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# Two examples of reaction-diffusion front propagation in heterogeneous media

Antoine Pauthier

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# THÈSE

En vue de l'obtention du

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

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Présentée et soutenue le *20/06/2016* par :

**ANTOINE PAUTHIER**

**Two examples of reaction-diffusion front propagation in  
heterogeneous media**

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*À mes premiers professeurs,  
Suzanne, Andrée, Marcel, Michel*



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# Introduction

## 1 Thesis subject

The main purpose of this thesis is to study the asymptotic behaviour of some reaction-diffusion equations in heterogeneous media. It contains two parts that deal with two different models.

### The influence of a line of fast diffusion and nonlocal exchanges

**Starting model** The purpose of the first part is to understand the effects of nonlocal interactions in a recent model of coupled parabolic equations in different dimensions. The initial model (1) introduced by Berestycki, Roquejoffre and Rossi in 2013 is the following.

$$\begin{cases} \partial_t u - D\partial_{xx}u = \bar{\nu}v(x, 0, t) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = v(1 - v) & (x, y) \in \mathbb{R} \times \mathbb{R}_+^*, t > 0 \\ -d\partial_y v(x, 0, t) = \bar{\mu}u(x, t) - \bar{\nu}v(x, 0, t) & x \in \mathbb{R}, t > 0. \end{cases} \quad (1)$$

A two-dimensional environment, the half-plane  $\{y > 0\}$ , is lined with a fast diffusion environment. For the sake of simplicity, as a reference to the biological situations, we will refer to the plane as "the field" and the line as "the road." The density of population is designated by  $v$  in the field,  $u$  on the road. Reaction and usual diffusion only occur on the field, outside the road. The reaction term is of logistic type  $f(v) = v - v^2$ , we denote  $d$  the diffusivity in the field and  $D$  on the road. Exchanges of population take place between the field and the road; a fraction  $\nu$  of individuals from the field at the road (i.e.  $v(t, x, 0)$ ) joins the road while a fraction  $\mu$  of the population on the road joins the field. It yields a Robin boundary condition for  $v$  in (1). This boundary condition is natural in the sense that it is mass conservative without reaction. Model (1) will sometimes be called "road-dield model."

**Biological motivations** This system was proposed in order to model and study the influence of a transportation network on biological invasions. It is well known by the historians that the "Black Death" plague in the middle of the 14<sup>th</sup> century in Europe spread along commercial roads connecting the cities that had trade fairs. An account of the spread of diseases and epidemics is provided by [93].

More recent studies highlighted the propagation of parasites along rivers (see [66] for instance). In France, the invasion speed of the pine processionary suggests an influence of the road network (see [3] and the ANR project URTICLIM). The Canadian forest provides an other example of heterogeneous biological environment. Oil companies have constructed



(a) Seismic lines in Alberta forest. Copyright (c) Province of British Columbia. All rights reserved. Reproduced with permission of the Province of British Columbia.



(b) Wolf. Original picture by Santiago Atienza, licence Cc-by-2.0

Figure 1: Example of heterogeneous environment.

a network of seismic lines, used to set off explosive charges to locate oil deposits (see figure 1a). It is assumed that the wolves' behaviour is modified, leading to their higher expansion (see [65]). Recent works of McKenzie et al. [77] based on GPS observations suggest that wolves move and concentrate along these lines.

**Mathematical issue** These examples rise the mathematical question of the influence of lines with fast diffusion on the spreading speed in reaction-diffusion models. The system (1) is a first step in this analysis. We are interested in the case  $D > d$ , in which the species go faster on the road.

Without the fast diffusion line it is well known, we will come back to this later, that a species whose dynamics is given by a reaction-diffusion equation such as (1) invades the environment at speed  $2\sqrt{d}$ . In [22], the authors show that a necessary and sufficient condition for the line to enhance the spreading in the direction of the road is  $D > 2d$ . This threshold was obtained with algebraic computations that allow to provide travelling supersolutions. The main issue in this thesis is to extend these results to nonlocal interactions between the road and the field, and to investigate qualitative properties of the spreading speed.

**Model under study** The model studied in this thesis is based on the system (1) with interactions that are no longer localised at the road. We look at the following system:

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0. \end{cases} \quad (2)$$

The density  $v$  is defined on the whole plane  $\mathbb{R}^2$ . Interactions between the road and the field are defined by two nonnegative exchange functions  $\mu$  and  $\nu$ , with the notation  $\bar{\mu} = \int_{\mathbb{R}} \mu$ ,  $\bar{\nu} = \int_{\mathbb{R}} \nu$ . This provides nonlocal interactions between these equations. Once again, interactions are chosen so that, in the absence of reaction, the system preserves the total

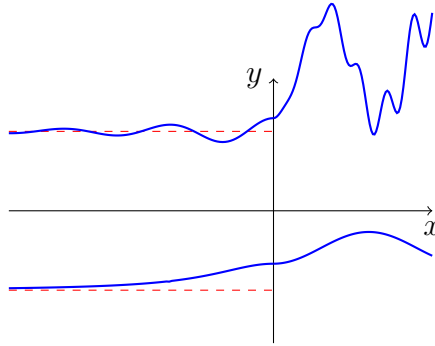


Figure 2: Example of asymptotically cylindrical domain

mass  $\|u(t)\|_{L^1(\mathbb{R})} + \|v(t)\|_{L^1(\mathbb{R}^2)}$ . The reaction term  $f$  is of KPP-type (i.e.  $f(s) \leq f'(0)s$ ) as in the initial model (1).

The initial model (1) can then be considered as a singular limit of the integral model (2). At least formally, one can retrieve the system (1) from (2) when considering Dirac masses as exchange functions,  $\mu = \bar{\mu}\delta_{y=0}$ ,  $\nu = \bar{\nu}\delta_{y=0}$ . This singular limit is studied in the second chapter of this thesis. We will also investigate two intermediate models, with one nonlocal interaction, the other one being localised.

In the first chapter, we show the robustness of the results and methods given in the initial paper [22]. We then study the two intermediate models and some specific properties of the model (2). The second chapter is concerned with the singular limit of exchange functions that tend to Dirac masses. Then, in a short third chapter, we study the converse problem, i.e. long range exchanges with support that goes to infinity.

## Entire solution in domains that tend to a cylinder

In the second part of this thesis we investigate the existence of a nontrivial entire solution of the following problem

$$\begin{cases} \partial_t u(t, x, y) - \Delta u(t, x, y) = f(u), & t \in \mathbb{R}, (x, y) \in \Omega, \\ \partial_\nu u(t, x, y) = 0, & t \in \mathbb{R}, (x, y) \in \partial\Omega. \end{cases} \quad (3)$$

We consider a domain  $\Omega$  of the form:

$$\Omega = \{(x, y), x \in \mathbb{R}, y \in ]Y^-(x), Y^+(x)[\}, \quad (4)$$

where  $Y^-$  and  $Y^+$  are two ordered functions defined on  $\mathbb{R}$  such that

$$Y^\pm(x) \xrightarrow{x \rightarrow -\infty} \pm Y^\infty. \quad (5)$$

The domain  $\Omega$  is asymptotically cylindrical, in the sense that as  $x$  goes to  $-\infty$ , the domain  $\Omega$  tends to the cylinder  $\mathbb{R} \times ]-Y^\infty, Y^\infty[$ . We give an example of such a domain in fig. 2. The reaction term  $f$  is a bistable nonlinearity, which means that both 0 and 1 are stable states of the equation  $\frac{du}{dt} = f(u(t))$ . We will come back to the specificity of such a nonlinearity later.

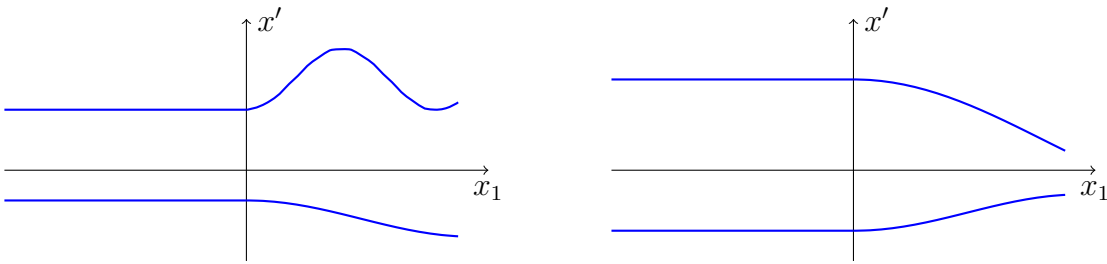


Figure 3: Two example of domains considered in [5]

This study is a sequel of previous works of Berestycki, Bouhours, and Chapuisat [5]. In their paper they study the problem (3) with a domain which is a straight cylinder in a half-space, that is

$$\Omega \cap \{(x, y) \in \mathbb{R}^2, x < 0\} = \mathbb{R}^- \times \omega, \quad \omega \subsetneq \mathbb{R}. \quad (6)$$

We give two examples of such a domain in fig. 3.

They prove existence and uniqueness of a natural entire solution of (3), i.e. a solution defined for  $t \in \mathbb{R}$  which is the bistable wave at  $t = -\infty$ . This solution characterises the propagation in such a domain. It is then possible, by studying it, to exhibit blocking phenomena, or partial or total invasion, depending on the geometry of the domain. However, the assumption (6) was essential for the theoretical study and seems quite restrictive. The natural question is then to wonder how to extend these results given in [5] to domains that tend to cylinders. This is the main purpose of the second part of this thesis.

**Biological motivations** Many biological situations may fit with this kind of model, as soon as a bistable reaction is relevant. One can think of marine species through straits, or spreads through an isthmus. However, the main motivation comes from medical modelling, and more specifically the study of Cortical Spreading Depression (CSD). In order to be able to receive and deliver nerve impulses, neurons have to be polarised. CSD are large depolarisation phenomena in the brain that look like travelling waves which propagate slowly in the grey matter. They are easily observable in rodent but their existence in human is still a matter of debate, even if a medical consensus has gained ground ([72]). Therapies aiming at blocking them are inefficient in humans.

A simplified model for CSD could be provided by a reaction-diffusion equation like (3) [95, 31]. The polarised state is represented by the state 0, whereas the depolarised state is the state 1. The wave propagates into the grey matter of the brain and is blocked by the white matter.

Differences between the human and rodent brain (see fig. 4) geometries could explain the blocking of the wave as well as therapies. It has been numerically observed, see [40, 51], and mathematically studied in some specific cases [32, 6]. It is worth mentioning that the hypothesis of the blocking of CSD by the geometry appears for the first time, to our knowledge, in the work of Chapuisat and Grenier [32].

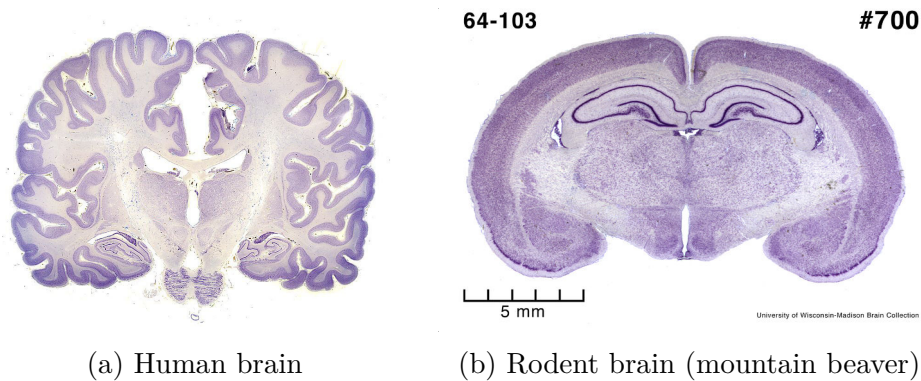


Figure 4: These images are from the University of Wisconsin and Michigan State Comparative Mammalian Brain Collections, and from the National Museum of Health and Medicine, available at the following website <http://www.brainmuseum.org/sections/index.html>. Preparation of all these images and specimens have been funded by the National Science Foundation, as well as by the National Institutes of Health.

## 2 An overview of propagation phenomena and transition fronts in reaction-diffusion equations

This section introduces a brief overview of the mathematical background of this thesis. As a vast literature is available on this subject, the only aspects and references especially relevant to this thesis are studied. We focus on two aspects: spreading speed for the reaction term considered in the first part of this thesis, and transition fronts for scalar reaction-diffusion equations on the other hand. We first give a short introduction on reaction-diffusion equations.

### Reaction-diffusion equations

In population dynamics, the evolution of the density of a species can be modelled by the ordinary differential equation

$$\frac{du}{dt} = f(u), \quad t \geq 0 \quad (7)$$

where  $f$  accounts for birth and death of the individuals. Such a model by logistic type functions dates back the mid-19<sup>th</sup> century and the works of Pierre Franois Verhulst [97]. Application examples are given in the book of Murray [81]. A simple way to model the movement is to suppose that individuals move locally and at random. Up to a rescaling it yields a classical diffusion equation

$$\frac{\partial u}{\partial t} = d\Delta u, \quad t > 0, x \in \mathbb{R}^N$$

where  $d$  is the diffusivity of the domain and  $\Delta u = \sum_{i=0}^N \frac{\partial^2 u}{\partial x_i^2}$  is the usual spatial Laplace operator. Combining these two terms, we get the homogeneous reaction-diffusion equation

$$\frac{\partial u}{\partial t}(t, x) - d\Delta u(t, x) = f(u(t, x)), \quad t > 0, x \in \mathbb{R}^N. \quad (8)$$

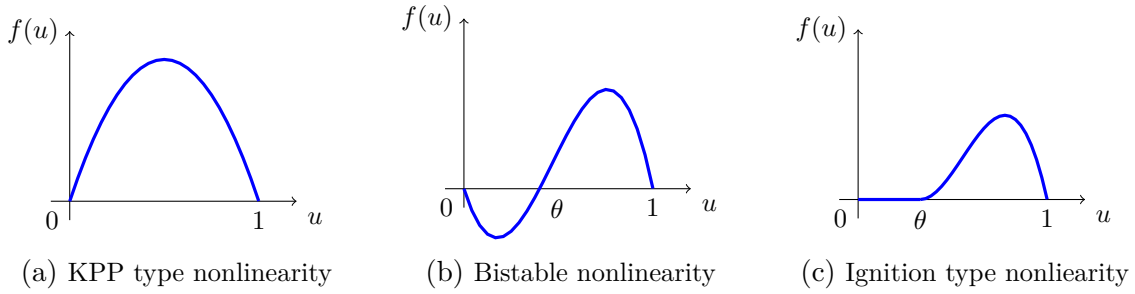


Figure 5: Reaction terms.

This equation was introduced and studied in 1937 by Fisher [45] and Kolmogorov, Petrovsky and Piskunov [70]. The reaction term  $f$  can take various forms. In this thesis, we focus on two specific cases.

- In the first part, the reaction term  $f$  is of KPP-type, as a reference to the pioneering works of Kolmogorov, Petrovsky and Piskunov [70]. The function  $f$  satisfies the following assumptions:

$$f(0) = f(1) = 0, \text{ et } 0 < f'(0)s \leq f(s) \text{ for } s \in (0, 1).$$

The archetype of such a reaction term is given by the logistic law  $f(s) = \lambda s(1 - s)$ . Equation (7) has two steady states, the unstable state 0 and the stable state 1. See fig.5a for a graph.

- In the second part, the reaction  $f$  is of bistable kind ; there exists  $\theta \in (0, 1)$  such that

$$f(0) = f(\theta) = f(1) = 0, \quad f < 0 \text{ on } (0, \theta), \quad f > 0 \text{ on } (\theta, 1), \quad f'(0), f'(1) < 0.$$

The usual graphical representation is given in fig. 5b. Equation (7) has three steady states, the unstable state  $\theta$  and the two stable states 0 and 1. We will assume  $\int_0^1 f(s)ds > 0$ . The state 1 is then in some sense the "more stable."

There are many other types of reaction that have been studied and are relevant in various contexts. In this thesis we will sometimes refer to the ignition type reaction, which is equal to 0 on  $(0, \theta)$ , and is positive on  $(\theta, 1)$ . The critical value  $\theta$  can be seen as the ignition temperature of a reagent(see fig. 5c).

In their seminal 1937 article Kolmogorov, Petrovsky et Piskunov introduce equation (8) in the general case and study it in one space dimension with a reaction  $f$  of KPP type. They prove the existence of a one-parameter family of propagation fronts (or travelling waves) of the form  $u(t, x) = U(x - ct)$ . More precisely, the equation

$$\begin{cases} U'' + cU' + f(U) = 0, \\ U(-\infty) = 1, \quad U(+\infty) = 0 \end{cases} \quad (9)$$

admits a unique solution  $U_c$  up to translation for all  $c \geq 2\sqrt{df'(0)} := c_{KPP}$ . They also demonstrate that, starting from  $u_0 = \mathbf{1}_{\mathbb{R}^-}$ , the solution  $u(t, x)$  of (8) satisfies

$$|u(t, x) - U_{c_{KPP}}(x - c_{KPP}t - s(t))| \xrightarrow[t \rightarrow +\infty]{} 0 \quad (10)$$

Where  $s(t) = \underset{t \rightarrow \infty}{o}(t)$ . The state 1 invades the state 0 at speed  $c^* = c_{KPP}$ , the minimal speed of a travelling wave of type (9). These works are the milestone of the two aspects that we develop.

## 2.1 Spreading phenomena for reaction-diffusion equations of KPP-type

**Spreading in the homogeneous framework** The relation (10) asserts that, for a KPP reaction term and starting from a step function, the state 1 invades the state 0 with an asymptotic speed of spreading  $c_{KPP}$  given by the linearised operator. In 1978 Aronson and Weinberger proved first in a one-dimension setting [1] then in every space dimension [2] that propagation occurs for a wide class of initial data and spread at the minimal speed of the travelling waves. In the KPP case, for any non-negative compactly supported initial datum we have

$$\begin{cases} \forall c < c_{KPP}, \lim_{t \rightarrow +\infty} \inf_{|x| \leq ct} u(t, x) = 1 \\ \forall c > c_{KPP}, \lim_{t \rightarrow +\infty} \sup_{|x| \geq ct} u(t, x) = 0. \end{cases} \quad (11)$$

Propagation spread at speed  $c_{KPP}$  in every directions. Such a propagation result remains available in the bistable case (where the speed depends on the whole nonlinearity  $f$ ), provided that the initial datum is sufficiently large on a certain ball. On the contrary, if  $u_0$  is too small extinction occurs [2].

Property (11) gives a definition of the spreading speed in every direction. It asserts that the level sets of the solution  $u$  of (8) behave like  $c_{KPP}t + o(t)$  as  $t$  goes to infinity. In a one-dimensional setting many works have improved this result since then. Uchiyama [96] is the first to show that the shift given in (10) satisfies  $s(t) \underset{t \rightarrow \infty}{\sim} -\frac{3}{2} \log t$  with  $d = f'(0) = 1$ . The sharpest estimates are given by Bramson [25, 26] with a probabilist approach. A recent proof using analytical tools is given in [60]. Let us also mention the formal approach through pulled fronts by Ebert and Van Saarloos [42]. Thus, in the KPP case, even if the solutions converges to an outline of the travelling wave, there is an increasing in time delay between the solution and the wave.

**Spreading in the heterogeneous framework** All the results given above are available in a homogeneous framework: the underlying domain is the whole space  $\mathbb{R}^N$ , neither the diffusivity nor the reaction depend on the space variable. In a heterogeneous framework they could be sensibly more difficult to obtain. The pioneering works on the propagation for heterogeneous reaction-diffusion equations are those of Freidlin and Gärtner [47] in 1979. They study the equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (A(x)\nabla u) = f(x, u), \quad t > 0, \quad x \in \mathbb{R}^N$$

where both the diffusivity matrix  $A$  and the reaction  $f(., s)$  are 1-periodic in space, and  $f(x, .)$  is of KPP-type for all  $x$ . They prove with probabilist arguments that the spreading speed in the sense of (11)  $w^*(e)$  in some direction  $e \in \mathbb{S}^{N-1}$  is given by

$$w^*(e) = \min_{e' \in \mathbb{S}^{N-1}, e \cdot e' > 0} \frac{c^*(e')}{e \cdot e'},$$



where  $c^*(e')$  is the minimal speed of exponential waves in the direction  $e'$  for the linearised equation, i.e. the smallest  $c$  such that a solution of the form  $(t, x) \mapsto \psi(x)e^{-\lambda(x \cdot e' - ct)}$  with  $\lambda > 0$  exists. Thus the propagation is no longer isotropic and depends on the linear waves in every directions. The Bramson result for a one-dimensional periodic equation has also been investigated in [61].

These results in a periodic framework are generalised by Weinberger in [98] using a discrete formalism. Propagation for a fully periodic heterogeneous equation is announced in [10] and analysed by Berestycki, Hamel and Nadirashvili in [15] ; the domain is also periodic in some directions. In a second paper [16], the same authors study the speed of propagation for homogeneous KPP equation but in general domains, without periodic assumption. They prove that the dependence of the spreading speed in a given direction on the geometry of the underlying domain is rather intricate; it can take values between 0 and  $+\infty$ .

On the other hand, Berestycki, Hamel and Nadin study propagation properties for equations of the type:

$$\frac{\partial u}{\partial t} - \nabla \cdot (A(t, x)\nabla u) + q(t, x) \cdot \nabla u = f(t, x, u), \quad t > 0, \quad x \in \mathbb{R}^N$$

where  $A, q$  and  $f$  are space and time heterogeneous and the hypotheses on  $f$  cover a large class of reaction, including the KPP case. They establish spreading properties, and give an analytical proof of the results of Freidlin and Gärtner.

In the context of reaction-diffusion equations in a cylinder with an advection field, Constantin, Kiselev, Oberman and Ryzhik prove spreading properties in [33] with the more general notion of bulk burning rate, that includes the notion of spreading speed. Finally, let us mention the recent results of Liang, Lin, and Matano [74] who study the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = b(x)u(1 - u) \tag{12}$$

under the constraints that  $b$  is 1-periodic, with fixed mass. They prove that the spreading speed, given by the minimal speed for a travelling pulse, is maximised with a Dirac comb for  $b$ .

## 2.2 Travelling waves and transition fronts for scalar reaction-diffusion equations

This subsection is devoted to the general notions of fronts and entire solutions for reaction-diffusions of type (8). Once again, we do not intend to be exhaustive and we focus on some aspects that we think are relevant for this thesis.

### Travelling waves and pulsating fronts

**One-dimensional waves** The works of Kolmogorov, Petrovski and Piskunov [70] provide for the one-dimensional KPP equation a two-parameters family of travelling waves given by (9). A wave is thus defined by its speed  $c \geq c_{KPP}$  and a shift. In the bistable case (as well as in the ignition case) there is uniqueness of the two-uple  $(c, \varphi)$  such that

$$\begin{cases} \varphi'' + c\varphi' + f(\varphi) = 0, \\ \varphi(-\infty) = 1, \quad \varphi(+\infty) = 0, \quad \varphi(0) = \theta. \end{cases} \tag{13}$$

We denote  $c^*$  the corresponding speed, or simply  $c$  when there is no possible confusion. We call this function or the two-uple the bistable wave. Contrary to the KPP case this travelling wave leads the dynamics. In the bistable case, Fife and McLeod prove [44] in one dimension that if the initial condition is front-like, that is  $\limsup_{x \rightarrow -\infty} u_0(x) > \theta$  and  $\liminf_{x \rightarrow +\infty} u_0(x) < \theta$  then the solution  $u(t, x)$  converges exponentially fast to a translated bistable wave: there exist  $\gamma, C > 0, x_0 \in \mathbb{R}$  such that

$$|u(t, x) - \varphi(x - ct + x_0)| \leq Ce^{-\gamma t}. \quad (14)$$

This result could be generalised to a wide class of initial condition. If  $u_0$  is compactly supported and large enough, the solution will converge exponentially fast to a couple of waves, one going to  $-\infty$ , the other one going to  $+\infty$ . Let us also mention the pioneering works of Kanel' in this context [67, 68] where the author studies (8) in one dimension with initial condition of the type  $\mathbb{1}_{[-L, L]}$  in the ignition case. He proves that if  $L < L_0$  extinction occurs, and if  $L > L_1$  the state 1 invades the whole line and converges to a couple of waves. This kind of dichotomy result extinction/invasion was considerably refined afterwards in both ignition and bistable cases [101, 41].

The above convergence result (14) ensures the stability of the bistable wave. For KPP-type waves, stability results are also available in some weighted space functions. The works of Sattinger [92] give a stability result for the waves in a more general framework, depending on the linearised equation, that includes non-critical KPP waves. This stability can also be obtained in the critical KPP case, see the works of Gallay [46].

**Travelling waves in higher dimensions, non-planar waves** With the travelling waves (9) and (13) one can define planar waves in any space dimension when considering solutions of the form  $u(t, x) = \varphi(x \cdot e - ct)$  for any  $e \in \mathbb{S}^{N-1}$ . The stability of such bistable planar waves have been investigated by Xin [100], Levermore and Xin [73] and Kapitula [69]. A general result is also given in [90].

The study of non-planar waves has also received much attention. Berestycki, Larouturou and Lions [19] and then Berestycki and Nirenberg [20] prove existence and eventual uniqueness of non-planar waves in cylinder with an advection field. Thus Roquejoffre [87, 88, 89] prove stability results for these non-planar waves for bistable and ignition reactions. Conversely, Berestycki and Hamel [9] prove that in a cylinder with a given advection field, the geometry of the domain can prevent the existence of propagation fronts for a bistable equation.

In the whole space  $\mathbb{R}^N$ , non-planar waves for the homogeneous equation are given by solutions  $u(x, y)$  of

$$\Delta u + c\partial_y u + f(u) = 0, \quad (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}, \quad (15)$$

for some speed  $c$  (not necessarily the one defined by the one-dimensional wave), where  $u$  converges to 1 (resp. to 0) as  $y$  goes to  $-\infty$  (resp.  $+\infty$ ). Thus, the wave  $(t, x, y) \mapsto u(x, y - ct)$  is a solution of (8). In the bistable case, the existence of waves with conical shape has been proved, first for small angles [43] and then for any angle [56] for any space dimension. A complete classification of travelling waves of the type (15) in two dimensions is given in [57] and a stability result for these conical waves is given in [55]. The higher dimensional case is more intricate and various profiles can exist; the more general result is

given by Monneau, Roquejoffre and Roussier-Michon in [80] on the force mean curvature motion which is, in some sense, an approximation to (15). Let us emphasize on the fact that, contrary to the one-dimensional setting, there is no convergence result in higher dimensions. Examples of solutions oscillating between two front profiles have been given by Roquejoffre and Roussier-Michon [90].

Finally, let us mention that the problem of travelling waves in the balanced case ( $\int_0^1 f = 0$ ) may lead to various and different behaviours. See, for instance, the works of Del Pino, Kowalczyk and Wei [35].

**Pulsating waves** Travelling waves of the form (15) no longer exist in the presence of heterogeneities. In the periodic framework, the natural generalisation of the notion of travelling wave is the notion of pulsating wave. The general considered equation is thus given by

$$\frac{\partial u}{\partial t} - \nabla \cdot (A(x, y) \nabla u) + q(x, y) \cdot \nabla u = f(x, y, u), \quad t \in \mathbb{R}, \quad (x, y) \in \Omega \quad (16)$$

where  $A, q$  and  $\Omega$  are periodic in the  $x$ -directions, and  $\Omega$  is bounded in the  $y$ -directions, with  $x \in \mathbb{R}^d$  and  $y \in \mathbb{R}^{N-d}$ . A pulsating wave is then a solution  $u(t, x, y)$  of (16) of the form  $u(t, x, y) = \phi(x.e - ct, x, y)$  where  $\phi(s, \cdot, \cdot)$  is periodic in the  $x$ -directions for all  $s \in \mathbb{R}$  and  $\phi(s, x, y)$  tends to 0 (resp. to 1) as  $s$  goes to  $-\infty$  (resp. to  $+\infty$ ). In the case  $d = 1$ , an equivalent and more intuitive expression is  $u\left(t + \frac{L}{c}, x, y\right) = u(t, x + L, y)$ .

With  $d = N$  and a homogeneous reaction term, Xin [99] prove the existence of periodic fronts an ignition equation. Later, Berestycki and Hamel [10] study the existence of pulsating fronts for (16), with a constant diffusivity  $A = Id$  for KPP an ignition reaction terms. Hamel [53] proves the existence and qualitative properties of pulsating fronts for the whole equation (16), still in the KPP and ignition cases. Uniqueness and convergence to these fronts is proved by Hamel and Roques in [62] in the KPP case. Existence for a reaction term of KPP-type that could be negative is also investigated in [18].

In a one-dimensional framework with just a periodic reaction term, Ducrot, Giletti and Matano prove, under some assumptions on  $f$ , the existence of pulsating fronts for the three considered nonlinearities, including the bistable case.

### The notion of transition fronts

It is in this mathematical context that the notion of generalised transition front is defined and studied [76, 11, 12]. The global aim is to study fully heterogeneous reaction-diffusion equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (A(t, x) \nabla u) + q(t, x) \cdot \nabla u = f(t, x, u), & t \in \mathbb{R}, \quad x \in \Omega \\ \frac{\partial u}{\partial n} = 0, & t \in \mathbb{R}, \quad x \in \partial\Omega \end{cases} \quad (17)$$

where  $\Omega \subset \mathbb{R}^N$  is an unbounded connected domain. A generalised transition front connecting two stationary solutions  $p^+$  and  $p^-$  of (17) is an entire solution  $u(t, x)$  of (17), i.e. defined for all  $t \in \mathbb{R}$ , such that there exist two non-empty, disjoint sets  $\Omega_t^\pm \subset \Omega$  satisfying for all  $t$

$$\begin{cases} \partial\Omega_t^+ \cap \Omega = \partial\Omega_t^- \cap \Omega = \Gamma_t, \quad \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t = \Omega, \\ \sup \left\{ d_\Omega(x, \Gamma_t), \quad x \in \Omega_t^\pm \right\} = +\infty \end{cases}$$

with  $u(t, x) - p^\pm(x) \rightarrow 0$  as  $d_\Omega(x, \Gamma_t) \rightarrow +\infty$ , uniformly in  $t \in \mathbb{R}$  and  $x \in \Omega_t^\pm$ . Here,  $d_\Omega$  refers to the geodesic distance in  $\Omega$ .

In our context, we usually have  $p^+ = 1$ ,  $p^- = 0$ . This notion of generalised transition fronts includes the above notions of travelling waves and pulsating waves, which are transition fronts connecting 0 to 1. It also provides a theory for the study of more general entire solutions in the homogeneous case as well as for the investigation of the heterogeneous equation (17). The general notion of global mean speed of the front is also closely related to the speed of a travelling wave. We refer to the above references for more details on these notions.

**Entire solutions in the homogeneous framework** The question of transition fronts for the homogeneous KPP equation other than the planar fronts given by (9) has been investigated by Hamel and Nadirashvili. In a first paper [58] they construct non-trivial entire solutions for the one-dimensional equation. In a second paper [59] they prove the existence of an infinite-dimensional manifold of entire solutions. Furthermore, up to additional assumption they provide a classification of the entire solutions for the KPP equation in  $\mathbb{R}^N$ . In a recent work [63], Hamel and Rossi give a complete classification for the one-dimensional case.

Transition fronts for the homogeneous bistable equation that are not travelling waves are investigated by Hamel in [54]. The author gives properties that such fronts have to satisfy and explicitly constructs different type of fronts.

Finally, let us mention the works of Berestycki, Hamel and Matano [14]. They construct an almost planar transition front in an exterior domain for a homogeneous bistable equation.

**Transition fronts in heterogeneous framework** For a one dimensional equation we can give a rather simplified definition of transition fronts. A transition front is a global in time solution  $u(t, x)$  such that for all  $t \in \mathbb{R}$  we have

$$\lim_{x \rightarrow -\infty} u(t, x) = 1, \quad \lim_{x \rightarrow +\infty} u(t, x) = 0,$$

and such that, for all  $\varepsilon > 0$ , there exists  $L_\varepsilon > 0$ , for all  $t$ ,

$$\text{diam} \{x \in \mathbb{R}, \varepsilon \leq u(t, x) \leq 1 - \varepsilon\} < L_\varepsilon.$$

Transition fronts in equations of the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, u) \tag{18}$$

have been widely study over the last few years. Existence and uniqueness for ignition-type reactions have been simultaneously proved by Mellet, Roquejoffre, and Sire [79] and by Nolen and Ryzhik [83], and they have come together to prove the stability of such fronts [78]. On the other hand, for a KPP-type reaction term, sufficiently strong heterogeneities can prevent the existence of such fronts [82]. Finally, in a recent contribution [102], Zlatoš proves existence and uniqueness as well as stability of transition fronts for equation like (18) with space and time heterogeneities, with ignition and bistable reactions. He also provides an example of periodic bistable reaction term such that no transition front exists for (18).

This thesis has been performed within this general context and has focused on three types of heterogeneous reaction-diffusion equations: firstly with a coupled system with a line of fast diffusion; then with the study of a bistable equation in a domain which is cylindrical at infinity. This second work leads us to a one dimensional equation of type (18), homogeneous at infinity in one direction.

### 3 Focus on the models studied in the manuscript

In this section we provide the state of the art of the two considered models.

#### Propagation driven by a line of fast diffusion

The model (1) was introduced by Berestycki, Roquejoffre and Rossi in [22] in 2013 with a KPP-type reaction term. They first prove that it is a well-posed system with a comparison principle: ordered solutions remain ordered, and can be dominated (resp. by above) by a super-solution (resp. sub-solution). We refer to the next chapters for a precise statement of these notions. The main result of this seminal article lies in the following propagation theorem.

**Theorem 3 .1.** [22]

1. *There exists an asymptotic speed of spreading  $c^* = c^*(\bar{\mu}, \bar{\nu}, d, D) > 0$  such that, if  $(u, v)$  is a solution of (1) with non-zero compactly supported non-negative initial condition, then*

- $\forall c > c^*, \lim_{t \rightarrow \infty} \sup_{|x| > ct} (u(t, x), v(t, x, y)) = (0, 0) ;$
- $\forall c < c^*, \lim_{t \rightarrow \infty} \inf_{|x| < ct} (u(t, x), v(t, x, y)) = \left( \frac{\bar{\nu}}{\bar{\mu}}, 1 \right) .$

2. *Moreover, with fixed parameters  $d, \bar{\mu}, \bar{\nu}$ , the spreading speed satisfies:*

- *if  $D \leq 2d$ , then  $c^* = c_{KPP} = 2\sqrt{df'(0)}$  ;*
- *if  $D > 2d$ , then  $c^* > c_{KPP}$ . Moreover,  $\lim_{t \rightarrow \infty} \frac{c^*(D)}{\sqrt{D}}$  exists and is positive.*

Without the presence of the road and with Neumann boundary conditions, the spreading speed in every direction would be the classical KPP speed  $2\sqrt{df'(0)}$ . Theorem 3 .1 says that the spreading is enhanced by the road. It is done if the threshold  $D = 2d$  on the diffusivity is reached, and the propagation speed in the direction of the road behaves as  $\sqrt{D}$  as  $D$  goes to infinity. If the threshold may seem surprising, the asymptotics is quite usual in KPP propagation. To get this spreading speed, they construct exponential super-solutions that travel at speed  $c^*$  thanks to algebraic computations on the linearised system.

In a second paper [21], the same authors study further aspects of the model, and particularly the influence of a reaction and an advection term on the road. The threshold  $2d$  for the diffusivity is thus explained by the lack of a reaction term on the road.

Finally in a third paper [23] they study more precisely the spacial impact of the enhancement of the spreading. Indeed, the spreading result given by Theorem 3.1 is not uniform in  $y$ . They prove that the road enhances the spreading in every direction up to a critical angle with the road, and give a description of the expansion shape which is asymptotically strictly convex.

The spreading being driven by the road it is quite natural to consider the above system in a strip of width  $L$ . In the KPP case, it has been investigated by Tellini with a Dirichlet boundary condition [94], and then by Rossi, Tellini and Valdinoci with a Neumann boundary condition [91], which gives the system (19).

$$\begin{cases} \partial_t u - D\partial_{xx}u = \bar{v}v(x, 0, t) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) & (x, y) \in \mathbb{R} \times ]0, L[, t > 0 \\ -d\partial_y v(x, 0, t) = \bar{\mu}u(x, t) - \bar{v}v(x, 0, t), \quad \partial_y v(x, L, t) = 0 & x \in \mathbb{R}, t > 0. \end{cases} \quad (19)$$

They investigate how the width of the strip influences the spreading speed. It is noticeable that it goes to 0 as  $L$  goes to 0; in the asymptotics  $L \rightarrow +\infty$ , they recover the spreading speed of the initial model.

Still in a KPP case, Giletti, Monsaingeon, and Zhou [49] study the case of spatially periodic exchanges, extending the theory given in [18, 15] to the model (1) as well as the system in a strip (19).

The problem of an ignition type reaction was investigated by Dietrich. The fact that the reaction degenerates at 0 does not allow to use the linearised system as in the KPP case. The author proves existence and uniqueness of travelling waves solutions of the system (19), i.e. three-uple  $(c, \phi(x), \psi(x, y))$  such that  $(t, x, y) \mapsto (\phi(x - ct), \psi(x - ct, y))$  is a solution of (19)[37]. In a second work, the author investigates the behaviour of these fronts as  $D$  goes to infinity [38]. Finally he proves in his thesis [36] and in a recent paper with Roquejoffre [39] that these waves attract a large class of initial data.

At last, the influence of a fractional diffusion on the road in the KPP case was studied by Coulon-Chalmin in her thesis [34] and in some other papers [7, 8]. The equation on the road becomes

$$\partial_t u + D(-\partial_{xx})^\alpha u = \bar{v}v(x, 0, t) - \bar{\mu}u.$$

The propagation is therefore exponentially enhanced by the road, which is consistent with the results of fractional propagation in homogeneous and periodic framework [28, 27].

## Propagation in cylinder-like domains

As already explained in the previous section, for a bistable reaction  $f$  in homogeneous media ( $\mathbb{R}^N$  or a cylinder) there exists a unique planar front  $(c, \varphi)$  satisfying (13). The problem of the propagation of bistable fronts in cylinder-like domains has been recently investigated by Berestycki, Bouhours and Chapuisat [5]. The equation under study is the following:

$$\begin{cases} \partial_t u(t, x) - \Delta u(t, x) = f(u), & t \in \mathbb{R}, x \in \Omega, \\ \partial_\nu u(t, x) = 0, & t \in \mathbb{R}, x \in \partial\Omega \end{cases} \quad (20)$$

where the domain  $\Omega$  is supposed to be infinite in the  $x_1$ -direction, that is

$$\Omega = \{(x_1, x'), x_1 \in \mathbb{R}, x' \in \omega(x_1) \subset \mathbb{R}^{N-1}\}. \quad (21)$$

Moreover, they also make the following assumption on  $\Omega$  :

$$\Omega \cap \{x \in \mathbb{R}^N, x_1 < 0\} = \mathbb{R}^- \times \omega, \omega \in \mathbb{R}^{N-1}. \quad (22)$$

Thus, the domain is a strict cylinder in a half plane. The reaction is of bistable type, with  $\int_0^1 f(s)ds > 0$  which implies that the speed  $c$  is positive.

The main objective is to investigate propagation and blocking phenomena induced by such domains. In dimensions two and three with cylinders with rectangular cross section Chapuisat and Grenier proved in [32] that a sudden increase of the diameter could block the propagation.

The first theorem in [5] is concerned with the well-posedness of the problem, i.e. the existence of a front that could characterise the propagation phenomena. They prove the following result.

**Theorem 3 .2.** *Let  $(c, \varphi)$  be the unique bistable wave satisfying (13). There exists a unique function  $u(t, x)$ , defined on  $t \in \mathbb{R}, x \in \Omega$ , entire solution of (20) such that*

$$u(t, x) - \varphi(x_1 - ct) \xrightarrow[t \rightarrow -\infty]{} \text{uniformly in } x \in \bar{\Omega}.$$

Moreover, we have  $0 < u < 1$  and  $u_t > 0$ , for all  $(t, x) \in \mathbb{R} \times \bar{\Omega}$ .

The function  $u$  being nondecreasing in time and uniformly bounded, it converges to a steady state  $u_\infty$  as  $t$  goes to infinity. Properties of this steady state allow to characterise the propagation in  $\Omega$ , which is investigated in numerous theorems in [5] (also see [24]) that generalise the results of Chapuisat and Grenier.

It is noticeable that the arguments used to prove Theorem 3 .2 rely on results by Berestycki, Hamel and Matano in [14]. In this paper, they prove the existence of an entire solution of the same kind in an exterior domain in  $\mathbb{R}^N$ . The idea is to construct suitable sub and super-solutions for (20) starting from negative time and having the good asymptotic properties as  $t \rightarrow -\infty$ . The entire solution is thus devised from a compactness argument.

## 4 Result of the thesis

### Road-field model with nonlocal exchanges

The starting point of our research was to generalise Theorem 3 .1 to the system (2). The integral terms for the exchanges do not allow to use algebraic computations on the linearised system. Moreover, the question of the existence and uniqueness of a stationary solution is quite intricate. The first results, published in [84] and developed in the first chapter of this thesis, give, under some technical assumptions on the exchange functions  $\mu$  and  $\nu$ , the following theorem

**Theorem 1.** *There exists  $c^*$  depending on the parameters of the system such that, if  $(u, v)$  is a solution of (2) starting from non-zero compactly supported nonnegative initial condition, we have:*

- for all  $c > c^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$ ;

- for all  $c < c^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \leq ct} |(u(x, t), v(x, y, t)) - (U_s, V_s(y))| = (0, 0)$  where  $(U_s, V_s(y))$  is the unique non-zero bounded nonnegative stationary solution of (2).
- Id  $D \leq 2d$ , then  $c^* = c_{KPP} = 2\sqrt{df'(0)}$ . If  $D > 2d$ , then  $c^* > c_{KPP}$ .

These results are obtained through a nonlinear eigenvalue problem. They are very similar to those given by Berestycki, Roquejoffre and Rossi in [22]. It may be surprising as it was not expected for the same threshold to arise. This shows the robustness of the model and method, but also led us to investigate more specific properties, particularly how this spreading speed is related to those of the initial model (1). It is natural to consider  $\bar{\nu}$  and  $\bar{\mu}$  as fixed parameters and see how the repartition of the densities influences the spreading speed. A first approach was to study these variations only on one nonlocal exchange  $\nu$  or  $\mu$ , the other being fixed and local. The results, also exposed in the first chapter, are the following:

- when the function  $\nu$  is a Dirac measure (i.e.  $\nu = \bar{\nu}\delta_{y=0}$ ), the spreading speed, depending on the repartition of the function  $\mu$ , is maximised by a local exchange;
- when the function  $\mu$  is a Dirac measure, when considering self-similar functions for  $\nu$ , i.e. of the form  $\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu(\frac{y}{\varepsilon})$ , the Dirac mass is a local minimizer for the spreading speed;
- the above result does not generalise to any approximation of a Dirac mass. More precisely, when considering exchanges of the form

$$\nu(y) = (1 - \varepsilon)\delta_0 + \varepsilon v(y) \tag{23}$$

where  $v$  is a compactly supported function with mass 1, we have the following theorem:

**Theorem 2.** *For some  $v$ ,  $\varepsilon > 0$ , let us consider an exchange function of the form (23). Let  $c^*(\nu)$  be the associated spreading speed and  $c_0^*$  the one associated to (1) with parameters  $\bar{\nu}, \bar{\mu}, D, f'(0)$ . There exist  $m_1 > 2$  depending on  $f'(0)$ ,  $M_1$  depending on  $\bar{\mu}$  such that:*

1. if  $D < m_1$  there exist  $\varepsilon_0$  and  $v$  such that  $\forall \varepsilon < \varepsilon_0$ ,  $c_0^* < c^*(\nu)$ ;
2. if  $\bar{\mu} > 4$  and  $D, f'(0) > M_1$  there exists  $\varepsilon_0$  such that  $\forall v, \forall \varepsilon < \varepsilon_0$ ,  $c_0^* > c^*(\nu)$ .

This last theorem sheds light on a significant difference on the behaviour of the spreading speed between auto-similar exchanges and more general approximations of a Dirac mass. This is reminiscent to a situation discovered in [50] in a different context: with a singular limit on the reaction term for a combustion system, he showed different behaviours depending on how to approach a Dirac mass.

The above results focus on maximisation of the spreading speed. The third chapter is concerned with minimisation of the speed. We prove that minimizing sequences for the spreading speed was given by vanishing exchange functions in both cases. More precisely, the result is the following:



**Theorem 3.** *Let us consider the nonlocal system (2) with fixed exchange masses  $\bar{\mu}$  and  $\bar{\nu}$ . Let  $c^*$  be the spreading speed given by Theorem 3.1, depending on the repartition of  $\mu$  or  $\nu$ .*

1. *If  $D \in \left[ 2d, d \left( 2 + \frac{\bar{\mu}}{f'(0)} \right) \right]$ ,  $\inf c^* = 2\sqrt{df'(0)}$ .*
2. *Fix  $D > d \left( 2 + \frac{\bar{\mu}}{f'(0)} \right)$ , then  $\inf c^* > 2\sqrt{df'(0)}$ .*

*Moreover, in both cases, minimizing sequences can be given by long range exchange terms of the form  $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$  or  $\nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right)$  with  $R \rightarrow \infty$ .*

The first and third chapters concentrate on the difference between the initial model (1) and the nonlocal one (2). In the second chapter I study the links between them. One way to retrieve the local model from the nonlocal one is to consider integral terms tending to Dirac measures. A natural question is that of the stability of the dynamics when considering exchange terms of the form

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right), \quad \mu_\varepsilon(y) = \frac{1}{\varepsilon}\mu\left(\frac{y}{\varepsilon}\right) \quad (24)$$

where  $\mu$  and  $\nu$  are given functions. The stability result proved in the second chapter and published in [86] is the following:

**Theorem 4.** *For  $\varepsilon > 0$ , let us denote  $(u_\varepsilon, v_\varepsilon)$  the solution of system (2) with exchange functions (24) starting from a common initial condition  $(u_0, v_0)$ . We also denote  $c_0^*$  the spreading speed for the initial model (1). There exists  $m > 0$  such that if  $(u_0, v_0) \leq \left(\frac{m}{\bar{\mu}}, m\right)$  we have:*

- $\forall c > c_0^*, \forall \eta > 0, \exists T_0, \varepsilon_0$  such that  $\forall t > T_0, \forall \varepsilon < \varepsilon_0, \sup_{|x| > ct} |u_\varepsilon(t, x)| < \eta$ .
- $\forall c < c_0^*, \forall \eta > 0, \exists T_0, \varepsilon_0$  such that  $\forall t > T_0, \forall \varepsilon < \varepsilon_0, \sup_{|x| < ct} \left| u_\varepsilon(t, x) - \frac{1}{\bar{\mu}} \right| < \eta$ .

This is done by showing the convergence of the solutions of (2) to the solutions of (1) as  $\varepsilon$  goes to 0. The convergence is global in space and local in time, and is performed with an argument from geometric theory of parabolic equations.

## Entire solutions in asymptotically cylindrical domains

In the second part of this thesis we generalise Theorem 3.2 to domains that are asymptotically cylindrical in space dimension two. The considered problem is given by (3)-(5). We prove the following theorem under some additional assumptions on the domain.

**Theorem 5.** *There exists a function  $u(t, x, y)$  defined for  $t \in \mathbb{R}$  and  $(x, y) \in \Omega$  such that*

$$\sup \{ |u(t, x, y) - \varphi(x - ct)|, (x, y) \in \Omega \} \xrightarrow{t \rightarrow -\infty} 0$$

*where  $(c, \varphi)$  is the unique bistable wave satisfying (13).*

The domain being heterogeneous everywhere, this result cannot be obtained with the same kind of sub and super-solutions as in [5]. We get it through a stability argument for the bistable wave under the perturbation induced by the domain.

To have a better intuition of our proof we introduce as a case study the following one dimensional inhomogeneous problem:

$$\partial_t u - \partial_{xx} u = f(u) (1 + g(x)), \quad t \in \mathbb{R}, x \in \mathbb{R}, \quad (25)$$

where  $g$  is a perturbation that tends to 0 as  $x$  goes to  $-\infty$ . If the existence of fronts has been recently investigated for this kind of problem [102], the question of the asymptotic behaviour remains open to our knowledge. We prove the following theorem under certain assumptions on the convergence of  $g$ .

**Theorem 6.** *There exists a positive constant  $\varpi$  depending only on  $f$  such that if  $g > -\varpi$ , then there exists an entire solution  $u_\infty = u_\infty(t, x)$  defined on  $t \in \mathbb{R}, x \in \mathbb{R}$  solution of (25) such that*

$$\|u_\infty(t, \cdot) - \varphi(\cdot - ct)\|_{L^\infty(\mathbb{R})} \xrightarrow{t \rightarrow -\infty} 0. \quad (26)$$

The constant  $\varpi$  is given by  $\varpi = \frac{\rho_1}{\|f'\|_\infty}$  where  $\rho_1$  is the spectral gap of the linearised operator associated with the travelling wave (13).

Once again, the proof relies on the stability of the bistable wave. We prove it with energy estimates on the solution.

## Outline of the manuscript

The first part is concerned with the road field model.

- The first chapter is published in [84]. It is concerned with the enhancement of a Fisher-KPP propagation driven by a line of fast diffusion with nonlocal exchanges. We prove theorems 1 and 2.
- The second chapter is the article [86]. It is concerned with the singular limit of exchanges tending to Dirac masses. We prove theorem 4.
- The third chapter is the note [85] and focuses on long range exchanges. We prove theorem 3.

The second part is concerned with nontrivial entire solutions in heterogeneous bistable equations. We prove theorems 5 and 6.

Finally, we present some perspectives in the last part.



