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Equidistribution of zeros of random holomorphic sections for moderate measures

Guokuan Shao

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Guokuan SHAO

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Abstract

This thesis investigates the equidistributions of zeros of random holomorphic sections of line bundles for moderate measures. It consists of two parts.

In the first part, we construct a large family of singular moderate measures on projective spaces. These measures are generated by quasi-plurisubharmonic functions with Hölder potentials.

The second part deals with an equidistribution property in general settings. We establish an equidistribution theorem in the case of several big line bundles endowed with singular metrics. A precise convergence speed for the equidistribution is obtained.

Key words: positive closed current, plurisubharmonic function, moderate measure, Dinh-Sibony equidistribution theorem, meromorphic transform, intermediate degree, multi-projective space, random holomorphic section, Hölder potential, big line bundle, Fubini-Study current.

Classification AMS 2010: 32A60, 32L10, 32U40.

Résumé

Cette thèse étudie les équidistributions de zéros de sections holomorphes aléatoires de fibrés en droites pour les mesures modérées. Elle consiste en deux parties.

Dans la première partie, nous construisons une famille étendue de mesures singulières modérées sur des espaces projectifs. Ces mesures sont générées par des fonctions quasi-plurisousharmoniques avec les potentiels höldériens.

Le deuxième partie traite une propriété d'équidistribution dans un contexte général. Nous établissons un théorème d'équidistribution dans le cas de quelques fibrés en droites gros munis de métriques singulières. Une vitesse de convergence précise pour l'équidistribution est obtenue.

Mots-clés: courant positif fermé, fonction plurisousharmonique, mesure modérée, théorème d'équidistribution de Dinh-Sibony, transformation méromorphe, degré intermédiaire, espace multi-projectif, section holomorphe aléatoire, potentiel höldérien, fibré en droites gros, courant de Fubini-Study.

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Chapter 0

Introduction.

Distribution of zeros of random polynomials is a classical subject. Waring [58] used a probabilistic method to determine the number of imaginary zeros of an algebraic polynomial. More rigorous and systematic research started with the paper of Bloch-Pólya [4] in 1930s. They gave an order of the expected number of real roots of certain random algebraic polynomial equations. Kac [40] obtained an exact distribution of real roots for random polynomials with normal distribution coefficients. Results about polynomials with other distributions or settings were extended notably. We refer the reader to [3, 2, 6, 7, 25, 41, 56] and references therein for more results and further discussions.

A classical theorem due to Hammersley [33] asserts that the normalized zeros of complex Gaussian random polynomials of large degree tend to accumulate on the unit circle with uniformly distribution. An analogous result holds for random $SU(2)$ polynomials whose expected distribution of zeros is uniform on \mathbb{P}^1 (cf. [3, Appendix C]). The general results about the holomorphic sections of a positive line bundle associated to the Lebesgue measures were obtained by Shiffman-Zelditch[57]. Note that the result covers the equidistribution property of random $SU(2)$ polynomials when $X = \mathbb{P}^1$ and $L = \mathcal{O}(1)$. Dinh-Sibony[23] extended the equidistribution property in the case of general measures and obtained a good estimate of the convergence speed. The potential-theoretic approach from Fornæss-Sibony[26] was used in the previous works.

0.1 Motivation and applications

The general measures in Dinh-Sibony's equidistribution theorems [23, Theorem 4.1, Theorem 7.3] have quite strict conditions. One explicit non-trivial example of a singular measure with real coefficients was given in [23, Corollary 7.4]. But it seems difficult to provide a wide class of general measures to satisfy the equidistribution theorems. Our motivation of the first part is to construct a large family of singular moderate measures which satisfies the equidistribution property. Roughly speaking, we prove that the normalized currents defined by the zeros of random sections of high powers of a positive

line bundle on a projective manifold converge weakly to the curvature form associated to certain family of singular moderate measures. Our method follows the techniques of exponential estimates for plurisubharmonic functions by Dinh-Nguy en-Sibony [21], see Section 2.3.

In the second part, we are motivated to generalize the equidistribution property in the case of several big line bundles endowed with singular metrics. We follow the approaches of Dinh-Sibony [23], Coman-Marinescu-Nguy en [14] and Section 2 [54]. To achieve our results, we establish sharp estimates on the constants related to Alexander-Dinh-Sibony capacity on multi-projective spaces [55], see Section 3.4.

The Alexander-Dinh-Sibony capacity is closely related to the global extremal function (cf. [23, A.2] and [32, Section 5]). The L^2 -method plays also an important role in the study of equidistribution properties. For example, we apply L^2 -method to explore the asymptotic behavior of Bergman kernel functions. Hence we can obtain equidistribution theorems with good convergence speed (cf. [10], [14], [15] etc). The equidistribution property provides a tool to study the Unique Ergodicity conjecture by Rudnick-Sarnak, see [51], [39], [49]. It can be also applied to study quantum chaos in statistical physics (cf. [3], [48] etc).

0.2 Part I

We give the basic setting in order to introduce the main theorems. For every complex vector space V of finite dimension, let ω_{FS} be the standard K ahler form induced by the Fubini-Study metric on its projective space $\mathbb{P}(V)$ normalized by $\int_{\mathbb{P}(V)} \omega_{FS}^{\dim \mathbb{P}(V)} = 1$. Let X be a projective manifold of dimension n , L be an ample line bundle over X . Fix a Hermitian metric h on L such that the curvature form ω is K ahler on X . Let L^p be the p th tensor product of L . Denote by $H^0(X, L^p)$ the space of all holomorphic sections of L^p . Let $\mathbb{P}H^0(X, L^p)$ be the associated projective space with ω_{FS} as its normalized Fubini-Study form. Set $n_p := \dim \mathbb{P}H^0(X, L^p)$. Let $[s_p = 0]$ be the current defined by the zero set of s_p , where $s_p \in \mathbb{P}H^0(X, L^p)$. Set $\mathbb{P}^X := \prod_{p \geq 1} \mathbb{P}H^0(X, L^p)$. See Section 1.2.3 for the notion of moderate measures.

Fix some exponent $0 < \rho < 1$, a function $u : M \rightarrow \mathbb{R}$ defined on a compact metric space (M, dist) is said to be of class \mathcal{C}^ρ with modulus c if

$$\sup_{\substack{x, y \in M \\ x \neq y}} \frac{|u(x) - u(y)|}{\text{dist}(x, y)^\rho} \leq c.$$

See Section 1.2 for the definition of γ -p.s.h. functions.

Let σ_p be the probability Lebesgue measure on $\mathbb{P}H^0(X, L^p)$ and σ be the product measure of these ones on \mathbb{P}^X . Shiffman-Zelditch [57] proved that the sequence of currents $\{\frac{1}{p}[s_p = 0]\}$ converges weakly to ω for σ -almost everywhere $(s_p) \in \mathbb{P}^X$. Dinh-Sibony [23] generalized the result and obtained a good estimate of the convergence speed over a projective manifold endowed with

a smooth positively-curved metric. They also constructed a singular measure with real coefficients which satisfies equidistribution property (cf. [23, Corollary 7.4]). When the Lebesgue measures in Shiffman-Zelditch's result are replaced by moderate measures with Hölder potentials (see Sections 2.3, 2.4), we have our main theorem as follows which gives a concrete large family of singular moderate measures that satisfies equidistribution property. It can be regarded as a perturbation of standard measures induced by Fubini-Study metric.

Theorem 0.2.1. *Let L be an ample line bundle over a projective manifold X of dimension n and $0 < \rho < 1$ be an exponent. Then there exists a constant $c = c(X, L, \rho) > 1$ with the following property. For each $p \geq 1$, $1 \leq j \leq n_p$, let $u_{p,j} : \mathbb{P}H^0(X, L^p) \rightarrow \mathbb{R}$ be a function and $\xi_p, \epsilon_p > 0$ two numbers such that*

- (i) $u_{p,j}$ is of class \mathcal{C}^ρ with modulus ξ_p , $\forall 1 \leq j \leq n_p$;
- (ii) $u_{p,j}$ is $\epsilon_p \omega_{FS}$ -p.s.h., $\forall 1 \leq j \leq n_p$;
- (iii) $\xi_p \leq 1/c^{p^n}$, $\epsilon_p \leq 1/c^{p^n}$.

Let $\sigma_p = (dd^c u_{p,1} + \omega_{FS}) \wedge (dd^c u_{p,2} + \omega_{FS}) \wedge \cdots \wedge (dd^c u_{p,n_p} + \omega_{FS})$ be the probability measure on $\mathbb{P}H^0(X, L^p)$. Endow \mathbb{P}^X with the product measure $\sigma = \prod_{p \geq 1} \sigma_p$. Then for almost every $s = (s_p) \in \mathbb{P}^X$ with respect to σ , the sequence of currents $\{\frac{1}{p}[s_p = 0]\}$ converges weakly to ω .

The following result gives a convergence speed for the equidistribution in Theorem 0.2.1.

Theorem 0.2.2. *In the setting of Theorem 0.2.1, there exist subsets $E_p \subset \mathbb{P}H^0(X, L^p)$ and a positive constant C depending only on X, L such that for all p sufficiently large, we have*

$$\sigma_p(E_p) \leq \frac{C}{p^2} \quad \text{and} \quad \left| \left\langle \frac{1}{p}[s_p = 0] - \omega, \psi \right\rangle \right| \leq \frac{C \log p}{p} \|\psi\|_{\mathcal{C}^2},$$

for any point $s_p \in \mathbb{P}H^0(X, L^p) \setminus E_p$ and any $(n-1, n-1)$ -form ψ of class \mathcal{C}^2 .

0.3 Part II

Two natural questions arise after solving the first part:

1. Is it possible to consider the case of more general line bundles, e.g. big line bundles?
2. Could we extend the zeros of sections of a single line bundle to the common zeros of sections of several ones?

Fortunately, the questions are solved in some directions in the context of probability Lebesgue measures. Such equidistribution problems and convergence speeds of holomorphic sections of singular Hermitian holomorphic line bundles have been intensively explored recently. Coman-Marinescu [10] extended

the equidistribution results of [57] and [23] in the case of a singular holomorphic line bundle endowed with a strictly positive-curved metric. Dinh-Marinescu [18] investigated the equidistribution for big line bundles endowed with semipositive-curved metrics. Coman-Marinescu [13] established the equidistribution results for singular holomorphic line bundles on a compact normal Kähler complex space. Our second part has been studied by Coman-Marinescu-Nguyễn [14] in the context of probability Lebesgue measures. We refer the reader to [11, 12] for a more detailed discussion.

Coman-Marinescu-Nguyễn [14] studied the equidistribution of common zeros of sections of several big line bundles. The measures of the equidistribution theorem in [14] are the standard ones induced by the Fubini-Study metric. On the other hand, our work [54] investigated the equidistribution of zeros of sections of a single positive holomorphic line bundle associated to moderate measures. The metric on the line bundle in the latter work is smooth. In this part, the normalized currents are defined by the common zeros of m -tuples of random sections of high powers of m singular Hermitian big line bundles on a compact Kähler manifold. When the measures in [14] are replaced by suitable moderate ones, we show that the above currents still distribute asymptotically to the wedge product of the curvature currents of the singular metrics. Consequently, we generalize both the main theorems in [14] and [54].

Our method follows the approach of Coman-Marinescu-Nguyễn [14]. Adapting their work, we prove that the intersections of currents of integration along subvarieties are well-defined almost everywhere with respect to a finite product of moderate measures (see Section 3.3). Moreover, their strategy of using Dinh-Sibony equidistribution theory leads us to obtain an estimate for the convergence speed (see Section 3.4 and Section 3.5). Then we combine the above with the technical analysis of moderate measures to achieve our results. Here our hard core work consists of estimating efficiently some constants which are intimately associated with multi-projective spaces (see Section 3.4 below).

We start with the basic settings of this part. Let X be a compact Kähler manifold of dimension n with a fixed smooth Kähler form ω . Recall that a singular Hermitian holomorphic line bundle (L, h) is a holomorphic line bundle L with a Hermitian metric which is given in any trivialization by a weight function $e^{-\varphi}$ such that φ is locally integrable (cf. Definition 1.3.13). Let $c_1(L, h)$ be its curvature current which represents the first Chern class. To be precise, if e_L is a holomorphic frame of L on an open subset $U \subset X$, then $|e_L|_h^2 = e^{-2\varphi}$, $c_1(L, h) = dd^c\varphi$ on U . The case when $c_1(L, h) \geq 0$ as a current is particularly interesting. We say that a holomorphic line bundle L is *big* if it admits a singular metric h with that $c_1(L, h) \geq \epsilon\omega$ for some constant $\epsilon > 0$ (cf. Theorem 1.3.23).

Let (L_k, h_k) , $1 \leq k \leq m \leq n$, be m singular Hermitian holomorphic line bundles on X . Let L_k^p be the p th tensor powers of L_k . Denote by $H_{(2)}^0(X, L_k^p)$ the Bergman space of L^2 -holomorphic sections of L_k^p relative to the metric $h_{k,p} := h_k^{\otimes p}$ induced by h_k and the volume form ω^n on X , endowed with the

inner product

$$\langle S, S' \rangle_{k,p} := \int_X h_{k,p}(S, S') \omega^n,$$

$\forall S, S' \in H_{(2)}^0(X, L_k^p)$. Let $\mathbb{P}H_{(2)}^0(X, L_k^p)$ be the associated projective space. Set $d_{k,p} := \dim H_{(2)}^0(X, L_k^p) - 1$. We have $p^n/C \leq d_{k,p} \leq Cp^n$, where $C > 0$ is a constant independent of k and p (cf. Theorem 3.4.1). Now we consider the multi-projective space

$$\mathbb{X}_p := \mathbb{P}H_{(2)}^0(X, L_1^p) \times \dots \times \mathbb{P}H_{(2)}^0(X, L_m^p)$$

endowed with a probability measure σ_p for every $p \geq 1$. Let $\pi_{k,p} : \mathbb{X}_p \rightarrow \mathbb{P}H_{(2)}^0(X, L_k^p)$ be the natural projections. Denote by $[S = 0]$ the current defined by the zero set of $S \in H^0(X, L_k^p)$. Set

$$[S_p = 0] := [S_{p1} = 0] \wedge \dots \wedge [S_{pm} = 0], \quad \forall S_p = (S_{p1}, \dots, S_{pm}) \in \mathbb{X}_p,$$

whenever it is well-defined. Let

$$\mathbb{P}^X := \prod_{p=1}^{\infty} \mathbb{X}_p.$$

It is a probability space with the product measure $\sigma = \prod_{p=1}^{\infty} \sigma_p$.

We denote by dist the distance on X induced by the fixed Kähler form ω . Let $\phi : U \rightarrow [-\infty, \infty)$ be a function on an open subset $U \subset X$, $A \subset X$ a proper analytic subset. Following the terminology in [14], ϕ is called *Hölder with singularities* along A if there are positive constants c, δ and $0 < \nu \leq 1$ satisfying that

$$|\phi(z) - \phi(w)| \leq \frac{c \text{dist}(z, w)^\nu}{\min\{\text{dist}(z, A), \text{dist}(w, A)\}^\delta}$$

for all $z, w \in U \setminus A$. A singular metric h of L is defined to be *Hölder with singularities* along A if every local weight of h is Hölder with singularities along A . For motivations as well as examples of such metrics, we refer the readers to [14].

The multi-projective space \mathbb{X}_p in [14] is equipped with the probability measure σ_p^0 which is the product of the Lebesgue measures induced by Fubini-Study metrics on the components. In this part, we define singular moderate measures σ_p as perturbations of σ_p^0 on \mathbb{X}_p . For each $p \geq 1, 1 \leq k \leq m, 1 \leq j \leq d_{k,p}$, let $u_j^{k,p} : \mathbb{P}H_{(2)}^0(X, L_k^p) \rightarrow \mathbb{R}$ be an upper-semi continuous function. Fix $0 < \rho < 1$ and a sequence of positive constants $\{c_p\}_{p \geq 1}$. We call $\{u_j^{k,p}\}$ a *family of (c_p, ρ) -functions* if all $u_j^{k,p}$ satisfy the following two conditions:

- $u_j^{k,p}$ is of class \mathcal{C}^ρ with modulus c_p ,
- $u_j^{k,p}$ is a $c_p \omega_{FS}$ -p.s.h.

Then for each $p \geq 1$, there is a probability measure

$$\sigma_p = \prod_{k=1}^m \bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^* (dd^c u_j^{k,p} + \omega_{FS})$$

on \mathbb{X}_p . By Theorem 0.2.1 and Remark 2.3.7, $\bigwedge_{j=1}^{d_{k,p}} (dd^c u_j^{k,p} + \omega_{FS})$ is a moderate measure on $\mathbb{P}H_{(2)}^0(X, L_k^p)$ when $c_p \leq 1/c^{p^n}$ for a suitable constant $c > 1$, $\forall 1 \leq k \leq m, p \geq 1$. The probability measure on \mathbb{P}^X

$$\sigma = \prod_{p=1}^{\infty} \sigma_p = \prod_{p=1}^{\infty} \prod_{k=1}^m \bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^* (dd^c u_j^{k,p} + \omega_{FS}) \quad (1)$$

is said to be generated by a family of (c_p, ρ) -functions $\{u_j^{k,p}\}$ on $\{\mathbb{P}H_{(2)}^0(X, L_k^p)\}$.

Here is our main theorem.

Theorem 0.3.1. *Let (X, ω) be a compact Kähler manifold of dimension n , (L_k, h_k) , $1 \leq k \leq m \leq n$, be m singular Hermitian holomorphic big line bundles on X . The metric h_k is continuous outside a proper analytic subset $A_k \subset X$, $c_1(L_k, h_k) \geq \epsilon \omega$ on X for some constant $\epsilon > 0$, and A_1, \dots, A_m are in general position. Let $0 < \rho < 1$. Then there exists a constant $c > 1$ which depends only on X, L_k, ρ with the following property: If σ is the probability measure on \mathbb{P}^X generated by a family of $(1/c^{p^n}, \rho)$ -functions $\{u_{k,p}^j\}$ on $\{\mathbb{P}H_{(2)}^0(X, L_k^p)\}$ defined by (2), then for almost every $\{S_p\}_{p \geq 1} \in \mathbb{P}^X$ with respect to σ , we have in the weak sense of currents as $p \rightarrow \infty$ on X ,*

$$\frac{1}{p^m} [S_p = 0] \rightarrow c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m).$$

Remark 0.3.2. *When all $u_{k,p}^j \equiv 0$, then σ_p are the Lebesgue measures σ_p^0 on \mathbb{X}_p and we obtain [14, Theorem 1.2]. In addition, the constant c is independent of the choices of singular metrics on the big line bundles.*

When the metrics h_k are all Hölder with singularities, we can also extend the result in [14] about the estimate of the speed of the above convergence associated to the moderate measures defined by (1). This theorem is also a generalization of Theorem 0.2.2.

Theorem 0.3.3. *We keep the notations and the hypotheses of Theorem 0.3.1. Suppose, moreover, that h_k is Hölder with singularities along A_k , $1 \leq k \leq m$. Then there exist a positive constant ξ which depends only on m , and another positive constant C which depends on $X, (L_1, h_1), \dots, (L_m, h_m)$ with the following property: Given any sequence of positive numbers $\{\lambda_p\}_{p=1}^{\infty}$ with the following conditions*

$$\liminf_{p \rightarrow \infty} \frac{\lambda_p}{\log p} > (1 + \xi n)C \quad \text{and} \quad \lim_{p \rightarrow \infty} \frac{\lambda_p}{p^n} = 0,$$

there exist subsets $E_p \subset \mathbb{X}_p$ such that for all p sufficiently large,

(i)

$$\sigma_p(E_p) \leq Cp^{\xi n} \exp\left(-\frac{\lambda_p}{C}\right),$$

(ii) for any point $S_p \in \mathbb{X}_p \setminus E_p$ and any $(n-m, n-m)$ -form ϕ of class \mathcal{C}^2 ,

$$\left| \left\langle \frac{1}{p^m} [S_p = 0] - \bigwedge_{k=1}^m c_1(L_k, h_k), \phi \right\rangle \right| \leq \frac{C\lambda_p}{p} \|\phi\|_{\mathcal{C}^2}.$$

Remark 0.3.4. *The primary difference between Theorem 0.3.3 and [14, Theorem 1.4] is that the measures in Theorem 0.3.3 are only moderate. The optimal estimate of the convergence speed in Theorem 0.3.3 (ii) is of order $O(\log p/p)$. In this case, the measures $\sigma_p(E_p)$ are polynomially small. So Theorem 0.3.3 generalizes Theorem 0.2.2 and [18, Theorem 1.2].*

Note that in Theorem 0.3.1 and all other equidistribution theorems studied in this thesis, the limit of the considered convergence sequence cannot be an arbitrarily given positive closed current. For example, there is a condition that $c_1(L_k, h_k) \geq \epsilon\omega$ in Theorem 0.3.1. However, it is possible when $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $m = 1$ with a probability measure σ chosen properly.

Theorem 0.3.5. *Given any positive closed current T of bidegree $(1, 1)$ with mass 1 on \mathbb{P}^n . Let $d_p = \dim H^0(\mathbb{P}^n, \mathcal{O}(p)) - 1$. Then there exists a family of smooth probability measures $\sigma_p = (\omega_{FS} + dd^c u_p)^{d_p}$ on $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$ for some smooth real functions u_p with the following property: For almost every $S = (S_p) \in \prod_{p \geq 1} \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$ with respect to $\sigma = \prod_{p \geq 1} \sigma_p$, we have in the weak sense of currents as $p \rightarrow \infty$ on \mathbb{P}^n ,*

$$\frac{1}{p} [S_p = 0] \rightarrow T.$$

Introduction. (French version)

Distribution des zéros de polynômes aléatoires est un sujet classique. Waring [58] a utilisé une méthode probabiliste pour déterminer le nombre de zéros imaginaires d'un polynôme algébrique. Des recherches plus rigoureuse et systématique a commencé avec le papier de Bloch-Pólya [4]. Ils ont donné un ordre du nombre espérance de racines réelles de certaines équations polynômes aléatoires. Kac [41] a obtenu une distribution exacte des racines réelles pour les polynômes aléatoires avec des coefficients de distribution normaux. Résultats sur polynômes avec d'autres distributions ou paramètres ont été étendues notamment. Voir [3, 2, 6, 7, 25, 41, 56] et références qui y sont pour plus de résultats.

Un théorème classique de Hammersley [33] affirme que les zéros normalisés de gaussien complexe polynômes aléatoires de degré élevé ont tendance à accumuler sur le cercle unité avec une distribution uniforme. Un résultat analogue est valable pour $SU(2)$ polynômes aléatoire dont la distribution espérance de zéros est uniforme sur \mathbb{P}^1 (cf. [3, Appendix C]). Les résultats généraux sur les sections holomorphes de fibrés en droites positive associée aux mesures de Lebesgue ont été obtenues par Shiffman-Zelditch [57]. Grosso modo, presque partout séquence de courants normalisés définis par sections holomorphes converge faiblement vers la forme Kählerienne par rapport aux mesures de Lebesgue. Notez que le résultat couvre la propriété équidistribution de $SU(2)$ polynômes aléatoire lorsque $X = \mathbb{P}^1$ et $L = \mathcal{O}(1)$. Dinh-Sibony [23] étendu la propriété équidistribution dans le cas de mesures générales et obtenu une bonne estimation de la vitesse de convergence. L'approche de potentiel théorique de Fornæss-Sibony [26] a été utilisé dans les travaux précédent.

Motivation et applications

Les mesures générales dans les théorèmes de équidistribution de Dinh-Sibony [23, Theorem 4.1, Theorem 7.3] avoir des conditions très strictes. Un exemple explicite non-trivial d'une mesure singulière avec des coefficients réels a été donnée dans [23, Corollary 7.4]. Mais il semble difficile de fournir une

large classe de mesures générales pour satisfaire les théorèmes équidistribution.

Notre motivation de la première partie est de construire une grande famille des mesures modérées singulières qui satisfait la propriété équidistribution. Nous montrons que les courants normalisés définis par les zéros de sections aléatoires de puissances élevées de un fibré en droites positive sur une variété projective convergent faiblement à la forme de courbure associée à certaines familles de mesures modérées singulières. Notre méthode suit les techniques d'estimations exponentielles pour les fonctions plurisousharmoniques par Dinh-Nguyễn-Sibony [21], voir la section 2.3.

Dans la deuxième partie, nous sommes motivés à généraliser la propriété équidistribution dans le cas de plusieurs fibrés en droites gros dotés de métriques singulières. Nous suivons les approches de Dinh-Sibony [23], Coman-Marinescu-Nguyễn [14] et [54, section 2]. Pour atteindre nos résultats, nous établissons des estimations pointues sur les constantes liées à la capacité Alexander-Dinh-Sibony sur les espaces multi-projectives [55], voir la section 3.4.

La capacité Alexander-Dinh-Sibony est étroitement liée à la fonction globale extrémal (cf. [23, A.2] et [32, Section 5]). Le L^2 -méthode joue aussi un rôle important dans l'étude des propriétés équidistribution. Par exemple, nous appliquons L^2 -méthode pour explorer le comportement asymptotique des fonctions du noyau de Bergman. Par conséquent, nous pouvons obtenir des théorèmes de équidistribution avec une bonne vitesse de convergence (cf. [10], [14], [15] etc). La propriété équidistribution fournit un outil pour étudier la conjecture Unique Ergodicité par Rudnick-Sarnak, voir [51], [39], [49]. Il peut aussi être appliquée à l'étude du chaos quantique en physique statistique (cf. [3], [48] etc).

Partie I

Nous donnons le fond de base, afin d'introduire les principaux théorèmes. Pour chaque espace vectoriel complexe V de dimension finie, soit ω_{FS} la forme Kählerienne standard induite par la métrique de Fubini-Study sur son espace projectif $\mathbb{P}(V)$ normalisé par $\int_{\mathbb{P}(V)} \omega_{FS}^{\dim \mathbb{P}(V)} = 1$. Soit X une variété projective de dimension n , L un fibré en droites ample sur X . Fixez d'une métrique hermitienne h sur L telle que la forme de courbure ω est Kähler sur X . Alors ω représente la première classe de Chern $c_1(L)$ avec $\int_X \omega^k = c_1(L)^k \in \mathbb{Z}^+$.

Soit L^p le p -produit tensoriel de L avec la métrique $h_p := h^{\otimes p}$. Désignons par $H^0(X, L^p)$ l'espace de toutes les sections holomorphes de L^p . Soit $\mathbb{P}H^0(X, L^p)$ l'espace projectif associé à ω_{FS} comme sa forme Fubini-Study normalisée. Laissez $n_p := \dim \mathbb{P}H^0(X, L^p)$. Définissez un produit intérieur sur $H^0(X, L^p)$ de la façon suivante

$$\langle s, s' \rangle := \int_X h_p(s, s') \omega^n,$$

$\forall s, s' \in H^0(X, L^p)$. Nous pouvons choisir une base orthonormée $\{s_{pj}\}_{j=0}^{n_p}$ par rapport au produit intérieur. Pour toute $s_p \in \mathbb{P}H^0(X, L^p)$, écrivez

$$s_p := \sum_{j=0}^{n_p} a_{pj} s_{pj}.$$

Ainsi, la section s_p peut être identifié avec les coefficients $(a_{p0}, \dots, a_{pn_p})$. La notion de sections holomorphes aléatoires découle des variables aléatoires $[a_{p0}, \dots, a_{pn_p}]$ avec une distribution induites par la mesure de probabilité $\omega_{FS}^{n_p}$ sur $\mathbb{P}H^0(X, L^p)$. Soit $[s_p = 0]$ le courant défini par l'ensemble zéro de s_p , où $s_p \in \mathbb{P}H^0(X, L^p)$. Laissez $\mathbb{P}^X := \prod_{p \geq 1} \mathbb{P}H^0(X, L^p)$.

Nous avons besoin de la notion de mesures modérées localement qui a été introduite par Dinh-Sibony [24]. La notion découle d'un résultat classique [36, Theorem 4.4.5].

Définition. *Soit X une variété complexe. Une mesure positive μ est modérée localement si pour tout ouvert $U \subset X$, un sous-ensemble compact $K \subset U$ et toute la famille compacte \mathcal{U} de q.p.s.h. fonctions sur U , il y a des constantes $\alpha > 0, c > 0$ tel que*

$$\int_K \exp(-\alpha\phi) d\mu \leq c, \quad \forall \phi \in \mathcal{U}.$$

Dinh-Nguyễn-Sibony [21] ont prouvé que la mesure de l'équilibre de endomorphisme de \mathbb{P}^n est modérées localement. Dans la thèse, nous traitons des mesures modérées dans une variété compacte Kählerienne X de dimension n avec une forme Kählerienne ω fixe tel que ω^n est la forme de volume de probabilité. Considérons une mesure positive μ sur X , μ est dit PLB si tout les q.p.s.h. fonctions sont μ -intégrable. Lorsque $\dim X = 1$, μ est PLB si et seulement si elle admet un potentiel borné locale [24].

Laisser

$$\mathcal{F} := \{\phi \text{ q.p.s.h. on } X : dd^c \phi \geq -\omega, \max_X \phi = 0\}.$$

L'ensemble \mathcal{F} est compact dans $L^p(X)$ et bornée dans $L^1(\mu)$ quand μ est une mesure PLB par Théorème 1.2.33 et Proposition 1.2.47. Rappelons que toute

q.p.s.h. fonctions sur X sont $c\omega$ -p.s.h. pour une constante $c \geq 0$ et bornée uniformément d'en haut. Ensuite, l'étude de q.p.s.h. fonctions peuvent être réduits à l'étude de la famille \mathcal{F} .

Définition. Soit μ un PLB mesure sur X . Nous disons que μ est (c, α) -modérée pour certains constantes $c > 0, \alpha > 0$ si

$$\int_X \exp(-\alpha\phi) d\mu \leq c$$

pour tous les $\phi \in \mathcal{F}$. La mesure μ est appelé modérée s'il existe des constantes $c > 0, \alpha > 0$ tel qu'il est (c, α) -modérée.

Par exemple, la forme de volume ω^n est modérée par Théorème 1.2.42. Voir Section 1.2 pour la notion de mesures modérées en détail.

Afin d'exprimer notre théorème, nous devons aussi les terminologies suivantes. Fixez de quelque exposant $0 < \rho < 1$, une fonction $u : M \rightarrow \mathbb{R}$ définie sur un espace métrique compact (M, dist) est dit être de la classe \mathcal{C}^ρ avec module c si

$$\sup_{\substack{x, y \in M \\ x \neq y}} \frac{|u(x) - u(y)|}{\text{dist}(x, y)^\rho} \leq c.$$

Considérons une variété complexe M avec une forme de volume fixe, soit γ un courant réel clos de bidegré $(1, 1)$ sur M . Une fonction semi-continue supérieurement $u : M \rightarrow [-\infty, \infty)$ en $L_{loc}^1(M)$ est dit γ -p.s.h. si $dd^c u + \gamma \geq 0$.

Soit σ_p la mesure probabilité de Lebesgue sur $\mathbb{P}H^0(X, L^p)$ et σ la mesure produit de celles-ci sur \mathbb{P}^X . Shiffman-Zelditch [57] a prouvé que la séquence des courants $\{\frac{1}{p}[s_p = 0]\}$ converge faiblement vers ω pour σ -presque partout $(s_p) \in \mathbb{P}^X$. C'est-à-dire,

$$\frac{1}{p}[s_p = 0] \rightarrow \omega$$

pour σ -presque partout $(s_p) \in \mathbb{P}^X$. Les ingrédients clés pour prouver le théorème sont le théorème de Tian [9, 60] (autres résultats sur les comportements asymptotiques des noyaux Bergman) et les estimations des variances des courants $[s_p = 0]$. Quand $X = \mathbb{P}^n, \omega = \omega_{FS}$ et $L = \mathcal{O}(1)$ le dual de fibré en droites tautologique, le théorème est d'étudier la distribution des zéros de polynômes homogènes aléatoire ou les polynômes complexes correspondants. La distribution du $SU(2)$ polynômes est un cas particulier, lorsque $n = 1$. Si on considère les polynômes aléatoires complexes d'une variable et le produit scalaire défini sur le cercle unitaire de \mathbb{C} avec la mesure de Haar, on obtient alors le résultat classique de Hammersley [33].

Dinh-Sibony [23] ont généralisé le résultat dans le cas de mesures PLB et obtenu une bonne estimation de la vitesse de convergence sur une variété projective doté d'une métrique positivement courbe lisse. Voir Section 2.1 et Section 2.3 pour les définitions et notations en détail.

Théorème. *Dans le cadre du théorème de Shiffman-Zelditch, les mesures de probabilité $\omega_{FS}^{n_p}$ sur $\mathbb{P}H^0(X, L^p)$ sont remplacés par des mesures de probabilité PLB σ_p avec les conditions suivantes:*

$$\sum_{p=1}^{\infty} \Delta_p(pt) < \infty, \forall t > 0,$$

$$\frac{R_p}{p} \rightarrow 0.$$

Alors la convergence faible

$$\frac{1}{p} [s_p = 0] \rightarrow \omega$$

est toujours vrai pour σ -presque partout $(s_p) \in \mathbb{P}^X$.

Les ingrédients clés pour prouver le théorème sont quelques méthodes pluripotentes, notion de transformations méromorphes et dd^c -méthode. Ils ont aussi construit une mesure singulière à coefficients réels qui satisfait la propriété équidistribution (cf. [23, Corollaire 7.4]).

