^1$.*

For the definition of Lattès maps, see Example 1.2.4.

We can ask the same question in higher dimension.

Problem 2.1.3. Describe all pairs of holomorphic maps $f, g : \mathbb{P}^k \to \mathbb{P}^k$ of degree at least two satisfying (2.1.1) and not sharing an iterate.

Maybe it is too optimistic to hope for a finite list of examples as in Theorem 2.1.2. Nevertheless, under an additional condition on the degrees of f and g, Dinh and Sibony obtained a structure theorem for this kind of maps.

Theorem 2.1.4 (Dinh-Sibony,[DS02])**.** *Let f and g be two commuting holomorphic endomorphisms of $\mathbb{P}^k$ of degree $d_f, d_g \geq 2$. Assume that*

$$d_f^n \neq d_g^m, \text{ for all } n, m \geq 1. \tag{2.1.2}$$

Then we have a commutative diagram of the following type

$$\begin{tikzcd} \mathbb{C}^k \arrow[r, "\Lambda_f, \Lambda_g"] \arrow[d, "\varphi"'] & \mathbb{C}^k \arrow[d, "\varphi"] \\ \mathbb{P}^k \arrow[r, "f, g"] & \mathbb{P}^k \end{tikzcd}$$

where $\varphi : \mathbb{C}^k \to \mathbb{P}^k$ is a holomorphic map with discrete fibers, $\varphi(\mathbb{C}^k)$ is a Zariski open subset of $\mathbb{P}^k$ and $\Lambda_f, \Lambda_g : \mathbb{C}^k \to \mathbb{C}^k$ are holomorphic affine maps. Moreover, there is a discrete group of affine transformations of $\mathbb{C}^k$ acting transitively on the fibers of φ.

As a consequence we get

Corollary 2.1.5. *Let f and g be as in Theorem 2.1.4. Then f and g are critically finite, that is, their critical set are pre-periodic.*

Let us illustrate the above result with some examples. The examples below are given in dimension 2, but they can easily be generalized to any dimension.

Example 2.1.6. Let $p_f(z), q_f(z)$ be commuting polynomials of degree d_f and $p_g(w), q_g(w)$ be commuting polynomials of degree d_g. Then the maps

$$f(z,w) = (p_f(z), q_f(w)) \text{ and } g(z,w) = (p_g(z), q_g(w))$$

are commuting polynomial endomorphisms of $\mathbb{C}^2$ that extend holomorphically to $\mathbb{P}^2$. The conditions ($\star$) and (2.1.2) are satisfied by f and g as long as they are satisfied by one of the pairs (p_f, p_g) or (q_f, q_g).

Example 2.1.7 (Lifts of commuting pairs)**.** Suppose we are given a commuting pair $h_f, h_g : \mathbb{P}^1 \to \mathbb{P}^1$. We can lift them to $\mathbb{P}^2$ in the following way. Write $h_f = [P_f : Q_f]$ where P_f and Q_f are homogeneous polynomials of two variables and same degree d_f. Similarly, write $h_g = [P_g : Q_g]$ for some homogeneous polynomials of degree d_g.

Define $f, g : \mathbb{P}^2 \to \mathbb{P}^2$ by the following formula

$$f[z : w : t] = [P_f(z,w) : Q_f(z,w) : t^{d_f}] \text{ and } g[z : w : t] = [P_g(z,w) : Q_g(z,w) : t^{d_g}].$$

It is clear that f and g commute and that the conditions ($\star$) and (2.1.2) for f and g are equivalent to same conditions for h_f and h_g.

We can make a slight modification of the above example in order to produce a non-example to Theorem 2.1.4.

Example 2.1.8. Let $h_f, h_g : \mathbb{P}^1 \to \mathbb{P}^1$ be as in Example 2.1.7 and suppose that they do not share an iterate. Suppose further that $d_f = d_g = d$. Let R be a one-variable polynomial of degree d. Define maps $f, g : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(z,w,t) = (P_f(z,w), Q_f(z,w), R(t)) \text{ and } g(z,w,t) = (\lambda P_g(z,w), \lambda Q_g(z,w), R(t)).$$

Then, for a suitable λ, f and g extend to commuting pairs $f, g : \mathbb{P}^3 \to \mathbb{P}^3$ of the *same degree* satisfying ($\star$). If we choose R in such a way that it has an infinite critical orbit, then f and g won't be critically finite. Hence, by Corollary 2.1.5, the conclusion of Theorem 2.1.4 cannot hold for this pair.

Our main theorem in this chapter says that the above phenomenon cannot occur if the dimension is two.

Theorem 2.1.9. *Let f and g be two commuting holomorphic endomorphisms of $\mathbb{P}^2$ of degree $d_f, d_g \geq 2$. Assume that $f^n \neq g^m$ for all $n, m \geq 1$ and that $d_f^k = d_g^\ell$ for some $k, \ell \geq 1$. Then both f and g are of one of the following types:*

1. *Commuting Lattès maps of $\mathbb{P}^2$.*
2. *Lifts of commuting Lattès maps of $\mathbb{P}^1$: f and g are homogeneous polynomial endomorphisms of $\mathbb{C}^2$ that extend holomorphically to $\mathbb{P}^2$ and such that the induced maps on the hyperplane at infinity are Lattès maps of $\mathbb{P}^1$.*

Notice that the above theorem together with Theorem 2.1.4 complete the solution to Problem 2.1.3 in dimension 2.

For a two-dimensional Lattès maps, the group $\mathcal{A}$ appearing in its definition is such that $\mathbb{C}^2/\mathcal{A} \simeq \mathbb{P}^2$. These groups were classified in [TY82] and using this classification it is possible to give a precise description of Lattès maps of $\mathbb{P}^2$ (see [Dup03] and [Ron10]). There is also a complete classification of Lattès maps of $\mathbb{P}^1$, which can be found in [Mil06b].

The strategy of the proof, which is inspired by the methods in [DS02] and [Ere89], relies on the study of the lamination of the Julia set J_2 and the Green measure μ of the pair f, g. We may suppose at first that $d_f = d_g$. We start by considering the Poincaré map $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$ of f and g associated with a common repelling periodic point. This map gives a semiconjugation between f, g and triangular automorphisms Λ_f, Λ_g of $\mathbb{C}^2$. We consider then the closed abelian Lie group of triangular automorphisms of $\mathbb{C}^2$ generated by $\Lambda_f \circ \Lambda_g^{-1}$, which preserves several dynamical objects. Condition ($\star$) insures that this group is positive dimensional whereas the condition that $d_f = d_g$ implies that it is compact. This procedure can be applied to different periodic points, giving to J_2 and μ a laminar structure induced by the orbits of these groups.

The next step consists of a detailed analysis of the geometry of the lamination defined above. We begin by showing that the dimension of its leaves is $m = 3$ or 4, see Proposition 2.5.4. When $m = 4$ the measure μ is smooth in some open set and we conclude, using a theorem of Berteloot and Dupont, that both f and g are Lattès maps of $\mathbb{P}^2$. The case $m = 3$ is more delicate and Section 2.6 is devoted to it. We prove in this setting that f and g possess a totally invariant line. This is done by analyzing separately the cases when the leaves are strictly pseudoconvex and when they are Levi-flat. Once this step is achieved, we reduce the problem to the case where f and g are polynomial maps of $\mathbb{C}^2$, see Theorem 2.6.6. We then apply a theorem of Dinh saying that a polynomial map of $\mathbb{C}^k$ that extends holomorphically to $\mathbb{P}^k$ and whose maximal order Julia set contains a piece of hypersurface is a lift of a Lattès maps of $\mathbb{P}^{k-1}$.

2.2 Holomorphic endomorphisms of $\mathbb{P}^k$: general facts

We recall here the basic dynamical objects associated with a holomorphic endomorphism of $\mathbb{P}^k$. For further details the reader may consult [Sib99], [DS10] and Chapter

1.

We start by considering a polynomial endomorphism $f : \mathbb{C}^k \to \mathbb{C}^k$ of algebraic degree $d \geq 2$ which can be extended holomorphically to $\mathbb{P}^k$. We define its *Green function* by

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^n(z)\|.$$

It is a Hölder continuous plurisubharmonic function on $\mathbb{C}^k$ vanishing exactly at the points of bounded orbit. Moreover we can show that $K = \{G = 0\}$ is a compact subset of $\mathbb{C}^k$. The *Green current of* f is the positive closed (1,1)-current defined by $T = dd^c G$.

For a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of degree $d \geq 2$ its *Green current* is defined as the limit

$$T = \lim_{n\to\infty} \frac{1}{d^n} (f^n)^* \omega,$$

where ω is the Fubini-Study form. We get the same limiting current if we replace ω in the above formula by any positive closed $(1,1)$-current of mass 1 and of bounded local potential. As in the polynomial case we can show that the local potentials of the Green current are Hölder continuous (see Proposition 1.2.8).

In both polynomial and holomorphic cases we get higher order Green currents by taking wedge products of T with itself: $T^\ell = T \wedge \cdots \wedge T$ (ℓ times) for $1 \leq \ell \leq k$. They are well-defined since T admits continuous local potentials. When $\ell = k$ we call $\mu = T^k$, the *Green measure* of f. It is the unique invariant probability measure of maximal entropy for the dynamical system defined by f, see Theorem 1.2.10.

The *Julia set of order* s of f, denoted by J_s, is by definition the support of the current T^s, $1 \leq \ell \leq k$. In this work we will focus mainly on the maximal order Julia set J_k.

The following result will be useful to us. Its proof is inspired by [DS08].

Proposition 2.2.1. *Let S be a positive closed $(1,1)$-current on $\mathbb{P}^k$. Assume that there are $n_i \nearrow +\infty$ and positive closed $(1,1)$-currents S_{n_i} of unit mass supported by J_k such that $S = \lim_{i\to\infty} d^{-n_i}(f^{n_i})^* S_{n_i}$. Then $S = T$. In particular, if S is a positive closed $(1,1)$-current of unit mass supported by J_k and $f^*S = d \cdot S$, then $S = T$.*

Proof. Recall that a function $u : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ is called p.s.h. modulo T if u is locally a difference of a p.s.h. function and a potential of T. Such a function is upper semi-continuous, it belongs to $L^1(\mathbb{P}^k)$ and $T + \mathrm{dd}^c u$ is a positive current. Modulo T p.s.h. functions satisfy the inequalities

$$\max_{\mathbb{P}^k} u \leq c(1 + \|u\|_{L^1(\mathbb{P}^k)}) \text{ and } \int u \, d\mu \leq c(1 + \|u\|_{L^1(\mathbb{P}^k)})$$

where c is a dimensional constant (see [DS08]) and μ is the Green measure.

By the dd^c-Lemma there are functions u_i satisfying $T = S_{n_i} - \mathrm{dd}^c u_i$ and $\max_{\mathbb{P}^k} u_i = 0$. The family $\{u_i\}$ is bounded in $L^1(\mathbb{P}^k)$ and $d^{-n_i} u_i \circ f^{n_i}$ converges in $L^1(\mathbb{P}^k)$ to a modulo

T p.s.h. function u satisfying $T = S - \mathrm{dd}^c u$.

Since $u_i \leq 0$ we have $u \leq 0$. From the second inequality above, the sequence u_i is bounded in $L^1(\mu)$. By the invariance of μ by f we get $\|d^{-n_i} u_i \circ f^{n_i}\|_{L^1(\mu)} = d^{-n_i}\|u_i\|_{L^1(\mu)} \to 0$, so by Fatou's Lemma $\int u \, d\mu \geq 0$. Together with the fact that $u \leq 0$ we get $\int u \, d\mu = 0$ and by upper semicontinuity $u = 0$ on supp $\mu = J_k$. In particular u is finite over J_k.

On the other hand, since supp $S \subset J_k$ we have $T|_{\mathbb{P}^k \setminus J_k} = (-\mathrm{dd}^c u)|_{\mathbb{P}^k \setminus J_k}$ showing that u is locally bounded outside J_k. We conclude that u is finite on $\mathbb{P}^k$. In particular the current S has zero Lelong number everywhere on $\mathbb{P}^k$.

From the upper semicontinuity of the Lelong number we get

$$\lim_{i\to\infty} \sup_{p\in\mathbb{P}^k} \nu(d^{-n_i}(f^{n_i})^* S_{n_i}, p) \leq \sup_{p\in\mathbb{P}^k} \nu(S, p) = 0.$$

An equidistribuition theorem (see [Gue03]), [DS08], [Taf11]) gives $\lim_{i\to\infty} d^{-n_i}(f^{n_i})^* S_{n_i} = T$, so $S = T$. □

Remark 2.2.2. The existence of a totally invariant $(1,1)$-current supported by J_k as in the above proposition is quite special. Indeed, in this case we have that $J_1 = \text{supp } T \subset J_k$, so $J_k = J_1$ and the Julia sets of all orders coincide. Some examples of this phenomenon appear in [FS01].

The next lemma is well known and it is useful in producing totally invariant currents.

Lemma 2.2.3. *Let R be a positive closed $(1,1)$-current of mass one on $\mathbb{P}^k$. Consider the sequence of Cesàro sums $S_n = \frac{1}{n}\sum_{j=0}^{n-1} d^{-j}(f^j)^* R$. Then $(S_n)_{n\geq 1}$ is relatively compact and every cluster value S is totally invariant, that is, $f^* S = d \cdot S$.*

Proof. The first statement follows directly from the fact that each S_n is of mass one. Let S be a cluster value and take $n_\ell \nearrow +\infty$ such that $S_{n_\ell} \to S$ as $\ell \to \infty$. Notice that $nS_n = \sum_{j=0}^{n-1} d^{-j}(f^j)^* R$ for every n, so that

$$d^{-1} f^* S_{n_\ell} = \frac{1}{n_\ell} \sum_{j=0}^{n_\ell - 1} d^{-j+1}(f^{j+1})^* R = \frac{1}{n_\ell} \sum_{j=1}^{n_\ell} d^{-j}(f^j)^* R = \frac{n_\ell + 1}{n_\ell} S_{n_\ell + 1} - \frac{1}{n_\ell} R.$$

Since the term on the left-hand side converges to $d^{-1} f^* S$ and the last term on the right-hand side converges to 0 we need to show that $\frac{n_\ell+1}{n_\ell} S_{n_\ell+1} \to S$ as $\ell \to \infty$. Writing

$$\frac{n_\ell + 1}{n_\ell} S_{n_\ell+1} = \frac{1}{n_\ell}[(n_\ell + 1) S_{n_\ell+1} - n_\ell S_{n_\ell}] + S_{n_\ell}$$

this amounts to showing that $\frac{1}{n_\ell}[(n_\ell + 1) S_{n_\ell+1} - n_\ell S_{n_\ell}] \to 0$.

We have that

$$\frac{1}{n_\ell}[(n_\ell+1)S_{n_\ell+1}-n_\ell S_{n_\ell}]=\frac{1}{n_\ell}[\sum_{j=0}^{n_\ell}d^{-j}(f^j)^*R-\sum_{j=0}^{n_\ell-1}d^{-j}(f^j)^*R]=\frac{1}{n_\ell}d^{-n_\ell}(f^{n_\ell})^*R\to 0,$$

since the currents $d^{-n_\ell}(f^{n_\ell})^*R$ have all mass one. This concludes the proof. □

The following theorem shows that the Green measure describes the distribution of pre-images of points outside an exceptional set, giving also a precise description of this set, see [DS10].

Theorem 2.2.4 (Equidistribution of pre-images)**.** *Let $f:\mathbb{P}^k\to\mathbb{P}^k$ be a holomorphic endomorphism of degree $d\geq 2$ and μ its Green measure. Then there is a proper algebraic subset $\mathcal{E}$ of $\mathbb{P}^k$ such that the measures*

$$\mu_n^a=\frac{1}{d^{nk}}(f^n)^*\delta_a=\frac{1}{d^{nk}}\sum_{x\in f^{-n}(a)}\delta_x$$

converge to μ as $n\to\infty$ if and only if $a\notin\mathcal{E}$.

Moreover, the set $\mathcal{E}$ is totally invariant $f^{-1}(\mathcal{E})=f(\mathcal{E})=\mathcal{E}$ and is maximal in the sense that if E is a proper algebraic subset of $\mathbb{P}^n$ such that $f^{-n}(E)\subset E$ for some $n\geq 1$, then $E\subset\mathcal{E}$.

It is known (see [FS94], [CLN00]) that the codimension one components of $\mathcal{E}$ form a hypersurface of degree at most $k+1$ and it is conjectured that they are all linear. In dimension 2 the situation is completely understood.

Theorem 2.2.5 (Fornæss-Sibony [FS94], Cerveau-Lins Neto [CLN00])**.** *Let $f:\mathbb{P}^2\to\mathbb{P}^2$ be a holomorphic endomorphism of degree $d\geq 2$ and let $\mathcal{E}$ be its exceptional set. Then the one dimensional part of $\mathcal{E}$ is a union of at most three lines. In particular, if no iterate of f is conjugated to a polynomial map then $\mathcal{E}$ is finite.*

We state a last general result for further use.

Proposition 2.2.6 ([Din01])**.** *Let f be a polynomial endomorphism of $\mathbb{C}^k$ of degree $d\geq 2$ that extends to a holomorphic endomorphism of $\mathbb{P}^k$. Then there is no complex manifold contained in the maximal order Julia set J_k which passes through a repelling periodic point.*

Remark 2.2.7. For non-polynomial endomorphisms, the above proposition is no longer true, as the Lattès maps show. Other non-Lattès examples can be found in [FS01]. Proposition 2.3.3 gives a related result for holomorphic endomorphisms of $\mathbb{P}^2$.

Commuting pairs

For a commuting pair of endomorphisms, the several dynamical objects introduced above coincide. This is made precise by the two propositions below.

Given a commuting pair $f, g : \mathbb{P}^k \to \mathbb{P}^k$ it is easy to see that we can always find commuting polynomial lifts $F, G : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$, that is, F and G are commuting homogeneous polynomial mappings, $\pi \circ F = \pi \circ f$ and $\pi \circ G = \pi \circ g$, where $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection.

Proposition 2.2.8 ([DS02])**.** *Let $f, g : \mathbb{P}^k \to \mathbb{P}^k$ be a commuting pair of endomorphisms of degree $d_f, d_g \geq 2$. Choose lifts $F, G : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ such that $F \circ G = G \circ F$ and denote by G_F and G_G their corresponding Green functions. Then $G_F = G_G$. In particular $T_f = T_g$ and $J_s(f) = J_s(g)$ for every $s = 1, \ldots, k$.*

Proposition 2.2.9. *Let $f, g : \mathbb{P}^k \to \mathbb{P}^k$ be a commuting pair of endomorphisms. Denote by $\mathcal{E}_f$ and $\mathcal{E}_g$ the corresponding exceptional sets. Then $\mathcal{E}_f = \mathcal{E}_g$.*

Proof. We will show that $\mathcal{E}_f \subset \mathcal{E}_g$, the other inclusion being analogous. By the maximality of $\mathcal{E}_g$ it suffices to show that $g^{-1}(\mathcal{E}_f) \subset \mathcal{E}_f$ and by the maximality of $\mathcal{E}_f$ this will follow if we show that $f^{-1}(g^{-1}(\mathcal{E}_f)) \subset g^{-1}(\mathcal{E}_f)$.

Let V be an irreducible component of $f^{-1}(g^{-1}(\mathcal{E}_f))$. We have that $f(V) \subset g^{-1}(\mathcal{E}_f)$, so $f(g(V)) = g(f(V)) \subset \mathcal{E}_f$, meaning that $g(V) \subset f^{-1}(\mathcal{E}_f) = \mathcal{E}_f$. This shows that $V \subset g^{-1}(\mathcal{E}_f)$ finishing the proof. □

Lattès maps

Lattès maps (see Example 1.2.4 for the definition) form a special class of holomorphic endomorphisms of $\mathbb{P}^k$. They possess a certain rigidity and therefore can be characterized by several extremal properties, see for instance [BL01] and [BD05]. One of these characterizations is in terms of Green currents and measures.

Theorem 2.2.10 (Berteloot-Loeb, Berteloot-Dupont)**.** *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$.*

1. *If the Green current of f is smooth and strictly positive in some nonempty open subset of $\mathbb{P}^k$, then f is a Lattès map.*
2. *If the Green measure of f is nonzero and absolutely continuous with respect to the Lebesgue measure in some open subset of $\mathbb{P}^k$, then f is a Lattès map.*

Corollary 2.2.11. *Let $f, g : \mathbb{P}^k \to \mathbb{P}^k$ be a commuting pair of endomorphisms of degree $d_f, d_g \geq 2$. If f is a Lattès map then so is g.*

Proof. Since f is a Lattès map its Green current is smooth and strictly positive in some nonempty open subset of $\mathbb{P}^k$. From Proposition 2.2.8 the same holds for the Green current of g. The conclusion follows from Theorem 2.2.10. □

2.3 Entire curves and Ahlfors currents

When dealing with non-normal families of meromorphic functions of one variable, a useful tool is the classical Zalcman's renormalization lemma. By applying it to the several variable setting we get the following result.

Lemma 2.3.1 ("Zalcman's Lemma", [DS02])**.** *Let $\{\psi_n\}$ be a non-equicontinuous family of germs of holomorphic maps from a neighborhood of the origin in $\mathbb{C}^k$ taking values in $\mathbb{P}^\ell$. Then there is a nonzero vector $v \in \mathbb{C}^k$, a sequence of complex numbers $z_i \to 0$, a sequence of real numbers $\rho_i \to 0$ and an increasing sequence of integers $n_i \to \infty$ such that the maps $\psi_{n_i}(z_i v + \rho_i \xi v)$ converge to a non-constant holomorphic map $\phi(\xi)$ from $\mathbb{C}$ to $\mathbb{P}^\ell$.*

In the same spirit of renormalization there is another useful tool which is the notion of Ahlfors current. It is a positive closed current constructed from the image of an increasing sequence of discs by an entire map.