## Part I

Fisher-KPP propagation driven by a  
line of fast diffusion: non-local  
exchanges



# Chapter 1

## The influence of non-local exchange terms on Fisher-KPP propagation driven by a line of fast diffusion

A new model to describe biological invasion influenced by a line with fast diffusion has been introduced by H. Berestycki, J.-M. Roquejoffre and L. Rossi in 2012. The purpose of this article is to present a related model where the line of fast diffusion has a nontrivial range of influence, i.e. the exchanges between the line and the surrounding space has a nontrivial support. We show the existence of a spreading velocity depending on the diffusion on the line. Two intermediate model are also discussed. Then, we try to understand the influence of different exchange terms on this spreading speed. We show that various behaviour may happen, depending on the considered exchange distributions.

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## 1.1 Introduction

### 1.1 .1 Models

The purpose of this study is a continuation of [22] in which was introduced, by H. Berestycki, J.-M. Roquejoffre and L. Rossi, a new model to describe biological invasions in the plane when a strong diffusion takes place on a line, given by (1.1).

$$\begin{cases}
 \partial_t u - D\partial_{xx}u = \bar{\nu}v(t, x, 0) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\
 \partial_t v - d\Delta v = f(v) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\
 v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\
 -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) - \bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0.
 \end{cases}
 \tag{1.1}$$

A two-dimensional environment (the plane  $\mathbb{R}^2$ ) includes a line (the line  $\{(x, 0), x \in \mathbb{R}\}$ ) in which fast diffusion takes place while reproduction and usual diffusion only occur outside the line. For the sake of simplicity, we will refer to the plane as “the field“ and the line as “the road“, as a reference to the biological situations. The density of the population is designated by  $v = v(t, x, y)$  in the field, and  $u = u(t, x)$  on the road. Exchanges of densities take place between the field and the road: a fraction  $\nu$  of individuals from the field at the road (i.e.  $v(x, 0, t)$ ) joins the road, while a fraction  $\mu$  of the population on the road joins the field. The diffusion coefficient in the field is  $d$ , on the road  $D$ . Of course, the aim is to study the case  $D > d$ . The nonlinearity  $f$  is of Fisher-KPP type, i.e. strictly concave with  $f(0) = f(1) = 0$ . Considering a nonnegative, compactly supported initial datum  $(u_0, v_0) \neq (0, 0)$ , the main result of [22] was the existence of an asymptotic speed of spreading  $c^*$  in the direction of the road. They also explained the dependence of  $c^*$  on  $D$ , the coefficient of diffusion on the road. In their model, the line separates the plane in two half-planes which do not interact with each other, but only with the line. Moreover, interactions between a half-plane and the line occur only with the limit-condition in (1.1). That is why, in [22], the authors consider only a half-plane as the field.

New results on (1.1) have been recently proved. Further effects like a drift or a killing term on the road have been investigated in [21]. The case of a fractional diffusion on the

road was studied and explained by the three authors and A.-C. Coulon in [7] and [34]. Models with an ignition-type nonlinearity are also studied by L. Dietrich in [37] and [38]. An approach using a variational formula based on the principal eigenvalue similar to the one studied in [15] has recently been explored in [49].

Our aim is to understand what happens when local interactions are replaced by integral-type interactions: exchanges of populations may happen between the road and a point of the field, not necessarily at the road. The density of individuals who jump from a point of the field to the road is represented by  $y \mapsto \nu(y)$ , from the road to a point of the field by  $y \mapsto \mu(y)$ . This is a more general model than the previous one, but interactions still only occur in one dimension, the  $y$ -axis. We are led to the following system:

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0, \end{cases} \quad (1.2)$$

where  $\bar{\mu} = \int \mu(y)dy$ , the parameters  $d$  and  $D$  are supposed constant positive,  $\mu$  and  $\nu$  are supposed nonnegative, and  $f$  is a reaction term of KPP type. Using the notation  $\bar{\nu} = \int \nu$ , we can generalise this to exchanges given by boundary conditions, with  $\mu = \bar{\mu}\delta_0$  and  $\nu = \bar{\nu}\delta_0$ . Hence, in the same vein as (1.2), it is natural to consider the following semi-limit model

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) & x \in \mathbb{R}, t > 0 \end{cases} \quad (1.3)$$

where interactions from the road to the field are local whereas interactions from the field to the road are still nonlocal. We also introduce the symmetrised semi-limit model, where nonlocal interactions are only from the road to the field.

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t, x) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = -\bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0. \end{cases} \quad (1.4)$$

All these models are connected with each other, setting the scaling

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right), \quad \mu_\varepsilon(y) = \frac{1}{\varepsilon}\mu\left(\frac{y}{\varepsilon}\right).$$

With this scaling, exchanges functions tends to Dirac functions, and integral exchanges tends formally to boundary conditions. For example, the limit  $\varepsilon \rightarrow 0$  in (1.2) leads to the dynamics of (1.1). This result will be investigating in [86]. A similar study would yield to the same kind of convergence of systems (1.3) or (1.4) to (1.1).

Reaction-diffusion equations of the type

$$\partial_t u - d\Delta u = f(u)$$

have been introduced in the celebrated articles of Fisher [45] and Kolmogorov, Petrovsky and Piskounov [70] in 1937. The initial motivation came from population genetics. The



reaction term are that of a logistic law, whose archetype is  $f(u) = u(1 - u)$  for the simplest example. In their works in one dimension, Kolmogorov, Petrovsky and Piskounov revealed the existence of propagation waves, together with an asymptotic speed of spreading of the dominating gene, given by  $2\sqrt{df'(0)}$ . The existence of an asymptotic speed of spreading was generalised in  $\mathbb{R}^n$  by D. G. Aronson and H. F. Weinberger in [2] (1978). Since these pioneering works, front propagation in reaction-diffusion equations have been widely studied. Let us cite, for instance, the works of Freidlin and Gärtner [47] for an extension to periodic media, or [98], [15] and [16] for more general domains.

### 1.1 .2 Assumptions

We always assume that  $u_0$  and  $v_0$  are nonnegative, bounded and uniformly continuous, with  $(u_0, v_0) \not\equiv (0, 0)$ . Our assumptions on the reaction term are of KPP-type:

$$f \in C^1([0, 1]), \quad f(0) = f(1) = 0, \quad \forall s \in (0, 1), \quad 0 < f(s) \leq f'(0)s.$$

We extend it to quadratic negative function outside  $[0, 1]$ . Our assumptions on the exchange terms will differ depending on the sections. For the parts concerning the robustness of the results of [22], that is Proposition 1.1 .1 and Theorem 1.1 .2, they are the following:

- $\mu$  is supposed to be nonnegative, continuous, and decreasing faster than an exponential function:  $\exists M > 0, a > 0$  such that  $\forall y \in \mathbb{R}, \mu(y) \leq M \exp(-a|y|)$ .
- $\nu$  is supposed to be nonnegative, continuous and twice integrable, both in  $+\infty$  and  $-\infty$ , id est

$$\int_0^{+\infty} \int_x^{+\infty} \nu(y) dy dx < +\infty, \quad \int_{-\infty}^0 \int_{-\infty}^x \nu(y) dy dx < +\infty \quad (1.5)$$

- We suppose  $\mu, \nu \not\equiv 0, \nu(0) > 0$ , and that both  $\nu$  and  $\mu$  tend to 0 as  $|y|$  tends to  $+\infty$ .

For the parts dealing with variations on the spreading speed, we suppose that  $\nu$  and  $\mu$  are either nonnegative, continuous, compactly supported even functions, either given by a Dirac measure, either the sum of a Dirac measure and a nonnegative, continuous, compactly supported even function.

### 1.1 .3 Results of the paper

#### Persistence of the results of [22]

We start with the results that are similar in flavour to those of [22] concerning the system (1.1) and showing the robustness of the threshold  $D = 2d$  which was brought out in the paper. The first one concerns the stationary solutions of (1.2) and the convergence of the solutions to this equilibrium.

**Proposition 1.1 .1.** *under the assumptions on  $f, \nu$ , and  $\mu$ , then:*

1. *problem (1.2) admits a unique positive bounded stationary solution  $(U_s, V_s)$ , which is  $x$ -independent ;*

2. for all nonnegative and uniformly continuous initial condition  $(u_0, v_0)$ , the solution  $(u, v)$  of (1.2) starting from  $(u_0, v_0)$  satisfies  $(u(t, x), v(t, x, y)) \xrightarrow[t \rightarrow \infty]{} (U_s, V_s)$  locally uniformly in  $(x, y) \in \mathbb{R}^2$ .

The second and main result deals with the spreading in the  $x$ -direction: we show the existence of an asymptotic speed of spreading  $c^*$  such that the following Theorem holds

**Theorem 1.1 .2.** *Let  $(u, v)$  be a solution of (1.2) with a nonnegative, compactly supported initial datum  $(u_0, v_0)$ . Then, pointwise in  $y$ , we have:*

- for all  $c > c^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$  ;
- for all  $c < c^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \leq ct} |(u(x, t), v(x, y, t)) - (U_s, V_s(y))| = (0, 0)$ .

Because  $f$  is a KPP-type reaction term, it is natural to look for positive solutions of the linearised system

$$\begin{cases} \partial_t u - D \partial_{xx} u = -\bar{\mu} u + \int \nu(y) v(t, x, y) dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d \Delta v = f'(0) v + \mu(y) u(t, x) - \nu(y) v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0. \end{cases} \quad (1.6)$$

We will construct exponential traveling waves and use them to compute the asymptotic speed of spreading in the  $x$ -direction. Theorem 1.1 .2 relies on the following Proposition:

**Proposition 1.1 .3.** *1. There exists a limiting velocity  $c_*$ , depending on  $D$  and  $d$ , such that  $\forall c > c^*$ ,  $\exists \lambda > 0$ ,  $\exists \phi \in H^1(\mathbb{R})$  positive such that  $(t, x, y) \mapsto e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}$  is a solution of (1.6). No such solution exists if  $c < c^*$ .*

2. If  $D \leq 2d$ , then  $c_* = c_{KPP} = 2\sqrt{df'(0)}$ . If  $D > 2d$ , then  $c_* > c_{KPP}$ .

These three results easily extend to the two semi-limit models (1.3) and (1.4). We will develop some proofs only for the system (1.3), the other being easier.

### Effect of the nonlocal exchanges on the spreading speed

Given all these connected models, a natural question is to understand how different exchange terms influence the propagation. One possible way to see it is to ask if, with similar parameters, some exchange functions give slower or faster spreading speed than other. Our results deal with maximal or locally maximal spreading speed. Throughout the end of the paper, we consider the set of admissible exchange functions from the road to the field for fixed  $\bar{\mu}$

$$\Lambda_{\bar{\mu}} = \left\{ \mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \bar{\mu}, \mu \text{ is even} \right\}.$$

Of course, we define  $\Lambda_{\bar{\nu}}$  in a similar fashion. The first result is devoted to the semi-limit case (1.4), where the exchange  $\nu$  is a Dirac measure at  $y = 0$ , and  $\mu$  is nonlocal. For fixed constants  $d, D, \bar{\nu}, f'(0)$ , for any function  $\mu \in \Lambda_{\bar{\mu}}$ , let  $c^*(\mu)$  be the spreading speed associated to the semi-limit system (1.4) with exchange function from the road to the field  $\mu$ . Then we have the following property.

**Proposition 1.1 .4.** *Let  $c_0^*$  the spreading speed associated with the limit system (1.1) with the same parameters and exchange rate from the road to the field  $\bar{\mu}$ . Then:*

$$c_0^* = \sup\{c^*(\mu), \mu \in \Lambda_{\bar{\mu}}\}.$$

The second main result is concerned with the other semi-limit case (1.3), where the exchange  $\mu$  is a Dirac measure, and  $\nu$  is nonlocal ; in our study, we consider  $\nu$  close to a Dirac measure. Let the exchange term  $\nu$  be of the form

$$\nu(y) = (1 - \varepsilon)\delta_0 + \varepsilon v(y) \tag{1.7}$$

where

$$v \in \Lambda_1 := \{v \in C_0(\mathbb{R}), v \geq 0, \int v = 1, v \text{ is even}\}.$$

**Theorem 1.1 .5.** *For some  $v \in \Lambda_1$ ,  $\varepsilon > 0$ , let us consider an exchange function of the form (1.7). Let  $c^*(\nu)$  be the spreading speed associated to (1.3) with exchange function  $\nu$ , and  $c_0^*$  the one associated to (1.1) with same parameters. There exist  $m_1 > 2$  depending on  $f'(0)$ ,  $M_1$  depending on  $\bar{\mu}$  such that:*

1. *if  $D < m_1$  there exist  $\varepsilon_0$  and  $v \in \Lambda_1$  such that  $\forall \varepsilon < \varepsilon_0$ ,  $c_0^* < c^*(\nu)$ ;*
2. *if  $\bar{\mu} > 4$  and  $D, f'(0) > M_1$  there exists  $\varepsilon_0$  such that  $\forall v \in \Lambda_1$ ,  $\forall \varepsilon < \varepsilon_0$ ,  $c_0^* > c^*(\nu)$ .*

## 1.1 .4 Outline and discussion

The following section is concerned with the Cauchy problem, stationary solutions and the long time behaviour. Its conclusion is the proof of Proposition 1.1 .1. The third section is devoted to the proof of Proposition 1.1 .3, and we prove Theorem 1.1 .2 in the fourth. Our results and methods in these two sections shed a new light on those of [22] and [21]. It is striking to find the same condition on  $D$  and  $d$  for the enhancement of the spreading in one direction. The stationary solutions are nontrivial and more complicated to bring out. The computation of the spreading speed  $c^*$  comes from a nonlinear spectral problem, and not from an algebraic system which could be solved explicitly. It also involves some tricky arguments of differential equations.

In the fifth section, we investigate the semi-limit model (1.3). This underlines the robustness of the method for this kind of system.

We study in the sixth section the asymptotics  $D \rightarrow +\infty$  in all cases, which has already been done for the initial model in [22]. Such an asymptotics for a nonlinearity has also been studied by L. Dietrich in [38].

We prove Proposition 1.1 .4 in the seventh section. We show that in the semi-limit case (1.4), the spreading speed is maximal for a concentrate exchange term, that is for the initial limit system (1.1). Such a result may be linked to the case of a periodic framework found in [74].

It could be expected a similar result in the other semi-limit case (1.3). We prove by two different ways that it is not true. We first investigate the case of a self-similar approximation of a Dirac measure for the nonlocal exchange  $\nu$ . For these kind of exchange functions, the Dirac measure is a local minimizer for the spreading speed. This is the purpose of the eighth section.

Considering that, a natural guess would be that in the semi-limit case (1.3) the Dirac measure is a local minimizer anyway. Once again, this is not true. This is the purpose of the last section: we prove that any behaviour may happen in a neighbourhood of concentrate exchange term. More precisely, we prove in Theorem 1.1 .5 that if  $c_0^*$  is the spreading speed associated to the limit system (1.1), considering a pertubated exchange function of the form  $\nu = (1 - \varepsilon)\delta_0 + \varepsilon\nu$ , that is mainly boundary conditions with a small integral contribution, then

- for some ranges of parameters  $D, \bar{\mu}, f'(0)$ , in the neighbourhood of  $\varepsilon = 0$ , the maximal speed is  $c_0^*$ ;
- for other ranges of these parameters and some integral exchange  $\nu$ , a perturbation as above enhances the spreading for  $\varepsilon$  small enough.

Such a difference between self-similar approximations and general approximations of a Dirac measure may be surprising, but a phenomenon of the same kind has already been observed by L. Glangetas in [50] in a totally different context. We can also notice that these results underline how different are the influences of the two exchange functions.

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## 1.2 Stationary solutions and long time behaviour

In this section, we are concerned with the well-posedness of the system (1.2) combined with the initial condition

$$\begin{cases} u|_{t=0} = u_0 & \in \mathbb{R} \\ v|_{t=0} = v_0 & \in \mathbb{R}^2. \end{cases} \quad (1.8)$$

### 1.2 .1 Existence, uniqueness and comparison principle

The system (1.2) is standard, in the sense that the coupling does not appear in the diffusion nor the reaction term. Anyway, well-posedness still has to be mentioned.

**Proposition 1.2 .1.** *Under the above assumptions on  $f, \mu$  and  $\nu$ , the Cauchy problem (1.2)-(1.8) admits a unique nonnegative bounded solution.*

Using the formalism of [64], it is easy to show that the linear part on (1.2) defines a sectorial operator, and that the non-linear is globally Lipschitz on  $X := C_{unif}(\mathbb{R}) \times C_{unif}(\mathbb{R}^2)$ , which gives the existence and uniqueness of the solution of (1.2).

We can also derive the uniqueness of the solution of (1.2) by showing that comparison between subsolutions and supersolutions is preserved during the evolution. Moreover, the

following property will also be the key point in our later study of the spreading. Throughout this article, we will call a subsolution (resp. a supersolution) a couple satisfying the system (in the classical sense) with the equal signs replaced by  $\leq$  (resp.  $\geq$ ) signs, which is also continuous up to time 0.

**Proposition 1.2 .2.** *Let  $(\underline{u}, \underline{v})$  and  $(\bar{u}, \bar{v})$  be respectively a subsolution bounded from above and a supersolution bounded from below of (1.2) satisfying  $\underline{u} \leq \bar{u}$  and  $\underline{v} \leq \bar{v}$  at  $t = 0$ . Then, either  $\underline{u} < \bar{u}$  and  $\underline{v} < \bar{v}$  for all  $t > 0$ , or there exists  $T > 0$  such that  $(\underline{u}, \underline{v}) = (\bar{u}, \bar{v})$ ,  $\forall t \leq T$ .*

Once again, the proof is quite classical and omitted here. This comparison principle extend immediately to generalised sub and supersolutions given by the supremum of subsolutions and the infimum of supersolutions. For our spreading result, we will need a more general class of subsolutions, already used for several results in this context. See for instance Proposition 3.3 in [22].

## 1.2 .2 Long time behaviour and stationary solutions

The main purpose of this section is to prove that any (nonnegative) solution of (1.2) converges locally uniformly to a unique stationary solution  $(U_s, V_s)$ , which is bounded, positive,  $x$ -independent, and solution of the stationary system of equations (1.9):

$$\begin{cases} -DU''(x) &= -\bar{\mu}U(x) + \int \nu(y)V(x, y)dy \\ -d\Delta V(x, y) &= f(V) + \mu(y)U(x) - \nu(y)V(x, y). \end{cases} \quad (1.9)$$

In the same way as above, we call a subsolution (resp. a supersolution) of (1.9) a couple satisfying the system (in the classical sense) with the equal signs replaced by  $\leq$  (resp.  $\geq$ ).

**Proposition 1.2 .3.** *Let  $(u, v)$  be the solution of (1.2) starting from  $(u_0, v_0) \not\equiv (0, 0)$ . then there exist two positive, bounded,  $x$ -independent, stationary solutions  $(U_1, V_1)$  and  $(U_2, V_2)$  such that*

$$U_1 \leq \liminf_{t \rightarrow +\infty} u(x, t) \leq \limsup_{t \rightarrow +\infty} u(x, t) \leq U_2,$$

$$V_1(y) \leq \liminf_{t \rightarrow +\infty} v(x, y, t) \leq \limsup_{t \rightarrow +\infty} v(x, y, t) \leq V_2(y),$$

locally uniformly in  $(x, y) \in \mathbb{R}^2$ .

*Proof.* The proof is adapted from [21]. We first need a  $L^\infty$  a priori estimate.

**A priori estimate** Considering the hypothesis on the reaction term  $f$ , there exists  $K > 0$  such that

$$\forall s \geq K, f(s) \leq s\left(\frac{\bar{\nu}}{\mu}\mu(y) - \nu(y)\right), \forall y \in \mathbb{R}.$$

Thus, for all constant  $V \geq K$ ,  $V\left(\frac{\bar{\nu}}{\mu}, 1\right)$  is a supersolution of (1.2).

**Construction of  $(U_1, V_1)$**  Let  $R > 0$  large enough in such a way that the first eigenvalue of the Laplace operator with Dirichlet boundary condition in  $B_R \subset \mathbb{R}^2$  is less than  $\frac{f'(0)}{3d}$ ,  $\phi_R$  the associated eigenfunction. We extend  $\phi_R$  to 0 outside  $B_R$ .  $\phi_R$  is continuous, bounded, and satisfies

$$-d\Delta\phi_R \leq \frac{1}{3}f'(0)\phi_R \text{ in } \mathbb{R}^2.$$

Let us choose  $\varepsilon > 0$  such that if  $0 < x \leq \varepsilon$ ,  $f(x) > \frac{2}{3}f'(0)x$ . Then define  $M > R$  such that  $\forall y / |y| > M - R$ ,  $\nu(y) \leq \frac{1}{3}f'(0)$ . Since  $(u_0, v_0) \not\equiv (0, 0)$  and  $(0, 0)$  is a solution, the comparison principle implies that  $u, v > 0$ ,  $\forall t > 0$ . Now, let us define  $\eta$  such that  $\eta\phi_R(x, |y| - M) < v(x, y, 1)$  and  $\eta\|\phi_R\|_\infty \leq \varepsilon$ . Define  $\underline{V}(x, y) := \eta\phi_R(x, |y| - M)$ , and, up to a smaller  $\eta$ ,  $(0, \underline{V})$  is a subsolution of (1.2) which is strictly below  $(u, v)$  at  $t = 1$ . Let  $(u_1, v_1)$  be the solution of (1.2) starting from  $(0, \underline{V})$  at  $t = 1$ ;  $(u_1, v_1)$  is strictly increasing in time, bounded by  $K(\frac{\bar{v}}{\mu}, 1)$ , and converges to a positive stationary solution  $(U_1, V_1)$ , satisfying

$$U_1 \leq \liminf_{t \rightarrow +\infty} u \quad V_1 \leq \liminf_{t \rightarrow +\infty} v$$

locally uniformly in  $(x, y) \in \mathbb{R}^2$ .

It remains to show that  $(U_1, V_1)$  is invariant in  $x$ . For  $h \in \mathbb{R}$ , let us denote  $\tau_h$  the translation by  $h$  in the  $x$ -direction:  $\tau_h w(x, y) = w(x+h, y)$ . Since  $\underline{V}$  is compactly supported, there exists  $\varepsilon > 0$  such that

$$\forall h \in (-\varepsilon, \varepsilon), \tau_h \underline{V} < V_1 \text{ and } \tau_h \underline{V} < v \text{ at } t = 1.$$

Thus, because of the  $x$ -invariance of the system (1.2), the solution  $(\tilde{u}_1, \tilde{v}_1)$  of (1.2) starting from  $(0, \tau_h \underline{V})$  at  $t = 1$  is equal to the translated  $(\tau_h u_1, \tau_h v_1)$ . So,  $(\tilde{u}_1, \tilde{v}_1)$  converges to  $(\tau_h U_1, \tau_h V_1)$ . But, since  $(\tilde{u}_1, \tilde{v}_1)$  is below  $(U_1, V_1)$  at  $t = 1$  and  $(U_1, V_1)$  is a (stationary) solution, from the comparison principle given by Proposition 1.2 .2 we deduce  $(\tilde{u}_1, \tilde{v}_1) < (U_1, V_1)$ ,  $\forall t > 1$ , and then

$$(\tau_h U_1, \tau_h V_1) \leq (U_1, V_1), \quad \forall h \in (-\varepsilon, \varepsilon).$$

Namely,  $(U_1, V_1)$  does not depend on  $x$ .

**Construction of  $(U_2, V_2)$**  Let  $\bar{V} = \max(\|v_0\|_\infty, K)$  and  $\bar{U} = \max(\|u_0\|_\infty, \bar{V}\frac{\bar{v}}{\mu})$ . Let  $(u_2, v_2)$  be the solution of (1.2) with initial datum  $(\bar{U}, \bar{V})$ . From the comparison principle (1.2 .2),  $(u, v)$  is strictly below  $(u_2, v_2)$ , for all  $t > 0$ ,  $(x, y) \in \mathbb{R}^2$ . Moreover, since  $(\bar{U}, \bar{V})$  is a supersolution of (1.2) it is clear that  $\partial_t u_2, \partial_t v_2 \leq 0$  at  $t = 0$ . Still using Proposition 1.2 .2, it is true for all  $t \geq 0$ , and  $u_2$  and  $v_2$  are nonincreasing in  $t$ , bounded from below by  $(U_1, V_1)$ . Thus,  $(u_2, v_2)$  converges as  $t \rightarrow \infty$  to a stationary solution  $(U_2, V_2)$  of (1.2) satisfying

$$\limsup_{t \rightarrow +\infty} u(t, x) \leq U_2 \quad \limsup_{t \rightarrow +\infty} v(t, x, y) \leq V_2(y),$$

locally uniformly in  $(x, y) \in \mathbb{R}^2$ . From the construction of  $(U_2, V_2)$ , which is totally independent of the  $x$ -variable, it is easy to see that  $(U_2, V_2)$  does not depend in  $x$ .  $\square$

**Uniqueness of the stationary solution** The previous proposition provides a theoretical proof of the existence of stationary solutions. It also means that a solution is either converging to a stationary solution, or will remain between two stationary solutions. In order to obtain a more precise description of the long time behaviour, we need the following uniqueness result.

**Proposition 1.2 .4.** *There is a unique positive, bounded, stationary solution of (1.2), denoted  $(U_s, V_s)$ .*

To prove the uniqueness, we first need the following intermediate lemma which is the key to all uniqueness properties in this kind of problem. The idea that a bound from below implies uniqueness appeared for the first time in [17].

**Lemma 1.2 .5.** *Let  $(U, V)$  be a positive, bounded stationary solution of (1.2). Then there exists  $m > 0$  such that*

$$\forall (x, y) \in \mathbb{R}^2, U(x) \geq m, V(x, y) \geq m.$$

*Proof.* Let  $(U, V)$  be such a stationary solution.

*First step:* there exists  $M > 0$  such that

$$m_1 := \inf\{V(x, y), |y| > M\} > 0.$$

We will state the proof for positive  $y$ . Let  $R > 0$  large enough in such a way that the first eigenvalue of the Laplace operator with Dirichlet boundary condition in  $B_R \subset \mathbb{R}^2$  is less than  $\frac{f'(0)}{3d}$ ,  $\phi_R$  the associated eigenfunction. We extend  $\phi_R$  to 0 outside  $B_R$ .  $\phi_R$  is continuous, bounded in  $\mathbb{R}^2$ , positive in  $B_R$ . For  $M > 0$ , we define  $\tau_M \phi_R(x, y) = \phi_R(x, y - M)$ . As above, let us define  $M_0 > R$  such that  $\forall y / |y| > M_0 - R, \nu(y) \leq \frac{1}{3}f'(0)$ . Then, there exists  $\varepsilon > 0$  such that  $\forall M > M_0, (0, \varepsilon \tau_M \phi_R)$  is a subsolution of (1.9). As  $V$  is positive, up to smaller  $\varepsilon$ , we can suppose that  $\varepsilon \tau_{M_0} \phi_R < V$ . Now, we claim that

$$\forall y > M_0, V(0, y) > \varepsilon \phi_R(0, 0).$$

Indeed, let us define

$$M_1 := \sup\{M \geq M_0, \forall K \in [M_0, M], \varepsilon \tau_K \phi_R < V\}.$$

Since  $V$  and  $\phi_R$  are continuous,  $M_1 > M_0$ . Suppose that  $M_1 < +\infty$ . Then  $(U, V) \geq (0, \varepsilon \tau_{M_1} \phi_R)$  and there exists  $(x_0, y_0), V(x_0, y_0) = \varepsilon \tau_{M_1} \phi_R(x_0, y_0)$ . Considering that the dynamical system starting from  $(0, \varepsilon \tau_{M_1} \phi_R)$ , which is a subsolution, we get a contradiction from Proposition 1.2 .2. Hence  $M_1 = +\infty$  and our claim is proved. Using the same argument in the  $x$ -direction, we get that  $m_1 \geq \varepsilon \phi_R(0, 0)$ .

*Second step:*

$$m_2 := \inf\{V(x, y), (x, y) \in \mathbb{R}^2\} > 0.$$

If  $m_2 = m_1$ , the assumption is proved. It is obvious that  $m_2 \geq 0$ . Let us assume by way of contradiction that  $m_2 = 0$ . We consider  $(x_n, y_n)$  such that  $V(x_n, y_n) \rightarrow 0$  with  $n \rightarrow \infty$ . Now, we set

$$U_n := U(\cdot + x_n), V_n := V(\cdot + x_n, \cdot + y_n), \mu_n := \mu(\cdot + y_n), \nu_n := \nu(\cdot + y_n).$$

Using the fact that  $U$  and  $V$  are smooth and bounded, by standard elliptic estimates (see [48] for example), there exists  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  strictly increasing such that  $(U_{\varphi(n)})_n, (V_{\varphi(n)})_n$  converge locally uniformly to some functions  $\tilde{U}, \tilde{V}$  satisfying

$$\begin{cases} -D\tilde{U}''(x) &= -\bar{\mu}\tilde{U}(x) + \int \tilde{\nu}(y)\tilde{V}(x, y)dy \\ -d\Delta\tilde{V}(x, y) &= f(\tilde{V}) + \tilde{\mu}(y)\tilde{U}(x) - \tilde{\nu}(y)\tilde{V}(x, y) \end{cases}$$

where  $\tilde{\mu}, \tilde{\nu}$  are some translated of  $\mu, \nu$ . Furthermore,  $\tilde{V} \geq 0$  and  $\tilde{V}(0, 0) = 0$ . Thus in a neighbourhood of  $(0, 0)$  we have

$$-d\Delta\tilde{V}(x, y) + \tilde{\nu}(y)\tilde{V}(x, y) \geq 0, \quad \min(\tilde{V}) = 0.$$

From the strong elliptic maximum principle, we deduce  $\tilde{V} \equiv 0$ . But by step 1  $\tilde{V}(\cdot, 2M) \geq m_1 > 0$ , and we get a contradiction. Hence the result stated above,  $m_2 := \inf(V) > 0$ .

Third step:  $U$  is also bounded from below by a positive constant. Indeed, if we set  $\phi(x) = \frac{1}{D} \int \nu(y)V(x, y)dy$ ,  $U$  is solution of

$$-U'' + \frac{\bar{\mu}}{D}U = \phi, \quad (1.10)$$

with  $\phi$  continuous and  $\phi \geq m_2\|\nu\|_{L^1}$ . Using  $\Phi(x) = \frac{D}{2\bar{\mu}} \exp(-\sqrt{\frac{\bar{\mu}}{D}}|x|)$  which is the fundamental solution of (1.10) we get

$$U(x) = \phi * \Phi(x) \geq \|\Phi\|_{L^1} \cdot m_2 \cdot \|\nu\|_{L^1} := m_3 > 0.$$

Now, set  $m = \inf(m_1, m_2, m_3)$  and the proof is concluded.  $\square$

**Proof of proposition 1.2 .4** It remains now to prove the uniqueness of the stationary solution of (1.2). The difficulties come from the fact that it is a coupled system in an unbounded domain: for bounded domains, uniqueness was proved in [4]. Let  $(U_1, V_1), (U_2, V_2)$  be two bounded, positive solutions of (1.9), and let us show that  $(U_1, V_1) = (U_2, V_2)$ . From Lemma 1.2 .5, there exists  $m > 0$  such that  $(U_i, V_i) \geq m, i = 1..2$ . Hence, for  $T$  large enough,  $T(U_1, V_1) > (U_2, V_2)$ . Let

$$T_1 = \inf\{T, \forall T' > T, T'(U_1, V_1) > (U_2, V_2)\} > 0,$$

and

$$(\delta U, \delta V) = T_1(U_1, V_1) - (U_2, V_2).$$

Up to take  $T_1(U_2, V_2) - (U_1, V_1)$  if needed, we can suppose  $T_1 \geq 1$ . The couple  $(\delta U, \delta V)$  satisfies the following system:

$$\begin{cases} -D\delta U''(x) &= -\bar{\mu}\delta U(x) + \int \nu(y)\delta V(x, y)dy \\ -d\Delta\delta V(x, y) &= T_1 f(V_1) - f(V_2) + \mu(y)\delta U(x) - \nu(y)\delta V(x, y) \end{cases}$$

and  $\inf(\delta U) = 0$  or  $\inf(\delta V) = 0$ . In order to show that  $(\delta U, \delta V) \equiv 0$  we have to distinguish five cases.



Case 1: there exists  $(x_0, y_0) \in \mathbb{R}^2$ ,  $\delta V(x_0, y_0) = 0$ . Then, using the fact that  $f(0) = 0$  and that  $f$  is strictly concave, we can easily check that  $T_1 f(V_1) - f(V_2) \geq 0$  in a neighbourhood of  $(x_0, y_0)$ . Thus, because  $\delta U \geq 0$ ,  $\delta V$  is solution of the inequality system

$$\begin{cases} -d\Delta\delta V + \nu\delta V & \geq 0 \\ \delta V \geq 0, & \delta V(x_0, y_0) = 0. \end{cases}$$

From the elliptic maximum principle, we infer  $\delta V \equiv 0$ . Because  $\mu \neq 0$ , we immediately get  $\delta U \equiv 0$ . So  $(U_2, V_2) = T_1(U_1, V_1)$ ; subtracting the two systems (1.9) in  $(U_1, V_1)$  and  $T_1(U_1, V_1)$  yields  $T_1 f(V_1) = f(V_1)$  and  $V_1 > 0$ . So  $T_1 = 1$ , and  $(U_2, V_2) = (U_1, V_1)$ .

Case 2: there exists  $x_0$  such that  $\delta U(x_0) = 0$ . In the same way we infer  $\delta U \equiv 0$ . Then,  $\forall x \in \mathbb{R}$ ,  $\int \nu \delta V = 0$ . In particular, there exists  $y_0$  such that  $\delta V(x_0, y_0) = 0$ , and the problem is reduced to the (solved) first case:  $T_1 = 1$ , and  $(U_2, V_2) = (U_1, V_1)$ .

Case 3: there is a contact point for  $U$  at infinite distance. Formally, there exists  $(x_n)_n$ ,  $|x_n| \rightarrow \infty$  such that  $\delta U(x_n) \rightarrow 0$  with  $n \rightarrow \infty$ . We set

$$U_i^n := U_i(\cdot + x_n), \quad V_i^n := V_i(\cdot + x_n, \cdot), \quad i = 1, 2.$$

In the same way as above, there exist  $\tilde{U}_i, \tilde{V}_i$  such that, up to a subsequence,  $(U_i^n, V_i^n)$  converges locally uniformly to  $(\tilde{U}_i, \tilde{V}_i)$ , and the couples  $(\tilde{U}_1, \tilde{V}_1)$  and  $(\tilde{U}_2, \tilde{V}_2)$  both satisfy (1.9) and

$$\begin{cases} T_1 = \inf\{T, \forall T' > T, T'(\tilde{U}_1, \tilde{V}_1) > (\tilde{U}_2, \tilde{V}_2)\}, \\ (T_1 \tilde{U}_1 - \tilde{U}_2)(0) = 0. \end{cases}$$

The problem is once again reduced to the first case, and  $T_1 = 1$ .

Case 4: there is a contact point for  $V$  at infinite distance in  $x$ , finite distance in  $y$ , say  $y_0$ . We use the same trick as above, the limit problem is this time reduced to the second case, and we still get  $T_1 = 1$ .

Case 5: there is a contact point for  $V$  at infinite distance in  $y$ . That is to say there exist  $(x_n)_n, (y_n)_n$ , with  $|y_n| \rightarrow \infty$  such that  $\delta V(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0$ . Once again, we set

$$V_i^n := V_i(\cdot + x_n, \cdot + y_n), \quad i = 1, 2.$$

Now, considering that  $U_1, U_2$  are bounded and that  $\mu, \nu \xrightarrow{|y| \rightarrow \infty} 0$ ,  $(V_1^n)_n, (V_2^n)_n$  converge locally uniformly to some functions  $\tilde{V}_1, \tilde{V}_2$  which satisfy

$$\begin{cases} -d\Delta\tilde{V}_i & = f(\tilde{V}_i) \\ (T_1 \tilde{V}_1 - \tilde{V}_2)(0, 0) & = 0 \end{cases}$$

and  $(T_1 \tilde{V}_1 - \tilde{V}_2) \geq 0$  in a neighbourhood of  $(0, 0)$ . Thus, using the concavity of  $f$  as in the first case, we get  $T_1 = 1$ .

From the five cases considered above, whatever may happen,  $T_1 = 1$ , and the proof is complete.  $\square$

The proof of Proposition 1.1 .1 is now a consequence of Propositions 1.2 .3 and 1.2 .4.

### 1.3 Exponential solutions of the linearised system

Looking for supersolution of the system (1.2) lead us to search positive solutions of the linearised system (1.6), hence we are looking for solutions of the form:

$$\begin{pmatrix} u(x, t) \\ v(x, y, t) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}, \quad (1.11)$$

where  $\lambda, c$  are positive constants, and  $\phi$  is a nonnegative function in  $H^1(\mathbb{R})$ . The system on  $(\lambda, \phi)$  reads:

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y). \end{cases} \quad (1.12)$$

The first equation of (1.12) gives the graph of a function  $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \bar{\mu}$ , which, if (1.11) is a solution of (1.6), is equal to  $\int \nu(y)\phi(y)dy$ .

The second equation of (1.12) gives, under some assumptions on  $\lambda$ , a unique solution  $\phi = \phi(y; \lambda, c)$  in  $H^1(\mathbb{R})$ . To this unique solution we associate the function  $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$ . Let us denote  $\Gamma_1$  the graph of  $\Psi_1$  in the  $(\lambda, \Psi_1(\lambda))$  plane, and  $\Gamma_2$  the graph of  $\Psi_2$ . So, (1.12) amounts to the investigation of  $\lambda, c > 0$  such that  $\Gamma_1$  and  $\Gamma_2$  intersect.

The graph of  $\lambda \mapsto \Psi_1(\lambda)$  is a parabola. As we are looking for a nonnegative function  $\phi$ , we are interested in the positive part of the graph. The function  $\lambda \mapsto \Psi_1(\lambda)$  is nonnegative for  $\lambda \in [\lambda_1^-(c), \lambda_1^+(c)]$ , with  $\lambda_1^\mp(c) = \frac{c \mp \sqrt{c^2 + 4D\bar{\mu}}}{2D}$ .

It reaches its maximum value in  $\lambda = \frac{c}{2D}$ , with  $\Psi_1(\frac{c}{2D}) = \bar{\mu} + \frac{c^2}{4D} > \bar{\mu}$ .

We also have

$$\Psi_1(0) = \Psi_1\left(\frac{C}{D}\right) = \bar{\mu}, \quad (1.13)$$

which will be quite important later.

We may observe that: with  $D$  fixed,  $(\lambda_1^-(c), \lambda_1^+(c)) \xrightarrow{c \rightarrow +\infty} (0^-, +\infty)$ ;  $\lambda \mapsto \Psi_1(\lambda)$  is strictly

concave;  $\frac{d\Psi_1}{d\lambda} \Big|_{\lambda=c/D} = -c$ . We can summarize it in fig. (1.1).

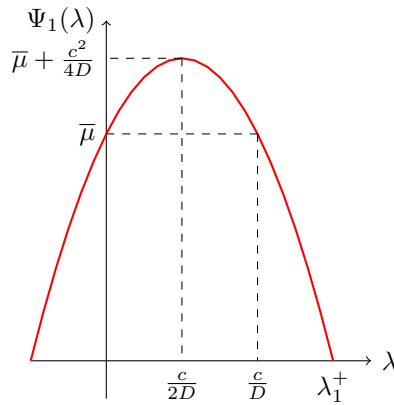


Figure 1.1: representation of  $\Gamma_1$

### 1.3 .1 Study of $\Psi_2$

The study of  $\Psi_2$  relies on the investigation of the solution  $\phi = \phi(\lambda; c)$  of

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}) \quad \phi \geq 0. \end{cases} \quad (1.14)$$

Since  $\mu$  is continuous and decays no slower than an exponential,  $\mu$  belongs to  $L^2(\mathbb{R})$ . Since  $\nu$  is nonnegative and bounded, the Lax-Milgram theorem assures us that (1.14) admits a unique solution if  $\lambda c - d\lambda^2 - f'(0) > 0$ , that is to say if  $\lambda$  belongs to  $]\lambda_2^-(c), \lambda_2^+(c)[$ , where

$$\lambda_2^\mp(c) = \frac{c \mp \sqrt{c^2 - c_{KPP}^2}}{2d},$$

with

$$c_{KPP} = 2\sqrt{df'(0)}.$$

As in [22], the KPP-asymptotic spreading speed will have a certain importance in the study of the spreading in our model. Moreover, since  $\nu, \mu$  tend to 0 with  $|y| \rightarrow \infty$ , an easy computation will show that, for  $\lambda < \lambda_2^-$  or  $\lambda > \lambda_2^+$ , equation (1.14) cannot have a constant sign solution. Moreover, we look for  $H^1$  solutions. We will see in Lemma 1.3 .3 that it prevents the existence of a solution for  $c = c_{KPP}$ . Thus,

$$\Gamma_2 \text{ exists if and only if } c > c_{KPP}. \quad (1.15)$$

The main properties of  $\Psi_2$  are the following:

**Proposition 1.3 .1.** *If  $c > c_{KPP}$ , then:*

1.  $\lambda \mapsto \Psi_2(\lambda)$  defined on  $]\lambda_2^-, \lambda_2^+[$  is positive, smooth, strictly convex and symmetric with respect to the line  $\{\lambda = \frac{c}{2d}\}$ . With  $\lambda$  fixed we also have  $\frac{d}{dc}\Psi_2(\lambda; c) < 0$ .
2.  $\Psi_2(\lambda) \xrightarrow{\lambda \rightarrow \lambda_2^\mp} \bar{\mu}$ .
3.  $\frac{d\Psi_2}{d\lambda}(\lambda) \xrightarrow[\lambda > \lambda_2^-]{\lambda \rightarrow \lambda_2^-} -\infty$ .

The graph  $\Gamma_2$  looks like fig. (1.2).

**Proof of the first part of proposition (1.3 .1)**

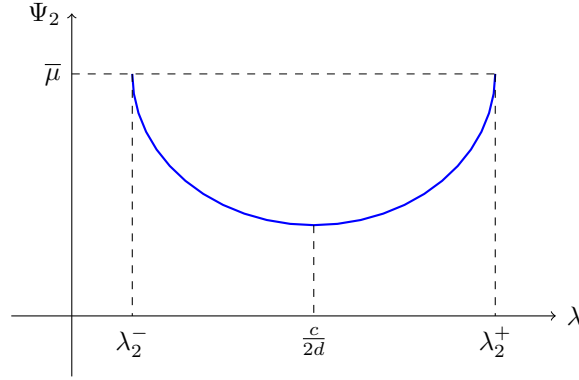
**Positivity, smoothness** For all  $\lambda$  in  $]\lambda_2^-, \lambda_2^+[$ ,

$$P(\lambda) := \lambda c - d\lambda^2 - f'(0) > 0. \quad (1.16)$$

Consequently,  $\forall \lambda \in ]\lambda_2^-, \lambda_2^+[$ ,  $\forall y \in \mathbb{R}$ ,  $P(\lambda) + \nu(y) > 0$ . From the elliptic maximum principle, as  $\mu$  is nonnegative, we deduce that  $\phi(y) > 0$ ,  $\forall y \in \mathbb{R}$ . Hence, since  $\nu$  is nonnegative, we have  $\Psi_2(\lambda) = \int \phi(y; \lambda)\nu(y)dy > 0$ , and  $\Psi_2$  is positive.

Considering that  $\lambda \mapsto P(\lambda)$  is polynomial, with the analytic implicit function theorem, we see immediately that  $\lambda \mapsto \phi(y; \lambda)$  is analytic (see [30], Theorem 3.7.1). Since  $\nu$  is integrable,  $\lambda \mapsto \Psi_2(\lambda)$  is also analytic.

From the symmetry of  $\lambda \mapsto P(\lambda)$  and the uniqueness of the solution, we deduce the symmetry of  $\Gamma_2$  with respect to the line  $\{\lambda = \frac{c}{2d}\}$ .


 Figure 1.2: representation of  $\Gamma_2$ 

**Monotonicity, convexity** Denote by  $\phi_\lambda$  the derivative of  $\phi$  with respect to  $\lambda$ . Then, if we differentiate (1.14) with respect to  $\lambda$ , we can see that  $\phi_\lambda$  satisfies:

$$-d\phi_\lambda''(y) + (P(\lambda) + \nu(y))\phi_\lambda(y) = (2d\lambda - c)\phi(y). \quad (1.17)$$

In the same way as equation (1.14), (1.17) has a unique solution in  $H^1(\mathbb{R})$  for all  $\lambda \in ]\lambda_2^-, \lambda_2^+[$ . Since  $\phi$  is positive,  $\phi_\lambda$  is of constant sign, with the sign of  $(2d\lambda - c)$ . Hence we have that  $\Psi_2$  is decreasing on  $]\lambda_2^-, \frac{c}{2d}[$  and increasing on  $]\frac{c}{2d}, \lambda_2^+[$ .

Differentiating once again (1.17) with respect to  $\lambda$ , the second derivative of  $\phi$  with respect to  $\lambda$  satisfies:

$$-d\phi_{\lambda\lambda}''(y) + (P(\lambda) + \nu(y))\phi_{\lambda\lambda}(y) = 2d\phi(y) + 2(2d\lambda - c)\phi_\lambda(y). \quad (1.18)$$

In the same way,  $\phi$  is positive for all  $\lambda \in ]\lambda_2^-, \lambda_2^+[$ , and  $\phi_\lambda(\lambda)$  has the positivity of  $(2d\lambda - c)$ . Hence the left term of equation (1.18) is positive, for all  $\lambda \in ]\lambda_2^-, \lambda_2^+[$ , and  $\Psi_2$  is strictly convex on  $]\lambda_2^-, \lambda_2^+[$ .

With the same arguments we see that  $\phi_c$ , the derivative of  $\phi$  with respect to  $c$ , satisfies

$$-d\phi_c'' + (P(\lambda) + \nu)\phi_c = -\lambda\phi < 0,$$

and then we get  $\int_{\mathbb{R}} \phi_c(y)\nu(y)dy = \frac{d}{dc}\Psi_2(\lambda; c) < 0$ .

In order to end the proof of the proposition (1.3 .1), we need to study behaviour of  $\Psi_2$  near  $\lambda_2^-$ . Setting  $\varepsilon = P(\lambda)$ , it is sufficient to study the behaviour of the solution  $\phi = \phi(y; \varepsilon)$  of

$$\begin{cases} -\phi''(y) + (\varepsilon + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}), \varepsilon > 0, \varepsilon \rightarrow 0. \end{cases} \quad (1.19)$$

The main lemma here is the following, which will evidently conclude Proposition 1.3 .1:

**Lemma 1.3 .2.** 1. If  $\phi$  is solution of (1.19) then  $\int_{\mathbb{R}} \phi(y)\nu(y)dy \xrightarrow[\varepsilon > 0]{\varepsilon \rightarrow 0} \bar{\mu}$  holds true.

Moreover,  $\|\phi\|_{L^\infty}$  is uniformly bounded on  $\varepsilon$ .

2. The derivative of  $\phi$  with respect to  $\varepsilon$ , denoted  $\phi_\varepsilon$ , satisfies  $\int_{\mathbb{R}} \phi_\varepsilon(y)\nu(y)dy \xrightarrow[\varepsilon > 0]{\varepsilon \rightarrow 0} -\infty$ .

**Proof of the first part of the Lemma 1.3 .2** An explicit computation is needed. We use a boxcar function for this. Under the assumptions on  $\nu$  and  $\mu$ , there exist  $\alpha$ ,  $M$ ,  $m_1 > 0$  such that:

- $\nu(y) \geq \alpha \mathbf{1}_{[-m_1, m_1]}$ ,  $\forall y \in \mathbb{R}$  (because  $\nu(0) > 0$ , and  $\nu$  is continuous);
- $\mu(y) \leq Me^{-a|y|}$ ,  $\forall y \in \mathbb{R}$  (from the exponential decay of  $\mu$ ).

Denoting  $\psi = \psi(y; \varepsilon)$  the solution of

$$-\psi'' + (\varepsilon + \alpha \mathbf{1}_{[-m_1, m_1]})\psi = Me^{-a|y|}, \quad (1.20)$$

$\psi$  is a supersolution for (1.19) and

$$\forall \varepsilon > 0, \forall y \in \mathbb{R}, 0 < \phi(y; \varepsilon) \leq \psi(y; \varepsilon). \quad (1.21)$$

We have already seen that  $\forall \varepsilon > 0$ ,  $\int_{\mathbb{R}} \phi''(y; \varepsilon) dy = 0$ . Consequently, the assumption  $\int_{\mathbb{R}} \phi(y) \nu(y) dy \xrightarrow{\varepsilon \rightarrow 0} \bar{\mu}$  is equivalent to  $\varepsilon \int_{\mathbb{R}} \phi(y; \varepsilon) dy \xrightarrow{\varepsilon \rightarrow 0} 0$ . To conclude, it remains to compute the solution  $\psi$  and to show that  $\varepsilon \int_{\mathbb{R}} \psi(y; \varepsilon) dy \xrightarrow{\varepsilon \rightarrow 0} 0$ . But the solution of (1.20) can be explicitly computed, which gives that  $\|\phi(\varepsilon)\|_{L^\infty(\mathbb{R})}$  is uniformly bounded on  $\varepsilon$  and that there exists  $C > 0$  such that for  $\varepsilon > 0$  small and  $y > m_1$ ,

$$\psi(y; \varepsilon) < Ce^{-\sqrt{\varepsilon}y},$$

so

$$\int_{\mathbb{R}} \psi(y; \varepsilon) dy = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ as } \varepsilon \rightarrow 0$$

and

$$\varepsilon \int_{\mathbb{R}} \psi(y; \varepsilon) dy \xrightarrow{\varepsilon \rightarrow 0} 0,$$

which concludes the proof of the first statement in Lemma 1.3 .2. Notice that we also get that there exist two constant  $C_1, C_2$  not depending on  $\varepsilon$  such that for all  $y$  in  $\mathbb{R}$ ,  $\psi(y; \varepsilon) \leq C_1 e^{-\sqrt{\varepsilon}|y|} + C_2 e^{-a|y|}$ , that will be useful later.  $\square$

Let us prove the second part of Lemma (1.3 .2). In order to prove it, we will first deal with the study of the homogeneous limit differential equation.