Lorsque les mesures de Lebesgue dans le résultat de Shiffman-Zelditch sont remplacées par des mesures modérées avec Hölder potentiels (voir sections 2.3, 2.4), nous avons notre théorème principal de la manière suivante qui donne une grande famille de mesures modérées singulières qui satisfait la propriété équidistribution. Il peut être considéré comme une perturbation des mesures standard induites par la métrique Fubini-Study.

Théorème. *Soit L un fibré en droites ample sur une variété projective X de dimension n et $0 < \rho < 1$ un exposant. Alors il existe une constante $c = c(X, L, \rho) > 1$ avec la propriété suivante. Pour chaque $p \geq 1$, $1 \leq j \leq n_p$, soit $u_{p,j} : \mathbb{P}H^0(X, L^p) \rightarrow \mathbb{R}$ une fonction et $\xi_p, \epsilon_p > 0$ deux nombres tels que:*

- (i) $u_{p,j}$ est de classe \mathcal{C}^ρ avec module ξ_p , $\forall 1 \leq j \leq n_p$;
- (ii) $u_{p,j}$ est $\epsilon_p \omega_{FS}$ -p.s.h., $\forall 1 \leq j \leq n_p$;
- (iii) $\xi_p \leq 1/c^{p^n}$, $\epsilon_p \leq 1/c^{p^n}$.

Soit $\sigma_p = (dd^c u_{p,1} + \omega_{FS}) \wedge (dd^c u_{p,2} + \omega_{FS}) \wedge \cdots \wedge (dd^c u_{p,n_p} + \omega_{FS})$ la mesure de probabilité sur $\mathbb{P}H^0(X, L^p)$. Doter \mathbb{P}^X avec la mesure produit $\sigma = \prod_{p \geq 1} \sigma_p$. Ensuite, pour presque partout $s = (s_p) \in \mathbb{P}^X$ par rapport à σ , la séquence des courants $\{\frac{1}{p} [s_p = 0]\}$ converge faiblement vers ω .

Le résultat suivant donne une vitesse de convergence pour l'équidistribution dans le théorème 0.2.1.

Théorème. *Dans le cadre du théorème 0.2.1, il existe des sous-ensembles $E_p \subset \mathbb{P}H^0(X, L^p)$ et une constante positive C dépendant seulement X, L tel que pour tout p suffisamment grand, nous avons*

$$\sigma_p(E_p) \leq \frac{C}{p^2}$$

$$|\langle \frac{1}{p}[s_p = 0] - \omega, \psi \rangle| \leq \frac{C \log p}{p} \|\psi\|_{\mathcal{C}^2},$$

pour tout point $s_p \in \mathbb{P}H^0(X, L^p) \setminus E_p$ et des $(n-1, n-1)$ -forme ψ de la classe \mathcal{C}^2 .

Partie II

Deux questions se posent naturelles après avoir résolu la première partie:

1. Il est possible de considérer le cas de plusieurs fibrés en droites générales, par exemple fibrés en droites gros?
2. On peut étendre les zéros de sections d'un seul fibré en droites aux zéros communs des sections de plusieurs ceux?

Heureusement, les questions sont résolues dans certaines directions dans le contexte de mesures de probabilité de Lebesgue. Ces problèmes déquidistribution et des vitesses de convergence des sections holomorphes de fibrés en droites singuliers holomorphes hermitiennes ont été intensivement exploré récemment.

Coman-Marinescu [10] ont étendu les résultats déquidistribution de [57] et [23] dans le cas d'un fibré en droites holomorphe singulier doté une métrique strictement positive-courbe. Dinh-Ma-Marinescu [18] étudié l'équidistribution pour fibrés en droites gros dotés métriques semipositif-courbées. Coman-Ma-Marinescu [13] ont établi les résultats déquidistribution pour fibrés en droites holomorphes singuliers sur une espace complexe compacte normale Kählerienne. Notre deuxième partie a été étudiée par Coman-Marinescu-Nguyên [14] dans le contexte de mesures de probabilité de Lebesgue. Voir [11, 12] pour une discussion plus détaillée.

Coman-Marinescu-Nguyên [14] étudié l'équidistribution de zéros communs de sections de plusieurs fibrés en droites gros. La mesure du théorème déquidistribution dans [14] est la seule norme induite par la métrique de Fubini-Study. D'autre part, notre travail [54] a étudié l'équidistribution des zéros de sections d'un seul fibré en droites holomorphe positif associé à des mesures modérées. La

métrique sur le fibré en droites dans ce dernier ouvrage est lisse.

Dans cette partie, les courants normalisés sont définis par les zéros communs de m tuples des sections aléatoires de puissances élevées de m fibrés en droites gros hermitiens singulier sur une variété compacte Kählerienne. Nous montrons qu'ils distribuent asymptotiquement vers le produit des courants de courbure des métriques singulières associées à certaines mesures modérées. Par conséquent, nous généralisons les deux théorèmes principaux [14] et [54].

Notre méthode suit l'approche de Coman-Marinescu-Nguyên [14]. Adapter leur travail, nous montrons que les intersections des courants d'intégration le long de sous-variétés sont bien définis presque partout par rapport à un produit fini de mesures modérées (voir Section 3.3). De plus, leur stratégie d'utilisation de la théorie déquidistribution Dinh-Sibony nous conduit à obtenir une estimation de la vitesse de convergence (voir Section 3.4 et Section 3.5). Ensuite, nous combinons ce qui précède à l'analyse technique des mesures modérées pour atteindre nos résultats, dans lequel notre travail de base est dans les estimations des constantes sur les espaces multi-projectives (voir section 3.4).

Nous commençons avec les fonds de base de cette partie. Soit X une variété compacte Kählerienne de dimension n avec une forme Kählerienne lisse ω . Rappelons qu'un fibré en droites singulier hermitienne holomorphe (L, h) est un fibré en droites holomorphe L avec une métrique hermitienne qui est donnée dans toute trivialisations par un poids fonction $e^{-\varphi}$ tels que φ est localement intégrable (cf. Définition 1.3.13). Soit $c_1(L, h)$ son courant de courbure qui représente la première classe de Chern. Pour être précis, si e_L est un cadre holomorphe de L sur un sous-ensemble ouvert $U \subset X$, alors $|e_L|_h^2 = e^{-2\varphi}$, $c_1(L, h) = dd^c\varphi$ sur U . Ici $d = \partial + \bar{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$. Le cas où $c_1(L, h) \geq 0$ comme un courant est particulièrement intéressant. On dit qu'un fibré en droites holomorphe L est *gros* si elle admet une métrique singulière h avec $c_1(L, h) \geq \epsilon\omega$ pour une constante $\epsilon > 0$ (cf. Théorème 1.3.23).

Soit (L_k, h_k) , $1 \leq k \leq m \leq n$, m fibrés en droites singuliers hermitiennes holomorphes sur X . Soit L_k^p le p -produit tensoriel de L_k . Désignons par $H_{(2)}^0(X, L_k^p)$ l'espace Bergman de L^2 -sections holomorphes de L_k^p par rapport au métrique $h_{k,p} := h_k^{\otimes p}$ induite par h_k et le volume forme ω^n sur X , muni du produit intérieur

$$\langle S, S' \rangle_{k,p} := \int_X h_{k,p}(S, S') \omega^n,$$

$\forall S, S' \in H_{(2)}^0(X, L_k^p)$. Soit $\mathbb{P}H_{(2)}^0(X, L_k^p)$ l'espace projectif associé. Laissez $d_{k,p} := \dim H_{(2)}^0(X, L_k^p) - 1$. Il est bien connu (voir Proposition 1.3.24) que

$$d_{k,p} = O(p^n).$$

Maintenant, nous considérons l'espace multi-projective

$$\mathbb{X}_p := \mathbb{P}H_{(2)}^0(X, L_1^p) \times \dots \times \mathbb{P}H_{(2)}^0(X, L_m^p)$$

doté d'une mesure de probabilité σ_p pour chaque $p \geq 1$. Soit $\pi_{k,p} : \mathbb{X}_p \rightarrow \mathbb{P}H_{(2)}^0(X, L_k^p)$ les projections naturelles. Désignons par $[S = 0]$ le courant défini par l'ensemble zéro de $S \in H^0(X, L_k^p)$. Soit

$$[S_p = 0] := [S_{p1} = 0] \wedge \dots \wedge [S_{pm} = 0], \quad \forall S_p = (S_{p1}, \dots, S_{pm}) \in \mathbb{X}_p,$$

chaque fois qu'il est bien définie. Laisser

$$\mathbb{P}^X := \prod_{p=1}^{\infty} \mathbb{X}_p.$$

Il est un espace de probabilité avec la mesure produit $\sigma = \prod_{p=1}^{\infty} \sigma_p$.

On note dist la distance sur X induite par la forme Kählerienne ω . Soit $\phi : U \rightarrow [-\infty, \infty)$ une fonction sur un sous-ensemble ouvert $U \subset X$, $A \subset X$ un sous-ensemble analytique approprié. Conformément à la terminologie dans [14], ϕ est appelé *Hölder avec singularités* le long A s'il y a des constantes positives c, δ et $0 < \nu \leq 1$ satisfaisant qui

$$|\phi(z) - \phi(w)| \leq \frac{c \text{dist}(z, w)^\nu}{\min\{\text{dist}(z, A), \text{dist}(w, A)\}^\delta}$$

pour tout $z, w \in U \setminus A$. Une métrique singulière h de L est définie comme *Hölder avec singularités* le long A si chaque poids local h est Hölder avec singularités le long de A . Pour motivations ainsi que des exemples de ces mesures, voir [14].

Le multi-espace projectif \mathbb{X}_p dans [14] est équipé de la mesure de probabilité σ_p^0 qui est le produit des mesures de Lebesgue induites par les mesures Fubini-Study sur les composants. Dans cette partie, nous définissons des mesures modérées singulières σ_p comme des perturbations de σ_p^0 sur \mathbb{X}_p . Pour chaque $p \geq 1, 1 \leq k \leq m, 1 \leq j \leq d_{k,p}$, laissez $u_j^{k,p} : \mathbb{P}H_{(2)}^0(X, L_k^p) \rightarrow \mathbb{R}$ une fonction semi-continue supérieurement. Fixez $0 < \rho < 1$ et une séquence de constantes positives $\{c_p\}_{p \geq 1}$. Nous appelons $\{u_j^{k,p}\}$ une famille de (c_p, ρ) -fonctions si tout $u_j^{k,p}$ satisfont les deux conditions suivantes:

- $u_j^{k,p}$ est de classe \mathcal{C}^ρ avec module c_p ,
- $u_j^{k,p}$ est $c_p \omega_{FS}$ -p.s.h..

Ensuite, pour chaque $p \geq 1$, il est une mesure de probabilité

$$\sigma_p = \prod_{k=1}^m \bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^* (dd^c u_j^{k,p} + \omega_{FS})$$

sur \mathbb{X}_p . D'après Théorème 0.2.1 et Remarque 2.3.7, $\bigwedge_{j=1}^{d_{k,p}} (dd^c u_j^{k,p} + \omega_{FS})$ est une mesure modérée sur $\mathbb{P}H_{(2)}^0(X, L_k^p)$ quand $c_p \leq 1/c^{p^n}$ pour un constant approprié $c > 1$, $\forall 1 \leq k \leq m, p \geq 1$. La mesure de probabilité sur \mathbb{P}^X

$$\sigma = \prod_{p=1}^{\infty} \sigma_p = \prod_{p=1}^{\infty} \prod_{k=1}^m \bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^* (dd^c u_j^{k,p} + \omega_{FS})$$

est dit être généré par une famille de (c_p, ρ) -fonctions $\{u_j^{k,p}\}$ sur $\{\mathbb{P}H_{(2)}^0(X, L_k^p)\}$.

Voici notre théorème principal.

Théorème. *Soit (X, ω) une variété compacte Kählerienne de dimension n , (L_k, h_k) , $1 \leq k \leq m \leq n$, soit m fibrés en droites gros singulières hermitiennes holomorphes sur X . La métrique h_k est continue en dehors d'un sous-ensemble analytique propre $A_k \subset X$, $c_1(L_k, h_k) \geq \epsilon \omega$ sur X pour une constante $\epsilon > 0$, et A_1, \dots, A_m sont en position générale. Soit $0 < \rho < 1$. Alors il existe une constante $c > 1$ qui ne dépend que X, L_k, ρ avec la propriété suivante: Si σ est la mesure de probabilité sur \mathbb{P}^X généré par une famille de $(1/c^{p^n}, \rho)$ -fonctions $\{u_{k,p}^j\}$ sur $\{\mathbb{P}H_{(2)}^0(X, L_k^p)\}$ défini par (2), alors pour presque partout $\{S_p\}_{p \geq 1} \in \mathbb{P}^X$ par rapport à σ , nous avons dans le sens faible des courants comme $p \rightarrow \infty$ sur X ,*

$$\frac{1}{p^m} [S_p = 0] \rightarrow c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m).$$

Remarque. *Quand tout $u_{k,p}^j \equiv 0$, alors σ_p sont les mesures de Lebesgue σ_p^0 sur \mathbb{X}_p et nous obtenons [14, Théorème 1.2]. En outre, la constante c est indépendante des choix de métriques singulières sur les fibrés en droites gros.*

Lorsque la métrique h_k sont tous Hölder avec singularités, nous pouvons aussi étendre le résultat dans [14] sur l'estimation de la vitesse de la convergence ci-dessus associée aux mesures modérées définies par (1). Ce théorème est aussi une généralisation du théorème 0.2.2.

Théorème. *Nous gardons les notations et les hypothèses du Théorème 0.3.1. Supposons, en outre, que h_k est Hölder avec singularités le long A_k , $1 \leq k \leq m$. Alors il existe une constante ξ positive qui ne dépend que de m , et une autre constante positive C qui dépend $X, (L_1, h_1), \dots, (L_m, h_m)$ avec la propriété suivante: Compte tenu de toute séquence des nombres positifs $\{\lambda_p\}_{p=1}^{\infty}$ avec les conditions suivantes*

$$\liminf_{p \rightarrow \infty} \frac{\lambda_p}{\log p} > (1 + \xi n)C \quad \text{and} \quad \lim_{p \rightarrow \infty} \frac{\lambda_p}{p^n} = 0,$$

il existe des sous-ensembles $E_p \subset \mathbb{X}_p$ tel que pour tout p suffisamment grand,

(i)

$$\sigma_p(E_p) \leq Cp^{\xi n} \exp\left(-\frac{\lambda_p}{C}\right),$$

(ii) pour tout point $S_p \in \mathbb{X}_p \setminus E_p$ et des $(n-m, n-m)$ -forme ϕ de classe \mathcal{C}^2 ,

$$\left| \left\langle \frac{1}{p^m} [S_p = 0] - \bigwedge_{k=1}^m c_1(L_k, h_k), \phi \right\rangle \right| \leq \frac{C\lambda_p}{p} \|\phi\|_{\mathcal{C}^2}.$$

Remarque. La principale différence entre Théorème 0.3.3 et [14, Théorème 1.4] est que les mesures en Théorème 0.3.3 ne sont que modérées. L'estimation optimale de la vitesse de convergence dans Théorème 0.3.3 (ii) est d'ordre $O(\log p/p)$. Dans ce cas, les mesures $\sigma_p(E_p)$ sont polynomialement petite. Donc Théorème 0.2.2 est un cas particulier du Théorème 0.3.3.

Notez que dans Théorème 0.3.1 et toutes les autres théorèmes équidistribution étudiés dans cette thèse, la limite de la séquence de convergence ne peut pas être un courant positif fermé arbitrairement donné. Par exemple, il y a une condition que $c_1(L_k, h_k) \geq \epsilon\omega$ dans Théorème 0.3.1. Cependant, il est possible lorsque $X = \mathbb{P}^n, L = \mathcal{O}(1), m = 1$ avec une mesure de probabilité σ choisi correctement.

Théorème. Compte tenu de tout courant positif fermé T de bidegré $(1, 1)$ de la masse 1 sur \mathbb{P}^n . Soit $d_p = \dim H^0(\mathbb{P}^n, \mathcal{O}(p)) - 1$. Alors il existe une famille de mesures de probabilité lisses $\sigma_p = (\omega_{FS} + dd^c u_p)^{d_p}$ sur $\mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$ pour certaines fonctions réelles lisses u_p avec la propriété suivante: Pour presque partout $S = (S_p) \in \prod_{p \geq 1} \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$ par rapport à $\sigma = \prod_{p \geq 1} \sigma_p$, nous avons dans le sens faible des courants comme $p \rightarrow \infty$ sur \mathbb{P}^n ,

$$\frac{1}{p} [S_p = 0] \rightarrow T.$$

Chapter 1

Preliminaries.

This chapter supplies the definitions, notations and background needed in the thesis. We mainly introduce currents, plurisubharmonic functions and Hermitian holomorphic line bundles. This chapter follows basically Demailly's book [16]. See [16], [22], [28], [34], [42] and [45] for complete references.

1.1 Currents.

The notion of currents was introduced by Georges de Rham. It generalizes the notion of distributions.

1.1.1 Currents on complex manifolds

Let X be a smooth complex manifold of dimension n . A differential form ϕ of bidegree (p, q) is a section of $\bigwedge^{p,q} T^*X$. In a local coordinate, we denote by (dz_1, \dots, dz_n) the corresponding basis of the cotangent space at a point. Let (x_1, \dots, x_{2n}) be the local real coordinate of (z_1, \dots, z_n) . We can write

$$\phi(z) = \sum_{|I|=p, |J|=q} \phi_{I,J} dz_I \wedge d\bar{z}_J,$$

where $\phi_{I,J}$ are complex smooth functions, $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ when $I = (i_1, \dots, i_p)$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$ when $J = (j_1, \dots, j_q)$. Let $\mathcal{D}^{p,q}(X)$ be the set of all differential forms of bidegree (p, q) with compact support. We

introduce a topology on $\mathcal{D}^{p,q}(X)$. If Ω is a subset of X , we denote $\mathcal{D}^{p,q}(\Omega)$ the space of all elements $\phi \in \mathcal{D}^{p,q}(X)$ with compact support in Ω . Let $\{\Omega_j\}_{j=1}^\infty$ be a sequence of relatively compact open subsets of X with $\bar{\Omega}_j \subset \Omega_{j+1}$ for every j , and $\cup_{j=1}^\infty \Omega_j = X$. To every compact subset K in a local coordinate and every integer $l \in \mathbb{N}$, we define a semi-norm

$$\|\phi\|_{l,K} := \sup_{z \in K} \max_{\substack{|I|=p, |J|=q \\ |\alpha| \leq l}} |D^\alpha \phi_{I,J}(z)|,$$

where $\alpha = (\alpha_1, \dots, \alpha_{2n})$ run over \mathbb{N}^{2n} and $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_{2n}^{\alpha_{2n}}$ is a derivation of order $|\alpha|$. Since X is assumed to be separable, we can equip the space $\mathcal{D}^{p,q}(\bar{\Omega})$ with the topology induced by a countable set of semi-norm $\|\cdot\|_{l,K}$ with K contained in $\bar{\Omega}$. Such topology is called the topology of the uniform convergence of coefficients and all the derivatives. Then $\mathcal{D}^{p,q}(\bar{\Omega})$ is a Fréchet space with the topology. Hence we equip $\mathcal{D}^{p,q}(X)$ with the topology of the strict inductive limit of the spaces $\mathcal{D}^{p,q}(\bar{\Omega}_j)$.

Definition 1.1.1. A current of bidegree (p, q) (or (p, q) -current) on X is a linear continuous form $T : \mathcal{D}^{n-p, n-q}(X) \rightarrow \mathbb{C}$. Let ϕ be a form in $\mathcal{D}^{n-p, n-q}(X)$, the value of T at ϕ is denoted by $\langle T, \phi \rangle$. The form ϕ is called a test form.

A (p, q) -current can also be called a current of bidimension $(n - p, n - q)$.

Definition 1.1.2. A sequence of (p, q) -currents $\{T_j\}$ converges weakly (or converges in the sense of currents) to a current T if $\langle T_j, \phi \rangle \rightarrow \langle T, \phi \rangle$ for any test form $\phi \in \mathcal{D}^{n-p, n-q}(X)$.

Denote by $\mathcal{D}'^{p,q}(X)$ the set of all (p, q) -currents. It is the dual of $\mathcal{D}^{p,q}(X)$. Note that a (p, q) -current T can be regarded as a form with distribution coefficients:

$$T = \sum_{|I|=p, |J|=q} T_{I,J} dz_I \wedge d\bar{z}_J,$$

where $T_{I,J}$ are distributions.

Example 1.1.3. A form ψ of bidegree (p, q) is a (p, q) -current T_ψ via exterior product as follows.

$$\langle T_\psi, \phi \rangle = \int_X \psi \wedge \phi, \quad \forall \phi \in \mathcal{D}^{n-p, n-q}(X).$$

Example 1.1.4. Let Y be a closed complex submanifold of X of dimension p , then we can define the current of integration over Y by

$$\langle [Y], \phi \rangle = \int_Y \phi, \quad \forall \phi \in \mathcal{D}^{p,p}(X).$$

If $X = \mathbb{C}^n$ and $Y = \{z_1 = \dots = z_{n-p} = 0\}$, then we have

$$[Y] = \left(\frac{i}{2}\right)^p \delta_0(z_1, \dots, z_{n-p}) \otimes 1(z_I) dz_I \wedge d\bar{z}_I,$$

where $I = (n - p + 1, \dots, n)$, δ_0 is the Dirac measure at the origin of the space $\{z_I = 0\}$.

The above examples explain the terminology of dimension and degree used for a current.

Since a current is a continuous form in the topology defined above, we have the following property.

Proposition 1.1.5. *Let T be a (p, q) -current on X and $K \subset X$ be a compact subset. Then there exist a positive integer l and a positive constant C such that*

$$|\langle T, \phi \rangle| \leq C \|\phi\|_{l, K},$$

for every $\phi \in \mathcal{D}^{n-p, n-q}(X)$ with $\text{supp}(\phi) \subset K$.

Definition 1.1.6. *If the integer l in Proposition 1.1.5 can be chosen independently of K , the current T is called of finite order. The smallest integer l satisfying the property is called the order of T .*

It is easy to see that the currents in the above two examples are both of order 0.

Definition 1.1.7. *The support of a (p, q) -current T is defined to be the smallest closed subset $\text{supp}(T)$ of X such that T vanishes on $X \setminus \text{supp}(T)$. That is to say, $\langle T, \phi \rangle = 0$ for every test form $\phi \in \mathcal{D}^{n-p, n-q}(X \setminus \text{supp}(T))$.*

Note that the current $[Y]$ in Example 1.1.4 has support Y .

1.1.2 Operators on currents

There are several operators on currents which are similar to those on differential forms. Recall that $d = \partial + \bar{\partial}$. In a local coordinate, we have

$$\begin{aligned} \partial\phi &:= \sum_{I, J} \sum_{i=1}^k \frac{\partial\phi_{I, J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{d}d\bar{z}_J, \\ \bar{\partial}\phi &:= \sum_{I, J} \sum_{i=1}^k \frac{\partial\phi_{I, J}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_I \wedge d\bar{d}d\bar{z}_J. \end{aligned}$$

Define $d^c := \frac{1}{2\pi i}(\partial - \bar{\partial})$. It is a real operator. It follows that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. The normalization of d^c is convenient for many purposes. For example, it simplifies the Lelong-Poincaré formula. We define the exterior derivative of currents as follows. Let T be a (p, q) -current, the $(p+1, q)$ -current ∂T and the $(p, q+1)$ -current $\bar{\partial}T$ is defined by

$$\begin{aligned} \langle \partial T, \phi \rangle &:= (-1)^{p+q+1} \langle T, \partial\phi \rangle, \\ \langle \bar{\partial}T, \phi \rangle &:= (-1)^{p+q+1} \langle T, \bar{\partial}\phi \rangle. \end{aligned}$$

The current dT can be defined similarly. The maps $T \rightarrow dT$, $T \rightarrow \partial T$ and $T \rightarrow \bar{\partial}T$ are continuous for the topology of currents. T is called *closed* if $dT = 0$. In particular, we have $\langle dd^c T, \phi \rangle = \langle T, dd^c \phi \rangle$. An application of the classical Stokes' formula yields

$$d[Y] = (-1)^{n-p+1} [\partial Y],$$

where ∂Y denotes the boundary of the complex manifold Y .

Definition 1.1.8. Let T be a (p, q) -current and ψ be a form of bidegree (p_1, q_1) . We define the wedge product $T \wedge \psi$ by

$$\langle T \wedge \psi, \phi \rangle := \langle T, \phi \wedge \psi \rangle, \forall \phi \in \mathcal{D}^{n-p-p_1, n-q-q_1}(X).$$

Let X' be another smooth complex manifold of dimension n' . Let $f : X \rightarrow X'$ be a holomorphic map which is proper on the support of T . We introduce the notions of direct image and inverse image of a current.

Definition 1.1.9. The direct image of T by f is defined by

$$\langle f_*(T), \phi \rangle := \langle T, f^*(\phi) \rangle, \quad \forall \phi \in \mathcal{D}^{n-p, n-q}(X').$$

The current $f_*(T)$ is of bidegree $(n' - n + p, n' - n + q)$ and of bidimension $(n - p, n - q)$. So the operator f_* preserves the dimension of currents.

Proposition 1.1.10. The direct image operator f_* is continuous. The support of $f_*(T)$ is contained in $f(\text{supp}T)$. Moreover, it commutes with the exterior derivative, i.e.

$$d(f_*(T)) = f_*(dT).$$

Assume in addition that f is a submersion. Let ϕ be a form of bidegree (p, q) (even with L^1_{loc} coefficients) with the condition that f is proper. Then $f_*\phi$ is a form of bidegree $(n' - n + p, n' - n + q)$. Moreover, $f_*\phi$ is calculated by integration over the fibers of f , i.e.

$$f_*\phi(w) = \int_{z \in f^{-1}(w)} \phi(z).$$

So we can define the inverse image of a current.

Definition 1.1.11. Let T be a (p, q) -current on X' . The inverse image of T by f is defined by

$$\langle f^*(T), \phi \rangle := \langle T, f_*(\phi) \rangle, \forall \phi \in \mathcal{D}^{n-p, n-q}(X).$$

The operator f^* preserves the degree of currents. For example, we have $f^*[Y] = [f^{-1}(Y)]$.

Assume that the support of T or a smooth function g on X is compact, We can define the convolution $T \star g$ by

$$T \star g := \sum_{I, J} T_{I, J} \star g,$$

where $T_{I, J}$ are the distribution coefficients of T . A convolution of a distribution and a smooth function with compact support is a smooth function, which is a direct consequence of Taylor's formula with compactness condition. So $T \star g$ is a smooth form. Hence we can approximate currents by smooth forms in the weak sense. Refer to [37] for more information about convolutions of distributions or currents.

1.1.3 De Rham theorem

By an analogous argument, we can define the following cohomology group associated to currents

$$H_c^p(X, \mathbb{R}) := \frac{\text{real valued closed } p\text{-currents on } X}{\text{real valued exact } p\text{-currents on } X}.$$

When X is paracompact, it follows from the theory of sheaf cohomology [16, Chapter IV] that the de Rham cohomology group $H^p(X, \mathbb{R})$ is isomorphic to $H_c^p(X, \mathbb{R})$. In particular every closed p -current is cohomologous to a closed smooth p -form.

Denote by $H^{p,q}(X, \mathbb{C})$ the Dolbeault cohomology group. If X is a complex projective space \mathbb{P}^n of dimension n , by Hodge decomposition theorem [16, Chapter VI], we deduce that

$$\begin{aligned} H^{p,q}(\mathbb{P}^n, \mathbb{C}) &= 0, \text{ for } p \neq q, \\ H^{p,p}(\mathbb{P}^n, \mathbb{C}) &\simeq \mathbb{C}. \end{aligned}$$

We will see later that the generator of $H^{p,p}(\mathbb{P}^n, \mathbb{C})$ is the p times tensor of the Fubini-Study form.

It is easy to calculate the Dolbeault cohomology groups of product spaces of several complex projective spaces by Künneth formula. The above results will be used in cohomological arguments in the proofs of our main theorems.

1.1.4 Positive closed currents

The notion of positive closed currents was introduced by Pierre Lelong [44] in 1957. It generalized analytic subsets in complex manifolds, since every analytic subset can be associated to a current by integration over its set of regular points and all such currents are proved to be positive closed. It has many applications in complex analysis (especially pluripotential theory) and dynamical systems in higher dimensions.

Definition 1.1.12. *A (p, p) -form ϕ is called positive if it is equal to a finite combination of forms $(i\alpha_1 \wedge \bar{\alpha}_1) \wedge \dots \wedge (i\alpha_p \wedge \bar{\alpha}_p)$ at each point, where α_j are $(1, 0)$ -forms. The form ϕ is said to be weakly positive if $\phi \wedge \psi$ is positive for any positive $(n - p, n - p)$ -form ψ . A (p, p) -current T is said to be positive (resp. weakly positive) if $\langle T, \phi \rangle \geq 0$ for every weakly positive (resp. positive) test form of bidegree $(n - p, n - p)$.*

Note that all positive closed currents are real by duality. The two notions of positivity and weakly positivity coincide when $p = 0, 1, n - 1, n$. Moreover, the operators f_* and f^* preserve the positivity.

Proposition 1.1.13. *Let T be a positive (p, p) -current. Then T is of order zero. The coefficients $T_{I,J}$ are distributions of order zero, i.e. complex measures.*

Proof. We give a sketch proof here. If T is a distribution and it is positive, then it can be extended to a positive linear functional on the space of complex continuous functions. Hence T is a positive measure. Note that the real vector space $\bigwedge^{p,p}(\mathbb{C}^n, \mathbb{R})$ admits a basis consisting of positive forms. We can choose such a basis $\{\psi_J\}$ for $\bigwedge^{n-p, n-p}(\mathbb{C}^n, \mathbb{R})$. Then we set $\{\phi_I\}$ to be the basis for $\bigwedge^{p,p}(\mathbb{C}^n, \mathbb{R})$ which is dual to $\{\psi_J\}$. So $T = T_I \phi_I$. If g is a non-negative test function, we have

$$T_I(g) = T(g\phi_I) \geq 0.$$

By the previous argument, T_I is a positive measure. Then the coefficients of T are complex measures if it is expressed in terms of an arbitrary basis. Hence T is of order 0. \square

Definition 1.1.14. Let $\beta = i\partial\bar{\partial}\|z\|^2$, define

$$\sigma_T := \frac{1}{2^{n-p}(n-p)!} T \wedge \beta^{n-p}.$$

Then σ_T is said to be the trace measure of T .

Recall that a Hermitian metric on X is a smooth positive Hermitian form $h = \sum h_{jk} dz_j \otimes d\bar{z}_k$. The form $\omega = i \sum h_{jk} dz_j \wedge d\bar{z}_k$ is the associated positive $(1, 1)$ -form. The following is a famous theorem due to Wirtinger.

Theorem 1.1.15. Let X be equipped with a Hermitian metric h . Deont by ω the associated form of h . Let Y be an oriented real submanifold of class \mathcal{C}^1 and real dimension $2p$ in X . Denote by dV_Y the Riemannian volume form on Y with respect to the induced metric $h|_Y$. There exists a continuous function α in Y such that

$$\alpha dV_Y = \frac{1}{2^p p!} \omega|_Y^p.$$

Then $|\alpha| \leq 1$ and the equality holds if and only if Y is a complex analytic submanifold of X .

We have defined currents of integration over a complex submanifold. This notion can be also defined over analytic subsets. The following Lelong's theorem claims that such currents are positive closed ones.

Theorem 1.1.16. Let Y be an analytic subset of pure dimension $n-p$ of X . Denote by $\text{reg}(Y)$ be the regular part of Y . Define the (p, p) -current $[Y]$ by

$$\langle [Y], \phi \rangle = \int_{\text{reg}(Y)} \phi, \forall \phi \in \mathcal{D}^{n-p, n-p}(X).$$

Then the current $[Y]$ is well-defined and positive closed.

The theorem shows that positive closed currents extend the notion of analytic subsets in complex manifolds. Moreover, the volume of $\text{reg}(Y)$ near singular points is locally bounded. Now we introduce a support theorem which is useful in latter chapters.

Definition 1.1.17. A current T is called normal if T and dT are both of order 0.

Every positive closed current is normal.

Theorem 1.1.18. Let T be a normal (p, p) -current. If the support of T is contained in an analytic subset V of dimension less than p , then $T = 0$.

Another useful support result is the following

Theorem 1.1.19. Let V be an analytic subset of X with global irreducible components V_j of pure dimension p . Assume that T is a closed (p, p) -current of order 0 with support contained in V . Then T can be written as the form $\sum \lambda_j [V_j]$ where $\lambda_j \in \mathbb{C}$.

1.2 Plurisubharmonic functions

In this section, we introduce the notions of plurisubharmonic functions and their basic properties. The positive closed $(1, 1)$ -currents can be studied by plurisubharmonic functions locally. Then we define Monge-Ampère operators. Some continuity properties of Monge-Ampère operators are needed in our thesis. The quasi-plurisubharmonic functions can be defined in compact complex manifolds. We conclude this section with the notion of moderate measures with respect to a compact family of quasi-plurisubharmonic functions.

1.2.1 Plurisubharmonic functions

Plurisubharmonic (p.s.h. for short) functions were introduced by Lelong and Oka in 1942. They play a central role in the study of complex analysis. For example, there are analogies between plurisubharmonicity and pseudoconvexity. The p.s.h. functions are the natural counterpart of the subharmonic functions of one variable. Note that, to some extent, the class of subharmonic functions of several complex variables is quite large. The p.s.h. functions are precisely the ones among subharmonic functions which are invariant under compositions with biholomorphic maps. That is why p.s.h. functions make sense on complex manifolds.