Theorem 2.3.2. *Let $\phi : \mathbb{C} \to X$ be a non constant holomorphic map onto a complex manifold X. Denote by D_r the closed disc of radius r and center 0 in $\mathbb{C}$. Then there is a sequence $r_n \nearrow \infty$ such that the positive currents*

$$R_n = \frac{\phi_*[D_{r_n}]}{\|\phi_*[D_{r_n}]\|}$$

converge to a positive closed current R of mass 1 whose support is contained in $\overline{\phi(\mathbb{C})}$.

As an application we have the following.

Proposition 2.3.3. *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism of degree $d \geq 2$. Suppose that the maximal order Julia set J_2 contains a holomorphic disc passing through a repelling periodic point. Then $J_1 = J_2$.*

Proof. Let $\Sigma \subset J_2$ be a holomorphic disc passing through a repelling periodic point $a \in J_2$. After replacing f by an iterate we may suppose that a is a fixed point. Since a is a repelling fixed point, the family $\{f^n|_\Sigma : n \geq 1\}$ of maps from Σ to $\mathbb{P}^k$ is not normal. Using Lemma 2.3.1 we can produce an entire curve whose image is contained in $\overline{\bigcup_{n\geq 1} f^n(\Sigma)} \subset J_2$. The last inclusion holds because J_2 is closed and invariant by f. By Theorem 2.3.2 there is a positive closed $(1,1)$-current R supported by J_2.

The sequence of Cesàro sums $S_n = \frac{1}{n}\sum_{j=1}^n d^{-j}(f^j)^*R$ admits a subsequence converging to a positive closed $(1,1)$-current of unit mass S that is totally invariant $f^*S = d{\cdot}S$ (Lemma 2.2.3). Since J_2 is closed and totally invariant, the current S is supported by J_2, so $S = T$ by Proposition 2.2.1. In particular $J_1 = \operatorname{supp} S \subset J_2$. As $J_1 \supset J_2$, the result follows. $\square$

2.4 Repelling periodic points and Poincaré maps

A germ of holomorphic map $\Lambda : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ is called *triangular* if it is invertible and has the form

$$\Lambda(z) = \big(\lambda_1 z_1, \lambda_2 z_2 + P_2(z_1), \dots, \lambda_k z_k + P_k(z_1, \dots, z_{k-1})\big),$$

where the P_j are polynomials containing only the monomials $z_1^{\ell_1} \cdots z_{j-1}^{\ell_{j-1}}$ for which $\lambda_j = \lambda_1^{\ell_1} \cdots \lambda_{j-1}^{\ell_{j-1}}$. In other words, Λ may contain non-linear terms in the presence of resonances of the k-uple $\lambda = (\lambda_1, \dots, \lambda_k)$.

In the case $k = 2$ the triangular maps take the form $\Lambda(z) = (\lambda_1 z_1, \lambda_2 z_2 + \alpha z_1^\ell)$ if $\lambda_2 = \lambda_1^\ell$ and we call them resonant of index ℓ, or $\Lambda(z) = (\lambda_1 z_1, \lambda_2 z_2)$ if $\lambda_2 \neq \lambda_1^\ell$ for every $\ell = 1, 2, \dots$ and we call them non-resonant.

The following statements follow from straightforward computations.

Lemma 2.4.1. *Let Λ and Γ be two triangular maps of $\mathbb{C}^2$.*

1. *Suppose that $\Lambda \circ \Gamma = \Gamma \circ \Lambda$ and that one of Λ or Γ is non-diagonal. Then Λ and Γ have the same resonance index.*
2. *Suppose that Λ and Γ are ℓ-resonant, so that $\Lambda(z_1, z_2) = (\lambda_1 z_1, \lambda_1^\ell z_2 + \alpha z_1^\ell)$ and $\Gamma(z_1, z_2) = (\gamma_1 z_1, \gamma_1^\ell z_2 + \beta z_1^\ell)$. Denoting $\xi = \lambda_1 \gamma_1^{-1}$ we have*

$$\Lambda \circ \Gamma^{-1}(z_1, z_2) = (\xi z_1, \xi^\ell z_2 + \gamma_1^{-\ell}(\alpha - \beta \xi^\ell)\, z_1^\ell)$$

and

$$(\Lambda \circ \Gamma^{-1})^n(z_1, z_2) = (\xi^n z_1, \xi^{n\ell} z_2 + n\xi^{(n-1)\ell} \gamma_1^{-\ell}(\alpha - \beta \xi^\ell)\, z_1^\ell).$$

In particular, the group generated by Λ and Γ is contained in the two-dimensional complex Lie group of maps of the form

$$\Lambda_{t,s}(z_1, z_2) = (t z_1, t^\ell z_2 + s z_1^\ell), \quad t \in \mathbb{C}^*, s \in \mathbb{C}.$$

We recall a classical theorem of Sternberg.

Theorem (Sternberg)**.** *Let $h : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ be a germ of holomorphic map. Suppose that 0 is a repelling fixed point for h. Then there is an invertible holomorphic germ $\varphi : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ such that $\varphi^{-1} \circ h \circ \varphi$ is triangular.*

Using the dilation nature of the triangular map it is possible to obtain a global version of the above theorem.

Proposition 2.4.2 ([DS02])**.** *Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$. Let b be a repelling fixed point of f. Then there is a holomorphic map $\varphi : \mathbb{C}^k \to \mathbb{P}^k$ with $\varphi(0) = b$ and $D\varphi(0)$ invertible and a triangular map Λ such that $f \circ \varphi = \varphi \circ \Lambda$.*

If furthermore $b \in J_k$ then the complement $\mathbb{P}^k \setminus \varphi(\mathbb{C}^k)$ is the exceptional set $\mathcal{E}$ of f.

Definition 2.4.3. The map $\varphi : \mathbb{C}^k \to \mathbb{P}^k$ as in Proposition 2.4.2 is called the **Poincaré map** associated with b.

The following statement, together with Proposition 2.2.8 imply that two commuting endomorphisms admit a common Poincaré map.

Proposition 2.4.4. *[DS02] Let $f, g : \mathbb{P}^k \to \mathbb{P}^k$ be two commuting endomorphisms of degree $d_f, d_g \geq 2$. Let b be a common fixed point for f and g which is repelling for f. Then there exist a holomorphic map $\varphi : \mathbb{C}^k \to \mathbb{P}^k$ with $\varphi(0) = b$ and $D\varphi(0)$ invertible and commuting triangular maps Λ_f, Λ_g such that $f \circ \varphi = \varphi \circ \Lambda_f$ and $g \circ \varphi = \varphi \circ \Lambda_g$.*

In the next sections, we will need to apply the above proposition to different periodic points on J_k. For this purpose we will the use next lemma together with the fact that the repelling periodic points belonging to J_k are dense in J_k [BD99].

Lemma 2.4.5. *[DS02] Let $f, g : \mathbb{P}^k \to \mathbb{P}^k$ be a commuting pair of endomorphisms. Then f and g possess infinitely many common periodic points in J_k which are repelling for f. Furthermore, given a repelling periodic point a of f there is an $m \geq 0$ such that $g^m(a)$ is periodic for f and g and repelling for f.*

2.5 Laminations of the equilibrium measure

In this section, we follow [DS02] and show that the existence of the maps Λ_f and Λ_g as in Proposition 2.4.4 enables us to produce laminations of the Julia set and of the measure μ.

We start by recalling a notion of laminarity that will be useful to us. It is worth mentioning that there are several non-equivalent definitions of this concept.

Definition 2.5.1. Let X be a real analytic manifold of real dimension n. Let $J \subset X$ be a closed subset and $b \in J$. We say that J is *m-laminated* at b ($0 \leq m \leq n$) if there is a neighborhood of b of the form $\Omega = U \times V$ with $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-m}$ open subsets and a real analytic coordinate system $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ defined on Ω such that $\Omega \cap J = U \times K$ for some closed set $K \subset V$.

Let ν be a measure on X. We say that ν is *m-laminated* at b if there are Ω and coordinates $(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ as above such that ν is a product of the Lebesgue measure Leb_m on U and a measure ν' on V.

Remark 2.5.2. Notice that if a measure ν is m-laminated at a point b then the constant vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_m$ on $\mathbb{R}^n$ induce vector fields near b whose corresponding local flows preserve ν.

The following lemma is an adaptation of a result by Eremenko ([Ere89], Proposition 1).

Lemma 2.5.3. *Let ν be an m-laminated measure of the form $\nu = \mathrm{Leb}_m \otimes \nu'$ on $\Omega = U \times V \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$. Let $\vartheta : \Omega \to \mathbb{R}^n$ be a non-vanishing vector field and denote by Γ^t its local flow. If $(\Gamma^t)^*\nu = \nu$ and if $\vartheta(p)$ is transversal to $\mathbb{R}^m$, $p \in \Omega$ then ν is $(m+1)$-laminated at p.*

The main result of this section is a description of the local structure of the maximal order Julia set and the Green measure of a commuting pair of same degree.

Proposition 2.5.4. *Let $f, g : \mathbb{P}^2 \to \mathbb{P}^2$ be commuting endomorphisms of degree d such that $f^n \neq g^m$ for all $n, m \geq 1$. Then the maximal order Julia set $J := J_2(f) = J_2(g)$ and the equilibrium measure $\mu = \mu_f = \mu_g$ are laminated outside an analytic subset of $\mathbb{P}^2$. If m denotes the maximal integer such that J and μ are m-laminated in some open subset of $\mathbb{P}^2$, then $m = 3$ or 4.*

Before proceeding to the proof let us introduce some notation and state some preliminary results.

Fix a common repelling fixed point a of f and g. Let $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$ be its associated Poincaré map and Λ_f, Λ_g the corresponding triangular lifts. Define the current $T^* = \varphi^* T$ and the measure $\mu^* = \varphi^*\mu$. When f and g are polynomial, set $G^* = G \circ \varphi$, where G is the common Green function of f and g.

We have the following invariance relations:

$$\Lambda_f^* T^* = d \cdot T^* = \Lambda_g T^*, \quad \Lambda_f^* \mu^* = d^2 \cdot \mu^* = \Lambda_g^* \mu^*$$

and when f and g are polynomial we have also $G^* \circ \Lambda_f = d \cdot G^* = G^* \circ \Lambda_g$.

Lemma 2.5.5. *The closed group generated by $\Lambda_f \circ \Lambda_g^{-1}$, i.e. the closure of $\{\Lambda_f^n \circ \Lambda_g^{-n}\}_{n\in\mathbb{N}}$ in the group of triangular maps, is a compact, positive dimensional Lie group.*

Proof. Notice that the $\Lambda_f^n \circ \Lambda_g^{-n}$ are pairwise distinct by condition ($\star$). The maps Λ_f and Λ_g belong to a two dimensional complex Lie group composed by maps of the form $\Lambda_{t,s}(z) = (tz_1, t^\ell z_2 + sz_1^\ell)$ if there are resonances or $\Lambda_{t,s} = (tz_1, sz_2)$ otherwise (see Lemma 2.4.1).

We claim that sequence $\{\Lambda_f^n \circ \Lambda_g^{-n}\}$ is bounded. Suppose that this is not the case, so the coefficients of $\Lambda_f^n \circ \Lambda_g^{-n}$ are unbounded. In particular the family $\{\Lambda_f^n \circ \Lambda_g^{-n}\}$ is not equicontinuous, neither is the family $\varphi \circ \Lambda_f^n \circ \Lambda_g^{-n} : \mathbb{C}^2 \to \mathbb{P}^2$. From Lemma 2.3.1 we can find a sequence of integers $\{n_i\}$, sequences $\{z_i\}$ and $\{\rho_i\}$ converging to 0 and a vector v in $\mathbb{C}^2$ such that $\varphi \circ \Lambda_f^{n_i} \circ \Lambda_g^{-n_i}(z_i v + \rho_i \zeta v)$ converges to a non-constant holomorphic map $h(\zeta)$ from $\mathbb{C}$ to $\mathbb{P}^2$.

We claim that $h^*T = 0$. Set $\phi_i(\zeta) = z_i v + \rho_i \zeta v$ and $h_i = \varphi \circ \Lambda_f^{n_i} \circ \Lambda_g^{-n_i} \circ \phi_i$, so $h = \lim h_i$. As $(\Lambda_f^{n_i} \circ \Lambda_g^{-n_i})^* T^* = T^*$ we have $h_i^* T = \phi_i^* (\Lambda_f^{n_i} \circ \Lambda_g^{-n_i})^* \varphi^* T = \phi_i^* T^*$. It suffices then to show that $\phi_i^* T^* \to 0$ as $i \to \infty$. Writing $T^* = \mathrm{dd}^c u$ for some continuous p.s.h. function on $\mathbb{C}^2$ with $u(0) = 0$, it is easy to see that $u \circ \phi_i \to 0$ in L^1_{loc}, as the image of a fixed compact set by the ϕ_i tends to 0. This yields $\phi_i^* T^* \to 0$ proving the claim.

Write $T = \omega + \mathrm{dd}^c \mathfrak{g}$, where ω is the Fubini-Study form and $\mathfrak{g}$ is a bounded function. Since $h^*T = 0$ we have $\mathrm{dd}^c(-\mathfrak{g} \circ h) = h^*\omega \geq 0$ on $\mathbb{C}$. In particular $-\mathfrak{g} \circ h$ is a bounded subharmonic function on $\mathbb{C}$, so it must be constant. We have then that $h^*\omega = 0$, which is absurd since the $(1,1)$-form ω is strictly positive.

The above contradiction shows that the sequence $\{\Lambda_f^n \circ \Lambda_g^{-n}\}$ is bounded. As it is infinite, it must have an accumulation point. In particular, its closure is a compact, positive dimensional Lie group. □

Corollary 2.5.6. *Let λ_i and γ_i, $i = 1, 2$ be the eigenvalues of $\Lambda_f'(0)$ and $\Lambda_g'(0)$ respectively, ordered by their absolute value. Then $|\lambda_i| = |\gamma_i|$ for $i = 1, 2$.*

Proof. From the above lemma the sequence $\{\lambda_i^n \gamma_i^{-n}\}$ is bounded. Let ξ be one of its cluster points and $n_j \nearrow +\infty$ such that $\lambda_i^{n_j} \gamma_i^{-n_j} \to \xi$. Since ξ^k, $k = \pm 1, \pm 2, \ldots$ are also cluster points and they should be bounded we must have $|\xi| = 1$. We thus have $|\lambda_i^{n_j} \gamma_i^{-n_j}| \to 1$ which is possible only if $|\lambda_i| = |\gamma_i|$. □

Lemma 2.5.7. *Let m be the maximal integer such that μ is m-laminated in some open set of $\mathbb{P}^2$. Then $m \geq 2$.*

Proof. Pick a common fixed point $a \in J_2$ which is repelling for f. Consider the associated Poincaré map φ and the triangular lifts Λ_f, Λ_g. By Lemma 2.5.5 there is a 1-parameter group of automorphisms preserving μ^* and J^*. This shows that the set J^* and the measure μ^* are at least 1-laminated. By pushing the lamination forward by φ we see that J and μ are at least 1-laminated outside the set of critical values of φ.

Let m be as in the statement of the lemma. Notice that each leaf of the m-lamination of J^* is invariant by the Lie group $\mathcal{G}$ generated by $\Lambda_f \circ \Lambda_g^{-1}$, where Λ_f and Λ_g are the triangular lifts of f and g at some common repelling periodic point. Indeed, if the orbit under $\mathcal{G}$ of some point in J^* connects two different leaves, the fact that $\mathcal{G}$ is positive dimensional together with Lemma 2.5.3 would allow us to produce an $(m+1)$-lamination of μ in some open set, contradicting the maximality of m.

Let us show that $m \geq 2$. Suppose that $m = 1$. Using Lemma 2.4.5 and the density of repelling periodic points in J_2 we may suppose, after replacing f and g by some iterates, that the leaf of J_2 passing through a is smooth. We claim that Λ_f and Λ_g are diagonal. If none of them is resonant the result follows from Sternberg's Theorem, so we may suppose that Λ_f is resonant.

By assumption, there is a real analytic curve C_0 contained in J^* passing through 0. Notice that C_0 is invariant by Λ_f and Λ_g. Indeed, if this were not the case then the image of C_0 by Λ_f would cut the nearby leaves transversally, but then using the invariance of μ^* by Λ_f we would get a 2-lamination of μ^* at the intersection point, contradicting the maximality of m. The same reasoning applies to Λ_g.

After a linear change of coordinates that does not change the triangular form of Λ_f and Λ_g we may assume that $L = T_0C_0 = \{z_1 = 0, \mathrm{Im}\ z_2 = 0\}$. Since C_0 is invariant by Λ_f and Λ_g we get that L by is invariant by $\Lambda_f'(0)$ and $\Lambda_g'(0)$. In particular λ_2 and γ_2 are real. By Corollary 2.5.6 we get $\lambda_2 = \pm\gamma_2$. Replacing Λ_f and Λ_g by Λ_f^2 and Λ_g^2 if necessary we may assume that $\lambda_2 = \gamma_2$.

If Λ_f and Λ_g are both resonant, the relation $|\lambda_1| = |\gamma_1|$ shows that they have the same resonance index, so $\lambda_1^\ell = \lambda_2 = \gamma_2 = \gamma_1^\ell$ for some $\ell \in \mathbb{N}$. Replacing Λ_f and Λ_g by their ℓ-th iterate we may assume that $\lambda_1 = \gamma_1$, so they have the form $\Lambda_f(z_1, z_2) = (\lambda z_1, \lambda^\ell z_2 + \alpha z_1^\ell)$ and $\Lambda_g(z_1, z_2) = (\lambda z_1, \lambda^\ell z_2 + \beta z_1^\ell)$. In this case $\Lambda_f \circ \Lambda_g^{-1} = (z_1, z_2 + \lambda^{-\ell}(\alpha - \beta) z_1^\ell)$ generates an unbounded group unless $\alpha = \beta$ (see Lemma 2.4.1). But then $\Lambda_f = \Lambda_g$, contradicting condition ($\star$). This contradiction shows that Λ_g cannot be resonant. By the commutativity of Λ_f and Λ_g we get that Λ_f is diagonal.

By the above discussion Λ_f and Λ_g are diagonal with $|\lambda_1| = |\gamma_1|$ and $\lambda_2 = \gamma_2$, so the automorphism group generated by $\Lambda_f \circ \Lambda_g^{-1}$ must be of the type $\Lambda^t(z_1, z_2) = (e^{it} z_1, z_2)$. If we choose a different repelling periodic point $a' \in J_2$ in C_0 and consider the corresponding Poincaré map φ' and the triangular lifts Λ_f', Λ_g' we can repeat the above arguments and get another group $(\Lambda')^t$ of the type above, whose orbits will be transversal to the orbits of Λ^t. This will produce a 2-lamination of μ^* at some point, contradicting the maximality of m. □

Lemma 2.5.8. *Let m be the maximal integer such that μ is m-laminated in some open set Ω of $\mathbb{P}^2$ and suppose that $m = 2$. Then, after reducing Ω if necessary, all the leaves of the lamination are totally real analytic surfaces.*

Proof. Pick a common repelling periodic point $a \in J_2$ at which the leaf passing through it is smooth (we can do so using by Lemma 2.4.5 and the density of repelling periodic points in J_2). After replacing f and g by some iterate we may suppose that a is a common fixed point. Let $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$ be the associated Poincaré map, Λ_f and Λ_g the corresponding triangular lifts and $J^* = \varphi^{-1}(J_2)$. Denote by S_c the leaves of the lamination of J^*.

If there is one leaf that is totally real then, by continuity, the nearby leaves are also totally real (it may as well happen that there is only one leaf). In this case we can shrink Ω so it contains only totally real leaves.

Suppose by contradiction that all the leaves are complex curves and denote by S_0 the leaf of J^* through the origin in $\mathbb{C}^2$. Notice that there is no open subset U of $\mathbb{C}^2$ containing only one leaf, otherwise $J^* \cap U$ would be pluripolar. Therefore there infinitely many leaves

accumulating around S_0.

If Λ_f and Λ_g are both resonant, the relations $|\lambda_i| = |\gamma_i|$, $i = 1, 2$ imply that they have the same resonance index, say ℓ. Writing $\Lambda_1(z_1, z_2) = (\lambda_1 z_1, \lambda_1^\ell z_2 + \alpha z_1^\ell)$ and $\Lambda_2(z_1, z_2) = (\gamma_1 z_1, \gamma_1^\ell z_2 + \beta z_1^\ell)$, Lemma 2.4.1 shows that $\Lambda_f \circ \Lambda_g^{-1}$ generate an unbounded group unless $\alpha = \beta\xi^\ell$, where $\xi = \lambda_1\gamma_1^{-1}$ and in this case the Lie group they generate contains a 1-parameter subgroup of the form $\Lambda^t(z_1, z_2) = (e^{it}z_1, e^{i\ell t}z_2)$. It is easy to see, by looking at the equation defining them, that the only complex curves invariant by this group are the axes $\{z_1 = 0\}$, $\{z_2 = 0\}$ and $z_2 = \kappa z_1^\ell$, $\kappa \in \mathbb{C}$. Since all these curves pass through 0, the leaves S_c with $c \neq 0$ small must cut them transversally, which is not possible by Lemma 2.5.3 and the maximality of m.