**Lemma 1.3 .3.** *Let us consider the scalar homogeneous equation (1.22):*

$$-\psi'' + \nu \cdot \psi = 0. \quad (1.22)$$

*Under the assumptions on  $\nu$ , there exist  $\phi_1, \phi_2$  satisfying*

- $\phi_1(x) \xrightarrow{x \rightarrow +\infty} 0$ , and, for  $x$  large enough,  $\phi_1(x) \geq 0$  ;
- $\exists C_1, C_2 > 0$  such that  $C_1 x \leq \phi_2(x) \leq C_2 x$  when  $x$  goes to  $+\infty$  (notation:  $\phi_2(x) = \Theta(x)$  ) ;

*such that*

$$\begin{cases} \psi_1 := 1 + \phi_1 \\ \psi_2 := \phi_2(1 + \phi_1) \end{cases}$$

*is a fundamental system of solutions of (1.22).*

*Proof. Construction of  $\phi_1$ :* let  $\psi := 1 + \phi_1$  be a solution of (1.22). Thus,  $\phi_1$  must satisfy

$$-\phi_1'' + \nu + \nu\phi_1 = 0. \quad (1.23)$$

Let us show that there exists a solution of (1.23) which is nonnegative for  $x$  large enough and tends to 0 as  $x$  goes to  $+\infty$ . Let  $M \geq 0$  such that  $\int_M^\infty \int_x^\infty \nu(y) dy dx < 1$  which is possible thanks to the assumption (1.5) on  $\nu$ . Now, define

$$\mathcal{E} := \{\phi \in C([M, +\infty[) / \forall x \geq M, \phi(x) \geq 0 \text{ and } \phi(x) \xrightarrow{x \rightarrow \infty} 0\}$$

and

$$F \begin{cases} \mathcal{E} \rightarrow \mathcal{E} \\ \phi \mapsto F\phi : x \mapsto \int_x^\infty \int_y^\infty (1 + \phi(z)) \nu(z) dz dy. \end{cases}$$

From the hypothesis on  $\mathcal{E}$  and  $\nu$ ,  $F$  is well defined.  $\mathcal{E}$  is a closed subset of the Banach space  $C_0([M, \infty[)$ . The choice of  $M$  implies that  $F$  is a contraction. From a classical Banach fixed point argument, there exists a unique positive solution  $\phi_1$  in  $C([M, +\infty[)$  of 1.23 satisfying  $\phi(x) \xrightarrow{x \rightarrow +\infty} 0$ .

Moreover, without loss of generality, we can only consider the case  $M = 0$ .

**Construction of  $\phi_2$ :** we are looking for a second solution of (1.22) in the form  $\psi_2 = \phi_2 \cdot \psi_1$ . Integrating the equation we get for  $x \geq 0$ :

$$\phi_2(x) = \int_0^x \frac{dy}{(1 + \phi_1(y))^2},$$

and  $\psi_2 := \phi_2(1 + \phi_1)$  is a second solution of the homogeneous equation (1.22). Finally, considering that  $\phi_1(x) \rightarrow 0$  with  $x \rightarrow +\infty$ , we get the desired estimate for  $\phi_2$ .  $\square$

Of course, we have a similar result for  $x \rightarrow -\infty$ . This lemma first allows us to give a useful lower bound of  $\phi(y; \varepsilon)$  at the limit  $\varepsilon = 0$ .

**Corollary 1.3 .4.** *Let  $\phi = \phi(y; \varepsilon)$  be the solution of (1.19). There exists  $k > 0$  such that,  $\forall y \in \mathbb{R}, \exists \varepsilon_y, \forall \varepsilon < \varepsilon_y, \phi(y; \varepsilon) \geq k$ , and this uniformly on every compact set in  $y$ .*

*Proof.* Since  $\mu \neq 0$  there exists a nonnegative compactly supported function  $\mu_c \neq 0$  such that  $0 \leq \mu_c \leq \mu$ . Let us now consider the (unique) solution  $\underline{\phi} = \underline{\phi}(y; \varepsilon)$  of

$$\begin{cases} -\underline{\phi}''(y) + (\varepsilon + \nu(y))\underline{\phi}(y) = \mu_c(y) \\ \underline{\phi} \in H^1(\mathbb{R}), \varepsilon > 0. \end{cases} \quad (1.24)$$

From the first part of Lemma 1.3 .2, we know that  $\exists K > 0, \forall y \in \mathbb{R}, \forall \varepsilon > 0, 0 < \underline{\phi}(y; \varepsilon) \leq \phi(y; \varepsilon) < K$ . Let us recall that for fixed  $y \in \mathbb{R}$ ,  $\underline{\phi}(y; \varepsilon)$  is increasing with  $\varepsilon \rightarrow 0$  and bounded by  $K$ . Hence there exists a positive function  $\underline{\phi}_0$  such that  $\underline{\phi}(y; \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \underline{\phi}_0(y)$ . Moreover, from the uniform boundedness of  $\underline{\phi}(\varepsilon)$  and Ascoli's theorem, the convergence is uniform for  $\underline{\phi}$  and  $\underline{\phi}'$  in every compact set. Thus,  $\underline{\phi}_0$  satisfies in the classical sense

$$\begin{cases} -\underline{\phi}_0''(y) + \nu(y)\underline{\phi}_0(y) = \mu_c(y) \\ 0 < \underline{\phi}_0 \leq K. \end{cases}$$

As  $\mu_c$  is compactly supported, for  $|y|$  large enough, let us say greater than  $A > 0$ ,  $\underline{\phi}_0$  is a solution of (1.22), that is to say, in the positive semi-axis

$$\begin{cases} -\underline{\phi}_0''(y) + \nu(y)\underline{\phi}_0(y) = 0, & y > A \\ 0 < \underline{\phi}_0(y) \leq K < +\infty & y > A. \end{cases}$$

Thus, there exist  $\alpha^+, \beta^+$  such that

$$\forall y > A, \underline{\phi}_0(y) = \alpha^+(1 + \phi_1(y)) + \beta^+\phi_2(y)(1 + \phi_1(y)),$$

where  $\phi_1$  and  $\phi_2$  are defined in Lemma 1.3 .3. Now considering that  $\phi_1(y) = o(1)$  and  $\phi_2(y) = \Theta(y)$  in  $y \rightarrow +\infty$ , as  $\underline{\phi}_0$  is bounded,  $\beta^+ = 0$ . Then, as  $\underline{\phi}_0 > 0$ ,  $\alpha^+ > 0$ . We have a similar result for  $y < -A$ , with  $\beta^- = 0$  and  $\alpha^- > 0$ . Finally, define

$$k = \frac{1}{2} \min(\alpha^-, \alpha^+, \min\{\underline{\phi}_0(y), y \in [-A, A]\}) > 0$$

and the proof is concluded.  $\square$

**Proof of the second part of Lemma 1.3 .2** Differentiating equation (1.19) with respect to  $\varepsilon$ , we get for the derivative  $\phi_\varepsilon$

$$-\phi_\varepsilon''(y; \varepsilon) + (\varepsilon + \nu(y))\phi_\varepsilon(y; \varepsilon) = -\phi(y; \varepsilon). \quad (1.25)$$

Since  $\phi$  is positive, we get that  $\phi_\varepsilon$  is negative. Let us denote

$$\varphi(y) = \varphi(y; \varepsilon) := -\phi_\varepsilon(y; \varepsilon) > 0.$$

We have previously seen (in the proof of the first part of Proposition 1.3 .1) that  $\forall y \in \mathbb{R}$ ,  $\frac{d}{d\varepsilon}\varphi(y; \varepsilon) < 0$ , *i.e.*  $\varphi$  is increasing with  $\varepsilon \rightarrow 0, \varepsilon > 0$ . Our purpose is to show that in a neighbourhood of 0,  $\inf(\varphi(\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} +\infty$ . For all  $\varepsilon > 0$ , define the function  $\bar{\varphi} = \bar{\varphi}(y; \varepsilon)$  as the unique solution of

$$\begin{cases} -\bar{\varphi}''(y; \varepsilon) + (\varepsilon + \nu(y))\bar{\varphi}(y; \varepsilon) = \min(k, \phi(y; \varepsilon)) \\ \bar{\varphi} \in H^1(\mathbb{R}). \end{cases} \quad (1.26)$$

The function  $\bar{\varphi}$  is obviously well-defined. By its definition, the elliptic maximum principle ensures us that  $0 < \bar{\varphi} \leq \varphi$ ,  $\forall y \in \mathbb{R}, \varepsilon > 0$ . We have also to notice that uniformly on every compact set in  $y$ ,  $\min(k, \phi(y; \varepsilon)) = k$  for  $\varepsilon$  small enough (consequence of corollary 1.3 .4). Assume by way of contradiction that

$$\left( \min_{y \in [-1, 1]} (\bar{\varphi}(y; \varepsilon)) \right)_\varepsilon \text{ is bounded.} \quad (1.27)$$

Let us show that it is inconsistent with the fact that  $\bar{\varphi} > 0, \forall \varepsilon > 0$ . As  $\min(k, \phi(y; \varepsilon))$  is uniformly bounded, from Harnack inequalities (see [48], Theorem 8.17 and 8.18) we know that for all  $R > 0$ , there exist  $C_1 = C_1(R), C_2 = C_2(R)$ , independent of  $\varepsilon$ , such that for all  $\varepsilon > 0$ ,

$$\sup_{[-R, R]} \bar{\varphi} \leq C_1 \inf_{[-R, R]} (\bar{\varphi} + C_2).$$

Combining this and hypothesis (1.27), we get that  $(\bar{\varphi}(y; \varepsilon))_{\varepsilon > 0}$  is increasing with  $\varepsilon \rightarrow 0$  and uniformly in every compact set in  $y$ . Using the same argument as in the proof of Corollary 1.3 .4,  $(\bar{\varphi}(\varepsilon))_\varepsilon$  converges locally uniformly to some function  $\bar{\varphi}_0$  which satisfies in the classical sense

$$\begin{cases} -\bar{\varphi}_0''(y) + \nu(y)\bar{\varphi}_0(y) = k \\ \bar{\varphi}_0(y) \geq 0, \forall y \in \mathbb{R}. \end{cases} \quad (1.28)$$

So there exist  $\alpha, \beta \in \mathbb{R}$  such that  $\bar{\varphi}_0 = \alpha(1 + \phi_1) + \beta\phi_2(1 + \phi_1) + \phi_s$ , where  $\phi_1, \phi_2$  are defined in Lemma 1.3 .3 and  $\phi_s$  is a particular solution of (1.28). Thus, for  $x \geq 0$ ,

$$\phi_s(y) = -k(1 + \phi_1(y))(1 + \phi_1(0)) \left( \int_0^y \int_z^y \frac{1 + \phi_1(z)}{(1 + \phi_1(t))^2} dt dz \right).$$

Now, recall that  $\phi_1 > 0, \phi_1(y) = o(y)$  as  $y$  goes to  $+\infty$ . So there exists  $\gamma > 0, \phi_s(y) \underset{y \rightarrow \infty}{\sim} -\gamma.y^2$ . As a result, for  $y \rightarrow \infty$ ,

$$\begin{cases} \bar{\varphi}_0(y) = -\gamma.y^2 + o(y^2) \xrightarrow{y \rightarrow +\infty} -\infty \\ \bar{\varphi}_0 \geq 0, \forall y \in \mathbb{R}, \end{cases}$$

which is obviously a contradiction. So the first hypothesis (1.27) is false, which gives, combined with the monotonicity in  $\varepsilon$ ,

$$\min_{y \in [-1, 1]} (\bar{\varphi}(y; \varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} +\infty,$$

and then, as  $\nu$  is continuous and  $\nu(0) > 0$ ,

$$\int_{\mathbb{R}} \phi_\varepsilon(y; \varepsilon) \nu(y) dy \xrightarrow{\varepsilon \rightarrow 0} -\infty,$$

and the proof of the main Lemma 1.3 .2 is complete.  $\square$

### 1.3 .2 Intersection of $\Gamma_1$ and $\Gamma_2$ , supersolution

**First case:**  $D > 2d$ . If  $D > 2d$ , we have of course  $\frac{c}{D} < \frac{c}{2d}, \forall c \geq c_{KPP}$ . Thus, for  $c$  close enough to  $c_{KPP}$ ,  $\Gamma_2$  does not intersect the closed convex hull of  $\Gamma_1$ . But since

$$\frac{c}{D} \xrightarrow{c \rightarrow +\infty} +\infty \text{ and } \lambda_2^-(c) \xrightarrow{c \rightarrow +\infty} 0^+,$$

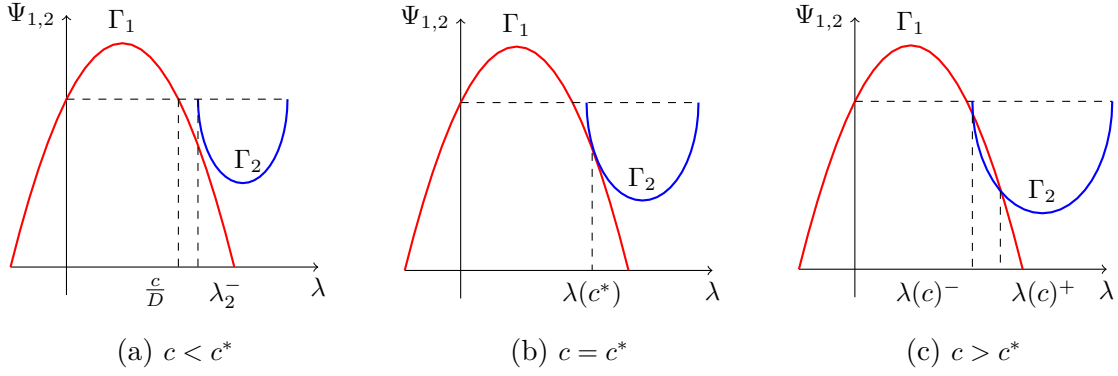
there exists

$$c_* = c_*(D) > c_{KPP}$$

such that  $\forall c > c_*$ ,  $\Gamma_1$  and  $\Gamma_2$  intersect, and  $\forall c < c_*$ ,  $\Gamma_2$  does not intersect the closed convex hull of  $\Gamma_1$ . Moreover, the strict concavity of  $\Gamma_1$  and the strict convexity of  $\Gamma_2$  allow us to assert that for  $c = c_*$ ,  $\Gamma_1$  and  $\Gamma_2$  are tangent on  $\lambda = \lambda(c_*)$  and for  $c > c_*$ ,  $c$  close to  $c_*$ ,  $\Gamma_1$  and  $\Gamma_2$  intersect twice, at  $\lambda(c)^+$  and  $\lambda(c)^-$ . The different situations are illustrated in fig. (1.3).

When  $c$  is such that  $\lambda_2^- \leq \frac{c}{D}$ , i.e.  $c \geq \frac{D}{2\sqrt{dD} - d^2} c_{KPP}$  there is only one solution for  $\lambda = \lambda(c)$ .




 Figure 1.3: Case  $D > 2d$  : intersection of  $\Gamma_1$  and  $\Gamma_2$ .

**Second case:**  $D = 2d$ . If  $D = 2d$ , then the point  $(\frac{c}{2d}, \bar{\mu})$  belongs to  $\Gamma_1$ . Therefore, for all  $c > c_{KPP}$ ,  $\Gamma_1$  and  $\Gamma_2$  intersect once at  $\lambda = \lambda(c)$ . We set:

$$c_*(2d) := c_{KPP}.$$

**Third case:**  $D < 2d$ . If  $D < 2d$ , we have  $\frac{c}{D} > \frac{c}{2d}$ . Then,  $\forall c > c_{KPP}$ ,  $\lambda_2^-(c) < \frac{c}{D}$ , the left part of  $\Gamma_2$  is strictly below  $\Gamma_1$ , and every  $c > c_{KPP}$  gives a super-solution. We set again:

$$c_*(D) := c_{KPP}.$$

All of this concludes the proof of Proposition 1.1 .3. Moreover, we can assert from geometrical considerations that

$$\frac{c_*}{D} \leq \frac{c_* - \sqrt{c_*^2 - c_{KPP}^2}}{2d} \leq \frac{c_* + \sqrt{c_*^2 + 4D\bar{\mu}}}{2D}. \quad (1.29)$$

It was proved in [22] that (1.29) implies that

$$\sqrt{4\bar{\mu}^2 + f'(0)^2} - 2\bar{\mu} \leq \liminf_{D \rightarrow +\infty} \frac{c_*^2}{D} \leq \limsup_{D \rightarrow +\infty} \frac{c_*^2}{D} \leq f'(0).$$

### 1.3 .3

Explicit computation of  $\Psi_2 := \Psi_2^0$  in the reference case (1.1) In the limit case, (1.14) can be written as follows, setting  $P(\lambda) = -d\lambda^2 + c\lambda - f'(0)$ :

$$-d\phi''(y) + (P(\lambda) + \bar{\nu}\delta_0) \phi(y) = \bar{\mu}\delta_0. \quad (1.30)$$

Thus, an explicit computation (see [22] or [86]) gives

$$\Psi_2^0(\lambda) := \bar{\nu}\phi(0) = \frac{\bar{\nu}\bar{\mu}}{\bar{\nu} + 2\sqrt{dP(\lambda)}}. \quad (1.31)$$

Notice that this function satisfies all properties given by Proposition 1.3 .1.

## 1.4 Spreading

In order to prove that solutions spread at least at speed  $c_*$ , we are looking for compactly supported general stationary subsolution in the moving framework at velocity  $c < c_*$ , arbitrarily close to  $c_*$ . We consider the linearised system penalised by  $\delta > 0$  in the moving framework :

$$\begin{cases} \partial_t u - D\partial_{xx}u + c\partial_x u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v + c\partial_x v = (f'(0) - \delta)v + \mu(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0. \end{cases} \quad (1.32)$$

The main result is here the following:

**Proposition 1.4 .1.** *Let  $c_* = c_*(D)$  be as in the previous section. Then, for  $c < c_*$  close enough to  $c_*$ , there exists  $\delta > 0$  such that (1.32) admits a nonnegative, compactly supported, generalised stationary subsolution  $(\underline{u}, \underline{v}) \not\equiv (0, 0)$ .*

As in the previous section, we will study separately the case  $D > 2d$ , which is the most interesting, and the case  $D \leq 2d$ .

### 1.4 .1 Construction of subsolutions: $D > 2d$

In order to keep the notation as light as possible, we will use the notation  $\tilde{f}'(0) := f'(0) - \delta$  and  $\tilde{P}(\lambda) := -d\lambda^2 + c\lambda - \tilde{f}'(0)$ , because all the results will perturb for small  $\delta > 0$ . We just have to keep in mind that  $\tilde{f}'(0) < f'(0)$  and  $\delta \ll 1$ , hence  $\tilde{P}(\lambda) > P(\lambda)$  and  $\tilde{P}(\lambda) - P(\lambda) \ll 1$ .

Our method is to devise a stationary solution of (1.32) not in  $\mathbb{R}^2$  anymore, but in the horizontal strip  $\Omega^L = \mathbb{R} \times (-L, L)$ , with  $L > 0$  large enough. Thus, we are solving

$$\begin{cases} -DU'' + cU' = -\bar{\mu}U + \int_{(-L, L)} \nu(y)V(x, y)dy & x \in \mathbb{R} \\ -d\Delta V + c\partial_x V = \tilde{f}'(0)V + \mu(y)U(x) - \nu(y)V(x, y) & (x, y) \in \Omega^L \\ V(x, L) = V(x, -L) = 0 & x \in \mathbb{R}. \end{cases} \quad (1.33)$$

In a similar fashion as in the previous section, we are looking for solutions of the form

$$\begin{pmatrix} U(x) \\ V(x, y) \end{pmatrix} = e^{\lambda x} \begin{pmatrix} 1 \\ \varphi(y) \end{pmatrix}, \quad (1.34)$$

where  $\varphi$  belongs to  $H_0^1(-L, L)$ . The system on  $(\lambda, \varphi)$  reads:

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \int_{(-L, L)} \nu(y)\phi(y)dy \\ -d\varphi''(y) + (\tilde{P}(\lambda) + \nu(y))\varphi(y) = \mu(y) \quad \varphi(-L) = \varphi(L) = 0. \end{cases} \quad (1.35)$$

The first equation of (1.35) gives a function  $\lambda \mapsto \Psi_1(\lambda; c) = -D\lambda^2 + \lambda c + \bar{\mu}$ . The second equation of (1.35) gives a unique solution  $\varphi = \varphi(y; \lambda, c; L) \in H_0^1(-L, L)$ . We associate this unique solution with the function  $\Psi_2^L(\lambda; c) = \int_{(-L, L)} \nu(y)\varphi(y)dy$ . A solution of the form (1.34) exists if and only if  $\Psi_1(\lambda; c) = \Psi_2^L(\lambda; c)$  for some  $\lambda, c$ , that is to say if and only if  $\Gamma_1$  and  $\Gamma_2^L$  intersect (with straightforward notations). In this section, the game is to make them intersect not with real but with complex  $\lambda$ .

**Study of  $\Gamma_1$**  The function  $\lambda \mapsto \Psi_1$  is exactly the same as in the search for supersolutions. In particular, it does not depend in  $L$ . Thus, the curve  $\Gamma_1$  is the same as in the previous section: it is a parabola, symmetric with respect to the line  $\{\lambda = \frac{c}{2D}\}$ . Notice that being a parabola, its curvature is positive at any point ; it will be important later.

**Study of  $\Gamma_2^L$**  The study of  $\Gamma_2^L$  is quite similar to that of  $\Gamma_2$ . It amounts to studying the solutions of

$$\begin{cases} -d\varphi''(y) + (\tilde{P}(\lambda) + \nu(y))\varphi(y) = \mu(y) & y \in (-L, L) \\ \varphi \in H_0^1(-L, L). \end{cases} \quad (1.36)$$

For real  $\lambda$ , (1.36) admits solution for  $\lambda \in [\lambda_{2,\delta}^-, \lambda_{2,\delta}^+]$ , with  $\lambda_{2,\delta}^\pm = \frac{c \pm \sqrt{c^2 - 4d\tilde{f}'(0)}}{2d}$ . We may notice that  $\lambda_{2,\delta}^- < \lambda_2^-$ ,  $\lambda_{2,\delta}^+ > \lambda_2^+$ , and of course  $\lambda_{2,\delta}^\pm \rightarrow \lambda_2^\pm$  as  $\delta \rightarrow 0$ . With a simple study of (1.36) and using what we proved in proposition (1.3 .1), we can assert :

$$\lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} \Psi_2^L(\lambda; c) = \lim_{\delta \rightarrow 0} \lim_{L \rightarrow \infty} \Psi_2^L(\lambda; c) = \Psi(\lambda; c), \quad (1.37)$$

and this uniformly on every compact set in  $] \lambda_2^-, \lambda_2^+ [ \times ] 2\sqrt{df'(0)}, +\infty [$ .

As a consequence, the picture is analogous to the case described in fig. (1.3): there exists a unique  $c_*^L$  (which depends on  $\delta$ ) such that  $\Gamma_2^L$  intersects  $\Gamma_1$  twice if  $c > c_*^L$ , close to  $c_*^L$ , once if  $c = c_*^L$ , and never if  $c < c_*^L$  (for real  $\lambda$ ).

Moreover, since  $\Gamma_2^L$  is below  $\Gamma_2$ , we have  $c_{KPP} < c_*^L < c_*$ . We also have  $c_*^L \rightarrow c_*$  as  $L \rightarrow \infty$ ,  $\delta \rightarrow 0$ .

**Complex solutions** We use the same argument as in [22]. Let us call  $\beta$  the ordinate of the plane  $(\lambda, \Psi_{1,2}^L)$ . For  $c = c_*^L$ , call  $(\lambda_*^L, \beta_*^L)$  the tangent point between  $\Gamma_1$  et  $\Gamma_2^L$ . The functions  $\Psi_1$  and  $\Psi_2^L$  are both analytical in  $\lambda$  at this point, and  $\frac{d}{d\lambda}\Psi_1(\lambda)$ ,  $\frac{d}{d\lambda}\Psi_2^L(\lambda) \neq 0$ , for  $(c, \lambda)$  in a neighbourhood of  $(c_*^L, \lambda_*^L)$ . Due to the implicit function theorem, there exist  $\lambda_1(c, \beta)$ ,  $\lambda_2^L(c, \beta)$  defined in a neighbourhood  $V_1$  of  $(c_*^L, \beta_*^L)$ , analytical in  $\beta$ , such that

$$\begin{cases} \Psi_1(\lambda_1(c, \beta); c) = \beta & \forall (c, \beta) \in V_1 \\ \Psi_2^L(\lambda_2^L(c, \beta); c) = \beta & \forall (c, \beta) \in V_1. \end{cases} \quad (1.38)$$

Then, set

$$h^L(c, \beta) = \lambda_2^L(c, \beta) - \lambda_1(c, \beta), \quad \text{for } (c, \beta) \in V_1,$$

and we get:

$$\begin{cases} \partial_\beta h^L(c_*^L, \beta_*^L) = 0. \\ \partial_{\beta\beta} h^L(c_*^L, \beta_*^L) := 2a > 0. \\ \partial_c h^L(c_*^L, \beta_*^L) := -e < 0. \end{cases} \quad (1.39)$$

The first point is obvious. The second comes from the fact that  $\Gamma_2^L$  is concave and  $\Gamma_1$  has a positive curvature at any point. The third is obvious given the first equation of (1.35).

Now, because we are working in a vicinity of  $(c_*^L, \beta_*^L)$ , set :

$$\xi := c_*^L - c, \quad \tau = \beta - \beta_*^L.$$

Call  $b := \partial_{c\beta} h^L(c_*^L, \beta_*^L)$ . From (1.38) and (1.39), we can assert that there exists a neighbourhood  $V_2 \subseteq V_1$  of  $(c_*^L, \beta_*^L)$ , there exists  $\eta = \eta(\tau, \xi)$  analytical in  $\tau$  in  $V_2 - (c_*^L, \beta_*^L)$ , vanishing at  $(0, 0)$  like  $|\tau|^3 + \xi^2$ , such that

$$(h^L(c, \beta) = 0, (c, \beta) \in V_2) \Leftrightarrow (a\tau^2 + b\xi\tau + e\xi = \eta(\tau, \xi)). \quad (1.40)$$

Recall that  $a$  and  $e$  are positive, so the discriminant  $\Delta = (b\xi)^2 - 4ae\xi$  is negative for  $\xi > 0$  small enough. The trinomial  $a\tau^2 + b\xi\tau + e\xi$  has two roots  $\tau_{\pm} = \frac{-b\xi \pm i\sqrt{4ea\xi - (b\xi)^2}}{2a}$ . Then, from an adaptation of Rouché's theorem (see [22]), the right handside of (1.40) has two roots, still called  $\tau_{\pm}$ , satisfying  $\tau_{\pm} = \pm i\sqrt{(e/a)\xi} + O(\xi)$ . Reverting to the full notation, we can see that for  $c$  strictly less than and close enough to  $c_*^L$ , there exist  $\beta, \lambda \in \mathbb{C}$ ,  $\varphi \in H_0^1((-L, L), \mathbb{C})$  satisfying (1.35). Since  $\beta = \Psi_1(\lambda) = -D\lambda^2 + c\lambda + \bar{\mu}$  and  $\beta$  has nonzero imaginary part,  $\lambda$  has also nonzero imaginary part. We can therefore write  $(\lambda, \beta) = (\lambda_1 + i\lambda_2, \beta_1 + i\beta_2)$  and:

$$\begin{pmatrix} U \\ V \end{pmatrix} = e^{(\lambda_1 + i\lambda_2)x} \begin{pmatrix} 1 \\ \varphi_1(y) + i\varphi_2(y) \end{pmatrix}$$

with

$$\begin{cases} \lambda_2, \beta_2 \neq 0 \\ \int \nu(y)\varphi_1(y)dy = \beta_1 = \beta_*^L + O(c_*^L - c) \\ \int \nu(y)\varphi_2(y)dy = \beta_2 = O(\sqrt{c_*^L - c}). \end{cases}$$

Thus :

- $\mathcal{R}e(U) > 0$  on  $(-\frac{\pi}{2\lambda_2}, \frac{\pi}{2\lambda_2})$  and vanishes at the ends ;
- $\mathcal{R}e(V) > 0 \Leftrightarrow \varphi_1 \cos(\lambda_2 x) > \varphi_2 \sin(\lambda_2 x)$ .

The set where  $\mathcal{R}e(V) > 0$  is periodic of period  $\frac{2\pi}{\lambda_2}$  in the direction of the road. Its connected components intersecting the strip  $\mathbb{R} \times (-L, L)$  are bounded. The function  $\varphi_2$  is continuous in  $c$ , hence the functions  $y \mapsto \varphi(y; c)$  are uniformly equicontinuous for  $c$  near  $c_*^L$ . Since  $\nu(0) > 0$  and  $\int \nu\varphi_2 = O(\sqrt{c_*^L - c})$ , we have  $\varphi_2(0) = O(\sqrt{c_*^L - c})$ , and we can make one of the connected components of  $\{\mathcal{R}e(V) > 0\}$ , denoted by  $F$ , satisfy the property that  $\{(x, 0) \in \bar{F}\}$  is arbitrary close to  $[-\frac{\pi}{2\lambda_2}, \frac{\pi}{2\lambda_2}]$ . We can now define the following functions:

$$\begin{aligned} \underline{u}(x) &:= \begin{cases} \max(\mathcal{R}e(U(x)), 0) & \text{if } |x| \leq \frac{\pi}{2\lambda_2} \\ 0 & \text{otherwise} \end{cases} \\ \underline{v}(x, y) &:= \begin{cases} \max(\mathcal{R}e(V(x, y)), 0) & \text{if } (x, y) \in \bar{F} \\ 0 & \text{otherwise} . \end{cases} \end{aligned} \quad (1.41)$$

The choice of  $F$  implies that  $(\underline{u}, \underline{v})$  is a subsolution of (1.32).

### 1.4.2 Subsolution: case $D \leq 2d$

Now assume that  $0 \leq D \leq 2d$ . In the previous section, we define  $c_*(D) = c_{KPP} = 2\sqrt{df'(0)}$ . Let  $c \leq c_{KPP}$ . Thus,  $4df'(0) - c^2 > 0$ . Let  $\delta$  be such that  $0 < 2\delta < \frac{4df'(0) - c^2}{4d} = f'(0) - \frac{c^2}{4d}$ .

With  $\omega = \frac{\sqrt{4d(f'(0)-2\delta)-c^2}}{2d}$ , we define

$$\phi(x) = e^{\frac{c}{2d}x} \cos(\omega x) \mathbf{1}_{(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})}.$$

The function  $\phi$  is continuous and satisfies

$$-d\phi'' + c\phi = (f'(0) - 2\delta)\phi \quad \text{on } \left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right).$$

Then, let us choose  $R > 0$  such that the first eigenvalue of  $-\partial_{yy}$  in  $(-R, R)$  is equal to  $\frac{\delta}{d} - \alpha$ , and  $\psi_R$  an associated nonnegative eigenfunction in  $H_0^1(-R, R)$ , where  $0 < \alpha < \delta$ . The function  $\psi_R$  satisfies

$$-d\psi_R'' = (\delta - \alpha)\psi_R \text{ in } (-R, R), \quad \psi_R(y) > 0, \quad \forall |y| < R, \quad \psi_R(R) = \psi_R(-R) = 0.$$

We extend  $\psi_R$  by 0 outside  $(-R, R)$ . Let  $M > 0$  such that  $\forall |y| > M - R$ ,  $\nu(y) \leq \alpha$ , which is possible since  $\nu(y) \rightarrow 0$  with  $y \rightarrow \pm\infty$ . The function

$$\underline{V}(x, y) := \phi(x)\psi_R(|y| - M)$$

is a solution of

$$\begin{cases} -d\Delta V + c\partial_x V = (f'(0) - \delta)V - \alpha V \\ x \in \left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right), \quad |y| \in (M - R, M + R), \end{cases}$$

vanishing on the boundary. Hence, from the choice of  $M$  and  $\alpha$ ,  $(0, \underline{V})$  is a nonnegative compactly supported subsolution of (1.32), non identically equal to  $(0, 0)$ ; which concludes the proof of Proposition 1.4 .1. The proof of the main Theorem 1.1 .2 follows as in [22].

## 1.5 The intermediate model (1.3)

**Formal derivation of the semi-limit model** Starting from the full model (1.2), we consider normal (i.e. integral) exchange from the field to the road but localised exchange from the road to the field. Formally, we define  $\mu_\varepsilon = \frac{1}{\varepsilon}\mu(\frac{y}{\varepsilon})$  and take the limit with  $\varepsilon \rightarrow 0$  of the system (1.42) :

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, \quad t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu_\varepsilon(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, \quad t > 0. \end{cases} \quad (1.42)$$

There is no influence in the first equation (the dynamic on the road), which is the same in the limit system. Though the second equation in (1.42) tends to

$$\partial_t v - d\Delta v = f(v) - \nu(y)v(t, x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}, \quad t > 0.$$

It remains to determine the limit condition between at the road. We may assume that for  $\varepsilon = 0$   $v$  is still continuous at  $y = 0$ . Now set  $\xi = y/\varepsilon$  and  $\tilde{v}(t, x, \xi) := v(t, x, y)$ . The second equation in (1.42) becomes in the  $(t, x, \xi)$ -variables

$$\varepsilon^2 (\partial_t \tilde{v} - d\partial_{xx} \tilde{v} - f(\tilde{v}) + \nu(\xi)\tilde{v}(t, x, \xi)) - d\partial_{\xi\xi} \tilde{v} = \varepsilon\mu(\xi)u(t, x).$$

Passing to the limit, it yields, in the  $y$ -variable:

$$-d \left( \partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-) \right) = \bar{\mu} u(t, x).$$

Consequently, the formal limit system of (1.42) should be (1.3) presented in the Introduction, which is the system we will study from now. Our assumptions on  $\nu$  and  $f$  are the same as above. The investigation is similar to the one done for the model (1.2), and we will only develop the parts which differ.

**Comparison principle** Throughout this section, we will call a supersolution of (1.3) a couple  $(\bar{u}, \bar{v})$  satisfying, in the classical sense, the following system:

$$\begin{cases} \partial_t \bar{u} - D\partial_{xx} \bar{u} \geq v(x, 0, t) - \bar{\mu} u + \nu(y)v(t, x, y) & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v \geq f(v) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d \{ \partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-) \} \geq \bar{\mu} u(t, x) & x \in \mathbb{R}, t > 0, \end{cases} \quad (1.43)$$

which is also continuous up to time 0. Similarly, we will call a subsolution of (1.3) a couple  $(\underline{u}, \underline{v})$  satisfying (1.43) with the inverse inequalities (*i.e.* the  $\geq$  signs replaced by  $\leq$ ). We now need a comparison principle in order to get monotonicity for solutions :

**Proposition 1.5 .1.** *Let  $(\underline{u}, \underline{v})$  and  $(\bar{u}, \bar{v})$  be respectively a subsolution bounded from above and a supersolution bounded from below of (1.3) satisfying  $\underline{u} \leq \bar{u}$  and  $\underline{v} \leq \bar{v}$  at  $t = 0$ . Then, either  $\underline{u} < \bar{u}$  and  $\underline{v} < \bar{v}$  for all  $t > 0$ , or there exists  $T > 0$  such that  $(\underline{u}, \underline{v}) = (\bar{u}, \bar{v})$ ,  $\forall t \leq T$ .*

We omit the proof.

**Long time behaviour and stationary solutions** We want to show that any (nonnegative) solution of (1.3) converges locally uniformly to a unique stationary solution  $(U_s, V_s)$ , which is bounded, positive,  $x$ -independent, and of course is solution of the stationary system of equations (1.44):

$$\begin{cases} -DU''(x) = -\bar{\mu}U + \int \nu(y)V(x, y) \\ d\Delta V(x, y) = f(V) - \nu(y)V(x, y) \\ V(x, 0^+) = V(x, 0^-) \\ -d \{ \partial_y V(x, 0^+) - \partial_y V(x, 0^-) \} = \bar{\mu}U(x). \end{cases} \quad (1.44)$$

Proofs of Propositions 1.2 .3 and 1.2 .4 can be easily adapted to this new system. The only nontrivial point lies in the existence of an  $L^\infty$  a priori estimate. Set  $\lambda = \frac{\bar{\nu}}{d}$ . From conditions on the reaction term, there exists  $M_1$  such that  $\forall s > M_1$ ,  $f(s) < -\frac{\bar{\nu}^2}{d}s$ . Now, set

$$M = \max(M_1, \frac{\bar{\nu}}{\bar{\mu}} \|u_0\|_\infty, \|v_0\|_\infty)$$

and the couple  $(\bar{U}, \bar{V})$  given by

$$\bar{V}(y) = M(1 + e^{-\lambda|y|}), \quad \bar{U} = \frac{1}{\bar{\mu}} \int_{\mathbb{R}} \nu(y)\bar{V}(y) dy$$

is a supersolution of (1.44) which is above  $(u_0, v_0)$ .

The proof of the corresponding Proposition 1.1 .1 follows easily.

**Exponential solutions, spreading** We are looking for solutions of the linearised system:

$$\begin{cases} \partial_t u - D\partial_{xx}u = v(x, 0, t) - \bar{\mu}u + \nu(y)v(t, x, y) & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f'(0)v - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) & x \in \mathbb{R}, t > 0, \end{cases} \quad (1.45)$$

and these solutions will be looked for under the form

$$\begin{pmatrix} u(t, x) \\ v_1(t, x, y) \\ v_2(t, x, y) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi_1(y) \\ \phi_2(y) \end{pmatrix} \quad (1.46)$$

where  $\lambda, c$  are positive constants and  $\phi$  is a nonnegative function in  $H^1(\mathbb{R})$ , with  $v = v_1, \phi = \phi_1$  for  $y \geq 0$  and  $v = v_2, \phi = \phi_2$  for  $y \leq 0$ . The system in  $(\lambda, \phi)$  reads

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi_1''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_1(y) = 0 & y \geq 0. \\ -d\phi_2''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi_2(y) = 0 & y \leq 0. \\ \phi_1(0) = \phi_2(0) & i.e. \phi \text{ is continuous.} \\ -\phi_1'(0) + \phi_2'(0) = \frac{\bar{\mu}}{d}. \end{cases} \quad (1.47)$$

The study is exactly the same as in the third section. The only point which deserves some explanation is the well-posedness of (1.47). For  $M > 0$  let us consider  $\varphi_M$  the unique solution of

$$\begin{cases} -d\varphi_M''(y) + (P(\lambda) + \nu(y))\varphi_M(y) = 0 & y \in ]0, +\infty[ \\ \varphi_M(0) = M & \varphi_M \in H^1(\mathbb{R}^+). \end{cases} \quad (1.48)$$

Let us show the following lemma, which will prove the well-posedness of (1.47):

**Lemma 1.5 .2.** 1.  $M \mapsto \varphi_M'(0)$  is decreasing ;

2.  $\varphi_M'(0) \xrightarrow{M \rightarrow 0} 0$  ;

3.  $\varphi_M'(0) \xrightarrow{M \rightarrow +\infty} -\infty$ .

*Proof.* Let us consider  $M_1, M_2$  with  $0 < M_1 < M_2$ ,  $\varphi_{M_1}, \varphi_{M_2}$  the associated solutions of (1.48). The elliptic maximum principle yields  $0 < \varphi_{M_1}(y) < \varphi_{M_2}(y)$ ,  $\forall y \geq 0$  and Hopf's lemma gives  $0 > \varphi_{M_1}'(0) > \varphi_{M_2}'(0)$ , which proves the first point.

Then, if we integrate (1.48) we get

$$\varphi_M'(0) = -\frac{1}{d} \int_0^\infty (P(\lambda) + \nu(y))\varphi_M(y)dy.$$

Let us now consider  $\bar{\varphi}_M$  the (unique) solution of

$$\begin{cases} -d\bar{\varphi}_M''(y) + P(\lambda)\bar{\varphi}_M(y) = 0 & y \in ]0, +\infty[ \\ \bar{\varphi}_M(0) = M & \bar{\varphi}_M \in H^1(\mathbb{R}^+). \end{cases}$$

The function  $\bar{\varphi}_M$  is a supersolution of (1.48). Thus,  $\bar{\varphi}_M(y) \geq \varphi_M(y)$ ,  $\forall y \geq 0$ . Moreover we have an explicit expression for  $\bar{\varphi}_M$ :  $\bar{\varphi}_M(y) = M \exp(-\sqrt{\frac{P(\lambda)}{d}}y)$ . Hence,

$$0 \leq -\varphi'_M(0) \leq \frac{M}{d} \int_0^\infty (P(\lambda) + \nu(y)) e^{-\sqrt{\frac{P(\lambda)}{d}}y} dy$$

and

$$-\varphi'_M(0) \xrightarrow{M \rightarrow 0} 0$$

uniformly in  $\lambda$ , which proves the second point.

In the same way, the unique solution  $\underline{\varphi}$  of

$$\begin{cases} -d\underline{\varphi}''(y) + (P(\lambda) + \|\nu\|_\infty)\underline{\varphi}_M(y) = 0 & y \in ]0, +\infty[ \\ \underline{\varphi}_M(0) = M & \underline{\varphi}_M \in H^1(\mathbb{R}^+). \end{cases}$$

is a subsolution of (1.48), and  $\underline{\varphi}_M(y) \leq \varphi_M(y)$ ,  $\forall y \geq 0$ . Hence,

$$\begin{aligned} -\varphi'_M(0) &\geq \frac{1}{d} \int_0^\infty (P(\lambda) + \nu(y)) \underline{\varphi}_M(y) dy \\ &\geq \frac{M}{d} \int_0^\infty (P(\lambda) + \nu(y)) e^{-\sqrt{\frac{P(\lambda) + \|\nu\|_\infty}{d}}y} dy \\ -\varphi'_M(0) &\rightarrow +\infty \text{ as } M \rightarrow +\infty, \end{aligned}$$

which concludes the proof of Lemma 1.5 .2.  $\square$

The corresponding Proposition 1.1 .3 and Theorem 1.1 .2 follows as in the previous part.

## 1.6 The large diffusion limit $D \rightarrow +\infty$

The behaviour of the spreading speed  $c^*$  as  $D$  goes to  $+\infty$  has already been investigated in [22] for the initial model (1.1). It has been shown that there exists  $c_\infty > 0$  such that

$$\frac{c^*(D)}{\sqrt{D}} \xrightarrow{D \rightarrow +\infty} c_\infty.$$

In the following Proposition, we show the robustness of this result and extend it to the general cases (1.1)-(1.4). We also give an asymptotic behaviour as  $f'(0)$  tends to  $+\infty$ .

**Proposition 1.6 .1.** *Let us consider any of the systems (1.1)-(1.4) with fixed parameters  $d, \bar{\nu}, \bar{\mu}$ . Let  $c^*(D, f'(0))$  be the associated spreading speed given by Theorem 1.1 .2.*

1. There exists  $c_\infty$ ,  $\frac{c^*(D, f'(0))}{\sqrt{D}} \xrightarrow{D \rightarrow +\infty} c_\infty$ .

2.  $c_\infty$  satisfies  $c_\infty \underset{f'(0) \rightarrow +\infty}{\sim} \sqrt{f'(0)}$ .

That is, with  $D \rightarrow +\infty$  and  $f'(0) \rightarrow +\infty$ , we have  $c_0^* \sim \sqrt{f'(0)D}$ , i. e. half of the KPP spreading speed for a reaction-diffusion on the road.



**Proof of Proposition 1.6 .1** We prove the result for the nonlocal system (1.2), the other cases being similar. We set

$$\tilde{u}(t, x) = u(t, \sqrt{D}x), \quad \tilde{v}(t, x, y) = v(t, \sqrt{D}x, y).$$

The system in the rescaled variables becomes

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx} \tilde{u} = -\bar{\mu} \tilde{u} + \int \nu(y) \tilde{v}(t, x, y) dy \\ \partial_t \tilde{v} - d \left( \partial_{yy} \tilde{v} + \frac{1}{D} \partial_{xx} \tilde{v} \right) = f(\tilde{v}) + \mu(y) \tilde{u}(t, x) - \nu(y) \tilde{v}(t, x, y). \end{cases} \quad (1.49)$$

The  $(c, \lambda, \phi)$ -system associated to (1.49) is then

$$\begin{cases} \lambda c - \lambda^2 + \bar{\mu} = \int \nu \phi \\ -d\phi''(y) + \left( \lambda c - \frac{d}{D} \lambda^2 - f'(0) + \nu(y) \right) \phi(y) = \mu(y). \end{cases}$$

Hence we get that  $c^* = \sqrt{D}\tilde{c}$  where  $\tilde{c}$  is the first  $c$  such that the graphs of  $\tilde{\Psi}_1$  and  $\tilde{\Psi}_2$  intersect, where  $\tilde{\Psi}_1$  and  $\tilde{\Psi}_2$  are defined as follows:

$$\tilde{\Psi}_1 : \lambda \mapsto \lambda c - \lambda^2 + \bar{\mu}$$

and

$$\tilde{\Psi}_2 : \begin{cases} ]\tilde{\lambda}^-, \tilde{\lambda}^+[ & \longrightarrow \mathbb{R} \\ \lambda & \longmapsto \int \nu \phi \end{cases}$$

where  $\phi$  is the unique  $H^1$  solution of

$$-d\phi''(y) + \left( \lambda c - \frac{d}{D} \lambda^2 - f'(0) + \nu(y) \right) \phi(y) = \mu(y) \quad (1.50)$$

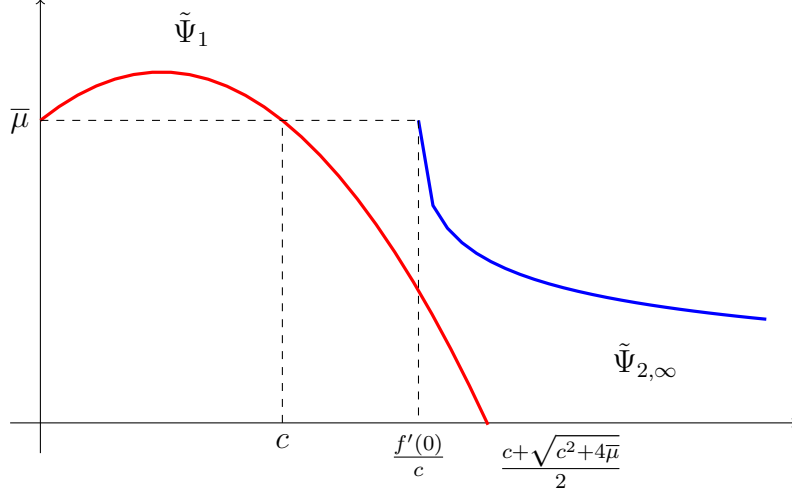
and  $\tilde{\lambda}^\pm = \frac{D}{2d} \left( c \pm \sqrt{c^2 - 4 \frac{df'(0)}{D}} \right)$ . We can see that as  $D$  tends to  $+\infty$ ,  $\tilde{\lambda}^- = \frac{f'(0)}{c} + o(1)$  and  $\tilde{\lambda}^+ \rightarrow +\infty$ . Behaviours of  $\tilde{\Psi}_{1,2}$  have already been studied above.  $\tilde{\Psi}_1$  is a concave parabola,  $\tilde{\Psi}_2$  is strictly convex, symmetric with respect to  $\{\lambda = \frac{cD}{2d}\}$ . Moreover, it has been showed that the solution  $\phi$  of (1.50) is bounded in  $L^\infty$ , uniformly in  $\lambda, c, D$ . It is also pointwise strictly decreasing in  $\left( \lambda c - \frac{d}{D} \lambda^2 - f'(0) \right)$ . Now, let  $\varphi$  be the  $H^1$  solution of the limit system defined for  $\lambda > \frac{f'(0)}{c}$

$$-d\varphi''(y) + (\lambda c - f'(0) + \nu(y)) \varphi(y) = \mu(y). \quad (1.51)$$

From the maximum principle and the monotonicity of  $\phi$  with respect to the nonlinear eigenvalue, we can easily see that  $\|\varphi - \phi\|_{L^\infty} \rightarrow 0$  as  $D \rightarrow \infty$ , locally uniformly in  $\lambda, c$ . Hence,  $\tilde{\Psi}_2$  tends to  $\tilde{\Psi}_{2,\infty}$  defined by

$$\tilde{\Psi}_{2,\infty} : \begin{cases} ]\frac{f'(0)}{c}, +\infty[ & \longrightarrow \mathbb{R} \\ \lambda & \longmapsto \int \nu \varphi \end{cases}$$

where  $\varphi$  is the unique solution of (1.51), and  $\tilde{c}$  tends to  $c_\infty$ , where  $c_\infty$  is the first  $c$  such that the graphs of  $\tilde{\Psi}_1$  and  $\tilde{\Psi}_{2,\infty}$  intersect. This concludes the proof of the first part of Proposition 1.6 .1.


 Figure 1.4: Curves in the limit case  $D \rightarrow \infty$ 

For the second part of Proposition 1.6 .1, we can see from geometric considerations (see figure 1.4) that  $c_\infty$  must satisfy

$$c_\infty \leq \frac{f'(0)}{c_\infty} \leq \frac{c_\infty + \sqrt{c_\infty^2 + 4\bar{\mu}}}{2}. \quad (1.52)$$

Passing to the limit  $f'(0) \rightarrow +\infty$  in (1.52) yields the expected result.  $\square$

## 1.7 Enhancement of the spreading speed in the semi-limit case (1.4)

This section is devoted to the semi-limit model (1.4) and the proof of Proposition 1.1 .4. For  $\bar{\mu} > 0$ , let

$$\Lambda_{\bar{\mu}} = \{\mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \bar{\mu}, \mu \text{ is even}\}.$$

Now, for fixed constants  $d, D, \bar{\nu}, f'(0)$ , for any function  $\mu \in \Lambda_{\bar{\mu}}$ , let  $c^*(\mu)$  be the spreading speed associated to the semi-limit system (1.4) with exchange function from the road to the field  $\mu$ . Let  $c_0^*$  the spreading speed associated with the limit system (1.1) with the same parameters and exchange rate from the road to the field  $\bar{\mu}$ .