Definition 1.2.1. Let Ω be an open subset of \mathbb{C}^n . A function $u : \Omega \rightarrow [-\infty, \infty)$ is said to be plurisubharmonic function if it is upper semicontinuous, not identically $-\infty$ on each connected component of Ω and satisfy

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + we^{i\theta}) d\theta,$$

for each $z \in \Omega$ and $w \in \mathbb{C}^n$ such that

$$\{z + w\lambda : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega.$$

The set of p.s.h. functions (resp. subharmonic functions) on Ω is denoted by $Psh(\Omega)$ (resp. $Sh(\Omega)$). If in addition $u \in \mathcal{C}^2(\Omega)$, then it is easy to see that $u \in Sh(\Omega)$. Most of properties for subharmonic functions in \mathbb{R}^{2n} can carry over to the case of plurisubharmonic functions in \mathbb{C}^n . Note that the semicontinuity implies that p.s.h. functions are locally bounded from above. Denote by $\{\chi_\epsilon\}$ the standard smoothing kernels [16, 2.D.3]. The following result is the approximation theorem for p.s.h. functions.

Theorem 1.2.2. *Let $u \in Psh(\Omega)$. Set $\Omega_\epsilon := \{z \in \Omega : \text{dist}(z, \partial\Omega) > \epsilon\}$ for $\epsilon > 0$. Then $u \star \chi_\epsilon \in \mathcal{C}^\infty(\Omega_\epsilon) \cap Psh(\Omega_\epsilon)$. Moreover, the family of $\{u \star \chi_\epsilon\}$ is non decreasing, and for each $z \in \Omega$, we have*

$$\lim_{\epsilon \rightarrow 0} u \star \chi_\epsilon(z) = u(z).$$

In general, we can not find a decreasing sequence $\{u_j\} \subset \mathcal{C}^\infty(\Omega) \cap Psh(\Omega)$ which converges pointwise to u . One counterexample was shown by Fornæss [42, Example 2.9.4].

Proposition 1.2.3. *Let Ω be an open subset of \mathbb{C}^n . Then we have $Psh(\Omega) \subset Sh(\Omega) \subset L^1_{loc}(\Omega)$.*

P.s.h. functions satisfy the maximum principle in bounded domains.

Proposition 1.2.4. *Let Ω be a bounded connected open subset of \mathbb{C}^n . Let $u \in Psh(\Omega)$. Then either u is constant or, for each $z \in \Omega$,*

$$u(z) < \sup_{w \in \partial\Omega} \{ \limsup_{y \rightarrow w, y \in \Omega} u(y) \}.$$

Proposition 1.2.5. *If $u \in Psh(\mathbb{C}^n)$ is bounded above, then u is constant.*

Theorem 1.2.6. [42, Theorem 2.9.12] *Let Ω (resp. Ω') be an open subset of \mathbb{C}^n (resp. $\mathbb{C}^{n'}$). If $u \in Psh(\Omega)$ and $f : \Omega' \rightarrow \Omega$ is a holomorphic map, then the composition $u \circ f$ is a p.s.h. function in Ω' . Conversely, $u \in Psh(\Omega)$ if and only if $u \circ g$ is subharmonic in $g^{-1}(\Omega)$ for every complex linear isomorphism $g : \Omega \rightarrow \Omega$.*

Proposition 1.2.7. *Let Ω be an open subset of \mathbb{C}^n .*

(i) *The set $Psh(\Omega)$ is a convex cone. That is to say, if $a > 0, b > 0$ and $u, v \in Psh(\Omega)$, then $au + bv \in Psh(\Omega)$.*

(ii) *If $\{u_j\}$ is a decreasing sequence of p.s.h. functions in a connected open subset Ω , then $u := \lim_{j \rightarrow \infty} u_j$ is p.s.h. or $u \equiv -\infty$.*

(iii) *If $u : \Omega \rightarrow \mathbb{R}$ and the sequence of p.s.h. functions in Ω converges uniformly to u on compact subsets of Ω , then $u \in Psh(\Omega)$.*

(iv) *Let $u_1, \dots, u_m \in Psh(\Omega)$. Let $\chi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a convex function and $\chi(t_1, \dots, t_m)$ is non decreasing in each variable t_j . Then $\chi(u_1, \dots, u_m) \in Psh(\Omega)$. In particular $u_1 + \dots + u_m, \max\{u_1, \dots, u_m\}, \log(e^{u_1} + \dots + e^{u_m})$ are p.s.h. functions.*

Example 1.2.8. *Since $\log |z|$ is a subharmonic function on \mathbb{C} , $\log |f| \in Psh(X)$, for any holomorphic function f on X . For any holomorphic functions f_j and $\alpha_j \geq 0$, $1 \leq j \leq m$,*

$$\log(|f_1|^{\alpha_1} + \dots + |f_m|^{\alpha_m}) \in Psh(X).$$

Let $\{u_\alpha\}_{\alpha \in \Lambda}$ be a family of upper semicontinuous functions from Ω to $[-\infty, \infty)$. Assume that $\{u_\alpha\}$ are locally uniformly bounded from above. Then we can define the upper envelope by $u := \sup u_\alpha$. Note that u may not be upper semicontinuous, so we consider its upper semicontinuous regularization as follows,

$$u^*(z) = \lim_{\epsilon \rightarrow 0} \sup_{B(z, \epsilon)} u \geq u(z).$$

It is easy to see that u^* is upper semicontinuous. The following property is called Choquet's lemma.

Lemma 1.2.9. *Every family $\{u_\alpha\}$ admits a countable subfamily $\{v_j\} = \{u_{\alpha_j}\}$ with its upper envelope v satisfying $v \leq u \leq u^* = v^*$.*

Proposition 1.2.10. *Let $\{u_\alpha\} \subset Psh(\Omega)$ be locally uniformly bounded from above and u be the upper envelope. Then its upper semicontinuous regularization u^* is also p.s.h. and is equal to u almost everywhere.*

Proposition 1.2.11. *The set $Psh(\Omega)$ is closed in $L^1_{loc}(\Omega)$ and every bounded subset is relatively compact. More precisely, if $\{u_j\} \subset Psh(\Omega)$ is a sequence which is locally bounded from above, then either it converges locally uniformly to ∞ , or there exists a subsequence $\{u_{j_k}\}$ which converges to a p.s.h. function in $L^p_{loc}(\Omega)$ for any $1 \leq p < \infty$.*

Definition 1.2.12. *A function u is called pluriharmonic if u and $-u$ are both plurisubharmonic.*

A pluriharmonic function u verifies the condition that $dd^c u = 0$. It is easy to check that the real part and image part of a holomorphic function are both pluriharmonic. Conversely, a pluriharmonic function is locally the real part of a holomorphic function. We have the following theorem in the global case.

Theorem 1.2.13. *If the first De Rham cohomology group of X is zero, then every plurisubharmonic function is the real part of some holomorphic function on X .*

Pluriharmonic functions are the counterpart of harmonic functions in several complex variables. But when we define harmonic functions by using the following way: A subharmonic function u is said to be harmonic if for every relatively compact open subset V of Ω and every upper semicontinuous function v in \bar{V} , we have

$$v \in Psh(\Omega) \text{ and } v \leq u \text{ in } \partial\Omega \text{ imply } v \leq u \text{ in } \Omega.$$

The counterpart of harmonic functions following this definition is called maximal plurisubharmonic functions. Note that harmonic functions are solutions of homogeneous Laplace equations. We will see later that the corresponding equations for maximal plurisubharmonic functions is called Monge-Ampère equations.

Definition 1.2.14. *A subset V of X is said to be pluripolar if for every point $z \in X$ there is a neighbourhood U of z and a p.s.h. function u on U such that $V \cap U \subset \{u = -\infty\}$.*

We know that if u is a p.s.h. function on a connected subset, then either $u \equiv -\infty$ or u is locally integrable. Hence a pluripolar set is of zero Lebesgue measure.

Proposition 1.2.15. *Any proper analytic subset of X is pluripolar. The Hausdorff dimension of a pluripolar set is smaller or equal to $2n - 2$.*

Theorem 1.2.16. *Let V be a closed pluripolar subset of X and u be a p.s.h. function on $X \setminus V$. If u is locally bounded from above near V . Then there is a unique extension $\tilde{u} \in Psh(X)$ such that $\tilde{u} = u$ on $X \setminus V$.*

There are close relationships between p.s.h. functions and positive closed $(1, 1)$ -currents. If $u \in Psh(\Omega)$, then $dd^c u$ is a positive closed $(1, 1)$ -current. This can be deduced by the approximation theorem of currents and dominated convergence theorem. Conversely, we have the following result,

Proposition 1.2.17. *Any positive closed $(1, 1)$ -current T can be locally written as $dd^c u$, where u is a p.s.h. function. The function u is called the local potential of T .*

Note that two local potentials differ by a pluriharmonic function, so the study of the singularities of positive closed $(1, 1)$ -currents can be reduced to study the local potentials. The Lelong-Poincaré formula offers a good example of potentials of currents by integration over hypersurfaces.

Theorem 1.2.18. *Let f be a holomorphic function on X which does not vanish identically on any connected component of X . Then the function $\log |f|$ is p.s.h. and verify the equation*

$$dd^c \log |f| = \sum m_j [Z_j],$$

where $\sum m_j Z_j$ is the divisor of f .

The above equation holds even when f is meromorphic, then $\log |f|$ is locally integrable and m_j can be negative numbers. We recall here an useful result, the so-called dd^c -lemma.

Proposition 1.2.19. *Let X be a compact Kähler manifold and T be a closed (p, q) -current on X . Then T is dd^c -exact if and only if it is exact (or ∂ -exact or $\bar{\partial}$ -exact).*

1.2.2 Intersection of currents and Monge-Ampère operators

Let u be a p.s.h. function and T be a positive closed (p, p) -current, $p \leq n-1$. Recall that σ_T is its trace measure. Due to the classical result by Bedford-Taylor [8], we can define

$$dd^c u \wedge T := dd^c(uT),$$

when u is locally σ_T -integrable, in particular when u is continuous or locally bounded. It follows from the approximation theorem that $dd^c u \wedge T$ is a positive closed current. When u_1, \dots, u_q are all locally bounded, we can define inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T).$$

It is a positive closed current.

Definition 1.2.20. *If u is a locally bounded p.s.h. function, then $(dd^c u)^n$ is called Monge-Ampère operator.*

Sometimes the map

$$(u_1, \dots, u_n) \rightarrow dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_n.$$

is also called Monge-Ampère operator. We have the following Chern-Levine-Nirenberg inequality.

Theorem 1.2.21. *Let L and K be compact subsets of X . Let L be relatively compact in K° . Assume that v is a locally σ_T -integrable p.s.h. function. Then there exists a constant C_{LK} independent of T, u_j and v such that*

$$\begin{aligned} \|dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T\|_L &\leq C_{LK} \|T\|_K \|u_1\|_{\mathcal{L}^\infty(K)} \dots \|u_q\|_{\mathcal{L}^\infty(K)}, \\ \|v dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T\|_L &\leq C_{LK} \|vT\|_K \|u_1\|_{\mathcal{L}^\infty(K)} \dots \|u_q\|_{\mathcal{L}^\infty(K)}. \end{aligned}$$

One of the most properties of Monge-Ampère operator is its continuity on decreasing sequences of p.s.h. functions.

Theorem 1.2.22. *Let u_1, \dots, u_q be locally bounded p.s.h. functions. Let $\{u_1^j\}, \dots, \{u_q^j\}$ be decreasing sequences of p.s.h. functions which converge pointwise to u_1, \dots, u_q respectively. Then we have in the weak sense of currents*

$$\begin{aligned} u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T &\rightarrow u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T, \\ dd^c u_1^j \wedge \dots \wedge dd^c u_q^j \wedge T &\rightarrow dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T. \end{aligned}$$

In the following chapters we will deal with the intersections of several currents by integration over analytic subvarieties. So we introduce Monge-Ampère operators on unbounded p.s.h. functions.

Definition 1.2.23. *Let u be a p.s.h. function. The unbounded locus $L(u)$ is defined to be the set of points $z \in X$ such that u is unbounded in any neighborhood of z .*

We can define Monge-Ampère operators on unbounded p.s.h. functions when the intersections of unbounded loci are sufficiently small measured by Hausdorff dimensions.

Theorem 1.2.24. *Let T be a (p, p) -current and u_1, \dots, u_q be p.s.h. functions on X , $q \leq n - p$. If the $(2n - 2p - 2k + 1)$ -Hausdorff dimension of the set $L(u_{j_1}) \cap \dots \cap L(u_{j_k}) \cap \text{supp}T$ is equal to 0 for all indices $j_1 < \dots < j_k$ in $\{1, \dots, q\}$, then the currents $u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ and $dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$ are well defined with locally finite mass.*

The continuity property of Monge-Ampère operators is also valid for unbounded p.s.h. functions.

Definition 1.2.25. *The analytic subsets V_1, \dots, V_q of X is said to be in general position if $\text{codim } V_{j_1} \cap \dots \cap V_{j_k} \geq k$ for all indices $j_1 < \dots < j_k$ in $\{1, \dots, q\}$.*

When T is of bidegree $(0, 0)$, we have the following useful criterion

Corollary 1.2.26. *If the unbounded locus $L(u_j)$ is contained in an analytic subset V_j such that V_1, \dots, V_q are in general position, then $dd^c u_1 \wedge \dots \wedge dd^c u_q$ is well defined.*

In particular, the current $[V_1] \wedge \dots \wedge [V_q]$ is well defined when V_1, \dots, V_q are in general position. Fornæss and Sibony defined Monge-Ampère operators and obtained continuity properties in other assumptions on u_1, \dots, u_q and T . We refer the reader to [27, Section 3] for a more detailed discussion.

Definition 1.2.27. *Let Ω be an open subset in \mathbb{C}^n and V be a closed subset in Ω . We say that V is in the envelope of p -pseudoconvexity of $\Omega \setminus V$ with respect to Ω if every point in V can be reached by pushing polydiscs of dimension $(n - p)$ by using biholomorphic images of $(n - p, p)$ Hartogs figures with hulls in Ω .*

Theorem 1.2.28. *Let T be a positive closed (p, p) -current and u_1, \dots, u_q be p.s.h. functions on Ω , $q \leq n - p$. If $u_j \leq 0, \forall 1 \leq j \leq q$, and $L(u_{j_1}) \cap \dots \cap L(u_{j_k}) \cap \text{supp}T$ is in the envelope of $(n - p - k + 1)$ -pseudoconvexity of the complement for all indices $j_1 < \dots < j_k$ in $\{1, \dots, q\}$, then the currents $u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T$ and $dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T$ are well defined with locally finite mass. Moreover, if u_j^l converges to u_j in $L_{loc}^1(\Omega)$ and $u_j^l \geq u_j$, we have*

$$\begin{aligned} u_1^j dd^c u_2^j \wedge \dots \wedge dd^c u_q^j \wedge T &\rightarrow u_1 dd^c u_2 \wedge \dots \wedge dd^c u_q \wedge T, \\ dd^c u_1^j \wedge \dots \wedge dd^c u_q^j \wedge T &\rightarrow dd^c u_1 \wedge \dots \wedge dd^c u_q \wedge T. \end{aligned}$$

1.2.3 Quai-plurisubharmonic functions and moderate measures

The notion of quai-plurisubharmonic (q.p.s.h. for short) functions was introduced by Yau. It plays an important role in the following chapters. P.s.h. functions have local properties. By the maximum principle we know that p.s.h. functions in compact complex manifolds are always constants. But

q.p.s.h. function is a global notion which can be defined in compact complex manifolds.

It can describe the Hermitian metrics on complex line bundles. In this subsection, we assume that X is a compact Kähler manifold of dimension n with Kähler form ω such that ω^n is a probability measure. The case when X is only a complex manifold will be specified.

Definition 1.2.29. *A q.p.s.h. function on X is locally the difference of a p.s.h. function and a smooth one.*

Definition 1.2.30. *Let γ be a closed real $(1, 1)$ -current on X . An upper-semi continuous function $u : X \rightarrow [-\infty, \infty)$ in $L^1_{loc}(X)$ is said to be γ -p.s.h. if $dd^c u + \gamma \geq 0$.*

Note that a function u on X is q.p.s.h. if and only if it is $c\omega$ -p.s.h. for some constant $c > 0$. Denote by $Psh(X, \gamma)$ the set of all γ -p.s.h. functions. Observe that $Psh(X, \gamma)$ is nonempty if and only if there is a positive closed $(1, 1)$ -current which is cohomologous to γ . There are some special cases when all γ -p.s.h. functions are constants.

Example 1.2.31. *Let E be the exceptional divisor of a smooth blow-up X . Then $Psh(X, [E]) \cong \mathbb{R}$. See [34, Chapter 2.5] for the definition of a blow-up. To see this, let $\pi : X \rightarrow X_1$ be a blow-up. The smooth center of π is Y of codimension ≥ 2 . The exceptional divisor E is the subset $\pi^{-1}(Y)$. Let $u \in Psh(X, [E])$. Since $\pi : X \setminus E \rightarrow X_1 \setminus Y$ is biholomorphic, by considering all test forms with support in $X \setminus E$, we obtain that $dd^c(u \circ \pi^{-1}) \geq 0$ in $X_1 \setminus Y$. Since $\text{codim } Y \geq 2$, we can extend $u \circ \pi^{-1}$ trivially through Y to a global p.s.h. function on X_1 . The maximum principle implies that $u \circ \pi^{-1}$ is constant, so is u . By dd^c -lemma, there are no other positive closed $(1, 1)$ -currents which are cohomologous to $[E]$.*

We can easily see some properties of q.p.s.h. functions from those of p.s.h. functions.

Proposition 1.2.32. *Let u and $\{u_j\}$ be q.p.s.h. functions, we have*

- (i) u belongs to $L^p(X)$ for every $1 \leq p < \infty$.
- (ii) If $\{u_j\}$ is a decreasing sequence satisfying $dd^c u_j \geq -\omega$, then the limit of $\{u_j\}$ is also a q.p.s.h. function.
- (iii) If $\{u_j\}$ is uniformly bounded from above and $dd^c u_j \geq -\omega$, then either it converges uniformly to ∞ , or there exists a subsequence u_{j_k} which converges to a q.p.s.h. function v in $L^p(X)$ for every $1 \leq p < \infty$ and $dd^c v \geq -\omega$.
- (iv) The functions $u_1 + \dots + u_m$, $\max\{u_1, \dots, u_m\}$, $\log(e^{u_1} + \dots + e^{u_m})$ are also q.p.s.h. functions.

The third result of the above property shows also the compactness of q.p.s.h. functions. More generally we have

Theorem 1.2.33. *The family of the q.p.s.h. functions which satisfy $dd^c u \geq -\omega$ and one of the following three normalization conditions*

- (i) $\max_X u = 0$,
- (ii) $\int_X u \omega^n = 0$,
- (iii) $\int_X |u| \omega^n \leq C$, where C is a constant

is compact in $L^p(X)$ for every $1 \leq p < \infty$. Moreover, the family of these q.p.s.h. functions is uniformly bounded from above .

Proof. The result with condition (i) follows from (iii) of the above proposition. Let u_j verify the condition (ii). Since a q.p.s.h. function is bounded from above in compact manifolds, set $a_j := \sup_X u_j$. Then there are no subsequences of $\{u_j - a_j\}$ which converges uniformly to $-\infty$. Hence the sequence $\{u_j - a_j\}$ is bounded in $L^p(X)$. Otherwise, if there is a subsequence $\{u_{j_k} - a_{j_k}\}$ which converges to a q.p.s.h. function v and $\|u_{j_k} - a_{j_k}\|_{L^p} \rightarrow \infty$, then it yields contradiction with $v \in L^p(X)$. Note that

$$a_j = a_j \int_X \omega^n = - \int_X (u_j - a_j) \omega^n.$$

So $\{a_j\}$ is bounded. The family of q.p.s.h. function u_j is bounded in $L^p(X)$ and there is a convergent subsequence. The result with condition (ii) holds. Let u_j verify the condition (iii). Let $\int_X (u_j - b_j) \omega^n = 0$. Then b_j is bounded. Then the result is a consequence of the result with condition (ii). \square

By the dd^c -lemma on compact Kähler manifolds, we can easily deduce the following result [23, Proposition 2.2].

Proposition 1.2.34. *There exists a constant $r > 0$ such that for any positive closed current T of bidegree $(1, 1)$ with mass 1 on (X, ω) , there is a smooth $(1, 1)$ -form α which depends only on the cohomology class of T and a q.p.s.h. function u satisfying that*

$$-r\omega \leq \alpha \leq r\omega, \quad dd^c u - T = \alpha.$$

Denote by $r(X, \omega)$ the smallest number of r which satisfy the above property.

We have an analogous regularization result for q.p.s.h. functions on compact Kähler manifolds. The following theorem is due to Demailly.

Theorem 1.2.35. *Let u be a q.p.s.h. function on X . Then there is a decreasing sequence of smooth functions u_j satisfying $dd^c u_j \geq -\epsilon \omega$ for some positive constant ϵ , which converges pointwise to u .*

Q.p.s.h. functions can be also defined on any complex manifolds. Błocki-Kołodziej obtained a generalization of regularization result.

Theorem 1.2.36. *Let X be a complex manifold with a fixed Hermitian form ω and $K \subset X$ be a compact subset. Assume that γ is a real closed form of bidegree $(1, 1)$ and u is a locally bounded γ -p.s.h. function. Then for any open neighborhood U of K , there exists a decreasing sequence of smooth functions u_j on U such that*

- (i) u_j converges pointwise to u ,
- (ii) $dd^c u_j + \gamma \geq -\epsilon_j \omega$, where ϵ_j tends to 0.

Remark 1.2.37. *In the above theorem, if γ is positive and u is any γ -p.s.h. function, then the same conclusion holds. Since γ is positive, constant functions are γ -p.s.h. functions. Then we consider the locally bounded γ -p.s.h. functions $\max\{u, -j\}$, $j \geq 1$ and use diagonal arguments.*

In general the global regularization for q.p.s.h. functions on a complex manifold fails.

Our core work through the thesis lies in some estimates of constants on complex projective spaces. The complex projective space \mathbb{P}^n of dimension n is a compact Kähler manifold.

Definition 1.2.38. *A projective manifold is a complex submanifold of some complex projective space.*

Projective manifolds are algebraic by Chow's theorem.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the canonical projection and $[z_0, \dots, z_n]$ be the homogeneous coordinate of \mathbb{P}^n . Denote by U_i the set of points $[z_0, \dots, z_n]$ such that $z_i \neq 0$. It is a local chart on \mathbb{P}^n . All the charts can cover \mathbb{P}^n .

Definition 1.2.39. *The Fubini-Study form is defined to be*

$$\omega_{FS} := dd^c \log \left(\sum_{j=0}^n |z_j/z_i|^2 \right)^{1/2}$$

in the local chart U_i .

In other words, we have

$$\pi^*(\omega_{FS}) = dd^c \log \left(\sum_{j=0}^n |z_j|^2 \right)^{1/2}$$

In the local chart U_0 , assume that $z_0 = 1$. The corresponding Fubini-Study metric has the following Hermitian components

$$h_{i\bar{j}} = \frac{1}{\pi} \frac{(1 + |z|^2) \delta_{i\bar{j}} - z_i \bar{z}_j}{(1 + |z|^2)^2},$$

where $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. Note that ω_{FS}^n is a probability measure on \mathbb{P}^n . This also enlighten the role of normalization in the definition of the operator d^c .

The cohomology class of ω_{FS}^p is a generator of the group $H^{p,p}(\mathbb{P}^n, \mathbb{R})$. Any positive closed $(1, 1)$ -current T of mass 1 is cohomologous to ω_{FS} . Then we obtain a q.p.s.h. function u on \mathbb{P}^n such that $dd^c u = T - \omega_{FS}$. We will write ω_{FS} to be the Fubini-Study form of the complex projective space \mathbb{P}^N of any dimension N . We give two examples of q.p.s.h. functions on \mathbb{P}^n .

Example 1.2.40. Set $|z|^2 = |z_0|^2 + \dots + |z_n|^2$. Let μ be a probability measure on \mathbb{P}^n .

(i)

$$v_1(z) = \max(\log \frac{|z_0|}{|z|}, \dots, \log \frac{|z_n|}{|z|})$$

is a ω_{FS} -p.s.h. function.

(ii)

$$v_2(z) = \int_{\mathbb{P}^n} \log \left(\frac{\|z \wedge w\|}{|z||w|} \right) d\mu(w)$$

is a ω_{FS} -p.s.h. function. This function can be used to define some capacities on \mathbb{P}^n [46].

Positive closed $(1, 1)$ -currents on \mathbb{P}^n can be associated to p.s.h. functions on \mathbb{C}^{n+1} with certain homogeneity properties. See [26, Section 4] and [53, A.5]. Let L_1 be the set of positive closed $(1, 1)$ -currents on \mathbb{P}^n . Consider a p.s.h. function v on \mathbb{C}^{n+1} satisfying the condition

$$v(\lambda z) = c \log |\lambda| + v(z)$$

for some constant $c \geq 0$ and all $z \in \mathbb{C}^{n+1}$. Let $[v]$ be the class of functions which is equal to v up to a constant. Denote by L_2 the set of the classes of p.s.h. functions with the above condition.

Proposition 1.2.41. *The two sets L_1 and L_2 are isomorphic. If $T \in L_1$ has the associated p.s.h. function v such that $v(\lambda z) = c \log |\lambda| + v(z)$ for some constant $c \geq 0$, then the mass of T is c .*

Positive closed $(1, 1)$ -currents on \mathbb{P}^n which are cohomologous to ω_{FS} can be characterized by p.s.h. functions on \mathbb{C}^n [32, Example 2.2]. To be precise, there is a one-to-one correspondence between $Psh(\mathbb{P}^n, \omega_{FS})$ and the Lelong class:

$$\mathcal{L}(\mathbb{C}^n) := \{v \in Psh(\mathbb{C}^n) : v(z) \leq \frac{1}{2} \log(1 + |z|^2) + C_v, C_v \text{ is a constant}\}.$$

Now we introduce the notion of moderate measures which is essential in the thesis. First we recall a classical result about a uniform estimate for certain compact family of p.s.h. functions in the unit ball of \mathbb{C}^n [36, Theorem 4.4.5].

Theorem 1.2.42. *There is a constant $c > 0$ such that for any p.s.h. function u in the unit ball of \mathbb{C}^n with $u(0) > -1$ and $u(z) \leq 0$, we have*

$$\int_{|z| < 1/2} \exp(-u(z)) d\lambda \leq c,$$

where $d\lambda$ denotes the Lebesgue measure on the unit ball.

We have a generalization for any compact family of p.s.h. functions.

Corollary 1.2.43. *Let \mathcal{U} be a compact family of p.s.h. functions in a complex manifold X of dimension n with a Hermitian form ω . For any compact subset K of X , there exist constants $c > 0, \alpha > 0$ such that*

$$\int_K \exp(-\alpha u) \omega^k \leq c,$$

for all $u \in \mathcal{U}$.

Dinh-Sibony [24] introduced the notion of locally moderate measures.

Definition 1.2.44. *Let X be a complex manifold. A positive measure μ is locally moderate if for any open set $U \subset X$, any compact subset $K \subset U$ and any compact family \mathcal{U} of q.p.s.h. functions on U , there are constants $\alpha > 0, c > 0$ such that*

$$\int_K \exp(-\alpha \phi) d\mu \leq c, \quad \forall \phi \in \mathcal{U}.$$

Dinh-Nguyễn-Sibony [21] proved that the equilibrium measure of endomorphism of \mathbb{P}^n is locally moderate.

In the thesis, we deal with the moderate measures in a compact Kähler manifold X of dimension n with a fixed Kähler form ω such that ω^n is the standard probability volume form. Consider a positive measure μ on X , μ is said to be PLB if all the q.p.s.h. functions are μ -integrable. When $\dim X = 1$, μ is PLB if and only if it admits a local bounded potential [24]. Let

$$\mathcal{F} := \{\phi \text{ q.p.s.h. on } X : dd^c \phi \geq -\omega, \max_X \phi = 0\}. \quad (1.1)$$

The set \mathcal{F} is compact in $L^p(X)$ and bounded in $L^1(\mu)$ when μ is a PLB measure by Theorem 1.2.33 and Proposition 1.2.47.

Recall that any q.p.s.h. functions on X are $c\omega$ -p.s.h. for some constant $c \geq 0$ and uniformly bounded from above. Then the study of q.p.s.h. functions can be reduced to the study of the family \mathcal{F} . Observe that there is a large family of positive singular measures which are PLB.

Example 1.2.45. *Let μ_0 be a smooth probability measure and T be a positive $(n-1, n-1)$ -current on X . Set $\mu := \mu_0 + dd^c T$. Then μ is a PLB measure. Consider a smooth function $\phi \in \mathcal{F}$, we have*

$$\begin{aligned} 0 \leq \int_X (-\phi) d\mu &= \int_X (-\phi) d\mu_0 + \int_X (-\phi) dd^c T \\ &\leq C \|\phi\|_{L^1} + (T, -dd^c \phi) \\ &\leq C \|\phi\|_{L^1} + (T, \omega) < \infty. \end{aligned}$$

Then the result follows from the regularization of q.p.s.h. functions.

Definition 1.2.46. Let μ be a PLB measure on X . We say that μ is (c, α) -moderate for some constants $c > 0, \alpha > 0$ if

$$\int_X \exp(-\alpha\phi) d\mu \leq c$$

for all $\phi \in \mathcal{F}$. The measure μ is called moderate if there exist constants $c > 0, \alpha > 0$ such that it is (c, α) -moderate.

For example, the standard volume form ω^n is moderate by Theorem 1.2.42. Let us recall the following proposition in [23].

Proposition 1.2.47. Let μ be a PLB measure on X . The family of the *q.p.s.h.* functions which satisfy $dd^c u \geq -\omega$ and one of the following three normalization conditions

- (i) $\max_X u = 0$,
- (ii) $\int_X u d\mu = 0$,
- (iii) $\int_X |u| d\mu \leq C$, where C is a constant

is bounded in $L^1(\mu)$ and bounded from above. In particular, there exists a constant $c > 0$ independent of u such that $\mu(u < -t) \leq \frac{c}{t}$ for any $t > 0$.

When $X = \mathbb{P}^n$, we recall the following proposition [23, Corollary A.5] which plays a crucial role in the following chapters,

Proposition 1.2.48. There are constants $c_0 > 0$ and $\alpha_0 > 0$ independent of n such that

$$\int_{\mathbb{P}^n} \exp(-\alpha_0\phi) \omega_{FS}^n \leq c_0 n, \quad \forall \phi \in \mathcal{F}.$$

The following lemma gives an alternative definition of moderate measures [20].

Lemma 1.2.49. A PLB measure μ is moderate if and only if there exist two constants $c' > 0, \alpha' > 0$ such that

$$\mu\{z \in K : \phi(z) < -M\} \leq c' e^{-\alpha' M}$$

for any $M \geq 0$ and $\phi \in \mathcal{F}$.

Remark 1.2.50. We can take $c' = c, \alpha' = \alpha$ when c, α are given and take $c = 2c', \alpha = \alpha'/2$ when c', α' are given.

1.3 Holomorphic line bundles

We introduce some basic notions and properties about holomorphic line bundles. There are closed relationships between holomorphic line bundles and divisors. The Kodaira maps are needed which can be regarded as meromorphic transforms in the following chapters. The Kodaira embedding theorem asserts that the notions of positive line bundles and ample line bundles are equivalent on compact Kähler manifolds. Then we introduce big line bundles with several criteria and Nadel vanishing theorem. See [16] for the knowledge of connections, analytic sets and sheaf cohomology.

1.3.1 Holomorphic line bundles and first Chern class

Recall that a complex line bundle is a complex vector bundle of rank 1. Let X be a complex manifold of dimension n and $\pi : L \rightarrow X$ be a complex line bundle. Denote by $L_z := \pi^{-1}(z)$ the fiber at a point z . There exists an open covering (U_α) of X and smooth diffeomorphisms

$$\theta_\alpha : L|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C},$$

such that for every $z \in U_\alpha$ the restriction map $\theta_\alpha : L_z \rightarrow \{z\} \times \mathbb{C}$ is a linear isomorphism. Such smooth diffeomorphisms are called local trivializations. Let $g_{\alpha\beta}$ be the transition functions satisfying

$$\theta_\alpha \circ \theta_\beta^{-1}(z, \xi) = (z, g_{\alpha\beta}(z)\xi), \quad (z, \xi) \in (U_\alpha \cap U_\beta) \times \mathbb{C}.$$

The collection of the transition functions $\{g_{\alpha\beta}\}$ defines a Čech 1-cocycle with values in the multiplicative sheaf \mathcal{E}^* of invertible smooth functions on X , i.e. $\{g_{\alpha\beta}\} \in H^1(X, \mathcal{E}^*)$.