The above discussion shows that one of Λ_f or Λ_g is non-resonant. The commutativity of Λ_f and Λ_g implies that they are both diagonal. The Lie group generated by $\Lambda_f \circ \Lambda_g^{-1}$ contains then a 1-parameter subgroup of one of the following types (i) $\Lambda^t(z_1, z_2) = (e^{it}z_1, z_2)$, (ii) $\Lambda^t(z_1, z_2) = (z_1, e^{it}z_2)$ or (iii) $\Lambda^t(z_1, z_2) = (e^{ipt}z_1, e^{iqt}z_2)$ $p, q, \in \mathbb{N}$ or the 2-dimensional group $\Lambda^{s,t}(z_1, z_2) = (e^{is}z_1, e^{it}z_2)$.

It is clear that there is no complex curve invariant by $\Lambda^{s,t}$, so case (iv) cannot occur. For the group in case (iii) the only invariant complex curves are $z_2^p = \kappa z_1^q$, $\kappa \in \mathbb{C}$ and the coordinate axes. Since they all pass through 0 we argue as above to see that this contradicts the maximality of m, so case (iii) cannot occur either. For the group in case (i) it is easy to see that the only invariant holomorphic curves are of the form $\{z_2 = c\}$ for some constant c and similarly for the case (ii), replacing z_2 by z_1. We conclude that, up to permuting z_1 and z_2 we have $S_c = \{z_2 = c\}$, so

$$J^* = \bigcup_{c \in K} \{z_2 = c\} \tag{2.5.1}$$

for some closed set of parameters $K \subset \mathbb{C}$.

Consider now the groupoid

$$\mathcal{A} = \left\{ \tau : \begin{array}{c} \tau : W \to \tau(W) \text{ is a local biholomorphism} \\ \text{of } \mathbb{C}^2 \text{ such that } \varphi \circ \tau = \varphi \text{ and } W \cap J^* \neq \varnothing \end{array} \right\}.$$

Notice that the elements $\tau \in \mathcal{A}$ preserve μ^* because $\tau^*\mu^* = \tau^*\varphi^*\mu = \varphi^*\mu = \mu^*$. This implies that they also preserve the leaves of J^*. Indeed, suppose that there is a $c \in K$ such that $\tau(S_c)$ is different from any other $S_{c'}$, $c' \in K$. Since there is no isolated leaf we can see that $\tau(S_c)$ cuts a generic leaf transversally, but then Lemma 2.5.3 would allow us to produce a 3-lamination of $\tau^*\mu^* = \mu^*$ on an open set, contradicting the maximality of m.

Denote by $\mathcal{F}^*$ the foliation $\{z_2 = constant.\}$. We claim that an element $\tau : W \to \tau(W)$ in $\mathcal{A}$ takes a leaf of $\mathcal{F}^*$ intersecting W to another leaf of $\mathcal{F}^*$. To see this we write

$\tau(z_1, z_2) = (\tau_1(z_1, z_2), \tau_2(z_1, z_2))$. From the fact that τ preserves the leaves of J^* we get that $\frac{\partial \tau}{\partial z_1}$ vanishes over $J^* \cap W$ so it must be identically zero on W by the identity principle (recall that K has no isolated points). This shows that τ_2 is independent of z_1, proving the claim.

Let $A \subset \mathbb{P}^2$ be the complement of the image of the regular values of φ. This is a pluripolar set containing $\mathcal{E}$ and the above discussion shows that φ induces a nonsingular holomorphic foliation $\mathcal{F}$ on a neighborhood of $J_2 \setminus A$: the leaf of $\mathcal{F}$ through b is defined as the image by φ of the leaf of $\mathcal{F}^*$ through w, where w is any regular value in $\varphi^{-1}(b)$. From (2.5.1) we have that, outside A, J_2 is locally given by a union of leaves of $\mathcal{F}$ and that these leaves are images of $\mathbb{C}$. Lemma 2.5.10 below gives a contradiction. □

Let $\mathcal{F}$ be a non-singular foliation by Riemann surfaces on a complex surface X. We say that a $(1,1)$-current S on X is directed by $\mathcal{F}$ if $S \wedge \gamma = 0$, where γ is a local holomorphic $(1,0)$-form whose kernel defines $\mathcal{F}$. The following result is well known, see for instance [DG12].

Lemma 2.5.9. *Let S be a $(1,1)$-current with continuous local potentials on a complex surface. If S is directed by a non-singular foliation then $S \wedge S = 0$.*

Lemma 2.5.10. *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism of degree $d \geq 2$. Suppose that there is a pluripolar set A and a nonsingular f-invariant holomorphic foliation $\mathcal{F}$ on an open neighborhood $\mathcal{W}$ of $J_2 \setminus A$ such that, outside A, J_2 is locally a union of leaves of $\mathcal{F}$. Then there is no entire curve $\phi : \mathbb{C} \to J_2$ whose image is contained in a leaf of $\mathcal{F}$.*

Proof. Let L be a leaf of $\mathcal{F}$ contained in J_2 and suppose that there is a non-constant holomorphic map $\phi : \mathbb{C} \to L$. By the Ahlfors construction there is a positive closed $(1,1)$-current R supported on $\overline{\phi(\mathbb{C})} \subset J_2$. Taking a subsequence of the Cesàro sums $S_n = \frac{1}{n}\sum_{j=1}^{n} d^{-j}(f^j)^*R$ we get a totally invariant current S supported by J_2. By Proposition 2.2.1 we have $S = T$. In particular S has continuous potentials, $S \wedge S$ is well-defined and does not charge pluripolar sets.

Notice that, by construction, the current R is directed by $\mathcal{F}$ on $\mathcal{W}$. From the invariance of $\mathcal{F}$ by f, the currents S_n are also directed by $\mathcal{F}$ on $\mathcal{W}$, and the same holds for the limit value S. In particular $S \wedge S = 0$ on $\mathcal{W}$, so $S \wedge S$ is supported by A. This gives a contradiction, since A is pluripolar. □

Proof of Proposition 2.5.4. We know from Lemma 2.5.7 that $m \geq 2$. Suppose that $m = 2$. We keep the notation as in the proof of Lemma 2.5.8 and denote by S_0 be the leaf through 0. By Lemma 2.5.8, S_0 is a totally real surface. Using Lemma 2.5.3, the fact that Λ_f and Λ_g fix the origin and the maximality of m we see that S_0 is invariant by Λ_f and Λ_g. After a linear change of coordinates that does not change the triangular form of Λ_f and Λ_g we may assume that $H = T_0 S_0 = \{\mathrm{Im}\ z_1 = \mathrm{Im}\ z_2 = 0\}$. By the invariance of S_0, H is

invariant by $\Lambda_f'(0)$ and $\Lambda_g'(0)$. In particular the eigenvalues λ_i, γ_i, $i = 1, 2$ are real. From Corollary 2.5.6 we have $\gamma_i = \pm\lambda_i$. In particular $\Lambda_f^2 \circ \Lambda_g^{-2}(z_1, z_2) = (z_1, z_2 + \delta z_1^\ell)$ cannot generate a non-trivial compact group unless $\delta = 0$ (Lemma 2.4.1). In this case we have $\Lambda_f^2 = \Lambda_g^2$, which cannot happen in view of condition ($\star$). This contradiction shows that $m \geq 3$. □

2.6 Laminations by hypersurfaces

In this section we will describe the commuting pairs of endomorphisms of $\mathbb{P}^2$ such that the corresponding Green measure is 3-laminated. We will see later (Theorem 2.6.6) that the maps in question are both polynomial after some number of iterations. For this reason we begin by analyzing the polynomial case.

2.6.1 Polynomial case

Let $f, g : \mathbb{P}^2 \to \mathbb{P}^2$ be a commuting pair of holomorphic endomorphisms of same degree $d_f = d_g = d \geq 2$ satisfying ($\star$). As before we denote by $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ the Poincaré map associated with a common repelling fixed point $a \in J_2$ and Λ_f, Λ_g the corresponding triangular lifts.

Lemma 2.6.1. *Let m be the maximal integer such that μ is m-laminated in some open set of $\mathbb{P}^2$. Suppose that $m = 3$. Then*

1. *The maps Λ_f and Λ_g are linear and diagonal. In particular, the axes $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are invariant.*
2. *The group $\mathcal{G}$ generated by $\Lambda_f \circ \Lambda_g^{-1}$ is one-dimensional and after a linear change of coordinates of $\mathbb{C}^2$ the connected component of the identity $\mathcal{G}_0$ contains a subgroup of the form $\Lambda^t = \left(\begin{smallmatrix} e^{it} & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. Let S_0 the leaf through 0 coming from the lamination of μ^*. It is a real analytic hypersurface and since Λ_f and Λ_g preserve μ^* up to a constant factor, S_0 is invariant by Λ_f and Λ_g. Indeed, if S_0 is the only leaf then it coincides with the support of μ^*, so it must be preserved by Λ_f and Λ_g. If there is another leaf different from S_0, there are infinitely many leaves accumulating around S_0. In this case, if $\Lambda_f(S_0)$ or $\Lambda_g(S_0)$ is different from S_0 then they must cut the nearby leaves transversally, which is not possible by Lemma 2.5.3.

After a linear change of coordinates that does not change the triangular form of Λ_f and Λ_g we may assume that $H = T_0 S_0 = \{\text{Im } z_2 = 0\}$. Since Λ_f and Λ_g fix 0 and leave S_0 invariant, their derivatives preserve H. This implies that λ_2 and γ_2 are real. Since $|\lambda_2| = |\gamma_2|$ from Corollary 2.5.6, we have that $\lambda_2 = \pm\gamma_2$. We can now follow the arguments as in the the proof of Lemma 2.5.7 and conclude. □

The following result is implicitly contained in [Din01].

Theorem 2.6.2 (Dinh)**.** *Let f be a polynomial endomorphism of degree $d \geq 2$ of $\mathbb{C}^k$ that extends to a holomorphic endomorphism of $\mathbb{P}^k$. Suppose that there is an open set Ω such that $J_k \cap \Omega$ is a real analytic hypersurface of Ω. Then there is a repelling periodic point $a \in J_k \cap \Omega$ at which J_k is strictly pseudoconvex and there is a suitable coordinate system of $\mathbb{C}^k$ in which the Poincaré map associated with a has the form $\varphi(z', z_k) = \nu_1(z') \exp(2\pi i z_k)$ for some $\nu_1 : \mathbb{C}^{k-1} \to \mathbb{C}^k$. Furthermore, the map f is homogeneous and the restriction of f to the hyperplane $\mathbb{P}^k \setminus \mathbb{C}^k \simeq \mathbb{P}^{k-1}$ is a Lattès map.*

The above result shows that, in order to prove Theorem 2.1.9 in the polynomial case, it suffices to show that in some open set the lamination of J_2 contains only one leaf. This was also done in [Din01]. We give here however an alternative proof in the setting of commuting pairs of $\mathbb{C}^2$, since some of the arguments involved will be used later.

Proposition 2.6.3. *Let $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ be commuting polynomial endomorphisms of degree d that extend holomorphically to $\mathbb{P}^2$. Suppose that the maximal integer m such that $J_2 = J_2(f) = J_2(g)$ is m-laminated in some open set is $m = 3$. Then there is an open set Ω such that $J \cap \Omega$ is an analytic hypersurface. In other words, there is only one leaf of J_2 intersecting Ω.*

Proof. Let Ω be an open set where J_2 is 3-laminated. Notice that, after shrinking Ω if necessary, we may assume that every leaf in Ω is a strictly pseudoconvex hypersurface. Indeed, if there is one leaf in Ω that is stricly pseudoconvex at some point, then the nearby leaves are strictly pseudoconvex by continuity. If this is not the case then all the leaves in Ω are Levi-flat hypersurfaces, which are foliated by complex curves. In particular J_2 would contain a complex curve passing through a repelling periodic point, which is impossible (Proposition 2.2.6).

Fix a common repelling periodic point $a \in \Omega \cap J_2$. We may replace f and g by suitable iterates and assume that a is fixed. Let $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$ be the associated Poincaré map and Λ_f and Λ_g the corresponding triangular lifts. Similar arguments as in the proof of Lemma 2.6.1 show that Λ_f and Λ_g preserve the leaves of J^*.

Lemma 2.6.4. *Up to a holomorphic change of coordinates that does not change the diagonal form of Λ_f, Λ_g, the leaves are given by $S_c = \{\mathrm{Im}\ z_2 = |z_1|^2 + c\}$, where c is the leaf passing through $(0, ic)$. Furthermore $\lambda_2 = |\lambda_1|^2$ and $\gamma_2 = |\gamma_1|^2$.*

Proof. Keeping the notation of Lemma 2.6.1 we let S_0 be the leaf through 0, fixing coordinates such that $H = T_0 S_0 = \{\mathrm{Im}\ z_2 = 0\}$ and such that Λ_f and Λ_g are diagonal. Choose t such that $\Lambda := \Lambda_f \circ \Lambda^t = \left(\begin{smallmatrix} \kappa & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ with $\kappa, \lambda_2 \in \mathbb{R}$.

We will first show that S_0 has the desired form. Locally, we can write S_0 as a graph $\mathrm{Im}\ z_2 = h(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2)$, where h is a real analytic function vanishing to order two at 0. Since S_0 is strictly pseudoconvex at 0, the Taylor expansion of h contains a nontrivial degree two homogeneous polynomial $P_2(z_1, \overline{z}_1)$ as one of its terms. The invariance of S_0

by Λ gives $h \circ \Lambda = \lambda_2 \mathrm{Im}\ z_2 = \lambda_2 h$. Comparing each term of the Taylor expansion of h we can see that $\kappa^2 = \lambda_2$ and that $P_2(z_1, \overline{z}_1)$ is the only nonzero term. Therefore S_0 is given by $\mathrm{Im}\ z_2 = P_2(z_1, \overline{z}_1) = \alpha z_1^2 + \beta |z_1|^2 + \delta\, \overline{z}_1^2$. The invariance of S by Λ^t implies that $\alpha = \delta = 0$ and the fact that S_0 is strictly pseudoconvex guarantees that $\beta \neq 0$. Therefore, after the change of coordinates $z_1 \mapsto z_1/\sqrt{\beta}$ the leaf S_0 is given by $\mathrm{Im}\ z_2 = |z_1|^2$.

In order to clarify the notation let us denote $\lambda := \lambda_2$. From the above paragraph $\kappa = \sqrt{\lambda}$, so $\Lambda = \left(\begin{smallmatrix} \sqrt{\lambda} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$.

Since J^* is 3-laminated there exists a neighborhood of 0 such that the S_c form a disjoint union of graphs given by equations of the form $\mathrm{Im}\ z_2 = h_c(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2)$, where c varies on some closed subset $K \subset \mathbb{R}$. Here, the functions h_c are real analytic and depend analytically on c. In particular the coefficients of h_c are locally bounded uniformly in c. The value c is determined by the intersection of S_c with the real line $\{z_1 = 0, \mathrm{Re}\ z_2 = 0\}$, that is, $c = h_c(0)$. From the above discussion we know that $h_0(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2) = |z_1|^2$.

Since Λ takes $(0, ic)$ to $(0, i\lambda c)$, it maps S_c to $S_{\lambda c}$. This implies that

$$h_{\lambda c}(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2) = \lambda h_c(\lambda^{-1/2}\mathrm{Re}\ z_1, \lambda^{-1/2}\mathrm{Im}\ z_1, \lambda^{-1}\mathrm{Re}\ z_2). \tag{2.6.1}$$

Consider the Taylor expansion

$$h_c(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2) = c + P_{2,c}(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1) + P_{1,c}(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1)\mathrm{Re}\ z_2 + \alpha_c(\mathrm{Re}\ z_2)^2 + \text{h.o.t},$$

where $P_{j,c}$ are homogeneous of degree j. Equation (2.6.1) implies that $P_{1,c/\lambda} = \sqrt{\lambda} P_{1,c}$. Iterating we get $P_{1,c/\lambda^n} = \lambda^{n/2} P_{1,c}$ for every $n \geq 0$. This shows that $P_{1,c} = 0$ for every c, otherwise the coefficients of $P_{1,c}$ would go to infinity when c approaches 0, which is absurd since the h_c depend analytically on c. A similar reasoning shows that all other higher order terms must vanish. Therefore, the only term left is $P_{2,c}$. Since S_c is invariant by Λ^t, we have $P_{2,c}(z_1) = \gamma(c)|z_1|^2$ for some continuous function γ. Using equation (2.6.1) once again we get $\gamma(c) = \gamma(c/\lambda^n)$ for every c. Making $n \to \infty$ we see that γ is constant and equal to $\gamma(0) = 1$. This proves that the S_c have the desired form. □

We are now in position to finish the proof. Observe that $J^* \cap \{z_2 = 0\} = \{0\}$. Indeed, if there exists $(\xi, 0) \in J^* \cap \{z_2 = 0\}$ with $\xi \neq 0$, the invariance of J^* by Λ^t implies that $J^* \cap \{z_2 = 0\}$ contains the circle $\{|z_1| = |\xi|\} \times \{0\}$. In particular the non-negative subharmonic function $G^*(z_1, 0)$ vanishes for $|z_1| = |\xi|$ and by the maximum principle it vanishes for $|z_1| \leq |\xi|$. The invariance of G^* by Λ_f implies that $G^*(z_1, 0) \equiv 0$ and hence G vanishes over the entire curve $\varphi(\{z_2 = 0\})$, yielding a contradiction, since the set $\{G = 0\}$ is compact.

We now claim that $S = S_0$ is the only leaf of J^*. Suppose that J^* contains a leaf S_c with $c > 0$ and let $w = (0, ic) \in S_c$. Consider the cylinders $C_t = \{\mathrm{Im}\ z_2 = t\} \cap S_0 = \{(\xi, x + it) : |\xi|^2 = t, x \in \mathbb{R}\}$ and their inner sides $D_t = \{(\xi, x + it) : |\xi|^2 < t, x \in \mathbb{R}\}$. The

C_t are non-empty for $t \in (c-\varepsilon, c+\varepsilon)$ if $\varepsilon > 0$ is small. In particular G^* vanishes over C_t and by the maximum principle it must vanish on D_t. We conclude that G^* vanishes over $W := \bigcup_{t\in(c-\varepsilon,c+\varepsilon)} D_t$, which is an open neighborhood of w. This contradicts the fact that $J^* \subset \partial\{G^* = 0\}$.

Suppose now that J^* contains a leaf S_c with $c < 0$. Consider $C'_t = \{\text{Im } z_2 = t\} \cap S_c = \{(\xi, x+it) : |\xi|^2 + c = t, x \in \mathbb{R}\}$ and $D'_t = \{(\xi, x+it) : |\xi|^2 + c < t, x \in \mathbb{R}\}$ for $t \in (-\varepsilon, \varepsilon)$. Using the fact that $G^* = 0$ on S_c and the maximum principle we see that G^* vanishes on the neighborhood of the origin $W' := \bigcup_{t\in(-\varepsilon,\varepsilon)} D'_t$. This is impossible since $0 \in J^* \subset \partial\{G^* = 0\}$. □

Theorem 2.6.5. *Let f and g be two commuting polynomial endomorphisms of $\mathbb{C}^2$ of degree $d_f, d_g \geq 2$ that extend holomorphically to $\mathbb{P}^2$. Assume that $f^n \neq g^m$ for all $n, m \geq 1$ and that $d_f^k = d_g^\ell$ for some $k, \ell \geq 1$. Then, in suitable coordinates, f and g are homogeneous and the induced maps on the hyperplane at infinity are Lattès maps of $\mathbb{P}^1$.*

Proof. Notice that f^k and g^ℓ are commuting endomorphisms of the same degree. By Proposition 2.6.3 and from the fact that the Julia set of f and g are the same as those of f^k and g^ℓ we see that there is a nonempty subset Ω of $\mathbb{P}^2$ such that $J_2 \cap \Omega$ is a real analytic hypersurface. From Theorem 2.6.2 there is a coordinate system on $\mathbb{C}^2$ such that f is homogeneous and the restriction of f to the line at infinty L_∞ is a Lattès map. Notice that $\mathcal{E}_f = \{0\} \cup L_\infty$, so $\mathcal{E}_g = \{0\} \cup L_\infty$ from Proposition 2.2.9.

From the form of the Poincaré map given by Theorem 2.6.2 we see that the lines through the origin are given by the images of the parallel lines $\{z_1 = constant\}$ by φ. Since Λ_g preserves this family we see that g preserves the family of lines through the origin, so g is also homogenous in these coordinates. As $g|_{L_\infty}$ commutes with $f|_{L_\infty}$ we get from Corollary 2.2.11 that $g|_{L_\infty}$ is a Lattès map. □

2.6.2 Ruling out the non-polynomial case

We consider now a commuting pair of holomorphic endomorphisms $f, g : \mathbb{P}^2 \to \mathbb{P}^2$ having same degree $d_f = d_g = d \geq 2$ and satisfying $(\star)$.

As above, let m be the maximal integer such that μ is laminated in some open subset of $\mathbb{P}^2$. We will prove that if $m = 3$ then both f and g are polynomial maps after some number of iterations. We thus fall into the case treated in the last section.