**Proof of Proposition 1.1 .4** If  $D \leq 2d$ , then for all systems,  $c^* = 2\sqrt{df'(0)} = c_K$  and the result is obvious. We consider only the case  $D > 2d$ . Let  $c > 2\sqrt{df'(0)}$ ,  $\lambda_2^\pm = \frac{c \pm \sqrt{c^2 - c_K^2}}{2d}$ . Then, for all  $\lambda \in ]\lambda_2^-, \lambda_2^+[$ , the  $(c, \lambda, \phi)$ -equation (1.14) associated to the semi-limit system (1.4) can be written as follows:

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 + f'(0))\phi(y) = \mu(y) & y > 0 \\ -2d\phi'(0) = -\bar{\nu}\phi(0), & \phi \in H^1(\mathbb{R}^+). \end{cases} \quad (1.53)$$

We keep in mind that we are interested in the behaviour of

$$\Psi_2(\lambda; \mu) := \bar{\nu}\phi(0)$$

where  $\phi$  is the unique solution of (1.53). For the sake of simplicity, we set

$$P(\lambda) = \lambda c - d\lambda^2 - f'(0), \quad \alpha^2 = \frac{P(\lambda)}{d}.$$

From the variation of constants method and the boundary conditions in 0 and  $+\infty$  we have

$$\phi(y) = e^{\alpha y} \left( K_1 - \frac{1}{2\alpha} \int_0^y e^{-\alpha z} \frac{\mu(z)}{d} dz \right) + e^{-\alpha y} \left( K_2 + \frac{1}{2\alpha} \int_0^y e^{\alpha z} \frac{\mu(z)}{d} dz \right).$$

where

$$K_1 = \frac{1}{2\alpha} \int_0^\infty e^{-\alpha z} \frac{\mu(z)}{d} dz, \quad K_2 = \frac{2\alpha d - \bar{\nu}}{2\alpha(2\alpha d + \bar{\nu})} \int_0^\infty e^{-\alpha z} \frac{\mu(z)}{d} dz.$$

We finally get, returning in the  $(\lambda, c)$ -variables,

$$\Psi_2(\lambda; \mu) := \bar{\nu}\phi(0) = \frac{2\bar{\nu}}{\bar{\nu} + 2\sqrt{dP(\lambda)}} \int_0^\infty e^{-\sqrt{\frac{P(\lambda)}{d}}z} \mu(z) dz. \quad (1.54)$$

Now, since  $e^{-\sqrt{\frac{P(\lambda)}{d}}z} \leq 1$  for all  $z \geq 0$  and  $\mu$  being nonnegative and even, it is easily seen that

$$\Psi_2(\lambda; \mu) \leq \Psi_2^0(\lambda; \bar{\mu}) \quad (1.55)$$

where  $\Psi_2^0$  is given by the limit model (1.1) associated to the same constants and exchange term  $\bar{\mu}$ . Hence, the above inequality (1.55) allows us to assert that

$$\forall \mu \in \Lambda_{\bar{\mu}}, \quad c^*(\mu) \leq c_0^*.$$

Then, stating  $c = c_0^*$ , let us consider any approximation to the identity sequence in (1.54). For any  $\mu \in \Lambda_{\bar{\mu}}, \varepsilon > 0$ , set  $\mu_\varepsilon(y) = \frac{1}{\varepsilon} \mu\left(\frac{y}{\varepsilon}\right)$ . Then we get that  $\Psi_2(\lambda; \mu_\varepsilon)$  converges to  $\Psi_2^0(\lambda; \bar{\mu})$  as  $\varepsilon$  goes to 0, uniformly in any compact set in  $]\lambda_2^-, \lambda_2^+[$  in  $\lambda$ . Hence,

$$c^*(\mu_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} c_0^*$$

and the proof of Proposition 1.1 .4 is concluded.  $\square$

## 1.8 Self-similar exchanges for the semi-limit case (1.3)

Considering the above result, it may seem natural that in the opposite case (1.3), that is when exchanges from the road to the field are localised on the road, the spreading speed would also be maximum for localised exchange from the field to the road. In order to compare the spreading speed associated to the initial model (1.1) and the one given by an integral model (1.3), it is first natural to look for the behaviour of the spreading speed when replacing the exchange function  $\nu$  by a self-similar approximation of a Dirac mass

$\frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right)$ . Hence, for a fixed constant rate  $\bar{\nu}$ , we will consider an exchange function of the form

$$\nu \in \Lambda_{\bar{\nu}} := \{\nu \in C_0(\mathbb{R}), \nu \geq 0, \int \nu = \bar{\nu}, \nu \text{ is even}\}.$$

For fixed constant  $f'(0), d, D, \bar{\mu}$ , and  $\nu \in \Lambda_{\bar{\nu}}$  let  $c_0^*$  be the spreading speed associated to the limit system (1.1), and  $c^*(\varepsilon)$  the spreading speed associated to the semilimit model (1.3) with exchange term

$$\nu_\varepsilon : y \mapsto \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right).$$

The  $(c, \lambda, \phi)$ -equation (1.14) associated is

$$\begin{cases} -d\phi''(y) + (P(\lambda) + \nu_\varepsilon(y))\phi(y) = \bar{\mu}\delta_0 \\ \phi \in H^1(\mathbb{R}), \phi \text{ is continuous.} \end{cases} \quad (1.56)$$

The  $\Psi_2$  function is given by

$$\Psi_2(\lambda, c; \varepsilon) = \int_{\mathbb{R}} \nu_\varepsilon(y)\phi(y)dy$$

where  $\phi$  is the unique solution of (1.56) and  $P(\lambda) = \lambda c - d\lambda^2 - f'(0)$ . An integration of (1.56) yields the following expression for  $\Psi_2$

$$\Psi_2(\lambda, c; \varepsilon) = \bar{\mu} - P(\lambda) \int_{\mathbb{R}} \phi(y; \lambda, c, \varepsilon)dy \quad (1.57)$$

from which we get the next proposition.

**Proposition 1.8 .1.** *The function  $\Psi_2$ , defined by (1.57) and (1.56), is continuously differentiable in all variables  $\lambda, c, \varepsilon$  up to  $\varepsilon = 0$  and satisfies for all  $\lambda, c$*

$$\frac{d}{d\varepsilon}\Psi_2|_{\varepsilon=0} > 0.$$

Considering the monotonicity of  $\Psi_2$  with respect to  $c$ , this provides the corollary

**Corollary 1.8 .2.** *Let us consider  $c^*$  as a function of the  $\varepsilon$  variable. Then there exists  $\varepsilon_0$ ,*

$$\forall \varepsilon < \varepsilon_0, c^*(\varepsilon) > c_0^*$$

*In other words, the Dirac mass is a local minimizer for the spreading speed when considering approximation of Dirac functions.*

**Proof of Proposition 1.8 .1** Throughout the proof, the function  $\phi$ , depending on  $\varepsilon$ , will be the solution of (1.56), and we will denote

$$\varphi := \frac{d}{d\varepsilon}\phi$$

its derivative with respect to  $\varepsilon$ . Moreover, once again, we set  $d = 1$  for the sake of simplicity, and consider an exchange function  $\nu$  with support in  $[-1, 1]$ . Differentiating (1.56) we obtain that  $\varphi$  is the unique solution of

$$\begin{cases} -\varphi''(y) + \left(P(\lambda + \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right))\right)\varphi(y) = \frac{1}{\varepsilon^2}g\left(\frac{y}{\varepsilon}\right)\phi(y) \\ \varphi \in H^1(\mathbb{R}) \end{cases} \quad (1.58)$$

where the function  $g$ , with compact support in  $[-1, 1]$ , is defined by

$$g : z \mapsto \frac{d}{dz}[z \cdot \nu(z)]. \quad (1.59)$$

Thanks to (1.57), it is enough to prove that  $\varphi$  tends to a negative function as  $\varepsilon$  goes to 0, uniformly  $L^1$  near  $\varepsilon = 0$ . The proof is divided in four steps. We first recall the convergence of  $\phi$  as  $\varepsilon$  goes to 0. Then, the most important step is the convergence of the righthandside of (1.58) to a Dirac measure of negative mass. The third step is to find some uniform boundedness for the sequence  $(\varphi)_\varepsilon$ , in order to finally pass to the limit and conclude the proof.

**Convergence of  $\phi$**  It has been proved in [86] that  $\phi$  converges in the  $C^1$  norm to

$$\phi_0 : y \mapsto \frac{\bar{\mu}}{\bar{\nu} + 2\sqrt{P(\lambda)}} e^{-\sqrt{P(\lambda)}|y|} \quad (1.60)$$

as  $\varepsilon$  goes to 0, and this convergence is locally uniform in  $\lambda, c$ . Actually, the monotonicity of  $\phi$  with respect to  $y$  makes the proof easier.

**Convergence of the righthandside of (1.58) to a Dirac mass** As its support shrinks to 0, and thanks to the regularity of  $g$  and  $\phi$  uniformly in  $\varepsilon$ , it is enough to prove the convergence of the mass to get the convergence in the sense of distribution. Let us consider the integral

$$I(\varepsilon) := \int_{\mathbb{R}} \frac{1}{\varepsilon^2} g\left(\frac{y}{\varepsilon}\right) \phi(y; \varepsilon) dy.$$

Evenness of  $g$  and  $\phi$ , compact support of  $g$ , and a Taylor formula yield

$$\frac{1}{2}I(\varepsilon) = \frac{1}{\varepsilon} \int_0^1 g(z) \left( \phi(0) + \varepsilon z \phi'(0) + \int_0^{\varepsilon z} (\varepsilon z - t) \phi''(t) dt \right) dz. \quad (1.61)$$

Recall that  $g$  is defined by (1.59), so  $\int_0^1 g(z) dz = 0$ . An integration by parts gives

$$\int_0^1 z \cdot g(z) dz = - \int_0^1 z \cdot \nu(z) dz > 0.$$

It remains to determine the last term in (1.61) given by

$$I_2 = \frac{1}{\varepsilon} \int_0^1 g(z) \int_0^{\varepsilon z} (\varepsilon z - t) \phi''(t) dt dz.$$

Recall that  $\phi$  is a solution of (1.56) and we get locally in  $\lambda, c$

$$\begin{aligned} I_2 &= \varepsilon \int_0^1 g(z) \int_0^z (z - u) \left( P(\lambda) + \frac{1}{\varepsilon} \nu(u) \right) \phi(\varepsilon u) du dz \\ &= \int_0^1 g(z) \int_0^z (z - u) \nu(u) \phi(\varepsilon u) du dz + O(\varepsilon). \end{aligned} \quad (1.62)$$

With the uniform boundedness of  $\|\phi'\|_{L^\infty}$  in  $\varepsilon$ , (1.62) becomes

$$\begin{aligned} I_2 &= \phi(0) \left( \int_0^1 z.g(z) \int_0^z \nu(u) dudz - \int_0^1 g(z) \int_0^z u.\nu(u) dudz \right) + O(\varepsilon) \\ &= -\phi(0) \int_0^1 z\nu(z) \int_0^z \nu(u) dudz + O(\varepsilon). \end{aligned} \quad (1.63)$$

Inserting (1.63) in (1.61), and with the convergence together to its derivative of  $\phi$  to  $\phi_0$  defined by (1.60), we get, as  $\varepsilon$  tends to 0,

$$I \xrightarrow{\varepsilon \rightarrow 0} \bar{\mu} \int_0^1 \left( \frac{1}{\bar{\nu} + 2\sqrt{P(\lambda)}} \int_{-z}^z \nu(u) du - 1 \right) z.\nu(z) dz := I_0. \quad (1.64)$$

We notice that  $I_0 < 0$ .

**Uniform boundedness for  $\|\varphi\|_{L^\infty}$**  Once again, let us set  $\alpha^2 := P(\lambda)$ . As  $g$  is compactly supported and even, for all  $\varepsilon > 0$ , there exists  $K(\varepsilon)$  such that

$$\forall |y| > \varepsilon, \quad \varphi(y) = K(\varepsilon)e^{-\alpha|y|}. \quad (1.65)$$

We do the change of variable  $\xi = \frac{y}{\varepsilon}$  and use the same notation  $\varphi(\xi) = \varphi(y)$  for the sake of simplicity. Equation (1.56) becomes in the  $\xi$ -variable

$$\begin{cases} -\varphi''(\xi) + (\varepsilon^2\alpha^2 + \varepsilon\nu(\xi))\varphi(\xi) = g(\xi)\phi(\varepsilon\xi) \\ \varphi(\pm 1) = K(\varepsilon)e^{-\alpha\varepsilon}. \end{cases} \quad (1.66)$$

As for Theorem 1.1 .5, let us set

$$\varphi(\xi) = \varphi_0(\xi) + \varepsilon\varphi_1(\xi)$$

where  $\varphi_0$  is the unique solution of

$$\begin{cases} -\varphi_0''(\xi) = g(\xi)\phi(\varepsilon\xi) \\ \varphi_0(\pm 1) = K(\varepsilon)e^{-\alpha\varepsilon}. \end{cases}$$

This yields the following explicit formula for  $\varphi_0$

$$\varphi_0(\xi) = K(\varepsilon)e^{-\alpha\varepsilon} - \int_{-1}^{\xi} \int_0^z g(u)\phi(\varepsilon u) dudz. \quad (1.67)$$

Now we introduce the operator

$$\mathcal{L} : \begin{cases} X & \longrightarrow X \\ \psi & \longmapsto \left\{ \xi \mapsto \int_{-1}^{\xi} \int_0^z (\varepsilon\alpha^2 + \nu(u)) \psi(u) dudz \right\} \end{cases}$$

where  $X = \{\psi \in C^1(-1, 1), \psi \text{ is even}\}$  endowed with the  $C^1$  norm.  $\mathcal{L}$  is obviously a bounded operator and  $\varphi_1$  satisfies

$$(I - \varepsilon\mathcal{L})\varphi_1 = \mathcal{L}\varphi_0.$$

Hence there exists a constant  $C$ , for  $\varepsilon$  small enough,  $\|\varphi_1\|_{C^1(-1,1)} \leq C \|\varphi_0\|_{C^1(-1,1)}$ . We also have the integral equation for  $\varphi_1$

$$\varphi_1(\xi) = \int_{-1}^{\xi} \int_0^z (\varepsilon\alpha^2 + \nu(u)) (\varphi_0 + \varepsilon\varphi_1)(u) dudz.$$

The continuity of the derivative in 1 gives

$$\varphi_0'(1) + \varepsilon\varphi_1'(1) = -\varepsilon\alpha K(\varepsilon)e^{-\alpha\varepsilon}. \quad (1.68)$$

The computation done in the previous paragraph yields:

$$\begin{aligned} \varphi_0'(1) &= - \int_0^1 g(u)\phi(\varepsilon u)du \\ \varphi_0'(1) &= -\varepsilon\frac{1}{2}I_0 + o(\varepsilon) \end{aligned} \quad (1.69)$$

where  $I_0$  is defined by (1.64). Using the integral equation for  $\varphi_1$ , the previous domination, (1.67) and at last the convergence of  $\phi$  as  $\varepsilon$  goes to 0, there exists a constant  $M$  such that

$$\begin{aligned} \varphi_0'(1) &= \int_0^1 (\varepsilon\alpha^2 + \nu(u)) (\varphi_0 + \varepsilon\varphi_1)(u)du \\ &= \frac{\bar{\nu}}{2}K(\varepsilon) - \int_0^1 \nu(\xi) \int_{-1}^{\xi} \int_0^z g(u)\phi(\varepsilon u)dudzd\xi + O(\varepsilon.K(\varepsilon)) \\ \varphi_0'(1) &= \frac{\bar{\nu}}{2}K(\varepsilon) - M + O(\varepsilon(1 + K(\varepsilon))). \end{aligned} \quad (1.70)$$

Insert (1.69) and (1.70) in (1.68) and we get

$$\limsup_{\varepsilon \rightarrow 0} |K(\varepsilon)| < +\infty, \quad (1.71)$$

which provides with (1.67) and (1.65) the boundedness of  $\|\varphi\|_{L^\infty(\mathbb{R})}$  as  $\varepsilon$  goes to 0. Moreover, the bound is locally uniform on  $\lambda, c$ .

**Convergence of  $\varphi$ , conclusion of the proof** We return to the initial variable. Let  $K_0$  be any limit point of  $(K(\varepsilon))_\varepsilon$ . Then a subsequence of  $(\varphi)_\varepsilon$  converges in the sense of distributions to

$$\varphi_l(y) = K_0 e^{-\sqrt{P(\lambda)}|y|}$$

and  $\varphi_l$  satisfies in the sense of distributions

$$-\varphi_l''(y) + (P(\lambda) + \bar{\nu}\delta_0)\varphi_l(y) = I_0\delta_0$$

whose unique solution is

$$\varphi_l : y \longmapsto \frac{I_0}{\bar{\nu} + 2\sqrt{P(\lambda)}} e^{-\sqrt{P(\lambda)}|y|}. \quad (1.72)$$

Being the only possible limit point, (1.72) is the limit of  $(\varphi)$  as  $\varepsilon$  goes to 0.  $I_0$  is negative, and so is  $\varphi_l$ . The uniform boundedness allows the derivation in (1.57), and the proof is concluded.  $\square$

### 1.8 .1 Proof of Corollary 1.8 .2

Let  $c_0^*$  the spreading speed associated to the limit model (1.1),  $(c_0^*, \lambda_0^*, \phi_0(c_0^*, \lambda_0^*))$  the corresponding linear traveling wave. We consider  $\Psi_2$  as a function of  $(\lambda, c; \varepsilon)$ ,  $\Psi_1$  as a function of  $(\lambda, c)$ . We have  $\Psi_1(\lambda_0^*, c_0^*) = \Psi_2(\lambda_0^*, c_0^*; 0)$  and  $\Psi_1(\lambda_2^-, c_0^*) < \Psi_2(\lambda_2^-, c_0^*; 0)$ . Hence, there exists  $\bar{\lambda}$ ,  $\lambda_2^- < \bar{\lambda} < \lambda_0^*$ ,  $\Psi_2(\bar{\lambda}, c_0^*; 0) = \Psi_1(\lambda_2^-, c_0^*)$ . Let  $V$  be any open set in  $]\lambda_2^-, \lambda_0^*[$  containing  $\bar{\lambda}$  and  $\lambda_0^*$ . From Proposition 1.8 .1, there exists  $\varepsilon_0$  such that  $\forall \varepsilon < \varepsilon_0, \forall \lambda \in V$ ,  $\Psi_2(\lambda, c_0^*; \varepsilon) > \Psi_2(\lambda, c_0^*; 0)$ . From the definition of  $\bar{\lambda}$ , it yields  $\Psi_2(\lambda, c_0^*; \varepsilon) > \Psi_1(\lambda, c_0^*), \forall \lambda$ . The monotonicity of  $\Psi_1$  and  $\Psi_2$  with respect to  $c$  concludes the proof.  $\square$

## 1.9 The semi-limit case (1.3): non optimality of concentrated exchanges

Considering the above result, it may seem natural that in the case (1.3), that is when exchanges from the road to the field are localised on the road, the spreading speed would be minimal for localised exchange from the field to the road. The purpose of this section is the proof of Theorem 1.1 .5 in which we show that any behaviour may happen in the neighbourhood of a Dirac measure. For the sake of convenience, throughout this section we set

$$d = \bar{v} = 1.$$

Let us recall that we consider exchange terms  $\nu$  of the form

$$\nu(y) = (1 - \varepsilon)\delta_0 + \varepsilon\nu(y) \quad (1.73)$$

where

$$v \in \Lambda_1 := \{v \in C_0(\mathbb{R}), v \geq 0, \int v = 1, v \text{ is even}\}.$$

Let  $c_0^*$  the spreading speed associated to the limit model (1.1),  $(c_0^*, \lambda_0^*, \phi_0(c_0^*, \lambda_0^*))$  the corresponding linear traveling waves. The  $(c, \lambda, \phi)$ -equation associated to the system (1.3) with exchange term of the form (1.7) is as follows, completed by evenness:

$$\begin{cases} -\phi''(y) + (-f'(0) + \lambda c - \lambda^2 + v(y))\phi(y) = 0 & y > 0 \\ \phi'(0) = \frac{1}{2}((1 - \varepsilon)\phi(0) - \bar{\mu}), & \phi \in H^1(\mathbb{R}^+). \end{cases} \quad (1.74)$$

The associated function  $\Psi_2$  is given by

$$\Psi_2(\lambda, c) = (1 - \varepsilon)\phi(0) + \varepsilon \int_{\mathbb{R}} v\phi \quad (1.75)$$

where  $\phi$  is the unique solution of (1.74). What we have to show is that, in a neighbourhood of  $(\lambda_0^*, c_0^*)$ , the difference  $(\Psi_2^0(\lambda, c) - \Psi_2(\lambda, c))$  is of constant sign for  $\varepsilon$  small enough, and that this sign can be different depending on the parameters  $D, \bar{\mu}, f'(0)$ . Once again, for the sake of simplicity and as long as there is no possible confusion, we set

$$P(\lambda) = -f'(0) + \lambda c - \lambda^2, \quad \alpha = \sqrt{P(\lambda)}.$$

Of course, we are looking for function  $\phi$  of the form

$$\phi = \phi_0 + \varepsilon\phi_1 \quad (1.76)$$



where  $\phi_0$  is solution of (1.30). Hence,  $\phi_1$  satisfies

$$-\phi_1'' + (\delta_0 + \alpha^2)\phi_1 = (\delta_0 - v)(\phi_0 + \varepsilon\phi_1). \quad (1.77)$$

**Lemma 1.9 .1.** *Let  $\alpha_0 > 0$ . There exist  $\varepsilon_0 > 0$ ,  $K > 0$ , depending only on  $\alpha_0$ , such that  $\forall \varepsilon < \varepsilon_0$ ,*

$$\|\phi_1\|_{L^\infty} \leq K \|\phi_0\|_{L^\infty}$$

where  $\phi_1$  is the solution of (1.77). We may also keep in mind that  $\|\phi_0\|_{L^\infty} = \phi_0(0)$ . We can see in (1.31) that it is uniformly bounded in  $\alpha, D, f'(0)$ .

*Proof.* We introduce the operator

$$\mathcal{L} : \begin{cases} X & \longrightarrow X \\ \psi & \longmapsto \varphi \end{cases}$$

where  $X = \{\psi \in BUC(\mathbb{R}), \psi \text{ is even}\}$  and  $\varphi$  is the only bounded solution of

$$-\varphi'' + (\alpha^2 + \delta_0)\varphi = (\delta_0 - v)\psi. \quad (1.78)$$

From (1.77), it is easy to see that  $\phi_1$  satisfies  $\phi_1 = \mathcal{L}\phi_0 + \varepsilon\mathcal{L}\phi_1$ . As  $v$  and  $\phi_0$  are even, we focus on  $\mathcal{L}$  defined for bounded, uniformly continuous even functions. Let  $\psi \in BUC(\mathbb{R})$  be any even function, and  $\varphi := \mathcal{L}\psi$ . That is,  $\varphi$  satisfies

$$\begin{cases} -\varphi'' + \alpha^2\varphi = -v\psi & y > 0 \\ \varphi'(0) = \frac{1}{2}(\varphi(0) - \psi(0)). \end{cases} \quad (1.79)$$

As in the previous section, a simple computation gives

$$\begin{aligned} \varphi(y) = & -\frac{e^{\alpha y}}{2\alpha} \int_y^\infty e^{-\alpha z} (v\psi)(z) dz \\ & + e^{-\alpha y} \left( \frac{\psi(0)}{1+2\alpha} + \frac{1-2\alpha}{2\alpha(1+2\alpha)} \int_0^\infty e^{-\alpha z} (v\psi)(z) dz - \frac{1}{2\alpha} \int_0^y e^{\alpha z} (v\psi)(z) dz \right). \end{aligned} \quad (1.80)$$

Recall that  $v$  is nonnegative and of weight 1, and  $\alpha = \sqrt{P(\lambda)} > 0$ . A rough majoration in (1.80) yields

$$\|\varphi\|_{L^\infty} \leq \|\psi\|_{L^\infty} \left( \frac{1}{1+2\alpha} + \frac{|1-2\alpha|}{4\alpha(1+2\alpha)} + \frac{1}{4\alpha} \right). \quad (1.81)$$

Hence  $\mathcal{L}$  is a bounded linear operator, with norm  $\|\mathcal{L}\|$  depending on  $\alpha$ , and uniformly bounded on  $\alpha > \alpha_0 > 0$ . For  $\varepsilon$  small enough,  $(I - \varepsilon\mathcal{L})$  is invertible with bounded inverse and

$$\phi_1 = (I - \varepsilon\mathcal{L})^{-1} \mathcal{L}\phi_0. \quad (1.82)$$

Moreover,  $\phi_1$  satisfies the integral equation  $\phi_1 = \mathcal{L}(\phi_0 + \varepsilon\phi_1)$  given by (1.80). Combining (1.82) with (1.81) concludes the proof of Lemma 1.9 .1.  $\square$

**The difference  $\Psi_2^0 - \Psi_2$**  The function  $\Psi_2$  is given by (1.75) with  $\phi$  of the form (1.76). Then, using Lemma 1.9 .1, for all  $\alpha > \alpha_0$ ,

$$\begin{aligned}\Psi_2^0 - \Psi_2 &= \phi_0(0) - (1 - \varepsilon)(\phi_0(0) + \varepsilon\phi_1(0)) - \varepsilon \int_{\mathbb{R}} (\phi_0 + \varepsilon\phi_1) v \\ &= \varepsilon \left( \phi_0(0) - \phi_1(0) - \int_{\mathbb{R}} v\phi_0 \right) + o(\varepsilon).\end{aligned}\tag{1.83}$$

It appears necessary to compute  $\phi_1(0)$ . Equation (1.80) gives

$$\begin{aligned}\phi_1(0) &= \left( \frac{1 - 2\alpha}{2\alpha(1 + 2\alpha)} - \frac{1}{2\alpha} \right) \int_{\mathbb{R}} e^{-\alpha y} v(y) (\phi_0(y) + \varepsilon\phi_1(y)) + \frac{1}{1 + 2\alpha} (\phi_0(0) + \varepsilon\phi_1(0)) \\ &= \frac{\phi_0(0)}{1 + 2\alpha} - \frac{2}{1 + 2\alpha} \int_0^\infty e^{-\alpha y} v(y) \phi_0(y) dy + O(\varepsilon) \\ &= \frac{\phi_0(0)}{1 + 2\alpha} \left( 1 - \int_{\mathbb{R}} e^{-2\alpha|y|} v(y) dy \right) + O(\varepsilon).\end{aligned}\tag{1.84}$$

Now recall that  $v$  is of mass 1 and, using (1.84) in (1.83),

$$\begin{aligned}\Psi_2^0 - \Psi_2 &= \varepsilon\phi_0(0) \int_{\mathbb{R}} v(y) \left( 1 - e^{-\alpha|y|} - \frac{1}{1 + 2\alpha} (1 - e^{-2\alpha|y|}) \right) dy + o(\varepsilon) \\ &= \varepsilon\phi_0(0) \int_{\mathbb{R}} v(y) g(\alpha, y) dy + o(\varepsilon).\end{aligned}\tag{1.85}$$

The function  $g$  is obviously even in  $y$ , and smooth on  $\mathbb{R}_*^{+2}$ . We can easily see that:

- if  $\alpha \geq \frac{1}{2}$ , then  $\forall y > 0$ ,  $g(\alpha, y) > 0$ .
- If  $\alpha < \frac{1}{2}$ , then there exists  $y(\alpha)$  such that, in a neighbourhood of  $\alpha$ ,  $\forall |y| < y(\alpha)$ ,  $g(\cdot, y) < 0$ .

We are interested in the local behaviour near  $(\lambda^*, c_0^*)$ . Hence,  $g(\alpha, y)$  has to be considered near  $\alpha^* := \sqrt{-f'(0) + c_0^* \lambda^* - \lambda^{*2}} = \sqrt{P(\lambda^*)}$ .

**Perturbation enhancing the velocity:**  $\alpha^* < 1/2$   $P$  achieves its maximum at  $\lambda = \frac{c}{2}$  and  $c \mapsto P(\frac{c}{2})$  is nondecreasing. From [22] we know that  $c_0^*$  satisfies

$$\frac{c_0^*}{D} \leq \frac{c_0^* - \sqrt{c_0^{*2} - c_K^2}}{2}$$

where  $c_K = 2\sqrt{f'(0)}$  is the classical spreading speed for KPP-type reaction-diffusion. It follows easily that  $c_0^* \leq \frac{D\sqrt{f'(0)}}{\sqrt{D-1}}$  which, combined with the two upper remarks, yields the following sufficient condition for  $\alpha^* = \sqrt{P(\lambda^*)}$  to be less than  $1/2$ :

$$D < 2 + \frac{1}{2f'(0)} + \frac{1}{2} \sqrt{12 + \left(\frac{1}{f'(0)}\right)^2 + \frac{7}{f'(0)}} := m_1.\tag{1.86}$$

Hence, provided the condition (1.86) holds,  $\alpha^* < 1/2$  and there exists  $y(\alpha^*)$  and a neighbourhood  $\mathcal{V}$  of  $\alpha^*$  such that  $g(\alpha, y) < 0$  for  $|y| < y(\alpha^*)$  and  $\alpha \in \mathcal{V}$ . Take  $v$  such that  $\text{supp}(v) \subset ] - y(\alpha^*), y(\alpha^*)[$  and, for all  $\alpha \in \mathcal{V}$ , that is for all  $\lambda$  in a neighbourhood of  $\lambda^*$ , for  $\varepsilon$  small enough,

$$\left( \Psi_2^0 - \Psi_2 \right) (\lambda, c_0^*) < 0.$$

The result follows from the monotonicity of  $\Psi_2$  with respect to  $c$ .

**Locally maximal velocity for  $\nu = \delta_0 : \alpha^* > 1/2$ : proof of Theorem 1.1 .5** It remains to show that  $\alpha^*$  can be greater than  $\frac{1}{2}$ . We will need the second part of Proposition 1.6 .1. From now, we fix an exchange rate  $\bar{\mu} > 4$ . We will use the fact that, at  $(c_0^*, \lambda^*)$ ,

$$\frac{d}{d\lambda} (\Psi_1 - \Psi_2^0) (\lambda) = 0. \quad (1.87)$$

Explicit computation gives

$$\begin{cases} \frac{d}{d\lambda} \Psi_1(\lambda) &= -2D\lambda + c \\ \frac{d}{d\lambda} \Psi_2^0(\lambda) &= -\frac{\bar{\mu}(c-2\lambda)}{\sqrt{P(\lambda)}(1+2\sqrt{P(\lambda)})^2}. \end{cases} \quad (1.88)$$

Recall that  $\lambda^*$  has to satisfy

$$\frac{c_0^*}{D} \leq \lambda^* \leq \lambda_1^+ := \frac{c_0^* + \sqrt{c_0^{*2} + 4D\bar{\mu}}}{2D}.$$

Now applying Lemma 1.6 .1, for all  $\delta > 0$  there exists  $M > 0$ ,  $f'(0), D > M$  entails  $|\Psi_1'(\lambda^*) - c_0^*| < \delta$  and  $|\lambda_2^-| < \delta$  (recall that  $\lambda_2^- = \frac{c - \sqrt{c^2 - c_K^2}}{2}$ ). To prove that  $\alpha^* = \sqrt{P(\lambda^*)} > 1/2$ , we distinguish two cases.

First case:  $\lambda^* > \frac{1}{2} \left( \lambda_2^- + \frac{c_0^*}{2} \right)$ . Thus  $\lambda^* > \frac{c_0^*}{4} - \delta$  which yields with Lemma 1.6 .1

$$P(\lambda^*) = Df'(0) \frac{3}{16} - f'(0) + O(\delta Df'(0)) > \frac{1}{4}.$$

Second case:  $\lambda_2^- < \lambda^* < \frac{1}{2} \left( \lambda_2^- + \frac{c_0^*}{2} \right)$ . Thus, from (1.87), (1.88) and the above inequalities given by Lemma 1.6 .1,

$$c_0^* + \delta > |\Psi_2^{0'}(\lambda^*)| = \left| \frac{\bar{\mu}(c_0^* - 2\lambda^*)}{\alpha^*(1 + 2\alpha^*)^2} \right| > \frac{2(c_0^* - 2\delta)}{\alpha^*(1 + 2\alpha^*)}$$

which implies  $\alpha^* > 1/2$  for  $\delta$  small enough. This concludes the proof of Theorem 1.1 .5.  $\square$

# Chapter 2

## Uniform dynamics for Fisher-KPP propagation driven by a line of fast diffusion under a singular limit

The purpose of this paper is to understand the links between the model (1) introduced in 2012 by H. Berestycki, J.-M. Roquejoffre and L. Rossi and the nonlocal model studied by in the first chapter. The general question is to investigate the influence of a line of fast diffusion on Fisher-KPP propagation. In the initial model, the exchanges are modeled by a Robin boundary condition, whereas in the nonlocal model the exchanges are described by integral terms. For both models was showed the existence of an enhanced spreading in the direction of the line. One way to retrieve the local model from the nonlocal one is to consider integral terms tending to Dirac masses. The question is then how the dynamics given by the nonlocal model resembles the local one. We show here that the nonlocal dynamics tends to the local one in a rather strong sense.

This chapter is a paper published in Nonlinearity.

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## 2.1 Introduction

**Presentation of the models** This paper is concerned with the large time behaviour and propagation phenomena for reaction-diffusion equations with a line of fast diffusion. Our model will degenerate, when a small parameter tends to 0, to a singular limit. The results that we will present will be uniform with respect to this small parameter. The model under study (2.1) was introduced in 2014 by the author in [84].

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu_\varepsilon(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu_\varepsilon(y)u(t, x) - \nu_\varepsilon(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0. \end{cases} \quad (2.1)$$

A two-dimensional environment (the plane  $\mathbb{R}^2$ ) includes a line (the line  $\{(x, 0), x \in \mathbb{R}\}$ ) in which fast diffusion takes place while reproduction and usual diffusion only occur outside the line. For the sake of simplicity, we will refer to the plane as “the field“ and the line as “the road“, as a reference to the biological situations. The density of the population is designated by  $v = v(t, x, y)$  in the field, and  $u = u(t, x)$  on the road. Exchanges of population between the road and field are defined by two nonnegative compactly supported functions  $\nu$  and  $\mu$ . These functions will be called the *exchange functions*. The density of individuals who jump from a point of the field to the road is represented by  $y \mapsto \nu_\varepsilon(y)$ , from the road to a point of the field by  $y \mapsto \mu_\varepsilon(y)$ , with the following scaling with  $\varepsilon > 0$  :

$$\nu_\varepsilon(y) = \frac{1}{\varepsilon}\nu\left(\frac{y}{\varepsilon}\right), \quad \mu_\varepsilon(y) = \frac{1}{\varepsilon}\mu\left(\frac{y}{\varepsilon}\right).$$

We use the notation  $\bar{\mu} = \int \mu, \bar{\nu} = \int \nu$ .

It is easy to see that  $\nu_\varepsilon \rightarrow \bar{\nu}\delta$  and  $\mu_\varepsilon \rightarrow \bar{\mu}\delta$  as  $\varepsilon \rightarrow 0$  in the distribution sense, where  $\delta = \delta_0$ , the Dirac function in 0. Hence, at least formally, the above system (2.1) tends to the following system (2.2) where exchanges of population are localised on the road:

$$\begin{cases} \partial_t u - D\partial_{xx}u = \bar{\nu}v(t, x, 0) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) - \bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0. \end{cases} \quad (2.2)$$

This model was introduced in 2013 in [22] by H. Berestycki, J.-M. Roquejoffre and L. Rossi to describe biological invasions in a plane when a strong diffusion takes place on a line. Considering a nonnegative, compactly supported initial datum  $(u_0, v_0) \neq (0, 0)$ , the authors proved the existence of an asymptotic speed of spreading  $c_0^*$  in the direction of the road for the system (2.2) for a KPP-type nonlinearity. They also explained the dependence of  $c_0^*$  on  $D$ , the coefficient of diffusion on the road.

The same kind of results was investigated in [84] for our system (2.1) with fixed  $\varepsilon$ , say  $\varepsilon = 1$  for instance. The main theorem was the following spreading result:

**Theorem 2.1 .1.** *We consider the nonlocal system (2.1) with a KPP-type nonlinearity and two nonnegative compactly supported exchange functions  $\nu$  and  $\mu$ , with  $\bar{\nu}, \bar{\mu} > 0$ . Let  $(u_\varepsilon, v_\varepsilon)$  be a solution of (2.1) with a nonnegative, compactly supported initial datum  $(u_0, v_0)$ . Then, for all  $\varepsilon > 0$ , there exists an asymptotic speed of spreading  $c_\varepsilon^*$  and a unique positive bounded stationary solution of (2.1)  $(U_\varepsilon, V_\varepsilon)$  such that, pointwise in  $y$ , we have:*

- for all  $c > c_\varepsilon^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \geq ct} (u(x, t), v(x, y, t)) = (0, 0)$  ;
- for all  $c < c_\varepsilon^*$ ,  $\lim_{t \rightarrow \infty} \inf_{|x| \leq ct} (u(x, t), v(x, y, t)) = (U_\varepsilon, V_\varepsilon)$ .

This result is similar to the one showed in [22], where the steady state is given by  $(U_0, V_0) = \left(\frac{\bar{\nu}}{\bar{\mu}}, 1\right)$ . So, a natural question is: are the limits in Theorem 2.1 .1 uniform in  $\varepsilon$ ? A first reasonable guess is that the spreading speed  $c_\varepsilon^*$  tends to the spreading speed  $c_0^*$  associated to the limit model (2.2). This point will be developed in the second section of this paper. The problem of the commutation of the two limits  $\varepsilon \rightarrow 0$  and  $t \rightarrow +\infty$  in Theorem 2.1 .1 is more intricate. It is the main purpose of this paper. It involves both tools from functional analysis concerning singular limits and reaction-diffusion methods about spreading phenomena.

**Assumptions** We always assume that the initial datum  $u_0, v_0$  is nonnegative, continuous and compactly supported, with  $(u_0, v_0) \not\equiv (0, 0)$ . The reaction term  $f$  satisfies:

$$f \in C^2([0, 1]), \quad f(0) = f(1) = 0, \quad \forall s \in (0, 1), \quad 0 < f(s) \leq f'(0)s. \quad (2.3)$$

Such a reaction term is, as usual, referred to as of KPP type (from the article of Kolmogorov, Petrovsky and Piskounov [70]). We extend it to a uniformly Lipschitz and concave function outside  $(0, 1)$ , satisfying

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{s} < -2\frac{\bar{\nu}^2}{d}. \quad (2.4)$$

In particular, denoting with an abuse of notation the Lipschitz constant of  $f$  on  $\mathbb{R}^+$  by  $\|f\|_{Lip}$ , we suppose  $2\frac{\bar{\nu}^2}{d} < \|f\|_{Lip} < +\infty$ . The assumption (2.4) seems to be technical, but such a kind of uniform coercivity appears to be crucial in order to get a uniform bound for the stationary solutions.

We make the following assumptions on the exchange functions.

- The exchange functions  $\nu$  and  $\mu$  are smooth, nonnegative and compactly supported. Without loss of generality we will assume

$$\text{supp}(\mu, \nu) \subset (-1, 1). \quad (2.5)$$

- For the sake of simplicity, we will take  $\bar{\nu} = 1$ , as in [22].

The parameters  $d, D, \bar{\mu}, f'(0)$  and the two functions  $\nu$  and  $\mu$  are fixed once and for all.

**Main results of the paper** The main result of the paper is that spreading in the  $x$ -direction is indeed uniform in  $\varepsilon$ . Set  $c_0^*$  the asymptotic speed of spreading associated to the initial model (2.2).

**Theorem 2.1 .2.** *For  $\varepsilon > 0$ , let us denote  $(u_\varepsilon, v_\varepsilon)$  the solution of system (2.1). There exists  $m > 0$  such that if  $(u_0, v_0) \leq \left(\frac{m}{\bar{\mu}}, m\right)$  we have:*

- $\forall c > c_0^*, \forall \eta > 0, \exists T_0, \varepsilon_0$  such that  $\forall t > T_0, \forall \varepsilon < \varepsilon_0, \sup_{|x| > ct} |u_\varepsilon(t, x)| < \eta$ .
- $\forall c < c_0^*, \forall \eta > 0, \exists T_0, \varepsilon_0$  such that  $\forall t > T_0, \forall \varepsilon < \varepsilon_0, \sup_{|x| < ct} \left| u_\varepsilon(t, x) - \frac{1}{\bar{\mu}} \right| < \eta$ .

The idea of the proof is to show that every solution  $(u_\varepsilon, v_\varepsilon)$  is above some travelling subsolutions in finite time. Then, we use the convergence of the spreading speed  $c_\varepsilon^*$  to  $c_0^*$ , which yields travelling subsolutions at some speed close to  $c_0^*$ . Hence, our main tool relies on the following convergence theorem.

**Theorem 2.1 .3.** *Let  $(u, v)$  be the solution of the limit system (2.2) and  $(u_\varepsilon, v_\varepsilon)$  be the solution of the  $\varepsilon$ -system (2.1) for  $\varepsilon > 0$ . Let  $(u_0, v_0)$  be a common initial datum for both systems. Then:*

$$\|(u - u_\varepsilon)(t)\|_{L^\infty(\mathbb{R})} + \|(v - v_\varepsilon)(t)\|_{L^\infty(\mathbb{R}^2)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

*The above convergence is uniform in every compact set in  $t$  included in  $(0, +\infty)$ .*

Notice that the convergence is global in space, but local in time.

**Bibliographical background** Reaction-diffusion equations of the type

$$\partial_t u - d\Delta u = f(u)$$

have been introduced in the celebrated articles of Fisher [45] and Kolmogorov, Petrovsky and Piskounov [70] in 1937. The initial motivation came from population genetics. The reaction term are that of a logistic law, whose archetype is  $f(u) = u(1 - u)$  for the simplest example. In their works in one dimension, Kolmogorov, Petrovsky and Piskounov revealed the existence of propagation waves, together with an asymptotic speed of spreading of the dominating gene, given by  $2\sqrt{df'(0)}$ . The existence of an asymptotic speed of spreading was generalised in  $\mathbb{R}^n$  by D. G. Aronson and H. F. Weinberger in [2] (1978). Since these pioneering works, front propagation in reaction-diffusion equations have been widely studied. Let us cite, for instance, the works of Freidlin and Gärtner [47] for an extension to periodic media, or [98], [15] and [16] for more general domains. An overview of the subject can be found in [13].

New results on the model under study (2.2) have been recently proved. Further effects like a drift or a killing term on the road have been investigated in [21]. The case of a fractional diffusion on the road was studied and explained by the three authors and A.-C. Coulon in [7] and [34]. See [8] for a summary of the results on this model. Models with an ignition-type nonlinearity are also studied by L. Dietrich in [37] and [38].

**Organisation of the paper** The second section of the paper is devoted to the convergence of the asymptotic speed of propagation  $c_\varepsilon^*$  with  $\varepsilon$  goes to 0. For this, we will also investigate some useful convergence result concerning travelling supersolutions. The third section deals with the convergence of the stationary solutions, which will be helpful for the control of the long time behaviour for Theorem 2.1 .2. Theorem 2.1 .3 is proved in sections 2.4 and 2.5 , with an argument from geometric theory of parabolic equations. Section 2.4 , which is the most technical, is devoted to resolvent bounds. They are used in section 2.5 to prove some convergence properties for the linear systems. Then, we use them in a Gronwall argument to deal with the nonlinearity. We prove Theorem 2.1 .2 in the last section.

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## 2.2 Asymptotic spreading speed

Let  $c_0^*$  be the asymptotic spreading speed associated with the above system (2.2), and  $c_\varepsilon^*$  the spreading speed given by Theorem 2.1 .1 associated with the system (2.1). Under our assumptions, the main result of this section is

**Proposition 2.2 .1.**  *$c_\varepsilon^*$  converges to  $c_0^*$  as  $\varepsilon$  goes to 0, locally uniformly in  $d, D, \bar{\mu}$ .*

For the sake of simplicity, we will consider that  $\nu$  is an even function. The general case is similar but heavier.

### 2.2 .1 Prerequisite - linear travelling waves and speed of propagation

All the results given in this subsection have been investigated in [22] by H. Berestycki, J.-M. Roquejoffre, and L. Rossi for the local system or in [84] by the author for the nonlocal. Our purpose is to recall the derivation of the asymptotic speed of spreading for both systems. Because  $f$  is of KPP-type (that is, satisfies  $f(v) \leq f'(0)v$  for nonnegative  $v$ ), we are interested in the linearised systems:

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu_\varepsilon(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f'(0)v + \mu_\varepsilon(y)u(t, x) - \nu_\varepsilon(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0 \end{cases} \quad (2.6)$$

for the nonlocal case, and

$$\begin{cases} \partial_t u - D\partial_{xx}u = v(t, x, 0) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f'(0)v & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) - \bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0 \end{cases} \quad (2.7)$$



for the local one. This motivates the following definition.

**Definition 2.2 .1.** For any of the two systems (2.6)-(2.7), we call a **linear travelling wave** a 3-tuple  $(c, \lambda, \phi)$  with  $c > 0$ ,  $\lambda > 0$ , and  $\phi \in H^1(\mathbb{R})$  a positive function such that

$$\begin{pmatrix} u(t, x) \\ v(t, x, y) \end{pmatrix} = e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}$$

be a solution of the corresponding linearised system (2.6) or (2.7). The quantity  $c$  is the speed of the linear travelling wave.

*Remark 2.2 .1.* From the KPP assumption (2.3) on  $f$ , a linear travelling wave for the linearised system (2.6) (resp. (2.7)) provides a supersolution for the nonlinear system (2.1) (resp. (2.2)). This will be a powerful tool to get  $L^\infty$  and decay estimates in the sequel.

The previous definition for travelling waves provides us a helpful characterisation for spreading speed.

**Proposition 2.2 .2.** [22, 84] For any of the systems (2.1)-(2.2), for all  $\varepsilon > 0$ , the **spreading speed**  $c^* = c_0^*$  or  $c_\varepsilon^*$  given by Theorem 2.1 .1 can be defined as follows:

$$c^* = \inf\{c > 0 \mid \text{a linear travelling wave with speed } c \text{ exists}\}.$$

1. If  $D \leq 2d$ , then  $c_0^* = c_\varepsilon^* = c_{KPP} = 2\sqrt{df'(0)}$ .
2. If  $D > 2d$ , then  $c_0^*, c_\varepsilon^* > c_{KPP}$  and the infimum is reached by a linear travelling wave, denoted  $(c_0^*, \lambda_0^*, \phi^*)$  or  $(c_\varepsilon^*, \lambda_\varepsilon^*, \phi_\varepsilon^*)$ .

Proposition 2.2 .2 provides the construction of  $c^*$  thanks to a nonlinear eigenvalue problem. We give an outline of a proof in both cases to make the sequel easier to read. We focus only on the case  $D > 2d$ , the other being trivial.

**Resolution for the local case** Inserting the definition supplied by Proposition 2.2 .2 into (2.7), we obtain the following system in  $(c, \lambda, \phi)$  :

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \phi(0) \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0))\phi(y) = 0 & y \in \mathbb{R}^* \\ \phi(0^+) = \phi(0^-), \phi \geq 0, \phi \in H^1(\mathbb{R}), \\ -d(\phi'(0^+) - \phi'(0^-)) = \bar{\mu} - \phi(0). \end{cases} \quad (2.8)$$

From now, we set

$$P(\lambda) = \lambda c - d\lambda^2 - f'(0), \quad \lambda_2^\pm(c) = \frac{c \pm \sqrt{c^2 - c_{KPP}^2}}{2d}$$

and

$$\mathcal{D} = \left\{ (c, \lambda), c > c_{KPP} \text{ and } \lambda \in (\lambda_2^-(c), \lambda_2^+(c)) \right\}.$$

Hence, the existence of a linear travelling wave is equivalent to the following system in  $(c, \lambda, \phi(0))$  provided that  $c > c_{KPP}$  and  $\lambda \in [\lambda_2^-(c), \lambda_2^+(c)]$ :

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \phi(0) \\ \phi(0) = \frac{\bar{\mu}}{1+2\sqrt{dP(\lambda)}}. \end{cases} \quad (2.9)$$

The first equation of (2.9) gives the graph of a function

$$\lambda \mapsto \Psi_1^0(c, \lambda) := -D\lambda^2 + c\lambda + \bar{\mu} \quad (2.10)$$

which is intended to be equal to  $\phi(0)$ , provided  $(c, \lambda, \phi)$  defines a linear travelling wave. Let us denote  $\Gamma_1$  its graph, depending on  $c$ , in the  $(\lambda, \Psi_1^0(\lambda))$ -plane.

The second equation of (2.9) gives the graph of a function

$$\Psi_2^0 : \begin{cases} \bar{\mathcal{D}} & \longrightarrow \mathbb{R} \\ (c, \lambda) & \longmapsto \frac{\bar{\mu}}{1+2\sqrt{dP(\lambda)}}. \end{cases}$$

Let us denote  $\Gamma_2^0$  its graph in the same plane, still depending on  $c$ . Hence, (2.9) amounts to looking for the first  $c \geq c_{KPP}$  such that the two graphs  $\Gamma_1$  and  $\Gamma_2^0$  intersect.