Theorem 1.3.1. *There is a one-to-one correspondence between the group of isomorphism classes of complex line bundles and the Čech cohomology group $H^1(X, \mathcal{E}^*)$. The group structure of the former group is the tensor product of line bundles.*

Definition 1.3.2. *A complex line bundle L is said to be Hermitian if there is a positive definite Hermitian metric h on each fiber L_z varying smoothly with $z \in X$.*

Let D be a connection on L and $\Theta(D)$ be the corresponding curvature. $\Theta(D)$ is a closed form of degree 2. The De Rham cohomology class $\{\Theta(D)\}$ is of complex coefficients and independent of the choice of D . When L is Hermitian, we can choose a connection D such that $i\Theta(D)$ is a real form, i.e. $\{i\Theta(D)\} \in H^2(X, \mathbb{R})$. Such connection is called Hermitian connection. The exponential exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{E} \rightarrow \mathcal{E}^* \rightarrow 1$$

gives a coboundary map in the long exact sequence of Čech cohomology

$$\delta : H^1(X, \mathcal{E}^*) \rightarrow H^2(X, \mathbb{Z}).$$

Since $H^1(X, \mathcal{E}) = H^2(X, \mathcal{E}) = 0$, the map δ is an isomorphism.

Definition 1.3.3. *The first Chern class of a complex line bundle L is defined to $c_1(L) := \delta(\{g_{\alpha\beta}\})$, where $\{g_{\alpha\beta}\}$ is the Čech cohomology class of the 1-cocycle associated to L .*

It follows from the definition that a complex line bundle is determined up to smooth isomorphism by its first Chern class. As usual the first Chern class is defined to be the image of $c_1(L)$ under the following natural map

$$H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R}).$$

Theorem 1.3.4. *The first Chern class $c_1(L) \in H^2(X, \mathbb{R})$ is equal to the De Rham cohomology class $\{\frac{i}{2\pi}\Theta(D)\}$.*

Definition 1.3.5. *A holomorphic line bundle L is a complex line bundle with a holomorphic map $\pi : L \rightarrow X$ and biholomorphic trivializations.*

It follows that the transition functions are also holomorphic. Let $H^0(X, L)$ be the space of holomorphic sections of L . Denote by \mathcal{O}^* the multiplicative sheaf of invertible holomorphic functions on X . Similarly there is a one-to-one correspondence between the group of isomorphism classes of holomorphic line bundles and the Čech cohomology group $H^1(X, \mathcal{O}^*)$. The group $H^1(X, \mathcal{O}^*)$ is called the Picard group of X .

One notable difference between smooth line bundles and holomorphic ones is that the operator $\bar{\partial}$ can be well-defined globally on holomorphic line bundles. We are interested in Hermitian holomorphic line bundles. There exists a unique Hermitian connection D such that its $(0, 1)$ -connection is $\bar{\partial}$ in a Hermitian holomorphic line bundle.

Definition 1.3.6. *The above unique Hermitian connection D is called Chern connection and the curvature $\Theta(D)$ of D is called Chern curvature.*

1.3.2 Divisors and Lelong-Poincaré formula

Since hypersurfaces are given by the zeros of a global holomorphic sections of a holomorphic line bundle, there are close relationships between divisors and holomorphic line bundles. Recall that an analytic hypersurface of X is an analytic subvariety of codimension one.

Definition 1.3.7. *A divisor D on X is a locally finite formal linear combination $D = \sum a_i Y_i$, where Y_i are irreducible analytic hypersurfaces and a_i are integers.*

Denote by $Div(X)$ the set of all divisors on X . The sum is finite if X is a compact complex manifold. A divisor is said to be *effective* if all a_i are nonnegative. Every hypersurface defines an effective divisor $\sum Y_i$. Recall that the order $ord_{Y,z}(f)$ of a meromorphic function f along Y at $z \in Y$ is the largest integer a such that $f = g^a h$ with $h \in \mathcal{O}_{X,z}^*$. If Y is irreducible, then the order is independent of $z \in Y$. Hence we can define $ord_Y(f)$ along an irreducible hypersurface.

Definition 1.3.8. *The divisor associated to a meromorphic function f is*

$$div(f) := \sum ord_Y(f)Y,$$

where the index of the sum runs over all irreducible hypersurfaces in X . Such divisor is called *principle*.

Denote by \mathcal{M}^* the multiplicative sheaf of invertible meromorphic functions on X .

Proposition 1.3.9. *There exist the following group homomorphisms*

$$H^0(X, \mathcal{M}^*/\mathcal{O}^*) \cong Div(X) \rightarrow Pic(X).$$

A divisor $D \in Div(X)$ is assigned to a holomorphic line bundle $\mathcal{O}(D)$. In fact D corresponds to an element $f \in H^0(X, \mathcal{M}^*/\mathcal{O}^*)$, which is represented by a family of functions $f_\alpha \in \mathcal{M}^*(U_\alpha)$ with respect to an open covering $\{U_\alpha\}$. Then the transition functions $g_{\alpha\beta} = f_\alpha f_\beta^{-1}$ defines the line bundle $\mathcal{O}(D)$. Note that the line bundle $\mathcal{O}(D)$ can be identified to the sheaf of germs of meromorphic functions f satisfying $div(f) + D \geq 0$.

Definition 1.3.10. *A meromorphic section of L is a section s defined by a collection of meromorphic functions $f_\alpha = \theta_\alpha(s) \in \mathcal{M}(U_\alpha)$.*

Since $\{f_\alpha\}$ corresponds to an element in $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$, then the meromorphic section s defines a divisor $div(s)$ which is written locally as $div(f_\alpha)$. The divisor $div(s)$ is effective if and only if s is a holomorphic section.

The current defined by $div(s)$ is written as $[div(s)]$ or $[s = 0]$ when s is holomorphic. The definition of $div(s)$ implies that $\mathcal{O}(div(s))$ is isomorphic to L for any nonzero meromorphic section s . It follows from Lelong-Poincaré formula that

$$dd^c \log |f_\alpha| = div(f_\alpha).$$

Note that L is a Hermitian holomorphic line bundle with the Hermitian metric h . There exists a local holomorphic frame e_L of L over U_α such that $|e_L|_h^2 = h(e_L, e_L) = e^{-2\varphi}$, where φ is a real function on U_α . It is possible since we can shrink U_α properly. Then $|s|_h^2 = |f_\alpha|^2 e^{-2\varphi}$. There is a characterization of Chern curvature by the Hermitian metric.

Proposition 1.3.11. *We have locally on U_α*

$$\frac{i}{2\pi} \Theta(L) = dd^c \varphi.$$

In particular, the current $dd^c \varphi$ represents the first Chern class $c_1(L)$.

Following the above arguments, we obtain

$$dd^c \log |s|_h^2 = \operatorname{div}(s) - \frac{i}{2\pi} \Theta(L).$$

The following theorem is the Lelong-Poincaré formula for meromorphic sections of line bundles.

Theorem 1.3.12. *Let L be a Hermitian holomorphic bundle and s be a meromorphic section of L which does not vanish identically on any component of X . Then*

$$\mathcal{O}(\operatorname{div}(s)) \cong L, \quad c_1(L) = \{[\operatorname{div}(s)]\}.$$

Consequently we have $c_1(\mathcal{O}(\operatorname{div}(s))) = \{[\operatorname{div}(s)]\}$.

We extend the notion of Hermitian line bundle to the singular case.

Definition 1.3.13. *Let L be a holomorphic line bundle on a complex manifold X . A singular Hermitian metric h on L is a sesquilinear Hermitian-symmetric form on each fiber such that for any local holomorphic frame e_L of L on $U \subset X$, we have $|e_L|_h^2 = e^{-2\varphi} \in [0, \infty]$, where $\varphi \in L_{loc}^1(U)$ is called local weight. A singular Hermitian holomorphic line bundle is a holomorphic line bundle with a singular Hermitian metric.*

If φ is smooth, the L is a Hermitian line bundle. It is easy to see that $dd^c\varphi$ is independent of the choice of open coverings. We can define the curvature current by $\Theta(L) = -2\pi i dd^c\varphi$. The current $c_{L,h} =: \frac{i}{2\pi} \Theta(L) = dd^c\varphi$ represents the first Chern class of L . The Lelong-Poincaré formula for meromorphic sections of singular Hermitian holomorphic line bundles also holds.

Given an arbitrary divisor D on X , we have $c_1(\mathcal{O}(D)) = \{[D]\}$. Indeed, let $D = a_j D_j$, the line bundle $\mathcal{O}(D)$ can be endowed with a singular Hermitian metric h such that $|f|_h = |f|$, where f is a meromorphic function with $\operatorname{div}(f) + D \geq 0$. Let g_j be the defining holomorphic function of D_j on U_α . Then there is a trivialization $\theta_\alpha(f) = f \prod g_j^{a_j}$ of $\mathcal{O}(D)$ on U_α . The singular Hermitian metric has local weight $\varphi = \sum a_j \log |g_j|$, The Lelong-Poincaré formula yields the equation $c_1(\mathcal{O}(D)) = \{[D]\}$.

Definition 1.3.14. *A Hermitian holomorphic line bundle L is called positive if there exists a smooth Hermitian metric on L with the Chern curvature $\Theta(L)$ such that $i\Theta(L)$ is a positive $(1, 1)$ -form.*

A divisor D is *positive* if the line bundle $\mathcal{O}(D)$ is positive. The following result shows that the positivity of a line bundle is a topological property.

Proposition 1.3.15. *Let ϕ be a real closed $(1, 1)$ -form whose cohomology class is $c_1(L)$. Then there exists a smooth Hermitian metric on L such that $\frac{i}{2\pi} \Theta(L) = \phi$. Hence L is positive if and only if $c_1(L)$ can be represented by a positive closed $(1, 1)$ -form.*

Example 1.3.16. *The hyperplane bundle $\mathcal{O}(1)$ on \mathbb{P}^n is a positive line bundle. The dual of $\mathcal{O}(1)$ is the tautological line bundle $\mathcal{O}(-1)$ whose fiber at a point $z = [z_0, \dots, z_n] \in \mathbb{P}^n$ is the complex line $\{\lambda z : \lambda \in \mathbb{C}\} \subset \mathbb{C}^{n+1}$. There is a natural Hermitian metric h on $\mathcal{O}(-1)$ such that $|z|_h^2 = \sum |z_i|^2$. Then the curvature form Θ of $\mathcal{O}(1)$ satisfy*

$$\frac{i}{2\pi}\Theta = dd^c \log \|z\|^2.$$

Note that $\frac{i}{2\pi}\Theta$ is just the associated $(1,1)$ -form of the Fubini-Study metric, which is positive.

1.3.3 Kodaira embedding theorem

We assume X is a compact Kähler manifold and L is a holomorphic line bundle on X in this subsection. Recall that $H^0(X, L)$ denotes the space of holomorphic sections of L . The k th tensor product of L is L^k . It follows from Hodge theory that the dimension of $H^0(X, L)$ is finite. Let $N = \dim H^0(X, L)$.

Definition 1.3.17. *A point $x \in X$ is said to be a base point if $s(x) = 0$ for all $s \in H^0(X, L)$. The base locus $Bs(L)$ is the set of all base points.*

Let s_0, \dots, s_N be a basis of $H^0(X, L)$, then $Bs(L) = (s_0 = 0) \cap \dots \cap (s_N = 0)$ which is an analytic subvariety. Denote by $H^0(X, L)^*$ the dual space of $H^0(X, L)$.

Definition 1.3.18. *The Kodaira map associated to L is defined by*

$$\begin{aligned} \Phi : X \setminus Bs(L) &\rightarrow \mathbb{P}(H^0(X, L)^*), \\ \Phi(x) &= \{s \in H^0(X, L) : s(x) = 0\}. \end{aligned}$$

We give a local analytic description of the Kodaira map. Let s_0, \dots, s_N be a basis of $H^0(X, L)$ and e_L be a local holomorphic frame of L on U , where U can be chosen as a contractible Stein open subset. Then there exist holomorphic functions f_j such that $s_j = f_j e_L$. By an identification via the basis, the Kodaira map can be expressed locally as

$$\Phi : X \rightarrow \mathbb{P}^N, \quad \Phi(x) = [f_0(x), \dots, f_N(x)].$$

This map does depend on the choice of the basis. But two such maps differ only by a linear transformation of \mathbb{P}^N for two different choices of basis. The Kodaira map is a meromorphic map on X , but it is holomorphic on $X \setminus Bs(L)$.

Definition 1.3.19. *A holomorphic line bundle is called ample if the Kodaira map associated to L^k is an embedding for some integer $k > 0$.*

A compact Kähler manifold is projective if and only if it admits an ample line bundle. A natural question is when the Kodaira map defines an embedding of X . If it is an embedding map, the X is a projective manifold. We have the following important theorem called Kodaira embedding theorem.

Theorem 1.3.20. *Let L be a holomorphic line bundle over a compact Kähler manifold X . Then L is positive if and only if L is ample.*

1.3.4 Big line bundles

We will use Kodaira maps associated to high tensor powers of a holomorphic line bundle as follows,

$$\Phi_p : X \setminus Bs(L^p) \rightarrow \mathbb{P}(H^0(X, L^p)^*)$$

Define $\varrho_p := \{\text{rank}_x \Phi_p : x \in X \setminus Bs(L^p)\}$. If $H^0(X, L^p)$ is equal to $\{0\}$, we take the convention that $\varrho_p = -\infty$.

Definition 1.3.21. *The Kodaira-Iitaka dimension of L is $\kappa(L) := \max\{\varrho_p : p \geq 1\}$.*

Note that $\kappa(L) \leq \dim X$.

Definition 1.3.22. *A big line bundle is a holomorphic line bundle L such that $\kappa(L) = \dim X$.*

There are several criterions for big line bundles [45, 2.2.1, 2.3.3].

Theorem 1.3.23. *Let L be a holomorphic line bundle over a connected compact complex manifold X of dimension n . The following are equivalent:*

- (i) L is big;
- (ii) $\limsup_{p \rightarrow \infty} p^{-n} \dim H^0(X, L^p) > 0$;
- (iii) L admits a singular Hermitian metric such that the curvature current is strictly positive;
- (iv) L admits a singular Hermitian metric which is smooth outside a proper analytic set of X such that the curvature current is strictly positive.

If X is compact Kähler manifold with a fixed Kähler form ω , then the big line bundle L admits a singular Hermitian metric h such that $c_1(L, h) \geq \epsilon \omega$ for some constant $\epsilon > 0$.

The asymptotic expansion of dimensions of $H^0(X, L^p)$ is crucial through our thesis. Let $d_p = \dim H^0(X, L^p)$. If L is positive, then the Hirzebruch-Riemann-Roch theorem and Kodaira-Serre vanishing theorem show that d_p is given by a Hilbert polynomial whose dominate term is $p^n/n! \int_X c_1(L)^n$, see [35, Section 20] or the proof of [45, Theorem 2.2.20]. The Siegel's lemma gives an upper bound of d_p [45, Lemma 2.2.6].

Proposition 1.3.24. *Let L be a holomorphic line bundle over a connected compact complex manifold X . There there exists a constant $C > 0$ satisfying*

$$\dim H^0(X, L^p) \leq Cp^{\varrho_p} \leq Cp^{\kappa(L)}, \forall p \geq 1.$$

The Kodaira-Iitaka dimension $\kappa(L)$ is the optimal constant independent of p .

Theorem 1.3.25. *If X is a projective manifold, then the natural group homomorphism $\text{Div}(X) \rightarrow \text{Pic}(X)$ is surjective.*

Theorem 1.3.25 [34, Corollary 5.3.7] shows that any holomorphic line bundle in a projective manifold is isomorphic to a line bundle associated to a divisor. More generally, this theorem holds also for Moishezon manifolds, see [45, Theorem 2.2.20] for more information.

We will use the notion of multiplier ideal sheaf introduced by Nadel [47]. It can be applied to the lower estimate of dimensions associated to big line bundles.

Definition 1.3.26. *Given a real function φ which is locally integrable on X , the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the ideal subsheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in a local coordinate near x .*

The zero variety of $\mathcal{I}(\varphi)$ is the set of points x such that $e^{-2\varphi}$ is not integrable in a neighborhood of x . Let h be a singular Hermitian metric on L and φ be the local weight of h in an open subset. Then the multiplier ideal sheaf of h is defined by $\mathcal{I}(h) := \mathcal{I}(\varphi)$. This definition does not depend on the choice of local trivializations. Recall that K_X is the canonical line bundle of X . We have the following Nadel vanishing theorem, see also [15, Theorem 4.5].

Theorem 1.3.27. *Let L be a big line bundle over a compact Kähler manifold (X, ω) . Assume that L is endowed with a singular Hermitian metric h such that $c_1(L, h) \geq \epsilon \omega$ for some constant $\epsilon > 0$. Then we have*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(h)) = 0$$

for all $q \geq 1$.

Chapter 2

Equidistribution of zeros of random holomorphic sections for moderate measures.

In this chapter, we establish an equidistribution theorem for the zeros of random holomorphic sections of high powers of a positive holomorphic line bundle. The equidistribution is associated to a family of singular moderate measures. We also give a convergence speed for the equidistribution which shows an alternative proof of the main theorem.

2.1 Dinh-Sibony equidistribution theory.

The Dinh-Sibony equidistribution theory [23] is the cornerstone in the thesis. Our main theorems are proved based on the techniques and theorems from this theory. They studied the equidistribution problem associated to PLB measures, which is a generalization of the result by Shiffman-Zelditch [57]. Appropriate conditions are posed on PLB measures to satisfy the equidistribution property.

2.1.1 Constants associated to PLB measures

Let (X, ω) be a compact Kähler manifold of dimension n and μ be a PLB measure on X . Recall that $r(X, \omega)$ is the smallest number of r in Proposition

1.2.34. Following Proposition 1.2.47, we can define the notations

$$\begin{aligned}
Q(X, \omega) &:= \{\varphi \text{ q.p.s.h. on } X, dd^c\varphi \geq -r(X, \omega)\omega\}, \\
R(X, \omega, \mu) &:= \sup\{\max_X \varphi : \varphi \in Q(X, \omega), \int_X \varphi d\mu = 0\} \\
&= \sup\{-\int_X \varphi d\mu : \varphi \in Q(X, \omega), \max_X \varphi = 0\}, \\
S(X, \omega, \mu) &:= \sup\{|\int_X \varphi d\mu| : \varphi \in Q(X, \omega), \int_X \varphi \omega^n = 0\}, \\
\Delta(X, \omega, \mu, t) &:= \sup\{\mu(\varphi < -t) : \varphi \in Q(X, \omega), \int_X \varphi d\mu = 0\}.
\end{aligned}$$

Here $t > 0$. When $\mu = \omega^n$, let $R^0(X, \omega) := R(X, \omega, \mu)$. These constants are related to Alexander-Dinh-Sibony capacity, see [23, A. 2] and [32, Section 5]. The following elementary inequality comes from [23, Proposition 2.4].

Proposition 2.1.1. *We have*

$$S(X, \omega, \mu) \leq R(X, \omega, \mu) + R^0(X, \omega).$$

The following estimate is very fundamental in Dinh-Sibony equidistribution theory, see [23, Proposition A.3].

Proposition 2.1.2.

$$R^0(\mathbb{P}^k, \omega_{FS}) \leq \frac{1}{2}(1 + \log k).$$

Proof. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ (or $\pi : S^{2k+1} \rightarrow \mathbb{P}^k$) be the natural projection. Let $d\lambda_{2k+1}$ be the probability spherical measure on S^{2k+1} . Given $\varphi \in \mathcal{F}$, then $dd^c\varphi + \omega_{FS}$ is a positive closed current of mass 1. By Proposition 1.2.41, there exists a corresponding p.s.h. function v on \mathbb{C}^{k+1} such that

$$v(\lambda z) = \log |\lambda| + v(z), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Note that $v(z) = \varphi \circ \pi + \log \|z\|$ and $v(0) = -\infty$. Moreover, the Fubini-Study volume form ω_{FS}^k and $d\lambda_{2k+1}$ are probability Haar measures on \mathbb{P}^k and S^{2k+1} respectively. By the uniqueness of Haar measures, the following equality holds

$$\omega_{FS}^k(A) = d\lambda_{2k+1}(\pi^{-1}(A)),$$

where A is a measurable set in \mathbb{P}^k . Hence

$$\begin{aligned}
\int_{\mathbb{P}^k} \varphi \omega_{FS}^k &= \int_{S^{2k+1}} \varphi \circ \pi d\lambda_{2k+1} \\
&= \int_{S^{2k+1}} v d\lambda_{2k+1}.
\end{aligned}$$

Since $\max_{S^{2k+1}} v = \max_{\mathbb{P}^k} \varphi = 0$, by a theorem due to Alexander [1, Theorem 2.2], we have

$$\begin{aligned} \int_{S^{2k+1}} \varphi \circ \pi d\lambda_{2k+1} &\geq \max_{S^{2k+1}} v + \int_{S^{2k+1}} \log |z_1| d\lambda_{2k+1} \\ &= -\frac{1}{2} \sum_{j=1}^{j=k} \frac{1}{j} \geq -\frac{1}{2}(1 + \log k). \end{aligned}$$

Then the proof is completed. \square

2.1.2 Meromorphic transforms

The meromorphic transforms play a crucial role in Dinh-Sibony equidistribution theory. Roughly speaking, a meromorphic transform between two complex manifolds is a surjective multivalued map with an analytic graph.

Definition 2.1.3. *Let $(X_1, \omega_1), (X_2, \omega_2)$ be two compact Kähler manifolds of dimension n_1 and n_2 respectively, a meromorphic transform $F : X_1 \rightarrow X_2$ is the data of an analytic subset $\Gamma \subset X_1 \times X_2$ of pure dimension $n_2 + l$ such that the natural projections $\pi_1 : X_1 \times X_2 \rightarrow X_1$ and $\pi_2 : X_1 \times X_2 \rightarrow X_2$ restricted to each irreducible component of Γ are surjective. Γ is called the graph of F .*

Note that a meromorphic transform is determined by its graph. We write $F = \pi_2 \circ (\pi_1|_{\Gamma})^{-1}$. The dimension of the fiber $F^{-1}(x_2) := \pi_1(\pi_2^{-1}|_{\Gamma}(x_2))$ is equal to l for a point $x_2 \in X_2$ generic. This is the codimension of the meromorphic transform F . If T is a current of bidegree (m, m) on X_2 , $n_2 + l - n_1 \leq m \leq n_2$, we define $F^*(T) := (\pi_1)_*(\pi_2^*(T) \wedge [\Gamma])$, where $[\Gamma]$ is the current of integration over Γ . The intermediate degree of order m of a meromorphic transform $F : X_1 \rightarrow X_2$ is defined by

$$\lambda_m(F) = \int_{X_1} F^*(\omega_2^m) \wedge \omega_1^{n_2+l-m} = \int_{X_2} \omega_2^m \wedge F_*(\omega_1^{n_2+l-m}).$$

Set $d(F) := \lambda_{n_2}(F)$ and $\delta(F) := \lambda_{n_2-1}(F)$. These two notations are useful in the following arguments. We give some examples of meromorphic transforms.

Example 2.1.4. *The examples are constructed based on Grassmannian manifolds.*

(i) *Let $G(k-l+1, k+1)$ be the Grassmannian manifold which parameterizes all the projective subspace of dimension $k-l$ of \mathbb{P}^k . For a point $\hat{s} \in G(k-l+1, k+1)$, there is a corresponding projective subspace $\mathbb{P}_{\hat{s}}^{k-l}$ of dimension $k-l$. Set*

$$\Gamma_1 := \{(z, \hat{s}) \in \mathbb{P}^k \times G(k-l+1, k+1), z \in \mathbb{P}_{\hat{s}}^{k-l}\}.$$

The meromorphic transform $F_1 : \mathbb{P}^k \rightarrow G(k-l+1, k+1)$ is defined by the graph Γ_1 . Since $F_1^{-1}(\hat{s}) = \mathbb{P}_{\hat{s}}^{k-l}$, it is of codimension $k-l$.

We can give another description of F_1 by duality. Note that $\mathbb{P}^{k*} := G(k, k+1)$ is the dual of \mathbb{P}^k . Denote by $G^*(l, k+1)$ the Grassmannian manifold which parameterizes all the projective subspace of dimension $l-1$ of \mathbb{P}^{k*} . This Grassmannian manifold is biholomorphic to $G(k-l+1, k+1)$. For a point $\check{s} \in G^*(l, k+1)$, there is a corresponding projective subspace $\mathbb{P}_{\check{s}}^{(l-1)*}$ of dimension $l-1$. Choose l points s_1, \dots, s_l which generate $\mathbb{P}_{\check{s}}^{(l-1)*}$. Let $\mathbb{P}_{s_j}^{(k-1)}$ be the corresponding hyperplane in \mathbb{P}^k of the point s_j . Note that $\mathbb{P}_{\check{s}}^{k-l} := \bigcap_{j=1}^{j=l} \mathbb{P}_{s_j}^{(k-1)}$ is independent of the choices of s_j . Set

$$\Gamma_2 := \{(z, \check{s}) \in \mathbb{P}^k \times G^*(l, k+1), z \in \mathbb{P}_{\check{s}}^{k-l}\}.$$

The meromorphic transform $F_2 : \mathbb{P}^k \rightarrow G^*(l, k+1)$ is defined by the graph Γ_2 . Since $F_2^{-1}(\check{s}) = \mathbb{P}_{\check{s}}^{k-l}$, it is of codimension $k-l$.

(ii) Let $\mathbb{P}^{k,l*} := \mathbb{P}^{k*} \times \dots \times \mathbb{P}^{k*}$ (l times) be a multi-projective space. Write $s = (s_1, \dots, s_l) \in \mathbb{P}^{k,l*}$. Set

$$\Gamma_3 := \{(s, \check{s}) \in \mathbb{P}^{k,l*} \times G^*(l, k+1), \mathbb{P}_{\check{s}}^{k-l} \subset \mathbb{P}_{s_j}^{(k-1)}, \text{ for } j = 1, \dots, l\}.$$

The meromorphic transform $\Psi_l : \mathbb{P}^{k,l*} \rightarrow G^*(l, k+1)$ is defined by the graph Γ_3 . Denote by $\bar{\Psi}_l$ the adjoint of Ψ_l . The composition $F_3 := \bar{\Psi}_l \circ F_2 : \mathbb{P}^k \rightarrow \mathbb{P}^{k,l*}$ is a meromorphic transform. For a point $s \in \mathbb{P}^{k,l*}$ generic, $F_2^{-1}(s)$ is the subspace $\mathbb{P}_s^{k-l} := \bigcap_{j=1}^{j=l} \mathbb{P}_{s_j}^{(k-1)}$. Then F_3 is of codimension $k-l$.

We will see later the graph induced by a Kodaira map defines a meromorphic transform. For more information about meromorphic transforms, refer to [23, Section 3].

2.1.3 General equidistribution theorem

Let $(X, \omega), (X_p, \omega_p)$ be compact Kähler manifolds of dimension n and n_p respectively. In addition, $\int_{X_p} \omega_p^{n_p} = 1$. Let $\mathbb{X} := \prod_{p=1}^{\infty} X_p$. It follows from Tychonoff's theorem that \mathbb{X} is also a compact space with respect to the product topology. A point $x \in \mathbb{X}$ is denoted by $(x_1, x_2, \dots) = (x_p)$. The measure δ_{x_p} is the Dirac measure at the point x_p on X_p .

Consider a probability PLB measure σ_p on each X_p . Endow \mathbb{X} with the product measure $\sigma := \prod_{p=1}^{\infty} \sigma_p$. Let $F_p : X \rightarrow X_p$ be a family of meromorphic transforms with the same codimension $l < n$. By the definition of the pullback of F_p on currents, we see that $F_p^*(\delta_{x_p})$ and $F_p^*(\sigma_p)$ are positive closed currents of bidimension (l, l) on X . Moreover, $F_p^*(\delta_{x_p})$ is well defined for $x_p \in X_p$ generic. To simplify the notations, set

$$\begin{aligned} R_p &:= R(X_p, \omega_p, \sigma_p), \\ S_p &:= S(X_p, \omega_p, \sigma_p), \\ \Delta_p(t) &:= \Delta(X_p, \omega_p, \sigma_p, t), \\ \delta_p &:= \delta(F_p), \quad d_p := d(F_p). \end{aligned}$$

The following is the main theorem in Dinh-Sibony equidistribution theory [23, Theorem 4.1].

Theorem 2.1.5. *Suppose that the sequence $\{R_p\delta_p d_p^{-1}\}$ tends to 0 and one of the following two conditions is verified*

$$\begin{aligned} \sum_{p \geq 1} S_p \delta_p d_p^{-1} &< \infty, \\ \sum_{p \geq 1} \Delta_p(\delta_p^{-1} d_p t) &< \infty, \quad \forall t > 0. \end{aligned}$$

Then for almost every $x = (x_p) \in \mathbb{X}$ with respect to σ , the sequence $\langle d_p^{-1}(F_p^(\delta_{x_p}) - F_p^*(\sigma_p)), \psi \rangle$ converges to 0 uniformly on the bounded set of (l, l) -forms on X of class \mathcal{C}^2 .*

Theorem 2.1.6. *Suppose that the sequence $\{S_p\delta_p d_p^{-1}\}$ tends to 0. Then $\langle d_p^{-1}(F_p^*(\sigma_p) - F_p^*(\omega_p^{n_p})), \psi \rangle$ converges to 0 uniformly on the bounded set of (l, l) -forms on X of class \mathcal{C}^2 . To be more precise, we have*

$$\langle d_p^{-1}(F_p^*(\sigma_p) - F_p^*(\omega_p^{n_p})), \psi \rangle \leq 2S_p \delta_p d_p^{-1} \|\phi\|_{\mathcal{C}^2}$$

for any (l, l) -form of class \mathcal{C}^2 on X .

There is a crucial estimate that we mention as follows. First we define for every $p > 0, \epsilon > 0$,

$$E_p(\epsilon) := \bigcup_{\|\phi\|_{\mathcal{C}^2} \leq 1} \{x_p \in X_p : |\langle F_p^*(\delta_{x_p}) - F_p^*(\sigma_p), \phi \rangle| \geq d_p \epsilon\}.$$

Theorem 2.1.7. *Let $\eta_{\epsilon, p} := \epsilon \delta_p^{-1} d_p - 3R_p$, then*

$$\sigma_p(E_p(\epsilon)) \leq \Delta_p(\eta_{\epsilon, p}).$$

2.2 Estimate for moderate measures on \mathbb{P}^k

In this section, we give an estimate for moderate measures on \mathbb{P}^k . Our work is inspired by the techniques of exponential estimates for p.s.h. functions from [21]. Such estimate, combined with Dinh-Sibony equidistribution theory, implies the main theorems. Let (X, ω) be a compact Kähler manifold of dimension k and ω^k be its standard volume form such that ω^k is a probability measure on X .

2.2.1 Locally moderate currents

Let S be a positive closed current of bidegree (p, p) on X , the trace measure is $\sigma_S = S \wedge \omega^{k-p}$ for a fixed Hermitian form ω on X . Here X may not be compact. S is said to be *locally moderate* if its trace measure is locally

moderate. If u is a continuous real-valued function and uS defines a current on X (for example, if $\text{supp } u \subset \text{supp } S$), then $dd^c(uS)$ is well defined (cf. Section 1.2.2). We say that u is S -p.s.h. if $dd^c(uS)$ is a positive current. Dinh-Nguyên-Sibony [21, Theorem 1.1] proved the following theorem. We improve their method quantitatively in this section.

Theorem 2.2.1. *Let S be a locally moderate positive closed (p, p) -current on a complex manifold X . If u is a Hölder continuous S -p.s.h. function, then $dd^c(uS)$ is locally moderate.*

Corollary 2.2.2. *Let u be a Hölder continuous p.s.h. function on X . Then the Monge-Ampère currents $(dd^c u)^p$ are locally moderate.*

The following lemma is needed [21, Lemma 2.3].

Lemma 2.2.3. *Let T be a positive closed current of bidegree $(k-1, k-1)$ and u be a T -p.s.h. function on a neighbourhood U of the unit ball B_1 in \mathbb{C}^k . Suppose that u is smooth on $B_{1-r} \setminus B_{1-4r}$ for a fixed number $0 < r < 1/4$. If ϕ is a q.p.s.h. function on U , χ is a smooth function with compact support on B_{1-r} , $0 \leq \chi_k \leq 1$ and $\chi_k \equiv 1$ on B_{1-2r} . Then*

$$\begin{aligned} \int_{B_1} \chi \phi dd^c(uT) &= - \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \phi u T \\ &- \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \phi d^c u \wedge T + \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \phi du \wedge T \\ &+ \int_{B_{1-r}} \chi u dd^c \phi \wedge T. \end{aligned}$$

2.2.2 Covering lemma on \mathbb{P}^k

Denote by S^k the unit sphere on \mathbb{R}^{k+1} , B_1 the unit ball in \mathbb{C}^k . Let $\pi : S^{2k+1} \rightarrow \mathbb{P}^k$ be the natural projection map. More precisely, set $z_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $0 \leq j \leq k$, when $\sum_{j=0}^k |z_j|^2 = 1$, we have $\pi(x_0, y_0, \dots, x_k, y_k) = [z_0, \dots, z_k]$. Let $U_0 = \{[z_0, \dots, z_k] \in \mathbb{P}^k, z_0 \neq 0\}$. There is a natural isomorphism

$$\theta : U_0 \rightarrow \mathbb{C}^k, [z_0, \dots, z_k] \rightarrow (z_1/z_0, \dots, z_k/z_0) \quad (2.1)$$

Let $K_0 = \theta^{-1}(B_1)$. K_0 is a neighbourhood of $[1, 0, \dots, 0]$ in \mathbb{P}^k . $\pi^{-1}(K_0) = \{(x_0, y_0, \dots, x_k, y_k) \in S^{2k+1}, \sum_{j=1}^k |z_j|^2 \leq |z_0|^2\}$. Let $S_0 = \{(x_0, y_0, \dots, x_k, y_k) \in S^{2k+1}, x_0 > \frac{1}{\sqrt{2}}\}$. It's obvious that $S_0 \subset \pi^{-1}(K_0)$ and $\pi(S_0)$ is a neighbourhood of $[1, 0, \dots, 0]$.