Theorem 2.6.6. *Let $f, g : \mathbb{P}^2 \to \mathbb{P}^2$ be commuting endomorphisms of degree d such that $f^n \neq g^m$ for all $n, m \geq 1$ and let μ be their common Green measure. Suppose that there is an open set Ω of $\mathbb{P}^2$ such that μ is 3-laminated in Ω and that μ is nowhere 4-laminated (i.e. smooth). Then f and g are polynomial after some number of iterations.*

Recall from Theorem 2.2.5 that if the exceptional set $\mathcal{E}$ of a holomorphic endomorphism f of $\mathbb{P}^2$ is positive-dimensional then the one-dimensional part of $\mathcal{E}$ is a union of 1, 2

or 3 lines. In this case f^3 possess a totally invariant line, so it must be a polynomial endomorphism. Therefore, in order to prove Theorem 2.6.6, we need to show that under the above hypothesis, the common exceptional set $\mathcal{E}$ of f and g is positive-dimensional.

The proof will be divided in two parts. In the first part we will show that the leaves coming from the lamination of μ^* cannot be Levi-flat. This leaves us to the case where the leaves are strictly pseudoconvex at a generic point, which will be considered in section 2.6.2.

Remark 2.6.7. The leaves of the lamination of μ and J are either all Levi-flat or all strictly pseudoconvex at a generic point. Indeed, suppose that there is one Levi-flat leaf $\mathcal{L}$ in Ω and take a point $a \in \mathcal{L} \setminus \mathcal{E}$. If b any point in J, the equidistribution of pre-images (Theorem 2.2.4) says that there is a point b' in a small neighborhood of b that is mapped to a by some iterate f^{n_0}. Since $(f^{n_0})^*\mu = d^{2n_0}\mu$, the map f^{n_0} must preserve the leaves, so it maps a piece of the leaf $\mathcal{L}'$ through b' to $\mathcal{L}$. Therefore $\mathcal{L}'$ is Levi-flat in a neighborhood of b'. Recall that $\mathcal{L}'$ is real analytic. Since b' can be taken arbitrarily close to b we have, by continuity, that the leaf through b is Levi-flat.

Lamination by Levi-flat hypersurfaces

The aim of this section is to show that the Green measure cannot be laminated by Levi-flat hypersurfaces.

Let f, g, μ and Ω be as in Theorem 2.6.6. Pick a common repelling periodic point $a \in \Omega \cap J$ and consider the corresponding Poincaré map $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$.

Lemma 2.6.8. *Suppose that μ is laminated by Levi-flat hypersurfaces. Then the leaves of the lamination of $\mu^* = \varphi^*\mu$ are given by $S_c = \{\mathrm{Im}\ z_2 = c\}$ where c varies on a closed set $K \subset \mathbb{R}$.*

Proof. Let us first show that the central leaf is $S_0 = \{\mathrm{Im}\ z_2 = 0\}$. We can choose coordinates as in Lemma 2.6.1 such that $H = T_0 S_0 = \{\mathrm{Im}\ z_2 = 0\}$, $\Lambda_f(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$ with $\lambda_2 \in \mathbb{R}$ and $\Lambda^t = (e^{it} z_1, z_2)$. We can show as in the proof of Lemma 2.6.1 that S_0 is invariant by Λ_f and Λ_g. In particular, S_0 is invariant by $\Lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ for some $\lambda \in \mathbb{R}$.

We can write S_0 as a graph of the form $\mathrm{Im}\ z_2 = h(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2)$ for some real analytic function h vanishing to order 2 at 0. The invariance of S_0 by Λ^t implies that $h \circ \Lambda^t = h$. Comparing the terms of the Taylor expansion of both sides we see that h cannot contain any monomial of the form $z_1^n \bar{z}_1^m (\mathrm{Re}\ z_2)^k$ with $n \neq m$. We can write then $h(\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \mathrm{Re}\ z_2) = \psi(|z_1|^2, \mathrm{Re}\ z_2)$ for some real analytic function ψ that vanishes at the origin. The invariance of S_0 by Λ implies that $\psi \circ \Lambda = \lambda_2 \psi$. Using this relation and comparing each term of the Taylor expansion of ψ it is possible to show that ψ has the form $\psi = \alpha |z_1|^{2\ell}$ for some $\alpha \in \mathbb{R}$, if there is an integer ℓ satisfying $\lambda^{2\ell} = \lambda_2$ or $\psi \equiv 0$ if no such ℓ exists. Since S_0 is Levi-flat the first case is only possible if $\alpha = 0$, so $\psi \equiv 0$ also in this case. This shows that S_0 has de desired form.

We can follow similar arguments as in the end of the proof of Lemma 2.6.4. In the present case we get that $S_c = \{\text{Im } z_2 = c\}$. □

Proposition 2.6.9. *Let f, g, μ and Ω as in Theorem 2.6.6. Then the leaves of the lamination of μ in Ω are strictly pseudoconvex at a generic point.*

Proof. Suppose by contradiction that there is a Levi flat leaf $\mathcal{L} \subset J \cap \Omega$ and take a repelling periodic point $a \in \mathcal{L}$. Consider the associated Poincaré map $\varphi : \mathbb{C}^2 \to \mathbb{P}^2$. From Remark 2.6.7 all the leaves are Levi-flat, so from Lemma 2.6.8 we have $J^* = \varphi^{-1}(J_2) = \cup_{c \in K}\{\text{Im } z_2 = c\}$.

Claim: We have $J^* = \mathbb{C}^2$ and $J_2 = \mathbb{P}^2$.

Since $\mathcal{L}$ is foliated by complex curves, Proposition 2.3.3 implies that $J_1 = J_2$. If there is a $c_0 \in \mathbb{R} \setminus K$ then $\varphi(\{z_2 = ic_0\})$ is an entire curve contained in $\mathbb{P}^2 \setminus (\mathcal{E} \cup J_1)$, which is in turn contained in the Fatou set of f. This is absurd because the Fatou set is hyperbolic [DS10]. We must have then $K = \mathbb{R}$ and $J^* = \mathbb{C}^2$. Since J_2 is closed we have $J_2 = \overline{\varphi(\mathbb{C}^2)} = \mathbb{P}^2$.

Considering the groupoid $\mathcal{A}$ of local biholomorphic maps acting on the fibers of φ (as in the proof of Lemma 2.5.8) we see that its elements preserve the real foliation $\{\text{Im } z_2 = cst.\}$, so they must also preserve the holomorphic foliation $\{z_2 = cst.\}$. If we let A be complement of the image the of regular values of φ we get then an f-invariant non-singular foliation $\mathcal{F}$ on $\mathbb{P}^2 \setminus A$. This is impossible by Lemma 2.5.10 (since $J_2 = \mathbb{P}^2$ in our case). This gives the contradiction we were looking for. □

Lamination by strictly pseudoconvex hypersurfaces

In this section we suppose that the Green measure is 3-laminated by leaves that are strictly pseudoconvex at a generic point.

Observe that on the proof Lemma 2.6.4 have not used the fact that the maps are polynomial. We only needed the strict pseudoconvexity of the central leaf at 0. By assumption we can choose a repelling periodic point $a \in \Omega \cap J$ at which the leaf that passes through it is strictly pseudoconvex. From this fact and Lemmas 2.6.1 and 2.6.4 we can assume, after a holomorphic change of coordinates and taking iterates, that:

a. The closed Lie group generated by $\Lambda_f \circ \Lambda_g^{-1}$ contains $\Lambda^t = \left(\begin{smallmatrix} e^{it} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as a 1-parameter subgroup.
b. The maps Λ_f and Λ_g are linear and diagonal.
c. The eigenvalues of Λ_f and Λ_g satisfy $|\lambda_1|^2 = |\gamma_1|^2 = \lambda_2 = \gamma_2$.
d. The leaves of the lamination of μ^* are given by $S_c = \{\text{Im } z_2 = |z_1|^2 + c\}$, where the parameter c varies on a closed set $K \subset \mathbb{R}$.

As in the previous section, we set $\lambda = |\lambda_1|^2$ and $\Lambda = \left(\begin{smallmatrix} \sqrt{\lambda} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Since $\Lambda = \Lambda_f \circ \Lambda^\theta$ for some $\theta \in \mathbb{R}$ we have the invariance relations $\Lambda^* T^* = d \cdot T^*$, $\Lambda^* \mu^* = d^2 \cdot \mu^*$ and $\Lambda(J^*) = J^* = \Lambda^{-1}(J^*)$.

The following result is classical. It follows for instance from [Ale74] and the classification of automorphisms of the unit ball of $\mathbb{C}^2$, which can be found in [Rud08].

Proposition 2.6.10. *Let W be an open subset of $\mathbb{C}^2$ intersecting J^* and $\tau : W \to \tau(W)$ a holomorphic map preserving the family $S_c = \{\mathrm{Im}\ z_2 = |z_1|^2 + c\}$, $c \in K$.*

1. *If $0 \in W$ and $\tau(0) = 0$ then τ is linear of the form $\tau = \left(\begin{smallmatrix} \sqrt{\delta}e^{i\rho} & 0 \\ 0 & \delta \end{smallmatrix}\right)$ for some $\delta > 0$ and some $\rho \in \mathbb{R}$.*
2. *If $0 \in W$ and $\tau(0) = (0, ic)$ then τ is affine of the form $\tau = t_{ic} \circ A$ where $A = \left(\begin{smallmatrix} \sqrt{\delta}e^{i\rho} & 0 \\ 0 & \delta \end{smallmatrix}\right)$ for some $\delta > 0$ and some $\rho \in \mathbb{R}$ and $t_a : (z_1, z_2) \mapsto (z_1, z_2 + a)$.*
3. *If $0 \in W$ then $\tau = \tau_u \circ t_{ic} \circ A$, where A and t_{ic} are as above and τ_u is the Heisenberg map:*
$$\tau_u(z_1, z_2) = (z_1 + u_1, z_2 + 2iz_1\overline{u}_1 + u_2), \tag{2.6.2}$$
with $u = (u_1, u_2) \in S_0$.
4. *In general, if τ takes a point of S_c to a point of $S_{c'}$ we have*
$$\tau = \tau_{u'} \circ t_{ic'} \circ A \circ t_{-ic} \circ \tau_u^{-1},$$
where A, t_{ic} $t_{ic'}$, τ_u, $\tau_{u'}$ are as above with $u, u' \in S_0$.

In particular the map $\tau : W \to \tau(W)$ extends to a globally defined affine map of $\mathbb{C}^2$.

Consider now the groupoid of local biholomorphisms acting on the fibers of φ. More precisely, set

$$\mathcal{A} = \{\tau : \tau \text{ is a local biholomorphism of } \mathbb{C}^2 \text{ such that } \varphi \circ \tau = \varphi\}.$$

Proposition 2.6.11. *The set $\mathcal{A}$ is a discrete subgroup of the group $\mathrm{Aff}(\mathbb{C}^2)$ of affine transformations of $\mathbb{C}^2$. Furthermore $\mathcal{A}$ is contained in the group generated by the following types of maps:*

i) purely imaginary translations along the z_2-axis: $t_{ic}(z_1, z_2) = (z_1, z_2 + ic)$,
ii) rotations around the z_2-axis: $r_\rho(z_1, z_2) = (e^{i\rho}z_1, z_2)$ and
iii) Heisenberg maps: $\tau_u(z_1, z_2) = (z_1 + u_1, z_2 + 2iz_1\overline{u}_1 + u_2)$ with $u = (u_1, u_2) \in S_0$.

Proof. Let $\tau \in \mathcal{A}$. Notice that τ must preserve μ^* so, by similar arguments as in the proof of Lemma 2.6.1, it must preserve the leaf structure of μ^*, i.e. , the family $S_c = \{\mathrm{Im}\ z_2 = |z_1|^2 + c\}$, $c \in K$. From Proposition 2.6.10, τ is a globally defined invertible affine map. This shows that $\mathcal{A}$ is subgroup of $\mathrm{Aff}(\mathbb{C}^2)$. The discreteness of $\mathcal{A}$ follows from the fact that φ has discrete fibers.

Set $S_c = \tau(S_0)$. Then Proposition 2.6.10 says that $\tau = \tau_u \circ t_{ic} \circ A$ with $A = \left(\begin{smallmatrix} \sqrt{\delta}e^{i\rho} & 0 \\ 0 & \delta \end{smallmatrix}\right)$. To conclude the proof we must show that $\delta = 1$.

Suppose that $\delta \neq 1$. Replacing τ by τ^{-1} if necessary we may assume that $\delta < 1$. In this case the map τ has a fixed point $z \in \mathbb{C}^2$ which is attracting. In particular, there is an orbit $\tau^n(w)$ converging to z. This is absurd because the fibers of φ are discrete. $\square$

We are now in position to give the proof of Theorem 2.6.6.

Proof of Theorem 2.6.6. From the remarks made after the statement of Theorem 2.6.6 the conclusion will follow if we show that $\dim \mathcal{E} = 1$.

By Proposition 2.6.9, we can pick a repelling periodic point at which the leaf passing through it is strictly pseudoconvex. From the considerations made at the beginning of this section we have that the leaves of the lamination of μ^* are given by $S_c = \{\mathrm{Im}\ z_2 = |z_1|^2 + c\}$, where the parameter c varies on a closed set $K \subset \mathbb{R}$. There are two cases to consider.

Case I: The lifted Julia set J^* consists of only one leaf, that is, $K = \{0\}$.

In this case the group $\mathcal{A}$ from Proposition 2.6.11 is generated only by rotations r_ρ and Heisenberg maps τ_u. Therefore the p.s.h. function $v(z_1, z_2) = |z_1|^2 - \mathrm{Im}\ z_2$ on $\mathbb{C}^2$ is invariant by $\mathcal{A}$, so it descends to a p.s.h. function u on $\mathbb{P}^2 \setminus \mathcal{E}$. If $\dim \mathcal{E} = 0$ then u extends to a p.s.h. function on $\mathbb{P}^2$, which is absurd because $\mathbb{P}^2$ is compact. This contradiction shows that $\dim \mathcal{E} = 1$.

Case II: The lifted Julia set J^* contains more than one leaf, that is, K contains at least 2 points.

Consider the p.s.h. function $v(z_1, z_2) = |z_1|^2 - \mathrm{Im}\ z_2$ on $\mathbb{C}^2$. It satisfies $t_{ic}^* v = v + c$, $\tau_u^* v = v$ and $r_\rho^* v = v$. These relations show that the positive closed $(1,1)$-current $S^* = \mathrm{dd}^c v$ is invariant under t_{ic}, r_ρ and τ_u. Proposition 2.6.11 implies then that S^* is invariant by the whole group $\mathcal{A}$, so it descends to a current S in $\mathbb{P}^2 \setminus \mathcal{E}$. More precisely, there is a positive closed $(1,1)$-current S on $\mathbb{P}^2 \setminus \mathcal{E}$ such that $\varphi^* S = S^*$.

Suppose that $\dim \mathcal{E} = 0$. Then S extends to a positive closed $(1,1)$-current on $\mathbb{P}^2$, which we still denote by S.

Claim. The eigenvalues of Λ_f and Λ_g satisfy $|\lambda_1|^2 = |\gamma_1|^2 = \lambda_2 = \gamma_2 = d$. The current S satisfies $f^* S = d \cdot S$.

Firstly we observe that $f^* S = \lambda \cdot S$ on $\mathbb{P}^2 \setminus \mathcal{E}$. Indeed, we have the relation $\Lambda_f^* S^* = \lambda \cdot S^*$, which gives

$$\varphi^* f^* S = \Lambda_f^* \varphi^* S = \Lambda_f^* S^* = \lambda \cdot S^* = \varphi^*(\lambda \cdot S),$$

so $f^* S = \lambda S$ on $\mathbb{P}^2 \setminus \mathcal{E}$. Now, the $(1,1)$-current $f^* S - \lambda S$ is supported by $\mathcal{E}$, so it must be identically zero. Therefore $f^* S = \lambda S$ on $\mathbb{P}^2$. On one hand we have $\|f^* S\| = \|\lambda S\| = \lambda \|S\|$. On the other hand, since f^* acts by multiplication by d on $H^{1,1}(\mathbb{P}^2)$ we have $\|f^* S\| = d\|S\|$. Hence $\lambda = d$, proving the claim (recall that $\lambda = |\lambda_1|^2$).

After multiplying S by a constant we may assume that S is of mass 1. Write $T = S - \mathrm{dd}^c u$, where T is the Green current of f and u is a p.s.h. function modulo T defined up to an additive constant. We may normalize it so that $\int u \, d\mu = 0$. Since T is of bounded

local potential, the function u is locally bounded from above. The invariance of S and T implies that $u \circ f^n = d^n \cdot u$, so we must have $u \leq 0$. The chosen normalization implies that $u = 0$ on J_2.

Consider the function $u^* = u \circ \varphi$. It vanishes on J^* and satisfies $T^* = S^* - \mathrm{dd}^c u^*$. Notice that $S^* = \mathrm{dd}^c(|z_1|^2 - \mathrm{Im}\ z_2) = \mathrm{dd}^c|z_1|^2 = \frac{i}{\pi} dz_1 \wedge d\bar{z}_1$ up to a constant multiple. In particular S^* restricts to zero on $L_0 = \{z_1 = 0\}$. We have then that $T^*|_{L_0} = -\mathrm{dd}^c u_0^*$, where $u_0^*(z_2) = u^*(0, z_2)$. The function u_0 is then a continuous negative superharmonic function on L_0 satisfying $u_0^*(dz_2) = d \cdot u_0^*(z_2)$ and vanishing on the lines $\{\mathrm{Im}\ z_2 = c\}, c \in K$. Lemma 2.6.12 below and the fact that K contains more then one point imply that $u_0^* \equiv 0$. Therefore u^* vanishes over L_0 and u vanishes over $\varphi(L_0)$. Since $u \leq 0$ and is upper semi-continuous it vanishes on $\overline{\varphi(L_0)}$.

The restriction of the Poincaré map to L_0 gives an entire curve $\varphi|_{L_0} : L_0 \simeq \mathbb{C} \to \mathbb{P}^2$. The Ahlfors construction (Theorem 2.3.2) produces a positive closed $(1,1)$-current R on $\mathbb{P}^2$ that is supported in $\overline{\varphi(L_0)}$. Consider the current $R^* = \varphi^* R$. It is a limit of currents R_n supported in $\varphi^{-1}(\varphi(\Delta_n)) = \cup_{\tau \in \mathcal{A}} \tau(\Delta_n)$, where Δ_n is an increasing sequence of discs contained in L_0. By Proposition 2.6.11 this set is a discrete union of discs contained in vertical lines $\{z_1 = c_\tau\}$, $\tau \in \mathcal{A}$. In particular $i dz_1 \wedge d\bar{z}_1 \wedge R_n = 0$ for every n, so $i dz_1 \wedge d\bar{z}_1 \wedge R = 0$. Therefore

$$\varphi^*(S \wedge R) = C^{st} \cdot i dz_1 \wedge d\bar{z}_1 \wedge R^* = 0,$$

so the measure $S \wedge R$ is supported by $\mathcal{E}$.

On the other hand we have seen that u vanishes on $\overline{\varphi(L_0)}$. In particular, u vanishes on the support of R. This implies that $T \wedge R = S \wedge R$. The above discussion shows that $T \wedge R$ is supported by $\mathcal{E}$. This gives a contradiction, because, since T has Hölder continuous local potentials, the measure $T \wedge R$ cannot charge points (see [Sib99]). □

The following result can be found in [DS02], Lemme 5.3.

Lemma 2.6.12. *Let $v \geq 0$ be a continuous subharmonic function defined on the closed upper half-plane $\overline{\mathbb{H}}$. Suppose that v vanishes on $\{\mathrm{Im}\ z = 0\}$ and that there is a $d > 1$ such that $v(dz) = d \cdot v(z)$ for every $z \in \overline{\mathbb{H}}$. Then there is a constant $\beta \geq 0$ such that $v(z) = \beta \mathrm{Im}\ z$.*

2.7 Proof of the Main Theorem

We are able now to give a proof of Theorem 2.1.9.

Proof of Theorem 2.1.9. Let f and g be commuting holomorphic endomorphisms of $\mathbb{P}^2$ such that $f^n \neq g^m$ for all $n, m \geq 1$ and that $d_f^k = d_g^\ell$ for some $k, \ell \geq 1$. Denote by J_2 the common Julia set of f and g and by μ their Green measure. Notice that f^k and g^ℓ are

commuting endomorphisms of same degree. By Proposition 2.5.4 and from the fact that the Julia set and the Green measure of f and g are the same as those of f^k and g^ℓ we see that J_2 and μ are laminated in some open subset of $\mathbb{P}^2$. Furthermore, the maximal m such that J_2 and μ are m-laminated in some open subset Ω of $\mathbb{P}^2$ is $m = 3$ or 4.

If $m = 3$, Theorem 2.6.6 implies that f^N and g^M are polynomial for some $N, M \geq 1$, and Theorem 2.6.5 shows that they are lifts of Lattès maps of $\mathbb{P}^1$. In particular, the common exceptional set of f and g, which is the same as the exceptional set of f^N and g^M, is a union of a point and a line. This shows that f and g are also polynomial, so by Theorem 2.6.5 again, f and g are lifts of Lattès maps of $\mathbb{P}^1$.

If $m = 4$, the Green measure is analytically equivalent to the Lebesgue measure in Ω so, by Theorem 2.2.10, f and g are Lattès maps. $\square$

Chapitre 3

Cycles Feuilletés

In this chapter we study foliated cycles on complex manifolds. These are positive closed currents give locally by an average of currents of integration along disjoint holomorphic graphs. We prove that the cohomological self-intersection of such currents vanish as long as they do not contain integration currents along compact manifolds. This result allows us to deduce the non-existence of foliated cycles of low codimension and without compact leaves on projective spaces. We also apply it to describe explicitly the Lipschitz foliated cycles in certain manifolds such as complex tori and Hirzebruch surfaces. The results of this chapter are presented in [Kau16b].