It was shown that there exists  $(c_0^*, \lambda_0^*) \in \mathcal{D}$  and an exponential function

$$\phi^* : y \mapsto \phi^*(0)e^{-\sqrt{P(\lambda_0^*)}|y|}$$

such that  $(c_0^*, \lambda_0^*, \phi^*)$  defines a linear travelling wave for (2.7), and  $c_0^*$  is the first  $c$  such that (2.8) admits a solution. Hence,  $c_0^*$  is the speed defined by Proposition 2.2 .2.

**Resolution for the nonlocal case** In this case, Proposition 2.2 .2 yields the following system in  $(c, \lambda, \phi)$ .

$$\begin{cases} -D\lambda^2 + c\lambda + \bar{\mu} = \int_{\mathbb{R}} \nu_\varepsilon(y)\phi(y)dy \\ -d\phi''(y) + (c\lambda - d\lambda^2 - f'(0) + \nu_\varepsilon(y))\phi(y) = \mu_\varepsilon(y), \quad \phi \in H^1(\mathbb{R}). \end{cases} \quad (2.11)$$

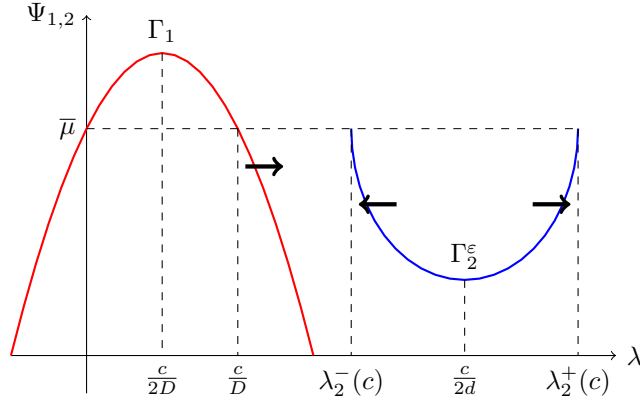
Once again, the first equation of (2.11) gives the function  $\Psi_1^0$  defined by (2.10), which is intended to be equal to  $\int \nu_\varepsilon \phi$  provided  $(c, \lambda, \phi)$  defines a linear travelling wave for (2.6). The second equation of (2.11) defines implicitly the function

$$\Psi_2^\varepsilon : \begin{cases} \mathcal{D} & \longrightarrow \mathbb{R} \\ (c, \lambda) & \longmapsto \int \nu_\varepsilon(y)\phi(y; \varepsilon, c, \lambda)dy \end{cases}$$

where  $\phi(\cdot; \varepsilon, c, \lambda)$  is the unique solution in  $H^1(\mathbb{R})$  of

$$-d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu_\varepsilon(y))\phi(y) = \mu_\varepsilon(y), \quad y \in \mathbb{R}. \quad (2.12)$$

For fixed  $c$ , we denote  $\Gamma_2^\varepsilon$  the graph of  $\Psi_2^\varepsilon$  in the  $(\lambda, \Psi_2^\varepsilon(\lambda))$ -plane.  $\Psi_2^\varepsilon$  is smooth in  $\mathcal{D}$  and can be continuously extended in  $\bar{\mathcal{D}}$ . It has also been proved that  $\phi$ , hence  $\Psi_2^\varepsilon$  is decreasing in  $c$ . It was shown that for all  $\varepsilon > 0$ , there exists  $(c_\varepsilon^*, \lambda_\varepsilon^*) \in \mathcal{D}$  and  $\phi_\varepsilon^* = \phi(\cdot; \varepsilon, c_\varepsilon^*, \lambda_\varepsilon^*)$  solution of (2.12) such that  $(c_\varepsilon^*, \lambda_\varepsilon^*)$   $\Gamma_1^0$  and  $\Gamma_2^\varepsilon$  intersect, and  $c_\varepsilon^*$  is the first  $c$  such that it occurs. The main ingredients of this proof are:


 Figure 2.1: representation of  $\Gamma_1$  and  $\Gamma_2^\varepsilon$ , behaviours as  $c$  increases

- the functions  $c \mapsto \Psi_1^0(c, \lambda)$  and  $c \mapsto \lambda_2^-(c)$  are respectively increasing and decreasing;
- the function  $\lambda \mapsto \Psi_2^\varepsilon(c, \lambda)$  is strictly concave.

These behaviours summed up in Figure 2.1.

The main purpose of the rest of this section will be to show that the curve  $\Gamma_2^\varepsilon$  converges locally uniformly in  $(c, \lambda)$  to  $\Gamma_2^0$  as  $\varepsilon$  goes to 0.

## 2.2 .2 Convergence of the spreading speed

### Uniform boundedness of $\phi$ in $C^1(\mathbb{R})$

**Lemma 2.2 .3.** *Let us consider  $\phi := \phi(y; \varepsilon, c, \lambda)$  the unique solution of (2.12) for  $\varepsilon > 0$  and  $(c, \lambda)$  in  $\mathcal{D}$ , that is to say  $c > c_{KPP}$  and  $\lambda$  in  $]\lambda_2^-(c), \lambda_2^+(c)[$ . Then the family  $(\|\phi\|_{L^\infty(y)} + \|\phi'\|_{L^\infty(y)})_\varepsilon$  is uniformly bounded in  $\varepsilon > 0$  and every compact set on  $\overline{\mathcal{D}}$ .*

*Proof.* We consider  $\phi = \phi(y; \varepsilon, \lambda, c)$  defined for  $\varepsilon > 0$ ,  $(c, \lambda)$  in  $\mathcal{D}$ , and  $y \in \mathbb{R}$ . We know that  $\phi$  can be continuously extended to  $\overline{\mathcal{D}}$ . Moreover, from the elliptic maximum principle, it is easy to see that  $\phi > 0$  on  $\mathbb{R}$ , for all admissible parameters. Considering the hypotheses (2.5) on  $\mu, \nu$ , we get that:

- $\forall \varepsilon > 0, \text{supp}(\nu_\varepsilon, \mu_\varepsilon) \subset (-\varepsilon, \varepsilon)$  ;
- $y \mapsto \phi(y)$  is even.

Hence there exists  $K = K(\varepsilon, c, \lambda)$  such that

$$\forall |y| > \varepsilon, \phi(y; \varepsilon, c, \lambda) = K(\varepsilon, c, \lambda) e^{-\sqrt{\frac{P(\lambda)}{d}}|y|}. \quad (2.13)$$

*Step 1.* It is enough to bound  $K$  to get the uniform boundedness of  $\|\phi(\varepsilon, c, \lambda)\|_{L^\infty(\mathbb{R})}$ . Indeed, using the scaling  $\xi = \frac{y}{\varepsilon}$ , with  $\psi(\xi) = \phi(y)$ , the function  $\psi$  satisfies

$$\begin{cases} -d\psi''(\xi) + (\varepsilon^2 P(\lambda) + \varepsilon \nu(\xi))\psi(\xi) = \varepsilon \mu(\xi) \\ \psi(\pm 1) = K e^{-\sqrt{\frac{P(\lambda)}{d}}}. \end{cases} \quad (2.14)$$

Now, let us recall that  $\mu, \nu$  are continuous and compactly supported. Then, by the Harnack inequality (theorem 8.17 and 8.18 in [48] for instance), there exist  $C_1, C_2 \geq 0$ , independent of  $\varepsilon, c, \lambda$ , such that

$$\sup_{[-1,1]} \psi \leq C_1 (\inf_{[-1,1]} \psi + C_2),$$

which gives immediately

$$\sup_{\mathbb{R}} \phi \leq C_1 (K(\varepsilon, c, \lambda) + C_2).$$

*Step 2.* Let us prove a uniform bound for  $K$ . Set  $c_1 \in ]c_{KPP}, +\infty[$ , and assume by contradiction that

$$\limsup_{\varepsilon \rightarrow 0} \left\{ K(\varepsilon, c, \lambda), (c, \lambda) \in \overline{\mathcal{D}}, c \leq c_1 \right\} = +\infty. \quad (2.15)$$

That is, there exist  $(\varepsilon_n)_n, (\lambda_n)_n, (c_n)_n$  with  $\varepsilon_n \rightarrow 0$  and  $c \leq c_1, \lambda_2^-(c_1) \leq \lambda \leq \lambda_2^+(c_1)$  such that

$$K(\varepsilon_n, \lambda_n, c_n) := K_n \xrightarrow{n \rightarrow \infty} +\infty.$$

Set

$$\tilde{\phi}_n = \frac{\phi(\cdot; \varepsilon_n, \lambda_n, c_n)}{K_n}.$$

The function  $\tilde{\phi}_n$  satisfies

$$\begin{cases} -d\tilde{\phi}_n'' + (P(\lambda_n) + \nu_{\varepsilon_n})\tilde{\phi}_n = \frac{\mu_{\varepsilon_n}}{K_n} & y \in \mathbb{R} \\ \tilde{\phi}_n(y) = e^{-\sqrt{\frac{P(\lambda_n)}{d}}|y|}, & \forall |y| > \varepsilon_n. \end{cases} \quad (2.16)$$

Again by the Harnack inequality,  $(\|\tilde{\phi}_n\|_{\infty})_n$  is bounded, and  $\tilde{\phi}_n$  is positive by the elliptic maximum principle. Integrating (2.16) between  $-\infty$  and  $y$  gives

$$d\tilde{\phi}_n'(y) = P(\lambda_n) \int_{-\infty}^y \tilde{\phi}_n + \int_{-\infty}^y \nu_{\varepsilon_n} \tilde{\phi}_n - \frac{1}{K_n} \int_{-\infty}^y \mu_{\varepsilon_n},$$

so

$$\begin{aligned} d\|\tilde{\phi}_n'\|_{\infty} &\leq P(\lambda_n) \int_{\mathbb{R}} e^{-\sqrt{\frac{P(\lambda_n)}{d}}|y|} dy + P(\lambda_n) \int_{-\varepsilon}^{+\varepsilon} \tilde{\phi}_n(y) dy + \|\tilde{\phi}_n\|_{\infty} + \frac{\bar{\mu}}{K_n} \\ &\leq 2\sqrt{dP(\lambda_n)} + \|\tilde{\phi}_n\|_{\infty} (1 + 2\varepsilon P(\lambda_n)) + \frac{\bar{\mu}}{K_n}. \end{aligned}$$

Hence  $(\tilde{\phi}_n)_n$  is uniformly Lipschitz. Specializing to  $y = 1$ , we get:

$$-d\tilde{\phi}_n'(1) = \frac{\bar{\mu}}{K_n} - P(\lambda_n) \int_{-\infty}^1 \tilde{\phi}_n - \int_{-\infty}^1 \nu_{\varepsilon_n} \tilde{\phi}_n. \quad (2.17)$$

But on the other hand, we have:

- $-d\tilde{\phi}_n'(1) = \sqrt{dP(\lambda_n)} e^{-\sqrt{\frac{P(\lambda_n)}{d}}} \geq 0$  from (2.13) ;
- $\frac{\bar{\mu}}{K_n} \rightarrow 0$  as  $n \rightarrow \infty$  by assumption (2.15) ;

- $-P(\lambda_n) \int_{-\infty}^1 \tilde{\phi}_n \leq 0$  ;
- $-\int_{-\infty}^1 \nu_{\varepsilon_n} \tilde{\phi}_n \rightarrow -1$  as  $n \rightarrow \infty$ . Indeed, the sequence  $\nu_{\varepsilon_n}$  tends to the Dirac measure, we have  $\tilde{\phi}_n(\pm\varepsilon_n) = 1 + O(\varepsilon_n)$  from (2.16), and  $(\tilde{\phi}_n)_n$  is uniformly Lipschitz.

For  $n \rightarrow \infty$ , this contradicts (2.17) since the left term is nonnegative and the right term tends to a negative limit. Hence, there is a contradiction. That is,  $K(\varepsilon, c, \lambda)$  is bounded for  $(c, \lambda) \in \overline{\mathcal{D}}$  satisfying  $c \leq c_1$ . Recall that  $c \mapsto \phi$  is nonincreasing, and  $\phi$  is uniformly bounded as  $\varepsilon \rightarrow 0$  in  $(c, \lambda) \in \overline{\mathcal{D}}$ .

*Step 3.* Boundedness of  $\|\phi'\|_\infty$  with  $\varepsilon \rightarrow 0$ . We integrate (2.12) from  $-\infty$  to  $y$  which gives:

$$d\phi'(y) = P(\lambda) \int_{-\infty}^y \phi + \int_{-\infty}^y \nu_\varepsilon \phi - \int_{-\infty}^y \mu_\varepsilon.$$

Now, the explicit formula for  $\phi$  and its uniform boundedness obtained in step 2 yields:

$$d\|\phi'\|_\infty \leq \|\phi\|_\infty \left(1 + 2\sqrt{dP(\lambda)} + 2\varepsilon\right) + \bar{\mu}$$

and the family  $(\|\phi(\varepsilon, c, \lambda)\|_\infty)$  is equicontinuous for all  $\varepsilon$  close enough to 0 and for all compact set in  $\overline{\mathcal{D}}$ . This implies the uniform boundedness of  $\phi$  in  $C^1(\mathbb{R})$ .  $\square$

**Convergence of  $\phi(\cdot; \varepsilon)$  with  $\varepsilon \rightarrow 0$ , continuity of  $\Psi_2^\varepsilon$ .** From Lemma 2.2 .3 the set  $(\phi(\cdot; \varepsilon))_\varepsilon$  is included in  $C_b(\mathbb{R})$  and equicontinuous, locally uniformly in  $c, \lambda$ . The Arzelà-Ascoli theorem (combined with Cantor's diagonal argument) yields the existence of a sequence  $(\varepsilon_n)_n \subset \mathbb{R}$ ,  $\varepsilon_n \rightarrow 0$  and a function  $\phi_0 \in C_b(\mathbb{R})$  such that  $\phi(\varepsilon_n) \xrightarrow[n \rightarrow \infty]{} \phi_0$  uniformly on compact sets. Passing to the limit in (2.12), we obtain that  $\phi_0$  satisfies

$$-d\phi_0'' + \phi_0(P(\lambda) + \delta_0) = \bar{\mu}\delta_0 \quad (2.18)$$

in the distribution sense. Moreover,  $K(\varepsilon_n, c, \lambda) \rightarrow K_0(c, \lambda) = \phi_0(0; c, \lambda)$  with  $n \rightarrow \infty$ , and

$$\phi_0(y) = K_0 \exp\left(-\sqrt{\frac{P(\lambda)}{d}}|y|\right).$$

It remains to show the uniqueness of the limit function. Let  $\phi_1$  be another accumulation point for  $(\phi(\cdot; \varepsilon))_\varepsilon$ . Then  $\phi_1$  also satisfies (2.18) in the distribution sense, and  $\phi_1(y) = \phi_1(0) \exp\left(-\sqrt{\frac{P(\lambda)}{d}}|y|\right)$ . Then  $\psi = \phi_0 - \phi_1$  is a solution of

$$-d\psi'' + \psi(P(\lambda) + \delta_0) = 0.$$

Now, let us consider  $(\psi_n)_n \subset \mathcal{D}(\mathbb{R})$  such that  $\psi_n \rightarrow \psi$  uniformly on every compact and  $\psi_n' \rightarrow \psi'$  on every compact of  $\mathbb{R} \setminus \{0\}$ . We get

$$d\|\psi'\|_{L^2}^2 + P(\lambda)\|\psi\|_{L^2}^2 + \psi(0)^2 = 0,$$

and  $\psi \equiv 0$ . So  $\phi_0$  is the unique accumulation point of  $(\phi(\cdot; \varepsilon))_\varepsilon$ , hence  $\phi(\cdot; \varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} \phi_0$  uniformly on compact sets. Hence we have proved the following lemma.

**Lemma 2.2 .4.** *Let us consider  $\phi$  as a function of  $\varepsilon, c, \lambda$ , extended to  $\phi_0$  for  $\varepsilon = 0$ . Then  $\phi(\cdot; \varepsilon, c, \lambda)$  is continuous from  $[0, 1] \times \overline{\mathcal{D}}$  to  $C^0(\mathbb{R})$  and bounded in  $C^1$  as a  $y$ -function on every compact set on it.*

It is now easy to see that the function  $\Psi_2^\varepsilon$  converges continuously in  $\lambda, c$ , to a limit function

$$\Psi_2^l : \begin{cases} ]\lambda_2^-, \lambda_2^+[ & \longrightarrow \mathbb{R} \\ \lambda & \longmapsto \phi_0(c, \lambda) \end{cases}$$

and the curve  $\Gamma_2^\varepsilon$  converges to the graph of  $\Psi_2^l$ . Let us denote it  $\Gamma_2^l$ .

**The limit curve  $\Gamma_2^l$ .** Now let us show that  $\Gamma_2^l = \Gamma_2^0$ . The limit function  $\phi_0$  is of the form  $\phi_0(y) = K_0 \exp(-\sqrt{\frac{P(\lambda)}{d}}|y|)$  and satisfies (2.18) in the distribution sense. Applying the two sides of (2.18) to  $\phi_0$  (or, to be strictly rigorous, to a sequence  $(\phi_n)$  that tends to  $\phi_0$ ), we get

$$d\|\phi_0'\|_{L^2}^2 + P(\lambda)\|\phi_0\|_{L^2}^2 + \phi_0(0)^2 = \bar{\mu}\phi_0(0).$$

Using the explicit formula for  $\phi_0$ , we obtain:

$$2\sqrt{dP(\lambda)}\phi_0(0)^2 + \phi_0(0)^2 = \bar{\mu}\phi_0(0).$$

Hence, because 0 is not a solution,

$$\phi_0(0) = \frac{\bar{\mu}}{1 + 2\sqrt{dP(\lambda)}}.$$

Hence,  $\Psi_2^l = \Psi_2^0$  and  $\Gamma_2^l = \Gamma_2^0$  which concludes the proof of Proposition 2.2 .1.  $\square$

*Remark 2.2 .2.* Actually, the spreading speed in the local case was not devised exactly with the system (2.9), but with the following system.

$$\begin{cases} -D\lambda^2 + c\lambda = \frac{\bar{\mu}}{1+2d\beta} - \bar{\mu} \\ -d\lambda^2 + c\lambda = f'(0) + d\beta^2. \end{cases}$$

But, setting  $\Phi(\beta) = \frac{\bar{\mu}}{1+2d\beta}$ , they are equivalent.

## 2.3 Convergence of the stationary solutions

We have already showed in [84] that, for all  $\varepsilon > 0$ , there exists a unique positive and bounded stationary solution of (2.1), which will be denoted  $(U_\varepsilon, V_\varepsilon)$ . Moreover, this stationary solution is  $x$ -independent and satisfies  $\lim_{y \rightarrow \pm\infty} V_\varepsilon(y) = 1$ . The corresponding equation is

$$\begin{cases} \bar{\mu}U_\varepsilon = \int \nu_\varepsilon(y)V_\varepsilon(y)dy \\ -dV_\varepsilon''(y) = f(V_\varepsilon) + \mu_\varepsilon(y)U_\varepsilon - \nu_\varepsilon(y)V_\varepsilon(y). \end{cases} \quad (2.19)$$

Solutions of (2.19) depend only of the  $y$ -variable, hence  $U$  is constant and  $V$  is entirely determined by the following integro-differential equation

$$-dV_\varepsilon''(y) = f(V_\varepsilon) + \frac{\mu_\varepsilon(y)}{\bar{\mu}} \int \nu_\varepsilon(z)V_\varepsilon(z)dz - \nu_\varepsilon(y)V_\varepsilon(y). \quad (2.20)$$

The main result of this section lies in the next proposition.

**Proposition 2.3 .1.**  $(V_\varepsilon)_\varepsilon$  converges uniformly to the constant function 1 when  $\varepsilon$  goes to 0.

Of course, this implies that  $U_\varepsilon \rightarrow \frac{\bar{\nu}}{\bar{\mu}}$  as  $\varepsilon \rightarrow 0$ . As a result, the stationary solutions of (2.1) converge to the stationary solutions of (2.2) which were identified in [22].

The difficulties lie in the singularity in  $(-\varepsilon, \varepsilon)$ . Outside this interval,  $V_\varepsilon$  satisfies

$$\begin{cases} -dV_\varepsilon'' = f(V_\varepsilon) \\ 0 < V_\varepsilon < +\infty, V_\varepsilon(\pm\infty) = 1. \end{cases} \quad (2.21)$$

As  $(V_\varepsilon, V_\varepsilon') = (1, 0)$  is a saddle point for the system (2.21), it is easy to see that solutions of (2.21) belong to one of the two integral curves that tend to  $(V_\varepsilon, V_\varepsilon') = (1, 0)$ . We can also notice that  $V_\varepsilon(\varepsilon) > 1$  (resp.  $< 1$ ) implies that  $V_\varepsilon$  is decreasing (resp. increasing) on  $(\varepsilon, +\infty)$ . Thus important estimates have to be found inside  $(-\varepsilon, \varepsilon)$ . From now and without lack of generality, we will assume  $d = 1$ .

**A uniform  $L^\infty$  boundary for  $(V_\varepsilon)$ .** Set  $z = \frac{y}{\varepsilon}$  and  $W_\varepsilon(z) := V_\varepsilon(y)$ . Then, we have  $\|V_\varepsilon\|_\infty = \|W_\varepsilon\|_\infty$ ,  $\|V_\varepsilon'\|_\infty = \frac{1}{\varepsilon}\|W_\varepsilon'\|_\infty$ , and  $W_\varepsilon$  satisfies in  $(-1, 1)$

$$-W_\varepsilon'' = \varepsilon^2 f(W_\varepsilon) + \varepsilon \frac{\mu(z)}{\bar{\mu}} \int \nu(s) W_\varepsilon(s) ds - \varepsilon \nu(z) W_\varepsilon(z). \quad (2.22)$$

*Step 1.* Lower bound for  $W_\varepsilon(1)$ . From (2.21), there exists at least a point  $z_\varepsilon \in (-1, 1)$  such that  $W_\varepsilon'(z_\varepsilon) = 0$ . Hence,

$$\forall z \in (-1, 1), W_\varepsilon'(z) = \int_{z_\varepsilon}^z W_\varepsilon''(s) ds. \quad (2.23)$$

Integrating (2.22) yields the following rough bound, for all  $\varepsilon > 0$  :

$$\forall z \in (-1, 1), |W_\varepsilon'(z)| \leq 2\varepsilon (\|\nu\|_\infty + 1 + \varepsilon \|f\|_{Lip}) \|W_\varepsilon\|_{L^\infty(-1,1)}. \quad (2.24)$$

Let us set  $K = 4(\|\nu\|_\infty + \bar{\nu} + \|f\|_{Lip})$  and we get the following estimate for  $W_\varepsilon(1) = V_\varepsilon(\varepsilon)$  :

$$\left| \frac{W_\varepsilon(1) - \|W_\varepsilon\|_{L^\infty(-1,1)}}{\|W_\varepsilon\|_{L^\infty(-1,1)}} \right| \leq K\varepsilon. \quad (2.25)$$

*Step 2.* Lower bound for  $W_\varepsilon'(1)$ . Using once again (2.23), we have

$$W_\varepsilon'(1) = \varepsilon \int_{z_\varepsilon}^1 \nu(s) W_\varepsilon(s) ds - \varepsilon^2 \int_{z_\varepsilon}^1 f(W_\varepsilon(s)) ds - \left( \frac{\varepsilon}{\bar{\mu}} \int_{-1}^1 \nu(s) W_\varepsilon(s) ds \right) \int_{z_\varepsilon}^1 \mu(s) ds. \quad (2.26)$$

The first term in the right handside in (2.26) is nonnegative, hence  $W_\varepsilon'(1) \geq -\varepsilon(\bar{\nu} + \varepsilon\|f\|_{Lip})\|W_\varepsilon\|_\infty$  and, in the  $y$ -variable :

$$V_\varepsilon'(\varepsilon) \geq -\|V_\varepsilon\|_\infty(\bar{\nu} + \varepsilon\|f\|_{Lip}). \quad (2.27)$$

*Step 3.* Proof of the boundedness by contradiction. Let us suppose that

$$\limsup_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^\infty} = +\infty.$$

That is, there exists a sequence  $\varepsilon_n \rightarrow 0$  such that  $\sup_{\mathbb{R}} V_{\varepsilon_n} = \sup_{(-\varepsilon_n, \varepsilon_n)} V_{\varepsilon_n} \xrightarrow{n \rightarrow \infty} +\infty$ . From (2.25), we have  $V_{\varepsilon_n}(\varepsilon_n) \geq \|V_{\varepsilon_n}\|_{\infty}(1 - K\varepsilon_n)$ . Now, recall that on  $(\varepsilon_n, +\infty)$ , the function  $V_{\varepsilon_n}$  satisfies (2.21). Multiply it by  $V'_{\varepsilon_n}$  and integrate, we get  $V_{\varepsilon_n}^{\prime 2}(\varepsilon_n) = -2F(V_{\varepsilon_n}(\varepsilon_n))$  where  $F(t) := \int_1^t f(s)ds$  is an antiderivative of  $f$ . Considering the hypotheses on  $f$  this gives

$$V'_{\varepsilon_n}(\varepsilon_n) \underset{n \rightarrow \infty}{\sim} -\sqrt{\|f\|_{Lip}} V_{\varepsilon_n}(\varepsilon_n).$$

From hypothesis (2.4) on  $f$ ,  $\|f\|_{Lip} > 2$  and we get a contradiction with (2.27) and (2.25). As a result,  $(\|V_{\varepsilon}\|_{L^{\infty}(\mathbb{R})})_{\varepsilon}$  and  $(\|V'_{\varepsilon}\|_{L^{\infty}(\mathbb{R})})_{\varepsilon}$  are uniformly bounded as  $\varepsilon$  goes to 0.

**Convergence of the stationary solutions** From the previous paragraph and Ascoli's Theorem,  $(V_{\varepsilon})_{\varepsilon}$  admits at least one accumulation point, let say  $V_0$ , and the convergence is uniform on every compact set, thus uniform on  $\mathbb{R}$  (from the monotonicity of  $V_{\varepsilon}$  outside  $(-\varepsilon, \varepsilon)$ , or even a diagonal argument). So  $V_0$  is continuous, bounded, tends to 1 at infinity. Passing to the limit in (2.20), it satisfies in the distribution sense

$$-V_0'' = f(V_0) + \bar{\nu}\delta_0(V_0(0) - V_0).$$

As the support of the Dirac distribution is reduced to  $\{0\}$ , and because of the continuity of  $V_0$ , it satisfies in the classical sense

$$\begin{cases} -V_0''(y) = f(V_0(y)), & y \in \mathbb{R} \\ V_0(\pm\infty) = 1. \end{cases} \quad (2.28)$$

The only solution of (2.28) is  $V_0 \equiv 1$ . Hence, the set  $(V_{\varepsilon})_{\varepsilon}$  admits only one accumulation point, so  $V_{\varepsilon} \rightarrow 1$  as  $\varepsilon$  goes to 0 uniformly on  $\mathbb{R}$ , and the proof of Proposition 2.3 .1 is complete.  $\square$

This convergence allows us to assert that there exist  $0 < m < M < +\infty$  such that

$$m < V_{\varepsilon}(y) < M, \quad \forall \varepsilon > 0, \quad \forall y \in \mathbb{R}. \quad (2.29)$$

Thus, any solution of (2.1) starting from an initial datum  $(u_0, v_0) \leq (\frac{\bar{\nu}}{\mu}m, m)$  will remain below  $M$ , which gives a uniform supersolution in  $\varepsilon$ .

## 2.4 Uniform bounds on the resolvents

Consider the two linear models

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu_{\varepsilon}(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = \mu_{\varepsilon}(y)u(t, x) - \nu_{\varepsilon}(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0 \end{cases} \quad (2.30)$$

and

$$\begin{cases} \partial_t u - D\partial_{xx}u = v(x, 0, t) - \bar{\mu}u & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = 0 & (x, y) \in \mathbb{R} \times \mathbb{R}^*, t > 0 \\ v(t, x, 0^+) = v(t, x, 0^-), & x \in \mathbb{R}, t > 0 \\ -d\{\partial_y v(t, x, 0^+) - \partial_y v(t, x, 0^-)\} = \bar{\mu}u(t, x) - \bar{\nu}v(t, x, 0) & x \in \mathbb{R}, t > 0. \end{cases} \quad (2.31)$$



The general goal is to give a uniform bound on the difference between solutions of the two above linear systems. We choose a sectorial operators approach to get an integral representation of the semigroups generated by (2.30) and (2.31). They both can be written in the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} \quad (2.32)$$

where  $L = L_\varepsilon$  in (2.30) and  $L = L_0$  in (2.31) are linear unbounded operators defined by

$$L_\varepsilon : \begin{cases} \mathcal{D}(L_\varepsilon) \subset X & \longrightarrow & X \\ \begin{pmatrix} u \\ v \end{pmatrix} & \longmapsto & \begin{pmatrix} D\partial_{xx}u - \bar{\mu}u + \int \nu_\varepsilon v \\ d\Delta v + \mu_\varepsilon u - \nu_\varepsilon v \end{pmatrix} \end{cases}$$

$$L_0 : \begin{cases} \mathcal{D}(L_0) \subset X & \longrightarrow & X \\ \begin{pmatrix} u \\ v \end{pmatrix} & \longmapsto & \begin{pmatrix} D\partial_{xx}u - \bar{\mu}u + \bar{\nu}v(\cdot, 0) \\ d\Delta v \end{pmatrix} \end{cases}$$

with  $X$  the space of continuous functions decaying to 0 at infinity  $\mathcal{C}_0(\mathbb{R}) \times \mathcal{C}_0(\mathbb{R}^2)$ . The domains are those of the Laplace operator, with exchange conditions included in  $\mathcal{D}(L_0)$ .

We recall the definition of a sectorial operator:

**Definition 2.4 .1.** A linear operator  $A : \mathcal{D}(A) \subset X \rightarrow X$  is **sectorial** if there are constants  $\varphi \in \mathbb{R}$ ,  $\theta \in (\frac{\pi}{2}, \pi)$ , and  $M > 0$  such that

$$\begin{cases} (i) & \rho(A) \supset S_{\theta, \varphi} := \{\lambda \in \mathbb{C} : \lambda \neq \varphi, |\arg(\lambda - \varphi)| < \theta\} \\ (ii) & \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \varphi|}, \lambda \in S_{\theta, \varphi} \end{cases} \quad (2.33)$$

**Proposition 2.4 .1.** Let  $\varphi > \max(2, 3\bar{\mu})$ ,  $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ . Then  $(L_\varepsilon, \mathcal{D}(L_\varepsilon))$  and  $(L_0, \mathcal{D}(L_0))$  are sectorial with sector  $S_{\theta, \varphi}$ ,  $\forall \varepsilon > 0$ .

Let us denote  $M_0, M_\varepsilon$  the corresponding constant for the norm of the resolvents. The proof for  $L_\varepsilon$  is quite standard and omitted here. There is a general approach of the theory in [64]. A proof for  $L_0$  can be found in [34]. Assumptions on  $\varphi$  and  $\theta$  are only technical and can be improved.

From Proposition 2.4 .1, we know (see [64] for instance) that, for all  $t > 0$ , solutions of (2.32) have the form

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{tL} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

where

$$e^{tL} = \frac{1}{2\pi i} \int_{\Gamma_{r, \vartheta}} e^{t\lambda} R(\lambda, L) d\lambda \quad (2.34)$$

for any  $r > 0, \vartheta \in (\frac{\pi}{2}, \theta)$ , where  $\Gamma_{r, \vartheta} := \{\lambda, |\arg(\lambda - \varphi - r)| = \vartheta\}$  is a counterclockwise oriented curve which encloses the spectrum of  $L$ , and will be denoted  $\Gamma$  when there is no possible confusion. Let us fix from now and for all  $r > 0$  and the angle  $\vartheta$  as above. A parametrisation of  $\Gamma_{r, \vartheta}$  is then given by  $s \in \mathbb{R} \mapsto \varphi + r + se^{i\vartheta \cdot \text{sgn}(s)}$ .

For  $(U, V) \in X$ , we will denote in this section:

$$\begin{cases} (u, v) = R(\lambda, L_0)(U, V) \\ (u_\varepsilon, v_\varepsilon) = R(\lambda, L_\varepsilon)(U, V) \end{cases}$$

that is

$$\begin{cases} (\lambda + \bar{\mu})u - D\partial_{xx}u = v(\cdot, 0) + U & x \in \mathbb{R} \\ \lambda v - \Delta v = V & (x, y) \in \mathbb{R} \times \mathbb{R}^* \\ v(x, 0^+) = v(x, 0^-) & x \in \mathbb{R} \\ -(\partial_y v(x, 0^+) - \partial_y v(x, 0^-)) = \bar{\mu}u - v(x, 0) & x \in \mathbb{R} \end{cases} \quad (2.35)$$

and

$$\begin{cases} (\lambda + \bar{\mu})u(x) - D\partial_{xx}u(x) & = \int \nu_\varepsilon(y)v(x, y)dy + U(x) \\ (\lambda + \frac{1}{\varepsilon}\nu(\frac{y}{\varepsilon}))v(x, y) - \Delta v(x, y) & = \frac{1}{\varepsilon}\mu(\frac{y}{\varepsilon})u(x) + V(x, y). \end{cases} \quad (2.36)$$

The purpose of this section is to give some estimates on the resolvents, that is on the solutions of (2.35) and (2.36). They are given in Lemma 2.4 .2 and Corollary 2.4 .4.

Lastly, let us recall (see [64] or [75]) that the Laplace operator is also sectorial, with a sector strictly containing  $S_{\theta, \varphi}$ . Thus, there exists a constant  $M > 0$  such that for  $d \in \{1, 2\}$ ,

$$\forall w \in \mathcal{C}_0(\mathbb{R}^d), \forall \lambda \in S_{\theta, \varphi}, \|w\|_\infty \leq \frac{M}{|\lambda|} \|\Delta w - \lambda w\|_\infty. \quad (2.37)$$

### 2.4 .1 Large values of $|\lambda|$

**Lemma 2.4 .2.** *There exist  $\varepsilon_0 > 0$  and a constant  $C_1$  depending only on  $D, \bar{\mu}$  and  $\vartheta$  such that for all positive  $\varepsilon < \varepsilon_0$ , for  $\beta > \frac{1}{2}$ ,*

$$\text{if } \lambda \in \Gamma_{r, \vartheta} \text{ and } |\lambda| > \varepsilon^{-\beta}, \text{ then } \|R(\lambda, L_\varepsilon)\| \leq C_1 \max(\varepsilon^\beta, \varepsilon^{2\beta-1}).$$

*Proof.* Let  $(U, V) \in X$ , and  $(u_\varepsilon, v_\varepsilon) = R(\lambda, L_\varepsilon)(U, V)$  be a solution of (2.36). Assumptions on  $\lambda$  imply for  $\varepsilon$  small enough that  $|\operatorname{Im}(\lambda)| > \frac{1}{2} \sin(\vartheta) \varepsilon^{-\beta}$ . Thus,  $\nu$  being a real nonnegative function,  $\lambda + \frac{1}{\varepsilon} \nu(\frac{y}{\varepsilon}) \in S_{\theta, \varphi}, \forall y \in \mathbb{R}$  and

$$\left| \lambda + \frac{1}{\varepsilon} \nu\left(\frac{y}{\varepsilon}\right) \right| \geq \operatorname{Im}\left(\lambda + \frac{1}{\varepsilon} \nu\left(\frac{y}{\varepsilon}\right)\right) > \frac{1}{2} \sin(\vartheta) \varepsilon^{-\beta}.$$

In the same way, we get a similar lower bound for  $|\lambda + \bar{\mu}|$ . Now we use (2.37) in (2.36) with the above estimates and get

$$\begin{cases} \|u\|_\infty \leq \varepsilon^\beta \frac{2MD^2}{\sin \vartheta} (\|U\|_\infty + \|v\|_\infty) \\ \|v\|_\infty \leq \varepsilon^\beta \frac{2M}{\sin \vartheta} (\|V\|_\infty + \frac{\|\mu\|_\infty}{\varepsilon} \|u\|_\infty). \end{cases} \quad (2.38)$$

Using the first equation of (2.38) in the second one yields

$$\|v\|_\infty \left( 1 - \varepsilon^{2\beta-1} \left( \frac{2MD}{\sin \vartheta} \right)^2 \|\mu\|_\infty \right) \leq \varepsilon^\beta \frac{2M}{\sin \vartheta} \|V\|_\infty + \varepsilon^{2\beta-1} \|\mu\|_\infty \left( \frac{2MD}{\sin \vartheta} \right)^2 \|U\|_\infty$$

i.e., for  $\varepsilon$  small enough,

$$\|v\|_\infty \leq K_1 \max(\varepsilon^\beta, \varepsilon^{2\beta-1}) (\|V\|_\infty + \|U\|_\infty) \quad (2.39)$$

with  $K_1$  depending only on  $D, \bar{\mu}, \vartheta$ . In the same vein, using (2.39) in the second equation of (2.38) produces the same estimate, and the proof is concluded.  $\square$

## 2.4 .2 Small values of $|\lambda|$

The values are treated with the help of the Fourier transform in  $x$ -direction. For  $U \in \mathcal{C}_0(\mathbb{R}) \cap L^1(\mathbb{R})$ ,  $V \in \mathcal{C}_0(\mathbb{R}^2) \cap L^1(\mathbb{R}, L^\infty(\mathbb{R}))$ , we define

$$\hat{U}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} U(x) dx \quad \hat{V}(\xi, y) := \int_{\mathbb{R}} e^{-ix\xi} V(x, y) dx.$$

**Proposition 2.4 .3.** *There exist  $\varepsilon_1$  a constant  $C_2$  depending only on  $D$ ,  $\bar{\mu}$  and  $\vartheta$  such that for all  $\varepsilon < \varepsilon_1$ , for  $\beta > \frac{1}{2}$ ,  $\gamma > 0$ , such that  $1 - \frac{3}{2}(\beta + \gamma) > 0$ , if  $\lambda \in \Gamma_{r, \vartheta}$  and  $|\lambda| < \varepsilon^{-\beta}$  then*

$$\|(R(\lambda, L_\varepsilon) - R(\lambda, L_0))(U, V)\|_\infty \leq C_2 \varepsilon^{1 - \frac{3}{2}(\beta + \gamma)} \left( \|\hat{U}\|_{L^\infty(\mathbb{R})} + \|\hat{V}\|_{L^\infty(\mathbb{R}^2)} \right).$$

**Corollary 2.4 .4.** *Under assumptions of Proposition 2.4 .3,*

$$\|(R(\lambda, L_\varepsilon) - R(\lambda, L_0))(U, V)\|_\infty \leq C_2 \varepsilon^{1 - \frac{3}{2}(\beta + \gamma)} \left( \|U\|_{L^1(x)} + \| \|V\|_{L^\infty(y)} \|_{L^1(x)} \right).$$

The proof requires two lemmas. First, we deal with the high frequencies in Lemma 2.4 .5, i.e. for  $|\xi| \gg \varepsilon^{-\beta}$ , using Lemma 1 in Appendix. Then, in Lemma 2.4 .6, we make an almost explicit computation of the solutions for small values of  $\lambda$ .

**Fourier transform of the equations** Let us consider  $U \in \mathcal{C}_0(\mathbb{R}) \cap L^1(\mathbb{R})$  and  $V \in \mathcal{C}_0(\mathbb{R}^2) \cap L^1(\mathbb{R}, L^\infty(\mathbb{R}))$ . For  $\varepsilon > 0$  and  $\lambda \in \Gamma_{r, \vartheta}$ ,  $|\lambda| < \varepsilon^{-\beta}$ , Recall that

$$\begin{cases} (u, v) = R(\lambda, L_0)(U, V) \\ (u_\varepsilon, v_\varepsilon) = R(\lambda, L_\varepsilon)(U, V) \end{cases}$$

which leads to the spectral problems (2.35) and (2.36). The Fourier transforms  $(\hat{u}, \hat{v})$  and  $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$  solve

$$\begin{cases} (D\xi^2 + \lambda + \bar{\mu})\hat{u}(\xi) = \hat{v}(\xi, 0) + \hat{U}(\xi) \\ (\xi^2 + \lambda)\hat{v} - \partial_{yy}\hat{v}(\xi, y) = \hat{V}(\xi, y) \\ \hat{v}(\xi, 0^+) = \hat{v}(\xi, 0^-) \\ -(\partial_y v(\xi, 0^+) - \partial_y v(\xi, 0^-)) = \bar{\mu}\hat{u}(\xi) - \hat{v}(\xi, 0) \end{cases} \quad (2.40)$$

and

$$\begin{cases} (D\xi^2 + \lambda + \bar{\mu})\hat{u}_\varepsilon(\xi) = \int \nu_\varepsilon(y)\hat{v}_\varepsilon(\xi, y)dy + \hat{U}(\xi) \\ (\xi^2 + \lambda + \nu_\varepsilon(y))\hat{v}_\varepsilon(\xi, y) - \partial_{yy}\hat{v}_\varepsilon(\xi, y) = \mu_\varepsilon(y)\hat{u}_\varepsilon(\xi) + \hat{V}(\xi, y). \end{cases} \quad (2.41)$$

**Lemma 2.4 .5.** *There exist  $\varepsilon_2 > 0$  and a constant  $C_3$  depending only on  $\bar{\mu}$ ,  $D$  such that for all  $\varepsilon < \varepsilon_2$ , for all  $\xi$  with  $\xi^2 \geq \varepsilon^{-\beta - \gamma}$  and  $\lambda \in \Gamma_{r, \vartheta}$  with  $|\lambda| < \varepsilon^{-\beta}$ ,*

$$\begin{aligned} |\hat{u}_\varepsilon(\xi)| + \|\hat{v}_\varepsilon(\xi)\|_\infty &\leq \frac{C_3}{\xi^2} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_\infty \right) \\ |\hat{u}(\xi)| + \|\hat{v}(\xi)\|_\infty &\leq \frac{C_3}{\xi^2} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_\infty \right) \end{aligned}$$

where  $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(y)}$ .

*Proof.* We give the proof only for the nonlocal case, the local one being easier. Combining the two equations of (2.41),  $\hat{v}_\varepsilon$  satisfies

$$-\partial_{yy}\hat{v}_\varepsilon(\xi, y) + (\xi^2 + \lambda + \nu_\varepsilon(y)) \hat{v}_\varepsilon(\xi, y) = \hat{V}(\xi, y) + \frac{\mu_\varepsilon(y)}{D\xi^2 + \lambda + \bar{\mu}} (\hat{v}_\varepsilon(\xi, 0) + \hat{U}(\xi)). \quad (2.42)$$

As  $\gamma > 0$  and considering the hypotheses on  $\lambda$  and  $\xi$ , there exists  $k > 0$  such that, for  $\varepsilon$  small enough, we have:

$$\min \left\{ \mathcal{R}e(\xi^2 + \lambda), |D\xi^2 + \lambda + \bar{\mu}| \right\} > k^2 \xi^2.$$

Now, we apply Lemma 1 in Appendix. It gives:

$$|\hat{v}_\varepsilon(\xi, y)| \leq \frac{1}{2k|\xi|} \int_{\mathbb{R}} e^{-k\xi|z|} \left( \hat{V}(\xi, z - y) + \frac{\mu_\varepsilon(z - y)}{k^2 \xi^2} (\hat{v}_\varepsilon(\xi, 0) + \hat{U}(\xi)) \right) dz.$$

A rough majoration yields

$$\begin{aligned} \|\hat{v}_\varepsilon(\xi)\|_\infty &\leq \frac{1}{k^2 \xi^2} \|\hat{V}(\xi)\|_\infty + \frac{\bar{\mu}}{k^3 |\xi|^3} (|\hat{v}_\varepsilon(\xi, 0)| + |\hat{U}(\xi)|) \\ &\leq \frac{1}{k^2 \xi^2} \left( \|\hat{V}(\xi)\|_\infty + \frac{\bar{\mu}}{k|\xi|} |\hat{U}(\xi)| + \frac{\bar{\mu}}{k|\xi|} \|\hat{v}_\varepsilon(\xi)\|_\infty \right) \end{aligned}$$

which, as  $|\xi| > \varepsilon^{-\frac{1}{2}(\beta+\gamma)}$ , provides the desired estimate on  $\|\hat{v}_\varepsilon(\xi)\|_\infty$ . The estimate on  $|\hat{u}_\varepsilon|$  follows from the first equation of (2.41), and the proof of Lemma 2.4 .5 is concluded.  $\square$

Now that we are done with the high frequencies, to finish the proof of Proposition 2.4 .3, it remains to control the lower frequencies. This is the purpose of the following Lemma.

**Lemma 2.4 .6.** *There exist  $\varepsilon_3 > 0$  and a constant  $C_5$ , for all  $\varepsilon < \varepsilon_3$ , for all  $\lambda \in \Gamma_{r,\vartheta}$  with  $|\lambda| < \varepsilon^{-\beta}$ , for all  $\xi$  with  $\xi^2 < \varepsilon^{-(\beta+\gamma)}$ ,*

$$\begin{aligned} \|\hat{v}_\varepsilon(\xi) - \hat{v}(\xi)\|_{L^\infty(y)} &\leq C_5 \varepsilon^{1-(\beta+\gamma)} (|\hat{U}(\xi)| + \|\hat{V}(\xi)\|_{L^\infty(y)}) \\ |\hat{u}_\varepsilon(\xi) - \hat{u}(\xi)| &\leq C_5 \varepsilon^{1-(\beta+\gamma)} (|\hat{U}(\xi)| + \|\hat{V}(\xi)\|_{L^\infty(y)}). \end{aligned}$$

**Proof of Proposition 2.4 .3 thanks to Lemma 2.4 .6** With the same notations,

$$\|(R(\lambda, L_\varepsilon) - R(\lambda, L_0))(U, V)\|_\infty = \max \left( \|u - u_\varepsilon\|_{L^\infty(\mathbb{R})}, \|v - v_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \right).$$

Let us prove the domination for  $\|v - v_\varepsilon\|_{L^\infty(\mathbb{R}^2)}$ , the one in  $(u - u_\varepsilon)$  being similar. For  $(x, y) \in \mathbb{R}^2$ ,

$$\begin{aligned} (v - v_\varepsilon)(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} (\hat{v}(\xi, y) - \hat{v}_\varepsilon(\xi, y)) d\xi. \\ |(v - v_\varepsilon)(x, y)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{v}(\xi, y) - \hat{v}_\varepsilon(\xi, y)| d\xi \\ &\leq \frac{1}{2\pi} \int_{|\xi| \geq \varepsilon^{-\frac{1}{2}(\beta+\gamma)}} \|\hat{v}(\xi)\|_{L^\infty(y)} + \|\hat{v}_\varepsilon(\xi)\|_{L^\infty(y)} d\xi \quad (2.43) \end{aligned}$$

$$+ \frac{1}{2\pi} \int_{|\xi| < \varepsilon^{-\frac{1}{2}(\beta+\gamma)}} \|\hat{v}(\xi) - \hat{v}_\varepsilon(\xi)\|_{L^\infty(y)} d\xi. \quad (2.44)$$

Now, from Lemma 2.4 .5, we have:

$$(2.43) \leq \frac{C_3}{\pi} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_\infty \right) \int_{\varepsilon^{-\frac{1}{2}(\beta+\gamma)}}^{+\infty} \frac{d\xi}{\xi^2} \\ \leq \frac{C_3}{\pi} \varepsilon^{\frac{1}{2}(\beta+\gamma)} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_\infty \right).$$

From Lemma 2.4 .6, we have:

$$(2.44) \leq \frac{C_5}{2\pi} \varepsilon^{1-(\beta+\gamma)} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_{L^\infty(y)} \right) \int_{-\varepsilon^{-\frac{1}{2}(\beta+\gamma)}}^{\varepsilon^{-\frac{1}{2}(\beta+\gamma)}} 1d\xi \\ \leq \frac{C_5}{\pi} \varepsilon^{1-\frac{3}{2}(\beta+\gamma)} \left( |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_{L^\infty(y)} \right).$$

Finally, from the choice of  $\beta, \gamma, \frac{\beta+\gamma}{2} > 1 - \frac{3}{2}(\beta+\gamma)$  and we have the required estimate.  $\square$

## Proof of Lemma 2.4 .6

The proof requires some explicit computations of the solutions of the spectral problems and is a bit long. First, we compute  $(\hat{u}, \hat{v})$ , solution of (2.40), introducing four constants  $K_1^+, K_2^+, K_1^-, K_2^-$ . Secondly, we do the same for  $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$  outside the strip  $\mathbb{R} \times (-\varepsilon, \varepsilon)$ . This leads us to introduce four constants  $K_1^+(\varepsilon), K_2^+(\varepsilon), K_1^-(\varepsilon)$ , and  $K_2^-(\varepsilon)$  which determine the behaviour of  $\hat{v}_\varepsilon$  outside the strip. Then, we show that it is enough to focus on  $K_2^\pm(\varepsilon)$  to get a global control of  $\hat{v}_\varepsilon$ . In a short paragraph, we establish that the derivative  $\partial_y \hat{v}_\varepsilon$  is controlled by the norm of  $\hat{v}_\varepsilon$ . The last paragraph of the proof is devoted to the computation of  $K_2^\pm(\varepsilon)$  and an estimate of its difference with  $K_2^\pm$ .

**Explicit computation of  $(\hat{u}, \hat{v})$**  Our choice of  $\Gamma_{r,\vartheta}$  allows us to choose a unique determination of the complex logarithm for all systems. From now and until the end of this proof we set for all  $\xi$  and  $\lambda$  satisfying the hypotheses of Lemma 2.4 .6

$$\alpha := \sqrt{\xi^2 + \lambda}, \quad \mathcal{R}e(\alpha) > 0 \quad (2.45)$$

the unique complex root of  $(\xi^2 + \lambda)$  with positive real part and

$$\omega := D\xi^2 + \bar{\mu} + \lambda. \quad (2.46)$$

The choice of  $\varphi, \theta$  (see the hypotheses of Proposition 2.4 .1) yields  $\min_{\lambda \in \Gamma_{r,\vartheta}} |\lambda| > \max(\sqrt{2}, 2\bar{\mu})$ .