By the homogeneity of S^{2k+1} (resp. \mathbb{P}^k), there is a neighbourhood S'_0 (resp. $\pi(S'_0)$) of any point $(x_0, y_0, \dots, x_k, y_k)$ (resp. $[z_0, \dots, z_k]$) which is the image of S_0 (resp. $\pi(S_0)$) by rotations (resp. unitary transformations). We say that S'_0 (resp. $\pi(S'_0)$) is similar to S_0 (resp. $\pi(S_0)$). Since \mathbb{P}^k is compact, there are finitely many such neighbourhoods $\pi(S_0)$ that cover \mathbb{P}^k . Denote by M_k the minimum number of such neighbourhoods $\pi(S_0)$ that cover \mathbb{P}^k . We have the following lemma.

Lemma 2.2.4. *Let K_0 be as above. For any point $z \in \mathbb{P}^k$, there exists a neighbourhood K_z of z which is similar to K_0 . Denote by N_k the minimum number of such neighbourhoods K_0 that cover \mathbb{P}^k . Then $N_k = O(8^k)$.*

Proof. Since $\pi(S_0) \subset K_0$, then $M_k \geq N_k$. So it remains to prove that $M_k = O(8^k)$. We endow S^{2k+1} with the great-circle distance. S_0 can be regarded as an open ball with central point $[1, 0, \dots, 0]$ of radius $\frac{\pi}{4}$. Denote S_0 by $B([1, 0, \dots, 0], \frac{\pi}{4})$. Let $S_1 = B([1, 0, \dots, 0], \frac{\pi}{8}) = \{(x_0, y_0, \dots, x_k, y_k) \in S^{2k+1}, x_0 > \frac{\sqrt{2+\sqrt{2}}}{2}\}$. We first consider the open balls of radius $\frac{\pi}{8}$. All of them are similar to each other. We put the maximal number of balls $B(z_1, \frac{\pi}{8}), \dots, B(z_{m_k}, \frac{\pi}{8})$ in S^{2k+1} such that all of them are disjoint mutually. Then $S^{2k+1} = \bigcup_{j=1}^{m_k} B(z_j, \frac{\pi}{4})$. If there exists a point $w \in S^{2k+1} \setminus \bigcup_{j=1}^{m_k} B(z_j, \frac{\pi}{4})$, then the great-circle distance between w and z_j is larger than or equal to $\frac{\pi}{4}$ for all $1 \leq j \leq m_k$. Hence $B(w, \frac{\pi}{8}) \subset S^{2k+1} \setminus \bigcup_{j=1}^{m_k} B(z_j, \frac{\pi}{8})$, contradicts with the maximality. Then $M_k \leq m_k \leq \text{Vol}(S^{2k+1})/\text{Vol}(S_1)$, the last inequality is due to the mutual disjointedness. It means that $N_k = O(\text{Vol}(S^{2k+1})/\text{Vol}(S_1))$.

We now use the spherical coordinate for S^{2k+1} . Let $x_0 = \cos \theta_1, y_0 = \sin \theta_1 \cos \theta_2, \dots, x_k = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{2k} \cos \theta_{2k+1}, y_k = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{2k} \sin \theta_{2k+1}$. Then the volume element of S^{2k+1} is $d_{S^{2k+1}} V = \sin^{2k} \theta_1 \sin^{2k-1} \theta_2 \cdots \sin \theta_{2k} d\theta_1 d\theta_2 \cdots d\theta_{2k+1}$.

$$\begin{aligned} \text{Vol}(S^{2k+1}) &= \int_0^\pi \sin^{2k} \theta_1 d\theta_1 \int_0^\pi \sin^{2k-1} \theta_2 d\theta_2 \cdots \int_0^\pi \sin \theta_{2k} d\theta_{2k} \int_0^{2\pi} d\theta_{2k+1} \\ \text{Vol}(S_1) &= \int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1 \int_0^\pi \sin^{2k-1} \theta_2 d\theta_2 \cdots \int_0^\pi \sin \theta_{2k} d\theta_{2k} \int_0^{2\pi} d\theta_{2k+1}. \end{aligned}$$

This yields $O(\text{Vol}(S^{2k+1})/\text{Vol}(S_1)) = O(\int_0^\pi \sin^{2k} \theta_1 d\theta_1 / \int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1)$.

Then it suffices to show that $\int_0^\pi \sin^{2k} \theta_1 d\theta_1 / \int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1 \leq 8^{k+1}, \forall k \geq 7$. When $k = 7$, the inequality is right. By induction on k and the following integrals

$$\int \sin^{2k} \theta_1 d\theta_1 = -\frac{\sin^{2k-1} \theta_1 \cos \theta_1}{2k} + \frac{2k-1}{2k} \int \sin^{2k-2} \theta_1 d\theta_1,$$

the proof is reduced to show that $\int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1 \geq \frac{8}{7} \frac{1}{2k+1} \frac{\sqrt{2}}{4} \left(\frac{2-\sqrt{2}}{4}\right)^k$. By the relation between $\int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1$ and $\int_0^{\frac{\pi}{8}} \sin^{2k+6} \theta_1 d\theta_1$, we have

$$\begin{aligned} &\int_0^{\frac{\pi}{8}} \sin^{2k} \theta_1 d\theta_1 \\ &\geq \frac{1}{2k+1} \frac{\sqrt{2}}{4} \left(\frac{2-\sqrt{2}}{4}\right)^k \left(1 + \frac{2-\sqrt{2}}{4} \frac{2k+2}{2k+3} + \left(\frac{2-\sqrt{2}}{4}\right)^2 \frac{(2k+2)(2k+4)}{(2k+3)(2k+5)}\right) \end{aligned}$$

Then the proof is completed. \square

2.2.3 Local estimate for moderate measures

Let \mathcal{F} be defined in (1.1) when $X = \mathbb{P}^k$ and θ defined in (2.1). The following lemma is crucial for the main proposition in this section.

Lemma 2.2.5. *Let u be of class \mathcal{C}^ρ with modulus ϵ on a neighbourhood U of B_1 in \mathbb{C}^k with $dd^c u \geq 0$ in the sense of currents, $0 < \rho < 1$. Set $\omega := \frac{1}{2}dd^c \log(1 + \|z\|^2)$. Let $\mathcal{F}_0 = \{\phi \circ \theta^{-1} \text{ on } U : \phi \in \mathcal{F}\}$ and T a positive closed $(k-1, k-1)$ -current. If $T \wedge \omega$ is (c, α) -moderate on U , then*

$$\int_{B_1} \exp\left(-\frac{\alpha\rho}{4}\phi\right) dd^c(uT) \leq ck\epsilon(c_1 e^\alpha + \frac{c_2}{\alpha})$$

where c_1, c_2 are positive constants independent of k, ρ and T .

Proof. We modify the function u on U . Subtracting a constant, we assume that $u \leq -\epsilon/2$ on B_1 . Consider the function $v(z) = \max(u(z), \epsilon A \log |z|)$ for a constant $A > 0$ large enough such that v coincides with u near the origin and $v(z) = \epsilon A \log |z|$ near the boundary of B_1 . For example, $A = \frac{1}{2} \log \frac{1}{1-4r}$. A is independent of the choice of u . Fix $0 < r < 1/16$, we are allowed to assume that $u = \epsilon A \log |z|$ on $B_1 \setminus B_{1-4r}$. For the smooth function χ defined in Lemma 2.3.3, we can assume that $\|\chi\|_{\mathcal{C}^2} < c_3$ for some constant $c_3 > 1$ large enough independent of k , since the terms in the definition of the norm $\|\bullet\|_{\mathcal{C}^2}$ are smooth on the compact subset $\bar{B}_{1-r} \setminus B_{1-2r}$. Set $\sigma_T = T \wedge \omega, \sigma_{T'} = dd^c(uT), \phi_M = \max(\phi, -M), \psi_M = \phi_{M-1} - \phi_M$, for $\phi \in \mathcal{F}_0, M \geq 0$.

To prove the lemma, we need to estimate the mass of $dd^c(uT)$ on $\{\phi < -M\}$. Since $\text{supp} \chi \subset B_{1-r}$, hence

$$\sigma_{T'}\{\phi < -M\} \leq \int \chi \psi_M dd^c(uT).$$

Since T is (c, α) -moderate, then

$$\sigma_T\{z \in B_{1-r}, \phi(z) \leq -M + 1\} \leq ce^\alpha e^{-\alpha M}.$$

By Lemma 2.3.3, we have

$$\begin{aligned} \int_{B_1} \chi \psi_M dd^c(uT) &= - \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \psi_M u T \\ &\quad - \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \psi_M d^c u \wedge T + \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \psi_M du \wedge T \\ &\quad + \int_{B_{1-r}} \chi u dd^c \psi_M \wedge T \end{aligned} \quad (2.2)$$

We know that $\omega = \frac{1}{2}dd^c \log(1 + \|z\|^2) = \frac{i}{2\pi} \sum_{j,l=1}^k \left(\frac{dz_j \wedge d\bar{z}_l}{1 + \|z\|^2} - \frac{\bar{z}_j z_l dz_j \wedge d\bar{z}_l}{(1 + \|z\|^2)^2} \right)$. By simple computations, the eigenvalues of the corresponding Hermitian matrix of ω are $\frac{1}{\pi(1 + \|z\|^2)^2}$ and $\frac{1}{\pi(1 + \|z\|^2)}$ ($k-1$ times). On the other hand, the eigenvalues of the corresponding Hermitian matrix of $i \sum_{j,l=1}^k dz_j \wedge d\bar{z}_l$ are k and 0 ($k-1$ times). So

there exists a constant $m_1 > 0$ small enough such that $\omega - \frac{m_1}{k} i \sum_{j,l=1}^k dz_j \wedge d\bar{z}_l > 0$ on B_1 . Hence $|dd^c \chi \wedge u T| \leq |uc_3 i \sum_{j,l=1}^k dz_j \wedge d\bar{z}_l \wedge T| \leq \epsilon A |\log(1-3r)| c_3 \frac{k}{m_1} \sigma_T$. Observing that $0 \leq \psi_M \leq 1$, $\text{supp} \psi_M \subset \{\phi < -M + 1\}$, we obtain

$$\left| \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \psi_M u T \right| \leq \epsilon A |\log(1-3r)| c_3 \frac{k}{m_1} c e^\alpha e^{-\alpha M}.$$

Since we know u explicitly on $\text{supp}(d\chi)$, we obtain

$$\left| \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \psi_M d^c u \wedge T \right| \leq \frac{\epsilon A}{1-3r} c_3 k m_2 c e^\alpha e^{-\alpha M},$$

$$\left| \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \psi_M du \wedge T \right| \leq \frac{\epsilon A}{1-3r} c_3 k m_2 c e^\alpha e^{-\alpha M}.$$

for a constant $m_2 > 0$ large enough independent of k . The sum of the first three terms is less than

$$c_4 \epsilon k c e^\alpha e^{-\alpha M} \quad (2.3)$$

where $c_4 = A c_3 \left(\frac{|\log(1-3r)|}{m_1} + \frac{2m_2}{1-3r} \right)$ is independent of k and ρ .

For the last integral in (2.2), we use a regularization procedure and the condition of ρ -Hölder continuity of u . Let $\{u_\delta\}$ be the smooth approximation of u obtained by convolution. For some fixed $0 < \delta < 1$ small enough, u_δ is defined in a neighborhood of \bar{B}_{1-r} . There exists a suitable function u_δ satisfying that $\|u_\delta\|_{\mathcal{C}^2} \leq \epsilon \delta^{-(2-\rho)}$ and $\|u - u_\delta\|_\infty \leq \epsilon \delta^\rho$, where the latter inequality follows from that u is of class \mathcal{C}^ρ with modulus ϵ . The above two inequalities are independent of k . We write

$$\begin{aligned} & \int_{B_1} \chi u d d^c \psi_M \wedge T \\ &= \int \chi d d^c \psi_M \wedge T u_\delta + \int \chi (d d^c \phi_{M-1} - d d^c \phi_M) \wedge T (u - u_\delta). \end{aligned}$$

Since

$$\left| \int \chi d d^c (\phi T) \right| = \left| \int d d^c \chi \wedge \phi T \right| \leq k \|\chi\|_{\mathcal{C}^2} \int_{B_{1-r}} |\phi| d\sigma_T,$$

We obtain

$$\begin{aligned} & \left| \int \chi (d d^c \phi_{M-1} - d d^c \phi_M) \wedge T \right| \leq 2k \|\chi\|_{\mathcal{C}^2} \int_{B_{1-r}} |\phi| d\sigma_T \\ & \leq 2k \|\chi\|_{\mathcal{C}^2} \frac{1}{\alpha} \int_{B_{1-r}} \exp(-\alpha \phi) d\sigma_T \leq 2c_3 k \frac{c}{\alpha}. \end{aligned}$$

Then

$$\left| \int \chi (d d^c \phi_{M-1} - d d^c \phi_M) \wedge T (u - u_\delta) \right| \leq 2c_3 k \frac{c}{\alpha} \epsilon \delta^\rho \quad (2.4)$$

Using Lemma 2.3.3 again, we obtain

$$\begin{aligned}
& \int \chi dd^c \psi_M \wedge Tu_\delta \\
&= \int_{B_{1-r} \setminus B_{1-3r}} dd^c \chi \wedge \psi_M T u_\delta + \int_{B_{1-r} \setminus B_{1-3r}} d\chi \wedge \psi_M T \wedge d^c u_\delta \\
&\quad - \int_{B_{1-r} \setminus B_{1-3r}} d^c \chi \wedge \psi_M T \wedge du_\delta + \int_{B_{1-r}} \chi \psi_M T \wedge dd^c u_\delta.
\end{aligned}$$

By the same argument, the first three integrals have the same dominant constant

$$c_4 \epsilon k c e^\alpha e^{-\alpha M}. \quad (2.5)$$

The final term

$$\begin{aligned}
\left| \int \chi \psi_M T \wedge dd^c u_\delta \right| &\leq c e^\alpha e^{-\alpha M} \|u_\delta\|_{\mathcal{E}^2} \\
&\leq c e^\alpha e^{-\alpha M} \epsilon \delta^{-(2-\rho)}.
\end{aligned} \quad (2.6)$$

Let $\delta = e^{-\alpha M/2}$ small enough, since it is sufficient to consider M big. Then $e^{-\alpha M \rho/2} = e^{-\alpha M} e^{\alpha M(2-\rho)/2}$. Combining (2.3), (2.4), (2.5), (2.6), we have

$$\sigma_{T'}\{z \in B_1, \phi < -M\} \leq \epsilon c k (2c_4 e^\alpha + \frac{e^\alpha}{k} + 2\frac{c_3}{\alpha}) e^{-\frac{\alpha M}{2}\rho}.$$

So by Remark 1.2.50 we have

$$\begin{aligned}
& \int_{B_1} \exp(-\frac{\alpha \rho}{4} \phi) dd^c(uT) \\
&\leq 2\epsilon c k (2c_4 e^\alpha + \frac{e^\alpha}{k} + 2\frac{c_3}{\alpha}) \leq \epsilon c k (c_1 e^\alpha + \frac{c_2}{\alpha}),
\end{aligned}$$

where $c_1 = 4c_4 + 2, c_2 = 4c_3$. □

2.2.4 Main result and its proof

The following proposition is our main result about the estimate for moderate measures on \mathbb{P}^k .

Proposition 2.2.6. *Suppose that u_j is of class \mathcal{C}^ρ with modulus ϵ on \mathbb{P}^k for some $0 < \rho < 1, 0 < \epsilon < 1$, and that u_j is an $\epsilon \omega_{FS}$ -p.s.h. function for all $1 \leq j \leq k$. Assume that $\epsilon < \beta_0 k^{-3} (\frac{\rho}{12})^{2k}$, where β_0 is a positive constant independent of k and ρ . Then there exists a positive constant c_5 independent of k and ρ , such that*

$$\int_{\mathbb{P}^k} \exp(-\alpha_0 (\frac{\rho}{4})^k \phi) (\wedge_{j=1}^{j=k} (dd^c u_j + \epsilon \omega_{FS} + \omega_{FS}) - \omega_{FS}^k) \leq c_5 (\frac{\rho}{4})^k \quad (2.7)$$

for all $\phi \in \mathcal{F}$, where α_0 is the constant in Proposition 1.2.48. In other words, $(\wedge_{j=1}^{j=k} (dd^c u_j + \epsilon \omega_{FS} + \omega_{FS}) - \omega_{FS}^k)$ is $(c_5 (\frac{\rho}{4})^k, \alpha_0 (\frac{\rho}{4})^k)$ -moderate.

Proof. We pull back the integral (2.7) locally to that on \mathbb{C}^k . There is a potential $v = \frac{\epsilon}{2} \log(1 + \|z\|^2)$ on \mathbb{C}^k such that $(\theta^{-1})^*(\epsilon\omega_{FS}) = dd^c v$, where the map θ is defined in (2.1). Set $\tilde{u}_j := u_j \circ \theta^{-1} + v$. Note that u_j is $\epsilon\omega_{FS}$ -p.s.h., then $dd^c \tilde{u}_j \geq 0$. Since u_j is of class \mathcal{C}^ρ with modulus ϵ on \mathbb{P}^k , $\log(1 + \|z\|^2)$ is of class \mathcal{C}^a on \mathbb{C}^k for all $0 < a < 1$, then we may assume that \tilde{u}_j is of class \mathcal{C}^ρ with modulus ϵ on B_1 . Hence $\|\tilde{u}_j\|_{\mathcal{C}^\rho(B_1)} \leq \epsilon$. Let $\omega = \frac{1}{2} dd^c \log(1 + \|z\|^2)$, we have

$$\begin{aligned} & \int_{K_0} \exp(-\alpha\phi) \wedge_{j=1}^{j=k} (dd^c u_j + \epsilon\omega_{FS} + \omega_{FS}) \\ &= \int_{B_1} \exp(-\alpha\phi \circ \theta^{-1}) (\theta^{-1})^* \wedge_{j=1}^{j=k} (dd^c u_j + \epsilon\omega_{FS} + \omega_{FS}) \\ &= \int_{B_1} \exp(-\alpha\phi \circ \theta^{-1}) (dd^c \tilde{u}_1 + \omega) \wedge \cdots \wedge (dd^c \tilde{u}_k + \omega) \end{aligned} \quad (2.8)$$

We replace \tilde{u}_j (resp. $\phi \circ \theta^{-1}$) by u_j (resp. ϕ) in the sequel. Since there are two constants $c_0 > 0, \alpha_0 > 0$ independent of k and ρ , such that

$$\int_{\mathbb{P}^k} \exp(-\alpha_0 \tilde{\phi}) \omega_{FS}^k \leq c_0 k, \quad (2.9)$$

by pulling back the integral in B_1 with Lemma 2.3.5, we have

$$\int_{B_1} \exp(-\alpha_0 \frac{\rho}{4} \phi) (dd^c u_j) \wedge \omega^{k-1} \leq c_0 \epsilon k^2 (c_1 e^{\alpha_0} + \frac{c_2}{\alpha_0}).$$

By induction we can show that

$$\int_{B_1} \exp(-\alpha_0 (\frac{\rho}{4})^j \phi) dd^c u_{l_1} \wedge \cdots \wedge dd^c u_{l_j} \wedge \omega^{k-j} \leq c_0 k (\epsilon k)^j \prod_{l=0}^{j-1} (c_1 e^{\alpha_0 (\frac{\rho}{4})^l} + \frac{c_2}{\alpha_0 (\frac{\rho}{4})^l})$$

for all $1 \leq l_1 < \cdots < l_j \leq k$. Let $\beta_0 = 1/(c_1 e^{\alpha_0} + \frac{c_2}{\alpha_0})$, $\epsilon_0 = \beta_0 k^{-3} (\frac{1}{8})^k (\frac{\rho}{4})^{\frac{3k-1}{2}} > \epsilon$, $\epsilon_0 = \epsilon_1 \epsilon_2$, $\epsilon_2 = (\frac{\rho}{4})^k$. Here β_0 is independent of k and ρ . Let $\epsilon_1 = \epsilon_3 / (\frac{\rho}{4})^{\frac{k+1}{2}}$, then $\epsilon_3 = \beta_0 (\frac{\rho}{32})^k / k^3$. Hence

$$\begin{aligned} & \int_{B_1} \exp(-\alpha_0 (\frac{\rho}{4})^k \phi) ((dd^c u_1 + \omega) \wedge \cdots \wedge (dd^c u_k + \omega) - \omega^k) \\ &= \sum_{j=1}^k \binom{k}{j} \int_{B_1} \exp(-\alpha_0 (\frac{\rho}{4})^j \phi) dd^c u_1 \wedge \cdots \wedge dd^c u_j \wedge \omega^{k-j} \\ &\leq \sum_{j=1}^k \binom{k}{j} c_0 k (\epsilon_1 k)^j (\frac{1}{\beta_0})^j (\frac{\rho}{4})^{k+k-1+\cdots+k-(j-1)} \\ &\leq \sum_{j=1}^k \binom{k}{j} c_0 k (\epsilon_1 k)^j (\frac{1}{\beta_0})^j (\frac{\rho}{4})^{\frac{k+1}{2} j} \leq c_0 k \sum_{j=1}^k \binom{k}{j} (\frac{\epsilon_3 k}{\beta_0})^j \\ &\leq c_0 (\frac{\rho}{32})^k (\sum_{j=0}^{k-1} \frac{1}{k^j}) \leq 2c_0 (\frac{\rho}{32})^k. \end{aligned} \quad (2.10)$$

This is equivalent to

$$\int_{K_0} \exp(-\alpha_0(\frac{\rho}{4})^k \phi) ((dd^c u_1 + \epsilon \omega_{FS} + \omega_{FS}) \wedge \cdots \wedge (dd^c u_k + \epsilon \omega_{FS} + \omega_{FS}) - \omega_{FS}^k) \leq 2c_0(\frac{\rho}{32})^k.$$

By Lemma 2.3.4, there is a positive constant N' independent of k and ρ such that $N_k \leq N'8^k$. Let $c_5 = 2c_0N'$. Due to the homogeneity of \mathbb{P}^k , we have

$$\int_{\mathbb{P}^k} \exp(-\alpha_0(\frac{\rho}{4})^k \phi) ((dd^c u_1 + \epsilon \omega_{FS} + \omega_{FS}) \wedge \cdots \wedge (dd^c u_k + \epsilon \omega_{FS} + \omega_{FS}) - \omega_{FS}^k) \leq c_5(\frac{\rho}{4})^k.$$

The proof is completed. \square

Remark 2.2.7. Since $(dd^c u_j + \omega_{FS})^k \leq (dd^c u_j + \epsilon \omega_{FS} + \omega_{FS})^k$, the above proposition, combined with (2.9), gives the following estimate

$$\int_{\mathbb{P}^k} \exp(-\alpha_0(\frac{\rho}{4})^k \phi) (dd^c u_1 + \omega_{FS}) \wedge \cdots \wedge (dd^c u_k + \omega_{FS}) \leq c_0k + c_5(\frac{\rho}{4})^k \leq c_0k + c_5$$

for all $\phi \in \mathcal{F}$. In other words, $(dd^c u_1 + \omega_{FS}) \wedge \cdots \wedge (dd^c u_k + \omega_{FS})$ is $(c_0k + c_5, \alpha_0(\frac{\rho}{4})^k)$ -moderate.

2.3 Equidistribution on positive line bundles for moderate measures

In this section we prove Theorem 0.2.1 and 0.2.2. Consider the projective manifold X of dimension n and the ample line bundle L on X in Theorem 0.2.1. By Proposition 1.3.11, there exists a smooth Hermitian metric h such that

$$c_1(L, h) = -dd^c \log h(e_L, e_L)^{\frac{1}{2}}$$

is a strictly positive $(1, 1)$ -form, where e_L is a local holomorphic section on L . As we know, $c_1(L, h)$ represents the Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. Let $\omega = c_1(L, h)$ be the Kähler form, $\int_X \omega^n = c_1(L)^n \in \mathbb{Z}^+$.

The line bundle L^p of the p th tensor power of L has a natural Hermitian metric h_p induced by h . The space $H^0(X, L^p)$ of holomorphic sections of L^p has the following inner product,

$$\langle s_1, s_2 \rangle := \frac{1}{c_1(L)^n} \int_X h_p(s_1, s_2) \omega^n$$

$\forall s_1, s_2 \in H^0(X, L^p)$.

2.3.1 Meromorphic transforms induced by Kodaira maps

First we consider a meromorphic map $f : X_1 \rightarrow X_2$ between two complex manifolds X_1 and X_2 of dimension n_1 and n_2 respectively. It follows from [50] that there exists an analytic subset I of X_1 such that f is holomorphic on $X_1 \setminus I$

and the closure of the graph of f over $X_1 \setminus I$ is an irreducible analytic subset of dimension n_1 of $X_1 \times X_2$. The smallest set I is called the *indeterminacy set* of f . Note that I has codimension at least 2.

Now we consider the meromorphic transforms from X to $\mathbb{P}H^0(X, L^p)$ induced by the Kodaira maps. The meromorphic transform $F_p : X \rightarrow \mathbb{P}H^0(X, L^p)$ has the following graph

$$\Gamma_p = \{(x, s) \in X \times \mathbb{P}H^0(X, L^p) : s(x) = 0\}.$$

Since L is ample, $\dim H^0(X, L^p) \geq 2$ (cf. Theorem 3.4.1), for every point $x \in X$, there exists a point $s \in \mathbb{P}H^0(X, L^p)$ such that $s(x) = 0$. Hence the projection from Γ_p to X is surjective. Since L^p is not trivial, there are no nowhere vanishing sections. That is to say, every point $s \in \mathbb{P}H^0(X, L^p)$ must vanish at some point $x \in X$. Hence the projection from Γ_p to $\mathbb{P}H^0(X, L^p)$ is surjective. Then F_p is indeed a meromorphic transform of codimension $n - 1$.

Recall that ω_{FS} denotes the Fubini-Study form of $\mathbb{P}^{n_p} \cong \mathbb{P}H^0(X, L^p)$. The *Fubini-Study current* $\Phi_p^*(\omega_{FS})$ of $H^0(X, L^p)$ is defined to be the pullback of ω_{FS} by the Kodaira map Φ_p . It is actually a L^1 -form, which is smooth outside the indeterminacy set of Φ_p by the following lemma.

Lemma 2.3.1. *Let $f : X_1 \rightarrow X_2$ be a meromorphic map between two complex compact manifolds X_1 and X_2 of dimension n_1 and n_2 respectively. If ϕ is a smooth form of bidegree (p, q) on X_2 , then $f^*(\phi)$ is a L^1 -form, which is smooth outside the indeterminacy set of f .*

Proof. Let π_1, π_2 be the natural projections from $X_1 \times X_2$ to X_1 and X_2 respectively. Denote by Γ the graph of f . The regular part of Γ is $reg(\Gamma)$. By definition, we have

$$f^*(\phi) := (\pi_1)_*(\pi_2^*(\phi) \wedge [\Gamma]).$$

Then for any test form ψ on X_1 ,

$$\langle f^*(\phi), \psi \rangle = \int_{reg(\Gamma)} \pi_2^*(\phi) \wedge \pi_1^*(\psi).$$

Then $f^*(\phi)$ can be extended to act on the space of continuous forms. Hence it is a current of order 0. If V is a proper analytic subset of X_1 , then $\pi_1^{-1}(V) \cap \Gamma$ is also a proper analytic subset of Γ and $\pi_1^{-1}(x_1) \cap \Gamma$ is of measure 0 for x_1 generic. So $f^*(\phi)$ has no mass on V , hence on the indeterminacy set I .

Note that $\pi_1 : \Gamma \setminus \pi_1^{-1}(I) \rightarrow X_1 \setminus I$ is a biholomorphic, therefore, we obtain

$$\begin{aligned} \langle f^*(\phi), \psi \rangle &= \int_{X_1 \setminus I} \pi_{1*} \pi_2^*(\phi) \wedge \psi \\ &= \int_{X_1 \setminus I} g^*(\phi) \wedge \psi, \end{aligned}$$

where g is the restriction map of f on $X_1 \setminus I$, which is holomorphic. Then $f^*(\phi) = g^*(\phi)$ is a smooth form on $X_1 \setminus I$. In addition, $f^*(\phi)$ has measure coefficients and has no mass on I . Hence $g^*(\phi)$ has L^1 -coefficients and is equal to $f^*(\phi)$ in the sense of currents. The proof is completed. \square

Note that $\delta_p := \lambda_{n_p-1}(F_p)$ (resp. $d_p := \lambda_{n_p}(F_p)$) is the intermediate degree of order $n_p - 1$ (resp. n_p) of F_p .

Lemma 2.3.2. *In the above setting, δ_p is bounded and $d_p = pc_1(L)^n$. Moreover, $F_p^*(\omega_{FS}^{n_p}) = \Phi_p^*(\omega_{FS})$.*

Proof. The first assertion is proved in [23, Lemma 7.1] by using cohomological arguments. We prove the second one with the definition of F_p^* . For any test $(n-1, n-1)$ -form ψ , we have

$$\begin{aligned}
\langle F_p^*(\omega_{FS}^{n_p}), \psi \rangle &= \int_{\Gamma_p} \pi_1^*(\psi) \wedge \pi_2^*(\omega_{FS}^{n_p}) \\
&= \int_{\mathbb{P}H^0(X, L^p)} \pi_{2*} \pi_1^*(\psi) \wedge \omega_{FS}^{n_p} \\
&= \int_{\mathbb{P}H^0(X, L^p)} \int_{\pi_2^{-1}(s_p) \cap \Gamma_p} \pi_1^*(\psi) \omega_{FS}^{n_p}(s_p) \\
&= \int_{\mathbb{P}H^0(X, L^p)} \int_{\{x \in X : s_p(x)=0\}} \psi \omega_{FS}^{n_p}(s_p) \\
&= \int_{\mathbb{P}H^0(X, L^p)} \langle [s_p = 0], \psi \rangle \omega_{FS}^{n_p}(s_p) \\
&= \langle \Phi_p^*(\omega_{FS}), \psi \rangle.
\end{aligned}$$

The last equality follows from [10, Proposition 4.2]. This completes the proof. \square

2.3.2 Proof of Theorem 0.2.1

From now on we recall the notations from Section 2.2.1 in the special case. Suppose that μ is a PLB probability measure on \mathbb{P}^k . \mathcal{F} is defined in (1.1) when $X = \mathbb{P}^k$. Let

$$\begin{aligned}
Q(\mathbb{P}^k, \omega_{FS}) &= \{\phi \text{ q.p.s.h. on } \mathbb{P}^k : dd^c \phi \geq -\omega_{FS}\}, \\
R(\mathbb{P}^k, \omega_{FS}, \mu) &= \sup_{\phi} \left\{ - \int \phi d\mu, \phi \in \mathcal{F} \right\}, \\
S(\mathbb{P}^k, \omega_{FS}, \mu) &= \sup_{\phi} \left\{ \left| \int \phi d\mu \right|, \phi \in Q(\mathbb{P}^k, \omega_{FS}), \int \phi \omega_{FS}^k = 0 \right\}, \\
\Delta(\mathbb{P}^k, \omega_{FS}, \mu, t) &= \sup_{\phi} \left\{ \mu(\phi < -t), \phi \in Q(\mathbb{P}^k, \omega_{FS}), \int \phi d\mu = 0 \right\}
\end{aligned}$$

for any $t > 0$. When $\mu = \omega_{FS}^k$, we write $R^0(\mathbb{P}^k, \omega_{FS}) = R(\mathbb{P}^k, \omega_{FS}, \mu)$. Let σ_p be a PLB probability measure on $\mathbb{P}H^0(X, L^p)$. To simplify the notations, let

$$\begin{aligned}
R_p &:= R(\mathbb{P}H^0(X, L^p), \omega_{FS}, \sigma_p), \\
R_p^0 &:= R(\mathbb{P}H^0(X, L^p), \omega_{FS}, \omega_{FS}^{n_p}), \\
S_p &:= S(\mathbb{P}H^0(X, L^p), \omega_{FS}, \sigma_p), \\
\Delta_p(t) &:= \Delta(\mathbb{P}H^0(X, L^p), \omega_{FS}, \sigma_p, t).
\end{aligned}$$

Let $\mathbb{P}^X := \prod_{p \geq 1} \mathbb{P}H^0(X, L^p)$ endowed with its measure $\sigma = \prod_{p \geq 1} \sigma_p$. Denote by δ_z the Dirac measure at a point z .

Lemma 2.3.3. $F_p^*(\delta_{s_p}) = [s_p = 0]$.

Proof. By the definition of pullbacks of meromorphic transforms, we have

$$F_p^*(\delta_{s_p}) = \pi_{1*}(\pi_2^*(\delta_{s_p}) \wedge [\Gamma_p]).$$

For any test $(n-1, n-1)$ -form ϕ on X ,

$$\begin{aligned} \langle F_p^*(\delta_{s_p}), \phi \rangle &= \int_{X \times \mathbb{P}H^0(X, L^p)} \pi_2^*(\delta_{s_p}) \wedge [\Gamma_p] \wedge \pi_1^*(\phi) \\ &= \int_{X \times \mathbb{P}H^0(X, L^p)} [\pi_2^{-1}(s_p)] \wedge [\Gamma_p] \wedge \pi_1^*(\phi) \\ &= \int_{\pi_2^{-1}(s_p) \cap \Gamma_p} \pi_1^*(\phi) = \int_{\{x \in X : s_p(x) = 0\}} \phi \\ &= \langle [s_p = 0], \phi \rangle. \end{aligned}$$

The proof is completed. \square

We specify Theorem 2.2.5 and Theorem 2.2.6 for the above case.