3.1 Introduction

Let X be a compact complex manifold and $\mathcal{L}$ be a lamination of dimension q embedded in X (see Section 1.2.2 for the definition and some examples). When $q = 1$ we say that $\mathcal{L}$ is a *Riemann surface lamination* on X. These objects have been considered by numerous authors and they have connections to many other branches of Geometry and Dynamics such as complex dynamical systems of several variables and foliation theory. For an account of the subject the reader may consult the surveys [FS08], [Ghy99] and the reference therein.

Given an embedded lamination $\mathcal{L}$ of dimension q, a foliated cycle T directed by $\mathcal{L}$ is a positive closed current on X that is locally given by an average of currents of integration along the plaques with respect to a positive measure μ (see section 1.2.2). We say that T is diffuse if the measure μ has no atoms. This is equivalent to the fact that T has no mass on submanifolds of dimension q (see Remark 3.2.5). Alternatively, a foliated cycle may be described as a holonomy invariant transverse measure.

From a geometric point of view it is expected that the self-intersection of a diffuse foliated cycle should be zero. For the homological intersection this was proved by Hurder–Mitsumatsu, [HM91] in the context of (real) foliations of arbitrary dimension and codimension. For complex laminations, the main results concern one-dimensional cycles on

complex surfaces.

The vanishing of the self-intersection indicates that the existence of a foliated cycle is quite special. As a matter of fact, many laminations carry no foliated cycle at all. For laminations in $\mathbb{P}^2$ this was first noted by Camacho–Lins-Neto–Sad in [CLNS88], where they considered cycles supported by minimal sets of holomorphic foliations. For transversally smooth foliations the result follows from Hurder–Mitsumatsu's Theorem and for Lipschitz laminations this is due to Fornæss and Sibony (see [FS05] and [FS08]).

Our main result concerns the self–intersection of diffuse foliated cycles of arbitrary dimension and codimension on a compact Kähler manifold.

Theorem 3.1.1. *Let X be a compact Kähler manifold of dimension n and let T be a diffuse foliated cycle of dimension q directed by an embedded transversally Lipschitz lamination on X. If $\{T\} \in H^{n-q,n-q}(X)$ denotes its cohomology class then $\{T\} \smile \{T\} = 0$ in $H^{2n-2q,2n-2q}(X)$.*

The proof uses a notion of intersection of currents introduced by Dinh and Sibony in [DS14], which is denoted by the symbol $\curlywedge$. Our arguments also show that $T \curlywedge T = 0$ if $q \geq \frac{n}{2}$, see Section 3.4.2. This is related to a theorem of Dujardin [Duj04] saying that for a foliated cycle T defined on an open set of $\mathbb{C}^2$, the wedge product $T \wedge T$ vanishes as long as it is well-defined in the usual sense of Pluripotential Theory. Our theorem extends this result in two ways, firstly it works in any dimension and codimension (as long as $q \geq \frac{n}{2}$) and secondly, no a priori regularity of T is assumed. Nevertheless we need the underlying lamination to be transversally Lipschitz and the current T to be globally defined.

As an application of Theorem 3.1.1 we can show that laminations of low codimension on certain Kähler manifolds carry no diffuse foliated cycles (see section 3.5 for other examples).

Corollary 3.1.2. *Let $\mathcal{L}$ be an embedded transversally Lipschitz lamination of dimension $q \geq n/2$ on $\mathbb{P}^n$. Then there is no diffuse foliated cycle directed by $\mathcal{L}$. In particular, if $\mathcal{L}$ has no compact leaves then there is no foliated cycle directed by $\mathcal{L}$.*

The presence of a foliated cycles imposes strong conditions on the dynamics of the lamination and as noted above their existence is quite rare. Nevertheless, every lamination by Riemann surfaces carries a *directed harmonic current*, this is a positive dd^c- closed current directed by the leaves of $\mathcal{L}$ (see [FS08] for more details). This was proven in [Gar83] for foliations, in [BS02] for continuous laminations with singularities and in [Sib15] in greater generality. However, there are examples of laminations of dimension two that do not carry any directed harmonic current, see [FSW11]. We should also notice that the non-existence of foliated cycles for Riemann surface laminations can be used in some cases to show that directed harmonic currents are unique. This can be viewed as a type of unique ergodicity in this setting, see [FS05], [DS15] and also [PG14].

The proof of Theorem 3.1.1 consists of an application of the theory of densities of positive closed currents of Dinh and Sibony, [DS14]. This notion can be used to define the intersection of positive closed currents in a quite general setting and it seems to be a well suited tool to deal with intersections of directed currents (see [DS15] for a recent application to foliations in $\mathbb{P}^2$).

This chapter is organized as follows. In Section 3.2 we set some notation and state the basic facts about laminations and foliated cycles that we will use. Section 3.3 is devoted to a brief overview of the theory of densities. The main points are the fact that density classes encode the cohomological intersection (see Proposition 3.3.6) and that representatives of these classes can be computed using local coordinates, see Theorem 3.3.4. Therefore, Theorem 3.1.1 can be proved using the local expression of a foliated cycle in a flow box, which is done in Section 3.4. Finally, in Section 3.5 we apply our theorem to some concrete examples.

3.2 Preliminaries

3.2.1 Laminations by complex manifolds

Let X be a compact complex manifold. Recall from Section 1.2.2 that an embedded lamination *lamination* $\mathcal{L}$ by complex manifolds of dimension q on X is an atlas of homeomorphisms $\varphi_i : U_i \to \mathbb{D}^q \times \mathbb{D}^{n-q}$, such that the coordinate changes $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ are of the form

$$\varphi_{ij}(z,t) = (f_{ij}(z,t), \gamma_{ij}(t)), \tag{3.2.1}$$

with f_{ij} holomorphic in the first variable.

The open sets U_i are called *flow boxes* and the sets $\Pi_t = \varphi_i^{-1}(\mathbb{D}^q \times \{t\})$ are called *plaques*. A subset $L \subset \Sigma$ is called a *leaf* of $\mathcal{L}$ if it is a minimal connected set with the property that if L meets a plaque Π then $\Pi \subset L$.

The following lemma shows that, in holomorphic coordinates, the plaques are given by disjoint graphs of holomorphic functions.

Lemma 3.2.1. *Let $\mathcal{L}$ be a lamination of dimension q embedded on X. Then for every point p in a flow box U_i there is an open set $V_i \ni p$ and a holomorphic chart $\psi_i : V_i \to \mathbb{D}^q \times \mathbb{D}^{n-q}$ centered at p such that*

$$\psi_i(\Pi_t) = (z', h_t(z')),\ z \in \mathbb{D}^q,$$

where h_t are uniformly bounded holomorphic maps with values in $\mathbb{C}^{n-q}$ depending continuously in t such that $h_t(0) = t$ and $h_0 \equiv 0$.

Sketch of the proof. Let p be a point in a flow box U_i and let $\varphi_i : U_i \to \mathbb{D}^q \times \mathbb{D}^{n-q}$ be a foliated chart centered at p. This is a homeomorphism that restricts to a holomorphic map on each plaque $\Pi_t = \varphi_i^{-1}(\mathbb{D}^q \times \{t\})$. Choose a *holomorphic* chart $\psi_i : V_i \to \mathbb{D}^q \times \mathbb{D}^{n-q}$

centered at p such that $\psi_i(\Pi_0)$ is given by $z'' = 0$, where $z = (z', z'')$ are the standard coordinates in $\mathbb{D}^q \times \mathbb{D}^{n-q}$. The projection on the first coordinate $\mathbb{D}^q \times \mathbb{D}^{n-q} \to \mathbb{D}^q$ restricts to a one–to–one holomorphic map on each $\psi_i(\Pi_t)$, so these sets are given by disjoint holomorphic graphs $(z', f_t(z'))$ varying continuously with t. We can take $h_0 = f_0 \equiv 0$ and $h_t(z') = \psi_t(f_t(z'))$ for $t \neq 0$, where ψ_t is a continuous family of linear isomorphisms of $\mathbb{C}^{n-q}$ mapping $f_t(0)$ to t. □

Remark 3.2.2. The continuity of the h_t in t in the above lemma is with respect to the $\mathcal{C}^\infty$ topology. Notice that, since the h_t are holomorphic, this equivalent to the continuity with respect to the $\mathcal{C}^0$ topology. This can be seen from Cauchy's formula.

In what follows we will also call an open set V_i as in Lemma 3.2.1 a *flow box*.

3.2.2 Foliated cycles

Let $\mathcal{L}$ be a lamination of dimension q embedded on X. Recall from Section 1.2.2 that a *foliated cycle* directed by $\mathcal{L}$ is a positive closed current T on X of bi-dimension (q, q) which in a flow box $\varphi_i : U_i \to \mathbb{D}^q \times K_i$ is given by

$$\langle T, \alpha \rangle = \int_{K_i} \left(\int_{\mathbb{D}^q \times \{t\}} (\varphi_i)_* \alpha \right) \mathrm{d}\mu_i(t),$$

where α is a differential $(q, q)-$form with compact support in U_i. We will say that T is *diffuse* if in every flow box the corresponding measure μ_i is diffuse (i.e. does not charge points).

Remark 3.2.3. A current T as above is sometimes called strongly directed or uniformly laminar. There is also a notion of weakly directed current (cf. [FS08], Definition 6). These notions coincide for foliations due to a classical result of Sullivan, [Sul76]. In [FWW08] the authors show that every weakly directed current is a foliated cycle when $n = 2$. They also give a counter-example in dimension 3.

Example 3.2.4. Suppose that $\mathcal{L}$ has a compact leaf L. Then the current of integration along L is a foliated cycle directed by $\mathcal{L}$. The transverse measures in this case are Dirac masses.

Remark 3.2.5. Conversely, we can show that if in some flow box the transverse measure of a foliated cycle T contains an atom then T "contains" the current of integration along a compact leaf L of $\mathcal{L}$, that is, $T = T' + c[L]$ where T' is a foliated cycle directed by $\mathcal{L}$ and $c > 0$ is a constant. In particular, if a lamination $\mathcal{L}$ has no compact leaves then any foliated cycle directed by $\mathcal{L}$ is diffuse.

Example 3.2.6 (Ahlfors' currents)**.** Suppose that $\mathcal{L}$ is a lamination by Riemann surfaces, that is $q = 1$. If $\mathcal{L}$ admits a parabolic leaf L (i.e., a leaf uniformized by the complex plane)

then the uniformizing map $\phi : \mathbb{C} \to L$ gives rise to foliated cycles directed by $\mathcal{L}$. More precisely, let a_r be the area of $\phi(\mathbb{D}_r)$ and ℓ_r be the length of $\phi(\partial\mathbb{D}_r)$. If r_n is a sequence tending to infinity such that $\lim_{r_n\to+\infty} \frac{\ell_{r_n}}{a_{r_n}} = 0$, then the normalized currents of integration along $\phi(\mathbb{D}_{r_n})$ converge to a closed positive current directed by $\mathcal{L}$. Ahlfors showed that such sequences always exist. We can also produce foliated cycles using different notions of parabolicity (see Remark 3.5.6).

3.3 Density of positive closed currents

We review in this section the basic facts of the theory of densities of positive closed currents introduced by T.–C.Dinh and N. Sibony. This will be the main tool in the proof of Theorem 3.1.1. We refer to [DS14] for the complete exposition.

3.3.1 Tangent currents

Let T be a positive closed current of bi–degree (p,p) defined in a neighborhood of the origin in $\mathbb{C}^n$. The Lelong number of T at 0 is defined by

$$\nu(T,0) = \lim_{r\to 0} \frac{\|T\|_{B(0,r)}}{(2\pi)^{n-p} r^{2n-2p}},$$

where $\|T\| = T \wedge \beta^{n-p}$ is the trace measure with respect to the standard Kähler form β on $\mathbb{C}^n$.

The Lelong number can be characterized by the following geometric construction. Let $A_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $A_\lambda(z) = \lambda z$ for $\lambda \in \mathbb{C}^*$. Then the family of currents $T_\lambda = (A_\lambda)_* T$, $|\lambda| \geq 1$ is relatively compact and any limit current S is defined in all of $\mathbb{C}^n$. We can view S as a current in $\mathbb{P}^n$ by considering its trivial extension through the hyperplane at infinity.

Lemma 3.3.1. *The class of S in $H^{2p}(\mathbb{P}^n, \mathbb{C})$ is equal to $\nu(T,0)$ times the class of a linear subspace of dimension $n-p$.*

Now let T be a positive closed $(p,p)-$current defined on a compact Kähler manifold X and let $V \subset X$ be a compact complex submanifold. Denote by E the normal bundle to V in X and $\overline{E}$ its canonical compactification, defined as $\overline{E} = \mathbb{P}(E \oplus \mathbb{C})$. The multiplication by $\lambda \in \mathbb{C}^*$ along the fibers defines an endomorphism $A_\lambda : E \to E$ that extends trivially to $\overline{E}$.

Inspired by the above discussion we may want to define a tangent of T along V as a limit of T under the dilations A_λ. The main problem is that we cannot transport T to a current in E in a holomorphic way. Nevertheless, there are maps τ from a neighborhood of V in X and a neighborhood of the zero section in E that are diffeomorphisms and whose

differential at V is the identity in the normal direction (see [DS14], Definition 2.14). These maps are called admissible and they suffice to define tangent currents.

Theorem 3.3.2 (Existence of tangent currents, [DS14] Thm 1.1)**.** *Let X,V,T,E,$\overline{E}$, A_λ and τ be as above. Then the family of currents $T_\lambda = (A_\lambda)_*\tau_*T$ on E is relatively compact. If S is any of its cluster values, viewed as a positive closed current in E extended by zero to $\overline{E}$, then its cohomology class in $H^{2p}(\overline{E},\mathbb{C})$ is independent of the choice of τ and S.*

Definition 3.3.3. A current S on $\overline{E}$ as in Theorem 3.3.2 is called a *tangent current* to T along V. The class $\{S\} \in H^{2p}(\overline{E},\mathbb{C})$, which is independent of S, is called the **total tangent class** of T along V and it is denoted by $\kappa^V(T)$.

Local computation of tangent currents

An important feature of the notion of tangent currents is that they can be computed using local coordinates (even though the current T must be globally defined). This property will be crucial in the proof of Theorem 3.1.1.

Let $V \subset X$ as above and denote $\ell = \dim V$ and $n = \dim X$. Let V_0 be a small open set of V and U_0 a small neighborhood of V_0 in X. We may assume that U_0 is biholomorphic to $\mathbb{D}^n = \mathbb{D}^\ell \times \mathbb{D}^{n-\ell}$ via local holomorphic coordinates $x = (x', x'')$ in which V_0 is given by $\{x'' = 0\}$. In these coordinates the normal bundle E to V_0 is holomorphically identified with the trivial $\mathbb{C}^{n-\ell}-$ bundle over $\mathbb{D}^\ell$ and V_0 is identified with its zero section.

Denote by $\tau : U_0 \to \mathbb{D}^\ell \times \mathbb{D}^{n-\ell}$ the above holomorphic chart. Let $A_\lambda(x', x'') = (x', \lambda x'')$, $\lambda \in \mathbb{C}^*$ be the multiplication along the fibers of E and define

$$T_\lambda = \tau^* (A_\lambda)_* \tau_* T.$$

The following result follows from [DS14], Proposition 4.4.

Theorem 3.3.4 (Local computation)**.** *The family $\{T_\lambda, |\lambda| \geq 1\}$ is relatively compact. Moreover, if (λ_n) is a sequence converging to infinity such that $T_{\lambda_n} \to S$, then S is a positive closed $(p,p)-$current on $E|_{V_0}$ independent of the choice of the local chart τ as above.*

3.3.2 Density classes and intersection product

Let X be a compact Kähler manifold of dimension n. Given two positive closed currents T_1 and T_2 of bi–degrees (p_i, p_i), $i = 1, 2$ on X we will define the density of the pair (T_1, T_2). When T_2 is the current of integration along a submanifold V we recover above the notion of tangent current along V.

Let $\mathbf{T} = T_1 \otimes T_2$ be the product current on $X \times X$. Notice that $\mathbf{T}$ is of bi-degree (p,p), where $p = p_1 + p_2$. Denote by $\Delta \subset X \times X$ the diagonal. Let E be the normal bundle to

Δ in $X \times X$. It is a rank n vector bundle and, under the natural isomorphism $\Delta \simeq X$, E identifies with the tangent bundle of X.

Definition 3.3.5. A tangent current to $\mathbf{T}$ along Δ is called a density current associated with T_1, T_2. The tangent class $\kappa(T_1, T_2) := \kappa^{\Delta}(\mathbf{T})$ is called the **total density class** of T_1, T_2.

These ideas can be further developed and culminate in the definition of an intersection product of T_1 and T_2 which is compatible with the cohomological cup product and coincides with classical notions of intersection. The interested reader may consult Section 5 in [DS14]. We state here only a particular case that we will use.

Proposition 3.3.6. *Suppose that $p_1 + p_2 \leq n$. If the total density class $\kappa(T_1, T_2)$ vanishes then $\{T_1\} \smile \{T_2\} = 0$ in $H^{2n-2p_1-2p_2}(X, \mathbb{C})$.*

3.4 Proof of the main theorem

This section is devoted to the proof of Theorem 3.1.1. By Theorem 3.3.4 we will be able to deduce the global result from a computation using local coordinates, so we begin by considering the local case.

3.4.1 Local result

Let $\mathcal{L}$ be a Lipschitz lamination of dimension q embedded on a compact Kähler manifold X of dimension n. From Lemma 3.2.1 we can choose holomorphic coordinates $x = (x', x'')$ in a flow box $B \simeq \mathbb{D}^q \times \mathbb{D}^{n-q}$ in which the plaques of $\mathcal{L}$ in B are given by

$$L_a = \{x'' = h_a(x')\},$$

for some uniformly bounded holomorphic functions $h_a : \mathbb{D}^q \to \mathbb{D}^{n-q}$ depending continuously on $a = h_a(0)$ and $h_0 \equiv 0$.

Let T be a diffuse foliated cycle of dimension q on X directed by $\mathcal{L}$. In the flow box B, T is given by

$$T = \int [L_a] \, \mathrm{d}\mu(a),$$

where $[L_a]$ denotes the current of integration along L_a and μ is a diffuse positive measure on $\mathbb{D}^{n-q}$.

Let $\mathcal{L} \times \mathcal{L}$ denote the self-product lamination on $X \times X$. The coordinates on the flow box B induce natural holomorphic coordinates (x, y) on $B \times B$ in which the plaques of $\mathcal{L} \times \mathcal{L}$ are given by

$$L_{a,b} := L_a \times L_b = \{x'' = h_a(x'),\, y'' = h_a(y')\}.$$

The product $\mathbf{T} := T \otimes T$ is a foliated cycle of dimension $2q$ on $X \times X$ directed by $\mathcal{L} \times \mathcal{L}$ which is given, on $B \times B$, by

$$\mathbf{T} = \int [L_{a,b}] \, \mathrm{d}\mathfrak{M}(a,b), \text{ where } \mathfrak{M} = \mu \otimes \mu.$$

We will need the following simple result. It can be proven, for instance, using Fubini's Theorem.

Lemma 3.4.1. *Let μ be a finite measure on a separable Borel space $\mathcal{X}$. If μ has no atoms then $\mathfrak{M} = \mu \otimes \mu$ gives no mass to the diagonal $\Delta \subset \mathcal{X} \times \mathcal{X}$.*

Since we are interested in computing the tangent currents of $\mathbf{T}$ along the diagonal $\Delta = \{x = y\}$ it is convenient to work in new holomorphic coordinates given by

$$(z, w) = (x, y - x)$$

and new parameters given by $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1 = a$ and $\alpha_2 = b - a$.

Write $z = (z', z'')$ with $z' \in \mathbb{C}^q$ and $z'' \in \mathbb{C}^{n-q}$ and similarly $w = (w', w'')$. In this new coordinate system the diagonal is given by $\Delta = \{w = 0\}$ and $L_{a,b}$ transforms to

$$\Gamma_\alpha = \left\{(z', f_\alpha(z', w'), w', g_\alpha(z', w'))\right\},$$

where $f_\alpha(z', w') = h_{\alpha_1}(z')$ and $g_\alpha(z', w') = h_{\alpha_1+\alpha_2}(z' + w') - h_{\alpha_1}(z'))$. Notice that $f_\alpha(0, w') = \alpha_1$ and $g_\alpha(0, 0) = \alpha_2$.

The expression of $\mathbf{T}$ becomes

$$\mathbf{T} = \int [\Gamma_\alpha] \, \mathrm{d}\mathfrak{M}(\alpha).$$

We will need the following properties of f_α, g_α and $\mathfrak{M}$:

1. The parameter $\alpha \in \mathbb{D}^{2n-2q}$ is determined by $\alpha = (f_\alpha(0,0), g_\alpha(0,0))$,
2. (f_α, g_α) depends on α on a bi-Lipschitz way,
3. f_α depends only on α_1,
4. If $\alpha_2 = 0$ then $g_\alpha(z', 0) \equiv 0$ and if $\alpha_2 \neq 0$ then $g_\alpha(z', 0) \neq 0$ for every $z' \in \mathbb{D}^q$.
5. The measure $\mathfrak{M}$ gives no mass to the set $\{\alpha_2 = 0\}$, see Lemma 3.4.1.