Moreover, we have  $\xi^2 > 0$  and  $D\xi^2 + \bar{\mu} > 0$ . Then,

$$\forall \bar{\lambda} \in \Gamma_{r,\vartheta}, \quad \max(\sqrt{2}, 2\bar{\mu}) < \min_{\lambda \in \Gamma_{r,\vartheta}} |\lambda| \leq \min(D\xi^2 + \bar{\mu} + \bar{\lambda}, \xi^2 + \bar{\lambda}). \quad (2.47)$$

Hence, we can assert:

$$\forall \lambda \in \Gamma_{r,\vartheta}, \xi \in \mathbb{R}, \quad 2 < 2 + \frac{1}{\alpha} \left( 1 - \frac{\bar{\mu}}{\omega} \right) < 2 + 2^{-\frac{1}{4}} + \frac{1}{\sqrt{2\bar{\mu}}}. \quad (2.48)$$

Moreover, considering the hypotheses on  $\xi$  and  $\lambda$ , there exists a constant  $k > 0$  such that, for  $\varepsilon$  small enough,

$$1 < |\alpha| < k\varepsilon^{-\frac{1}{2}(\beta+\gamma)}, \quad 2\bar{\mu} \leq |\omega| < k\varepsilon^{-(\beta+\gamma)}, \quad |e^{\varepsilon\alpha}| < 2. \quad (2.49)$$

The first equation of (2.40) gives

$$\hat{u}(\xi) = \frac{1}{\omega} \left( \hat{v}(\xi, 0) + \hat{U}(\xi) \right).$$

Integrating the second equation of (2.40) yields the existence of four constants  $K_1^+$ ,  $K_2^+$ ,  $K_1^-$ ,  $K_2^-$ , depending on  $\xi$ , such that

$$\begin{cases} \hat{v}(\xi, y) = e^{\alpha y} \left( K_1^+ - \frac{1}{2\alpha} \int_0^y e^{-\alpha z} \hat{V}(\xi, z) dz \right) + e^{-\alpha y} \left( K_2^+ + \frac{1}{2\alpha} \int_0^y e^{\alpha z} \hat{V}(\xi, z) dz \right) & y > 0 \\ \hat{v}(\xi, y) = e^{-\alpha y} \left( K_1^- - \frac{1}{2\alpha} \int_y^0 e^{\alpha z} \hat{V}(\xi, z) dz \right) + e^{\alpha y} \left( K_2^- + \frac{1}{2\alpha} \int_y^0 e^{-\alpha z} \hat{V}(\xi, z) dz \right) & y < 0. \end{cases} \quad (2.50)$$

The integrability of  $\hat{v}$  in  $y$  gives

$$K_1^+ = \frac{1}{2\alpha} \int_0^\infty e^{-\alpha z} \hat{V}(\xi, z) dz, \quad K_1^- = \frac{1}{2\alpha} \int_{-\infty}^0 e^{\alpha z} \hat{V}(\xi, z) dz. \quad (2.51)$$

The continuity and exchange conditions at  $y = 0$  impose

$$\begin{cases} K_1^+ + K_2^+ = K_1^- + K_2^- \\ \alpha \left( K_2^+ - K_1^+ + K_2^- - K_1^- \right) = \frac{\bar{\mu}}{\omega} \left( K_1^+ + K_2^+ + \hat{U}(\xi) \right) - K_1^+ - K_2^+. \end{cases} \quad (2.52)$$

Combining these two equations yields

$$\begin{cases} K_2^+ \left( 2 + \frac{1}{\alpha} \left( 1 - \frac{\bar{\mu}}{\omega} \right) \right) = 2K_1^- + K_1^+ \frac{1}{\alpha} \left( \frac{\bar{\mu}}{\omega} - 1 \right) + \frac{\bar{\mu}}{\alpha\omega} \hat{U}(\xi) \\ K_2^- \left( 2 + \frac{1}{\alpha} \left( 1 - \frac{\bar{\mu}}{\omega} \right) \right) = 2K_1^+ + K_1^- \frac{1}{\alpha} \left( \frac{\bar{\mu}}{\omega} - 1 \right) + \frac{\bar{\mu}}{\alpha\omega} \hat{U}(\xi). \end{cases} \quad (2.53)$$

From (2.48), the above system (2.53) is well-posed. From (2.50), (2.51) and (2.53) we have an explicit formula for  $(\hat{u}(\xi), \hat{v}(\xi))$ .

### Study of $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$

**Explicit formula** In the same way as above, the first equation of (2.41) yields

$$\hat{u}_\varepsilon(\xi) = \frac{1}{\omega} \left( \int_{\mathbb{R}} \nu_\varepsilon(y) \hat{v}_\varepsilon(\xi, y) dy + \hat{U}(\xi) \right).$$

Integrating the second equation of (2.41) leads us to set four constants  $K_1^+(\varepsilon)$ ,  $K_2^+(\varepsilon)$ ,  $K_1^-(\varepsilon)$ ,  $K_2^-(\varepsilon)$ , depending on  $\varepsilon$  and  $\xi$ , such that

$$y > \varepsilon : \hat{v}_\varepsilon(\xi, y) = e^{\alpha y} \left( K_1^+(\varepsilon) - \frac{1}{2\alpha} \int_\varepsilon^y e^{-\alpha z} \hat{V}(\xi, z) dz \right) + e^{-\alpha y} \left( K_2^+(\varepsilon) + \frac{1}{2\alpha} \int_\varepsilon^y e^{\alpha z} \hat{V}(\xi, z) dz \right) \quad (2.54)$$

$$y < -\varepsilon : \hat{v}_\varepsilon(\xi, y) = e^{-\alpha y} \left( K_1^-(\varepsilon) - \frac{1}{2\alpha} \int_y^{-\varepsilon} e^{\alpha z} \hat{V}(\xi, z) dz \right) + e^{\alpha y} \left( K_2^-(\varepsilon) + \frac{1}{2\alpha} \int_y^{-\varepsilon} e^{-\alpha z} \hat{V}(\xi, z) dz \right). \quad (2.55)$$

For the same integrability reason as in the limit case, we already have an explicit formula for  $K_1^\pm(\varepsilon)$  :

$$K_1^+(\varepsilon) = \frac{1}{2\alpha} \int_\varepsilon^\infty e^{-\alpha z} \hat{V}(\xi, z) dz, \quad K_1^-(\varepsilon) = \frac{1}{2\alpha} \int_{-\infty}^{-\varepsilon} e^{\alpha z} \hat{V}(\xi, z) dz, \quad (2.56)$$

which immediately gives us a uniform boundary and, combining with (2.51), the first following estimate

$$\left| K_1^\pm - K_1^\pm(\varepsilon) \right| \leq \frac{\varepsilon}{2\alpha} \left\| \hat{V}(\xi) \right\|_{L^\infty(y)}. \quad (2.57)$$

It remains to determine  $K_2^\pm(\varepsilon)$ . We set

$$z = \frac{y}{\varepsilon} \text{ and } \hat{v}_\varepsilon(z) := \hat{v}_\varepsilon(y) \text{ for } z \in (-1, 1). \quad (2.58)$$

The equation for  $\hat{v}_\varepsilon(\xi, z)$ , now set for  $z \in (-1, 1)$ , is

$$(\varepsilon^2 \xi^2 + \varepsilon^2 \lambda + \varepsilon \nu(z)) \hat{v}_\varepsilon(\xi, z) - \partial_{zz} \hat{v}_\varepsilon(\xi, z) = \varepsilon \mu(z) \frac{1}{\omega} \left( \int_{\mathbb{R}} \nu(z) \hat{v}_\varepsilon(\xi, z) dz + \hat{U}(\xi) \right) + \varepsilon^2 \hat{V}(\xi, \varepsilon z). \quad (2.59)$$

Specifying (2.54) and (2.55) at  $y = \varepsilon$  and  $y = -\varepsilon$  gives us the two following boundary conditions for (2.59):

$$\begin{cases} \hat{v}_\varepsilon(1, \xi) &= K_1^+(\varepsilon) e^{\alpha \varepsilon} + K_2^+(\varepsilon) e^{-\alpha \varepsilon} \\ \hat{v}_\varepsilon(-1, \xi) &= K_1^-(\varepsilon) e^{\alpha \varepsilon} + K_2^-(\varepsilon) e^{-\alpha \varepsilon}. \end{cases} \quad (2.60)$$

$$\begin{cases} \partial_z \hat{v}_\varepsilon(1, \xi) &= \varepsilon \alpha \left( K_1^+(\varepsilon) e^{\alpha \varepsilon} - K_2^+(\varepsilon) e^{-\alpha \varepsilon} \right) \\ \partial_z \hat{v}_\varepsilon(-1, \xi) &= \varepsilon \alpha \left( K_2^-(\varepsilon) e^{-\alpha \varepsilon} - K_1^-(\varepsilon) e^{\alpha \varepsilon} \right). \end{cases} \quad (2.61)$$

**Blow-up condition for  $\hat{v}_\varepsilon$**  From now, we are only considering the rescaled equation (2.59) with the boundary conditions (2.60) and (2.61). Hence, all functions and derivatives are to be considered in these rescaled variables (2.58). We first show that the  $L^\infty(z)$ -norm of  $\hat{v}_\varepsilon$  is controlled by  $K_2^\pm(\varepsilon)$ . We have:

$$\begin{aligned} \hat{v}_\varepsilon(z) - \hat{v}_\varepsilon(-1) &= (z+1) \hat{v}'_\varepsilon(-1) + \int_{-1}^z \int_{-1}^s \hat{v}''_\varepsilon(u) du ds \\ &= (z+1) \hat{v}'_\varepsilon(-1) + \int_{-1}^z \int_{-1}^s \hat{v}_\varepsilon(u) \left( \varepsilon \nu(u) + \varepsilon^2 \xi^2 + \varepsilon^2 \lambda \right) du ds \\ &\quad - \int_{-1}^z \int_{-1}^s \varepsilon \frac{\mu(u)}{\omega} \left( \int \nu \hat{v}_\varepsilon + \hat{U}(\xi) \right) + \varepsilon^2 \hat{V}(\xi, \varepsilon u) du ds. \end{aligned}$$

Hence

$$\begin{aligned} \|\hat{v}_\varepsilon\|_{L^\infty(-1,1)} &\leq |\hat{v}_\varepsilon(-1)| + 2 |\hat{v}'_\varepsilon(-1)| + \varepsilon \|\hat{v}_\varepsilon\|_\infty \left( 2 + \frac{2\bar{\mu}}{|\omega|} + 4\varepsilon |\xi^2 + \lambda| \right) \\ &\quad + \varepsilon \frac{2\bar{\mu}}{|\omega|} |\hat{U}(\xi)| + 4\varepsilon^2 \|\hat{V}(\xi)\|_\infty \\ &\leq \left( |K_1^-(\varepsilon)| + |K_2^-(\varepsilon)| \right) (2 + 4|\varepsilon\alpha|) + \varepsilon \|\hat{v}_\varepsilon\|_\infty \left( 4 + 4k^2 \varepsilon^{1-(\beta+\gamma)} \right) \\ &\quad + 2\varepsilon |\hat{U}(\xi)| + 4\varepsilon^2 \|\hat{V}(\xi)\|_\infty. \end{aligned}$$

Now let us recall that  $K_1^-(\varepsilon)$  is uniformly bounded in  $\varepsilon$  from (2.57), we have  $\beta + \gamma < 1$ , and  $\xi \mapsto (\hat{U}(\xi), \hat{V}(\xi))$  is uniformly bounded. These facts, combined with the above inequality and the symmetry of the problem, allow us to assert that

$$\left( \limsup_{\varepsilon \rightarrow 0} \|\hat{v}_\varepsilon\|_{L^\infty(y)} = +\infty \right) \Leftrightarrow \left( \limsup_{\varepsilon \rightarrow 0} |K_2^-(\varepsilon)| = +\infty \right) \Leftrightarrow \left( \limsup_{\varepsilon \rightarrow 0} |K_2^+(\varepsilon)| = +\infty \right) \quad (2.62)$$

uniformly in  $\xi \in \mathbb{R}$ .

**Control of the derivative** In the same way as above we get a control of  $\|\hat{v}'_\varepsilon\|_\infty$  by  $\|\hat{v}_\varepsilon\|_\infty$  with a simple integration of (2.59):

$$\begin{aligned} \hat{v}'_\varepsilon(z) - \hat{v}'_\varepsilon(-1) &= \varepsilon \int_{-1}^z \hat{v}_\varepsilon(s) (\nu(s) + \varepsilon \xi^2 + \varepsilon \lambda) - \frac{\mu(s)}{\omega} \left( \int \nu \hat{v}_\varepsilon + \hat{U}(\xi) \right) - \varepsilon \hat{V}(\xi, \varepsilon s) ds \\ |\hat{v}'_\varepsilon(s) - \hat{v}'_\varepsilon(-1)| &\leq \varepsilon \|\hat{v}_\varepsilon\|_\infty (2 + k^2 \varepsilon^{1-(\beta+\gamma)}) + \varepsilon (|\hat{U}(\xi)| + 4\varepsilon \|\hat{V}(\xi)\|_\infty). \end{aligned}$$

So for  $\varepsilon$  small enough,

$$\|\hat{v}'_\varepsilon\|_{L^\infty(-1,1)} \leq 8\varepsilon (\|\hat{v}_\varepsilon\|_\infty + 1) + 4\varepsilon (|\hat{U}(\xi)| + \varepsilon \|\hat{V}(\xi)\|_\infty). \quad (2.63)$$

**Explicit computation of  $K_2^\pm(\varepsilon)$**  We are now ready to prove Lemma 2.4 .6. The only argument we will use is, once again, an integration of (2.59). It yields:

$$\begin{aligned} \hat{v}_\varepsilon(1) - \hat{v}_\varepsilon(-1) &= 2\hat{v}'_\varepsilon(-1) + \varepsilon \int_{-1}^1 \int_{-1}^z \hat{v}_\varepsilon(s) (\nu(s) + \varepsilon \xi^2 + \varepsilon \lambda) ds dz \\ &\quad - \varepsilon \int_{-1}^1 \int_{-1}^z \frac{\mu(s)}{\omega} \left( \int \nu \hat{v}_\varepsilon + \hat{U}(\xi) \right) + \varepsilon \hat{V}(\xi, \varepsilon s) ds dz. \end{aligned}$$

Hence, with (2.60) and the estimate (2.49) on  $\alpha$ , we have

$$\begin{aligned} &\left| e^{-\alpha\varepsilon} (K_2^+(\varepsilon) - K_2^-(\varepsilon)) - e^{\alpha\varepsilon} (K_1^-(\varepsilon) - K_1^+(\varepsilon)) \right| \\ &\leq 2|\hat{v}'_\varepsilon(-1)| + \varepsilon \|\hat{v}_\varepsilon\|_\infty (6 + 4k^2 \varepsilon^{1-(\beta+\gamma)}) + \varepsilon (4|\hat{U}(\xi)| + 4\varepsilon \|\hat{V}(\xi)\|_\infty). \end{aligned} \quad (2.64)$$

In the same fashion,

$$\hat{v}'_\varepsilon(1) - \hat{v}'_\varepsilon(-1) = \varepsilon \int_{-1}^1 \hat{v}_\varepsilon(z) (\nu(z) + \varepsilon(\xi^2 + \lambda)) - \frac{\mu(z)}{\omega} \left( \int \nu \hat{v}_\varepsilon + \hat{U} \right) - \hat{V}(\varepsilon z) dz.$$

Hence, using (2.61),

$$\begin{aligned} &\left| \alpha e^{\alpha\varepsilon} (K_1^+(\varepsilon) + K_1^-(\varepsilon)) - \alpha e^{-\alpha\varepsilon} (K_2^+(\varepsilon) + K_2^-(\varepsilon)) - \left(1 - \frac{\bar{\mu}}{\omega}\right) \int \nu \hat{v}_\varepsilon + \frac{\bar{\mu}}{\omega} \hat{U}(\xi) \right| \\ &\leq 2k^2 \varepsilon^{1-(\beta+\gamma)} \|\hat{v}_\varepsilon\|_\infty + 2\varepsilon \|\hat{V}(\xi)\|_\infty. \end{aligned} \quad (2.65)$$

Now we just do a Taylor-Lagrange expansion: for all  $\varepsilon, \xi, \lambda$ , there exists a function  $c : [-1, 1] \mapsto [-1, 1]$  such that

$$\int_{-1}^1 \nu(z) \hat{v}_\varepsilon(z) dz = (K_1^+(\varepsilon) e^{\alpha\varepsilon} + K_2^+(\varepsilon) e^{-\alpha\varepsilon}) + \int_{-1}^1 \hat{v}'_\varepsilon(c(z)) (z-1) \nu(z) dz. \quad (2.66)$$



Using (2.66) in (2.65) gives

$$\begin{aligned} & \left| e^{-\alpha\varepsilon} K_2^+(\varepsilon) \left( \alpha + \left( 1 - \frac{\bar{\mu}}{\omega} \right) \right) + \alpha e^{-\alpha\varepsilon} K_2^-(\varepsilon) + e^{\alpha\varepsilon} K_1^+(\varepsilon) \left( \left( 1 - \frac{\bar{\mu}}{\omega} \right) - \alpha \right) \right. \\ & \quad \left. - \alpha e^{\alpha\varepsilon} K_1^-(\varepsilon) - \frac{\bar{\mu}}{\omega} \hat{U}(\xi) \right| \leq 2 \|\hat{v}'_\varepsilon\|_\infty + 2k^2 \varepsilon^{1-(\beta+\gamma)} \|\hat{v}_\varepsilon\|_\infty + 2\varepsilon \|\hat{V}(\xi)\|_\infty. \end{aligned} \quad (2.67)$$

At this point, we have a system of two inequations (2.64) and (2.67) which will allow us to compute an approximation of  $K_2^+(\varepsilon)$  and  $K_2^-(\varepsilon)$ . We will give details only for  $K_2^+(\varepsilon)$ , the other case being similar. Let us consider  $e^{\alpha\varepsilon} \left[ (2.64) - \frac{1}{\alpha}(2.67) \right]$ , still using (2.49). This reads:

$$\begin{aligned} & \left| K_2^+(\varepsilon) \left( 2 + \frac{1}{\alpha} \left( 1 - \frac{\bar{\mu}}{\omega} \right) \right) - 2e^{2\alpha\varepsilon} K_1^-(\varepsilon) - e^{2\alpha\varepsilon} K_1^+(\varepsilon) \frac{1}{\alpha} \left( \frac{\bar{\mu}}{\omega} - 1 \right) - e^{\alpha\varepsilon} \frac{\bar{\mu}}{\alpha\omega} \hat{U}(\xi) \right| \\ & \leq 4 |\hat{v}'_\varepsilon(-1)| + \varepsilon \|\hat{v}_\varepsilon\|_\infty \left( 6 + 4k^2 \varepsilon^{1-(\beta+\gamma)} \right) + 2\varepsilon \left( 4 |\hat{U}(\xi)| + 4\varepsilon \|\hat{V}(\xi)\|_\infty \right) + \\ & \quad 4 \|\hat{v}'_\varepsilon\|_\infty + 4k^2 \varepsilon^{1-(\beta+\gamma)} \|\hat{v}_\varepsilon\|_\infty + 4\varepsilon \|\hat{V}(\xi)\|_\infty. \end{aligned}$$

Let us recall that (2.63) gives us a control of  $\|\hat{v}'_\varepsilon\|_{L^\infty(z)}$  by  $\varepsilon \|\hat{v}\|_{L^\infty(z)}$  in the strip  $[-1, 1]$ . Thus, for some constant  $C_6$ , we have:

$$\begin{aligned} & \left| K_2^+(\varepsilon) \left( 2 + \frac{1}{\alpha} \left( 1 - \frac{\bar{\mu}}{\omega} \right) \right) - 2K_1^-(\varepsilon) - K_1^+(\varepsilon) \frac{1}{\alpha} \left( \frac{\bar{\mu}}{\omega} - 1 \right) - \frac{\bar{\mu}}{\alpha\omega} \hat{U}(\xi) \right| \\ & \leq C_6 \left( |\varepsilon\alpha| \left( |K_1^+(\varepsilon)| + |K_1^-(\varepsilon)| \right) + \varepsilon^{1-(\beta+\gamma)} \|\hat{v}_\varepsilon\|_\infty + \varepsilon |\hat{U}(\xi)| + \varepsilon \|\hat{V}(\xi)\|_\infty \right). \end{aligned} \quad (2.68)$$

The last expression (2.68) combined with the control of  $\|\hat{v}_\varepsilon\|_\infty$  by  $K_2^+(\varepsilon)$  given in (2.62) allows us to assert that  $\left( \|z \mapsto \hat{v}_\varepsilon(\xi, z)\|_{L^\infty(-1,1)} \right)_\varepsilon$  is uniformly bounded on  $\varepsilon, \xi, \lambda$  under assumptions of Lemma 2.4 .6, and so is  $\left( \|y \mapsto \hat{v}_\varepsilon(\xi, y)\|_{L^\infty(\mathbb{R})} \right)_\varepsilon$  with (2.54) and (2.55). Comparing (2.68) with (2.53) and using the previous estimate (2.57) and the explicit formula for  $K_1^\pm(\varepsilon)$  yields, for some constant  $C_7$  :

$$\left| K_2^+(\varepsilon) - K_2^+ \right| \leq C_7 \varepsilon^{1-\beta-\gamma} \left( \|\hat{v}(\xi)\|_{L^\infty(y)} + |\hat{U}(\xi)| + \|\hat{V}(\xi)\|_{L^\infty(y)} \right). \quad (2.69)$$

Now we are done with the rescaled variables. To conclude the proof of Lemma 2.4 .6, we compute directly the difference from (2.50) and (2.54), (2.55). We have the explicit formulas (2.51) and (2.56), the other terms being treated by (2.69). As for our previous estimate (2.69), we will only focus on the case  $y > 0$ , the other one  $y < 0$  being similar. All in all, we have

$$\begin{aligned} |\hat{v}_\varepsilon(\xi, y) - \hat{v}(\xi, y)| & \leq e^{-\alpha y} \left| K_2^+(\varepsilon) - K_2^+ \right| + \frac{1}{|2\alpha|} \int_0^\varepsilon \left| e^{\alpha(z-y)} \hat{V}(\xi, z) \right| dz \\ & \leq C_7 \varepsilon^{1-\beta-\gamma} \left( \|\hat{v}(\xi)\|_\infty + |\hat{U}(\xi)| + 2\|\hat{V}(\xi)\|_\infty \right) \end{aligned}$$

and the proof of Lemma 2.4 .6 is finished.  $\square$

## 2.5 Finite time convergence

In this section, we finish the proof of Theorem 2.1 .3. The first ingredient is Proposition 2.5 .1 which is a corollary of our estimates on the resolvents. It gives a control of the

semigroups generated by the linear operators. The second ingredient is Lemma 2.5 .2 which gives some  $L^\infty$  and decay estimates on the solutions. At last, we will use these two ingredients in a Gronwall argument to deal with the nonlinearity.

### 2.5 .1 Difference between the semigroups

Combined with the explicit formula (2.34), the results given in Lemma 2.4 .2 and Corollary 2.4 .4 allow us to assert the following estimate on the difference between the analytic semigroups.

**Proposition 2.5 .1.** *Let  $t \in (0, T)$ ,  $\beta > \frac{1}{2}$  and  $\gamma > 0$  such that  $1 - \frac{3}{2}(\beta + \gamma) > 0$ . There exists a constant  $C_8$  depending only on  $r, \vartheta, D, \bar{\mu}$  such that, for  $\varepsilon$  small enough,*

$$\begin{aligned} \left\| \left( e^{tL_0} - e^{tL_\varepsilon} \right) (U, V) \right\|_\infty &\leq \varepsilon^{1 - \frac{3}{2}(\beta + \gamma)} \left( \|U\|_{L^1(x)} + \| \|V\|_{L^\infty(y)} \|_{L^1(x)} \right) C_8 \left( e^{TC_8} + \frac{1}{t} \right) \\ &\quad + \varepsilon^{2\beta - 1} \|(U, V)\|_\infty \frac{C_8}{t} e^{-\frac{t}{C_8 \varepsilon^\beta}}. \end{aligned}$$

*Proof.*

$$\begin{aligned} \left\| \left( e^{tL_0} - e^{tL_\varepsilon} \right) (U, V) \right\|_\infty &\leq \frac{1}{2\pi} \int_\Gamma |e^{t\lambda}| \cdot \|(R(\lambda, L_0) - R(\lambda, L_\varepsilon)) (U, V)\|_\infty d\lambda \\ &\leq \frac{1}{2\pi} \int_{\lambda \in \Gamma, |\lambda| > \varepsilon^{-\beta}} |e^{t\lambda}| \cdot (\|R(\lambda, L_0)\| + \|R(\lambda, L_\varepsilon)\|) \|(U, V)\|_\infty d\lambda \end{aligned} \quad (2.70)$$

$$+ \frac{1}{2\pi} \int_{\lambda \in \Gamma, |\lambda| < \varepsilon^{-\beta}} |e^{t\lambda}| \cdot \|(R(\lambda, L_0) - R(\lambda, L_\varepsilon)) ((U, V))\|_\infty d\lambda. \quad (2.71)$$

We recall that  $\frac{\pi}{2} < \vartheta < \frac{3\pi}{4}$ . Hence, for large  $\lambda$ , the curve  $\Gamma_{r, \vartheta}$  lies in the half-plane  $\{z, \operatorname{Re}(z) < 0\}$ . From Lemma 2.4 .2 (and Proposition 2.4 .1 for  $L_0$ ), the first term of the right handside of the above inequality satisfies, for some constant  $C$ ,

$$(2.70) \leq 2 \int_{s > \varepsilon^{-\beta}} C e^{ts \cos \vartheta} \varepsilon^{2\beta - 1} \|(U, V)\|_\infty ds.$$

The second term satisfies from Corollary 2.4 .4

$$(2.71) \leq \varepsilon^{1 - \frac{3}{2}(\beta + \gamma)} \left( \|U\|_{L^1(x)} + \| \|V\|_{L^\infty(y)} \|_{L^1(x)} \right) \int_{\lambda \in \Gamma, |\lambda| < \varepsilon^{-\beta}} |C e^{t\lambda}| d\lambda.$$

It remains to notice that  $\Gamma \cap \{z, \operatorname{Re}(z) \geq 0\}$  is bounded, and the proof of Proposition 2.5 .1 is complete.  $\square$

Remark that  $e^{tL_\varepsilon}, e^{tL_0} \rightarrow Id$  as  $t \rightarrow 0$ . So the estimate given in Proposition 2.5 .1 is far from optimal, especially for small  $t$ . But it will be enough for our purpose.

## 2.5 .2 Uniform decay in $x$

**Lemma 2.5 .2.** *Let  $(u, v)(t)$  be the solution of (2.2), and, for all  $\varepsilon \in (0, 1)$ ,  $(u_\varepsilon, v_\varepsilon)(t)$  the solution of (2.1), both with initial datum  $(u_0, v_0)$ . Then, there exists  $K_2, \bar{\lambda}, \bar{c} > 0$  independent of  $\varepsilon$  such that for all  $(x, y) \in \mathbb{R}^2, t > 0$ ,*

$$\max (u(t, x), u_\varepsilon(t, x), v(t, x, y), v_\varepsilon(t, x, y)) \leq K_2 e^{-\bar{\lambda}(|x| - \bar{c}t)}.$$

*Proof. First case:  $D > 2d$ .* Let  $\bar{c}$  be the unique positive solution of the equation in  $c$

$$\lambda_2^-(c) := \frac{c - \sqrt{c^2 - c_{KPP}^2}}{2d} = \frac{c}{D}.$$

To this velocity  $\bar{c}$  we associate the decay rate  $\bar{\lambda} := \frac{\bar{c}}{D}$ . Thus, from simple geometric considerations (see Figure 2.1), we have for all  $\varepsilon > 0$  that  $c_\varepsilon^*, c_0^* < \bar{c}$  and  $\lambda_\varepsilon^*, \lambda_0^* > \bar{\lambda}$ . Hence, for all  $t > 0, x \in \mathbb{R}, \varepsilon > 0$ ,

$$e^{-\lambda_\varepsilon^*(|x| - c_\varepsilon^*t)} < e^{-\bar{\lambda}(|x| - \bar{c}t)}.$$

We recall that the linear travelling waves  $(c_\varepsilon^*, \lambda_\varepsilon^*, \phi_\varepsilon^*)$  and  $(c_0^*, \lambda_0^*, \phi^*)$  are supersolutions for (2.1) and (2.2). From Lemma 2.2 .4, as  $(u_0, v_0)$  is continuous and compactly supported, we know that there exists a constant  $K_1$  such that

$$\begin{cases} u_0(x) \leq K_1 e^{-\bar{\lambda}|x|} & \forall x \in \mathbb{R} \\ v_0(x, y) \leq K_1 e^{-\bar{\lambda}|x|} \min\{\phi^*(y), \inf_{\varepsilon \in (0,1)} \phi_\varepsilon^*(y)\} & \forall (x, y) \in \mathbb{R}^2. \end{cases}$$

From Lemma 2.2 .3, there exists a constant  $K_2$  such that

$$\sup_{\varepsilon \geq 0} \sup_{y \in \mathbb{R}} \phi_\varepsilon(y) \leq K_2.$$

up to replace  $K_2$  by  $\max(K_1, K_2)$ , the proof is completed.

*Second case:  $D \leq 2d$ .* In this case, for all  $\varepsilon > 0$ ,  $c_0^* = c_\varepsilon^* = c_{KPP}$ . Let us choose  $\bar{c} > c_{KPP}$  and  $\bar{\lambda} \in \left(\lambda_2^-(\bar{c}), \frac{\bar{c}}{D}\right)$  and we conclude in the same fashion.  $\square$

## 2.5 .3 Proof of Theorem 2.1 .3

For  $(u, v) \in X$ , set  $F(u, v) := (0, f(v))$  the nonlinear term in the studied systems. From the regularity of  $F$  and Proposition 2.4 .1, the solution of (2.2)  $(u, v)$  and of (2.1)  $(u_\varepsilon, v_\varepsilon)$  can be written in the form

$$\begin{aligned} (u, v)(t) &= e^{tL_0}(u_0, v_0) + \int_0^t e^{(t-s)L_0} F(u(s), v(s)) ds \\ (u_\varepsilon, v_\varepsilon)(t) &= e^{tL_\varepsilon}(u_0, v_0) + \int_0^t e^{(t-s)L_\varepsilon} F(u_\varepsilon(s), v_\varepsilon(s)) ds. \end{aligned}$$

Set  $0 < \tau_1 < T$ . For all  $t \in (\tau_1, T)$ ,

$$\begin{aligned}
 (u, v)(t) - (u_\varepsilon, v_\varepsilon)(t) &= (e^{tL_0} - e^{tL_\varepsilon})(u_0, v_0) \\
 &\quad + \int_0^t e^{(t-s)L_0} F(u(s), v(s)) - e^{(t-s)L_\varepsilon} F(u_\varepsilon(s), v_\varepsilon(s)) ds \\
 &= (e^{tL_0} - e^{tL_\varepsilon})(u_0, v_0) \\
 &\quad + \int_0^t (e^{(t-s)L_0} - e^{(t-s)L_\varepsilon}) F(u_\varepsilon(s), v_\varepsilon(s)) ds \\
 &\quad + \int_0^t e^{(t-s)L_0} (F(u(s), v(s)) - F(u_\varepsilon(s), v_\varepsilon(s))) ds.
 \end{aligned}$$

It is easy to see (cf [22]) that for all  $t > 0$ ,  $\|e^{tL_0}\| \leq \max(1, \frac{1}{\mu})$ . Set

$$\delta(t) := \|(u, v)(t) - (u_\varepsilon, v_\varepsilon)(t)\|_\infty, \quad \alpha = \min(1 - \frac{3}{2}(\beta + \gamma), 2\beta - 1) > 0.$$

and  $\delta(t)$  satisfies the following inequation:

$$\delta(t) \leq \|(e^{tL_0} - e^{tL_\varepsilon})(u_0, v_0)\|_\infty \tag{2.72}$$

$$+ \int_0^t \|(e^{(t-s)L_0} - e^{(t-s)L_\varepsilon}) F(u_\varepsilon(s), v_\varepsilon(s))\|_\infty ds \tag{2.73}$$

$$+ \int_0^{\tau_1} \max\left(1, \frac{1}{\mu}\right) \|F\|_{Lip} (\|(u, v)(s)\|_\infty + \|(u_\varepsilon, v_\varepsilon)(s)\|_\infty) ds \tag{2.74}$$

$$+ \int_{\tau_1}^t \max\left(1, \frac{1}{\mu}\right) \|F\|_{Lip} \delta(s) ds \tag{2.75}$$

From Proposition 2.5 .1, we can assert that for some constant  $C_9$  depending only on  $r, \vartheta, D, \bar{\mu}, \tau_1$ , and  $T$ , the first term (2.72) satisfies

$$\|(e^{tL_0} - e^{tL_\varepsilon})(u_0, v_0)\|_\infty \leq \varepsilon^\alpha C_9 ((\|u_0\|_\infty + \|v_0\|_\infty) (1 + |supp(v_0)| + |supp(u_0)|)). \tag{2.76}$$

From Lemma 2.5 .2, we get that for all  $t > 0, \varepsilon > 0$ ,

$$\|u_\varepsilon(t)\|_\infty, \|v_\varepsilon(t)\|_\infty \leq K_2 e^{\bar{\lambda}ct}, \quad \left\| \|v_\varepsilon(t)\|_{L^\infty(y)} \right\|_{L^1(x)} \leq 2 \frac{K_2}{\lambda} e^{\bar{\lambda}ct}, \tag{2.77}$$

and the same estimates holds for  $(u, v)$ . So we get for the third term (2.74)

$$\int_0^{\tau_1} (\|(u, v)(s)\|_\infty + \|(u_\varepsilon, v_\varepsilon)(s)\|_\infty) ds \leq 4\tau_1 K_2 e^{\bar{\lambda}c\tau_1}. \tag{2.78}$$

Recall that  $f'(0) > 0$ . Hence, any supersolution of (2.1) is also a supersolution of the linear system (2.30). Thus, Lemma 2.5 .2 is also available for time-dependent solutions of the linear system (2.30), which in particular entails

$$\forall \varepsilon > 0, \forall t > 0, \|e^{tL_\varepsilon}\| \leq K_2 e^{\bar{\lambda}ct}. \tag{2.79}$$

Now we can deal with the second term (2.73). Let us choose  $\tau_2 > 0$  small enough, and set

$$\begin{aligned} & \int_0^t \left\| \left( e^{(t-s)L_0} - e^{(t-s)L_\varepsilon} \right) F(u_\varepsilon(s), v_\varepsilon(s)) \right\|_\infty ds \leq \\ & \int_0^{t-\tau_2} \left\| \left( e^{(t-s)L_0} - e^{(t-s)L_\varepsilon} \right) F(u_\varepsilon(s), v_\varepsilon(s)) \right\|_\infty ds + \\ & \int_{t-\tau_2}^t \left( \left\| e^{(t-s)L_0} \right\| + \left\| e^{(t-s)L_\varepsilon} \right\| \right) \|F\|_{Lip} \| (u_\varepsilon(s), v_\varepsilon(s)) \|_\infty ds. \end{aligned}$$

From (2.77) and (2.79),

$$\begin{aligned} & \int_{t-\tau_2}^t \left( \left\| e^{(t-s)L_0} \right\| + \left\| e^{(t-s)L_\varepsilon} \right\| \right) \|F\|_{Lip} \| (u_\varepsilon(s), v_\varepsilon(s)) \|_\infty ds \leq \\ & \tau_2 \|F\|_{Lip} \left( K_2 e^{\bar{\lambda}cT} + 1 + \frac{1}{\bar{\mu}} \right) \| (u_0, v_0) \|_\infty. \end{aligned} \quad (2.80)$$

From Proposition 2.5 .1 and (2.77),

$$\begin{aligned} & \int_0^{t-\tau_2} \left\| \left( e^{(t-s)L_0} - e^{(t-s)L_\varepsilon} \right) F(u_\varepsilon(s), v_\varepsilon(s)) \right\|_\infty ds \leq \\ & \varepsilon^\alpha TC_8 \left( e^{TC_8} + \frac{1}{\tau_2} \right) \left( 2 + \frac{4}{\bar{\lambda}} \right) (K_2 e^{\bar{\lambda}cT}). \end{aligned} \quad (2.81)$$

We can now conclude the proof of Theorem 2.1 .3 by a classical Gronwall argument in (2.72)-(2.75), choosing  $\tau_1$ , then  $\tau_2$  and at last  $\varepsilon$ . Let  $\eta > 0$  be any small quantity. Let  $\tau_1 > 0$  small enough such that  $4\tau_1 K_2 e^{\bar{\lambda}c\tau_1} \leq \frac{\eta}{4}$ . Let  $\tau_2 > 0$  such that

$$\tau_2 \|F\|_{Lip} \left( K_2 e^{\bar{\lambda}cT} + 1 + \frac{1}{\bar{\mu}} \right) \| (u_0, v_0) \|_\infty \leq \frac{\eta}{4}.$$

Now, let us choose  $\varepsilon > 0$  such that

$$\begin{cases} \varepsilon^\alpha C_9 (\|u_0\|_\infty + \|v_0\|_\infty) (1 + |supp(v_0)| + |supp(u_0)|) \leq \frac{\eta}{4} \\ \varepsilon^\alpha TC_8 \left( e^{TC_8} + \frac{1}{\tau_2} \right) \left( 2 + \frac{4}{\bar{\lambda}} \right) (K_2 e^{\bar{\lambda}cT}) \leq \frac{\eta}{4}. \end{cases}$$

Hence, (2.72) + (2.73) + (2.74)  $\leq \eta$  and we get from Gronwall's inequality for all  $t \in [\tau_1, T]$ :

$$\delta(t) \leq \eta e^{\|F\|_{Lip} \left( 1 + \frac{1}{\bar{\mu}} \right) (T-\tau_1)}.$$

□

## 2.6 Uniform spreading

Once again, we consider nonnegative compactly supported initial datum  $(u_0, v_0)$ . We also made the following assumption on  $(u_0, v_0)$  :

$$(u_0, v_0) \leq \left( \frac{1}{\bar{\mu}} m, m \right) \quad (2.82)$$

where  $m$  is given by (2.29). The purpose is now to prove Theorem 2.1 .2. Our notations are these of Section 2.2 .  $c_0^*$  is the asymptotic speed of spreading associated to the limit system (2.2),  $c_\varepsilon^*$  the one associated to (2.1). Recall that in the case  $D \leq 2d$ , the spreading is driven by the field, and  $c_0^* = c_\varepsilon^* = 2\sqrt{df'(0)}$ . In both systems, the spreading in this case is independent of the line, and the uniform spreading is easy to get. We will focus on the case  $D > 2d$ , where the spreading is enhanced by the line.

**First part:**  $c > c_0^*$ . This is the easiest case. Let  $c_1 = \frac{c+c_0^*}{2}$ . From Proposition 2.2 .1, there exists  $\varepsilon_0$  such that  $\forall \varepsilon < \varepsilon_0$ ,  $c_\varepsilon^* \leq c_1$ . From Lemma 2.5 .2, there exists  $K$  such that

$$\begin{cases} u_0(x) \leq K_1 e^{-\bar{\lambda}|x|} & \forall x \in \mathbb{R} \\ v_0(x, y) \leq K_1 e^{-\bar{\lambda}|x|} \inf_{\varepsilon \in [0,1]} \phi_\varepsilon(y; c_1) & \forall (x, y) \in \mathbb{R}^2. \end{cases}$$

Then, from Proposition 2.2 .2 and Lemma 2.5 .2,

$$\forall \varepsilon < \varepsilon_0, u_\varepsilon(t, x) \leq K e^{-\bar{\lambda}(x-c_1 t)},$$

which concludes the proof of the first part of Theorem 2.1 .2.  $\square$

**Second part:**  $c < c_0^*$

**Background on subsolutions** Let us recall that in [22] (resp. in [84]) the argument to prove the spreading was to devise stationary compactly supported subsolutions of (2.2) (resp. (2.1)) in a moving framework at some speed  $c$  less than and close to  $c_0^*$  (resp.  $c_\varepsilon^*$ ). More precisely, for  $L$  large enough, set  $\Omega^L := \mathbb{R} \times (-L, L)$  and let us consider the following systems for some  $\delta \ll 1$  :

$$\begin{cases} -DU'' + cU' = V(x, 0) - \bar{\mu}U & x \in \mathbb{R} \\ -d\Delta V + c\partial_x V = (f'(0) - \delta)V & (x, y) \in \Omega^L \\ -d(\partial_x V(x, 0^+) - \partial_x V(x, 0^-)) = \bar{\mu}U(x) - V(x, 0) & x \in \mathbb{R} \\ V(x, \pm L) = 0 & x \in \mathbb{R}. \end{cases} \quad (2.83)$$

$$\begin{cases} -DU'' + cU' = -\bar{\mu}U + \int_{(-L,L)} \nu_\varepsilon(y)V(x, y)dy & x \in \mathbb{R} \\ -d\Delta V + c\partial_x V = (f'(0) - \delta)V + \mu_\varepsilon(y)U(x) - \nu_\varepsilon(y)V(x, y) & (x, y) \in \Omega^L \\ V(x, \pm L) = 0 & x \in \mathbb{R}. \end{cases} \quad (2.84)$$

It was showed by an explicit computation in [22] and an analysis of a spectral problem in [84] that there exists a unique  $c := c_0^*(L)$  (resp.  $c_\varepsilon^*(L)$ ) such that (2.83) (resp. (2.84)) admits a unique solution of the form

$$\begin{pmatrix} U(x) \\ V(x, y) \end{pmatrix} = e^{\lambda x} \begin{pmatrix} 1 \\ \varphi(y) \end{pmatrix} \quad (2.85)$$

with  $c_0^*(L) < c_0^*$ ,  $c_\varepsilon^*(L) < c_\varepsilon^*$ , and

$$\begin{cases} \lim_{\delta \rightarrow 0} \lim_{L \rightarrow \infty} c_0^*(L) = \lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} c_0^*(L) = c_0^* \\ \lim_{\delta \rightarrow 0} \lim_{L \rightarrow \infty} c_\varepsilon^*(L) = \lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} c_\varepsilon^*(L) = c_\varepsilon^*. \end{cases}$$

Using (2.85), the system (2.84) reads on  $(c, \lambda, \varphi)$

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \int_{(-L,L)} \nu(y)\phi(y)dy \\ -d\varphi''(y) + (c\lambda - d\lambda^2 - f'(0) + \delta + \nu_\varepsilon(y))\varphi(y) = \mu_\varepsilon(y) \quad \varphi(\pm L) = 0 \end{cases} \quad (2.86)$$

and  $c_\varepsilon^*(L)$  was given as the first  $c$  such that the graphs of the two following functions intersect

$$\begin{cases} \Psi_1 : \lambda \mapsto -D\lambda^2 + \lambda c + \bar{\mu} \\ \Psi_2 : \lambda \mapsto \int_{(-L,L)} \nu_\varepsilon(y)\varphi(y)dy \end{cases}$$

where, for  $\Psi_2$ ,  $\varphi$  is given by the unique solution of second equation of (2.86). Let us call  $\Gamma_1$ , resp.  $\Gamma_2$ , the graph of  $\Psi_1$ , resp.  $\Psi_2$ . So we should keep in mind that in (2.85), both  $\lambda$  and  $\varphi$  depend on  $L, \delta, \varepsilon$ . Using the same kind of arguments as for Lemma 2.2 .3 and 2.2 .4, we can assert that this dependence is continuous for the  $L^\infty$ -topology. In particular, the subsolution (2.85) of (2.84) converges uniformly in  $\delta, L$  to the subsolution of (2.83) as  $\varepsilon$  goes to 0, and of course  $c_\varepsilon^*(L) \rightarrow c_0^*(L)$  as  $\varepsilon \rightarrow 0$ . Hence, the notations are not confusing, as we can continuously extend  $\varphi(\varepsilon, \delta, L)$  to  $\varphi(0, \delta, L)$  as  $\varepsilon$  goes to 0.

So we get that both  $\varphi$  and  $\Psi_2$  are:

- analytical in  $\lambda, c, \delta$ ;
- uniformly continuous in  $L$  and  $\varepsilon$ , up to  $\varepsilon = 0$ .

Then, a perturbative argument gives for some  $c$  less than but close to  $c^*(L)$  a compactly supported subsolution of (2.84), or (2.83) in the limit case  $\varepsilon = 0$ .

**Spreading** Let  $c < c_0^*$ . Let  $c_1 = \frac{c+c_0^*}{2}$ ,  $c_2 = \frac{c+c_0^*}{4}$ . From Proposition 2.2 .1, there exists  $\varepsilon_1$  such that for all  $\varepsilon < \varepsilon_1$ ,  $c_\varepsilon^* > c_2$ . Now, for some  $\delta$  small enough and some  $L$  large enough, [22] and [84] give us a family of subsolutions of (2.83) and (2.84) denoted  $(\underline{u}, \underline{v})$  and  $(\underline{u}_\varepsilon, \underline{v}_\varepsilon)$  for some  $c_\varepsilon > c_1$  for  $\varepsilon < \varepsilon_1$ . The uniform continuity of  $\Psi_2$  allows us to take same  $\delta$  and  $L$  for all  $\varepsilon \in [0, \varepsilon_1)$ . Hence, the convergence result given in Lemma 2.2 .4 adapted to this case gives that

$$(\underline{u}_\varepsilon, \underline{v}_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (\underline{u}, \underline{v})$$

for the  $L^\infty$ -norm. Set:

$$\begin{cases} u_1(x) = \inf\{\underline{u}_\varepsilon(x), \varepsilon \in [0, \varepsilon_1)\}, & v_1(x) = \inf\{\underline{v}_\varepsilon(x, y), \varepsilon \in [0, \varepsilon_1)\} \\ u_2(x) = \sup\{\underline{u}_\varepsilon(x), \varepsilon \in [0, \varepsilon_1)\}, & v_2(x) = \sup\{\underline{v}_\varepsilon(x, y), \varepsilon \in [0, \varepsilon_1)\}. \end{cases}$$

Both  $u_1, v_1, u_2, v_2$  are nonnegative, continuous and compactly supported. Let us set  $\gamma$  such that

$$\gamma(f'(0) - \delta) \|v_2\|_\infty \leq f(\|v_2\|_\infty) \quad \text{and} \quad \gamma \|u_2\|_\infty < \frac{1}{\bar{\mu}}.$$

We know that  $(u, v)(t)$  converges locally uniformly to the steady state  $(\frac{1}{\bar{\mu}}, 1)$ . So, let  $t_1$  such that  $(u, v)(t_1) > \gamma(u_2, v_2)$ . From Theorem 2.1 .3, there exists  $\varepsilon_2$  such that for all  $\varepsilon < \varepsilon_2$ ,  $(u_\varepsilon, v_\varepsilon)(t_1) > \gamma(u_2, v_2)$ . Up to replace  $\varepsilon_2$  by  $\min(\varepsilon_1, \varepsilon_2)$ , we get from comparison principle that

$$\forall t > t_1, \forall \varepsilon < \varepsilon_2, (u_\varepsilon(t, x), v_\varepsilon(t, x, y)) > \gamma(u_1(x - c_1(t - t_1)), v_1(x - c_1(t - t_1), y)).$$

Let  $(\tilde{u}, \tilde{v})$  and  $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$  the solutions of (2.2) and (2.1) starting at  $t = t_1$  from  $\gamma(u_1, v_1)$ . Then, considering the hypotheses on  $(u_0, v_0)$ , for all  $t > t_1$ , we have:

$$\begin{cases} (\tilde{u}, \tilde{v})(t) < (u, v)(t) < \left(\frac{1}{\mu}, 1\right) \\ (\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)(t) < (u_\varepsilon, v_\varepsilon)(t) < (U_\varepsilon, V_\varepsilon). \end{cases}$$

Now, from Proposition 2.3 .1, there exists  $\varepsilon_3$  such that if  $\varepsilon < \varepsilon_3$ ,  $\left|U_\varepsilon - \frac{1}{\mu}\right| < \frac{\eta}{3}$ . Let  $t_2$  such that

$$\forall x \in \text{supp}(u_1), \forall t > t_2, \left|\tilde{u}(t, x) - \frac{1}{\mu}\right| < \frac{\eta}{3}.$$

From Theorem 2.1 .3, there exists  $\varepsilon_4$  such that if  $\varepsilon < \varepsilon_4$ ,  $|(\tilde{u}_\varepsilon - \tilde{u})(t_2)| < \frac{\eta}{3}$ . Now, set  $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ ,  $T_0 = t_2 \frac{c_0^*}{c_0^* - c_2}$ , and the proof of Theorem 2.1 .2 is concluded.  $\square$



## Appendix

Here we prove for the convenience of the reader the lemma used in the proof of Lemma 2.4 .5. It relies on a Kato-type inequality.