Theorem 2.3.4. *Suppose that the sequence $\{R_p \delta_p d_p^{-1}\}$ tends to 0 and*

$$\sum_{p \geq 1} \Delta_p(\delta_p^{-1} d_p t) < \infty$$

for all $t > 0$. Then for almost everywhere $s = (s_p) \in \mathbb{P}^X$ with respect to σ , the sequence $\langle d_p^{-1}(F_p^*(\delta_{s_p}) - F_p^*(\sigma_p)), \psi \rangle$ converges to 0 uniformly on the bounded set of $(n-1, n-1)$ -forms on X of class \mathcal{C}^2 .

Theorem 2.3.5. *Suppose that the sequence $\{S_p \delta_p d_p^{-1}\}$ tends to 0. Then $\langle d_p^{-1}(F_p^*(\sigma_p) - F_p^*(\omega_{FS}^{n_p})), \psi \rangle$ converges to 0 uniformly on the bounded set of $(n-1, n-1)$ -forms on X of class \mathcal{C}^2 .*

The following theorem is due to Tian, Ruan, Catlin and Zelditch, see [60, 9, 52].

Theorem 2.3.6. *For all $r \geq 0$, $\|p^{-1}\Phi_p^*(\omega_{FS}) - \omega\|_{\mathcal{C}^r} = O(p^{-1})$.*

In order to prove the main theorem, we write

$$\begin{aligned} |\langle p^{-1}[s_p = 0] - \omega, \psi \rangle| &\leq |\langle p^{-1}[s_p = 0] - p^{-1}F_p^*(\sigma_p), \psi \rangle| \\ &\quad + |\langle p^{-1}F_p^*(\sigma_p) - p^{-1}F_p^*(\omega_{FS}^{n_p}), \psi \rangle| + |\langle p^{-1}F_p^*(\omega_{FS}^{n_p}) - \omega, \psi \rangle|, \end{aligned}$$

for any test form ψ of bidegree $(n-1, n-1)$ on X . It is sufficient to prove that the three terms in the right side of the inequality all tend to 0 when $p \rightarrow \infty$. The third one is right due to Theorem 2.3.6. The first one holds under the conditions that $R_p = o(p)$, $\sum_{p \geq 1} \Delta(pt) < \infty$, $\forall t > 0$ by Theorem

2.3.4. The second one is valid when $S_p = o(p)$ by Theorem 2.3.5. By applying Proposition 2.1.1 and Proposition 2.1.2, the proof is reduced to the estimates of R_p/p and $\sum_{p \geq 1} \Delta(pt)$ for any $t > 0$.

End of the proof of Theorem 0.2.1. We have $F_p^*(\omega_{FS}^{n_p}) = (\Phi_p)^* \omega_{FS}$ by Lemma 2.3.2. It follows from Theorem 2.3.6 that

$$p^{-1}F_p^*(\omega_{FS}^{n_p}) \rightarrow \omega \quad (2.11)$$

in the weak sense of currents. We write $\mu_{1,p} = \omega_{FS}^{n_p}$, $\mu_{2,p} = \wedge_{j=1}^{n_p} (dd^c u_{p,j} + \epsilon_p \omega_{FS} + \omega_{FS}) - \mu_{1,p}$. Then $\sigma_p \leq \mu_{1,p} + \mu_{2,p}$. Note that $n_p = c_1(L)^n p^n / n! + O(p^{n-1})$. Let $c > (\frac{12}{\rho})^{2c_1(L)^n / n!} > 1$ such that $c^{p^n} \geq \frac{1}{\beta_0} n_p^3 (\frac{12}{\rho})^{2n_p}$, then c depends only on X, L and ρ . Hence $\mu_{2,p}$ is a positive moderate measure satisfying Proposition 2.2.6. To estimate Δ_p , we consider any q.p.s.h. function ϕ on \mathbb{P}^{n_p} such that $dd^c \phi \geq -\omega_{FS}$ and $\int \phi d\sigma_p = 0$. Set $\varphi := \phi - \max_{\mathbb{P}^{n_p}} \phi$. It is obvious that $\varphi \in \mathcal{F}$ by definition in (1.1). Since $\int \phi d\sigma_p = 0$, $\max_{\mathbb{P}^{n_p}} \phi \geq 0$. Hence $\varphi \leq \phi$. Then we have

$$\begin{aligned} \sigma_p(\phi < -pt) &\leq \sigma_p(\varphi < -pt) \\ &\leq \mu_{1,p}(\varphi < -pt) + \mu_{2,p}(\varphi < -pt) \\ &\leq \int \exp(\alpha_0(-pt - \varphi)) d\mu_{1,p} + \int \exp(\alpha_0(\frac{\rho}{4})^{n_p}(-pt - \varphi)) d\mu_{2,p} \\ &\leq c_0 n_p \exp(-\alpha_0 pt) + c_5 (\frac{\rho}{4})^{n_p} \exp(-\alpha_0 (\frac{\rho}{4})^{n_p} pt). \end{aligned}$$

The last inequality follows from Proposition 1.2.48 and Proposition 2.2.6. Then by the definition of Δ_p , we have

$$\sum_{p \geq 1} \Delta_p(pt) \leq \sum_{p \geq 1} c_0 n_p \exp(-\alpha_0 pt) + \sum_{p \geq 1} c_5 (\frac{\rho}{4})^{n_p} \exp(-\alpha_0 (\frac{\rho}{4})^{n_p} pt). \quad (2.12)$$

It is obvious that $\sum_{p \geq 1} p^n \exp(-pt) < \infty$ and that $\exp(-(\frac{\rho}{4})^{n_p} pt)$ tends to 1 when p tends to infinity, $\forall t > 0$. This yields $\sum_{p \geq 1} \Delta_p(pt) < \infty$. By Proposition 2.1.2 and Proposition 2.2.6,

$$\limsup_{p \rightarrow \infty} R_p^0/p \leq \lim_{p \rightarrow \infty} \frac{1 + \log n_p}{2p} = 0. \quad (2.13)$$

$$\begin{aligned} \limsup_{p \rightarrow \infty} R_p/p &\leq \limsup_{p \rightarrow \infty} \left\{ - \int \phi d\mu_{1,p} - \int \phi d\mu_{2,p} \right\} / p \\ &\leq \limsup_{p \rightarrow \infty} R_p^0/p + \lim_{p \rightarrow \infty} c_5 (\frac{\rho}{4})^{n_p} / (\alpha_0 (\frac{\rho}{4})^{n_p} p) = 0 \end{aligned} \quad (2.14)$$

By Proposition 2.1.1, (2.13) and (2.14), $\limsup_{p \rightarrow \infty} S_p/p = 0$. Note that $\delta_p d_p^{-1} = O(\frac{1}{p})$ by Lemma 2.3.2. Hence by applying Theorem 2.3.5, the following sequence

$$p^{-1}F_p^*(\sigma_p) - p^{-1}F_p^*(\omega_{FS}^{n_p}) \rightarrow 0 \quad (2.15)$$

in the weak sense of currents. We know that $F_p^*(\delta_{s_p}) = [s_p = 0]$ by Lemma 2.3.3. Combined with (2.12) and (2.14), Theorem 2.3.4 implies that for σ -almost everywhere $s \in \mathbb{P}^X$, the following sequence

$$p^{-1}[s_p = 0] - p^{-1}F_p^*(\sigma_p) \rightarrow 0 \quad (2.16)$$

in the weak sense of currents. Then we deduce from (2.11), (2.15) and (2.16) that for σ -almost everywhere $s \in \mathbb{P}^X$,

$$\begin{aligned} |\langle p^{-1}[s_p = 0] - \omega, \psi \rangle| &\leq |\langle p^{-1}[s_p = 0] - p^{-1}F_p^*(\sigma_p), \psi \rangle| \\ &+ |\langle p^{-1}F_p^*(\sigma_p) - p^{-1}F_p^*(\omega_{FS}^{n_p}), \psi \rangle| + |\langle p^{-1}F_p^*(\omega_{FS}^{n_p}) - \omega, \psi \rangle| \rightarrow 0, \end{aligned}$$

for any test form ψ of bidegree $(n-1, n-1)$ on X when p tends to ∞ . That is to say, $p^{-1}[s_p = 0]$ converges weakly to ω . The proof is completed. \square

2.3.3 Nontrivial examples of moderate measures

Now given X and L in Theorem 0.2.1, we construct a concrete example of a sequence of functions $(u_{p,j})$ satisfying the conditions of the theorem. We require that $u_{p,1} = \dots = u_{p,n_p} = u_p$. Notice that we can perturbate u_p so that the constants ξ_p, ϵ_p do not change and the perturbed functions still satisfy the conditions in Theorem 0.2.1.

Example 2.3.7. *Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k$ be the natural map. Consider the map $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ with $f[z_0, \dots, z_k] = [z_0^k, \dots, z_k^k]$. From [53, Example 1.6.4], its Green function is $s(z) = \max(\log |z_0|, \dots, \log |z_k|)$. Moreover, s is a Hölder continuous function with any exponent $0 < \rho < 1$. We obtain a well-defined function*

$$v := \max\left(\log \frac{|z_0|}{|z|}, \dots, \log \frac{|z_k|}{|z|}\right) \quad (2.17)$$

on \mathbb{P}^k . Since $\pi^*(dd^c v + \omega_{FS}) = dd^c s \geq 0$, then v is ω_{FS} -p.s.h. and Hölder continuous with any exponent $0 < \rho < 1$. Denote by d_{FS} the distance induced by Fubini-Study metric. Let $d_k = \sup_{\substack{z, w \in \mathbb{P}^k \\ z \neq w}} \frac{|v(z) - v(w)|}{d_{FS}(z, w)^\rho}$. We will show that

$$d_k \leq \sqrt{\pi} k \quad (2.18)$$

at the end of the example.

For each p , we obtain a corresponding function v_p by using (2.17) and identifying $\mathbb{P}H^0(X, L^p)$ with \mathbb{P}^{n_p} . Consider the functions $u_p = c'_p v_p$ with suitable constants $c'_p = O(\frac{1}{p^n c^{p^n}}) < 1/c^{p^n}$, where $c = (145)^{c_1(L)^n/n!}$. Let $\epsilon_n := c'_n$. Since $k_n = O(n^k)$, it follows from (2.18) that $d_{k_n} = O(n^k)$. Consequently, u_n is of class \mathcal{C}^ρ with modulus $1/c^{p^n}$. Moreover, since v_p is ω_{FS} -p.s.h., we infer that u_p is $\epsilon_p \omega_{FS}$ -p.s.h.. So $\{u_n\}$ satisfy the three conditions in Theorem 0.2.1. From the above proof, we see that $\sigma = \prod_{p \geq 1} \sigma_p = (dd^c u_p + \omega_{FS})^{n_p}$ satisfies the equidistribution property.

Finally we prove (2.18). It is sufficient to consider the special case when $|z_0| \geq \max\{|z_1|, \dots, |z_k|\}$, $|w_0| \geq \max\{|w_1|, \dots, |w_k|\}$. Then

$$d_k = \frac{1}{2} \sup_{\substack{z, w \in K \\ z \neq w}} \frac{|\log(1 + |z|^2) - \log(1 + |w|^2)|}{d_{FS}(z, w)^\rho}$$

where $z = (\frac{z_1}{z_0}, \dots, \frac{z_k}{z_0})$, $w = (\frac{w_1}{w_0}, \dots, \frac{w_k}{w_0}) \in \mathbb{C}^k$ and $K = \{z \in \mathbb{C}^k : |z_i| \leq 1, 1 \leq i \leq k\}$. Let $g = \sum_{i,j=1}^{2k} g_{ij} dx^i \otimes dx^j$ be the associated Riemannian metric with $g_{11} = \frac{1}{\pi} \frac{1+|z|^2-|z_1|^2}{(1+|z|^2)^2}$. When $r_1 = |z|$, $r_2 = |w|$ are fixed, $d_{FS}(z, w)$ takes its minimum only when z and w are at the same line through the origin in \mathbb{R}^{2k} . The distance is invariant with respect to the orthogonal group $O(2k)$ in this case since the Fubini-Study metric is invariant with respect to the unitary group $U(k)$ on \mathbb{P}^k . So we take the simple case when $z = (r_1, 0, \dots, 0)$, $w = (r_2, 0, \dots, 0)$. Hence

$$\begin{aligned} d_k &= \frac{\sqrt{\pi}}{2} \sup_{0 \leq r_1 < r_2 \leq k} \frac{\log(1 + r_2^2) - \log(1 + r_1^2)}{(\arctan r_2 - \arctan r_1)^\rho} \\ &= \frac{\sqrt{\pi}}{2} \sup_{0 \leq s_1 < s_2 \leq \arctan k} \frac{\log(1 + \tan^2 s_2) - \log(1 + \tan^2 s_1)}{(s_2 - s_1)^\rho} \\ &\leq \frac{\sqrt{\pi}}{2} \max(\log(1 + k^2), \sup_{\substack{s_2 - s_1 < 1 \\ 0 \leq s_1 < s_2 \leq \arctan k}} \frac{\log(1 + \tan^2 s_2) - \log(1 + \tan^2 s_1)}{s_2 - s_1}). \end{aligned}$$

The function $y = \log(1 + \tan^2 x)$ is increasing and convex on $[0, \infty)$. So the second term in the last inequality is equal to $(\log(1 + \tan^2 s))' \Big|_{s=\arctan k} = 2k$. This completes the proof of (2.18).

2.3.4 Proof of Theorem 0.2.2

Now we are in a position to prove Theorem 0.2.2.

Proof. It follows from Lemma 2.3.2 and Theorem 2.3.6 that

$$|\langle p^{-1} F_p^*(\omega_{FS}^{n_p}) - \omega, \psi \rangle| \leq \frac{C_1}{p} \|\psi\|_{\mathcal{E}^2} \quad (2.19)$$

for some positive constant C_1 depending only on X, L . We know that $S_p = O(\log p)$ by using Proposition 2.2.1, (2.13) and (2.14), then Theorem 2.3.5 and Theorem 2.2.6 imply that

$$|\langle p^{-1} F_p^*(\sigma_p) - p^{-1} F_p^*(\omega_{FS}^{n_p}), \psi \rangle| \leq \frac{C_2 \log p}{p} \|\psi\|_{\mathcal{E}^2} \quad (2.20)$$

for some positive constant C_2 depending only on X, L . Set

$$E_p(\epsilon_0) := \bigcup_{\|\psi\|_{\mathcal{E}^2} \leq 1} \{s_p \in \mathbb{P}H^0(X, L^p) : |\langle p^{-1}[s_p = 0] - p^{-1} F_p^*(\sigma_p), \psi \rangle| \geq \epsilon_0\}$$

for any $\epsilon_0 > 0$. We define $E_p := E_p(\frac{C_3 \log p}{p})$, where C_3 is some positive constant depending only on X, L . Note that $R_p = O(\log p)$ from inequalities (2.13) and (2.14). By applying Theorem 2.1.7, we deduce that

$$\sigma_p(E_p) \leq \Delta_p(C_4 \log p).$$

Here C_4 is a positive constant depending only on X, L . Moreover, C_4 is sufficiently large such that $\alpha_0 C_4 > k + 2$ since C_3 can be chosen sufficiently large. Recall that α_0 is the constant defined in Proposition 1.2.48. Then by (2.12), we obtain

$$\begin{aligned} \sigma_p(E_p) &\leq \Delta_p(C_4 \log p) \\ &\leq c_0 n_p \exp(-\alpha_0 C_4 \log p) + c_5 \left(\frac{\rho}{4}\right)^{n_p} \exp(-\alpha_0 C_4 \left(\frac{\rho}{4}\right)^{n_p} \log p) \\ &\leq (c_0 + c_5) n_p \frac{1}{p^{\alpha_0 C_4}} \leq \frac{C}{p^2}. \end{aligned} \quad (2.21)$$

Here C is a positive constant sufficiently large which depends only on X, L . Note that the third inequality of (2.21) follows from a direct calculation when p is big enough. The fact that $n_p = O(p^n)$ yields the last inequality of (2.21). By definition of E_p , we obtain for any point $s_p \in \mathbb{P}H^0(X, L^p) \setminus E_p$,

$$|\langle p^{-1}[s_p = 0] - p^{-1}F_p^*(\sigma_p), \psi \rangle| \leq \frac{C_3 \log p}{p} \|\psi\|_{\mathcal{G}^2}. \quad (2.22)$$

It follows from (2.19),(2.20) and (2.22) that

$$|\langle p^{-1}[s_p = 0] - \omega, \psi \rangle| \leq \frac{C \log p}{p} \|\psi\|_{\mathcal{G}^2}. \quad (2.23)$$

The proof is completed. \square

Remark 2.3.8. Since $\sum_{p=1}^{\infty} \sigma_p(E_p) < \infty$, Theorem 0.2.2 gives an alternative proof of Theorem 0.2.1. This is a standard proof which is analogous to that of Borel-Cantelli lemma. Note that

$$\sum_{p=1}^{\infty} \sigma_p(E_p) < \infty.$$

Define

$$E := \{s = \{s_p\}_{p=1}^{\infty} \in \mathbb{P}^X : s_p \in E_p \text{ for infinitely many } p\}.$$

It is easy to see that E is contained in the following set

$$\tilde{E}_N := \{s = \{s_p\}_{p=1}^{\infty} \in \mathbb{P}^X : s_p \in E_p \text{ for at least one } p \geq N\}$$

for each integer $N \geq 1$. Hence we have

$$\sigma(E) \leq \sigma(\tilde{E}_N) \leq \sum_{p=N}^{\infty} \sigma_p(E_p) \leq C \sum_{p=N}^{\infty} \frac{1}{p^2}.$$

The proof is completed by letting N tend to ∞ .

Chapter 3

Equidistribution on big line bundles for moderate measures.

In this chapter, we establish an equidistribution theorem for the common zeros of random sections of high powers of several singular Hermitian big line bundles associated to moderate measures [55]. The basic setting is taken from Part II in Chapter 0.

3.1 Intersection of Fubini-Study currents

In this section, we introduce some results about the intersection of the Fubini-Study currents associated to m line bundles. We will see that the current $c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)$ in Theorem 0.3.1 is well-defined. Based on the elementary techniques in [14], we also show that for almost all the zero-divisors of sections of high powers of the bundles with respect to a moderate measure are in general position. Then it follows from Corollary 1.2.26 (see also [17]) that the currents $[S_p = 0]$ are well-defined for almost all S_p with respect to moderate measures σ in Theorem 0.3.1.

We keep the notations and hypotheses in Section 0.3. Consider the *Kodaira map*

$$\Phi_{k,p} : X \rightarrow \mathbb{P}(H_{(2)}^0(X, L_k^p)^*).$$

Here $H_{(2)}^0(X, L_k^p)^*$ is the dual space of $H_{(2)}^0(X, L_k^p)$. By a similar argument in Section 1.3.3, choose $\{S_{k,p}^j\}_{j=0}^{d_{k,p}}$ as an orthonormal basis of $H_{(2)}^0(X, L_k^p)$. By an identification via the basis, it boils down to a meromorphic map

$$\Phi_{k,p} : X \rightarrow \mathbb{P}^{d_{k,p}}.$$

Now we give a local analytic description of the above map. Let $U \subset X$ be a contractible Stein open subset, e_k a local holomorphic frame of L_k on U . Then there exists a holomorphic function $s_j^{k,p}$ on U such that $S_{k,p}^j = s_j^{k,p} e_k^{\otimes p}$. Then the map is expressed locally as

$$\Phi_{k,p}(x) = [s_0^{k,p}(x) : \dots : s_{d_{k,p}}^{k,p}(x)], \quad \forall x \in U \quad (3.1)$$

It is called the Kodaira map defined by the basis $\{S_{k,p}^j\}_{j=0}^{d_{k,p}}$. Denote by $P_{k,p}$ the *Bergman kernel function* defined by

$$P_{k,p}(x) = \sum_{j=0}^{d_{k,p}} |S_{k,p}^j(x)|_{h_{k,p}}^2, \quad |S_{k,p}^j(x)|_{h_{k,p}}^2 = h_{k,p}(S_{k,p}^j(x), S_{k,p}^j(x)).$$

It is easy to see that this definition is independent of the choice of basis.

Recall that ω_{FS} is the normalized Fubini-Study form on $\mathbb{P}^{d_{k,p}}$. The Fubini-Study current $\gamma_{k,p}$ of $H_{(2)}^0(X, L_k^p)$ is

$$\gamma_{k,p} = \Phi_{k,p}^*(\omega_{FS}).$$

Lemma 2.4.1 says that the Fubini-Study current is a L^1 -form, which is smooth outside the indeterminacy set of $\Phi_{k,p}$. We have in the local Stein open subset U ,

$$\gamma_{k,p}|_U = \frac{1}{2} dd^c \log \sum_{j=0}^{d_{k,p}} |s_j^{k,p}|^2.$$

Let u_k be the local weight of the Hermitian metric h_k on U . Then the following p.s.h. function verifies

$$u_{k,p} := \frac{1}{2p} \log \sum_{j=0}^{d_{k,p}} |s_j^{k,p}|^2 = u_k + \frac{1}{2p} \log P_{k,p}. \quad (3.2)$$

It implies that $dd^c u_{k,p} = \frac{1}{p} \gamma_{k,p}$ and

$$\frac{1}{p} \gamma_{k,p} = c_1(L_k, h_k) + \frac{1}{2p} dd^c \log P_{k,p}.$$

Since $\log P_{k,p}$ is a global function which belongs to $L^1(X, \omega^n)$, $\frac{1}{p} \gamma_{k,p}$ has the same cohomology class as $c_1(L_k, h_k)$. Define the base locus of $H_{(2)}^0(X, L_k^p)$ as

$$A_{k,p} = \{x \in X : S_{k,p}^j = 0, \forall 0 \leq j \leq d_{k,p}\}, \quad 1 \leq k \leq m.$$

Note that $A_{k,p}$ is an analytic subset, $A_{k,p} \cap U = \{u_{k,p} = -\infty\}$ and $\{u_k = -\infty\} \subset A_k \cap U$. By [10, Theorem 5.1] and its proof, we know that there exist constants $C > 0, p_0 \in \mathbb{N}$, such that

$$P_{k,p}(x) \geq C, \forall p \geq p_0, 1 \leq k \leq m, x \in X \setminus A_k.$$

Then we have

$$u_{k,p}(x) \geq u_k(x) + \frac{\log C}{2p}, \forall p \geq p_0, 1 \leq k \leq m, x \in U. \quad (3.3)$$

Hence $A_{k,p} \subset A_k$ for $p \geq p_0$. Moreover,

$$\frac{1}{p} \log P_{k,p}(x) \rightarrow 0 \quad (3.4)$$

in $L^1(X, \omega^n)$.

We have the following result [14, Proposition 3.1].

Proposition 3.1.1. *We keep the notations and hypotheses of Theorem 0.3.1, then*

(i) *For every $J \subset \{1, \dots, m\}$, $J' = \{1, \dots, m\} \setminus J$, the analytic subsets $A_{k,p}$ and A_l , for $k \in J, l \in J'$, are in general position, when p is sufficiently large.*

(ii) *The currents*

$$\bigwedge_{k \in J} \gamma_{k,p} \wedge \bigwedge_{l \in J'} c_1(L_l, h_l)$$

are well defined for every $J \subset \{1, \dots, m\}$, when p is sufficiently large.

Proof. Since the analytic subsets A_1, \dots, A_m are in general position, the fact that $A_{k,p} \subset A_k$ for all p sufficiently large yields (i). Hence the currents

$$\bigwedge_{k \in J} \gamma_{k,p} \wedge \bigwedge_{l \in J'} c_1(L_l, h_l)$$

are well defined by Corollary 1.2.26. Then (ii) is proved. \square

3.2 Bertini theorem associated to moderate measures

Lemma 3.2.1. *Let σ be a moderate measure on \mathbb{P}^N . Then every proper analytic subset of \mathbb{P}^N has measure zero with respect to σ .*

Proof. By the homogeneity of \mathbb{P}^N , it is sufficient to prove that

$$\sigma([z_0]) = 0$$

for some homogeneous coordinate $[z_0, \dots, z_N]$. Recall that for a moderate measure σ , there exist constants $c > 0, \alpha > 0$ such that

$$\int_{\mathbb{P}^N} \exp(-\alpha\phi) d\sigma \leq c,$$

$\forall \phi \in \mathcal{F}$, where \mathcal{F} is defined in (1.1). It follows that

$$\int_{\mathbb{P}^N} |\phi| d\sigma < \infty.$$

Let $\phi = \log \frac{|z_0|}{|z|}$, where $|z|^2 = \sum_{j=0}^N |z_j|^2$. This function is well defined on \mathbb{P}^N . Note that ϕ is ω_{FS} -p.s.h., $\max_{z \in \mathbb{P}^N} \phi(z) = 0$. Then $\phi \in \mathcal{F}$. So we have that

$$\int_{[z_0=0]} \left| \log \frac{|z_0|}{|z|} \right| d\sigma < \infty.$$

Hence $\sigma([z_0]) = 0$. The proof is completed. \square

Adapting the proof of [14, Proposition 3.2], we obtain the following Bertini type theorem in the context of moderate measures.

Proposition 3.2.2. *Let X be a compact complex manifold of dimension n . Let $L_k, 1 \leq k \leq m \leq n$, be m holomorphic line bundles on X . If*

(i) V_k is a vector subspace of $H^0(X, L_k)$ with basis $S_{k,0}, \dots, S_{k,d_k}$, the base loci BsV_1, \dots, BsV_m are in general position, where $BsV_k := \{x \in X : S_{k,0}(x) = \dots = S_{k,d_k}(x) = 0\}$.

(ii) For each $t_k = [t_{k,0} : \dots : t_{k,d_k}] \in \mathbb{P}^{d_k}$, we set

$$Z(t_k) := \{x \in X : \sum_{j=0}^{d_k} t_{k,j} S_{k,j}(x) = 0\}.$$

(iii) $\sigma = \sigma_1 \times \dots \times \sigma_m$ is the product measure on the multi-projective space $\mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_m}$, where σ_k is a probability moderate measure on \mathbb{P}^{d_k} .

Then the analytic subsets $Z(t_1), \dots, Z(t_m)$ are in general position for almost all $(t_1, \dots, t_m) \in \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_m}$ with respect to σ .

Proof. Let $\sigma_{l_1 \dots l_k} = \sigma_{l_1} \times \dots \times \sigma_{l_k}$ be the product measure on $\mathbb{P}^{d_{l_1}} \times \dots \times \mathbb{P}^{d_{l_k}}$ for every $1 \leq l_1 < \dots < l_k \leq m$. Set

$$U_k = \{(t_{l_1}, \dots, t_{l_k}) \in \mathbb{P}^{d_{l_1}} \times \dots \times \mathbb{P}^{d_{l_k}} : \dim Z(t_{l_1}) \cap \dots \cap Z(t_{l_k}) \cap \tilde{V}_j \leq n - k - j\},$$

where $1 \leq l_1 < \dots < l_k \leq m, j = 0$ and $\tilde{V}_0 = X, \tilde{V}_j = BsV_{i_1} \cap \dots \cap BsV_{i_j}$ for some $i_1 < \dots < i_j$ in $\{1, \dots, m\} \setminus \{l_1, \dots, l_k\}$ for $1 \leq j \leq m - k$. Note that the sets U_k depend on the choices of l_1, \dots, l_k, j and \tilde{V}_j .

It is sufficient to prove that

$$\sigma_{l_1 \dots l_k}(U_k) = 1$$

by induction on k for every subset $U_k, 1 \leq l_1 < \dots < l_k \leq m, 0 \leq j \leq m - k$. We only consider the case when $\{l_1, \dots, l_k\} = \{1, \dots, k\}$. Write $\sigma'_k = \sigma_{1 \dots k}$ for short. We first consider the case when $k = 1$. If $j = 0$, then

$$U_1 = \{t_1 \in \mathbb{P}^{d_1} : \dim Z(t_1) \leq n - 1\} = \mathbb{P}^{d_1}.$$

If $1 \leq j \leq m - 1$, let $\tilde{V}_j = \bigcup_{l=1}^N D_l \cup B$, where the subsets D_l are the irreducible components of \tilde{V}_j of dimension $n - j$ and B is of dimension less than $n - j$. So $\{t_1 \in \mathbb{P}^{d_1} : D_l \subset Z(t_1)\}$ is a proper linear subspace of \mathbb{P}^{d_1} . If not, $D_l \subset BsV_1$ implies that $\dim \tilde{V}_j \cap BsV_1 = n - j$, which contradicts the condition that BsV_1, \dots, BsV_m are in general position. We know that $\dim Z(t_1) \cap \tilde{V}_j \geq n - j$ if $t_1 \in \mathbb{P}^{d_1} \setminus U_1$. Since $Z(t_1) \cap \tilde{V}_j$ is an analytic subset in \tilde{V}_j , then $D_l \subset Z(t_1) \cap \tilde{V}_j$ for some l . It follows that

$$\mathbb{P}^{d_1} \setminus U_1 = \bigcup_{l=1}^N \{t_1 \in \mathbb{P}^{d_1} : D_l \subset Z(t_1)\}.$$

Hence we have that $\sigma_1(\mathbb{P}^{d_1} \setminus U_1) = 0$ by Lemma 3.2.1.

Now we can assume that $\sigma'_k(U_k) = 1$ for any U_k defined as above. Set

$$\begin{aligned} U_{k+1} &= \{(t_1, \dots, t_{k+1}) \in \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_{k+1}} : \\ &\quad \dim Z(t_1) \cap \dots \cap Z(t_{k+1}) \cap \tilde{V}_j \leq n - k - 1 - j\}, \\ U' &= \{(t_1, \dots, t_k) \in \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_k} : \\ &\quad \dim Z(t_1) \cap \dots \cap Z(t_k) \cap \tilde{V}_j \leq n - k - j\}, \\ U'' &= \{(t_1, \dots, t_k) \in \mathbb{P}^{d_1} \times \dots \times \mathbb{P}^{d_k} : \\ &\quad \dim Z(t_1) \cap \dots \cap Z(t_k) \cap \tilde{V}_j \cap BsV_{k+1} \leq n - k - 1 - j\}, \end{aligned}$$

where $0 \leq j \leq m - k - 1$, $\tilde{V}_j = BsV_{i_1} \cap \dots \cap BsV_{i_j}$ for $k+2 \leq i_1 < \dots < i_j \leq m$. Let $U = U' \cap U''$. By using the induction on k , we know that $\sigma'_k(U') = \sigma'_k(U'') = 1$, thus $\sigma'_k(U) = 1$.

We need to prove that

$$\sigma'_{k+1}(U_{k+1}) = 1.$$

It is enough to prove that

$$\sigma'_{k+1}(W) = 0, \quad W := (U \times \mathbb{P}^{d_{k+1}}) \setminus U_{k+1}.$$

Given some $t = (t_1, \dots, t_k) \in U$, set

$$\begin{aligned} Z(t) &:= Z(t_1) \cap \dots \cap Z(t_k), \\ W(t) &:= \{t_{k+1} \in \mathbb{P}^{d_{k+1}} : \dim Z(t) \cap \tilde{V}_j \cap Z(t_{k+1}) \geq n - k - j\}, \end{aligned}$$

then it is sufficient to show that $\sigma_{k+1}(W(t)) = 0$.