Remark 3.4.2. (a) Property (2) means that for every $0 < \rho < 1$ there is a constant $c = c(\rho) > 0$ such that

$$c^{-1}\|\alpha - \beta\| \leq \max\{\|f_\alpha(z', w') - f_\beta(z', w')\|, \|g_\alpha(z', w') - g_\beta(z', w')\|\} \leq c\,\|\alpha - \beta\|, \quad (3.4.1)$$

for every (z', w') such that $\|z'\|, \|w'\| \leq \rho$.

(b) From Cauchy formula, we have analogous upper bounds for the derivatives of f_α and g_α of any order.

(c) Property (4) corresponds to the geometric fact that the plaques $L_{a,b}$ with $a \neq b$ don't intersect the diagonal and the plaques $L_{a,a}$ intersect the diagonal along a q-dimensional manifold.

Let $A_\lambda(z,w) = (z, \lambda w)$ be the dilation in the direction normal to Δ and set

$$\mathbf{T}_\lambda = (A_\lambda)_* \mathbf{T}, \quad \text{for} \quad |\lambda| \geq 1.$$

The main result in this section is the following.

Proposition 3.4.3. *Let $\mathcal{L}$, Γ_α, $\mathbf{T}$ and $\mathbf{T}_\lambda$ be as above. Then*

$$\lim_{|\lambda| \to \infty} \mathbf{T}_\lambda = 0$$

in a neighborhood of the origin.

The proof will follow from the following basic estimates.

Lemma 3.4.4. *Fix $0 < \rho < 1$. There are constants $c_1, c_2, c_3 > 0$ such that the following estimates hold for every (z', w') such that $\|z'\|, \|w'\| \leq \rho$*

$$c_1 \cdot (\|\alpha_2\| - c_2 \|\alpha\| \|w'\|) \leq \|g_\alpha(z', w')\| \leq c_3 \cdot (\|\alpha_2\| + \|\alpha\| \|w'\|). \tag{3.4.2}$$

Proof. From Property (4) and the Lipschitz condition (3.4.1) we get

$$\|g_\alpha(z', 0)\| = \|g_{\alpha_1,\alpha_2}(z', 0)\| = \|g_{\alpha_1,\alpha_2}(z', 0) - g_{\alpha_1,0}(z', 0)\| \leq c\|\alpha_2\|.$$

From Remark 3.4.2–(b) we have that $\|Dg_\alpha(z', w')\| \leq k\|\alpha\|$ for $\|z'\|, \|w'\| \leq \rho$ and some constant $k > 0$, so by the mean value theorem we get

$$\|g_\alpha(z', w') - g_\alpha(z', 0)\| \leq k \|\alpha\| \|w'\|.$$

Combining these two inequalities gives

$$\|g_\alpha(z', w')\| \leq c_3(\|\alpha_2\| + \|\alpha\| \|w'\|)$$

for some constant $c_3 > 0$.

To prove the left inequality in (3.4.2) we use Property (3) and the estimate (3.4.1).

We get

$$\begin{aligned}\|g_\alpha(z',0)\| &= \|g_{\alpha_1,\alpha_2}(z',0)\| = \|g_{\alpha_1,\alpha_2}(z',0) - g_{\alpha_1,0}(z',0)\| \\ &= \max\left\{\|g_{\alpha_1,\alpha_2}(z',0) - g_{\alpha_1,0}(z',0)\|, \|f_{\alpha_1,\alpha_2}(z',0) - f_{\alpha_1,0}(z',0)\|\right\} \\ &\geq c^{-1}\|\alpha_2\|.\end{aligned}$$

Using again the fact that $\|Dg_\alpha(z',w')\| \leq k\|\alpha\|$ for $\|z'\|, \|w'\| \leq \rho$ and the mean value theorem we get

$$\|g_\alpha(z',w')\| \geq c_1 \cdot (\|\alpha_2\| - c_2\,\|\alpha\|\,\|w'\|),$$

for some constants $c_1, c_2 > 0$. This completes the proof of the Lemma. □

We will also need the following standard result, saying that the volume of a bounded family of graphs is uniformly bounded. We first recall some basic facts.

Let M be a complex manifold equipped with a hermitian metric ω. If Y is a complex submanifold of dimension m of M then the metric ω induces a metric on Y. By Wirtinger's Theorem the corresponding volume form on Y is $\frac{1}{m!}\omega^m$. This means that the $2m$-dimensonal volume of Y over a Borel set K, denoted by $\mathrm{vol}_{2m}(A \cap K)$, is given by $\frac{1}{m!}\int_{A\cap K}\omega^m$. This quantity is equal to $\frac{1}{m!}$ times the mass of the trace measure $[A] \wedge \omega^m$ over K.

Lemma 3.4.5. *Let Ω be a domain in $\mathbb{C}^m$, $\mathcal{F}$ be a family of holomorphic maps $f : \Omega \to \mathbb{C}^N$ and denote by $\Gamma_f \subset \mathbb{C}^{m+N}$ the graph of f. Suppose that there exists an $M > 0$ such that $\|f(z)\| \leq M$ for every $z \in \Omega$ and every $f \in \mathcal{F}$. Then for every compact $K \subset \Omega$ there is a constant $C > 0$ such that*

$$\mathrm{vol}_{2m}(\Gamma_f \cap (K \times \mathbb{C}^N)) \leq C \quad \textit{for every} \;\; f \in \mathcal{F},$$

where the volume is computed with respect to the standard Hermitian metric on $\mathbb{C}^{m+N}$.

Proof. Let $\omega = \sum_{i=1}^{m+N} \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i$ be standard Hermitian form on $\mathbb{C}^{m+N}$. For each $f \in \mathcal{F}$ consider the parametrization $\phi_f : \Omega \to \Gamma_f$, $\phi_f(x) = (x, f(x))$. From Wirtinger's Theorem we get that

$$\mathrm{vol}_{2m}(\Gamma_f \cap (K \times \mathbb{C}^N)) = \frac{1}{m!}\int_{\Gamma_f \cap (K\times\mathbb{C}^N)} \omega^m = \frac{1}{m!}\int_K \phi_f^*\omega^m.$$

Since the functions f are uniformly bounded it follows from Cauchy's estimates that the first derivatives of f are uniformly bounded over K . This implies that the coefficients of the $(m,m)-$form $\phi_f^*\omega^m$ are bounded in K, uniformly in f. This gives the result. □

To prove Proposition 3.4.3 we will look at the action of the dilation A_λ on each plaque $[\Gamma_\alpha]$ and consider the complementary cases when $\|\alpha_2\| \leq \frac{1}{|\lambda|}$ and when $\|\alpha_2\| > \frac{1}{|\lambda|}$.

Lemma 3.4.6. *The family of currents* $\mathcal{F} = \{(A_\lambda)_*[\Gamma_\alpha] : |\lambda| \geq 1, \|\alpha_2\| \leq \frac{1}{|\lambda|}\}$ *is relatively compact. In other words the mass of* $(A_\lambda)_*[\Gamma_\alpha]$ *on a compact set is bounded uniformly in* (λ, α) *when* $\|\alpha_2\| \leq \frac{1}{|\lambda|}$.

Proof. We will consider only the pairs (λ, α) satisfying $\|\alpha_2\| \leq |\lambda|^{-1}$. Recall that the para–meters α vary on a bounded set. The support Γ_α^λ of $(A_\lambda)_*[\Gamma_\alpha]$ is a graph over the (z', w') coordinates given by

$$\Gamma_\alpha^\lambda = \left\{(z', f_\alpha(z', \lambda^{-1}w'), w', \lambda g_\alpha(z', \lambda^{-1}w'))\right\}, \quad \|z'\| < 1, \ \|w'\| < |\lambda| \tag{3.4.3}$$

and the mass of $(A_\lambda)_*[\Gamma_\alpha]$ over a compact set K is $(2q)!$ times the $4q$-dimensional volume of $\Gamma_\alpha^\lambda \cap K$.

We claim that, for a fixed $0 < \rho < 1$, the above graphs are uniformly bounded over the set $\|z'\|, \|w'\| \leq \rho$. Indeed, the functions $f_\alpha(z', \lambda^{-1}w')$ are bounded uniformly in α and λ, and Lemma 3.4.4 gives

$$\|\lambda g_\alpha(z', \lambda^{-1}w')\| \leq |\lambda|\, c_3 \cdot (\|\alpha_2\| + \|\alpha\|\, |\lambda|^{-1}\|w'\|) = c_3 \cdot (|\lambda|\, \|\alpha_2\| + \|\alpha\|\, \|w'\|) \leq c_3'(1+\rho)$$

for $\|z'\|, \|w'\| \leq \rho$, since we are assuming $\|\alpha_2\| \leq |\lambda|^{-1}$.

It follows from Lemma 3.4.5 that the $2q$-dimensional volumes of the Γ_α^λ are locally boun–ded, uniformly in (λ, α) when $\|\alpha_2\| \leq \frac{1}{|\lambda|}$. □

Lemma 3.4.7. *There exists a compact neighborhood of the origin* K *such that* $(A_\lambda)_*[\Gamma_\alpha]$ *has no mass on* K *for every pair* (λ, α) *such that* $\|\alpha_2\| > \frac{1}{|\lambda|}$.

Proof. The left inequality in (3.4.2) together with the assumption that $\|\alpha_2\| > |\lambda|^{-1}$ gives

$$\|\lambda g_\alpha(z', \lambda^{-1}w')\| \geq c_1\, |\lambda| \cdot (\|\alpha_2\| - c_2\, |\lambda|^{-1}\, \|\alpha\|\, \|w'\|) \geq c_1 - k\|w'\|,$$

for some constant $k > 0$.

Hence, if $\|w'\| \leq \frac{c_1}{3k}$ then $\|\lambda g_\alpha(z', \lambda^{-1}w')\| > \frac{c_1}{2}$, meaning that the support Γ_α^λ of $(A_\lambda)_*[\Gamma_\alpha]$ (see (3.4.3)) lies outside the compact set

$$K = \left\{\|z'\| \leq \frac{1}{2}, \|w'\| \leq \frac{c_1}{3k}, \|z''\| \leq \frac{1}{2}, \|w''\| \leq \frac{c_1}{2}\right\}.$$

The lemma follows. □

Proof of Proposition 3.4.3. Dividing the integral defining $\mathbf{T}_\lambda$ in two parts we get

$$\begin{aligned}\mathbf{T}_\lambda = \int (A_\lambda)_*[\Gamma_\alpha]\, \mathrm{d}\mathfrak{M}(\alpha) &= \int_{\left\{\|\alpha_2\| \leq \frac{1}{|\lambda|}\right\}} (A_\lambda)_*[\Gamma_\alpha]\, \mathrm{d}\mathfrak{M}(\alpha) + \int_{\left\{\|\alpha_2\| > \frac{1}{|\lambda|}\right\}} (A_\lambda)_*[\Gamma_\alpha]\, \mathrm{d}\mathfrak{M}(\alpha) \\ &= \mathbf{T}_\lambda' + \mathbf{T}_\lambda''.\end{aligned}$$

Take a compact neighborhood of the origin K as in Lemma 3.4.7, so that $\mathbf{T}''_\lambda$ is zero on K for every λ. It follows from Lemma 3.4.6 that there is a constant $C > 0$, independent of λ, such that the mass of $\mathbf{T}'_\lambda$ over K is bounded by C times $\mathfrak{M}(\{\|\alpha_2\| < |\lambda|^{-1}\})$. The fact that $\mathfrak{M}$ gives no mass to the set $\{\alpha_2 = 0\}$ shows that $\mathbf{T}'_\lambda \to 0$ on K as $|\lambda| \to \infty$. The proposition follows. □

3.4.2 Global result

The local computations of the last section allow us to prove our main theorem.

Proof of Theorem 3.1.1. Notice first that if $q < \frac{n}{2}$ then $2n-2q > n$, so $H^{2n-2q,2n-2q}(X) = 0$ for dimensional reasons. Therefore the product $\{T\} \smile \{T\}$ is trivially zero in this case. We may assume then that $q \geq \frac{n}{2}$. According to Proposition 3.3.6 it suffices to show that the total density class $\kappa(T,T)$ vanishes.

Let $\mathbf{S}$ be a tangent current to $\mathbf{T}$ along Δ. If U is a small open set intersecting the support of $\mathbf{T}$ and contained in a flow box $B \times B \subset X \times X$ we can choose coordinates (z,w) in U as in Section 3.4.1, so that Δ is given by $\{w = 0\}$. From Theorem 3.3.4 we can compute the restriction of $\mathbf{S}$ to U as the limit

$$\mathbf{S} = \lim_{\lambda_n \to \infty} \mathbf{T}_{\lambda_n},$$

where $A_\lambda(z,w) = (z, \lambda w)$, $\mathbf{T}_\lambda = (A_\lambda)_* \mathbf{T}$ and $\lambda_n \to \infty$.

From Proposition 3.4.3 we have that $\lim_{|\lambda| \to \infty} \mathbf{T}_\lambda = 0$, so any tangent current to $\mathbf{T}$ along Δ vanishes on U. Since U is arbitrary we conclude that the zero current is the only tangent current to $\mathbf{T}$ along Δ. Therefore the tangent class $\kappa^\Delta(\mathbf{T}) = \kappa(T,T)$ is zero, concluding the proof. □

Remark 3.4.8. For $q \geq \frac{n}{2}$, the vanishing of the class $\kappa(T,T)$ implies that $T \curlywedge T = 0$, where the wedge product is in the sense of Dinh–Sibony (see [DS14], Section 5).

3.5 Applications

We give in this section some consequences of Theorem 3.1.1.

Foliated cycles on projective spaces and other manifolds

Recall that for a compact Kähler manifold X, the Hodge Decomposition Theorem says that the de Rham cohomology groups of X decompose as $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, where $H^{p,q}(X)$ is the set of classes that can be represented by a closed $(p,q)-$form (see [Voi07]). The numbers $h^{p,q}(X) = \dim H^{p,q}(X)$ are called the *Hodge numbers* of X. For $1 \leq p \leq \dim X$ the space $H^{p,p}(X)$ is never zero since it contains the class of the $p-$th power of a Kähler form.

Theorem 3.5.1. *Let X be a compact Kähler manifold of dimension n and let $\mathcal{L}$ be an embedded Lipschitz lamination on X of dimension $q \geq n/2$. Suppose that $h^{n-q,n-q}(X) = 1$. Then there is no diffuse foliated cycle directed by $\mathcal{L}$.*

Proof. Fix a Kähler form ω in X. Let T be a diffused foliated cycle of dimension q directed by $\mathcal{L}$. From Theorem 3.1.1 we have that $\{T\} \smile \{T\} = 0$. On the other hand, since $h^{n-q,n-q}(X) = 1$ the class $\{T\} \in H^{n-q,n-q}(X)$ equals to $c\{\omega^{n-q}\}$ for some $c > 0$. In particular $\{T\} \smile \{T\} = c^2\{\omega^{2n-2q}\} \neq 0$, a contradiction. □

Corollary 3.5.2. *Let X be a compact Kähler manifold with $b_2(X) = 1$. Then there is no diffuse foliated cycle on X directed by a Lipschitz lamination of codimension 1.*

From the fact that $h^{p,p}(\mathbb{P}^n) = 1$ for every p we get

Corollary 3.5.3. *Let $\mathcal{L}$ be an embedded Lipschitz lamination of dimension $q \geq n/2$ on $\mathbb{P}^n$. Then there is no diffuse foliated cycle directed by $\mathcal{L}$. In particular, if $\mathcal{L}$ has no compact leaf then there is no foliated cycle directed by $\mathcal{L}$.*

The above result is proven in [FS05] for $n = 2$ and $q = 1$. In their proof, a crucial ingredient is the fact that $\mathbb{P}^2$ is homogeneous under the action of its automorphism group. This is used in order to regularize the foliated cycle. Our approach allows us to bypass this difficulty.

In what follows a Riemann surface is called parabolic if it is uniformized by the complex plane and hyperbolic if it is uniformized by the the unit disc. The following corollary is consequence of the Ahflors' construction (Example 3.2.6).

Corollary 3.5.4. *Let X be a Kähler surface with $h^{1,1}(X) = 1$. Let $\mathcal{L}$ be an embedded Lipschitz lamination by Riemann surfaces on X without compact leaves. Then all the leaves of $\mathcal{L}$ are hyperbolic.*

Corollary 3.5.5. *Let X be a Kähler surface with $h^{1,1}(X) = 1$. Let $\mathcal{F}$ be a (singular) holomorphic foliation of X without compact leaves. If $\mathcal{F}$ admits a parabolic leaf L then $\overline{L}$ intersects the singular set of $\mathcal{F}$.*

Proof. Suppose that there is a parabolic leaf L whose closure does not meet the singular set of $\mathcal{F}$. In this case, the foliation $\mathcal{F}$ restricts to a non-singular lamination $\mathcal{L}$ in a neighborhood of $\overline{L}$ and the Ahlfors' construction (Example 3.2.6) gives a foliated cycle directed by $\mathcal{L}$, which is impossible in view of Corollary 3.5.4. □

The above results hold in particular when X is the complex projective plane. In this case, the condition of having no compact leaves is generic, see [Jou79].

Remark 3.5.6. Some authors consider different notions of parabolicity and use them to produce d−closed and dd^c–closed currents in the same spirit as Ahlfors (see [BS12], [Thé10] and [PS14]). We can also apply the above results to these types of parabolic leaves.

Foliated cycles on complex tori

Let X be a complex torus. A lamination on X is called linear if it is given by a union of leaves of a linear foliation on X.

Remark 3.5.7. Let $\mathcal{L}$ be a lamination of dimension q on a complex manifold X that extends to a holomorphic foliation $\mathcal{F}$ on X. Then the foliated cycles directed by $\mathcal{L}$ are precisely the positive closed $(n-q, n-q)-$currents T whose support is contained in the support of $\mathcal{L}$ and satisfy $T \wedge \gamma_i = 0$, where γ_i are local holomorphic $(1,0)-$forms whose kernel defines $\mathcal{F}$.

Proposition 3.5.8. *Let X be a complex torus and let $\mathcal{L}$ be a Lipschitz lamination of codimension one embedded on X. If $\mathcal{L}$ is linear then it directs a foliated cycle. Conversely, if there is a diffuse foliated cycle T directed by $\mathcal{L}$ then the support of T is a linear lamination.*

Proof. Suppose that $\mathcal{L}$ is linear. In particular, every leaf of $\mathcal{L}$ is the image of an affine hyperplane $H \simeq \mathbb{C}^{n-1}$ by the canonical projection $\pi : \mathbb{C}^n \to X$. A result of de Thélin [Thé10] allows us to construct a positive closed current T of bi-dimension $(n-1, n-1)$ as a limit of normalized currents of integration along $\pi(\mathbb{D}_{r_n})$, where $\mathbb{D}_r$ is an $(n-1)-$dimensional disc of radius r in H and r_n is a sequence tending to infinity. By construction, the current T is a foliated cycle directed by $\mathcal{L}$.

Let us prove the converse. Recall that, using Hodge theory, we can see that the cohomology ring $H^\bullet(X, \mathbb{C})$ is isomorphic to the exterior algebra $\bigwedge^\bullet(\mathbb{C}^n)^*$. Under this identification, a positive $(1,1)-$class $\mathbf{c}$ has vanishing square if and only if it has the from $\mathbf{c} = i\gamma \wedge \overline{\gamma}$ for some linear form γ of type $(1,0)$.

Let $\mathcal{L}$ be a Lipschitz lamination of codimension one on X and let T be a diffuse foliated cycle directed by $\mathcal{L}$. Denote by $\mathbf{c}$ the class of T. By Theorem 3.1.1 we have $\mathbf{c}^2 = 0$, so by the above remark $\mathbf{c} = i\gamma \wedge \overline{\gamma}$ for some linear form γ of type $(1,0)$. We claim that T is directed by the linear foliation defined by γ. Indeed we have

$$0 = \{T\}^2 = \{T\} \smile i\gamma \wedge \overline{\gamma} = \{T \wedge i\gamma \wedge \overline{\gamma}\},$$

and since $T \wedge i\gamma \wedge \overline{\gamma}$ is a positive current, it must be zero. It is not hard to show from this fact that $T \wedge \gamma = 0$, so T is directed by the linear foliation defined by $\ker \gamma$. □

Foliated cycles on Hirzebruch surfaces

We now apply Theorem 3.1.1 to describe the foliated cycles on a Hirzebruch surface. Let us first recall some facts about currents on projective bundles. We refer to [DS14], Section 3 for more details.

Let V be a Kähler manifold of dimension ℓ with Kähler form ω_V and let E be a rank r holomorphic vector bundle over V. The projectivization of E, denoted by $\mathbb{P}(E)$, is the $\mathbb{P}^{r-1}-$bundle over V whose fiber over $x \in V$ is the projectivization of E_x.

Definition 3.5.9. Let S be a non-zero positive $(p,p)-$current on $\mathbb{P}(E)$. The *horizontal dimension* (or $h-$dimension for short) of S is the largest integer j such that $S \wedge \omega_V^j \neq 0$. When this dimension is 0 we say that S is vertical.

Given a cohomology class $\mathbf{c} \in H^{2p}(\mathbb{P}(E), \mathbb{C})$ we can apply the Leray-Hisrch Theorem and get a decomposition

$$\mathbf{c} = \sum_{j=\max(0,\ell-p)}^{\min(\ell,\ell+r-1-p)} \pi^*(\kappa_j(\mathbf{c})) \smile h_{\mathbb{P}(E)}^{j-\ell+p},$$

where $\kappa_j(\mathbf{c})$ is a class in $H^{2\ell-2j}(V, \mathbb{C})$ and $h_{\mathbb{P}(E)} \in H^2(\mathbb{P}(E), \mathbb{C})$ is the tautological class of the associated $\mathcal{O}_{\mathbb{P}(E)}(1)-$bundle over $\mathbb{P}(E)$.