**Lemma 1.** *Let  $\phi, m, f$  be functions in  $BUC(\mathbb{R}, \mathbb{C})$  such that*

$$-\phi''(y) + m(y)\phi(y) = f(y), \quad y \in \mathbb{R}. \quad (\text{A.1})$$

*If there exists  $\kappa > 0$  such that for all  $y \in \mathbb{R}$ ,  $\mathcal{R}e(m(y)) \geq \kappa^2$  then  $|\phi|$  satisfies*

$$|\phi(y)| \leq \frac{1}{2\kappa} \int_{\mathbb{R}} e^{-\kappa|z|} |f(y-z)| dz. \quad (\text{A.2})$$

*Proof.* Let us first compute the second derivative of the modulus of a complex valued function. Let  $\phi \in BUC(\mathbb{R}, \mathbb{C}) \cap \mathcal{C}^2$ . An easy computation yields

$$|\phi|'' = \frac{\mathcal{R}e(\bar{\phi}\phi'')}{|\phi|} + \frac{|\phi'|^2}{|\phi|} - \frac{\mathcal{R}e(\bar{\phi}\phi')^2}{|\phi|^3}.$$

Hence, for all smooth enough complex-valued function of the real variable, we get

$$-|\phi|'' \leq -\frac{\mathcal{R}e(\bar{\phi}\phi'')}{|\phi|}. \quad (\text{A.3})$$

Now, let us multiply (A.1) by  $\frac{\bar{\phi}}{|\phi|}$  and take the real part. It gives

$$-\frac{\mathcal{R}e(\bar{\phi}\phi'')}{|\phi|} + \mathcal{R}e(m)\phi = \mathcal{R}e\left(f \frac{\bar{\phi}}{|\phi|}\right).$$

Using (A.3) in the above inequality yields the following inequation for  $|\phi|$  :

$$-|\phi(y)|'' + \mathcal{R}e(m(y))|\phi(y)| \leq |f(y)|.$$

Now we are reduced to an inequation with real functions. If  $\varphi$  is the unique solution in  $H^1(\mathbb{R})$  of

$$-\varphi''(y) + \kappa^2\varphi(y) = |f(y)|,$$

from the elliptic maximum principle, we get  $|\phi| \leq \varphi$ , which is exactly the desired inequality (A.2).  $\square$

# Chapter 3

## Road-field reaction-diffusion system: a new threshold for long range exchanges

We consider the coupled system of reaction-diffusion equations of KPP type in a presence of a line of fast diffusion with non-local exchange terms between the line and the framework presented in the first chapter. Our study deals with the infimum of the spreading speed depending on the exchange functions. We exhibit a new threshold in the limit of long range exchange terms for the line to influence the propagation.

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### 3.1 Introduction

The purpose of this note is to study some properties concerning the spreading speed of the following reaction-diffusion system, introduced in [84]:

$$\begin{cases} \partial_t u - D\partial_{xx}u = -\bar{\mu}u + \int \nu(y)v(t, x, y)dy & x \in \mathbb{R}, t > 0 \\ \partial_t v - d\Delta v = f(v) + \mu(y)u(t, x) - \nu(y)v(t, x, y) & (x, y) \in \mathbb{R}^2, t > 0. \end{cases} \quad (3.1)$$

The initial road-field system was introduced in [22]. It was generalised to nonlocal exchange terms in [84] and [86]. We refer to these papers for more informations. We use the notation  $\bar{\mu} = \int \mu$ ,  $\bar{\nu} = \int \nu$ . Thus, it is easy to check that without reaction, the above system is mass-conservative. Our assumptions are the following.

- The reaction term  $f$  is of KPP type, i.e. strictly concave with  $f(0) = f(1) = 0$ , and quadratic outside  $[0, 1]$ .
- The two exchange functions  $\mu$  and  $\nu$  are continuous, nonnegative, even. For the sake of simplicity, we will consider compactly supported functions, but our results can easily be extended to a mere general class of functions. See [84] for the optimal (to our knowledge) hypothesis.

The purpose of the model (3.1) is to study a propagation driven by the line. This is the main motivation of the following Theorem, which also gives a definition of the **spreading speed** for this kind of model. It was proved in [84].

**Theorem 3.1 .1.** *Let  $(u, v)$  be a solution of (3.1) with a nonnegative, compactly supported initial datum  $(u_0, v_0)$ . Then, there exists an asymptotic speed of spreading  $c^*$  and a unique positive bounded stationary solution of (3.1)  $(U, V)$  such that, pointwise in  $y$ , we have:*

- for all  $c > c^*$ ,  $\lim_{t \rightarrow \infty} \sup_{|x| \geq ct} (u(t, x), v(t, x, y)) = (0, 0)$  ;
- for all  $c < c^*$ ,  $\lim_{t \rightarrow \infty} \inf_{|x| \leq ct} (u(t, x), v(t, x, y)) = (U, V)$ .

## 3.2 Infimum for the spreading speed

For fixed parameters  $d, D, f'(0), \bar{\mu}, \bar{\nu}$  we consider the set of admissible exchanges

$$\Lambda_{\bar{\mu}} = \left\{ \mu \in C_0(\mathbb{R}), \mu \geq 0, \int \mu = \bar{\mu}, \mu \text{ even} \right\}.$$

We define  $\Lambda_{\bar{\nu}}$  in a similar fashion. For  $\mu \in \Lambda_{\bar{\mu}}$  and  $\nu \in \Lambda_{\bar{\nu}}$ , there exists a spreading speed  $c^*(\mu, \nu)$ . A natural question is to wonder about the existence of maximal or minimal spreading speed for  $\mu, \nu$  admissible exchanges. This note is devoted to the existence of an infimum for the spreading speed. Thus, we also prove that there is no minimal spreading speed. The main result relies on the following theorem.

**Theorem 3.2 .1.** *Let us consider the nonlocal system (3.1) with fixed exchange masses  $\bar{\mu}$  and  $\bar{\nu}$ . Let  $c^*$  be the spreading speed given by Theorem 3.1 .1, depending on the repartition of  $\mu$  or  $\nu$ .*

1. If  $D \in \left[ 2d, d \left( 2 + \frac{\bar{\mu}}{f'(0)} \right) \right]$ ,  $\inf c^* = 2\sqrt{df'(0)}$ .
2. Fix  $D > d \left( 2 + \frac{\bar{\mu}}{f'(0)} \right)$ , then  $\inf c^* > 2\sqrt{df'(0)}$ .

Moreover, in both cases, minimizing sequences can be given by long range exchange terms of the form  $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$  or  $\nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right)$  with  $R \rightarrow \infty$ .

Let us recall that in [22] and [84] was exhibited the threshold  $D = 2d$  for the spreading in the direction of the road, whatever be the considered road-field system:

- if  $D \leq 2d$ ,  $c^* = c_K := 2\sqrt{df'(0)}$ ;

- if  $D > 2d$ ,  $c^* > c_K$ .

We show that the threshold  $D = 2d + d\frac{\bar{\mu}}{f'(0)}$  has an important effect.

- If  $D > 2d + d\frac{\bar{\mu}}{f'(0)}$ , the speed  $c^*$  is strictly greater than  $c_K$  in the  $x$ -direction, with a bound independent of the exchange functions.
- If  $D < 2d + d\frac{\bar{\mu}}{f'(0)}$ ,  $c^*$  tends to  $c_K$  as the exchange functions vanish.

### 3.3 Background on the computation of the spreading speed

The importance of the linearised system for KPP-type model motivates the following definition for travelling waves.

**Definition 3.3 .1.** We call a **linear travelling wave** a 3-uple  $(c, \lambda, \phi)$  with  $c > 0$ ,  $\lambda > 0$ , and  $\phi \in H^1(\mathbb{R})$  a positive function such that

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto e^{-\lambda(x-ct)} \begin{pmatrix} 1 \\ \phi(y) \end{pmatrix}$$

be a solution of the corresponding linearised system in 0.  $c$  is the speed of the exponential traveling waves.

The previous definition for travelling waves provides us a helpful characterisation for spreading speed.

**Proposition 3.3 .1.** *The **spreading speed**  $c^*$  given by Theorem 3.1 .1 can be defined as follows:*

$$c^* = \inf\{c > 0 \mid \text{linear travelling waves with speed } c \text{ exists}\}.$$

Inserting definition supplied by Proposition 3.3 .1 into (3.1) yields the following system in  $(c, \lambda, \phi)$  :

$$\begin{cases} -D\lambda^2 + \lambda c + \bar{\mu} = \int \nu(y)\phi(y)dy \\ -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y). \end{cases} \quad (3.2)$$

These equations and integrals have to be understood in a distribution sense if needed. As explained in [84], the first equation of (3.2) gives the graph of a function  $\lambda \mapsto \Psi_1(\lambda, c) := -D\lambda^2 + \lambda c + \bar{\mu}$ , which means to be equal to  $\int \nu(y)\phi(y)dy$ , provided  $(c, \lambda, \phi)$  defines an exponential travelling waves.

The second equation of (3.2) gives, under some assumptions on  $\lambda$ , a unique nonnegative solution  $\phi = \phi(y; \lambda, c)$  in  $H^1(\mathbb{R})$ . To this unique solution we associate the function  $\Psi_2(\lambda, c) := \int \nu(y)\phi(y)dy$ . Let us denote  $\Gamma_1$  the graph of  $\Psi_1$  in the  $(\lambda, \Psi_1(\lambda))$  plane, and  $\Gamma_2$  the graph of  $\Psi_2$ . So, (3.2) amounts to the investigation of  $\lambda, c > 0$  such that  $\Gamma_1$  and  $\Gamma_2$  intersect.

### 3.3 .1 Resolution of the $(c, \lambda, \phi)$ -system: general remarks

Thereafter we recall some facts on the two functions  $\Psi_{1,2}$ . For more details and proofs, we refer to [84].

**Behaviour of  $\Psi_1$**  Let us recall that in a  $(\lambda, \Psi_1(\lambda))$  plane,  $\Psi_1$  defines parabola, nonnegative for  $\lambda \in [\lambda_1^-, \lambda_1^+]$  with  $\lambda_1^\pm = \frac{c \pm \sqrt{c^2 + 4D\bar{\mu}}}{2D}$ , and that

$$\Psi_1(0) = \Psi_1\left(\frac{c}{D}\right) = \bar{\mu}.$$

Notice that  $\Psi_1$  depends on  $c, D$ , and  $\bar{\mu}$ , but does not depend on  $\nu, d$ , neither on the repartition of  $\mu$ . Only the mass matters. This will be useful for the sequel.

**Behaviour of  $\Psi_2$**  The function  $\Psi_2$  is defined implicitly by the solution of the following system (3.3)

$$\begin{cases} -d\phi''(y) + (\lambda c - d\lambda^2 - f'(0) + \nu(y))\phi(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}), \phi \geq 0. \end{cases} \quad (3.3)$$

If it exists, to the solution of (3.3) we associate  $\Psi_2(\lambda) := \int_{\mathbb{R}} \nu(y)\phi(y)dy$ . It has been shown that  $\Psi_2$  is defined for  $c > c_K$  and  $\lambda \in ]\lambda_2^-, \lambda_2^+[$ , with

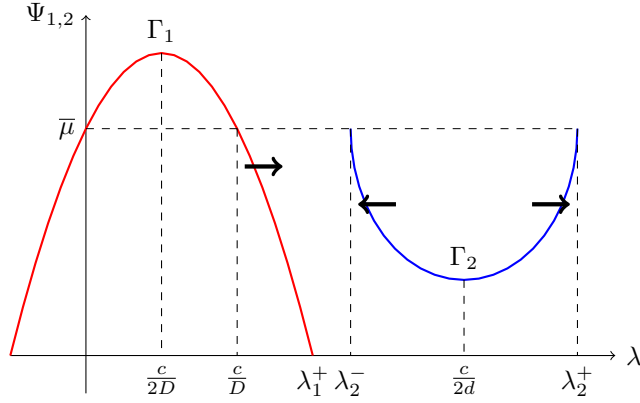
$$\lambda_2^\pm = \frac{c \pm \sqrt{c^2 - c_K^2}}{2d}.$$

Recall that the classical Fisher-KPP speed is given by  $c_K = 2\sqrt{df'(0)}$ .  $\Psi_2$  is a smooth convex function, symmetric with respect to the line  $\{\lambda = \frac{c}{2d}\}$ , and can be continuously extended to  $\lambda_2^\pm$  by  $\Psi_2(\lambda_2^\pm) = \bar{\mu}$ , with vertical tangents at these points.

Notice that, contrary to  $\Psi_1$ ,  $\Psi_2$  is highly dependent on the two exchange functions  $\nu$  and  $\mu$ . Thus, this paper will mainly focus on this function and how its variations depend on these exchange functions. However, the extreme points  $(\lambda_2^\pm, \Psi_2(\lambda_2^\pm))$  do not depend on these functions, but only on  $c, d, f'(0)$ , and  $\bar{\mu}$ . Global behaviours of  $\Psi_1$  and  $\Psi_2$  are summarised in Figure 3.1.

**Intersection of  $\Gamma_1$  and  $\Gamma_2$**  We focus on the case  $D > 2d$ , the other one leading to dynamics with no influence of the road. The functions of the  $c$ -variable  $c \mapsto \Psi_1$  and  $c \mapsto \lambda_2^-$  are respectively increasing and decreasing. Hence, there exists  $c^*$  such that  $\forall c > c^*$ ,  $\Gamma_1$  and  $\Gamma_2$  intersect, and  $\forall c < c^*$ ,  $\Gamma_2$  does not intersect the closed convex hull of  $\Gamma_1$ . Moreover, the strict concavity of  $\Gamma_1$  and the strict convexity of  $\Gamma_2$  give that for  $c = c^*$ ,  $\Gamma_1$  and  $\Gamma_2$  are tangent on  $\lambda^*$  and for  $c > c^*$ ,  $c$  close to  $c^*$ ,  $\Gamma_1$  and  $\Gamma_2$  intersect twice.

As the extreme points  $(\lambda_2^\pm, \Psi_2(\lambda_2^\pm))$  are independent of  $\nu$  and the repartition of  $\mu$ , the study of the spreading speed associated to two exchange functions amounts to analyse the relative position of the corresponding  $\Psi_2$  functions. Given the monotonicity in  $c$ , a  $\Psi_2$  function under another leads to a slower spreading speed, and vice versa. The convexity of  $\Psi_2$  even allows us to study the variations in a neighbourhood of  $(\lambda, c^*)$  to get a local result.


 Figure 3.1: representation of  $\Gamma_1$  and  $\Gamma_2$ , behaviours as  $c$  increases

### 3.4 A new threshold

We show how long range exchange terms tend to slow down the dynamics. More precisely, for any given functions  $\mu, \nu$ , we set

$$\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right), \quad \nu_R(y) = \frac{1}{R}\nu\left(\frac{y}{R}\right). \quad (3.4)$$

This asymptotics yields to a new threshold in order to get or be greater than the KPP spreading speed. Moreover, this provides minimizing sequences for the spreading speed, as asserted in Theorem 3.2 .1.

In the system (3.1), replace at least one exchange function by a long range exchange function given by (3.4), and let us denote  $c^*(R)$  the corresponding spreading speed in the sense of Proposition 3.3 .1.

The proof relies on a very simple remark: for  $\Gamma_1$  and  $\Gamma_2$  to intersect, it is necessary to have  $\lambda_1^+ > \lambda_2^-$  (see Figure 3.1). Let us see  $\lambda_1^+$  and  $\lambda_2^-$  as functions of the speed  $c$  given by

$$\lambda_1^+ : c \mapsto \frac{c + \sqrt{c^2 + 4D\bar{\mu}}}{2D} \quad (3.5)$$

$$\lambda_2^- : c \mapsto \frac{c - \sqrt{c^2 - c_K^2}}{2d}. \quad (3.6)$$

They are both continuous.  $\lambda_1^+$  is increasing on  $[0, +\infty[$ ,  $\lambda_2^-$  is decreasing on  $[c_K, +\infty[$ . We may also notice that  $\lambda_1^+$  is decreasing with respect to  $D$ . An explicit computation gives

$$d\left(2 + \frac{\bar{\mu}}{f'(0)}\right) = \inf\{D > 0, \lambda_1^+(c_K) < \lambda_2^-(c_K)\}. \quad (3.7)$$

**Proof of the first part of Theorem 3.2 .1** For the sake of simplicity we will focus on the general model (3.1), the other being similar and even easier - see [84]. Let  $D$  be less than  $d\left(2 + \frac{\bar{\mu}}{f'(0)}\right)$ ,  $\varepsilon > 0$ . Let  $c \in ]c_K, c_K + \varepsilon[$ . Then  $\lambda_1^+(c) > \lambda_2^-(c)$ . Choose any  $\lambda_0 \in ]\lambda_2^-, \lambda_1^+[$ .

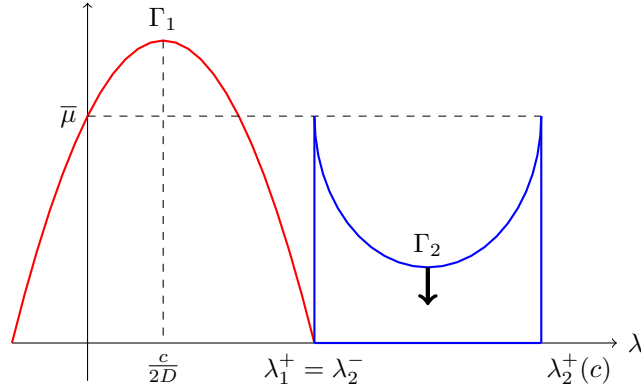


Figure 3.2: Behaviour of  $\Gamma_2$  as  $R \rightarrow +\infty$ , critical case  $\lambda_1^+ = \lambda_2^-$

*First case: long range for  $\mu$ .* Let us replace  $\mu$  by  $\mu_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right)$  in (3.1). The  $(c, \lambda, \phi)$  associated equation (3.3) is

$$\begin{cases} -d\phi_R''(y) + (P(\lambda) + \nu(y))\phi_R(y) = \frac{1}{R}\mu\left(\frac{y}{R}\right) \\ \phi \in H^1(\mathbb{R}), \phi \geq 0 \end{cases} \quad (3.8)$$

where as usual  $P(\lambda) = \lambda c - d\lambda^2 - f'(0)$ . The curve  $\Gamma_2$  is defined as the graph of

$$\Psi_2 : \lambda \mapsto \int_{\mathbb{R}} \nu \phi_R$$

where  $\phi_R$  is the unique solution of (3.8). From the choice of  $\lambda_0$ ,  $P(\lambda_0) > 0$ . The maximum principle yields for  $\lambda = \lambda_0$

$$\|\phi_R\|_{\infty} \leq \frac{1}{RdP(\lambda_0)} \|\mu\|_{\infty}$$

which gives

$$\Psi_2(\lambda_0) \leq \frac{\bar{\nu}}{RdP(\lambda_0)} \|\mu\|_{\infty} \xrightarrow{R \rightarrow +\infty} 0.$$

Thus, there exists  $R_0$ ,  $\forall R > R_0$ ,  $\Psi_1(\lambda_0) > \Psi_2(\lambda_0)$ , so  $c^*(R) < c$ .

*Second case: long range for  $\nu$ .* The study is quite similar. The  $(c, \lambda, \phi)$  associated equation (3.3) is

$$\begin{cases} -d\phi_R''(y) + (P(\lambda) + \frac{1}{R}\nu\left(\frac{y}{R}\right))\phi_R(y) = \mu(y) \\ \phi \in H^1(\mathbb{R}), \phi \geq 0 \end{cases} \quad (3.9)$$

Let  $\varphi$  be the only  $H^1$  solution of

$$-d\varphi''(y) + P(\lambda_0)\varphi(y) = \mu(y).$$

This provides a supersolution for (3.9) with  $\lambda = \lambda_0$ . As  $\mu$  is compactly supported,  $\varphi$  belongs to  $L^1(\mathbb{R})$ . Hence

$$\Psi_2(\lambda_0) = \frac{1}{R} \int_{\mathbb{R}} \nu\left(\frac{y}{R}\right)\phi_R(y)dy \leq \frac{\|\nu\|_{\infty}}{R} \int_{\mathbb{R}} \varphi(y)dy \xrightarrow{R \rightarrow +\infty} 0$$

and we conclude as in the previous case.

**Proof of the second part of Theorem 3.2 .1** Let  $D$  be greater than the threshold  $d \left( 2 + \frac{\bar{\mu}}{f'(0)} \right)$ . Thus,  $\lambda_1^+(c_K) < \lambda_2^-(c_K)$ . From the above argument, the minimal speed  $c_{min}$  is given by

$$c_{min} = \inf\{c, \lambda_1^+(c) > \lambda_2^-(c)\}.$$

Continuity and monotonicity of  $\lambda_1^+, \lambda_2^-$  given by (3.5)-(3.6) warrant the existence of  $c_{min}$  and the inequality  $c_{min} > c_K$ . Moreover, the above study guarantees that it is optimal. See Figure 3.2.  $\square$

### 3.5 Remarks and open questions

The above result can easily be extended to the semi-limit models presented in [84], with one nonlocal exchange and the other exchange by boundary condition.

Using the same kind of geometric considerations, it is easy to give the following upper bound for the spreading speed.

**Proposition 3.5 .1.** *For fixed parameters  $d, D, \bar{\mu}, \bar{\nu}, f'(0)$ , then for all admissible exchanges  $\mu \in \Lambda_{\bar{\mu}}$  and  $\nu \in \Lambda_{\bar{\nu}}$  we have*

$$c^*(\mu, \nu) \leq D \sqrt{\frac{f'(0)}{D-d}}.$$

An open question is to know if this bound is reached for some exchanges. Moreover, this bound ensures us the existence of minimizing sequences for the spreading speed. Hence, it is questionable whether these sequences converge, in which sense.

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## **Part II**

### **Bistable entire solutions**



# Chapter 4

## Bistable entire solutions in cylinder-like domains

In a recent paper, Berestycki, Bouhours and Chapuisat proved the existence and uniqueness of an entire solution for a bistable equation in a class of unbounded domains. The key assumptions in their study was the "cylinder-like" assumption ; their domains are supposed to be straight cylinders in a half space. The purpose of this chapter is to generalised this assumption, and to consider domains that tend to a straight cylinder in one direction. We also prove the existence for a one-dimensional problem with a non-homogeneous linear term.

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## 4.1 Introduction

The purpose of this paper is to construct a nontrivial entire solution for a bistable equation in an unbounded domain in one direction. The equation under study is the following parabolic problem with Neumann boundary condition:

$$\begin{cases} \partial_t u(t, x, y) - \Delta u(t, x, y) = f(u), & t \in \mathbb{R}, (x, y) \in \Omega, \\ \partial_\nu u(t, x, y) = 0, & t \in \mathbb{R}, (x, y) \in \partial\Omega. \end{cases} \quad (4.1)$$

The domain  $\Omega$  is supposed to be smooth enough and infinite in the  $x$ -direction and tends to a straight cylinder when  $x$  goes to  $-\infty$ . More precisely, we suppose that there exist two functions  $Y^-, Y^+$  and a positive number  $Y^\infty$  such that

$$\begin{cases} \Omega = \{(x, y), x \in \mathbb{R}, y \in ]Y^-(x), Y^+(x)[\}, \\ Y^\pm(x) \xrightarrow{x \rightarrow -\infty} \pm Y^\infty. \end{cases} \quad (4.2)$$

Throughout the paper, the nonlinear term  $f$  is assumed to be of bistable kind, *i.e.* there exists  $\theta \in (0, 1)$  such that

$$f \in C^3([0, 1]), \quad f(0) = f(\theta) = f(1) = 0, \quad f'(0), f'(1) < 0 \quad (4.3a)$$

$$f(s) < 0 \text{ for all } s \in (0, \theta), \quad f(s) > 0 \text{ for all } s \in (\theta, 1). \quad (4.3b)$$

Moreover, we assume that the invader is the state  $u = 1$ , that is

$$\int_0^1 f(s) ds > 0. \quad (4.4)$$

Let us recall (see [44, 1]) that for such a kind of nonlinearity, there exists a unique two-uple  $(c, \varphi) \in \mathbb{R} \times C^5(\mathbb{R})$  satisfying

$$\begin{cases} \varphi'' + c\varphi' + f(\varphi) = 0, \\ \varphi(-\infty) = 1, \quad \varphi(+\infty) = 0, \quad \varphi(0) = \theta. \end{cases} \quad (4.5)$$

The function  $(t, x) \mapsto \varphi(x - ct)$  is, up to translation, the unique planar travelling wave for reaction-diffusion equation of the form

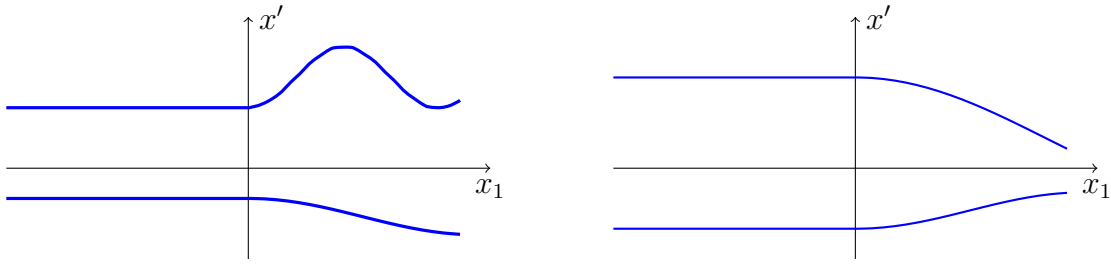
$$\partial_t u - \Delta u = f(u), \quad t \in \mathbb{R}, x \in \mathbb{R}^N. \quad (4.6)$$

**Motivations** When the underlying domain has a spatial dependence, travelling waves of the form (4.5) no longer exist. However the notion of transition waves has been introduced in [12] by Berestycki and Hamel. The specific problem of generalised transition waves for a bistable equation in cylinders with varying cross section has been treated by Berestycki, Bouhours and Chapuisat in [5]. They considered the parabolic problem (4.1) where, this time,  $\Omega$  is assumed to be a smooth infinite cylinder in the  $x_1$ -direction, *i.e.*

$$\Omega = \{(x_1, x'), x_1 \in \mathbb{R}, x' \in \omega(x_1) \subset \mathbb{R}^{N-1}\}. \quad (4.7)$$

However, they also made the following assumption on  $\Omega$  :

$$\Omega \cap \{x \in \mathbb{R}^N, x_1 < 0\} = \mathbb{R}^- \times \omega, \omega \in \mathbb{R}^{N-1}. \quad (4.8)$$


 Figure 4.1: Two examples of domains  $\Omega$  considered in [5]

This assumption asserts that the domain is equal to a cylinder in the left half space. See fig. 4.1 for examples. For such a kind of domains they obtained many properties concerning the propagation of a bistable wave, the first of them being the existence and uniqueness of an entire solution: if  $(c, \varphi)$  is the unique solution of (4.5), they showed that there exists a unique entire solution  $u$  of (4.1) such that

$$|u(t, x) - \varphi(x_1 - ct)| \xrightarrow[t \rightarrow -\infty]{} 0 \text{ uniformly in } \bar{\Omega}.$$

The assumption (4.8) was essential, but it is quite restrictive. It is therefore a relevant question to consider domains that **converge** to a cylinder as  $x_1$  tends to  $-\infty$ . This is the main topic of this paper.

**The domain** Throughout our study, we make the following assumptions on the domain.

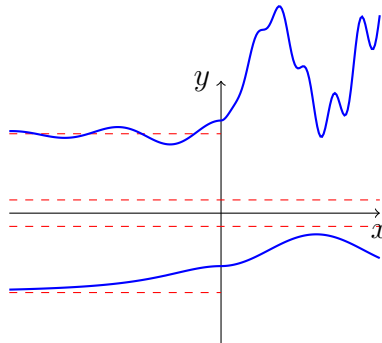
$$\text{There exist } \kappa > 0, \text{ for all } x \in \mathbb{R}, \left\| Y^\pm(\cdot) \mp Y^\infty \right\|_{C^3(-\infty, x)} \leq e^{\kappa x}. \quad (4.9a)$$

$$\text{There exists } l > 0, \text{ for all } x \in \mathbb{R}, Y^-(x) < -l < l < Y^+(x). \quad (4.9b)$$

$$\text{There exists } L > 0, \left\| Y^\pm \right\|_{C^3(\mathbb{R})} < L. \quad (4.9c)$$

$$\text{Without loss of generality, we take } 0 < \kappa < -\lambda - \frac{c}{2}. \quad (4.9d)$$

Contrary to what was done in [5], we only consider a two dimensional-domain. Hypothesis


 Figure 4.2: An example of domain  $\Omega$  considered in our paper

(4.9a) assert that the convergence to a straight cylinder is at some exponential rate and

up to the third derivative. However, it can be any exponential rate, hence it is a plausible guess that this assumption may be lightened. We also ask for our domain to contain strictly a small cylinder in (4.9b), and to be bounded. The last hypothesis (4.9d) is just technical and not restrictive. An example of such a domain is given in figure 4.2.

**Results and organisation of the paper** We prove the following existence theorem.

**Theorem 4.1 .1.** *Under the assumptions (4.2) and (4.9) on the domain  $\Omega$ , there exists a function  $u_\infty$  defined for  $t \in \mathbb{R}$ ,  $(x, y) \in \Omega$  solution of (4.1) which satisfies*

$$\sup \{|u_\infty(t, x, y) - \varphi(x - ct)|, (x, y) \in \Omega\} \xrightarrow[t \rightarrow -\infty]{} 0. \quad (4.10)$$

The proof of Theorem 4.1 .1 amounts to proving stability of the travelling wave (4.5) under the perturbation induced by the inhomogeneity of the domain. Thus it is quite natural to first consider the following inhomogeneous problem in one dimension as a case study:

$$\partial_t u - \partial_{xx} u = f(u) (1 + g(x)), \quad t \in \mathbb{R}, x \in \mathbb{R} \quad (4.11)$$

where  $g$  is a bounded perturbation that satisfies the assumption

$$\text{there exists } \kappa > 0, |g(x)| \leq e^{\kappa x} \text{ for all } x \in \mathbb{R}. \quad (4.12)$$

This model seems easier than the one considered previously. The non-homogeneous perturbation is only on the nonlinearity and the problem is one dimensional. Hypothesis (4.12) is closely related to our hypothesis (4.3a); once again, we ask for an exponential rate, but it is arbitrary. Existence of transition waves has already been proved for similar nonlinearities (see [102] for instance, and references given in it). However, as far as we know, the question of an entire solution that converges to the bistable wave as  $t$  goes to  $-\infty$  is not treated. It is the purpose of the next theorem, of independent interest.

**Theorem 4.1 .2.** *There exists a positive constant  $\varpi$  which depends only on  $f$  such that if  $g$  satisfies (4.12) and  $g > -\varpi$ , then there exists a function  $u_\infty = u_\infty(t, x)$  defined for  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}$  solution of (4.11) which satisfies*

$$\|u_\infty(t, \cdot) - \varphi(\cdot - ct)\|_{L^\infty(\mathbb{R})} \xrightarrow[t \rightarrow -\infty]{} 0. \quad (4.13)$$

The constant  $\varpi$  is given by  $\varpi = \frac{\rho_1}{\|f'\|_\infty}$  where  $\rho_1$  is the spectral gap of the linearised operator associated with the travelling wave (4.5).

In the last paragraph of this introduction we recall some well-known facts about bistable nonlinearities. Namely, the spectral decomposition associated to the linearised operator will be the most important tool of our study. The next section is concerned with the proof of Theorem 4.1 .2. It is done using a perturbative argument. We study the Cauchy problem associated to (4.11) starting from a translated bistable wave and prove that the solution can stay arbitrary close to the wave up to a certain time, and then we use a compactness argument. The stability result is obtained thanks to a Lyapunov-Schmidt decomposition. We project the equation onto the kernel and the range of the operator. Equation on the kernel involves only quadratic terms, and the linear perturbation in the range is treated with an energy method. In order to get coercivity in the equation we need

the hypothesis concerning the lower bound on the perturbation. Then we use a bootstrap argument between these two equations.

The last section is concerned with the proof of Theorem 4.1 .1. The proof follows the same steps, and differs in one major point: perturbative terms coming from the right are controlled with travelling supersolution that we construct in the beginning of this last section. Then, the arguments are similar.

**Bibliographical remarks** Existence of travelling waves for reaction-diffusion equations have been studied since the well known works of Kolmogorov, Petrovskii and Piskunov [70]. It has been generalised in the late seventies by Aronson and Weinberger [2] and Fife and McLeod [44] for the specific problem of a one dimensional bistable wave. Since these pioneering works, an important effort has been done to study front propagation and transition waves in inhomogeneous reaction-diffusion equations and it is impossible to cite them all. For our purpose let us mention the paper of Berestycki and Nirenberg [20] where they showed existence of travelling waves in cylinders with an inhomogeneous advection field. The notion of transition fronts, or invasion fronts, has been introduced in [76] and precisely defined and studied in [12]. The specific problem of transition waves when heterogeneities come from the boundary have received a much more recent attention. In [32], the authors showed that for a bistable nonlinearity, if the domain  $\Omega$  is a succession of two half rectangular cylinders, one can find conditions on them width for the propagation to be blocked. The problem of transition fronts for exterior domains has been treated in [14] where the authors devised geometrical conditions for the invasion to be complete. Finally, as already said, cylinder with varying cross sections has just been studied in [5].

Entire solutions for inhomogeneous one dimensional reaction-diffusion equations like (4.11) has been much studied in the few last years. Existence for an ignition nonlinearity has been devised independently in [79] and [83]. A generalisation for time and space inhomogeneous reaction-diffusion equations of both ignition and bistable kind has, as said before, recently been given in [102].

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### Some preliminary material

**Behaviour at infinity** As is shown in [44, 52], there exist two constants  $C_1, C_2$  such that  $\varphi$  and  $\varphi'$  satisfy

$$\begin{cases} C_1 e^{\mu z} \leq 1 - \varphi(z) \leq C_2 e^{\mu z} & z \leq 0 \\ C_1 e^{\lambda z} \leq \varphi(z) \leq C_2 e^{\lambda z} & z > 0 \\ C_1 e^{\mu z} \leq -\varphi'(z) \leq C_2 e^{\mu z} & z \leq 0 \\ C_1 e^{\lambda z} \leq -\varphi'(z) \leq C_2 e^{\lambda z} & z > 0 \end{cases} \quad (4.14)$$



where

$$\mu = \frac{-c + \sqrt{c^2 - 4f'(1)}}{2}, \quad \lambda = \frac{-c - \sqrt{c^2 - 4f'(0)}}{2}. \quad (4.15)$$

**Spectral decomposition** Throughout this paper, we will make a large use of a classical Lyapunov-Schmidt decomposition. We recall here some well-known facts of the involved spectral theory. We consider the Banach space  $X = UC_0(\mathbb{R})$ . As we are looking for a stability result, the linearised operator that will naturally appear in the moving framework  $\xi = x - ct$  is given by

$$\mathcal{L} : \begin{cases} \mathcal{D}(\mathcal{L}) \subset X & \longrightarrow X \\ v & \longmapsto c\partial_\xi v + \partial_{\xi\xi} v + f'(\varphi)v. \end{cases}$$

It is a common result (see [92] or [87] for instance) that there exists  $X_1 \simeq \mathcal{R}(\mathcal{L})$  a closed subspace of  $X$  such that

$$X = X_1 \oplus \mathcal{N}(\mathcal{L}). \quad (4.16)$$

The null space of  $\mathcal{L}$  satisfies  $\mathcal{N}(\mathcal{L}) = \mathcal{N}(\mathcal{L}^2) = \varphi'\mathbb{R}$ . As we consider a bistable nonlinearity, 0 is the first and an isolated eigenvalue in the spectrum of  $\mathcal{L}$ . We denote  $\rho_1$  the spectral gap between 0 and the second eigenvalue. The projection on  $\mathcal{N}(\mathcal{L})$  is given by

$$P\psi(\xi) = \langle e_*, \psi \rangle \varphi'(\xi) = \frac{1}{\Lambda} \left( \int_{\mathbb{R}} e^{cz} \varphi'(z) \psi(z) dz \right) \varphi'(\xi) \quad (4.17)$$

with the normalisation  $\Lambda = \int_{\mathbb{R}} e^{cx} \varphi'^2(x) dx$ . The projection on  $X_1$  is then given by

$$Q\psi = \psi - P\psi.$$

The operator  $\mathcal{L}|_{X_1}$  generates an analytic semigroup on  $X_1$  endowed with the  $L^\infty$  norm that satisfies for all  $t \geq 0$

$$\|e^{t\mathcal{L}}\| \leq Ce^{-\rho t} \quad (4.18)$$

where  $\rho$  is any positive constant smaller than the spectral gap  $\rho_1$  of  $\mathcal{L}$  and  $C$  is a positive constant.

## 4.2 One-dimensional model: proof of Theorem 4.1 .2

Our study deals with the following parabolic problem indexed by  $M$ :

$$\begin{cases} \partial_t u(t, x) - \partial_{xx} u(t, x) = f(u) (1 + r(x)) & t > 0, x \in \mathbb{R} \\ u(0, x) = \varphi(x) \\ r(x) = g(x - M). \end{cases} \quad (4.19)$$

Therefore, the perturbation term satisfies

$$\|r(x)\| \leq e^{\kappa(x-M)}, \quad r > -\varpi. \quad (4.20)$$

We have the following result.

**Proposition 4.2 .1.** *Let  $u$  be the solution of the Cauchy problem (4.19). Under the assumptions (4.3) and (4.20) on  $f$  and  $r$ , there exists  $\gamma > 0$ , there exist  $M_0 > 0, K > 0, N_0 > 0$  such that, for all  $M \geq M_0$ , for all  $t \in \left(0, \frac{M}{c} - N_0\right)$ , for all  $x \in \mathbb{R}$ ,*

$$|u(t, x) - \varphi(x - ct)| \leq Ke^{\gamma(ct-M)}.$$

#### 4.2 .1 Proof of Theorem 4.1 .2 with Proposition 4.2 .1

The argument is quite classical, see [14] for instance. For all integer  $n$ , let  $u_n$ , defined for  $t \geq -n$  and  $x \in \mathbb{R}$ , be the solution of the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u_n(t, x) - \partial_{xx} u_n(t, x) = f(u_n)(1 + g(x)) & t > -n, x \in \mathbb{R} \\ u_n(-n, x) = \varphi(x + cn). \end{cases} \quad (4.21)$$

From parabolic estimates,  $(u_n)_n$  converges up to an extraction locally uniformly to some function  $u_\infty$  defined for  $t \in \mathbb{R}, x \in \mathbb{R}$ . Then we apply Theorem 4.2 .1:

$$\forall n > \frac{M_0}{c}, \forall t < -N_0, \|u_n(t, \cdot) - \varphi(\cdot - ct)\|_{L^\infty(\mathbb{R})} \leq Ke^{\gamma ct}, \quad (4.22)$$

and the proof of Theorem 4.1 .2 is concluded by letting  $n$  tend to  $+\infty$ .

#### 4.2 .2 Proof of Proposition 4.2 .1: splitting of the problem

It is natural to consider the equation in the moving framework. Hence we make the following change of variable

$$\xi = x - ct, \quad \tilde{u}(t, \xi) = u(t, x) = \tilde{u}(t, x - ct).$$

The problem under study becomes

$$\begin{cases} \partial_t \tilde{u}(t, \xi) - c\partial_\xi \tilde{u}(t, \xi) - \partial_{\xi\xi} \tilde{u}(t, \xi) = f(\tilde{u})(1 + r(\xi + ct)) & t > 0, \xi \in \mathbb{R} \\ \tilde{u}(0, \xi) = \varphi(\xi). \end{cases} \quad (4.23)$$

Using the decomposition (4.16) we have the next lemma.

**Lemma 4.2 .2.** *Let  $u$  be the solution of the Cauchy problem (4.23). There exists  $\varepsilon_1 > 0$  such that if*

$$T_{max} = \sup\{T \geq 0, \forall t \in [0, T], \|\tilde{u}(t) - \varphi\|_\infty \leq \varepsilon_1\},$$

*there exist two functions  $\chi \in C([0, T_{max}))$  and  $v \in C([0, T_{max}), X)$  such that, for all  $t \in (0, T_{max})$ ,*

$$\tilde{u}(t, \xi) = \varphi(\xi + \chi(t)) + v(t, \xi) \quad (4.24)$$

*where  $v$  satisfies, for all  $t \in [0, T_{max})$ ,*

$$\langle e_*, v(t) \rangle = 0.$$

The proof is a consequence of the implicit function theorem. See [64] for instance for a guideline of the proof.

Inserting the ansatz (4.24) in our problem (4.23) yields the following equation:

$$\chi' \varphi'(\xi + \chi) + v_t - c \varphi'(\xi + \chi) - c v_\xi - \varphi''(\xi + \chi) - v_{\xi\xi} = f(\varphi(\xi + \chi) + v) (1 + r(\xi + ct)) \quad (4.25)$$

We make a Taylor expansion for the terms  $\chi' \varphi'(\xi + \chi)$  and  $f(\varphi(\xi + \chi) + v)$ . From now, for the sake of simplicity and when there is no possible confusion, we will sometimes omit the variables. In particular, we will use the following notations:

$$\varphi := \varphi(\xi) \quad \varphi_\chi := \varphi(\xi + \chi(t)) \quad r := r(\xi + ct). \quad (4.26)$$

Equation (4.25) becomes

$$\chi' \varphi' + v_t - c v_\xi - v_{\xi\xi} - f'(\varphi)v - r(\xi + ct)f'(\varphi)v - r(\xi + ct)f(\varphi_\chi) = R(t, \xi, \chi, v)$$

where the right term is given by

$$R(t, \xi, \chi, v) = v \chi \varphi'(b_1) f''(b_3) - \chi \chi' \varphi''(b_2) + v^2 f''(b_4) (1 + r) \quad (4.27)$$

with

$$b_1, b_2 \in (\xi - |\chi|, \xi + |\chi|), \quad b_3, b_4 \in (-\|\varphi\|_\infty - |v|, \|\varphi\|_\infty + |v|).$$

We write this term in a more convenient form. As soon as  $\chi < 1$ , there exist  $\Phi_1, \Phi_2, \Phi_3$  uniformly bounded functions of  $(t, \xi)$  with bounds depending only on  $\|f\|_{C^2}$ ,  $\|\varphi\|_{C^2}$ , and  $\|r\|_\infty$  such that

$$R(t, \xi, \chi, v) = v \chi \varphi' \Phi_1 + \chi \chi' (\varphi' + f(\varphi)) \Phi_2 + v^2 \Phi_3. \quad (4.28)$$

Hence, using the decomposition (4.16), the problem (4.23) is equivalent to the following system on  $[0, T_{max})$ :

$$\begin{cases} \chi' - \langle e_*, r(\cdot + ct) (f'(\varphi)v + f(\varphi_\chi)) \rangle = \langle e_*, R(t, \cdot, \chi, v) \rangle & t \in [0, T_{max}) \\ v_t - \mathcal{L}v - Q[r(\cdot + ct) (f'(\varphi)v + f(\varphi_\chi))] = Q[R(\cdot, \chi, v)] & t \in [0, T_{max}), \xi \in \mathbb{R} \\ \chi(0) = 0, v(0, \xi) = 0. \end{cases} \quad (4.29)$$

For all  $\alpha > 0$ , for all  $M > 0$ , let us define  $T(\alpha, M)$  by

$$T(\alpha, M) := \sup\{T > 0, \forall t < T, \max\{|\chi(t)|, |\chi'(t)|, \|v(t)\|_\infty\} \leq e^{\alpha(ct-M)}\}. \quad (4.30)$$

### 4.2 .3 Equation on $\mathcal{R}(\mathcal{L})$

**Lemma 4.2 .3.** *Let  $\chi, v$  be solution of (4.29). There exist  $\alpha, \gamma > 0$  with  $\gamma > \alpha$ , there exist  $M_1 > 0$  and  $N_1 > 0$  and a constant  $C_3 > 0$  such that for all  $M > M_1$ , for all  $t < \inf\{T(\alpha, M), \frac{M}{c} - N_1, T_{max}\}$ ,*

$$\|v(t)\|_{L^\infty(\mathbb{R})} \leq C_1 e^{\gamma(ct-M)}.$$

**Proof of Lemma 4.2 .3** The equation under study is the following:

$$\begin{cases} v_t - \mathcal{L}v - (rf'(\varphi)v - \langle e_*, rf'(\varphi)v \rangle \varphi') = Q [R + rf(\varphi_\chi)] \\ v(0, \xi) = 0. \end{cases} \quad (4.31)$$

**Symmetrisation of the problem** Let  $v$  be the solution of (4.31) on  $[0, T_{max})$ . We define for  $t \in [0, T_{max})$  and  $\xi \in \mathbb{R}$

$$w(t, \xi) := e^{\frac{c}{2}\xi} v(t, \xi). \quad (4.32)$$

The function  $w$  satisfies the following equation:

$$w_t - w_{\xi\xi} + \left( \frac{c^2}{4} - f'(\varphi) \right) w - rf'(\varphi)w = -\langle e_*, rf'(\varphi)v \rangle \varphi' e^{\frac{c}{2}\xi} + e^{\frac{c}{2}\xi} Q [R + rf(\varphi_\chi)]. \quad (4.33)$$

**Energy estimates on  $w$**  Multiply (4.33) by  $w$  and integrate by parts ; as  $v$  is orthogonal with  $\varphi'$ , it comes  $\int_{\mathbb{R}} e^{c\xi} \varphi'(\xi) v(t, \xi) d\xi = 0$  for all  $t < T_{max}$ . We get:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} w^2 + \int_{\mathbb{R}} w_{\xi}^2 + \int_{\mathbb{R}} \left( \frac{c^2}{4} - f'(\varphi) \right) w^2 - \int_{\mathbb{R}} rf'(\varphi)w^2 = \int_{\mathbb{R}} e^{c\xi} v Q [R + rf(\varphi_\chi)]. \quad (4.34)$$

Since  $v = e^{-\frac{c}{2}\xi} w$  lies in the orthogonal of  $N(\mathcal{L})$ , and knowing that 0 is an isolated eigenvalue in the spectrum of  $\mathcal{L}$ , we have for all  $t \in [0, T_{max})$ :

$$\int_{\mathbb{R}} w_{\xi}^2 + \int_{\mathbb{R}} \left( \frac{c^2}{4} - f'(\varphi) \right) w^2 \geq \rho_1 \int_{\mathbb{R}} w^2. \quad (4.35)$$

Let us set  $\zeta > 0$  such that

$$r > -\frac{\rho_1}{\|f'\|_{\infty}} + \frac{\zeta}{\|f'\|_{\infty}}. \quad (4.36)$$

The function  $f$  is  $C^1$  with  $f'(0) < 0$ . Hence, there exists  $X_1$  such that for all  $\xi > X_1$ ,  $f'(\varphi(\xi)) < 0$ . Hence we get for all  $\xi \in \mathbb{R}$ , for all  $t \leq \frac{1}{c} \left( M - X_1 + \log \left( \frac{\rho_1 - \zeta}{\|f'\|_{\infty}} \right) \right)$ ,

$$r(\xi + ct) f'(\varphi(\xi)) \leq \rho_1 - \zeta. \quad (4.37)$$

Let us set

$$N_1 = \max \left\{ 0, \frac{1}{c} \left( X_1 - \log \left( \frac{\rho_1 - \zeta}{\|f'\|_{\infty}} \right) \right) \right\}, \quad t \leq \inf \left\{ T(\alpha, M), \frac{M}{c} - N_1, T_{max} \right\}. \quad (4.38)$$

Using (4.35) and (4.37) in (4.34) and a Young inequality yield:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_2^2 + \zeta \|w\|_2^2 &\leq \int_{\mathbb{R}} \left| e^{\frac{c}{2}\xi} w(t, \xi) r(\xi + ct) f(\varphi_\chi) \right| d\xi + \int_{\mathbb{R}} \left| e^{\frac{c}{2}\xi} w(t, \xi) R(\xi, \chi, v) \right| d\xi \\ &\leq \frac{\zeta}{2} \|w\|_2^2 + \frac{2}{\zeta} \int_{\mathbb{R}} e^{c\xi} r^2 f^2(\varphi_\chi) d\xi + \int_{\mathbb{R}} \left| e^{\frac{c}{2}\xi} w R \right| d\xi. \end{aligned} \quad (4.39)$$

The second term in (4.39) satisfies:

$$\begin{aligned}
 \int_{\mathbb{R}} e^{c\xi} r^2(\xi + ct) f^2(\varphi_\chi) &\leq \|f\|_\infty^2 \int_{-\infty}^0 e^{c\xi} r^2(\xi + ct) d\xi + \int_0^{+\infty} r^2(\xi + ct) e^{c\xi} f^2(\varphi_\chi) d\xi \quad (4.40) \\
 &\leq \frac{\|f\|_\infty^2}{2\kappa + c} e^{2\kappa(ct-M)} \\
 &\quad + e^{2\kappa(ct-M)} \int_0^{M-ct} e^{(2\kappa+c)\xi} f^2(\varphi(\xi + \chi)) d\xi \\
 &\quad + \|r\|_\infty \int_{M-ct}^{+\infty} e^{c\xi} f^2(\varphi(\xi + \chi)) d\xi.
 \end{aligned}$$

We linearise the function  $f$  at 0 and use (4.14). There exists a constant  $C_4$  such that, as soon as  $|\chi(t)| \leq 1$  and  $\xi > 0$ ,

$$|f(\varphi(\xi + \chi(t)))| \leq C_4 e^{\lambda\xi}. \quad (4.41)$$

This gives

$$\begin{aligned}
 e^{2\kappa(ct-M)} \int_0^{M-ct} e^{(2\kappa+c)\xi} f^2(\varphi(\xi + \chi)) d\xi &\leq e^{2\kappa(ct-M)} \frac{C_4^2}{2\kappa + c + 2\lambda} \left( e^{(2\kappa+c+2\lambda)(M-ct)} - 1 \right) \\
 &\leq e^{2\kappa(ct-M)} \frac{2C_4^2}{-2\kappa - c - 2\lambda} \quad (4.42)
 \end{aligned}$$

and

$$\int_{M-ct}^{+\infty} e^{c\xi} f^2(\varphi(\xi + \chi)) d\xi \leq \frac{C_4^2}{-c - 2\lambda} e^{(-c-2\lambda)(ct-M)}. \quad (4.43)$$

From (4.9c),  $-c - 2\lambda > 2\kappa$ . We insert (4.42) and (4.43) in (4.40). There exists a constant  $K_1$  such that

$$\int_{\mathbb{R}} e^{c\xi} r^2(\xi + ct) f^2(\varphi_\chi) d\xi \leq K_1 e^{2\kappa(ct-M)}.$$

Let us go back to (4.39). It yields:

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 + \frac{\zeta}{2} \|w(t)\|_2^2 &\leq K_1 e^{2\kappa(ct-M)} + \int_{\mathbb{R}} |e^{\frac{c}{2}\xi} w R| d\xi \\
 &\leq K_1 e^{2\kappa(ct-M)} + \frac{\zeta}{4} \|w(t)\|_2^2 + \frac{4}{\zeta} \int_{\mathbb{R}} e^{c\xi} R^2(t, \xi) d\xi. \quad (4.44)
 \end{aligned}$$

Recall that  $R$  is defined by (4.28). From (4.14), the functions  $\xi \mapsto e^{c\xi} \varphi'^2(\xi)$  and  $\xi \mapsto e^{c\xi} f^2(\varphi(\xi))$  belong to  $L^1(\mathbb{R})$ . Hence, as soon as  $t$  satisfies (4.38), the first term in (4.44) satisfies for some constant  $C$

$$\begin{aligned}
 \int_{\mathbb{R}} e^{c\xi} R^2(t, \xi) d\xi &\leq C e^{4\alpha(ct-M)} + 3 \int_{\mathbb{R}} e^{c\xi} v^4(t, \xi) \Phi_3^2(t, \xi) d\xi \\
 &\leq C e^{4\alpha(ct-M)} + 3 \|\Phi_3\|_\infty^2 \int_{\mathbb{R}} w^2(t, \xi) v^2(t, \xi) d\xi \\
 &\leq C e^{4\alpha(ct-M)} + 3 \|\Phi_3\|_\infty^2 \|w(t)\|_2^2 e^{-2\alpha N_1}. \quad (4.45)
 \end{aligned}$$

We set:

$$\alpha = \frac{\kappa}{2}. \quad (4.46)$$

Under the assumptions of Lemma 4.2 .3, from (4.44) and (4.45), up to a greater  $N_1$  we get that the function  $w$  satisfies

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 + \frac{\zeta}{8} \|w(t)\|_2^2 \leq e^{4\alpha(ct-M)} (K_1 + C). \quad (4.47)$$

With a Gronwall argument in (4.47), we get that for  $M$  large enough, for some constant  $K_2$  which does not depend on  $M$ ,

$$\|w(t)\|_{L^2} \leq K_2 e^{2\alpha(ct-M)}, \quad 0 \leq t < \inf\{T(\alpha, M), T_{max}, \frac{M}{c} - N_1\}. \quad (4.48)$$

**Parabolic estimates on  $w$**  From the energy estimates on  $w$  we derive in this paragraph  $L^\infty$  estimates. From (4.33), the function  $w$  satisfies a parabolic equation of the form

$$w_t - w_{\xi\xi} + a(t, \xi)w = g(t, \xi) \quad (4.49)$$

with

$$\begin{cases} a(t, \xi) = \frac{c^2}{4} - f'(\varphi(\xi)) (1 + r(\xi + ct)) - v(t, \xi)\Phi_3(t, \xi) \\ g(t, \xi) = -\langle e_*, r f'(\varphi)v \rangle \varphi'(\xi) e^{\frac{c}{2}\xi} + e^{\frac{c}{2}\xi} Q [R + r f(\varphi_\chi)] - v(t, \xi)\Phi_3(t, \xi)w(t, \xi). \end{cases} \quad (4.50)$$

Under the assumptions of Lemma 4.2 .3 the function  $a$  clearly belongs to  $L^\infty([0, T] \times \mathbb{R})$  uniformly in  $T$  for all  $T < \inf\{T(\alpha, M), T_{max}, \frac{M}{c} - N_1\}$ . Let us prove that the function  $g$  also satisfies this property. The function  $a$  clearly belongs to  $L^\infty([0, +\infty[ \times \mathbb{R})$ . Let us show that the function  $g$  belongs to  $L^\infty([0, T] \times \mathbb{R})$ , uniformly in  $T$ . From (4.14) and the smoothness of  $f$  the functions  $x \mapsto e^{\frac{c}{2}\xi} \varphi'(\xi)$  and  $\xi \mapsto e^{\frac{c}{2}\xi} f(\varphi(\xi + \chi))$  are uniformly bounded provided that  $\chi$  remains bounded. There exists  $K_3 > 0$ ,

$$\left| e^{\frac{c}{2}\xi} f(\varphi(\xi + \chi)) \right| + \left| e^{\frac{c}{2}\xi} \varphi'(\xi) \right| \leq K_3, \quad \forall \xi \in \mathbb{R}.$$

The function  $g$  is given by

$$g(t, \xi) = e^{\frac{c}{2}\xi} (\varphi' \chi v \Phi_1 + \chi \chi' (\varphi' + f(\varphi)) \Phi_2 + r f(\varphi_\chi) - \langle e_*, R + r f(\varphi_\chi) + r v f'(\varphi) \rangle).$$

Hence we have

$$\begin{aligned} |g(t, \xi)| &\leq K_3 (\|\Phi_1\|_\infty |\chi v| + \|\Phi_2\|_\infty (1 + \|f'\|_\infty) |\chi \chi'| + \|r\|_\infty \|f\|_\infty) \\ &\quad + K_3 (\|e_*\| (\|R\| + \|r\|_\infty \|f\|_{C^1} (1 + |v|))). \end{aligned}$$

This inequality provides the desired  $L^\infty$  estimate provided that  $v, \chi, \chi'$  remain bounded. Now we can apply classical parabolic estimates (see [71], Thm 8.1 p.192 for instance) for (4.49) with our previous estimate (4.48). There exists a constant  $K_4$  such that, for all  $T < \inf\{T(\alpha, M), T_{max}, \frac{M}{c} - N_1\}$ ,

$$\|w\|_{L^\infty([0, T] \times \mathbb{R})} \leq K_4 e^{2\alpha(cT-M)}. \quad (4.51)$$

**From  $w$  to  $v$**  The function  $v$  is given by  $v = e^{-\frac{\sigma}{2}\xi}w$ . Let us set

$$\sigma = \frac{\kappa}{2c}, \quad T < \inf\{T(\alpha, M), T_{max}, \frac{M}{c} - N_1\}, \quad A(T) = \sigma(M - cT) > 0. \quad (4.52)$$

The function  $v$  satisfies for  $t \in [0, T]$  on the half line  $(-\infty, -A(T))$

$$\begin{cases} v_t - cv_\xi - v_{\xi\xi} - f'(\varphi)(1+r)v = \langle e_*, rf'(\varphi)v \rangle \varphi' + Q[R + rf(\varphi_\chi)] \\ v(0, \xi) = 0, \quad v(t, -\infty) = 0, \quad v(t, -A) = w(t, -A)e^{\frac{\sigma}{2}A(T)}. \end{cases} \quad (4.53)$$

We have that  $f'(\varphi(\xi)) \rightarrow f'(1) < 0$  as  $\xi \rightarrow -\infty$ . Hence, up to a larger  $N_1$ ,

$$-f'(\varphi)(1+r(\xi+ct)) \geq -\frac{1}{2}f'(1) > 0, \quad \forall t \in [0, T], \forall \xi \in (-\infty, -A(T)).$$

We recall that  $e_*$  is defined by (4.17) and is continuous, and so is  $Q$ . The function  $\varphi'$  satisfies (4.14). The perturbative term satisfies (4.20). Hence, for some constant  $K_5$ , for all  $t \in [0, T]$ , for all  $\xi \in (-\infty, -A(T))$ , considering hypotheses of Lemma 4.2 .3,

$$\begin{aligned} |\langle e_*, rf'(\varphi)v \rangle \varphi' + Q[R + rf(\varphi_\chi)]| &\leq K_5 \left( \|v\|_\infty e^{-\mu A} + \|v\|_\infty^2 + \chi^2 + |\chi\chi'| + e^{\kappa(-A+cT-M)} \right) \\ &\leq K_5 \left( e^{(\alpha+\sigma\mu)(cT-M)} + 3e^{2\alpha(cT-M)} + e^{2\alpha(1+\sigma)(cT-M)} \right). \end{aligned} \quad (4.54)$$

Indeed, we have fixed  $2\alpha = \kappa$  in (4.46). Finally, using (4.51), we have

$$|v(t, A(T))| \leq K_4 e^{(2\alpha - \frac{\sigma\kappa}{2})(cT-M)}. \quad (4.55)$$

Combining (4.54) and (4.55) in (4.53) with the parabolic maximum principle and our previous estimate (4.51) for  $\xi > -A$ , we get, for some constant  $C_3$ :

$$\|v\|_{L^\infty([0, T] \times \mathbb{R})} \leq C_3 e^{\gamma(cT-M)}, \quad \text{with } \gamma = \inf\{2\alpha - \frac{3\alpha}{2}, 2\alpha(1+\sigma), \alpha + \sigma\mu\} \quad (4.56)$$

which concludes the proof of Lemma 4.2 .3.  $\square$

#### 4.2 .4 Equation on $\mathcal{N}(\mathcal{L})$

**Lemma 4.2 .4.** *Let  $\chi, v$  be solution of (4.29). Let  $\alpha = \frac{\kappa}{2}, \gamma, M_1, N_1$  given by Lemma 4.2 .3. There exists a constant  $C_4$  such that for all  $M > M_1$ , for all  $t < \inf\{T(\alpha, M), T_{max}, \frac{M}{c} - N_1\}$ ,*

$$\max\{|\chi(t)|, |\chi'(t)|\} \leq C_4 e^{\gamma(ct-M)}.$$

*Proof.* The shift function  $\chi$  satisfies the following equation, still using the notations given by (4.26):

$$\begin{cases} \chi'(t) = \langle e_*, R \rangle + \langle e_*, r(f'(\varphi)v + f(\varphi_\chi)) \rangle \\ \chi(0) = 0. \end{cases} \quad (4.57)$$

Hence

$$|\chi'(t)| \leq |\langle e_*, R \rangle| + |\langle e_*, rf'(\varphi)v \rangle| + |\langle e_*, rf(\varphi_\chi) \rangle|. \quad (4.58)$$

The higher order term  $R$  is given by (4.28). Under the assumptions on  $T(\alpha, M)$  it yields

$$\begin{aligned} |\langle e_*, R \rangle| &\leq \|e_*\| \cdot \|R\|_\infty \\ &\leq \|e_*\| e^{2\alpha(ct-M)} (C_2(\|\Phi_1\|_\infty + \|\Phi_2\|_\infty) + \|\Phi_3\|_\infty). \end{aligned} \quad (4.59)$$

From Lemma 4.2 .3 we get

$$|\langle e_*, r f'(\varphi)v \rangle| \leq C_3 \|e_*\| \cdot \|r\|_\infty \|f'\|_\infty e^{\gamma(ct-M)}. \quad (4.60)$$

Let us deal with the last term. We have:

$$\begin{aligned} |\langle e_*, r f(\varphi_\chi) \rangle| &= \left| \int_{\mathbb{R}} e^{c\xi} \varphi'(\xi) r(\xi + ct) f(\varphi(\xi + \chi)) d\xi \right| \\ &\leq \|\varphi'\|_\infty \|f\|_\infty \int_{-\infty}^0 e^{c\xi} e^{\kappa(\xi+ct-M)} d\xi \\ &\quad + \int_0^{M-ct} \left| e^{c\xi} e^{\kappa(\xi+ct-M)} \varphi'(\xi) f(\varphi(\xi + \chi)) \right| d\xi \\ &\quad + \|r\|_\infty \int_{M-ct}^{+\infty} \left| e^{c\xi} \varphi'(\xi) f(\varphi(\xi + \chi)) \right| d\xi. \end{aligned} \quad (4.61)$$

Once again, we use (4.14) and (4.41). It yields

$$\int_0^{M-ct} \left| e^{c\xi} e^{\kappa(\xi+ct-M)} \varphi'(\xi) f(\varphi(\xi + \chi)) \right| d\xi \leq \frac{C_2 C_4}{-c - 2\lambda} e^{\kappa(ct-M)} \quad (4.62)$$

and

$$\int_{M-ct}^{+\infty} \left| e^{c\xi} \varphi'(\xi) f(\varphi(\xi + \chi)) \right| d\xi \leq \frac{C_2 C_4}{-c - 2\lambda} e^{(-c-2\lambda)(ct-M)}. \quad (4.63)$$

We have that  $\kappa \leq -c - 2\lambda$  by hypothesis (4.9c). Hence, using (4.62) and (4.63) in (4.61) gives that, for some constant  $K_5$  and under hypotheses of the lemma,

$$|\langle e_*, r f(\varphi_\chi) \rangle| \leq K_5 e^{\kappa(ct-M)}. \quad (4.64)$$

Hence, (4.59), (4.60) and (4.64) in (4.58) gives, for some constant  $K_6$ ,

$$|\chi'(t)| \leq K_6 e^{\gamma(ct-M)}. \quad (4.65)$$

It remains to integrate (4.65) to get the desired result.  $\square$

## 4.2 .5 Conclusion of the proof

Let  $\alpha = \frac{\kappa}{2}, \gamma, M_1, N_1$  given by Lemma 4.2 .3. We set

$$N_2 = \frac{1}{c(\gamma - \alpha)} \log(\max\{1, C_3, C_4\}). \quad (4.66)$$

From Lemma 4.2 .3 and Lemma 4.2 .4, we get that, for all  $M > \max\{M_1, cN_2\}$ ,

$$T(\alpha, M) \geq \min \left\{ T_{max}, \frac{M}{c} - N_2, \frac{M}{c} - N_1 \right\}.$$



Now we deal with  $T_{max}$  given by Lemma 4.2 .2. We have, for all  $t, M$  satisfying hypotheses of Lemma 4.2 .3,

$$\begin{aligned}
 \|u(t) - \varphi\|_\infty &= \|\varphi(\cdot + \chi(t)) + v(t) - \varphi\|_\infty \\
 &\leq \|\varphi(\cdot + \chi(t)) - \varphi\|_\infty + \|v(t)\|_\infty \\
 &\leq |\chi(t)| \|\varphi'\|_\infty + \|v(t)\|_\infty \\
 &\leq (\|\varphi'\|_\infty + 1) \max\{C_3, C_4\} e^{\gamma(ct-M)}.
 \end{aligned} \tag{4.67}$$

Let us set

$$N_3 = \frac{1}{\gamma c} \log \left( \frac{(\|\varphi'\|_\infty + 1) \max\{C_3, C_4\}}{\varepsilon_1} \right), \quad N_0 = \max\{N_1, N_2, N_3\}, \quad M_0 = \max\{M_1, cN_0\}$$

and, according to Lemma 4.2 .2, for all  $M > M_0$ ,  $T_{max} > \frac{M}{c} - N_0$ . The proof is then concluded by (4.67).  $\square$

### 4.3 Cylinder-like domains: proof of Theorem 4.1 .1

The proof of Theorem 4.1 .1 relies on the same kind of arguments that are used in the previous section. Hence we will mainly focus on the difficulties coming from the inhomogeneity of the domain. The point is again a stability result. We consider the following problem, indexed by  $M$  :

$$\begin{cases} \partial_t u - \Delta u = f(u) & t > 0, (x, y) \in \Omega \\ \partial_\nu y = 0 & t > 0, (x, y) \in \partial\Omega \\ u(0, x, y) = \varphi(x + M) := \tau_M \varphi(x). \end{cases} \tag{4.68}$$

**Proposition 4.3 .1.** *Under assumptions (4.3) on  $f$  and (4.2) and (4.9) on the domain  $\Omega$ , there exist  $\gamma > 0$  and three constants  $M_0, K, N_0$  such that for all  $M > M_0$ , for all  $t \in \left(0, \frac{M}{c} - N_0\right)$ , for all  $(x, y) \in \Omega$ , the solution  $u$  of (4.68) satisfies*

$$|u(t, x, y) - \tau_M \varphi(x - ct)| \leq K e^{\gamma(ct-M)}. \tag{4.69}$$

Using the same compactness argument as in the previous section with the regularity of the domain, this proposition concludes the proof of Theorem 4.1 .1.

Throughout this section, we will denote by  $C$  a generic positive constant, which may differ from place to place even in the same chain of inequalities. Moreover, we will use classical notations concerning Hölder spaces:

- For any  $\delta \in (0, 1)$  and integer  $k$ , if  $\Omega_0$  is a spatial domain included in  $\mathbb{R}$  or  $\mathbb{R}^2$ ,  $C^{k, \delta}(\Omega_0) = C^{k+\delta}$  is the space of functions whose derivatives up to order  $k$  lie in  $C^\delta(\Omega_0)$ , the space of  $\delta$ -Hölder functions.
- For parabolic problems with functions depending on the time  $t$  and the space, the space  $C^{k, \delta}([T^-, T^+] \times \Omega_0)$  is the space of functions that are  $C^{\frac{k}{2} + \frac{\delta}{2}}$  in time and  $C^{k, \delta}$  in space.

### 4.3 .1 A preliminary result: travelling supersolutions

To control the tail of our solution when the domain is heterogeneous, we will use the following lemma.

**Lemma 4.3 .2.** *There exist  $\alpha_1 > 0$  and a positive function  $\psi$  such that for all  $\alpha \leq \alpha_1$ , for all  $\varepsilon \leq 1$ ,*

$$\bar{u} : (t, x, y) \mapsto \varepsilon \psi(y) e^{-\alpha(x-ct)}$$

is a supersolution of the problem (4.1) on the set  $\Omega \cap \{x > ct\}$ .

*Proof.* Let us recall that a supersolution in a domain with boundary such as (4.1) is a function  $\bar{u}$  which satisfies the following inequality

$$\begin{cases} \partial_t \bar{u} - \Delta \bar{u} - f(\bar{u}) \geq 0 & t \geq 0, (x, y) \in \Omega \\ \partial_\nu \bar{u} \geq 0 & t \geq 0, (x, y) \in \partial\Omega \end{cases} \quad (4.70)$$

where  $\vec{\nu}$  is the outward unit normal to  $\Omega$  in  $\partial\Omega$ .

Let us fix  $\delta > 0$  small enough. For  $\alpha \in \left(0, \frac{c + \sqrt{c^2 - 4(f'(0) + \delta)}}{2}\right)$  we define  $\omega(\alpha) = \sqrt{\alpha c - \alpha^2 - f'(0) - \delta}$ . Let us choose  $\alpha_1 > 0$  small enough such that

$$\omega(\alpha_1) \sinh(\omega(\alpha_1)l) \geq L\alpha_1 \cosh(\omega(\alpha_1)L)$$

where  $l$  and  $L$  are given in hypothesis (4.9). Finally, as  $f$  is  $C^1$  in 0 there exists  $\varepsilon_1$  such that for all  $s \in [0, \varepsilon_1 \cosh(\omega(\alpha_1)L)]$ ,  $f(s) \leq (f'(0) + \delta)s$ , and the function

$$\bar{u} : (t, x, y) \mapsto \varepsilon_1 \cosh(\omega(\alpha_1)y) e^{-\alpha_1(x-ct)}$$

is a supersolution in the desired domain. The result follows easily.  $\square$

### 4.3 .2 From $\Omega$ to a straight cylinder

For the sake of simplicity and without loss of generality we state  $Y^\infty = 1$ . In order to make the same kind of decomposition as for the one dimensional model, we modify the domain  $\Omega$  onto a straight cylinder. Let us consider the following change of variable:

$$\vec{\Phi} : \begin{cases} \Omega & \longrightarrow \Omega^\infty = \mathbb{R} \times ]-1, 1[ \\ (x, y) & \longmapsto \left(x, y \frac{2}{Y^+(x) - Y^-(x)} + \frac{Y^+(x) + Y^-(x)}{Y^-(x) - Y^+(x)}\right). \end{cases} \quad (4.71)$$

It provides a simple diffeomorphism between  $\Omega$  and the limit cylinder  $\Omega^\infty = \mathbb{R} \times ]-1, 1[$ . With an abuse of notation, we will also write  $\Phi(x, y)$  the second component of the vector  $\vec{\Phi}(x, y)$  and do the same with  $\vec{\Phi}^{-1}$ . We write  $\Phi(x, y) = z$  and  $y = \phi^{-1}(x, z)$ . The function  $\Phi$  inherits the regularity of  $Y^-, Y^+$  and from (4.9a) satisfies for some positive constant  $C$

$$\|\Phi_x\|_{C^2(-\infty, x)} + \|\Phi_y - 1\|_{C^2(-\infty, x)} \leq C \min\{e^{\kappa x}, 1\}. \quad (4.72)$$

Let us now combine the above change of variable  $z = \Phi(x, y)$  and the transformation to a moving coordinate framework  $\xi = x - ct$  and set

$$\tilde{u}(t, \xi, z) = u(t, x, y) = \tilde{u}(t, x - ct, \Phi(x, y)) = \tilde{u}(t, \xi, \Phi(\xi + ct, y))$$

where  $u$  is the solution of (4.68). The equation for  $\tilde{u}$  in the  $(\xi, z)$  coordinates is given by:

$$\begin{cases} \tilde{u}_t - c\tilde{u}_\xi - \Delta\tilde{u} - f(\tilde{u}) - \tilde{u}_z\Delta\Phi - 2\tilde{u}_{\xi z}\Phi_x + \tilde{u}_{zz}(1 - \Phi_x^2 - \Phi_y^2) = 0 & t > 0, (\xi, z) \in \Omega^\infty \\ \tilde{u}_z\left(\Phi_y + \frac{\Phi_x^2}{\Phi_y}\right) + \tilde{u}_\xi\frac{\Phi_x}{\Phi_y} = 0 & t > 0, (\xi, z) \in \partial\Omega^\infty \\ \tilde{u}(0, \xi, z) = \tau_M\varphi(\xi) \end{cases} \quad (4.73)$$

where  $\Phi$  and all its derivatives have to be considered in the  $(x, y)$ -variables:

$$\Phi_{x,y} = \Phi_{x,y}(x, y) = \Phi_{x,y}(\xi + ct, \Phi^{-1}(\xi + ct, z)). \quad (4.74)$$

As all the coefficients in (4.73) are bounded we can apply the same decomposition as that given by Lemma 4.2 .2:

**Lemma 4.3 .3.** *Let  $\tilde{u}$  be the solution of the Cauchy problem (4.73). There exists  $\varepsilon_1$  such that, if*

$$T_{max} = \sup \{T > 0, |\tilde{u}(t, \xi, z) - \tau_M\varphi(\xi)| \leq \varepsilon_1, \forall t < T, (\xi, z) \in \Omega^\infty\},$$

there exist two functions  $\chi \in C^{2,\delta}([0, T_{max}] \times (-1, 1))$  and  $v \in C^{2,\delta}([0, T_{max}] \times \Omega^\infty)$  such that for all  $t < T_{max}$ ,

$$\tilde{u}(t, \xi, z) = \tau_M\varphi(\xi + \chi(t, z)) + v(t, \xi, z) \quad (4.75)$$

where  $v$  satisfies  $\langle \tau_M e_*, v(t, \cdot, z) \rangle = 0$  for all  $t < T_{max}$  and  $z \in (-1, 1)$ .

*Remark 4.3 .1.* The considered decomposition is similar to (4.16) with a translation of  $M$ . In order not to overburden the notations, **we will omit to mention this translation** in the operators  $\mathcal{L}$ ,  $P$  and  $e_*$  and in the functions  $\varphi$  and  $\varphi'$  throughout the remainder of this paper. We write  $v \in \mathcal{R}(\mathcal{L})$ .

*Remark 4.3 .2.* Considering the regularity of the solution, operators  $\mathcal{L}$  and  $\partial_z$  commute, and so do  $e_*$  and  $\partial_z$ . Hence,  $\xi \mapsto \partial_z v(t, \xi, z)$  also belongs to  $\mathcal{R}(\mathcal{L})$ .

### 4.3 .3 splitting of the problem

We insert the ansatz provided by Lemma 4.3 .3 in (4.73). Using a Taylor decomposition and that  $\varphi$  is a solution of (4.5) it yields the following equation in  $(v, \chi)$  :

$$\begin{cases} \chi_t\varphi'(\xi + \chi) - \chi_{zz}\varphi'(\xi + \chi) + v_t - c\partial_\xi v - \Delta v - v f'(\varphi(\xi + \chi)) - R_1 = 0, & (\xi, z) \in \Omega^\infty \\ (\chi_z\varphi'(\xi + \chi) + v_z)\left(\Phi_y + \frac{\Phi_x^2}{\Phi_y}\right) + \tilde{u}_\xi\frac{\Phi_x}{\Phi_y} = 0, & (\xi, z) \in \partial\Omega^\infty \\ \tilde{u}(0, \xi, z) = \varphi(\xi) \end{cases} \quad (4.76)$$

where  $R_1$  is given by

$$R_1 = \chi_z^2\varphi'' + v^2 \int_0^1 (1-s)f''(\varphi + sv)ds + \tilde{u}_z\Delta\Phi + 2\tilde{u}_{\xi z}\Phi_x - \tilde{u}_{zz}(1 - \Phi_x^2 - \Phi_y^2) \quad (4.77)$$

and  $\varphi = \varphi(\xi + \chi)$ . We have to get rid of this translation in order to properly project this equation on  $\mathcal{R}(\mathcal{L})$  and  $\mathcal{N}(\mathcal{L})$ . Using once again a Taylor expansion, we have

$$\begin{cases} \chi_t \varphi' - \chi_{zz} \varphi' + v_t - \mathcal{L}v - v_{zz} - R_2 = 0 & (\xi, z) \in \Omega^\infty \\ \chi_z \varphi' + v_z = R_3 & (\xi, z) \in \partial\Omega^\infty \\ \tilde{u}(0, \xi, z) = \varphi(\xi) \end{cases} \quad (4.78)$$

where

$$R_2 = R_1 + (\chi\chi_{zz} - \chi\chi_t) \int_0^1 \varphi''(\xi + s\chi) ds + v\chi \int_0^1 \varphi'(\xi + s\chi) ds f''(b) \quad (4.79a)$$

$$R_3 = -\tilde{u}_\xi \frac{\Phi_x}{\Phi_x^2 + \Phi_y^2} - \chi\chi_z \int_0^1 \varphi''(\xi + s\chi) ds. \quad (4.79b)$$

Now, as for the one dimensional case, we set for some  $\alpha > 0$ , for all  $M > 0$ ,

$$T(\alpha, M) := \sup \left\{ T > 0, \forall t < T, \|\chi\|_{C^{2,\delta}([0,t] \times (-1,1))} + \|v\|_{C^{2,\delta}([0,t] \times \Omega^\infty)} \leq e^{\alpha(ct-M)} \right\}. \quad (4.80)$$

### 4.3 .4 Estimates on $R_1, R_2, R_3$

Let us define

$$\alpha := \frac{1}{4} \min\{\alpha_1, \kappa\}, \quad \varepsilon_2 = \inf \psi \quad (4.81)$$

where  $\kappa$  is set in (4.72) and  $\alpha_1, \psi$  are given in Lemma 4.3 .2.

**Lemma 4.3 .4.** *Let  $u$  be the solution of (4.68) and equivalently  $(v, \chi)$  the solution of (4.76). Then, there exist  $M_1 > 0$ ,  $N_1 > 0$  and some positive constant  $C$  such that for all  $M \geq M_1$ , for all  $T < \inf\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}$  we have the following estimates:*

$$\|R_2\|_{C^{0,\delta}([0,T] \times \Omega^\infty)} + \|R_3\|_{C^{1,\delta}([0,T] \times \Omega^\infty)} \leq C e^{2\alpha(ct-M)}. \quad (4.82)$$

We need two steps to prove the above lemma. In the next paragraph we use lemma 4.3 .2 to control the solution  $u$  for large  $x$ . Then we use these estimates to globally control  $R_{1,2,3}$ .

**Estimate of the tail of  $u$  thanks to Lemma 4.3 .2** Let us recall that as 0 and 1 are respectively sub and supersolution for (4.68),  $0 < u < 1$  for all  $t > 0$ ,  $(x, y) \in \Omega$ . From (4.14) we have for any  $K_0 > 0$

$$\begin{aligned} u(t = 0, x, y) &= \varphi(x + M) \\ &\leq C_2 e^{\lambda(x+M-K_0)} e^{\lambda K_0}. \end{aligned}$$

Let us choose  $K_0$  such that  $C_2 e^{\lambda K_0} < \frac{\varepsilon_2}{2}$  and  $M_1 > K_0$ . Knowing that  $0 < \alpha_1 < -\lambda$ , we can apply Lemma 4.3 .2. It yields

$$\begin{aligned} 0 < u(t, x, y) &< \|\psi\|_\infty e^{-\alpha_1(x-ct+M-K_0)}, \\ \forall x > ct + K_0 - M, \forall t &\in [0, \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}] \end{aligned}$$

where  $N_1 = \frac{1}{\alpha c} \log\left(\frac{2}{\varepsilon_2}\right) + 1$ . Indeed, this ensures that  $|v| < \frac{\varepsilon_2}{2}$ , hence the inequality  $u(t, ct - M + K_0, y) < \varepsilon_2 \leq \psi(y)$  is valid on the desired interval in time and  $(t, x, y) \mapsto \psi(y)e^{-\alpha_1(x-ct+M-K_0)}$  is strictly above  $u$  on the considered domain. This provides a  $L^\infty$  estimate for  $u$ . With the regularity of  $f$ ,  $u$  satisfies a parabolic equation with Neumann boundary condition and a second term satisfying the same  $L^\infty$  estimate. Hence, with classical parabolic estimates (see [71], theorem 10.1 p.204 for the Hölder regularity, then theorem 10.1 p.351 for the  $C^{2,\delta}$  regularity) we have for some constant  $C$  and some  $\delta \in (0, 1)$

$$\begin{aligned} \|u\|_{C^{2,\delta}([0,T] \times \Omega \cap \{x > X\})} &\leq C e^{-\alpha_1(X-ct+M-K_0)}, \\ \forall T < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}, \forall X > cT + K_0 - M + 1. \end{aligned}$$

Now,  $\vec{\Phi}$  is a  $C^3$ -diffeomorphism, hence a diffeomorphism with regularity  $C^{2,\delta}$  for all  $\delta \in (0, 1)$ . So the above estimate works on  $\tilde{u}$ . In the moving framework, it yields for some constant  $C$

$$\begin{aligned} \|\tilde{u}\|_{C^{2,\delta}([0,T] \times \Omega^\infty \cap \{\xi > \tilde{X}\})} &\leq C e^{-\alpha_1(\tilde{X}+M-K_0)}, \\ \forall T < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}, \forall \tilde{X} > K_0 - M + 1. \end{aligned} \quad (4.83)$$

**Global control of  $R_1, R_2, R_3$**  In this paragraph, we combine the estimate (4.83) with the assumption (4.72) to conclude the proof of lemma 4.3 .4. We give details only for  $R_3$ , the other being similar.

From our hypotheses (4.9) on  $\Omega$  the quantity  $\Phi_x^2 + \Phi_y^2$  is bounded by above and by below by positive constants. Similarly, as  $0 < u < 1$  for all  $t > 0$ ,  $(x, y) \in \Omega$ , parabolic estimates provide a uniform bound for  $|\tilde{u}_\xi|$ . Let us recall that, with (4.74), the derivatives of  $\Phi$  can be considered as functions of the  $(\xi, z)$ -variables. Hypothesis (4.72) gives, translated in the moving framework coordinates

$$\|\Phi_x(\cdot + ct, \cdot)\|_{C^2(\Omega \cap \{\xi < \tilde{X}\})} + \|\Phi_y(\cdot + ct, \cdot) - 1\|_{C^2(\Omega \cap \{\xi < \tilde{X}\})} \leq C e^{\kappa(\tilde{X}+ct)}. \quad (4.84)$$

We update the value of  $N_1$  and  $M_1$  by

$$N_1 \leftarrow \max\left(N_1, 2\frac{K_0 + 1}{c}\right), \quad M_1 \leftarrow \max(M_1, cN_1).$$

Thus, for all  $T < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}$ , we set in (4.83) and (4.84)

$$\tilde{X} = -M + \frac{1}{2}(M - cT).$$

It yields for some positive constant  $C$

$$\left\| \tilde{u}_\xi \frac{\Phi_x}{\Phi_x^2 + \Phi_y^2} \right\|_{C^{1,\delta}([0,T] \times \Omega^\infty)} \leq C \left( e^{-\frac{\alpha_1}{2}(M-cT)} + e^{-\frac{\kappa}{2}(M-cT)} \right).$$

We recall that  $\alpha$  is given by (4.81). Using the definition of  $T(\alpha, M)$  given by (4.80) and the regularity of  $\varphi$ , up to a smaller  $\delta$ , the proof of Lemma 4.3 .4 is completed for  $R_3$ . The other are similar.  $\square$

### 4.3 .5 Equation on $\mathcal{R}(\mathcal{L})$

We project the system (4.78) on  $\mathcal{R}(\mathcal{L})$ . Using Remark 4.3 .2 it provides the following system for  $v$

$$\begin{cases} \partial_t v - \mathcal{L}v - \partial_{zz}v = Q[R_2] & t > 0, (\xi, z) \in \Omega^\infty \\ \partial_z v = Q[R_3] & t > 0, (\xi, z) \in \partial\Omega^\infty = \mathbb{R} \times \{-1, 1\} \\ v(0, \xi, z) = 0. \end{cases} \quad (4.85)$$

We state a lemma similar to Lemma 4.2 .3 given in the previous section.

**Lemma 4.3 .5.** *Let  $v$  be the solution of (4.85). There exists a positive constant  $C$  such that for all  $M > M_1$ , for all  $T < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}$ ,*

$$\|v\|_{C^{2,\delta}([0,T] \times \Omega^\infty)} \leq C e^{2\alpha(cT-M)}$$

where  $\alpha, M_1, N_1$  are given by Lemma 4.3 .4.

Once again, the proof lies on parabolic estimates. The main point is to get the estimate in the  $L^\infty$ -norm. Then, using the continuity of  $Q$  with respect to the Hölder norms, the result follows from Theorem 10.1 p.351 in [71].

In order to easily control  $|v|$  we split it on  $v = v_1 + v_2$  where  $v_1$  and  $v_2$  are solutions of the following systems:

$$\begin{cases} \partial_t v_1 - \mathcal{L}v_1 - \partial_{zz}v_1 = 0 & t > 0, (\xi, z) \in \Omega^\infty \\ \partial_z v_1 = Q[R_3] & t > 0, (\xi, z) \in \partial\Omega^\infty = \mathbb{R} \times \{-1, 1\} \\ v_1(0, \xi, z) = 0 \end{cases} \quad (4.86)$$

and

$$\begin{cases} \partial_t v_2 - \mathcal{L}v_2 - \partial_{zz}v_2 = Q[R_2] & t > 0, (\xi, z) \in \Omega^\infty \\ \partial_z v_2 = 0 & t > 0, (\xi, z) \in \partial\Omega^\infty = \mathbb{R} \times \{-1, 1\} \\ v_2(0, \xi, z) = 0. \end{cases} \quad (4.87)$$

They are quite similar and we will not give all the details. The point is to see that it is a one-dimensional heat equation in the  $z$ -direction, and use the non-increasing property (4.18) on the semigroup generated by  $\mathcal{L}$  in the  $\xi$ -direction. Thanks to remark 4.3 .2, the operators  $e^{t\mathcal{L}}$  and  $e^{t\partial_{zz}}$  commute.

**$L^\infty$  control on  $v_1$**  As already said we can see that (4.86) is a heat equation in the  $z$ -direction with non-homogeneous Neumann boundary conditions. The explicit resolution is classical, see [29] for instance. For  $t \geq 0$ ,  $\xi \in \mathbb{R}$ ,  $z \in \mathbb{R}$  we set

$$\begin{aligned} g^-(t, \xi) &= Q[R_3(t, \xi, -1)], & g^+(t, \xi) &= Q[R_3(t, \xi, 1)] \\ G^-(t, \xi, z) &= -2g^-(t, \xi) \sum_{k=-\infty}^{+\infty} \delta_{z=4k-1}, & G^+(t, \xi, z) &= 2g^+(t, \xi) \sum_{k=-\infty}^{+\infty} \delta_{z=4k+1}. \end{aligned} \quad (4.88)$$

Duhamel's formula gives:

$$v_1(t) = \int_0^1 e^{(t-s)\mathcal{L}} e^{(t-s)\partial_{zz}} \left( G^-(s) + G^+(s) \right) ds \quad (4.89)$$

where

$$e^{(t-s)\partial_{zz}}G^\pm(s)(\xi, \cdot) = \frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{|\cdot|^2}{4(t-s)}} * G^\pm(s, \xi, \cdot). \quad (4.90)$$

Using Remark 4.3 .2 and the definition of  $G^\pm$  it is clear that quantities (4.90) still lie in  $\mathcal{R}(\mathcal{L})$ . Hence we just need to control quantities (4.90). We focus on  $G^+$ , the control of  $G^-$  being similar. From Lemma 4.3 .4 we have

$$|g^+(s)| \leq Ce^{2\alpha(cs-M)}.$$

Thus, a simple computation gives for some constant  $C$ :

$$\begin{aligned} e^{(t-s)\partial_{zz}}G^\pm(s)(\xi, z) &= \frac{2g^+(s, \xi)}{\sqrt{4\pi(t-s)}} \sum_{k=-\infty}^{+\infty} e^{-\frac{|z-4k-1|^2}{4(t-s)}} \\ |e^{(t-s)\partial_{zz}}G^\pm(s)(\xi, z)| &\leq Ce^{2\alpha(cs-M)} \left( \frac{1}{\sqrt{t-s}} + 1 \right). \end{aligned} \quad (4.91)$$

We insert (4.91) in the Duhamel's formula (4.89) and use (4.18). It gives for all  $M > M_1$  and for all  $t < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}$ ,

$$\begin{aligned} \|v_1(t)\|_{L^\infty(\Omega^\infty)} &\leq C \int_0^t e^{-\rho(t-s)} e^{2\alpha(cs-M)} \left( \frac{1}{\sqrt{t-s}} + 1 \right) ds \\ &\leq Ce^{2\alpha(ct-M)}. \end{aligned} \quad (4.92)$$

**$L^\infty$  control on  $v_2$**  We proceed in the same fashion. For all  $t > 0$ ,  $\xi \in \mathbb{R}$ ,  $z \in \mathbb{R}$  we set

$$\begin{cases} G(t, \xi, z) = Q[R_2(t, \xi, z)] \text{ if } z \in (-1, 1) \\ G(t, \xi, \cdot) \text{ is 2-periodic.} \end{cases} \quad (4.93)$$

Then, from Lemma 4.3 .4 and the continuity of  $Q$ ,  $G$  satisfies uniformly in  $(\xi, z)$ , for all  $M > M_1$  and for all  $t < \min\{T_{max}, T(\alpha, M), \frac{M}{c} - N_1\}$ ,

$$|G(t, \xi, z)| \leq Ce^{2\alpha(ct-M)}.$$

The function  $v_2$  is given by

$$v_2(t) = \int_0^t e^{(t-s)\mathcal{L}} e^{(t-s)\partial_{zz}} G(s) ds$$

and

$$\begin{aligned} |e^{(t-s)\partial_{zz}}G(s, \xi, z)| &= \left| \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{|z-\eta|^2}{4(t-s)}} G(s, \xi, \eta) d\eta \right| \\ &\leq Ce^{2\alpha(cs-M)}. \end{aligned}$$

Hence it gives for  $v_2$

$$\begin{aligned} \|v_2(t)\|_{L^\infty(\Omega^\infty)} &\leq C \int_0^t e^{-\rho(t-s)} e^{2\alpha(cs-M)} ds \\ &\leq Ce^{2\alpha(ct-M)}. \end{aligned}$$

The above control and (4.92) give the same for  $v$  which, combined with parabolic estimates, concludes the proof of Lemma 4.3 .5.  $\square$

### 4.3 .6 Equation on $\mathcal{N}(\mathcal{L})$

We project the system (4.78) on  $\mathcal{N}(\mathcal{L})$ . It yields the following system for  $\chi$  :

$$\begin{cases} \partial_t \chi - \partial_{zz} \chi = \langle e_*, R_2 \rangle, & t > 0, z \in (-1, 1) \\ \partial_z \chi = \langle e_*, R_3 \rangle, & t > 0, z \in \{-1, 1\} \\ \chi(0, z) = 0. \end{cases} \quad (4.94)$$

We have the following lemma:

**Lemma 4.3 .6.** *Let  $\chi$  be the solution of (4.94). There exists a positive constant  $C$  such that for all  $M > M_1$ , for all  $T < \min \left\{ T_{max}, T(\alpha, M), \frac{M}{c} - N_1 \right\}$ ,*

$$\|\chi\|_{C^{2,s}([0,T] \times (-1,1))} \leq C e^{2\alpha(ct-M)}$$

where  $\alpha, M_1, N_1$  are given by Lemma 4.3 .4.

The proof is completely similar to the proof of Lemma 4.3 .5 and we skip some details. We just need a  $L^\infty$  estimate for  $\chi$  and then parabolic estimates with Lemma 4.3 .4 give the result.

The form  $e_*$  is continuous with respect to the Hölder norms. Thus, from Lemma 4.3 .4 there exist  $g, g^+, g^-$  such that for all  $t < \min \left\{ T_{max}, T(\alpha, M), \frac{M}{c} - N_1 \right\}$ ,

$$\begin{aligned} \langle e_*, R_2(t, \cdot, z) \rangle &= g(t, z), & \langle e_*, R_3(t, \cdot, \pm 1) \rangle &= g^\pm(t), \\ |g^+(t)| + |g^-(t)| + \|g(t)\|_{L^\infty(-1,1)} &\leq C e^{2\alpha(ct-M)}. \end{aligned}$$

We then define  $G^\pm(t, z)$  and  $G(t, z)$  in the same way as that of (4.88) and (4.93) in the previous subsection. Thereafter we have an explicit formula for  $\chi$  :

$$\begin{aligned} \chi(t) &= \int_0^t e^{(t-s)\partial_{zz}} \left[ G^+(s) + G^-(s) + G(s) \right] ds \\ &= \int_0^t \frac{2}{\sqrt{4\pi(t-s)}} \sum_{k=-\infty}^{+\infty} \left( g^+(s) e^{-\frac{|z-4k-1|^2}{4(t-s)}} - g^-(s) e^{-\frac{|z-4k+1|^2}{4(t-s)}} \right) ds \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{-\frac{|z-\eta|^2}{4(t-s)}} G(s, \eta) d\eta ds. \end{aligned}$$

So

$$\begin{aligned} |\chi(t, z)| &\leq C \int_0^t e^{2\alpha(cs-M)} \left( 1 + \frac{1}{\sqrt{t-s}} \right) ds \\ \|\chi(t)\|_{L^\infty(-1,1)} &\leq C e^{2\alpha(ct-M)} \end{aligned}$$

which concludes the proof of Lemma 4.3 .6.  $\square$

### 4.3 .7 Conclusion of the proof of Proposition 4.3 .1

We conclude in the same way as for the proof of Proposition 4.2 .1. Thanks to the two previous Lemmas, it is easy to construct  $M_0 \geq M_1, N_0 \geq N_1$  such that for all  $M \geq M_0, T_{max} = T(\alpha, M) = \frac{M}{c} - N_0$ . Stating  $\gamma = \alpha$ , the proof is finished.  $\square$



## Remarks and questions

- In both theorem 4.1 .2 and theorem 4.1 .1, we do not prove the uniqueness of such an entire solution. However, if one is able to prove the monotonicity in time of the solution, or that these solutions are transition fronts results from [14] or [12] may provide the uniqueness.
- In both theorems we ask for an exponential decay of the perturbation, but it is arbitrary. Hence a natural guess is that it is not optimal and some weaker convergence rate may be sufficient. But the proof also suggests that too slow a convergence may prevent the existence of such entire solutions.
- Though theorem 4.1 .1 is given in a two-dimensional setting, there is no real barrier that prevent from extending it to a multi-dimensional setting. The two key arguments are the existence of a diffeomorphism that sends  $\Omega$  to a straight cylinder and lemma 4.3 .2 which gives a global control of the solution for large  $x$ . This lemma can be extended in a higher dimensional setting provided that  $\Omega$  is star-shaped with respect to a thin contained cylinder.
- Some of the properties proved in [5] are still valid for these domains. The invasion properties, theorems 1.5, 1.7, 1.8, 1.9 apply in our context. The blocking property, theorem 1.6, is more intricate. In their paper, the proof rely on the existence of a blocking stationary solution. To extend this result to our context it would be sufficient to get a decay estimate of this stationary solution. A decay weaker than the one of the bistable wave would give the result.

## Part III

# Conclusion and perspectives



We present some natural questions and other long term projects risen by the research presented in this manuscript.

## Road-field system with nonlocal exchanges

**Including reaction on the road: persistence of the differences ?** Further qualitative properties have been studied for the initial model (1) in a second paper on the subject by Berestycki, Roquejoffre and Rossi[21]. It could be interesting to see how these properties extend to the nonlocal model. In particular, if we add a strong enough reaction term on the road, I suppose that the spreading speed is maximal for local exchanges. The intuition is given by a work of Liang, Lin and Matano in [74] in the one dimensional heterogeneous case (12): the spreading speed is maximal with a reaction term given by a Dirac comb. It could be interesting to see if it is true, and if there is a critical reaction term for this.

**Transition between classical and enhanced spreading for long range exchanges** In chapter 3 is given a new threshold in the limit of long range exchange terms for the line to influence the propagations. Hence, for  $D$  large enough, even in the case of very extended exchanges, there exists an asymptotic speed of spreading in the direction of the road uniformly greater than the classical spreading speed. A natural question is the study of the transition between the local dynamics and the asymptotic one for such exchanges.

**More general kernels** The integral terms involved throughout part I have no dependence in the direction of the road. For the modelling aspect, it is important to release this assumption and consider  $x$ -dependent kernels, or even a two dimensional kernel. A first study with periodic boundary conditions has been done in [49]. It may also lead to other aspects, like the crossing of two roads.

**Supremum for the spreading speed: a general result ?** Even if we are not able to give a precise general result for exchange functions that maximise the spreading speed in chapter 1, we can exhibit bounds on it that provides existence of maximizing sequences. Hence, a different approach would be to study the convergence properties of such sequences.

## Generalised transition waves for asymptotically cylindrical domains

**Properties of the entire solutions** For both the one dimensional model and the problem of asymptotic cylinders given in 4 holds the question of the properties of the entire solutions. Intuition suggests that they are generalised front propagations, depending on the geometry of the domain. It remains to be proved. Other properties may be investigated: convergence as  $t$  goes to  $+\infty$ , or transition between two travelling waves when heterogeneities go to 0 as  $x$  goes to infinity.

**Limiting convergence rate for the domain** The proofs given in chapter 4 suggest that there might exist no such entire solution if the convergence of the domain to a cylinder is too slow. A natural question is the sharpness of the assumptions on the domain to expect existence of propagation fronts, as well as what happens when the convergence to the straight cylinder occurs like, say,  $\frac{1}{|x|^\alpha}$ .

**Time delay coming from the variation of the cross section** In the case of a domain passing from a cylinder to a thinner one it is natural to expect the front to pass from the classical travelling wave to a translated one (see [5]), even if it has not been proved yet. The question of this translation, that is the time delay induced by the geometry, remains an open and interesting question.

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# Deux exemples de propagation de fronts de réaction-diffusion en milieu hétérogène

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**Résumé :** L'objet de cette thèse est l'étude de deux exemples de propagation pour des équations de réaction-diffusion hétérogènes.

Le but de la première partie est de déterminer quels sont les effets d'échanges non locaux entre une ligne de diffusion rapide et un environnement bidimensionnel dans lequel a lieu un phénomène de réaction-diffusion de type KPP usuel. Dans le premier chapitre nous étudions comment ce couplage non local entre la ligne et le plan accélère la propagation dans la direction de la ligne ; on détermine aussi comment différentes fonctions d'échanges maximisent ou non la vitesse d'invasion. Le deuxième chapitre est consacré à la limite singulière de termes d'échanges qui convergent vers des masses de Dirac. On montre alors que la dynamique converge avec une certaine uniformité. Dans le troisième chapitre nous étudions la limite d'échanges étalés à l'infini. Ils permettent de donner un infimum sur la vitesse de propagation pour ce type de modèle qui peut cependant être supérieure à la vitesse KPP usuelle.

La seconde partie de cette thèse est consacrée à l'étude de solutions entières (ou éternelles) pour des équations bistables hétérogènes. On considère un domaine bidimensionnel infini dans une direction, borné dans l'autre, qui converge vers un cylindre quand  $x$  tend vers  $-\infty$ . On montre alors l'existence d'une solution entière dans un tel domaine qui est égal à l'onde bistable en  $t = -\infty$ . Cela nous conduit à étudier un modèle unidimensionnel avec un terme de réaction hétérogène, pour lequel on obtient le même résultat.

**Mots-clés :** réaction-diffusion, échanges non locaux, fronts de transition, propagation, vitesse.

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**Abstract :** The aim of this thesis is to study two examples of propagation phenomena in heterogeneous reaction-diffusion equations.

The purpose of the first part is to understand the effect of nonlocal exchanges between a line of fast diffusion and a two dimensional environment in which reaction-diffusion of KPP type occurs. The initial model was introduced in 2013 by Berestycki, Roquejoffre, and Rossi. In the first chapter we investigate how the nonlocal coupling between the line and the plane enhances the spreading in the direction of the line; we also investigate how different exchange functions may maximize or not the spreading speed. The second chapter is concerned with the singular limit of nonlocal exchanges that tend to Dirac masses. We show the convergence of the dynamics in a rather strong sense. In the third chapter we study the limit of long range exchanges with constant mass. It gives an infimum for the asymptotic speed of spreading for these models that still could be bigger than the usual KPP spreading speed.

The second part of this thesis is concerned with entire solutions for heterogeneous bistable equations. We consider a two dimensional domain infinite in one direction, bounded in the other, that converges to a cylinder as  $x$  goes to  $-\infty$ . We prove the existence of an entire solution in such a domain which is the bistable wave for  $t = -\infty$ . It also lead us to investigate a one dimensional model with a non-homogeneous reaction term, for which we prove the same property.

**Keywords :** reaction-diffusion, nonlocal exchanges, transition fronts, propagation, speed.