Let $Z(t) \cap \tilde{V}_j = \bigcup_{l=1}^N D_l \cup B$, where D_l are irreducible components of $Z(t) \cap \tilde{V}_j$ of dimension $n - k - j$, $\dim B \leq n - k - 1 - j$ as $t \in U \subset U'$. By the same argument in the above, if $t_{k+1} \in W(t)$, then $Z(t) \cap \tilde{V}_j \cap Z(t_{k+1})$ is an analytic subset of $Z(t) \cap \tilde{V}_j$ of dimension $n - k - j$, hence there exists some l such that $D_l \subset Z(t) \cap \tilde{V}_j \cap Z(t_{k+1})$. We obtain that

$$W(t) = \bigcup_{l=1}^N F_l(t), \quad F_l(t) := \{t_{k+1} \in \mathbb{P}^{d_{k+1}} : D_l \subset Z(t_{k+1})\}.$$

We claim that not all the sections of V_{k+1} can vanish on D_l . If not, that is to say, $D_l \subset BsV_{k+1}$, this implies that

$$\dim Z_t \cap \tilde{V}_j \cap BsV_{k+1} = n - k - j,$$

which contradicts the fact that $t \in U''$. Hence we can suppose that $S_{k+1, d_{k+1}} \not\equiv 0$ on D_l . So

$$\begin{aligned} F_l(t) &\subset \{t_{k+1,0} = 0\} \cup G_l(t), \\ G_l(t) &:= \{[1 : t_{k+1,1} : \dots : t_{k+1, d_{k+1}}] \in \mathbb{P}^{d_{k+1}} : D_l \subset Z([1 : t_{k+1,1} : \dots : t_{k+1, d_{k+1}}])\}. \end{aligned}$$

There exists at most one $\eta \in \mathbb{C}$ such that $[1 : t_{k+1,1} : \dots : t_{k+1,d_{k+1}-1} : \eta] \in G_l(t)$ for any $(t_{k+1,1}, \dots, t_{k+1,d_{k+1}-1}) \in \mathbb{C}^{d_{k+1}-1}$. Otherwise, if there exist two complex numbers $\eta \neq \eta'$, which satisfy the property, then we have on D_l ,

$$\begin{aligned} S_{k+1,0} + t_{k+1,1}S_{k+1,1} + \dots + t_{k+1,d_{k+1}-1}S_{k+1,d_{k+1}-1} + \eta S_{k+1,d_{k+1}} &\equiv 0, \\ S_{k+1,0} + t_{k+1,1}S_{k+1,1} + \dots + t_{k+1,d_{k+1}-1}S_{k+1,d_{k+1}-1} + \eta' S_{k+1,d_{k+1}} &\equiv 0. \end{aligned}$$

Then we have a contradiction with that $S_{k+1,d_{k+1}} \not\equiv 0$ on D_l . This implies that $\sigma_{k+1}(G_l(t)) = 0$. Moreover, $\sigma_{k+1}(F_l(t)) = 0$. It follows that $\sigma_{k+1}(W(t)) = 0$. This completes the proof. \square

In the setting of Theorem 0.3.1, let $V_{k,p} = H_{(2)}^0(X, L_k^p)$ with orthonormal basis $\{S_{k,p}^j\}_{j=0}^{d_{k,p}}$. Then the base locus of $V_{k,p}$ is $A_{k,p}$. Let $Z(t_k)$ be an analytic hypersurface for any $t_k = [t_{k,0} : \dots : t_{k,d_{k,p}}] \in \mathbb{P}^{d_{k,p}}$, defined in Proposition 3.2.2 (ii). Let σ_p be the product measure of probability moderate measures on $\mathbb{P}^{d_{1,p}} \times \dots \times \mathbb{P}^{d_{m,p}}$ in Theorem 0.3.1. Arguing as in the proof of [14, Proposition 3.3], we obtain the following

Proposition 3.2.3. *In the above setting,*

(i) *The analytic subsets $Z(t_1), \dots, Z(t_m)$ are in general position, for almost every $(t_1, \dots, t_m) \in \mathbb{P}^{d_{1,p}} \times \dots \times \mathbb{P}^{d_{m,p}}$ with respect to σ_p , when p is sufficiently large.*

(ii) *$Z(t_{i_1}) \cap \dots \cap Z(t_{i_k})$ is of pure dimension $n - k$ for each $1 \leq k \leq m, 1 \leq i_1 < \dots < i_k \leq m$.*

Proof. The base loci $A_{1,p}, \dots, A_{m,p}$ are in general position for all p sufficiently large, by Proposition 3.1.1. Then (i) follows by Proposition 3.2.2. We fix such p and consider the current $[Z(t_k) = 0]$. Recall that e_k is a local holomorphic frame for L_k and $S_{k,p}^j = s_j^{k,p} e_k^{\otimes p}$, where $s_j^{k,p}$ is a holomorphic function on a contractible Stein open subset in X . By Poincaré-Lelong formula, we have locally on U

$$[Z(t_k) = 0] = dd^c \log \left| \sum_{j=0}^{d_{k,p}} t_{k,j} s_j^{k,p} \right| = pc_1(L_k, h_k) + dd^c \log \left| \sum_{j=0}^{d_{k,p}} t_{k,j} S_{k,p}^j \right|_{h_{k,p}}.$$

Since $\log \left| \sum_{j=0}^{d_{k,p}} t_{k,j} S_{k,p}^j \right|_{h_{k,p}}$ is a global p.s.h. function, $d^c \log \left| \sum_{j=0}^{d_{k,p}} t_{k,j} S_{k,p}^j \right|_{h_{k,p}}$ is a current of order at most 1. Then $dd^c \log \left| \sum_{j=0}^{d_{k,p}} t_{k,j} S_{k,p}^j \right|_{h_{k,p}}$ is closed. Hence $[Z(t_k) = 0]$ has the same cohomology class as $pc_1(L_k, h_k)$. By (i), the current $[Z(t_{i_1}) = 0] \wedge \dots \wedge [Z(t_{i_k}) = 0]$ is well defined and supported in $Z(t_{i_1}) \cap \dots \cap Z(t_{i_k})$, for each $1 \leq k \leq m, 1 \leq i_1 < \dots < i_k \leq m$ and almost all $(t_1, \dots, t_m) \in \mathbb{P}^{d_{1,p}} \times \dots \times \mathbb{P}^{d_{m,p}}$ with respect to σ_p . Since L_k is a big line bundle and $c_1(L_k, h_k) \geq \epsilon \omega$ for some constant $\epsilon > 0$, then we have

$$\begin{aligned} \int_{Z(t_{i_1}) \cap \dots \cap Z(t_{i_k})} \omega^{n-k} &= \int_X [Z(t_{i_1}) = 0] \wedge \dots \wedge [Z(t_{i_k}) = 0] \wedge \omega^{n-k} \\ &= p^k \int_X c_1(L_{i_1}, h_{i_1}) \wedge \dots \wedge c_1(L_{i_k}, h_{i_k}) \wedge \omega^{n-k} > p^k \epsilon^k \int_X \omega^n > 0. \end{aligned}$$

It follows that $Z(t_{i_1}) \cap \dots \cap Z(t_{i_k}) \neq \emptyset$. By Chapter III in [43],

$$\text{codim} (Z(t_{i_1}) \cap \dots \cap Z(t_{i_k})) \leq \sum_{j=1}^k \text{codim} Z(t_{i_j}) = k.$$

Moreover, $Z(t_{i_1}), \dots, Z(t_{i_k})$ are in general position. Hence it is of pure dimension $n - k$. (ii) is then proved. \square

3.3 Estimate on multi-projective spaces

In this section we give our core work about some estimates on multi-projective spaces. This leads to our main theorem and convergence speed for equidistributions.

3.3.1 Meromorphic transforms for several line bundles

We recall some results in Dinh-Sibony equidistribution theory in the setting of Theorem 0.3.1. Let Φ_p be a sequence of meromorphic transforms from a projective manifold (X, ω) into the compact Kähler manifolds (\mathbb{X}_p, ω_p) of the same codimension k . Let

$$d_{0,p} = d_{1,p} + \dots + d_{m,p}$$

be the dimension of \mathbb{X}_p . Consider a PLB probability measure μ_p on \mathbb{X}_p , for every $p > 0, \epsilon > 0$, we define

$$E_p(\epsilon) := \bigcup_{\|\phi\|_{\mathcal{C}^2} \leq 1} \{x_p \in \mathbb{X}_p : |\langle \Phi_p^*(\delta_{x_p}) - \Phi_p^*(\mu_p), \phi \rangle| \geq d(\Phi_p)\epsilon\},$$

where δ_{x_p} is the Dirac measure at the point x_p . Let us restate Theorem 2.2.7 in the setting of Part II.

Theorem 3.3.1. *Let $\eta_{\epsilon,p} := \epsilon \delta(\Phi_p)^{-1} d(\Phi_p) - 3R(\mathbb{X}_p, \omega_p, \mu_p)$, then*

$$\mu_p(E_p(\epsilon)) \leq \Delta(\mathbb{X}_p, \omega_p, \mu_p, \eta_{\epsilon,p}).$$

Another one is the estimate from Theorem 2.2.6.

Theorem 3.3.2. *We have*

$$|\langle d(\Phi_p)^{-1}(\Phi_p^*(\mu_p) - \Phi_p^*(\omega_p^{d_{0,p}})), \phi \rangle| \leq 2S(\mathbb{X}_p, \omega_p, \mu_p) \delta(\Phi_p) d(\Phi_p)^{-1} \|\phi\|_{\mathcal{C}^2}$$

for any (k, k) -form ϕ of class \mathcal{C}^2 on X .

From now on, we study the special case when the meromorphic transforms are induced by Kodaira maps. We already know that the Kodaira map in (3.1) is a meromorphic transform with the graph

$$\Gamma_{k,p} = \{(x, S) \in X \times \mathbb{P}H_{(2)}^0(X, L_k^p) : S(x) = 0\}.$$

Refer to Section 2.4.1.

Note that

$$\begin{aligned}\Phi_{k,p}(x) &= \{S \in \mathbb{P}H_{(2)}^0(X, L_k^p) : S(x) = 0\}, \\ \Phi_{k,p}^{-1}(S) &= \{x \in X : S(x) = 0\}.\end{aligned}$$

Let

$$\Phi_p : X \rightarrow \mathbb{X}_p$$

be the product map of $\Phi_{1,p}, \dots, \Phi_{m,p}$. We claim that Φ_p is also a meromorphic transform with the graph

$$\Gamma_p = \{(x, S_{p1}, \dots, S_{pm}) \in X \times \mathbb{X}_p : S_{p1}(x) = \dots = S_{pm}(x) = 0\}.$$

It is obvious that the projection $\Pi_1 : \Gamma_p \rightarrow X$ is surjective. The projection $\Pi_2 : \Gamma_p \rightarrow \mathbb{X}_p$ is proper, then the image $\Pi_2(\Gamma_p)$ is an analytic subvariety of \mathbb{X}_p by Remmert's proper mapping theorem [29]. Note that the zero set of every $S_{pk} \in H_{(2)}^0(X, L_k^p)$ is represented by $Z(t_k)$ for some t_k defined in Proposition 3.2.2 (ii). Then by Proposition 3.2.3 for almost every $(S_{p1}, \dots, S_{pm}) \in \mathbb{X}_p$ with respect to σ_p , the common zero set of S_{p1}, \dots, S_{pm} is of pure dimension $n - m \geq 0$. Then there exists some point $x \in X$ such that $(x, S_{p1}, \dots, S_{pm}) \in \Gamma_p$. So $\sigma_p(\Pi_2(\Gamma_p)) = 1$. Hence Π_2 is surjective. Indeed, if Π_2 is not surjective, then $\Pi_2(\Gamma_p)$ is a proper analytic subvariety of \mathbb{X}_p , Lemma 3.2.1 implies that $\sigma_p(\Pi_2(\Gamma_p)) = 0$, a contradiction. Hence Φ_p is a meromorphic transform of codimension $n - m$ with fibers for every $S_p = (S_{p1}, \dots, S_{pm}) \in \mathbb{X}_p$,

$$\Phi_p^{-1}(S_p) = \{x \in X : S_{p1}(x) = \dots = S_{pm}(x) = 0\}.$$

Considering the product map of any $\Phi_{i_1,p}, \dots, \Phi_{i_k,p}$, $1 \leq i_1 < \dots < i_k \leq m$, it follows from Proposition 3.2.3 that, the analytic subsets $(S_{p1} = 0), \dots, (S_{pm} = 0)$ are in general position for $S_p = (S_{p1}, \dots, S_{pm}) \in \mathbb{X}_p$ generic. Then by Corollary 1.2.26, the current $[S_p = 0] = [S_{p1} = 0] \wedge \dots \wedge [S_{pm} = 0]$ of bidegree (m, m) is well defined for almost all $S_p \in \mathbb{X}_p$ with respect to σ_p .

3.3.2 Intermediate degrees

It follows from Lemma 2.3.3 that $\Phi_p^*(\delta_{S_p}) = [S_p = 0]$ for a point $S_p \in \mathbb{X}_p$ generic.

Remark 3.3.3. *By the same argument, note that $\Phi_{k,p}^*(\delta_{S_{pk}}) = [S_{pk} = 0]$ for each $1 \leq k \leq m$. This yields*

$$\begin{aligned}\Phi_p^*(\delta_{S_p}) &= [S_p = 0] = [S_{p1} = 0] \wedge \dots \wedge [S_{pm} = 0] \\ &= \Phi_{1,p}^*(\delta_{S_{p1}}) \wedge \dots \wedge \Phi_{m,p}^*(\delta_{S_{pm}}).\end{aligned}$$

Recall that $\pi_{k,p} : \mathbb{X}_p \rightarrow \mathbb{P}H_{(2)}^0(X, L_k^p)$ is the natural projection. Set

$$\omega_p := c_{0,p}(\pi_{1,p}^* \omega_{FS} + \dots + \pi_{m,p}^* \omega_{FS}).$$

We always assume that $\omega_p^{d_{0,p}}$ is a probability measure on \mathbb{X}_p . Then $c_{0,p}$ satisfies the following condition,

$$(c_{0,p})^{-d_{0,p}} = \frac{d_{0,p}!}{d_{1,p}! \dots d_{m,p}!}.$$

The sequence $\{c_{0,p}\}$ has a lower bound by using Stirling's formula (cf. [18, p9] and [14, Lemma 4.3]).

Lemma 3.3.4. *There exists a positive constant c_0 such that $c_{0,p} \geq c_0$ for all $p \geq 1$.*

To simplify the notations, we write

$$\begin{aligned} d_p &= d(\Phi_p) := \int_X \Phi_p^*(\omega_p^{d_{0,p}}) \wedge \omega^{n-m}, \\ \delta_p &= \delta(\Phi_p) := \int_X \Phi_p^*(\omega_p^{d_{0,p}-1}) \wedge \omega^{n-m+1}. \end{aligned}$$

Using the classical cohomological arguments, d_p and δ_p can be calculated as follows.

Proposition 3.3.5. *In the above setting, we have*

$$\begin{aligned} d_p &= p^m \|c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)\|, \\ \delta_p &= \frac{p^{m-1}}{c_{0,p}} \sum_{k=1}^m \frac{d_{k,p}}{d_{0,p}} \left\| \bigwedge_{l=1, l \neq k}^m c_1(L_l, h_l) \right\|. \end{aligned}$$

Proof. We replace $\omega_p^{d_{0,p}}$ by a Dirac measure, since

$$\mathcal{H}^{2d_{0,p}}(\mathbb{X}_p, \mathbb{C}) \cong \mathbb{C}.$$

Choose a Dirac measure δ_S , where $S = (S_1, \dots, S_m) \in \mathbb{X}_p$, such that the analytic subsets $(S_1 = 0), \dots, (S_m = 0)$ are in general position. By Remark 3.3.3, the current

$$\Phi_p^*(\delta_S) = [S_1 = 0] \wedge \dots \wedge [S_m = 0]$$

is well defined. By the same argument in the proof of Proposition 3.2.3, we see that $[S_k = 0]$ has the same cohomology class as $pc_1(L_k, h_k)$. By proposition 3.1.1, the current

$$c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)$$

is well defined. Thus by Stokes' theorem,

$$\int_X \Phi_p^*(\delta_S) \wedge \omega^{n-m} = p^m \int_X c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m) \wedge \omega^{n-m}.$$

Hence

$$\begin{aligned} d_p &= \int_X \Phi_p^*(\omega_p^{d_{0,p}}) \wedge \omega^{n-m} = \int_X \Phi_p^*(\delta_S) \wedge \omega^{n-m} \\ &= p^m \|c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)\|. \end{aligned}$$

A direct computation gives the following equality

$$\omega_p^{d_{0,p}-1} = \sum_{k=1}^m \frac{c_p^{d_{0,p}-1} (d_{0,p}-1)!}{d_{1,p}! \dots (d_{k,p}-1)! \dots d_{m,p}!} \pi_{1,p}^* (\omega_{FS}^{d_{1,p}}) \wedge \dots \wedge \pi_{k,p}^* (\omega_{FS}^{d_{k,p}-1}) \wedge \dots \wedge \pi_{m,p}^* (\omega_{FS}^{d_{m,p}}).$$

Repeating the cohomological argument, we replace $\omega_{FS}^{d_{k,p}}$ (resp. $\omega_{FS}^{d_{1,p}-1}$) by a generic point S_k (resp. a generic complex line \mathcal{D}_k) in $\mathbb{P}H_{(2)}^0(X, L_k^p)$. By the definition of $\Phi_{k,p}^*$,

$$\Phi_{k,p}^*([\mathcal{D}_k]) = [\Pi_{1k}(\{(x, S_k) \in X \times \mathcal{D}_k, S_k(x) = 0\})].$$

Here Π_{1k} is the natural projection from $X \times \mathbb{P}H_{(2)}^0(X, L_k^p)$ to X . We show that $\Phi_{k,p}^*([\mathcal{D}_k]) = [X]$. Otherwise, if there exists a point $x_0 \in X$ such that $S_k(x_0) \neq 0$ for all sections $S_k \in \mathcal{D}_k$, then by the genericity of \mathcal{D}_k , take $S = S_{2k}(x_0)S_{1k} - S_{1k}(x_0)S_{2k}$ for some $S_{1k}, S_{2k} \in \mathcal{D}_k$. But $S(x_0) = 0$, we get a contradiction. So we have

$$\Phi_p^*([\{S_1\} \times \dots \times \mathcal{D}_k \times \dots \times \{S_m\}]) = \bigwedge_{l=1, l \neq k}^m [S_l = 0].$$

Hence

$$\|\Phi_p^*([\{S_1\} \times \dots \times \mathcal{D}_k \times \dots \times \{S_m\}])\| = p^{m-1} \left\| \bigwedge_{l=1, l \neq k}^m c_1(L_l, h_l) \right\|.$$

Now we can replace $\omega_p^{d_{0,p}-1}$ by the following current

$$T := \sum_{k=1}^m \frac{d_{k,p}}{c_p d_{0,p}} [\{S_1\} \times \dots \times \mathcal{D}_k \times \dots \times \{S_m\}].$$

So

$$\begin{aligned} \delta_p &= \int_X \Phi_p^*(\omega_p^{d_{0,p}-1}) \wedge \omega^{n-m+1} = \int_X \Phi_p^*(T) \wedge \omega^{n-m+1} \\ &= \frac{p^{m-1}}{c_p} \sum_{k=1}^m \frac{d_{k,p}}{d_{0,p}} \left\| \bigwedge_{l=1, l \neq k}^m c_1(L_l, h_l) \right\|. \end{aligned}$$

This completes the proof. \square

Remark 3.3.6. Lemma 3.3.4 implies that $\delta_p \leq Cp^{m-1}$ for some constant $C > 0$ which depends on $(L_k, h_k), 1 \leq k \leq m$.

Recall that $\gamma_{k,p} = \Phi_{k,p}^* \omega_{FS}$ is the Fubini-Study current.

Proposition 3.3.7. [14, Lemma 4.5] $\Phi_p^*(\omega_p^{d_{0,p}}) = \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p}$ for all p sufficiently large.

Proof. Denote by $\sigma_{k,p}^0$ the standard volume $\omega_{FS}^{d_{k,p}}$ on $\mathbb{P}H_{(2)}^0(X, L_k^p)$. Note that $\omega_p^{d_{0,p}}$ is a smooth form. By the definition of Φ_p^* , for a test $(n-m, n-m)$ -form ϕ on X , we have

$$\langle \Phi_p^*(\omega_p^{d_{0,p}}), \phi \rangle = \int_{\Gamma_p} \Pi_2^*(\omega_p^{d_{0,p}}) \wedge \Pi_1^*(\phi) = \int_{\mathbb{X}_p} \Pi_{2*} \Pi_1^*(\phi) \wedge \omega_p^{d_{0,p}}.$$

By the definition of direct image (Section 1.1.2), for a point $S_p \in \mathbb{X}_p$,

$$\Pi_{2*} \Pi_1^*(\phi(S_p)) = \int_{\Pi_2^{-1}(S_p)} \Pi_1^*(\phi) = \int_{(S_p=0)} \phi = \langle [S_p = 0], \phi \rangle.$$

Then

$$\langle \Phi_p^*(\omega_p^{d_{0,p}}), \phi \rangle = \int_{\mathbb{X}_p} \langle [S_p = 0], \phi \rangle \omega_p^{d_{0,p}}(S_p).$$

Then Proposition 4.2 in [10] and the proof of Theorem 1.2 in [10] imply the following

$$\begin{aligned} & \langle \Phi_p^*(\omega_p^{d_{0,p}}), \phi \rangle \\ &= \int_{\mathbb{P}H_{(2)}^0(X, L_m^p)} \cdots \int_{\mathbb{P}H_{(2)}^0(X, L_1^p)} \langle [S_{p1} = 0] \wedge \dots \wedge [S_{pm} = 0], \phi \rangle \\ & \quad d\sigma_{1,p}^0(S_{p1}) \dots d\sigma_{m,p}^0(S_{pm}) \\ &= \int_{\mathbb{P}H_{(2)}^0(X, L_m^p)} \cdots \int_{\mathbb{P}H_{(2)}^0(X, L_2^p)} \langle \gamma_{1,p} \wedge [S_{p2} = 0] \wedge \dots \wedge [S_{pm} = 0], \phi \rangle \\ & \quad d\sigma_{2,p}^0(S_{p2}) \dots d\sigma_{m,p}^0(S_{pm}) \\ &= \dots = \langle \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p}, \phi \rangle. \end{aligned}$$

Then the proof is completed. \square

3.3.3 Main result and its proof

We recall the construction of moderate measures in the settings of Theorem 0.3.1. Consider the functions $u_j : \mathbb{P}^N \rightarrow \mathbb{R}$, $1 \leq j \leq N$. Fix an exponent $0 < \rho < 1$. Let $\{u_j\}_{j=1}^N$ be a family of (c_N, ρ) -functions (cf. Part II) where $\{c_N\}_{N=1}^\infty$ is a sequence of positive numbers. Set

$$\sigma_N := \wedge_{j=1}^N (dd^c u_j + \omega_{FS}).$$

This is a probability measure on \mathbb{P}^N . Remark 2.3.7 shows that σ_N is a moderate measure for suitable c_N depending only on ρ and N (e.g. $c_N = O(1/c^N)$, where the constant $c > 1$ depends only on ρ). We reformulate the above result and some estimates from the proofs of Theorem 0.2.1 and Theorem 0.2.2.

Proposition 3.3.8. *In the above setting, there exists a constant $0 < c_N < 1$ for the measure σ_N which depends only on ρ and N such that*

$$\int_{\mathbb{P}^N} \exp(-\alpha_0 (\frac{\rho}{4})^N \phi) d\sigma_N \leq \beta_0 N \quad (3.5)$$

for all $\phi \in \mathcal{F}$, where $\alpha_0 < 1, \beta_0$ are universal positive constants. That is to say, σ_N is $(\beta_0 N, \alpha_0 (\frac{\rho}{4})^N)$ -moderate.

Proposition 3.3.9. *Under the above hypotheses, there exist universal positive constants $\beta_1, \beta_2, \beta_3$ such that $\forall t \in \mathbb{R}$,*

$$R(\mathbb{P}^N, \omega_{FS}, \sigma_N) \leq \beta_2 + \frac{1}{2} \log N,$$

$$\Delta(\mathbb{P}^N, \omega_{FS}, \sigma_N, t) \leq \beta_0 N \exp(-\alpha_0 t) + \beta_1 \left(\frac{\rho}{4}\right)^N \exp(-\alpha_0 \left(\frac{\rho}{4}\right)^N t).$$

When $t \leq \frac{1}{\alpha_0}(\log N + N \log \frac{4}{\rho})$, we have

$$\Delta(\mathbb{P}^N, \omega_{FS}, \sigma_N, t) \leq \beta_3 N \exp(-\alpha_0 t).$$

Proof. By Proposition 2.1.2,

$$R(\mathbb{P}^N, \omega_{FS}, \omega_{FS}^N) \leq \frac{1}{2}(1 + \log N).$$

We write

$$\mu_{1,N} = \omega_{FS}^N, \quad \mu_{2,N} = \bigwedge_{j=1}^N (dd^c u_j + c_N \omega_{FS} + \omega_{FS}) - \omega_{FS}^N.$$

The measure $\mu_{2,N}$ is positive since u_j is $c_N \omega_{FS}$ -p.s.h.. By Proposition 2.2.6, there exists a universal positive constant β_1 such that for all $\phi \in \mathcal{F}$,

$$\int_{\mathbb{P}^N} \exp(-\alpha_0 \left(\frac{\rho}{4}\right)^N \phi) d\mu_{2,N} \leq \beta_1 \left(\frac{\rho}{4}\right)^N. \quad (3.6)$$

By applying (3.6) and Proposition 1.2.48, we obtain

$$\begin{aligned} R(\mathbb{P}^N, \omega_{FS}, \sigma_N) &\leq \sup_{\phi \in \mathcal{F}} \left\{ - \int \phi d\mu_{1,N} - \int \phi d\mu_{2,N} \right\} \\ &\leq R(\mathbb{P}^N, \omega_{FS}, \omega_{FS}^N) + \sup_{\phi \in \mathcal{F}} \left\{ \int \exp(-\alpha_0 \left(\frac{\rho}{4}\right)^N \phi) d\mu_{2,N} / (\alpha_0 \left(\frac{\rho}{4}\right)^N) \right\} \\ &\leq \frac{1}{2}(1 + \log N) + \frac{\beta_1}{\alpha_0} \leq \beta_2 + \frac{1}{2} \log N, \end{aligned}$$

It follows from (2.12) that

$$\Delta(\mathbb{P}^N, \omega_{FS}, \sigma_N, t) \leq \beta_0 N \exp(-\alpha_0 t) + \beta_1 \left(\frac{\rho}{4}\right)^N \exp(-\alpha_0 \left(\frac{\rho}{4}\right)^N t).$$

Let

$$N \exp(-\alpha_0 t) = \left(\frac{\rho}{4}\right)^N \exp(-\alpha_0 \left(\frac{\rho}{4}\right)^N t),$$

then

$$t = \frac{\log N + N \log \frac{4}{\rho}}{\alpha_0 (1 - (\frac{\rho}{4})^N)}.$$

Hence

$$\Delta(\mathbb{P}^N, \omega_{FS}, \sigma_N, t) \leq \beta_3 N \exp(-\alpha_0 t),$$

when $t \leq \frac{1}{\alpha_0}(\log N + N \log \frac{4}{\rho})$. This completes the proof. \square

Now we study the estimates on multi-projective spaces. Let $\mathbb{P}^{\ell_1}, \dots, \mathbb{P}^{\ell_m}$ be m projective spaces. Let $\pi_k : \mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m} \rightarrow \mathbb{P}^{\ell_k}$ be the natural projection map. Let σ_k be a probability moderate measure with respect to a family of (c_{ℓ_k}, ρ) -functions $\{u_{k,j}\}_{j=1}^{\ell_k}$ on \mathbb{P}^{ℓ_k} . In the sequel of this section, c_{ℓ_k} is always chosen such that the probability measure σ_k satisfies the property of Proposition 3.3.8 (hence Proposition 3.3.9). Let $\ell = \ell_1 + \dots + \ell_m$ and

$$\omega_{MP} := c_{1m}(\pi_1^*(\omega_{FS}) + \dots + \pi_m^*(\omega_{FS})), \quad c_{1m}^{-\ell} = \frac{\ell!}{\ell_1! \dots \ell_m!}.$$

It is equivalent to that ω_{MP}^ℓ is a probability measure. Recall that the notation $r(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP})$ is defined after Proposition 1.2.34. We have the following lemma [14, Lemma 4.6].

Lemma 3.3.10. *Under the above hypotheses,*

$$r(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP}) \leq r(\ell_1, \dots, \ell_m) := \max_{1 \leq k \leq m} \frac{\ell}{\ell_k}.$$

Proof. Consider a positive closed current T of bidegree $(1, 1)$ with mass 1 on $\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}$. It has a nontrivial cohomology class. By Künneth formula,

$$\mathcal{H}^2(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \mathbb{C}) \cong \mathbb{C}^{\oplus m}.$$

Then T is in the cohomology class of $\lambda = a_1 \pi_1^*(\omega_{FS}) + \dots + a_m \pi_m^*(\omega_{FS})$, for some constants $a_k \geq 0, 1 \leq k \leq m$. Then we have

$$0 \leq \lambda \leq \left(\max_{1 \leq k \leq m} \frac{a_k}{c_{1m}} \right) \omega_{MP}$$

in the sense of currents. Since

$$\begin{aligned} 1 &= \|T\| = \int_{\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}} \lambda \wedge \omega_{MP}^{\ell-1} \\ &= c_{1m}^{\ell-1} \sum_{k=1}^m a_k \int_{\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}} \pi_k^*(\omega_{FS}) \wedge (\pi_1^*(\omega_{FS}) + \dots + \pi_m^*(\omega_{FS}))^{\ell-1} \\ &= c_{1m}^{\ell-1} \sum_{k=1}^m a_k \frac{(\ell-1)!}{\ell_1! \dots (\ell_k-1)! \dots \ell_m!} \\ &= \sum_{k=1}^m \frac{a_k \ell_k}{c_{1m} \ell}, \end{aligned}$$

Then $a_k/c_{1m} \leq \ell/\ell_k, 1 \leq k \leq m$. This implies that

$$r(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP}) \leq \max_{1 \leq k \leq m} \frac{\ell}{\ell_k}.$$

The proof is completed. \square

We first consider the case when $m = 2$. The corresponding result of estimates in a simpler case was proved in [23, Proposition A.8]. Set $\omega_{12} := \omega_{MP}$ as the the Kähler form on $\mathbb{P}^{\ell_1} \times \mathbb{P}^{\ell_2}$. Denote by σ the product of σ_1 and σ_2 . Write $r := r(\mathbb{P}^{\ell_1} \times \mathbb{P}^{\ell_2}, \omega_{12})$. Lemma 3.3.10 guarantees the existence of sufficiently large ℓ_1, ℓ_2 such that

$$\frac{r \log(\ell_1 + \ell_2)}{\min(\ell_1, \ell_2)} \ll 1.$$

Proposition 3.3.11. *In the above setting, let \mathbb{P}^{ℓ_1} (resp. \mathbb{P}^{ℓ_2}) be a projective space endowed with a probability moderate measure σ_1 (resp. σ_2) satisfying Proposition 3.3.9. Suppose that ℓ_1, ℓ_2 are chosen sufficiently large such that*

$$\begin{aligned} \frac{r \log(\ell_1 + \ell_2)}{\min(\ell_1, \ell_2)} &\ll 1, \\ \left(\frac{\rho}{4}\right)^{\min(\ell_1, \ell_2)} (\ell_1 + \ell_2) &\ll 1. \end{aligned} \tag{3.7}$$

Then there exist universal positive constants β_4, β_5 such that for $0 \leq t \leq \min(\ell_1, \ell_2)$, we have

$$\begin{aligned} \Delta(\mathbb{P}^{\ell_1} \times \mathbb{P}^{\ell_2}, \omega_{12}, \sigma, t) &\leq \beta_4 (\ell_1 + \ell_2)^{1 + \frac{\alpha_0}{2}} \exp\left(-\frac{\alpha_0}{2r}t\right), \\ R(\mathbb{P}^{\ell_1} \times \mathbb{P}^{\ell_2}, \omega_{12}, \sigma) &\leq \beta_5 r (1 + \log(\ell_1 + \ell_2)). \end{aligned} \tag{3.8}$$

Proof. To simplify the notations, let $X_1 = \mathbb{P}^{\ell_1}, X_2 = \mathbb{P}^{\ell_2}, X = X_1 \times X_2$. Denote by ω_1 (resp. ω_2) the normalized Fubini-Study form ω_{FS} in \mathbb{P}^{ℓ_1} (resp. \mathbb{P}^{ℓ_2}). Consider a function ψ on X with the conditions that $\max_X \psi = 0, dd^c \psi \geq -r\omega_{12}$. Fix a point (a, b) such that $\psi(a, b) = 0$. Let E be the set of all points with $\psi < -t$ for $t \geq 0$. We write $E = \{\psi < -t\}$. Set

$$\begin{aligned} F &:= \{x_2 \in X_2, \psi(a, x_2) < -t/2\}, \\ E_{x_2} &:= \{x_1 \in X_1, \psi(x_1, x_2) < -t\}. \end{aligned}$$

Define

$$E' := \bigcup_{x_2 \in X_2 \setminus F} (E_{x_2} \times \{x_2\}).$$

Note that $E \subset \pi_2^{-1}(F) \cup E'$.