When $\mathbb{P}(E) = \mathbb{P}(N_V \oplus \mathbb{C}) = \overline{N_V}$ is the compactification of the normal bundle to V and $\mathbf{c} = \kappa^V(T)$ is the total tangent class to T along V (see Definition 3.3.3) the above classes are denoted by $\kappa_j^V(T)$.

Recall that a cohomology class is pseudo-effective if it contains a positive closed current.

Lemma 3.5.10 ([DS14], Lemma 3.8)**.** *Let S be a $(p,p)-$current on $\mathbb{P}(E)$. If s denotes the $h-$dimension of S then $\kappa_s(\{S\}) \in H^{2\ell-2s}(V, \mathbb{C})$ is pseudo-effective.*

Lemma 3.5.11. *Let X be a compact Kähler surface, $V \subset X$ be a smooth curve and T be a $(1,1)-$ current without mass on V. If S is a tangent current to T along V then the h-dimension of S in $\overline{N_V}$ is zero. In particular $\kappa_0^V(T)$ is pseudo-effective.*

Proof. We have to show that $S \wedge \pi^*\omega_V = 0$, where ω_V is a Kähler form in V. We can work locally. Choose coordinates (z, w) with values in the unit bi-disc in which V is given by $\{w = 0\}$ and $N_V \to V$ identifies with the trivial bundle. We can take $\omega_V = i\mathrm{d}z \wedge \mathrm{d}\overline{z}$.

In these coordinates $S = \lim_{\lambda_n \to \infty} T_{\lambda_n}$, where $T_\lambda = (A_\lambda)_* T$ and $A_\lambda(z, w) = (z, \lambda w)$. Let $0 < \rho < 1$. Since $i\mathrm{d}z \wedge \mathrm{d}\overline{z}$ is invariant under A_λ, the mass of $T_\lambda \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z}$ over $\mathbb{D}(\rho)$ is given by

$$\int_{\mathbb{D}(\rho)} T_\lambda \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z} = \int_{\mathbb{D}(\rho)} (A_\lambda)_*(T \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z}) = \int_{\mathbb{D}(\rho,\lambda^{-1}\rho)} T \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z} \leq \|T\|_{\mathbb{D}(\rho,\lambda^{-1}\rho)}.$$

Making $\lambda \to \infty$ and using the fact that T has no mass on V gives $\lim_{|\lambda|\to\infty} T_\lambda \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z} = 0$. By semicontinuity $S \wedge i\mathrm{d}z \wedge \mathrm{d}\overline{z} = 0$, proving the first part of the Lemma. The last statement follows from Lemma 3.5.11. □

Corollary 3.5.12. *Let X, T and V be as above. Then $\{T\} \smile \{V\} \geq 0$.*

Proof. It follows from the results in Section 5 of [DS14] that the class $\{T\} \smile \{V\} \in H^4(X, \mathbb{C})$ is the canonical image of $\kappa_0^V(T) \in H^2(V, \mathbb{C})$ in $H^4(X, \mathbb{C})$. So by the above lemma $\{T\} \smile \{V\}$ is a pseudo-effective class in $H^4(X, \mathbb{C}) \simeq \mathbb{C}$, hence it is given by a positive real number. □

Recall the the Hirzebruch surface Σ_n, $n \geq 0$ is defined by

$$\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)),$$

where $\mathcal{O}_{\mathbb{P}^1}$ denotes the trivial line bundle over $\mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1}(-n)$ is the nth exterior power of the tautological line bundle on $\mathbb{P}^1$.

Every compact complex surface that is a $\mathbb{P}^1-$bundle over $\mathbb{P}^1$ is isomorphic to Σ_n for some $n \geq 0$. The surface Σ_0 is equal to $\mathbb{P}^1 \times \mathbb{P}^1$ and Σ_1 is isomorphic to the the projective plane blown-up at a point.

Proposition 3.5.13. *Let $\mathcal{L}$ be a Lipschitz lamination by Riemann surfaces embedded in Σ_n, $n \neq 0$. Then every diffuse foliated cycle T directed by $\mathcal{L}$ is given by $T = \pi^*\nu$ for some diffuse measure on $\mathbb{P}^1$. In particular, the lamination $\mathcal{L}$ is a union of fibers of $\pi : \Sigma_n \to \mathbb{P}^1$.*

Proof. The group $H^2(\Sigma_n, \mathbb{C})$ is generated by F and C , where F is the class of a fiber and C is the class of the section at infinity. Their intersections are given by

$$F^2 = 0, \quad F \smile C = 1 \quad \text{and} \quad C^2 = -n.$$

If ν is any probabilty measure on $\mathbb{P}^1$ then the class of the current $\pi^*\nu$ is equal to F. In particular, if ω_{FS} denotes the Fubini-Study form on $\mathbb{P}^1$ then $\{\pi^*\omega_{FS}\} = F$.

Let T be a diffuse foliated cycle directed by $\mathcal{L}$ and denote by $\mathbf{c}$ its class on $H^2(\Sigma_n, \mathbb{C})$. Write $\mathbf{c} = aF + bC$ for some $a, b \in \mathbb{R}$. Notice that, from Corollary 3.5.12, we have $b = C \smile F \geq 0$. From Theorem 3.1.1 we have

$$0 = \mathbf{c}^2 = 2ab - b^2\, n = b\,(2a - bn).$$

Assume that $b \neq 0$, then $2a = bn$ and

$$\mathbf{c} \smile C = \left(\frac{bn}{2}F + bC\right) \smile C = \frac{bn}{2} - bn = -\frac{bn}{2} < 0,$$

which contradicts Corollary 3.5.12. Therefore we must have $b = 0$ and $\mathbf{c} = aF$.

We claim that $T \wedge \pi^*\omega_{FS} = 0$. Indeed, as noted above the class of $\pi^*\omega_{FS}$ equals to F, so the class the positive current $T \wedge \pi^*\omega_{FS}$ is $\mathbf{c} \smile F = a\,F^2 = 0$. The claim follows from the fact that the class of a positive closed current S is zero if and only if $S = 0$.

The fact that $T \wedge \pi^*\omega_{FS} = 0$ means that the current T is vertical, i.e. its $h-$dimension is zero. It follows that T is given by an average of currents of integration along the fibers of π (see [DS14], Lemma 3.3), that is, $T = \pi^*\nu$ for some measure on $\mathbb{P}^1$. □

The above result shows in particular that a Lipschitz lamination on Σ_n without compact leaves carries no directed positive closed current. It is worth mentioning that every

non-singular holomorphic foliation globally defined on Σ_n is given by the canonical fibration (see [Bru15], p. 37).

For $n = 0$ we have that $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Arguing as above we can show that the only diffuse foliated cycles are given by $T = \pi_1^* \nu$ or $T = \pi_2^* \nu$ for some diffuse positive measure ν on $\mathbb{P}^1$, where π_i denote the projection on each factor. A similar result was obtained by Perez-Garrandes in [PG14], where he also showed that for a Lipschitz lamination in $\mathbb{P}^1 \times \mathbb{P}^1$ the absence of foliated cycles implies the existence of a unique directed harmonic current of mass one.

Chapitre 4

Mesures de Monge-Ampère

This chapter is concerned with the study of Monge–Ampère measures associated with Hölder continuous potentials. These measures appear for instance as the Green measure of a holomorphic endomorphism of $\mathbb{P}^k$ or as solutions of some singular complex Monge–Ampère equations. We give here a version of the classical Skoda's integrability theorem in this setting. As a consequence we obtain an estimate of the mass of sublevel sets of plurisubharmonic functions, which can be interesting in applications. The results presented here are contained in [Kau16a].

4.1 Introduction and main result

Let Ω be an open subset of $\mathbb{C}^n$ and let φ be a p.s.h. function on Ω. Recall that the *Lelong number* of φ at $a \in \Omega$ can be defined as

$$\nu(\varphi; a) := \liminf_{z \to a,\, z \neq a} \frac{\varphi(z)}{\log|z-a|}$$

and it somewhat measures the singularity of φ at a. This number can be characterized as $\nu(\varphi; a) = \sup\{\gamma : \varphi(z) \leq \gamma \log|z-a| + O(1) \text{ as } z \to a\}$ and is one of the most basic quantities associated to the singularity of φ at a. The function $z \mapsto \nu(\varphi; z)$ is upper semicontinuous with respect to the usual topology. It is a deep theorem of Y-T. Siu that this function is also upper semicontinuous with respect to the Zariski topology, i.e. for evey $c > 0$ the set $\{a \in \Omega\,;\, \nu(\varphi, a) \geq c\}$ is a closed analytic subvariety of Ω. For further properties and equivalent definitions of the Lelong number the reader may consult [Dem] and [Hör07].

Another way of measuring the singularity of a p.s.h. function at a point was introduced in [DK01] (see also [Tia87]) as tool for studying several types of algebraic and analytic objects (holomorphic functions, divisors, coherent ideal sheaves, currents, etc.). They

define the *integrability index* of φ at a by

$$c(\varphi; a) = \sup\{c \geq 0 : e^{-2c\varphi} \text{ is Lebesgue integrable in a neighborhood of } a\}.$$

A classical theorem of Skoda states that if $\nu(\varphi; a) < 2$ then $e^{-\varphi}$ is integrable in a neighborhood of a with respect to the Lebesgue measure, see Theorem 1.1.14. In terms of the above quantities this translates as $c(\varphi; a) \geq \nu(\varphi; a)^{-1}$. On the other direction, another theorem of Skoda says that $c(\varphi; a) \leq n\nu(\varphi; a)^{-1}$.

Our main result in this chapter is a generalization of Skoda's theorem where the Lebesgue measure is replaced by a general Monge–Ampère mass with a Hölder continuous local potential. See Section 4.1.1 for the definition and some examples. This class of measures is important in many applications of Pluripotential Theory and has been the topic of recent research. The reader may consult [Dem+14] and also [DN14] for a complete classification of Monge–Ampère measures with a Hölder continuous local potential.

Theorem 4.1.1. *Let $0 < \alpha \leq 1$, let u be an α-Hölder continuous p.s.h. function in the domain $\Omega \subset \mathbb{C}^n$ and let $z \in \Omega$. If φ is a p.s.h. function in Ω and if $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$, then there is a neighborhood $K \subset \Omega$ of z such that the integral*

$$\int_K e^{-\varphi}(\mathrm{dd}^c u)^n$$

is finite. In other words, $e^{-\varphi}$ is locally integrable in $U := \{\xi \in \Omega\,;\, \nu(\varphi; \xi) < \frac{2\alpha}{\alpha+n(2-\alpha)}\}$ with respect to the positive measure $(\mathrm{dd}^c u)^n$.

The above result gives as a corollary an estimate of the mass of sublevel sets of a p.s.h. function with respect to the measure $(\mathrm{dd}^c u)^n$, see Corollary 4.3.3 below.

4.1.1 Monge–Ampère measures with Hölder continuous potentials

Let Ω be a domain in $\mathbb{C}^n$ and let u be an $\alpha-$Hölder continuous p.s.h. on Ω for some $0 < \alpha \leq 1$. From Bedford–Taylor's construction (see Section 1.1.3) we can define the positive measure

$$\mu = (\mathrm{dd}^c u)^n.$$

We call a measure of the above type a *Monge–Ampère measure of Hölder continuous potential.*

Example 4.1.2. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$. Let T be its Green current (see Definition 1.2.7) and $\mu = T^k$ be its Green measure. Then T has Hölder continuous local potentials (Proposition 1.2.8), so μ is, locally, a Monge–Ampère measure of Hölder continuous potential.

Example 4.1.3. Measures which are absolutely continuous with respect to the Lebesgue measure with a density in L^p for some $p > 1$ are examples of Monge–Ampère measures of Hölder continuous potential.

More precisely, let Ω be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$. Denote by λ_{2n} the Lebesgue measure on $\mathbb{C}^n$. Let $\phi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$. By a result of Guedj–Zeriahi–Kolodziej, [GKZ08] the Dirichlet problem

$$\begin{cases} (\mathrm{dd}^c u)^n = f\,\lambda_{2n} & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ which is $\alpha-$Hölder continuous on $\overline{\Omega}$. The exponent α is explicit in terms of p and n. For a global counterpart of this result, see [Dem+14].

4.1.2 Locally moderate currents

The main tool in the proof of the Theorem 4.1.1 is the notion of locally moderate currents, which we now introduce. Recall that the set of all p.s.h. functions in X is closed in the space $L^1_{loc}(\Omega)$ and that every family of p.s.h. functions that is bounded in $L^1_{loc}(\Omega)$ is relatively compact in $L^1_{loc}(\Omega)$ (see Theorem 3.2.12 in [Hör07]). For the sake of simplicity, such a family is called a compact family.

The notion of locally moderate currents and measures was introduced by Dinh-Sibony, see [DNS10] and it is useful in the study of complex dynamical systems (see for instance [DS10])

Definition 4.1.4. A measure μ on a complex manifold X is called *locally moderate* if for any open set $U \subset X$, any compact set $K \subset U$ and any compact family $\mathcal{F}$ of p.s.h. functions on U there are constants $\beta > 0$ and $C > 0$ such that

$$\int_K e^{-\beta\psi} d\mu \leq C, \quad \text{for every} \quad \psi \in \mathcal{F}.$$

It follows immediately from the definition that for any $\mathcal{F}$ and μ as above, $\mathcal{F}$ is bounded in $L^p_{loc}(\mu)$ for $1 \leq p < \infty$ and that μ does not charge pluripolar sets.

A positive closed current S of type (p,p) on X is said to be *locally moderate* if the trace measure $\sigma_S = S \wedge \omega^{n-p}$ is locally moderate. Here $n = \dim X$ and ω is the fundamental form of a fixed Hermitian metric on X.

Theorem 4.1.5 (Dinh-Nguyen-Sibony [DNS10])**.** *If $1 \leq p \leq n$ and $u_1, \ldots, u_p$ are Hölder continuous p.s.h. functions on X then the Monge–Ampère current $\mathrm{dd}^c u_1 \wedge \ldots \wedge \mathrm{dd}^c u_p$ is locally moderate.*

In particular that measures of the form $\mu = (\mathrm{dd}^c u)^n$ with u an $\alpha-$Hölder continuous are locally moderate. It seems interesting to know if the converse is true

Question 4.1.6. Let μ be a locally moderate measure. Is μ locally of the form $\mu = (\mathrm{dd}^c u)^n$ for $\alpha-$Hölder continuous p.s.h. function u?

The above question has a positive answer whenever the measure μ is smooth outside a finite set and if around this set it has radial or toric symmetries, see [Dem+14].

4.2 Proof of the integrability theorem

This section is devoted to the proof of Theorem 4.1.1 and some related results.

We first state two lemmas used in the proof of Theorem 4.1.5. We denote B_r the ball of radius r centered at the origin of $\mathbb{C}^n$ and fix a fundamental form ω as above.

Lemma 4.2.1 ([DNS10])**.** *Let S be a locally moderate closed positive current of type $(n-1, n-1)$ on B_r. If $\mathcal{G}$ is a compact family of p.s.h. functions on B_r then $\mathcal{G}$ is bounded in $L^1_{loc}(\sigma_S)$. Moreover, the mass of the measures $\mathrm{dd}^c\varphi \wedge S$, $\varphi \in \mathcal{G}$ are locally bounded in B_r uniformly on φ.*

Proof. Let K be a compact subset of B_r. After subtracting a fixed constant we may assume that every element of $\mathcal{G}$ is negative on K. Since σ_S is locally moderate we can choose $\beta, C > 0$ such that $\int_K e^{-\beta\varphi} d\sigma_S \leq C$ for every $\varphi \in \mathcal{G}$. We thus have $\int_K \beta|\varphi|\, d\sigma_S \leq \int_K e^{-\beta\varphi} d\sigma_S \leq C$ for every $\varphi \in \mathcal{G}$ which proves the first assertion.

For the second assertion let K be a compact subset of B_r and consider a cut-off function χ which is equal to 1 in a neighborhood of K and which is supported on a larger compact $L \subset B_r$. We have, for $\varphi \in \mathcal{G}$

$$\int_K \mathrm{dd}^c\varphi \wedge S \leq \int_L \chi \mathrm{dd}^c\varphi \wedge S = \int_L \mathrm{dd}^c\chi \wedge \varphi S \leq \|\chi\|_{\mathcal{C}^2} \int_L |\varphi|\, d\sigma_S,$$

which is uniformly bounded by the first part of the lemma. □

Lemma 4.2.2 ([DNS10])**.** *Let $r > 0$, S be a locally moderate closed positive current of type $(n-1, n-1)$ on B_{2r} and u be an α-Hölder continuous p.s.h. function on B_r which is smooth on $B_r \setminus B_{r-4\rho}$ for some $0 < \rho < r/4$. Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B_{r-2\rho}$.*

If φ is a p.s.h. function on B_{2r}, then

$$\begin{aligned}\int_{B_r} \chi\varphi \mathrm{dd}^c(uS) = &-\int_{B_r\setminus B_{r-3\rho}} \mathrm{dd}^c\chi \wedge \varphi u S - \int_{B_r\setminus B_{r-3\rho}} d\chi \wedge \varphi d^c u \wedge S \\ &+ \int_{B_r\setminus B_{r-3\rho}} d^c\chi \wedge \varphi du \wedge S + \int_{B_{r-\rho}} \chi u\, \mathrm{dd}^c\varphi \wedge S.\end{aligned}$$

Notice that the smoothness of u in $B_r \setminus B_{r-4\rho}$ makes the second and third integrals meaningful.

Proof. (ref. [DNS10]) The case when φ is smooth follows from a direct computation using integration by parts. The general case follows by approximating φ by a decreasing sequence of smooth p.s.h. functions. See [DNS10] for the complete proof. □

We will also need a volume estimate of the sublevel sets of p.s.h. functions due to M. Kiselman. We include Kiselman's argument here for the the reader's convenience.

Lemma 4.2.3. *([Kis00]) Let φ be a p.s.h. function on an open set $\Omega \subset \mathbb{C}^n$ and $K \subset \Omega$ be a compact subset. Then, for every $\gamma < 2/\sup_{z\in K} \nu(\varphi; z)$ there is a constant $C_\gamma = C_\gamma(\varphi, \Omega, K)$ such that*

$$\lambda_{2n}(K \cap \{\varphi \leq -M\}) \leq C_\gamma\, e^{-\gamma M}, \quad M \in \mathbb{R},$$

where λ_{2n} denotes the Lebesgue measure in $\mathbb{C}^n$.

Proof. Since $e^{\gamma(-M-\varphi)} \geq 1$ on $K \cap \{\varphi \leq -M\}$ we have

$$\mu(K \cap \{\varphi \leq -M\}) \leq \int_K e^{\gamma(-M-\varphi(z))} d\lambda_{2n}(z) = e^{-\gamma M} \int_K e^{-\gamma\varphi} d\lambda_{2n}.$$

It suffices then to take $C_\gamma = \int_K e^{-\gamma\varphi} d\lambda_{2n}$, which is finite by Skoda's Theorem since $\nu(\gamma\varphi; z) < 2$ for every $z \in K$. □

Before getting to the proof of of Theorem 4.1.1, we will need the following simple extension of the second part of Lemma 4.2.1.

Lemma 4.2.4. *Let $1 \leq p \leq n$ and let S be a locally moderate closed positive current of type $(n-p-1, n-p-1)$ on B_r. If $\mathcal{G}$ is a compact family of p.s.h. functions on B_r and $\mathcal{H}$ is locally uniformly bounded family of p.s.h. functions on B_r then the mass of the measures $\mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S$, $\varphi \in \mathcal{G}, u \in \mathcal{H}$ are locally bounded in B_r uniformly on φ and u.*

Proof. Fix a compact subset K of B_r and let $L_0 = K, L_1, \ldots, L_p$ be compact subsets of B_r such that L_i is contained in the interior of L_{i+1}. Let χ_i, $i = 1, \ldots, p$ be smooth cut-off functions such that $0 \leq \chi_i \leq 1$, $\chi_i \equiv 1$ in L_{i-1} and χ_i is supported in L_i. Then, for $\varphi \in \mathcal{G}$

and $u \in \mathcal{H}$, the mass of $\mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S$ over K is bounded by

$$\begin{aligned}
\int_{L_1} \chi_1 \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S &= \int_{L_1} u(\mathrm{dd}^c\chi_1) \wedge \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \\
&\leq \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_1} \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \wedge \omega \\
&\leq \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_2} \chi_2 \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \wedge \omega \\
&= \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_2} u\, \mathrm{dd}^c\chi_2 \wedge \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-2} \wedge S \wedge \omega \\
&\leq \|\chi_1\|_{\mathcal{C}^2} \|\chi_2\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \|u\|_{L^\infty(L_2)} \int_{L_2} \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-2} \wedge S \wedge \omega^2 \\
&\leq \dots \\
&\leq \|\chi_1\|_{\mathcal{C}^2} \cdots \|\chi_p\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \cdots \|u\|_{L^\infty(L_p)} \int_{L_p} \mathrm{dd}^c\varphi \wedge S \wedge \omega^p,
\end{aligned}$$

where $\omega = \mathrm{dd}^c\|z\|^2$ is the standard fundamental form on $\mathbb{C}^n$. The result now follows from Lemma 4.2.1 and the fact that $\|u\|_{L^\infty(L_i)}$ is bounded independently of u. □

Proof of Theorem 4.1.1. There is no loss of generality in assuming that $z = 0$ and since φ is locally bounded from above we may also assume that φ is negative. As before $\omega = \mathrm{dd}^c\|z\|^2$.