We first estimate the measure of $\pi_2^{-1}(F)$. Let $\psi_1(x_2) := \psi(a, x_2)$, then $\max_{X_2} \psi_1 = \psi_1(b) = 0$. Define a new function $\psi_2 := \psi_1 - \int \psi_1 d\sigma_2$. Note that

$$\int \psi_2 d\sigma_2 = 0, \quad \psi_2 \geq \psi_1 \quad dd^c \psi_2 \geq -r\omega_2.$$

Let $R := \beta_2 + \frac{1}{2} \log(\ell_1 + \ell_2)$. Since $r(X_2, \omega_2) = 1$, then by Proposition 3.3.9, we have

$$-\int \psi_1 d\sigma_2 = \max_{X_2} \psi_2 \leq rR(X_2, \omega_2, \sigma_2) \leq rR.$$

Hence by hypotheses,

$$\begin{aligned}
\sigma_2(F) &\leq \sigma_2(\psi_2 \leq rR - t/2) \\
&= \sigma_2(r^{-1}\psi_2 \leq R - r^{-1}t/2) \leq \Delta(X_2, \omega_2, \sigma_2, r^{-1}t/2 - R) \\
&\leq \beta_0 \ell_2 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\
&\quad + \beta_1 \left(\frac{\rho}{4}\right)^{\ell_2} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_2} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_2} t\right).
\end{aligned} \tag{3.9}$$

When

$$\frac{t}{2r} - R \leq \frac{1}{\alpha_0} (\log \ell_2 + \ell_2 \log \frac{4}{\rho}),$$

i.e.

$$t \leq \frac{2r}{\alpha_0} \log \ell_2 + \frac{2r}{\alpha_0} \ell_2 \log \frac{4}{\rho} + 2r\beta_2 + r \log(\ell_1 + \ell_2),$$

it yields

$$\sigma_2(F) \leq \beta_3 \ell_2 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right). \tag{3.10}$$

Since $r \geq 1$ (cf. Lemma 3.3.10), $\alpha_0 < 1$, $\log \frac{4}{\rho} > 1$, inequality (3.10) holds obviously when $0 \leq t \leq \ell_2$. By Fubini theorem, we obtain

$$\begin{aligned}
\sigma(\pi_2^{-1}(F)) &\leq \beta_0 \ell_2 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\
&\quad + \beta_1 \left(\frac{\rho}{4}\right)^{\ell_2} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_2} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_2} t\right).
\end{aligned} \tag{3.11}$$

When $0 \leq t \leq \ell_2$,

$$\sigma(\pi_2^{-1}(F)) \leq \beta_3 \ell_2 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right). \tag{3.12}$$

We secondly estimate the measure of E' . For any $x_2 \in X_2 \setminus F$, let $\psi_3(x_1) := \psi(x_1, x_2)$, then $\psi_3 \leq 0$, $\max_{X_1} \psi_3 \geq \psi(a, x_2) \geq -t/2$ and $dd^c \psi_3 \geq -r\omega_1$. Define a new function $\psi_4 := \psi_3 - \int_{X_1} \psi_3 d\sigma_1$. Then

$$\begin{aligned}
-\int \psi_3 d\sigma_1 &\leq \max_{X_1} \psi_4 + t/2 \\
&\leq rR(X_1, \omega_1, \sigma_1) + t/2 \leq rR + t/2.
\end{aligned}$$

Hence by the same argument,

$$\begin{aligned}
\sigma_1(E_{x_2}) &\leq \sigma_1(\psi_4 \leq rR - t/2) \\
&\leq \beta_0 \ell_1 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\
&\quad + \beta_1 \left(\frac{\rho}{4}\right)^{\ell_1} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_1} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_1} t\right).
\end{aligned} \tag{3.13}$$

When $0 \leq t \leq \ell_1$,

$$\sigma_1(E_{x_2}) \leq \beta_3 \ell_1 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right). \tag{3.14}$$

By Fubini theorem, we obtain

$$\begin{aligned}\sigma(E') &\leq \beta_0 \ell_1 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\ &\quad + \beta_1 \left(\frac{\rho}{4}\right)^{\ell_1} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_1} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_1} t\right).\end{aligned}\tag{3.15}$$

When $0 \leq t \leq \ell_1$,

$$\sigma(E') \leq \beta_3 \ell_1 \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right).\tag{3.16}$$

So by estimates (3.11) and (3.15) for $t \geq 0$,

$$\begin{aligned}\sigma(\psi < -t) &\leq \beta_0(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\ &\quad + \sum_{j=1}^2 \beta_1 \left(\frac{\rho}{4}\right)^{\ell_j} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_j} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_j} t\right).\end{aligned}$$

When $0 \leq t \leq t_0 =: \min(\ell_1, \ell_2)$, (3.12) and (3.16) yield

$$\sigma(\psi < -t) \leq \beta_3(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right).$$

It is obvious that the above inequality is also valid for $t < 0$ since $\psi \leq 0$. By the definition of $\Delta(X, \omega_{12}, \sigma, t)$, we need to consider a function φ on X with the conditions that $dd^c\varphi \geq -r\omega$ and $\int \varphi d\sigma = 0$. Define a new function $\psi := \varphi - \max_X \varphi$. The fact that $\int \varphi d\sigma = 0$ implies that $\max_X \varphi \geq 0$. Then $\psi \leq \varphi$. Moreover, $\max_X \psi = 0$. Then

$$\sigma(\varphi < -t) \leq \sigma(\psi < -t).$$

Hence

$$\begin{aligned}\Delta(X, \omega_{12}, \sigma, t) &\leq \beta_0(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\ &\quad + \sum_{j=1}^2 \beta_1 \left(\frac{\rho}{4}\right)^{\ell_j} \exp\left(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_j} R\right) \exp\left(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_j} t\right).\end{aligned}\tag{3.17}$$

When $0 \leq t \leq t_0$,

$$\begin{aligned}\Delta(X, \omega_{12}, \sigma, t) &\leq \beta_3(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp\left(-\frac{\alpha_0}{2r}t\right) \\ &= \beta_4(\ell_1 + \ell_2)^{1+\frac{\alpha_0}{2}} \exp\left(-\frac{\alpha_0}{2r}t\right),\end{aligned}\tag{3.18}$$

where $\beta_4 =: \beta_3 \exp(\alpha_0 \beta_2)$ is a universal positive constant.

To estimate $R(X, \omega_{12}, \sigma)$, we consider a function ψ on X with the conditions that $\max_X \psi = 0$, $dd^c \psi \geq -r\omega_{12}$. For any $0 \leq \tilde{t} \leq t_0$,

$$\begin{aligned}
-\int \psi d\sigma &= \int_0^\infty \sigma(\psi \leq -t) dt \\
&= \int_0^{\tilde{t}} \sigma(\psi \leq -t) dt + \int_{\tilde{t}}^{t_0} \sigma(\psi \leq -t) dt + \int_{t_0}^\infty \sigma(\psi \leq -t) dt \\
&\leq \int_0^{\tilde{t}} dt + \int_{\tilde{t}}^\infty (\beta_0 + \beta_3)(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp(-\frac{\alpha_0}{2r} t) dt \\
&\quad + \sum_{j=1}^2 \beta_1 \int_{t_0}^\infty \left(\frac{\rho}{4}\right)^{\ell_j} \exp(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_j} R) \exp(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_j} t) dt \\
&= \tilde{t} + \frac{2r}{\alpha_0} (\beta_0 + \beta_3)(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp(-\frac{\alpha_0}{2r} \tilde{t}) \\
&\quad + \sum_{j=1}^2 \frac{2r}{\alpha_0} \beta_1 \exp(\alpha_0 \left(\frac{\rho}{4}\right)^{\ell_j} R) \exp(-\frac{\alpha_0}{2r} \left(\frac{\rho}{4}\right)^{\ell_j} t_0).
\end{aligned} \tag{3.19}$$

The above inequality follows from (3.17) and (3.18). By the hypotheses in (3.7), the last term in the last equality is less than $\frac{5r}{\alpha_0} \beta_1$ for ℓ_1, ℓ_2 sufficiently large. Hence

$$-\int \psi d\sigma \leq \tilde{t} + \frac{2r}{\alpha_0} (\beta_0 + \beta_3)(\ell_1 + \ell_2) \exp(\alpha_0 R) \exp(-\frac{\alpha_0}{2r} \tilde{t}) + \frac{5r}{\alpha_0} \beta_1.$$

Take $\tilde{t} = 2rR + \frac{2r}{\alpha_0} \log((\beta_0 + \beta_3)(\ell_1 + \ell_2))$. By the hypotheses in (3.7), $\tilde{t} \leq t_0$ for ℓ_1, ℓ_2 sufficiently large. We deduce that

$$\begin{aligned}
-\int \psi d\sigma &\leq 2rR + 2rR + \frac{2r}{\alpha_0} \log((\beta_0 + \beta_3)(\ell_1 + \ell_2)) + \frac{2r}{\alpha_0} + \frac{5r}{\alpha_0} \beta_1 \\
&\leq \beta_5 r (1 + \log(\ell_1 + \ell_2)),
\end{aligned}$$

where β_5 is a universal positive constant. This completes the proof. \square

The following proposition shows the main estimates in this section.

Proposition 3.3.12. *In the above setting, let \mathbb{P}^{ℓ_k} be a projective space endowed with a probability moderate measure σ_k satisfying Proposition 3.3.9, $\forall 1 \leq k \leq m$. Set $\sigma := \sigma_1 \times \dots \times \sigma_m$. Suppose that ℓ_1, \dots, ℓ_m are chosen sufficiently large such that*

$$\begin{aligned}
\frac{r(\ell_1, \dots, \ell_m) \log \ell}{\min(\ell_1, \dots, \ell_m)} &\ll 1, \\
\left(\frac{\rho}{4}\right)^{\min(\ell_1, \dots, \ell_m)} \ell &\ll 1.
\end{aligned} \tag{3.20}$$

Then there exist positive constants β_6, β, ξ depending only on m such that for $0 \leq t \leq \min(\ell_1, \dots, \ell_m)$, we have

$$\begin{aligned} R(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP}, \sigma) &\leq \beta_6 r(\ell_1, \dots, \ell_m)(1 + \log \ell), \\ S(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP}, \sigma) &\leq \beta_6 r(\ell_1, \dots, \ell_m)(1 + \log \ell), \\ \Delta(\mathbb{P}^{\ell_1} \times \dots \times \mathbb{P}^{\ell_m}, \omega_{MP}, \sigma, t) &\leq \beta_6 \ell^\xi \exp(-\beta t / r(\ell_1, \dots, \ell_m)). \end{aligned}$$

Proof. When $m = 2$, the estimates on R and Δ are proved in Proposition 3.3.11. When $m = 3$, following the notations in the proof of Proposition 3.3.11, we write $X_1 = \mathbb{P}^{\ell_1} \times \mathbb{P}^{\ell_2}$, $X_2 = \mathbb{P}^{\ell_3}$, $X = X_1 \times X_2$. The estimates on R and Δ for X_1 (resp. X_2) are showed in Proposition 3.3.11 and (3.17) (resp. Proposition 3.3.9). Consequently, the results of estimates on R and Δ for X are proved by using the analogous arguments in (3.9), (3.10), (3.13), (3.14) and (3.19) with the hypotheses (3.20). For the general case, the results can be deduced inductively by using the analogous arguments in the proof of Proposition 3.3.11. The estimate on S follows from Proposition 2.2.1 and [14, Lemma 4.6]. \square

3.4 Proof of main theorems

In this section we will prove the main theorems.

3.4.1 Lower bound of dimensions

First we give an estimate of the dimension $d_{k,p}$. The lower estimate is proved by construction of a new metric on the line bundle with only one singularity and application of vanishing theorem relative to multiplier ideal sheaves.

Theorem 3.4.1. *Let (X, ω) be a compact Kähler manifold of dimension n . Suppose that (L, h) is a singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \epsilon \omega$ for some positive constant ϵ . Moreover, h is continuous outside a proper analytic subset A of X . Then there exist a constant $C > 1$ and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$*

$$p^n / C \leq \dim H_{(2)}^0(X, L^p) \leq Cp^n.$$

Proof. By Proposition 1.3.24, there exist a positive constant C_0 and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$

$$\dim H_{(2)}^0(X, L^p) \leq C_0 p^n.$$

It is sufficient to show that there exist a positive constant C_1 and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$

$$\dim H_{(2)}^0(X, L^p) \geq C_1 p^n.$$

Now we fix a point $x_0 \in X \setminus A$ and $r > 0$ such that $B(x_0, 2r) \cap A = \emptyset$. Consider a smooth cut-off function $0 \leq \chi \leq 1$ such that it is equal to 1 on the closed set $\bar{B}(x_0, r)$ and supported in $B(x_0, 2r)$. Define a new function

$$\psi : X \rightarrow [-\infty, \infty), \quad \psi(x) = \eta \chi(x) \log |x - x_0|.$$

Here η is some positive constant. This function has only one singular point. Moreover, we consider a new metric on L as follows

$$h_0 = h \exp(-\psi).$$

η can be chosen sufficiently small such that on X ,

$$c_1(L, h_0) \geq \frac{\epsilon}{2}\omega.$$

Indeed it suffices to show the following inequality

$$dd^c(\chi(x) \log |x - x_0|) + \frac{\epsilon}{2\eta}\omega \geq 0.$$

Since $dd^c(\log |x - x_0|) = [x = x_0]$ is positive, we only prove the inequality in a small neighborhood of $\bar{B}(x_0, 2r) \setminus B(x_0, r)$, which is a simple conclusion by straightforward computations.

Denote by $\mathcal{I}(h^p)$ the multiplier ideal sheaf associated to the metric h^p . We have $H_{(2)}^0(X, L^p) = H^0(X, L^p \otimes \mathcal{I}(h^p))$, $\mathcal{I}(h_0^p) = \mathcal{I}(h^p) \otimes \mathcal{I}(p\psi)$. There is an identification of sheaves

$$L^p \otimes \mathcal{I}(h_0^p) \cong K_X \otimes K_X^* \otimes L^p \otimes \mathcal{I}(h_0^p).$$

Recall that the first Chern class of K_X^* is $Ric(\omega) = -\partial\bar{\partial} \log \det(\omega)$ which is smooth on X . Then

$$c_1(K_X^* \otimes L^p) = Ric(\omega) + pc_1(L, h_0) \geq \epsilon'\omega$$

for some $\epsilon' > 0$ when p is sufficiently large. Hence it follows from the Nadel vanishing theorem (cf. Theorem 1.3.27) that there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$

$$H^1(X, L^p \otimes \mathcal{I}(h_0^p)) = 0. \quad (3.21)$$

The following short exact sequence

$$0 \rightarrow L^p \otimes \mathcal{I}(h^p) \otimes \mathcal{I}(p\psi) \rightarrow L^p \otimes \mathcal{I}(h^p) \rightarrow L^p \otimes \mathcal{I}(h^p) \otimes \mathcal{O}_X/\mathcal{I}(p\psi) \rightarrow 0$$

with (3.21) implies that

$$H^0(X, L^p \otimes \mathcal{I}(h^p)) \rightarrow H^0(X, L^p \otimes \mathcal{I}(h^p) \otimes \mathcal{O}_X/\mathcal{I}(p\psi)) \rightarrow 0$$

for all $p \geq p_0$. Here \mathcal{O}_X is the sheaf of holomorphic functions on X . Note that $\mathcal{I}(p\psi)_x = \mathcal{O}_{X,x}$ for every point $x \neq x_0$. That is to say, $\mathcal{O}_{X,x}/\mathcal{I}(p\psi)_x = 0$. Since h is continuous at x_0 , then $\mathcal{I}(h^p)_{x_0} = \mathcal{O}_{X,x_0}$. So we have

$$\begin{aligned} & H^0(X, L^p \otimes \mathcal{I}(h^p) \otimes \mathcal{O}_X/\mathcal{I}(p\psi)) \\ &= L_{x_0}^p \otimes \mathcal{I}(h^p)_{x_0} \otimes \mathcal{O}_{X,x_0}/\mathcal{I}(p\psi)_{x_0} \\ &= L_{x_0}^p \otimes \mathcal{O}_{X,x_0}/\mathcal{I}(p\psi)_{x_0}. \end{aligned}$$

Hence

$$H^0(X, L^p \otimes \mathcal{I}(h^p)) \rightarrow L_{x_0}^p \otimes \mathcal{O}_{X,x_0}/\mathcal{I}(p\psi)_{x_0} \rightarrow 0 \quad (3.22)$$

for all $p \geq p_0$. Let \mathcal{M}_{X,x_0} be the maximal ideal of \mathcal{O}_{X,x_0} . Consider a germ $f \in \mathcal{I}(p\psi)_{x_0}$ in a local coordinate $z = (z_1, \dots, z_n)$ with $x_0 = 0$. Let $f = \sum a_J z^J$ be its Taylor expansion. Note that only the terms $|a_J|^2 |z|^{2J}$ contribute to the following integral

$$\int_{B(0,a)} |f|^2 |z|^{-2p\eta} dV,$$

where $B(0,a)$ is a neighborhood of the origin and dV denotes the Lebesgue measure. Hence for each multi-index J , we obtain

$$\int_{B(0,a)} |z|^{2J} |z|^{-2p\eta} dV < \infty.$$

It is equivalent to

$$\int_0^a r^{(|J| - p\eta + n - 1)} dr < \infty \iff |J| \geq [p\eta] - n + 1.$$

Then

$$\mathcal{I}(p\psi)_{x_0} \subset \mathcal{M}_{X,x_0}^{[p\eta] - n + 1}, \quad \dim \mathcal{O}_{X,x_0} / \mathcal{M}_{X,x_0}^{k+1} = \binom{n+k}{k}. \quad (3.23)$$

Then the estimate of lower bound of the dimensions follows from (3.22) and (3.23). The proof is completed. \square

3.4.2 Proof of Theorem 0.3.1

Recall that $\gamma_{k,p}$ is the Fubini-Study current defined in Section 3.1. With techniques from [10] and [27], we can show that the sequence of wedge products of these Fubini-Study currents converges weakly to the wedge product of the curvature currents of the line bundles in Theorem 0.3.1, see [14, Proposition 3.1].

Proposition 3.4.2. *In the setting of Theorem 0.3.1, we have*

$$\frac{1}{p^m} \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p} \rightarrow c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)$$

when p tends to ∞ .

Proof. we consider the p.s.h. functions $u_k, u_{k,p}$ on the contractible Stein open subset U defined in Section 3.1. Recall that $dd^c u_k = c_1(L_k, h_k)$, $dd^c u_{k,p} = \frac{1}{p} \gamma_{k,p}$ on U . We know that $\frac{1}{p} \log P_{k,p} \rightarrow 0$ in $L^1(X, \omega^n)$ by Theorem 5.1 in [10]. So we have that $u_{k,p} \rightarrow u_k$ in $L^1_{loc}(U)$, $\forall 1 \leq k \leq m$. By (3.3), the inequality $u_{k,p} \geq u_k - \frac{C}{p}$ holds on U for some constant $C > 0$ and all p sufficiently large. By Theorem 1.2.28, we have

$$dd^c u_{1,p} \wedge \dots \wedge dd^c u_{m,p} \rightarrow dd^c u_1 \wedge \dots \wedge dd^c u_m$$

in the weak sense of currents as $p \rightarrow \infty$. This completes the proof. \square

We also need the following convergence property.

Proposition 3.4.3. *In the setting of Theorem 0.3.1, there exists a positive constant C depending only on $X, (L_1, h_1), \dots, (L_m, h_m)$ such that*

$$\left| \left\langle \frac{1}{p^m} (\Phi_p^*(\sigma_p) - \Phi_p^*(\omega_p^{d_0, p})), \phi \right\rangle \right| \leq \frac{C \log p}{p} \|\phi\|_{\mathcal{E}^2}$$

for any $(n - m, n - m)$ -form of class \mathcal{E}^2 on X and p sufficiently large. In particular, $\frac{1}{p^m} (\Phi_p^*(\sigma_p) - \Phi_p^*(\omega_p^{d_0, p}))$ converges weakly to 0 as $p \rightarrow \infty$.

Proof. By Theorem 3.4.1, there exist a positive constant C_1 and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0, 1 \leq k \leq m$, we have

$$p^n / C_1 \leq d_{k, p} \leq C_1 p^n.$$

Then by Lemma 3.3.10, $r(\mathbb{X}_p, \omega_p) \leq m C_1^2$. Moreover, $d_{1, p}, \dots, d_{m, p}$ satisfy the conditions in (3.20) for p sufficiently large. Hence it follows from Proposition 3.3.12 that there exists a positive constant C_2 ,

$$S(\mathbb{X}_p, \omega_p, \sigma_p) \leq C_2 \log p.$$

Thanks to Proposition 3.3.5 and Remark 3.3.6, we can deduce that

$$\delta_p d_p^{-1} \leq C_3 \frac{1}{p}$$

for some positive constant C_3 . Note that the constants C_1, C_2, C_3 all depend only on $X, (L_1, h_1), \dots, (L_m, h_m)$. Then

$$2S(\mathbb{X}_p, \omega_p, \sigma_p) \delta_p d_p^{-1} \leq \frac{C \log p}{p}$$

for some positive constant C depending only on $X, (L_1, h_1), \dots, (L_m, h_m)$. Hence the proof is completed by applying Theorem 3.3.2. \square

The basic proof of the main theorem will end with the following theorem, which extends [18, Corollary 3.9] and [14, Theorem 4.2].

Theorem 3.4.4. *In the setting of Theorem 0.3.1, there exist a positive constant ξ which depends only on m and a positive constant C which depends only on $X, (L_1, h_1), \dots, (L_m, h_m)$ with the following property: Given any sequence of positive numbers $\{\lambda_p\}_{p=1}^\infty$ with the following conditions*

$$\begin{aligned} \liminf_{p \rightarrow \infty} \frac{\lambda_p}{\log p} &> (1 + \xi n) C, \\ \lim_{p \rightarrow \infty} \frac{\lambda_p}{p^n} &= 0, \end{aligned}$$

there exist subsets $E_p \subset \mathbb{X}_p$ such that for all p sufficiently large,

(i)

$$\sigma_p(E_p) \leq Cp^{\xi n} \exp\left(-\frac{\lambda_p}{C}\right),$$

(ii) for any point $S_p \in \mathbb{X}_p \setminus E_p$ and any $(n-m, n-m)$ -form ϕ of class \mathcal{C}^2 ,

$$\left| \frac{1}{p^m} \langle [S_p = 0] - \Phi_p^*(\sigma_p), \phi \rangle \right| \leq \frac{C\lambda_p}{p} \|\phi\|_{\mathcal{C}^2}.$$

Proof. To simplify the notations, let

$$\begin{aligned} R_p &:= R(\mathbb{X}_p, \omega_p, \sigma_p), \\ \Delta_p(t) &:= \Delta(\mathbb{X}_p, \omega_p, \sigma_p, t), \\ E_p(\epsilon) &:= \bigcup_{\|\phi\|_{\mathcal{C}^2} \leq 1} \{S_p \in \mathbb{X}_p : |\langle [S_p = 0] - \Phi_p^*(\sigma_p), \phi \rangle| \geq d_p \epsilon\}, \end{aligned}$$

where $t \geq 0, \epsilon > 0$. By Theorem 0.2.1 with its proof, there exists a constant $c > 1$ which depends only on X, L_1, \dots, L_m, ρ such that each component $\bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^* (dd^c u_j^{k,p} + \omega_{FS})$ of σ_p is a probability moderate measure satisfying Proposition 3.3.9. Theorem 3.4.1 implies that $d_{1,p}, \dots, d_{m,p}$ satisfy the conditions in (3.20) for p sufficiently large. Hence σ_p satisfy Proposition 3.3.12. Let \tilde{C} be a positive constant depending only on $X, (L_1, h_1), \dots, (L_m, h_m)$ such that for all $p \geq p_0, 1 \leq k \leq m$, we have

$$p^n / \tilde{C} \leq d_{k,p} \leq \tilde{C} p^n,$$

Here p_0 is a positive integer large enough. Then we have for $p \geq p_0$ and $0 \leq t \leq p^n / \tilde{C}$,

$$\begin{aligned} R_p &\leq m\beta_6 \tilde{C}^2 (1 + \log(m\tilde{C}p^n)) \leq C_1 \log p, \\ \Delta_p(t) &\leq \beta_6 (m\tilde{C}p^n)^\xi \exp\left(\frac{-\beta t}{m\tilde{C}^2}\right) \leq C_1 p^{\xi n} \exp\left(-\frac{t}{C_1}\right). \end{aligned}$$

Here we can see that C_1 is some constant depending only on $X, (L_1, h_1), \dots, (L_m, h_m)$.

Let

$$\epsilon_p := \frac{\lambda_p}{p}, \quad \eta_p := \epsilon_p d_p \delta_p^{-1} - 3R_p.$$

It follows from Proposition 3.3.5 and Remark 3.3.6 that for $p \geq p_0$,

$$\eta_p \geq C_2 \lambda_p - 3C_1 \log p.$$

Here C_2 is some constant depending only on $X, (L_1, h_1), \dots, (L_m, h_m)$. If there is a condition that

$$\liminf_{p \rightarrow \infty} \frac{\lambda_p}{\log p} > \frac{6C_1}{C_2},$$

then for all p sufficiently large, $\eta_p > \frac{C_2}{2} \lambda_p$. Since

$$\lim_{p \rightarrow \infty} \frac{\lambda_p}{p^n} = 0,$$

η_p can be always chosen such that $\frac{C_2}{2}\lambda_p < \eta_p < p^n/\tilde{C}$ for p sufficiently large. By applying Theorem 3.3.1 to the subset $E_p \subset \mathbb{X}_p$, we obtain

$$\sigma_p(E_p) \leq \Delta_p(\eta_p) \leq C_1 p^{\xi n} \exp\left(\frac{-C_2}{2C_1}\lambda_p\right),$$

where $E_p = E_p(\epsilon_p)$. Now we set

$$C = \max\left(\frac{6C_1}{C_2(1+\xi n)}, \frac{2C_1}{C_2}, C_1, \|c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)\|\right).$$

Then for all p sufficiently large,

$$\sigma_p(E_p) \leq C p^{\xi n} \exp\left(\frac{-\lambda_p}{C}\right)$$

under the conditions that

$$\begin{aligned} \liminf_{p \rightarrow \infty} \frac{\lambda_p}{\log p} &> (1 + \xi n)C, \\ \lim_{p \rightarrow \infty} \frac{\lambda_p}{p^n} &= 0. \end{aligned}$$

By the definition of E_p , it is obvious that for any $S_p \in \mathbb{X}_p \setminus E_p$ and any $(n-m, n-m)$ -form ϕ of class \mathcal{C}^2 ,

$$\left| \frac{1}{p^m} \langle [S_p = 0] - \Phi_p^*(\sigma_p), \phi \rangle \right| \leq \frac{d_p}{p^m} \frac{\lambda_p}{p} \|\phi\|_{\mathcal{C}^2} \leq \frac{C\lambda_p}{p} \|\phi\|_{\mathcal{C}^2}.$$

This completes the proof. \square

Proposition 3.4.5. *In the setting of Theorem 0.3.1, for almost all $S = \{S_p\}_{p=1}^\infty \in \mathbb{P}^X$ with respect to σ , we have*

$$\frac{1}{p^m} [S_p = 0] - \frac{1}{p^m} \Phi_p^*(\sigma_p) \rightarrow 0$$

in the weak sense of currents as $p \rightarrow \infty$ on X .

Proof. Note that

$$\sum_{p=1}^\infty \sigma_p(E_p) \leq C_3 \sum_{p=1}^\infty \frac{1}{p^\eta} < \infty$$

for some positive constant C_3 and $\eta > 1$. The proof follows from the same argument in Remark 2.3.8. \square

End of the proof of Theorem 0.3.1. By Theorem 0.2.1 with its proof, there exists a constant $c > 1$ which depends only on X, L_1, \dots, L_m, ρ such that each component $\bigwedge_{j=1}^{d_{k,p}} \pi_{k,p}^*(dd^c u_j^{k,p} + \omega_{FS})$ of σ_p is a probability moderate measure satisfying Proposition 3.3.9. Hence σ_p satisfy Proposition 3.3.12.

Note that c is independent of the choices of the metrics h_1, \dots, h_m . It follows from Proposition 3.3.7 and Proposition 3.4.3 that

$$\frac{1}{p^m}(\Phi_p^*(\sigma_p) - \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p}) \rightarrow 0$$

in the weak sense of currents as $p \rightarrow \infty$. Then Proposition 3.4.5 implies that for almost all $S = \{S_p\}_{p=1}^\infty \in \mathbb{P}^X$ with respect to σ

$$\frac{1}{p^m}([S_p = 0] - \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p}) \rightarrow 0$$

in the weak sense of currents as $p \rightarrow \infty$. The proof is finally completed by application of Proposition 3.4.2. \square

Theorem 0.3.3 follows from Proposition 3.4.3 and Theorem 3.4.4 with the following theorem [14, Proposition 5.1].

Theorem 3.4.6. *In the setting of Theorem 0.3.3, there exists a positive constant C which depends only on $X, (L_1, h_1), \dots, (L_m, h_m)$ such that for all p sufficiently large and any $(n - m, n - m)$ -form ϕ of class \mathcal{C}^2 , we have*

$$\left| \left\langle \frac{1}{p^m} \gamma_{1,p} \wedge \dots \wedge \gamma_{m,p} - c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m), \phi \right\rangle \right| \leq \frac{C \log p}{p} \|\phi\|_{\mathcal{C}^2}.$$

3.4.3 Proof of Theorem 0.3.5

To prove Theorem 0.3.5, we need the following result [38, Theorem 15.1.6].

Theorem 3.4.7. *Let P_A be the set of all functions of the form $p^{-1} \log |f(z)|$ where p is a positive integer and f an entire function $\not\equiv 0$ in \mathbb{C}^n . Then the closure of P_A in $L_{loc}^1(\mathbb{C}^n)$ consists of all plurisubharmonic functions.*

Let ω_0 be the Fubini-Study form with mass 1 in \mathbb{P}^n . Given any positive closed current T of bidegree $(1, 1)$ with mass 1 in \mathbb{P}^n . Then by Proposition 1.2.19, there exists a q.p.s.h. function φ such that

$$T - \omega_0 = dd^c \varphi.$$

Therefore T corresponds to an entire plurisubharmonic function ψ in \mathbb{C}^n which belongs to the Lelong class (cf. [32, Example 2.2]). By Theorem 3.4.7, there exists a sequence $\{p^{-1} \log |f_p|\}$ which converges to ψ . Since holomorphic functions in \mathbb{C}^n can be approximated by polynomials, by using diagonal argument, we can choose a sequence of polynomials g_p of degree $\leq p$ such that $\{p^{-1} \log |g_p|\}$ converges to ψ . It is possible since all such $p^{-1} \log |g_p|$ and ψ belong to the Lelong class. Since g_p can be regarded as a homogeneous polynomial of degree p in \mathbb{C}^{n+1} , it induces a global section $S_p \in H^0(\mathbb{P}^n, \mathcal{O}(p))$. Hence by Lelong-Poincaré formula, $\frac{1}{p}[S_p = 0]$ converges weakly to T . We obtain the following result due to Oka.

Proposition 3.4.8. *Given any positive closed current T of bidegree $(1, 1)$ with mass 1 in \mathbb{P}^n , there exists a sequence of $\{S_p\}_{p \geq 1}$, $S_p \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$, such that*

$$\frac{1}{p}[S_p = 0] \rightarrow T$$

in the weak sense of currents.

This section concludes with the proof of Theorem 0.3.5.

Proof. By Proposition 3.4.8, there exists a sequence of $\{S_p\}_{p \geq 1}$, $S_p \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$, such that

$$\frac{1}{p}[S_p = 0] \rightarrow T.$$

Denote by δ_{S_p} the Dirac measure at the point $S_p \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$. Choose a sequence of smooth probability measures $\{\mu_{pj}\}$ which is an approximation of δ_{S_p} . Note that the map

$$\begin{aligned} \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p)) &\rightarrow \mathbb{C} \\ V_p &\mapsto \langle [V_p = 0], \phi \rangle \end{aligned}$$

is continuous. Then there exists a neighborhood $E_p \subset \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(p))$ of S_p such that

$$|\langle [V_p = 0], \phi \rangle - \langle [S_p = 0], \phi \rangle| \leq 1,$$

$\forall V_p \in E_p$ and $(n-1, n-1)$ -form ϕ of class \mathcal{C}^2 with $\|\phi\|_{\mathcal{C}^2} \leq 1$. Hence

$$\begin{aligned} &|\langle \frac{1}{p}[V_p = 0] - T, \phi \rangle| \\ &\leq |\langle \frac{1}{p}[V_p = 0] - \frac{1}{p}[S_p = 0], \phi \rangle| + |\langle \frac{1}{p}[S_p = 0] - T, \phi \rangle| \\ &\leq \frac{1}{p}\|\phi\|_{\mathcal{C}^2} + |\langle \frac{1}{p}[S_p = 0] - T, \phi \rangle| \rightarrow 0, \end{aligned}$$

$\forall V_p \in E_p$ and $(n-1, n-1)$ -form ϕ of class \mathcal{C}^2 . Since $\mu_{pj} \rightarrow \delta_{S_p}$ as measures when $j \rightarrow \infty$, there exists an index j_p satisfying

$$|\mu_{pj_p}(E_p) - \delta_{S_p}(E_p)| = |\mu_{pj_p}(E_p) - 1| \leq \frac{1}{p^2}.$$

Denote by E_p^c the complement of E_p . Set $\sigma_p := \mu_{pj_p}$. So $\sigma_p(E_p^c) \leq \frac{1}{p^2}$. Yau's theorem [59] implies that there exists a smooth real function u_p with $(\omega_{FS} + dd^c u_p)^{d_p} = \sigma_p$. Then the theorem follows from the same argument in the proof of Remark 2.3.8. This completes the proof. \square

3.5 Further problems

We continue studying the equidistribution property and some approximations of currents by intersection of divisors. Several questions are posed as follows.

1) We will investigate the equidistribution of zeros of random holomorphic sections over more general spaces, such as non-compact manifolds [19] and normal Kähler complex spaces [13]. With emphasis on certain conditions of curvature forms of line bundles, I believe the equidistribution property would still hold, even in the context of moderate measures. We will also study some equidistribution properties with a limit of currents related to weighted global extremal functions (cf. [8]) and convergence results for Bergman kernels with equilibrium measures (cf. [5] and therein).

2) Approximation of currents by divisors is an interesting subject which was explored by many mathematicians, for example, Demailly [15], Duval-Sibony [24], Guedj [30] and Coman-Marinescu [11] etc. We would like to show that some positive closed current of bidegree (m, m) could be approximated by a sequence of intersections of divisors defined by m line bundles. It could be regarded as a generalization of results obtained in [11].

3) Following Theorem 0.3.5, we consider the following question. In the setting of Theorem 0.3.3, given any positive closed current T of bidegree (m, m) , could we construct a family of diffuse measures σ_p such that for σ -a.e. $\{S_p\}_{p \geq 1} \in \mathbb{P}^X$, $\frac{1}{p^m}[S_p = 0] \rightarrow T$? The key problem is the existence of approximation of any current by a sequence of currents (e.g. those currents defined by integration on varieties of codimension m).

4) We will study the equidistribution property of random sections with real coefficients, which is linked to the recent work by Gayet-Welschinger [31].

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