The proof is inspired by the methods in [DNS10], Theorem 1.1. It will consist of successive applications of integration by parts formulas (Lemma 4.2.2) together with a regularization procedure.

For $N > 0$ define $\varphi_N = \max\{\varphi, -N\}$ and $\psi_N = \varphi_{N-1} - \varphi_N$. Notice that $0 \leq \psi_N \leq 1$, ψ_N is supported in $\{\varphi < -N+1\}$ and $\psi_N \equiv 1$ in $\{\varphi < -N\}$.

Observe that

$$\begin{aligned}
\int e^{-\varphi}(\mathrm{dd}^c u)^n &= \sum_{N=0}^{\infty} \int_{\{-N \leq \varphi < -N+1\}} e^{-\varphi}(\mathrm{dd}^c u)^n \leq \sum_{N=0}^{\infty} e^N \int_{\{-N \leq \varphi < -N+1\}} (\mathrm{dd}^c u)^n \\
&\leq \sum_{N=0}^{\infty} e^N \int \psi_{N-1}(\mathrm{dd}^c u)^n.
\end{aligned} \tag{4.2.1}$$

From the hypothesis that $\nu(\varphi; 0) < \frac{2\alpha}{\alpha + n(2-\alpha)}$ and from the upper semicontinuity of the function $z \mapsto \nu(\varphi; z)$ there is an $r > 0$ such that $\sup_{z \in B_{2r}} \nu(\varphi, z) \leq \frac{2\alpha}{\alpha+n(2-\alpha)} - \sigma$ for a small constant $\sigma > 0$. From Lemma 4.2.3 we get that

$$\lambda_{2n}(B_{2r} \cap \{\varphi \leq -N+1\}) \lesssim e^{-(\frac{\alpha+n(2-\alpha)}{\alpha}+\delta)N} = e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}, \tag{4.2.2}$$

where $\delta > 0$ is a small constant (depending on φ). Here and in what follows the sign $\lesssim$

means that the left-hand sign is smaller or equal than a constant times the right-hand side, the constant being independent from N.

Taking a smaller r if necessary we may assume that u is defined on B_{2r}. Subtracting a constant we may assume that $u \leq -1$. Consider the function $v(z) = \max(u(z), A \log \|z\|)$. If we choose $A > 0$ sufficiently small, we see that v coincides with u near the origin and that $v(z) = A \log \|z\|$ near the boundary of B_r. This allows us to assume that $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$ for some fixed $\rho < r/4$. Notice that, in particular, u is smooth on $B_r \setminus B_{r-4\rho}$.

Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B_{r-2\rho}$. Applying Lemma 4.2.2 to ψ_{N-1} and $(\mathrm{dd}^c u)^{n-1}$ and noticing that $(\mathrm{dd}^c u)^n = \mathrm{dd}^c(u(\mathrm{dd}^c u)^{n-1})$ we get

$$\begin{aligned}
&\int_{B_r} \chi \psi_{N-1} (\mathrm{dd}^c u)^n = \\
&- \int_{B_r \setminus B_{r-3\rho}} \mathrm{dd}^c \chi \wedge \psi_{N-1} u (\mathrm{dd}^c u)^{n-1} - \int_{B_r \setminus B_{1-3\rho}} \mathrm{d}\chi \wedge \psi_{N-1} \mathrm{d}^c u \wedge (\mathrm{dd}^c u)^{n-1} \\
&+ \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}^c \chi \wedge \psi_{N-1} \mathrm{d}u \wedge (\mathrm{dd}^c u)^{n-1} + \int_{B_{r-\rho}} \chi u \, \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1}.
\end{aligned} \tag{4.2.3}$$

Observing that u is smooth in $B_r \setminus B_{r-3\rho}$, that the support of ψ_{N-1} is contained in $\{\varphi \leq -N+1\}$ and using the volume estimate (4.2.2) we get that the absolute values of the first three integrals on the right-hand side are $\leq c_1 e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha}N}$, where $c_1 > 0$ does not depend on N.

For $N \geq 1$ set $\varepsilon = \varepsilon(N) = e^{-(\frac{1}{\alpha}+c)N}$, where $0 < c < \frac{\delta}{n(2-\alpha)}$. Using a convolution with a smooth $U(n)$-invariant approximation of identity one can choose for N large a regularization u_ε of u defined on $B_{r-\rho}$ in such a way that $\|u - u_\varepsilon\|_\infty \lesssim \varepsilon^\alpha = e^{-(1+c\alpha)N}$ and $\|u_\varepsilon\|_{\mathcal{C}^2} := \|u_\varepsilon\|_{\mathcal{C}^2(B_{r-\rho})} \lesssim \varepsilon^{\alpha-2}$.

Writing $u = u_\varepsilon + (u - u_\varepsilon)$ the last integral in (4.2.3) is equal to

$$\int_{B_{r-\rho}} \chi u_\varepsilon \, \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1} + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) \, \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1}.$$

Since $\{\varphi_N\}_{N \geq 0}$ is a compact family of p.s.h. functions and since the current $(\mathrm{dd}^c u)^{n-1}$ is locally moderate (Theorem 4.1.5), we see from Lemma 4.2.1 that the modulus of the second integral above is less than $c_2 \|u - u_\varepsilon\|_\infty \leq c_2' \, e^{-(1+c\alpha)N}$ where $c_2' > 0$ does not depend on N.

To deal with the remaining integral we apply Lemma 4.2.2 for u_ε instead of u. Noticing

that $\mathrm{dd}^c(u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}) = \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}$ we get

$$\int_{B_{r-\rho}} \chi u_\varepsilon \, \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1} =$$
$$\int_{B_r \setminus B_{r-3\rho}} \mathrm{dd}^c \chi \wedge \psi_{N-1} u_\varepsilon (\mathrm{dd}^c u)^{n-1} + \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}\chi \wedge \psi_{N-1} \mathrm{d}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}$$
$$- \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}^c \chi \wedge \psi_{N-1} \mathrm{d} u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1} + \int_{B_r} \chi \psi_{N-1} \, \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}.$$

Since $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$ the $\mathcal{C}^2$ norm of u_ε on $B_r \setminus B_{r-3\rho}$ does not depend on $\varepsilon = \varepsilon(N)$. Together with the volume estimate (4.2.2) this implies that the first three integrals in the right-hand side have absolute values less than $c_3 \, e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha} N}$ where $c_3 > 0$ does not depend on N.

For the last integral we write $\mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1} = \mathrm{dd}^c(u(\mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2})$ and apply Lemma 4.2.2 for $S = \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2}$. This gives us four integrals. Three of them are integrals over $B_r \setminus B_{r-3\rho}$ involving u, u_ε, ψ_{N-1} and its derivatives. As above, the absolute value of each one of them is $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha} N}$. The remaining integral is

$$\int_{B_{r-\rho}} \chi u \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2},$$

which we write again as

$$\int_{B_{r-\rho}} \chi u_\varepsilon \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2} + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2}. \tag{4.2.4}$$

Since u_ε converges to u in L^∞, Lemma 4.2.4 implies that the mass of $\mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2}$ is bounded independently of N and ε. Therefore, the modulus of the second integral above is less than $c_4 \|u - u_\varepsilon\|_\infty \leq c_4' \, e^{-(1+c\alpha)N}$ where $c_4' > 0$ does not depend on N.

To deal with the the first integral in (4.2.4) we apply Lemma 4.2.2, obtaining three integrals over $B_r \setminus B_{r-3\rho}$ whose absolute values are $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha} N}$ and the integral

$$\int_{B_{r-\rho}} \chi \psi_{N-1} (\mathrm{dd}^c u_\varepsilon)^2 \wedge (\mathrm{dd}^c u)^{n-2}.$$

We can repeat the above procedure in order "move" the dd^c's from u to u_ε. We get at each step integrals with absolute values $\lesssim e^{-(1+\delta)N} e^{-\frac{n(2-\alpha)}{\alpha} N}$ or $\lesssim e^{-(1+c\alpha)N}$ (where the constants involved don't depend on N) and at the final step we get the integral

$$\int_{B_{r-\rho}} \chi \psi_{N-1} (\mathrm{dd}^c u_\varepsilon)^n,$$

whose absolute value is less than

$$c_5\|u_\varepsilon\|^n_{\mathcal{C}^2}\cdot\lambda_{2n}(B_r\cap\{\varphi\le -N+1\})\le c_5'\,\varepsilon^{n(\alpha-2)}e^{-(1+\delta)N}e^{-\frac{n(2-\alpha)}{\alpha}N}=c_5'\,e^{(cn(2-\alpha)-(1+\delta))N},$$

with c_5' independent from N.

Altogether the above estimates yield

$$\int_{B_{r-\rho}}\chi\psi_{N-1}(\mathrm{dd}^c u)^n\lesssim e^{-(1+\delta)N}e^{-\frac{n(2-\alpha)}{\alpha}N}+e^{-(1+c\alpha)N}+e^{(cn(2-\alpha)-(1+\delta))N}.$$

Inserting these estimates in (4.2.1) we finally get

$$\begin{aligned}\int_{B_r}e^{-\varphi}(\mathrm{dd}^c u)^n&\lesssim\sum_{N=0}^{\infty}e^N\left[e^{-(1+\delta)N}e^{-\frac{n(2-\alpha)}{\alpha}N}+e^{-(1+c\alpha)N}+e^{(cn(2-\alpha)-(1+\delta))N}\right]\\&=\sum_{N=0}^{\infty}\left[e^{-\delta N-\frac{n(2-\alpha)}{\alpha}N}+e^{-c\alpha N}+e^{(cn(2-\alpha)-\delta)N}\right].\end{aligned}$$

By the choice of c all the factors of N in the exponentials above are negative, so the series converges and hence the integral $\int e^{-\varphi}(\mathrm{dd}^c u)^n$ is finite. □

Remark 4.2.5. Using the same kind of computation as in the above proof we can obtain an analogous result for mixed Monge–Ampère masses of the type $\mathrm{dd}^c u_1\wedge\cdots\wedge\mathrm{dd}^c u_n$. One can show that, if u_j is α_j-Hölder continuous ($0<\alpha_j\le 1$), then the integral

$$\int e^{-\varphi}\mathrm{dd}^c u_1\wedge\cdots\wedge\mathrm{dd}^c u_n$$

is locally finite in the set of points $z\in\Omega$ satisfying $\nu(\varphi;z)<\frac{2}{1-n+2(\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_n})}$.

4.3 Related results and examples

A theorem of Zeriahi [Zer01] improves Kiselman's result (Lemma 4.2.3) by making the volume estimates of the sets $K\cap\{\varphi\le -M\}$ uniform in φ when these vary in a compact family of p.s.h. functions. Using this fact we can prove a uniform version of Theorem 4.1.1. The proof follows from Zeriahi's result and the fact that for a compact family of p.s.h. functions $\mathcal{F}$, the family $\{\varphi_N:\varphi\in\mathcal{F},N\in\mathbb{N}\}$ is again compact, where $\varphi_N=\max\{\varphi,-N\}$. I would like to thank A. Zerahi for bringing [Zer01] to my attention and for raising the question of the uniform bound provided by the following extension of Theorem 4.1.1.

Theorem 4.3.1. *Let Ω, $0<\alpha\le 1$ and u as in Theorem 4.1.1. Let $\mathcal{F}$ be a compact family of p.s.h. functions in Ω and $K\subset\Omega$ a compact subset. Set $\nu\doteq\sup\{\nu(\varphi;z):z\in K,\ \varphi\in\mathcal{F}\}$.*

If $\nu < \frac{2\alpha}{\alpha+n(2-\alpha)}$, *then there is a neighborhood* E *of* K *and a constant* $C > 0$ *such that*

$$\int_E e^{-\varphi}(\mathrm{dd}^c u)^n \leq C,$$

for every $\varphi \in \mathcal{F}$.

Remark 4.3.2. Let $\Omega, u, \mathcal{F}$ and K be as above and set $\mu = (\mathrm{dd}^c u)^n$. From the fact that μ is locally moderate (Theorem 4.1.5) there are constants $\beta > 0$ and $C > 0$ such that $\int_K e^{-\beta\psi} d\mu \leq C$ for every $\psi \in \mathcal{F}$. The above theorem allows us to estimate the value of the constant β in terms of α and the Lelong numbers of the functions of $\mathcal{F}$ over K.

From Theorem 4.1.1 and a computation analogous to the one made in the proof of Lemma 4.2.3 follows an estimate of the measure of the sub-level sets of p.s.h. functions with respect to Monge–Ampère masses with Hölder continuous potential. Informations about the size sublevel sets of p.s.h. functions in terms of measures and capacities are useful in many applications (see [Kis00], [Zer01], [FG01], [Gue04]). As above, this estimate can be made uniform in φ as long as these vary in a compact family.

Corollary 4.3.3. *Let* φ *be a p.s.h. function on an open set* $\Omega \subset \mathbb{C}^n$ *and* $\mu = (dd^c u)^n$ *a Monge–Ampère mass on* Ω *with* u *an* α*-Hölder continuous p.s.h. function. If* $K \subset \Omega$ *is a compact subset then for every* $\gamma < \frac{2\alpha}{\alpha+n(2-\alpha)} \frac{1}{\sup_{z\in K} \nu(\varphi;z)}$ *there is a constant* $C_\gamma = C_\gamma(\varphi, \Omega, K)$ *such that*

$$\mu(K \cap \{\varphi \leq -M\}) \leq C_\gamma e^{-\gamma M}, \quad M \in \mathbb{R}.$$

Another theorem of Skoda concerns the non-integrability of a p.s.h. function with large Lelong number: if $\nu(\varphi; 0) > 2n$ then $e^{-\varphi}$ is not integrable in any neighborhood of the origin with respect to Lebesgue measure (see [Hör07] Lemma 4.3.1). One cannot hope for a similar result with respect to every Monge–Ampère measure with Hölder continuous potential, because the measure $\mu = (\mathrm{dd}^c u)^n$ can be arbitrarily small near 0 (and even zero), making the integral $\int e^{-\varphi} d\mu$ finite. We may note however the following fact.

Proposition 4.3.4. *Fix* $0 < \alpha \leq 1$. *There exists a Monge–Ampère mass* $\mu = (\mathrm{dd}^c u)^n$ *where* u *is an* α*-Hölderian p.s.h. function such that for every p.s.h. function* φ *defined near* 0 *with* $\nu(\varphi; 0) > n\alpha$ *we have* $\int_K e^{-\varphi} d\mu = +\infty$ *for every neighborhood* K *of the origin.*

Proof. Let $\gamma = \nu(\varphi; 0) > n\alpha$. Since $\varphi(z) \leq \gamma \log \|z\| + O(1)$ near 0 (see Introduction) we have that $e^{-\varphi(z)} \geq C \frac{1}{\|z\|^\gamma} \geq C \frac{1}{\|z\|^{n\alpha}}$. If we take $u(z) = \|z\|^\alpha$ then a direct computation shows that $(\mathrm{dd}^c u)^n = C^{st} \|z\|^{n(\alpha-2)} \cdot \lambda_{2n}$ in the sense of currents. We thus have

$$\int e^{-\varphi}(\mathrm{dd}^c u)^n \geq C^{st} \int \frac{1}{\|z\|^{n\alpha}} \frac{1}{\|z\|^{n(2-\alpha)}} d\lambda_{2n} = C^{st} \int \frac{1}{\|z\|^{2n}} d\lambda_{2n},$$

and the last integral diverges in any neighborhood of the origin.

□

Remark 4.3.5. For $n = 1$ the condition on the Lelong number of φ on the hypothesis of Theorem 4.1.1 is $\nu(\varphi; z) < \alpha$. This bound is sharp as Proposition 4.3.4 shows.

Remark 4.3.6. For $n \geq 2$ the condition $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$ in Theorem 4.1.1 is probably no longer optimal, as the example below suggest.

Let $0 < \alpha \leq 1$ and $c > 0$. Consider the potential $u(z) = |z_1|^\alpha + \cdots + |z_n|^\alpha$ and the p.s.h. function $\varphi(z) = c \log |z_1|$ defined on $\mathbb{C}^n$. We have then that $(\mathrm{dd}^c u)^n = n! \left(\frac{\alpha}{2}\right)^{2n} |z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2} \cdot \lambda_{2n}$ as measures on $\mathbb{C}^n$. Notice that this expression makes sense, since $|z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2} \in L^1_{loc}(\mathbb{C}^n)$ for $\alpha > 0$.

We thus have

$$\begin{aligned}\int_{B_1} e^{-\varphi} (\mathrm{dd}^c u)^n &= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^c} |z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2} \, d\lambda_{2n} \\ &= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^{c-\alpha+2}} |z_2|^{\alpha-2} \cdots |z_n|^{\alpha-2} \, d\lambda_{2n},\end{aligned}$$

which is finite if and only if $\nu(\varphi; 0) = c < \alpha$.

Let us finish this chapter with some other examples where the critical integrability exponents can be explicitly computed.

Example 4.3.7 (Haar measure on the real torus)**.** Consider the function $u(z) = \log^+ |z_1| + \ldots + \log^+ |z_n|$ in $\mathbb{C}^n$. It is a Lipschitz p.s.h. function and $(\mathrm{dd}^c u)^n = \mu_{Haar}$ is the Haar measure on the n-dimensional torus $\mathbb{T}^n = \{|z_1| = \ldots = |z_n| = 1\} = \{(e^{\theta_1}, \ldots, e^{\theta_n}) : \theta_j \in [-\pi, \pi]\}$. Theorem 4.1.1 says that the integral $\int e^{-\varphi} d\mu_{Haar}$ near a point a is finite as long as $\nu(\varphi; a) < \frac{2}{1+n}$.

Let $p = (1, \ldots, 1)$ and consider the p.s.h. function $\psi(z) = c \log |z_1 - 1|$, its restriction to $\mathbb{T}^n$ is given by $\psi(\theta_1, \ldots, \theta_n) = c \log |e^{i\theta_1} - 1|$, so

$$\int e^{-\psi} d\mu_{Haar} = \int \frac{d\theta_1 \cdots d\theta_n}{|e^{i\theta_1} - 1|^c} = \int \frac{d\theta_1 \cdots d\theta_n}{2^c \sin(\frac{\theta_1}{2})^c},$$

which is finite if and only if $\nu(\varphi; p) = c < 1$.

Example 4.3.8. Consider the function $u(z, w) = |z|(1 + |w|^2)$ in $\mathbb{C}^2$. It is a Lipschitz continuous p.s.h. function and the associated Monge–Ampère measure $\mu = (\mathrm{dd}^c u)^2$ is a multiple of the Lebesgue measure in $\mathbb{C}^2$, so the critical exponents in this case are given by Skoda's Theorem. In other words the integral $\int e^{-\varphi} d\mu$ converges around a if $\nu(\varphi; a) < 2$ and diverges if $\nu(\varphi; a) > 4$.

Example 4.3.9. There are even less regular p.s.h. functions whose associated Monge–Ampère measure is smooth. In $\mathbb{C}^n$ write $z = (z', z'')$ with $z' \in \mathbb{C}^{n-k}$ and $z'' \in \mathbb{C}^k$ for some

$1 \leq k \leq n-1$. Consider the function

$$u_k(z) = \|z'\|^{2-2\frac{k}{n}} \left(1 + \|z''\|^2\right).$$

Then u_k is a continuous p.s.h. function in $\mathbb{C}^n$ (because $\log u$ is p.s.h.) and

$$(\mathrm{dd}^c u_k)^n = c_{n,k}(1 + \|z''\|^2)^{n-k-1}\lambda_{2n},$$

where $c_{n,k}$ is some positive constant. Since $\mu_k = (\mathrm{dd}^c u_k)^n$ is absolutely continuous with respect to λ_{2n} with a smooth density function, the critical exponents for μ_k are the same as those for λ_{2n}, that is, the integral $\int e^{-\varphi} d\mu_k$ converges around a if $\nu(\varphi; a) < 2$ and it diverges if $\nu(\varphi; a) > 2n$. Notice that for $k > \frac{n}{2}$, the function u_k is Hölder continuous with an exponent strictly smaller than 1. I would like to thank S. Dinew for bringing this example to my attention.

Example 4.3.10 (Lebesgue measure on the unit sphere)**.** Consider the function $u(z) = \log^+ \|z\|$ in $\mathbb{C}^n$. It is a Lipschitz p.s.h. function whose associated Monge–Ampère measure $\mu = (\mathrm{dd}^c u)^n$ is the Lebesgue measure on the unit sphere $S^{2n-1} = \{\|z\| = 1\}$.

Consider the function $\psi(z) = c \log |z_1| = c \log \mathrm{dist}(z, H)$, where $H = \{z_1 = 0\}$. We claim that $e^{-\psi}$ is integrable in a neighborhood of $q = (0, \ldots, 0, 1)$ if and only if $\nu(\psi; q) = c < 2$. Notice that $\Sigma = H \cap S^{2n-1}$ is a sphere of dimension $2n-3$ passing through q. The restriction of the function $\mathrm{dist}(\cdot, H)$ to a neighborhood of q in S^{2n-1} is comparable to $\mathrm{dist}(\cdot, \Sigma)$, where the later is the distance with respect to the standard Riemannian metric on S^{2n-1}. Choosing local (real) coordinates in S^{2n-1} around q we reduce the problem to study the integrability of the function $\rho = 1/\mathrm{dist}(\cdot, V)^c$ in a neighborhood of the origin in $\mathbb{R}^{2n-1}$ where V is a linear subspace of dimension $2n-3$. It is not hard to check that such a ρ is integrable if and only if $c < 2$. This proves the above claim.

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